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Nicolas Schaeffer

Sous la direction de :

Aurélien Deya et Laurent Thomann

Etude d'équations aux dérivées partielles dirigées par une perturbation stochastique

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Membres du jury :

Président du jury : Nicolas Burq, Professeur, Université Paris-Saclay et IUF, Paris
Directeur de thèse : Aurélien Deya, Chargé de recherche CNRS, IECL, Vandœuvre-lès-Nancy
Directeur de thèse : Laurent Thomann, Professeur UL, IECL, Vandœuvre-lès-Nancy
Rapportrice : Anne-Sophie de Suzzoni, maître de Conférences HDR, USPN, détachée à l'école Polytechnique, Palaiseau
Rapporteur : Paul Gassiat, maître de conférences HDR, Université Paris Dauphine, Paris
Examinateur : David Dos Santos Fereira, Professeur UL, IECL, Vandœuvre-lès-Nancy

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Résumé de la thèse

Le sujet de cette thèse porte sur l'étude de certaines équations aux dérivées partielles non linéaires et dirigées par une perturbation aléatoire.

Au Chapitre 1, on définit la notion de bruit blanc et de bruit fractionnaire. On décrit ensuite la procédure générale permettant de prouver le caractère bien posé des modèles considérés. Après avoir présenté un état de l'art, on détaille et commente les différents résultats obtenus en insistant sur les nouveautés et en précisant les éventuelles perspectives.

Au Chapitre 2, on présente les outils probabilistes dont on aura besoin tout au long de notre étude. On commence par définir le mouvement brownien fractionnaire. On rappelle ensuite les notions essentielles concernant l'intégrale de Wiener et l'intégration contre la transformée de Fourier d'un bruit blanc. On établit alors la formule de la représentation harmonisable du champ brownien fractionnaire qui sera un outil de calcul précieux. On énonce également les deux résultats phares relatifs à la régularité des termes aléatoires, à savoir le critère de Kolmogorov et l'inégalité de Garsia-Rodemich-Rumsey. Pour terminer, on définit les polynômes d'Hermite qui nous permettront de renormaliser nos équations et on développe la notion de Chaos de Wiener afin de bénéficier de la traditionnelle inégalité de contrôle des moments d'ordre p.

Au Chapitre 3, on étudie une équation de la chaleur stochastique (SNLH) avec une non-linéarité quadratique, perturbée par un bruit fractionnaire en temps et en espace. On distingue deux types de régimes, dépendant des valeurs prises par l'indice de Hurst $H = (H_0, ..., H_d) \in (0,1)^{d+1}$. En particulier, on montre que le caractère localement bien posé de (SNLH), résultant de l'astuce de Da Prato et Debussche, est obtenu facilement quand $2H_0 + \sum_{i=1}^{d} H_i > d$. Au contraire, (SNLH) est plus difficile à traiter quand $2H_0 + \sum_{i=1}^{d} H_i \leq d$. Dans ce cas, le modèle doit être interprété au sens de Wick, grâce à une renormalisation dépendant du temps. Aidé par l'effet régularisant du semi-groupe de la chaleur, on établit le caractère localement bien posé de (SNLH) en toute dimension $d \geq 1$.

Au Chapitre 4, on étudie une équation de Schrödinger stochastique avec une non-linéarité quadratique et une perturbation fractionnaire en temps et en espace. Quand l'indice de Hurst est suffisamment grand, ce qui se traduit par l'inégalité $2H_0 + \sum_{i=1}^{d} H_i > d + 1$, on prouve le caractère localement bien posé du modèle en utilisant des arguments classiques. Cependant, quand l'indice de Hurst est petit, c'est-à-dire quand $2H_0 + \sum_{i=1}^{d} H_i \leq d + 1$, même l'interprétation de l'équation a besoin d'un intérêt particulier. Dans ce cas, une procédure de renormalisation doit être mise en place, conduisant à une interprétation du modèle au sens de Wick. Notre argument de point fixe met alors en jeu des propriétés spécifiques de régularisation du groupe de Schrödinger qui nous permettent de traiter la forte irrégularité de la solution.

Au Chapitre 5, on étudie une équation de Schrödinger stochastique (SNLS), avec une non-linéarité quadratique, perturbée par une dérivée fractionnaire en espace (d'ordre $-\alpha < 0$) d'un bruit blanc espace-temps. Quand $\alpha < \frac{d}{2}$, la convolution stochastique est une fonction du temps à valeurs dans un espace de Sobolev d'ordre négatif et le modèle doit être interprété au sens de Wick au moyen d'une renormalisation dépendant du temps. Quand $1 \leq d \leq 3$, combinant les inégalités de Strichartz et un effet de régular-

isation local déterministe, on établit le caractère localement bien posé de (SNLS) pour une petite rangée de α. On revisite ensuite nos arguments et on démontre un gain de régularité multilinéaire au niveau du terme stochastique d'ordre deux. Ceci nous permet d'améliorer notre résultat d'existence et d'unicité local pour certaines valeurs de α.

Summary of the thesis

The subject of this thesis is the study of some nonlinear partial differential equations driven by a stochastic perturbation.

In Chapter 1, we define the notion of white noise and fractional noise. We then describe the general procedure to prove the local well-posedness of the models under consideration. After having presented a state of the art, we detail and comment the different results obtained, we insist on the novelties and we precise the possible perspectives.

In Chapter 2, we present the stochastic tools we will need throughout our study. We start by defining the fractional Brownian motion. We then recall the essential notions concerning Wiener integral and the integration against the Fourier transform of a white noise. We also establish the harmonizable representation formula of the fractional Brownian motion that will be a precious tool when doing computations. We state the main results related to the regularity of stochastic terms, namely Kolmogorov's criterion and the Garsia-Rodemich-Rumsey inequality. To end with, we define Hermite polynomials that will allow us to renormalize our equations and we develop the notion of Wiener Chaoses in order to benefit from the classical inequality of control of moments of order p.

In Chapter 3, we study a stochastic nonlinear heat equation (SNLH) with a quadratic nonlinearity, forced by a fractional space-time white noise. Two types of regimes are exhibited, depending on the ranges of the Hurst index $H = (H_0, ..., H_d) \in (0,1)^{d+1}$. In particular, we show that the local well-posedness of (SNLH) resulting from the Da Prato-Debussche trick, is easily obtained when $2H_0 + \sum_{i=1}^{d} H_i > d$. On the contrary, (SNLH) is much more difficult to handle when $2H_0 + \sum_{i=1}^{d} H_i \leq d$. In this case, the model has to be interpreted in the Wick sense, thanks to a time-dependent renormalization. Helped with the regularising effect of the heat semigroup, we establish local well-posedness results for (SNLH) for all dimension $d \geq 1$.

In Chapter 4, we study a stochastic Schrödinger equation with a quadratic nonlinearity and a space-time fractional perturbation. When the Hurst index is large enough, precisely when $2H_0 + \sum_{i=1}^{d} H_i > d+1$, we prove local well-posedness of the problem using classical arguments. However, for a small Hurst index, that is when $2H_0 + \sum_{i=1}^{d} H_i \leq d$, even the interpretation of the equation needs some care. In this case, a renormalization procedure must come into the picture, leading to a Wick-type interpretation of the model. Our fixed-point argument then involves some specific regularization properties of the Schrödinger group, which allows us to cope with the strong irregularity of the solution.

In Chapter 5, we study a stochastic quadratic nonlinear Schrödinger equation (SNLS), driven by a fractional derivative (of order $-\alpha < 0$) of a space-time white noise. When $\alpha < \frac{d}{2}$, the stochastic convolution is a function of time with values in a negative-order Sobolev space and the model has to be interpreted in the Wick sense by means of a time-dependent renormalization. When $1 \leq d \leq 3$, combining both the Strichartz estimates and a deterministic local smoothing, we establish the local well-posedness of (SNLS) for

a small range of α. Then, we revisit our arguments and establish multilinear smoothing on the second order stochastic term. This allows us to improve our local well-posedness result for some α.

Chapter 1

Introduction

Le sujet de cette thèse porte sur l'étude de certaines équations aux dérivées partielles non linéaires et dirigées par une perturbation aléatoire. Le premier modèle est une équation de la chaleur quadratique perturbée par un bruit fractionnaire et fait l'objet du chapitre 3. Les deux autres modèles sont des équations de type Schrödinger et font l'objet des chapitres 4 et 5. Ils sont également quadratiques. L'aléa est représenté par un bruit fractionnaire dans la première équation et par une dérivée fractionnaire en espace d'un bruit blanc espace-temps dans la seconde. Avant de décrire plus précisément les équations considérées, on se propose de mentionner le contexte général dans lequel ces dernières s'inscrivent. Les équations de la chaleur non linéaires perturbées par un terme aléatoire sont de la forme suivante :

$$\begin{cases} \partial_t u - \Delta u = P(u) + \xi \,, \quad t \in [0,T] \,, \, x \in \mathbb{R}^d \,, \\ u_0 = \phi \,, \end{cases}$$

où T désigne un temps strictement positif, où P est un polynôme à coefficients réels à une variable et où ϕ est la condition initiale déterministe. Le terme ξ représente quant à lui un bruit. Les équations de Schrödinger non linéaires perturbées par un terme aléatoire sont elles de la forme suivante :

$$\begin{cases} i\partial_t u - \Delta u = P(u, \overline{u}) + \xi \,, \quad t \in [0,T] \,, \, x \in \mathbb{R}^d \,, \\ u_0 = \phi \,, \end{cases}$$

où P est un polynôme à coefficients réels à deux variables. L'originalité de ces modèles, par rapport au cas déterministe, est la présence du bruit additif ξ. Revenons plus en détails sur ce dernier.

1.1 Définition du bruit

Le bruit le plus standard et le plus couramment utilisé est ce qu'on appelle le bruit blanc espace-temps, noté ξ. Pour des raisons de modélisation, on souhaite que ξ soit une fonction aléatoire telle que la famille $(\xi(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ soit composée de variables aléatoires gaussiennes centrées et indépendantes et donc en particulier non corrélées, ce qui peut se résumer de manière informelle par la formule :

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta_{\{t=s\}}\delta_{\{x=y\}}.$$

En fait, il n'existe pas de telle fonction aléatoire. La théorie des distributions développées par Laurent Schwartz permet néanmoins de contourner cette difficulté. La définition rigoureuse du bruit blanc espace-temps est la suivante :

Définition 1.1.1. *Soit $T > 0$ et $d \geq 1$. On dit que ξ est un bruit blanc espace-temps si ξ est une variable aléatoire sur l'espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$ à valeurs dans l'espace des distributions $\mathcal{D}'(]0,T[\times\mathbb{R}^d)$ telle que $(\langle\xi,\phi\rangle, \phi \in \mathcal{D}(]0,T[\times\mathbb{R}^d))$ est une famille de variables gaussiennes centrées dont les covariances sont données par les relations :*

$$\mathbb{E}[\langle\xi,\phi\rangle\langle\xi,\psi\rangle] = \int_{]0,T[\times\mathbb{R}^d} \phi(t,x)\psi(t,x)dtdx.$$

Ce bruit blanc espace-temps peut être interprété comme la dérivée en temps et en espace, au sens des distributions, du champ brownien espace-temps. Il existe une généralisation du bruit blanc espace-temps appelée bruit fractionnaire espace-temps. Il peut être défini comme une variable aléatoire à valeurs dans l'espace des distributions ou encore comme la dérivée en temps et en espace (toujours au sens des distributions) du mouvement brownien fractionnaire dont la définition est la suivante :

Définition 1.1.2. *Soit $T > 0$ et $d \geq 1$. Soit $H = (H_0, H_1, ..., H_d) \in (0,1)^{d+1}$. On appelle mouvement brownien fractionnaire en temps et en espace d'indice de Hurst H sur l'espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$ un processus gaussien centré $B : \Omega \times ([0,T] \times \mathbb{R}^d) \to \mathbb{R}$ dont les covariances sont données par les relations :*

$$\mathbb{E}\Big[B(s,x_1,...,x_d)B(t,y_1,...,y_d)\Big] = R_{H_0}(s,t)\prod_{i=1}^{d} R_{H_i}(x_i,y_i)\,, \quad s,t \in [0,T]\,,\ x,y \in \mathbb{R}^d\,,$$

où

$$R_{H_i}(x,y) = \frac{1}{2}(|x|^{2H_i} + |y|^{2H_i} - |x-y|^{2H_i})\ .$$

Il s'agit donc d'un objet complexe à manipuler. Une étude du mouvement brownien fractionnaire est proposée à la section 2.1.
Dans cette thèse, la première étape pour résoudre une EDPS consiste à régulariser le bruit. Il existe plusieurs manières de procéder. La technique la plus classique est de convoler le bruit avec une fonction $\mathcal{C}^\infty$ à support compact. En effet, la théorie générale des distributions assure que le produit de convolution entre une distribution et une fonction $\mathcal{C}^\infty$ à support compact est une fonction $\mathcal{C}^\infty$. On peut également réaliser une troncature dans l'espace des fréquences via la transformée de Fourier ou encore directement régulariser le champ brownien pour pouvoir le dériver.

1.2 Ce qu'on appelle "solution" d'une équation différentielle stochastique

Dans cette section, on se propose de préciser la notion de solution d'une équation différentielle stochastique. On présente également de manière concise la stratégie générale développée dans les chapitres 3, 4 et 5. Pour ce faire, considérons à titre d'exemple le modèle suivant :

$$\begin{cases} \partial_t u - \Delta u = P(u) + \xi\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases}$$

où P est un polynôme à coefficients réels à une variable, ϕ est la condition initiale déterministe et où ξ est un bruit quelconque.
Dans cette thèse, les équations considérées seront définies sur $\mathbb{R}^d$. Afin de ramener les calculs à un domaine compact, nous aurons besoin d'introduire une fonction $\rho : \mathbb{R}^d \to \mathbb{R}$ $\mathcal{C}^\infty$ à support compact jouant le rôle d'un cut-off. On considère donc dans la suite plutôt le modèle suivant :

$$\begin{cases} \partial_t u - \Delta u = P(\rho u) + \xi\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,. \end{cases} \tag{1.2.1}$$

La première chose à faire consiste à réécrire cette équation sous forme intégrale. On parle également de formule de Duhamel ou de forme mild. On considère l'égalité, pour tout $t \in [0,T]$,

$$u_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(P(\rho u)_\tau) d\tau + \int_0^t e^{(t-\tau)\Delta}(\xi_\tau)\, d\tau.$$

L'égalité précédente est pour le moment formelle. Décrivons maintenant les différentes étapes nous conduisant à la notion de solution.

• Première étape : Définition de la convolution stochastique.

On aimerait donner un sens rigoureux, pour tout $t \in [0,T]$, à $\text{⚲}(t,.) = \displaystyle\int_0^t e^{(t-\tau)\Delta}(\xi_\tau)\, d\tau$. On commence par régulariser le bruit par une des méthodes décrites plus haut. Cela nous conduit à la considération d'une suite de processus $(\text{⚲}_n)_{n\in\mathbb{N}}$. On recherche alors un espace fonctionnel dans lequel, pour toute fonction test χ, les suites $(\chi\text{⚲}_n)_{n\in\mathbb{N}}$ sont de Cauchy. On note alors $\chi\text{⚲}$ les limites associées. La connaissance de ces limites permet de définir rigoureusement l'objet ⚲. On renvoie le lecteur aux chapitres 3 et 4 pour plus de détails. Soulignons seulement ici que c'est la régularité de $\chi\text{⚲}$ que l'on sait bien mesurer. En particulier, prenant $\chi = \rho$, on montre que $\rho\text{⚲}$ est une fonction du temps, à valeurs dans un espace de fonctions ou de distributions.

• Deuxième étape : Réécriture de l'équation en utilisant l'astuce de Da Prato et Debussche.

L'égalité vérifiée par u s'écrit, pour tout $t \in [0,T]$,

$$u_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(P(\rho u)_\tau) d\tau + \text{⚲}_t.$$

L'idée est d'isoler le terme dont on s'attend à ce qu'il possède la pire régularité, ici ⚲. On cherche alors à écrire u sous la forme $u = v + \text{⚲}$ avec v un terme de reste plus régulier que Ψ. L'équation vérifiée par v est la suivante, pour tout $t \in [0, T]$,

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(P(\rho v + \rho \Psi)_\tau) d\tau.$$

• <u>Troisième étape :</u> Etude des termes dont la définition pose problème.

Prenons $P(X) = X^k$ avec $k \geq 2$. v vérifie, pour tout $t \in [0, T]$,

$$v_t = e^{t\Delta}\phi + \sum_{i=0}^{k} \binom{k}{i} \int_0^t e^{(t-\tau)\Delta}((\rho^i v^i \cdot \rho^{k-i} \Psi^{k-i})_\tau) d\tau. \tag{1.2.2}$$

L'équation précédente fait apparaître $\rho^i \Psi^i$ avec $2 \leq i \leq k$. Lorsque le bruit est irrégulier, $\rho \Psi$ sera une fonction du temps à valeurs dans un espace de distributions. Il ne sera donc pas possible de définir les puissances $\rho^i \Psi^i$, $2 \leq i \leq k$, comme simples produits de fonctions.

• <u>Quatrième étape :</u> Constructions stochastiques des termes Ψ^i .

On revient à notre suite régularisée $(\Psi_n)_{n\in\mathbb{N}}$. On note, pour tout $t \in]0, T]$, pour tout $x \in \mathbb{R}^d$, $\sigma_n(t, x) = \mathbb{E}[\Psi_n^2(t, x)]$. On considère alors, pour tout $i \geq 1$, H_i le $i^{\text{ième}}$ polynôme d'Hermite et $H_i(x; \sigma)$ le polynôme défini par $H_i(x; \sigma) = i!\, \sigma^{\frac{i}{2}} H_i(\frac{x}{\sqrt{\sigma}})$. Le lecteur trouvera une étude de ces polynômes à la section 2.7. On définit maintenant le normalisé de Wick de Ψ_n^i de la manière suivante :

$$: \Psi_n^i(x, t) := H_i(\Psi_n(x, t), \sigma_n(x, t)).$$

La notation $: \Psi_n^i :$ est classique. Elle désigne un produit de Wick de variable aléatoire (ce qui justifie la terminologie "renormalisé de Wick") et est empruntée à la théorie des champs. Le lecteur intéressé trouvera plus de détails à la section 2.8. A nouveau, on cherche des espaces fonctionnels dans lesquels, pour toute fonction test χ, $(\chi : \Psi_n^i :)_{n\in\mathbb{N}}$ est une suite de Cauchy et on appelle $\chi : \Psi^i :$, $1 \leq i \leq k$, les limites correspondantes. La donnée de ces limites permet la définition rigoureuse de $\chi : \Psi^i :$. On remplace alors Ψ^i par $: \Psi^i :$ dans l'équation (1.2.2) pour obtenir

$$v_t = e^{t\Delta}\phi + \sum_{i=0}^{k} \binom{k}{i} \int_0^t e^{(t-\tau)\Delta}((\rho^i v^i \cdot \rho^{k-i} : \Psi^{k-i} :)_\tau) d\tau. \tag{1.2.3}$$

Effectuons dès à présent quelques précisions.
• Au chapitre 3, on étudiera une équation de la chaleur fractionnaire avec une non linéarité quadratique. $P(X)$ sera alors X^2 et $: \Psi^2 :$ sera noté $\vee$.
• Aux chapitres 4 et 5, dans le cas des équations de Schrödinger, on aura $P(u, \overline{u}) = |u|^2$. On pourra procéder à une renormalisation analogue en remplaçant Ψ_n^2 par $|\Psi_n|^2$ et $\sigma_n(t, x)$ par $\mathbb{E}[|\Psi_n(t, x)|^2]$. En fait, dans le cas complexe, ce sont les polynômes de Laguerre qui

jouent un rôle analogue aux polynômes d'Hermite (voir par exemple [18]).

- Cinquième étape : Mise en place d'un point fixe.

On fixe une fois pour toutes un ensemble mesurable $\tilde{\Omega}$ tel que $\mathbb{P}(\tilde{\Omega}) = 1$ et sur lequel vivent les processus $(\rho^i : \Psi^i :)_{1\leq i\leq k}$. Lorsque ces derniers sont des fonctions du temps à valeurs dans un espace de distributions, précisément, un espace de Sobolev d'ordre négatif, il peut y avoir une difficulté à définir les produits avec les v^i, $1 \leq i \leq k$. Rappelons la règle permettant de définir le produit dans les espaces de Sobolev. Si f possède β dérivées et g possède $-\alpha$ dérivées, alors on peut définir le produit $f \cdot g$ si $\beta > \alpha$ et, $f \cdot g$ possède $-\alpha$ dérivées, héritant de la pire régularité. Ainsi, pour donner un sens aux produits apparaissant dans l'équation (1.2.3), on a besoin d'un effet régularisant de l'opérateur $e^{t\Delta}$. Une fois les constructions stochastiques réalisées, ce problème de gain de régularité sera la difficulté majeure. Il est mis en avant et détaillé dans les chapitres 3, 4 et 5. Supposons que nous sommes en moyen de donner un sens à tous les termes de (1.2.3). Notons F la fonction définie par

$$F(v)_t = e^{t\Delta}\phi + \sum_{i=0}^{k} \binom{k}{i} \int_0^t e^{(t-\tau)\Delta}((\rho^i v^i \cdot \rho^{k-i} : \Psi^{k-i} :)_\tau) d\tau.$$

On appellera solution de (1.2.1) le processus $u = \Psi + v$ où v est l'unique point fixe de F. Ce point fixe sera obtenu en appliquant le théorème de Picard sur une partie stable et complète sur laquelle F est contractante.

1.3 Etude d'une équation de la chaleur fractionnaire

Au chapitre 3, on s'intéresse à une équation de la chaleur avec une non linéarité quadratique et perturbée par un bruit fractionnaire. Cette dernière s'écrit de la manière suivante :

$$\begin{cases} \partial_t u - \Delta u = \rho^2 u^2 + \dot{B}\,, & t \in [0,T]\,,\ x \in \mathbb{R}^d\,, \\ u(0,.) = \phi\,, \end{cases} \tag{1.3.1}$$

où ϕ vit dans un espace de Sobolev, $\rho : \mathbb{R}^d \to \mathbb{R}$ est une fonction $\mathcal{C}^\infty$ à support compact et $\dot{B} = \partial_t \partial_{x_1} ... \partial_{x_d} B^H$, B^H étant un mouvement brownien fractionnaire en temps et en espace d'indice de Hurst $H = (H_0, H_1, ..., H_d) \in (0,1)^{d+1}$.

Avant de détailler nos résultats, commençons par présenter un état de l'art. Le modèle le plus classique, et sans perturbation aléatoire, est le suivant :

$$\partial_t u = \Delta u + u^p\,, \quad t \in [0,T]\,,\ x \in \mathbb{R}^d \tag{1.3.2}$$

où $p > 1$. Ce modèle est appelé modèle de Fujita. En particulier, quand $p = 2$, il décrit un système de réaction-diffusion. De nombreux résultats ont été obtenus pour cette équation. Par exemple, on sait que, dès lors que la condition initiale, notée ϕ, est un élément de $L^q(\mathbb{R}^d)$ avec $q \geq 1$ et $q > \frac{d(p-1)}{2}$, il existe une constante $T = T(\phi) > 0$ et une unique

fonction $u \in \mathcal{C}([0,T], L^q(\mathbb{R}^d))$ qui est une solution classique de (1.3.2) sur $[0,T]\times\mathbb{R}^d$ alors que, pour $q < \frac{d(p-1)}{2}$, il n'y a pas de théorie générale d'existence (voir [1, 19]).
De nombreux travaux ont cherché à généraliser le modèle (1.3.2) en y introduisant un terme stochastique représentant un aléa. Da Prato et Debbusche [3] ont par exemple étudié la dynamique suivante de réaction-diffusion :

$$dX = (\Delta X - X + :P(X):)dt + dW(t), \quad x \in [0, 2\pi]^2,$$

où P est un polynôme à coefficients réels à une variable et où W désigne un bruit blanc espace-temps. Ils ont démontré l'existence et l'unicité d'une solution forte dans un espace de Besov convenable. Depuis, de nombreux progrès ont été réalisé dans le domaine des équations paraboliques. Un cap majeur a été franchi par Hairer grâce à ses travaux sur les structures de régularité. Cette théorie permet par exemple de traiter le modèle suivant, appelé Φ^4_3 :

$$\partial_t\Phi = \Delta\Phi + C\Phi - \Phi^3 + \xi, \quad x \in \mathbb{R}^3,$$

où C est une constante de renormalisation et ξ désigne un bruit blanc espace-temps. On renvoie le lecteur à [11] pour plus de détails. D'autres équations ont fait leur apparition dans la littérature, la non-linéarité polynomiale étant remplacée par une non-linéarité sinusoïdale puis exponentielle. Grâce aux structures de régularité, Hairer-Shen [12] et Chandra-Hairer-Shen [2] ont par exemple étudié le modèle sine-Gordon sur le tore $\mathbb{T}^2$:

$$\partial_t u = \frac{1}{2}\Delta u + \sin(\beta u) + \xi,$$

où ξ est un bruit blanc espace-temps. Les auteurs ont établi le caractère bien posé et démontré qu'il dépend essentiellement du paramètre clé $\beta^2 > 0$. Un modèle de type exponentiel a été considéré par Oh, Robert et Wand dans [16]. Ces derniers se sont concentrés sur l'équation parabolique ci-dessous, également définie sur le tore $\mathbb{T}^2$:

$$\partial_t u + \frac{1}{2}(1-\Delta)u + \frac{1}{2}\lambda\beta e^{\beta u} = \xi,$$

où $\beta, \lambda \in \mathbb{R}\backslash\{0\}$. Ils ont démontré que le caractère bien posé dépend à nouveau sensiblement de la valeur de $\beta^2 > 0$, ainsi que du signe de λ.
Pour terminer notre historique, mentionnons le travail récent de Gubinelli et Hofmanová (voir [10]) qui ont étudié le modèle Φ^4 ci-dessous :

$$(\partial_t - \Delta + \mu)\phi + \phi^3 = \xi, \quad x \in \mathbb{R}^d, \tag{1.3.3}$$

où ξ est un bruit blanc en espace et en temps et $\mu \in \mathbb{R}$. Les deux auteurs ont prouvé l'existence et l'unicité d'une solution pour (1.3.3) lorsque $d = 2$ et $d = 3$.

Revenons maintenant au sujet du Chapitre 3, à savoir le modèle (1.3.1). Ce travail est à mettre en relation avec l'article [17] de Oh et Okamoto où les auteurs ont étudié la dynamique suivante :

$$\partial_t u + (1-\Delta)u + u^2 = \langle\nabla\rangle^\alpha\xi, \tag{1.3.4}$$

avec $\alpha > 0$ et où ξ est un bruit blanc espace-temps sur $\mathbb{T}^2 \times \mathbb{R}^+$. Bien que nous ayons prouvé le caractère localement bien posé de (1.3.1) en toute dimension $d \geq 1$ (voir Chapitre 3), nous supposerons dans la suite que $d = 2$ afin de pouvoir comparer les deux équations précédentes. Il y a essentiellement deux différences. Notre équation est définie sur $\mathbb{R}^2$ alors que celle de Oh et Okamoto est définie sur le tore et nos bruits diffèrent, le bruit fractionnaire étant plus général que le bruit blanc. Effectuons une première comparaison au niveau des convolutions stochastiques :

$$\Psi^H_t = \int_0^t e^{(t-\tau)\Delta}(\dot{B}_\tau)d\tau \quad \text{et} \quad \tilde{\Psi}_t = \int_{-\infty}^t e^{-(t-\tau)(1-\Delta)}\langle\nabla\rangle^\alpha dW(\tau).$$

Suite à une procédure de régularisation, c'est-à-dire en considérant une suite $(\Psi_n)_{n\geq 0}$ de processus réguliers, on montre le résultat suivant :

Proposition 1.3.1. *Soit $T > 0$. Pour tout $(H_0, H_1, H_2) \in (0,1)^3$ et toute fonction test $\chi : \mathbb{R}^2 \to \mathbb{R}$, presque sûrement, la suite $(\chi\Psi_n)_{n\geq 0}$ converge dans l'espace $\mathcal{C}([0,T]; \mathcal{W}^{-s,p}(\mathbb{R}^2))$ dès que $2 \leq p < \infty$ et*

$$s > 2 - \Big(2H_0 + H_1 + H_2\Big).$$

La limite de cette suite est notée $\chi\Psi$.

On remarque l'importance d'introduire une fonction cut-off χ afin d'établir la convergence. Cela n'est pas nécessaire dans le cas du tore qui est compact, comme l'affirme la proposition ci-dessous.

Proposition 1.3.2. *Soit $T > 0$. Soit α et s deux réels tels que*

$$s < -\alpha.$$

Presque sûrement, la suite $(\tilde{\Psi}_n)_{n\geq 0}$ converge dans l'espace $\mathcal{C}([0,T]; \mathcal{C}^s(\mathbb{T}^2))$. En notant $\tilde{\Psi}$ la limite de cette suite, pour tout $\varepsilon > 0$,

$$\tilde{\Psi} \in \mathcal{C}([0,T]; \mathcal{C}^{-\alpha-\varepsilon}(\mathbb{T}^2)).$$

En prenant, pour tout $i \in [\![0,2]\!]$, $H_i = \frac{1}{2}$ et $\alpha = 0$, on retrouve bien des conditions similaires.
Dans la suite, on se limite au cas où $2H_0 + H_1 + H_2 \leq 2$. Dans cette situation, $\rho\Psi$ est une fonction du temps à valeurs dans un espace de distributions. Afin de définir $:\!\Psi^2\!:$, on procède, comme expliqué à la section 1.2 à une renormalisation de Wick. Précisément, on introduit :

$$:\!\Psi^2\!:_n(t,x) = \Psi_n(t,x)^2 - \sigma_n(t,x) \quad \text{où } \sigma_n(t,x) = \mathbb{E}\Big[\Psi_n(t,x)^2\Big] \tag{1.3.5}$$

et on démontre le résultat de convergence suivant :

Proposition 1.3.3. *Soit $T > 0$. Pour tout $(H_0, H_1, H_2) \in (0,1)^3$ tel que*

$$0 < H_1 < \frac{3}{4} \quad , \quad 0 < H_2 < \frac{3}{4} \quad \textit{et} \quad \frac{3}{2} < 2H_0 + H_1 + H_2 \leq \frac{7}{4} \ ,$$

pour toute fonction test $\chi : \mathbb{R}^2 \to \mathbb{R}$*, presque sûrement, la suite* $(\chi^2 \curlyvee_n)_{n\geq 0}$ *(définie par* (1.3.5)*) converge dans l'espace* $\mathcal{C}([0,T]; \mathcal{W}^{-2s,p}(\mathbb{R}^2))$ *dès que* $2 \leq p < \infty$ *et*

$$s > 2 - (2H_0 + H_1 + H_2) \ .$$

La limite de cette suite est notée $\chi^2 \curlyvee$*.*

On peut également réaliser cette construction lorsque $2H_0 + H_1 + H_2 > \frac{7}{4}$ en retirant les deux hypothèses sur H_1 et sur H_2. Cette proposition 1.3.3 est en fait un raffinement en dimension 2 d'un résultat obtenu en dimension d quelconque (voir Chapitre 3, Proposition 3.1.8). On peut se demander si la condition $2H_0 + H_1 + H_2 > \frac{3}{2}$ est optimale. C'est en effet le cas comme l'assure la proposition suivante.

Proposition 1.3.4. *Soit* $(H_0, H_1, H_2) \in (0,1)^3$ *tel que*

$$2H_0 + H_1 + H_2 \leq \frac{3}{2}$$

et soit $\chi : \mathbb{R}^2 \to \mathbb{R}$ *une fonction* $\mathcal{C}^\infty$ *à support compact et non nulle. Alors, pour tout* $s > 0$ *et tout* $t > 0$*,*

$$\mathbb{E}\Big[\|\chi \cdot \curlyvee_n(t,.)\|^2_{H^{-2s}(\mathbb{R}^2)}\Big] \underset{n\to+\infty}{\to} \infty.$$

Regardons maintenant les résultats obtenus par Oh et Okamoto.

Proposition 1.3.5. *Soit* $T > 0$*. Soit* α *et* s *deux réels tels que*

$$0 < \alpha < \frac{1}{2} \quad et \quad s < -2\alpha.$$

Presque sûrement, la suite $(\tilde{\curlyvee}_n)_{n\geq 0}$ *converge dans l'espace* $\mathcal{C}([0,T]; \mathcal{C}^s(\mathbb{T}^2))$*. En notant* $\tilde{\curlyvee}$ *la limite de cette suite, pour tout* $\varepsilon > 0$*,*

$$\tilde{\curlyvee} \in \mathcal{C}([0,T]; \mathcal{C}^{-2\alpha-\varepsilon}(\mathbb{T}^2)).$$

De plus, Oh et Okamoto montre la condition d'explosion suivante :

Proposition 1.3.6. *Pour tout* $\alpha \geq \frac{1}{2}$*, presque sûrement, la suite* $(\tilde{\curlyvee}_n)_{n\geq 0}$ *est une suite divergente de* $\mathcal{C}([0,T]; \mathcal{D}'(\mathbb{T}^2))$ *pour tout* $T > 0$*.*

Ainsi, comme $2H_0 + H_1 + H_2 \leq \frac{3}{2}$ si et seulement si $2 - (2H_0 + H_1 + H_2) \geq \frac{1}{2}$, il y a une analogie entre α et $2 - (2H_0 + H_1 + H_2)$.
On précise maintenant ce que l'on entend par solution de notre modèle.

Définition 1.3.7. *Un processus stochastique* $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ *est une solution de Wick (sur* $[0,T]$*) de l'équation* (1.3.1) *si, presque sûrement, le processus* $v := u - \mathsf{I}$ *est une solution de l'équation mild*

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2) d\tau + 2\int_0^t e^{(t-\tau)\Delta}((\rho v_\tau)\cdot(\rho \mathsf{I}_\tau))\, d\tau$$
$$+ \int_0^t e^{(t-\tau)\Delta}(\rho^2 \curlyvee_\tau)\, d\tau\,, \quad t \in [0,T]\,.$$

En négligeant l'action de la convolution, on s'attend à ce que v hérite de la régularité du pire terme, ici $\rho^2 \cherry$, qui est une fonction du temps à valeurs dans un espace de Sobolev d'ordre $-2s < 0$. Il y a donc une difficulté à définir les produits $\rho^2 v^2$ et $\rho v \cdot \rho \lollipop$. On se sert alors de la propriété de régularisation de l'opérateur de la chaleur.

Proposition 1.3.8. *Soit $1 < p < \infty$. Pour tout $s_1 < s_2$, il existe une constante $C(s_1, s_2) < \infty$ telle que l'inégalité*

$$\|e^{t\Delta}\phi\|_{\mathcal{W}^{s_2,p}(\mathbb{R}^d)} \le C(s_1, s_2)(1 + t^{-\frac{s_2 - s_1}{2}})\|\phi\|_{\mathcal{W}^{s_1,p}(\mathbb{R}^d)}$$

soit vraie pour chaque $\phi \in \mathcal{W}^{s_1,p}(\mathbb{R}^d)$ et $t > 0$.

Autrement dit, le noyau de la chaleur permet d'obtenir un gain de 2 dérivées (au sens des espaces de Sobolev). On est donc en mesure de donner un sens aux produits évoqués plus haut. On démontre alors le résultat suivant :

Théorème 1.3.9. *Supposons que $d = 2$ et que $p \ge 2$. Soit $(H_0, H_1, H_2) \in (0,1)^3$ tel que*

$$0 < H_1 < \frac{3}{4} \quad , \quad 0 < H_2 < \frac{3}{4} \quad , \quad \frac{3}{2} < 2H_0 + H_1 + H_2 \le \frac{7}{4} \, .$$

Soit $\alpha > 0$ tel que

$$2 - (2H_0 + H_1 + H_2) < s < \frac{1}{2}. \tag{1.3.6}$$

Alors,
(i) On peut trouver $\beta > 0$ tel que, presque sûrement, pour tout $\phi \in \mathcal{W}^{-s,p}(\mathbb{R}^2)$, il existe $T_0(\omega) > 0$ tel que l'équation (1.3.1) *admet une unique solution de Wick u dans l'ensemble*

$$\mathcal{S}_{T_0} := \lollipop + X^{s,\beta}(T_0) \, ,$$

où

$$X^{s,\beta}(T_0) := \mathcal{C}([0,T_0]; \mathcal{W}^{-s,p}(\mathbb{R}^2)) \cap \mathcal{C}((0,T_0]; \mathcal{W}^{\beta,p}(\mathbb{R}^2)).$$

(ii) Pour tout $n \ge 1$, notons $\widetilde{u}_n$ la solution de Wick de (1.3.1)*, c'est-à-dire que $\widetilde{u}_n$ est la solution (au sens de la définition 1.3.7) associée à la paire $(\rho \lollipop_n, \rho^2 \cherry_n)$. Alors, pour tout s vérifiant (1.3.6) et pour toute fonction test $\chi : \mathbb{R}^2 \to \mathbb{R}$, la suite $(\chi \widetilde{u}_n)_{n \ge 1}$ converge presque sûrement dans $\mathcal{C}([0,T_0]; \mathcal{W}^{-s,p}(\mathbb{R}^2))$ vers χu, où u est la solution de Wick de l'item (i).*

Le résultat précédent montre que, même avec un bruit et des données initiales très irrégulières (ces dernières vivant dans des espaces de Sobolev d'ordre négatif), on peut établir le caractère bien posé de notre modèle (1.3.1). En fait, grâce à l'opérateur de la chaleur, la solution v est même une fonction du temps à valeurs dans un espace de fonctions (et non seulement un espace de distributions) pour tout temps t strictement positif.

Un prolongement naturel de ce travail serait d'obtenir un résultat analogue au théorème 1.3.9 pour $d \ge 3$. La difficulté principale consisterait à généraliser la construction stochastique de la proposition 1.3.3 et proviendrait, comme souvent, de la technicité sous-jacente au calcul fractionnaire.

Un autre prolongement possible serait de reprendre l'étude de notre modèle (1.3.1) en retirant la fonction ρ dont le but est de permettre la construction des différents objets aléatoires dans des espaces de fonctions continues par rapport au temps à valeurs dans des espaces de Sobolev. Pour compenser la présence de cette fonction jouant le rôle d'un cut-off, on pourrait par exemple envisager de construire les processus stochastiques dans des espaces à poids.

1.4 Etude de deux équations de Schrödinger stochastiques

Aux chapitres 4 et 5, on s'intéresse aux deux équations de Schrödinger quadratiques suivantes :

$$\begin{cases} i\partial_t u - \Delta u = \rho^2 |u|^2 + \dot{B}\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases} \tag{1.4.1}$$

et

$$\begin{cases} i\partial_t u - \Delta u = \rho^2 |u|^2 + \langle \nabla \rangle^{-\alpha} \dot{W}\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases} \tag{1.4.2}$$

où $\rho : \mathbb{R}^d \to \mathbb{R}$ est une fonction $\mathcal{C}^\infty$ à support compact. Dans le modèle (1.4.1), $\dot{B}$ désigne la dérivée en temps et en espace d'un mouvement brownien fractionnaire d'indice de Hurst $H = (H_0, H_1, ..., H_d) \in (0,1)^{d+1}$ tandis que dans le modèle (1.4.2), $\langle \nabla \rangle^{-\alpha} \dot{W}$ est la dérivée fractionnaire d'ordre $-\alpha$ en espace d'un bruit blanc espace-temps.

Avant de détailler nos travaux, on se propose de présenter un état de l'art. L'équation de Schrödinger a toujours suscité l'intérêt des mathématiciens, notamment à cause de son lien important avec la mécanique quantique. Considérons le modèle de Schrödinger cubique suivant :

$$i\partial_t u - \Delta u = |u|^2 u, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d.$$

L'équation précédente permet par exemple d'étudier un modèle simplifié des condensats de Bose-Einstein [7]. Elle permet également de décrire la propagation d'ondes lumineuses dans des fibres optiques ou la propagation de certains types de vagues dans l'océan [13]. De nombreux travaux ont cherché à généraliser le modèle (1.3.2) en y introduisant un terme stochastique représentant un aléa. Comme pour l'équation de la chaleur, on se limite ici à des bruits additifs. Les tous premiers travaux portant sur des équations de Schrödinger avec perturbation stochastique additive sont des articles de De Bouard et Debussche (voir [4, 5]) où les auteurs ont étudié le modèle ci-dessous :

$$i\partial_t u + \Delta u = |u|^{2\sigma} u + \dot{\xi}\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \tag{1.4.3}$$

pour certaines valeurs de $\sigma > 0$ (dépendant de d) et où $\dot{\xi}$ désigne un bruit blanc en temps avec une forte régularité en espace. Leurs résultats ont ensuite été étendus à des bruits moins réguliers en espace. Dans le cas où $\sigma = 1$, Forlano, Oh et Wang [8] ont démontré le caractère bien posé de l'équation de Schrödinger cubique sur le tore :

$$\begin{cases} i\partial_t u - \partial_x^2 u = |u|^2 u + \langle \nabla \rangle^{-\alpha} \dot{W}\,, & t \in [0,T]\,,\, x \in \mathbb{T}\,, \\ u_0 = \phi\,, \end{cases}$$

où $\alpha > 0$ et où $\dot{W}$ désigne un bruit blanc espace-temps. Ils ont traité cette équation dans le cas d'un bruit "presque" blanc, c'est-à-dire pour tout $\alpha > 0$. Effectuons un commentaire concernant leur résultat. On note Ψ la convolution stochastique définie par, pour tout $t \in [0,T]$, $\Psi(t,.) = -i \int_0^t \frac{e^{-i(t-t')\Delta}}{\langle \nabla \rangle^{\alpha}} \dot{W}(t',.)dt'$. Un de leurs arguments majeurs a été de mesurer la régularité de Ψ dans des espaces de Fourier-Lebesgue. Dans ces derniers, Ψ a une régularité $s < \alpha - \frac{1}{p}$ en espace et donc s peut être choisi strictement positif dès lors que p est suffisamment grand. Par conséquent, Ψ est une fonction du temps à valeurs dans un espace de fonctions.
Le modèle (1.4.3) a également été étendu à des bruits moins réguliers en espace, cette fois sur $\mathbb{R}^d$ pour certaines valeurs de $\sigma > 0$ (voir [14, 15]). Remarquons que, dans tous ces travaux, la convolution stochastique est une fonction du temps à valeurs dans un espace de fonctions.
Mentionnons pour terminer le travail de Deng, Nahmod et Yue. Ces trois auteurs ont développé la théorie des tenseurs aléatoires afin d'étudier la propagation de l'aléa dans des équations dispersives. Grâce à ces nouveaux outils, ils ont été capables de prouver le caractère bien posé d'équations de Schrödinger semi-linéaires déterministes avec conditions initiales aléatoires dans des espaces qui sont sous-critiques dans l'échelle probabiliste. On renvoie le lecteur à [6] pour plus de détails.

Revenons maintenant à nos deux modèles (1.4.1) et (1.4.2). On se propose d'établir le caractère bien posé de ces deux équations ainsi que de les comparer. Commençons par observer qu'en prenant les H_i, $0 \leq i \leq d$ égaux à $\frac{1}{2}$, on a $\dot{B} = \dot{W}$. En fait, dans le second cas, on régularise en plus $\dot{W}$ grâce à l'opérateur $\langle \nabla \rangle^{-\alpha}$, ce qui va permettre d'aller plus loin dans les constructions stochastiques. Les définitions formelles des convolutions stochastiques sous-jacentes sont :

$$\Psi_t^H = -i \int_0^t e^{-i(t-\tau)\Delta} (\dot{B}_\tau)\, d\tau \quad \text{et} \quad \tilde{\Psi}_t = -i \int_0^t \frac{e^{-i(t-t')\Delta}}{\langle \nabla \rangle^{\alpha}} \dot{W}(t',.)dt'.$$

Dans le cas du modèle (1.4.1), à l'aide d'une procédure de régularisation basée sur la représentation harmonisable du mouvement brownien fractionnaire espace-temps menant à la considération d'une suite $(\Psi_n)_{n\in\mathbb{N}}$ de processus réguliers, on démontre le résultat suivant :

Proposition 1.4.1. *Soit $d \geq 1$ et $(H_0, \ldots, H_d) \in (0,1)^{d+1}$. Alors, pour toute fonction test $\chi : \mathbb{R}^d \to \mathbb{R}$, presque sûrement, la suite $(\chi\Psi_n)_{n\in\mathbb{N}}$ converge dans l'espace $\mathcal{C}([0,T];\mathcal{W}^{-s,p}(\mathbb{R}^d))$, dès que $2 \leq p < \infty$ et*

$$s > d + 1 - \left(2H_0 + \sum_{i=1}^{d} H_i\right). \tag{1.4.4}$$

On note la limite de cette suite $\chi\Psi$.

Dans le cas du bruit blanc espace-temps avec $-\alpha$ dérivées en espace, on obtient l'énoncé qui suit.

Proposition 1.4.2. *Soit $d \geq 1$ et $T > 0$. Soit α un nombre réel et*

$$s > \frac{d}{2} - \alpha.$$

Soit $\chi : \mathbb{R}^d \to \mathbb{R}$ une fonction test. Alors, pour tout $2 \leq p < \infty$, la suite $(\chi\tilde{\Psi}_n)_{n\in\mathbb{N}}$ converge presque sûrement dans l'espace $\mathcal{C}([0,T];\mathcal{W}^{-s,p}(\mathbb{R}^d))$.
Notant $\chi\tilde{\Psi}$ la limite presque sûre, on a

$$\chi\tilde{\Psi} \in \mathcal{C}([0,T];\mathcal{W}^{-s,p}(\mathbb{R}^d)).$$

On voit que l'action de l'opérateur $\langle\nabla\rangle^{-\alpha}$ apparaît dans l'inégalité $s > \frac{d}{2} - \alpha$. On peut maintenant établir un lien entre les deux propositions précédentes. En prenant les H_i, $0 \leq i \leq d$ tous égaux à $\frac{1}{2}$, la combinaison linéaire $d + 1 - \left(2H_0 + \sum_{i=1}^{d} H_i\right)$ devient $\frac{d}{2}$. En prenant alors $\alpha = 0$, ce qui correspond à annuler l'action de $\langle\nabla\rangle^{-\alpha}$, l'inégalité $s > \frac{d}{2} - \alpha$ devient $s > \frac{d}{2}$ et on constate qu'on a bien deux énoncés analogues.
Dans la suite, on se concentre sur les deux cas suivants :

$$2H_0 + \sum_{i=1}^{d} H_i \leq d + 1 \quad \text{et} \quad \alpha < \frac{d}{2}$$

qui correspondent au cas où la convolution stochastique est une fonction du temps à valeurs dans un espace de Sobolev d'ordre négatif. Afin de définir $\Psi^{\diamond 2}$, on procède, comme expliqué à la section 1.2 à une renormalisation de Wick. Précisément, on introduit :

$$\Psi^{\diamond 2}_n(t,x) = |\Psi_n(t,x)|^2 - \sigma_n(t,x) \quad \text{où } \sigma_n(t,x) = \mathbb{E}\Big[|\Psi_n(t,x)|^2\Big] \tag{1.4.5}$$

et on démontre les résultats de convergence suivants. Dans le cas du mouvement brownien fractionnaire espace-temps, on obtient la proposition ci-dessous.

Proposition 1.4.3. *Soit $d \geq 1$ et $(H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$ tel que*

$$d + \frac{3}{4} < 2H_0 + \sum_{i=1}^{d} H_i \leq d + 1\,.$$

Alors, pour toute fonction test $\chi : \mathbb{R}^d \to \mathbb{R}$, la suite $(\chi^2\Psi^{\diamond 2}_n)_{n\in\mathbb{N}}$ (définie par (1.4.5)) converge presque sûrement dans l'espace $\mathcal{C}([0,T];\mathcal{W}^{-2s,p}(\mathbb{R}^d))$, pour tout $2 \leq p < \infty$ et $s > 0$ vérifiant (1.4.4). On note la limite de cette suite $\chi^2\Psi^{\diamond 2}$.

Dans le cas du bruit blanc espace-temps avec $-\alpha$ dérivées en espace, le résultat de convergence s'énonce ainsi :

Proposition 1.4.4. *Soit $d \geq 1$ et $T > 0$. Soit α et s deux nombres réels tels que*

$$\frac{d}{4} < \alpha < \frac{d}{2} \quad \text{et} \quad s > \frac{d}{2} - \alpha.$$

Soit $\chi : \mathbb{R}^d \to \mathbb{R}$ une fonction test. Alors, pour tout $2 \leq p < \infty$, la suite $(\chi^2 \widetilde{\curlyvee}_n)_{n\in\mathbb{N}}$ converge presque sûrement dans l'espace $\mathcal{C}([0,T]; \mathcal{W}^{-2s,p}(\mathbb{R}^d))$.
Notant $\chi^2\widetilde{\curlyvee}$ la limite presque sûre, on a

$$\chi^2\widetilde{\curlyvee} \in \mathcal{C}([0,T]; \mathcal{W}^{-2s,p}(\mathbb{R}^d)).$$

Prenons à nouveau les H_i, $0 \leq i \leq d$ égaux à $\frac{1}{2}$. Alors $2H_0 + \sum_{i=1}^{d} H_i = 1 + \frac{d}{2} < d + \frac{3}{4}$. Autrement dit, dans le cas du bruit blanc espace-temps, on ne sait pas construire $\curlyvee$. La proposition 1.4.4 montre néanmoins qu'en régularisant le bruit blanc avec $-\alpha$ dérivées en espace, on y parvient.
On précise maintenant ce que l'on entend par solution de nos modèles.

Définition 1.4.5. *Un processus stochastique $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ est une solution de Wick (sur $[0,T]$) de l'équation (1.4.1) si, presque sûrement, le processus $v := u - \Psi$ est une solution de l'équation mild*

$$\begin{aligned} v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau))\,d\tau \\ - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\Psi_\tau}))\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2\curlyvee_\tau)\,d\tau\,, \quad t\in[0,T]\,. \end{aligned} \tag{1.4.6}$$

On a évidemment une définition analogue pour l'équation (1.4.2).

En négligeant l'action de la convolution, on s'attend à ce que v hérite de la régularité du pire terme, ici $\rho^2\curlyvee$, qui est une fonction du temps à valeurs dans un espace de Sobolev d'ordre $-2s < 0$ que ce soit dans le cas de l'équation (1.4.1) ou celui de l'équation (1.4.2). Dans tous les cas, il y a donc une difficulté à définir les produits $\rho^2|v|^2$, $\rho\overline{v}\cdot\rho\Psi$ et $\rho v\cdot\overline{\rho\Psi}$. On sait que l'opérateur de Schrödinger ne permet pas d'obtenir de gain de régularité global. En revanche, il peut offrir un gain local, par le biais de l'effet de Kato (voir Proposition 4.5.1). En fait, on peut obtenir un raffinement de ce dernier. Commençons par prendre $\rho : \mathbb{R}^d \to \mathbb{R}$ de la forme

$$\rho(x_1,\ldots,x_d) = \rho_1(x_1)\cdots\rho_d(x_d) \tag{$\mathbf{F}_\rho$}$$

pour des fonctions $\rho_1,\ldots,\rho_d$ $\mathcal{C}^\infty$ à support compact sur $\mathbb{R}$. On démontre alors le résultat ci-dessous :

Lemme 1.4.6. *Soit $d \geq 1$. Soit $\rho : \mathbb{R}^d \to \mathbb{R}$ de la forme $(\mathbf{F}_\rho)$, $0 \leq s,\eta \leq \frac{1}{2}$ et $0 \leq T \leq 1$. Supposons que $\phi \in H^{-s}(\mathbb{R}^d)$, $F \in L^1([0,T]; H^{-s}(\mathbb{R}^d))$, et considérons la solution u de l'équation de Schrödinger inhomogène sur $\mathbb{R}^d$ suivante :*

$$\begin{cases} i\partial_t u(t,x) - \Delta u(t,x) = F(t,x)\,, \quad t\in[0,T]\,,\ x\in\mathbb{R}^d\,, \\ u_0 = \phi\,. \end{cases}$$

Alors, l'inégalité ci-dessous est vérifiée

$$\|u\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}} \lesssim \|\phi\|_{H^{-s}(\mathbb{R}^d)} + \|F\|_{L_T^1 H^{-s}} ,$$

où la constante de proportionnalité dépend seulement de ρ, s et η.

Autrement dit, la solution u gagne η dérivée en espace, quitte à mesurer sa régularité dans un espace de Sobolev à poids. Grâce au lemme précédent, on est en mesure de donner un sens aux produits évoqués plus haut. On peut donc maintenant énoncer nos deux résultats principaux. Dans le cas du modèle (1.4.1), on obtient l'énoncé suivant:

Théorème 1.4.7. *Supposons que $1 \leq d \leq 3$ et que la fonction test $\rho : \mathbb{R}^d \to \mathbb{R}$ est de la forme ($\mathbf{F}_\rho$). En outre, supposons que*

$$-s_d < 2H_0 + \sum_{i=1}^{d} H_i - (d+1) \leq 0 , \quad \text{où} \quad s_d := \begin{cases} \frac{3}{20} & \text{si } d = 1 \\ \frac{1}{10} & \text{si } d = 2 \\ \frac{1}{24} & \text{si } d = 3 \end{cases} .$$

Soit $s > 0$ tel que $d + 1 - \left(2H_0 + \sum_{i=1}^d H_i\right) < s < s_d$. Alors,
(i) On peut trouver des paramètres $\eta \in [2s, 1/2]$ et $p, q \geq 2$ tels que, presque sûrement, pour tout $\phi \in H^{-2s}(\mathbb{R}^d)$, il existe un temps $T_0 > 0$ pour lequel l'équation (1.4.1) *admet une unique solution de Wick u (au sens de la définition 1.4.5) dans l'ensemble*

$$\mathcal{S}_{T_0} := + X_\rho^{s,\eta,(p,q)}(T_0) ,$$

où

$$X_\rho^{s,\eta,(p,q)}(T) := \mathcal{C}([0,T]; H^{-2s}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-2s,q}(\mathbb{R}^d)) \cap L_T^{\frac{1}{\eta}} H_\rho^{-2s+\eta} .$$

(ii) Pour tout $n \geq 1$, notons $\tilde{u}_n$ la solution de Wick de (1.4.1)*, associée à la paire (ρ_n, ρ^2_n). Alors, pour tout s vérifiant* (1.4.4) *et pour toute fonction test $\chi : \mathbb{R}^d \to \mathbb{R}$, la suite $(\chi \tilde{u}_n)_{n \geq 1}$ converge presque sûrement dans $\mathcal{C}([0,T_0]; H^{-2s}(\mathbb{R}^d))$ vers χu, où u est la solution de Wick de l'item (i).*

Le résultat précédent montre que, même avec un bruit et des données initiales très irrégulières (ces dernières vivant dans des espaces de Sobolev d'ordre négatif), on peut établir le caractère bien posé de notre modèle (1.4.1) grâce au gain local du lemme 1.4.6. Précisons tout de même que le paramètre α mesurant la régularité est petit et qu'il diminue lorsque la dimension augmente.
Dans le cas du modèle (1.4.2), on obtient l'énoncé analogue ci-dessous :

Théorème 1.4.8. *Supposons que $1 \leq d \leq 3$ et que la fonction test $\rho : \mathbb{R}^d \to \mathbb{R}$ est de la forme ($\mathbf{F}_\rho$). En outre, supposons que*

$$\alpha_d < \alpha < \frac{d}{2} , \quad \text{où} \quad \alpha_d = \begin{cases} 7/20 & \text{si } d = 1 \\ 18/20 & \text{si } d = 2 \\ 29/20 & \text{si } d = 3 \end{cases} .$$

Soit $s > 0$ tel que $\frac{d}{2} - \alpha < s < s_d$, où

$$s_d = \begin{cases} 3/20 & \text{si } d = 1 \\ 1/10 & \text{si } d = 2 \\ 1/24 & \text{si } d = 3 \end{cases} .$$

Alors,
(i) On peut trouver des paramètres $\eta \in [2s, 1/2]$ et $p, q \geq 2$ tels que, presque sûrement, pour tout $\phi \in H^{-2s}(\mathbb{R}^d)$, il existe un temps $T_0 > 0$ pour lequel l'équation (1.4.2) admet une unique solution de Wick u (au sens de la définition 1.4.5) dans l'ensemble

$$\mathcal{S}_{T_0} := \tilde{\mathsf{I}} + X_\rho^{s,\eta,(p,q)}(T_0) ,$$

où

$$X_\rho^{s,\eta,(p,q)}(T) := \mathcal{C}([0,T]; H^{-2s}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-2s,q}(\mathbb{R}^d)) \cap L_T^{\frac{1}{\eta}} H_\rho^{-2s+\eta} .$$

(ii) Pour tout $n \geq 1$, notons $\tilde{u}_n$ la solution de Wick de (1.4.2), associée à la paire $(\rho\tilde{\mathsf{I}}_n, \rho^2\tilde{\mathsf{V}}_n)$. Alors, pour toute fonction test $\chi : \mathbb{R}^d \to \mathbb{R}$, la suite $(\chi\tilde{u}_n)_{n\geq 1}$ converge presque sûrement dans $\mathcal{C}([0,T_0]; H^{-2s}(\mathbb{R}^d))$ vers χu, où u est la solution de Wick de l'item (i).

Les énoncés sont essentiellement similaires comme le montrent les valeurs de s_d. En fait, une fois les constructions stochastiques des modèles (1.4.1) et (1.4.2) réalisées (qui diffèrent car les bruits diffèrent), la mise en place du point fixe (déterministe) est exactement la même dans les deux modèles. Cette dernière résulte essentiellement des inégalités de Strichartz (voir Lemma 4.3.4) combinées au Lemme 1.4.6. On renvoie le lecteur au Chapitre 4 pour plus de détails.

Dans le cas du modèle (1.4.2), on se propose d'aller plus loin en ce qui concerne les constructions stochastiques. On a vu que $\rho^2\tilde{\mathsf{V}}$ était une fonction du temps à valeurs dans un espace de Sobolev d'ordre $-2s < 0$ et donc, qu'a priori, l'élément

$$-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2\tilde{\mathsf{V}}(\tau,.))d\tau$$

était aussi une fonction du temps, toujours à valeurs dans un espace de Sobolev d'ordre $-2s < 0$ (ou $-2s+\eta$ en considérant l'effet de régularisation local). En fait, on peut construire directement $-i\int_0^t e^{-i(t-\tau)\Delta}(\tilde{\mathsf{V}}(\tau,.))d\tau$ à l'aide d'arguments probabilistes. On commence par se donner une suite de processus réguliers $\left(\tilde{\mathsf{Y}}_n\right)_{n\in\mathbb{N}}$ de la manière suivante.

Définition 1.4.9. *Pour tout $t \in [0,T]$,*

$$\tilde{\mathsf{Y}}_n(t,.) = -i\int_0^t e^{-i(t-\tau)\Delta}(\tilde{\mathsf{V}}_n(\tau,.))d\tau.$$

On démontre alors le résultat de convergence suivant qui met en avant un gain de régularité multilinéaire.

Proposition 1.4.10. *Soit $1 \leq d \leq 3$ et $T > 0$. Soit α et s deux nombres réels tels que*
$$\frac{d}{4} < \alpha < \frac{d}{2} \quad et \quad s > \frac{d}{2} - \alpha.$$
Supposons que
$$\kappa = \begin{cases} 1-\alpha & si\ d=1 \\ \frac{3}{2}-\alpha & si\ d=2 \\ 2-\alpha & si\ d=3\ et\ \alpha \geq 1 \\ 1 & si\ d=3\ et\ \alpha < 1\ . \end{cases}$$
Soit $\chi : \mathbb{R}^d \to \mathbb{R}$ une fonction test et $2 \leq p < \infty$. Alors, la suite $\left(\chi \widetilde{\Upsilon}_n\right)_{n\in\mathbb{N}}$ converge presque sûrement dans l'espace $\mathcal{C}([0,T];\mathcal{W}^{-2s+\kappa,p}(\mathbb{R}^d))$. Notant $\chi\widetilde{\Upsilon}$ la limite presque sûre, on a
$$\chi\widetilde{\Upsilon} \in \mathcal{C}([0,T];\mathcal{W}^{-2s+\kappa,p}(\mathbb{R}^d)).$$

Ce gain de régularité est une nouveauté de cette thèse. Il résulte d'arguments probabilistes combinés à l'effet dispersif de l'équation de Schrödinger. Ce résultat est à rapprocher des travaux de Gubinelli, Koch, Oh et Okamoto [9, 17] qui ont démontré le même type d'énoncé dans le cas de l'équation des ondes. Revenons maintenant à l'équation (1.4.6). En remplaçant $-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2 \widetilde{\vee}_\tau)\, d\tau$ par $\rho^2\widetilde{\Upsilon}$, on obtient
$$\begin{aligned} v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\rho\tilde{\mathsf{I}}_\tau))\, d\tau \\ - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\tilde{\mathsf{I}}_\tau}))\, d\tau + \rho^2\widetilde{\Upsilon}, \quad t\in[0,T]\,. \end{aligned} \tag{1.4.7}$$
Remarquons que remplacer $-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2 \widetilde{\vee}_\tau)\, d\tau$ par $\rho^2\widetilde{\Upsilon}$ dans (1.4.6) change le modèle (1.4.2) puisqu'on a sorti la fonction test ρ^2 de l'intégrale. Précisément, l'égalité précédente correspond à la version déformée suivante de (1.4.2):
$$\begin{cases} i\partial_t u - \Delta u = \rho^2|u|^2 + C_\rho + \langle\nabla\rangle^{-\alpha}\dot{W}\,, \quad t\in[0,T]\,,\ x\in\mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases} \tag{1.4.8}$$
où C_ρ est une variable de renormalisation dont l'expression est donnée par
$$C_\rho = (i\partial_t - \Delta)\left(\rho^2\widetilde{\Upsilon}\right) - \rho^2\widetilde{\vee} - \rho^2\mathbb{E}[|\tilde{\mathsf{I}}|^2].$$
Définissons ce qu'on appelle solution de l'équation (1.4.8) :

Définition 1.4.11. *Un processus stochastique $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ est une solution sur $[0,T]$ de l'équation (1.4.8) si, presque sûrement, le processus $v := u - \tilde{\mathsf{I}}$ est une solution de l'équation mild*
$$\begin{aligned} v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\rho\tilde{\mathsf{I}}_\tau))\, d\tau \\ - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\tilde{\mathsf{I}}_\tau}))\, d\tau + \rho^2\widetilde{\Upsilon}, \quad t\in[0,T]\,. \end{aligned}$$

On peut donc maintenant énoncer notre ultime résultat.

Théorème 1.4.12. *Supposons que $1 \leq d \leq 3$ et que la fonction test $\rho : \mathbb{R}^d \to \mathbb{R}$ est de la forme $(\mathbf{F}_\rho)$. En outre, supposons que*

$$\alpha_d < \alpha < \frac{d}{2}\,, \quad \text{où} \quad \alpha_d = \begin{cases} 1/4 & \text{si } d = 1 \\ 5/6 & \text{si } d = 2 \\ 17/12 & \text{si } d = 3 \end{cases}.$$

Alors,
(i) Il existe $\varepsilon > 0$ suffisamment petit tel que, notant $s > 0$ le réel $s = \frac{d}{2} - \alpha + \varepsilon$, on peut trouver des paramètres $\eta \in [s, 1/2]$ et $p, q \geq 2$ tels que, presque sûrement, pour tout $\phi \in H^{-s}(\mathbb{R}^d)$, il existe un temps $T_0 > 0$ pour lequel l'équation (1.4.8) admet une unique solution u (au sens de la définition 1.4.11) dans l'ensemble

$$\mathcal{S}_{T_0} := \tilde{} + Y_\rho^{s,\eta,(p,q)}(T_0)\,,$$

où

$$Y_\rho^{s,\eta,(p,q)}(T) := \mathcal{C}([0,T]; H^{-s}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-s,q}(\mathbb{R}^d)) \cap L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}\,.$$

(ii) Pour tout $n \geq 1$, notons $\tilde{u}_n$ la solution de (1.4.8), associée à la paire $(\rho\tilde{}_n, \rho^2\tilde{}_n)$. Alors, pour toute fonction test $\chi : \mathbb{R}^d \to \mathbb{R}$, la suite $(\chi\tilde{u}_n)_{n\geq 1}$ converge presque sûrement dans $\mathcal{C}([0,T_0]; H^{-s}(\mathbb{R}^d))$ vers χu, où u est la solution de l'item (i).

En comparant les α_d des théorèmes 1.4.12 et 1.4.8, on observe que les α_d du théorème 1.4.12 sont inférieurs, à d fixé, aux α_d du théorème 1.4.8 et donc qu'on est en mesure de résoudre l'équation (1.4.8) pour des processus plus irréguliers que dans le cas du modèle (1.4.2). En particulier, en dimension $d = 1$, on couvre toute la palette des α $\left(\text{ie } \frac{1}{4} < \alpha < \frac{1}{2}\right)$ pour lesquels on sait construire le terme stochastique $\rho^2\tilde{}$.

On se propose maintenant de mentionner plusieurs perspectives qui prolongeraient les travaux précédents.
Tout d'abord, on constate que la fonction ρ jouant le rôle d'un cut-off est omniprésente. En effet, d'une part, elle permet la construction des différents objets aléatoires dans des espaces de fonctions continues par rapport au temps à valeurs dans des espaces de Sobolev et, d'autre part, elle permet un gain de régularité local provenant de l'opérateur de Schrödinger. Un objectif important serait donc de retirer la fonction ρ des modèles étudiés. Dans le cas du premier point développé, on pourrait envisager de construire les processus stochastiques dans des espaces à poids. Dans le cas du second point, i.e. en ce qui concerne le gain de régularité local, le rôle de la fonction ρ semble essentiel.

Une autre perspective serait d'étudier les modèles (1.4.1) et (1.4.2) avec une non-linéarité cubique, i.e. de la forme $|u|^2 u$. De nouvelles difficultés seraient soulevées par ce modèle, tant lors de la construction des processus stochastiques que dans la mise en place du point fixe déterministe puisque les techniques développées dans cette thèse sont spécifiques aux contrôles des termes quadratiques.

Une dernière perspective serait de considérer une version déformée du modèle (1.4.1) et d'essayer de construire le processus $\rho^2\widetilde{\Upsilon}$ dans le cas d'un bruit fractionnaire en temps et en espace afin de bénéficier d'un gain multilinéaire. A nouveau, on se heurterait probablement à la haute technicité sous-jacente au calcul fractionnaire.

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Chapter 2

Tools from stochastic analysis

The aim of this chapter is to present the different tools that will be of constant use when proving the different results of chapters 3, 4 and 5. To begin with, let us develop some classical results about the fractional Brownian motion. The interested reader will find more information in the classical book [3].

2.1 The fractional Brownian motion and some of its major properties

The definition of the fractional Brownian motion is the following:

Definition 2.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space and $H \in (0,1]$. A fractional Brownian motion of Hurst index H is a centered continuous Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with covariance function given for all $s,t \geq 0$ by*

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

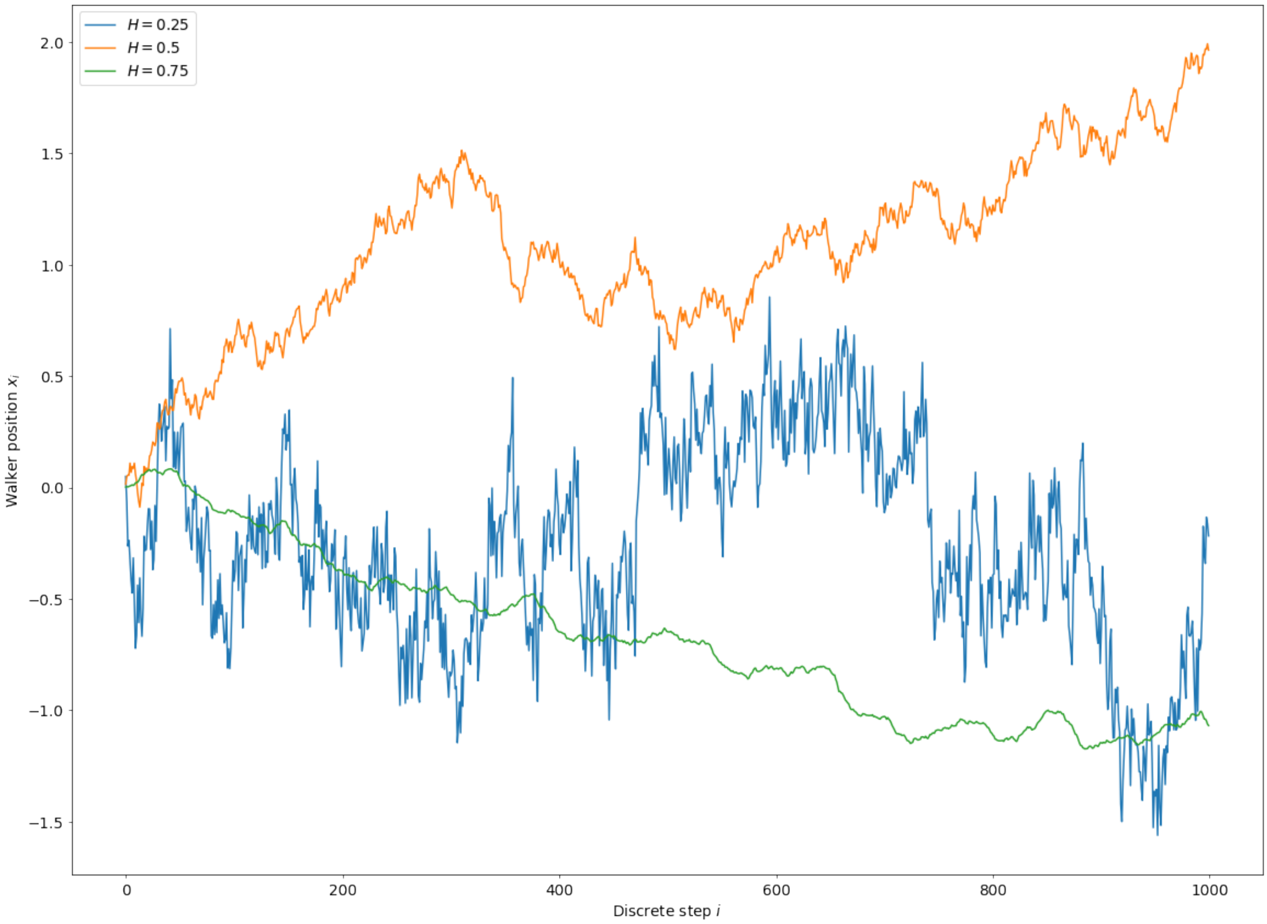
One may wonder whether such a processus does exist. It is indeed the case, according to Kolmogorov theorem (see Theorem 1.2 p 7 in [3]).

Proposition 2.1.2. *Let $H > 0$ be a real parameter. Then, there exists a continuous centered Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with covariance function given for all $s,t \geq 0$ by*

$$\Gamma_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

if, and only if, $H \leq 1$. In this case, the sample paths of B^H are, for any α in $(0,H)$, α-Hölder continuous on each compact set.

The latter proposition specifies the regularity of the fractional Brownian motion. Let us illustrate this.



We can notice that the more the parameter H is close to 0, the more the trajectories are oscillating.
To end this short presentation, let us mention that when $H = \frac{1}{2}$, a fractional Brownian motion is only a classical Brownian motion.

Proposition 2.1.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space and B^H be a fractional Brownian motion of Hurst index $H \in (0,1]$ on Ω.*
1) If $H = \dfrac{1}{2}$, then B^H is a classical Brownian motion.
2) If $H = 1$, then $B_t^H = tB_1^H$ almost surely for all $t \geq 0$. As a result, we will always assume $H \in (0,1)$ in the following.

Proof. 1) In this case, for all $s, t \geq 0$, $\mathbb{E}[B_s^{\frac{1}{2}} B_t^{\frac{1}{2}}] = \frac{1}{2}(t + s - |t-s|)$, and we immediately recognize the covariance function of a classical Brownian motion.
2) Fix $t \geq 0$.

$$\begin{aligned} \mathbb{E}[(B_t^H - tB_1^H)^2] &= \mathbb{E}[(B_t^H)^2] + t^2\mathbb{E}[(B_1^H)^2] - 2t\mathbb{E}[B_t^H B_1^H] \\ &= t^2 + t^2 - t(1 + t^2 - (t-1)^2) \\ &= 0. \end{aligned}$$

Consequently, almost surely, $B_t^H = tB_1^H$. □

2.2 Wiener integral

In this section, we recall the definition of Wiener integral. We begin by considering real-valued functions.

Definition 2.2.1. *Let $d \geq 1$ be a space dimension and $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Let W be a space white noise on $\mathbb{R}^d$. Let $(f_t)_{t\geq 0}$ be a family of measurable functions in $L^2(\mathbb{R}^d, \mathbb{R})$. We call Wiener integral of the family $(f_t)_{t\geq 0}$ the real centered Gaussian process $\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)_{t\geq 0}$ whose covariance function is given for all $s, t \geq 0$ by*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)W(d\eta)\right)\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)\right] = \int_{\mathbb{R}^d} f_s(\eta)f_t(\eta)d\eta.$$

Let us remark that, since the integrated functions are deterministic, the definition of $\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)_{t\geq 0}$ is quite easy compared with Itô's integral. Let us extend this definition to complex-valued functions.

Definition 2.2.2. *Let $d \geq 1$ be a space dimension and $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Let W be a space white noise on $\mathbb{R}^d$. Let $(f_t)_{t\geq 0}$ be a family of measurable functions in $L^2(\mathbb{R}^d, \mathbb{C})$. We call Wiener integral of the family $(f_t)_{t\geq 0}$ the process*

$$\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)_{t\geq 0} = \left(\int_{\mathbb{R}^d} Re(f_t(\eta))W(d\eta) + i\int_{\mathbb{R}^d} Im(f_t(\eta))W(d\eta)\right)_{t\geq 0}.$$

In this case, $\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)_{t\geq 0}$ is a complex Gaussian process as stated by the following proposition.

Proposition 2.2.3. *Under the assumptions of the latter definition, the process $\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)_{t\geq 0}$ is a complex centered Gaussian process verifying for all $s, t \geq 0$,*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)W(d\eta)\right)\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)\right] = \int_{\mathbb{R}^d} f_s(\eta)f_t(\eta)d\eta$$

and

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)W(d\eta)\right)\overline{\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)}\right] = \int_{\mathbb{R}^d} f_s(\eta)\overline{f_t(\eta)}d\eta.$$

Proof. It is clear that the process under consideration is Gaussian and centered. Let

$s, t \geq 0$. It holds that:

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)W(d\eta)\right)\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)\right] &= \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_t(\eta))W(d\eta)\right)\right] \\
&+ i\mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_t(\eta))W(d\eta)\right)\right] \\
&+ i\mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_t(\eta))W(d\eta)\right)\right] \\
&- \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_t(\eta))W(d\eta)\right)\right] \\
= \int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))\mathrm{Re}(f_t(\eta))d\eta + i\int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))\mathrm{Im}(f_t(\eta))d\eta & \\
+ i\int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))\mathrm{Re}(f_t(\eta))d\eta - \int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))\mathrm{Im}(f_t(\eta))d\eta & \\
= \int_{\mathbb{R}^d} \left(\mathrm{Re}(f_s(\eta)) + i\mathrm{Im}(f_s(\eta))\right)\left(\mathrm{Re}(f_t(\eta)) + i\mathrm{Im}(f_t(\eta))\right)d\eta & \\
= \int_{\mathbb{R}^d} f_s(\eta)f_t(\eta)d\eta. &
\end{aligned}
$$

And the second identity results from the following computation:

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)W(d\eta)\right)\overline{\left(\int_{\mathbb{R}^d} f_t(\eta)W(d\eta)\right)}\right] &= \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_t(\eta))W(d\eta)\right)\right] \\
&- i\mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_t(\eta))W(d\eta)\right)\right] \\
&+ i\mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Re}(f_t(\eta))W(d\eta)\right)\right] \\
&+ \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))W(d\eta)\right)\left(\int_{\mathbb{R}^d} \mathrm{Im}(f_t(\eta))W(d\eta)\right)\right] \\
= \int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))\mathrm{Re}(f_t(\eta))d\eta - i\int_{\mathbb{R}^d} \mathrm{Re}(f_s(\eta))\mathrm{Im}(f_t(\eta))d\eta & \\
+ i\int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))\mathrm{Re}(f_t(\eta))d\eta + \int_{\mathbb{R}^d} \mathrm{Im}(f_s(\eta))\mathrm{Im}(f_t(\eta))d\eta & \\
= \int_{\mathbb{R}^d} \left(\mathrm{Re}(f_s(\eta)) + i\mathrm{Im}(f_s(\eta))\right)\left(\mathrm{Re}(f_t(\eta)) - i\mathrm{Im}(f_t(\eta))\right)d\eta & \\
= \int_{\mathbb{R}^d} f_s(\eta)\overline{f_t(\eta)}d\eta. &
\end{aligned}
$$

□

2.3 Integral against the Fourier transform of a white noise

In this section, we define the notion of integral against the Fourier transform of a white noise. This will permit us to consider what is called the harmonizable representation of the fractional Brownian motion, a precious tool to make computations.

Definition 2.3.1. *Let $d \geq 1$ be a space dimension and $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Let W be a space white noise on $\mathbb{R}^d$. Let $(f_t)_{t\geq 0}$ be a family of measurable functions in $L^2(\mathbb{R}^d, \mathbb{C})$. We call integral against the Fourier transform of W of the family $(f_t)_{t\geq 0}$ the complex centered Gaussian process*

$$\left(\int_{\mathbb{R}^d} f_t(\eta)\widehat{W}(d\eta)\right)_{t\geq 0} = \left(\int_{\mathbb{R}^d} \hat{f}_t(\eta)W(d\eta)\right)_{t\geq 0}.$$

As $\left(\int_{\mathbb{R}^d} f_t(\eta)\widehat{W}(d\eta)\right)_{t\geq 0}$ is a complex Gaussian process, it is caracterized by its mean and its covariance functions.

Proposition 2.3.2. *Under the assumptions of the latter definition, the process $\left(\int_{\mathbb{R}^d} f_t(\eta)\widehat{W}(d\eta)\right)_{t\geq 0}$ is a complex centered Gaussian process verifying for all $s, t \geq 0$,*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)\widehat{W}(d\eta)\right)\left(\int_{\mathbb{R}^d} f_t(\eta)\widehat{W}(d\eta)\right)\right] = \int_{\mathbb{R}^d} f_s(\eta)\overline{f_t(-\eta)}d\eta$$

and

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)\widehat{W}(d\eta)\right)\overline{\left(\int_{\mathbb{R}^d} f_t(\eta)\widehat{W}(d\eta)\right)}\right] = \int_{\mathbb{R}^d} f_s(\eta)\overline{f_t(\eta)}d\eta.$$

Proof. Let $s, t \geq 0$. On the one hand, denoting by $\tilde{f}$ the function defined for all $\eta \in \mathbb{R}^d$ by $\tilde{f}(\eta) = f(-\eta)$ and resorting to Parseval's identity, we write:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)\widehat{W}(d\eta)\right)\left(\int_{\mathbb{R}^d} f_t(\eta)\widehat{W}(d\eta)\right)\right] = \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \hat{f}_s(\eta)W(d\eta)\right)\left(\int_{\mathbb{R}^d} \hat{f}_t(\eta)W(d\eta)\right)\right]\\
&= \int_{\mathbb{R}^d} \hat{f}_s(\eta)\hat{f}_t(\eta)d\eta = \int_{\mathbb{R}^d} \hat{f}_s(\eta)\overline{\overline{\hat{f}_t(\eta)}}d\eta\\
&= \int_{\mathbb{R}^d} \hat{f}_s(\eta)\overline{\hat{\tilde{f}}_t(\eta)}d\eta = \int_{\mathbb{R}^d} f_s(\eta)\overline{\tilde{f}_t(\eta)}d\eta\\
&= \int_{\mathbb{R}^d} f_s(\eta)\overline{f_t(-\eta)}d\eta.
\end{aligned}$$

On the other hand, using Parseval's identity again, we derive:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f_s(\eta)\widehat{W}(d\eta)\right)\overline{\left(\int_{\mathbb{R}^d} f_t(\eta)\widehat{W}(d\eta)\right)}\right] = \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \hat{f}_s(\eta)W(d\eta)\right)\overline{\left(\int_{\mathbb{R}^d} \hat{f}_t(\eta)W(d\eta)\right)}\right]\\
&= \int_{\mathbb{R}^d} \hat{f}_s(\eta)\overline{\hat{f}_t(\eta)}d\eta = \int_{\mathbb{R}^d} f_s(\eta)\overline{f_t(\eta)}d\eta.
\end{aligned}$$

□

Remark 2.3.3. We define

$$L^2_{even}(\mathbb{C}) := \left\{\phi \in L^2(\mathbb{R}^d, \mathbb{C}) : \forall \eta \in \mathbb{R}^d, \phi(-\eta) = \overline{\phi(\eta)}\right\}.$$

For all $t \geq 0$ and for every $\phi_t \in L^2_{even}(\mathbb{C})$, the Fourier transform of ϕ_t is real-valued and, consequently, the stochastic process $\left(\int_{\mathbb{R}^d} \phi_t(\eta)\widehat{W}(d\eta)\right)_{t\geq 0}$ is a real centered Gaussian process. Indeed,

$$\begin{aligned}
\overline{\int_{\mathbb{R}^d} \phi_t(\eta)\widehat{W}(d\eta)} &= \overline{\int_{\mathbb{R}^d} \hat{\phi}_t(\eta)W(d\eta)} \\
&= \int_{\mathbb{R}^d} \overline{\hat{\phi}_t(\eta)}W(d\eta) = \int_{\mathbb{R}^d} \hat{\phi}_t(\eta)W(d\eta) \\
&= \int_{\mathbb{R}^d} \phi_t(\eta)\widehat{W}(d\eta).
\end{aligned}$$

2.4 Harmonizable representation of the fractional Brownian motion

In this section, we focus on the case where the space dimension d equals 1. We have seen that the fractional Brownian motion is a centered Gaussian process. Now, we would like a formula that will allow us to make computations easily. That is precisely the objective of the harmonizable representation below.

Proposition 2.4.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Let W be a real Brownian motion on Ω. We consider the centered Gaussian process R defined for all $t \geq 0$ by:*

$$R_t := \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+\frac{1}{2}}} \widehat{W}(d\xi).$$

Then, we can find $c_H > 0$ such that $c_H R$ is a fractional Brownian motion of Hurst index H.

Remark 2.4.2. The definition of the harmonizable representation can easily be extended to space dimensions $d \geq 1$. See for instance equality (3.4.1) in Chapter 3 or equality (4.2.1) in Chapter 4.

Proof. First of all, let us show that R is well-defined. Let ϕ the function defined for all $t \geq 0$ and for every $\xi \in \mathbb{R}$ by the formula

$$\phi_t(\xi) := \frac{e^{it\xi} - 1}{|\xi|^{H+\frac{1}{2}}}.$$

Let $t \geq 0$. Let us check that $\phi_t \in L^2_{even}(\mathbb{C})$. It holds that for all $\xi \in \mathbb{R}$, $\phi_t(-\xi) = \dfrac{e^{-it\xi} - 1}{|\xi|^{H+\frac{1}{2}}} = \dfrac{\overline{e^{it\xi} - 1}}{|\xi|^{H+\frac{1}{2}}} = \overline{\phi_t(\xi)}$. Besides, $|\phi_t(\xi)|^2 \leq \dfrac{4}{|\xi|^{2H+1}}$. As $2H + 1 > 1$ (since $H > 0$), $|\phi_t|^2$ is integrable on $[1, +\infty[$ and on $]-\infty, -1]$. For ξ near to 0,

$$\phi_t(\xi) = \frac{(1 + it\xi + o(\xi)) - 1}{|\xi|^{H+\frac{1}{2}}} = \frac{it\xi + o(\xi)}{|\xi|^{H+\frac{1}{2}}}.$$

As a result, $|\phi_t(\xi)| \leq \dfrac{C_t}{|\xi|^{H-\frac{1}{2}}}$ that is integrable on $]0, 1]$ and on $[-1, 0[$ because $2\Big(H - \frac{1}{2}\Big) = 2H - 1 < 1$ if, and only if, $H < 1$. Finally, for all $t \geq 0$,

$$\phi_t \in L^2_{even}(\mathbb{C}).$$

Now, fix $s, t \geq 0$ such that $s \leq t$.

$$\begin{aligned}
\mathbb{E}[R_s R_t] &= \mathbb{E}\left[\left(\int_{\mathbb{R}} \phi_s(\xi)\widehat{W}(d\xi)\right)\left(\int_{\mathbb{R}} \phi_t(\xi)\widehat{W}(d\xi)\right)\right] \\
&= \int_{\mathbb{R}} \phi_s(\xi)\overline{\phi_t(\xi)}d\xi \\
&= \int_{\mathbb{R}} \frac{(e^{is\xi}-1)(e^{-it\xi}-1)}{|\xi|^{2H+1}}d\xi \\
&= \int_{\mathbb{R}} \frac{e^{i\xi(s-t)} - e^{is\xi} - e^{-it\xi} + 1}{|\xi|^{2H+1}}d\xi \\
&= \int_{\mathbb{R}} \frac{e^{i\xi(s-t)}-1}{|\xi|^{2H+1}}d\xi - \int_{\mathbb{R}} \frac{e^{is\xi}-1}{|\xi|^{2H+1}}d\xi - \int_{\mathbb{R}} \frac{e^{-it\xi}-1}{|\xi|^{2H+1}}d\xi \\
&= (t-s)^{2H}\left(\int_{\mathbb{R}} \frac{e^{iu}-1}{|u|^{2H+1}}du\right) - s^{2H}\left(\int_{\mathbb{R}} \frac{e^{iu}-1}{|u|^{2H+1}}du\right) - t^{2H}\left(\int_{\mathbb{R}} \frac{e^{iu}-1}{|u|^{2H+1}}du\right).
\end{aligned}$$

This computation leads to the following identity

$$\mathbb{E}[R_s R_t] = \left(2\int_{\mathbb{R}} \frac{1-e^{iu}}{|u|^{2H+1}}du\right)\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

As

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1-e^{iu}}{|u|^{2H+1}}du &= \int_0^{+\infty} \frac{1-e^{iu}}{u^{2H+1}}du + \int_{-\infty}^0 \frac{1-e^{iu}}{(-u)^{2H+1}}du \\
&= \int_0^{+\infty} \frac{1-e^{iu}}{u^{2H+1}}du + \int_0^{+\infty} \frac{1-e^{-iu}}{u^{2H+1}}du \\
&= 2\int_0^{+\infty} \frac{1-\cos(u)}{u^{2H+1}}du > 0,
\end{aligned}$$

we get the desired result with

$$c_H = \frac{1}{2\sqrt{\int_0^{+\infty} \frac{1-\cos(u)}{u^{2H+1}}du}}.$$

□

Thanks to the harmonizable representation, we are in a position to define, for every $t \geq 0$,

$$B_n(t) := \int_{|\xi|\leq 2^n} \frac{e^{it\xi}-1}{|\xi|^{H+\frac{1}{2}}}\widehat{W}(d\xi),$$

a smooth approximation of the fractional Brownian motion.

Proposition 2.4.3. *Fix $t \geq 0$. Then,*

$$\mathbb{E}[|B_n(t) - B(t)|^2] \underset{n\to+\infty}{\longrightarrow} 0.$$

To put it differently,

$$B_n(t) \underset{n\to\infty}{\overset{L^2(\Omega)}{\longrightarrow}} B(t).$$

Proof. We see that

$$\begin{aligned}
\mathbb{E}[|B_n(t)-B(t)|^2] &= \mathbb{E}\left[\left(c_H\int_{\mathbb{R}}(\mathbb{1}_{\{|\xi|\le 2^n\}}-1)\frac{e^{it\xi}-1}{|\xi|^{H+\frac{1}{2}}}\widehat{W}(d\xi)\right)^2\right]\\
&= c_H^2\int_{\mathbb{R}}|\mathbb{1}_{\{|\xi|\le 2^n\}}-1|^2\frac{|e^{it\xi}-1|^2}{|\xi|^{2H+1}}d\xi,
\end{aligned}$$

that tends to 0 by virtue of Lebesgue's theorem. □

In fact, we can even show an almost sure convergence as developed below.

Proposition 2.4.4. *Let $T>0$. Almost surely, the sequence $(B_n(\omega))_{n\ge 1}$ uniformly converges to $B(\omega)$ on $[0,T]$.*

Proof. The key of the proof lies in a smart use of the Garsia-Rodemich-Rumsey inequality (see Theorem 2.6.1) coupled with the Borel-Cantelli lemma. Let f_n the function defined for all $n\ge 1$, for all $t\in[0,T]$ by the formula: $f_n(t):=B_n(t)-B(t)$. For the sake of simplicity, we will suppose that c_H equals 1. We write

$$\begin{aligned}
\mathbb{E}[|f_n(s)-f_n(t)|^2] &= \mathbb{E}[|B_n(s)-B(s)-B_n(t)+B(t)|^2]\\
&= \mathbb{E}\left[\left(\int_{\mathbb{R}}\mathbb{1}_{\{|\xi|>2^n\}}\frac{(e^{is\xi}-1)-(e^{it\xi}-1)}{|\xi|^{H+\frac{1}{2}}}\widehat{W}(d\xi)\right)^2\right]\\
&= \int_{\mathbb{R}}\mathbb{1}_{\{|\xi|>2^n\}}\frac{|e^{is\xi}-e^{it\xi}|^2}{|\xi|^{2H+1}}d\xi\\
&= 2\int_{2^n}^{+\infty}\frac{|e^{is\xi}-e^{it\xi}|^2}{\xi^{2H+1}}d\xi.
\end{aligned}$$

Now, fix $\eta>\in]0,1[$. Resorting to the mean value theorem, we get

$$\begin{aligned}
|e^{is\xi}-e^{it\xi}|^2 &= |e^{is\xi}-e^{it\xi}|^{2\eta}|e^{is\xi}-e^{it\xi}|^{2(1-\eta)}\\
&\le 4|s-t|^{2\eta}|\xi|^{2\eta}.
\end{aligned}$$

Consequently, as soon as $\eta<H$, it holds that

$$\begin{aligned}
\mathbb{E}[|f_n(s)-f_n(t)|^2] &\le 8|s-t|^{2\eta}\int_{2^n}^{+\infty}\xi^{2\eta-2H-1}d\xi\\
&= 8|s-t|^{2\eta}\left[\frac{\xi^{2\eta-2H}}{2\eta-2H}\right]_{2^n}^{+\infty}\\
&= \frac{4}{H-\eta}|s-t|^{2\eta}\frac{1}{4^{n(H-\eta)}}\\
&= C_{\eta,H}|s-t|^{2\eta}\frac{1}{4^{n(H-\eta)}}.
\end{aligned}$$

With the help of the Garsia-Rodemich-Rumsey inequality, we obtain that, for all $0\le s,t\le T$,

$$|f_n(s)-f_n(t)|^p\le C_{\alpha,p}|s-t|^{\alpha p-2}\int_0^T\int_0^T\frac{|f_n(x)-f_n(y)|^p}{|x-y|^{\alpha p}}dxdy.$$

Picking $s = 0$ and taking the supremum on $t \in [0, T]$ yields

$$\sup_{t \in [0,T]} |f_n(t)|^p \leq C_{\alpha,p} T^{\alpha p - 2} \int_0^T \int_0^T \frac{|f_n(x) - f_n(y)|^p}{|x-y|^{\alpha p}} dx dy,$$

when $\alpha > \frac{2}{p}$. A standard application of Tonelli's theorem entails:

$$\begin{aligned}
\mathbb{E}[\sup_{t \in [0,T]} |f_n(t)|^p] &\leq C_{\alpha,p,T} \int_0^T \int_0^T \frac{\mathbb{E}[|f_n(x) - f_n(y)|^p]}{|x-y|^{\alpha p}} dx dy \\
&\leq C_{\alpha,p,T,\eta,H} \left(\int_0^T \int_0^T \frac{1}{|x-y|^{p(\alpha-\eta)}} dx dy \right) \frac{1}{2^{n(H-\eta)p}} \\
&\leq C_{\alpha,p,T,\eta,H} \frac{1}{2^{n(H-\eta)p}},
\end{aligned}$$

as soon as $p(\alpha - \eta) < 1$, ie $\alpha < \frac{1}{p} + \eta$.
Let us fix p large enough once and for all so that $\frac{2}{p} < \alpha < \frac{1}{p} + \eta$. We are now in a position to prove the uniform convergence of f_n to 0. Let $\varepsilon > 0$. Markov's inequality immediately implies

$$\begin{aligned}
\mathbb{P}\left(\sup_{t \in [0,T]} |f_n(t)| > \varepsilon \right) &\leq \frac{\mathbb{E}\left[\sup_{t \in [0,T]} |f_n(t)|^p \right]}{\varepsilon^p} \\
&\leq C_{\alpha,p,T,\eta,H,\varepsilon} \frac{1}{2^{n(H-\eta)p}}.
\end{aligned}$$

The Borel-Cantelli lemma ensures that

$$\mathbb{P}\left(\limsup_{n \to +\infty} \left\{ \sup_{t \in [0,T]} |f_n(t)| > \varepsilon \right\} \right) = 0.$$

As a result, $\mathbb{P}\left(\liminf_{n \to +\infty} \left\{ \sup_{t \in [0,T]} |f_n(t)| \leq \varepsilon \right\} \right) = 1$. For all $k \geq 1$, denoting

$$\Omega_k := \liminf_{n \to +\infty} \left\{ \sup_{t \in [0,T]} |f_n(t)| \leq \frac{1}{k} \right\}$$

and considering $\tilde{\Omega} := \cap_{k \geq 1} \Omega_k$, we see that $\mathbb{P}(\tilde{\Omega}) = 1$ and that for all $\omega \in \tilde{\Omega}$, for all $k \geq 1$, there exists $N_k(\omega)$ such that for all $n \geq N_k(\omega)$,

$$\sup_{t \in [0,T]} |f_n(t)|(\omega) \leq \frac{1}{k}.$$

Finally, almost surely, $\sup_{t \in [0,T]} |f_n(t)|(\omega)$ tends to 0, that is the desired conclusion. □

2.5 Kolmogorov criterion

The main result to measure the regularity of processes is Kolmogorov criterion whose statement is the following. A proof can be found in [1].

Theorem 2.5.1. *Let T be a positive time and $(X_t)_{t\in[0,T]}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in (E, d) a complete metric space. Assume that there exists constants $\alpha, \beta, K > 0$ satisfying the inequality*

$$\mathbb{E}[d(X_t, X_s)^\alpha] \leq K|t-s|^{1+\beta},$$

for all $0 \leq s \leq t \leq T$. Then, X_t has a version $\tilde{X}_t$ which is Hölder-continuous with exponent γ for every $\gamma \in \left(0, \frac{\beta}{\alpha}\right)$, i.e. almost surely, for all $0 \leq s \leq t \leq T$,

$$d(\tilde{X}_t(\omega), \tilde{X}_s(\omega)) \leq C_\gamma(\omega)|t-s|^\gamma.$$

2.6 Garsia, Rodemich and Rumsey inequality

Another useful tool, often combined with Kolmogorov criterion is Garsia, Rodemich and Rumsey inequality (see the proof of Proposition 4.1.2 for instance). It can be interpreted as some kind of Sobolev embedding (see Example 2.6.3). We propose a detailed proof of this result.

Theorem 2.6.1. *Let Ψ and p be two continuous strictly increasing functions on $[0, \infty)$ with $p(0) = \Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$. Assume that f is a continuous function on $[0, T]$ with values in (E, d) a complete metric space which satisfies*

$$\int_0^T \int_0^T \Psi\left(\frac{d(f(t), f(s))}{p(|t-s|)}\right) ds dt = B < \infty. \tag{2.6.1}$$

Then, for all $0 \leq s \leq t \leq T$, it holds

$$d(f(t), f(s)) \leq 8 \int_0^{t-s} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u). \tag{2.6.2}$$

In particular, the modulus of continuity $\omega_f(\delta)$ satisfies

$$\omega_f(\delta) = \sup_{\substack{0\leq s,t\leq T \\ |t-s|\leq\delta}} d(f(t), f(s)) \leq 8 \int_0^\delta \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u). \tag{2.6.3}$$

Remark 2.6.2. We observe that the double integral in (2.6.1) has a singularity on the diagonal and its finiteness depends on f, p and Ψ. The integral in (2.6.2) has a singularity at $u = 0$ and thus its convergence requires some conditions on Ψ and p. The existence of a pair (Ψ, p) satisfying these conditions will turn out to imply some regularity on f.

Proof. For simplicity, we will assume for the whole proof that $T = 1$.
Step 1: Let us show that

$$d(f(1), f(0)) \leq 8 \int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u). \tag{2.6.4}$$

We define

$$I(t) = \int_0^1 \Psi\left(\frac{d(f(t), f(s))}{p(|t-s|)}\right) ds \quad \textit{and} \quad B = \int_0^1 I(t)dt.$$

There exists $t_0 \in (0, 1)$ such that $I(t_0) \leq B$. Indeed, suppose on the contrary that for all $t \in (0, 1)$, $I(t) > B$ and consider, for all $k \geq 1$, the subset $E_k := \left\{t \in (0, 1); I(t) \geq B + \frac{1}{k}\right\}$. So, the sequence (E_k) is increasing with limit $\cup_{k\geq 1} E_k = \{t \in (0, 1); I(t) > B\} = (0, 1)$. As the measure of the latter subset is strictly positive, we can find $k_0 \geq 1$ such that $m(E_{k_0}) > 0$ and

$$\int_0^1 I(t)dt \geq \left(B + \frac{1}{k_0}\right) m(E_{k_0}) + B(1 - m(E_{k_0})) = \frac{1}{k_0} m(E_{k_0}) + B > B,$$

which is false.
We shall prove that

$$d(f(t_0), f(0)) \leq 4 \int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u). \tag{2.6.5}$$

By a similar argument, it could be shown that

$$d(f(1), f(t_0)) \leq 4 \int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u). \tag{2.6.6}$$

Combining the two latter inequalities immediately yields to the desired conclusion, namely (2.6.4). To proof (2.6.5), we will pick recursively two sequences (u_n) and (t_n) satisfying

$$t_0 > u_1 > t_1 > u_2 > t_2 > ... > u_n > t_n > ...$$

in the following manner. By induction, suppose that $u_{n-1} > t_{n-1} > 0$ have already been constructed. We define $d_n = p(t_{n-1})$ and, since p is continuous, according to the intermediate value theorem, there exists $0 < u_n < t_{n-1}$ so that $p(u_n) = \frac{d_n}{2}$. Then, since $u_n \leq 1$,

$$\int_0^{u_n} I(t)dt \leq B \quad \textit{and} \quad \int_0^{u_n} \Psi\left(\frac{d(f(t_{n-1}), f(s))}{p(|t_{n-1} - s|)}\right) ds \leq I(t_{n-1}). \tag{2.6.7}$$

We select $0 < t_n < u_n$ so that

$$I(t_n) \leq \frac{2B}{u_n} \quad \textit{and} \quad \Psi\left(\frac{d(f(t_{n-1}), f(t_n))}{p(|t_{n-1} - t_n|)}\right) \leq \frac{2I(t_{n-1})}{u_n}. \tag{2.6.8}$$

It is always possible to find t_n such that the two previous inequalities hold simultaneously, since each of the two inequalities can be violated only on a set of measure strictly less than $\frac{u_n}{2}$.
a) Assume that the first inequality fails on the set X, that is

$$I(t) > \frac{2B}{u_n} \quad \text{for all } t \in X.$$

Thus, if $m(X) \geq \frac{u_n}{2}$, then we consider, for all $k \geq 1$, the subset $A_k := \left\{t \in (0, u_n); I(t) \geq \frac{2B}{u_n} + \frac{1}{k}\right\}$. So, the sequence (A_k) is increasing with limit $\cup_{k\geq 1} A_k = X$. As $m(X) > 0$, we can find $k_0 \geq 1$ such that $m(A_{k_0}) > 0$ and

$$\begin{aligned}\int_0^{u_n} I(t)dt \geq& \left(\frac{2B}{u_n} + \frac{1}{k_0}\right) m(A_{k_0}) + \frac{2B}{u_n}(m(X) - m(A_{k_0}))\\ =& \frac{1}{k_0} m(A_{k_0}) + \frac{2B}{u_n} m(X) \geq \frac{1}{k_0} m(A_{k_0}) + B > B,\end{aligned}$$

which contradicts (2.6.7).
b) Assume that the second inequality fails on the set Y, that is

$$\Psi\left(\frac{d(f(t_{n-1}), f(t))}{p(|t_{n-1} - t|)}\right) > \frac{2I(t_{n-1})}{u_n} \quad \text{for all } t \in Y.$$

Thus, if $m(Y) \geq \frac{u_n}{2}$, then as above

$$\int_0^{u_n} \Psi\left(\frac{d(f(t_{n-1}), f(s))}{p(|t_{n-1} - s|)}\right) ds > \frac{2I(t_{n-1})}{u_n} m(Y) \geq I(t_{n-1}),$$

which contradicts (2.6.7). Finally, $m(X \cup Y) \leq m(X) + m(Y) < u_n$ and the complementary of $X \cup Y$ in $(0, u_n)$ is of measure strictly positive and is consequently not empty.
Coming back to (2.6.8), one has

$$\Psi\left(\frac{d(f(t_{n-1}), f(t_n))}{p(|t_{n-1} - t_n|)}\right) \leq \frac{2I(t_{n-1})}{u_n} \leq \frac{4B}{u_{n-1}u_n} \leq \frac{4B}{u_n^2}.$$

It entails

$$d(f(t_{n-1}), f(t_n)) \leq \Psi^{-1}\left(\frac{4B}{u_n^2}\right) p(t_{n-1} - t_n) \leq \Psi^{-1}\left(\frac{4B}{u_n^2}\right) p(t_{n-1}).$$

We now remark that, since $p(u_{n+1}) = \frac{p(t_n)}{2} \leq \frac{p(u_n)}{2}$,

$$p(t_{n-1}) = 2p(u_n) = 4(p(u_n) - \frac{1}{2}p(u_n)) \leq 4(p(u_n) - p(u_{n+1})).$$

Then,

$$\begin{aligned} d(f(t_{n-1}), f(t_n)) &\leq 4\Psi^{-1}\left(\frac{4B}{u_n^2}\right)(p(u_n) - p(u_{n+1})) \\ &\leq 4\Psi^{-1}\left(\frac{4B}{u_{n+1}^2}\right)(p(u_n) - p(u_{n+1})) \\ &\leq 4\int_{u_{n+1}}^{u_n} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u). \end{aligned}$$

For all $N \geq 1$, we deduce that

$$d(f(t_0), f(t_N)) \leq \sum_{n=1}^{N} d(f(t_{n-1}), f(t_n)) \leq 4\sum_{n=1}^{N} \int_{u_{n+1}}^{u_n} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u) \leq 4\int_{u_{N+1}}^{u_1} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u).$$

A quick observation reveals that the sequences (u_n) and (t_n) are decreasing and greater than 0 and therefore converge. Moreover, by construction, they have the same limit l, which is in fact 0 since, as p is continuous, $p(l) = \frac{p(l)}{2}$. Taking the limit over N, we can conclude that

$$d(f(t_0), f(0)) \leq 4\int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u).$$

Step 2: Let us prove that, if $0 \leq T_1 \leq T_2 \leq 1$, then

$$d(f(T_2), f(T_1)) \leq 4\int_0^{T_2-T_1} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u). \tag{2.6.9}$$

We denote

$$B_{0,1} = \int_0^1 \int_0^1 \Psi\left(\frac{d(f(t), f(s))}{p(|t-s|)}\right) dsdt < \infty \quad and \quad B_{T_1,T_2} = \int_{T_1}^{T_2} \int_{T_1}^{T_2} \Psi\left(\frac{d(f(t), f(s))}{p(|t-s|)}\right) dsdt.$$

The key argument is to map the interval $[T_1, T_2]$ into $[0, 1]$ by $t' = \frac{t-T_1}{T_2-T_1}$ and to redefine

$$\tilde{f} := f(T_1 + (T_2 - T_1)t) \quad and \quad \tilde{p} := p((T_2 - T_1)t).$$

Then,

$$\int_0^1 \int_0^1 \Psi\left(\frac{d(\tilde{f}(t), \tilde{f}(s))}{\tilde{p}(|t-s|)}\right) dsdt = \frac{1}{(T_2 - T_1)^2} \int_{T_1}^{T_2} \int_{T_1}^{T_2} \Psi\left(\frac{d(f(t), f(s))}{p(|t-s|)}\right) dsdt = \frac{B_{T_1,T_2}}{(T_2 - T1)^2}.$$

The latter quantity is finite because smaller than $\frac{B_{0,1}}{(T_2-T_1)^2}$. Applying the first step, we deduce that

$$\begin{aligned} d(f(T_2), f(T_1)) &= d(\tilde{f}(1), \tilde{f}(0)) \\ &\leq 8 \int_0^1 \Psi^{-1}\left(\frac{4B_{T_1,T_2}}{(T_2-T_1)^2 u^2}\right) d\tilde{p}(u) \\ &= 8 \int_0^{T_2-T_1} \Psi^{-1}\left(\frac{4B_{T_1,T_2}}{u^2}\right) dp(u) \\ &\leq 8 \int_0^{T_2-T_1} \Psi^{-1}\left(\frac{4B_{0,1}}{u^2}\right) dp(u), \end{aligned}$$

which is the desired bound. □

Example 2.6.3.
• Let us apply the previous theorem to continuous processes defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfy

$$\mathbb{E}[|X_t - X_s|^\beta] \leq C|t-s|^{1+\alpha}$$

on $[0,T]$, where $\alpha, \beta, C > 0$. We insist on the fact that the continuity is a hypothesis (often verified thanks to Kolmogorov criterion). Now, with $\Psi(x) = x^\beta$ and $p(u) = u^{\frac{\gamma}{\beta}}$ (with $2 \leq \gamma < \alpha + 2$),

$$\begin{aligned} \mathbb{E}\left[\int_0^T \int_0^T \Psi\left(\frac{|X_t - X_s|}{p(|t-s|)}\right) dsdt\right] &= \int_0^T \int_0^T \mathbb{E}\left[\frac{|X_t - X_s|^\beta}{|t-s|^\gamma}\right] dsdt \\ &\leq C \int_0^T \int_0^T |t-s|^{1+\alpha-\gamma} dsdt \\ &\leq C_\delta < \infty \end{aligned}$$

where $\delta = 2 + \alpha - \gamma > 0$. By Tonelli's theorem, almost surely,

$$\int_0^T \int_0^T \Psi\left(\frac{|X_t - X_s|(\omega)}{p(|t-s|)}\right) dsdt = B(\omega) < \infty.$$

We deduce that

$$\begin{aligned} \sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq \delta}} |X_t - X_s|(\omega) \leq 8 \int_0^\delta \left(\frac{4B(\omega)}{u^2}\right)^{\frac{1}{\beta}} dp(u) &= 8\frac{\gamma}{\beta}(4B(\omega))^{\frac{1}{\beta}} \int_0^\delta u^{\frac{\gamma-2}{\beta}-1} du \\ &= 8\frac{\gamma}{\gamma-2}(4B(\omega))^{\frac{1}{\beta}} \delta^{\frac{\gamma-2}{\beta}}. \end{aligned}$$

This proves the Hölder continuity of X with exponent $\frac{\gamma-2}{\beta}$ which can be anything less than $\frac{\alpha}{\beta}$. We have thus obtained the second point of the Kolmogorov criterion in a different way.

• The Garsia-Rodemich-Rumsey theorem allows us to derive the following Sobolev embedding inequality:

$$|f(s) - f(t)| \leq C_{\alpha,p}|t-s|^{\alpha-\frac{1}{p}} \left(\int_0^1 \int_0^1 \frac{|f(u)-f(v)|^p}{|u-v|^{\alpha p+1}} dudv \right)^{\frac{1}{p}},$$

where $p\alpha > 1$. It suffices to consider $\Psi(x) = x^p$ and $p(u) = u^{\alpha+\frac{1}{p}}$.

2.7 Hermite polynomials

We begin this section by presenting a well-known family of polynomials, namely Hermite polynomials whose relationship with Gaussian random variables will be of great interest. This section is directly inspired from the classical reference [4].

Definition 2.7.1. *Fix $x \in \mathbb{R}$. For all $n \geq 1$, we denote by $H_n(x)$ the nth Hermite polynomial, which is defined by the formula*

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \partial_x^n (e^{-\frac{x^2}{2}})$$

and $H_0(x) = 1$.

The main advantage of these polynomials is that they can be interpreted as the coefficients in the Taylor series of a quite simple function. More precisely,

Theorem 2.7.2. *Hermite polynomials are the coefficients of the expansion in powers of t of the function F defined for all $x \in \mathbb{R}$, for all $t \in \mathbb{R}$ by*

$$F(x,t) = \exp\left(tx - \frac{t^2}{2}\right).$$

Proof. We first recall the following basic fact: if f is a function which has a Taylor series whose radius of convergence is $+\infty$, then for all $t \in \mathbb{R}$,

$$f(t) = \sum_{n=0}^{+\infty} \frac{f^n(0)}{n!} t^n.$$

We then write:

$$\begin{aligned} F(x,t) &= \exp\left(\frac{x^2}{2} - \frac{1}{2}(x-t)^2\right) \\ &= e^{\frac{x^2}{2}} \sum_{n=0}^{+\infty} \frac{t^n}{n!} \left(\partial_t^n e^{-\frac{(x-t)^2}{2}}\right)_{|t=0} \\ &= \sum_{n=0}^{+\infty} t^n H_n(x). \end{aligned}$$

□

Using the latter representation, we can derive the following classical properties:

Proposition 2.7.3. *Fix $x \in \mathbb{R}$ and $n \geq 1$. Then, it holds*

$$H_n'(x) = H_{n-1}(x), \tag{2.7.1}$$

$$(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x), \tag{2.7.2}$$

$$H_n(-x) = (-1)^n H_n(x). \tag{2.7.3}$$

Proof. *i*) We resort to Theorem 2.7.2. We observe that

$$\partial_x F = tF.$$

Now, on the one hand,

$$\partial_x F(x,t) = \sum_{n=0}^{+\infty} t^n H_n'(x)$$

whereas, on the other hand,

$$tF(x,t) = \sum_{n=0}^{+\infty} t^{n+1} H_n(x) = \sum_{n=1}^{+\infty} t^n H_{n-1}(x)$$

leading to the desired conclusion by uniqueness of the coefficients in a Taylor series.
ii) The second equality (2.7.2) comes from the relationship

$$\partial_t F = (x-t)F.$$

Indeed,

$$\partial_t F(x,t) = \sum_{n=0}^{+\infty} (n+1)H_{n+1}(x)t^n,$$

whereas

$$\begin{aligned}(x-t)F(x,t) &= x\sum_{n=0}^{+\infty} H_n(x)t^n - \sum_{n=0}^{+\infty} H_n(x)t^{n+1} \\ &= x + \sum_{n=1}^{+\infty} (xH_n(x) - H_{n-1}(x))t^n.\end{aligned}$$

iii) To end with, (2.7.3) is a straightforward consequence of $F(-x,t) = F(x,-t)$. □

We can easily compute the first Hermite polynomials which are:

Example 2.7.4.

$$\begin{aligned} H_1(x) &= x, \\ H_2(x) &= \frac{1}{2}(x^2-1), \\ H_3(x) &= \frac{x^3}{6} - \frac{x}{2}. \end{aligned}$$

Remark 2.7.5. Hermite polynomials have others properties which are worth mentioning. For instance, (2.7.2) shows that the highest-order term of $H_n(x)$ is $\frac{x^n}{n!}$. Then, a quick view on the expansion of $F(0,t) = \exp(-\frac{t^2}{2})$ in powers of t reveals that $H_{2k+1}(0) = 0$ whereas $H_{2k}(0) = \frac{(-1)^k}{2^k k!}$ for all $k \geq 0$. Also, thanks to (2.7.2), we observe that x^n can be expressed as a linear combination of the Hermite polynomials $H_r(x), 0 \leq r \leq n$.

The major result about Hermite polynomials is their profound link with Gaussian random variables. Indeed,

Theorem 2.7.6. *Let X and Y be two random variables with joint Gaussian distribution such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = 1$. Then, for all $n, m \geq 0$, we have*

$$\mathbb{E}(H_n(X)H_m(Y)) = \delta_{n,m}\frac{1}{n!}\mathbb{E}(XY)^n.$$

Proof. We recall the following basic fact: if V is a Gaussian random variable of variance σ^2, then it holds, for every $t \in \mathbb{R}$,

$$\mathbb{E}(\exp(tV)) = \exp\left(\frac{1}{2}\sigma^2 t^2\right).$$

By definition, for all $s, t \in \mathbb{R}$, $sX + tY$ is a Gaussian random variable which readily entails

$$\mathbb{E}\left(\exp\left(sX - \frac{s^2}{2}\right)\exp\left(tY - \frac{t^2}{2}\right)\right) = \exp(st\mathbb{E}(XY)),$$

which can be rewritten as

$$\mathbb{E}(F(X,s)F(Y,t)) = \exp(st\mathbb{E}(XY)).$$

Now, taking the $(n+m)$ th partial derivative $\partial_s^n \partial_t^m$ at $s = t = 0$ in both sides of the above equality, we deduce that if $n \neq m$, then $\mathbb{E}(n!m!H_n(X)H_n(Y)) = 0$, whereas if $n = m$,

$$\mathbb{E}(n!m!H_n(X)H_n(Y)) = n!\mathbb{E}(XY)^n.$$

□

The aim of the next subsection is to give a physical meaning to the previous theorem through the notion of "Wick product".

2.8 Wick product of random variables

In this section, we focus our attention on Wick product of random variables, which is a term taken from field theory. The interested reader will find further informations about this topic in the classical book of Simon [5]. Fix $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space.

Definition 2.8.1. *Let X be a random variable with finite moments. We define $: X^n :$, the nth Wick power of X in the following recursive manner:*

$$: X^0 := 1, \tag{2.8.1}$$
$$\partial_X : X^n := n : X^{n-1} : \tag{2.8.2}$$
$$\mathbb{E}(: X^n :) = 0, \tag{2.8.3}$$

for all $n \geq 1$.

Let us compute the first Wick powers of X.

Example 2.8.2.

$$\begin{aligned} : X : &= X - \mathbb{E}(X), \\ : X^2 : &= X^2 - 2\mathbb{E}(X)X - \mathbb{E}(X^2) + 2\mathbb{E}(X)^2. \end{aligned}$$

A useful tool to derive some properties of Wick powers is the formal generating function:

$$: \exp(tX) := \sum_{n=0}^{+\infty} t^n \frac{: X^n :}{n!}. \tag{2.8.4}$$

With the help of (2.8.2), we have

$$\partial_X : \exp(tX) := t : \exp(tX) :$$

and (2.8.3) implies

$$\mathbb{E}(: \exp(tX) :) = 1.$$

This leads us to

$$: \exp(tX) := \frac{\exp(tX)}{\mathbb{E}(\exp(tX))} \tag{2.8.5}$$

in the sense of formal power series in t. Now, let us suppose that X is a centered Gaussian random variable of variance σ^2. Then, the formal previous series converges in $L^1(\Omega)$ and, moreover, $\mathbb{E}(\exp(tX)) = \exp\left(\frac{1}{2}\sigma^2 t^2\right)$, which readily entails

$$: \exp(tX) := \exp\left(tX - \frac{1}{2}\sigma^2 t^2\right). \tag{2.8.6}$$

We now state a theorem which allows to write a Wick power of a centered Gaussian random variable in term of usual powers of this variable and conversely.

Theorem 2.8.3. *Let $n \geq 1$ and X a centered Gaussian random variable of variance σ^2. We have:*

i)

$$: X^n := \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!} X^{n-2k} \left(-\frac{1}{2}\sigma^2\right)^k,$$

ii)

$$X^n = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!} : X^{n-2k} : \left(\frac{1}{2}\sigma^2\right)^k.$$

Proof. *i*) Thanks to (2.8.6), it suffices to multiply the series for $\exp(tX)$ and $\exp\left(-\frac{1}{2}\sigma^2t^2\right)$ and to identify the coefficients.
ii) We rewrite (2.8.6) as $\exp(tX) =: \exp(tX) : \exp\left(\frac{1}{2}\sigma^2t^2\right)$ and we adopt the previous strategy. □

Remark 2.8.4. If $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) = 1$, then $: X^n := n!H_n(X)$. This follows from (2.8.6) and the fact that $F(x,t) = \exp\left(tx - \frac{t^2}{2}\right)$ is the generating function for Hermite polynomials.

To end with, we reformulate and prove Theorem 2.7.6 in terms of Wick products.

Theorem 2.8.5. *Let X and Y be two random variables with joint Gaussian distribution such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$. Then*

$$\mathbb{E}(: X^n :: Y^m :) = \delta_{n,m} n! \mathbb{E}(XY)^n.$$

Proof. Using the identity (2.8.6), we write

$$\begin{aligned} : \exp(sX) :: \exp(tY) : &= \exp\left(sX - \frac{1}{2}s^2\sigma_X\right)\exp\left(tY - \frac{1}{2}t^2\sigma_Y\right) \\ &= \exp(sX + tY)\exp\left(-\frac{1}{2}[s^2\sigma_X + t^2\sigma_Y]\right) \\ &= \, : \exp(sX + tY) : \exp\left(\frac{1}{2}\mathbb{V}(sX + tY)\right)\exp\left(-\frac{1}{2}[s^2\sigma_X + t^2\sigma_Y]\right) \\ &= \, : \exp(sX + tY) : \exp(st\mathbb{E}(XY)). \end{aligned}$$

Taking the expectation, we obtain

$$\mathbb{E}[: \exp(sX) :: \exp(tY) :] = \exp(st\mathbb{E}(XY)).$$

We get the desired conclusion by developing the exponentials. □

2.9 The Wiener chaos decomposition

In this section, we develop the notion of isonormal Gaussian process which is inspired from [4]. First of all, we fix $(H, \langle ., . \rangle)$ a real separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. The norm of an element $h \in H$ will be denoted by $\|h\|$.

Definition 2.9.1. *A stochastic process $W = \{W(h), h \in H\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be an isonormal Gaussian process (or a Gaussian process on H) if W is a centered Gaussian family of random variables such that for all $h, g \in H$,*

$$\mathbb{E}(W(h)W(g)) = \langle h, g \rangle.$$

Remark 2.9.2. 1) We can observe that the mapping $h \mapsto W(h)$ is linear. Indeed, let $\alpha, \beta \in \mathbb{R}$ and $h, g \in H$. The aim is to show that

$$W(\alpha h + \beta g) = \alpha W(h) + \beta W(g),$$

or $W(\alpha h+\beta g)-\alpha W(h)-\beta W(g)=0$ in $L^2(\Omega)$, which can be obtained by proving that the norm equals 0.

$$\begin{aligned}
&\mathbb{E}((W(\alpha h+\beta g)-\alpha W(h)-\beta W(g))^2)=\mathbb{E}(W(\alpha h+\beta g)^2)+\alpha^2\mathbb{E}(W(h)^2)+\beta^2\mathbb{E}(W(g)^2)\\
&-2\alpha\mathbb{E}(W(\alpha h+\beta g)W(h))-2\beta\mathbb{E}(W(\alpha h+\beta g)W(g))+2\alpha\beta\mathbb{E}(W(h)W(g))\\
&=\|\alpha h+\beta g\|^2+\alpha^2\|h\|^2+\beta^2\|g\|^2-2\alpha\langle\alpha h+\beta g,h\rangle-2\alpha\beta\langle\alpha h+\beta g,g\rangle+2\alpha\beta\langle h,g\rangle\\
&=0.
\end{aligned}$$

Therefore, the mapping $h\mapsto W(h)$ is a linear isometry from H to $\{W(h),h\in H\}$ which is a closed subspace of $L^2(\Omega)$ that we will denote by $\mathcal{H}_1$ and whose elements are zero-mean Gaussian random variables.
2) In Definition 2.9.1, there is in fact no need to assume that W is a Gaussian process. Thanks to the linearity of the previous map, the Gaussian caracter of each random variable $W(h),h\in H$ is enough.
3) One may wonder if, given the Hilbert space H, we can always construct a probability space and a Gaussian process $W=\{W(h),h\in H\}$ verifying the above conditions. The positive answer comes from Kolmogorov's theorem.

We denote by $\mathcal{G}$ the $\sigma-$field generated by the random variables $\{W(h),h\in H\}$.

Definition 2.9.3. *Let $n\geq 1$. The closed linear subspace of $L^2(\Omega,\mathcal{F},\mathbb{P})$ generated by the random variables $\{H_n(W(h)),h\in H,\|h\|=1\}$ will be denoted by $\mathcal{H}_n$. $\mathcal{H}_0$ will be the set of constants. The space $\mathcal{H}_n$ is called the Wiener chaos of order n.*

Remark 2.9.4. For $n=1$, $\mathcal{H}_1$ coincides with the set of random variables $\{W(h),h\in H\}$.

Lemma 2.9.5. *For all $n\neq m$, the subspaces $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal.*

Proof. It is an immediate consequence of Theorem 2.7.6. □

A major result about these particular subspaces is the following orthogonal decomposition:

Theorem 2.9.6. *The space $L^2(\Omega,\mathcal{G},\mathbb{P})$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_n$:*

$$L^2(\Omega,\mathcal{G},\mathbb{P})=\oplus_{n=0}^{\infty}\mathcal{H}_n.$$

Proof. In order to prove this theorem, we will resort to the Hilbert space nature of $L^2(\Omega,\mathcal{G},\mathbb{P})$. Let also $X\in L^2(\Omega,\mathcal{G},\mathbb{P})$ such that X is orthogonal to $\mathcal{H}_n$ for every $n\geq 0$. The aim is to show that $X=0$. By virtue of the orthogonality, we have $\mathbb{E}(XH_n(W(h)))=0$ for all $h\in H$ with $\|h\|=1$. Using Remark 2.7.5, since x^n can be expressed as a linear combination of the Hermite polynomials $H_r(x),0\leq r\leq n$, it holds $\mathbb{E}(XW(h)^n)=0$, for all $n\geq 0$. Now, fix $t\in\mathbb{R}$ and $h\in H$ of norm one. We write:

$$\mathbb{E}(X\exp(tW(h)))=\sum_{n=0}^{+\infty}t^n\frac{\mathbb{E}(XW(h)^n)}{n!}=0.$$

The lemma below allows us to conclude:

Lemma 2.9.7. *The random variables $\{e^{W(h)}, h \in H\}$ form a total subset of $L^2(\Omega, \mathcal{G}, \mathbb{P})$.*

$\square$

Definition 2.9.8. *Let $n \geq 1$. We denote by $\mathcal{P}_n^0$ the space formed by the random variables $p(W(h_1), ..., W(h_k))$, where $k \geq 1$, $h_1, ..., h_k \in H$, and p is a real polynomial in k variables of degree less than or equal to n. Then, $\mathcal{P}_n$ will be the closure of $\mathcal{P}_n^0$ in L^2.*

The following decomposition of $\mathcal{P}_n$ occurs:

Proposition 2.9.9.

$$\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_n = \mathcal{P}_n.$$

Proof. The inclusion $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_n \subset \mathcal{P}_n$ is quite immediate. In order to prove the converse inclusion, we are going to check that $\mathcal{P}_n$ is orthogonal to $\mathcal{H}_m$ for every $m > n$. Let also fix p a real polynomial in k variables of degree less than or equal to n, $h_1, ..., h_k \in H$ and $h \in H$ with norm one and prove that

$$\mathbb{E}(p(W(h_1), ..., W(h_k))H_m(W(h))) = 0.$$

We call V the linear subspace generated by $h, h_1, ..., h_k$. As $h \neq 0$, according to the incomplete basis theorem, there exists a base $(h, e_1, ..., e_j)$ of V. Now, we can apply the Gram-Schmidt process to obtain an orthonormal family of V which still contains h (because h is of norm one) and that we still denote by $(h, e_1, ..., e_j)$. Consequently, there exists a polynomial q of degree less than or equal to n such that

$$\mathbb{E}(p(W(h_1), ..., W(h_k))H_m(W(h))) = \mathbb{E}(q(W(h), W(e_1), ..., W(e_j))H_m(W(h))).$$

Then, we rewrite $q(W(h), W(e_1), ..., W(e_j))$ under the form $p_1(W(e_1), ..., W(e_j))p_2(W(h))$. By orthonormality of $(h, e_1, ..., e_j)$ and since W is an isonormal Gaussian process, the random variables $W(h), W(e_1), ..., W(e_j)$ are independent leading to

$$\mathbb{E}(p(W(h_1), ..., W(h_k))H_m(W(h))) = \mathbb{E}(p_1(W(e_1), ..., W(e_j)))\mathbb{E}(p_2(W(h))H_m(W(h))).$$

It only remains to show that for all $r \leq n < m$,

$$\mathbb{E}(W(h)^r H_m(W(h))) = 0,$$

which is an easy consequence of the fact that x^r can be expressed as a linear combination of the Hermite polynomials $H_q(x), 0 \leq q \leq r$ combined with Theorem 2.7.6. $\square$

2.10 Hypercontractivity of the Wiener Chaoses

Fix $d \geq 1$. Keeping in mind the notations of the previous section, we now consider $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$, where $d\mathbb{P}(x_1, ..., x_d) = (2\pi)^{-\frac{d}{2}} \exp(-\frac{|x|^2}{2})dx$ and $H = \mathbb{R}^d$ endowed with the usual canonical dot product. We define the stochastic process W in the following way: for all $h = (h_1, ..., h_d) \in H$, for all $x = (x_1, ..., x_d) \in \Omega$,

$$W(h)(x_1, ..., x_d) = \sum_{i=1}^{d} h_i x_i = \sum_{i=1}^{d} h_i X_i(x_1, ..., x_d)$$

where X_i is the projection on the ith coordinate defined on $\Omega = \mathbb{R}^d$. Under this form, as a combination of independent Gaussian random variables (whose law is $\mathcal{N}(0,1)$), $W(h)$ is clearly a Gaussian random variable. Moreover,

$$\begin{aligned}\mathbb{E}(W(h)W(g)) &= \mathbb{E}\left(\sum_{i=1}^{d} h_i X_i \sum_{j=1}^{d} g_j X_j\right)\\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} h_i g_j \mathbb{E}(X_i X_j)\\ &= \sum_{i=1}^{d} h_i g_i\\ &= \langle h, g\rangle.\end{aligned}$$

Thanks to Remark 2.9.2, we deduce that $W = \{W(h), h \in H\}$ is an isonormal Gaussian process. Theorem 2.9.6 can be rewritten as:

Theorem 2.10.1. *The space $L^2(\mathbb{R}^d, \mu_d)$, where $d\mu_d = (2\pi)^{-\frac{d}{2}} \exp(-\frac{|x|^2}{2})dx$, can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_n$:*

$$L^2(\mathbb{R}^d, \mu_d) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

Let us introduce the Ornstein-Uhlenbeck operator $L := \Delta - x.\nabla$ and the Ornstein-Uhlenbeck semigroup $U(t) := e^{tL}$ which verifies the following hypercontractivity property due to Nelson [2]:

Lemma 2.10.2. *Let $q > 1$ and $p \geq q$. Then, for every $u \in L^q(\mathbb{R}^d, \mu_d)$ and $t \geq \frac{1}{2}\log(\frac{p-1}{q-1})$, it holds*

$$\|U(t)u\|_{L^p(\mathbb{R}^d,\mu_d)} \leq \|u\|_{L^q(\mathbb{R}^d,\mu_d)}. \tag{2.10.1}$$

Remark 2.10.3. It is worth mentioning that (2.10.1) is independent of the dimension d.

As a consequence, we obtain the following corollary:

Lemma 2.10.4. *Let $F \in \mathcal{H}_k$. Then, for every $p \geq 2$, we have*

$$\|F\|_{L^p(\mathbb{R}^d,\mu_d)} \leq (p-1)^{\frac{k}{2}} \|F\|_{L^2(\mathbb{R}^d,\mu_d)}. \tag{2.10.2}$$

Proof. It is known that any element in $\mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$. Thus, F is an eigenfunction of $U(t) = e^{tL}$ with eigenvalue e^{-kt}. It now suffices to set $q = 2$ and $t = \frac{1}{2}\log(p-1)$ in the equation (2.10.1) to obtain the desired inequality. □

We finish by the lemma below which has been very useful in the recent probabilistic study of dispersive PDEs and related areas.

Lemma 2.10.5. *Fix $k \in \mathbb{N}$ and $c(n_1, ..., n_k) \in \mathbb{C}$. Given $d \in \mathbb{N}$, let $\{g_n\}_{n=1}^d$ be a sequence of independent standard complex-valued Gaussian random variables and set $g_{-n} = \overline{g_n}$. Define $S_k(\omega)$ by*

$$S_k(\omega) = \sum_{\Gamma(k,d)} c(n_1, ..., n_k) g_{n_1}(\omega) ... g_{n_k}(\omega),$$

where $\Gamma(k,d)$ *is defined by*

$$\Gamma(k,d) = \{(n_1,...,n_k) \in \{\pm 1,...,\pm d\}^k\}.$$

Then, for every $p \geq 2$*, it holds:*

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \|S_k\|_{L^2(\Omega)}.$$

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Chapter 3

Study of a fractional stochastic heat equation

In this chapter, we study a d-dimensional stochastic nonlinear heat equation (SNLH) with a quadratic nonlinearity, forced by a fractional space-time white noise:

$$\begin{cases} \partial_t u - \Delta u = \rho^2 u^2 + \dot{B}\,, & t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,. \end{cases}$$

Two types of regimes are exhibited, depending on the ranges of the Hurst index $H = (H_0, ..., H_d) \in (0,1)^{d+1}$. In particular, we show that the local well-posedness of (SNLH) resulting from the Da Prato-Debussche trick, is easily obtained when $2H_0 + \sum_{i=1}^d H_i > d$. On the contrary, (SNLH) is much more difficult to handle when $2H_0 + \sum_{i=1}^d H_i \leq d$. In this case, the model has to be interpreted in the Wick sense, thanks to a time-dependent renormalization. Helped with the regularising effect of the heat semigroup, we establish local well-posedness results for (SNLH) for all dimension $d \geq 1$.

3.1 Introduction and main results

3.1.1 General introduction

This chapter is devoted to the study of a heat equation with a quadratic nonlinearity, driven by an additive fractional space-time white noise forcing. More precisely, we consider the following nonlinear stochastic heat model:

$$\begin{cases} \partial_t u - \Delta u = \rho^2 u^2 + \dot{B}\,, & t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u(0,.) = \phi\,, \end{cases} \tag{3.1.1}$$

where ϕ is a (deterministic) initial condition living in some appropriate Sobolev space and $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth compactly-supported function fixed once and for all. Before going any further, let us specify the roughness of the stochastic term, namely, $\dot{B} := \partial_t \partial_{x_1} ... \partial_{x_d} B$ where $B = B^H$ is a space-time fractional Brownian motion of Hurst index $H = (H_0, H_1, ..., H_d) \in (0,1)^{d+1}$. The definition of this process is the following:

Definition 3.1.1. *Fix $d \geq 1$ a space dimension, $T \geq 0$ a positive time and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete filtered probability space. On this space, and for any fixed $H = (H_0, H_1, ..., H_d) \in (0,1)^{d+1}$, a centered Gaussian process $B : \Omega \times ([0,T] \times \mathbb{R}^d) \to \mathbb{R}$ is said to be a space-time fractional Brownian motion of Hurst index H if its covariance function is given by the formula:*

$$\mathbb{E}\Big[B(s, x_1, ..., x_d)B(t, y_1, ..., y_d)\Big] = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i)\,, \quad s,t \in [0,T]\,,\ x, y \in \mathbb{R}^d\,,$$

where

$$R_{H_i}(x,y) := \frac{1}{2}(|x|^{2H_i} + |y|^{2H_i} - |x-y|^{2H_i})\ .$$

Stochastic partial differential equations (SPDEs) perturbed by fractional noise terms have become increasingly popular in the recent years. Indeed, these equations can be interpreted as a more flexible alternative to classical SPDEs driven by a space-time white noise, and consequently they can be used to model more complex physical phenomena which are subjects to random perturbations. For instance, fractional noises appear in lots of practical situations, whether in biophysics (see [18]), in the study of financial time series (see [1]) or simply in electrical engineering (see [5]).

The major motivation of this work is that equation (3.1.1) is both a physical and mathematically challenging model. In fact, it can be seen as a stochastic reaction diffusion model with $p = 2$. Recall that the equation:

$$\partial_t u = \Delta u + u^p, \quad p > 1 \tag{3.1.2}$$

is known under the name of Fujita model and has been the subject of many questions. It has already been established (see [2, 24]) that when $\phi \in L^q(\mathbb{R}^d)$ with $q \geq 1$ and $q > \frac{d(p-1)}{2}$, there exists a constant $T = T(\phi) > 0$ and a unique function $u \in \mathcal{C}([0,T], L^q(\mathbb{R}^d))$ that is a classical solution to (1.1) on $[0,T] \times \mathbb{R}^d$ whereas when $q < \frac{d(p-1)}{2}$, there is no general theory of existence. In this work, we introduce a cut-off function ρ to bring back computations to compact domains, a fractional random perturbation $\dot{B}$ and we are interested in the well-posedness of the associated equation (3.1.1).

Two classical obstacles have to be taken over. First of all, as often in the theory of SPDEs, a standard problem is the roughness of the fractional noise which prevents us from solving directly the associated linear part of the equation, namely:

$$\begin{cases} \partial_t \mathring{|} - \Delta \mathring{|} = \dot{B}, \quad t \in [0,T], \ x \in \mathbb{R}^d, \\ \mathring{|}(0,.) = 0. \end{cases} \tag{3.1.3}$$

Then, the quadratic nonlinearity will require us to give a precise meaning to the notion of solution. Before going into details, let us recall that, over the last decade, we have seen numerous developments in the study of singular stochastic PDEs, in particular in the parabolic setting. Remember that there are three main components to consider: the space dimension d, the nature of the nonlinearity and the irregularity of the noise. For example, polynomial nonlinearities (see [4, 9] or the more recent work of Gubinelli and Hofmanová [12]) have been progressively replaced by sinusoidal and exponential ones.

As an application of the theory of regularity structures, Hairer-Shen [16] and Chandra-Hairer-Shen [3] studied the following parabolic sine-Gordon model on $\mathbb{T}^2$:

$$\partial_t u = \frac{1}{2}\Delta u + \sin(\beta u) + \xi, \tag{3.1.4}$$

where ξ is a space-time white noise and proved that the local well-posedness depends essentially on the value of the key parameter $\beta^2 > 0$. Oh, Robert and Wand [20] focused on the parabolic equation in dimension 2:

$$\partial_t u + \frac{1}{2}(1-\Delta)u + \frac{1}{2}\lambda\beta e^{\beta u} = \xi, \tag{3.1.5}$$

where $\beta, \lambda \in \mathbb{R}\backslash\{0\}$ and they proved that the local well-posedness depends again sensitively on the value of $\beta^2 > 0$ as well as the sign of λ. To end with, resorting to a paracontrolled approach, Oh and Okamoto [19] worked on a stochastic heat equation with a quadratic nonlinearity but forced by a more irregular noise:

$$\partial_t u + (1-\Delta)u + u^2 = \langle\nabla\rangle^\alpha \xi, \tag{3.1.6}$$

where $\langle\nabla\rangle^\alpha\xi$ denotes a α-derivative (in space) of a (Gaussian) space-time white noise on $\mathbb{T}^2 \times \mathbb{R}^+$. Our objective is to go one step further that is to study a stochastic heat equation on $\mathbb{R}^d$ (instead of $\mathbb{T}^d$) driven by the derivative (in space *and in time*) of a space-time fractional Brownian motion for $d \geq 1$. Helped with the regularising effect of the heat semigroup which allows a gain of two derivatives in the fractional Sobolev setting, we will be able to derive existence results about (3.1.1).

Our results can be summed up in the following way:

Theorem 3.1.2. *Suppose that $d \geq 1$ and set $\alpha_H := \left(2H_0 + \sum_{i=1}^d H_i\right) - d$. The picture below holds true:*

(i) ***Case*** $\alpha_H > 0$. *The equation (3.1.1) is almost surely locally well-posed in $\mathcal{W}^{\beta,p}(\mathbb{R}^d)$ for some $\beta > 0$ and $p \geq 2$.*

(ii) ***Case*** $\alpha_H \leq 0$. *If $\alpha_H > -\frac{1}{4}$, then the equation (3.1.1) can be interpreted in the Wick (renormalized) meaning and it is almost surely locally well-posed in $\mathcal{W}^{-\beta,p}(\mathbb{R}^d)$ for some $\beta > 0$ and $p \geq 2$.*

Let us now come back to the analysis of equation (3.1.1). We first rewrite it under the mild formulation, that is the (formal) equation

$$u_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 u_\tau^2)\,d\tau + ⚲_t \tag{3.1.7}$$

where $e^{t\Delta}$ stands for the heat semigroup, and (morally) $⚲_t := \int_0^t e^{(t-\tau)\Delta}(\dot{B}_\tau)\,d\tau$. Here, we adopt Hairer's convention (see [15]) to denote the stochastic terms by trees; the vertex $\circ$ in ⚲ represents the random noise $\dot{B}$ and the edge corresponds to the Duhamel integral operator:

$$\mathcal{I} = (\partial_t - \Delta)^{-1}.$$

Now, in order to isolate the more irregular term Ψ, we resort to the so-called Da Prato-Debussche trick to rewrite the equation under the form:

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}((\rho v_\tau)\cdot(\rho\Psi_\tau))\,d\tau \\ + \int_0^t e^{(t-\tau)\Delta}(\rho^2\Psi_\tau^2)\,d\tau\,, \quad t\in[0,T]\,, \tag{3.1.8}$$

where $v := u - \Psi$. Our first and main objective is to define properly the solution Ψ related to the linear equation:

$$\begin{cases} \partial_t\Psi - \Delta\Psi = \dot{B}, \quad t\in[0,T], \quad x\in\mathbb{R}^d, \\ \Psi(0,.) = 0. \end{cases} \tag{3.1.9}$$

Our strategy is to consider $(\dot{B}_n)_{n\geq 0}$ a smooth approximation of $\dot{B}$ and to show that the associated solutions $(\Psi_n)_{n\geq 0}$ verifying

$$\begin{cases} \partial_t\Psi_n - \Delta\Psi_n = \dot{B}_n\,, \quad t\in[0,T]\,, \quad x\in\mathbb{R}^d\,, \\ \Psi_n(0,.) = 0\,, \end{cases} \tag{3.1.10}$$

form a Cauchy sequence in a convenient subspace (see Section 3.4 for more details). This leads us to the construction of the first order stochastic process Ψ.

Proposition 3.1.3. *Suppose that $d\geq 1$. For all $(H_0,H_1,...,H_d)\in(0,1)^{d+1}$ and every test (i.e., smooth compactly-supported) function $\chi:\mathbb{R}^d\to\mathbb{R}$, almost surely the sequence $(\chi\Psi_n)_{n\geq 0}$ converges in the space $\mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$ as soon as $2\leq p\leq\infty$ and*

$$\alpha > d - \left(2H_0 + \sum_{i=1}^d H_i\right) = -\alpha_H\,. \tag{3.1.11}$$

The limit of this sequence will be denoted by $\chi\Psi$.

It is now clear that the sign of α_H will play a major role in the resolution of (3.1.1). Indeed, when $\alpha_H > 0$, $\chi\Psi(t)$ will be seen as a function defined on $\mathbb{R}^d$ whereas, when $\alpha_H < 0$, $\chi\Psi(t)$ will be seen as a spatial distribution. It is important to remark that the precise nature of Ψ is known up to multiplication by $\chi\in\mathcal{C}_c^\infty(\mathbb{R}^d)$. To put it differently, the stochastic process $\Psi(t)$ is (almost surely) only a distribution on $\mathbb{R}^d$ but we have the more refined result : $\chi\Psi\in\mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$ (see section 3.4.6 for more details). In the study of our equation, we will thus take $\chi=\rho$.

The regular case

In the following, we will call *regular case* the situation where

$$2H_0 + \sum_{i=1}^d H_i > d\,. \tag{\textbf{H1}}$$

When this hypothesis is realized, $\alpha < 0$ will be picked such that condition (3.1.11) is satisfied, and therefore, resorting to Proposition 3.1.3, every element $\chi \Psi$ ($\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$) will be considered as a function of both time and space variables (with probability one) and consequently $\rho^2 \Psi^2(t)$ will perfectly make sense as a classical function defined on $\mathbb{R}^d$. We thus propose the following natural interpretation of the model:

Definition 3.1.4. *Let $d \geq 1$ and suppose that condition $(\mathbf{H1})$ is verified. Remember that for all test function $\chi : \mathbb{R}^d \to \mathbb{R}$, $\chi \Psi$ is the process provided by Proposition 3.1.3.*

A stochastic process $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ is said to be a solution (on $[0,T]$) of equation (3.1.1) if, almost surely, the process $v := u - \Psi$ is a solution of the mild equation

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}((\rho v_\tau)\cdot(\rho \Psi_\tau))\, d\tau + \int_0^t e^{(t-\tau)\Delta}(\rho^2 \Psi_\tau^2)\, d\tau\,, \quad t \in [0,T]\,. \tag{3.1.12}$$

We are now in a position to state our first existence result which will be a consequence of a standard fixed-point theorem:

Theorem 3.1.5 (Local well-posedness under (H1)). *Suppose that $d \geq 1$ and that condition $(\mathbf{H1})$ is satisfied. Let $p \geq 2$ and β be such that $0 < \beta < 2H_0 + \sum_{i=1}^d H_i - d$ and $\frac{d}{2p} < 1 + \frac{\beta}{2}$. Finally, assume that $\phi \in \mathcal{W}^{\beta,p}(\mathbb{R}^d)$. In this case, the assertions below hold true:*

(i) Almost surely, there exists a time $T_0(\omega) > 0$ such that equation (3.1.1) admits a unique solution u (in the meaning of Definition 3.1.4) in the subset

$$\mathcal{S}_{T_0} := \Psi + X^{\beta,p}(T_0), \quad \text{where } X^{\beta,p}(T_0) := \mathcal{C}([0,T_0];\mathcal{W}^{\beta,p}(\mathbb{R}^d)).$$

(ii) For any $n \geq 1$, let us note u_n the smooth *solution of (3.1.1), that is u_n is the solution (in the meaning of Definition 3.1.4) related to $\rho \Psi_n$. Then, for all*

$$0 < \beta < 2H_0 + \sum_{i=1}^d H_i - d$$

and for any test function $\chi : \mathbb{R}^d \to \mathbb{R}$, the sequence $(\chi u_n)_{n\geq 1}$ converges almost surely in $\mathcal{C}([0,T_0];\mathcal{W}^{\beta,p}(\mathbb{R}^d))$ to χu, where u is the solution from item (i).

The rough case

Let us now focus on the second situation, where

$$2H_0 + \sum_{i=1}^d H_i \leq d\,. \tag{3.1.13}$$

In this situation, $\rho\mathring{|}(t)$ is only a distribution and not a function anymore. So the classical difficulty to properly define $\rho^2\mathring{|}^2(t)$ appears. A Wick-renormalization procedure permits to overcome this issue. To begin with, let us introduce the Wick-renormalized product

$$\mathring{\vee}_n(t,x) := \mathring{|}_n(t,x)^2 - \sigma_n(t,x) \quad \text{where } \sigma_n(t,x) := \mathbb{E}\Big[\mathring{|}_n(t,x)^2\Big]\,. \tag{3.1.14}$$

Before looking for a convenient subspace in which $(\rho^2\mathring{\vee}_n)_{n\geq 0}$ would be a Cauchy sequence, let us have a look at the renormalization constant.

Proposition 3.1.6. *Let $d \geq 1$ and suppose that $2H_0 + \sum_{i=1}^d H_i \leq d$. Then, the value of $\mathbb{E}\Big[\mathring{|}_n(t,x)^2\Big]$ does not depend on x. Setting $\sigma_n(t) := \mathbb{E}\Big[\mathring{|}_n(t,x)^2\Big]$, the asymptotic equivalence property below is obtained: for every $(t,x) \in (0,T] \times \mathbb{R}^d$,*

$$\sigma_n(t,x) := \mathbb{E}\Big[\mathring{|}_n(t,x)^2\Big] \underset{n\to\infty}{\sim} \begin{cases} c_H^1\, n & \text{if } 2H_0 + \sum_{i=1}^d H_i = d, \\ c_H^2\, 2^{2n(d-[2H_0+\sum_{i=1}^d H_i])} & \text{if } 2H_0 + \sum_{i=1}^d H_i < d\,, \end{cases} \tag{3.1.15}$$

where $c_H^1, c_H^2 > 0$ are two constants.

Remark 3.1.7. It is interesting to note that the nature of the equivalent (that is linear when $2H_0 + \sum_{i=1}^d H_i = d$ and geometric when $2H_0 + \sum_{i=1}^d H_i < d$) is the same as in the wave setting (see [6]) and in the Schrödinger setting (see [8]). Let us add that, in this case, the term in the right hand-side of (3.1.15) has no dependence on t. In fact, this results from a dissipative phenomenon of the heat equation (see the proof of Proposition 3.4.4) contrary to an oscillating one in the case of the Schrödinger equation (see [8]). This kind of phenomenon is well-known in the parabolic setting and has already been observed in [17] for instance in its study of the KPZ equation with fractional derivatives of white noise.

Our definition of $\mathring{\vee}$, the second order stochastic term, is the following:

Proposition 3.1.8. *Fix $d \geq 1$ and $(H_0, H_1, ..., H_d) \in (0,1)^{d+1}$ such that*

$$d - \frac{1}{4} < 2H_0 + \sum_{i=1}^d H_i \leq d\,. \tag{\textbf{H2}}$$

Then, for all test function $\chi : \mathbb{R}^d \to \mathbb{R}$, almost surely the sequence $(\chi^2\mathring{\vee}_n)_{n\geq 1}$ (defined by (3.1.14)) converges in the space $\mathcal{C}([0,T]; \mathcal{W}^{-2\alpha,p}(\mathbb{R}^d))$ as soon as $2 \leq p \leq \infty$ and α satisfying (3.1.11).

The limit of this sequence will be denoted by $\chi^2\mathring{\vee}$.

With the above constructed stochastic processes, the following *Wick interpretation* of the model naturally comes to mind:

Definition 3.1.9. *Let $d \geq 1$ and assume that condition (**H2**) is verified. Remember that for all test function $\chi : \mathbb{R}^d \to \mathbb{R}$, $\chi\mathring{|}$ and $\chi^2\mathring{\vee}$ are the processes defined in Proposition 3.1.3 and Proposition 3.1.8, respectively.*

In this setting, a stochastic process $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ *is said to be a* Wick *solution (on* $[0,T]$*) of equation* (3.1.1) *if, almost surely, the process* $v := u - \mathbf{I}$ *is a solution of the mild equation*

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}((\rho v_\tau)\cdot(\rho \mathbf{I}_\tau))\,d\tau + \int_0^t e^{(t-\tau)\Delta}(\rho^2 \mathbf{V}_\tau)\,d\tau\,, \quad t\in[0,T]\,. \quad (3.1.16)$$

In this case, a major difficulty is the treatment of the term $(\rho v)\cdot(\rho\mathbf{I})$ that will be understood as the product of a distribution $\rho\mathbf{I}$ with a regular enough function ρv. Indeed, should $\rho\mathbf{I}$ be of Sobolev regularity $-\alpha$ and ρv be a function of Sobolev regularity β with $\beta > \alpha$, $(\rho v)\cdot(\rho\mathbf{I})$ can be defined as a distribution of order $-\alpha$ (see Section 3.3 for more details). Again, the regularising effect of the heat semigroup will help us to find a stable subspace. The main result of this chapter can be stated in the following way:

Theorem 3.1.10 (Local well-posedness under (H2)). *Suppose that* $d \geq 1$ *and* $p \geq 2$ *verifies that* $p \geq \frac{2d}{3}$. *Assume that* $d - \frac{1}{4} < 2H_0 + \sum_{i=1}^d H_i \leq d$. *Fix* $\alpha > 0$ *such that*

$$d - \left(2H_0 + \sum_{i=1}^d H_i\right) < \alpha < \frac{1}{4}. \quad (3.1.17)$$

Then the assertions below hold true:

(i) *One can find* $\beta > 0$ *such that almost surely, for every* $\phi \in \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)$, *there exists a time* $T_0(\omega) > 0$ *for which equation* (3.1.1) *admits a unique Wick solution* u *(in the meaning of Definition 3.1.9) in the set*

$$\mathcal{S}_{T_0} := \mathbf{I} + X^{\alpha,\beta}(T_0)\,,$$

where

$$X^{\alpha,\beta}(T_0) := \mathcal{C}([0,T_0];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)) \cap \mathcal{C}((0,T_0];\mathcal{W}^{\beta,p}(\mathbb{R}^d)).$$

(ii) *For any* $n \geq 1$, *let us note* $\tilde{u}_n$ *the* smooth *Wick solution of* (3.1.1), *that is* $\tilde{u}_n$ *is the solution (in the meaning of Definition 3.1.9) related to the pair* $(\rho\mathbf{I}_n, \rho^2\mathbf{V}_n)$. *Then, for all* α *satisfying (3.1.17) and for any test function* $\chi : \mathbb{R}^d \to \mathbb{R}$, *the sequence* $(\chi\tilde{u}_n)_{n\geq1}$ *converges almost surely in* $\mathcal{C}([0,T_0];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$ *to* χu, *where* u *is the Wick solution from item* (i).

Remark 3.1.11. Thanks to the regularising effect of the heat semigroup, we are able to solve equation (3.1.1) for all dimension $d \geq 1$ as soon as $\alpha < \frac{1}{4}$. This situation highly contrasts with the Schrödinger setting (see [8]) where the restriction comes from the deterministic part of the equation. Nonetheless, although the restriction in Theorem 3.1.10 comes from the stochastic constructions, we will be able to improve this result in dimension $d = 2$. See subsection 3.1.1 below.

Remark 3.1.12. Let us go back to the role of the cut-off function ρ in equation (3.1.1). Contrary to the Schrödinger setting where ρ played an essential role to obtain local regularity, in our case, ρ is only needed to measure precisely the regularity of the stochastic processes $\rho\mathbf{I}(t,.)$, $\rho^2\mathbf{V}(t,.)$ in some $\mathcal{W}^{\alpha,p}(\mathbb{R}^d)$ spaces.

Further study of the model when d=2

At this point of the analysis, one may wonder whether the restriction $2H_0+\sum_{i=1}^d H_i > d-\frac{1}{4}$ in (**H2**) (that comes from our technical computations) is optimal. The following proposition provides us with a first partial result in this direction.

Proposition 3.1.13. *Fix $d \geq 1$. Let $(H_0, H_1, ..., H_d) \in (0,1)^{d+1}$ be such that*

$$2H_0 + H_1 + \cdots + H_d \leq \frac{3}{4}d \tag{3.1.18}$$

and let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a non-zero, smooth and compactly-supported function. Then, for all $\alpha > 0$ and $t > 0$, one has $\mathbb{E}\Big[\|\chi \cdot \Psi_n(t,.)\|^2_{H^{-2\alpha}(\mathbb{R}^d)}\Big] \to \infty$ as $n \to \infty$.

Based on the latter explosion phenomenon for Ψ_n, we can at least conclude that the condition (**H2**) in Proposition 3.1.8 and Theorem 3.1.10 is optimal when $d = 1$.

When $d \geq 2$, one can observe that the situation where $\frac{3}{4}d < 2H_0 + \sum_{i=1}^d H_i \leq d - \frac{1}{4}$ is not covered by the above results. In fact, we must admit that we are not able, for the moment, to offer a complete picture of the situation, that is a general well-posedness result for every $d \geq 2$ and for all indexes such that $2H_0 + \sum_{i=1}^d H_i > \frac{3}{4}d$.

However, we propose to make a first step toward this ambitious challenge by going further with the analysis in the two-dimensional setting $d = 2$. In this case, it turns out that several of our arguments and estimates behind the construction of Ψ can be slightly refined, leading us to the following statement:

Proposition 3.1.14. *Let $(H_0, H_1, H_2) \in (0,1)^3$ be such that*

$$0 < H_1 < \frac{3}{4} \quad , \quad 0 < H_2 < \frac{3}{4} \quad , \quad \frac{3}{2} < 2H_0 + H_1 + H_2 \leq \frac{7}{4}\ . \tag{3.1.19}$$

Then, for all smooth compactly-supported function $\chi : \mathbb{R}^2 \to \mathbb{R}$, $T > 0$, $2 \leq p \leq \infty$ and

$$\alpha > 2 - (2H_0 + H_1 + H_2)\ , \tag{3.1.20}$$

the sequence $(\chi^2 \Psi_n)_{n\geq 1}$ (defined by (3.1.14)) converges almost surely in the space $\mathcal{C}([0,T]; \mathcal{W}^{-2\alpha,p}(\mathbb{R}^2))$.

We still denote the limit of this sequence by $\chi^2 \Psi$.

According to Proposition 3.1.13 with $d = 2$, this construction is optimal. We are now in a position to state our more general result when $d = 2$:

Theorem 3.1.15 (Roughest case). *Suppose that $d = 2$ and $p \geq 2$. Let $(H_0, H_1, H_2) \in (0,1)^3$ be such that*

$$0 < H_1 < \frac{3}{4} \quad , \quad 0 < H_2 < \frac{3}{4} \quad , \quad \frac{3}{2} < 2H_0 + H_1 + H_2 \leq \frac{7}{4}\ . \tag{3.1.21}$$

Fix $\alpha > 0$ such that

$$2 - (2H_0 + H_1 + H_2) < \alpha < \frac{1}{2}. \tag{3.1.22}$$

Then the assertions below hold true:

(i) One can find $\beta > 0$ such that almost surely, for every $\phi \in \mathcal{W}^{-\alpha,p}(\mathbb{R}^2)$, there exists a time $T_0(\omega) > 0$ for which equation (3.1.1) *admits a unique Wick solution u (in the meaning of Definition 3.1.9) in the set*

$$\mathcal{S}_{T_0} := \Psi + X^{\alpha,\beta}(T_0)\,,$$

where

$$X^{\alpha,\beta}(T_0) := \mathcal{C}([0,T_0];\mathcal{W}^{-\alpha,p}(\mathbb{R}^2)) \cap \mathcal{C}((0,T_0];\mathcal{W}^{\beta,p}(\mathbb{R}^2)).$$

(ii) For any $n \geq 1$, let us note $\tilde{u}_n$ the smooth *Wick solution of* (3.1.1)*, that is $\tilde{u}_n$ is the solution (in the meaning of Definition 3.1.9) related to the pair $(\rho\Psi_n, \rho^2\Psi^{\diamond 2}_n)$. Then, for all α satisfying (3.1.22) and for any test function $\chi : \mathbb{R}^2 \to \mathbb{R}$, the sequence $(\chi\tilde{u}_n)_{n\geq 1}$ converges almost surely in $\mathcal{C}([0,T_0];\mathcal{W}^{-\alpha,p}(\mathbb{R}^2))$ to χu, where u is the Wick solution from item (i).*

Remark 3.1.16. Again, the restriction comes from the stochastic part of the study (see Proposition 3.1.14) that tackles with the existence of the stochastic element $\Psi^{\diamond 2}$. This situation is analogous to those in the wave setting (see for example [6, 13, 14, 19]), where the strategy is limited by the domain of validity of the stochastic constructions.

The above well-posedness theorem can be viewed as the main novelty of our work. We can sum up our results in the following way ; equation (3.1.1) is well-posed in some $\mathcal{W}^{\alpha,p}(\mathbb{R}^2)$ space when

$$\frac{3}{2} < 2H_0 + H_1 + H_2$$

and is ill-posed as soon as

$$2H_0 + H_1 + H_2 \leq \frac{3}{2}$$

in the sense that the second order stochastic term is not a continuous function of time with values in Sobolev spaces. These results can be viewed as a generalization of those obtained by Oh and Okamoto in [19] where the parameter α (related to the derivatives of their Gaussian noise) has to satisfy analogous constraints to those of the combination $2H_0 + H_1 + H_2$.

Remark 3.1.17. It is interesting to compare the above regularity restriction (3.1.11) for Ψ in Proposition 3.1.3 with the analogous results of [6] in the wave setting and of [8] in the Schrödinger setting, that is when replacing $\partial_t\Psi - \Delta\Psi$ with $\partial_t^2\Psi - \Delta\Psi$ (respectively $i\partial_t\Psi - \Delta\Psi$) . In these situations, the conditions on α are

$$\alpha_{\text{wave}} > d - \frac{1}{2} - \sum_{i=0}^{d} H_i \quad \textit{and} \quad \alpha_{\text{schröd}} > d + 1 - \Big(2H_0 + \sum_{i=1}^{d} H_i\Big).$$

In fact, the method presented in this chapter would allow us to define the linear solution Ψ of several other fractional problems. Let us briefly describe some results that could be obtained along the same procedure (see Section 3.4). First of all, for fractional Schrödinger-type equations of the form:

$$i\partial_t\Psi(t,x) + |\nabla|^{\sigma}\Psi(t,x) = 0,$$

where $|\nabla|^\sigma$ is the Fourier multiplier by $|\xi|^\sigma$ and $\sigma \in (0,+\infty)$, that generalize the classical one (take $\sigma = 2$), one could prove that the associated condition on α becomes

$$\alpha_{\text{ frac schröd}} > d + \frac{\sigma}{2} - \Big(\sigma H_0 + \sum_{i=1}^{d} H_i\Big).$$

Likewise, we could construct the linear solution Ψ for fractional wave-type equations of the form:

$$\partial_t^2 \Psi(t,x) + |\nabla|^{2\sigma} \Psi(t,x) = 0,$$

that generalize the standard one (take $\sigma = 1$). In this case, the associated condition on α would be

$$\alpha_{\text{ frac wave}} > d - \frac{\sigma}{2} - \Big(\sigma H_0 + \sum_{i=1}^{d} H_i\Big).$$

To end with, it is possible to construct the first order stochastic process Ψ for fractional heat-type equations of the form:

$$\partial_t \Psi(t,x) + |\nabla|^{\sigma} \Psi(t,x) = 0,$$

as soon as

$$\alpha_{\text{ frac heat}} > d - \Big(\sigma H_0 + \sum_{i=1}^{d} H_i\Big).$$

These considerations allow us to understand that the key parameter in the construction of the linear solution is the exponent of the Fourier multiplier that appears in front of H_0 (that described the irregularity of B with respect to the time t) whereas the combination $\sum_{i=1}^{d} H_i$ stays unchanged.

3.1.2 Notations

Let $d \geq 1$. Throughout the chapter, a test function (on $\mathbb{R}^d$) will be any function $\rho : \mathbb{R}^d \to \mathbb{R}$ that is smooth and compactly-supported. Naturally, let us denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions on $\mathbb{R}^d$.

The main functional spaces of this chapter are the Sobolev spaces defined for all $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ as

$$\mathcal{W}^{s,p}(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{\mathcal{W}^{s,p}} = \|\mathcal{F}^{-1}(\{1+|.|^2\}^{\frac{s}{2}} \mathcal{F} f) | L^p(\mathbb{R}^d)\| < \infty \Big\} ,$$

where the Fourier transform $\mathcal{F}$, resp. the inverse Fourier transform $\mathcal{F}^{-1}$, is defined with the following convention: for every $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\mathcal{F}(f)(x) = \hat{f}(x) := \int_{\mathbb{R}^d} f(y) e^{-i\langle x,y\rangle} dy\,, \quad \text{resp. } \mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(y) e^{i\langle x,y\rangle} dy\,.$$

We also set $H^s(\mathbb{R}^d) := \mathcal{W}^{s,2}(\mathbb{R}^d)$.

Now, as soon as space-time functions (or distributions) are concerned, we may use the following shortcut notation: for every $T \geq 0$, $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$,

$$L^p_T \mathcal{W}^{s,q} := L^p([0,T]; \mathcal{W}^{s,q}(\mathbb{R}^d))\,, \qquad \|.\|_{L^p_T \mathcal{W}^{s,q}} := \|.\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\,. \tag{3.1.23}$$

The notation $\mathcal{C}([0,T]; \mathcal{W}^{s,q}(\mathbb{R}^d))$ will refer to the set of continuous functions on $[0,T]$ taking values in $\mathcal{W}^{s,q}(\mathbb{R}^d)$.

3.1.3 Outline of the study

The rest of the chapter is organized as follows. In Section 3.2, we prove the local well-posedness result in the regular case. Section 3.3 is devoted to the analysis in the irregular case. Then, in Section 3.4 and Section 3.5, we develop the stochastic constructions at the core of the equation. To end with, in Section 3.6, we combined the results from the previous sections to prove Theorem 3.1.5, Theorem 3.1.10 and Theorem 3.1.15.

Throughout the chapter, the notation $A \lesssim B$ will be used to signify that there exists an irrelevant constant c such that $A \leq cB$.

3.2 Analysis of the deterministic equation under condition (H1)

The aim of this section is to deal with the model in the *regular* case, that is when assumption (**H1**) on the Hurst index is verified, and the linear solution $\rho\Upsilon$ (defined by Proposition 3.1.3) is a function of both time *and* space. Recall that in this situation, the equation is interpreted through Definition 3.1.4. Thus, what we would like to solve in this section is the equation

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \rho\Upsilon_\tau)\,d\tau + \int_0^t e^{(t-\tau)\Delta}(\rho^2\Upsilon_\tau^2)\,d\tau\,, \quad t \in [0,T]\,. \tag{3.2.1}$$

Contrary to the next sections where several probabilistic arguments will be used, our strategy towards a (local) solution v for (3.2.1) will be based on deterministic estimates only. To put it differently, we henceforth consider $\rho\Upsilon$ as a fixed (i.e., non-random) element in the space $\mathcal{C}([0,T];\mathcal{W}^{\beta,p}(\mathbb{R}^d))$ for some appropriate $0<\beta<1$ (where $\beta=-\alpha$ is given by Proposition 3.1.3) and $p \geq 2$, and try to solve the deterministic equation below: for $\mathbf{\Psi} \in \mathcal{C}([0,T];\mathcal{W}^{\beta,p}(\mathbb{R}^d))$,

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\,d\tau + \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}_\tau^2)\,d\tau\,, \quad t \in [0,T]\,. \tag{3.2.2}$$

Let us present the tools related to the two main operations in (3.2.2).

3.2.1 $\mathcal{W}^{\alpha,p} - \mathcal{W}^{\alpha,q}$ estimate

We recall the well-known integrability property of the heat semigroup resulting from the Riesz-Thorin theorem.

Proposition 3.2.1. *Let $1 \leq q \leq p \leq \infty$, $\alpha \in \mathbb{R}$ and $u_0 \in \mathcal{W}^{\alpha,q}(\mathbb{R}^d)$. Then for all $t > 0$,*

$$\|e^{t\Delta}u_0\|_{\mathcal{W}^{\alpha,p}} \leq (4\pi t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{\mathcal{W}^{\alpha,q}}.$$

Proof. Suppose $u_0 \in \mathcal{S}(\mathbb{R}^d)$. We first notice that: $e^{t\Delta}(Id-\Delta)^{\frac{\alpha}{2}}u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^2}(1+|\xi|^2)^{\frac{\alpha}{2}}\widehat{u_0}) = \mathcal{F}^{-1}((1+|\xi|^2)^{\frac{\alpha}{2}}e^{-t|\xi|^2}\widehat{u_0}) = (Id-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u_0$. Resorting to Lemma 3.2.2 below, we then deduce

$$\begin{aligned}
\|e^{t\Delta}u_0\|_{\mathcal{W}^{\alpha,p}} &= \|(Id-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u_0\|_{L^p} \\
&= \|e^{t\Delta}(Id-\Delta)^{\frac{\alpha}{2}}u_0\|_{L^p} \\
&\leq (4\pi t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|(Id-\Delta)^{\frac{\alpha}{2}}u_0\|_{L^q} \\
&= (4\pi t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{\mathcal{W}^{\alpha,q}}.
\end{aligned}$$

Lemma 3.2.2. *Let $1 \leq q \leq p \leq \infty$ and $u_0 \in L^q(\mathbb{R}^d)$. Then*

$$e^{t\Delta}u_0 \in \mathcal{C}(\mathbb{R}^+; L^q(\mathbb{R}^d)) \cap \mathcal{C}^\infty(\mathbb{R}^+_*; L^p(\mathbb{R}^d))$$

and for all $t > 0$,

$$\|e^{t\Delta}u_0\|_{L^p} \leq (4\pi t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q}.$$

□

3.2.2 Pointwise multiplication

The second tool to deal with equation (3.2.2) is a refined fractional Leibniz rule that can be found in [11].

Lemma 3.2.3 (Kato-Ponce inequality)**.** *Let $1 \leq r < \infty$ and $1 < p_1, p_2, q_1, q_2 < \infty$ verifying*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Let $s \geq 0$. Then one has

$$\|u \cdot v\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \lesssim \|u\|_{\mathcal{W}^{s,p_1}(\mathbb{R}^d)}\|v\|_{L^{p_2}(\mathbb{R}^d)} + \|u\|_{L^{q_1}(\mathbb{R}^d)}\|v\|_{\mathcal{W}^{s,q_2}(\mathbb{R}^d)}.$$

3.2.3 About the resolution of the deterministic equation

Our main result regarding equation (3.2.2) can be formulated in the following way:

Theorem 3.2.4. *Fix $\beta \in (0,1)$. Assume that $d \geq 1$ and $p \geq 2$ is such that $\frac{d}{2p} < 1 + \frac{\beta}{2}$. For every $T > 0$, define the space $X^{\beta,p}(T)$ as*

$$X^{\beta,p}(T) := \mathcal{C}([0,T]; \mathcal{W}^{\beta,p}(\mathbb{R}^d)), \tag{3.2.3}$$

equipped with the norm

$$\|v\|_{X(T)} := \|v\|_{X^{\beta,p}(T)} = \|v\|_{L^\infty_T \mathcal{W}^{\beta,p}}.$$

Then for every $\phi \in \mathcal{W}^{\beta,p}(\mathbb{R}^d)$, one can find a time $T_0 > 0$ such that, for every $\mathbf{\Psi} \in X^{\beta,p}(T_0)$, equation (3.2.2) *admits a unique solution in $X^{\beta,p}(T_0)$.*

This local well-posedness result will be the consequence of a classical fixed-point argument. In order to implement this procedure, let us introduce the map Γ defined by the right-hand side of (3.2.2), that is: for all $\phi \in \mathcal{W}^{\beta,p}(\mathbb{R}^d)$, $v, \mathbf{\Psi} \in X^{\beta,p}(T)$, $T \geq 0$, set

$$\begin{aligned}\Gamma_{T,\mathbf{\Psi}}(v)_t := e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\, d\tau \\ + \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}_\tau^2)\, d\tau\,, \quad t \in [0,T]\,.\end{aligned}$$

Proposition 3.2.5. *In the setting of Theorem 3.2.4, the bounds below hold true: one can find $\varepsilon > 0$ such that for all $0 \leq T \leq 1$, $\phi \in \mathcal{W}^{\beta,p}(\mathbb{R}^d)$, $\mathbf{\Psi}_1, \mathbf{\Psi}_2, v, v_1, v_2 \in X^{\beta,p}(T) := X(T)$,*

$$\|\Gamma_{T,\mathbf{\Psi}_1}(v)\|_{X(T)} \lesssim \|\phi\|_{\mathcal{W}^{\beta,p}(\mathbb{R}^d)} + T^\varepsilon\Big[\|v\|^2_{X(T)} + \|\mathbf{\Psi}_1\|_{X(T)}\|v\|_{X(T)} + \|\mathbf{\Psi}_1\|^2_{X(T)}\Big], \tag{3.2.4}$$

and

$$\begin{aligned}&\|\Gamma_{T,\mathbf{\Psi}_1}(v_1) - \Gamma_{T,\mathbf{\Psi}_2}(v_2)\|_{X(T)} \\ &\lesssim T^\varepsilon\Big[\|v_1 - v_2\|_{X(T)}\Big\{\|v_1\|_{X(T)} + \|v_2\|_{X(T)}\Big\} + \|\mathbf{\Psi}_1 - \mathbf{\Psi}_2\|_{X(T)}\|v_1\|_{X(T)} \\ &\qquad + \|\mathbf{\Psi}_2\|_{X(T)}\|v_1 - v_2\|_{X(T)} + \|\mathbf{\Psi}_1 - \mathbf{\Psi}_2\|_{X(T)}\Big\{\|\mathbf{\Psi}_1\|_{X(T)} + \|\mathbf{\Psi}_2\|_{X(T)}\Big\}\Big],\end{aligned} \tag{3.2.5}$$

where the proportional constants only depend on β, p and ρ.

Before we focus on the proof of this proposition, let us briefly remember that, once endowed (3.2.4)-(3.2.5), the result of Theorem 3.2.4 follows from a classical application of the Picard fixed-point theorem. Precisely, using (3.2.4), we can first show that for all $T = T(\phi, \mathbf{\Psi}) > 0$ small enough, there exists a ball in $X(T)$ that is stable through the action of $\Gamma_{T,\mathbf{\Psi}}$. Then, helped with (3.2.5) (used with $\mathbf{\Psi}_1 = \mathbf{\Psi}_2 = \mathbf{\Psi}$), the fact that $\Gamma_{T,\mathbf{\Psi}}$ is actually a contraction on this ball is easily deduced (for $T > 0$ possibly even smaller), which completes the proof of the theorem.

Let us remark that the continuity of $\Gamma_{T,\mathbf{\Psi}}$ with respect to $\mathbf{\Psi}$ (a direct consequence of (3.2.5)) will be a major ingredient toward item (*ii*) of Theorem 3.1.5.

We begin by establishing some estimates.

Lemma 3.2.6. *Fix $\beta \in (0,1)$. Assume that $d \geq 1$ and $p \geq 2$ is such that $\frac{d}{2p} < 1 + \frac{\beta}{2}$. There exists $\varepsilon > 0$ such that for all $0 \leq T \leq 1$, $f_i = \rho v$ or $f_i = \rho$,*

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(f_1 \cdot f_2(\tau))d\tau \Big\|_{X^{\beta,p}(T)} \lesssim T^\varepsilon \|f_1\|_{X^{\beta,p}(T)} \|f_2\|_{X^{\beta,p}(T)}.$$

Proof. Let $0 \leq t \leq T, 1 \leq q < p$ and $1 \leq r < \infty$ be such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Using successively Minkowsky inequality, Proposition 3.2.1 and Lemma 3.2.3, we get

$$\begin{aligned} \Big\| \int_0^t e^{(t-\tau)\Delta}(f_1 \cdot f_2(\tau)) d\tau \Big\|_{\mathcal{W}^{\beta,p}} &\leq \int_0^t \| e^{(t-\tau)\Delta}(f_1 \cdot f_2(\tau)) \|_{\mathcal{W}^{\beta,p}} d\tau \\ &\lesssim \int_0^t (t-\tau)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})} \| f_1 \cdot f_2(\tau) \|_{\mathcal{W}^{\beta,q}} d\tau \\ &\lesssim \int_0^t (t-\tau)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})} [\| f_1(\tau)\|_{\mathcal{W}^{\beta,p}} \| f_2(\tau)\|_{L^r} + \| f_2(\tau)\|_{\mathcal{W}^{\beta,p}} \| f_1(\tau)\|_{L^r}] d\tau. \end{aligned}$$

Now, as soon as $\frac{1}{q} > \max\left(\frac{2}{p} - \frac{\beta}{d}, \frac{1}{p}\right)$, observe that $\beta \geq d(\frac{1}{p} - \frac{1}{r})$ and consequently we have the Sobolev embedding $\mathcal{W}^{\beta,p}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ that leads to

$$\begin{aligned} \Big\| \int_0^t e^{(t-\tau)\Delta}(f_1 \cdot f_2(\tau)) d\tau \Big\|_{\mathcal{W}^{\beta,p}} &\lesssim \int_0^t (t-\tau)^{-\frac{d}{2r}} \| f_1(\tau)\|_{\mathcal{W}^{\beta,p}} \| f_2(\tau)\|_{\mathcal{W}^{\beta,p}} d\tau \\ &\lesssim T^{1-\frac{d}{2r}} \| f_1 \|_{X^{\beta,p}(T)} \| f_2 \|_{X^{\beta,p}(T)} \\ &\lesssim T^{1-\frac{d}{2p}+\frac{\beta}{2}} \| f_1 \|_{X^{\beta,p}(T)} \| f_2 \|_{X^{\beta,p}(T)}. \end{aligned}$$

Taking the supremum over $t \in [0, T]$, we obtain the desired conclusion:

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(f_1 \cdot f_2(\tau)) d\tau \Big\|_{X^{\beta,p}(T)} \lesssim T^{1-\frac{d}{2p}+\frac{\beta}{2}} \| f_1 \|_{X^{\beta,p}(T)} \| f_2 \|_{X^{\beta,p}(T)}.$$

□

Proof of Proposition 3.2.5. Let us bound each of the four terms in the expression of $\Gamma_{T,\mathbf{\Psi}}$:

$$\begin{aligned} \Gamma_{T,\mathbf{\Psi}}(v)_t := e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2) d\tau + 2 \int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\, d\tau \\ + \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}_\tau^2)\, d\tau\,, \quad t \in [0, T]\,. \end{aligned}$$

On the one hand, for all $0 \leq t \leq T$, using the expression of the heat semigroup, it holds that $\| e^{t\Delta}\phi \|_{\mathcal{W}^{\beta,p}(\mathbb{R}^d)} \lesssim \| \phi \|_{\mathcal{W}^{\beta,p}(\mathbb{R}^d)}$, yielding to

$$\| e^{t\Delta}\phi \|_{X^{\beta,p}(T)} \lesssim \| \phi \|_{\mathcal{W}^{\beta,p}(\mathbb{R}^d)}. \tag{3.2.6}$$

On the other hand, the bound concerning the three other terms is a straight consequence from Lemma 3.2.6 and provides us with the bound (3.2.4) we are looking for. The second one (3.2.5) can be obtained with quite the same arguments. □

3.3 Analysis of the deterministic equation under condition (H2)

The objective of this section is to cope with the wellposedness issue in the rough situation, that is when condition (**H2**) on the Hurst indexes is verified. We recall that in this rough case, the model is understood in the meaning of Definition 3.1.9, that is as

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \rho\overset{\circ}{|}_\tau)\, d\tau + \int_0^t e^{(t-\tau)\Delta}(\rho^2 \overset{\circ\,\circ}{\vee}_\tau)\, d\tau\,, \quad t \in [0,T]\,. \tag{3.3.1}$$

where the processes $\rho\overset{\circ}{|}$ and $\rho^2\overset{\circ\,\circ}{\vee}$ are those defined in Proposition 3.1.3 and Proposition 3.1.8.

In order to handle (3.3.1), we intend to follow the same deterministic strategy as in Section 3.2. To put it differently, we will consider the pair $(\rho\overset{\circ}{|}, \rho^2\overset{\circ\,\circ}{\vee})$ as a fixed element in the subspace

$$\mathcal{R}_{\alpha,p} := L^\infty\Big([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)\Big) \times L^\infty\Big([0,T]; \mathcal{W}^{-2\alpha,p}(\mathbb{R}^d)\Big)\,, \tag{3.3.2}$$

where $0 < \alpha < \frac{1}{4}$ (coming from Propositions 3.1.3 and 3.1.8), $p \geq 2$ and then the aim will be to solve the more general deterministic equation: for $(\mathbf{\Psi}, \mathbf{\Psi^2}) \in \mathcal{R}_{\alpha,p}$,

$$v_t = e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau + 2\int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\, d\tau + \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)\, d\tau\,, \quad t \in [0,T]\,. \tag{3.3.3}$$

The main originality of the situation (compared to Section 3.2) is the lack of regularity of $\mathbf{\Psi}_\tau$ and $\mathbf{\Psi}^{\mathbf{2}}_\tau$, that can only be seen as negative-order Sobolev terms (remark indeed that $\alpha > 0$ in (3.3.2)).

3.3.1 Regularising effect of the heat semigroup

The use of the fractional Sobolev spaces allows us to quantify the regularising effect of the heat semigroup (see [21]).

Proposition 3.3.1. *Let $1 < p < \infty$. For any $s_1 < s_2$, there exists a constant $C(s_1, s_2) < \infty$ such that*

$$\|e^{t\Delta}\phi\|_{\mathcal{W}^{s_2,p}(\mathbb{R}^d)} \leq C(s_1,s_2)(1 + t^{-\frac{s_2-s_1}{2}})\|\phi\|_{\mathcal{W}^{s_1,p}(\mathbb{R}^d)}$$

holds for all $\phi \in \mathcal{W}^{s_1,p}(\mathbb{R}^d)$ and $t > 0$.

Proof. According to [21, Section 2.6], for all $s \geq 0$ and every $\phi \in L^p$, one has

$$\|e^{t\Delta}\phi\|_{\mathcal{W}^{s,p}} \lesssim (1 + t^{-\frac{s}{2}})\|\phi\|_{L^p}.$$

Now, fix $s_1 < s_2$. Using the previous inequality, we easily obtain

$$\begin{aligned}
\|e^{t\Delta}\phi\|_{\mathcal{W}^{s_2,p}} &= \|(Id-\Delta)^{\frac{s_1}{2}} e^{t\Delta}\phi\|_{\mathcal{W}^{s_2-s_1,p}} \\
&= \|e^{t\Delta}(Id-\Delta)^{\frac{s_1}{2}}\phi\|_{\mathcal{W}^{s_2-s_1,p}} \\
&\lesssim (1+t^{-\frac{s_2-s_1}{2}})\|(Id-\Delta)^{\frac{s_1}{2}}\phi\|_{L^p} \\
&\lesssim (1+t^{-\frac{s_2-s_1}{2}})\|\phi\|_{\mathcal{W}^{s_1,p}},
\end{aligned}$$

that is the desired conclusion. □

3.3.2 Pointwise multiplication

The solution we propose to deal with the product $(\rho v_\tau)\cdot(\mathbf{\Psi}_\tau)$ in (3.3.3) will be to resort to the following general multiplication property in Sobolev spaces (see e.g. [22, Section 4.4.3] for a proof of this result) which garantees that $(\rho v_\tau)\cdot(\mathbf{\Psi}_\tau)$ exists and inherits the worst regularity. More precisely:

Lemma 3.3.2. *Fix $d \geq 1$. Let $\alpha, \beta > 0$ and $1 \leq p, p_1, p_2 < \infty$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad 0 < \alpha < \beta\,.$$

If $f \in \mathcal{W}^{-\alpha,p_1}(\mathbb{R}^d)$ and $g \in \mathcal{W}^{\beta,p_2}(\mathbb{R}^d)$, then $f \cdot g \in \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)$ and

$$\|f\cdot g\|_{\mathcal{W}^{-\alpha,p}} \lesssim \|f\|_{\mathcal{W}^{-\alpha,p_1}}\|g\|_{\mathcal{W}^{\beta,p_2}}\ .$$

3.3.3 About the resolution of the auxiliary deterministic equation

For all $T \geq 0$, $\alpha, \beta > 0$ and $p \geq 2$, introduce the space

$$X^{\alpha,\beta,p}(T) := \mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)) \cap \mathcal{C}((0,T];\mathcal{W}^{\beta,p}(\mathbb{R}^d))\,, \tag{3.3.4}$$

equipped with the norm

$$\|v\|_{X^{\alpha,\beta,p}(T)} := \|v\|_{X(T)} = \|v\|_{L^\infty_T\mathcal{W}^{-\alpha,p}} + \|v\|_{Y(T)}$$

where the $Y(T)$-seminorm is given by

$$\|v\|_{Y(T)} = \sup_{0<t\leq T} t^{\frac{\beta+\alpha}{2}}\|v(t)\|_{\mathcal{W}^{\beta,p}(\mathbb{R}^d)}.$$

Also, remember that the subspace $\mathcal{R}_{\alpha,p}$ has been defined in (3.3.2).

To end with, we are in a position to formulate (and prove) the main result of this section:

Theorem 3.3.3. *Suppose that $d \geq 1$ and $p \geq 2$ is such that $\frac{d}{2p} < 1$. In addition, assume that $\alpha, \beta > 0$ verify*

$$\alpha < \beta < \min\left(2 - \alpha - \frac{d}{p}, 2 - 2\alpha\right). \tag{3.3.5}$$

Then for every $\phi \in \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)$ and $(\mathbf{\Psi}, \mathbf{\Psi^2}) \in \mathcal{R}_{\alpha,p}$, one can find a time $T > 0$ for which equation (3.3.3) admits a unique solution in the above-defined set $X^{\alpha,\beta,p}(T)$.

In the same way as in Section 3.2.3, the proof of Theorem 3.3.3 is of course a direct consequence of the estimates below for the map $\Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}$ defined for all $T \geq 0$ and $(\mathbf{\Psi}, \mathbf{\Psi^2}) \in \mathcal{R}_{\alpha,p}$ by

$$\Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}(v) := e^{t\Delta}\phi + \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2) d\tau + 2\int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\, d\tau \\ + \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi^2_\tau})\, d\tau\,, \quad t \in [0,T]\,. \tag{3.3.6}$$

Proposition 3.3.4. *Suppose that $d \geq 1$ and $p \geq 2$ is such that $\frac{d}{2p} < 1$. In addition, assume that $\alpha, \beta > 0$ verify condition (3.3.5). Then there exists $\varepsilon > 0$ such that, setting $X(T) := X^{\alpha,\beta,p}(T)$, the following bounds hold true: for all $0 \leq T \leq 1$, $\phi \in \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)$, $(\mathbf{\Psi}_1, \mathbf{\Psi^2_1}) \in \mathcal{R}_{\alpha,p}$, $(\mathbf{\Psi}_2, \mathbf{\Psi^2_2}) \in \mathcal{R}_{\alpha,p}$ and $v, v_1, v_2 \in X(T)$,*

$$\|\Gamma_{T,\mathbf{\Psi}_1,\mathbf{\Psi^2_1}}(v)\|_{X(T)} \lesssim \|\phi\|_{\mathcal{W}^{-\alpha,p}} + T^\varepsilon\Big[\|v\|^2_{X(T)} + \|\mathbf{\Psi}_1\|_{L^\infty_T\mathcal{W}^{-\alpha,p}}\|v\|_{X(T)} + \|\mathbf{\Psi^2_1}\|_{L^\infty_T\mathcal{W}^{-2\alpha,p}}\Big]\,, \tag{3.3.7}$$

and

$$\|\Gamma_{T,\mathbf{\Psi}_1,\mathbf{\Psi^2_1}}(v_1) - \Gamma_{T,\mathbf{\Psi}_2,\mathbf{\Psi^2_2}}(v_2)\|_{X(T)} \\ \lesssim T^\varepsilon\Big[\|v_1 - v_2\|_{X(T)}\{\|v_1\|_{X(T)} + \|v_2\|_{X(T)}\} + \|\mathbf{\Psi}_1 - \mathbf{\Psi}_2\|_{L^\infty_T\mathcal{W}^{-\alpha,p}}\|v_1\|_{X(T)} \\ + \|\mathbf{\Psi}_2\|_{L^\infty_T\mathcal{W}^{-\alpha,p}}\|v_1 - v_2\|_{X(T)} + \|\mathbf{\Psi^2_1} - \mathbf{\Psi^2_2}\|_{L^\infty_T\mathcal{W}^{-2\alpha,p}}\Big]\,, \tag{3.3.8}$$

where the proportional constants depend only on ρ, p, α and β.

Proof of Proposition 3.3.4. Let us bound each of the four terms in the expression of $\Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}$ separately.

Bound on $e^{t\Delta}\phi$: For all $0 \leq t \leq T$, using the expression of the heat semigroup, it holds that $\|e^{t\Delta}\phi\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)} \lesssim \|\phi\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)}$, yielding to

$$\|e^{t\Delta}\phi\|_{L^\infty_T\mathcal{W}^{-\alpha,p}} \lesssim \|\phi\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)}.$$

Now, thanks to Proposition 3.3.1, for any $0 < t \leq T$, $\|e^{t\Delta}\phi\|_{\mathcal{W}^{\beta,p}(\mathbb{R}^d)} \lesssim t^{-\frac{\beta+\alpha}{2}}\|\phi\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)}$, which implies $t^{\frac{\beta+\alpha}{2}}\|e^{t\Delta}\phi\|_{\mathcal{W}^{\beta,p}(\mathbb{R}^d)} \lesssim \|\phi\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)}$ and $\|e^{t\Delta}\phi\|_{Y(T)} \lesssim \|\phi\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)}$ leading to

$$\|e^{t\Delta}\phi\|_{X(T)} \lesssim \|\phi\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)}. \tag{3.3.9}$$

Bound on $\int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau$: Let $0 \leq t \leq T$. Using successively Minkowsky inequality, Proposition 3.2.1 and Lemma 3.3.2 (since $\alpha < \beta$), we get

$$\begin{aligned}
\left\| \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau \right\|_{\mathcal{W}^{-\alpha,p}} &\leq \int_0^t \|e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{d}{2p}}} \|\rho^2 v_\tau^2\|_{\mathcal{W}^{-\alpha,\frac{p}{2}}} d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{d}{2p}}} \|v_\tau\|_{\mathcal{W}^{\beta,p}} \|v_\tau\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{d}{2p}} \tau^{-\frac{\beta+\alpha}{2}} d\tau \|v\|_{Y(T)} \|v\|_{L_T^\infty \mathcal{W}^{-\alpha,p}} \\
&\lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)}^2.
\end{aligned}$$

Taking the supremum over $t \in [0,T]$, we deduce:

$$\left\| \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau \right\|_{L_T^\infty \mathcal{W}^{-\alpha,p}} \lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)}^2.$$

Now, as $e^{(t-\tau)\Delta} = e^{\frac{t-\tau}{2}\Delta} e^{\frac{t-\tau}{2}\Delta}$, the heat kernel allows a gain of both regularity (Proposition 3.3.1) and integrability (Proposition 3.2.1) showing that for all $0 < t \leq T$,

$$\begin{aligned}
\left\| \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau \right\|_{\mathcal{W}^{\beta,p}} &\leq \int_0^t \|e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)\|_{\mathcal{W}^{\beta,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}} \|e^{\frac{t-\tau}{2}\Delta}(\rho^2 v_\tau^2)\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}-\frac{d}{2p}} \|\rho^2 v_\tau^2\|_{\mathcal{W}^{-\alpha,\frac{p}{2}}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}-\frac{d}{2p}} \|v_\tau\|_{\mathcal{W}^{\beta,p}} \|v_\tau\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}-\frac{d}{2p}} \tau^{-\frac{\beta+\alpha}{2}} d\tau \|v\|_{Y(T)} \|v\|_{L_T^\infty \mathcal{W}^{-\alpha,p}} \\
&\lesssim t^{1-\frac{d}{2p}-\beta-\alpha} \|v\|_{X(T)}^2,
\end{aligned}$$

leading to

$$\begin{aligned}
t^{\frac{\beta+\alpha}{2}} \left\| \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau \right\|_{\mathcal{W}^{\beta,p}} &\lesssim t^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)}^2 \\
&\lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)}^2.
\end{aligned}$$

Taking the supremum over $t \in (0,T]$, it holds

$$\left\| \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2)d\tau \right\|_{Y(T)} \lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)}^2,$$

which entails

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(\rho^2 v_\tau^2) d\tau \Big\|_{X(T)} \lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)}^2. \tag{3.3.10}$$

Bound on $\int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\, d\tau$: Fix $0 \le t \le T$. Resorting successively to Minkowsky inequality, Proposition 3.2.1 and Lemma 3.3.2 (since $\alpha < \beta$), we derive

$$\begin{aligned}
\Big\| \int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau) d\tau \Big\|_{\mathcal{W}^{-\alpha,p}} &\le \int_0^t \| e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{d}{2p}}} \|\rho v_\tau \cdot \mathbf{\Psi}_\tau\|_{\mathcal{W}^{-\alpha,\frac{p}{2}}} d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{d}{2p}}} \|\rho v_\tau\|_{\mathcal{W}^{\beta,p}} \|\mathbf{\Psi}_\tau\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{d}{2p}} \tau^{-\frac{\beta+\alpha}{2}} d\tau \|v\|_{Y(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}} \\
&\lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}}.
\end{aligned}$$

Taking the supremum over $t \in [0,T]$, we deduce:

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau) d\tau \Big\|_{L_T^\infty \mathcal{W}^{-\alpha,p}} \lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}}.$$

Now, for every $0 < t \le T$, combining Proposition 3.3.1 and Proposition 3.2.1 with Lemma 3.3.2, we write

$$\begin{aligned}
\Big\| \int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau) d\tau \Big\|_{\mathcal{W}^{\beta,p}} &\le \int_0^t \| e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\|_{\mathcal{W}^{\beta,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}} \| e^{\frac{t-\tau}{2}\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau)\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}-\frac{d}{2p}} \|\rho v_\tau \cdot \mathbf{\Psi}_\tau\|_{\mathcal{W}^{-\alpha,\frac{p}{2}}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}-\frac{d}{2p}} \|\rho v_\tau\|_{\mathcal{W}^{\beta,p}} \|\mathbf{\Psi}_\tau\|_{\mathcal{W}^{-\alpha,p}} d\tau \\
&\lesssim \int_0^t (t-\tau)^{-\frac{\beta+\alpha}{2}-\frac{d}{2p}} \tau^{-\frac{\beta+\alpha}{2}} d\tau \|v\|_{Y(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}} \\
&\lesssim t^{1-\frac{d}{2p}-\beta-\alpha} \|v\|_{X(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}}.
\end{aligned}$$

Thus, $t^{\frac{\beta+\alpha}{2}} \Big\| \int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau) d\tau \Big\|_{\mathcal{W}^{\beta,p}} \lesssim T^{1-\frac{d}{2p}-\frac{\alpha+\beta}{2}} \|v\|_{X(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}}$ and

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau) d\tau \Big\|_{Y(T)} \lesssim T^{1-\frac{d}{2p}-\frac{\alpha+\beta}{2}} \|v\|_{X(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}}.$$

Finally,

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(\rho v_\tau \cdot \mathbf{\Psi}_\tau) d\tau \Big\|_{X(T)} \lesssim T^{1-\frac{d}{2p}-\frac{\beta+\alpha}{2}} \|v\|_{X(T)} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-\alpha,p}}. \tag{3.3.11}$$

Bound on $\int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)d\tau$: Fix $0 \le t \le T$. Using successively Minkowsky inequality and Proposition 3.3.1, we write

$$\begin{aligned}\Big\| \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)d\tau \Big\|_{\mathcal{W}^{-\alpha,p}} &\le \int_0^t \|e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)\|_{\mathcal{W}^{-\alpha,p}} d\tau \\ &\lesssim \int_0^t (t-\tau)^{-\frac{\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}_\tau\|_{\mathcal{W}^{-2\alpha,p}} d\tau \\ &\lesssim T^{1-\frac{\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}\|_{L^\infty_T \mathcal{W}^{-2\alpha,p}},\end{aligned}$$

which entails

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)d\tau \Big\|_{L^\infty_T \mathcal{W}^{-\alpha,p}} \lesssim T^{1-\frac{\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}\|_{L^\infty_T \mathcal{W}^{-2\alpha,p}}. \tag{3.3.12}$$

Then, for all $0 < t \le T$, thanks to the regularising effect of the heat semigroup and the assumption $\beta + 2\alpha < 2$,

$$\begin{aligned}\Big\| \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)d\tau \Big\|_{\mathcal{W}^{\beta,p}} &\le \int_0^t \|e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)\|_{\mathcal{W}^{\beta,p}} d\tau \\ &\lesssim \int_0^t (t-\tau)^{-\frac{\beta+2\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}_\tau\|_{\mathcal{W}^{-2\alpha,p}} d\tau \\ &\lesssim t^{1-\frac{\beta+2\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}\|_{L^\infty_T \mathcal{W}^{-2\alpha,p}},\end{aligned}$$

which leads to $t^{\frac{\beta+\alpha}{2}} \Big\| \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)d\tau \Big\|_{\mathcal{W}^{\beta,p}} \lesssim T^{1-\frac{\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}\|_{L^\infty_T \mathcal{W}^{-2\alpha,p}}$. Thus,

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)d\tau \Big\|_{Y(T)} \lesssim T^{1-\frac{\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}\|_{L^\infty_T \mathcal{W}^{-2\alpha,p}} \tag{3.3.13}$$

and, combining the identities (3.3.12) and (3.3.13),

$$\Big\| \int_0^t e^{(t-\tau)\Delta}(\mathbf{\Psi}^{\mathbf{2}}_\tau)d\tau \Big\|_{X(T)} \lesssim T^{1-\frac{\alpha}{2}} \|\mathbf{\Psi}^{\mathbf{2}}\|_{L^\infty_T \mathcal{W}^{-2\alpha,p}}. \tag{3.3.14}$$

Putting together the four inequalities (3.3.9), (3.3.10), (3.3.11) and (3.3.14) provides us with the desired bound (3.3.7). The second one (3.3.8) can be obtained with similar arguments.

□

3.4 On the construction of the relevant stochastic objects

The objective of this section is to prove Proposition 3.1.3, Proposition 3.1.8 and to give an asymptotic estimate of the quantity $\mathbb{E}\Big[\overset{\circ}{\underset{\bullet}{|}}_n(t,x)^2\Big]$ (Proposition 3.1.6). We recall that the space-time fractional white noise B has been defined as a Gaussian process whose mean and covariance function are well-known (see Definition 3.1.1). This definition can seem a bit abstract and, in order to realize computations with this noise, we need a representation formula. We will resort to the harmonizable representation of the fractional Brownian motion presented in [23]. Let us recall the procedure. First of all, we fix, on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a space-time white noise W on $\mathbb{R}^{d+1}$. Then let $H = (H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$ and set, for any $t \in [0,T]$ and $x \in \mathbb{R}^d$,

$$B(t, x_1, \ldots, x_d) := c_H \int_{\xi \in \mathbb{R}} \int_{\eta \in \mathbb{R}^d} \frac{e^{it\xi} - 1}{|\xi|^{H_0 + \frac{1}{2}}} \prod_{i=1}^{d} \frac{e^{ix_i \eta_i} - 1}{|\eta_i|^{H_i + \frac{1}{2}}} \widehat{W}(d\xi, d\eta)\,, \tag{3.4.1}$$

where $c_H > 0$ is a constant, and where $\widehat{W}$ stands for the Fourier transform of W. Then, for an appropriate value of c_H, B is a space-time fractional Brownian motion of index H (in the sense of Definition 3.1.1).

Unfortunately, we know that B suffers from a lack of regularity measured by its Hurst index. That is why we are looking for a smooth approximation $(B_n)_{n \geq 0}$. Our strategy is based on a truncation of the integration domain in (3.4.1): precisely, we set $B_0 \equiv 0$, and for $n \geq 1$,

$$B_n(t, x_1, \ldots, x_d)(\omega) := c_H \int_{|\xi| \leq 2^{2n}} \int_{|\eta| \leq 2^n} \frac{e^{it\xi} - 1}{|\xi|^{H_0 + \frac{1}{2}}} \prod_{i=1}^{d} \frac{e^{ix_i \eta_i} - 1}{|\eta_i|^{H_i + \frac{1}{2}}} \widehat{W}(d\xi, d\eta)\,. \tag{3.4.2}$$

It is then clear than B_n defines a smooth process for any $n \geq 0$ and that for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $B_n(t,x) \overset{n \to \infty}{\longrightarrow} B(t,x)$ in $L^2(\Omega)$.

3.4.1 Linear solution associated with the model

With a view to constructing the first order process Ψ, the solution of the linear equation

$$\begin{cases} \partial_t \Psi - \Delta \Psi = \dot{B}, \quad t \in [0,T], \quad x \in \mathbb{R}^d, \\ \Psi(0,.) = 0, \end{cases} \tag{3.4.3}$$

we are now interested in giving a rigorous meaning to the sequence $(\Psi_n)_{n \geq 0}$ of the regularised version of (3.4.3), that is the sequence of solutions to

$$\begin{cases} \partial_t \Psi_n - \Delta \Psi_n = \dot{B}_n\,, \quad t \in [0,T]\,, \quad x \in \mathbb{R}^d\,, \\ \Psi_n(0,.) = 0\,, \end{cases} \tag{3.4.4}$$

where, for all $n \geq 0$, $\dot{B}_n$ denotes the classical derivative $\dot{B}_n := \partial_t \partial_{x_1} \cdots \partial_{x_d} B_n$. Despite the regularity of $\dot{B}_n$, its *a priori* lack of integrability over the whole space $\mathbb{R}^d$ prevents us from using some general theorem which would guarantee the existence of Ψ_n. We thus propose a stochastic construction of Ψ_n. Should this latter exist, its expression would by given the formula

$$\Psi_n(t,x) = \int_0^t e^{(t-s)\Delta}(\dot{B}_n(s))(x) ds.$$

A formal computation leads us to the following expression:

$$
\begin{aligned}
♀_n(t,x) &= \int_0^t e^{(t-s)\Delta}(\dot{B}_n(s))(x)ds \\
&= \int_0^t \mathcal{F}^{-1}(e^{-|\xi|^2(t-s)}\widehat{\dot{B}_n}(s,\xi))(x)ds \\
&= \int_0^t \mathcal{F}^{-1}(e^{-|\xi|^2(t-s)}) \star \dot{B}_n(s,x)ds \\
&= \int_0^t \int_{\mathbb{R}^d} \mathcal{F}^{-1}(e^{-|\xi|^2(t-s)})(x-y)\dot{B}_n(s,y)dyds \\
&= c_H \int_0^t ds \int_{\mathbb{R}^d} dy \int_{|\xi|\leq 2^{2n}} \int_{|\eta|\leq 2^n} \mathcal{F}^{-1}(e^{-|\xi|^2(t-s)})(x-y) i^{d+1} \frac{\xi}{|\xi|^{H_0+\frac{1}{2}}} \prod_{i=1}^d \frac{\eta_i}{|\eta_i|^{H_i+\frac{1}{2}}} e^{i\xi s} e^{i\langle \eta, y\rangle} \widehat{W}(d\xi, d\eta) \\
&= c_H i^{d+1} \int_{|\xi|\leq 2^{2n}} \int_{|\eta|\leq 2^n} \frac{\xi}{|\xi|^{H_0+\frac{1}{2}}} \prod_{i=1}^d \frac{\eta_i}{|\eta_i|^{H_i+\frac{1}{2}}} e^{i\langle \eta, x\rangle} . \\
&\qquad \left[\int_0^t ds\, e^{i\xi s} \left(\int_{\mathbb{R}^d} dy\, \mathcal{F}^{-1}(e^{-|\xi|^2(t-s)})(x-y) e^{-i\langle \eta, x-y\rangle}\right)\right] \widehat{W}(d\xi, d\eta) \\
&= c_H i^{d+1} \int_{|\xi|\leq 2^{2n}} \int_{|\eta|\leq 2^n} \frac{\xi}{|\xi|^{H_0+\frac{1}{2}}} \prod_{i=1}^d \frac{\eta_i}{|\eta_i|^{H_i+\frac{1}{2}}} e^{i\langle \eta, x\rangle} . \\
&\qquad \left[\int_0^t ds\, e^{i\xi(t-s)} \left(\int_{\mathbb{R}^d} dy\, \mathcal{F}^{-1}(e^{-|\xi|^2 s})(y) e^{-i\langle \eta, y\rangle}\right)\right] \widehat{W}(d\xi, d\eta).
\end{aligned}
$$

We have finally obtained that

$$
♀_n(t,x) = c_H i^{d+1} \int_{|\xi|\leq 2^{2n}} \int_{|\eta|\leq 2^n} \frac{\xi}{|\xi|^{H_0+\frac{1}{2}}} \prod_{i=1}^d \frac{\eta_i}{|\eta_i|^{H_i+\frac{1}{2}}} e^{i\langle \eta, x\rangle} \gamma_t(\xi, |\eta|)\, \widehat{W}(d\xi, d\eta),
$$

where for every $t \geq 0$, $\xi \in \mathbb{R}$ and $r > 0$, we introduce the quantity $\gamma_t(\xi, r)$ as

$$
\gamma_t(\xi, r) := e^{i\xi t} \int_0^t e^{-sr^2} e^{-i\xi s} ds\,. \tag{3.4.5}
$$

Moreover, let us observe that $♀$ is real-valued. Indeed, let $f_{t,x}$ the function defined for all $\xi \in \mathbb{R}, \eta \in \mathbb{R}^d$ by

$$
f_{t,x}(\xi,\eta) := c_H i^{d+1} \mathbb{1}_{|\xi|\leq 2^{2n}} \mathbb{1}_{|\eta|\leq 2^n} \frac{\xi}{|\xi|^{H_0+\frac{1}{2}}} \prod_{i=1}^d \frac{\eta_i}{|\eta_i|^{H_i+\frac{1}{2}}} e^{i\langle \eta, x\rangle} \gamma_t(\xi, |\eta|),
$$

in such a way that $♀_n(t,x) = \int_{\mathbb{R}^{d+1}} f_{t,x}(\xi,\eta)\widehat{W}(d\xi, d\eta)$. As $\overline{f_{t,x}(\xi,\eta)} = f_{t,x}(-\xi,-\eta)$, the

Fourier transform of f is real-valued that implies

$$\begin{aligned}
\overline{\Psi_n(t,x)} &= \overline{\int_{\mathbb{R}^{d+1}} f_{t,x}(\xi,\eta)\widehat{W}(d\xi,d\eta)} \\
&= \overline{\int_{\mathbb{R}^{d+1}} \widehat{f_{t,x}}(\xi,\eta) W(d\xi,d\eta)} \\
&= \int_{\mathbb{R}^{d+1}} \overline{\widehat{f_{t,x}}(\xi,\eta)} W(d\xi,d\eta) \\
&= \int_{\mathbb{R}^{d+1}} \widehat{f_{t,x}}(\xi,\eta) W(d\xi,d\eta) \\
&= \int_{\mathbb{R}^{d+1}} f_{t,x}(\xi,\eta)\widehat{W}(d\xi,d\eta) \\
&= \Psi_n(t,x).
\end{aligned}$$

We are now in a position to define properly Ψ_n.

Definition 3.4.1. *We call a solution of equation* (3.4.4) *(or* linear solution associated with (3.1.1)*) any centered real Gaussian process*

$$\left\{\Psi_n(s,x),\ n\geq 1,\ s\geq 0,\ x\in\mathbb{R}^d\right\}$$

whose covariance function is given by the relation: for all $n,m\geq 1$, $s,t\geq 0$ *and* $x,y\in\mathbb{R}^d$,

$$\mathbb{E}\Big[\Psi_n(s,x)\Psi_m(t,y)\Big] = c_H^2 \int_{(\xi,\eta)\in D_n\cap D_m} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \gamma_s(\xi,|\eta|)\overline{\gamma_t(\xi,|\eta|)} e^{i\langle \eta, x-y\rangle}\, d\xi d\eta\,, \tag{3.4.6}$$

where $D_n := B^1_{2n}\times B^d_n$ *with* $B^k_\ell := \left\{\lambda\in\mathbb{R}^k : |\lambda|\leq 2^\ell\right\}$.

3.4.2 Some preliminary estimates on γ

Before proving that $\chi\Psi_n$ is a Cauchy sequence in a convenient subspace, we need to establish some bounds on the quantity $\gamma_t(\xi,r)$ which contains all the information with respect to the heat semigroup:

$$\gamma_t(\xi,r) := e^{i\xi t}\int_0^t e^{-sr^2} e^{-i\xi s} ds\,. \tag{3.4.7}$$

We first state some estimates on the variation

$$\gamma_{s,t}(\xi,r) := \gamma_t(\xi,r) - \gamma_s(\xi,r). \tag{3.4.8}$$

Lemma 3.4.2. *For all* $0\leq s\leq t\leq 1$, $\xi\in\mathbb{R}$, $r>0$ *and* $\varepsilon\in[0,1]$, *the following bound holds true:*

$$|\gamma_{s,t}(\xi,r)| \lesssim \min\left(|\xi|^\varepsilon|t-s|^\varepsilon + |t-s|, \frac{|t-s|r^2}{|\xi|} + \frac{|t-s|^\varepsilon\{1+r^2\}}{|\xi|^{1-\varepsilon}}, \frac{|t-s|^\varepsilon\{r^{2\varepsilon}+|\xi|^\varepsilon\}}{\sqrt{r^4+\xi^2}}\right).$$

Proof. First of all, we write

$$\gamma_{s,t}(\xi,r)=\{e^{i\xi t}-e^{i\xi s}\}\int_0^s e^{-ur^2}e^{-i\xi u}du+e^{i\xi t}\int_s^t e^{-ur^2}e^{-i\xi u}du\,,$$

and so

$$|\gamma_{s,t}(\xi,r)|\lesssim|e^{i\xi t}-e^{i\xi s}|\left|\int_0^s e^{-ur^2}e^{-i\xi u}du\right|+\left|\int_s^t e^{-ur^2}e^{-i\xi u}du\right|\lesssim|\xi|^\varepsilon|t-s|^\varepsilon+|t-s|\,.$$

Then observe that

$$\gamma_t(\xi,r)=e^{i\xi t}\int_0^t e^{-sr^2}e^{-i\xi s}ds=-\frac{e^{-r^2t}-e^{i\xi t}}{i\xi}+\frac{ie^{i\xi t}r^2}{\xi}\int_0^t e^{-sr^2}e^{-i\xi s}ds\,,$$

which readily entails

$$\begin{aligned}&\gamma_{s,t}(\xi,r)\\&=-\frac{\{e^{-r^2t}-e^{-r^2s}\}-\{e^{i\xi t}-e^{i\xi s}\}}{i\xi}+\frac{ir^2}{\xi}\{e^{i\xi t}-e^{i\xi s}\}\int_0^s e^{-ur^2}e^{-i\xi u}du+\frac{ie^{i\xi t}r^2}{\xi}\int_s^t e^{-ur^2}e^{-i\xi u}du\,.\end{aligned}$$

Thus,

$$\begin{aligned}|\gamma_{s,t}(\xi,r)|&\lesssim r^2\frac{|t-s|}{|\xi|}+\frac{|t-s|^\varepsilon}{|\xi|^{1-\varepsilon}}+r^2\frac{|t-s|^\varepsilon}{|\xi|^{1-\varepsilon}}+\frac{r^2}{|\xi|}|t-s|\\&\lesssim r^2\frac{|t-s|}{|\xi|}+\{1+r^2\}\frac{|t-s|^\varepsilon}{|\xi|^{1-\varepsilon}}\,.\end{aligned}$$

To end with, it can be verified that

$$\gamma_t(\xi,r)=\frac{e^{i\xi t}-e^{-r^2t}}{r^2+i\xi},$$

which yields

$$\begin{aligned}|\gamma_{s,t}(\xi,r)|&=\frac{1}{\sqrt{r^4+\xi^2}}\left|\{e^{-r^2t}-e^{-r^2s}\}-\{e^{i\xi t}-e^{i\xi s}\}\right|\\&\lesssim\frac{|t-s|^\varepsilon}{\sqrt{r^4+\xi^2}}\{r^{2\varepsilon}+|\xi|^\varepsilon\}\ .\end{aligned}$$

□

Corollary 3.4.3. *For any* $0\leq s\leq t\leq 1$, $H\in(0,1)$, $r>0$ *and* $\varepsilon\in[0,H)$, *it holds that*

$$\int_{\mathbb{R}}\frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\,d\xi\lesssim\frac{|t-s|^{2\varepsilon}}{1+r^{4(H-\varepsilon)}}\,.$$

Proof. The desired inequality comes of course from a relevant use of the estimates shown in Lemma 3.4.2.

For $0 < r < 1$, we have

$$\int_{\mathbb{R}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi \lesssim |t-s|^{2\varepsilon}\left[\int_{|\xi|\leq 1}\frac{d\xi}{|\xi|^{2H-1}} + \int_{|\xi|\geq 1}\frac{d\xi}{|\xi|^{2(H-\varepsilon)+1}}\right] \lesssim |t-s|^{2\varepsilon}\,.$$

Then, for $r > 1$, it holds that

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi &\lesssim |t-s|^{2\varepsilon}\int_{\mathbb{R}}\frac{r^{4\varepsilon}+|\xi|^{2\varepsilon}}{(r^4+\xi^2)|\xi|^{2H-1}}\, d\xi \\ &\lesssim \frac{|t-s|^{2\varepsilon}}{r^{4(H-\varepsilon)}}\int_{\mathbb{R}}\frac{1+|\xi|^{2\varepsilon}}{(1+\xi^2)|\xi|^{2H-1}}\, d\xi \lesssim \frac{|t-s|^{2\varepsilon}}{r^{4(H-\varepsilon)}}\,.\end{aligned}$$

□

3.4.3 Construction of the first order stochastic process

Proof of Proposition 3.1.3. Without loss of generality, we will suppose during the whole proof that $T = 1$ and we set, for all $m, n \geq 0$, $\Psi_{n,m} := \Psi_m - \Psi_n$. Also, along the statement of the proposition, we fix α verifying (3.1.11).

Step 1: The first objective is to show that for any $p \geq 1$, $1 \leq n \leq m$, $0 \leq s \leq t \leq 1$ and $\varepsilon > 0$ small enough, it holds

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\right|^{2p}\right] dx \lesssim 2^{-4n\varepsilon p}|t-s|^{2\varepsilon p}\,, \quad (3.4.9)$$

where the proportional constant only depends on p, α and χ.

Resorting to the same kind of arguments as those used in [8, Proof of Prop 1.2] , we obtain the estimate

$$\begin{aligned}&\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\right|^{2p}\right] \\ &\lesssim \left(\int_{(\xi,\eta)\in D_{n,m}} \frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\, d\xi d\eta\right)^p.\end{aligned}$$

Now we can split the latter integral into

$$\begin{aligned}&\left(\int_{(\xi,\eta)\in D_{n,m}} \frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\, d\xi d\eta\right)^p \\ &\lesssim \left(\int_{2^{2n}\leq|\xi|\leq 2^{2m}}\int_{|\eta|\leq 2^m} \frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\, d\xi d\eta\right)^p \\ &\qquad + \left(\int_{|\xi|\leq 2^{2m}}\int_{2^n\leq|\eta|\leq 2^m} \frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}\{1+|\eta|^2\}^{-\alpha}|\gamma_{s,t}(\xi,|\eta|)|^2\, d\xi d\eta\right)^p \\ &=: (\mathbb{I}_{n,m}(s,t))^p + (\mathbb{II}_{n,m}(s,t))^p\,. \qquad (3.4.10)\end{aligned}$$

Let us focus on the estimation of $\mathbb{I}_{n,m}(s,t)$. To this end, we fix $\varepsilon > 0$, so that

$$\mathbb{I}_{n,m}(s,t) \leq 2^{-4n\varepsilon} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}^d} d\eta \, \frac{1}{|\xi|^{2H_0-2\varepsilon-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \{1+|\eta|^2\}^{-\alpha} |\gamma_{s,t}(\xi,|\eta|)|^2 \,, \quad (3.4.11)$$

which, after a classical hyperspherical change of variables, leads us to

$$\mathbb{I}_{n,m}(s,t) \lesssim 2^{-4n\varepsilon} \int_0^\infty dr \, \frac{\{1+r^2\}^{-\alpha}}{r^{2(H_1+\cdots+H_d)-2d+1}} \left(\int_{\mathbb{R}} d\xi \, \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H_0-2\varepsilon-1}} \right) . \quad (3.4.12)$$

We can now apply Corollary 3.4.3 with $H := H_0 - \varepsilon$, which gives, for all $0 < \varepsilon < \frac{H_0}{2}$,

$$\begin{aligned} &\mathbb{I}_{n,m}(s,t) \\ &\lesssim 2^{-4n\varepsilon} |t-s|^{2\varepsilon} \int_0^\infty dr \, \frac{\{1+r^2\}^{-\alpha}}{r^{2(H_1+\cdots+H_d)-2d+1}} \frac{1}{1+r^{4H_0-8\varepsilon}} \\ &\lesssim 2^{-4n\varepsilon} |t-s|^{2\varepsilon} \left(\int_0^1 \frac{1}{r^{2(H_1+\cdots+H_d)-2d+1}} dr + \int_1^\infty \frac{1}{r^{2\alpha+2(2H_0+H_1+\cdots+H_d)-2d+1-8\varepsilon}} dr \right) . \end{aligned} \quad (3.4.13)$$

Due to our assumption (3.1.11), we can in fact pick ε small enough so that

$$4\varepsilon < \alpha - \left[d - \left(2H_0 + \sum_{i=1}^{d} H_i \right) \right] ,$$

and for such a parameter, the two integrals in (3.4.13) are finite, yielding

$$\mathbb{I}_{n,m}(s,t) \lesssim 2^{-4n\varepsilon} |t-s|^{2\varepsilon} .$$

It is not difficult to see that the previous estimates could also be used to control $\mathbb{III}_{n,m}(s,t)$, yielding the very same estimate

$$\mathbb{III}_{n,m}(s,t) \lesssim 2^{-4n\varepsilon} |t-s|^{2\varepsilon} .$$

Coming back to (3.4.10), we get the desired bound (3.4.9).

Step 2: The estimate we proved in the previous step can be reformulated in the following way

$$\mathbb{E}\Big[\big\| \chi^{\mathring{|}}_{n,m}(t,.) - \chi^{\mathring{|}}_{n,m}(s,.) \big\|_{\mathcal{W}^{-\alpha,2p}}^{2p} \Big] \lesssim 2^{-4n\varepsilon p} |t-s|^{2\varepsilon p} \,, \quad (3.4.14)$$

for all $p \geq 1$, $1 \leq n \leq m$, $0 \leq s \leq t \leq 1$ and $\varepsilon > 0$ small enough.

Combining Kolmogorov continuity criterion with the classical Garsia-Rodemich-Rumsey estimate (see [10]), we easily show that for any $p \geq 1$ large enough,

$$\| \chi^{\mathring{|}}_{n,m} \|_{L^{2p}(\Omega;\mathcal{C}_T\mathcal{W}^{-\alpha,2p})} \lesssim 2^{-2n\varepsilon} . \quad (3.4.15)$$

In particular, $(\chi\<1>_n)_{n\geq 1}$ is a Cauchy sequence in $L^{2p}(\Omega;\mathcal{C}([0,T];\mathcal{W}^{-\alpha,2p}(\mathbb{R}^d)))$ (for any $p\geq 1$ large enough). As $L^{2p}(\Omega;\mathcal{C}([0,T];\mathcal{W}^{-\alpha,2p}(\mathbb{R}^d)))$ is a Banach space, we deduce the convergence of $(\chi\<1>_n)_{n\geq 1}$ in this space to a limit $\chi\<1>$. Coming back to (3.4.15), we also have

$$\|\chi\<1> - \chi\<1>_n\|_{L^{2p}(\Omega;\mathcal{C}_T\mathcal{W}^{-\alpha,2p})} \lesssim 2^{-2n\varepsilon} ,$$

and from there, a classical use of the Borell-Cantelli lemma justifies the desired almost sure convergence of $(\chi\<1>_n)_{n\geq 1}$ to $\chi\<1>$ in $\mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$, for every $2\leq p<\infty$. To end with, the convergence in $\mathcal{C}([0,T];\mathcal{W}^{-\alpha,\infty}(\mathbb{R}^d))$ is the result of the Sobolev embedding $\mathcal{W}^{-\alpha+\frac{d}{p}+\eta,p}(\mathbb{R}^d)\subset\mathcal{W}^{-\alpha,\infty}(\mathbb{R}^d)$, for any $\eta>0$.

□

3.4.4 Construction of the second order stochastic process

Proof of Proposition 3.1.8. We will follow the same general strategy as in the proof of Proposition 3.1.3. Let us again suppose that $T=1$, and set, for every $m,n\geq 0$, $\<2>_{n,m} := \<2>_m - \<2>_n$.

Step 1: Our first objective here is to prove that for every $p\geq 1$, $0\leq n\leq m$, $0\leq s\leq t\leq 1$, $\varepsilon>0$ small enough, and for every α verifying

$$d-\left(2H_0+\sum_{i=1}^d H_i\right)<\alpha<\frac{1}{4}\,, \tag{3.4.16}$$

it holds

$$\int_{\mathbb{R}^d}\mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\<2>_{n,m}(t,.)-\<2>_{n,m}(s,.)\Big]\Big)\Big)(x)\right|^{2p}\right]dx\lesssim 2^{-4n\varepsilon p}|t-s|^{\varepsilon p}\,, \tag{3.4.17}$$

where the proportional constant only depends on p,α, and χ.

In order to reduce the length of the proof, we will only show estimate (3.4.17) for $n=0$, that is we will focus on the bound for the time-variation $\<2>_m(t,.)-\<2>_m(s,.)$, with $m\geq 1$. The extended result to all $m\geq n\geq 0$ could in fact be easily deduced from the estimates below combined with the bounding argument used (for example) in (3.4.11).

Resorting to the same kind of arguments as those used in [6, Proof of Prop 1.6], we obtain the estimate

$$\int_{\mathbb{R}^d}dx\,\mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\<2>_m(t,.)-\<2>_m(s,.)\Big]\Big)\Big)(x)\right|^2\right]^p\lesssim\left(\sum_{\ell=1}^4\mathcal{J}^\ell_{m;s,t}\right)^p \tag{3.4.18}$$

where

$$\mathcal{J}^\ell_{m;s,t}:=\int_{(\xi,\eta)\in D_m}d\xi d\eta\int_{(\tilde\xi,\tilde\eta)\in D_m}d\tilde\xi d\tilde\eta\,\frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^d\frac{1}{|\eta_i|^{2H_i-1}}\frac{1}{|\tilde\xi|^{2H_0-1}}\prod_{i=1}^d\frac{1}{|\tilde\eta_i|^{2H_i-1}}\\ \Big|\Gamma^\ell_{m;s,t}(\xi,|\eta|;\tilde\xi,|\tilde\eta|)\Big|\Big\{1+|\eta-\tilde\eta|^2\Big\}^{-2\alpha}, \tag{3.4.19}$$

with $\Gamma^{\ell}_{m;s,t} = \Gamma^{\ell}_{m;s,t}(\xi, |\eta|; \tilde{\xi}, |\tilde{\eta}|)$ given by

$$\Gamma^1_{m;s,t} := \gamma_t(\xi,|\eta|)\,\overline{\gamma_{s,t}(\xi,|\eta|)}\,|\gamma_t(\tilde{\xi},|\tilde{\eta}|)|^2\,, \quad \Gamma^2_{m;s,t} := \gamma_t(\xi,|\eta|)\,\overline{\gamma_{s,t}(\xi,|\eta|)}\,\overline{\gamma_s(\tilde{\xi},|\tilde{\eta}|)}\,\gamma_t(\tilde{\xi},|\tilde{\eta}|)\,,$$
$$\Gamma^3_{m;s,t} := \gamma_s(\xi,|\eta|)\,\overline{\gamma_{s,t}(\xi,|\eta|)}\,\overline{\gamma_t(\tilde{\xi},|\tilde{\eta}|)}\,\gamma_s(\tilde{\xi},|\tilde{\eta}|)\,, \quad \Gamma^4_{m;s,t} := \overline{\gamma_{t,s}(\xi,|\eta|)}\,\gamma_s(\xi,|\eta|)|\gamma_s(\tilde{\xi},|\tilde{\eta}|)|^2\ .$$

Let us focus on the treatment of $\mathcal{J}^1_{m;s,t}$.

$$\begin{aligned}
\mathcal{J}^1_{m;s,t} &\lesssim \int_{(\xi,\eta)\in D_m} d\xi d\eta \int_{(\tilde{\xi},\tilde{\eta})\in D_m} d\tilde{\xi} d\tilde{\eta} \left\{1+|\eta-\tilde{\eta}|^2\right\}^{-2\alpha} \\
&\quad \left(\frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi,|\eta|)||\gamma_{s,t}(\xi,|\eta|)|\right) \cdot \left(\frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_t(\tilde{\xi},|\tilde{\eta}|)|^2\right) \\
&\lesssim \int_{(\mathbb{R}\times\mathbb{R}^d)^2} d\xi d\eta d\tilde{\xi} d\tilde{\eta} \left\{1+||\eta|-|\tilde{\eta}||^2\right\}^{-2\alpha} \\
&\quad \left(\frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi,|\eta|)||\gamma_{s,t}(\xi,|\eta|)|\right) \cdot \left(\frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_t(\tilde{\xi},|\tilde{\eta}|)|^2\right),
\end{aligned} \tag{3.4.20}$$

where we have used the trivial bound $|\eta-\tilde{\eta}| \geq ||\eta|-|\tilde{\eta}||$.

Now, let us decompose the latter integration domain into $(\mathbb{R}\times\mathbb{R}^d)^2 := D_1 \cup D_2$, where

$$D_1 := \left\{(\xi,\eta,\tilde{\xi},\tilde{\eta}) :\ 0 \leq |\tilde{\eta}| \leq \frac{|\eta|}{2} \text{ or } |\tilde{\eta}| \geq \frac{3|\eta|}{2}\right\}$$

and

$$D_2 := \left\{(\xi,\eta,\tilde{\xi},\tilde{\eta}) :\ \frac{|\eta|}{2} < |\tilde{\eta}| < \frac{3|\eta|}{2}\right\}.$$

Concerning the integral on D_1, the inequality $||\eta|-|\tilde{\eta}|| \geq \max(\frac{|\eta|}{2}, \frac{|\tilde{\eta}|}{3})$ (valid for all

$(\xi, \eta, \tilde{\xi}, \tilde{\eta}) \in D_1)$ allows us to write

$$\begin{aligned}
\mathcal{A}_1 &:= \int_{D_1} \frac{d\xi d\eta d\tilde{\xi} d\tilde{\eta}}{\{1 + ||\eta| - |\tilde{\eta}||^2\}^{2\alpha}} \left(\frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi, |\eta|)||\gamma_{s,t}(\xi, |\eta|)| \right) \cdot \\
&\qquad \left(\frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_t(\tilde{\xi}, |\tilde{\eta}|)|^2 \right) \\
&\lesssim \left(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\xi d\eta}{\{1 + |\eta|^2\}^{\alpha}} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi, |\eta|)||\gamma_{s,t}(\xi, |\eta|)| \right) \cdot \\
&\qquad \left(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\tilde{\xi} d\tilde{\eta}}{\{1 + |\tilde{\eta}|^2\}^{\alpha}} \frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_t(\tilde{\xi}, |\tilde{\eta}|)|^2 \right) \\
&\lesssim \left(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\xi d\eta}{\{1 + |\eta|^2\}^{\alpha}} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi, |\eta|)|^2 \right)^{\frac{1}{2}} \cdot \\
&\qquad \left(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\xi d\eta}{\{1 + |\eta|^2\}^{\alpha}} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_{s,t}(\xi, |\eta|)|^2 \right)^{\frac{1}{2}} \cdot \\
&\qquad \left(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\tilde{\xi} d\tilde{\eta}}{\{1 + |\tilde{\eta}|^2\}^{\alpha}} \frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_t(\tilde{\xi}, |\tilde{\eta}|)|^2 \right),
\end{aligned}$$

where Cauchy-Schwarz inequality has been used to get the last estimate.

Now remark that we are here coping with the same integral as in the proof of Proposition 3.1.3 (see in particular (3.4.11)) and therefore we can reproduce the arguments in (3.4.12)-(3.4.13) to obtain the estimate we are looking for, namely: for all $\varepsilon > 0$ small enough,

$$\mathcal{A}_1 \lesssim |t - s|^{\varepsilon} .$$

Concerning the integral over D_2, a hyperspherical change of variable entails

$$\begin{aligned}
&\int_{\frac{|\eta|}{2} < |\tilde{\eta}| < \frac{3|\eta|}{2}} \frac{d\tilde{\eta}}{\{1 + ||\eta| - |\tilde{\eta}||^2\}^{2\alpha}} |\gamma_t(\tilde{\xi}, |\tilde{\eta}|)|^2 \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} \\
&= |\eta|^{-2(H_1+\ldots+H_d)+2d} \int_{\frac{1}{2} < |\tilde{\eta}| < \frac{3}{2}} \frac{d\tilde{\eta}}{\{1 + |\eta|^2(1 - |\tilde{\eta}|)^2\}^{2\alpha}} |\gamma_t(\tilde{\xi}, |\eta||\tilde{\eta}|)|^2 \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} \\
&\lesssim |\eta|^{-2(H_1+\ldots+H_d)+2d} \int_{\frac{1}{2} < r < \frac{3}{2}} \frac{dr}{\{1 + |\eta|^2(1 - r)^2\}^{2\alpha}} |\gamma_t(\tilde{\xi}, |\eta| r)|^2 .
\end{aligned}$$

As a consequence,

$$\mathcal{A}_2 := \int_{D_2} \frac{d\xi d\eta d\tilde{\xi} d\tilde{\eta}}{\{1+||\eta|-|\tilde{\eta}||^2\}^{2\alpha}} \bigg(\frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi,|\eta|)||\gamma_{s,t}(\xi,|\eta|)| \bigg) \cdot \bigg(\frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_t(\tilde{\xi},|\tilde{\eta}|)|^2 \bigg)$$

$$\lesssim \int_{\mathbb{R}^d} \frac{d\eta}{|\eta|^{2(H_1+\ldots+H_d)-2d}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} \int_{\frac12<r<\frac32} \frac{dr}{\{1+|\eta|^2(1-r)^2\}^{2\alpha}} \cdot \bigg(\int_{\mathbb{R}} d\xi \, \frac{|\gamma_t(\xi,|\eta|)||\gamma_{s,t}(\xi,|\eta|)|}{|\xi|^{2H_0-1}} \bigg) \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi} \, \frac{|\gamma_t(\tilde{\xi},|\eta| r)|^2}{|\tilde{\xi}|^{2H_0-1}} \bigg)$$

$$\lesssim \int_{\mathbb{R}^d} \frac{d\eta}{|\eta|^{2(H_1+\ldots+H_d)-2d}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} \int_{\frac12<r<\frac32} \frac{dr}{\{1+|\eta|^2(1-r)^2\}^{2\alpha}} \cdot \bigg(\int_{\mathbb{R}} d\xi \, \frac{|\gamma_t(\xi,|\eta|)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac12} \cdot \bigg(\int_{\mathbb{R}} d\xi \, \frac{|\gamma_{s,t}(\xi,|\eta|)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac12} \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi} \, \frac{|\gamma_t(\tilde{\xi},|\eta| r)|^2}{|\tilde{\xi}|^{2H_0-1}} \bigg).$$

Using again a hyperspherical change of variable (with respect to η), we obtain

$$\mathcal{A}_2 \lesssim \int_0^\infty \frac{d\rho}{\rho^{4(H_1+\ldots+H_d)-4d+1}} \int_{\frac12<r<\frac32} \frac{dr}{\{1+\rho^2(1-r)^2\}^{2\alpha}} \bigg(\int_{\mathbb{R}} d\xi \, \frac{|\gamma_t(\xi,\rho)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac12} \cdot \bigg(\int_{\mathbb{R}} d\xi \, \frac{|\gamma_{s,t}(\xi,\rho)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac12} \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi} \, \frac{|\gamma_t(\tilde{\xi},\rho r)|^2}{|\tilde{\xi}|^{2H_0-1}} \bigg),$$

and we can now use Corollary 3.4.3 to assert that

$$\mathcal{A}_2 \lesssim |t-s|^\varepsilon \bigg[\int_0^1 \frac{d\rho}{\rho^{4(H_1+\ldots+H_d)-4d+1}} + \int_1^\infty \frac{d\rho}{\rho^{4(2H_0+H_1+\ldots+H_d)-4d+1-8\varepsilon}} \int_{\frac12<r<\frac32} \frac{dr}{\{1+\rho^2(1-r)^2\}^{2\alpha}} \bigg]$$

for all $0<\varepsilon<H_0$.

Now, remark that

$$\int_1^\infty \frac{d\rho}{\rho^{4(2H_0+H_1+\ldots+H_d)-4d+1-8\varepsilon}} \int_{\frac12<r<\frac32} \frac{dr}{\{1+\rho^2(1-r)^2\}^{2\alpha}} \leq \int_1^\infty \frac{d\rho}{\rho^{4\alpha+4(2H_0+H_1+\ldots+H_d)-4d+1-8\varepsilon}} \int_{\frac12<r<\frac32} \frac{dr}{(1-r)^{4\alpha}}. \tag{3.4.21}$$

Thanks to our hypothesis on α (see (3.4.16)), we know that $4\alpha<1$ and we can choose $\varepsilon>0$ such that

$$4\alpha+4(2H_0+H_1+\ldots+H_d)-4d+1-8\varepsilon>1\,.$$

For such a choice of parameter, the two integrals in the right-hand side of (3.4.21) are clearly finite, and finally we have proved that

$$\mathcal{A}_2 \lesssim |t-s|^\varepsilon\,.$$

Going back to (3.4.20), we have thus shown that, uniformly over m,

$$\mathcal{J}^1_{m;s,t} \lesssim |t-s|^{\varepsilon}\,.$$

It is now easy to realize that the other three integrals $\{\mathcal{J}^i_{m;s,t},\ i=2,3,4\}$ (as defined in (3.4.19)) could be controlled with the very same arguments (with the very same resulting bound).

Injecting the above estimates into (3.4.18) provides us with the desired conclusion (3.4.17).

Step 2: Conclusion. Let α satisfying (3.1.11), that is $\alpha > d - \Big(2H_0 + \sum_{i=1}^d H_i\Big)$.

If $\alpha < \frac{1}{4}$, then condition (3.4.16) is satisfied, and so the moment estimate (3.4.17) holds true. Thanks to this bound, we can reproduce the arguments used in Step 2 of Section 3.4.3 to get that $(\chi^2\vee_n)_{n\geq 1}$ converges almost surely in the space $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha,p}(\mathbb{R}^d))$, for all $2 \leq p < \infty$.

If $\alpha \geq \frac{1}{4}$, remark that, due to assumption (**H2**), we can choose α' verifying $\alpha' < \alpha$ and $d - \Big(2H_0 + \sum_{i=1}^d H_i\Big) < \alpha' < \frac{1}{4}$ (that is, α' satisfies (3.4.16)). By executing the above scheme again, we deduce that the sequence $(\chi^2\vee_n)_{n\geq 1}$ converges almost surely in $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha',p}(\mathbb{R}^d))$, and therefore it converges almost surely in $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha,p}(\mathbb{R}^d))$ as well, for all $2 \leq p < \infty$.

Finally, the (a.s.) convergence in $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha,\infty}(\mathbb{R}^d))$ is the result of the Sobolev embedding $\mathcal{W}^{-2\alpha+\frac{d}{p}+\eta,p}(\mathbb{R}^d) \subset \mathcal{W}^{-2\alpha,\infty}(\mathbb{R}^d)$, for any $\eta > 0$, which ends the proof of Proposition 3.1.8. $\square$

3.4.5 Asymptotic estimate of the renormalization constant

Fix $d \geq 1$ and $(H_0, ..., H_d) \in (0,1)^{d+1}$ such that

$$2H_0 + \sum_{i=1}^{d} H_i \leq d\,.$$

The objective of this subsection is to give an equivalent of the quantity $\sigma_n(t,x) = \mathbb{E}[?_n(t,x)^2]$. Let us rewrite it under the integral form:

$$\begin{aligned}\sigma_n(t,x) \;=\; \mathbb{E}[?_n(t,x)^2] \;&=\; c^2 \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{2H_0-1}} \int_{|\eta|\leq 2^n} \prod_{i=1}^{d} \frac{d\eta_i}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi,|\eta|)|^2 \\ &=\; C \int_0^{2^n} \frac{dr}{r^{2(H_1+\ldots+H_d)-2d+1}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{2H_0-1}} |\gamma_t(\xi,r)|^2\,,\end{aligned}$$

where the previous identity is obtained after a hyperspherical change of variables. As remarked in Proposition 3.1.6, the above formula shows the surprising fact that σ_n does not depend on x.

The desired estimate (3.1.15) is now a consequence of the following technical result (applied with $\alpha := 2H_0 \in (0,2)$ and $\kappa := d - [2H_0 + \sum_{i=1}^d H_i] \geq 0$):

Proposition 3.4.4. *Fix $t > 0$. For all $\alpha \in (0,2)$ and $\kappa \geq 0$, there exists two constants c_1 and c_2 depending only on α and κ such that*

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+1}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 \underset{n\to\infty}{\sim} \begin{cases} c_1 4^{n\kappa} & \text{if } \kappa > 0 \\ c_2 n & \text{if } \kappa = 0 \end{cases}.$$

Proof. It is quite easy to verify from the expression

$$\gamma_t(\xi,r) := e^{i\xi t} \int_0^t e^{-sr^2} e^{-i\xi s} ds \tag{3.4.22}$$

that

$$|\gamma_t(\xi,r)|^2 = \frac{1 - 2\cos(\xi t) e^{-r^2 t} + e^{-2r^2 t}}{r^4 + \xi^2}\,.$$

With the change of variables $u = r^2$ and using the parity in ξ of the function $\frac{|\gamma_t(\xi,r)|^2}{|\xi|^{\alpha-1}}$, we rewrite the integral we are dealing with as

$$\begin{aligned}\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+1}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 &= \int_0^{4^n} \frac{dr}{r^{-\alpha-\kappa+1}} \int_0^{4^n} \frac{d\xi}{\xi^{\alpha-1}} \frac{1 - 2\cos(\xi t)e^{-rt} + e^{-2rt}}{r^2+\xi^2} \\ &= 4^{n\kappa} \int_0^1 \frac{dr}{r^{-\alpha-\kappa+1}} \int_0^1 \frac{d\xi}{\xi^{\alpha-1}} \frac{1 - 2\cos(4^n \xi t)e^{-4^n rt} + e^{-2.4^n rt}}{r^2+\xi^2}.\end{aligned}$$

We perform the change of variables $(r,\xi) \mapsto T(r,\xi)$ defined by

$$T : (r,\xi) \mapsto T(r,\xi) = \Big(\frac{r}{\xi}, \xi\Big).$$

It is readily checked that T is a one-to-one map of $(0,1)^2$ on

$$\{(x,y)\in\mathbb{R}^2, x>0, 0<y<\min\Big(1,\frac{1}{x}\Big)\}.$$

Moreover, its reverse is explicitly given by $T^{-1}:(x,y)\mapsto(xy,y)$ whose absolute value of the Jacobian equals y.

$$\begin{aligned}&\int_0^{2^n}\frac{dr}{r^{-2\alpha-2\kappa+1}}\int_{|\xi|\le 4^n}\frac{d\xi}{|\xi|^{\alpha-1}}|\gamma_t(\xi,r)|^2=\\
&4^{n\kappa}\int_0^1\frac{x^{\alpha+\kappa-1}}{1+x^2}dx\int_0^1\frac{dy}{y^{1-\kappa}}\Big(1-2\cos(4^nyt)e^{-4^nxyt}+e^{-2.4^nxyt}\Big)\\
&+4^{n\kappa}\int_1^{+\infty}\frac{x^{\alpha+\kappa-1}}{1+x^2}dx\int_0^{\frac{1}{x}}\frac{dy}{y^{1-\kappa}}\Big(1-2\cos(4^nyt)e^{-4^nxyt}+e^{-2.4^nxyt}\Big).\end{aligned}$$

First case: $\kappa>0$. By Lebesgue's dominated convergence theorem, we observe that

$$\begin{aligned}&\int_0^1\frac{x^{\alpha+\kappa-1}}{1+x^2}dx\int_0^1\frac{dy}{y^{1-\kappa}}\Big(-2\cos(4^nyt)e^{-4^nxyt}+e^{-2.4^nxyt}\Big)\\
&+\int_1^{+\infty}\frac{x^{\alpha+\kappa-1}}{1+x^2}dx\int_0^{\frac{1}{x}}\frac{dy}{y^{1-\kappa}}\Big(-2\cos(4^nyt)e^{-4^nxyt}+e^{-2.4^nxyt}\Big)\underset{n\to\infty}{\to}0,\end{aligned}$$

leading to

$$\int_0^{2^n}\frac{dr}{r^{-2\alpha-2\kappa+1}}\int_{|\xi|\le 4^n}\frac{d\xi}{|\xi|^{\alpha-1}}|\gamma_t(\xi,r)|^2\underset{n\to\infty}{\sim}c_1 4^{n\kappa},$$

where

$$\begin{aligned}c_1&=\int_0^1\frac{x^{\alpha+\kappa-1}}{1+x^2}dx\int_0^1\frac{dy}{y^{1-\kappa}}+\int_1^{+\infty}\frac{x^{\alpha+\kappa-1}}{1+x^2}dx\int_0^{\frac{1}{x}}\frac{dy}{y^{1-\kappa}}\\
&=\frac{1}{\kappa}\left(\int_0^1\frac{x^{\alpha+\kappa-1}}{1+x^2}dx+\int_1^{+\infty}\frac{x^{\alpha-1}}{1+x^2}dx\right).\end{aligned}$$

A more visual expression of c_1 can be found in the appendix.
Second case: $\kappa=0$. Let us recall that

$$\begin{aligned}\int_0^{2^n}\frac{dr}{r^{-2\alpha+1}}\int_{|\xi|\le 4^n}\frac{d\xi}{|\xi|^{\alpha-1}}|\gamma_t(\xi,r)|^2&=\int_0^1\frac{x^{\alpha-1}}{1+x^2}dx\int_0^1\frac{dy}{y}\Big(1-2\cos(4^nyt)e^{-4^nxyt}+e^{-2.4^nxyt}\Big)\\
&+\int_1^{+\infty}\frac{x^{\alpha-1}}{1+x^2}dx\int_0^{\frac{1}{x}}\frac{dy}{y}\Big(1-2\cos(4^nyt)e^{-4^nxyt}+e^{-2.4^nxyt}\Big).\end{aligned}$$

Let us introduce the function f defined for all $T>0$ by

$$\begin{aligned}f(T)=&\int_0^1\frac{x^{\alpha-1}}{1+x^2}dx\int_0^1\frac{dy}{y}\Big(1-2\cos(Ty)e^{-Txy}+e^{-2Txy}\Big)\\
&+\int_1^{+\infty}\frac{x^{\alpha-1}}{1+x^2}dx\int_0^{\frac{1}{x}}\frac{dy}{y}\Big(1-2\cos(Ty)e^{-Txy}+e^{-2Txy}\Big).\end{aligned}$$

Then, one has by derivation

$$\begin{aligned} f'(T) &= \int_0^1 \frac{x^{\alpha-1}}{1+x^2}dx \int_0^1 dy\Big(2\sin(Ty)e^{-Txy} + 2x\cos(Ty)e^{-Txy} - 2xe^{-2Txy}\Big) \\ &+ \int_1^{+\infty} \frac{x^{\alpha-1}}{1+x^2}dx \int_0^{\frac{1}{x}} dy\Big(2\sin(Ty)e^{-Txy} + 2x\cos(Ty)e^{-Txy} - 2xe^{-2Txy}\Big). \end{aligned}$$

To deal with this derivative with ease, we will resort to the following technical lemma:

Lemma 3.4.5. *Let a, x and T three positive numbers. It holds that:*

$$\begin{aligned} \int_0^a \sin(Ty)e^{-Txy}dy &= -\frac{xe^{-Tax}\sin(Ta)}{(1+x^2)T} + \frac{1-e^{-Tax}\cos(Ta)}{(1+x^2)T}; \\ \int_0^a \cos(Ty)e^{-Txy}dy &= \frac{e^{-Tax}\sin(Ta)}{(1+x^2)T} + \frac{x(1-e^{-Tax}\cos(Ta))}{(1+x^2)T}; \\ \int_0^a e^{-2Txy}dy &= \frac{1-e^{-2Txa}}{2Tx}. \end{aligned}$$

Lemma 3.4.5 with $a = 1$ and $a = \frac{1}{x}$ readily entails:

$$\begin{aligned} f'(T) = \frac{1}{T}\Bigg(&\int_0^{+\infty} \frac{x^{\alpha-1}}{1+x^2}dx + \int_0^1 \frac{x^{\alpha-1}}{1+x^2}dx\Big(e^{-2Tx} - 2\cos(T)e^{-Tx}\Big) \\ &+ \int_1^{+\infty} \frac{x^{\alpha-1}}{1+x^2}dx\Big(e^{-2T} - 2\cos\left(\frac{T}{x}\right)e^{-T}\Big)\Bigg). \end{aligned}$$

By Lebesgue's dominated convergence theorem, we deduce since $\alpha \in (0,2)$ that

$$\int_0^1 \frac{x^{\alpha-1}}{1+x^2}dx\Big(e^{-2Tx} - 2\cos(T)e^{-Tx}\Big) + \int_1^{+\infty} \frac{x^{\alpha-1}}{1+x^2}dx\Big(e^{-2T} - 2\cos\left(\frac{T}{x}\right)e^{-T}\Big) \underset{T\to\infty}{\to} 0$$

and consequently

$$f'(T) \underset{T\to\infty}{\sim} \frac{c_2}{T}$$

where $c_2 = \displaystyle\int_0^{+\infty} \frac{x^{\alpha-1}}{1+x^2}dx$. For the sake of beauty, let us compute the value of the constant c_2. We recall the classical result, coming from complex analysis, stating that for all $\beta \in]0,1[$,

$$\int_0^{+\infty} \frac{dt}{t^\beta(1+t)} = \frac{\pi}{\sin \pi\beta}.$$

It yields:

$$f'(T) \underset{T\to\infty}{\sim} \frac{\pi}{2\sin(\frac{\alpha\pi}{2})T}.$$

We can finally use a standard comparison argument (see Lemma 3.4.6 below) to assert that

$$f(T) \underset{T\to\infty}{\sim} \frac{\pi}{2\sin(\frac{\alpha\pi}{2})}\ln(T).$$

We obtain the equivalent we are looking for, namely

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha+1}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 \underset{n\to\infty}{\sim} \frac{\pi \ln(2)}{\sin(\frac{\alpha\pi}{2})} n.$$

$\square$

Lemma 3.4.6. *Fix $a \in \mathbb{R}$ and let $g : [a,+\infty[\to \mathbb{R}$, $h : [a,+\infty[\to (0,\infty)$, be two continuous functions. If $g(t) \underset{t\to\infty}{\sim} h(t)$ and $\int_a^{+\infty} h(t)dt = \infty$, then*

$$\int_a^T g(t)dt \underset{T\to\infty}{\sim} \int_a^T h(t)dt \,.$$

3.4.6 Details about the definition of the linear solution

As in [8], the statement of Proposition 3.1.3 provides us with a *local* definition of the process Ψ, that is, up to multiplication by $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. To put it differently, what is actually given by the proposition is the set of the limit elements $\{\chi\Psi,\ \chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}$. Let us briefly recall how those elements can be gathered into a single process Ψ.

Fix $p \geq 2$ and α verifying (3.1.11). We denote by $\mathcal{P}$ the set of sequences $\sigma = (\sigma_k)_{k\geq 1}$ such that for every $k \geq 1$, $\sigma_k : \mathbb{R}^d \to \mathbb{R}$ is a smooth function verifying

$$\sigma_k(x) = \begin{cases} 1 & \text{if } \|x\| \leq k\,, \\ 0 & \text{if } \|x\| \geq k+1\,. \end{cases}$$

Let us fix such a sequence σ, and for all $k \geq 1$, we call $\Psi^{(\sigma_k)}$ the limit of the sequence $(\sigma_k \Psi_n)_{n\geq 1}$ in the space $\mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$, as given by Proposition 3.1.3. As $\Psi^{(\sigma_k)}$ is defined on a probability space $\Omega^{(\sigma_k)}$ of full measure 1, $\Omega^{(\sigma)} := \cap_{k\geq 1}\Omega^{(\sigma_k)}$ is still of measure 1.

For each time $t \in [0,T]$, we are now able to define the random distribution

$$\Psi^{(\sigma)}(t) : \Omega^{(\sigma)} \to \mathcal{D}'(\mathbb{R}^d)$$

as follows: for all smooth compactly-supported function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\operatorname{supp}(\varphi) \subset B(0,k)$ (for some $k \geq 1$),

$$\langle \Psi^{(\sigma)}(t), \varphi\rangle := \langle \Psi^{(\sigma_k)}(t), \varphi\rangle\,.$$

Proposition 3.4.7. 1. *The above distribution $\Psi^{(\sigma)}$ is well defined, i.e. for every $1 \leq k \leq \ell$ and for all smooth compactly-supported function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $\operatorname{supp}(\varphi) \subset B(0,k) \subset B(0,\ell)$, one has*

$$\langle \Psi^{(\sigma_k)}(t), \varphi\rangle = \langle \Psi^{(\sigma_\ell)}(t), \varphi\rangle \quad \textit{on} \quad \Omega^{(\sigma)}\,.$$

2. *For any smooth compactly-supported function $\chi : \mathbb{R}^d \to \mathbb{R}$, one has, on $\Omega^{(\sigma)}$,*

$$\chi \cdot \Psi_n \underset{n\to\infty}{\to} \chi \cdot \Psi^{(\sigma)} \quad \textit{in} \quad \mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))\,.$$

3. If $\sigma, \gamma \in \mathcal{P}$, it holds that

$$\RSI^{(\sigma)} = \RSI^{(\gamma)} \quad on \quad \Omega^{(\sigma)} \cap \Omega^{(\gamma)} .$$

Thanks to the latter identification property, we can set $\RSI := \RSI^{(\sigma)}$, as soon as $\sigma \in \mathcal{P}$ is a fixed element.

Remark 3.4.8. The latter procedure can of course be used to give a rigorous meaning to the second-order process $\chi^2 \RSV$ as a distribution-valued function $\RSV$.

3.5 Rougher stochastic constructions when the space dimension equals 2

The aim of this section is to establish the proofs of Proposition 3.1.14 and Proposition 3.1.13, that is to construct the second order stochastic process $\RSV$ in the roughest case, namely when $\frac{3}{2} < 2H_0 + H_1 + H_2 \leq \frac{7}{4}$, and to show that this construction is in some sense optimal insofar as

$$\mathbb{E}\Big[\|\chi \cdot \RSV^n(t,.)\|^2_{H^{-2\alpha}}\Big] \underset{n\to+\infty}{\longrightarrow} +\infty,$$

for all compactly-supported function χ and $t > 0$ when $2H_0 + H_1 + H_2 \leq \frac{3}{2}$.

3.5.1 Additional notations

To begin with, let us introduce some notations that will be frequently used in the proofs of Proposition 3.1.14 and Proposition 3.1.13. For all $\tau \in \Big\{\RSI, \RSV\Big\}$, $0 \leq n \leq m$ and $0 \leq s, t \leq 1$, let us set $\tau^{n,m} := \tau^m - \tau^n$, and then, for $f \in \{\tau^n, \tau^m, \tau^{n,m}\}$, $f_{s,t} := f_t - f_s$.

Then, we set for all $H = (H_0, H_1, H_2) \in (0,1)^3$, $\eta \in \mathbb{R}^2$,

$$K^H(\eta) := \frac{|\eta_1|^{1-2H_1}|\eta_2|^{1-2H_2}}{1+|\eta|^{4H_0}} . \tag{3.5.1}$$

For all $a = (a_1, a_2)$, resp. $b = (b_1, b_2)$, with $a_i \in \{n, m, \{n,m\}\}$, resp. $b_i \in \{s, t, \{s,t\}\}$, we write

$$L^{H,a}_b(\eta) := \frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}} \int_{(\xi,\eta)\in D^{a_1}\cap D^{a_2}} d\xi \, \frac{\gamma_{b_1}(\xi,|\eta|)\overline{\gamma_{b_2}(\xi,|\eta|)}}{|\xi|^{2H_0-1}}$$

where

$$D_n := B^1_{2n} \times B^2_n \quad \text{with} \quad B^k_\ell := \Big\{\lambda \in \mathbb{R}^k : |\lambda| \leq 2^\ell\Big\} \quad \text{and} \quad D^{n,m} := D^m \backslash D^n ,$$

in such a way that for all $y, \tilde{y} \in \mathbb{R}^2$,

$$\mathbb{E}\Big[\RSI^{a_1}_{b_1}(y)\overline{\RSI^{a_2}_{b_2}(\tilde{y})}\Big] = c_H^2 \int_{\mathbb{R}^2} d\eta \, e^{i\langle\eta,y\rangle} e^{-i\langle\eta,\tilde{y}\rangle} L^{H,a}_b(\eta) . \tag{3.5.2}$$

In the following, we will resort to several technical lemmas whose statements and proofs can be found in the Appendix. In particular, Lemma 3.7.2 is of major interest since it describes in some way the role of the cut-off function χ which allows a gain of integrability.

3.5.2 Proof of Proposition 3.1.14

The strategy is exactly the same as the one developed in the proof of Proposition 3.1.3 and results from the combination of Kolmogorov criterion and Garsia-Rodemich-Rumsey lemma. Let us write

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\cdot \curlyvee^{n,m}_{s,t}\Big)\Big)(x)\Big|^2\Big] \\
&= \frac{1}{(2\pi)^4}\iint_{(\mathbb{R}^2)^2}\frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\alpha}\{1+|\tilde{\lambda}|^2\}^{\alpha}}e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\iint_{(\mathbb{R}^2)^2}d\xi d\tilde{\xi}\,\widehat{\chi^2}(\lambda-\xi)\overline{\widehat{\chi^2}(\tilde{\lambda}-\tilde{\xi})}\mathcal{Q}_{n,m;s,t}(\xi,\tilde{\xi}),
\end{aligned} \tag{3.5.3}$$

with

$$\begin{aligned}
\mathcal{Q}_{n,m;s,t}(\xi,\tilde{\xi}) &:= \mathbb{E}\Big[\mathcal{F}\Big(\curlyvee^{n,m}_{s,t}\Big)(\xi)\overline{\mathcal{F}\Big(\curlyvee^{n,m}_{s,t}\Big)(\tilde{\xi})}\Big] \\
&= \iint_{(\mathbb{R}^2)^2}dyd\tilde{y}\,e^{-i\langle\xi,y\rangle}e^{i\langle\tilde{\xi},\tilde{y}\rangle}\mathbb{E}\Big[\curlyvee^{n,m}_{s,t}(y)\overline{\curlyvee^{n,m}_{s,t}(\tilde{y})}\Big].
\end{aligned}$$

Resorting to Wick formula, we derive

$$\begin{aligned}
\mathbb{E}\Big[\curlyvee^{n,m}_{s,t}(y)\overline{\curlyvee^{n,m}_{s,t}(\tilde{y})}\Big] &= \sum_{(a,b)\in\mathcal{A}}c_{a,b}\,\mathbb{E}\Big[\mathsf{I}^{a_1}_{b_1}(y)\overline{\mathsf{I}^{a_2}_{b_2}(\tilde{y})}\Big]\mathbb{E}\Big[\mathsf{I}^{a_3}_{b_3}(y)\overline{\mathsf{I}^{a_4}_{b_4}(\tilde{y})}\Big] \\
&= \sum_{(a,b)\in\mathcal{A}}\tilde{c_{a,b}}\iint_{(\mathbb{R}^2)^2}d\eta d\tilde{\eta}\,e^{i\langle\eta,y\rangle}e^{-i\langle\eta,\tilde{y}\rangle}e^{i\langle\tilde{\eta},y\rangle}e^{-i\langle\tilde{\eta},\tilde{y}\rangle}L^{H,(a_1,a_2)}_{b_1,b_2}(\eta)L^{H,(a_3,a_4)}_{b_3,b_4}(\tilde{\eta}),
\end{aligned}$$

for some index set $\mathcal{A}$ such that $a_i\in\{n,m,\{n,m\}\}$, $b_i\in\{s,t,\{s,t\}\}$, and one has both $\{a_1,\ldots,a_4\}\cap\{\{n,m\}\}\neq\emptyset$ and $\{b_1,\ldots,b_4\}\cap\{\{s,t\}\}\neq\emptyset$. It leads us to

$$\mathcal{Q}_{n,m;s,t}(\xi,\tilde{\xi}) = \sum_{(a,b)\in\mathcal{A}}\tilde{c_{a,b}}\iint_{(\mathbb{R}^2)^2}d\eta d\tilde{\eta}\,L^{H,(a_1,a_2)}_{b_1,b_2}(\eta)L^{H,(a_3,a_4)}_{b_3,b_4}(\tilde{\eta})\delta_{\{\xi=\eta+\tilde{\eta}\}}\delta_{\{\tilde{\xi}=\eta+\tilde{\eta}\}}.$$

Consequently,

$$\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\cdot \curlyvee^{n,m}_{s,t}\Big)\Big)(x)\Big|^2\Big] = \sum_{(a,b)\in\mathcal{A}}\phi_{a,b},$$

with

$$\begin{aligned}
\phi_{a,b} = \tilde{c_{a,b}}&\iint_{(\mathbb{R}^2)^2}\frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\alpha}\{1+|\tilde{\lambda}|^2\}^{\alpha}}e^{i\langle x,\lambda-\tilde{\lambda}\rangle} \\
&\iint_{(\mathbb{R}^2)^2}d\eta d\tilde{\eta}L^{H,(a_1,a_2)}_{b_1,b_2}(\eta)L^{H,(a_3,a_4)}_{b_3,b_4}(\tilde{\eta})\widehat{\chi^2}(\lambda-(\eta+\tilde{\eta}))\overline{\widehat{\chi^2}(\tilde{\lambda}-(\eta+\tilde{\eta}))}.
\end{aligned}\tag{3.5.4}$$

With the help of Lemma 3.7.2 and the hypercontractivity of Gaussian chaoses, we obtain

$$\int_{\mathbb{R}^2}dx\,\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\cdot \curlyvee^{n,m}_{s,t}\Big)\Big)(x)\Big|^{2p}\Big] \lesssim \Big(\sum_{(a,b)\in\mathcal{A}}\psi_{a,b}\Big)^p,$$

where

$$\psi_{a,b} = \iint_{(\mathbb{R}^2)^2} d\eta d\tilde{\eta}\, \{1+|\eta+\tilde{\eta}|^2\}^{-2\alpha} |L^{H,(a_1,a_2)}_{b_1,b_2}(\eta)||L^{H,(a_3,a_4)}_{b_3,b_4}(\tilde{\eta})|. \tag{3.5.5}$$

Now, for $0<\varepsilon<H_0$, Lemma 3.7.1 provides us with the bound

$$\begin{aligned}\psi_{a,b} \lesssim 2^{-2n\varepsilon}|t-s|^{\varepsilon} &\iint_{(\mathbb{R}^2)^2} d\eta d\tilde{\eta}\, \{1+|\eta+\tilde{\eta}|^2\}^{-2\alpha} \\ &\Big\{K^{H_{\varepsilon,0}}(\eta)+K^{H_{\varepsilon,0,1}}(\eta)+K^{H_{\varepsilon,0,2}}(\eta)\Big\}\Big\{K^{H_{\varepsilon,0}}(\tilde{\eta})+K^{H_{\varepsilon,0,1}}(\tilde{\eta})+K^{H_{\varepsilon,0,2}}(\tilde{\eta})\Big\}.\end{aligned} \tag{3.5.6}$$

According to Lemma 3.7.3, the latter quantity is finite as soon as ε is small enough and we have finally obtained:

$$\int_{\mathbb{R}^2} dx\, \mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\cdot \curlyvee^{n,m}_{s,t}\Big)\Big)(x)\Big|^{2p}\Big] \lesssim 2^{-2n\varepsilon p}|t-s|^{\varepsilon p}.$$

We can mimic the arguments at the end of the proof of Proposition 3.1.3 to get the result.

3.5.3 Proof of Proposition 3.1.13

Suppose that $d\geq 1$ and that $(H_0,H_1,...,H_d)\in(0,1)^{d+1}$ verifies the condition $2H_0+H_1+\cdots+H_d\leq \frac{3}{4}d$. As usual, we will use the notation $A\gtrsim B$ whenever one can find a constant $c>0$ such that $A\geq cB$. Let us also introduce the additional notation

$$\Gamma^{H_0,n}_t(r) := \int_{-4^n}^{4^n} d\xi\, \frac{|\gamma_t(\xi,r)|^2}{|\xi|^{2H_0-1}}\ . \tag{3.5.7}$$

Fix $t>0$. Using (3.5.2) and then Wick formula, we get that

$$\begin{aligned}&\mathbb{E}\Big[\|\chi\cdot\curlyvee^n(t,.)\|^2_{H^{-2\alpha}}\Big] = \\ &c\int_{|\eta|\leq 2^n} d\eta \int_{|\tilde{\eta}|\leq 2^n} d\tilde{\eta} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}}\prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}}\Gamma^{H_0,n}_t(|\eta|)\Gamma^{H_0,n}_t(|\tilde{\eta}|)\int_{\mathbb{R}^d}\frac{d\xi}{\{1+|\xi|^2\}^{2\alpha}}\Big|\widehat{\chi}(\xi-(\eta-\tilde{\eta}))\Big|^2.\end{aligned}$$

With the change of variables $\tilde{\xi}:=\xi-(\eta-\tilde{\eta})$, it holds that

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{d\xi}{\{1+|\xi|^2\}^{2\alpha}}\Big|\widehat{\chi}(\xi-(\eta-\tilde{\eta}))\Big|^2 \\ &=\int_{\mathbb{R}^d}\frac{d\xi}{\{1+|\xi+(\eta-\tilde{\eta})|^2\}^{2\alpha}}\Big|\widehat{\chi}(\xi)\Big|^2 \gtrsim \frac{1}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}}\int_{\mathbb{R}^d}\frac{d\xi}{\{1+|\xi|^2\}^{2\alpha}}\Big|\widehat{\chi}(\xi)\Big|^2.\end{aligned}$$

As χ is a non-zero compactly-supported function, the support of $\widehat{\chi}$ contains a ball of radius $R>0$ that guarantees $\int_{\mathbb{R}^d}\frac{d\xi}{\{1+|\xi|^2\}^{2\alpha}}\Big|\widehat{\chi}(\xi)\Big|^2>0$. Performing the changes of variables

$r_1 = \frac{\eta_1 - \tilde{\eta}_1}{\eta_1}, \cdots, r_d = \frac{\eta_d - \tilde{\eta}_d}{\eta_d}$ (second inequality) followed by $\tilde{r_1} = \eta_1 r_1, \cdots, \tilde{r_d} = \eta_d r_d$ (third inequality), we write

$$\begin{aligned}
&\mathbb{E}\Big[\|\chi \cdot \curlyvee^n(t,.)\|^2_{H^{-2\alpha}}\Big] \\
&\gtrsim \int_0^{\frac{2^n}{\sqrt{d}}} d\eta_1 \int_{\frac{1}{2}\eta_1}^{\eta_1} d\tilde{\eta}_1 \cdots \int_0^{\frac{2^n}{\sqrt{d}}} d\eta_d \int_{\frac{1}{2}\eta_d}^{\eta_d} d\tilde{\eta}_d \frac{\Gamma_t^{H_0,n}(|\eta|)\Gamma_t^{H_0,n}(|\tilde{\eta}|)}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} \\
&\gtrsim \int_0^{\frac{2^n}{\sqrt{d}}} \cdots \int_0^{\frac{2^n}{\sqrt{d}}} d\eta_1 \cdots d\eta_d \prod_{i=1}^d \frac{1}{|\eta_i|^{4H_i-3}} \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \frac{dr_1 \cdots dr_d}{\{1+\eta_1^2 r_1^2 + \cdots + \eta_d^2 r_d^2\}^{2\alpha}} \\
&\qquad \Gamma_t^{H_0,n}(|\eta|)\Gamma_t^{H_0,n}\Big(\sqrt{\eta_1^2(1-r_1)^2 + \cdots + \eta_d^2(1-r_d)^2}\Big) \\
&\gtrsim \int_0^{\frac{2^n}{\sqrt{d}}} \cdots \int_0^{\frac{2^n}{\sqrt{d}}} d\eta_1 \cdots d\eta_d \prod_{i=1}^d \frac{1}{|\eta_i|^{4H_i-2}} \int_0^{\frac{1}{2}\eta_1} \cdots \int_0^{\frac{1}{2}\eta_d} \frac{dr_1 \cdots dr_d}{\{1+r_1^2 + \cdots + r_d^2\}^{2\alpha}} \\
&\qquad \Gamma_t^{H_0,n}(|\eta|)\Gamma_t^{H_0,n}\left(\sqrt{\eta_1^2\Big(1-\frac{r_1}{\eta_1}\Big)^2 + \cdots + \eta_d^2\Big(1-\frac{r_d}{\eta_d}\Big)^2}\right).
\end{aligned}$$

The hyperspherical change of variables below

$$\begin{cases}
|\eta| = r \\
\eta_1 = r\cos(\theta_1) \\
\eta_2 = r\sin(\theta_1)\cos(\theta_2) \\
. \\
. \\
. \\
\eta_{d-1} = r\sin(\theta_1)\cdots\sin(\theta_{d-2})\cos(\theta_{d-1}) \\
\eta_d = r\sin(\theta_1)\cdots\sin(\theta_{d-2})\sin(\theta_{d-1})
\end{cases}$$

whose absolute value of the Jacobian equals $r^{d-1}\prod_{i=1}^d |\sin(\theta_i)|^{d-1-i}$ entails that

$$\begin{aligned}
&\mathbb{E}\Big[\|\chi \cdot \curlyvee^n(t,.)\|^2_{H^{-2\alpha}}\Big] \\
&\gtrsim \int_{[\frac{\pi}{8},\frac{\pi}{4}]^{d-1}} d\theta_1 \cdots d\theta_{d-1} \int_2^{2^n} \frac{dr}{r^{4(H_1+\cdots+H_d)-3d+1}} \int_0^{\frac{1}{2}r\cos\theta_1} \cdots \int_0^{\frac{1}{2}r\sin(\theta_1)\cdots\sin(\theta_{d-1})} \frac{dr_1\cdots dr_d}{\{1+r_1^2+\cdots+r_d^2\}^{2\alpha}} \\
&\Gamma_t^{H_0,n}(r)\Gamma_t^{H_0,n}\left(\sqrt{r^2\cos^2\theta_1\Big(1-\frac{r_1}{r\cos\theta_1}\Big)^2 + \cdots + r^2\sin^2\theta_1\cdots\sin^2\theta_{d-1}\Big(1-\frac{r_d}{r\sin\theta_1\cdots\sin\theta_{d-1}}\Big)^2}\right).
\end{aligned}$$

A quick view on the integration domain reveals that

$$\begin{aligned}
&r \geq \tilde{r}_\theta := \sqrt{r^2\cos^2\theta_1\Big(1-\frac{r_1}{r\cos\theta_1}\Big)^2 + \cdots + r^2\sin^2\theta_1\cdots\sin^2\theta_{d-1}\Big(1-\frac{r_d}{r\sin\theta_1\cdots\sin\theta_{d-1}}\Big)^2} \\
&\geq \frac{1}{2}r \geq 1 \;.
\end{aligned}$$

Resorting to Lemma 3.5.1, the (forthcoming) lower bound (3.5.8) leads to

$$
\begin{aligned}
\mathbb{E}\Big[\|\chi\cdot \curlyvee^{n}(t,.)\|^2_{H^{-2\alpha}}\Big] \ \gtrsim\ & \int_{[\frac{\pi}{8},\frac{\pi}{4}]^{d-1}} d\theta_1\cdots d\theta_{d-1}\int_2^{2^n}\frac{dr}{r^{4(H_1+\cdots+H_d)-3d+1}} \\
& \int_0^{\frac{1}{2}r\cos\theta_1}\cdots\int_0^{\frac{1}{2}r\sin(\theta_1)\cdots\sin(\theta_{d-1})}\frac{dr_1\cdots dr_d}{\{1+r_1^2+\cdots+r_d^2\}^{2\alpha}}\frac{1}{r^{8H_0}}(1-2e^{-t}+e^{-2t})^2 \\
\gtrsim\ & (1-2e^{-t}+e^{-2t})^2\left(\int_0^{\cos\frac{\pi}{8}}\cdots\int_0^{\sin(\frac{\pi}{8})^{d-1}}\frac{dr_1\cdots dr_d}{\{1+r_1^2+\cdots+r_d^2\}^{2\alpha}}\right)\left(\int_2^{2^n}\frac{dr}{r^{4(2H_0+H_1+\cdots+H_d)-3d+1}}\right).
\end{aligned}
$$

As $2H_0+H_1+\cdots+H_d\leq\frac{3}{4}d$, $4(2H_0+H_1+\cdots+H_d)-3d+1\leq 1$, and consequently

$$
\int_2^{2^n}\frac{dr}{r^{4(2H_0+H_1+\cdots+H_d)-3d+1}}\ \underset{n\to+\infty}{\longrightarrow}\ +\infty\ .
$$

We then get the desired conclusion

$$
\mathbb{E}\Big[\|\chi\cdot \curlyvee^{n}(t,.)\|^2_{H^{-2\alpha}}\Big]\ \underset{n\to+\infty}{\longrightarrow}\ +\infty.
$$

Lemma 3.5.1. *For all $H_0\in(0,1)$, $n\geq 1$, $t>0$ and $r\in[1,2^n]$,*

$$
\Gamma_t^{H_0,n}(r)\gtrsim\frac{1}{r^{4H_0}}(1-2e^{-t}+e^{-2t})>0. \tag{3.5.8}
$$

Proof. Remember that

$$
|\gamma_t(\xi,r)|^2=\frac{1-2\cos(\xi t)e^{-r^2t}+e^{-2r^2t}}{r^4+\xi^2}\ .
$$

It is clear that $\Gamma_t^{H_0,n}(r)=\int_{-4^n}^{4^n}\frac{|\gamma_t(\xi,r)|^2}{|\xi|^{2H_0-1}}d\xi\geq\int_{-r^2}^{r^2}\frac{|\gamma_t(\xi,r)|^2}{|\xi|^{2H_0-1}}d\xi$. Now, by parity,

$$
\begin{aligned}
\int_{-r^2}^{r^2}\frac{|\gamma_t(\xi,r)|^2}{|\xi|^{2H_0-1}}d\xi &= 2\int_0^{r^2}\frac{|\gamma_t(\xi,r)|^2}{|\xi|^{2H_0-1}}d\xi \\
&= \frac{2}{r^{4H_0}}\int_0^1\frac{1-2\cos(r^2\xi t)e^{-r^2t}+e^{-2r^2t}}{(1+\xi^2)\xi^{2H_0-1}}d\xi \\
&\geq \frac{2}{r^{4H_0}}\left(\int_0^1\frac{d\xi}{(1+\xi^2)\xi^{2H_0-1}}\right)(1-2e^{-r^2t}+e^{-2r^2t})>0.
\end{aligned}
$$

Indeed, if we set for all $t\geq 0, r\geq 1$,

$$
h(t,r):=1-2e^{-r^2t}+e^{-2r^2t},
$$

it is quite easy to check that, when $r\geq 1$, $h(.,r)$ is strictly increasing on $[0,+\infty)$ and, since $h(0,r)=0$, for every $t>0$, $h(t,r)=1-2e^{-r^2t}+e^{-2r^2t}>0$. Moreover, we can verify that if $t\geq 0$, $h(t,.)$ is increasing on $[1,+\infty)$ that provides the conclusion.

□

3.6 Proof of the main results

3.6.1 Proof of Theorem 3.1.5

Suppose that $d \geq 1$. Let $p \geq 2$ and β be such that $0 < \beta < 2H_0 + \sum_{i=1}^d H_i - d$ and $\frac{d}{2p} < 1 + \frac{\beta}{2}$.
Recall that for every $T \geq 0$,

$$X^{\beta,p}(T) := \mathcal{C}([0,T]; \mathcal{W}^{\beta,p}(\mathbb{R}^d)) \,. \tag{3.6.1}$$

$i)$ The statement of Proposition 3.1.3 with $\alpha := -\beta$ and $\chi := \rho$ ensures the existence of a measurable set $\tilde{\Omega}$ of measure one on which $\rho\mathbf{I}(\omega) \in X^{\beta,p}(T)$. Now, the well-posedness result of Theorem 3.1.5 comes from the application of Theorem 3.2.4 (in an almost sure way) to $\mathbf{\Psi} := \rho\mathbf{I}$.

$ii)$ By reproducing the arguments that can be found in the proof of [6, Theorem 1.7], we see that the convergence property is a consequence from the continuity of $\Gamma_{T,\mathbf{\Psi}}$ with respect to $\mathbf{\Psi}$ (along (3.2.5)) and the almost sure convergence of $\chi\mathbf{I}_n$ to $\chi\mathbf{I}$.

3.6.2 Proof of Theorem 3.1.10

Suppose that $d \geq 1$ and $p \geq 2$ verifies that $\frac{d}{2p} \leq \frac{3}{4}$. Assume that $2H_0 + \sum_{i=1}^d H_i \leq d$. Fix $\alpha > 0$ such that

$$d - \left(2H_0 + \sum_{i=1}^d H_i\right) < \alpha < \frac{1}{4}.$$

As $\frac{d}{2p} \leq \frac{3}{4}$, observe that $\alpha < \frac{1}{4} \leq 1 - \frac{d}{2p}$ and we can pick $\alpha < \beta < \min\left(2 - \alpha - \frac{d}{p}, 2 - 2\alpha\right)$.
Recall that

$$\mathcal{R}_{\alpha,p} := L^\infty\Big([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)\Big) \times L^\infty\Big([0,T]; \mathcal{W}^{-2\alpha,p}(\mathbb{R}^d)\Big) \,. \tag{3.6.2}$$

$i)$ Proposition 3.1.3 and Proposition 3.1.8 applied with $\alpha > 0$ guarantees the existence of a measurable set $\tilde{\Omega}$ of measure one on which $(\rho\mathbf{I}(\omega), \rho^2\mathbf{V}(\omega)) \in \mathcal{R}_{\alpha,p}$. Now, the statement of Theorem 3.1.10 results from the application of Theorem 3.2.4 (in an almost sure way) to $(\mathbf{\Psi}, \mathbf{\Psi^2}) := (\rho\mathbf{I}, \rho^2\mathbf{V})$.

$ii)$ Again, we remark that the convergence property is a consequence from the continuity of $\Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}$ with respect to $(\mathbf{\Psi}, \mathbf{\Psi^2})$ and the almost sure convergence of $(\chi\mathbf{I}_n, \chi^2\mathbf{V}_n)$ to $(\chi\mathbf{I}, \chi^2\mathbf{V})$.

3.6.3 Proof of Theorem 3.1.15

Suppose that $d = 2$ and $p \geq 2$. Let $(H_0, H_1, H_2) \in (0,1)^3$ be such that

$$0 < H_1 < \frac{3}{4} \quad , \quad 0 < H_2 < \frac{3}{4} \quad , \quad \frac{3}{2} < 2H_0 + H_1 + H_2 \leq \frac{7}{4} \,. \tag{3.6.3}$$

Fix $\alpha > 0$ such that

$$2 - (2H_0 + H_1 + H_2) < \alpha < \frac{1}{2}.$$

As $p \geq 2$, observe that $\alpha < \frac{1}{2} \leq 1 - \frac{2}{2p}$ and we can pick $\alpha < \beta < \min\left(2 - \alpha - \frac{d}{p}, 2 - 2\alpha\right)$.

i) Proposition 3.1.3 and Proposition 3.1.8 applied with $\alpha > 0$ guarantees the existence of a measurable set $\tilde{\Omega}$ of measure one on which $(\rho(\omega), \rho^2(\omega)) \in \mathcal{R}_\alpha$. Again, Theorem 3.1.10 results from the application of Theorem 3.2.4 (in an almost sure way) to $(\mathbf{\Psi}, \mathbf{\Psi^2}) := (\rho, \rho^2)$.

ii) As before, the convergence property is a consequence from the continuity of $\Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}$ with respect to $(\mathbf{\Psi}, \mathbf{\Psi^2})$ and the almost sure convergence of (χ_n, χ^2_n) to (χ, χ^2).

3.7 Appendix

3.7.1 Technical lemmas

In this subsection, we state the three technical lemmas at the core of the proof of Proposition 3.1.14.

Lemma 3.7.1. *For all $H = (H_0, H_1, H_2) \in (0,1)^3$, $\varepsilon \in (0, H_0)$, $0 \leq n \leq m$, $0 \leq s, t, u \leq 1$ and $\eta \in \mathbb{R}^2$, it holds that*

$$\left|L^{H,(m,m)}_{t,t}(\eta)\right| \lesssim K^{H_{\varepsilon,0}}(\eta) \tag{3.7.1}$$

and

$$\left|L^{H,((n,m),m)}_{(s,t),u}(\eta)\right| \lesssim 2^{-n\varepsilon}|t-s|^{\frac{\varepsilon}{2}}\left\{K^{H_{\varepsilon,0}}(\eta) + K^{H_{\varepsilon,0,1}}(\eta) + K^{H_{\varepsilon,0,2}}(\eta)\right\}, \tag{3.7.2}$$

where $H_{\varepsilon,0} := (H_0 - \varepsilon, H_1, H_2)$, $H_{\varepsilon,0,1} := (H_0 - \varepsilon, H_1 - \varepsilon, H_2)$, $H_{\varepsilon,0,2} := (H_0 - \varepsilon, H_1, H_2 - \varepsilon)$, and the proportional constants do no depend on (n,m), (s,t), u and η.

Proof. The first inequality is a straight consequence of Corollary 3.4.3 :

$$\begin{aligned}
\left|L^{H,(m,m)}_{t,t}(\eta)\right| &= \left|\frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}}\int_{(\xi,\eta)\in D^m} d\xi\, \frac{|\gamma_t(\xi,|\eta|)|^2}{|\xi|^{2H_0-1}}\right| \\
&\lesssim \frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}}\int_{\mathbb{R}} d\xi\, \frac{|\gamma_t(\xi,|\eta|)|^2}{|\xi|^{2H_0-1}} \lesssim K^{H_{\varepsilon,0}}(\eta).
\end{aligned}$$

The second one is a bit more technical. Recall that one has

$$L^{H,((n,m),m)}_{(s,t),u}(\eta) := \frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}}\int_{(\xi,\eta)\in D^{n,m}\cap D^m} d\xi\, \frac{\gamma_{s,t}(\xi,|\eta|)\overline{\gamma_u(\xi,|\eta|)}}{|\xi|^{2H_0-1}}$$

which leads to

$$\begin{aligned}
\left|L^{H,((n,m),m)}_{(s,t),u}(\eta)\right| &\lesssim \frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}}\int_{(\xi,\eta)\in D^{n,m}} d\xi\, \frac{|\gamma_{s,t}(\xi,|\eta|)||\gamma_u(\xi,|\eta|)|}{|\xi|^{2H_0-1}} \\
&\lesssim \frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}}\int_{|\xi|\geq 2^{2n}} d\xi\, \frac{|\gamma_{s,t}(\xi,|\eta|)||\gamma_u(\xi,|\eta|)|}{|\xi|^{2H_0-1}} \\
&\quad + \frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}}\mathbb{1}_{|\eta|\geq 2^n}\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{s,t}(\xi,|\eta|)||\gamma_u(\xi,|\eta|)|}{|\xi|^{2H_0-1}}.
\end{aligned}$$

Let us call $\mathbb{I}_{m,n}(s,t,u)$ (resp. $\mathbb{II}_{m,n}(s,t,u)$) the first (resp. the second) integral. Since $0 < \varepsilon < H_0$, Corollary 3.4.3 combined with Cauchy-Schwarz inequality entails:

$$\mathbb{I}_{m,n}(s,t,u) \lesssim 2^{-2n\varepsilon}|t-s|^{\frac{\varepsilon}{2}} K^{H_{\varepsilon,0}}(\eta),$$

whereas, keeping in mind the identity

$$\{|\eta| \geq 2^n\} \subset \{|\eta| \geq 2^n, |\eta_1| \geq \frac{1}{\sqrt{2}}|\eta|\} \cup \{|\eta| \geq 2^n, |\eta_2| \geq \frac{1}{\sqrt{2}}|\eta|\},$$

it holds

$$\frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2H_2-1}}\mathbb{1}_{|\eta|\geq 2^n} \lesssim 2^{-2n\varepsilon}\frac{1}{|\eta_1|^{2(H_1-\varepsilon)-1}|\eta_2|^{2H_2-1}} + 2^{-2n\varepsilon}\frac{1}{|\eta_1|^{2H_1-1}|\eta_2|^{2(H_2-\varepsilon)-1}}$$

which, thanks to Corollary 3.4.3, immediately implies

$$\mathbb{II}_{m,n}(s,t,u) \lesssim 2^{-n\varepsilon}|t-s|^{\varepsilon}\Big\{K^{H_{\varepsilon,0,1}}(\eta) + K^{H_{\varepsilon,0,2}}(\eta)\Big\}.$$

□

The lemma below brings back computations on compact domains and its proof can be found in [8, Lemma 2.6].

Lemma 3.7.2. *Let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a test function and fix $\sigma \in \mathbb{R}$. Then, for every $p \geq 1$ and for all $\eta_1, \dots, \eta_p \in \mathbb{R}^d$, it holds that*

$$\left|\int_{\mathbb{R}^d} dx \prod_{i=1}^{p}\int_{(\mathbb{R}^d)^2}\frac{d\lambda_i d\tilde{\lambda}_i}{\{1+|\lambda_i|^2\}^{\frac{\sigma}{2}}\{1+|\tilde{\lambda}_i|^2\}^{\frac{\sigma}{2}}} e^{i\langle x,\lambda_i-\tilde{\lambda}_i\rangle}\widehat{\chi}(\lambda_i-\eta_i)\overline{\widehat{\chi}(\tilde{\lambda}_i-\eta_i)}\right| \lesssim \prod_{i=1}^{p}\frac{1}{\{1+|\eta_i|^2\}^{\sigma}},$$

where the proportional constant only depends on χ and σ.

We end this subsection by a highly technical lemma that permits us to construct the second order stochastic process Υ when $d = 2$ in the roughest case.

Lemma 3.7.3. *For all $H = (H_0, H_1, H_2), \tilde{H} = (\tilde{H}_0, \tilde{H}_1, \tilde{H}_2) \in (0,1)^3$ verifying*

$$0 < H_1, \tilde{H}_1 < \frac{3}{4} \quad , \quad 0 < H_2, \tilde{H}_2 < \frac{3}{4} \quad , \quad \frac{3}{2} < 2H_0+H_1+H_2 \leq \frac{7}{4} \quad , \quad \frac{3}{2} < 2\tilde{H}_0+\tilde{H}_1+\tilde{H}_2 \leq \frac{7}{4}, \tag{3.7.3}$$

and any

$$\alpha \in \Big(\max\Big(2-(2H_0+H_1+H_2), 2-(2\tilde{H}_0+\tilde{H}_1+\tilde{H}_2)\Big), \frac{1}{2}\Big), \tag{3.7.4}$$

it holds that

$$\iint_{(\mathbb{R}^2)^2}\frac{d\eta d\tilde{\eta}}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}} K^{H}(\eta)K^{\tilde{H}}(\tilde{\eta}) \ < \ \infty .$$

Proof.
As in [7], it suffices to prove the convergence of the four integrals below:

$$\mathcal{J}_1 = \int_{\mathbb{R}^2} d\eta \int_{\mathbb{R}^2} d\tilde{\eta}\frac{1}{\{1+|\eta|^2\}^{\alpha}}\frac{1}{\{1+|\tilde{\eta}|^2\}^{\alpha}} K^{H}(\eta)K^{\tilde{H}}(\tilde{\eta}),$$

$$\mathcal{J}_2 = \int_0^\infty d\eta_1 \int_0^\infty d\tilde{\eta}_1 \int_0^\infty d\eta_2 \int_{\eta_2}^{2\eta_2} d\tilde{\eta}_2 \, \frac{1}{\{1+\eta_1^2+(\eta_2-\tilde{\eta}_2)^2\}^\alpha} \frac{1}{\{1+\tilde{\eta}_1^2+(\eta_2-\tilde{\eta}_2)^2\}^\alpha} K^H(\eta) K^{\tilde{H}}(\tilde{\eta}) \, ,$$

$$\mathcal{J}_3 = \int_0^\infty d\eta_1 \int_{\eta_1}^{2\eta_1} d\tilde{\eta}_1 \int_0^\infty d\eta_2 \int_{\eta_2}^{2\eta_2} d\tilde{\eta}_2 \frac{1}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}} K^H(\eta) K^{\tilde{H}}(\tilde{\eta}) \, ,$$

and

$$\mathcal{J}_4 = \int_0^\infty d\eta_1 \int_{\eta_1}^{2\eta_1} d\tilde{\eta}_1 \int_0^\infty d\tilde{\eta}_2 \int_{\tilde{\eta}_2}^{2\tilde{\eta}_2} d\eta_2 \frac{1}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}} K^H(\eta) K^{\tilde{H}}(\tilde{\eta}) \, .$$

Estimation of $\mathcal{J}_1$. It is immediate that $\mathcal{J}_1$ can be seen as

$$\mathcal{J}_1 = \left(\int_{\mathbb{R}^2} \frac{d\eta}{\{1+|\eta|^2\}^\alpha} K^H(\eta) \right) \left(\int_{\mathbb{R}^2} \frac{d\tilde{\eta}}{\{1+|\tilde{\eta}|^2\}^\alpha} K^{\tilde{H}}(\tilde{\eta}) \right) \, , \tag{3.7.5}$$

and the result is a direct consequence from the fact that $\alpha > \max\Big(2-(2H_0+H_1+H_2), 2-(2\tilde{H}_0+\tilde{H}_1+\tilde{H}_2)$ (see the proof of Proposition 3.1.3 with $d=2$).

Estimation of $\mathcal{J}_2$. Let us deal with

$$\mathcal{J}_2 = \int_0^\infty d\eta_1 \int_0^\infty d\tilde{\eta}_1 \int_0^\infty d\eta_2 \int_{\eta_2}^{2\eta_2} d\tilde{\eta}_2 \, \frac{1}{\{1+\eta_1^2+(\eta_2-\tilde{\eta}_2)^2\}^\alpha} \frac{1}{\{1+\tilde{\eta}_1^2+(\eta_2-\tilde{\eta}_2)^2\}^\alpha} K^H(\eta) K^{\tilde{H}}(\tilde{\eta}) \, .$$

Executing the change of variables $r = \frac{\tilde{\eta}_2-\eta_2}{\eta_2}$ and applying several times Cauchy-Schwarz inequality, we derive

$$\begin{aligned}
\mathcal{J}_2 &= \int_0^\infty d\eta_1 \int_0^\infty d\tilde{\eta}_1 \int_0^\infty d\eta_2\, \eta_2 \int_0^1 dr \, \frac{1}{\{1+\eta_1^2+\eta_2^2 r^2\}^\alpha} \frac{1}{\{1+\tilde{\eta}_1^2+\eta_2^2 r^2\}^\alpha} \\
&\qquad \frac{|\eta_1|^{1-2H_1}|\tilde{\eta}_1|^{1-2\tilde{H}_1}}{|\eta_2|^{2(H_2+\tilde{H}_2)-2}(1+r)^{2\tilde{H}_2-1}} \frac{1}{\{1+|\eta|^{4H_0}\}} \frac{1}{\{1+(\tilde{\eta}_1^2+\eta_2^2(1+r)^2)^{2\tilde{H}_0}\}} \\
&\lesssim \int_0^\infty d\eta_1 \int_0^\infty d\tilde{\eta}_1 \int_0^\infty d\eta_2 \, \frac{|\eta_1|^{1-2H_1}|\tilde{\eta}_1|^{1-2\tilde{H}_1}}{|\eta_2|^{2(H_2+\tilde{H}_2)-3}} \frac{1}{\{1+|\eta|^{4H_0}\}\{1+(\tilde{\eta}_1^2+\eta_2^2)^{2\tilde{H}_0}\}} \\
&\qquad \int_0^1 dr \, \frac{1}{\{1+\eta_1^2+\eta_2^2 r^2\}^\alpha} \frac{1}{\{1+\tilde{\eta}_1^2+\eta_2^2 r^2\}^\alpha} \\
&\lesssim \int_0^\infty \frac{d\eta_2}{|\eta_2|^{2(H_2+\tilde{H}_2)-3}} \left[\int_0^\infty \frac{d\eta_1}{|\eta_1|^{2H_1-1}\{1+(\eta_1^2+\eta_2^2)^{2H_0}\}} \left(\int_0^1 \frac{dr}{\{1+\eta_1^2+\eta_2^2 r^2\}^{2\alpha}} \right)^{\frac12} \right] \\
&\qquad \left[\int_0^\infty \frac{d\tilde{\eta}_1}{|\tilde{\eta}_1|^{2\tilde{H}_1-1}\{1+(\tilde{\eta}_1^2+\eta_2^2)^{2\tilde{H}_0}\}} \left(\int_0^1 \frac{dr}{\{1+\tilde{\eta}_1^2+\eta_2^2 r^2\}^{2\alpha}} \right)^{\frac12} \right] \\
&\lesssim \int_0^\infty \frac{d\eta_2}{|\eta_2|^{2(H_2+\tilde{H}_2)-3}} \left(\int_0^\infty \frac{d\eta_1}{|\eta_1|^{4H_1-2}\{1+(\eta_1^2+\eta_2^2)^{4H_0}\}} \right)^{\frac12} \\
&\qquad \left(\int_0^\infty \frac{d\tilde{\eta}_1}{|\tilde{\eta}_1|^{4\tilde{H}_1-2}\{1+(\tilde{\eta}_1^2+\eta_2^2)^{4\tilde{H}_0}\}} \right)^{\frac12} \left(\int_0^\infty d\lambda \int_0^1 \frac{dr}{\{1+\lambda^2+\eta_2^2 r^2\}^{2\alpha}} \right) \\
&\lesssim \int_0^\infty \frac{d\eta_2}{|\eta_2|^{2(H_2+\tilde{H}_2)-2}} \left(\int_0^\infty \frac{d\eta_1}{|\eta_1|^{4H_1-2}\{1+(\eta_1^2+\eta_2^2)^{4H_0}\}} \right)^{\frac12} \\
&\qquad \left(\int_0^\infty \frac{d\tilde{\eta}_1}{|\tilde{\eta}_1|^{4\tilde{H}_1-2}\{1+(\tilde{\eta}_1^2+\eta_2^2)^{4\tilde{H}_0}\}} \right)^{\frac12} \left(\int_0^\infty d\lambda \int_0^{\eta_2} \frac{dr}{\{1+\lambda^2+r^2\}^{2\alpha}} \right) \\
&\lesssim \left(\int_0^\infty \frac{d\eta_2}{|\eta_2|^{4H_2-2}} \int_0^\infty \frac{d\eta_1}{|\eta_1|^{4H_1-2}\{1+|\eta|^{8H_0}\}} \left(\int_0^\infty d\lambda \int_0^{\eta_2} \frac{dr}{\{1+\lambda^2+r^2\}^{2\alpha}} \right) \right)^{\frac12} \\
&\qquad \left(\int_0^\infty \frac{d\tilde{\eta}_2}{|\tilde{\eta}_2|^{4\tilde{H}_2-2}} \int_0^\infty \frac{d\tilde{\eta}_1}{|\tilde{\eta}_1|^{4\tilde{H}_1-2}\{1+|\tilde{\eta}|^{8\tilde{H}_0}\}} \left(\int_0^\infty d\lambda \int_0^{\tilde{\eta}_2} \frac{dr}{\{1+\lambda^2+r^2\}^{2\alpha}} \right) \right)^{\frac12} .
\end{aligned}$$

Now, according to (3.7.3), $\frac14 < \alpha < \frac12$ and one can find $\varepsilon > 0$ such that $2\alpha - \frac12 - \varepsilon > 0$ and obtain

$$\int_0^\infty d\lambda \int_0^{\eta_2} \frac{dr}{\{1+\lambda^2+r^2\}^{2\alpha}} \leq \int_0^\infty \frac{d\lambda}{\{1+\lambda^2\}^{\frac12+\varepsilon}} \int_0^{\eta_2} \frac{dr}{\{1+r^2\}^{2\alpha-\frac12-\varepsilon}} \lesssim 1+|\eta_2|^{2-4\alpha+2\varepsilon} ,$$

in such a way that

$$\begin{aligned}
&\int_0^\infty \frac{d\eta_2}{|\eta_2|^{4H_2-2}} \int_0^\infty \frac{d\eta_1}{|\eta_1|^{4H_1-2}\{1+|\eta|^{8H_0}\}} \left(\int_0^\infty d\lambda \int_0^{\eta_2} \frac{dr}{\{1+\lambda^2+r^2\}^{2\alpha}} \right) \\
&\lesssim \int_0^\infty \frac{d\eta_1}{|\eta_1|^{4H_1-2}\{1+|\eta_1|^{8H_0}\}} \int_0^1 \frac{d\eta_2}{|\eta_2|^{4H_2-2}} + \int_0^\infty d\eta_1 \int_1^\infty d\eta_2 \, \frac{1}{|\eta_1|^{4H_1-2}|\eta_2|^{4H_2+4\alpha-4-2\varepsilon}|\eta|^{8H_0}} .
\end{aligned}$$

Since $2H_0 + H_1 + H_2 > \frac{3}{2}$ and $0 < H_2 < \frac{3}{4}$, $2H_0 + H_1 > \frac{3}{4}$ which assures the convergence of the first integral. After a change of polar variables, we see that for ε small enough

$$4(2H_0 + H_1 + H_2) - 7 + 4\alpha - 2\varepsilon > 1,$$

and the second integral is finite.

We can mimic these arguments to bound the second term in (3.7.6), which prove that $\mathcal{J}_2$ is finite.

Estimation of $\mathcal{J}_3$. Recall that

$$\mathcal{J}_3 = \int_0^\infty d\eta_1 \int_{\eta_1}^{2\eta_1} d\tilde{\eta}_1 \int_0^\infty d\eta_2 \int_{\eta_2}^{2\eta_2} d\tilde{\eta}_2 \frac{1}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}} K^H(\eta) K^{\tilde{H}}(\tilde{\eta}) \ .$$

With the help of the changes of variables $r_1 = \frac{\tilde{\eta}_1-\eta_1}{\eta_1}$ and $r_2 = \frac{\tilde{\eta}_2-\eta_2}{\eta_2}$ (first equality) followed by $\tilde{r_1} = \eta_1 r_1$ and $\tilde{r_2} = \eta_2 r_2$ (second inequality), we write

$$\begin{aligned}
\mathcal{J}_3 &= \int_0^\infty d\eta_1 \int_0^\infty d\eta_2\, \eta_1\eta_2 \int_0^1 dr_1 \int_0^1 dr_2 \frac{1}{\{1+\eta_1^2 r_1^2 + \eta_2^2 r_2^2\}^{2\alpha}} \\
&\quad \frac{(1+r_1)^{1-2\tilde{H}_1}(1+r_2)^{1-2\tilde{H}_2}}{|\eta_1|^{2(H_1+\tilde{H}_1)-2}|\eta_2|^{2(H_2+\tilde{H}_2)-2}} \frac{1}{1+|\eta|^{4H_0}} \frac{1}{1+(\eta_1^2(1+r_1)^2+\eta_2^2(1+r_2)^2)^{2\tilde{H}_0}} \\
&\lesssim \int_0^\infty d\eta_1 \int_0^\infty d\eta_2\, \eta_1\eta_2 \int_0^1 dr_1 \int_0^1 dr_2 \frac{1}{\{1+\eta_1^2 r_1^2 + \eta_2^2 r_2^2\}^{2\alpha}} \\
&\quad \frac{1}{|\eta_1|^{2(H_1+\tilde{H}_1)-2}|\eta_2|^{2(H_2+\tilde{H}_2)-2}} \frac{1}{1+|\eta|^{4H_0}} \frac{1}{1+|\eta|^{4\tilde{H}_0}} \\
&\lesssim \int_0^\infty \int_0^\infty \frac{d\eta_1 d\eta_2}{|\eta_1|^{2(H_1+\tilde{H}_1)-2}|\eta_2|^{2(H_2+\tilde{H}_2)-2}\{1+|\eta|^{4(H_0+\tilde{H}_0)}\}} \int_0^{\eta_1} dr_1 \int_0^{\eta_2} dr_2 \frac{1}{\{1+r_1^2+r_2^2\}^{2\alpha}} \\
&\lesssim \int_0^1 \int_0^1 \frac{d\eta_1 d\eta_2}{|\eta_1|^{2(H_1+\tilde{H}_1)-3}|\eta_2|^{2(H_2+\tilde{H}_2)-3}} \\
&\quad + \int\int_{|\eta|\geq 1} \frac{d\eta_1 d\eta_2}{|\eta_1|^{2(H_1+\tilde{H}_1)-2}|\eta_2|^{2(H_2+\tilde{H}_2)-2}\{1+|\eta|^{4(H_0+\tilde{H}_0)}\}} \iint_{[0,|\eta|]^2} \frac{dr_1 dr_2}{\{1+r_1^2+r_2^2\}^{2\alpha}} \ .
\end{aligned}$$

As $H_1 + \tilde{H}_1 < 2$ and $H_2 + \tilde{H}_2 < 2$, it is clear that the first integral is finite. Then, since $\alpha < \frac{1}{2}$,

$$\iint_{[0,|\eta|]^2} \frac{dr_1 dr_2}{\{1+r_1^2+r_2^2\}^{2\alpha}} \lesssim \int_0^{|\eta|} d\rho \, \frac{\rho}{\{1+\rho^2\}^{2\alpha}} \lesssim 1 + |\eta|^{2-4\alpha} \ ,$$

and consequently, the fact that $H_i, \tilde{H}_i \in (0, \frac{3}{4})$ for $i \in \{1,2\}$ implies

$$\begin{aligned}
\int\int_{|\eta|\geq 1} \frac{d\eta_1 d\eta_2}{|\eta_1|^{2(H_1+\tilde{H}_1)-2}|\eta_2|^{2(H_2+\tilde{H}_2)-2}\{1+|\eta|^{4(H_0+\tilde{H}_0)}\}} \iint_{[0,|\eta|]^2} \frac{dr_1 dr_2}{\{1+r_1^2+r_2^2\}^{2\alpha}} \\
\lesssim \int_1^\infty \frac{dr}{r^{2(2H_0+H_1+H_2)+2(2\tilde{H}_0+\tilde{H}_1+\tilde{H}_2)+4\alpha-7}} \ .
\end{aligned}$$

The hypothesis (3.7.4) on α guarantees that the latter integral is finite and so is $\mathcal{J}_3$.

Estimation of $\mathcal{J}_4$. Remember that

$$\mathcal{J}_4 = \int_0^{\infty} d\eta_1 \int_{\eta_1}^{2\eta_1} d\tilde{\eta}_1 \int_0^{\infty} d\tilde{\eta}_2 \int_{\tilde{\eta}_2}^{2\tilde{\eta}_2} d\eta_2 \frac{1}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}} K^H(\eta) K^{\tilde{H}}(\tilde{\eta}) \ .$$

Thus, we can rewrite $\mathcal{J}_4$ as

$$\mathcal{J}_4 = \int_0^{\infty} d\eta_1 \int_{\eta_1}^{2\eta_1} d\tilde{\eta}_1 \int_0^{\infty} d\eta_2 \int_{\frac{\eta_2}{2}}^{\eta_2} d\tilde{\eta}_2 \frac{1}{\{1+|\eta-\tilde{\eta}|^2\}^{2\alpha}} K^H(\eta) K^H(\tilde{\eta}) \ .$$

and from there we can reproduce the previous arguments to show that this integral is finite. □

3.7.2 Estimation of the constant c_1 from Proposition 3.4.4

Let us compute the value of c_1. By developing the integrand in Taylor series, we write

$$\begin{aligned}
\int_0^1 \frac{x^{\alpha+\kappa-1}}{1+x^2} dx + \int_1^{+\infty} \frac{x^{\alpha-1}}{1+x^2} dx &= \int_0^1 \frac{x^{\alpha+\kappa-1}}{1+x^2} dx + \int_0^1 \frac{x^{1-\alpha}}{1+x^2} dx \\
&= \sum_{n=0}^{+\infty} (-1)^n \int_0^1 x^{2n+\alpha+\kappa-1} dx + \sum_{n=0}^{+\infty} (-1)^n \int_0^1 x^{2n+1-\alpha} dx \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+\alpha+\kappa} + \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+2-\alpha} \\
&= \sum_{n=0}^{+\infty} \frac{2}{(4n+\alpha+\kappa)(4n+2+\alpha+\kappa)} + \sum_{n=0}^{+\infty} \frac{2}{(4n+2-\alpha)(4n+4-\alpha)} \\
&= \frac{1}{8} \sum_{n=0}^{+\infty} \frac{1}{(n+\frac{\alpha+\kappa}{4})(n+\frac{\alpha+\kappa+2}{4})} + \frac{1}{8} \sum_{n=0}^{+\infty} \frac{1}{(n+\frac{2-\alpha}{4})(n+1-\frac{\alpha}{4})}.
\end{aligned} \tag{3.7.6}$$

To continue the computation, we need to introduce two well-known special functions, namely:

Definition 3.7.4. *The Gamma function Γ and the digamma function Ψ are defined for all $x > 0$ by the formulas*

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

We are now in a position to recall the following classical result from analytic number theory:

Lemma 3.7.5. *For all $a, b > 0$, it holds that:*

$$\sum_{n=0}^{+\infty} \frac{1}{(n+a)(n+b)} = \frac{\Psi(b)-\Psi(a)}{b-a}.$$

Coming back to (3.7.6), with the help of Lemma 3.7.5, we deduce

$$\int_0^1 \frac{x^{\alpha+\kappa-1}}{1+x^2}dx+\int_1^{+\infty}\frac{x^{\alpha-1}}{1+x^2}dx=\frac{1}{4}\Big[\Psi\left(\frac{\alpha+\kappa+2}{4}\right)-\Psi\left(\frac{\alpha+\kappa}{4}\right)+\Psi\left(\frac{4-\alpha}{4}\right)-\Psi\left(\frac{2-\alpha}{4}\right)\Big]$$

and, finally, for your viewing pleasure,

$$c_1=\frac{1}{4\kappa}\left[\frac{\Gamma'(\frac{\alpha+\kappa+2}{4})}{\Gamma(\frac{\alpha+\kappa+2}{4})}+\frac{\Gamma'(\frac{4-\alpha}{4})}{\Gamma(\frac{4-\alpha}{4})}-\frac{\Gamma'(\frac{\alpha+\kappa}{4})}{\Gamma(\frac{\alpha+\kappa}{4})}-\frac{\Gamma'(\frac{2-\alpha}{4})}{\Gamma(\frac{2-\alpha}{4})}\right].$$

3.7.3 Heuristic argument to justify the change of regime in Proposition 3.4.4

Let us give a formal argument to explain the change of regime in Proposition 3.4.4. First of all, as κ tends to zero,

$$\frac{1}{4\kappa}\Big[\Psi\left(\frac{\alpha+\kappa+2}{4}\right)-\Psi\left(\frac{\alpha+\kappa}{4}\right)+\Psi\left(\frac{4-\alpha}{4}\right)-\Psi\left(\frac{2-\alpha}{4}\right)\Big]$$

is equivalent to

$$\frac{1}{4\kappa}\Big[\Psi\left(\frac{\alpha+2}{4}\right)-\Psi\left(\frac{\alpha}{4}\right)+\Psi\left(\frac{4-\alpha}{4}\right)-\Psi\left(\frac{2-\alpha}{4}\right)\Big].$$

Using Euler's reflection formula

$$\Gamma(\beta)\Gamma(1-\beta)=\frac{\pi}{\sin(\pi\beta)}\qquad(\beta\in(0,1)),$$

we deduce that, for all $\beta\in(0,1)$, $\Gamma'(\beta)\Gamma(1-\beta)-\Gamma(\beta)\Gamma'(1-\beta)=-\frac{\pi^2\cos(\pi\beta)}{\sin(\pi\beta)^2}$, leading to

$$\Psi(\beta)-\Psi(1-\beta)=-\pi\mathrm{cotan}(\pi\beta)$$

and

$$\begin{aligned}\frac{1}{4}\Big[\Psi\left(\frac{\alpha+2}{4}\right)-\Psi\left(\frac{\alpha}{4}\right)+\Psi\left(\frac{4-\alpha}{4}\right)-\Psi\left(\frac{2-\alpha}{4}\right)\Big] &= -\frac{\pi}{4}\Big[\mathrm{cotan}\Big(\frac{\pi(\alpha+2)}{4}\Big)-\mathrm{cotan}\Big(\frac{\alpha\pi}{4}\Big)\Big]\\ &= \frac{\pi}{4}\Big[\mathrm{cotan}\Big(\frac{\alpha\pi}{4}\Big)+\tan\Big(\frac{\alpha\pi}{4}\Big)\Big]\\ &= \frac{\pi}{4}\frac{1}{\cos\left(\frac{\alpha\pi}{4}\right)\sin\left(\frac{\alpha\pi}{4}\right)}\\ &= \frac{\pi}{2}\frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)}.\end{aligned}$$

Consequently,

$$c_1 4^{n\kappa}\underset{\kappa\to 0}{\sim}\frac{\pi}{2\kappa\sin\left(\frac{\alpha\pi}{2}\right)}4^{n\kappa}\underset{\kappa\to 0}{\sim}\frac{\pi}{2\kappa\sin\left(\frac{\alpha\pi}{2}\right)}\Big(1+n\kappa\ln(4)\Big).$$

Finally,

$$c_1 4^{n\kappa}\underset{n\to\infty}{\sim}\frac{\pi\ln(2)}{\sin\left(\frac{\alpha\pi}{2}\right)}n$$

with a formal and illicit use of the equivalents.

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Chapter 4

A Schrödinger equation with fractional noise

We study a stochastic Schrödinger equation with a quadratic nonlinearity and a space-time fractional perturbation, in space dimension $d \leq 3$. When the Hurst index is large enough, we prove local well-posedness of the problem using classical arguments. However, for a small Hurst index, even the interpretation of the equation needs some care. In this case, a renormalization procedure must come into the picture, leading to a Wick-type interpretation of the model. Our fixed-point argument then involves some specific regularization properties of the Schrödinger group, which allows us to cope with the strong irregularity of the solution.

4.1 Introduction and main results

4.1.1 General introduction

In this chapter we study the following d-dimensional stochastic Schrödinger equation with a quadratic nonlinearity and a space-time fractional perturbation:

$$\begin{cases} i\partial_t u - \Delta u = \rho^2 |u|^2 + \dot{B}\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases} \tag{4.1.1}$$

where $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth cut-off function in space and $\dot{B}$ stands for the derivative of a space-time fractional Brownian motion of Hurst index $H = (H_0, \ldots, H_d) \in (0,1)^{d+1}$.

We first show that, when $2H_0 + \sum_{i=1}^d H_i > d+1$ (the so-called *regular* case), the interpretation and local well-posedness of (4.1.1) can be derived from quite direct arguments, based on a first-order expansion and the use of Strichartz inequalities.

The equation behaves less favorably when $2H_0 + \sum_{i=1}^d H_i \leq d+1$ (the irregular or *rough* case). In this situation, we first need a Wick-type renormalization procedure in order to interpret the model. The fixed-point argument then relies on the smoothing properties of the Schrödinger equation, and in particular on its local regularization effect.

We can loosely sum up our results as follows (see the subsequent Sections 4.1.2-4.1.4 for precise statements).

Theorem 4.1.1. *Assume that* $1 \leq d \leq 3$ *and set* $\alpha_H := \Big(2H_0 + \sum_{i=1}^d H_i\Big) - (d+1)$. *The following picture holds true:*

(i) ***Case*** $\alpha_H > 0$. *The equation* (4.1.1) *is almost surely locally well-posed in* $H^\beta(\mathbb{R}^d)$ *for some* $\beta > 0$.

(ii) ***Case*** $\alpha_H \leq 0$. *There exists* $\alpha_d < 0$ *such that if* $\alpha_H > \alpha_d$ *then the equation* (4.1.1) *can be interpreted in the Wick (renormalized) sense and it is almost surely locally well-posed in* $H^{-\beta}(\mathbb{R}^d)$ *for some* $\beta > 0$.

We refer to Definition 4.1.4 and Theorem 4.1.10 for precise statements in the regular case (i), and to Definition 4.1.8 and Theorem 4.1.11 in the rough case (ii) (see in particular the condition (**H2'**) for the exact value of α_d, depending on d). To our knowledge, and although the statement is still restricted to values of α_H close to 0, Theorem 4.1.1 (ii) is the first result in the context of nonlinear Schrödinger equations where both renormalization arguments and local regularization properties are used to control an irregular noise (in Sobolev spaces of negative order).

The stochastic Schrödinger equation is a widely studied model in the SPDE literature. Just as for stochastic heat or wave equations, the stochastic Schrödinger model admits numerous possible variants and is known to be the source of many challenging questions, whose treatment can only be achieved through the sophisticated combination of PDE tools with probabilistic analysis.

The study of nonlinear stochastic Schrödinger models includes for instance the consideration of a multiplicative noise, that is (see e.g. [10, 13, 16, 17])

$$i\partial_t u - \Delta u = u|u|^{2\sigma} + u \cdot \dot{\xi}\,, \quad t \in [0,T]\,,\, x \in \mathbb{T}^d \text{ or } x \in \mathbb{R}^d\,,\, \sigma > 0\,,$$

or the consideration of a random dispersion, that is (see e.g. [14])

$$i\partial_t u - \Delta u \cdot \dot{\xi} = u|u|^{2\sigma}\,, \quad t \in [0,T]\,,\, x \in \mathbb{T}^d \text{ or } x \in \mathbb{R}^d\,,\, \sigma > 0\,.$$

In these equations, $\dot{\xi}$ stands for a random noise which, in most situations, is taken as white in time.

In the additive-noise framework (which our model (4.1.1) belongs to), the first mathematical study of nonlinear stochastic Schrödinger equations essentially traces back to a series of works by De Bouard and Debussche (see e.g. [11, 12]), where the authors considered the general dynamics

$$i\partial_t u + \Delta u = |u|^{2\sigma} u + \dot{\xi}\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \tag{4.1.2}$$

for suitable values of $\sigma > 0$ (depending on d), and with $\dot{\xi}$ a white noise in time admitting a high regularity in space. These results have been recently extended to noises $\dot{\xi}$ less regular in space, first for the equation on the one-dimensional torus $\mathbb{T}$ with $\sigma = 1$ (see [23]), and then for the equation on the whole Euclidean space $\mathbb{R}^d$ with suitable values of $\sigma > 0$ (see [34, 36]).

In spite of these substantial advances, it can be observed that the analysis of equation (4.1.2) is so far limited to situations where $\dot{\xi}$ is a white noise in time.

In the present chapter, we propose to investigate a new direction in this field, by considering the model of a nonlinear Schrödinger equation with quadratic perturbation and additive forcing term given by a *space-time fractional noise*. To be more specific, the dynamics we will focus on can be described as follows:

$$\begin{cases} i\partial_t u - \Delta u = \rho^2 |u|^2 + \dot{B}\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases} \tag{4.1.3}$$

where

- $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth compactly-supported function of the space variable, allowing us to bring the analysis back to compact domains,
- ϕ is a deterministic initial condition, the regularity of which will be specified later on,
- one has $\dot{B} := \partial_t \partial_{x_1} \cdots \partial_{x_d} B$, that is $\dot{B}$ is the space-time derivative (in the sense of distributions) of B, with B a *fractional sheet* of Hurst index $H = (H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$ (see Definition 4.2.1 below for details).

The consideration of such a fractional noise $\dot{B}$ represents the main specificity of our analysis, and the main novelty with respect to the above-reported (white-noise) literature. The other new contribution brought by our analysis will consist in the *possible treatment of a distributional-valued solution u for* (4.1.3), which enables us to cover more irregular noises in the model (see the description of the so-called rough case in Sections 4.1.3 and 4.1.4).

In fact, our aim in the sequel will be to offer as much flexibility as possible regarding the choice of the Hurst index $H \in (0,1)^{d+1}$ that governs $\dot{B}$. Thus, for a (hopefully) large range of such indexes, we intend to - at least - prove local well-posedness of equation (4.1.3). Note that this objective was already at the core of the investigations of the first author in [20, 21] for a quadratic fractional wave equation.

Before we go further, let us recall that over the last decade, tremendous developments have been observed in the field of singular stochastic PDEs. This progress has been especially prominent in the parabolic (SPDE) setting, with the introduction of the theory of regularity structures [29] or the paracontrolled approach [26]. Among other contributions, those theories provide a convenient framework towards renormalization procedures, thus paving the way to a rigorous treatment of many long-standing problems. If one focuses on additive-noise models only (in the vein of (4.1.3)), let us quote for instance, among a flourishing literature, the work of Catellier and Chouk [3] about the stochastic quantization equation on the three-dimensional torus

$$\partial_t u - \Delta u = -u^3 + \xi\,, \quad t \in [0,T]\,,\, x \in \mathbb{T}^3\,, \tag{4.1.4}$$

which extends the pioneering results of Da Prato and Debussche [9] for the two-dimensional equation (let us also mention [22] about the consideration of a quadratic nonlinearity - similar to ours - in this parabolic setting, for $d = 3$). The regularity-structure approach has also proven to be highly effective (see [15, 30]) for the study of the parabolic sine-Gordon model

$$\partial_t u + \frac{1}{2}(1-\Delta)u + \sin(\beta u) = \xi\,, \quad t \in [0,T]\,,\, x \in \mathbb{T}^2\,, \tag{4.1.5}$$

with ξ a space-time white noise (see also [37, 38] for recent hyperbolic versions of the model).

Unfortunately, the application of those new groundbreaking approaches beyond the parabolic setting has proved to be very limited so far (this holds true for both regularity structures and paracontrolled theories). To our knowledge, the only attempt to extend such a strategy to a non-parabolic setting is due to Gubinelli, Koch and Oh in their recent work [28], dealing with a stochastic wave model. In particular, we are not aware of any similar extension to a singular stochastic Schrödinger equation.

As regards the deterministic Schrödinger equation with polynomial nonlinearities, its well-posedness in positive Sobolev spaces was established long ago using Strichartz estimates (see [25, 5] and also the monography [4]). More recent developments, applying in particular to NLS with quadratic nonlinearities, also led to well-posedness results for the model in negative Sobolev spaces, thanks to subtle bilinear estimates in the so-called Bourgain spaces (see [7, 41, 2, 31], and also Remark 4.4.7 below).

With this background in mind, let us now go back to the analysis of equation (4.1.3). The starting point of the study will be the mild formulation of (4.1.3), that is the equation

$$u_t = S_t\phi - i\int_0^t S_{t-\tau}(\rho^2|u_\tau|^2)\,d\tau + \Psi_t\,, \tag{4.1.6}$$

where S stands for the Schrödinger group, and where we have set

$$\Psi_t := -i\int_0^t S_{t-\tau}(\dot{B}_\tau)\,d\tau\,. \tag{4.1.7}$$

Note that Ψ can also be seen as the solution of the following "linear" counterpart of (4.1.3):

$$\begin{cases} i\partial_t \Psi - \Delta \Psi = \dot{B}, & t \in [0,T], \ \ x \in \mathbb{R}^d, \\ \Psi(0,.) = 0\,. \end{cases} \tag{4.1.8}$$

For this reason, we will henceforth refer to Ψ as the linear solution of the problem.

A first essential part of our work will be to give a precise meaning to both definition (4.1.7) and equation (4.1.6). In order to initiate this analysis, let us proceed with a standard transformation of the problem, by considering the equation satisfied by the process $v := u - \Psi$, that is the equation (which is still formal at this point)

$$\begin{aligned} v_t = S_t(\phi) - i\int_0^t S_{t-\tau}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau))\,d\tau \\ - i\int_0^t S_{t-\tau}((\rho v_\tau)\cdot(\overline{\rho\Psi_\tau}))\,d\tau - i\int_0^t S_{t-\tau}(|\rho\Psi_\tau|^2)\,d\tau\,. \end{aligned} \tag{4.1.9}$$

The main idea behind this transition from u to v is that Ψ is expected to behave as some "first-order expansion" of u. In other words, due to the specific properties of the group S (which we will detail later on), we expect the process v to be more regular than u and Ψ, making equation (4.1.9) more tractable than (4.1.6). Following this idea, we will focus our subsequent investigations on equation (4.1.9).

Let us now elaborate on the successive steps of our reasoning, and introduce our main results. Note that these steps are overall similar to those developed in [20, 21, 27] for the corresponding quadratic wave model. Nevertheless, when going into the technical details, some new fundamental difficulties arise in the analysis of the Schrödinger model, as we will try to highlight it in the presentation below.

4.1.2 Step 1: Study of the linear equation

Recall that the noise $\dot{B}$ involved in (4.1.3) is defined as the derivative of a fractional sheet B, which is a non-differentiable process (in the standard sense). Consequently, just as the white noise ξ in (4.1.4)-(4.1.5), $\dot{B}$ can only be understood as some random negative-order distribution, and thus the interpretation of the convolution in (4.1.7) requires some clarification.

To do so, let us start from a smooth approximation $(B_n)_{n\geq 0}$ of B, that is, for each fixed $n \geq 0$, $(t,x) \mapsto B_n(t,x)$ is a.s. smooth, and $B_n(t,x) \overset{n\to\infty}{\longrightarrow} B(t,x)$ for every $(t,x) \in [0,T]\times\mathbb{R}^d$ (the choice of such an approximation process will be specified in Section 4.2 below). Then consider the corresponding sequence of linear solutions, that is the sequence Ψ_n of solutions to the equation

$$\begin{cases} i\partial_t \Psi_n - \Delta \Psi_n = \dot{B}_n\,, \quad t\in[0,T]\,,\ x\in\mathbb{R}^d\,, \\ \Psi_n(0,.) = 0\,, \end{cases} \tag{4.1.10}$$

where $\dot{B}_n := \partial_t \partial_{x_1}\cdots\partial_{x_d} B_n$. For every fixed $n \geq 0$, the smoothness of B_n (and accordingly the smoothness of $\dot{B}_n$) makes the analysis of (4.1.10) considerably easier than the one of (4.1.8), and readily allows us to define a unique Gaussian solution process $\big\{\Psi_n(t,x),\, t\in [0,T], x\in\mathbb{R}^d\big\}$ (see Definition 4.2.3).

The solution Ψ of the rough equation (4.1.8) is then interpreted through the following convergence result, which can be seen as our first main contribution:

Proposition 4.1.2. *Let $d \geq 1$ and fix $(H_0,\ldots,H_d)\in(0,1)^{d+1}$. Let $(B_n)_{n\geq 0}$ be the sequence of smooth processes defined by formula* (4.2.2)*, and let Ψ_n be the solution of* (4.1.10) *associated with B_n.*

Then, for every test function $\chi : \mathbb{R}^d \to \mathbb{R}$ (i.e., smooth and compactly-supported), the sequence $(\chi\Psi_n)_{n\geq 0}$ converges almost surely in the space $\mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$, for all $2\leq p\leq\infty$ and

$$\alpha > d+1-\Big(2H_0+\sum_{i=1}^d H_i\Big)\,. \tag{4.1.11}$$

We denote the limit of this sequence by $\chi\Psi$.

The proof of this convergence result relies on the stochastic properties of $\dot{B}$, and will be developed in Section 4.2.2. As the reader might expect it, the resulting regularity property (i.e., the fact that $\chi ⫯ \in \mathcal{C}([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$, for every α satisfying (4.1.11)) will be of crucial importance in the analysis of (4.1.9).

Using a standard patching argument, the limit elements $\{\chi ⫯,\ \chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}$ provided by Proposition 4.1.2 can then be merged into a single locally-controlled distribution $⫯$ (see Proposition 4.2.16), which we will refer to in the sequel.

Remark 4.1.3. We can compare the above regularity restriction (4.1.11) for $⫯$ with the corresponding result of [20] in the wave setting, that is when replacing $i\partial_t ⫯ - \Delta ⫯$ with $\partial_t^2 ⫯ - \Delta ⫯$ in (4.1.8). In the latter situation, and according to [20, Proposition 1.2], one must have

$$\alpha_{\text{wave}} > d - \frac{1}{2} - \left(H_0 + \sum_{i=1}^{d} H_i \right). \tag{4.1.12}$$

Observe in particular that the Hurst indexes are involved through the combination $H_0 + \sum_{i=1}^d H_i$ in (4.1.12), in contrast with the combination $2 \times H_0 + \sum_{i=1}^d H_i$ in (4.1.11). This difference is in fact a direct echo of the behaviour of the underlying linear operators: in the Schrödinger case, time variable somehow "counts twice" with respect to space.

Besides, although such a property cannot be found in the existing literature, we could show along the same pattern that in the heat situation (that is with $\partial_t ⫯ - \Delta ⫯$ instead of $i\partial_t ⫯ - \Delta ⫯$ in (4.1.8)), the restriction for α becomes

$$\alpha_{\text{heat}} > d - \left(2H_0 + \sum_{i=1}^{d} H_i \right),$$

which, compared to (4.1.11), reflects the stronger regularization properties of the heat kernel.

4.1.3 Step 2: Interpretation of the main equation

Now equipped with a proper definition of $⫯$ (as well as a sharp control on its regularity), we can go back to our interpretation issue for the main equation (4.1.6) (or equivalently (4.1.9)). In fact, according to the result of Proposition 4.1.2, two cases need to be distinguished.

The regular case

In the sequel, we will call *regular case* the situation where

$$2H_0 + \sum_{i=1}^{d} H_i > d + 1. \tag{\textbf{H1}}$$

In this case, we can pick $\alpha < 0$ such that condition (4.1.11) is satisfied, and therefore, thanks to Proposition 4.1.2, we are allowed to consider every element $\chi ⫯$ ($\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$)

as a function *of both time and space variables* (in an almost sure way). As a result, the square element $|\rho\Psi|^2$ involved in (4.1.9) simply makes sense as a standard pointwise product of functions. This immediately leads us to the following direct interpretation of the equation:

Definition 4.1.4. *Fix $d \geq 1$ and assume that condition* (**H1**) *is satisfied. Recall that for every test function $\chi : \mathbb{R}^d \to \mathbb{R}$, $\chi\Psi$ is the process defined in Proposition 4.1.2.*

Then we call a solution (on $[0,T]$) of equation (4.1.3) *any stochastic process $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ such that, almost surely, the process $v := u - \Psi$ is a solution of the mild equation*

$$v_t = S_t(\phi) - i\int_0^t S_{t-\tau}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau))\,d\tau \\ - i\int_0^t S_{t-\tau}((\rho v_\tau)\cdot(\overline{\rho\Psi_\tau}))\,d\tau - i\int_0^t S_{t-\tau}(|\rho\Psi_\tau|^2)\,d\tau\,, \quad t\in[0,T]\,.$$

The rough case

Let us now turn to the second situation, where

$$2H_0 + \sum_{i=1}^d H_i \leq d+1\,. \tag{4.1.13}$$

In this case, we can no longer pick $\alpha < 0$ such that condition (4.1.11) is satisfied, and so, referring to Proposition 4.1.2, the element $\rho\Psi$ involved in (4.1.9) must be considered as a function of time *with values in a space of negative-order distribution*. We will call this situation the *rough case*, to signify that we are here working with very irregular processes.

Naturally, the fact that $\rho\Psi_\tau$ must be seen as a distribution (for every fixed τ) makes the interpretation of $|\rho\Psi_\tau|^2$ in (4.1.9) unclear.

This problem can be emphasized through a regularization approach. Namely, let us go back to the sequence of approximated linear solutions $(\Psi_n)_{n\geq 0}$ satisfying (4.1.10). For every fixed $n \geq 0$, Ψ_n is (almost surely) a function of time and space, and so, for every $(t,x)\in(0,T]\times\mathbb{R}^d$, we can compute the moment $\mathbb{E}\Big[|\Psi_n(t,x)|^2\Big]$. The following asymptotic result (which will be proved in Section 4.2.4) rules out any possibility to extend the pointwise interpretation to the limit element $|\Psi|^2$:

Proposition 4.1.5. *Fix $d \geq 1$ and assume that $2H_0 + \sum_{i=1}^d H_i \leq d+1$. Then the quantity $\mathbb{E}\Big[|\Psi_n(t,x)|^2\Big]$ does not depend on x. Denoting $\sigma_n(t) := \mathbb{E}\Big[|\Psi_n(t,x)|^2\Big]$, the following asymptotic equivalence property holds true: for every $(t,x)\in(0,T]\times\mathbb{R}^d$,*

$$\sigma_n(t) \underset{n\to\infty}{\sim} \begin{cases} c_H^1\, t\, n & \text{if} \quad 2H_0 + \sum_{i=1}^d H_i = d+1, \\ c_H^2\, t\, 2^{2n(d+1-[2H_0+\sum_{i=1}^d H_i])} & \text{if} \quad 2H_0 + \sum_{i=1}^d H_i < d+1\,, \end{cases} \tag{4.1.14}$$

for some constants $c_H^1, c_H^2 > 0$.

A natural way to circumvent this divergence issue and to offer a possible interpretation of $|\<1>|^2$ is to turn to a *renormalization procedure*. In fact, it will here be sufficient to consider the most elementary of those methods, namely the Wick renormalization. Thus, for all fixed $n \geq 0$, $(t,x) \in [0,T] \times \mathbb{R}^d$, we set

$$\<2>_n(t,x) := |\<1>_n(t,x)|^2 - \sigma_n(t) \quad \text{where } \sigma_n(t) := \mathbb{E}\Big[|\<1>_n(t,x)|^2\Big]\,. \tag{4.1.15}$$

Our second main contribution now reads as follows:

Proposition 4.1.6. *Let $d \geq 1$ and consider $(H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$ such that*

$$d + \frac{3}{4} < 2H_0 + \sum_{i=1}^{d} H_i \leq d+1\,. \tag{\textbf{H2}}$$

Then, for every test function $\chi : \mathbb{R}^d \to \mathbb{R}$, the sequence $(\chi^2 \<2>_n)_{n\geq 1}$ (defined by (4.1.15)) converges almost surely in the space $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha,p}(\mathbb{R}^d))$, for all $2 \leq p \leq \infty$ and $\alpha > 0$ satisfying (4.1.11).

We denote the limit of this sequence by $\chi^2 \<2>$.

Just as the proof of Proposition 4.1.2, the proof of Proposition 4.1.6 will strongly lean on the stochastic properties of $\dot{B}$. It will be the topic of Section 4.2.3 below.

Remark 4.1.7. The restriction $2H_0 + \sum_{i=1}^d H_i > d + \frac{3}{4}$ in (**H2**) (which will stem from our technical computations) can be compared with the restriction $\sum_{i=0}^d H_i > d - \frac{3}{4}$ in [20, Proposition 1.4] for the quadratic wave model. We suspect that this condition might not be optimal, that is, we can certainly extend the construction of $\chi^2 \<2>$ below this threshold. However, condition (**H2**) will prove to be sufficient for our purpose, as it can be seen from the comparison with the more restrictive assumption (**H2'**) in our main theorem (see also Remark 4.1.12).

With the above constructions in hand, the following *Wick interpretation* of the equation naturally arises:

Definition 4.1.8. *Fix $d \geq 1$ and assume that condition* (**H2**) *is satisfied. Recall that for every test function $\chi : \mathbb{R}^d \to \mathbb{R}$, $\chi \<1>$ and $\chi^2 \<2>$ are the processes defined in Proposition 4.1.2 and Proposition 4.1.6, respectively.*

In this setting, we call a Wick *solution (on $[0,T]$) of equation* (4.1.3) *any stochastic process $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ such that, almost surely, the process $v := u - \<1>$ is a solution of the mild equation*

$$\begin{aligned} v_t = S_t(\phi) - i \int_0^t S_{t-\tau}(\rho^2 |v_\tau|^2)\, d\tau - i \int_0^t S_{t-\tau}((\rho \overline{v}_\tau) \cdot (\rho \<1>_\tau))\, d\tau \\ - i \int_0^t S_{t-\tau}((\rho v_\tau) \cdot (\overline{\rho \<1>_\tau}))\, d\tau - i \int_0^t S_{t-\tau}(\rho^2 \<2>_\tau)\, d\tau\,, \quad t \in [0,T]\,. \end{aligned} \tag{4.1.16}$$

Remark 4.1.9. As the reader may have noticed, our tree notation $\<1>$ and $\<2>$ for the main stochastic processes follows the convention used in [29]: the circle $\circ$ therein stands for the random noise $\dot{B}$, while the line $\vdots$ represents the Duhamel integral operator $\mathcal{I} = (i\partial_t - \Delta)^{-1}$.

4.1.4 Step 3: Local wellposedness results

At this stage of the procedure, the stochastic part of our analysis can be considered as complete, and our aim now is to solve equation (4.1.3) (understood along Definition 4.1.4 or Definition 4.1.8) by means of a *deterministic* fixed-point argument.

As in the previous section, and for the sake of clarity, let us separate the regular and rough situations in the presentation of our results.

The regular case

Let us first handle the regular case, where condition (**H1**) is satisfied and Definition 4.1.4 of a solution prevails. By relying on the most standard estimates associated with the Schrödinger group S (the so-called *Strichartz inequalities*, summed up in Lemma 4.3.3 below), we are here able to establish the following result:

Theorem 4.1.10 (**Local well-posedness under** (**H1**))**.** *Assume that* $1 \leq d \leq 4$ *and that condition* (**H1**) *is satisfied. Let* β *be such that* $0 < \beta < 2H_0 + \sum_{i=1}^{d} H_i - (d+1)$*, and consider the pair* (p,q) *given by the formulas*

$$p = \frac{12}{d-\beta}\ ,\ q = \frac{6d}{2d+\beta}\,.$$

Assume finally that $\phi \in H^\beta(\mathbb{R}^d)$*. In this setting, the following assertions hold true:*

(i) *Almost surely, there exists a time* $T_0 > 0$ *such that equation* (4.1.3) *admits a unique solution* u *(in the sense of Definition 4.1.4) in the set*

$$\mathcal{S}_{T_0} := ⫯ + X^\beta(T_0), \qquad \textit{where } X^\beta(T_0) := \mathcal{C}([0,T_0]; H^\beta(\mathbb{R}^d)) \cap L^p([0,T_0]; \mathcal{W}^{\beta,q}(\mathbb{R}^d)).$$

(ii) *For every* $n \geq 1$*, let* u_n *denote the* smooth *solution of* (4.1.3)*, that is* u_n *is the solution (in the sense of Definition 4.1.4) associated with* $\rho⫯_n$*. Then, for every*

$$0 < \beta < 2H_0 + \sum_{i=1}^{d} H_i - (d+1)$$

and for every test function $\chi : \mathbb{R}^d \to \mathbb{R}$*, the sequence* $(\chi u_n)_{n\geq 1}$ *converges almost surely in* $\mathcal{C}([0,T_0]; H^\beta(\mathbb{R}^d))$ *to* χu*, where* u *is the solution exhibited in item* (i)*.*

Let us be slightly more specific about the convergence property in item (ii). In fact, what we will establish in the sequel (see Theorem 4.3.4 below) is that for every element $\mathbf{\Psi^1}$ in a suitable space of functions, the equation obtained by replacing $\rho⫯$ with $\mathbf{\Psi^1}$ in (4.1.9) admits a unique solution v, on some time interval depending only on $\mathbf{\Psi^1}$. Besides, this solution is a continuous function of $\mathbf{\Psi^1}$ (see Proposition 4.3.5). Item (i) in the above statement is then an application of these general results to $\mathbf{\Psi^1} = \rho⫯$, while item (ii) corresponds to taking $\mathbf{\Psi^1} = \rho⫯_n$, which provides us with the desired time $T_0 > 0$.

The rough case

Let us conclude this presentation of our main results by considering the wellposedness issue for the equation in the rough case. To be more specific, we assume from now on that the assumptions in (**H2**) are satisfied, so that the two processes $\rho\Psi$ and $\rho^2\curlyvee$ are well defined and the equation can be understood in the sense of Definition 4.1.8. In other words, we now focus on the analysis of equation (4.1.16).

In order to describe the major technical difficulty arising in this case, recall first that under assumption (**H2**) and following Proposition 4.1.2, the element $\rho\Psi_\tau$ must here be treated as a distribution of negative Sobolev regularity (for every fixed time τ). Consequently, the term $(\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau)$ (or $(\rho v_\tau)\cdot(\overline{\rho\Psi_\tau})$) in (4.1.16) can only be understood as the product of a distribution, namely $\rho\Psi_\tau$, with a function, that is $\rho\overline{v}_\tau$. Such a product is known to obey the following simple rule (see Lemma 4.4.1 for a precise statement): if $\rho\Psi_\tau$ is of Sobolev regularity $-\alpha$ (with $\alpha>0$), then $\rho\overline{v}_\tau$ must be a function of Sobolev regularity β with $\beta>\alpha$, and in this case $(\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau)$ is indeed well defined as a distribution of negative order $-\alpha$.

Going back to equation (4.1.16), our only hope to settle a fixed-point argument thus lies in the possibility to control the terms

$$\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau))\,d\tau\ ,\ \int_0^t S_{t-\tau}((\rho v_\tau)\cdot(\rho\overline{\Psi}_\tau))\,d\tau$$

as functions of Sobolev regularity $\beta>\alpha>0$. Otherwise stated, we (morally) need convolution with S to produce a regularization effect and allow the transition from $H^{-\alpha}(\mathbb{R}^d)$ to $H^\beta(\mathbb{R}^d)$.

Unfortunately, such a regularization property, which corresponds to a well-controlled phenomenon in the heat or in the wave situation, is not as standard in the Schrödinger setting. In particular, the most classical estimates on S (the so-called Strichartz inequalities, which we have already mentioned in the regular situation) cannot provide us with the desired smoothing effect (see Lemma 4.3.3 for more details).

To overcome this obstacle, we propose to turn to more specific *local* regularization properties, similar to those exhibited by Constantin and Saut in [8]. It indeed appears that if we only focus on the local regularity of the distributions at stake (meaning that a cut-off function is inserted within the usual Sobolev norm, see (4.1.19)), then a small gain can be expected from the convolution with S. This will be the topic of our intermediate Lemma 4.4.3, which can be seen as an extension of the main result in [8].

For this technical property to be implemented here, an additional condition must be imposed on the function $\rho:\mathbb{R}^d\to\mathbb{R}$ in (4.1.3) (or in (4.1.16)): namely, we need ρ to be of the form

$$\rho(x_1,\ldots,x_d)=\rho_1(x_1)\cdots\rho_d(x_d) \tag{$\mathbf{F}_\rho$}$$

for smooth compactly-supported functions $\rho_1,\ldots,\rho_d$ on $\mathbb{R}$.

Note also that for stability reasons (toward a fixed-point argument), the consideration of a local Sobolev norm is of course not without consequences: it will have to be counterbalanced by a commutator-type estimate, that is a suitable control on switching ρ with the fractional Laplacian in Sobolev norms, which will be the purpose of Lemma 4.4.4.

With the above elements in mind, we are finally in a position to state our main result in the rough situation (the spaces involved in this statement will all be introduced into details in the subsequent Section 4.1.5):

Theorem 4.1.11 (**Local well-posedness under** (**H2'**))**.** *Assume that* $1 \leq d \leq 3$ *and that the cut-off function* $\rho : \mathbb{R}^d \to \mathbb{R}$ *in* (4.1.3) *is of the form* $(\mathbf{F}_\rho)$*. Besides, assume that*

$$-\,\alpha_d < 2H_0 + \sum_{i=1}^{d} H_i - (d+1) \leq 0\,, \quad \text{where} \quad \alpha_d := \begin{cases} \frac{3}{20} & \text{if } d=1 \\ \frac{1}{10} & \text{if } d=2 \\ \frac{1}{24} & \text{if } d=3 \end{cases}\,. \tag{H2'}$$

Fix $\alpha > 0$ *such that* $d+1-\Big(2H_0+\sum_{i=1}^d H_i\Big) < \alpha < \alpha_d$*. Then the following assertions hold true:*

(i) *One can find parameters* $\kappa \in [2\alpha, 1/2]$ *and* $p, q \geq 2$ *such that almost surely, and for every* $\phi \in H^{-2\alpha}(\mathbb{R}^d)$*, there exists a time* $T_0 > 0$ *for which equation* (4.1.3) *admits a unique Wick solution* u *(in the sense of Definition 4.1.8) in the set*

$$\mathcal{S}_{T_0} := \; + X_\rho^{\alpha,\kappa,(p,q)}(T_0)\,,$$

where

$$X_\rho^{\alpha,\kappa,(p,q)}(T) := \mathcal{C}([0,T]; H^{-2\alpha}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-2\alpha,q}(\mathbb{R}^d)) \cap L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}\,.$$

(ii) *For every* $n \geq 1$*, let* $\widetilde{u}_n$ *denote the* smooth *Wick solution of* (4.1.3)*, that is* $\widetilde{u}_n$ *is the solution (in the sense of Definition 4.1.8) associated with the pair* $(\rho\,{}_n, \rho^2\,{}_n)$*. Then, for every* α *satisfying* (4.1.11) *and every test function* $\chi : \mathbb{R}^d \to \mathbb{R}$*, the sequence* $(\chi \widetilde{u}_n)_{n\geq 1}$ *converges almost surely in* $\mathcal{C}([0,T_0]; H^{-2\alpha}(\mathbb{R}^d))$ *to* χu*, where* u *is the Wick solution exhibited in item* (i)*.*

The above wellposedness result can be considered as the main novelty of our work. We are indeed not aware of any previous study of a nonlinear Schrödinger equation involving such an irregular noise, and thus forcing us to rely on both renormalization arguments and local regularization properties.

Remark 4.1.12. It is interesting to note that the conditions in (**H2'**), which stem from the deterministic part of the study (as emphasized by Theorem 4.4.5 below), are more restrictive than those in (**H2**) ensuring the existence of the stochastic element . This observation contrasts with the situation described for instance in [20, 21, 27, 35], where the application of a similar strategy is, on the contrary, limited by the scope of validity of the stochastic objects.

Remark 4.1.13. Due to the limitations of the local regularization effect of the Schrödinger group (as reported in Lemma 4.4.3), further expansions of the strategy, in the spirit of the "second-order analysis" developed e.g. in [21, 35], seem difficult to set up in this Schrödinger setting.

Remark 4.1.14. As can be seen from the above description of our methodology, the cut-off function ρ in (4.1.3) is to play two fundamental roles in the study:

• First, it will allow us to bring the analysis of the equation back to compact domains, where the regularity of the driving processes is well controlled (by Propositions 4.1.2 and 4.1.6). The situation, in this regard, is somehow equivalent to studying equation (4.1.3) on a torus (although the definition of the space-time fractional noise on a torus is not as clear as in the current Euclidean setting).

• Secondly, thanks to the involvement of ρ, we can appeal to the specific local regularization properties of S, which, as we have explained it above, will be our key ingredient toward stability and fixed-point arguments. Observe that no similar regularization result would be available for a study of the equation on a torus.

This being said, in spite of the restriction ($\mathbf{F}_\rho$), the function ρ can still be taken equal to 1 on any arbitrary compact domain, and so, at least locally (in time and in space), our solution u of (4.1.3) can be regarded as a viable model for the (formal) dynamics

$$i\partial_t u - \Delta u = |u|^2 + \dot{B}\,. \tag{4.1.17}$$

A direct analysis of equation (4.1.17) may be possible through an extension of the subsequent methods to weighted Sobolev spaces (allowing to control the asymptotic behaviour of the processes), but such adaptations are clearly beyond our reach for the moment.

Remark 4.1.15. Our arguments and results could certainly be extended to the nonlinearity $\rho^2 u^2$ or $\rho^2 \overline{u}^2$ (instead of $\rho^2|u|^2$) through minor modifications of the stochastic constructions of Section 4.2 (the deterministic well-posedness procedure would then clearly follow the same lines). Besides, the considerations in our study may be seen as a first step towards the treatment of more general power nonlinearities of the form $\rho^{i+2j} u^i |u|^{2j}$ in (4.1.1) (instead of $\rho^2|u|^2$). This would include in particular the celebrated cubic model $u|u|^2$, with its many physical significances. Nevertheless, it is clear to us that any such extension would be the source of new technical difficulties at every step of the subsequent analysis (stochastic constructions, fixed-point argument,...). Therefore, at this point, we are not able to provide any specific conjecture in this direction.

Remark 4.1.16. In their recent work [18], Deng, Nahmod and Yue introduced the fundamental notion of *random tensors*, presented as a dispersive counterpart of the new parabolic theories (regularity structures, paracontrolled approach and renormalization group techniques). The applications of this promising approach to stochastic PDEs are so far limited to the treatment of irregular random initial conditions within deterministic dynamics. However, it is reasonable to expect that this sophisticated machinery could be extended to dispersive PDEs with additive stochastic noise, such as our model (4.1.3), and it may offer a way to handle rougher situations regarding $\dot{B}$ (just as the paracontrolled method in [28] allow to cover a space-time white noise situation in dimension 3). We hope to get the opportunity to investigate these questions in a future work.

4.1.5 Notations

Fix a (space) dimension $d \geq 1$. Throughout the chapter, we will call a *test function* (on $\mathbb{R}^d$) any function $\rho : \mathbb{R}^d \to \mathbb{R}$ that is smooth and compactly-supported. Besides, we denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions on $\mathbb{R}^d$.

We will also refer to the scale of Sobolev spaces defined for all $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ as

$$\mathcal{W}^{s,p}(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{\mathcal{W}^{s,p}} = \|\mathcal{F}^{-1}(\{1+|.|^2\}^{\frac{s}{2}}\mathcal{F}f)|L^p(\mathbb{R}^d)\| < \infty \Big\} ,$$

where the Fourier transform $\mathcal{F}$, resp. the inverse Fourier transform $\mathcal{F}^{-1}$, is defined along the convention: for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\mathcal{F}(f)(x) = \hat{f}(x) := \int_{\mathbb{R}^d} f(y) e^{-i\langle x,y\rangle} dy , \quad \text{resp. } \mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(y) e^{i\langle x,y\rangle} dy .$$

We then set $H^s(\mathbb{R}^d) := \mathcal{W}^{s,2}(\mathbb{R}^d)$.

Now, as far as space-time functions (or distributions) are concerned, and for the sake of clarity, we will occasionally use the following shortcut notation: for all $T \geq 0$, $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$,

$$L^p_T \mathcal{W}^{s,q} := L^p([0,T]; \mathcal{W}^{s,q}(\mathbb{R}^d)) , \qquad \|.\|_{L^p_T \mathcal{W}^{s,q}} := \|.\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))} . \tag{4.1.18}$$

The notation $\mathcal{C}([0,T]; \mathcal{W}^{s,q}(\mathbb{R}^d))$ will refer to the set of continuous functions on $[0,T]$ with values in $\mathcal{W}^{s,q}(\mathbb{R}^d)$.

Let us finally introduce the aforementioned *local* Sobolev spaces, that will play a prominent role in the analysis of the rough situation. Namely, for all test function $\rho : \mathbb{R}^d \to \mathbb{R}$ and $s \in \mathbb{R}$, we set

$$H^s_\rho(\mathbb{R}^d) := \{ v \in \mathcal{S}'(\mathbb{R}^d);\ \|\rho \cdot (\mathrm{Id} - \Delta)^{\frac{s}{2}}(v)\|_{L^2(\mathbb{R}^d)} < \infty \} , \tag{4.1.19}$$

where the operator $(\mathrm{Id} - \Delta)^{\frac{s}{2}}$ is understood (as usual) through the Fourier transform formula

$$\mathcal{F}\Big((\mathrm{Id} - \Delta)^{\frac{s}{2}}(v) \Big)(\xi) := \{1 + |\xi|^2\}^{\frac{s}{2}} \mathcal{F}v(\xi) .$$

We endow $H^s_\rho(\mathbb{R}^d)$ with the natural seminorm $\|v\|_{H^s_\rho} := \|\rho \cdot (\mathrm{Id} - \Delta)^{\frac{s}{2}}(v)\|_{L^2(\mathbb{R}^d)}$, and finally set, along the convention in (4.1.18),

$$L^p_T H^s_\rho := L^p([0,T]; H^s_\rho(\mathbb{R}^d)) , \qquad \|.\|_{L^p_T H^s_\rho} := \|.\|_{L^p([0,T];H^s_\rho(\mathbb{R}^d))} .$$

4.1.6 Outline of the study

The rest of the chapter is organized as follows. In Section 4.2 we perform the stochastic constructions which allow to give a sense to the equation. In Section 4.3 we prove the local well-posedness result in the regular case, while Section 4.4 is devoted to the analysis in the irregular case. Finally, Section 4.5 is an appendix in which we gather the proofs of some technical results.

Throughout the chapter, and for the sake of clarity, we will use the notation $A \lesssim B$ in order to signify that there exists an irrelevant constant c such that $A \leq cB$.

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4.2 Stochastic constructions

As emphasized in Section 4.1, our analysis of equation (4.1.3) will be clearly splitted into a stochastic part (essentially corresponding to the construction of and) and a deterministic part (devoted to the fixed-point procedure).

In this section, we propose to deal with the stochastic step of the study. In other words, we focus here on the proofs of Proposition 4.1.2, Proposition 4.1.5 and Proposition 4.1.6.

Before we go into the details, and for the sake of clarity, let us recall the specific definition of the process at the core of the model, that is the space-time fractional Brownian motion:

Definition 4.2.1. *Let $d \geq 1$ be a space dimension, $T \geq 0$ a positive time and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete filtered probability space. On this space, and for every fixed $H = (H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$, we call a space-time fractional Brownian motion of Hurst index H any centered Gaussian process $B : \Omega \times ([0,T] \times \mathbb{R}^d) \to \mathbb{R}$ with covariance function given by the formula:*

$$\mathbb{E}\Big[B(s, x_1, \ldots, x_d)B(t, y_1, \ldots, y_d)\Big] = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i)\,, \quad s,t \in [0,T]\,,\ x, y \in \mathbb{R}^d\,,$$

where

$$R_{H_i}(x,y) := \frac{1}{2}(|x|^{2H_i} + |y|^{2H_i} - |x-y|^{2H_i})\ .$$

Now remember that our strategy to initiate the construction of both and consists in the introduction of a smooth approximation $(B_n)_{n\geq 0}$ of B (leading immediately to a smooth approximation $(\dot{B}_n)_{n\geq 0}$ of the noise $\dot{B}$). We will rely here on a sequence derived from the so-called harmonizable representation of B, and which happens to be especially suited for Fourier analysis and computations in Sobolev spaces (a similar choice is made in [20], for the same reasons).

Along this idea, let us first introduce, on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a space-time white noise W on $\mathbb{R}^{d+1}$. Then fix $H = (H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$ and set, for all $t \in [0,T]$ and $x \in \mathbb{R}^d$,

$$B(t, x_1, \ldots, x_d) := c \int_{\xi \in \mathbb{R}} \int_{\eta \in \mathbb{R}^d} \frac{e^{\imath t\xi} - 1}{|\xi|^{H_0 + \frac{1}{2}}} \prod_{i=1}^{d} \frac{e^{\imath x_i \eta_i} - 1}{|\eta_i|^{H_i + \frac{1}{2}}}\, \widehat{W}(d\xi, d\eta)\,, \qquad (4.2.1)$$

for some constant c, and where $\widehat{W}$ stands for the Fourier transform of W.

It is a well-established fact (see e.g. [40]) that for some appropriate value $c = c_H$ of the constant, the so-defined process B is a space-time fractional Brownian motion of index H (in the sense of Definition 4.2.1). Our approximation of B (for every fixed $H = (H_0, H_1, \dots, H_d) \in (0,1)^{d+1}$) is now obtained through a basic truncation of the integration domain in (4.2.1): namely, we set $B_0 \equiv 0$, and for $n \geq 1$,

$$B_n(t, x_1, \dots, x_d)(\omega) := c_H \int_{|\xi| \leq 2^{2n}} \int_{|\eta| \leq 2^n} \frac{e^{it\xi} - 1}{|\xi|^{H_0 + \frac{1}{2}}} \prod_{i=1}^{d} \frac{e^{ix_i\eta_i} - 1}{|\eta_i|^{H_i + \frac{1}{2}}} \widehat{W}(d\xi, d\eta) \,. \tag{4.2.2}$$

By a quick examination of the possible singularities in (ξ, η), it is not hard to see that, owing to the restricted integration domain, B_n indeed defines a smooth process, for every fixed $n \geq 0$.

Remark 4.2.2. The choice of the scaling $\{|\xi| \leq 2^{2n}, |\eta| \leq 2^n\}$ in (4.2.2) is directly related to the structure of the Schrödinger operator, and will prove to be essential in the estimation of the renormalization constant (see Section 4.2.4). This choice naturally contrasts with the "hyperbolic" scaling used in [20] for the corresponding wave model (see also Remark 4.1.3). Note also that the approximation (4.2.2) is the same as the one used in [19] for the study of a (rough) parabolic model.

With approximation $(B_n)_{n \geq 0}$ in hand, we now would like to consider the sequence $(\Psi_n)_{n \geq 0}$ of approximated linear solutions, that is the sequence of solutions to

$$\begin{cases} i\partial_t \Psi_n - \Delta \Psi_n = \dot{B}_n \,, \quad t \in [0,T] \,, \;\; x \in \mathbb{R}^d \,, \\ \Psi_n(0,.) = 0 \,, \end{cases} \tag{4.2.3}$$

where, for every $n \geq 0$, $\dot{B}_n$ is defined as the standard derivative $\dot{B}_n := \partial_t \partial_{x_1} \cdots \partial_{x_d} B_n$.

Note however that, without further integrability assumptions, the smoothness of B_n (for each fixed $n \geq 0$) is not a sufficient condition to apply classical deterministic results immediately ensuring existence and uniqueness of Ψ_n. A possible way to circumvent the problem in this case is to rely on some stochastic interpretation of Ψ_n, based on the Gaussianity of the processes under consideration.

In order to justify this interpretation (that is, Definition 4.2.3 below), let us go back to formula (4.2.2) for B_n. Denoting by S the d-dimensional Schrödinger group and applying a Fubini-type theorem, the solution Ψ_n can (at least formally) be written as

$$\begin{aligned}
\Psi_n(t,x) &= -i \int_0^t ds \int_{\mathbb{R}^d} dy \, S_{t-s}(x-y) \dot{B}_n(s,y) \\
&= c_H i^d \int_0^t ds \int_{\mathbb{R}^d} dy \int_{|\xi| \leq 2^{2n}} \int_{|\eta| \leq 2^n} S_{t-s}(x-y) \frac{\xi}{|\xi|^{H_0 + \frac{1}{2}}} \prod_{i=1}^{d} \frac{\eta_i}{|\eta_i|^{H_i + \frac{1}{2}}} e^{i\xi s} e^{i\langle \eta, y \rangle} \widehat{W}(d\xi, d\eta) \\
&= c_H i^d \int_{|\xi| \leq 2^{2n}} \int_{|\eta| \leq 2^n} \frac{\xi}{|\xi|^{H_0 + \frac{1}{2}}} \prod_{i=1}^{d} \frac{\eta_i}{|\eta_i|^{H_i + \frac{1}{2}}} e^{i\langle \eta, x \rangle} \left[\int_0^t ds \, e^{i\xi s} \left(\int_{\mathbb{R}^d} dy \, S_{t-s}(x-y) e^{-i\langle \eta, x-y \rangle} \right) \right] \widehat{W}(d\xi, d\eta) \\
&= c_H i^d \int_{|\xi| \leq 2^{2n}} \int_{|\eta| \leq 2^n} \frac{\xi}{|\xi|^{H_0 + \frac{1}{2}}} \prod_{i=1}^{d} \frac{\eta_i}{|\eta_i|^{H_i + \frac{1}{2}}} e^{i\langle \eta, x \rangle} \left[\int_0^t ds \, e^{i\xi(t-s)} \left(\int_{\mathbb{R}^d} dy \, S_s(y) e^{-i\langle \eta, y \rangle} \right) \right] \widehat{W}(d\xi, d\eta) \,.
\end{aligned}$$

At this point, remember that the spatial Fourier transform of S is explicitly given by

$$\int_{\mathbb{R}^d} dx\, e^{-i\langle \xi, x\rangle} S_t(x) = e^{it|\xi|^2}\,,$$

and so we end up with the (a priori formal) representation

$$\text{⚲}_n(t,x) = c_H i^d \int_{|\xi|\le 2^{2n}} \int_{|\eta|\le 2^n} \frac{\xi}{|\xi|^{H_0+\frac{1}{2}}} \prod_{i=1}^{d} \frac{\eta_i}{|\eta_i|^{H_i+\frac{1}{2}}} e^{i\langle \eta, x\rangle} \gamma_t(\xi,|\eta|)\, \widehat{W}(d\xi,d\eta)\,, \tag{4.2.4}$$

where for all $t \geq 0$, $\xi \in \mathbb{R}$ and $r > 0$, we define the quantity $\gamma_t(\xi, r)$ as

$$\gamma_t(\xi,r) := e^{i\xi t} \int_0^t e^{is\{r^2-\xi\}} ds\,. \tag{4.2.5}$$

Now, let $f_{t,x,n}$ the function defined for all $\xi \in \mathbb{R}, \eta \in \mathbb{R}^d$ by

$$f_{t,x,n}(\xi,\eta) := c_H i^{d+1} \mathbb{1}_{|\xi|\le 2^{2n}} \mathbb{1}_{|\eta|\le 2^n} \frac{\xi}{|\xi|^{H_0+\frac{1}{2}}} \prod_{i=1}^{d} \frac{\eta_i}{|\eta_i|^{H_i+\frac{1}{2}}} e^{i\langle \eta, x\rangle} \gamma_t(\xi,|\eta|),$$

in such a way that $\text{⚲}_n(t,x) = \int_{\mathbb{R}^{d+1}} f_{t,x,n}(\xi,\eta)\widehat{W}(d\xi,d\eta)$.
On the one hand, resorting to Parseval's identity, we write:

$$\begin{aligned}
\mathbb{E}\Big[\text{⚲}_n(s,x)\overline{\text{⚲}_m(t,y)}\Big] = &= \mathbb{E}\Big[\int_{\mathbb{R}^{d+1}} f_{s,x,n}(\xi,\eta)\widehat{W}(d\xi,d\eta)\overline{\int_{\mathbb{R}^{d+1}} f_{t,y,m}(\xi,\eta)\widehat{W}(d\xi,d\eta)}\Big]\\
&= \mathbb{E}\Big[\int_{\mathbb{R}^{d+1}} \widehat{f_{s,x,n}}(\xi,\eta)W(d\xi,d\eta)\overline{\int_{\mathbb{R}^{d+1}} \widehat{f_{t,y,m}}(\xi,\eta)W(d\xi,d\eta)}\Big]\\
&= \mathbb{E}\Big[\int_{\mathbb{R}^{d+1}} \widehat{f_{s,x,n}}(\xi,\eta)W(d\xi,d\eta)\int_{\mathbb{R}^{d+1}} \overline{\widehat{f_{t,y,m}}(\xi,\eta)}W(d\xi,d\eta)\Big]\\
&= \int_{\mathbb{R}^{d+1}} \widehat{f_{s,x,n}}(\xi,\eta)\overline{\widehat{f_{t,y,m}}(\xi,\eta)}d\xi d\eta\\
&= \int_{\mathbb{R}^{d+1}} f_{s,x,n}(\xi,\eta)\overline{f_{t,y,m}(\xi,\eta)}d\xi d\eta\\
&= c_H^2 \int_{(\xi,\eta)\in D_n\cap D_m} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \gamma_s(\xi,|\eta|)\overline{\gamma_t(\xi,|\eta|)} e^{i\langle \eta, x-y\rangle}\, d\xi d\eta,
\end{aligned}$$

whereas, on the other hand, denoting $\tilde{f}_{t,y,m}(\xi,\eta) := \overline{f_{t,y,m}(-\xi,-\eta)}$, and with the help of

Parseval's identity again, we derive

$$
\begin{aligned}
\mathbb{E}\Big[\mathsf{I}_n(s,x)\mathsf{I}_m(t,y)\Big] = \;&= \mathbb{E}\Big[\int_{\mathbb{R}^{d+1}} f_{s,x,n}(\xi,\eta)\widehat{W}(d\xi,d\eta)\int_{\mathbb{R}^{d+1}} f_{t,y,m}(\xi,\eta)\widehat{W}(d\xi,d\eta)\Big]\\
&= \mathbb{E}\Big[\int_{\mathbb{R}^{d+1}} \widehat{f_{s,x,n}}(\xi,\eta)W(d\xi,d\eta)\int_{\mathbb{R}^{d+1}} \widehat{f_{t,y,m}}(\xi,\eta)W(d\xi,d\eta)\Big]\\
&= \int_{\mathbb{R}^{d+1}} \widehat{f_{s,x,n}}(\xi,\eta)\widehat{f_{t,y,m}}(\xi,\eta)d\xi d\eta\\
&= \int_{\mathbb{R}^{d+1}} \widehat{f_{s,x,n}}(\xi,\eta)\overline{\overline{\widehat{f_{t,y,m}}(\xi,\eta)}}d\xi d\eta\\
&= \int_{\mathbb{R}^{d+1}} \widehat{f_{s,x,n}}(\xi,\eta)\overline{\widehat{\tilde{f}_{t,y,m}}(\xi,\eta)}d\xi d\eta\\
&= \int_{\mathbb{R}^{d+1}} f_{s,x,n}(\xi,\eta)\overline{\tilde{f}_{t,y,m}(\xi,\eta)}d\xi d\eta\\
&= -c_H^2\int_{(\xi,\eta)\in D_n\cap D_m}\frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^d\frac{1}{|\eta_i|^{2H_i-1}}\gamma_s(\xi,|\eta|)\gamma_t(-\xi,|\eta|)e^{i\langle\eta,x-y\rangle}\,d\xi d\eta.
\end{aligned}
$$

Thanks to formula (4.2.4) and to the two previous computations, we are in a position to offer the following natural (and rigorous) stochastic definition for the solution of (4.2.3):

Definition 4.2.3. *We call a solution of equation* (4.2.3) *(or* linear solution associated with (4.1.3)*) any centered complex Gaussian process*

$$\Big\{\mathsf{I}_n(s,x),\ n\geq 1,\ s\geq 0,\ x\in\mathbb{R}^d\Big\}$$

whose covariance is given by the relations: for all $n,m\geq 1,\ s,t\geq 0$ *and* $x,y\in\mathbb{R}^d$,

$$\mathbb{E}\Big[\mathsf{I}_n(s,x)\overline{\mathsf{I}_m(t,y)}\Big] = c_H^2\int_{(\xi,\eta)\in D_n\cap D_m}\frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^d\frac{1}{|\eta_i|^{2H_i-1}}\gamma_s(\xi,|\eta|)\overline{\gamma_t(\xi,|\eta|)}e^{i\langle\eta,x-y\rangle}\,d\xi d\eta\,, \tag{4.2.6}$$

$$\mathbb{E}\Big[\mathsf{I}_n(s,x)\mathsf{I}_m(t,y)\Big] = -c_H^2\int_{(\xi,\eta)\in D_n\cap D_m}\frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^d\frac{1}{|\eta_i|^{2H_i-1}}\gamma_s(\xi,|\eta|)\gamma_t(-\xi,|\eta|)e^{i\langle\eta,x-y\rangle}\,d\xi d\eta\,, \tag{4.2.7}$$

where $D_n := B^1_{2n}\times B^d_n$ *with* $B^k_\ell := \Big\{\lambda\in\mathbb{R}^k : |\lambda|\leq 2^\ell\Big\}$.

4.2.1 Preliminary estimates

As a first step toward Proposition 4.1.2 and Proposition 4.1.6, let us collect some essential estimates on the quantity $\gamma_t(\xi,r)$ at the core of the covariance formula (4.2.6) (and explicitly defined by (4.2.5)).

To this end, we set, for all $0\leq s\leq t$, $\xi\in\mathbb{R}$ and $r>0$,

$$\gamma_{s,t}(\xi,r) := \gamma_t(\xi,r)-\gamma_s(\xi,r)\,.$$

Lemma 4.2.4. *For all $T \geq 0$, $0 \leq s \leq t \leq T$, $\xi \in \mathbb{R}$, $r > 0$ and $\kappa, \lambda \in [0,1]$, it holds that*

$$|\gamma_{s,t}(\xi,r)| \lesssim \min\left(|\xi|^\kappa|t-s|^\kappa + |t-s|, \frac{|t-s|r^2}{|\xi|} + \frac{|t-s|^\kappa\{1+r^2\}}{|\xi|^{1-\kappa}}, \frac{|t-s|^\kappa\{r^{2\kappa}+|\xi|^\kappa\}}{||\xi|-r^2|^{1-\lambda(1-\kappa)}}\right),$$

where the proportional constant in $\lesssim$ only depends on T.

Proof. To begin with, let us write

$$\gamma_{s,t}(\xi,r) = \{e^{i\xi t} - e^{i\xi s}\}\int_0^s e^{iu\{r^2-\xi\}}du + e^{i\xi t}\int_s^t e^{iu\{r^2-\xi\}}du\,,$$

and so

$$|\gamma_{s,t}(\xi,r)| \lesssim |e^{i\xi t} - e^{i\xi s}|\left|\int_0^s e^{iu\{r^2-\xi\}}du\right| + \left|\int_s^t e^{iu\{r^2-\xi\}}du\right| \lesssim |\xi|^\kappa|t-s|^\kappa + |t-s|\,.$$

Then observe that

$$\gamma_t(\xi,r) = e^{i\xi t}\int_0^t e^{is\{r^2-\xi\}}ds = -\frac{e^{ir^2t}-e^{i\xi t}}{i\xi} + \frac{e^{i\xi t}r^2}{\xi}\int_0^t e^{is\{r^2-\xi\}}ds\,,$$

which readily entails

$$\begin{aligned}&\gamma_{s,t}(\xi,r)\\ &= -\frac{\{e^{ir^2t}-e^{ir^2s}\} - \{e^{i\xi t}-e^{i\xi s}\}}{i\xi} + \frac{r^2}{\xi}\{e^{i\xi t}-e^{i\xi s}\}\int_0^s e^{iu\{r^2-\xi\}}du + \frac{e^{i\xi t}r^2}{\xi}\int_s^t e^{iu\{r^2-\xi\}}du\,.\end{aligned}$$

Thus,

$$\begin{aligned}|\gamma_{s,t}(\xi,r)| &\lesssim r^2\frac{|t-s|}{|\xi|} + \frac{|t-s|^\kappa}{|\xi|^{1-\kappa}} + r^2\frac{|t-s|^\kappa}{|\xi|^{1-\kappa}} + \frac{r^2}{|\xi|}|t-s|\\ &\lesssim r^2\frac{|t-s|}{|\xi|} + \{1+r^2\}\frac{|t-s|^\kappa}{|\xi|^{1-\kappa}}\,.\end{aligned}$$

Finally, it can be checked that

$$\gamma_t(\xi,r) = \frac{i}{r^2-\xi}\{e^{i\xi t} - e^{ir^2t}\}\,,$$

which yields

$$\begin{aligned}|\gamma_{s,t}(\xi,r)| &= \frac{1}{|\xi-r^2|}\left|\{e^{ir^2t}-e^{ir^2s}\} - \{e^{i\xi t}-e^{i\xi s}\}\right|^\kappa\left|\{e^{ir^2t}-e^{i\xi t}\} - \{e^{ir^2s}-e^{i\xi s}\}\right|^{1-\kappa}\\ &\lesssim \frac{1}{|\xi-r^2|}\{|e^{ir^2t}-e^{ir^2s}| + |e^{i\xi t}-e^{i\xi s}|\}^\kappa\{|e^{ir^2t}-e^{i\xi t}| + |e^{ir^2s}-e^{i\xi s}|\}^{1-\kappa}\\ &\lesssim \frac{|t-s|^\kappa}{|\xi-r^2|}\{r^2+|\xi|\}^\kappa\{|e^{ir^2t}-e^{i\xi t}|^\lambda + |e^{ir^2s}-e^{i\xi s}|^\lambda\}^{1-\kappa}\\ &\lesssim \frac{|t-s|^\kappa|\xi-r^2|^{\lambda(1-\kappa)}}{|\xi-r^2|}\{r^2+|\xi|\}^\kappa\{t^\lambda+s^\lambda\}^{1-\kappa}\\ &\lesssim \frac{|t-s|^\kappa}{|\xi-r^2|^{1-\lambda(1-\kappa)}}\{r^{2\kappa}+|\xi|^\kappa\}\\ &\lesssim \frac{|t-s|^\kappa}{||\xi|-r^2|^{1-\lambda(1-\kappa)}}\{r^{2\kappa}+|\xi|^\kappa\}\,.\end{aligned}$$

□

Corollary 4.2.5. *For all $T \geq 0$, $0 \leq s \leq t \leq T$, $H \in (0,1)$, $r > 0$, $\varepsilon \in (0,1)$ and $\kappa \in [0, \min(H, \frac{1-\varepsilon}{2}))$, it holds that*

$$\int_{\mathbb{R}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi \lesssim \frac{|t-s|^{2\kappa}}{1+r^{4(H-\kappa)-2-2\varepsilon}}\,,$$

where the proportional constant in $\lesssim$ only depends on T.

Proof. We will naturally lean on the estimates exhibited in Lemma 4.2.4.

For $0 < r < 1$, we have

$$\int_{\mathbb{R}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi \lesssim |t-s|^{2\kappa}\left[\int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{2H-1}} + \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{2(H-\kappa)+1}}\right] \lesssim |t-s|^{2\kappa}\,.$$

Then, for $r > 1$, let us consider the decomposition

$$\int_{\mathbb{R}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi = \int_{||\xi|-r^2|\geq \frac{|\xi|}{2}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi + \int_{||\xi|-r^2|\leq \frac{|\xi|}{2}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi\,.$$

On the one hand, it holds that

$$\begin{aligned}\int_{||\xi|-r^2|\geq \frac{|\xi|}{2}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}}\, d\xi &\lesssim |t-s|^{2\kappa} \int_{\left\{|\xi|\leq \frac{2}{3}r^2\right\}\cup\{|\xi|\geq 2r^2\}} \frac{r^{4\kappa}+|\xi|^{2\kappa}}{|\xi|^{2H-1}||\xi|-r^2|^2}\, d\xi \\ &\lesssim \frac{|t-s|^{2\kappa}}{r^{4(H-\kappa)}} \int_{\left\{|\xi|\leq \frac{2}{3}\right\}\cup\{|\xi|\geq 2\}} \frac{1+|\xi|^{2\kappa}}{|\xi|^{2H-1}||\xi|-1|^2}\, d\xi \lesssim \frac{|t-s|^{2\kappa}}{r^{4(H-\kappa)}}\,.\end{aligned}$$

On the other hand, setting $\lambda = \frac{1+\varepsilon}{2(1-\kappa)}$, one has

$$\begin{aligned}\int_{||\xi|-r^2|\leq \frac{|\xi|}{2}} \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H-1}} d\xi &\lesssim |t-s|^{2\kappa} \int_{\left\{\frac{2}{3}r^2\leq|\xi|\leq 2r^2\right\}} \frac{r^{4\kappa}+|\xi|^{2\kappa}}{|\xi|^{2H-1}||\xi|-r^2|^{2-2\lambda(1-\kappa)}}\, d\xi \\ &\lesssim \frac{|t-s|^{2\kappa}}{r^{4(H-\kappa)-4\lambda(1-\kappa)}} \int_{\left\{\frac{2}{3}\leq|\xi|\leq 2\right\}} \frac{1}{|\xi|^{2H-1}||\xi|-1|^{2-2\lambda(1-\kappa)}}\, d\xi \\ &\lesssim \frac{|t-s|^{2\kappa}}{r^{4(H-\kappa)-2-2\varepsilon}}\,.\end{aligned}$$

□

Let us also take advantage of this preliminary section to introduce the following lemma, which accounts for the technical simplifications offered by the test function χ in the forthcoming computations.

Lemma 4.2.6. *Let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a test function and fix $\sigma \in \mathbb{R}$. Then, for every $p \geq 1$ and for all $\eta_1, \ldots, \eta_p \in \mathbb{R}^d$, it holds that*

$$\left|\int_{\mathbb{R}^d} dx \prod_{i=1}^{p} \int_{(\mathbb{R}^d)^2} \frac{d\lambda_i d\tilde{\lambda}_i}{\{1+|\lambda_i|^2\}^{\frac{\sigma}{2}}\{1+|\tilde{\lambda}_i|^2\}^{\frac{\sigma}{2}}} e^{i\langle x,\lambda_i-\tilde{\lambda}_i\rangle}\widehat{\chi}(\lambda_i-\eta_i)\overline{\widehat{\chi}(\tilde{\lambda}_i-\eta_i)}\right| \lesssim \prod_{i=1}^{p} \frac{1}{\{1+|\eta_i|^2\}^{\sigma}}\,,$$

where the proportional constant only depends on χ and σ.

Proof. Let us set, for all $\eta, \lambda \in \mathbb{R}^d$,

$$\Gamma_\eta(\lambda) := \widehat{\chi}(\lambda - \eta)\frac{1}{\{1+|\lambda|^2\}^{\frac{\sigma}{2}}}$$

and observe that the integral under consideration can then be written as

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx \prod_{i=1}^p \int_{(\mathbb{R}^d)^2} \frac{d\lambda_i d\tilde{\lambda}_i}{\{1+|\lambda_i|^2\}^{\frac{\sigma}{2}}\{1+|\tilde{\lambda}_i|^2\}^{\frac{\sigma}{2}}} e^{i\langle x, \lambda_i - \tilde{\lambda}_i\rangle}\widehat{\chi}(\lambda_i - \eta_i)\overline{\widehat{\chi}(\tilde{\lambda}_i - \eta_i)} \\
&= c\int_{\mathbb{R}^d} dx \prod_{i=1}^p \left|\mathcal{F}^{-1}\Big(\Gamma_{\eta_i}\Big)(x)\right|^2 \\
&= c\int_{\mathbb{R}^d} dx \left|\mathcal{F}^{-1}\Big(\Gamma_{\eta_1} * \cdots * \Gamma_{\eta_p}\Big)(x)\right|^2 \\
&\lesssim \left\|\Gamma_{\eta_1} * \cdots * \Gamma_{\eta_p}\right\|_{L^2(\mathbb{R}^d)}^2 \lesssim \left\|\Gamma_{\eta_1}\right\|_{L^1(\mathbb{R}^d)}^2 \cdots \left\|\Gamma_{\eta_{p-1}}\right\|_{L^1(\mathbb{R}^d)}^2 \left\|\Gamma_{\eta_p}\right\|_{L^2(\mathbb{R}^d)}^2 , \qquad (4.2.8)
\end{aligned}$$

where the last inequality is derived from Plancherel theorem.

The conclusion now comes from the fact that for all $\eta, \lambda \in \mathbb{R}^d$ and for every $\kappa > 0$,

$$\left|\widehat{\chi}(\lambda)\{1+|\lambda+\eta|^2\}^{-\frac{\sigma}{2}}\right| \leq c_{\sigma,\chi,\kappa}\{1+|\lambda|^2\}^{-\kappa}\{1+|\eta|^2\}^{-\frac{\sigma}{2}} . \qquad (4.2.9)$$

Let us briefly verify (4.2.9) for $\sigma \geq 0$ (the proof for $\sigma < 0$ being immediate). In fact, since χ is smooth and compactly-supported, one has

$$\begin{aligned}
&\left|\hat{\chi}(\lambda)\{1+|\lambda+\eta|^2\}^{-\frac{\sigma}{2}}\right| \\
&= \left|\hat{\chi}(\lambda)\{1+|\lambda+\eta|^2\}^{-\frac{\sigma}{2}}\right|\mathbf{1}_{\{|\lambda|\geq\frac{1}{2}|\eta|\}} + \left|\hat{\chi}(\lambda)\{1+|\lambda+\eta|^2\}^{-\frac{\sigma}{2}}\right|\mathbf{1}_{\{|\lambda|<\frac{1}{2}|\eta|\}} \\
&\leq c_\sigma\Big[|\hat{\chi}(\lambda)|\mathbf{1}_{\{|\lambda|\geq\frac{1}{2}|\eta|\}} + |\hat{\chi}(\lambda)|\{1+|\eta|^2\}^{-\frac{\sigma}{2}}\mathbf{1}_{\{|\lambda|<\frac{1}{2}|\eta|\}}\Big] \\
&\leq c_{\sigma,\chi,\kappa}\Big[\{1+|\lambda|^2\}^{-\kappa}\{1+|\lambda|^2\}^{-\frac{\sigma}{2}}\mathbf{1}_{\{|\lambda|\geq\frac{1}{2}|\eta|\}} + \{1+|\lambda|^2\}^{-\kappa}\{1+|\eta|^2\}^{-\frac{\sigma}{2}}\mathbf{1}_{\{|\lambda|<\frac{1}{2}|\eta|\}}\Big] \\
&\leq c_{\sigma,\chi,\kappa}\{1+|\lambda|^2\}^{-\kappa}\{1+|\eta|^2\}^{-\frac{\sigma}{2}} .
\end{aligned}$$

Going back to (4.2.8), and with estimate (4.2.9) in hand, we can check for instance that

$$\left\|\Gamma_{\eta_1}\right\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} d\lambda \left|\widehat{\chi}(\lambda)\right|\frac{1}{\{1+|\lambda+\eta_1|^2\}^{\frac{\sigma}{2}}} \leq \frac{c_{\sigma,\chi}}{\{1+|\eta_1|^2\}^{\frac{\sigma}{2}}} ,$$

which yields the desired bound. □

4.2.2 Proof of Proposition 4.1.2

For simplicity, we will assume for the whole proof that $T = 1$ and we set, for all $m, n \geq 0$, $\text{⚲}_{n,m} := \text{⚲}_m - \text{⚲}_n$. Besides, along the statement of the proposition, we fix α satisfying (4.1.11).

Step 1: Let us show that for all $p \geq 1$, $1 \leq n \leq m$, $0 \leq s \leq t \leq 1$ and $\varepsilon, \kappa > 0$ small enough, one has

$$\int_{\mathbb{R}^d} \mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\Big|^{2p}\Big]\,dx \lesssim 2^{-4n\varepsilon p}|t-s|^{2\kappa p}\,, \qquad (4.2.10)$$

where the proportional constant only depends on p, α and χ.

First, we can observe that the random variable under consideration is Gaussian, and so, for every $p \geq 1$, one has

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\Big|^{2p}\Big]\\
&\leq c_p\,\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\Big|^2\Big]^p\,, \qquad (4.2.11)
\end{aligned}$$

where the constant c_p only depends on p.

Let us then expand this variable as

$$\begin{aligned}
&\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\\
&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\lambda\, e^{i\langle x,\lambda\rangle}\{1+|\lambda|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)(\lambda)\\
&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\lambda\,\{1+|\lambda|^2\}^{-\frac{\alpha}{2}}e^{i\langle x,\lambda\rangle}\left(\int_{\mathbb{R}^d} d\beta\,\widehat{\chi}(\lambda-\beta)\mathcal{F}\Big(\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)(\beta)\right),
\end{aligned}$$

so that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}}\mathcal{F}\Big(\chi\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\Big|^2\Big]\\
&=\frac{1}{(2\pi)^{2d}}\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\frac{\alpha}{2}}\{1+|\tilde{\lambda}|^2\}^{\frac{\alpha}{2}}}e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\,\widehat{\chi}(\lambda-\beta)\overline{\widehat{\chi}(\tilde{\lambda}-\tilde{\beta})}\mathcal{Q}_{n,m;s,t}(\beta,\tilde{\beta})\,,
\end{aligned} \qquad (4.2.12)$$

with

$$\mathcal{Q}_{n,m;s,t}(\beta,\tilde{\beta}) := \mathbb{E}\Big[\mathcal{F}\Big(\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)(\beta)\overline{\mathcal{F}\Big(\Big[\Psi_{n,m}(t,.)-\Psi_{n,m}(s,.)\Big]\Big)(\tilde{\beta})}\Big]\,.$$

To expand the latter quantity, observe first that, using the covariance formula (4.2.6), we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big\{\Psi_{n,m}(t,y)-\Psi_{n,m}(s,y)\Big\}\overline{\Big\{\Psi_{n,m}(t,\tilde{y})-\Psi_{n,m}(s,\tilde{y})\Big\}}\Big]\\
&=c\int_{\{(\xi,\eta)\in D_{n,m}\}}\frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}|\gamma_{s,t}(\xi,|\eta|)|^2 e^{i\langle\eta,y\rangle}e^{-i\langle\eta,\tilde{y}\rangle}d\xi d\eta\,,
\end{aligned}$$

where $D_{n,m} := (B^1_{2m} \times B^d_m) \setminus (B^1_{2n} \times B^d_n)$, and hence

$$\mathcal{Q}_{n,m;s,t}(\beta, \tilde{\beta}) = c \int_{\{(\xi,\eta)\in D_{n,m}\}} d\xi d\eta \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_{s,t}(\xi, |\eta|)|^2 \delta_{\beta=\eta}\delta_{\tilde{\beta}=\eta} \,. \tag{4.2.13}$$

Combining (4.2.11)-(4.2.12)-(4.2.13) and using a standard Fubini argument, we end up with the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx\, \mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\frac{\alpha}{2}} \mathcal{F}\Big(\chi\big[\mathring{\vert}_{n,m}(t,.) - \mathring{\vert}_{n,m}(s,.)\big]\Big)\Big)(x)\Big|^{2p}\Big] \\
&\lesssim \int_{\mathbb{R}^d} dx \bigg(\int_{\{(\xi,\eta)\in D_{n,m}\}} d\xi d\eta \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_{s,t}(\xi,|\eta|)|^2 \\
&\qquad\qquad \iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\frac{\alpha}{2}}\{1+|\tilde{\lambda}|^2\}^{\frac{\alpha}{2}}} e^{i\langle x, \lambda-\tilde{\lambda}\rangle}\, \widehat{\chi}(\lambda-\eta)\overline{\widehat{\chi}(\tilde{\lambda}-\eta)} \bigg)^p \\
&\lesssim \bigg(\int_{(\xi,\eta)\in D_{n,m}} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \{1+|\eta|^2\}^{-\alpha} |\gamma_{s,t}(\xi,|\eta|)|^2 \, d\xi d\eta \bigg)^p \,,
\end{aligned}$$

where we have used Lemma 4.2.6 to get the last inequality.

Now we can obviously write

$$\begin{aligned}
&\bigg(\int_{(\xi,\eta)\in D_{n,m}} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \{1+|\eta|^2\}^{-\alpha} |\gamma_{s,t}(\xi,|\eta|)|^2 \, d\xi d\eta \bigg)^p \\
&\lesssim \bigg(\int_{2^{2n}\le|\xi|\le 2^{2m}} \int_{|\eta|\le 2^m} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \{1+|\eta|^2\}^{-\alpha} |\gamma_{s,t}(\xi,|\eta|)|^2 \, d\xi d\eta \bigg)^p \\
&\qquad + \bigg(\int_{|\xi|\le 2^{2m}} \int_{2^n\le|\eta|\le 2^m} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \{1+|\eta|^2\}^{-\alpha} |\gamma_{s,t}(\xi,|\eta|)|^2 \, d\xi d\eta \bigg)^p \\
&=: (\mathbb{I}_{n,m}(s,t))^p + (\mathbb{III}_{n,m}(s,t))^p \,.
\end{aligned} \tag{4.2.14}$$

Let us focus on the estimation of $\mathbb{I}_{n,m}(s,t)$. To this end, we fix $0 < \varepsilon < \min\big(H_0, \frac{1}{2}\big)$, so that

$$\mathbb{I}_{n,m}(s,t) \le 2^{-4n\varepsilon} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}^d} d\eta \, \frac{1}{|\xi|^{2H_0-2\varepsilon-1}} \prod_{i=1}^{d} \frac{1}{|\eta_i|^{2H_i-1}} \{1+|\eta|^2\}^{-\alpha} |\gamma_{s,t}(\xi,|\eta|)|^2 \,. \tag{4.2.15}$$

The hyperspherical change of variables below

$$\begin{cases}
|\eta| = r \\
\eta_1 = r\cos(\theta_1) \\
\eta_2 = r\sin(\theta_1)\cos(\theta_2) \\
. \\
. \\
. \\
\eta_{d-1} = r\sin(\theta_1)\ldots\sin(\theta_{d-2})\cos(\theta_{d-1}) \\
\eta_d = r\sin(\theta_1)\ldots\sin(\theta_{d-2})\sin(\theta_{d-1})
\end{cases}$$

whose absolute value of the Jacobian equals $r^{d-1}\prod_{i=1}^{d}|\sin(\theta_i)|^{d-1-i}$, leads to

$$\mathbb{I}_{m,n}(s,t) \leq 2^{-4n\varepsilon}\int_{\mathbb{R}} d\xi \int_0^{+\infty}\int_{[0,\pi]^{d-2}\times[0,2\pi]} \frac{1}{|\xi|^{2H_0-2\varepsilon-1}} \frac{\{1+r^2\}^{-\alpha}}{r^{2(H_1+\ldots+H_d)-2d+1}} |\gamma_{s,t}(\xi,r)|^2$$
$$\prod_{i=1}^{d-1} \frac{1}{|\cos(\theta_i)|^{2H_i-1}|\sin(\theta_i)|^{2(H_{i+1}+\ldots+H_d)-2d+2i+1}} d\theta_1 \ldots d\theta_{d-1} dr.$$

As $\max(2H_i-1, 2(H_{i+1}+\ldots+H_d)-2d+2i+1) < 1$ for all $i \in \{1,\ldots,d-1\}$, this entails

$$\mathbb{I}_{n,m}(s,t) \lesssim 2^{-4n\varepsilon}\int_0^{\infty} dr\, \frac{\{1+r^2\}^{-\alpha}}{r^{2(H_1+\cdots+H_d)-2d+1}} \left(\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H_0-2\varepsilon-1}}\right). \tag{4.2.16}$$

We can now apply Corollary 4.2.5 with $H := H_0 - \varepsilon$, which gives, for all $0 < \kappa < \min(H_0-\varepsilon, \frac{1}{2}-\varepsilon)$,

$$\begin{aligned}
&\mathbb{I}_{n,m}(s,t)\\
&\lesssim 2^{-4n\varepsilon}|t-s|^{2\kappa}\int_0^{\infty} dr\, \frac{\{1+r^2\}^{-\alpha}}{r^{2(H_1+\cdots+H_d)-2d+1}} \frac{1}{1+r^{4H_0-4\kappa-2-8\varepsilon}}\\
&\lesssim 2^{-4n\varepsilon}|t-s|^{2\kappa}\left(\int_0^1 \frac{1}{r^{2(H_1+\cdots+H_d)-2d+1}}dr + \int_1^{\infty} \frac{1}{r^{2\alpha+2(2H_0+H_1+\cdots+H_d)-2d-1-4\kappa-8\varepsilon}}dr\right).
\end{aligned} \tag{4.2.17}$$

Due to our assumption (4.1.11), we can in fact pick ε and κ sufficiently small so that

$$4\varepsilon + 2\kappa < \alpha - \left[d+1-\left(2H_0+\sum_{i=1}^{d} H_i\right)\right],$$

and for such a choice, the integrals involved in (4.2.17) are obviously finite, implying that

$$\mathbb{I}_{n,m}(s,t) \lesssim 2^{-4n\varepsilon}|t-s|^{2\kappa}.$$

Similar arguments can be used to bound $\mathbb{III}_{n,m}(s,t)$. Using the same hyperspherical change of variables as above, we can first write

$$\begin{aligned}
\mathbb{III}_{n,m}(s,t) &\lesssim \int_{2^n}^{2^m} dr\, \frac{\{1+r^2\}^{-\alpha}}{r^{2(H_1+\cdots+H_d)-2d+1}} \left(\int_{|\xi|\leq 2^{2m}} d\xi\, \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H_0-1}}\right)\\
&\lesssim 2^{-4n\varepsilon}\int_0^{\infty} dr\, \frac{\{1+r^2\}^{-\alpha}}{r^{2(H_1+\cdots+H_d)-4\varepsilon-2d+1}} \left(\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{s,t}(\xi,r)|^2}{|\xi|^{2H_0-1}}\right),
\end{aligned}$$

for any $\varepsilon > 0$. Now we are essentially in the same position as in (4.2.16), and so we can rely on the estimation strategy of (4.2.17) to derive that

$$\mathbb{III}_{n,m}(s,t) \lesssim 2^{-4n\varepsilon}|t-s|^{2\kappa},$$

for any $\varepsilon, \kappa > 0$ small enough.

Going back to (4.2.14), we deduce the desired bound (4.2.10).

Step 2: The estimate obtained in the previous step can naturally be rephrased as

$$\mathbb{E}\Big[\Big\|\chi^{♀}_{n,m}(t,.) - \chi^{♀}_{n,m}(s,.)\Big\|^{2p}_{\mathcal{W}^{-\alpha,2p}}\Big] \lesssim 2^{-4n\varepsilon p}|t-s|^{2\kappa p}, \tag{4.2.18}$$

for all $p \geq 1$, $1 \leq n \leq m$, $0 \leq s \leq t \leq 1$ and $\varepsilon, \kappa > 0$ small enough.

By choosing $p \geq 1$ large enough so that $2\kappa p > 1$, Kolmogorov continuity criterion allows us to assert that $\chi^{♀}_{n,m} \in \mathcal{C}([0,T];\mathcal{W}^{-\alpha,2p}(\mathbb{R}^d))$ almost surely. In turn, this puts us in a position to use the classical Garsia-Rodemich-Rumsey estimate (see [24]) and deduce that almost surely, for all $p \geq 1$, $0 < \kappa_0 < \kappa$, $0 \leq s \leq t \leq 1$, one has

$$\|\chi^{♀}_{n,m}(t,.) - \chi^{♀}_{n,m}(s,.)\|^{2p}_{\mathcal{W}^{-\alpha,2p}} \lesssim |t-s|^{2\kappa_0 p}\int_{[0,1]^2} \frac{\|\chi^{♀}_{n,m}(u,.) - \chi^{♀}_{n,m}(v,.)\|^{2p}_{\mathcal{W}^{-\alpha,2p}}}{|u-v|^{2\kappa_0 p+2}}\, dudv\,,$$

for some proportional constant that only depends on κ_0 and p.

Picking $s = 0$ and taking the supremum over $t \in [0,1]$, we derive

$$\|\chi^{♀}_{n,m}\|^{2p}_{\mathcal{C}_T\mathcal{W}^{-\alpha,2p}} \lesssim \int_{[0,1]^2} \frac{\|\chi^{♀}_{n,m}(u,.) - \chi^{♀}_{n,m}(v,.)\|^{2p}_{\mathcal{W}^{-\alpha,2p}}}{|u-v|^{2\kappa_0 p+2}} dudv\,,$$

and therefore, using (4.2.18) again, we obtain that

$$\mathbb{E}\Big[\|\chi^{♀}_{n,m}\|^{2p}_{\mathcal{C}_T\mathcal{W}^{-\alpha,2p}}\Big] \;\lesssim\; 2^{-4n\varepsilon p}\int_{[0,1]^2} \frac{dudv}{|u-v|^{-2(\kappa-\kappa_0)p+2}} \;\lesssim\; 2^{-4n\varepsilon p}\,,$$

for any $p \geq 1$ large enough so that $-2(\kappa-\kappa_0)p + 2 < 1$.

We can conclude that for any $p \geq 1$ large enough,

$$\|\chi^{♀}_{n,m}\|_{L^{2p}(\Omega;\mathcal{C}_T\mathcal{W}^{-\alpha,2p})} \lesssim 2^{-2n\varepsilon}\,. \tag{4.2.19}$$

In particular, $(\chi^{♀}_n)_{n\geq 1}$ is a Cauchy sequence in $L^{2p}(\Omega;\mathcal{C}([0,T];\mathcal{W}^{-\alpha,2p}(\mathbb{R}^d)))$ (for any $p \geq 1$ large enough), which entails convergence in this space to a limit $\chi^{♀}$. Going back to (4.2.19), we also have

$$\|\chi^{♀} - \chi^{♀}_n\|_{L^{2p}(\Omega;\mathcal{C}_T\mathcal{W}^{-\alpha,2p})} \lesssim 2^{-2n\varepsilon}\,,$$

and from there, a standard Borell-Cantelli argument provides us with the desired almost sure convergence of $(\chi^{♀}_n)_{n\geq 1}$ to $\chi^{♀}$ in $\mathcal{C}([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$, for every $2 \leq p < \infty$. Finally, the convergence in $\mathcal{C}([0,T];\mathcal{W}^{-\alpha,\infty}(\mathbb{R}^d))$ follows from the Sobolev embedding $\mathcal{W}^{-\alpha+\frac{d}{p}+\eta,p}(\mathbb{R}^d) \subset \mathcal{W}^{-\alpha,\infty}(\mathbb{R}^d)$, for any $\eta > 0$.

4.2.3 Proof of Proposition 4.1.6

We will follow the same general scheme as in the proof of Proposition 4.1.2. Just as in Section 4.2.2, let us assume that $T = 1$, and set, for all $m, n \geq 0$, $\Psi_{n,m} := \Psi_m - \Psi_n$.

Step 1: Our main objective here is to show that for all $p \geq 1$, $0 \leq n \leq m$, $0 \leq s \leq t \leq 1$, $\varepsilon, \kappa > 0$ small enough, and for every α satisfying

$$d + 1 - \Big(2H_0 + \sum_{i=1}^{d} H_i\Big) < \alpha < \frac{1}{4}\,, \tag{4.2.20}$$

one has

$$\int_{\mathbb{R}^d} \mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\Psi_{n,m}(t,.) - \Psi_{n,m}(s,.)\Big]\Big)\Big)(x)\Big|^{2p}\Big]dx \lesssim 2^{-4n\varepsilon p}|t-s|^{\kappa p}\,, \tag{4.2.21}$$

where the proportional constant only depends on p, α, and χ.

For the sake of conciseness, we shall only prove estimate (4.2.21) for $n = 0$, that is we will only focus on the estimate for the time-variation $\Psi_m(t,.) - \Psi_m(s,.)$, with $m \geq 1$. The extension of the result to all $m \geq n \geq 0$ could in fact be easily deduced from the combination of the subsequent estimates with the elementary bounding argument used in (4.2.15).

A first fundamental observation is that the contractivity argument used in (4.2.11) can be extended to the present setting. Indeed, the random variable under consideration clearly belongs to the first two chaoses generated by W (with representation (4.2.2) of the noise in mind), and therefore, due to the hypercontractivity property holding in such a space (see e.g. [33]), we can assert that

$$\begin{aligned}&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\Psi_m(t,.) - \Psi_m(s,.)\Big]\Big)\Big)(x)\Big|^{2p}\Big]\\ &\leq c_p\,\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\Psi_m(t,.) - \Psi_m(s,.)\Big]\Big)\Big)(x)\Big|^{2}\Big]^p\,,\end{aligned}$$

where the constant c_p only depends on p.

Let us then write

$$\begin{aligned}&\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\Psi_m(t,.) - \Psi_m(s,.)\Big]\Big)\Big)(x)\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\lambda\,\{1+|\lambda|^2\}^{-\alpha}e^{i\langle x,\lambda\rangle}\Big(\int_{\mathbb{R}^d} d\beta\,\widehat{\chi^2}(\lambda-\beta)\mathcal{F}\Big(\Big[\Psi_m(t,.) - \Psi_m(s,.)\Big]\Big)(\beta)\Big)\,,\end{aligned}$$

which immediately yields

$$\begin{aligned}&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\Psi_m(t,.) - \Psi_m(s,.)\Big]\Big)\Big)(x)\Big|^{2}\Big]\\ &= \frac{1}{(2\pi)^{2d}}\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\alpha}\{1+|\tilde{\lambda}|^2\}^{\alpha}}e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\,\widehat{\chi^2}(\lambda-\beta)\overline{\widehat{\chi^2}(\tilde{\lambda}-\tilde{\beta})}\mathcal{Q}^{(2)}_{m;s,t}(\beta,\tilde{\beta})\,,\end{aligned} \tag{4.2.22}$$

with

$$\mathcal{Q}^{(2)}_{m;s,t}(\beta,\tilde\beta) := \mathbb{E}\Big[\mathcal{F}\Big(\big[\mathbf{Y}_m(t,.)-\mathbf{Y}_m(s,.)\big]\Big)(\beta)\overline{\mathcal{F}\Big(\big[\mathbf{Y}_m(t,.)-\mathbf{Y}_m(s,.)\big]\Big)(\tilde\beta)}\Big].$$

In order to expand $\mathcal{Q}^{(2)}_{m;s,t}(\beta,\tilde\beta)$, let us first recall that the expansion of the ("renormalized") quantities

$$\mathbb{E}\Big[\mathbf{Y}_m(t,y)\overline{\mathbf{Y}_m(s,\tilde y)}\Big]$$

is governed by the following standard application of Wick formula (see e.g. [33]) :

Lemma 4.2.7. *For all $m,n\geq 1, s,t\geq 0$ and $y,\tilde y\in\mathbb{R}^d$, it holds that*

$$\mathbb{E}\Big[\mathbf{Y}_m(t,y)\overline{\mathbf{Y}_n(s,\tilde y)}\Big] = \Big|\mathbb{E}\Big[\mathbf{I}_m(t,y)\overline{\mathbf{I}_n(s,\tilde y)}\Big]\Big|^2 + \Big|\mathbb{E}\Big[\mathbf{I}_m(t,y)\mathbf{I}_n(s,\tilde y)\Big]\Big|^2.$$

Proof. We first write

$$\begin{aligned}\mathbb{E}\Big[\mathbf{Y}_m(t,y)\overline{\mathbf{Y}_n(s,\tilde y)}\Big] &= \mathbb{E}\Big[\Big(|\mathbf{I}_m(t,y)|^2-\mathbb{E}[|\mathbf{I}_m(t,y)|^2]\Big)\Big(|\mathbf{I}_n(s,\tilde y)|^2-\mathbb{E}[|\mathbf{I}_n(s,\tilde y)|^2]\Big)\Big]\\ &= \mathbb{E}\Big[|\mathbf{I}_m(t,y)|^2|\mathbf{I}_n(s,\tilde y)|^2\Big]-\mathbb{E}[|\mathbf{I}_m(t,y)|^2]\mathbb{E}[|\mathbf{I}_n(s,\tilde y)|^2]. \qquad (4.2.23)\end{aligned}$$

Resorting to Wick formula, we obtain

$$\begin{aligned}&\mathbb{E}\Big[|\mathbf{I}_m(t,y)|^2|\mathbf{I}_n(s,\tilde y)|^2\Big] = \mathbb{E}\Big[\mathbf{I}_m(t,y)\overline{\mathbf{I}_m(t,y)}\mathbf{I}_n(s,\tilde y)\overline{\mathbf{I}_n(s,\tilde y)}\Big]\\ =\;& \mathbb{E}[|\mathbf{I}_m(t,y)|^2]\mathbb{E}[|\mathbf{I}_n(s,\tilde y)|^2]+\mathbb{E}\Big[\mathbf{I}_m(t,y)\mathbf{I}_n(s,\tilde y)\Big]\mathbb{E}\Big[\overline{\mathbf{I}_m(t,y)\mathbf{I}_n(s,\tilde y)}\Big]\\ &+\mathbb{E}\Big[\mathbf{I}_m(t,y)\overline{\mathbf{I}_n(s,\tilde y)}\Big]\mathbb{E}\Big[\overline{\mathbf{I}_m(t,y)}\mathbf{I}_n(s,\tilde y)\Big]\\ =\;& \mathbb{E}[|\mathbf{I}_m(t,y)|^2]\mathbb{E}[|\mathbf{I}_n(s,\tilde y)|^2]+\Big|\mathbb{E}\Big[\mathbf{I}_m(t,y)\mathbf{I}_n(s,\tilde y)\Big]\Big|^2+\Big|\mathbb{E}\Big[\mathbf{I}_m(t,y)\overline{\mathbf{I}_n(s,\tilde y)}\Big]\Big|^2.\end{aligned}$$

By injecting the latter equality into the first one (4.2.23), we can conclude that

$$\mathbb{E}\Big[\mathbf{Y}_m(t,y)\overline{\mathbf{Y}_n(s,\tilde y)}\Big] = \Big|\mathbb{E}\Big[\mathbf{I}_m(t,y)\overline{\mathbf{I}_n(s,\tilde y)}\Big]\Big|^2 + \Big|\mathbb{E}\Big[\mathbf{I}_m(t,y)\mathbf{I}_n(s,\tilde y)\Big]\Big|^2.$$

□

Using Lemma 4.2.7 and the shortcut notation $\mathbf{I}_m(u,v;z) := \mathbf{I}_m(v,z)-\mathbf{I}_m(u,z)$, we easily verify that

$$\begin{aligned}&\mathbb{E}\Big[[\mathbf{Y}_m(t,y)-\mathbf{Y}_m(s,y)]\overline{[\mathbf{Y}_m(t,\tilde y)-\mathbf{Y}_m(s,\tilde y)]}\Big]\\ &=\Big[\mathbb{E}\Big[\mathbf{I}_m(s,t;y)\overline{\mathbf{I}_m(t,\tilde y)}\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(t,y)}\mathbf{I}_m(t,\tilde y)\Big]+\mathbb{E}\Big[\mathbf{I}_m(s,y)\overline{\mathbf{I}_m(t,\tilde y)}\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(s,t;y)}\mathbf{I}_m(t,\tilde y)\Big]\\ &\quad+\mathbb{E}\Big[\mathbf{I}_m(t,s;y)\overline{\mathbf{I}_m(s,\tilde y)}\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(t,y)}\mathbf{I}_m(s,\tilde y)\Big]+\mathbb{E}\Big[\mathbf{I}_m(s,y)\overline{\mathbf{I}_m(s,\tilde y)}\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(t,s;y)}\mathbf{I}_m(s,\tilde y)\Big]\Big]\\ &+\Big[\mathbb{E}\Big[\mathbf{I}_m(s,t;y)\mathbf{I}_m(t,\tilde y)\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(t,y)\mathbf{I}_m(t,\tilde y)}\Big]+\mathbb{E}\Big[\mathbf{I}_m(s,y)\mathbf{I}_m(t,\tilde y)\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(s,t;y)\mathbf{I}_m(t,\tilde y)}\Big]\\ &\quad+\mathbb{E}\Big[\mathbf{I}_m(t,s;y)\mathbf{I}_m(s,\tilde y)\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(t,y)\mathbf{I}_m(s,\tilde y)}\Big]+\mathbb{E}\Big[\mathbf{I}_m(s,y)\mathbf{I}_m(s,\tilde y)\Big]\cdot\mathbb{E}\Big[\overline{\mathbf{I}_m(t,s;y)\mathbf{I}_m(s,\tilde y)}\Big]\Big],\end{aligned}$$

which, combined with the covariance formulas (4.2.6)-(4.2.7), allows us to expand (4.2.22) as

$$\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}(\chi^2[\curlyvee_m(t,.)-\curlyvee_m(s,.)])\Big)(x)\Big|^2\Big]$$

$$= c\int_{(\xi,\eta)\in D_m} d\xi d\eta \int_{(\tilde{\xi},\tilde{\eta})\in D_m} d\tilde{\xi} d\tilde{\eta}\, \frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}\frac{1}{|\tilde{\xi}|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\tilde{\eta}_i|^{2H_i-1}}$$
$$\Big(\sum_{\ell=1}^{8}\Gamma^{\ell}_{m;s,t}(\xi,|\eta|;\tilde{\xi},|\tilde{\eta}|)\Big)$$
$$\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde{\lambda}}{\{1+|\lambda|^2\}^{\alpha}\{1+|\tilde{\lambda}|^2\}^{\alpha}} e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\widehat{\chi^2}(\lambda-(\eta-\tilde{\eta}))\overline{\widehat{\chi^2}(\tilde{\lambda}-(\eta-\tilde{\eta}))}\,,$$

with $\Gamma^{\ell}_{m;s,t}=\Gamma^{\ell}_{m;s,t}(\xi,|\eta|;\tilde{\xi},|\tilde{\eta}|)$ given by

$$\Gamma^1_{m;s,t} := \gamma_{s,t}(\xi,|\eta|)\,\overline{\gamma_t(\xi,|\eta|)}\,|\gamma_t(\tilde{\xi},|\tilde{\eta}|)|^2\,,\quad \Gamma^2_{m;s,t} := \gamma_s(\xi,|\eta|)\,\overline{\gamma_t(\xi,|\eta|)}\,\overline{\gamma_{s,t}(\tilde{\xi},|\tilde{\eta}|)}\,\gamma_t(\tilde{\xi},|\tilde{\eta}|)\,,$$
$$\Gamma^3_{m;s,t} := \gamma_{t,s}(\xi,|\eta|)\,\overline{\gamma_s(\xi,|\eta|)}\,\overline{\gamma_t(\tilde{\xi},|\tilde{\eta}|)}\,\gamma_s(\tilde{\xi},|\tilde{\eta}|)\,,\quad \Gamma^4_{m;s,t} := |\gamma_s(\xi,|\eta|)|^2\,\overline{\gamma_{t,s}(\tilde{\xi},|\tilde{\eta}|)}\,\gamma_s(\tilde{\xi},|\tilde{\eta}|)\,,$$
$$\Gamma^5_{m;s,t} := \gamma_{s,t}(\xi,|\eta|)\,\gamma_t(-\xi,|\eta|)\,\overline{\gamma_t(\tilde{\xi},|\tilde{\eta}|)\gamma_t(-\tilde{\xi},|\tilde{\eta}|)}\,,\quad \Gamma^6_{m;s,t} := \gamma_s(\xi,|\eta|)\,\gamma_t(-\xi,|\eta|)\,\overline{\gamma_{s,t}(\tilde{\xi},|\tilde{\eta}|)\gamma_t(-\tilde{\xi},|\tilde{\eta}|)}\,,$$
$$\Gamma^7_{m;s,t} := \gamma_{t,s}(\xi,|\eta|)\,\gamma_s(-\xi,|\eta|)\,\overline{\gamma_t(\tilde{\xi},|\tilde{\eta}|)\gamma_s(-\tilde{\xi},|\tilde{\eta}|)}\,,\quad \Gamma^8_{m;s,t} := \gamma_s(\xi,|\eta|)\,\gamma_s(-\xi,|\eta|)\,\overline{\gamma_{t,s}(\tilde{\xi},|\tilde{\eta}|)\gamma_s(-\tilde{\xi},|\tilde{\eta}|)}\,.$$

At this point, we can rely on the technical Lemma 4.2.6 to assert that

$$\int_{\mathbb{R}^d} dx\,\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\{1+|.|^2\}^{-\alpha}\mathcal{F}\Big(\chi^2\Big[\curlyvee_m(t,.)-\curlyvee_m(s,.)\Big]\Big)\Big)(x)\Big|^2\Big]^p \lesssim \Big(\sum_{\ell=1}^{8}\mathcal{J}^{\ell}_{m;s,t}\Big)^p \qquad (4.2.24)$$

where

$$\mathcal{J}^{\ell}_{m;s,t} := \int_{(\xi,\eta)\in D_m} d\xi d\eta \int_{(\tilde{\xi},\tilde{\eta})\in D_m} d\tilde{\xi} d\tilde{\eta}\, \frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}\frac{1}{|\tilde{\xi}|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\tilde{\eta}_i|^{2H_i-1}}$$
$$\Big|\Gamma^{\ell}_{m;s,t}(\xi,|\eta|;\tilde{\xi},|\tilde{\eta}|)\Big|\Big\{1+|\eta-\tilde{\eta}|^2\Big\}^{-2\alpha}. \qquad (4.2.25)$$

Using the trivial bound $|\eta-\tilde{\eta}|\geq ||\eta|-|\tilde{\eta}||$ and going back to the expression of $\Gamma^{\ell}_{m;s,t}(\xi,|\eta|;\tilde{\xi},|\tilde{\eta}|)$, we can first bound these integrals along the pattern

$$\mathcal{J}^{\ell}_{m;s,t} \lesssim \int_{(\mathbb{R}\times\mathbb{R}^d)^2}\frac{d\xi d\eta d\tilde{\xi} d\tilde{\eta}}{\{1+||\eta|-|\tilde{\eta}||^2\}^{2\alpha}}\Big(\frac{1}{|\xi|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\eta_i|^{2H_i-1}}|\gamma_{a_1}(\xi,|\eta|)||\gamma_{a_2}(\pm\xi,|\eta|)|\Big)\cdot$$
$$\Big(\frac{1}{|\tilde{\xi}|^{2H_0-1}}\prod_{i=1}^{d}\frac{1}{|\tilde{\eta}_i|^{2H_i-1}}|\gamma_{a_3}(\tilde{\xi},|\tilde{\eta}|)||\gamma_{a_4}(\pm\tilde{\xi},|\tilde{\eta}|)|\Big) \qquad (4.2.26)$$

for some $a_1,a_2,a_3,a_4\in\{s,t,\{s,t\}\}$ (depending on ℓ) such that exactly one of the a_k's is $\{s,t\}$.

Now, let us decompose the latter integration domain into $(\mathbb{R}\times\mathbb{R}^d)^2 := \mathfrak{D}_1\cup\mathfrak{D}_2$, where

$$\mathfrak{D}_1 := \Big\{(\xi,\eta,\tilde{\xi},\tilde{\eta}) : 0\leq|\tilde{\eta}|\leq\frac{|\eta|}{2} \text{ or } |\tilde{\eta}|\geq\frac{3|\eta|}{2}\Big\} \text{ and } \mathfrak{D}_2 := \Big\{(\xi,\eta,\tilde{\xi},\tilde{\eta}) : \frac{|\eta|}{2}<|\tilde{\eta}|<\frac{3|\eta|}{2}\Big\}.$$

For the integral over $\mathfrak{D}_1$, we can rely on the inequality $||\eta|-|\tilde{\eta}|| \geq \max(\frac{|\eta|}{2}, \frac{|\tilde{\eta}|}{3})$ (valid for all $(\xi,\eta,\tilde{\xi},\tilde{\eta}) \in \mathfrak{D}_1$) to write

$$\begin{aligned}
\mathcal{A}_1^\ell &:= \int_{\mathfrak{D}_1} \frac{d\xi d\eta d\tilde{\xi} d\tilde{\eta}}{\{1+||\eta|-|\tilde{\eta}||^2\}^{2\alpha}} \bigg(\frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_{a_1}(\xi,|\eta|)||\gamma_{a_2}(\pm\xi,|\eta|)| \bigg) \cdot \\
&\qquad\qquad \bigg(\frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_{a_3}(\tilde{\xi},|\tilde{\eta}|)||\gamma_{a_4}(\pm\tilde{\xi},|\tilde{\eta}|)| \bigg) \\
&\lesssim \bigg(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\xi d\eta}{\{1+|\eta|^2\}^\alpha} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_{a_1}(\xi,|\eta|)||\gamma_{a_2}(\pm\xi,|\eta|)| \bigg) \cdot \\
&\qquad \bigg(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\tilde{\xi} d\tilde{\eta}}{\{1+|\tilde{\eta}|^2\}^\alpha} \frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_{a_3}(\tilde{\xi},|\tilde{\eta}|)||\gamma_{a_4}(\pm\tilde{\xi},|\tilde{\eta}|)| \bigg) \\
&\lesssim \prod_{k=1}^4 \bigg(\int_{\mathbb{R}\times\mathbb{R}^d} \frac{d\xi d\eta}{\{1+|\eta|^2\}^\alpha} \frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_{a_k}(\xi,|\eta|)|^2 \bigg)^{\frac{1}{2}},
\end{aligned}$$

where we have merely used Cauchy-Schwarz inequality to derive the last estimate.

Observe that we are here dealing with the same integrals as in the proof of Proposition 4.1.2 (see in particular (4.2.15)), with α satisfying the same assumption (4.1.11). Therefore we can mimic the arguments in (4.2.16)-(4.2.17) to deduce the desired estimate, namely: for every $\ell = 1,\ldots,8$ and any $\kappa > 0$ small enough,

$$\mathcal{A}_1^\ell \lesssim |t-s|^\kappa,$$

where the $|t-s|^\kappa$-increment stems from the fact that one of the a_k's must be $\{s,t\}$.

Let us turn to the integral over $\mathfrak{D}_2$ and lean first on a hyperspherical change of variable (with respect to $\tilde{\eta}$) to write

$$\begin{aligned}
&\int_{\frac{|\eta|}{2}<|\tilde{\eta}|<\frac{3|\eta|}{2}} \frac{d\tilde{\eta}}{\{1+||\eta|-|\tilde{\eta}||^2\}^{2\alpha}} |\gamma_{a_3}(\tilde{\xi},|\tilde{\eta}|)||\gamma_{a_4}(\pm\tilde{\xi},|\tilde{\eta}|)| \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} \\
&= |\eta|^{-2(H_1+\cdots+H_d)+2d} \int_{\frac{1}{2}<|\tilde{\eta}|<\frac{3}{2}} \frac{d\tilde{\eta}}{\{1+|\eta|^2(1-|\tilde{\eta}|)^2\}^{2\alpha}} |\gamma_{a_3}(\tilde{\xi},|\eta||\tilde{\eta}|)||\gamma_{a_4}(\pm\tilde{\xi},|\eta||\tilde{\eta}|)| \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} \\
&\lesssim |\eta|^{-2(H_1+\cdots+H_d)+2d} \int_{\frac{1}{2}<r<\frac{3}{2}} \frac{dr}{\{1+|\eta|^2(1-r)^2\}^{2\alpha}} |\gamma_{a_3}(\tilde{\xi},|\eta|r)||\gamma_{a_4}(\pm\tilde{\xi},|\eta|r)|\,.
\end{aligned}$$

As a consequence,

$$
\begin{aligned}
\mathcal{A}_2^\ell &:= \int_{\mathfrak{D}_2} \frac{d\xi d\eta d\tilde{\xi} d\tilde{\eta}}{\{1+||\eta|-|\tilde{\eta}||^2\}^{2\alpha}} \bigg(\frac{1}{|\xi|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} |\gamma_{a_1}(\xi,|\eta|)||\gamma_{a_2}(\pm\xi,|\eta|)| \bigg) \cdot \\
&\qquad\qquad\qquad\qquad \bigg(\frac{1}{|\tilde{\xi}|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\tilde{\eta}_i|^{2H_i-1}} |\gamma_{a_3}(\tilde{\xi},|\tilde{\eta}|)||\gamma_{a_4}(\pm\tilde{\xi},|\tilde{\eta}|)| \bigg) \\
&\lesssim \int_{\mathbb{R}^d} \frac{d\eta}{|\eta|^{2(H_1+\cdots+H_d)-2d}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} \int_{\frac{1}{2}<r<\frac{3}{2}} \frac{dr}{\{1+|\eta|^2(1-r)^2\}^{2\alpha}} \cdot \\
&\qquad \bigg(\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{a_1}(\xi,|\eta|)||\gamma_{a_2}(\pm\xi,|\eta|)|}{|\xi|^{2H_0-1}} \bigg) \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi}\, \frac{|\gamma_{a_3}(\tilde{\xi},|\eta|r)||\gamma_{a_4}(\pm\tilde{\xi},|\eta|r)|}{|\tilde{\xi}|^{2H_0-1}} \bigg) \\
&\lesssim \int_{\mathbb{R}^d} \frac{d\eta}{|\eta|^{2(H_1+\cdots+H_d)-2d}} \prod_{i=1}^d \frac{1}{|\eta_i|^{2H_i-1}} \int_{\frac{1}{2}<r<\frac{3}{2}} \frac{dr}{\{1+|\eta|^2(1-r)^2\}^{2\alpha}} \cdot \\
&\bigg(\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{a_1}(\xi,|\eta|)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac{1}{2}} \cdot \bigg(\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{a_2}(\xi,|\eta|)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac{1}{2}} \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi}\, \frac{|\gamma_{a_3}(\tilde{\xi},|\eta|r)|^2}{|\tilde{\xi}|^{2H_0-1}} \bigg)^{\frac{1}{2}} \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi}\, \frac{|\gamma_{a_4}(\tilde{\xi},|\eta|r)|^2}{|\tilde{\xi}|^{2H_0-1}} \bigg)^{\frac{1}{2}} .
\end{aligned}
$$

Using again a hyperspherical change of variable (with respect to η), we obtain

$$
\begin{aligned}
\mathcal{A}_2^\ell &\lesssim \int_0^\infty \frac{d\rho}{\rho^{4(H_1+\cdots+H_d)-4d+1}} \int_{\frac{1}{2}<r<\frac{3}{2}} \frac{dr}{\{1+\rho^2(1-r)^2\}^{2\alpha}} \\
&\bigg(\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{a_1}(\xi,\rho)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac{1}{2}} \cdot \bigg(\int_{\mathbb{R}} d\xi\, \frac{|\gamma_{a_2}(\xi,\rho)|^2}{|\xi|^{2H_0-1}} \bigg)^{\frac{1}{2}} \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi}\, \frac{|\gamma_{a_3}(\tilde{\xi},\rho r)|^2}{|\tilde{\xi}|^{2H_0-1}} \bigg)^{\frac{1}{2}} \cdot \bigg(\int_{\mathbb{R}} d\tilde{\xi}\, \frac{|\gamma_{a_4}(\tilde{\xi},\rho r)|^2}{|\tilde{\xi}|^{2H_0-1}} \bigg)^{\frac{1}{2}} .
\end{aligned}
$$

We can now appeal to Corollary 4.2.5, together with the fact that one of the a_k's is $\{s,t\}$, to assert that

$$
\begin{aligned}
\mathcal{A}_2^\ell \lesssim |t-s|^\kappa \bigg[&\int_0^1 \frac{d\rho}{\rho^{4(H_1+\cdots+H_d)-4d+1}} \\
&+ \int_1^\infty \frac{d\rho}{\rho^{4(2H_0+H_1+\cdots+H_d)-4d-3-8\varepsilon-8\kappa}} \int_{\frac{1}{2}<r<\frac{3}{2}} \frac{dr}{\{1+\rho^2(1-r)^2\}^{2\alpha}} \bigg]
\end{aligned}
$$

for all $0<\varepsilon<\frac{1}{2}$ and $0<\kappa<\min(H_0,\frac{1}{2}-\varepsilon)$.

At this point, observe that

$$
\begin{aligned}
&\int_1^\infty \frac{d\rho}{\rho^{4(2H_0+H_1+\cdots+H_d)-4d-3-8\varepsilon-8\kappa}} \int_{\frac{1}{2}<r<\frac{3}{2}} \frac{dr}{\{1+\rho^2(1-r)^2\}^{2\alpha}} \\
&\leq \int_1^\infty \frac{d\rho}{\rho^{4\alpha+4(2H_0+H_1+\cdots+H_d)-4d-3-8\varepsilon-8\kappa}} \int_{\frac{1}{2}<r<\frac{3}{2}} \frac{dr}{(1-r)^{4\alpha}} . \qquad (4.2.27)
\end{aligned}
$$

Thanks to our assumption (4.2.20) on α, we know on the one hand that $4\alpha<1$, and on the other hand we can pick $\varepsilon,\kappa>0$ such that

$$
4\alpha+4(2H_0+H_1+\cdots+H_d)-4d-3-8\varepsilon-8\kappa>1 .
$$

For such a choice of ε, κ, the integrals in the right-hand side of (4.2.27) are clearly finite, and therefore we have shown that for every $\ell = 1, \ldots, 8$,

$$\mathcal{A}_2^\ell \lesssim |t-s|^\kappa .$$

Going back to (4.2.26), we have thus proved that for every $\ell = 1, \ldots, 8$, and uniformly over m,

$$\mathcal{J}_{m;s,t}^\ell \lesssim |t-s|^\kappa .$$

Injecting the above estimates into (4.2.24) provides us with the desired bound (4.2.21).

Step 2: Conclusion. Let α satisfying the condition in the statement of Proposition 4.1.6, that is $\alpha > d+1-\Big(2H_0+\sum_{i=1}^d H_i\Big)$.

If in addition one has $\alpha < \frac{1}{4}$, then condition (4.2.20) is satisfied, and so the moment estimate (4.2.21) holds true. Starting from this estimate, we can use the same arguments as in Step 2 of Section 4.2.2 to obtain that $(\chi^2 \curlyvee_n)_{n\geq 1}$ converges almost surely in the space $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha,p}(\mathbb{R}^d))$, for every $2 \leq p < \infty$.

If $\alpha \geq \frac{1}{4}$, observe that due to assumption (**H2**), we can pick α' satisfying $\alpha' < \alpha$ and $d+1-\Big(2H_0+\sum_{i=1}^d H_i\Big) < \alpha' < \frac{1}{4}$ (that is, α' satisfies (4.2.20)). By repeating the above arguments, we deduce that the sequence $(\chi^2 \curlyvee_n)_{n\geq 1}$ converges almost surely in $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha',p}(\mathbb{R}^d))$, and therefore it converges almost surely in $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha,p}(\mathbb{R}^d))$ as well, for $2 \leq p < \infty$.

Finally, the (a.s.) convergence in $\mathcal{C}([0,T];\mathcal{W}^{-2\alpha,\infty}(\mathbb{R}^d))$ can be easily derived from the Sobolev embedding $\mathcal{W}^{-2\alpha+\frac{d}{p}+\eta,p}(\mathbb{R}^d) \subset \mathcal{W}^{-2\alpha,\infty}(\mathbb{R}^d)$, for any $\eta > 0$, which completes the proof of Proposition 4.1.6.

4.2.4 Proof of Proposition 4.1.5

Fix $d \geq 1$ and $(H_0, \ldots, H_d) \in (0,1)^{d+1}$ such that

$$d + \frac{3}{4} < 2H_0 + \sum_{i=1}^d H_i \leq d+1 .$$

Using (4.2.6), the quantity under consideration can be written as

$$\begin{aligned}
\sigma_n(t,x) \;=\; \mathbb{E}\Big[|\Psi_n(t,x)|^2\Big] \;&=\; c_H^2 \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{2H_0-1}} \int_{|\eta|\leq 2^n} \prod_{i=1}^d \frac{d\eta_i}{|\eta_i|^{2H_i-1}} |\gamma_t(\xi,|\eta|)|^2 \\
&=\; C_H \int_0^{2^n} \frac{dr}{r^{2(H_1+\cdots+H_d)-2d+1}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{2H_0-1}} |\gamma_t(\xi,r)|^2 ,
\end{aligned}$$

for some constant C_H, and where the last identity is derived from a hyperspherical change of variables. The above formula shows in particular that σ_n does not depend on x, as stated in Proposition 4.1.5. Regarding the desired estimate (4.1.14), it is now a consequence of the following technical result (applied with $\alpha := 2H_0 \in (0,2)$ and $\kappa := d+1-[2H_0+\sum_{i=1}^d H_i] \geq 0$):

Proposition 4.2.8. *Fix $t > 0$. For all $\alpha \in (0,2)$ and $\kappa \geq 0$ verifying $\alpha + \kappa > 1$, one has*

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 \underset{n\to\infty}{\sim} \begin{cases} \frac{\pi}{\kappa} 4^{n\kappa} t & \text{if } \kappa > 0 \\ \pi \ln(4) \cdot nt & \text{if } \kappa = 0 \end{cases}.$$

Remark 4.2.9. Let us give a formal argument to explain the change of regime in Proposition 4.2.8. It holds that

$$\frac{\pi}{\kappa} 4^{n\kappa} t \underset{\kappa\to 0}{\sim} \frac{\pi t}{\kappa}\Big(1 + n\kappa \ln(4)\Big).$$

Thus,

$$\frac{\pi}{\kappa} 4^{n\kappa} t \underset{n\to\infty}{\sim} \pi t \ln(4) \cdot n$$

with a formal and illicit use of the equivalents.

First proof

Proof. An easy computation provides us with the identity

$$|\gamma_t(\xi,r)|^2 = 2\frac{1-\cos(t(\xi - r^2))}{|r^2 - \xi|^2}.$$

Then, with the help of two changes of variables and the well-known formula, for all $a \in \mathbb{R}$, $2\sin^2(a) = 1 - \cos(2a)$, we write

$$\begin{aligned}
\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 &= \int_0^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{(\xi-r)^2} \\
&= 4^{n\kappa} \int_0^1 \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(4^n t(\xi-r))}{4^n(\xi-r)^2} \\
&= 2\cdot 4^{n\kappa} \int_0^1 \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{\sin^2(\frac{4^n t}{2}(\xi-r))}{4^n(\xi-r)^2} \\
&= t\cdot 4^{n\kappa} \int_0^1 \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{\sin^2(\frac{4^n t}{2}(\xi-r))}{\frac{4^n t}{2}(\xi-r)^2}.
\end{aligned}$$

First case: $\kappa > 0$.

Let us introduce the function Φ defined by, for all $x \in \mathbb{R}^*$,

$$\Phi(x) := \frac{\sin^2(x)}{\pi x^2} \quad \text{and} \quad \Phi(0) = \frac{1}{\pi}.$$

It is readily checked that Φ is positive, even and that $\int_{\mathbb{R}} \Phi(x)dx = 1$. Thus, we can consider the approximation of unity Φ_n defined by, for all $x \in \mathbb{R}$,

$$\Phi_n(x) := \frac{4^n t}{2} \Phi\Big(\frac{4^n t}{2} x\Big),$$

leading to

$$t\cdot 4^{n\kappa} \int_0^1 \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{\sin^2(\frac{4^n t}{2}(\xi-r))}{\frac{4^n t}{2}(\xi-r)^2} = \pi t 4^{n\kappa} \int_0^1 \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha-1}} \Phi_n(\xi - r).$$

We observe that

$$\int_0^1 \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\le 1} \frac{d\xi}{|\xi|^{\alpha-1}} \Phi_n(\xi - r) = \int_{\mathbb{R}} f(r)(\Phi_n * g)(r)dr$$

where

$$f(r) := \mathbb{1}_{[0,1]}(r)\frac{1}{r^{-\alpha-\kappa+2}}$$

and

$$g(\xi) := \mathbb{1}_{[-1,1]}(\xi)\frac{1}{|\xi|^{\alpha-1}}.$$

As $\kappa > 0$, there exists a pair of conjugate exponents $p, p' > 1$ depending on α and κ (see the end of the proof about this technical point) such that

$$g \in L^p(\mathbb{R}) \quad \textit{and} \quad f \in L^{p'}(\mathbb{R}).$$

Since Φ_n is an approximation of unity, the following estimates show that

$$\begin{aligned}\left| \int_{\mathbb{R}} f(r)(\Phi_n * g)(r)dr - \int_{\mathbb{R}} f(r)g(r)dr \right| &= \left| \int_{\mathbb{R}} f(r)[(\Phi_n * g)(r) - g(r)]dr \right| \\ &\le \left(\int_{\mathbb{R}} |f(r)|^{p'} dr \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}} |(\Phi_n * g)(r) - g(r)|^p dr \right)^{\frac{1}{p}}\end{aligned}$$

tends to 0 as n tends to infinity. As $\int_{\mathbb{R}} f(r)g(r)dr = \frac{1}{\kappa}$, we obtain the desired conclusion

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi, r)|^2 \underset{n\to\infty}{\sim} \frac{\pi t}{\kappa} 4^{n\kappa}.$$

Before dealing with the more difficult case $\kappa = 0$, let us prove that there exists a pair of conjugate exponents $p, p' > 1$ depending on α and κ such that

$$g \in L^p(\mathbb{R}) \quad \textit{and} \quad f \in L^{p'}(\mathbb{R}).$$

We divide the proof into two subcases. On the one hand, if $-1 < -\alpha - \kappa + 2 \le 0$, α and κ verify the following constraints

$$2 \le \alpha + \kappa < 3, \quad 0 < \alpha < 2 \quad \textit{and} \quad 0 < \kappa < 1.$$

We easily see that $3 - \alpha$ is a possible choice for p and it suffices to take the associated p'. On the other hand, if $0 < -\alpha - \kappa + 2 < 1$, α and κ verify the following constraints

$$1 < \alpha + \kappa < 2, \quad 0 < \alpha < 2 \quad \textit{and} \quad 0 < \kappa < 1.$$

A brief observation shows that should p and p' exist, then

$$\alpha - 1 < \frac{1}{p} < \alpha + \kappa - 1.$$

Thus, if $\alpha > 1$, $p > 1$ defined by

$$\frac{1}{p} = \frac{\alpha - 1}{2} + \frac{\alpha + \kappa - 1}{2}$$

is a possible choice whereas, if $\alpha \leq 1$, $p > 1$ defined by

$$\frac{1}{p} = \frac{\alpha + \kappa - 1}{2}$$

is a solution. The associated p' gives the conclusion.

Second case: $\kappa = 0$. The following equality has already been obtained

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha+3}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi, r)|^2 = t \int_0^1 \frac{dr}{r^{-\alpha+2}} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{\sin^2(\frac{4^n t}{2}(\xi - r))}{\frac{4^n t}{2}(\xi - r)^2}.$$

Let us focus our attention on

$$I := \int_0^1 \frac{dr}{r^{-\alpha+2}} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{\sin^2(\frac{4^n t}{2}(\xi - r))}{\frac{4^n t}{2}(\xi - r)^2}. \tag{4.2.28}$$

Consider the function f defined, for all $T > 0$ by

$$f(T) := \int_0^1 \int_{|\xi|\leq 1} \frac{drd\xi}{r^{-\alpha+2}|\xi|^{\alpha-1}} \frac{\sin^2(T(\xi - r))}{(\xi - r)^2}.$$

By successive derivations,

$$\begin{aligned} f'(T) &= \int_0^1 \int_{|\xi|\leq 1} \frac{drd\xi}{r^{-\alpha+2}|\xi|^{\alpha-1}} \frac{2\sin(T(\xi - r))\cos(T(\xi - r))}{(\xi - r)} \\ &= \int_0^1 \int_{|\xi|\leq 1} \frac{drd\xi}{r^{-\alpha+2}|\xi|^{\alpha-1}} \frac{\sin(2T(\xi - r))}{(\xi - r)} \end{aligned}$$

and

$$\begin{aligned} f''(T) &= 2\int_0^1 \int_{|\xi|\leq 1} \frac{drd\xi}{r^{-\alpha+2}|\xi|^{\alpha-1}} \cos(2T(\xi - r)) \\ &= 2\int_0^1 \int_{0\leq\xi\leq 1} \frac{drd\xi}{r^{-\alpha+2}\xi^{\alpha-1}} \cos(2T(\xi - r)) + 2\int_0^1 \int_{-1\leq\xi\leq 0} \frac{drd\xi}{r^{-\alpha+2}(-\xi)^{\alpha-1}} \cos(2T(\xi - r)) \\ &:= J_1 + J_2. \end{aligned}$$

On the one hand, by a change of variables,

$$\begin{aligned} J_1 &= 8\int_0^1 \int_0^1 \left(\frac{v}{u}\right)^{3-2\alpha} \cos(2T(u^2 - v^2))dudv \\ &= \frac{4}{T}\int_0^{\sqrt{2T}} \int_0^{\sqrt{2T}} \left(\frac{v}{u}\right)^{3-2\alpha} \cos(u^2 - v^2)dudv, \end{aligned}$$

whereas on the other hand

$$\begin{aligned} J_2 &= 2\int_0^1 \int_{0\leq\xi\leq 1} \frac{drd\xi}{r^{-\alpha+2}\xi^{\alpha-1}} \cos(2T(\xi + r)) \\ &= 8\int_0^1 \int_0^1 \left(\frac{v}{u}\right)^{3-2\alpha} \cos(2T(u^2 + v^2))dudv \\ &= \frac{4}{T}\int_0^{\sqrt{2T}} \int_0^{\sqrt{2T}} \left(\frac{v}{u}\right)^{3-2\alpha} \cos(u^2 + v^2)dudv. \end{aligned}$$

Finally, the well-known formula, for all $a, b \in \mathbb{R}$,

$$\cos(a+b) + \cos(a-b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

leads us to

$$\begin{aligned} f''(T) &= J_1 + J_2 \\ &= \frac{8}{T}\int_0^{\sqrt{2T}}\int_0^{\sqrt{2T}}\left(\frac{v}{u}\right)^{3-2\alpha}\cos(u^2)\cos(v^2)dudv \\ &= \frac{8}{T}\Big(\int_0^{\sqrt{2T}}\frac{\cos(u^2)}{u^{3-2\alpha}}du\Big)\Big(\int_0^{\sqrt{2T}} v^{3-2\alpha}\cos(v^2)dv\Big). \end{aligned}$$

In order to go further in our estimate, we recall two technical results from complex analysis whose proofs can be found below:

Lemma 4.2.10. *For all $\beta \in \mathbb{C}$ such that $\mathcal{R}e(\beta) \in]-1,1[$,*

$$\int_0^{+\infty} t^\beta e^{-it^2}dt = e^{-i\frac{\pi}{4}(\beta+1)}\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2},$$

where

$$\Gamma(z) := \int_0^{+\infty} t^{z-1}e^{-t}dt,$$

for all $z \in \mathbb{C}$ such that $\mathcal{R}e(z) > 0$.

Lemma 4.2.11. *[complement (or reflection) formula] For all $z \in \mathbb{C}$ such that $0 < \mathcal{R}e(z) < 1$,*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Using Lemma 4.2.10, we see that (taking the real part)

$$\int_0^{+\infty}\frac{\cos(u^2)}{u^{3-2\alpha}}du = \cos\left(\frac{\pi}{2}(\alpha-1)\right)\frac{\Gamma(\alpha-1)}{2}$$

and

$$\int_0^{+\infty} v^{3-2\alpha}\cos(v^2)dv = \cos\left(\frac{\pi}{2}(2-\alpha)\right)\frac{\Gamma(2-\alpha)}{2}.$$

The reflection formula entails

$$\begin{aligned} \Big(\int_0^{+\infty}\frac{\cos(u^2)}{u^{3-2\alpha}}du\Big)\Big(\int_0^{+\infty} v^{3-2\alpha}\cos(v^2)dv\Big) &= \cos\left(\frac{\pi}{2}(\alpha-1)\right)\cos\left(\frac{\pi}{2}(2-\alpha)\right)\frac{\Gamma(\alpha-1)\Gamma(2-\alpha)}{4} \\ &= \cos\left(\frac{\pi}{2}(\alpha-1)\right)\cos\left(\frac{\pi}{2}(2-\alpha)\right)\frac{\pi}{4\sin(\pi(\alpha-1))} \\ &= \frac{\cos\left(\frac{\pi}{2}(\alpha-1)\right)\cos\left(\frac{\pi}{2}(2-\alpha)\right)\pi}{8\sin\left(\frac{\pi}{2}(\alpha-1)\right)\cos\left(\frac{\pi}{2}(\alpha-1)\right)} \\ &= \frac{\pi}{8}. \end{aligned}$$

Therefore,

$$f''(T) \underset{T\to\infty}{\sim} \frac{\pi}{T}.$$

Applying Theorem 4.2.12, $\int_1^T f''(t)dt \underset{T\to\infty}{\sim} \pi \int_1^T \frac{dt}{t}$. Consequently, $f'(T) \underset{T\to\infty}{\sim} \pi \ln(T)$. Resorting to Theorem 4.2.12 one more time, $\int_1^T f'(t)dt \underset{T\to\infty}{\sim} \pi \int_1^T \ln(t)dt$. As $t \mapsto t\ln(t) - t$ is a primitive of $t \mapsto \ln(t)$, we easily see that $f(T) \underset{T\to\infty}{\sim} \pi T \ln(T)$. To put it differently, $\frac{f(T)}{T} \sim \pi \ln(T)$. Now, coming back to (4.2.28), when n tends to infinity, $T := \frac{4^n t}{2}$ tends to infinity and

$$I \underset{n\to\infty}{\sim} \pi \ln\left(\frac{4^n t}{2}\right).$$

As a result,

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha+3}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 \underset{n\to\infty}{\sim} \pi t n \ln(4).$$

□

Lemma 4.2.12. *Fix $a \in \mathbb{R}$ and let $g : [a,+\infty[\to \mathbb{R}$, $h : [a,+\infty[\to (0,\infty)$, be two continuous functions. If $g(t) \underset{t\to\infty}{\sim} h(t)$ and $\int_a^{+\infty} h(t)dt = \infty$, then*

$$\int_a^T g(t)dt \underset{T\to\infty}{\sim} \int_a^T h(t)dt\,.$$

Proof of Lemma 4.2.10. Let us introduce the holomorphic function Ψ defined by, for all $z \in \mathbb{C}\backslash\mathbb{R}^-$,

$$\Psi(z) := z^\beta e^{-z^2}.$$

Let $R > 1$ and $0 < \varepsilon < 1$ two real numbers. Our objective is to calculate the integral of Ψ on the closed curve $\mathcal{C}$ defined by the four points $\varepsilon, R, e^{i\frac{\pi}{4}}R$ and $e^{i\frac{\pi}{4}}\varepsilon$. We write

$$\begin{aligned}\int_{\mathcal{C}} \Psi(z)dz &= \int_\varepsilon^R t^\beta e^{-t^2}dt + \int_0^{\frac{\pi}{4}} iR^{\beta+1}e^{it(\beta+1)}e^{-R^2\exp 2it}dt \\ &\quad - \int_\varepsilon^R t^\beta e^{i\frac{\pi}{4}(\beta+1)}e^{-it^2}dt - \int_0^{\frac{\pi}{4}} i\varepsilon^{\beta+1}e^{it(\beta+1)}e^{-\varepsilon^2\exp 2it}dt \\ &= I_1 + I_2 - I_3 - I_4.\end{aligned}$$

We study the limits of the previous integrals. As R tends to infinity and ε tends to 0, I_1 converges to

$$\begin{aligned}\int_0^{+\infty} t^\beta e^{-t^2}dt &= \frac{1}{2}\int_0^{+\infty} u^{\frac{\beta-1}{2}}e^{-u}du \\ &= \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2}\end{aligned}$$

which is well-defined since $\mathcal{R}e(\beta) > -1$. The second integral I_2 tends to 0 when R tends to infinity thanks to the assumption $\mathcal{R}e(\beta) < 1$, as shown by the following inequalities

$$\begin{aligned}
|I_2| &\leq \int_0^{\frac{\pi}{4}} R^{\mathcal{R}e(\beta)+1} e^{-R^2\cos(2t)} dt \\
&\leq \frac{1}{2}\int_0^{\frac{\pi}{2}} R^{\mathcal{R}e(\beta)+1} e^{-R^2\cos(u)} du \\
&\leq \frac{1}{2} R^{\mathcal{R}e(\beta)+1} \int_0^{\frac{\pi}{2}} e^{R^2(\frac{2u}{\pi}-1)} du \\
&\leq \frac{\pi}{4} R^{\mathcal{R}e(\beta)-1}(1-e^{-R^2})
\end{aligned}$$

where we have used the concavity inequality, for all $u \in [0, \frac{\pi}{2}]$,

$$1 - \frac{2u}{\pi} \leq \cos(u).$$

The same estimates lead us to

$$|I_4| \leq \frac{\pi}{4}\varepsilon^{\mathcal{R}e(\beta)-1}(1-e^{-\varepsilon^2})$$

which tends to 0 as ε tends to 0 since $\mathcal{R}e(\beta) > -1$. Now, since Ψ is holomorphic on $\mathcal{C}$, Cauchy integral theorem states that

$$\int_{\mathcal{C}} \Psi(z)dz = 0 = I_1 + I_2 - I_3 - I_4.$$

As I_1, I_2 and I_4 are convergent integrals, I_3 has a limit:

$$\lim_{\varepsilon\to 0,\ R\to+\infty} I_3 = e^{i\frac{\pi}{4}(\beta+1)} \int_0^{+\infty} t^\beta e^{-it^2} dt.$$

We have thus obtained the desired conclusion

$$\int_0^{+\infty} t^\beta e^{-it^2} dt = e^{-i\frac{\pi}{4}(\beta+1)} \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2}.$$

□

Proof of Lemma 4.2.11. We proceed in two steps. We first establish that, for all $\alpha \in]0,1[$,

$$\int_0^{+\infty} \frac{dt}{t^\alpha(1+t)} = \frac{\pi}{\sin \pi\alpha}.$$

To this end, let us introduce, for all $\alpha \in]0,1[$,

$$I_\alpha := \int_0^{+\infty} \frac{dt}{t^\alpha(1+t)}.$$

We easily see that $\frac{1}{t^\alpha(1+t)} \underset{t\to 0}{\sim} \frac{1}{t^\alpha}$ which is integrable in 0 since $\alpha<1$ and that $\frac{1}{t^\alpha(1+t)} \underset{t\to+\infty}{\sim} \frac{1}{t^{\alpha+1}}$ which is integrable in $+\infty$ since $\alpha>0$. Thus, I_α is finite. Let $\Omega=\mathbb{C}\backslash\mathbb{R}^+$ and the function f defined by, for all $z\in\Omega\backslash\{-1\}$,

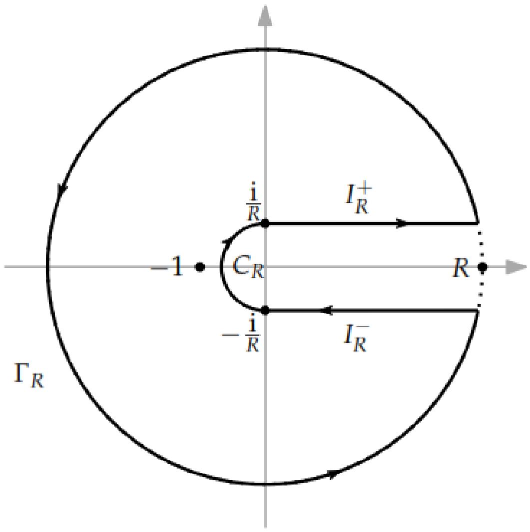
$$f(z):=\frac{1}{z^\alpha(1+z)},$$

where the determination used is $z^\alpha=r^\alpha e^{i\alpha\theta}$ if $z=re^{i\theta}$ with $\theta\in]0,2\pi[$. f is holomorphic on $\Omega\backslash\{-1\}$ and -1 is a simple pole such that:

$$\operatorname{Res}(f,-1)=\lim_{z\to-1}(1+z)f(z)=\frac{1}{(-1)^\alpha}=\mathrm{e}^{-\mathrm{i}\pi\alpha}.$$

For $R>1$, we consider the closed curve $\gamma_R=-C_R\cup I_R^+\cup\Gamma_R\cup -I_R^-$, where: $C_R=\left\{\frac{1}{R}e^{i\theta}|\theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right\}$, $I_R^\pm=\left[\pm\frac{i}{R},\pm\frac{i}{R}+\sqrt{R^2-\frac{1}{R^2}}\right]$, $\Gamma_R=\left\{Re^{i\theta}|\theta\in[\theta_R,2\pi-\theta_R]\right\}$, with $\theta_R=\arcsin\frac{\frac{1}{R}}{R}$, in such a way that

$$\int_{\gamma_R}f(z)dz=-\int_{C_R}f(z)dz+\int_{I_R^+}f(z)dz+\int_{\Gamma_R}f(z)dz-\int_{I_R^-}f(z)dz.$$



According to the residue theorem, for all $R>1$,

$$\int_{\gamma_R}f(z)dz=2i\pi e^{-i\pi\alpha}.$$

Let us study the four other integrals.

$$\left|\int_{C_R}f(z)dz\right|=\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}f\left(\frac{1}{R}e^{i\theta}\right)i\frac{1}{R}e^{i\theta}d\theta\right|\leqslant\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{\frac{1}{R}}{\left(\frac{1}{R}\right)^\alpha\left|1+\frac{1}{R}e^{i\theta}\right|}d\theta\leqslant\pi\frac{\left(\frac{1}{R}\right)^{1-\alpha}}{1-\frac{1}{R}}\underset{R\to+\infty}{\longrightarrow}0.$$

A similar computation shows that

$$\left|\int_{\Gamma_R}f(z)dz\right|=\left|\int_0^{2\pi}\mathbb{1}_{[\theta_R,2\pi-\theta_R]}(\theta)\frac{iRe^{i\theta}}{R^\alpha e^{i\alpha\theta}\left(1+Re^{i\theta}\right)}d\theta\right|\leqslant\int_0^{2\pi}\frac{R}{R^\alpha\left|1+Re^{i\theta}\right|}d\theta\leqslant 2\pi\frac{R^{1-\alpha}}{R-1}\underset{R\to+\infty}{\longrightarrow}0.$$

The third integral verifies

$$\int_{I_R^+} f(z)dz = \int_0^{\sqrt{R^2-\frac{1}{R^2}}} f\left(\frac{i}{R}+t\right)dt = \int_0^{\sqrt{R^2-\frac{1}{R^2}}} \frac{1}{\left(t+\frac{i}{R}\right)^\alpha\left(1+t+\frac{i}{R}\right)}dt.$$

We easily see that

$$\left(t+\frac{i}{R}\right)^\alpha = \left(\sqrt{t^2+\frac{1}{R^2}}\exp\left(i\arctan\frac{\frac{1}{R}}{t}\right)\right)^\alpha \underset{R\to+\infty}{\longrightarrow} t^\alpha,$$

leading to

$$\mathbb{1}_{\left]0,\sqrt{R^2-\frac{1}{R^2}}\right[}(t) f\left(\frac{i}{R}+t\right) \underset{R\to+\infty}{\longrightarrow} \mathbb{1}_{\mathbb{R}^{+*}}(t)\frac{1}{t^\alpha(1+t)}.$$

As $\left|\mathbb{1}_{\left]0,\sqrt{R^2-\frac{1}{R^2}}\right[}(t) f\left(\frac{i}{R}+t\right)\right| \leqslant \mathbb{1}_{\mathbb{R}^{+*}}(t)\frac{1}{t^\alpha(1+t)}$, by the dominated convergence theorem, we deduce

$$\lim_{R\to+\infty}\int_{I_R^+} f(z)\mathrm{d}z = I_\alpha.$$

Observing that $\left(t-\frac{i}{R}\right)^\alpha \underset{R\to+\infty}{\longrightarrow} t^\alpha e^{2i\pi\alpha}$, we get

$$\lim_{R\to+\infty}\int_{I_R^-} f(z)\mathrm{d}z = e^{-2i\pi\alpha} I_\alpha.$$

To sum up, $(1-e^{-2i\pi\alpha})\, I_\alpha = 2i\pi e^{-i\pi\alpha}$, that is

$$I_\alpha = \frac{\pi}{\sin\pi\alpha}.$$

We are now able to prove the desired identity. According to the principle of isolated zeros, it suffices to obtain

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

for $z=\alpha\in]0,1[$. We easily calculate

$$\begin{aligned}\Gamma(\alpha)\Gamma(1-\alpha) &= \left(\int_0^{+\infty} t^{\alpha-1}e^{-t}dt\right)\left(\int_0^{+\infty} s^{-\alpha}e^{-s}ds\right) = \int_0^{+\infty}\int_0^{+\infty} s^{-\alpha}t^{\alpha-1}e^{-t-s}dtds \\ &= \int_0^{+\infty}\int_0^{+\infty}\left(\frac{t}{s}\right)^\alpha e^{-(s+t)}ds\frac{dt}{t}.\end{aligned}$$

Let us perform the change of variables $u=s+t$ and $v=\frac{s}{t}$ whose Jacobian (in absolute value) equals $\frac{v+1}{t}$. Then,

$$\begin{aligned}\Gamma(\alpha)\Gamma(1-\alpha) &= \int_0^{+\infty}\int_0^{+\infty} v^{-\alpha}e^{-u}\frac{dudv}{v+1} = \int_0^{+\infty}\frac{1}{v^\alpha(v+1)}\int_0^{+\infty}\mathrm{e}^{-u}dudv = \int_0^{+\infty}\frac{dv}{v^\alpha(v+1)} \\ &= \frac{\pi}{\sin\pi\alpha}.\end{aligned}$$

$\square$

Second proof

We now propose an alternative proof to Proposition 4.2.8 that is more technical but more natural and that allows us to gather the cases $\kappa > 0$ and $\kappa = 0$.

Proposition 4.2.13. *Fix $t > 0$. For all $\alpha \in (0,2)$ and $0 \le \kappa < 1$ verifying $\alpha + \kappa > 1$, one has as n tends to infinity,*

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 = ct \int_1^{4^n} \frac{dr}{r^{1-\kappa}} + O(1)\,.$$

Proof. For the sake of clarity, we suppose $0 < t \le 1$. We first write

$$\begin{aligned}&\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 \\ &= \int_0^{1} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 + \int_1^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2\,.\end{aligned}$$

With the help of Lemma 4.2.4, we then observe that

$$\begin{aligned}&\left|\int_0^{1} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2\right| \\ &\le \int_0^{1} \frac{dr}{r^{-2\alpha-2\kappa+3}} \left[\int_{|\xi|\le 1} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 + \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2\right] \\ &\lesssim \int_0^{1} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 1} \frac{d\xi}{|\xi|^{\alpha-1}} + \int_0^{1} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{1+\alpha}}\,.\end{aligned}$$

Therefore,

$$\int_0^{1} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 = O(1)$$

and it suffices to focus on the estimation of the integral

$$\int_1^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2\,.$$

An easy computation provides us with the identity

$$|\gamma_t(\xi,r)|^2 = 2\frac{1-\cos(t(\xi-r^2))}{|r^2-\xi|^2}\,.$$

We now perform the change of variables $r^2 \leftarrow r$ and we split the integral into three terms:

$$\begin{aligned}\int_1^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 &= \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_{|\xi|\le 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{|r-\xi|^2} \\ &= \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_0^{r} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{|r-\xi|^2} \\ &\quad + \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_r^{4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{|r-\xi|^2} \\ &\quad + \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_0^{4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi+r))}{|r+\xi|^2} \\ &:= I^1_{n,t} + I^2_{n,t} + I^3_{n,t}.\end{aligned}$$

Study of $I^1_{n,t}$: By using the elementary changes of variables $u = \frac{\xi}{r}$ and $\tilde{\xi} = 1-u$, we have

$$\begin{aligned}
I^1_{n,t} &= \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_0^r \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{|r-\xi|^2} \\
&= \int_1^{4^n} \frac{dr}{r^{-\kappa+2}} \int_0^1 \frac{du}{|u|^{\alpha-1}} \frac{1-\cos(tr(1-u))}{(1-u)^2} \\
&= \int_1^{4^n} \frac{dr}{r^{-\kappa+2}} \int_0^{\frac{1}{2}} \frac{d\xi}{|1-\xi|^{\alpha-1}} \frac{1-\cos(tr\xi)}{\xi^2} + \int_1^{4^n} \frac{dr}{r^{-\kappa+2}} \int_{\frac{1}{2}}^1 \frac{d\xi}{|1-\xi|^{\alpha-1}} \frac{1-\cos(tr\xi)}{\xi^2} \\
&=: A_{n,t} + B_{n,t}\,.
\end{aligned}$$

The treatment of $B_{n,t}$ is easy, since

$$|B_{n,t}| \lesssim \int_1^\infty \frac{dr}{r^{-\kappa+2}} \int_{\frac{1}{2}}^1 \frac{d\xi}{|1-\xi|^{\alpha-1}} < \infty\,.$$

As far as $A_{n,t}$ is concerned, we have

$$\begin{aligned}
A_{n,t} &= t\int_1^{4^n} \frac{dr}{r^{1-\kappa}} \int_0^{\frac{rt}{2}} \frac{d\xi}{|1-\frac{\xi}{rt}|^{\alpha-1}} \frac{1-\cos(\xi)}{\xi^2} \\
&= t\int_0^\infty d\xi \frac{1-\cos(\xi)}{\xi^2} \int_1^{4^n} \frac{dr}{r^{1-\kappa}} \\
&\quad + t\int_1^{4^n} \frac{dr}{r^{1-\kappa}} \left[\int_0^{\frac{rt}{2}} \frac{d\xi}{|1-\frac{\xi}{rt}|^{\alpha-1}} \frac{1-\cos(\xi)}{\xi^2} - \int_0^\infty d\xi \frac{1-\cos(\xi)}{\xi^2} \right].
\end{aligned}$$

We now resort to the following result, proven in [20]:

Lemma 4.2.14. *Given $\alpha \in (0,2)$ and $\varepsilon \in (0,1)$, one has, for all $r > 0$,*

$$\left| \int_0^{\frac{r}{2}} \frac{d\xi}{|1-\frac{\xi}{r}|^{\alpha-1}} \frac{1-\cos(\xi)}{\xi^2} - \int_0^\infty d\xi \frac{1-\cos(\xi)}{\xi^2} \right| \leq c_{\alpha,\varepsilon} \left[\frac{1}{r} + \frac{1}{r^{1-\varepsilon}} \right].$$

Thus,

$$A_{n,t} = ct \int_1^{4^n} \frac{dr}{r^{1-\kappa}} + O(1)$$

and

$$I^1_{n,t} = ct \int_1^{4^n} \frac{dr}{r^{1-\kappa}} + O(1). \tag{4.2.29}$$

Study of $I^2_{n,t}$: Let us here write

$$\frac{1-\cos(t(\xi-r))}{r^2(r-\xi)^2} = 2\frac{1-\cos(t(\xi-r))}{r(\xi-r)^2(r+\xi)} + \frac{1-\cos(t(\xi-r))}{r^2(\xi-r)(r+\xi)}$$

and accordingly $I^2_{n,t} = 2I^{2,1}_{n,t} + I^{2,2}_{n,t}$, with

$$I^{2,1}_{n,t} := \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+1}} \int_r^{4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{(\xi-r)^2(r+\xi)}$$

and

$$I_{n,t}^{2,2} := \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_r^{4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{(\xi-r)(r+\xi)}.$$

On the one hand, we can remark that

$$\begin{aligned}
I_{n,t}^{2,1} &= \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+1}} \int_r^{4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi-r))}{(\xi-r)^2(r+\xi)} \\
&= \int_1^{4^n} \frac{dr}{r^{1-\alpha-\kappa}} \int_0^{4^n-r} \frac{d\xi}{|\xi+r|^{\alpha-1}|\xi+2r|} \frac{1-\cos(t\xi)}{\xi^2} \\
&= \frac{t}{2} \int_1^{4^n} \frac{dr}{r^{1-\kappa}} \int_0^{\infty} d\xi \frac{1-\cos(\xi)}{\xi^2} \\
&+ \int_1^{4^n} \frac{dr}{r^{1-\alpha-\kappa}} \left[\int_0^{4^n-r} \frac{d\xi}{|\xi+r|^{\alpha-1}|\xi+2r|} \frac{1-\cos(t\xi)}{\xi^2} - \frac{1}{2r^\alpha} \int_0^\infty d\xi \frac{1-\cos(t\xi)}{\xi^2} \right].
\end{aligned}$$

Let us here recall the following technical lemma, borrowed from [20]:

Lemma 4.2.15. *Given $\alpha \in (0,2)$ and $\varepsilon \in (0, \min(\alpha,1))$, one has, for all $t>0$, $n \geq 0$ and $1<r<4^n$,*

$$\left| \int_0^{4^n-r} \frac{d\xi}{|\xi+r|^{\alpha-1}|\xi+2r|} \frac{1-\cos(t\xi)}{\xi^2} - \frac{1}{2r^\alpha} \int_0^\infty d\xi \frac{1-\cos(t\xi)}{\xi^2} \right| \leq c_{\alpha,\varepsilon} \left[\frac{t^\varepsilon}{r^{1+\alpha-\varepsilon}} + \frac{t^\varepsilon}{r^\alpha |4^n-r|^{1-\varepsilon}} \right].$$

Now, fix $0<\varepsilon<\min(\alpha, \frac{1-\kappa}{2}, 1)$. Using the latter result, we obtain that

$$\begin{aligned}
&\left| \int_1^{4^n} \frac{dr}{r^{1-\alpha-\kappa}} \Big[\int_0^{4^n-r} \frac{d\xi}{|\xi+r|^{\alpha-1}|\xi+2r|} \frac{1-\cos(t\xi)}{\xi^2} - \frac{1}{2r^\alpha} \int_0^\infty d\xi \frac{1-\cos(t\xi)}{\xi^2} \Big] \right| \\
&\lesssim t^\varepsilon \int_1^{4^n} \frac{dr}{r^{-\kappa+2-\varepsilon}} + t^\varepsilon \int_1^{4^n} \frac{dr}{r^{-\kappa+1}|4^n-r|^{1-\varepsilon}} \\
&\lesssim t^\varepsilon \int_1^{\infty} \frac{dr}{r^{-\kappa+2-\varepsilon}} + t^\varepsilon 4^{-n(1-\kappa-2\varepsilon)} \int_0^1 \frac{dr}{r^{-\kappa+1-\varepsilon}|1-r|^{1-\varepsilon}}
\end{aligned}$$

and these integrals are clearly finite thanks to our assumptions on ε.

On the other hand, one has easily

$$\begin{aligned}
|I_{n,t}^{2,2}| &\lesssim t^\varepsilon \int_1^\infty \frac{dr}{r^{-\alpha-\kappa+2}} \int_r^\infty \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1}{|\xi-r|^{1-\varepsilon}(\xi+r)} \\
&\lesssim t^\varepsilon \int_1^\infty \frac{dr}{r^{2-\kappa-\varepsilon}} \int_1^\infty \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1}{|\xi-1|^{1-\varepsilon}(\xi+1)}
\end{aligned}$$

and the latter integrals are finite as soon as $\varepsilon \in (0, \min(\alpha, 1-\kappa))$. We finally obtain

$$I_{n,t}^2 = ct \int_1^{4^n} \frac{dr}{r^{1-\kappa}} + O(1)\,. \tag{4.2.30}$$

Study of $I^3_{n,t}$: By using the elementary change of variables $u = \frac{\xi}{r}$, we have

$$\begin{aligned}
|I^3_{n,t}| &= \left| \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_0^{4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1-\cos(t(\xi+r))}{|r+\xi|^2} \right| \\
&\lesssim \int_1^{4^n} \frac{dr}{r^{-\alpha-\kappa+2}} \int_0^{4^n} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1}{|r+\xi|^2} \\
&\lesssim \int_1^{4^n} \frac{dr}{r^{-\kappa+2}} \int_0^{\infty} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1}{|1+\xi|^2} \\
&\lesssim \int_1^{\infty} \frac{dr}{r^{-\kappa+2}} \int_0^{\infty} \frac{d\xi}{|\xi|^{\alpha-1}} \frac{1}{|1+\xi|^2} \, .
\end{aligned}$$

Thus,

$$I^3_{n,t} = O(1). \tag{4.2.31}$$

It now suffices to sum the three terms (4.2.29), (4.2.30) and (4.2.31) to obtain the asymptotic estimate we are looking for, that is

$$\int_0^{2^n} \frac{dr}{r^{-2\alpha-2\kappa+3}} \int_{|\xi|\leq 4^n} \frac{d\xi}{|\xi|^{\alpha-1}} |\gamma_t(\xi,r)|^2 = ct \int_1^{4^n} \frac{dr}{r^{1-\kappa}} + O(1).$$

□

4.2.5 Global definition of the linear solution

At first reading, the result of Proposition 4.1.2 only provides us with a *local* definition of the process Ψ. In other words, what is actually given by the statement is the set of the limit elements $\{\chi\Psi,\, \chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}$. For the sake of rigor, let us say a few words about how those elements can be glued together into a single process Ψ, which we can then inject into the transformation $v := u - \Psi$ of Definition 4.1.4 or Definition 4.1.8.

To this end, fix $p \geq 2$ and α satisfying (4.1.11). Let us denote by $\mathcal{P}$ the set of sequences $\sigma = (\sigma_k)_{k\geq 1}$ such that for each $k \geq 1$, $\sigma_k : \mathbb{R}^d \to \mathbb{R}$ is a smooth function satisfying

$$\sigma_k(x) = \begin{cases} 1 & \text{if } \|x\| \leq k\,, \\ 0 & \text{if } \|x\| \geq k+1\,. \end{cases}$$

Given such a sequence σ, and for each fixed $k \geq 1$, let us denote by $\Psi^{(\sigma_k)}$ the limit of the sequence $(\sigma_k \Psi_n)_{n\geq 1}$ in the space $\mathcal{C}([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$, as provided by Proposition 4.1.2. We know in particular that $\Psi^{(\sigma_k)}$ is defined on a probability space $\Omega^{(\sigma_k)}$ of full measure 1. Let us set $\Omega^{(\sigma)} := \cap_{k\geq 1} \Omega^{(\sigma_k)}$, and note that this space is still of measure 1.

For every fixed time $t \in [0,T]$, we now define the random distribution

$$\Psi^{(\sigma)}(t) : \Omega^{(\sigma)} \to \mathcal{D}'(\mathbb{R}^d)$$

as follows: for every test function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\mathrm{supp}(\varphi) \subset B(0,k)$ (for some $k \geq 1$),

$$\langle \Psi^{(\sigma)}(t), \varphi \rangle := \langle \Psi^{(\sigma_k)}(t), \varphi \rangle \, .$$

Proposition 4.2.16. *(i) The above distribution* $\Psi^{(\sigma)}$ *is well defined, i.e. for all* $1 \leq k \leq \ell$ *and for every test function* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *with* $\operatorname{supp}(\varphi) \subset B(0,k) \subset B(0,\ell)$*, one has*

$$\langle \Psi^{(\sigma_k)}(t), \varphi \rangle = \langle \Psi^{(\sigma_\ell)}(t), \varphi \rangle \quad \textit{on} \quad \Omega^{(\sigma)}\,.$$

(ii) For any test function $\chi : \mathbb{R}^d \to \mathbb{R}$*, one has, on* $\Omega^{(\sigma)}$*,*

$$\chi \cdot \Psi_n \underset{n\to\infty}{\to} \chi \cdot \Psi^{(\sigma)} \quad \textit{in} \;\; \mathcal{C}([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))\,.$$

(iii) If $\sigma, \gamma \in \mathcal{P}$*, it holds that*

$$\Psi^{(\sigma)} = \Psi^{(\gamma)} \quad \textit{on} \;\; \Omega^{(\sigma)} \cap \Omega^{(\gamma)}\,.$$

Due to the latter identification property, we simply set $\Psi := \Psi^{(\sigma)}$*, for some fixed element* $\sigma \in \mathcal{P}$*.*

Proof. (i) We first show that for $1 \leq k \leq \ell$

$$\Psi^{(\sigma_k)} = \sigma_k \Psi^{(\sigma_\ell)} \quad \text{on} \;\; \Omega^{(\sigma_k)} \cap \Omega^{(\sigma_\ell)}. \tag{4.2.32}$$

By Proposition 4.1.2, the sequence $\sigma_k \sigma_\ell \Psi_n = \sigma_k \Psi_n$ converges almost surely to $\Psi^{(\sigma_k)}$ in the space $\mathcal{C}([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$, on $\Omega^{(\sigma_k)}$. But by continuity of the product in $\mathcal{C}([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$ by the test function σ_k, we also have that $\sigma_k \sigma_\ell \Psi_n$ converges almost surely to $\sigma_k \Psi^{(\sigma_\ell)}$ in $\mathcal{C}([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$ on $\Omega^{(\sigma_\ell)}$, and we deduce (4.2.32) by uniqueness of the limit. Therefore by (4.2.32), for all $t \in [0,T]$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(\varphi) \subset B(0,k)$

$$\langle \Psi^{(\sigma_k)}(t), \varphi \rangle = \langle \sigma_k \Psi^{(\sigma_\ell)}(t), \varphi \rangle = \langle \Psi^{(\sigma_\ell)}(t), \sigma_k \varphi \rangle = \langle \Psi^{(\sigma_\ell)}(t), \varphi \rangle \quad \text{on} \;\; \Omega^{(\sigma)}.$$

(ii) Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, then there exists $k \geq 1$ such that $\operatorname{supp}(\chi) \subset B(0,k)$. According to Proposition 4.1.2, $\chi \Psi_n = \chi \sigma_k \Psi_n$ converges almost surely to $\chi \Psi^{(\sigma_k)}$ in $\mathcal{C}([0,T]; \mathcal{W}^{-\alpha,p}(\mathbb{R}^d))$ on $\Omega^{(\sigma_k)}$. But $\chi \Psi^{(\sigma_k)} = \chi \Psi^{(\sigma)}$ on $\Omega^{(\sigma_k)}$. Indeed, for all $t \in [0,T]$, if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\langle \chi \Psi^{(\sigma)}(t), \varphi \rangle = \langle \Psi^{(\sigma)}(t), \chi\varphi \rangle = \langle \Psi^{(\sigma_k)}(t), \chi\varphi \rangle = \langle \chi \Psi^{(\sigma_k)}(t), \varphi \rangle,$$

where we have used that $\operatorname{supp}(\chi\varphi) \subset B(0,k)$ to derive the second equality.

(iii) For all $t \in [0,T]$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(\varphi) \subset B(0,k)$, we have

$$\langle \sigma_k \Psi_n(t), \varphi \rangle = \langle \sigma_k \gamma_k \Psi_n(t), \varphi \rangle = \langle \gamma_k \Psi_n(t), \sigma_k \varphi \rangle = \langle \gamma_k \Psi_n(t), \varphi \rangle,$$

then by taking the limit $n \longrightarrow +\infty$, we get $\langle \Psi^{(\sigma_k)}(t), \varphi \rangle = \langle \Psi^{(\gamma_k)}(t), \varphi \rangle$ on $\Omega^{(\sigma_k)} \cap \Omega^{(\gamma_k)}$, hence the result. □

Remark 4.2.17. The above patching procedure could of course be applied to the second-order process $\chi^2 \mathcal{V}$ as well, leading to a well-defined distribution-valued function $\mathcal{V}$.

4.3 Deterministic analysis of the equation under condition (H1)

In this section, we propose to analyze the equation in the *regular* situation, that is when assumption (**H1**) on the Hurst index is satisfied, and the linear solution $\rho^{♀}$ (defined by Proposition 4.1.2) is known to be a function of time *and* space. Remember that in this case, the model is interpreted through Definition 4.1.4. Thus, what we wish to solve in this section is the equation

$$v_t = S_t(\phi) - i\int_0^t S_{t-\tau}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot(\rho^{♀}_\tau))\,d\tau \\ - i\int_0^t S_{t-\tau}((\rho v_\tau)\cdot(\overline{\rho^{♀}_\tau}))\,d\tau - i\int_0^t S_{t-\tau}(|\rho^{♀}_\tau|^2)\,d\tau\,, \quad t\in[0,T]\,. \quad (4.3.1)$$

As opposed to the stochastic arguments used in the previous section, our strategy towards a (local) solution v for (4.3.1) will rely on deterministic estimates only. In other words, we henceforth consider $\rho^{♀}$ as a fixed (i.e., non-random) element in the space

$$\mathcal{E}_\beta := \bigcap_{2\leq p\leq\infty} \mathcal{C}([0,T];\mathcal{W}^{\beta,p}(\mathbb{R}^d)) = \mathcal{C}([0,T];H^\beta(\mathbb{R}^d)) \cap \mathcal{C}([0,T];\mathcal{W}^{\beta,\infty}(\mathbb{R}^d))\,,$$

for some appropriate $0<\beta<1$ (where $\beta=-\alpha$ is given by Proposition 4.1.2), and try to solve the following deterministic equation: for $\mathbf{\Psi}\in\mathcal{E}_\beta$,

$$v_t = S_t(\phi) - i\int_0^t S_{t-\tau}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot\mathbf{\Psi}_\tau)\,d\tau \\ - i\int_0^t S_{t-\tau}((\rho v_\tau)\cdot\overline{\mathbf{\Psi}}_\tau)\,d\tau - i\int_0^t S_{t-\tau}(|\mathbf{\Psi}_\tau|^2)\,d\tau\,. \quad (4.3.2)$$

Let us set the stage for this solving procedure by reporting on fundamental estimates related to the two main operations in (4.3.2).

4.3.1 Pointwise multiplication

Lemma 4.3.1 (Fractional Leibniz rule, see [42, Proposition 1.1, p. 105])**.** *Let $s\geq 0$, $1<r<\infty$ and $1<p_1,p_2,q_1,q_2<\infty$ satisfying*

$$\frac{1}{r} = \frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{q_1}+\frac{1}{q_2}\,.$$

Then one has

$$\|u\cdot v\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \lesssim \|u\|_{\mathcal{W}^{s,p_1}(\mathbb{R}^d)}\|v\|_{L^{p_2}(\mathbb{R}^d)} + \|u\|_{L^{q_1}(\mathbb{R}^d)}\|v\|_{\mathcal{W}^{s,q_2}(\mathbb{R}^d)}\,.$$

4.3.2 Convolution with the Schrödinger group

Naturally, we also need some control on the operation $(\phi, F, t) \mapsto S_t\phi - i\int_0^t S_{t-s}(F_s)\,ds$, or otherwise stated some estimate on the solution u to the general Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) - \Delta u(t,x) = F(t,x)\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u(0,x) = \phi(x)\,. \end{cases} \tag{4.3.3}$$

Such a control is classically provided by the so-called Strichartz inequalities, which will prove to be sufficient for our purpose in this functional setting.

Definition 4.3.2. *A pair (p,q) is said to be Schrödinger admissible if*

$$(p,q) \in [2,\infty]^2,\ (p,q,d) \neq (2,\infty,2),\ \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

Lemma 4.3.3 (Strichartz inequalities, see [4, Paragraph 2.3])**.** *Fix $d \geq 1$, $s \in \mathbb{R}$, and let u stand for the mild solution of equation* (4.3.3)*.*

Then for all Schrödinger admissible pairs (p,q) and (a,b), it holds that

$$\|u\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))} \lesssim \|\phi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^{a'}([0,T];\mathcal{W}^{s,b'}(\mathbb{R}^d))}\,, \tag{4.3.4}$$

where the notations a', b' refer to the Hölder conjugates of a, b.

4.3.3 Solving the equation

Our main result regarding equation (4.3.2) can be stated as follows:

Theorem 4.3.4. *Assume that $1 \leq d \leq 4$ and fix $\beta \in (0,1)$. Consider the Schrödinger admissible pair (p,q) given by the formulas*

$$p = \frac{12}{d-\beta}\ ,\ q = \frac{6d}{2d+\beta}\,,$$

and for every $T > 0$, define the space $X^\beta(T)$ as

$$X^\beta(T) := \mathcal{C}([0,T]; H^\beta(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{\beta,q}(\mathbb{R}^d)).$$

Then for all $\phi \in H^\beta(\mathbb{R}^d)$ and $\mathbf{\Psi} \in \mathcal{E}_\beta$, there exists a time $T_0 > 0$ such that equation (4.3.2) *admits a unique solution in $X^\beta(T_0)$.*

This local well-posedness result will be derived from a standard fixed-point argument. To this end, we introduce the map Γ defined by the right-hand side of (4.3.2), that is: for all $\mathbf{\Psi} \in \mathcal{E}_\beta$, $\phi \in H^\beta(\mathbb{R}^d)$, $v \in X^\beta(T)$, $T \geq 0$ and $t \in [0,T]$, set

$$\begin{aligned}\Gamma_{T,\mathbf{\Psi}}(v)_t := S_t(\phi) - i\int_0^t S_{t-\tau}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot\mathbf{\Psi}_\tau)\,d\tau \\ - i\int_0^t S_{t-\tau}((\rho v_\tau)\cdot\overline{\mathbf{\Psi}}_\tau)\,d\tau - i\int_0^t S_{t-\tau}(|\mathbf{\Psi}_\tau|^2)\,d\tau\,.\end{aligned}$$

Proposition 4.3.5. *In the setting of Theorem 4.3.4, the following bounds hold true: there exists $\varepsilon > 0$ such that for all $0 \leq T \leq 1$, $\phi \in H^\beta(\mathbb{R}^d), (\mathbf{\Psi}_1, \mathbf{\Psi}_2) \in \mathcal{E}_\beta \times \mathcal{E}_\beta$ and $v, v_1, v_2 \in X^\beta(T)$,*

$$\|\Gamma_{T,\mathbf{\Psi}_1}(v)\|_{X^\beta(T)} \lesssim \|\phi\|_{H^\beta(\mathbb{R}^d)} + T^\varepsilon\Big[\|v\|^2_{X^\beta(T)} + \|\mathbf{\Psi}_1\|_{L^\infty_T\mathcal{W}^{\beta,q}}\|v\|_{X^\beta(T)} + \|\mathbf{\Psi}_1\|^2_{L^\infty_T\mathcal{W}^{\beta,q}}\Big], \tag{4.3.5}$$

and

$$\begin{aligned}
&\|\Gamma_{T,\mathbf{\Psi}_1}(v_1) - \Gamma_{T,\mathbf{\Psi}_2}(v_2)\|_{X^\beta(T)} \\
&\lesssim T^\varepsilon\Big[\|v_1 - v_2\|_{X^\beta(T)}\Big\{\|v_1\|_{X^\beta(T)} + \|v_2\|_{X^\beta(T)}\Big\} + \|\mathbf{\Psi}_1 - \mathbf{\Psi}_2\|_{L^\infty_T\mathcal{W}^{\beta,q}}\|v_1\|_{X^\beta(T)} \\
&\qquad + \|\mathbf{\Psi}_2\|_{L^\infty_T\mathcal{W}^{\beta,q}}\|v_1 - v_2\|_{X^\beta(T)} + \|\mathbf{\Psi}_1 - \mathbf{\Psi}_2\|_{L^\infty_T\mathcal{W}^{\beta,q}}\Big\{\|\mathbf{\Psi}_1\|_{L^\infty_T\mathcal{W}^{\beta,q}} + \|\mathbf{\Psi}_2\|_{L^\infty_T\mathcal{W}^{\beta,q}}\Big\}\Big],
\end{aligned} \tag{4.3.6}$$

where the proportional constants only depend on β and ρ.

Before we turn to the proof of this proposition, let us briefly recall that, once endowed (4.3.5)-(4.3.6), the statement of Theorem 4.3.4 follows from a standard two-step procedure. Namely, using (4.3.5), we can first establish that for any $T = T(\phi, \mathbf{\Psi}) > 0$ small enough, there exists a ball in $X^\beta(T)$ that is stable through the application of $\Gamma_{T,\mathbf{\Psi}}$. Then, thanks to (4.3.6) (applied with $\mathbf{\Psi}_1 = \mathbf{\Psi}_2 = \mathbf{\Psi}$), we can show that $\Gamma_{T,\mathbf{\Psi}}$ is actually a contraction on this ball (for $T > 0$ possibly even smaller), which completes the proof of the assertion.

Note also that the continuity of $\Gamma_{T,\mathbf{\Psi}}$ with respect to $\mathbf{\Psi}$ (an immediate consequence of (4.3.6)) will be the key ingredient toward item *(ii)* of Theorem 4.1.10.

Proof of Proposition 4.3.5. Let us set, for any suitable distribution u on $\mathbb{R}^{d+1}$,

$$\mathcal{G}(u)_t := -i\int_0^t S_{t-\tau}(u_\tau)\, d\tau\,,$$

which allows to recast $\Gamma_{T,\mathbf{\Psi}}$ as

$$\Gamma_{T,\mathbf{\Psi}}(v) = S(\phi) + \mathcal{G}(\rho^2|v|^2) + \mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}) + \mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}}) + \mathcal{G}(|\mathbf{\Psi}|^2)\,.$$

Let us now bound each of the four above terms separately.

Bound on $S(\phi)$: Since $(\infty, 2)$ and (p, q) are both Schrödinger admissible pairs, we can apply Lemma 4.3.3 to assert that

$$\|S(\phi)\|_{X^\beta(T)} \lesssim \|\phi\|_{H^\beta}\,. \tag{4.3.7}$$

Bound on $\mathcal{G}(\rho^2|v|^2)$: By Lemma 4.3.3, we can first assert that

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X^\beta(T)} \lesssim \|\rho^2|v|^2\|_{L^{p'}_T\mathcal{W}^{\beta,q'}}\,.$$

Let us now introduce the additional parameter $n := \frac{3d}{d-\beta} > 1$, in such a way that

$$\frac{1}{q'} = \frac{1}{q} + \frac{1}{n}\,.$$

Using the fractional Leibniz rule given by Lemma 4.3.1, we get that for all $t \geq 0$,

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{\beta,q'}} \lesssim \|\rho v(t,.)\|_{\mathcal{W}^{\beta,q}}\|\rho v(t,.)\|_{L^n}\,.$$

It is easy to check that $\beta \geq d(\frac{1}{q} - \frac{1}{n})$, and accordingly we can rely on the Sobolev embedding

$$\mathcal{W}^{\beta,q}(\mathbb{R}^d) \hookrightarrow L^n(\mathbb{R}^d) \tag{4.3.8}$$

to derive that

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{\beta,q'}} \lesssim \|\rho v(t,.)\|^2_{\mathcal{W}^{\beta,q}}\,.$$

Consider the parameter m defined through the relation

$$\frac{1}{p'} = \frac{2}{p} + \frac{1}{m}\,.$$

Since $1 \leq d \leq 4$ and $\beta \in (0,1)$, it can actually be verified that $\frac{1}{m} = 1 - \frac{d-\beta}{4} > 0$. Then, by Hölder inequality, we have

$$\|\rho^2|v|^2\|_{L^{p'}_T\mathcal{W}^{\beta,q'}} \lesssim T^{\frac{1}{m}}\|v\|^2_{L^p_T\mathcal{W}^{\beta,q}} \lesssim T^{1-\frac{d-\beta}{4}}\|v\|^2_{X^\beta(T)}\,,$$

and we have thus shown that

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X^\beta(T)} \lesssim T^{1-\frac{d-\beta}{4}}\|v\|^2_{X^\beta(T)}\,. \tag{4.3.9}$$

Bound on $\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi})$, $\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}})$: Just as above, we can first apply Lemma 4.3.3 to get that

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi})\|_{X^\beta(T)} + \|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}})\|_{X^\beta(T)} \lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L^{p'}_T\mathcal{W}^{\beta,q'}}\,.$$

Thanks to Lemma 4.3.1, it holds, for all $t \geq 0$,

$$\begin{aligned}\|\rho\overline{v}\cdot\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{\beta,q'}} &\lesssim \|\rho v(t,.)\|_{\mathcal{W}^{\beta,q}}\|\mathbf{\Psi}(t,.)\|_{L^n} + \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{\beta,q}}\|\rho v(t,.)\|_{L^n}\\ &\lesssim \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{\beta,q}}\|\rho v(t,.)\|_{\mathcal{W}^{\beta,q}}\,,\end{aligned}$$

where we have again used the Sobolev embedding (4.3.8).

Then, by Hölder inequality, we deduce

$$\begin{aligned}\|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L^{p'}_T\mathcal{W}^{\beta,q'}} &\lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{L^\infty_T\mathcal{W}^{\beta,q}}\|v\|_{L^p_T\mathcal{W}^{\beta,q}}\\ &\lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{L^\infty_T\mathcal{W}^{\beta,q}}\|v\|_{X^\beta(T)}\,,\end{aligned}$$

and we have thus established that

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi})\|_{X^\beta(T)} + \|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}})\|_{X^\beta(T)} \lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{L^\infty_T\mathcal{W}^{\beta,q}}\|v\|_{X^\beta(T)}\,. \tag{4.3.10}$$

Bound on $\mathcal{G}(|\mathbf{\Psi}|^2)$: By Lemma 4.3.3,

$$\|\mathcal{G}(|\mathbf{\Psi}|^2)\|_{X^\beta(T)} \lesssim \||\mathbf{\Psi}|^2\|_{L_T^{p'}\mathcal{W}^{\beta,q'}} .$$

Using Lemma 4.3.1 and the Sobolev embedding (4.3.8), we get that for every $t \geq 0$,

$$\||\mathbf{\Psi}|^2(t,.)\|_{\mathcal{W}^{\beta,q'}} \lesssim \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{\beta,q}}\|\mathbf{\Psi}(t,.)\|_{L^n} \lesssim \|\mathbf{\Psi}(t,.)\|^2_{\mathcal{W}^{\beta,q}} .$$

Then

$$\||\mathbf{\Psi}|^2\|_{L_T^{p'}\mathcal{W}^{\beta,q'}} \lesssim T^{\frac{1}{p'}}\|\mathbf{\Psi}\|^2_{L_T^\infty\mathcal{W}^{\beta,q}} ,$$

and finally

$$\|\mathcal{G}(|\mathbf{\Psi}|^2)\|_{X^\beta(T)} \lesssim T^{\frac{1}{p'}}\|\mathbf{\Psi}\|^2_{L_T^\infty\mathcal{W}^{\beta,q}} . \tag{4.3.11}$$

The combination of estimates (4.3.7), (4.3.9), (4.3.10) and (4.3.11) entails the desired bound (4.3.5).

It is then easy to see that (4.3.6) can be derived from similar arguments: for instance,

$$\|\mathcal{G}(\rho^2(|v_1|^2-|v_2|^2))\|_{X^\beta(T)} \lesssim \|\rho^2(|v_1|^2-|v_2|^2)\|_{L_T^{p'}\mathcal{W}^{\beta,q'}} \lesssim \||v_1|^2-|v_2|^2\|_{L_T^{p'}\mathcal{W}^{\beta,q'}} .$$

Combining again Lemma 4.3.1 and embedding (4.3.8), we obtain, for every $t \geq 0$,

$$\begin{aligned}\|(|v_1|^2-|v_2|^2)(t,.)\|_{\mathcal{W}^{\beta,q'}} &\lesssim \|(v_1-v_2)(t,.)\|_{\mathcal{W}^{\beta,q}}\{\|v_1(t,.)\|_{L^n}+\|v_2(t,.)\|_{L^n}\}\\ &\quad +\|(v_1-v_2)(t,.)\|_{L^n}\{\|v_1(t,.)\|_{\mathcal{W}^{\beta,q}}+\|v_2(t,.)\|_{\mathcal{W}^{\beta,q}}\}\\ &\lesssim \|(v_1-v_2)(t,.)\|_{\mathcal{W}^{\beta,q}}\{\|v_1(t,.)\|_{\mathcal{W}^{\beta,q}}+\|v_2(t,.)\|_{\mathcal{W}^{\beta,q}}\} ,\end{aligned}$$

and as a result

$$\||v_1|^2-|v_2|^2\|_{L_T^{p'}\mathcal{W}^{\beta,q'}} \lesssim T^{\frac{1}{m}}\|v_1-v_2\|_{L_T^p\mathcal{W}^{\beta,q}}\{\|v_1\|_{L_T^p\mathcal{W}^{\beta,q}}+\|v_2\|_{L_T^p\mathcal{W}^{\beta,q}}\} .$$

□

Proof of Theorem 4.3.4. Thanks to the bound (4.3.5), it holds that for all $0 \leq T \leq 1$ and for all $v \in X^\beta(T)$,

$$\|\Gamma_{T,\mathbf{\Psi}}(v)\|_{X^\beta(T)} \leq C_{\beta,\rho}\Big[\|\phi\|_{H^\beta(\mathbb{R}^d)} + T^\varepsilon(\|v\|^2_{X^\beta(T)} + \|\mathbf{\Psi}\|_{L_T^\infty\mathcal{W}^{\beta,q}}(\|v\|_{X^\beta(T)} + \|\mathbf{\Psi}\|_{L_T^\infty\mathcal{W}^{\beta,q}}))\Big].$$

Let $R > 0$ such that $R > C_{\beta,\rho}\|\phi\|_{H^\beta(\mathbb{R}^d)} + 1$. Then, for all $v \in \overline{B}_{X^\beta(T)}(0,R)$,

$$\|\Gamma_{T,\mathbf{\Psi}}(v)\|_{X^\beta(T)} \leq R - 1 + C_{\beta,\rho}T^\varepsilon\Big[R^2 + \|\mathbf{\Psi}\|_{L_T^\infty\mathcal{W}^{\beta,q}}(R + \|\mathbf{\Psi}\|_{L_T^\infty\mathcal{W}^{\beta,q}})\Big].$$

Besides, let $0 < T_1 \leq 1$ such that $C_{\beta,\rho}T_1^\varepsilon\Big[R^2 + \|\mathbf{\Psi}\|_{L_{T_1}^\infty\mathcal{W}^{\beta,q}}(R + \|\mathbf{\Psi}\|_{L_{T_1}^\infty\mathcal{W}^{\beta,q}})\Big] \leq \frac{1}{2}$. It implies that

$$\|\Gamma_{T_1,\mathbf{\Psi}}(v)\|_{X^\beta(T_1)} \leq R - 1 + \frac{1}{2} \leq R.$$

Thus, $\Gamma_{T_1,\mathbf{\Psi}}$ lets $\overline{B}_{X^\beta(T_1)}(0,R)$ stable. Now, thanks to the bound (4.3.6), it holds that for all $0 \leq T \leq 1$ and for all $v_1, v_2 \in X^\beta(T)$,

$$\|\Gamma_{T,\mathbf{\Psi}}(v_1) - \Gamma_{T,\mathbf{\Psi}}(v_2)\|_{X^\beta(T)} \leq C_{\beta,\rho} T^\varepsilon \|v_1 - v_2\|_{X^\beta(T)} \Big[\|v_1\|_{X^\beta(T)} + \|v_2\|_{X^\beta(T)} + \|\mathbf{\Psi}\|_{L^\infty_T \mathcal{W}^{\beta,q}} \Big].$$

Then, for all $v_1, v_2 \in \overline{B}_{X^\beta(T_1)}(0,R)$, we get

$$\|\Gamma_{T_1,\mathbf{\Psi}}(v_1) - \Gamma_{T_1,\mathbf{\Psi}}(v_2)\|_{X^\beta(T_1)} \leq C_{\beta,\rho} T_1^\varepsilon (2R + \|\mathbf{\Psi}\|_{L^\infty_{T_1} \mathcal{W}^{\beta,q}}) \|v_1 - v_2\|_{X^\beta(T_1)}.$$

Finally, with $0 < T_0 \leq T_1$ such that $C_{\beta,\rho} T_0^\varepsilon (2R + \|\mathbf{\Psi}\|_{L^\infty_{T_0} \mathcal{W}^{\beta,q}}) \leq \frac{1}{2}$, the inequality

$$\|\Gamma_{T_0,\mathbf{\Psi}}(v_1) - \Gamma_{T_0,\mathbf{\Psi}}(v_2)\|_{X^\beta(T_0)} \leq \frac{1}{2} \|v_1 - v_2\|_{X^\beta(T_0)}$$

allows us to apply the Picard fixed-point theorem and to conclude. □

4.3.4 Proof of Theorem 4.1.10

At this point, the statement of Theorem 4.1.10 (item (i)) is of course a mere combination of the construction of $\rho \mathring{\vert}$ as an element in $\mathcal{E}_\beta$ (Proposition 4.1.2) with the well-posedness result of Theorem 4.3.4. In brief, it suffices to apply Theorem 4.3.4 (in an almost sure way) to $\mathbf{\Psi} := \rho \mathring{\vert}$.

As for the convergence property in item (ii), it can easily be deduced from the continuity of $\Gamma_{T,\mathbf{\Psi}}$ with respect to $\mathbf{\Psi}$ (along (4.3.6)) and the almost sure convergence of $\chi \mathring{\vert}_n$ to $\chi \mathring{\vert}$. We propose to develop the details about this elementary procedure.
For every $n \geq 1$, let u_n denote the *smooth* solution of (4.1.3), that is u_n is the solution (in the sense of Definition 4.1.4) associated with $\rho \mathring{\vert}_n$. To put it differently, almost surely, the process $v_n := u_n - \mathring{\vert}_n$ is a solution of the mild equation

$$\begin{aligned} v_n(t,.) = S_t(\phi) - i \int_0^t S_{t-\tau}(\rho^2 |v_n(\tau,.)|^2)\, d\tau - i \int_0^t S_{t-\tau}((\rho \overline{v}_n(\tau,.)) \cdot (\rho \mathring{\vert}_n(\tau,.)))\, d\tau \\ - i \int_0^t S_{t-\tau}((\rho v_n(\tau,.)) \cdot (\overline{\rho \mathring{\vert}_n(\tau,.)}))\, d\tau - i \int_0^t S_{t-\tau}(|\rho \mathring{\vert}_n(\tau,.)|^2)\, d\tau\,, \quad t \in [0,T]\,. \end{aligned}$$

Resorting to Proposition 4.3.5, it holds that, for all $0 < T \leq 1$ (small enough, i.e. T is such that v and v_n are well-defined for all n, and T_0 will always satisfied this hypothesis in the following),

$$\|v_n\|_{X^\beta(T)} \lesssim \|\phi\|_{H^\beta(\mathbb{R}^d)} + T^\varepsilon \Big[\|v_n\|^2_{X^\beta(T)} + \|\rho \mathring{\vert}_n\|_{L^\infty_T \mathcal{W}^{\beta,q}} \|v_n\|_{X^\beta(T)} + \|\rho \mathring{\vert}_n\|^2_{L^\infty_T \mathcal{W}^{\beta,q}} \Big]. \quad (4.3.12)$$

We know that $\rho \mathring{\vert}_n$ is a convergent sequence (to $\rho \mathring{\vert}$). Consequently, almost surely, $\sup_{n \geq 0} \|\rho \mathring{\vert}_n\|_{L^\infty_T \mathcal{W}^{\beta,q}} < \infty$.
Denoting $f_n(T) := \|v_n\|_{X^\beta(T)}$, we obtain that for all $0 < T_0 \leq 1$ and $0 < T \leq T_0$,

$$f_n(T) \leq C_1 (A + T_0^\varepsilon f_n(T)^2),$$

where $C_1(\omega) \geq 1$ is a random constant and $A = 1 + \|\phi\|_{H^\beta(\mathbb{R}^d)}$. Let us focus our attention on the equation: $C_1 T_1^\varepsilon x^2 - x + C_1 A = 0$ where x is unknown and where T_1 is a parameter that will be adjusted. The discriminant of this second-degree polynomial equation is given by the formula: $1 - 4C_1^2 A T_1^\varepsilon$, that equals $\frac{1}{2}$ if, and only if, $4C_1^2 A T_1^\varepsilon = \frac{1}{2}$. In this case, the two roots are: $x_{1,T_1} = \frac{1-\frac{1}{\sqrt{2}}}{2C_1 T_1^\varepsilon}$ and $x_{2,T_1} = \frac{1+\frac{1}{\sqrt{2}}}{2C_1 T_1^\varepsilon}$ that verify $0 < x_{1,T_1} < x_{2,T_1}$. Let us observe that when $T_0 \leq T_1$, the equality $0 < x_{1,T_0} < x_{2,T_0}$ is still true. We have thus shown that there exists a random time $T_1(\omega) > 0$ such that for every $0 < T_0 \leq T_1$, the equation $C_1 T_0^\varepsilon x^2 - x + C_1 A = 0$ owns two solutions x_{1,T_0} and x_{2,T_0} satisfying $0 < x_{1,T_0} < x_{2,T_0}$. It entails that for all $0 < T_0 \leq \min(1, T_1)$ and $0 < T \leq T_0$, either $f_n(T) \leq x_{1,T_0}$, or $f_n(T) \geq x_{2,T_0}$. When the latter conditions are verified, $\sup_{T\in[0,T_0]} f_n(T) \leq x_{1,T_0}$ or $\inf_{T\in[0,T_0]} f_n(T) \geq x_{2,T_0}$. Denoting $T_2 := \sup\left\{0 \leq T : 2C_1 T^\varepsilon \|\phi\|_{H^\beta(\mathbb{R}^d)} \leq 1\right\}$, it can be checked that if $T_0 \leq T_2$, then $f_n(0) = \|\phi\|_{H^\beta(\mathbb{R}^d)} < x_{2,T_0}$. We now fix once and for all $T_0(\omega)$ satisfying $0 < T_0(\omega) \leq \min(1, T_1, T_2)$. Then, $\sup_{T\in[0,T_0]} f_n(T) \leq x_{1,T_0}$, ie $\sup_{T\in[0,T_0]} \|v_n\|_{X^\beta(T)} \leq x_{1,T_0}$. As a result, for all $0 < \overline{T_0} \leq T_0$, $\|v_n\|_{X^\beta(\overline{T_0})} \leq x_{1,T_0}$. We have finally obtained the following uniform in n bound, for every $0 < \overline{T_0} \leq T_0$,

$$\sup_{n\geq 1} \|v_n\|_{X^\beta(\overline{T_0})} \leq x_{1,T_0}.$$

By injecting this bound into (4.3.6), it holds that, for every $0 < \overline{T_0} \leq T_0$,

$$\|v_n - v\|_{X^\beta(\overline{T_0})} \lesssim \|\rho \Psi_n - \rho \Psi\|_{L^\infty_{\overline{T_0}} \mathcal{W}^{\beta,q}},$$

where the proportional constant does not depend on n. Now, let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a test function. Writing $\chi u_n - \chi u = (\chi v_n - \chi v) + (\chi \Psi_n - \chi \Psi)$, we deduce that

$$\begin{aligned} \|\chi u_n - \chi u\|_{\mathcal{C}([0,\overline{T_0}];H^\beta(\mathbb{R}^d))} &\lesssim \|\chi v_n - \chi v\|_{\mathcal{C}([0,\overline{T_0}];H^\beta(\mathbb{R}^d))} + \|\chi \Psi_n - \chi \Psi\|_{\mathcal{C}([0,\overline{T_0}];H^\beta(\mathbb{R}^d))} \\ &\lesssim \|v_n - v\|_{X^\beta(\overline{T_0})} + \|\chi \Psi_n - \chi \Psi\|_{\mathcal{C}([0,\overline{T_0}];H^\beta(\mathbb{R}^d))} \\ &\lesssim \|\rho \Psi_n - \rho \Psi\|_{L^\infty_{\overline{T_0}} \mathcal{W}^{\beta,q}} + \|\chi \Psi_n - \chi \Psi\|_{\mathcal{C}([0,\overline{T_0}];H^\beta(\mathbb{R}^d))}, \end{aligned}$$

and the latter quantities tend to 0 as n tends to $+\infty$, that entails the desired conclusion.

4.3.5 Small extension of Theorem 4.1.10 when the space dimension equals 5.

The aim of this subsection is to extend the results of Theorem 4.1.10 to the case when $d = 5$. Nonetheless, the condition $2H_0 + \sum_{i=1}^d H_i > d + 1$ has to be replaced by a more restrictive one:

$$2H_0 + \sum_{i=1}^d H_i > d + \frac{3}{2}.$$

Precisely, we propose to prove the following statement:

Theorem 4.3.6. *Assume that $d = 5$ and that*

$$2H_0 + \sum_{i=1}^{d} H_i > d + \frac{3}{2}.$$

Let β be such that $\frac{1}{2} < \beta < 2H_0 + \sum_{i=1}^{d} H_i - (d+1)$, and consider the pair (p,q) given by the formulas

$$p = \frac{12}{d-2\beta}\ ,\ q = \frac{3d}{d+\beta}\,.$$

Assume finally that $\phi \in H^{\beta}(\mathbb{R}^d)$. In this setting, the following assertions hold true:

(i) Almost surely, there exists a time $T_0 > 0$ such that equation (4.1.3) admits a unique solution u (in the sense of Definition 4.1.4) in the set

$$\mathcal{S}_{T_0} := \overset{\circ}{\vert} + X^{\beta}(T_0), \qquad \textit{where } X^{\beta}(T_0) := \mathcal{C}([0,T_0]; H^{\beta}(\mathbb{R}^d)) \cap L^p([0,T_0]; \mathcal{W}^{\beta,q}(\mathbb{R}^d)).$$

(ii) For every $n \geq 1$, let u_n denote the smooth *solution of (4.1.3), that is u_n is the solution (in the sense of Definition 4.1.4) associated with $\rho\overset{\circ}{\vert}_n$. Then, for every*

$$\frac{1}{2} < \beta < 2H_0 + \sum_{i=1}^{d} H_i - (d+1)$$

and for every test function $\chi : \mathbb{R}^d \to \mathbb{R}$, the sequence $(\chi u_n)_{n \geq 1}$ converges almost surely in $\mathcal{C}([0,T_0]; H^{\beta}(\mathbb{R}^d))$ to χu, where u is the solution exhibited in item (i).

The proof of this theorem comes from the proposition below where it suffices to take $s = \beta$.

Proposition 4.3.7. *Given $d \geq 1$, $0 < \beta < 1$ and $0 \leq s \leq \beta$ with $s_{crit} := \frac{d-4}{2} < s < \frac{d}{2}$, let*

$$p = \frac{12}{d-2s}, q = \frac{3d}{d+s}.$$

For all $T > 0$, we define $X^s(T)$ by

$$X^s(T) := \mathcal{C}([0,T]; H^s(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{s,q}(\mathbb{R}^d)).$$

Then, for all $T > 0$, $\phi \in H^s(\mathbb{R}^d), (\mathbf{\Psi}_1, \mathbf{\Psi}_2) \in \mathcal{E}_\beta^2$ and $v, v_1, v_2 \in X^s(T)$, the following bounds hold true:

$$\begin{aligned} \|\Gamma_{T,\mathbf{\Psi}_1}(v)\|_{X^s(T)} \lesssim \|\phi\|_{H^s(\mathbb{R}^d)} + T^{1-\frac{d-2s}{4}} \|v\|_{X^s(T)}^2 \\ + T^{1-\frac{d-2s}{4}+\frac{1}{p}} \|\mathbf{\Psi}_1\|_{\mathcal{C}([0,T];\mathcal{W}^{\beta,q}(\mathbb{R}^d))} \|v\|_{X^s(T)} \\ + T^{\frac{1}{p'}} \|\mathbf{\Psi}_1\|_{\mathcal{C}([0,T];\mathcal{W}^{\beta,q}(\mathbb{R}^d))}^2, \end{aligned} \tag{4.3.13}$$

and

$$\begin{aligned}\|\Gamma_{T,\mathbf{\Psi}_1}(v_1)-\Gamma_{T,\mathbf{\Psi}_2}(v_2)\|_{X^s(T)}&\\ \lesssim T^{1-\frac{d-2s}{4}}\|v_1-v_2\|_{X^s(T)}(\|v_1\|_{X^s(T)}+\|v_2\|_{X^s(T)})&\\ +T^{1-\frac{d-2s}{4}+\frac{1}{p}}\|\mathbf{\Psi}_1-\mathbf{\Psi}_2\|_{\mathcal{C}([0,T];\mathcal{W}^{\beta,q}(\mathbb{R}^d))}\|v_1\|_{X^s(T)}&\\ +T^{1-\frac{d-2s}{4}+\frac{1}{p}}\|\mathbf{\Psi}_2\|_{\mathcal{C}([0,T];\mathcal{W}^{\beta,q}(\mathbb{R}^d))}\|v_1-v_2\|_{X^s(T)}&\\ +T^{\frac{1}{p'}}\|\mathbf{\Psi}_1-\mathbf{\Psi}_2\|_{\mathcal{C}([0,T];\mathcal{W}^{\beta,q}(\mathbb{R}^d))}\Big[\|\mathbf{\Psi}_1\|_{\mathcal{C}([0,T];\mathcal{W}^{\beta,q}(\mathbb{R}^d))}+\|\mathbf{\Psi}_2\|_{\mathcal{C}([0,T];\mathcal{W}^{\beta,q}(\mathbb{R}^d))}\Big],&\end{aligned}\tag{4.3.14}$$

where the proportional constants depend only on s,β and ρ.

Proof. The strategy is again to bound each term in the expression of $\Gamma_{T,\mathbf{\Psi}}$ separately.
Bound on $S(\phi)$: It is readily checked that the pair (p,q) is Schrödinger admissible. The pair $(\infty,2)$ verifies the Schrödinger admissibility condition too. Lemma 4.3.3 immediately yields to

$$\|S(\phi)\|_{X^s(T)}\lesssim\|\phi\|_{H^s(\mathbb{R}^d)}.$$

Bound on $\mathcal{G}(\rho^2|v|^2)$: By Lemma 4.3.3,

$$\|\mathcal{G}(\rho^2|v|^2)\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\lesssim\|\rho^2|v|^2\|_{L^{p'}([0,T];\mathcal{W}^{s,q'}(\mathbb{R}^d))}.$$

Let us introduce the additional parameter n such that

$$\frac{1}{q'}=\frac{1}{q}+\frac{1}{n}.$$

Precisely, $n=\frac{q}{q-2}=\frac{dq}{d-sq}$ thanks to the expression of q. Now, using the fractional Leibniz rule 4.3.1, it holds that, for all $t\geq 0$,

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{s,q'}(\mathbb{R}^d)}\lesssim\|\rho v(t,.)\|_{\mathcal{W}^{s,q}(\mathbb{R}^d)}\|\rho v(t,.)\|_{L^n(\mathbb{R}^d)}.$$

As $s\geq d(\frac{1}{q}-\frac{1}{n})$, we have the following elementary Sobolev embedding

$$\mathcal{W}^{s,q}(\mathbb{R}^d)\hookrightarrow L^n(\mathbb{R}^d),$$

leading to

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{s,q'}(\mathbb{R}^d)}\lesssim\|\rho v(t,.)\|^2_{\mathcal{W}^{s,q}(\mathbb{R}^d)}.$$

Let us introduce the additional parameter m such that

$$\frac{1}{p'}=\frac{2}{p}+\frac{1}{m}.$$

Then, by Hölder's inequality, we have:

$$\begin{aligned}\|\rho^2|v|^2\|_{L^{p'}([0,T];\mathcal{W}^{s,q'}(\mathbb{R}^d))}&\lesssim T^{\frac{1}{m}}\|v\|^2_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\\&\lesssim T^{1-\frac{d-2s}{4}}\|v\|^2_{X^s(T)},\end{aligned}$$

where $\frac{1}{m} = 1 - \frac{d-2s}{4} > 0$ since $s > s_{crit}$. Similarly, $(\infty, 2)$ is a Schrödinger admissible pair which entails

$$\|\mathcal{G}(\rho^2|v|^2)\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} \lesssim T^{1-\frac{d-2s}{4}}\|v\|^2_{X^s(T)}.$$

Finally,

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X^s(T)} \lesssim T^{1-\frac{d-2s}{4}}\|v\|^2_{X^s(T)}.$$

Bound on $\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi})$, $\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}})$: As before,

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi})\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))} \lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L^{p'}([0,T];\mathcal{W}^{s,q'}(\mathbb{R}^d))}.$$

Thanks to the fractional Leibniz rule 4.3.1, it holds that, for all $t \geq 0$,

$$\begin{aligned}\|\rho\overline{v}\cdot\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{s,q'}(\mathbb{R}^d)} &\lesssim \|\rho v(t,.)\|_{\mathcal{W}^{s,q}(\mathbb{R}^d)}\|\mathbf{\Psi}(t,.)\|_{L^n(\mathbb{R}^d)}\\ &+ \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{s,q}(\mathbb{R}^d)}\|\rho v(t,.)\|_{L^n(\mathbb{R}^d)}\\ &\lesssim \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{s,q}(\mathbb{R}^d)}\|\rho v(t,.)\|_{\mathcal{W}^{s,q}(\mathbb{R}^d)},\end{aligned}$$

where we have used the same Sobolev embedding

$$\mathcal{W}^{s,q}(\mathbb{R}^d) \hookrightarrow L^n(\mathbb{R}^d),$$

to derive the last inequality. Then, by Hölder's inequality, we have:

$$\begin{aligned}\|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L^{p'}([0,T];\mathcal{W}^{s,q'}(\mathbb{R}^d))} &\lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\|v\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\\ &\lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\|v\|_{X^s(T)}.\end{aligned}$$

Similarly, $(\infty, 2)$ is a Schrödinger admissible pair which entails

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi})\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} \lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\|v\|_{X^s(T)}.$$

Finally,

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi})\|_{X^s(T)} \lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\|v\|_{X^s(T)}.$$

The same computations lead us to the following bound:

$$\|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}})\|_{X^s(T)} \lesssim T^{\frac{1}{p}+\frac{1}{m}}\|\mathbf{\Psi}\|_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}\|v\|_{X^s(T)}.$$

Bound on $\mathcal{G}(|\mathbf{\Psi}|^2)$: Likewise,

$$\|\mathcal{G}(|\mathbf{\Psi}|^2)\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))} \lesssim \||\mathbf{\Psi}|^2\|_{L^{p'}([0,T];\mathcal{W}^{s,q'}(\mathbb{R}^d))}.$$

The fractional Leibniz rule 4.3.1 states that for all $t \geq 0$,

$$\begin{aligned}\||\mathbf{\Psi}|^2(t,.)\|_{\mathcal{W}^{s,q'}(\mathbb{R}^d)} &\lesssim \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{s,q}(\mathbb{R}^d)}\|\mathbf{\Psi}(t,.)\|_{L^n(\mathbb{R}^d)}\\ &\lesssim \|\mathbf{\Psi}(t,.)\|^2_{\mathcal{W}^{s,q}(\mathbb{R}^d)}.\end{aligned}$$

Then,

$$\||\mathbf{\Psi}|^2\|_{L^{p'}([0,T];\mathcal{W}^{s,q'}(\mathbb{R}^d))} \lesssim T^{\frac{1}{p'}}\|\mathbf{\Psi}\|^2_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}.$$

Similarly, $(\infty, 2)$ is a Schrödinger admissible pair which entails

$$\|\mathcal{G}(|\mathbf{\Psi}|^2)\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} \lesssim T^{\frac{1}{p'}}\|\mathbf{\Psi}\|^2_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}.$$

Finally,

$$\|\mathcal{G}(|\mathbf{\Psi}|^2)\|_{X^s(T)} \lesssim T^{\frac{1}{p'}}\|\mathbf{\Psi}\|^2_{\mathcal{C}([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))}.$$

Combining the above estimates provides us with (4.3.13). It is then easy to see that (4.3.14) results from similar computations. □

4.4 Deterministic analysis of the equation under condition (H2')

It remains us to deal with the wellposedness issue in the rough case, that is to present the proof of Theorem 4.1.11. Therefore, we assume in this section that condition (**H2'**) on the Hurst indexes is satisfied. We recall that in this rough situation, the equation is understood in the sense of Definition 4.1.8, that is as

$$v_t = S_t(\phi) - i\int_0^t S_{t-\tau}(\rho^2|v_\tau|^2)\, d\tau - i\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot(\rho{}^{\circ}\!|_\tau))\, d\tau \\ - i\int_0^t S_{t-\tau}((\rho v_\tau)\cdot(\overline{\rho{}^{\circ}\!|_\tau}))\, d\tau - i\int_0^t S_{t-\tau}(\rho^2{}^{\circ}\!\!\vee\!\!{}^{\circ}_\tau)\, d\tau\,, \quad (4.4.1)$$

where the processes $\rho{}^{\circ}\!|$ and $\rho^2{}^{\circ}\!\!\vee\!\!{}^{\circ}$ are defined through Proposition 4.1.2 and Proposition 4.1.6.

In order to handle (4.4.1), we intend to follow the same deterministic approach as in Section 4.3. In other words, we will henceforth consider the pair $(\rho{}^{\circ}\!|, \rho^2{}^{\circ}\!\!\vee\!\!{}^{\circ})$ as a given element in the space

$$\mathcal{R}_\alpha := \bigcap_{2\leq p\leq\infty} L^\infty([0,T];\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)) \times L^\infty([0,T];H^{-2\alpha}(\mathbb{R}^d))\,, \quad (4.4.2)$$

for some $0<\alpha<1$ (provided by Propositions 4.1.2 and 4.1.6), and then try to solve the more general deterministic equation: for $(\mathbf{\Psi^1},\mathbf{\Psi^2})\in\mathcal{R}_\alpha$,

$$v_t = S_t(\phi) - i\int_0^t S_{t-\tau}(\rho^2|v_\tau|^2)\, d\tau - i\int_0^t S_{t-\tau}((\rho\overline{v}_\tau)\cdot \mathbf{\Psi}^{\mathbf{1}}_\tau)\, d\tau \\ - i\int_0^t S_{t-\tau}((\rho v_\tau)\cdot \overline{\mathbf{\Psi}^{\mathbf{1}}_\tau})\, d\tau - i\int_0^t S_{t-\tau}(\mathbf{\Psi}^{\mathbf{2}}_\tau)\, d\tau\,. \quad (4.4.3)$$

As we have already highlighted it in Section 4.1.4, the whole specificity of the situation (in comparison to Section 4.3) lies in the irregularity of $\mathbf{\Psi}^{\mathbf{1}}_\tau$ and $\mathbf{\Psi}^{\mathbf{2}}_\tau$, which can only be treated as negative-order distributions (note indeed that $\alpha>0$ in (4.4.2)). The technical ingredients towards a fixed-point argument need to be revised accordingly: this will be the purpose of the subsequent Sections 4.4.1-4.4.3, which lay the ground for our main wellposedness result, namely Theorem 4.4.5.

4.4.1 Pointwise multiplication and interpolation

In view of the above considerations, our only possibility to handle the product $(\rho v_\tau)\cdot(\mathbf{\Psi}^{\mathbf{1}}_\tau)$ in (4.4.3) will be to rely on the following general multiplication property in Sobolev spaces (see e.g. [39, Section 4.4.3] for a proof of this result):

Lemma 4.4.1. *Fix $d\geq 1$. Let $\alpha,\beta>0$ and $1\leq p,p_1,p_2<\infty$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad 0<\alpha<\beta.$$

If $f \in \mathcal{W}^{-\alpha,p_1}(\mathbb{R}^d)$ and $g \in \mathcal{W}^{\beta,p_2}(\mathbb{R}^d)$, then $f \cdot g \in \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)$ and

$$\|f \cdot g\|_{\mathcal{W}^{-\alpha,p}} \lesssim \|f\|_{\mathcal{W}^{-\alpha,p_1}} \|g\|_{\mathcal{W}^{\beta,p_2}} \ .$$

Let us also label the following classical interpolation result for further reference:

Lemma 4.4.2. *Fix $d \geq 1$. Let $s, s_1, s_2 \in \mathbb{R}$ and $1 \leq p, p_1, p_2 < \infty$ be such that, for some $\theta \in (0,1)$,*

$$s = \theta s_1 + (1-\theta)s_2 \quad \textit{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \,.$$

Then for every $v \in \mathcal{W}^{s_1,p_1}(\mathbb{R}^d) \cap \mathcal{W}^{s_2,p_2}(\mathbb{R}^d)$, it holds that $v \in \mathcal{W}^{s,p}(\mathbb{R}^d)$ and

$$\|v\|_{\mathcal{W}^{s,p}} \leq \|v\|_{\mathcal{W}^{s_1,p_1}}^{\theta} \|v\|_{\mathcal{W}^{s_2,p_2}}^{1-\theta} \,.$$

4.4.2 A local regularization property of the Schrödinger group S

It is a well-known fact that the classical Strichartz inequalities for the Schrödinger group (summed up in Lemma 4.3.3) do not offer any regularization effect, as can be seen from the constant derivative parameter s in (4.3.4). This phenomenon naturally becomes a fundamental obstacle in our rough setting, where, for stability reasons, the distribution $(\rho v_\tau) \cdot (\mathbf{\Psi}_\tau^{\mathbf{1}})$ in (4.4.3) is expected to turn into a function through the action of S.

A possible way to reach such a regularization property is to let *local* Sobolev topologies come into picture, through the consideration of the spaces $H^s_\rho(\mathbb{R}^d)$ defined by (4.1.19). Our main technical result in this direction can be stated as follows:

Lemma 4.4.3. *Fix $d \geq 1$. Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be of the form $(\mathbf{F}_\rho)$, $0 \leq \alpha, \kappa \leq \frac{1}{2}$ and $0 \leq T \leq 1$. Assume that $\phi \in H^{-\alpha}(\mathbb{R}^d)$, $F \in L^1([0,T]; H^{-\alpha}(\mathbb{R}^d))$, and consider the solution u of the following inhomogeneous Schrödinger equation on $\mathbb{R}^d$*

$$\begin{cases} i\partial_t u(t,x) - \Delta u(t,x) = F(t,x)\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,. \end{cases}$$

Then it holds that

$$\|u\|_{L_T^{\frac{1}{\kappa}} H_\rho^{-\alpha+\kappa}} \lesssim \|\phi\|_{H^{-\alpha}(\mathbb{R}^d)} + \|F\|_{L^1_T H^{-\alpha}} \,, \tag{4.4.4}$$

where the proportional constant only depends on ρ, α and κ.

The above property can in fact be seen as a slight extension of the result of [8, Theorem 3.1]. For the sake of clarity, we have postponed the proof of the lemma to Section 4.5.1.

4.4.3 A commutator estimate

Keeping our objective in mind (that is, to settle a fixed-point argument for (4.4.3)), the previous estimate (4.4.4) clearly lacks some stability: the left-hand side is indeed based on the consideration of a local Sobolev norm (in $H_\rho^{-\alpha+\kappa}$), while the right-hand side appeals to a standard Sobolev space ($H^{-\alpha}$).

Our strategy to overcome this problem will consist in using the presence of the cut-off function ρ *within the model* (4.4.3) (through ρv), which somehow allows us to turn global Sobolev norms into local ones. To implement this idea, an additional commutator-type estimate will be required:

Lemma 4.4.4. *For every $s > 0$ and for all test functions $\rho, g : \mathbb{R}^d \to \mathbb{R}$, it holds that*

$$\|(\mathrm{Id} - \Delta)^{\frac{s}{2}}(\rho \cdot g) - \rho \cdot (\mathrm{Id} - \Delta)^{\frac{s}{2}}(g)\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|_{H^{s-1}(\mathbb{R}^d)}\,, \tag{4.4.5}$$

where the proportional constant only depends on ρ and s.

As a consequence, for every test function $\rho : \mathbb{R}^d \to \mathbb{R}$ and for every $g \in H_\rho^s(\mathbb{R}^d) \cap H^{s-1}(\mathbb{R}^d)$, it holds that

$$\|\rho \cdot g\|_{H^s} \lesssim \|g\|_{H_\rho^s} + \|g\|_{H^{s-1}} \tag{4.4.6}$$

and

$$\|g\|_{H_\rho^s} \lesssim \|\rho \cdot g\|_{H^s} + \|g\|_{H^{s-1}}\,, \tag{4.4.7}$$

for some proportional constant depending only on ρ and s.

Proof. See Section 4.5.2. ☐

4.4.4 Solving the auxiliary deterministic equation

Let us fix (once and for all) a cut-off function $\rho : \mathbb{R}^d \to \mathbb{R}$ of the form ($\mathbf{F}_\rho$), and for all $T \geq 0$, $\alpha, \kappa > 0$, $p, q \geq 2$, define the space

$$X_\rho^{\alpha,\kappa,(p,q)}(T) := \mathcal{C}([0,T]; H^{-2\alpha}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-2\alpha,q}(\mathbb{R}^d)) \cap L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}\,. \tag{4.4.8}$$

Besides, recall that the space $\mathcal{R}_\alpha$ has been introduced in (4.4.2).

We are finally in a position to state (and prove) the main result of this section:

Theorem 4.4.5. *Assume that $1 \leq d \leq 3$ and that*

$$0 < \alpha < \begin{cases} \frac{3}{20} & \text{if } d = 1 \\ \frac{1}{10} & \text{if } d = 2 \\ \frac{1}{24} & \text{if } d = 3\,. \end{cases} \tag{4.4.9}$$

Then one can find parameters $\kappa \in [2\alpha, 1/2]$ and $p, q \geq 2$ such that for all $\phi \in H^{-2\alpha}(\mathbb{R}^d)$ and $(\mathbf{\Psi^1}, \mathbf{\Psi^2}) \in \mathcal{R}_\alpha$, there exists a time $T > 0$ for which equation (4.4.3) admits a unique solution in the above-defined set $X_\rho^{\alpha,\kappa,(p,q)}(T)$.

Remark 4.4.6. A review of our arguments in the below proof show that condition (4.4.9) on α is essentially optimal *with respect to the spaces and the tools that we have relied on* (the details of this procedure can be found to Remark 4.4.9). To be more specific, condition (4.4.9) is derived from an optimal choice of the four parameters α, κ, p, q in the scale of spaces (4.4.8), when using Lemmas 4.4.1-4.4.4 to estimate the right-hand side of (4.4.3).

We do not pretend that this restriction on α could not be alleviated by considering a different solution space, or using more sophisticated tools to control the equation.

Remark 4.4.7. As we mentionned it in the introduction, the well-posedness of similar (deterministic) quadratic NLS has already been studied in the literature. A recurrent ingredient consists of sharp bilinear estimates prevailing in the so-called Bourgain spaces (see [7, 2]). However, it seems to us that those techniques could not be directly applied to our problem, for two reasons.

Firstly, it is not clear how the term $\rho^2|u|^2$ could be treated through the bilinear estimates of [7], since the Bourgain spaces $X^{s,b}$ are not stable by multiplication with a $\mathcal{C}_c^\infty$ function[1]. Secondly, even if we replace ρ with 1 in the initial problem (4.1.1) (thus getting access to sharp bilinear estimates for $|u|^2$), it is unlikely that the stochastic terms and can then be injected into Bourgain spaces, owing to the spatial asymptotic behavior of those processes.

Just as in Section 4.3.3, the proof of Theorem 4.4.5 is in fact a straightforward consequence of the following estimates for the map $\Gamma_{T,\mathbf{\Psi^1},\mathbf{\Psi^2}}$ defined for all $T \geq 0$ and $(\mathbf{\Psi^1}, \mathbf{\Psi^2}) \in \mathcal{R}_\alpha$ by

$$\Gamma_{T,\mathbf{\Psi^1},\mathbf{\Psi^2}}(v) := S(\phi) + \mathcal{G}(\rho^2|v|^2) + \mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi^1}) + \mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi^1}}) + \mathcal{G}(\mathbf{\Psi^2})\,,$$

where the shortcut notation $\mathcal{G}$ refers to the operator

$$\mathcal{G}(u)_t := -i\int_0^t S_{t-\tau}(u_\tau)\,d\tau\,.$$

Proposition 4.4.8. *Assume that $1 \leq d \leq 3$ and that α satisfies condition* (4.4.9)*. Then one can find parameters $\kappa > 0$, $p, q \geq 2$ and $\varepsilon > 0$ such that, setting $X(T) := X_\rho^{\alpha,\kappa,(p,q)}(T)$, the following bounds hold true: for all $0 \leq T \leq 1$, $\phi \in H^{-2\alpha}(\mathbb{R}^d)$, $(\mathbf{\Psi}_1^{\mathbf{1}}, \mathbf{\Psi}_1^{\mathbf{2}}) \in \mathcal{R}_\alpha$, $(\mathbf{\Psi}_2^{\mathbf{1}}, \mathbf{\Psi}_2^{\mathbf{2}}) \in \mathcal{R}_\alpha$ and $v, v_1, v_2 \in X(T)$,*

$$\|\Gamma_{T,\mathbf{\Psi}_1^{\mathbf{1}},\mathbf{\Psi}_1^{\mathbf{2}}}(v)\|_{X(T)} \lesssim \|\phi\|_{H^{-2\alpha}} + T^\varepsilon\Big[\|v\|_{X(T)}^2 + \|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L_T^\infty\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)} + \|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L_T^\infty H^{-2\alpha}}\Big]\,, \tag{4.4.10}$$

and

$$\begin{aligned}&\|\Gamma_{T,\mathbf{\Psi}_1^{\mathbf{1}},\mathbf{\Psi}_1^{\mathbf{2}}}(v_1) - \Gamma_{T,\mathbf{\Psi}_2^{\mathbf{1}},\mathbf{\Psi}_2^{\mathbf{2}}}(v_2)\|_{X(T)}\\ &\lesssim T^\varepsilon\Big[\|v_1 - v_2\|_{X(T)}\{\|v_1\|_{X(T)} + \|v_2\|_{X(T)}\} + \|\mathbf{\Psi}_1^{\mathbf{1}} - \mathbf{\Psi}_2^{\mathbf{1}}\|_{L_T^\infty\mathcal{W}^{-\alpha,r}}\|v_1\|_{X(T)}\\ &\qquad + \|\mathbf{\Psi}_2^{\mathbf{1}}\|_{L_T^\infty\mathcal{W}^{-\alpha,r}}\|v_1 - v_2\|_{X(T)} + \|\mathbf{\Psi}_1^{\mathbf{2}} - \mathbf{\Psi}_2^{\mathbf{2}}\|_{L_T^\infty H^{-2\alpha}}\Big]\,,\end{aligned} \tag{4.4.11}$$

[1]We thank Jean-Marc Delort for this remark.

where r depends on α and κ and the proportional constants depend only on ρ and α.

The choice of the three parameters κ, p, q in the above proposition highly depends on the space dimension $d \in \{1,2,3\}$. For the sake of clarity, let us consider each value of d in a distinct subsection.

Proof of Proposition 4.4.8 when $d = 1$

In this situation, we pick κ such that $3\alpha < \kappa < \inf(\frac{1}{2}, \frac{3}{4} - 2\alpha)$ and $(p,q) := (\infty, 2)$, so that the space under consideration reduces to

$$X(T) := \mathcal{C}([0,T]; H^{-2\alpha}(\mathbb{R})) \cap L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}.$$

Also, we set $\theta := \frac{2\alpha}{\kappa} \in (0, \frac{2}{3})$.

We now bound each term in the expression of $\Gamma_{T,\mathbf{\Psi^1},\mathbf{\Psi^2}}$ separately. In the sequel we assume that $0 \leq T \leq 1$.

Bound on $S(\phi)$: As S is a unitary operator on $H^{-2\alpha}(\mathbb{R})$, one has

$$\|S(\phi)\|_{L_T^\infty H^{-2\alpha}} = \|\phi\|_{H^{-2\alpha}}.$$

Besides, since $\alpha \leq \frac{1}{4}$ and $\kappa \leq \frac{1}{2}$, we can apply Lemma 4.4.3 to assert that

$$\|S(\phi)\|_{L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}} \lesssim \|\phi\|_{H^{-2\alpha}},$$

and we have thus shown that

$$\|S(\phi)\|_{X(T)} \lesssim \|\phi\|_{H^{-2\alpha}}.$$

Bound on $\mathcal{G}(\rho^2|v|^2)$: Since $\kappa > 0$, and since ρ is smooth and compactly-supported, one has

$$\begin{aligned}\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} &= \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^\infty H^{-2\alpha}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}} \\ &\lesssim \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^\infty H^{-2\alpha+\kappa}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^{\frac{1}{\kappa}} H^{-2\alpha+\kappa}} \\ &\lesssim \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^\infty H^{-2\alpha+\kappa}}.\end{aligned}$$

From here we can apply Strichartz inequality (Lemma 4.3.3) to assert that

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} \lesssim \|\rho^2|v|^2\|_{L_T^{\frac{4}{3}} \mathcal{W}^{-2\alpha+\kappa,1}}. \tag{4.4.12}$$

By Lemma 4.3.1, one has, for every fixed $t \geq 0$,

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,1}} \lesssim \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}} \|\rho v(t,.)\|_{L^2},$$

and then, by Lemma 4.4.2,

$$\|\rho v(t,.)\|_{L^2} \leq \|\rho v(t,.)\|^{\theta}_{H^{-2\alpha+\kappa}} \|\rho v(t,.)\|^{1-\theta}_{H^{-2\alpha}} ,$$

which entails, for every fixed $t \geq 0$,

$$\begin{aligned} \|\rho^2 |v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,1}} &\lesssim \|\rho v(t,.)\|^{1+\theta}_{H^{-2\alpha+\kappa}} \|\rho v(t,.)\|^{1-\theta}_{H^{-2\alpha}} \\ &\lesssim \|v(t,.)\|^{1+\theta}_{H_\rho^{-2\alpha+\kappa}} \|v(t,.)\|^{1-\theta}_{H^{-2\alpha}} + \|v(t,.)\|^2_{H^{-2\alpha}} , \end{aligned}$$

where we have used Lemma 4.4.4 to derive the second inequality.

As a result,

$$\begin{aligned} &\int_0^T dt\, \|\rho^2 |v|^2(t,.)\|^{\frac{4}{3}}_{\mathcal{W}^{-2\alpha+\kappa,1}} \\ &\lesssim \|v\|^{\frac{4}{3}(1-\theta)}_{X(T)} \int_0^T dt\, \|v(t,.)\|^{\frac{4}{3}(1+\theta)}_{H_\rho^{-2\alpha+\kappa}} + T\|v\|^{\frac{8}{3}}_{X(T)} \\ &\lesssim T^{1-\frac{4}{3}(1+\theta)\kappa} \|v\|^{\frac{4}{3}(1-\theta)}_{X(T)} \bigg(\int_0^T dt\, \|v(t,.)\|^{\frac{1}{\kappa}}_{H_\rho^{-2\alpha+\kappa}} \bigg)^{\frac{4}{3}(1+\theta)\kappa} + T\|v\|^{\frac{8}{3}}_{X(T)} \\ &\lesssim T^{1-\frac{4}{3}(1+\theta)\kappa} \|v\|^{\frac{8}{3}}_{X(T)} , \end{aligned}$$

and thus, going back to (4.4.12), we have shown the desired estimate, that is

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} \lesssim T^{\frac{3}{4}-(\kappa+2\alpha)} \|v\|^2_{X(T)} .$$

Bound on $\mathcal{G}(\rho\overline{v}\cdot \mathbf{\Psi}^{\mathbf{1}}_1)$, $\mathcal{G}(\rho v\cdot \overline{\mathbf{\Psi}^{\mathbf{1}}_1})$: Since $\alpha \leq \frac{1}{4}$ and $\kappa \leq \frac{1}{2}$, we can appeal to Lemma 4.4.3 to assert that

$$\begin{aligned} \|\mathcal{G}(\rho\overline{v}\cdot \mathbf{\Psi}^{\mathbf{1}}_1)\|_{L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}} + \|\mathcal{G}(\rho v\cdot \overline{\mathbf{\Psi}^{\mathbf{1}}_1})\|_{L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}} &\lesssim \|\rho\overline{v}\cdot \mathbf{\Psi}^{\mathbf{1}}_1\|_{L^1_T H^{-2\alpha}} \\ &\lesssim \|\rho\overline{v}\cdot \mathbf{\Psi}^{\mathbf{1}}_1\|_{L^1_T H^{-\alpha}} . \end{aligned} \tag{4.4.13}$$

Let $s > \alpha$ and $2 \leq r, p < \infty$ be such that $1/2 = 1/r + 1/p$, then by Lemma 4.4.1, for every $t \geq 0$

$$\|(\rho\overline{v}\cdot \mathbf{\Psi}^{\mathbf{1}}_1)(t,.)\|_{H^{-\alpha}} \lesssim \|\mathbf{\Psi}^{\mathbf{1}}_1(t,.)\|_{\mathcal{W}^{-\alpha,r}} \|\rho v(t,.)\|_{\mathcal{W}^{s,p}} . \tag{4.4.14}$$

Next, observe that if

$$-2\alpha+\kappa-s = d(\frac{1}{2}-\frac{1}{p}) = \frac{d}{r} \quad \Leftrightarrow \quad s = -2\alpha+\kappa-\frac{d}{r} ,$$

we have the Sobolev embedding $H^{-2\alpha+\kappa}(\mathbb{R}^d) \hookrightarrow \mathcal{W}^{s,p}(\mathbb{R}^d)$. Therefore, since $-2\alpha+\kappa > \alpha$, if $r \geq 2$ is large enough, $s = -2\alpha+\kappa-\frac{d}{r} > \alpha$, and from (4.4.14) we deduce

$$\|(\rho\overline{v}\cdot \mathbf{\Psi}^{\mathbf{1}}_1)(t,.)\|_{H^{-\alpha}} \lesssim \|\mathbf{\Psi}^{\mathbf{1}}_1(t,.)\|_{\mathcal{W}^{-\alpha,r}} \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}} .$$

By applying Lemma 4.4.4, we obtain that for every $t \geq 0$,

$$\|\rho v(t,.)\|_{H^{-2\alpha+\kappa}} \lesssim \|v(t,.)\|_{H_\rho^{-2\alpha+\kappa}} + \|v(t,.)\|_{H^{-2\alpha}} ,$$

and so we deduce

$$\|\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^1_TH^{-\alpha}}+\|\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}}\|_{L^1_TH^{-\alpha}}\lesssim\|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^\infty_T\mathcal{W}^{-\alpha,r}}\{T^{1-\kappa}\|v\|_{L_T^{\frac{1}{\kappa}}H_\rho^{-2\alpha+\kappa}}+T\,\|v\|_{L^\infty_TH^{-2\alpha}}\}$$
$$\lesssim T^{1-\kappa}\|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^\infty_T\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)}\,,$$

which, going back to (4.4.13), leads us to

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{L_T^{\frac{1}{\kappa}}H_\rho^{-2\alpha+\kappa}}+\|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{L_T^{\frac{1}{\kappa}}H_\rho^{-2\alpha+\kappa}}\lesssim T^{1-\kappa}\|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^\infty_T\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)}\,. \tag{4.4.15}$$

On the other hand, by applying Lemma 4.3.3, we get

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{L^\infty_TH^{-2\alpha}}+\|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{L^\infty_TH^{-2\alpha}}\lesssim\|\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^1_TH^{-2\alpha}}\lesssim\|\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^1_TH^{-\alpha}}\,.$$

We are thus in the same position as in (4.4.13), and we can repeat the above arguments to obtain

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{L^\infty_TH^{-2\alpha}}+\|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{L^\infty_TH^{-2\alpha}}\lesssim T^{1-\kappa}\|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^\infty_T\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)}\,.$$

Combining this bound with (4.4.15), we can conclude that

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{X(T)}+\|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{X(T)}\lesssim T^{1-\kappa}\|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L^\infty_T\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)}\,.$$

Bound on $\mathcal{G}(\mathbf{\Psi}_1^{\mathbf{2}})$: First, according to Lemma 4.4.3, we know that

$$\|\mathcal{G}(\mathbf{\Psi}_1^{\mathbf{2}})\|_{L_T^{\frac{1}{\kappa}}H_\rho^{-2\alpha+\kappa}}\lesssim\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L^1_TH^{-2\alpha}}\lesssim T\,\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L^\infty_TH^{-2\alpha}}\,.$$

Then, by applying Lemma 4.3.3, we obtain that

$$\|\mathcal{G}(\mathbf{\Psi}_1^{\mathbf{2}})\|_{L^\infty_TH^{-2\alpha}}\lesssim\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L^1_TH^{-2\alpha}}\lesssim T\,\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L^\infty_TH^{-2\alpha}}\,,$$

and we have thus shown that

$$\|\mathcal{G}(\mathbf{\Psi}_1^{\mathbf{2}})\|_{X(T)}\lesssim T\,\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L^\infty_TH^{-2\alpha}}\,.$$

Combining the above estimates provides us with (4.4.10). It is then easy to see that (4.4.11) can be derived from similar arguments.

Proof of Proposition 4.4.8 when $d=2$

In this situation, we pick κ such that $3\alpha<\kappa<\frac{1}{2}-2\alpha$ and $(p,q):=(4,4)$, so that the space under consideration becomes

$$X(T):=\mathcal{C}([0,T];H^{-2\alpha}(\mathbb{R}^2))\cap L^4([0,T];\mathcal{W}^{-2\alpha,4}(\mathbb{R}^2))\cap L_T^{\frac{1}{\kappa}}H_\rho^{-2\alpha+\kappa}\,.$$

Note in particular that the so-defined pair (p,q) is Schrödinger admissible. Also, as in the previous section, we set $\theta := \frac{2\alpha}{\kappa} \in (0,\frac{2}{3})$, and we only focus on the derivation of (4.4.10) *(estimate* (4.4.11) *could be obtained along the same arguments).*

Bound on $S(\phi)$: The arguments are exactly the same as for $d=1$ (see Section 4.4.4), and yield

$$\|S(\phi)\|_{X(T)} \lesssim \|\phi\|_{H^{-2\alpha}}.$$

Bound on $\mathcal{G}(\rho^2|v|^2)$: Since $\kappa>0$, and since ρ is smooth and compactly-supported, one has

$$\begin{aligned}\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} &= \|\mathcal{G}(\rho^2|v|^2)\|_{L^\infty_T H^{-2\alpha}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L^4_T \mathcal{W}^{-2\alpha,4}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}}\\
&\lesssim \|\mathcal{G}(\rho^2|v|^2)\|_{L^\infty_T H^{-2\alpha+\kappa}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L^4_T \mathcal{W}^{-2\alpha+\kappa,4}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^{\frac{1}{\kappa}} H^{-2\alpha+\kappa}}\\
&\lesssim \|\mathcal{G}(\rho^2|v|^2)\|_{L^\infty_T H^{-2\alpha+\kappa}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L^4_T \mathcal{W}^{-2\alpha+\kappa,4}}\,,\end{aligned}$$

and from here we can apply Strichartz inequality (Lemma 4.3.3) to assert that

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} \lesssim \|\rho^2|v|^2\|_{L_T^{r'}\mathcal{W}^{-2\alpha+\kappa,s'}} \tag{4.4.16}$$

where we define the (Schrödinger admissible) pair (r,s) along the formula

$$(r,s) := \Big(\frac{4}{1+\theta},\frac{4}{1-\theta}\Big)\,.$$

By Lemma 4.3.1, one has, for every fixed $t\geq 0$,

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,s'}} \lesssim \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}}\|\rho v(t,.)\|_{L^{\frac{4}{1+\theta}}}\,.$$

Besides, by using Lemma 4.4.2, one can check that

$$\|\rho v(t,.)\|_{L^{\frac{4}{1+\theta}}} \leq \|\rho v(t,.)\|^\theta_{H^{-2\alpha+\kappa}}\|\rho v(t,.)\|^{1-\theta}_{\mathcal{W}^{-2\alpha,4}}\,.$$

Therefore, for every fixed $t\geq 0$,

$$\begin{aligned}\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,s'}} &\lesssim \|\rho v(t,.)\|^{1+\theta}_{H^{-2\alpha+\kappa}}\|\rho v(t,.)\|^{1-\theta}_{\mathcal{W}^{-2\alpha,4}}\\
&\lesssim \|v(t,.)\|^{1+\theta}_{H_\rho^{-2\alpha+\kappa}}\|v(t,.)\|^{1-\theta}_{\mathcal{W}^{-2\alpha,4}} + \|v(t,.)\|^{1+\theta}_{H^{-2\alpha}}\|v(t,.)\|^{1-\theta}_{\mathcal{W}^{-2\alpha,4}}\,,\end{aligned}$$

where we have used Lemma 4.4.4 to derive the second inequality.

Then, taking $\lambda := \frac{3-\theta}{2} > 1$, we get

$$\begin{aligned}&\int_0^T dt\,\|\rho^2|v|^2(t,.)\|^{r'}_{\mathcal{W}^{-2\alpha+\kappa,s'}}\\
&\lesssim \bigg(\int_0^T dt\,\|v(t,.)\|^{(1+\theta)r'\lambda}_{H_\rho^{-2\alpha+\kappa}}\bigg)^{\frac{1}{\lambda}}\bigg(\int_0^T dt\,\|v(t,.)\|^{(1-\theta)r'\lambda'}_{\mathcal{W}^{-2\alpha,4}}\bigg)^{\frac{1}{\lambda'}} + \|v\|^{r'(1+\theta)}_{X(T)}\int_0^T dt\,\|v_t\|^{r'(1-\theta)}_{\mathcal{W}^{-2\alpha,4}}\,.\end{aligned}$$

With our choices of parameters (remember that $\kappa < \frac{1}{2} - 2\alpha$ and $\theta = \frac{2\alpha}{\kappa}$), one has in fact

$$(1+\theta)r'\lambda = 2(1+\theta) < \frac{1}{\kappa} \quad \text{and} \quad (1-\theta)r'\lambda' = 4\,,$$

so that the above inequality yields

$$\begin{aligned}
&\int_0^T dt\, \|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,s'}}^{r'} \\
&\lesssim T^{\frac{1-2\kappa(1+\theta)}{\lambda}} \bigg(\int_0^T dt\, \|v(t,.)\|_{H_\rho^{-2\alpha+\kappa}}^{\frac{1}{\kappa}} \bigg)^{(1+\theta)r'\kappa} \|v\|_{X(T)}^{\frac{4}{\lambda'}} + T^{1-\frac{r'(1-\theta)}{4}} \|v\|_{X(T)}^{r'(1+\theta)} \|v\|_{X(T)}^{r'(1-\theta)} \\
&\lesssim T^{\frac{1-2\kappa-4\alpha}{\lambda}} \|v\|_{X(T)}^{(1+\theta)r'+\frac{4}{\lambda'}} + T^{1-\frac{r'(1-\theta)}{4}} \|v\|_{X(T)}^{2r'}\,.
\end{aligned}$$

It is now easy to check that this estimate can be rephrased as

$$\|\rho^2|v|^2\|_{L_T^{r'}\mathcal{W}^{-2\alpha+\kappa,s'}} \lesssim \{T^\varepsilon + T^{\frac{1}{2}}\}\|v\|_{X(T)}^2\,,$$

with $\varepsilon := \frac{1}{2}(1-2\kappa-4\alpha)$.

Going back to (4.4.16), we can conclude that

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} \lesssim \{T^\varepsilon + T^{\frac{1}{2}}\}\|v\|_{X(T)}^2\,.$$

Bound on $\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})$, $\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})$: Using the same arguments as for $d=1$, we can show first that

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{L_T^{\frac{1}{\kappa}}H_\rho^{-2\alpha+\kappa}} + \|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{L_T^{\frac{1}{\kappa}}H_\rho^{-2\alpha+\kappa}} \lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}}\|_{L_T^1H^{-\alpha}}\,,$$

and then

$$\|\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}}\|_{L_T^1H^{-\alpha}} + \|\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}}\|_{L_T^1H^{-\alpha}} \lesssim T^{1-\kappa}\|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L_T^\infty\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)}\,.$$

On the other hand, we can use Lemma 4.3.3 to assert that

$$\begin{aligned}
&\Big(\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{L_T^\infty H^{-2\alpha}} + \|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{L_T^\infty H^{-2\alpha}}\Big) + \Big(\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{L_T^4\mathcal{W}^{-2\alpha,4}} + \|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{L_T^4\mathcal{W}^{-2\alpha,4}}\Big) \\
&\lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}}\|_{L_T^1H^{-2\alpha}} \lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}}\|_{L_T^1H^{-\alpha}}\,.
\end{aligned}$$

Combining the above estimates easily provides us with the desired bound

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}_1^{\mathbf{1}})\|_{X(T)} + \|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}_1^{\mathbf{1}}})\|_{X(T)} \lesssim T^{1-\kappa}\|\mathbf{\Psi}_1^{\mathbf{1}}\|_{L_T^\infty\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)}\,.$$

Bound on $\mathcal{G}(\mathbf{\Psi}_1^{\mathbf{2}})$: The arguments are exactly the same as for $d=1$ (see Section 4.4.4), and lead us to

$$\|\mathcal{G}(\mathbf{\Psi}_1^{\mathbf{2}})\|_{X(T)} \lesssim T\,\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L_T^\infty H^{-2\alpha}}\,.$$

Proof of Proposition 4.4.8 when $d = 3$

In this situation, we pick $\kappa := 4\alpha$ and $(p, q) := (2, 6)$, so that one has

$$X(T) := \mathcal{C}([0,T]; H^{-2\alpha}(\mathbb{R}^3)) \cap L^2([0,T]; \mathcal{W}^{-2\alpha,6}(\mathbb{R}^3)) \cap L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}\,.$$

Observe here again that $(p, q) = (2, 6)$ defines a Schrödinger admissible pair.

Bound on $S(\phi)$: We can repeat the arguments used for $d = 1, 2$ to assert that

$$\|S(\phi)\|_{X(T)} \lesssim \|\phi\|_{H^{-2\alpha}(\mathbb{R}^3)}.$$

Bound on $\mathcal{G}(\rho^2|v|^2)$: Let us first write, just as for $d = 2$,

$$\begin{aligned}\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} &= \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^\infty H^{-2\alpha}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^2 \mathcal{W}^{-2\alpha,6}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^{\frac{1}{\kappa}} H_\rho^{-2\alpha+\kappa}} \\ &\lesssim \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^\infty H^{-2\alpha+\kappa}} + \|\mathcal{G}(\rho^2|v|^2)\|_{L_T^2 \mathcal{W}^{-2\alpha+\kappa,6}}\,,\end{aligned}$$

and then apply Strichartz inequality (Lemma 4.3.3) to obtain

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} \lesssim \|\rho^2|v|^2\|_{L_T^2 \mathcal{W}^{-2\alpha+\kappa,\frac{6}{5}}}\,. \tag{4.4.17}$$

By Lemma 4.3.1, one has, for every fixed $t \geq 0$,

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,\frac{6}{5}}} \lesssim \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}} \|\rho v(t,.)\|_{L^3}\,.$$

Besides, by using Lemma 4.4.2, one can check that

$$\|\rho v(t,.)\|_{L^3} \leq \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}}^{\frac{1}{2}} \|\rho v(t,.)\|_{\mathcal{W}^{-2\alpha,6}}^{\frac{1}{2}}\,.$$

Therefore, for every fixed $t \geq 0$,

$$\begin{aligned}\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,\frac{6}{5}}} &\lesssim \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}}^{\frac{3}{2}} \|\rho v(t,.)\|_{\mathcal{W}^{-2\alpha,6}}^{\frac{1}{2}} \\ &\lesssim \|v(t,.)\|_{H_\rho^{-2\alpha+\kappa}}^{\frac{3}{2}} \|v(t,.)\|_{\mathcal{W}^{-2\alpha,6}}^{\frac{1}{2}} + \|v(t,.)\|_{H^{-2\alpha}}^{\frac{3}{2}} \|v(t,.)\|_{\mathcal{W}^{-2\alpha,6}}^{\frac{1}{2}}\,,\end{aligned}$$

where we have used Lemma 4.4.4 to derive the second inequality.

This entails

$$\begin{aligned}&\int_0^T dt\, \|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-2\alpha+\kappa,\frac{6}{5}}}^2 \\ &\lesssim \left(\int_0^T dt\, \|v(t,.)\|_{H_\rho^{-2\alpha+\kappa}}^6\right)^{\frac{1}{2}} \left(\int_0^T dt\, \|v(t,.)\|_{\mathcal{W}^{-2\alpha,6}}^2\right)^{\frac{1}{2}} + \|v\|_{X(T)}^3 \int_0^T dt\, \|v_t\|_{\mathcal{W}^{-2\alpha,6}} \\ &\lesssim \{T^{\frac{1}{2}(1-24\alpha)} + T^{\frac{1}{2}}\} \|v\|_{X(T)}^4\,,\end{aligned}$$

which can obviously be recast as

$$\|\rho^2|v|^2\|_{L^2_T\mathcal{W}^{-2\alpha+\kappa,\frac{6}{5}}} \lesssim T^{\frac{1}{4}(1-24\alpha)}\|v\|^2_{X(T)}\,.$$

Going back to (4.4.17), we can conclude that

$$\|\mathcal{G}(\rho^2|v|^2)\|_{X(T)} \lesssim T^{\frac{1}{4}(1-24\alpha)}\|v\|^2_{X(T)}\,.$$

Bound on $\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}^{\mathbf{1}}_1)$, $\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}^{\mathbf{1}}_1})$: Using the same arguments as for $d=2$ (note indeed that $-2\alpha+\kappa=2\alpha>\alpha$), we get that

$$\|\mathcal{G}(\rho\overline{v}\cdot\mathbf{\Psi}^{\mathbf{1}}_1)\|_{X(T)}+\|\mathcal{G}(\rho v\cdot\overline{\mathbf{\Psi}^{\mathbf{1}}_1})\|_{X(T)} \lesssim T^{1-\kappa}\|\mathbf{\Psi}^{\mathbf{1}}_1\|_{L^\infty_T\mathcal{W}^{-\alpha,r}}\|v\|_{X(T)}\,.$$

Bound on $\mathcal{G}(\mathbf{\Psi}^{\mathbf{2}}_1)$: Here again, the arguments are exactly the same as for $d=1,2$ and entail

$$\|\mathcal{G}(\mathbf{\Psi}^{\mathbf{2}}_1)\|_{X(T)} \lesssim T\,\|\mathbf{\Psi}^{\mathbf{2}}_1\|_{L^\infty_T H^{-2\alpha}}\,.$$

Remark 4.4.9. We now propose to develop the heuristic considerations that, in some sense, justify the optimality with respect to α. These latter are initially due to Aurélien Deya.

Bound on $\mathcal{G}(\rho^2|v|^2)$: First of all, we write

$$\begin{aligned}\left\|\mathcal{G}(\rho^2|v|^2)\right\|_{L^{\frac{1}{\kappa}}_T H^{-2\alpha+\kappa}_\rho} &\lesssim \left\|\mathcal{G}(\rho^2|v|^2)\right\|_{L^{\frac{1}{\kappa}}_T H^{-2\alpha+\kappa}}\\ &\lesssim \left\|\mathcal{G}(\rho^2|v|^2)\right\|_{L^\infty_T H^{-2\alpha+\kappa}}\,.\end{aligned}$$

For any Schrödinger admissible pair (r,s), according to Lemma 4.3.3, one has then

$$\left\|\mathcal{G}(\rho^2|v|^2)\right\|_{L^\infty_T H^{-2\alpha+\kappa}} \lesssim \left\|\rho^2|v|^2\right\|_{L^{r'}_T\mathcal{W}^{-2\alpha+\kappa,s'}}\,. \tag{4.4.18}$$

Let n be such that

$$\frac{1}{s'}=\frac{1}{2}+\frac{1}{n}.$$

With the help of the fractional Leibniz rule from Lemma 4.3.1, for every fixed $t\geq 0$, it holds that

$$\left\|\rho^2|v|^2(t,.)\right\|_{\mathcal{W}^{-2\alpha+\kappa,s'}} \lesssim \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}}\|\rho v(t,.)\|_{L^n}.$$

Besides, setting $\theta:=\frac{2\alpha}{\kappa}$, and if n additionally satisfies

$$\frac{1}{n}=\frac{\theta}{2}+\frac{1-\theta}{q},$$

one has by interpolation (Lemma 4.4.2):

$$\|\rho v(t,.)\|_{L^n} \leq \|\rho v(t,.)\|^\theta_{H^{-2\alpha+\kappa}}\|\rho v(t,.)\|^{1-\theta}_{\mathcal{W}^{-2\alpha,q}}.$$

Therefore, for every fixed $t \geq 0$,

$$\begin{aligned}\left\|\rho^2|v|^2(t,.)\right\|_{\mathcal{W}^{-2\alpha+\kappa,s'}} &\lesssim \|\rho v(t,.)\|_{H^{-2\alpha+\kappa}}^{1+\theta}\|\rho v(t,.)\|_{\mathcal{W}^{-2\alpha,q}}^{1-\theta}\\ &\lesssim \|v(t,.)\|_{H_\rho^{-2\alpha+\kappa}}^{1+\theta}\|v(t,.)\|_{\mathcal{W}^{-2\alpha,q}}^{1-\theta}+\|v(t,.)\|_{H^{-2\alpha}}^{1+\theta}\|v(t,.)\|_{\mathcal{W}^{-2\alpha,q}}^{1-\theta},\end{aligned}$$

where we have used the commutator estimate (Lemma 4.4.4) to derive the second inequality. We can now go back to (4.4.18) and obtain, for every $\lambda \geq 1$,

$$\begin{aligned}&\int_0^T dt\,\left\|\rho^2|v|^2(t,.)\right\|_{\mathcal{W}^{-2\alpha+\kappa,s'}}^{r'}\\ &\lesssim \left(\int_0^T dt\|v(t,.)\|_{H_\rho^{-2\alpha+\kappa}}^{(1+\theta)r'\lambda}\right)^{\frac{1}{\lambda}}\left(\int_0^T dt\|v(t,.)\|_{\mathcal{W}^{-2\alpha,q}}^{(1-\theta)r'\lambda'}\right)^{\frac{1}{\lambda'}}+\|v\|_{X(T)}^{r'(1+\theta)}\int_0^T dt\,\|v(t,.)\|_{\mathcal{W}^{-2\alpha,q}}^{r'(1-\theta)}.\end{aligned}$$

The above reasoning thus drives us to consider the seven following constraints:

$$\begin{aligned}&(i)\quad (p,q)\quad \text{admissible},\\ &(ii)\quad (r,s)\quad \text{admissible},\\ &(iii)\quad \theta=\frac{2\alpha}{\kappa},\\ &(iv)\quad \frac{1}{s'}=\frac{1}{2}+\frac{\theta}{2}+\frac{1-\theta}{q},\\ &(v)\quad \lambda\geq 1,\\ &(vi)\quad (1+\theta)r'\lambda<\frac{1}{\kappa}\quad \text{and}\\ &(vii)\quad (1-\theta)r'\lambda'\leq p.\end{aligned}$$

Let us rewrite p and r under the form

$$p:=\frac{4}{d-\beta}\qquad \text{and}\qquad r:=\frac{4}{d-\eta}$$

for some parameters β,η satisfying $0 \leq \beta,\eta \leq 1$ for $d=1$, $\quad 0<\beta,\eta\leq 2$ for $d=2$ and $1\leq \beta,\eta\leq 3$ for $d=3$. With this notation, we can gather the constraints $(i)-(ii)-(iii)$ to successively derive:

$$\begin{aligned}\frac{d}{s'}=\frac{d}{2}+\frac{d\theta}{2}+\frac{d}{q}(1-\theta) &\Longleftrightarrow 2+\frac{d}{2}-\frac{2}{r'}=\frac{d}{2}+\frac{d\theta}{2}+\frac{d}{2}(1-\theta)-\frac{2}{p}(1-\theta)\\ &\Longleftrightarrow \frac{2}{r}=\frac{d}{2}-\frac{2}{p}(1-\theta)\\ &\Longleftrightarrow \frac{1}{r}=\frac{d}{4}-\frac{1}{p}(1-\theta)\\ &\Longleftrightarrow d-\eta=d-(d-\beta)(1-\theta)\\ &\Longleftrightarrow \eta=(d-\beta)(1-\theta).\end{aligned}$$

Moreover, condition (vi) can be reformulated as

$$4\kappa(1+\theta)<\frac{4-d+\eta}{\lambda}, \tag{4.4.19}$$

while condition (vii) becomes

$$\frac{4-d+\eta}{\lambda} \leqslant 4-d+\eta-(d-\beta)(1-\theta). \tag{4.4.20}$$

Combining (4.4.19)-(4.4.20) with the equality $\eta=(d-\beta)(1-\theta)$, we get the constraint

$$4\kappa(1+\theta) < \frac{4-d+\eta}{\lambda} \leq 4-d,$$

which finally yields the following condition on (α,κ) :

$$\kappa+2\alpha < 1-\frac{d}{4}. \tag{4.4.21}$$

• When $d=1$, if the latter condition (on (α,κ) only) is satisfied, then we can pick $\beta=1, \eta=0$ and $\lambda=1$ in the above analysis (any other choice for these parameters would not improve the restriction on (α,κ)).
• When $d=2$, the choice $\eta=0$ is impossible (this would correspond to the excluded limit point), but we can pick for instance $\beta=1$ and $\eta=1-\theta$ (this is still optimal as far as (α,κ) are concerned).
• When $d=3$,

$$1-\theta=\frac{\eta}{3-\beta} \geq \frac{1}{2}.$$

Therefore, $\theta=\frac{2\alpha}{\kappa} \leq \frac{1}{2}$ and the choice $\kappa:=4\alpha$ is optimal. According to condition (4.4.21), the case $d \geq 4$ cannot be treated with our strategy.

4.4.5 Proof of Theorem 4.1.11

With the statements of Proposition 4.1.6 and Theorem 4.4.5 in hand, we are of course in the same position as in Section 4.3.4, and so the desired properties follow again from an elementary combination of these results.

4.5 Appendix

We gather here the proofs of two technical lemmas that have been used in the analysis of the rough case, namely Lemma 4.4.3 and Lemma 4.4.4.

4.5.1 Proof of Lemma 4.4.3

The argument is based on an interpolation procedure, combined with the following result:

Proposition 4.5.1 (Constantin-Saut [8]). *Fix $d \geq 1$. Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be of the form ($\mathbf{F}_\rho$), $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq T \leq 1$. Assume that $\phi \in H^{-\alpha}(\mathbb{R}^d)$, $F \in L^1([0,T]; H^{-\alpha}(\mathbb{R}^d))$, and consider the solution u of the following inhomogeneous Schrödinger equation on $\mathbb{R}^d$*

$$\begin{cases} i\partial_t u(t,x) - \Delta u(t,x) = F(t,x)\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,. \end{cases}$$

Then it holds that

$$\|u\|_{L^2_T H_\rho^{-\alpha+\frac{1}{2}}} \lesssim \|\phi\|_{H^{-\alpha}} + \|F\|_{L^1_T H^{-\alpha}}\,, \tag{4.5.1}$$

where the proportional constant only depends on ρ and α.

Proof of Lemma 4.4.3. We define $v = (Id-\Delta)^{-\frac{\alpha}{2}}(u)$ such that $(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u) = (Id-\Delta)^{\frac{\kappa}{2}}(v)$. Let $t \in [0,T]$. By Lemma 4.4.4, it holds

$$\begin{aligned} \|\rho \cdot (Id-\Delta)^{\frac{\kappa}{2}} v(t,.)\|_{L^2(\mathbb{R}^d)} &\lesssim \|\rho v(t,.)\|_{H^\kappa(\mathbb{R}^d)} + \|v(t,.)\|_{H^{\kappa-1}(\mathbb{R}^d)} \\ &\lesssim \|\rho v(t,.)\|_{H^\kappa(\mathbb{R}^d)} + \|v(t,.)\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

since $\kappa - 1 \leq 0$. Thanks to Lemma 4.4.2,

$$\|\rho v(t,.)\|_{H^\kappa(\mathbb{R}^d)} \lesssim \|v(t,.)\|^{1-2\kappa}_{L^2(\mathbb{R}^d)} \|\rho v(t,.)\|^{2\kappa}_{H^{\frac{1}{2}}(\mathbb{R}^d)}.$$

Resorting to Lemma 4.4.4 again, we obtain

$$\begin{aligned} \|\rho v(t,.)\|_{H^{\frac{1}{2}}(\mathbb{R}^d)} &\lesssim \|\rho \cdot (Id-\Delta)^{\frac{1}{2}\cdot\frac{1}{2}} v(t,.)\|_{L^2(\mathbb{R}^d)} + \|v(t,.)\|_{H^{-\frac{1}{2}}(\mathbb{R}^d)} \\ &\lesssim \|\rho \cdot (Id-\Delta)^{\frac{1}{2}\cdot\frac{1}{2}} v(t,.)\|_{L^2(\mathbb{R}^d)} + \|v(t,.)\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

We easily deduce

$$\|\rho v(t,.)\|_{H^\kappa(\mathbb{R}^d)} \lesssim \|v(t,.)\|^{1-2\kappa}_{L^2(\mathbb{R}^d)} \|\rho \cdot (Id-\Delta)^{\frac{1}{2}\cdot\frac{1}{2}} v(t,.)\|^{2\kappa}_{L^2(\mathbb{R}^d)} + \|v(t,.)\|_{L^2(\mathbb{R}^d)}.$$

This yields

$$\|\rho \cdot (Id-\Delta)^{\frac{\kappa}{2}} v(t,.)\|_{L^2(\mathbb{R}^d)} \lesssim \|v(t,.)\|^{1-2\kappa}_{L^2(\mathbb{R}^d)} \|\rho \cdot (Id-\Delta)^{\frac{1}{2}\cdot\frac{1}{2}} v(t,.)\|^{2\kappa}_{L^2(\mathbb{R}^d)} + \|v(t,.)\|_{L^2(\mathbb{R}^d)}.$$

By integration on $t \in [0,T]$, it entails

$$\|\rho\cdot(Id-\Delta)^{\frac{\kappa}{2}}(v)\|_{L^{\frac{1}{\kappa}}([0,T];L^2(\mathbb{R}^d))} \lesssim \|v\|^{1-2\kappa}_{L^\infty([0,T];L^2(\mathbb{R}^d))} \|\rho\cdot(Id-\Delta)^{\frac{1}{2}\cdot\frac{1}{2}}(v)\|^{2\kappa}_{L^2([0,T];L^2(\mathbb{R}^d))} + \|v\|_{L^\infty([0,T];L^2(\mathbb{R}^d))}.$$

On the one hand, since $(\infty, 2)$ is a Schrödinger admissible pair, we can apply Lemma 4.3.3 to assert that

$$\|v\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} = \|(Id-\Delta)^{-\frac{\alpha}{2}}(u)\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} \lesssim \|\phi\|_{H^{-\alpha}(\mathbb{R}^d)} + \|F\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}^d))},$$

whereas, on the other hand, by Proposition 4.5.1,

$$\begin{aligned} \|\rho \cdot (Id-\Delta)^{\frac{1}{2}\cdot\frac{1}{2}}(v)\|_{L^2([0,T];L^2(\mathbb{R}^d))} &= \|\rho \cdot (Id-\Delta)^{\frac{-\alpha+\frac{1}{2}}{2}}(u)\|_{L^2([0,T];L^2(\mathbb{R}^d))} \\ &\lesssim \|\phi\|_{H^{-\alpha}(\mathbb{R}^d)} + \|F\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}^d))}. \end{aligned}$$

Combining the previous bounds provides us with the desired estimate, namely,

$$\|\rho \cdot (Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u)\|_{L^{\frac{1}{\kappa}}([0,T];L^2(\mathbb{R}^d))} \lesssim \|\phi\|_{H^{-\alpha}(\mathbb{R}^d)} + \|F\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}^d))}.$$

$\square$

We now propose to prove another local regularization result that is less precise than Lemma 4.4.3 which states that the solution of the non-linear Schrödinger equation is locally one quarter more regular than the initial condition and the non-linearity when we try to control it in the space $L^4([0,T];L^2(\mathbb{R}^d))$. Nonetheless, the advantage of its proof is that it does not rely on Lemma 4.4.4.

Proposition 4.5.2. *Let $\rho:\mathbb{R}^d\to\mathbb{R}$ be a smooth compactly-supported function of the form $(\mathbf{F}_\rho)$. Suppose $0\le\ \alpha\le\ \frac{1}{2}$, $0\le\kappa\le\frac{1}{4}$ and $T>0$. Assume that $\phi\in H^{-\alpha}(\mathbb{R}^d)$ and $F\in L^1([0,T];H^{-\alpha}(\mathbb{R}^d))$. Let us then consider the inhomogeneous linear Schrödinger equation on $\mathbb{R}^d$,*

$$\begin{cases} i\partial_t u(t,x)-\Delta u(t,x)=F(t,x)\,, \quad t\in[0,T]\,,\, x\in\mathbb{R}^d\,, \\ u_0=\phi. \end{cases} \tag{4.5.2}$$

Then the solution u to (4.5.2) verifies

$$\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u)\|_{L^4([0,T];L^2(\mathbb{R}^d))}\lesssim \max(1,T^{\frac{1}{4}})\{\|\phi\|_{H^{-\alpha}(\mathbb{R}^d)}+\|F\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}^d))}\},$$

where the proportional constant only depends on ρ.

As in the original paper of Constantin and Saut [8], we first need to establish what they call a lemma on the restriction of the Fourier Transform. We adapt their proof to obtain a result with $\mathcal{X}$ independent of time.

Lemma 4.5.3. *Let $\mathcal{X}:\mathbb{R}^d\to\mathbb{R}$ of the type $\mathcal{X}(x_1,...,x_d)=\mathcal{X}_1(x_1)\cdots\mathcal{X}_d(x_d)$ where $\mathcal{X}_1$,...,$\mathcal{X}_d$ are smooth compactly-supported functions. Suppose $0\le\alpha\le\frac{1}{2}$ and $T>0$. Then there exists a constant $C_\mathcal{X}$ depending only on $\mathcal{X}$ such that for every $f\in\mathcal{S}(\mathbb{R}^{d+1})$ which is supported in $[0,T]\times\mathbb{R}^d$*

$$\Big(\int_{\mathbb{R}^d}(1+|\xi|^2)^\alpha|\mathcal{F}(\mathcal{X}f)(-|\xi|^2,\xi)|^2d\xi\Big)^{\frac{1}{2}}\le C_\mathcal{X}\max(1,T^{\frac{1}{2}})\|f\|_{L^2([0,T];L^2(\mathbb{R}^d))}\,.$$

Proof. The above statement is a (slightly) modified version of the result of [8, Theorem 1.1], and therefore, for the sake of both rigour and completeness, we prefer to briefly review the arguments behind this estimate in the case $d=2$ which is representative of the difficulty of the dimension d.

Let us decompose the integral under consideration as

$$\int_{|\xi|\le1}(1+|\xi|^2)^\alpha|\mathcal{F}(\mathcal{X}f)(-|\xi|^2,\xi)|^2d\xi+\int_{|\xi|>1}(1+|\xi|^2)^\alpha|\mathcal{F}(\mathcal{X}f)(-|\xi|^2,\xi)|^2d\xi\,. \tag{4.5.3}$$

The first term can be controlled in the following way:

$$\begin{aligned}\int_{|\xi|\le1}(1+|\xi|^2)^\alpha|\mathcal{F}(\mathcal{X}f)(-|\xi|^2,\xi)|^2d\xi &\le \pi\sqrt{2}\|\mathcal{F}(\mathcal{X}f)\|^2_{L^\infty(\mathbb{R}^3)}\\ &\le \pi\sqrt{2}\|\mathcal{X}f\|^2_{L^1(\mathbb{R}^3)}\\ &\le \pi\sqrt{2}T\|\mathcal{X}\|^2_{L^2(\mathbb{R}^2)}\|f\|^2_{L^2([0,T];L^2(\mathbb{R}^2))}\,.\end{aligned}$$

In order to estimate the second integral in (4.5.3), consider the sets, for $j = 1, 2$,

$$\Gamma_j := \left\{ \xi \in \mathbb{R}^2, |\xi| > 1, |\xi_j| \geq \frac{1}{\sqrt{2}} |\xi| \right\}.$$

Then one has obviously

$$\{\xi \in \mathbb{R}^2, |\xi| > 1\} \subset \cup_{j=1}^2 \Gamma_j \, ,$$

and it is therefore sufficient to estimate the integral over each Γ_j.

Let us focus on the integral over Γ_1 (the integral over Γ_2 could be treated in a similar way). We split Γ_1 into

$$\Gamma_1 = \Gamma_1^+ \cup \Gamma_1^-$$

where $\Gamma_1^+ = \{\xi \in \Gamma_1, \xi_1 > 0\}$ and $\Gamma_1^- = \{\xi \in \Gamma_1, \xi_1 < 0\}$. Let us estimate

$$\int_{\Gamma_1^+} (1 + |\xi|^2)^\alpha |\mathcal{F}(\mathcal{X} f)(-|\xi|^2, \xi)|^2 \, d\xi \, .$$

We perform a change of variables $\xi \mapsto T(\xi)$ defined by

$$T : (\xi_1, \xi_2) \mapsto T(\xi_1, \xi_2) = (|\xi|^2, \xi_2) \, .$$

It is readily checked that T is a one-to-one map of Γ_1^+ on

$$G_1^+ := \{(u, v) \in \mathbb{R}^2, u > v^2, u > 1, \sqrt{u - v^2} \geq \frac{1}{\sqrt{2}} \sqrt{u}\} \, .$$

Moreover, its inverse is explicitly given by

$$T^{-1} : (u, v) \mapsto (\sqrt{u - v^2}, v) \, ,$$

whose Jacobian is given (in absolute value) by $\frac{1}{2\sqrt{u-v^2}}$. We are finally led to the consideration of the integral

$$\int_{G_1^+} (1 + u)^\alpha |\mathcal{F}(\mathcal{X} f)(-u, \sqrt{u - v^2}, v)|^2 \frac{1}{2\sqrt{u - v^2}} \, du dv$$

which can be bounded as follows:

$$\begin{aligned} &\int_{G_1^+} (1 + u)^\alpha |\mathcal{F}(\mathcal{X} f)(-u, \sqrt{u - v^2}, v)|^2 \frac{1}{2\sqrt{u - v^2}} \, du dv \\ &\quad \lesssim \int_{G_1^+} (1 + u)^{\alpha - \frac{1}{2}} |\mathcal{F}(\mathcal{X} f)(-u, \sqrt{u - v^2}, v)|^2 \, du dv \\ &\quad \lesssim \int_{G_1^+} |\mathcal{F}(\mathcal{X} f)(-u, \sqrt{u - v^2}, v)|^2 \, du dv \, , \end{aligned}$$

insofar as $\alpha \leq \frac{1}{2}$. Our strategy is now to prove that

$$\int_{\mathbb{R}^2} \sup_{w \in \mathbb{R}} |\mathcal{F}(\mathcal{X} f)(-u, w, v)|^2 \, du dv \lesssim \|f\|^2_{L^2([0,T]; L^2(\mathbb{R}^2))} \, ,$$

where the proportional constant depends only on $\mathcal{X}$. To this end, let us write

$$\mathcal{F}(\mathcal{X}f)(-u,w,v) = \int_{-\infty}^{+\infty} \hat{\mathcal{X}}_1(w-x)\mathcal{F}(\mathcal{X}_2 f)(-u,x,v)dx\,.$$

Using Cauchy-Schwartz inequality and Parseval's identity, we get that

$$|\mathcal{F}(\mathcal{X}f)(-u,w,v)|^2 \lesssim \|\mathcal{X}_1\|_{L^2(\mathbb{R})}^2 \int_{-\infty}^{+\infty} |\mathcal{F}(\mathcal{X}_2 f)(-u,x,v)|^2\, dx\,.$$

The latter bound being independent of w, we can conclude with the help of Parseval's identity again that

$$\begin{aligned}\int_{\mathbb{R}^2} \sup_{w\in\mathbb{R}} |\mathcal{F}(\mathcal{X}f)(-u,w,v)|^2\, dudv &\lesssim \|\mathcal{X}_1\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}^3} |\mathcal{F}(\mathcal{X}_2 f)(-u,x,v)|^2\, dxdudv\\ &\lesssim \|\mathcal{X}_1\|_{L^2(\mathbb{R})}^2 \|\mathcal{X}_2 f\|_{L^2(\mathbb{R}^3)}^2\\ &\lesssim \|\mathcal{X}_1\|_{L^2(\mathbb{R})}^2 \|\mathcal{X}_2\|_{L^\infty(\mathbb{R})}^2 \|f\|_{L^2([0,T];L^2(\mathbb{R}^2))}^2\,,\end{aligned}$$

which corresponds to the desired estimate. □

Proof. For the sake of simplicity, we limit ourselves to the case when $d=1$.
Step 1: Regularization.
By density, we can choose $(\phi_k) \in \mathcal{S}(\mathbb{R})$ and $(F_k) \in L^1([0,T];\mathcal{S}(\mathbb{R}))$ such that

$$\|\phi_k - \phi\|_{H^{-\alpha}(\mathbb{R})} \to 0, \int_0^T \|F_k(t) - F(t)\|_{H^{-\alpha}(\mathbb{R})} dt \to 0.$$

We consider the regularized version of (4.5.2), that is

$$\begin{cases} i\partial_t u_k(t,x) - \Delta u_k(t,x) = F_k(t,x)\,, \quad t\in[0,T]\,,\, x\in\mathbb{R}\,,\\ u_k(0,.) = \phi_k. \end{cases} \tag{4.5.4}$$

Then, the solution of (4.5.4) is expressed for all $0\le t\le T$ by

$$u_k(t,.) = S_t(\phi_k) - i\int_0^t S_{t-\tau}(F_k(\tau))d\tau.$$

We easily see that $u_k \in \mathcal{C}([0,T];\mathcal{S}(\mathbb{R}))$. Let us write

$$\begin{aligned}&\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u_k)\|_{L^4([0,T];L^2(\mathbb{R}))}\\ &\le \|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_.(\phi_k))\|_{L^4([0,T];L^2(\mathbb{R}))} + \|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k))\|_{L^4([0,T];L^2(\mathbb{R}))}.\end{aligned}$$

We treat each term of the sum separately. First, on the one hand,

$$\begin{aligned}
&\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_{\cdot}(\phi_k))\|^4_{L^4([0,T];L^2(\mathbb{R}))}\\
=\;& \int_0^T\left(\int_{\mathbb{R}}\rho(x)^2\Big|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x)\Big|^2dx\right)^2dt\\
=\;& \int_0^T\left(\int_{\mathbb{R}}\rho(x_1)^2\Big|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_1)\Big|^2dx_1\right)\\
&\qquad\times\left(\int_{\mathbb{R}}\rho(x_2)^2\Big|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_2)\Big|^2dx_2\right)dt\\
=\;& \int_0^T\left(\int_{\mathbb{R}^2}\rho(x_1)^2\rho(x_2)^2\Big|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_1)\Big|^2\Big|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_2)\Big|^2dx_1dx_2\right)dt\\
=\;& \|(t,x_1,x_2)\mapsto\mathcal{X}(x_1,x_2)[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_1)][(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_2)]\|^2_{L^2([0,T];L^2(\mathbb{R}^2))}
\end{aligned}$$

where $\mathcal{X}(x_1,x_2)=\rho(x_1)\rho(x_2)$. Aiming at using duality, we thus have to estimate the integral

$$I=\int_{[0,T]}\int_{\mathbb{R}^2}\mathcal{X}(x_1,x_2)[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_1)][(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))(x_2)]\overline{f(t,x_1,x_2)}dtdx_1dx_2$$

for all $f\in\mathcal{S}(\mathbb{R}^3)$ which is supported in $[0,T]\times\mathbb{R}^2$. Using Parseval's identity, we obtain

$$\begin{aligned}
I\;&=\;c\int_{[0,T]}\int_{\mathbb{R}^2}\mathcal{F}_x[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))]\otimes[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_t(\phi_k))](\xi)\overline{\mathcal{F}_x(\mathcal{X}\overline{f})}(t,\xi)d\xi dt\\
&=\;c\int_{[0,T]}\int_{\mathbb{R}^2}\{1+|\xi_1|^2\}^{\frac{-\alpha+\kappa}{2}}\{1+|\xi_2|^2\}^{\frac{-\alpha+\kappa}{2}}e^{it|\xi|^2}\widehat{\phi_k}(\xi_1)\widehat{\phi_k}(\xi_2)\overline{\mathcal{F}_x(\mathcal{X}\overline{f})}(t,\xi)d\xi dt.
\end{aligned}$$

By interchanging the order of integration,

$$I=c\int_{\mathbb{R}^2}[\{1+|\xi_1|^2\}^{-\frac{\alpha}{2}}\widehat{\phi_k}(\xi_1)][\{1+|\xi_2|^2\}^{-\frac{\alpha}{2}}\widehat{\phi_k}(\xi_2)][\{1+|\xi_1|^2\}^{\frac{\kappa}{2}}\{1+|\xi_2|^2\}^{\frac{\kappa}{2}}\mathcal{F}(\overline{\mathcal{X}}f)(-|\xi|^2,-\xi)]d\xi.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
|I|\;&\le\;c\left(\int_{\mathbb{R}}[\{1+|\xi_1|^2\}^{-\alpha}|\widehat{\phi_k}(\xi_1)|^2]d\xi_1\int_{\mathbb{R}}[\{1+|\xi_2|^2\}^{-\alpha}|\widehat{\phi_k}(\xi_2)|^2]d\xi_2\right)^{\frac12}\\
&\qquad\left(\int_{\mathbb{R}^2}\{1+|\xi|^2\}^{2\kappa}|\mathcal{F}(\overline{\mathcal{X}}f)(-|\xi|^2,-\xi)|^2d\xi\right)^{\frac12}.
\end{aligned}$$

We are now in position to apply Lemma 4.5.3 which leads to

$$\left(\int_{\mathbb{R}^2}\{1+|\xi|^2\}^{2\kappa}|\mathcal{F}(\overline{\mathcal{X}}f)(-|\xi|^2,-\xi)|^2d\xi\right)^{\frac12}\;\lesssim\;\max(1,T^{\frac12})\|f\|_{L^2([0,T];L^2(\mathbb{R}^2))}.$$

As

$$\int_{\mathbb{R}}[\{1+|\xi_1|^2\}^{-\alpha}|\widehat{\phi_k}(\xi_1)|^2]d\xi_1=\|\phi_k\|^2_{H^{-\alpha}(\mathbb{R})},$$

we deduce

$$|I|\lesssim\max(1,T^{\frac12})\|f\|_{L^2([0,T];L^2(\mathbb{R}^2))}\|\phi_k\|^2_{H^{-\alpha}(\mathbb{R})}.$$

By duality, we have thus proven

$$\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(S_{\cdot}(\phi_k))\|_{L^4([0,T];L^2(\mathbb{R}))}\lesssim \max(1,T^{\frac{1}{4}})\|\phi_k\|_{H^{-\alpha}(\mathbb{R})}.$$

Now, on the other hand, using a similar approach,

$$\begin{aligned}
&\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k))\|^4_{L^4([0,T];L^2(\mathbb{R}))}\\
=&\int_0^T\left(\int_{\mathbb{R}}\rho(x)^2\left|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x)\right|^2dx\right)^2dt\\
=&\int_0^T\left(\int_{\mathbb{R}}\rho(x_1)^2\left|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_1)\right|^2dx_1\right)\\
&\qquad\times\left(\int_{\mathbb{R}}\rho(x_2)^2\left|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_2)\right|^2dx_2\right)dt\\
=&\int_0^T\left(\int_{\mathbb{R}^2}\rho(x_1)^2\rho(x_2)^2\left|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_1)\right|^2\left|(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_2)\right|^2dx_1dx_2\right)dt\\
=&\|(t,x_1,x_2)\mapsto\mathcal{X}(x_1,x_2)[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_1)][(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_2)]\|^2_{L^2([0,T];L^2(\mathbb{R}^2))}.
\end{aligned}$$

Aiming at using duality, we thus have to estimate the integral

$$J=\int_{[0,T]}\int_{\mathbb{R}^2}\mathcal{X}(x_1,x_2)[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_1)][(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)(x_2)]\overline{f(t,x_1,x_2)}dtdx_1dx_2$$

for all $f\in\mathcal{S}(\mathbb{R}^3)$ which is supported in $[0,T]\times\mathbb{R}^2$. Using Parseval's identity and interchanging the order of integration, we obtain

$$\begin{aligned}
J\;=\;&c\int_{[0,T]}\int_{\mathbb{R}^2}\mathcal{F}_x[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)]\otimes[(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k)_t)](\xi)\overline{\mathcal{F}_x(\mathcal{X}\overline{f})}(t,\xi)d\xi dt\\
=\;&c\int_{[0,T]}\int_{\mathbb{R}^2}\{1+|\xi_1|^2\}^{\frac{-\alpha+\kappa}{2}}\{1+|\xi_2|^2\}^{\frac{-\alpha+\kappa}{2}}\int_0^te^{i(t-\tau_1)|\xi_1|^2}\mathcal{F}_{x_1}((F_k)_{\tau_1})(\xi_1)d\tau_1\\
&\int_0^te^{i(t-\tau_2)|\xi_2|^2}\mathcal{F}_{x_2}((F_k)_{\tau_2})(\xi_2)d\tau_2\overline{\mathcal{F}_x(\mathcal{X}\overline{f})}(t,\xi)d\xi dt\\
=\;&c\int_{[0,T]}\int_{[0,T]}\int_{\mathbb{R}^2}[\{1+|\xi_1|^2\}^{-\frac{\alpha}{2}}e^{-i\tau_1|\xi_1|^2}\mathcal{F}_{x_1}((F_k)_{\tau_1})(\xi_1)][\{1+|\xi_2|^2\}^{-\frac{\alpha}{2}}e^{-i\tau_2|\xi_2|^2}\mathcal{F}_{x_2}((F_k)_{\tau_2})(\xi_2)]\\
&[\{1+|\xi_1|^2\}^{\frac{\kappa}{2}}\{1+|\xi_2|^2\}^{\frac{\kappa}{2}}\int_{\max(\tau_1,\tau_2)}^Te^{it|\xi|^2}\overline{\mathcal{F}_x(\mathcal{X}\overline{f})}(t,\xi)dt]d\xi d\tau_1d\tau_2.
\end{aligned}$$

For $\tau_1\in[0,T],\tau_2\in[0,T]$, we define f_{τ_1,τ_2} by

$$f_{\tau_1,\tau_2}(t,x_1,x_2)=\mathbb{1}_{[\max(\tau_1,\tau_2),T]}(t)f(t,x_1,x_2).$$

Therefore,

$$\int_{\max(\tau_1,\tau_2)}^Te^{it|\xi|^2}\overline{\mathcal{F}_x(\mathcal{X}\overline{f})}(t,\xi)dt=\mathcal{F}(\overline{\mathcal{X}}f_{\tau_1,\tau_2})(-|\xi|^2,-\xi)$$

implying

$$\begin{aligned}
J\;=\;&c\int_{[0,T]}\int_{[0,T]}\int_{\mathbb{R}^2}[\{1+|\xi_1|^2\}^{-\frac{\alpha}{2}}e^{-i\tau_1|\xi_1|^2}\mathcal{F}_{x_1}((F_k)_{\tau_1})(\xi_1)][\{1+|\xi_2|^2\}^{-\frac{\alpha}{2}}e^{-i\tau_2|\xi_2|^2}\mathcal{F}_{x_2}((F_k)_{\tau_2})(\xi_2)]\\
&[\{1+|\xi_1|^2\}^{\frac{\kappa}{2}}\{1+|\xi_2|^2\}^{\frac{\kappa}{2}}\mathcal{F}(\overline{\mathcal{X}}f_{\tau_1,\tau_2})(-|\xi|^2,-\xi)]d\xi d\tau_1d\tau_2.
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}|J| &\leq c\int_{[0,T]}\int_{[0,T]}\left(\int_{\mathbb{R}}[\{1+|\xi_1|^2\}^{-\alpha}|\mathcal{F}_{x_1}((F_k)_{\tau_1})(\xi_1)|^2]d\xi_1\int_{\mathbb{R}}[\{1+|\xi_2|^2\}^{-\alpha}|\mathcal{F}_{x_2}((F_k)_{\tau_2})(\xi_2)|^2]d\xi_2\right)^{\frac{1}{2}}\\ &\quad\left(\int_{\mathbb{R}^2}\{1+|\xi|^2\}^{2\kappa}|\mathcal{F}(\overline{\mathcal{X}}f_{\tau_1,\tau_2})(-|\xi|^2,-\xi)|^2d\xi\right)^{\frac{1}{2}}d\tau_1d\tau_2.\end{aligned}$$

We are now in position to apply Lemma 4.5.3 which leads to

$$\begin{aligned}\left(\int_{\mathbb{R}^2}\{1+|\xi|^2\}^{2\kappa}|\mathcal{F}(\overline{\mathcal{X}}f_{\tau_1,\tau_2})(-|\xi|^2,-\xi)|^2d\xi\right)^{\frac{1}{2}} &\lesssim \max(1,T^{\frac{1}{2}})\|f_{\tau_1,\tau_2}\|_{L^2([0,T];L^2(\mathbb{R}^2))}\\ &\lesssim \max(1,T^{\frac{1}{2}})\|f\|_{L^2([0,T];L^2(\mathbb{R}^2))}.\end{aligned}$$

As

$$\int_{\mathbb{R}}[\{1+|\xi_1|^2\}^{-\alpha}|\mathcal{F}_{x_1}((F_k)_{\tau_1})(\xi_1)|^2]d\xi_1=\|F_k(\tau_1,.)\|^2_{H^{-\alpha}(\mathbb{R})},$$

we deduce

$$\begin{aligned}|J| &\lesssim \max(1,T^{\frac{1}{2}})\|f\|_{L^2([0,T];L^2(\mathbb{R}^2))}\int_{[0,T]}\int_{[0,T]}\|F_k(\tau_1,.)\|_{H^{-\alpha}(\mathbb{R})}\|F_k(\tau_2,.)\|_{H^{-\alpha}(\mathbb{R})}d\tau_1d\tau_2\\ &= \max(1,T^{\frac{1}{2}})\|f\|_{L^2([0,T];L^2(\mathbb{R}^2))}\left(\int_{[0,T]}\|F_k(\tau,.)\|_{H^{-\alpha}(\mathbb{R})}d\tau\right)^2\\ &\lesssim \max(1,T^{\frac{1}{2}})\|f\|_{L^2([0,T];L^2(\mathbb{R}^2))}\|F_k\|^2_{L^1([0,T];H^{-\alpha}(\mathbb{R}))}.\end{aligned}$$

By duality, we have thus proven

$$\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(\mathcal{G}(F_k))\|_{L^4([0,T];L^2(\mathbb{R}))}\lesssim\max(1,T^{\frac{1}{4}})\|F_k\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}))}.$$

To sum up, we have obtained

$$\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u_k)\|_{L^4([0,T];L^2(\mathbb{R}))}\lesssim\max(1,T^{\frac{1}{4}})\{\|\phi_k\|_{H^{-\alpha}(\mathbb{R})}+\|F_k\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}))}\}.$$

Step 2: Convergence and identification of the limit.
By linearity, u_k-u_l solves

$$\begin{cases} i\partial_t v(t,x)-\Delta v(t,x)=F_k(t,x)-F_l(t,x)\,, & t\in[0,T]\,,\ x\in\mathbb{R}\,,\\ v(0,.)=\phi_k-\phi_l. \end{cases}\tag{4.5.5}$$

It immediately entails

$$\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u_k-u_l)\|_{L^4([0,T];L^2(\mathbb{R}))}\lesssim\max(1,T^{\frac{1}{4}})\{\|\phi_k-\phi_l\|_{H^{-\alpha}(\mathbb{R})}+\|F_k-F_l\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}))}\}.$$

Therefore, $(\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u_k))_k$ is a Cauchy sequence in the Banach space $L^4([0,T];L^2(\mathbb{R}))$ and thus converges to a limit that we denote by U. As the convergence in $L^4([0,T];L^2(\mathbb{R}))$ implies the convergence in the sense of tempered distributions, $(\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u_k))_k$ tends to U in the sense of tempered distributions. Moreover, according to Strichartz inequalities, we know that $(u_k)_k$ tends to u the solution of (4.5.2) in the sense of tempered distributions. Hence, $(\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u_k))_k$ tends to $\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u)$. By uniqueness of the limit, we have thus proven that

$$\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u)=U$$

and we obtain the conclusion

$$\|\rho\cdot(Id-\Delta)^{\frac{-\alpha+\kappa}{2}}(u)\|_{L^4([0,T];L^2(\mathbb{R}))}\lesssim \max(1,T^{\frac14})\{\|\phi\|_{H^{-\alpha}(\mathbb{R})}+\|F\|_{L^1([0,T];H^{-\alpha}(\mathbb{R}))}\}.$$

□

4.5.2 Proof of Lemma 4.4.4

In order to establish this commutator estimate, we can essentially follow the arguments of Kato and Ponce in their proof of [32, Lemma X1]. However, *in the specific case where ρ is a test function* (which is the situation we would like to handle here), the bound (4.4.5) is clearly sharper than the general estimate in [32, Lemma X1]. For this reason, let us briefly review the main modifications leading to (4.4.5). The bound (4.4.5) also follows from the theory of pseudo-differential operators (see e.g. [1]) but the proof below only relies on the classical estimate by Coifman and Meyer ([6]).

Proposition 4.5.4. *Fix $d\geq 1$ and consider a function $\sigma\in\mathcal{C}^\infty((\mathbb{R}^d\times\mathbb{R}^d)\backslash(0,0);\mathbb{R})$ satisfying*

$$|\partial_\xi^\alpha\partial_\eta^\beta\sigma(\xi,\eta)|\leq C_{\alpha,\beta}(|\xi|+|\eta|)^{-|\alpha|-|\beta|} \tag{4.5.6}$$

for all $(\xi,\eta)\neq(0,0)$ and $\alpha,\beta\in\mathbb{N}^d$. Let us denote by $B(\sigma)$ the bilinear operator defined for all test functions $\varphi,\psi:\mathbb{R}^d\to\mathbb{R}$ as

$$B(\sigma)(\varphi,\psi)(x)=\int_{\mathbb{R}^d\times\mathbb{R}^d}d\xi d\eta\, e^{i\langle x,\xi+\eta\rangle}\sigma(\xi,\eta)\mathcal{F}\varphi(\xi)\mathcal{F}\psi(\eta)\,.$$

Then it holds that

$$\|B(\sigma)(\varphi,\psi)\|_{L^2(\mathbb{R}^d)}\lesssim\|\varphi\|_{L^\infty(\mathbb{R}^d)}\|\psi\|_{L^2(\mathbb{R}^d)}\,,$$

where the proportional constant only depends on the coefficients $(C_{\alpha,\beta})_{\alpha,\beta\in\mathbb{N}^d}$ in (4.5.6).

Proof of Lemma 4.4.4. The quantity under consideration can be written as

$$\begin{aligned}&\|(\mathrm{Id}-\Delta)^{\frac{s}{2}}(\rho\cdot g)-\rho\cdot(\mathrm{Id}-\Delta)^{\frac{s}{2}}(g)\|^2_{L^2(\mathbb{R}^d)}\\&=c\int_{\mathbb{R}^d}d\xi\left|\int_{\mathbb{R}^d}d\eta\left[\{1+|\xi|^2\}^{\frac{s}{2}}-\{1+|\eta|^2\}^{\frac{s}{2}}\right]\mathcal{F}\rho(\xi-\eta)\mathcal{F}g(\eta)\right|^2.\end{aligned}$$

Let us introduce a smooth function $\Phi:\mathbb{R}\to[0,1]$ with support in $\left[-\frac13,\frac13\right]$ such that $\Phi=1$ on $\left[-\frac14,\frac14\right]$. Then bound the above integral as

$$\int_{\mathbb{R}^d}d\xi\left|\int_{\mathbb{R}^d}d\eta\left[\{1+|\xi|^2\}^{\frac{s}{2}}-\{1+|\eta|^2\}^{\frac{s}{2}}\right]\mathcal{F}\rho(\xi-\eta)\mathcal{F}g(\eta)\right|^2\lesssim\mathcal{J}_1+\mathcal{J}_2\,,$$

where

$$\mathcal{J}_1 := \int_{\mathbb{R}^d} d\xi \bigg| \int_{\mathbb{R}^d} d\eta\, (1-\Phi)\Big(\frac{|\xi-\eta|}{|\eta|}\Big)\Big[\{1+|\xi|^2\}^{\frac{s}{2}} - \{1+|\eta|^2\}^{\frac{s}{2}}\Big]\mathcal{F}\rho(\xi-\eta)\mathcal{F}g(\eta)\bigg|^2$$

and

$$\mathcal{J}_2 := \int_{\mathbb{R}^d} d\xi \bigg| \int_{\mathbb{R}^d} d\eta\, \Phi\Big(\frac{|\xi-\eta|}{|\eta|}\Big)\Big[\{1+|\xi|^2\}^{\frac{s}{2}} - \{1+|\eta|^2\}^{\frac{s}{2}}\Big]\mathcal{F}\rho(\xi-\eta)\mathcal{F}g(\eta)\bigg|^2 .$$

Bound on $\mathcal{J}_1$. Using Cauchy-Schwarz inequality, we get first

$$\mathcal{J}_1 \leq \|g\|^2_{H^{s-1}(\mathbb{R}^d)} \bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} d\xi d\eta\, \Big|(1-\Phi)\Big(\frac{|\xi-\eta|}{|\eta|}\Big)\Big|^2 |\mathcal{F}\rho(\xi-\eta)|^2 \frac{\big|\{1+|\xi|^2\}^{\frac{s}{2}} - \{1+|\eta|^2\}^{\frac{s}{2}}\big|^2}{\{1+|\eta|^2\}^{s-1}} \bigg). \tag{4.5.7}$$

In order to show that the latter integral is indeed finite, observe that if $(1-\Phi)\big(|\xi-\eta|/|\eta|\big) \neq 0$, then $|\xi-\eta| \geq \frac{1}{4}|\eta|$, and so $|\xi-\eta| \geq \frac{1}{5}|\xi|$. Therefore, as ρ is smooth and compactly-supported, one has, for all $\lambda,\beta \geq 0$,

$$\Big|(1-\Phi)\Big(\frac{|\xi-\eta|}{|\eta|}\Big)\Big|^2 \Big|\mathcal{F}\rho(\xi-\eta)\Big|^2 \leq c_{\rho,\lambda,\beta}\{1+|\xi|^2\}^{-\lambda}\{1+|\eta|^2\}^{-\beta} ,$$

and the finiteness of the integral in (4.5.7) immediately follows.

Bound on $\mathcal{J}_2$. By Fourier isometry, we can write this quantity as $\mathcal{J}_2 = c\Big\|F\|^2_{L^2(\mathbb{R}^d)}$, with

$$F(x) := \int_{\mathbb{R}^d\times\mathbb{R}^d} d\xi d\eta\, e^{i\langle x,\xi+\eta\rangle} \Phi\Big(\frac{|\xi|}{|\eta|}\Big)\Big[\{1+|\xi+\eta|^2\}^{\frac{s}{2}} - \{1+|\eta|^2\}^{\frac{s}{2}}\Big]\mathcal{F}\rho(\xi)\mathcal{F}g(\eta) .$$

Then

$$\begin{aligned}
&\Phi\Big(\frac{|\xi|}{|\eta|}\Big)\Big[\{1+|\xi+\eta|^2\}^{\frac{s}{2}} - \{1+|\eta|^2\}^{\frac{s}{2}}\Big]\mathcal{F}\rho(\xi)\mathcal{F}g(\eta)\\
&= \Phi\Big(\frac{|\xi|}{|\eta|}\Big)\{1+|\eta|^2\}^{\frac{s}{2}}\bigg[\Big(1+\frac{\langle \xi,\xi+2\eta\rangle}{1+|\eta|^2}\Big)^{\frac{s}{2}} - 1\bigg]\mathcal{F}\rho(\xi)\mathcal{F}g(\eta)\\
&= \Phi\Big(\frac{|\xi|}{|\eta|}\Big)\{1+|\eta|^2\}^{\frac{1}{2}}\bigg[\Big(1+\frac{\langle \xi,\xi+2\eta\rangle}{1+|\eta|^2}\Big)^{\frac{s}{2}} - 1\bigg]\mathcal{F}\rho(\xi)\mathcal{F}\Big((\mathrm{Id}-\Delta)^{\frac{s-1}{2}}g\Big)(\eta) .
\end{aligned}$$

At this point, observe that if $\Phi\Big(\frac{|\xi|}{|\eta|}\Big) \neq 0$, then $|\xi| \leq \frac{1}{3}|\eta|$, and so $|\langle \xi,\xi+2\eta\rangle| \leq \frac{7}{9}|\eta|^2$. Therefore, we can rely on the pointwise expansion

$$\Phi\Big(\frac{|\xi|}{|\eta|}\Big)\{1+|\eta|^2\}^{\frac{1}{2}}\bigg[\Big(1+\frac{\langle \xi,\xi+2\eta\rangle}{1+|\eta|^2}\Big)^{\frac{s}{2}} - 1\bigg] = \sum_{k\geq 1} a_k(s)\Phi\Big(\frac{|\xi|}{|\eta|}\Big)\{1+|\eta|^2\}^{\frac{1}{2}-k}\langle \xi,\xi+2\eta\rangle^k ,$$

where $a_k(s) := \frac{s(s-1)\cdots(s-k+1)}{k!}$.

Since $s>0$ and ρ, g are assumed to be test functions, one has

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} d\xi d\eta \sum_{k\geq 1} \Big| a_k(s)\Phi\Big(\frac{|\xi|}{|\eta|}\Big)\{1+|\eta|^2\}^{\frac{1}{2}-k}\langle \xi,\xi+2\eta\rangle^k \mathcal{F}\rho(\xi)\mathcal{F}\Big((\mathrm{Id}-\Delta)^{\frac{s-1}{2}}g\Big)(\eta)\Big|$$
$$\lesssim \Big(\sum_{k\geq 1}\Big|a_k(s)\Big|\Big)\int_{\mathbb{R}^d\times\mathbb{R}^d} d\xi d\eta\,\{1+|\eta|^2\}^{\frac{1}{2}}\Big|\mathcal{F}\rho(\xi)\Big|\Big|\mathcal{F}\Big((\mathrm{Id}-\Delta)^{\frac{s-1}{2}}g\Big)(\eta)\Big| \;<\; \infty\,,$$

and accordingly we can write

$$F(x)=\sum_{k\geq 1} a_k(s)\int_{\mathbb{R}^d\times\mathbb{R}^d} d\xi d\eta\, e^{i\langle x,\xi+\eta\rangle}\,\Phi\Big(\frac{|\xi|}{|\eta|}\Big)\{1+|\eta|^2\}^{\frac{1}{2}-k}\langle \xi,\xi+2\eta\rangle^k\mathcal{F}\rho(\xi)\mathcal{F}((\mathrm{Id}-\Delta)^{\frac{s-1}{2}}g)(\eta)$$
$$=\sum_{k\geq 1} a_k(s)\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}^d} d\xi d\eta\, e^{i\langle x,\xi+\eta\rangle}\,\Phi\Big(\frac{|\xi|}{|\eta|}\Big)$$
$$\{1+|\eta|^2\}^{\frac{1}{2}-k}\langle \xi,\xi+2\eta\rangle^{k-1}(\xi_i+2\eta_i)\,[\xi_i\mathcal{F}\rho(\xi)]\mathcal{F}((\mathrm{Id}-\Delta)^{\frac{s-1}{2}}g)(\eta)\,.$$

Using the notation of Proposition 4.5.4 and the fact that $\xi_i\mathcal{F}\rho(\xi)=i\mathcal{F}\Big(\partial_{x_i}\rho\Big)(\xi)$, the latter identity can be rephrased as

$$F(x)=i\sum_{k\geq 1}a_k(s)\sum_{i=1}^d B(\sigma_{k,i})\Big(\partial_{x_i}\rho,(\mathrm{Id}-\Delta)^{\frac{s-1}{2}}g\Big)\,,$$

with

$$\sigma_{k,i}(\xi,\eta):=\Phi\Big(\frac{|\xi|}{|\eta|}\Big)\{1+|\eta|^2\}^{\frac{1}{2}-k}\langle \xi,\xi+2\eta\rangle^{k-1}(\xi_i+2\eta_i)\,.$$

It is not hard to check that for all $k\geq 1$ and $1\leq i\leq d$, the function $\sigma_{k,i}$ satisfies condition (4.5.6) with coefficients $(C_{\alpha,\beta})_{\alpha,\beta\in\mathbb{N}^d}$ independent of k and i. Consequently, we are in a position to apply Proposition 4.5.4 and conclude that

$$\|F\|_{L^2(\mathbb{R}^d)}\lesssim\sum_{k\geq 1}\Big|a_k(s)\Big|\sum_{i=1}^d\|\partial_{x_i}\rho\|_{L^\infty(\mathbb{R}^d)}\|g\|_{H^{s-1}(\mathbb{R}^d)}\lesssim\|g\|_{H^{s-1}(\mathbb{R}^d)}\,,$$

where we resort to Lemma 4.5.5 below to guarantee that $\sum_{k\geq 1}\Big|a_k(s)\Big|$ is finite. □

Lemma 4.5.5 (Duhamel's rule)**.** *Let $\sum x_n$ be a series with positive terms such that for all $n\in\mathbb{N}$, $x_n\neq 0$ and such that*

$$\frac{x_{n+1}}{x_n}=1-\frac{\lambda}{n}+o\Big(\frac{1}{n}\Big).$$

Then,
i) if $\lambda>1$, the series is convergent,
ii) if $\lambda<1$, the series is divergent.

Proof. Let us define $y_n=\frac{1}{n^\alpha}$. It holds that:

$$\frac{y_{n+1}}{y_n}=1-\frac{\alpha}{n}+o\Big(\frac{1}{n}\Big).$$

We deduce that, if $\alpha \neq \lambda$, then

$$\frac{x_{n+1}}{x_n} - \frac{y_{n+1}}{y_n} \underset{n \to +\infty}{\sim} \frac{\alpha - \lambda}{n}$$

has the sign of $\alpha - \lambda$ for n large enough.
If $\lambda < 1$, then we can pick α such that $\lambda < \alpha < 1$. For n large enough, $\frac{x_{n+1}}{x_n} \geq \frac{y_{n+1}}{y_n}$ and, as the series $\sum y_n$ diverges (since $\alpha < 1$), the series $\sum x_n$ diverges.
Now, if $\lambda > 1$, then we can pick α such that $\lambda > \alpha > 1$. For n large enough, $\frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n}$ and, as the series $\sum y_n$ converges (since $\alpha > 1$), the series $\sum x_n$ converges.

$\square$

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Chapter 5

Multilinear smoothing in a SNLS equation

In this chapter, we study a d-dimensional stochastic quadratic nonlinear Schrödinger equation (SNLS), driven by a fractional derivative (of order $-\alpha < 0$) of a space-time white noise:

$$\begin{cases} i\partial_t u - \Delta u = \rho^2 |u|^2 + \langle \nabla \rangle^{-\alpha} \dot{W} \,, \quad t \in [0,T] \,, \, x \in \mathbb{R}^d \,, \\ u_0 = \phi \,, \end{cases}$$

where $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth compactly-supported function. When $\alpha < \frac{d}{2}$, the stochastic convolution is a function of time with values in a negative-order Sobolev space and the model has to be interpreted in the Wick sense by means of a time-dependent renormalization. When $1 \leq d \leq 3$, combining both the classical Strichartz estimates and a deterministic local smoothing, we establish the local well-posedness of (SNLS) for a small range of α, in the spirit of [2]. Then, we revisit our arguments and establish multilinear smoothing on the second order stochastic term. This allows us to improve our local well-posedness result for some α. We point out that this is the first result concerning a Schrödinger equation on $\mathbb{R}^d$ driven by such an irregular noise and whose local well-posedness results from both a stochastic multilinear smoothing and a deterministic local one combined with Strichartz inequalities.

5.1 Introduction and main results

In this chapter, we study a d-dimensional stochastic quadratic nonlinear Schrödinger equation, driven by a fractional derivative (of order $-\alpha < 0$) of a space-time white noise:

$$\begin{cases} i\partial_t u - \Delta u = \rho^2 |u|^2 + \langle \nabla \rangle^{-\alpha} \dot{W} \,, \quad t \in [0,T] \,,\, x \in \mathbb{R}^d \,, \\ u_0 = \phi \,, \end{cases} \tag{5.1.1}$$

where, for any $\alpha > 0$, $\langle \nabla \rangle^{-\alpha}$ is the Bessel potential of order α, $\dot{W}$ denotes a space-time white noise on $\mathbb{R}^d$ and $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth compactly-supported function.
Schrödinger equations with polynomial nonlinearities and forced by white noises have aroused the curiosity of researchers for many years by giving birth to numerous challenging questions and because of their applications in physics. A major objective in this field would be for instance to obtain the local well-posedness of the following equation:

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u + \dot{W} \,, \quad t \in [0,T] \,,\, x \in D \,, \\ u_0 = \phi \,, \end{cases}$$

where $D = \mathbb{R}^d$ or $D = \mathbb{T}^d$. Indeed, the cubic nonlinear Schrödinger equation describes for instance the propagation of light waves in optical fibers or can be used to study a simplified model of Bose-Einstein condensates [4] or even freak waves in the ocean [9]. The space-time white noise $\dot{W}$ represents a stochastic perturbation whose values at each space-time points are independent and thus allows a generalization of the classical model. Some progress have been made in this direction. Forlano, Ho and Wang [5] have managed to prove the local well-posedness of a stochastic cubic nonlinear Schrödinger equation with almost space-time white noise on the one-dimensional torus, namely:

$$\begin{cases} i\partial_t u - \partial_x^2 u = |u|^2 u + \langle \nabla \rangle^{-\alpha} \dot{W} \,, \quad t \in [0,T] \,,\, x \in \mathbb{T} \,, \\ u_0 = \phi \,, \end{cases}$$

with $\alpha > 0$. In the following, let us denote by $\overset{\circ}{|}$ the associated stochastic convolution, defined by, for all $t \in [0,T]$, $\overset{\circ}{|}(t,.) = -i \int_0^t \frac{e^{-i(t-t')\Delta}}{\langle \nabla \rangle^{\alpha}} \dot{W}(t',.) dt'$. Their key arguments have been to measure the regularity of $\overset{\circ}{|}$ in terms of Fourier-Lebesgue spaces and, after a suitable renormalization of the equation, to establish a fixed-point argument in Bourgain spaces thanks to new trilinear estimates. We underline the fact that, in their paper, the stochastic convolution has for spatial regularity $s < \alpha - \frac{1}{p}$ so that, given $\alpha > 0$, they can choose sufficiently large p in such a way that $\overset{\circ}{|}$ has for spatial regularity $s > 0$ and is consequently a function of time with values in a space of functions. Then, the previous work has been extended to some nonlinearity of order $p > 1$, that is of the form $|u|^{p-1}u$ (see [12, 14]), on $\mathbb{R}^d$. Another model is the one proposed by Deya, Schaeffer and Thomann [2] and whose dynamic is described through the equation below:

$$\begin{cases} i\partial_t u - \Delta u = \rho^2 |u|^2 + \dot{B} \,, \quad t \in [0,T] \,,\, x \in \mathbb{R}^d \,, \\ u_0 = \phi \,, \end{cases} \tag{5.1.2}$$

for $1 \leq d \leq 3$ and with $\rho : \mathbb{R}^d \to \mathbb{R}$ a smooth compactly-supported function. In their article, the authors have considered a quadratic nonlinearity and a very rough noise, namely the derivative of a fractional (in time and in space) Brownian field. In what they call the rough case, that is when the stochastic convolution is only a function of time taking values in a space of distributions, there is a difficulty in defining $|\;|^2$. Resorting to a Wick renormalization combined with a local smoothing effect of the Schrödinger operator, they have established the local well-posedness of (5.1.2) in Sobolev spaces of negative order, namely $H^{-2\alpha}(\mathbb{R}^d)$, for small $\alpha > 0$. We will make a comparison between their model and ours in Subsection 5.1.2. In fact, the litterature concerning Schrödinger equations with nonlinearities and additive white noise is rather poor. The main reasons are that the stochastic convolution has very low regularity and that the Schrödinger operator, contrary to the heat or waves operator, does not offer any form of global regularization. To end with, let us mention the work of Deng, Nahmod and Yue [3]. In order to study the propagation of randomness under nonlinear dispersive equations, they have developed the theory of random tensors. Thanks to these new tools, they have been able to prove local well-posedness for semilinear Schrödinger equations in spaces that are subcritical in the probabilistic scaling. Before coming back to our model, let us present two other works that are not dealing with Schrödinger equations but with waves equations (another kind of dispersive equations) but that share an important common point with our model. The first one is the article of Gubinelli, Koch and Oh [8] in which they have used ideas from paracontrolled calculus to establish the local well-posedness of the following equation:

$$\partial_t^2 u + (1-\Delta)u = -|u|^2 + \dot{W}\,, \quad t \in [0,T]\,,\, x \in \mathbb{T}^3\,.$$

It is well-known that the waves operator provides a gain of one derivative (measured for instance in term of Sobolev norms). What interests us in this work is that the authors have proved a $\frac{1}{2}$-extra smoothing for the convolution between the waves operator and the second order stochastic term: $= (\partial_t^2 + (1-\Delta))^{-1}$ where denotes a renormalized version of ^{2}. That was the first time that such a smoothing resulting from stochastic tools combined with multilinear dispersive analysis was obtained. In a similar manner, Oh and Okamoto [13] have obtained a $\frac{1}{4}$-extra smoothing for the convolution between the waves operator and the second order stochastic term when studying the model below:

$$\partial_t^2 u + (1-\Delta)u = -|u|^2 + \langle\nabla\rangle^{\alpha}\dot{W}\,, \quad t \in [0,T]\,,\, x \in \mathbb{T}^2,$$

with $\alpha > 0$. The main novelty of our work lies in the fact that, as in the wave setting, we will be able to prove an extra smoothing for the convolution between the Schrödinger operator and the second order stochastic term (see Proposition 5.1.8 for more details) thanks to which we will be able to establish the local well-posedness of our model for some range of α. This is the first time that such a smoothing resulting both from stochastic and dispersive analysis is proved.

5.1.1 Interpretation of our model

In this chapter, our aim is to establish the local well-posedness of the following equation:

$$\begin{cases} i\partial_t u - \Delta u = \rho^2|u|^2 + \langle\nabla\rangle^{-\alpha}\dot{W}\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases} \tag{5.1.3}$$

for some $\alpha > 0$ and where $\rho : \mathbb{R}^d \to \mathbb{R}$ is a fixed smooth compactly-supported function. Here, $\dot{W}$ is a space-time white noise, that is a distribution-valued random variable such that for every test function f, $\langle \dot{W}, f \rangle$ is a centered Gaussian random variable with variance

$$\mathbb{E}[|\langle \dot{W}, f \rangle|^2] = \|f\|^2_{L^2([0,T]\times\mathbb{R}^d)}.$$

As usual, in order to isolate the expected worse term, we intend to resort to the Da Prato and Debbusche trick. We thus consider the stochastic convolution denoted by Ψ, the solution of the linear equation

$$\begin{cases} i\partial_t \Psi - \Delta \Psi = \langle \nabla \rangle^{-\alpha} \dot{W}\,, & t \in [0,T]\,,\ x \in \mathbb{R}^d\,, \\ \Psi(0,.) = 0\,. \end{cases}$$

Rewriting equation (5.1.3) under the mild form, we see that u is solution of

$$u_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|u_\tau|^2)\,d\tau + \Psi\,, \quad t \in [0,T]\,.$$

Consequently, $v := u - \Psi$ has to verify the equation below

$$\begin{aligned} v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau))\,d\tau \\ - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\Psi_\tau}))\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}(|\rho\Psi_\tau|^2)\,d\tau\,, \quad t \in [0,T]\,. \end{aligned} \tag{5.1.4}$$

We see that there are three major obstacles in the treatment of the latter equation, namely the proper definitions of the products $\rho v \cdot \overline{\rho\Psi}$, $\rho\overline{v}\cdot\rho\Psi$ and $|\Psi|^2$ and how to deal with the quadratic term $\rho^2|v|^2$. Let us first focus our attention on $|\Psi|^2$. We need to measure the regularity of the stochastic convolution that is formally defined for every $t \in [0,T]$ by:

$$\Psi(t,.) = -i\int_0^t \frac{e^{-i(t-t')\Delta}}{\langle \nabla \rangle^\alpha}\dot{W}(t',.)dt'.$$

In fact, by considering a sequence of smooth processes $(\Psi_n)_{n\in\mathbb{N}}$ (see Section 5.2), we will be able to propose a rigorous construction of Ψ and to prove the following result:

Proposition 5.1.1. *Let $d \geq 1$ be a space dimension and $T > 0$ a positive time. Fix α a positive number and*

$$s > \frac{d}{2} - \alpha.$$

Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function and $2 \leq p < \infty$. Then, the sequence $(\rho\Psi_n)_{n\in\mathbb{N}}$ converges almost surely in the space $\mathcal{C}([0,T];\mathcal{W}^{-s,p}(\mathbb{R}^d))$.
Denoting by $\rho\Psi$ the almost sure limit, it holds that

$$\rho\Psi \in \mathcal{C}([0,T];\mathcal{W}^{-s,p}(\mathbb{R}^d)).$$

The latter proposition sheds light on an important feature. Two regimes have to be distinguished. When $\alpha \geq \frac{d}{2}$, Ψ is a function of time with values in a space of functions (up to multiplication by ρ) whereas, when $\alpha < \frac{d}{2}$, Ψ is a function of time taking values in

a negative order Sobolev space. In the first case, for all $0 \leq t \leq T$, $|\Psi(t,.)|^2$ makes sense as product of two functions. But, in the second one, we cannot define $|\Psi|^2$ as a product of distributions of negative order. In the following, we will only deal with the case where $\alpha < \frac{d}{2}$ that is the hardest one.
In order to define $|\Psi|^2$, a first idea would be to use a sequential argument and to see $|\Psi|^2$ as the limit of the stochastic processes $(|\Psi_n|^2)_{n\in\mathbb{N}}$ in a convenient space. But, a quick computation shows that, for all $x \in \mathbb{R}^d$ and $t \in (0,T]$,

$$\mathbb{E}\Big[\big|\Psi_n(t,x)\big|^2\Big] = \frac{t}{(2\pi)^d}\int_{|\xi|\leq n}\frac{1}{(1+|\xi|^2)^\alpha}d\xi \underset{n\to+\infty}{\sim} C_\alpha t n^{d-2\alpha},$$

that tends to $+\infty$ as n goes to infinity, preventing us from applying our strategy. In fact, as in [2], we will proceed with a Wick renormalization and rather consider for every $x \in \mathbb{R}^d$ and $0 \leq t \leq T$,

$$\Psi^{\diamond 2}_n(t,x) = |\Psi_n(t,x)|^2 - \mathbb{E}\Big[\big|\Psi_n(t,x)\big|^2\Big].$$

Remark 5.1.2. The value of the renormalization constant $\mathbb{E}\Big[\big|\Psi_n(t,x)\big|^2\Big]$ does not depend on $x \in \mathbb{R}^d$, an already underlined point in [2] in the case of the fractional (in time and in space) noise.

We are now in a position to define our second order stochastic process $\Psi^{\diamond 2}$.

Proposition 5.1.3. *Let $d \geq 1$ be a space dimension and $T > 0$ a positive time. Fix α and s two positive numbers verifying*

$$\frac{d}{4} < \alpha < \frac{d}{2} \quad \textit{and} \quad s > \frac{d}{2} - \alpha.$$

Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function and $2 \leq p < \infty$. Then, the sequence $(\rho\Psi^{\diamond 2}_n)_{n\in\mathbb{N}}$ converges almost surely in the space $\mathcal{C}([0,T];\mathcal{W}^{-2s,p}(\mathbb{R}^d))$.
Denoting by $\rho\Psi^{\diamond 2}$ the almost sure limit, it holds that

$$\rho\Psi^{\diamond 2} \in \mathcal{C}([0,T];\mathcal{W}^{-2s,p}(\mathbb{R}^d)).$$

Replacing $|\Psi|^2$ by $\Psi^{\diamond 2}$, we rewrite equation (5.1.4) under the form

$$\begin{aligned} v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\rho\Psi_\tau))\,d\tau \\ - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\Psi_\tau}))\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2\Psi^{\diamond 2}_\tau)\,d\tau\,, \quad t\in[0,T]\,. \end{aligned} \tag{5.1.5}$$

We can now come back on the definitions of the products $\rho v \cdot \overline{\rho\Psi}$ and $\rho\overline{v}\cdot\rho\Psi$. Having a look on equation (5.1.5), we see that v is expected to live in $\mathcal{C}([0,T];H^{-2s}(\mathbb{R}^d))$, inheriting the bad regularity of $\rho^2\Psi^{\diamond 2}$. Again, we cannot define a product between two distributions of negative order. In fact, as $\rho : \mathbb{R}^d \to \mathbb{R}$ is a test function, we can benefit from a local smoothing effect of the Schrödinger operator described in Lemma 5.3.4. To be more precise, the latter lemma imposes an additionnal assumption on ρ, that is to take ρ of the form

$$\rho(x_1, \ldots, x_d) = \rho_1(x_1) \cdots \rho_d(x_d) \tag{$\mathbf{F}_\rho$}$$

for smooth compactly-supported functions $\rho_1, \ldots, \rho_d$ on $\mathbb{R}$. Thanks to this smoothing, up to multiplication by ρ, for all $0 \leq t \leq T$, $v(t, .)$ lives in $\mathcal{W}^{-2s+\eta,p}(\mathbb{R}^d)$ for small $\eta > 0$ such that $-2s + \eta > 0$ and where $p \geq 2$. Applying the classical rule (see Lemma 5.3.6) that states that we can define the product of two distributions of Sobolev regularities $-s < 0$ and $\beta > 0$ as a distribution of Sobolev regularity $-s$ as soon as $\beta > s$, we see that the two terms $\rho v \cdot \overline{\rho \Psi}$ and $\rho \overline{v} \cdot \rho \Psi$ make sense. And, thanks to the local smoothing again, the quadratic term $\rho^2 |v|^2$ can be interpreted as a product of two functions. Finally, each term in the expression of (5.1.5) is well-defined. We can now propose our interpretation of the model (5.1.1).

5.1.2 Interpretation and local well-posedness of equation (5.1.1)

Definition 5.1.4. *Let $d \geq 1$ be a space dimension and $T > 0$ a positive time. Fix α and s two real numbers verifying*

$$\frac{d}{4} < \alpha < \frac{d}{2} \quad \textit{and} \quad s > \frac{d}{2} - \alpha.$$

A stochastic process $(u(t,x))_{t \in [0,T], x \in \mathbb{R}^d}$ is said to be a Wick solution on $[0,T]$ of equation (5.1.1) if, almost surely, the process $v := u - \Psi$ is a solution of the mild equation

$$\begin{aligned} v_t = e^{-it\Delta}\phi - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2 |v_\tau|^2)\, d\tau - i \int_0^t e^{-i(t-\tau)\Delta}((\rho \overline{v}_\tau) \cdot (\rho \Psi_\tau))\, d\tau \\ - i \int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau) \cdot (\overline{\rho \Psi_\tau}))\, d\tau - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2 \mathcal{V}_\tau)\, d\tau\,, \quad t \in [0,T]\,. \end{aligned}$$

Let us now present our first result. As in [2], the key point is the introduction of the local Sobolev space below

$$H_\rho^{-2s+\eta}(\mathbb{R}^d) := \{v \in \mathcal{S}'(\mathbb{R}^d);\ \|\rho \cdot (\mathrm{Id} - \Delta)^{\frac{-2s+\eta}{2}}(v)\|_{L^2(\mathbb{R}^d)} < \infty\},$$

for $s \in \mathbb{R}$ and $\eta > 0$, that permits us to benefit from the local smoothing of the Schrödinger operator. We only present our result under the form of a proposition because we want to insist on the fact that the main novelty of this chapter lies in Theorem 5.1.12 resulting from the multilinear smoothing developed in the next subsection.

Proposition 5.1.5. *Let $1 \leq d \leq 3$ be a space dimension and $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function of the form $(\mathbf{F}_\rho)$. Besides, assume that*

$$\alpha_d < \alpha < \frac{d}{2}\,, \quad \textit{where} \quad \alpha_d = \begin{cases} 7/20 & \textit{if } d = 1 \\ 18/20 & \textit{if } d = 2 \\ 29/20 & \textit{if } d = 3 \end{cases}.$$

Fix $s > 0$ *such that* $\frac{d}{2} - \alpha < s < s_d$*, where*

$$s_d = \begin{cases} 3/20 & \text{if } d = 1 \\ 1/10 & \text{if } d = 2 \\ 1/24 & \text{if } d = 3 \end{cases} .$$

Then the following assertions hold true:
(i) *One can find parameters* $\eta \in [2s, 1/2]$ *and* $p, q \geq 2$ *such that, almost surely, for every* $\phi \in H^{-2s}(\mathbb{R}^d)$*, there exists a time* $T_0 > 0$ *for which equation* $(5.1.1)$ *admits a unique Wick solution* u *(in the sense of Definition 5.1.4) in the set*

$$\mathcal{S}_{T_0} := \<1> + X_\rho^{s,\eta,(p,q)}(T_0)\,,$$

where

$$X_\rho^{s,\eta,(p,q)}(T) := \mathcal{C}([0,T]; H^{-2s}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-2s,q}(\mathbb{R}^d)) \cap L_T^{\frac{1}{\eta}} H_\rho^{-2s+\eta}\,.$$

(ii) *For every* $n \geq 1$*, let* $\widetilde{u}_n$ *denote the smooth Wick solution of* $(5.1.1)$*, that is* $\widetilde{u}_n$ *is the solution (in the sense of Definition 5.1.4) associated with the pair* $(\rho\<1>_n, \rho^2\<2>_n)$*. For all* $\mathcal{C}^\infty$ *compactly-supported functions* $\chi : \mathbb{R}^d \to \mathbb{R}$ *, the sequence* $(\chi\widetilde{u}_n)_{n\geq 1}$ *converges almost surely in* $\mathcal{C}([0, T_0]; H^{-2s}(\mathbb{R}^d))$ *to* χu*, where* u *is the Wick solution exhibited in item* (i)*.*

This proposition can be proved with analogous arguments as those developed in [2]. Precisely, the stochastic processes $\<1>$ and $\<2>$ being constructed, it results from a deterministic fixed-point argument.

Remark 5.1.6. In [2], the authors have studied a similar model, the noise $\langle\nabla\rangle^{-\alpha}\dot{W}$ being replaced by the derivative of a fractional (in time and in space) Brownian field B. They have not constructed the second order stochastic process $\<2>$ when B is a white noise, that is when $H_0 = H_1 = \cdots = H_d = \frac{1}{2}$. Proposition 5.1.3 shows that this construction can be realized under a hypothesis of spatial regularization, namely when $\frac{d}{4} < \alpha < \frac{d}{2}$. In fact, we suspect that this condition on α is optimal and that $\<2>$ could not be defined as a function of time taking values in a space of distributions in the case of the white noise, that is when $\alpha = 0$.

We can observe that we are able to solve equation (5.1.1) only for a small range of α. This comes from the difficulty to bound $|v|^2$. Let us now follow a different strategy. We have seen that $\rho^2\<2> \in \mathcal{C}([0,T]; H^{-2s}(\mathbb{R}^d))$ and that, consequently, v inherits its bad regularity. That is why we propose a stochastic construction of $\<21>(t,.) = -i\int_0^t e^{-i(t-\tau)\Delta}(\<2>(\tau,.))d\tau$ and hope for a better regularity. Let us develop this point in the next subsections.

5.1.3 Multilinear smoothing

In order to construct the convolution between the Schrödinger operator and the second order stochastic process $\<2>$, we consider the following sequence:

Definition 5.1.7. *For every $t \in [0, T]$,*

$$\overset{\circ\circ}{\mathrm{Y}}_n(t, .) = -i \int_0^t e^{-i(t-\tau)\Delta}(\circ\!\smile\!\circ_n(\tau, .))d\tau.$$

As usual, the strategy is to show that the latter sequence is a Cauchy sequence in a convenient subspace. Our next result reads as follows.

Proposition 5.1.8. *Let $1 \leq d \leq 3$ be a space dimension and $T > 0$ a positive time. Fix α and s two real numbers verifying*

$$\frac{d}{4} < \alpha < \frac{d}{2} \quad \text{and} \quad s > \frac{d}{2} - \alpha.$$

Assume that

$$\kappa = \begin{cases} 1 - \alpha & \text{if } d = 1 \\ \frac{3}{2} - \alpha & \text{if } d = 2 \\ 2 - \alpha & \text{if } d = 3 \text{ and } \alpha \geq 1 \\ 1 & \text{if } d = 3 \text{ and } \alpha < 1 \ . \end{cases}$$

Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function and $2 \leq p < \infty$. Then, the sequence $\left(\rho \overset{\circ\circ}{\mathrm{Y}}_n\right)_{n\in\mathbb{N}}$ converges almost surely in the space $\mathcal{C}([0, T]; \mathcal{W}^{-2s+\kappa,p}(\mathbb{R}^d))$. Denoting by $\rho \overset{\circ\circ}{\mathrm{Y}}$ the almost sure limit, it holds that

$$\rho \overset{\circ\circ}{\mathrm{Y}} \in \mathcal{C}([0, T]; \mathcal{W}^{-2s+\kappa,p}(\mathbb{R}^d)).$$

Remark 5.1.9. Suppose that the assumptions of the latter proposition are verified.

• When $d = 1$, for all $0 \leq t \leq T$, $\rho \overset{\circ\circ}{\mathrm{Y}}(t, .) \in H^{\alpha-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$. In particular, if ε is small enough, $\rho \overset{\circ\circ}{\mathrm{Y}}$ is a function of time with values in a space of functions. Thanks to this multilinear smoothing, we will be able to settle a fixed-point argument for all the range of α, namely when $\frac{1}{4} < \alpha < \frac{1}{2}$.

• When $d = 2$, for all $0 \leq t \leq T$, $\rho \overset{\circ\circ}{\mathrm{Y}}(t, .) \in H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{R}^2)$ for every $\varepsilon > 0$. In particular, if ε is small enough, $\rho \overset{\circ\circ}{\mathrm{Y}}$ is again a function of time with values in a space of functions. This time, we will be able to settle a fixed-point argument only when $\frac{5}{6} < \alpha < 1$. As in [2], this constraint will result from the deterministic part of our study.

• When $d = 3$ and $\alpha > 1$, for all $0 \leq t \leq T$, $\rho \overset{\circ\circ}{\mathrm{Y}}(t, .) \in H^{\alpha-1-\varepsilon}(\mathbb{R}^3)$ for every $\varepsilon > 0$. In particular, if ε is small enough, $\rho \overset{\circ\circ}{\mathrm{Y}}$ is a function of time with values in a space of functions but, when $\alpha \leq 1$, despite the multilinear smoothing, $\rho \overset{\circ\circ}{\mathrm{Y}}$ is a function of time with values in a negativ order Sobolev space. Here, as in dimension 2, the deterministic part of our study will impose a more restrictive condition on α, namely $\frac{17}{12} < \alpha < \frac{3}{2}$.

Remark 5.1.10. The multilinear smoothing defined by the real $\kappa > 0$ of Proposition 5.1.8 is better than the local smoothing of Lemma 5.3.4 that only provides a gain of a half of a derivative (understood in the Sobolev meaning). One can observe that the smoothing effect is a decreasing function of α. In other words, the more the noise is regular, the less the extra smoothing is strong. This a common point shared with the waves setting. Indeed, when $d = 2$, Oh and Okamoto [13] proved a gain of $\min(\alpha, \frac{1}{4})$ of a derivative.

5.1.4 A deformed version of equation (5.1.1)

Let us go back to equation (5.1.5). Replacing $-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2 {}^{\circ}\!\vee\!{}^{\circ}_\tau)\,d\tau$ by $\rho^2\overset{\circ\circ}{\mathrm{Y}}$, we obtain

$$v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\bar{v}_\tau)\cdot(\rho\overset{\circ}{|}_\tau))\,d\tau$$
$$- i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\overset{\circ}{|}_\tau}))\,d\tau + \rho^2\overset{\circ\circ}{\mathrm{Y}}, \quad t\in[0,T]. \quad (5.1.6)$$

We underline the fact that replacing $-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2 {}^{\circ}\!\vee\!{}^{\circ}_\tau)\,d\tau$ by $\rho^2\overset{\circ\circ}{\mathrm{Y}}$ in (5.1.5) changes the model (5.1.1) since we have extracted the test function ρ^2 from the integral. Precisely, the previous equality corresponds to the following deformed version of (5.1.1):

$$\begin{cases} i\partial_t u - \Delta u = \rho^2|u|^2 + C_\rho + \langle\nabla\rangle^{-\alpha}\dot{W}\,, \quad t\in[0,T]\,,\ x\in\mathbb{R}^d\,, \\ u_0 = \phi\,, \end{cases} \quad (5.1.7)$$

where C_ρ is a renormalization *variable* whose expression is given by

$$C_\rho = (i\partial_t - \Delta)\Big(\rho^2\overset{\circ\circ}{\mathrm{Y}}\Big) - \rho^2\,{}^{\circ}\!\vee\!{}^{\circ} - \rho^2\mathbb{E}[|\overset{\circ}{|}|^2].$$

Now we define what we will call a solution of equation (5.1.7):

Definition 5.1.11. *Let $d\geq 1$ be a space dimension and $T>0$ a positive time. Fix α and s two real numbers verifying*

$$\frac{d}{4} < \alpha < \frac{d}{2} \quad \textit{and} \quad s > \frac{d}{2} - \alpha.$$

A stochastic process $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ is said to be a solution on $[0,T]$ of equation (5.1.7) if, almost surely, the process $v := u - \overset{\circ}{|}$ is a solution of the mild equation

$$v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\bar{v}_\tau)\cdot(\rho\overset{\circ}{|}_\tau))\,d\tau$$
$$- i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\overset{\circ}{|}_\tau}))\,d\tau + \rho^2\overset{\circ\circ}{\mathrm{Y}}, \quad t\in[0,T], \quad (5.1.8)$$

where $\rho:\mathbb{R}^d\to\mathbb{R}$ is a $\mathcal{C}^\infty$ compactly-supported function.

Let us study the regularity of each term in equation (5.1.8). Suppose that $\rho^2\overset{\circ\circ}{\mathrm{Y}}$ is a function of time with values in a space of functions (that is possible for some range of α according to the multilinear smoothing of Proposition 5.1.8). Then, the term with the worst regularity is $\rho\overset{\circ}{|}$ and v is expected to live in $\mathcal{C}([0,T];H^{-s}(\mathbb{R}^d))$. Again, there is an issue in making sense of $\rho v\cdot\overline{\rho\overset{\circ}{|}}$, $\rho\bar{v}\cdot\rho\overset{\circ}{|}$ and $\rho^2|v|^2$. In fact, resorting to the local smoothing effect of the Schrödinger operator, up to multiplication by ρ, for every $0\leq t\leq T$, $v(t,.)$ lives in $\mathcal{W}^{-s+\eta,p}(\mathbb{R}^d)$ for small $\eta>0$ such that $-s+\eta>0$ and where $p\geq 2$. Consequently, using the same arguments as in Subsection 5.1.2, the quadratic term $\rho^2|v|^2$ can be interpreted as a product of two functions and, under the additional assumption $-s+\eta>s$ allowing us to define $\rho v\cdot\overline{\rho\overset{\circ}{|}}$ and $\rho\bar{v}\cdot\rho\overset{\circ}{|}$ as two functions of time with values in a Sobolev space of order $-s$, we see that each term in equation (5.1.8) is well-defined.
We can now state our main local well-posedness result:

Theorem 5.1.12. *Let $1 \leq d \leq 3$ be a space dimension and $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function of the form $(\mathbf{F}_\rho)$. Besides, assume that*

$$\alpha_d < \alpha < \frac{d}{2}, \quad \text{where } \alpha_d = \begin{cases} 1/4 & \text{if } d = 1 \\ 5/6 & \text{if } d = 2 \\ 17/12 & \text{if } d = 3 \end{cases} .$$

Then the following assertions hold true:

(i) There exists $\varepsilon > 0$ small enough such that, denoting by $s > 0$ the real number $s = \frac{d}{2} - \alpha + \varepsilon$, one can find parameters $\eta \in [s, 1/2]$ and $p, q \geq 2$ such that, almost surely, for every $\phi \in H^{-s}(\mathbb{R}^d)$, there exists a time $T_0 > 0$ for which equation (5.1.7) admits a unique solution u (in the sense of Definition 5.1.11) in the set

$$\mathcal{S}_{T_0} := + Y_\rho^{s,\eta,(p,q)}(T_0) ,$$

where

$$Y_\rho^{s,\eta,(p,q)}(T) := \mathcal{C}([0,T]; H^{-s}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-s,q}(\mathbb{R}^d)) \cap L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta} .$$

(ii) For every $n \geq 1$, let $\widetilde{u}_n$ denote the smooth solution of (5.1.7), that is $\widetilde{u}_n$ is the solution (in the sense of Definition 5.1.11) associated with $(\rho{}_n, \rho^2{}_n)$. For all $\mathcal{C}^\infty$ compactly-supported functions $\chi : \mathbb{R}^d \to \mathbb{R}$, the sequence $(\chi \widetilde{u}_n)_{n \geq 1}$ converges almost surely in $\mathcal{C}([0,T_0]; H^{-s}(\mathbb{R}^d))$ to χu, where u is the solution exhibited in item (i).

Remark 5.1.13. The proof of this theorem results from a deterministic fixed-point argument. A precise description of the different tools we will resort to is given in Section 5.3.

Remark 5.1.14. Thanks to the multilinear smoothing, we see that, in dimension one, we are able to solve our model for all $\frac{1}{4} < \alpha < \frac{1}{2}$ (that is as soon as we are able to construct) and that, when $d = 2$ or $d = 3$, an additional constraint on α appears. This condition comes from the deterministic part of our work. See Section 5.3 for more details.

Remark 5.1.15. In all this study, the $\mathcal{C}^\infty$ compactly-supported function ρ plays a major role. Formally, when $\rho = 1$, Definition 5.1.11 is equivalent to the mild form of equation (5.1.1) (after renormalization). An important step would be to establish the local well-posedness of (5.1.1) without any cut-off but, insofar as the Schrödinger operator does not provide any form of global regularization, it currently lies beyond our ken.

5.1.5 Notations

Let $d \geq 1$ be a space dimension. Let us recall some classical notations.

- $\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz functions on $\mathbb{R}^d$.
- For all $s \in \mathbb{R}$ and $1 \leq p < \infty$,

$$\mathcal{W}^{s,p}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \ \|f\|_{\mathcal{W}^{s,p}} = \|\mathcal{F}^{-1}(\{1 + |.|^2\}^{\frac{s}{2}} \mathcal{F} f) | L^p(\mathbb{R}^d)\| < \infty \right\} ,$$

where the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are defined by the following formula: for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\mathcal{F}(f)(x) = \hat{f}(x) = \int_{\mathbb{R}^d} f(y)e^{-i\langle x,y\rangle}dy \quad \text{and} \quad \mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(y)e^{i\langle x,y\rangle}dy\,.$$

- For every $s \in \mathbb{R}$, $H^s(\mathbb{R}^d) = \mathcal{W}^{s,2}(\mathbb{R}^d)$.
- The operator $\langle\nabla\rangle^{-\alpha}$ is defined through the Fourier transform formula:

$$\mathcal{F}\Big(\langle\nabla\rangle^{-\alpha}(v)\Big)(\xi) = \{1+|\xi|^2\}^{-\frac{\alpha}{2}}\mathcal{F}v(\xi)\,.$$

5.1.6 Organization of the chapter

In section 5.2, we prove Propositions 5.1.1, 5.1.3 and 5.1.8. Then, in Section 5.3, we establish the deterministic fixed-point leading to the local well-posedness of our model.

5.2 On the construction of the relevant stochastic objects

In this section, we propose a rigorous construction of the three stochastic processes I, V and Y at the core of our problem. The first order stochastic process I is solution of

$$\begin{cases} i\partial_t \mathsf{I} - \Delta \mathsf{I} = \langle\nabla\rangle^{-\alpha}\dot{W}\,, \quad t\in[0,T]\,,\ x\in\mathbb{R}^d\,, \\ \mathsf{I}(0,.) = 0\,. \end{cases}$$

Formally, it is defined for every $t\in[0,T]$ by:

$$\mathsf{I}(t,.) = -i\int_0^t \frac{e^{-i(t-t')\Delta}}{\langle\nabla\rangle^\alpha}\dot{W}(t',.)dt'.$$

Let us introduce the sequence of truncated stochastic processes $(\mathsf{I}_n)_{n\in\mathbb{N}}$ defined for all $t\in[0,T]$ by:

$$\mathsf{I}_n(t,.) = -i\int_0^t \frac{e^{-i(t-t')\Delta}}{\langle\nabla\rangle^\alpha}\chi_n(\nabla)\dot{W}(t',.)dt',$$

where χ_n is the indicator function of the ball of radius n in $\mathbb{R}^d$. One may observe that an analogous regularization procedure has been used in the waves setting by Tolomeo (see [18]). Let us consider, for every $n\in\mathbb{N}$, the operator A_n defined for any $t\in[0,T]$ by:

$$A_n(t) = \frac{e^{-it\Delta}}{\langle\nabla\rangle^\alpha}\chi_n(\nabla).$$

It holds that, for all $\phi\in\mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} [A_n(t)\phi](x) &= \mathcal{F}^{-1}\bigg(\frac{e^{it|\xi|^2}}{(1+|\xi|^2)^{\frac{\alpha}{2}}}\chi_n(|\xi|)\mathcal{F}(\phi)\bigg)(x) \\ &= \mathcal{F}^{-1}\bigg(\frac{e^{it|\xi|^2}}{(1+|\xi|^2)^{\frac{\alpha}{2}}}\chi_n(|\xi|)\bigg)\star\phi(x) \\ &= a_n(t,.)\star\phi(x), \end{aligned}$$

where

$$a_n(t,x) = \frac{1}{(2\pi)^d}\int_{B_n} \frac{e^{it|\xi|^2}}{(1+|\xi|^2)^{\frac{\alpha}{2}}} e^{i\langle x,\xi\rangle} d\xi.$$

Consequently, Ψ_n can be properly defined in the following way: for all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\Psi_n(t,x) = -i\int_0^t \int_{\mathbb{R}^d} a_n(t-t',x-x')\dot{W}(t',x')dt'dx' = -i\int_0^t\int_{\mathbb{R}^d} a_n(t-t',x-x')W(dt',dx'), \tag{5.2.1}$$

and Ito's isometry allows us to compute the associated covariance function.

Definition 5.2.1. *Let $d \geq 1$ be a space dimension and $T > 0$ a positive time. We call regularized stochastic convolution any centered complex Gaussian process*

$$\left\{\Psi_n(s,x), n\in\mathbb{N}, 0\leq s\leq T, x\in\mathbb{R}^d\right\}$$

whose covariance function verifies: for all $(n,m)\in\mathbb{N}^2$, $(s,t)\in[0,T]^2$ and $(x,y)\in\mathbb{R}^d$,

$$\mathbb{E}\Big[\Psi_n(s,x)\overline{\Psi_m(t,y)}\Big] = \min(s,t)\frac{1}{(2\pi)^d}\int_{B_n\cap B_m}\frac{e^{i(s-t)|\xi|^2}}{(1+|\xi|^2)^\alpha}e^{-i\langle\xi,x-y\rangle}d\xi,$$
$$\mathbb{E}\Big[\Psi_n(s,x)\Psi_m(t,y)\Big] = \frac{1}{(2\pi)^d}\int_0^{\min(s,t)}\int_{B_n\cap B_m}\frac{e^{i(s+t-2t')|\xi|^2}}{(1+|\xi|^2)^\alpha}e^{-i\langle\xi,x-y\rangle}d\xi dt',$$

where $B_n = \Big\{\xi\in\mathbb{R}^d, |\xi|\leq n\Big\}$.

5.2.1 Technical lemmas

We begin by introducing our stochastic tools. The first one is Wick's formula that assures that the mean of a product of Gaussian random variables can be written as a sum of product of means of only two Gaussian random variables. The interested reader shall find more details in [11] for instance.

Lemma 5.2.2. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let $n\geq 1$ and $X_1,\dots,X_{2n}$ be real-valued jointly Gaussian random variables. Then, it holds that*

$$\mathbb{E}[X_1\cdots X_{2n}] = \sum_{\textit{pairings } \mathcal{P} \textit{ of } \{1,\dots,2n\}}\ \prod_{(i,j)\in\mathcal{P}}\mathbb{E}[X_iX_j]$$

where a pairing of $\{1,\dots,2n\}$ is a partition $\mathcal{P} = \{\{i_1,j_1\},\dots,\{i_n,j_n\}\}$ of this set into disjoint subsets of two elements.

The lemma below describes the hypercontractivity of Wiener Chaoses and deals with products of Gaussian random variables. It will permit us to turn $L^2(\Omega)$-bounds into $L^p(\Omega)$-bounds for $p\geq 2$. A classical reference on this topic is [11].

Lemma 5.2.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $k \geq 1$ and $c(n_1, \dots, n_k) \in \mathbb{C}$. Let $d \geq 1$ and $g_1, \dots, g_d$ a sequence of independent standard complex-valued Gaussian random variables. Let $S_k : \Omega \to \mathbb{C}$ the random variable defined by, for all $\omega \in \Omega$,*

$$S_k(\omega) = \sum_{\Gamma(k,d)} c(n_1, \dots, n_k) g_{n_1}(\omega) \cdots g_{n_k}(\omega),$$

where

$$\Gamma(k, d) = \{(n_1, \dots, n_k) \in \{1, \dots, d\}^k\}.$$

Then, for every $p \geq 2$, it holds that

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \|S_k\|_{L^2(\Omega)}.$$

Let us now present our two main deterministic tools. The following lemma will be of constant use to bring back computations on compact domains. In particular, we will be able to construct our stochastic processes in $\mathcal{W}^{-s,p}(\mathbb{R}^d)$ for some $s \in \mathbb{R}$ and for all $2 \leq p < \infty$, up to multiplication by a test function. A proof of this lemma can be found in [2].

Lemma 5.2.4. *Let $\alpha \in \mathbb{R}$ and $\rho : \mathbb{R}^d \to \mathbb{R}$ a $\mathcal{C}^\infty$ compactly-supported function. For all $p \geq 1$ and $(\eta_1, \dots, \eta_p) \in (\mathbb{R}^d)^p$, it holds that*

$$\left| \int_{\mathbb{R}^d} dx \prod_{i=1}^{p} \iint_{(\mathbb{R}^d)^2} \frac{d\lambda_i d\tilde{\lambda}_i}{(1+|\lambda_i|^2)^{\frac{\alpha}{2}}(1+|\tilde{\lambda}_i|^2)^{\frac{\alpha}{2}}} e^{i\langle x, \lambda_i - \tilde{\lambda}_i\rangle} \hat{\rho}(\lambda_i - \eta_i)\overline{\hat{\rho}(\tilde{\lambda}_i - \eta_i)} \right| \lesssim \prod_{i=1}^{p} \frac{1}{(1+|\eta_i|^2)^\alpha},$$

where the proportional constant only depends on ρ and α.

Our last lemma will allow us to control some continuous convolutions in the proof of and constructions. A proof can be found for instance in [7] for $d = 1$ or in [10, Lemma 4.1 p 26] for all $d \geq 1$ in the discrete case (it suffices to replace the summation by an integral to obtain the result below).

Lemma 5.2.5. *Let $d \geq 1$ be a space dimension and α and β be two non-negative real numbers. If*

$$\alpha < d, \quad \beta < d \quad \text{and} \quad \alpha + \beta > d\,,$$

then, for every $x \in \mathbb{R}^d$, the following bound holds true

$$\int_{\mathbb{R}^d} \frac{dy}{(1+|x-y|^2)^{\frac{\alpha}{2}}} \frac{1}{(1+|y|^2)^{\frac{\beta}{2}}} \lesssim \frac{1}{(1+|x|^2)^{\frac{\alpha+\beta-d}{2}}}.$$

5.2.2 Construction of the first order stochastic process

Proof of Proposition 5.1.1.
Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function and $2 \leq p < \infty$. Let s and t be two real numbers such that $0 \leq s \leq t \leq T$. In the following, for all $(m, n) \in \mathbb{N}^2$, we will use the shortcut notations: ${}_{m,n} = {}_m - {}_n$, $a_{m,n} = a_m - a_n$ and $B_{m,n} = B_m \backslash B_n$. As in

[2, 16], our aim is to prove that for all $(m,n) \in \mathbb{N}^2$ such that $1 \leq n \leq m$, the following bound holds true:

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-\frac{s}{2}}\mathcal{F}\Big(\rho[\Psi_{m,n}(t,.)-\Psi_{m,n}(s,.)]\Big)\Big)(x)\right|^{2p}\right] \lesssim \frac{(t-s)^{\varepsilon p}}{n^{2\varepsilon p}},$$

for $\varepsilon > 0$ small enough.
First of all, for every $x \in \mathbb{R}^d$, we write

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-\frac{s}{2}}\mathcal{F}\Big(\rho[\Psi_{m,n}(t,.)-\Psi_{m,n}(s,.)]\Big)\Big)(x)\right|^{2}\right]\\
&= \frac{1}{(2\pi)^{4d}}\iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{\frac{s}{2}}(1+|\tilde{\lambda}|^2)^{\frac{s}{2}}} e^{i\langle x,\lambda-\tilde{\lambda}\rangle} \iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\, \hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})}\\
&\quad \mathbb{E}\left[\mathcal{F}\Big(\Psi_{m,n}(t,.)-\Psi_{m,n}(s,.)\Big)(\beta)\overline{\mathcal{F}\Big(\Psi_{m,n}(t,.)-\Psi_{m,n}(s,.)\Big)(\tilde{\beta})}\right]\\
&= \frac{1}{(2\pi)^{4d}}\iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{\frac{s}{2}}(1+|\tilde{\lambda}|^2)^{\frac{s}{2}}} e^{i\langle x,\lambda-\tilde{\lambda}\rangle} \iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\, \hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})}\\
&\quad \int_{\mathbb{R}^d} dy\, e^{-i\langle\beta,y\rangle}\int_{\mathbb{R}^d} d\tilde{y}\, e^{i\langle\tilde{\beta},\tilde{y}\rangle}\mathbb{E}\left[\Big(\Psi_{m,n}(t,y)-\Psi_{m,n}(s,y)\Big)\Big(\overline{\Psi_{m,n}(t,\tilde{y})-\Psi_{m,n}(s,\tilde{y})}\Big)\right].
\end{aligned}$$

Let us focus on the estimation of the previous mean. Coming back to the definition of Ψ_n (5.2.1), we compute

$$\begin{aligned}
&\Psi_{m,n}(t,y)-\Psi_{m,n}(s,y)\\
&= -i\int_0^t\int_{\mathbb{R}^d} a_{m,n}(t-t',y-x')W(dt',dx') + i\int_0^s\int_{\mathbb{R}^d} a_{m,n}(s-t',y-x')W(dt',dx')\\
&= -i\int_s^t\int_{\mathbb{R}^d} a_{m,n}(t-t',y-x')W(dt',dx')\\
&\qquad - i\int_0^s\int_{\mathbb{R}^d}\Big[a_{m,n}(t-t',y-x')-a_{m,n}(s-t',y-x')\Big]W(dt',dx')
\end{aligned}$$

and, in a similar way,

$$\begin{aligned}
&\overline{\Psi_{m,n}(t,\tilde{y})-\Psi_{m,n}(s,\tilde{y})}\\
&= i\int_s^t\int_{\mathbb{R}^d} \overline{a_{m,n}(t-t',\tilde{y}-x')}W(dt',dx')\\
&\qquad + i\int_0^s\int_{\mathbb{R}^d}\Big[\overline{a_{m,n}(t-t',\tilde{y}-x')-a_{m,n}(s-t',\tilde{y}-x')}\Big]W(dt',dx').
\end{aligned}$$

Resorting successively to Ito's isometry and Plancherel's identity, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\Big(\Psi_{m,n}(t,y)-\Psi_{m,n}(s,y)\Big)\Big(\overline{\Psi_{m,n}(t,\tilde{y})-\Psi_{m,n}(s,\tilde{y})}\Big)\right] = \int_s^t\int_{\mathbb{R}^d} a_{m,n}(t-t',y-x')\overline{a_{m,n}(t-t',\tilde{y}-x')}dt'dx'\\
&+\int_0^s\int_{\mathbb{R}^d}\Big[a_{m,n}(t-t',y-x')-a_{m,n}(s-t',y-x')\Big]\Big[\overline{a_{m,n}(t-t',\tilde{y}-x')-a_{m,n}(s-t',\tilde{y}-x')}\Big]dt'dx'\\
&= \frac{1}{(2\pi)^d}\int_s^t\int_{B_{m,n}}\frac{1}{(1+|\xi|^2)^\alpha}e^{-i\langle\xi,y-\tilde{y}\rangle}dt'd\xi + \frac{1}{(2\pi)^d}\int_0^s\int_{B_{m,n}}\frac{\left|e^{i(t-t')|\xi|^2}-e^{i(s-t')|\xi|^2}\right|^2}{(1+|\xi|^2)^\alpha}e^{-i\langle\xi,y-\tilde{y}\rangle}dt'd\xi.
\end{aligned} \tag{5.2.2}$$

Now, using equality (5.2.2), it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^d} dy\, e^{-i\langle \beta, y\rangle} \int_{\mathbb{R}^d} d\tilde{y}\, e^{i\langle \tilde{\beta}, \tilde{y}\rangle} \mathbb{E}\Big[\big(\Psi_{m,n}(t,y) - \Psi_{m,n}(s,y)\big)\big(\overline{\Psi_{m,n}(t,\tilde{y}) - \Psi_{m,n}(s,\tilde{y})}\big)\Big] \\
&= (2\pi)^d\,(t-s) \int_{B_{m,n}} \delta_{\{\beta=\xi\}}\delta_{\{\tilde{\beta}=\xi\}} \frac{d\xi}{(1+|\xi|^2)^{\alpha}} \\
&\qquad + (2\pi)^d \int_0^s \int_{B_{m,n}} \delta_{\{\beta=\xi\}}\delta_{\{\tilde{\beta}=\xi\}} \frac{\left|e^{i(t-t')|\xi|^2} - e^{i(s-t')|\xi|^2}\right|^2}{(1+|\xi|^2)^{\alpha}} dt'd\xi
\end{aligned} \tag{5.2.3}$$

that leads us to

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-\frac{s}{2}}\mathcal{F}\Big(\rho[\Psi_{m,n}(t,.) - \Psi_{m,n}(s,.)]\Big)\Big)(x)\right|^2\right] \\
&= \frac{1}{(2\pi)^{3d}} \iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{\frac{s}{2}}(1+|\tilde{\lambda}|^2)^{\frac{s}{2}}} e^{i\langle x, \lambda-\tilde{\lambda}\rangle} \\
&\left[(t-s)\int_{B_{m,n}} \frac{\hat{\rho}(\lambda-\xi)\overline{\hat{\rho}(\tilde{\lambda}-\xi)}}{(1+|\xi|^2)^{\alpha}} d\xi + \int_0^s \int_{B_{m,n}} \hat{\rho}(\lambda-\xi)\overline{\hat{\rho}(\tilde{\lambda}-\xi)} \frac{\left|e^{i(t-t')|\xi|^2} - e^{i(s-t')|\xi|^2}\right|^2}{(1+|\xi|^2)^{\alpha}} dt'd\xi\right].
\end{aligned}$$

Using the hypercontractivity of Gaussian variables, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-\frac{s}{2}}\mathcal{F}\Big(\rho[\Psi_{m,n}(t,.) - \Psi_{m,n}(s,.)]\Big)\Big)(x)\right|^{2p}\right] \\
&\leq c_p \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-\frac{s}{2}}\mathcal{F}\Big(\rho[\Psi_{m,n}(t,.) - \Psi_{m,n}(s,.)]\Big)\Big)(x)\right|^{2}\right]^p
\end{aligned}$$

and summoning Lemma 5.2.4, we get the following estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-\frac{s}{2}}\mathcal{F}\Big(\rho[\Psi_{m,n}(t,.) - \Psi_{m,n}(s,.)]\Big)\Big)(x)\right|^{2p}\right] \\
&\lesssim \left(\int_{B_{m,n}} \frac{d\xi}{(1+|\xi|^2)^s}\left[\frac{t-s}{(1+|\xi|^2)^{\alpha}} + \int_0^s \frac{\left|e^{i(t-t')|\xi|^2} - e^{i(s-t')|\xi|^2}\right|^2}{(1+|\xi|^2)^{\alpha}} dt'\right]\right)^p = I_{m,n}^p.
\end{aligned} \tag{5.2.4}$$

Let $0 < \varepsilon < 1$. A straight application of the mean value theorem entails that

$$\begin{aligned}
I_{m,n} &\lesssim \int_{B_{m,n}} \frac{d\xi}{(1+|\xi|^2)^s}\left[\frac{t-s}{(1+|\xi|^2)^{\alpha}} + \int_0^s \frac{(t-s)^{\varepsilon}}{(1+|\xi|^2)^{\alpha-\varepsilon}} dt'\right] \\
&\lesssim (t-s)^{\varepsilon} \int_{B_{m,n}} \frac{d\xi}{(1+|\xi|^2)^{s+\alpha-\varepsilon}} \\
&\lesssim \frac{(t-s)^{\varepsilon}}{n^{2\varepsilon}}\left[1 + \int_1^{+\infty} \frac{dr}{r^{2s+2\alpha-4\varepsilon-d+1}}\right],
\end{aligned}$$

where we have performed a hyperspherical change of variables to get the last inequality. As $s > \frac{d}{2} - \alpha$, we can pick $\varepsilon > 0$ small enough so that $2s + 2\alpha - 4\varepsilon - d + 1 > 1$ and

$$I_{m,n} \lesssim \frac{(t-s)^{\varepsilon}}{n^{2\varepsilon}}. \tag{5.2.5}$$

Injecting (5.2.5) into (5.2.4), we deduce the desired bound

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-\frac{s}{2}}\mathcal{F}\Big(\rho[\mathsf{I}_{m,n}(t,.) - \mathsf{I}_{m,n}(s,.)]\Big)\Big)(x)\right|^{2p}\right] \lesssim \frac{(t-s)^{\varepsilon p}}{n^{2\varepsilon p}},$$

for $\varepsilon > 0$ small enough.

The end of the proof is classical and is the result of a combination of Kolmogorov's criterion and of the Garsia-Rodemich-Rumsey lemma (see [6]). The interested reader shall find the details in [2, 16]. $\square$

5.2.3 Construction of the second order stochastic process

Proof of Proposition 5.1.3. Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function and $2 \leq p < \infty$. Let s and t be two real numbers such that $0 \leq s \leq t \leq T$. As in [2, 16], our strategy is to obtain a bound of the form:

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-s}\mathcal{F}\Big(\rho[\curlyvee_n(t,.) - \curlyvee_n(s,.)]\Big)\Big)(x)\right|^{2p}\right] \lesssim (t-s)^{\varepsilon p},$$

where $\varepsilon > 0$ is small enough and uniformly in n. Indeed, a small adaptation of the proof leads to similar results concerning the variation $\curlyvee_{m,n} = \curlyvee_m - \curlyvee_n$, namely

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-s}\mathcal{F}\Big(\rho[\curlyvee_{m,n}(t,.) - \curlyvee_{m,n}(s,.)]\Big)\Big)(x)\right|^{2p}\right] \lesssim \frac{(t-s)^{\varepsilon p}}{n^{2\varepsilon p}}.$$

First of all, for every $x \in \mathbb{R}^d$, we write

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-s}\mathcal{F}\Big(\rho[\curlyvee_n(t,.) - \curlyvee_n(s,.)]\Big)\Big)(x)\right|^{2}\right]\\
&= \frac{1}{(2\pi)^{4d}} \iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^s(1+|\tilde{\lambda}|^2)^s} e^{i\langle x, \lambda - \tilde{\lambda}\rangle} \iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\, \hat{\rho}(\lambda - \beta)\overline{\hat{\rho}(\tilde{\lambda} - \tilde{\beta})}\\
&\quad \mathbb{E}\left[\mathcal{F}\Big(\curlyvee_n(t,.) - \curlyvee_n(s,.)\Big)(\beta)\overline{\mathcal{F}\Big(\curlyvee_n(t,.) - \curlyvee_n(s,.)\Big)(\tilde{\beta})}\right].
\end{aligned}$$

We focus on the estimation of the latter mean.

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{F}\Big(\curlyvee_n(t,.) - \curlyvee_n(s,.)\Big)(\beta)\overline{\mathcal{F}\Big(\curlyvee_n(t,.) - \curlyvee_n(s,.)\Big)(\tilde{\beta})}\right]\\
&= \int_{\mathbb{R}^d} dy\, e^{-i\langle \beta, y\rangle} \int_{\mathbb{R}^d} d\tilde{y}\, e^{i\langle \tilde{\beta}, \tilde{y}\rangle} \mathbb{E}\left[\Big(\curlyvee_n(t,y) - \curlyvee_n(s,y)\Big)\Big(\overline{\curlyvee_n(t,\tilde{y}) - \curlyvee_n(s,\tilde{y})}\Big)\right].
\end{aligned}$$

Resorting to Wick's formula (see Lemma 5.2.2), we can expand

$$\mathbb{E}\Big[\Big(\curlyvee_n(t,y)-\curlyvee_n(s,y)\Big)\Big(\overline{\curlyvee_n(t,\tilde{y})-\curlyvee_n(s,\tilde{y})}\Big)\Big]$$

in the following way

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\curlyvee_n(t,y)-\curlyvee_n(s,y)\Big)\Big(\overline{\curlyvee_n(t,\tilde{y})-\curlyvee_n(s,\tilde{y})}\Big)\Big]= \\
&\mathbb{E}\Big[\Big(\mathring{|}_n(t,y)-\mathring{|}_n(s,y)\Big)\overline{\mathring{|}_n(t,\tilde{y})}\Big]\mathbb{E}\Big[\overline{\mathring{|}_n(t,y)}\mathring{|}_n(t,\tilde{y})\Big]+\mathbb{E}\Big[\mathring{|}_n(s,y)\overline{\mathring{|}_n(t,\tilde{y})}\Big]\mathbb{E}\Big[\Big(\overline{\mathring{|}_n(t,y)-\mathring{|}_n(s,y)}\Big)\mathring{|}_n(t,\tilde{y})\Big] \\
&+\mathbb{E}\Big[\Big(\mathring{|}_n(s,y)-\mathring{|}_n(t,y)\Big)\overline{\mathring{|}_n(s,\tilde{y})}\Big]\mathbb{E}\Big[\overline{\mathring{|}_n(t,y)}\mathring{|}_n(s,\tilde{y})\Big]+\mathbb{E}\Big[\mathring{|}_n(s,y)\overline{\mathring{|}_n(s,\tilde{y})}\Big]\mathbb{E}\Big[\Big(\overline{\mathring{|}_n(s,y)-\mathring{|}_n(t,y)}\Big)\mathring{|}_n(s,\tilde{y})\Big] \\
&+\mathbb{E}\Big[\Big(\mathring{|}_n(t,y)-\mathring{|}_n(s,y)\Big)\mathring{|}_n(t,\tilde{y})\Big]\mathbb{E}\Big[\overline{\mathring{|}_n(t,y)\mathring{|}_n(t,\tilde{y})}\Big]+\mathbb{E}\Big[\mathring{|}_n(s,y)\mathring{|}_n(t,\tilde{y})\Big]\mathbb{E}\Big[\overline{\Big(\mathring{|}_n(t,y)-\mathring{|}_n(s,y)\Big)\mathring{|}_n(t,\tilde{y})}\Big] \\
&+\mathbb{E}\Big[\Big(\mathring{|}_n(s,y)-\mathring{|}_n(t,y)\Big)\mathring{|}_n(s,\tilde{y})\Big]\mathbb{E}\Big[\overline{\mathring{|}_n(t,y)\mathring{|}_n(s,\tilde{y})}\Big]+\mathbb{E}\Big[\mathring{|}_n(s,y)\mathring{|}_n(s,\tilde{y})\Big]\mathbb{E}\Big[\overline{\Big(\mathring{|}_n(s,y)-\mathring{|}_n(t,y)\Big)\mathring{|}_n(s,\tilde{y})}\Big].
\end{aligned} \tag{5.2.6}$$

Consequently, we have to bound eight integrals. Since their treatments are quite the same, we will only detail the computations related to the first one. Namely, our aim is to control the quantity:

$$\begin{aligned}
\mathbb{I}=\frac{1}{(2\pi)^{4d}}&\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^s(1+|\tilde{\lambda}|^2)^s}e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\iint_{(\mathbb{R}^d)^2}d\beta d\tilde{\beta}\,\hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})} \\
&\int_{\mathbb{R}^d}dy\,e^{-i\langle\beta,y\rangle}\int_{\mathbb{R}^d}d\tilde{y}\,e^{i\langle\tilde{\beta},\tilde{y}\rangle}\mathbb{E}\Big[\Big(\mathring{|}_n(t,y)-\mathring{|}_n(s,y)\Big)\overline{\mathring{|}_n(t,\tilde{y})}\Big]\mathbb{E}\Big[\overline{\mathring{|}_n(t,y)}\mathring{|}_n(t,\tilde{y})\Big].
\end{aligned} \tag{5.2.7}$$

Expanding the previous mean thanks to Ito's isometry and Plancherel's identity, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\mathring{|}_n(t,y)-\mathring{|}_n(s,y)\Big)\overline{\mathring{|}_n(t,\tilde{y})}\Big]\mathbb{E}\Big[\overline{\mathring{|}_n(t,y)}\mathring{|}_n(t,\tilde{y})\Big]=\Bigg[\int_s^t\int_{\mathbb{R}^d}a_n(t-t',y-x')\overline{a_n(t-t',\tilde{y}-x')}dt'dx' \\
&+\int_0^s\int_{\mathbb{R}^d}\Big[a_n(t-t',y-x')-a_n(s-t',y-x')\Big]\overline{a_n(t-t',\tilde{y}-x')}dt'dx'\Bigg] \\
&\qquad\qquad\qquad\times\Bigg[\int_0^t\int_{\mathbb{R}^d}\overline{a_n(t-t',y-x')}a_n(t-t',\tilde{y}-x')dt'dx'\Bigg] \\
&=\frac{1}{(2\pi)^{2d}}\Bigg[\int_s^t\int_{B_n}\frac{1}{(1+|\xi_1|^2)^\alpha}e^{-i\langle\xi_1,y-\tilde{y}\rangle}dt'd\xi_1+\int_0^s\int_{B_n}\frac{\Big(e^{i(t-t')|\xi_1|^2}-e^{i(s-t')|\xi_1|^2}\Big)}{(1+|\xi_1|^2)^\alpha} \\
&\qquad e^{-i(t-t')|\xi_1|^2}e^{-i\langle\xi_1,y-\tilde{y}\rangle}dt'd\xi_1\Bigg]\Bigg[\int_0^t\int_{B_n}\frac{1}{(1+|\xi_2|^2)^\alpha}e^{-i\langle\xi_2,y-\tilde{y}\rangle}dt'd\xi_2\Bigg]
\end{aligned} \tag{5.2.8}$$

and, consequently, injecting equality (5.2.8) into (5.2.7), we get that

$$
\begin{aligned}
\mathbb{I} = \frac{1}{(2\pi)^{4d}} \iint_{(\mathbb{R}^d)^2} & \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^s(1+|\tilde{\lambda}|^2)^s} e^{i\langle x, \lambda - \tilde{\lambda}\rangle} \\
& \Bigg[(t-s)t \int_{B_n}\int_{B_n} \frac{\hat{\rho}(\lambda-(\xi_1+\xi_2))\overline{\hat{\rho}(\tilde{\lambda}-(\xi_1+\xi_2))}}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha} d\xi_1 d\xi_2 \\
& + t\int_0^s \int_{B_n}\int_{B_n} \hat{\rho}(\lambda-(\xi_1+\xi_2))\overline{\hat{\rho}(\tilde{\lambda}-(\xi_1+\xi_2))} \frac{\left(e^{i(t-t')|\xi_1|^2} - e^{i(s-t')|\xi_1|^2}\right)}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha} e^{-i(t-t')|\xi_1|^2} dt' d\xi_1 d\xi_2 \Bigg].
\end{aligned}
$$

Combining the hypercontractivity of Wiener chaoses (see Lemma 5.2.3) and Lemma 5.2.4, we get the following estimate

$$
\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-s}\mathcal{F}\Big(\rho[{}^{\circ}\!\vee\!{}^{\circ}_n(t,.) - {}^{\circ}\!\vee\!{}^{\circ}_n(s,.)]\Big)\Big)(x)\right|^{2p}\right] \lesssim \left(\sum_{k=1}^{8} I_k\right)^p,
$$

where the eight integrals are the one related to the terms in (5.2.6). As above, let us focus on the treatment of the first term denoted by I_1. Let $0<\varepsilon<1$. A straight application of the mean value theorem entails that

$$
\begin{aligned}
I_1 = & \Bigg[(t-s)t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{(1+|\xi_1+\xi_2|^2)^{2s}} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha} \\
& + t\int_0^s\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{(1+|\xi_1+\xi_2|^2)^{2s}} \frac{\left|e^{i(t-t')|\xi_1|^2} - e^{i(s-t')|\xi_1|^2}\right|}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha} dt' d\xi_1 d\xi_2 \Bigg] \\
& \lesssim (t-s)^\varepsilon \iint_{(\mathbb{R}^d)^2} \frac{d\xi_1 d\xi_2}{(1+|\xi_1+\xi_2|^2)^{2s}} \frac{1}{(1+|\xi_1|^2)^{\alpha-\varepsilon}} \frac{1}{(1+|\xi_2|^2)^\alpha} \\
& \lesssim (t-s)^\varepsilon \int_{\mathbb{R}^d} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s}} \int_{\mathbb{R}^d} \frac{d\xi_2}{(1+|\xi_1-\xi_2|^2)^{\alpha-\varepsilon}} \frac{1}{(1+|\xi_2|^2)^\alpha} \\
& \lesssim (t-s)^\varepsilon \int_{\mathbb{R}^d} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s+2\alpha-\frac{d}{2}-\varepsilon}},
\end{aligned}
$$

where we have used Lemma 5.2.5 to derive the last inequality for small $\varepsilon>0$ verifying $4\alpha-2\varepsilon>d$. Now, resorting to a hyperspherical change of variables,

$$
\int_{\mathbb{R}^d} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s+2\alpha-\frac{d}{2}-\varepsilon}} \lesssim 1 + \int_1^{+\infty} \frac{dr}{r^{4s+4\alpha-2\varepsilon-2d+1}}.
$$

As $s>\frac{d}{2}-\alpha$, we can pick $\varepsilon>0$ so that $4s+4\alpha-2\varepsilon-2d+1>1$ and

$$
I_1 \lesssim (t-s)^\varepsilon.
$$

The end of the proof is classical and is the result of a combination of Kolmogorov's criterion and of the Garsia-Rodemich-Rumsey lemma (see [6]). The interested reader shall find the details in [2, 16]. □

5.2.4 Construction of the convolution with the second order stochastic process

Proof of Proposition 5.1.8. Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function and $2 \le p < \infty$. Let s and t be two real numbers such that $0 \le s \le t \le T$. Again, our objective is to obtain a bound of the form:

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\left(\left(1+|.|^2\right)^{-s+\frac{\kappa}{2}}\mathcal{F}\left(\rho\left[\mathbf{Y}_n(t,.)-\mathbf{Y}_n(s,.)\right]\right)\right)(x)\right|^{2p}\right] \lesssim (t-s)^{\varepsilon p},$$

where $\varepsilon > 0$ is small enough and uniformly in n. Indeed, a small adaptation of the proof leads to similar results concerning the variation $\mathbf{Y}_{m,n} = \mathbf{Y}_m - \mathbf{Y}_n$, namely

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}\left[\left|\mathcal{F}^{-1}\left(\left(1+|.|^2\right)^{-s+\frac{\kappa}{2}}\mathcal{F}\left(\rho\left[\mathbf{Y}_{m,n}(t,.)-\mathbf{Y}_{m,n}(s,.)\right]\right)\right)(x)\right|^{2p}\right] \lesssim \frac{(t-s)^{\varepsilon p}}{n^{2\varepsilon p}}.$$

First of all, for every $x \in \mathbb{R}^d$, we write

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{F}^{-1}\left(\left(1+|.|^2\right)^{-s+\frac{\kappa}{2}}\mathcal{F}\left(\rho\left[\mathbf{Y}_n(t,.)-\mathbf{Y}_n(s,.)\right]\right)\right)(x)\right|^{2}\right]\\
&= \frac{1}{(2\pi)^{4d}}\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde{\lambda}|^2)^{s-\frac{\kappa}{2}}}e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\,\hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})}\\
&\qquad\qquad \mathbb{E}\left[\mathcal{F}\left(\mathbf{Y}_n(t,.)-\mathbf{Y}_n(s,.)\right)(\beta)\overline{\mathcal{F}\left(\mathbf{Y}_n(t,.)-\mathbf{Y}_n(s,.)\right)(\tilde{\beta})}\right]\\
&= \frac{1}{(2\pi)^{4d}}\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde{\lambda}|^2)^{s-\frac{\kappa}{2}}}e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\,\hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})}\\
&\qquad\qquad \mathbb{E}\left[\mathcal{F}\left(\mathbf{Y}_n(t,.)-\mathbf{Y}_n(s,.)\right)(\beta)\overline{\mathcal{F}\left(\mathbf{Y}_n(t,.)\right)(\tilde{\beta})}\right]\\
&- \frac{1}{(2\pi)^{4d}}\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde{\lambda}|^2)^{s-\frac{\kappa}{2}}}e^{i\langle x,\lambda-\tilde{\lambda}\rangle}\iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\,\hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})}\\
&\qquad\qquad \mathbb{E}\left[\mathcal{F}\left(\mathbf{Y}_n(t,.)-\mathbf{Y}_n(s,.)\right)(\beta)\overline{\mathcal{F}\left(\mathbf{Y}_n(s,.)\right)(\tilde{\beta})}\right].
\end{aligned} \tag{5.2.9}$$

Based on Definition 5.1.7 of $\mathbf{Y}_n$, it holds that, for every $\beta \in \mathbb{R}^d$,

$$\begin{aligned}
\mathcal{F}\left(\mathbf{Y}_n(t,.)-\mathbf{Y}_n(s,.)\right)(\beta) &= -i\int_s^t e^{i(t-\tau)|\beta|^2}\mathcal{F}(\vee_n(\tau,.))(\beta)d\tau\\
&\quad - i\int_0^s\left(e^{i(t-\tau)|\beta|^2}-e^{i(s-\tau)|\beta|^2}\right)\mathcal{F}(\vee_n(\tau,.))(\beta)d\tau.
\end{aligned}$$

The previous identity entails that

$$
\begin{aligned}
&\mathbb{E}\Big[\mathcal{F}\Big(\Upsilon_n(t,.)-\Upsilon_n(s,.)\Big)(\beta)\overline{\mathcal{F}\Big(\Upsilon_n(t,.)\Big)(\tilde\beta)}\Big]-\mathbb{E}\Big[\mathcal{F}\Big(\Upsilon_n(t,.)-\Upsilon_n(s,.)\Big)(\beta)\overline{\mathcal{F}\Big(\Upsilon_n(s,.)\Big)(\tilde\beta)}\Big]\\
&=\int_s^t dt_1\int_0^t dt_2\, e^{i(t-t_1)|\beta|^2}e^{-i(t-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\mathbb{E}\Big[\curlyvee_n(t_1,y)\overline{\curlyvee_n(t_2,\tilde y)}\Big]\\
&+\int_0^s dt_1\int_0^t dt_2\,\Big(e^{i(t-t_1)|\beta|^2}-e^{i(s-t_1)|\beta|^2}\Big)e^{-i(t-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\mathbb{E}\Big[\curlyvee_n(t_1,y)\overline{\curlyvee_n(t_2,\tilde y)}\Big]\\
&-\int_s^t dt_1\int_0^s dt_2\, e^{i(t-t_1)|\beta|^2}e^{-i(s-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\mathbb{E}\Big[\curlyvee_n(t_1,y)\overline{\curlyvee_n(t_2,\tilde y)}\Big]\\
&-\int_0^s dt_1\int_0^s dt_2\,\Big(e^{i(t-t_1)|\beta|^2}-e^{i(s-t_1)|\beta|^2}\Big)e^{-i(s-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\mathbb{E}\Big[\curlyvee_n(t_1,y)\overline{\curlyvee_n(t_2,\tilde y)}\Big].
\end{aligned}
\tag{5.2.10}
$$

Resorting to Wick's formula (see Lemma 5.2.2), we can write

$$
\mathbb{E}\Big[\curlyvee_n(t_1,y)\overline{\curlyvee_n(t_2,\tilde y)}\Big]=\Big|\mathbb{E}\Big[\iota_n(t_1,y)\overline{\iota_n(t_2,\tilde y)}\Big]\Big|^2+\Big|\mathbb{E}\Big[\iota_n(t_1,y)\iota_n(t_2,\tilde y)\Big]\Big|^2. \tag{5.2.11}
$$

Consequently, combining (5.2.10) and (5.2.11), and coming back to (5.2.9), we obtain that

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-s+\frac{\kappa}{2}}\mathcal{F}\Big(\rho\Big[\Upsilon_n(t,.)-\Upsilon_n(s,.)\Big]\Big)\Big)(x)\Big|^2\Big]\\
&=\frac{1}{(2\pi)^{4d}}\iint_{(\mathbb{R}^d)^2}\frac{d\lambda d\tilde\lambda}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde\lambda|^2)^{s-\frac{\kappa}{2}}}e^{i\langle x,\lambda-\tilde\lambda\rangle}\iint_{(\mathbb{R}^d)^2}d\beta d\tilde\beta\,\hat\rho(\lambda-\beta)\overline{\hat\rho(\tilde\lambda-\tilde\beta)}\\
&\Big[\int_s^t dt_1\int_0^t dt_2\, e^{i(t-t_1)|\beta|^2}e^{-i(t-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\overline{\iota_n(t_2,\tilde y)}\Big]\Big|^2\\
&+\int_s^t dt_1\int_0^t dt_2\, e^{i(t-t_1)|\beta|^2}e^{-i(t-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\iota_n(t_2,\tilde y)\Big]\Big|^2\\
&+\int_0^s dt_1\int_0^t dt_2\,\Big(e^{i(t-t_1)|\beta|^2}-e^{i(s-t_1)|\beta|^2}\Big)e^{-i(t-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\overline{\iota_n(t_2,\tilde y)}\Big]\Big|^2\\
&+\int_0^s dt_1\int_0^t dt_2\,\Big(e^{i(t-t_1)|\beta|^2}-e^{i(s-t_1)|\beta|^2}\Big)e^{-i(t-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\iota_n(t_2,\tilde y)\Big]\Big|^2\\
&-\int_s^t dt_1\int_0^s dt_2\, e^{i(t-t_1)|\beta|^2}e^{-i(s-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\overline{\iota_n(t_2,\tilde y)}\Big]\Big|^2\\
&-\int_s^t dt_1\int_0^s dt_2\, e^{i(t-t_1)|\beta|^2}e^{-i(s-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\iota_n(t_2,\tilde y)\Big]\Big|^2\\
&-\int_0^s dt_1\int_0^s dt_2\,\Big(e^{i(t-t_1)|\beta|^2}-e^{i(s-t_1)|\beta|^2}\Big)e^{-i(s-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\overline{\iota_n(t_2,\tilde y)}\Big]\Big|^2\\
&-\int_0^s dt_1\int_0^s dt_2\,\Big(e^{i(t-t_1)|\beta|^2}-e^{i(s-t_1)|\beta|^2}\Big)e^{-i(s-t_2)|\tilde\beta|^2}\iint_{(\mathbb{R}^d)^2}dy d\tilde y\, e^{-i\langle\beta,y\rangle}e^{i\langle\tilde\beta,\tilde y\rangle}\Big|\mathbb{E}\Big[\iota_n(t_1,y)\iota_n(t_2,\tilde y)\Big]\Big|^2\Big].
\end{aligned}
\tag{5.2.12}
$$

The latter computation shows that $\mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-s+\frac{\kappa}{2}}\mathcal{F}\Big(\rho\Big[\Upsilon_n(t,.)-\Upsilon_n(s,.)\Big]\Big)\Big)(x)\Big|^2\Big]$ is a sum of eight integrals. Let us focus our attention on the two following since the treatment of the six others is quite the same:

$$I_1 = \frac{1}{(2\pi)^{4d}} \iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde{\lambda}|^2)^{s-\frac{\kappa}{2}}} e^{i\langle x,\lambda-\tilde{\lambda}\rangle} \iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\, \hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})}$$
$$\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\beta|^2} - e^{i(s-t_1)|\beta|^2}\right) e^{-i(t-t_2)|\tilde{\beta}|^2} \iint_{(\mathbb{R}^d)^2} dy d\tilde{y}\, e^{-i\langle \beta,y\rangle} e^{i\langle\tilde{\beta},\tilde{y}\rangle} \left|\mathbb{E}\left[\text{⚲}_n(t_1,y)\overline{\text{⚲}_n(t_2,\tilde{y})}\right]\right|^2$$

and

$$I_2 = \frac{1}{(2\pi)^{4d}} \iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde{\lambda}|^2)^{s-\frac{\kappa}{2}}} e^{i\langle x,\lambda-\tilde{\lambda}\rangle} \iint_{(\mathbb{R}^d)^2} d\beta d\tilde{\beta}\, \hat{\rho}(\lambda-\beta)\overline{\hat{\rho}(\tilde{\lambda}-\tilde{\beta})}$$
$$\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\beta|^2} - e^{i(s-t_1)|\beta|^2}\right) e^{-i(t-t_2)|\tilde{\beta}|^2} \iint_{(\mathbb{R}^d)^2} dy d\tilde{y}\, e^{-i\langle \beta,y\rangle} e^{i\langle\tilde{\beta},\tilde{y}\rangle} \left|\mathbb{E}\left[\text{⚲}_n(t_1,y)\text{⚲}_n(t_2,\tilde{y})\right]\right|^2.$$

- <u>Study of I_1</u>: Using Definition 5.2.1 of the covariance function of ⚲_n, we get that

$$\left|\mathbb{E}\left[\text{⚲}_n(t_1,y)\overline{\text{⚲}_n(t_2,\tilde{y})}\right]\right|^2 = \left|\min(t_1,t_2)\frac{1}{(2\pi)^d}\int_{B_n} \frac{e^{i(t_1-t_2)|\xi|^2}}{(1+|\xi|^2)^\alpha} e^{-i\langle\xi,y-\tilde{y}\rangle} d\xi\right|^2$$
$$= \min(t_1,t_2)^2 \frac{1}{(2\pi)^{2d}} \int_{B_n}\int_{B_n} d\xi_1 d\xi_2 \frac{e^{i(t_1-t_2)(|\xi_1|^2-|\xi_2|^2)}}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha} e^{-i\langle\xi_1-\xi_2,y-\tilde{y}\rangle}. \tag{5.2.13}$$

Injecting (5.2.13) into the definition of I_1, it holds that

$$I_1 = \frac{1}{(2\pi)^{4d}} \iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde{\lambda}|^2)^{s-\frac{\kappa}{2}}} e^{i\langle x,\lambda-\tilde{\lambda}\rangle}$$
$$\int_{B_n}\int_{B_n} \hat{\rho}(\lambda-(\xi_1-\xi_2))\overline{\hat{\rho}(\tilde{\lambda}-(\xi_1-\xi_2))} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha}$$
$$\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1-\xi_2|^2} - e^{i(s-t_1)|\xi_1-\xi_2|^2}\right) e^{-i(t-t_2)|\xi_1-\xi_2|^2} \min(t_1,t_2)^2 e^{i(t_1-t_2)(|\xi_1|^2-|\xi_2|^2)}.$$

- <u>Study of I_2</u>: Using Definition 5.2.1 of the covariance function of ⚲_n, we get that

$$\left|\mathbb{E}\left[\text{⚲}_n(t_1,y)\text{⚲}_n(t_2,\tilde{y})\right]\right|^2 = \left|\frac{1}{(2\pi)^d}\int_0^{\min(t_1,t_2)}\int_{B_n} \frac{e^{i(t_1+t_2-2t')|\xi|^2}}{(1+|\xi|^2)^\alpha} e^{-i\langle\xi,y-\tilde{y}\rangle} d\xi dt'\right|^2$$
$$= \frac{1}{(2\pi)^{2d}} \int_0^{\min(t_1,t_2)}\int_0^{\min(t_1,t_2)} dt_1' dt_2' \int_{B_n}\int_{B_n} d\xi_1 d\xi_2 \frac{e^{i(t_1+t_2)(|\xi_1|^2-|\xi_2|^2)} e^{-2it_1'|\xi_1|^2} e^{2it_2'|\xi_2|^2}}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha} e^{-i\langle\xi_1-\xi_2,y-\tilde{y}\rangle}. \tag{5.2.14}$$

Injecting (5.2.14) into the definition of I_2, it holds that

$$I_2 = \frac{1}{(2\pi)^{4d}} \iint_{(\mathbb{R}^d)^2} \frac{d\lambda d\tilde{\lambda}}{(1+|\lambda|^2)^{s-\frac{\kappa}{2}}(1+|\tilde{\lambda}|^2)^{s-\frac{\kappa}{2}}} e^{i\langle x,\lambda-\tilde{\lambda}\rangle}$$
$$\int_{B_n}\int_{B_n} \hat{\rho}(\lambda-(\xi_1-\xi_2))\overline{\hat{\rho}(\tilde{\lambda}-(\xi_1-\xi_2))} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^\alpha(1+|\xi_2|^2)^\alpha}$$
$$\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1-\xi_2|^2} - e^{i(s-t_1)|\xi_1-\xi_2|^2}\right) e^{-i(t-t_2)|\xi_1-\xi_2|^2} e^{i(t_1+t_2)(|\xi_1|^2-|\xi_2|^2)}$$
$$\int_0^{\min(t_1,t_2)}\int_0^{\min(t_1,t_2)} dt_1' dt_2'\, e^{-2it_1'|\xi_1|^2} e^{2it_2'|\xi_2|^2}.$$

• Come back to (5.2.12): Combining the hypercontractivity of Wiener chaoses (see Lemma 5.2.3) and Lemma 5.2.4, we get the following estimate

$$\int_{\mathbb{R}^d} dx\, \mathbb{E}\Big[\Big|\mathcal{F}^{-1}\Big(\big(1+|.|^2\big)^{-s+\frac{\kappa}{2}}\mathcal{F}\Big(\rho\Big[\mathrm{Y}_n(t,.)-\mathrm{Y}_n(s,.)\Big]\Big)\Big)(x)\Big|^{2p}\Big] \lesssim \Big(\sum_{k=1}^{8}|\tilde{I}_k|\Big)^p,$$

where the eight integrals are the one related to the terms in (5.2.12). As above, let us focus on the treatment of the two first terms denoted by $\tilde{I}_1$ and $\tilde{I}_2$.

• Bound on $\tilde{I}_1$: Two changes of variables lead to the following equalities:

$$\begin{aligned}
\tilde{I}_1 &= \int_{B_n}\int_{B_n} \frac{d\xi_1 d\xi_2}{(1+|\xi_1-\xi_2|^2)^{2s-\kappa}(1+|\xi_1|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}} \\
&\quad \int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1-\xi_2|^2}-e^{i(s-t_1)|\xi_1-\xi_2|^2}\right)e^{-i(t-t_2)|\xi_1-\xi_2|^2}\min(t_1,t_2)^2 e^{i(t_1-t_2)(|\xi_1|^2-|\xi_2|^2)} \\
&= \int_{B_n}\int_{B_n} \frac{d\xi_1 d\xi_2}{(1+|\xi_1+\xi_2|^2)^{2s-\kappa}(1+|\xi_1|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}} \\
&\quad \int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1+\xi_2|^2}-e^{i(s-t_1)|\xi_1+\xi_2|^2}\right)e^{-i(t-t_2)|\xi_1+\xi_2|^2}\min(t_1,t_2)^2 e^{i(t_1-t_2)(|\xi_1|^2-|\xi_2|^2)} \\
&= \int_{|\xi_1-\xi_2|\leq n}\int_{B_n} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}} \\
&\quad \int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1|^2}-e^{i(s-t_1)|\xi_1|^2}\right)e^{-i(t-t_2)|\xi_1|^2}\min(t_1,t_2)^2 e^{i(t_1-t_2)(|\xi_1-\xi_2|^2-|\xi_2|^2)}.
\end{aligned}$$

Let us introduce the additional notation: $\kappa(\underline{\xi}) = |\xi_1|^2-|\xi_1-\xi_2|^2+|\xi_2|^2$. Then, treating separately the cases where $t_1 \leq t_2$ and $t_2 < t_1$, performing an integration by part in the second term of the following sum and using the mean value theorem, we obtain:

$$\begin{aligned}
&\left|\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1|^2}-e^{i(s-t_1)|\xi_1|^2}\right)e^{-i(t-t_2)|\xi_1|^2}\min(t_1,t_2)^2 e^{i(t_1-t_2)(|\xi_1-\xi_2|^2-|\xi_2|^2)}\right| \\
&\leq \left|\int_0^s dt_1 \int_{t_1}^t dt_2 \left(e^{i(t-t_1)|\xi_1|^2}-e^{i(s-t_1)|\xi_1|^2}\right)e^{-i(t-t_2)|\xi_1|^2}t_1^2 e^{i(t_1-t_2)(|\xi_1-\xi_2|^2-|\xi_2|^2)}\right| \\
&\quad +\left|\int_0^s dt_1 \int_0^{t_1} dt_2 \left(e^{i(t-t_1)|\xi_1|^2}-e^{i(s-t_1)|\xi_1|^2}\right)e^{-i(t-t_2)|\xi_1|^2}t_2^2 e^{i(t_1-t_2)(|\xi_1-\xi_2|^2-|\xi_2|^2)}\right| \\
&= \left|\int_0^s dt_1\Big(e^{i(t-t_1)|\xi_1|^2}-e^{i(s-t_1)|\xi_1|^2}\Big)t_1^2 e^{-it|\xi_1|^2}e^{it_1(|\xi_1-\xi_2|^2-|\xi_2|^2)}\int_{t_1}^t dt_2\, e^{it_2\kappa(\underline{\xi})}\right| \\
&\quad +\left|\int_0^s dt_1\Big(e^{i(t-t_1)|\xi_1|^2}-e^{i(s-t_1)|\xi_1|^2}\Big)e^{-it|\xi_1|^2}e^{it_1(|\xi_1-\xi_2|^2-|\xi_2|^2)}\int_0^{t_1} dt_2\, t_2^2 e^{it_2\kappa(\underline{\xi})}\right| \\
&\lesssim \frac{(t-s)^{\varepsilon}}{(1+|\xi_1|^2)^{-\varepsilon}(1+|\kappa(\underline{\xi})|)},
\end{aligned}$$

for all $0<\varepsilon<1$.

Finally,

$$|\tilde{I}_1| \lesssim (t-s)^{\varepsilon}\iint_{(\mathbb{R}^d)^2}\frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa-\varepsilon}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}(1+|\kappa(\underline{\xi})|)}.$$

• Bound on $\tilde{I}_2$: Two changes of variables lead to the following equalities:

$$\tilde{I}_2 = \int_{B_n}\int_{B_n} \frac{d\xi_1 d\xi_2}{(1+|\xi_1-\xi_2|^2)^{2s-\kappa}(1+|\xi_1|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}}$$
$$\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1-\xi_2|^2} - e^{i(s-t_1)|\xi_1-\xi_2|^2}\right) e^{-i(t-t_2)|\xi_1-\xi_2|^2} e^{i(t_1+t_2)(|\xi_1|^2-|\xi_2|^2)}$$
$$\int_0^{\min(t_1,t_2)}\int_0^{\min(t_1,t_2)} dt_1' dt_2'\, e^{-2it_1'|\xi_1|^2} e^{2it_2'|\xi_2|^2}$$
$$= \int_{B_n}\int_{B_n} \frac{d\xi_1 d\xi_2}{(1+|\xi_1+\xi_2|^2)^{2s-\kappa}(1+|\xi_1|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}}$$
$$\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1+\xi_2|^2} - e^{i(s-t_1)|\xi_1+\xi_2|^2}\right) e^{-i(t-t_2)|\xi_1+\xi_2|^2} e^{i(t_1+t_2)(|\xi_1|^2-|\xi_2|^2)}$$
$$\int_0^{\min(t_1,t_2)}\int_0^{\min(t_1,t_2)} dt_1' dt_2'\, e^{-2it_1'|\xi_1|^2} e^{2it_2'|\xi_2|^2}$$
$$= \int_{B_n}\int_{B_n} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}}$$
$$\int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1|^2} - e^{i(s-t_1)|\xi_1|^2}\right) e^{-i(t-t_2)|\xi_1|^2} e^{i(t_1+t_2)(|\xi_1-\xi_2|^2-|\xi_2|^2)}$$
$$\int_0^{\min(t_1,t_2)}\int_0^{\min(t_1,t_2)} dt_1' dt_2'\, e^{-2it_1'|\xi_1-\xi_2|^2} e^{2it_2'|\xi_2|^2}.$$

Let us introduce the additional notation: $\kappa_2(\underline{\xi}) = |\xi_1|^2 + |\xi_1-\xi_2|^2 - |\xi_2|^2$. Then, treating separately the cases where $t_1 \le t_2$ and $t_2 < t_1$, changing the order of integration and using the mean value theorem, we obtain:

$$\begin{aligned}
&\left| \int_0^s dt_1 \int_0^t dt_2 \left(e^{i(t-t_1)|\xi_1|^2} - e^{i(s-t_1)|\xi_1|^2} \right) e^{-i(t-t_2)|\xi_1|^2} e^{i(t_1+t_2)(|\xi_1-\xi_2|^2 - |\xi_2|^2)} \right. \\
&\qquad\qquad \left. \int_0^{\min(t_1,t_2)} \int_0^{\min(t_1,t_2)} dt_1' dt_2' \, e^{-2it_1'|\xi_1-\xi_2|^2} e^{2it_2'|\xi_2|^2} \right| \\
&\le \left| \int_0^s dt_1 \int_{t_1}^t dt_2 \left(e^{i(t-t_1)|\xi_1|^2} - e^{i(s-t_1)|\xi_1|^2} \right) e^{-i(t-t_2)|\xi_1|^2} e^{i(t_1+t_2)(|\xi_1-\xi_2|^2 - |\xi_2|^2)} \right. \\
&\qquad\qquad \left. \int_0^{t_1} \int_0^{t_1} dt_1' dt_2' \, e^{-2it_1'|\xi_1-\xi_2|^2} e^{2it_2'|\xi_2|^2} \right| \\
&+ \left| \int_0^s dt_1 \int_0^{t_1} dt_2 \left(e^{i(t-t_1)|\xi_1|^2} - e^{i(s-t_1)|\xi_1|^2} \right) e^{-i(t-t_2)|\xi_1|^2} e^{i(t_1+t_2)(|\xi_1-\xi_2|^2 - |\xi_2|^2)} \right. \\
&\qquad\qquad \left. \int_0^{t_2} \int_0^{t_2} dt_1' dt_2' \, e^{-2it_1'|\xi_1-\xi_2|^2} e^{2it_2'|\xi_2|^2} \right| \\
&= \left| \int_0^s dt_1 \left(e^{i(t-t_1)|\xi_1|^2} - e^{i(s-t_1)|\xi_1|^2} \right) e^{-it|\xi_1|^2} e^{it_1(|\xi_1-\xi_2|^2 - |\xi_2|^2)} \right. \\
&\qquad\qquad \left. \int_0^{t_1} \int_0^{t_1} dt_1' dt_2' \, e^{-2it_1'|\xi_1-\xi_2|^2} e^{2it_2'|\xi_2|^2} \int_{t_1}^t dt_2 \, e^{it_2\kappa_2(\underline{\xi})} \right| \\
&\quad + \left| \int_0^s dt_1 \left(e^{i(t-t_1)|\xi_1|^2} - e^{i(s-t_1)|\xi_1|^2} \right) e^{-it|\xi_1|^2} e^{it_1(|\xi_1-\xi_2|^2 - |\xi_2|^2)} \right. \\
&\qquad\qquad \left. \int_0^{t_1} \int_0^{t_1} dt_1' dt_2' \, e^{-2it_1'|\xi_1-\xi_2|^2} e^{2it_2'|\xi_2|^2} \int_{\max(t_1',t_2')}^{t_1} dt_2 \, e^{it_2\kappa_2(\underline{\xi})} \right| \\
&\lesssim \frac{(t-s)^\varepsilon}{(1+|\xi_1|^2)^{-\varepsilon}(1+|\kappa_2(\underline{\xi})|)},
\end{aligned}$$

for all $0 < \varepsilon < 1$.
Finally,

$$\begin{aligned}
|\tilde{I}_2| &\lesssim (t-s)^\varepsilon \iint_{(\mathbb{R}^d)^2} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa-\varepsilon}(1+|\xi_1-\xi_2|^2)^\alpha(1+|\xi_2|^2)^\alpha(1+|\kappa_2(\underline{\xi})|)} \\
&= (t-s)^\varepsilon \iint_{(\mathbb{R}^d)^2} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa-\varepsilon}(1+|\xi_1-\xi_2|^2)^\alpha(1+|\xi_2|^2)^\alpha(1+|\kappa(\underline{\xi})|)}.
\end{aligned}$$

The end of the proof consists in showing that the integral

$$\mathbb{I} = \iint_{(\mathbb{R}^d)^2} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^\alpha(1+|\xi_2|^2)^\alpha(1+|\kappa(\underline{\xi})|)}$$

is finite.
We split $\mathbb{R}^d \times \mathbb{R}^d$ into two domains defined by

$$D_1 = \{(\xi_1,\xi_2) \in \mathbb{R}^d \times \mathbb{R}^d, |\xi_1| \le 1\} \quad \text{and} \quad D_2 = \{(\xi_1,\xi_2) \in \mathbb{R}^d \times \mathbb{R}^d, |\xi_1| > 1\}$$

and we denote by $\mathbb{I}_1$ and $\mathbb{I}_2$ the associated integrals.

• Bound on $\mathbb{I}_1$:
It holds that:

$$\begin{aligned}\mathbb{I}_1 &= \int_{|\xi_1|\le 1}\int_{\mathbb{R}^d} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}(1+|\kappa(\underline{\xi})|)} \\ &\lesssim \int_{|\xi_1|\le 1}\int_{\mathbb{R}^d} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}} \\ &\lesssim \int_{|\xi_1|\le 1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1|^2)^{2\alpha-\frac{d}{2}}} \\ &\lesssim 1 \end{aligned} \tag{5.2.15}$$

where we have used Lemma 5.2.5 with the condition $\frac{d}{4} < \alpha < \frac{d}{2}$ to derive the second inequality.

• Bound on $\mathbb{I}_2$:
We split D_2 into two domains defined by

$$D_2^1 = \{(\xi_1,\xi_2) \in D_2, |\xi_1-\xi_2| < |\xi_2|\} \quad \text{and} \quad D_2^2 = \{(\xi_1,\xi_2) \in D_2, |\xi_1-\xi_2| \ge |\xi_2|\},$$

and we denote by $\mathbb{I}_2^1$ and $\mathbb{I}_2^2$ the associated integrals.

• Bound on $\mathbb{I}_2^1$:
Let us observe that on D_2^1, the following bound holds true: $|\kappa(\underline{\xi})| \ge |\xi_1|^2$. Now, resorting to Lemma 5.2.5 again, one has

$$\begin{aligned}\mathbb{I}_2^1 &= \int_{|\xi_1|>1}\int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|\xi_1-\xi_2|<|\xi_2|\}} d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}(1+|\kappa(\underline{\xi})|)} \\ &\lesssim \int_{|\xi_1|>1}\int_{\mathbb{R}^d} \frac{d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa+1}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}} \\ &\lesssim \int_{|\xi_1|>1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+1+2\alpha-\frac{d}{2}}} \\ &\lesssim \int_1^{+\infty} \frac{dr}{r^{4s-2\kappa+2+4\alpha-2d+1}} \\ &\lesssim 1 \end{aligned} \tag{5.2.16}$$

where we have used the fact that $4s-2\kappa+2+4\alpha-2d+1 > 1$ resulting from $s > \frac{d}{2}-\alpha$ to obtain the last inequality.

• Bound on $\mathbb{I}_2^2$:
Using the law of cosines, we write

$$\kappa(\underline{\xi}) = |\xi_1|^2 - |\xi_1-\xi_2|^2 + |\xi_2|^2 = 2|\xi_1||\xi_2|\cos(\theta),$$

where $\theta = \angle(\xi_1,\xi_2) \in [0,2\pi[$.
We split D_2^2 into two domains defined by

$$D_2^{2,1} = \{(\xi_1,\xi_2) \in D_2^2, |\cos(\theta)| \gtrsim 1\} \quad \text{and} \quad D_2^{2,2} = \{(\xi_1,\xi_2) \in D_2^2, |\cos(\theta)| \ll 1\}$$

and we denote by $\mathbb{I}_2^{2,1}$ and $\mathbb{I}_2^{2,2}$ the associated integrals.

• Bound on $\mathbb{I}_2^{2,1}$:
The aim is to prove that the integral

$$\mathbb{I}_2^{2,1} = \int_{|\xi_1|>1} \int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|\xi_1-\xi_2|\geq|\xi_2|\}} \mathbb{1}_{\{|\cos(\theta)|\gtrsim 1\}} d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}(1+|\kappa(\underline{\xi})|)}$$

is finite. By dyadically decomposing ξ_2 into $|\xi_2| \sim N_2$ for dyadic numbers N_2 (ie $N_2 = 2^j$ for $j \geq 1$ or $N_2 = 2^{-j}$ for $j \geq 1$), it holds that

$$\begin{aligned}
\mathbb{I}_2^{2,1} &\lesssim \int_{|\xi_1|>1} \sum_{\substack{N_2 \\ \text{dyadic}}} \int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|\xi_2|\sim N_2\}} \mathbb{1}_{\{|\xi_1-\xi_2|\geq|\xi_2|\}} d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}(1+|\xi_1||\xi_2|)} \\
&\lesssim \int_{|\xi_1|>1} \sum_{\substack{N_2 \\ \text{dyadic}}} \int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|\xi_2|\sim N_2\}} d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa} \max(|\xi_1|, N_2)^{2\alpha}(1+N_2^2)^{\alpha}(1+|\xi_1|N_2)}, \\
&\lesssim \int_{|\xi_1|>1} d\xi_1 \sum_{\substack{N_2 \\ \text{dyadic}}} \frac{N_2^d}{(1+|\xi_1|^2)^{2s-\kappa} \max(|\xi_1|, N_2)^{2\alpha}(1+N_2^2)^{\alpha}(1+|\xi_1|N_2)},
\end{aligned} \tag{5.2.17}$$

where we haved used the fact that $|\xi_1-\xi_2| \sim \max(|\xi_1|, |\xi_2|)$ to derive the second inequality. Or, for $0 < \varepsilon < 1$,

$$\begin{aligned}
&\sum_{\substack{N_2 \\ \text{dyadic}}} \frac{N_2^d}{(1+|\xi_1|^2)^{2s-\kappa} \max(|\xi_1|, N_2)^{2\alpha}(1+N_2^2)^{\alpha}(1+|\xi_1|N_2)} \\
&\lesssim \sum_{\substack{N_2 \\ \text{dyadic}}} \frac{N_2^d}{(1+|\xi_1|^2)^{2s-\kappa} \max(|\xi_1|, N_2)^{2\alpha}(1+N_2^2)^{\alpha}|\xi_1|^{1-\varepsilon} N_2^{1-\varepsilon}} \\
&\lesssim \sum_{\substack{N_2 \\ \text{dyadic}}} \frac{N_2^{d-1+\varepsilon}}{(1+|\xi_1|^2)^{2s-\kappa} \max(|\xi_1|, N_2)^{2\alpha}(1+N_2^2)^{\alpha}|\xi_1|^{1-\varepsilon}} \\
&\lesssim \sum_{\substack{N_2\geq 1 \\ \text{dyadic}}} \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\frac{1}{2}-\frac{\varepsilon}{2}} \max(|\xi_1|, N_2)^{2\alpha} N_2^{2\alpha-d+1-\varepsilon}} \\
&\qquad + \sum_{\substack{N_2\leq 1 \\ \text{dyadic}}} \frac{N_2^{d-1+\varepsilon}}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}-\frac{\varepsilon}{2}}} \\
&\lesssim \sum_{\substack{1\leq N_2<|\xi_1| \\ \text{dyadic}}} \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}-\frac{\varepsilon}{2}} N_2^{2\alpha-d+1-\varepsilon}} + \sum_{\substack{N_2\geq|\xi_1| \\ \text{dyadic}}} \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\frac{1}{2}-\frac{\varepsilon}{2}} N_2^{4\alpha-d+1-\varepsilon}} \\
&\qquad + \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}-\frac{\varepsilon}{2}}}.
\end{aligned} \tag{5.2.18}$$

Let us focus our attention on the first term in (5.2.18). When $d = 1$ or $d = 2$, as $\alpha > \frac{d-1}{2}$, $2\alpha - d + 1 - \varepsilon \geq 0$ for $\varepsilon > 0$ small enough and this term is bounded by

$$\sum_{\substack{1 \leq N_2 < |\xi_1| \\ \text{dyadic}}} \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}-\frac{\varepsilon}{2}} N_2^{2\alpha-d+1-\varepsilon}} \lesssim \frac{\ln(|\xi_1|)}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}-\frac{\varepsilon}{2}}}.$$

Now, let us remark that, when $d = 3$, ε can be removed from the previous computations. When $\alpha \geq 1$, the following bound holds true

$$\sum_{\substack{1 \leq N_2 < |\xi_1| \\ \text{dyadic}}} \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}} N_2^{2\alpha-d+1}} \lesssim \frac{\ln(|\xi_1|)}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}}},$$

whereas, when $\alpha < 1$, one has

$$\sum_{\substack{1 \leq N_2 < |\xi_1| \\ \text{dyadic}}} \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}} N_2^{2\alpha-d+1}} \lesssim \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+2\alpha+1-\frac{d}{2}}}.$$

Now, for all $1 \leq d \leq 3$, as $\alpha > \frac{d-1}{4}$, $4\alpha - d + 1 - \varepsilon > 0$ for $\varepsilon > 0$ small enough and the second term in (5.2.18) is bounded by

$$\sum_{\substack{N_2 \geq |\xi_1| \\ \text{dyadic}}} \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+\frac{1}{2}-\frac{\varepsilon}{2}} N_2^{4\alpha-d+1-\varepsilon}} \lesssim \frac{1}{(1+|\xi_1|^2)^{2s-\kappa+2\alpha+1-\frac{d}{2}-\varepsilon}}.$$

Finally, coming back to (5.2.17), when $d = 1$ or $d = 2$,

$$\mathbb{I}_2^{2,1} \lesssim \int_{|\xi_1|>1} \frac{\ln(|\xi_1|) d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}-\frac{\varepsilon}{2}}} + \int_{|\xi_1|>1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+2\alpha+1-\frac{d}{2}-\varepsilon}} + \int_{|\xi_1|>1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}-\frac{\varepsilon}{2}}}.$$

When $d = 3$ and $\alpha \geq 1$,

$$\mathbb{I}_2^{2,1} \lesssim \int_{|\xi_1|>1} \frac{\ln(|\xi_1|) d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}}} + \int_{|\xi_1|>1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+2\alpha+1-\frac{d}{2}}} + \int_{|\xi_1|>1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}}},$$

and, when $d = 3$ and $\alpha < 1$,

$$\mathbb{I}_2^{2,1} \lesssim \int_{|\xi_1|>1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+2\alpha+1-\frac{d}{2}}} + \int_{|\xi_1|>1} \frac{d\xi_1}{(1+|\xi_1|^2)^{2s-\kappa+\alpha+\frac{1}{2}}}.$$

In all cases, according to the value of κ and since $s > \frac{d}{2} - \alpha$,

$$\mathbb{I}_2^{2,1} \lesssim 1 \tag{5.2.19}$$

as soon as $\varepsilon > 0$ is small enough.

• Bound on $\mathbb{I}_2^{2,2}$:
First of all, we can remark that, when $d = 1$, $\mathbb{I}_2^{2,2} = 0$ since, for all $(\xi_1, \xi_2) \in \mathbb{R}^2$,

$\cos(\angle(\xi_1,\xi_2)) = 1$ or -1. Consequently, in the following, $d=2$ or 3. As $|\cos(\theta)| \ll 1$, $\left|\frac{\pi}{2}-\theta\right| \ll 1$ or $\left|\frac{3\pi}{2}-\theta\right| \ll 1$. We will only deal with the case where $\left|\frac{\pi}{2}-\theta\right| \ll 1$ since the other case is quite the same (it suffices to replace $\left|\frac{\pi}{2}-\theta\right|$ by $\left|\frac{3\pi}{2}-\theta\right|$ in all the computations). By dyadically decomposing $\left|\frac{\pi}{2}-\theta\right| \sim 2^{-k}$ for $k \geq 0$ and ξ_2 into $|\xi_2| \sim N_2$ for dyadic numbers N_2 (ie $N_2 = 2^j$ for $j \geq 1$ or $N_2 = 2^{-j}$ for $j \geq 1$) again, we get that

$$\begin{aligned}\mathbb{I}_2^{2,2} &\lesssim \int_{|\xi_1|>1} \sum_{\substack{N_2 \\ \text{dyadic}}} \sum_{k=0}^{+\infty} \int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|\xi_2|\sim N_2\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\ll 1\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\sim 2^{-k}\}}\mathbb{1}_{\{|\xi_1-\xi_2|\geq|\xi_2|\}}d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}(1+|\xi_1-\xi_2|^2)^{\alpha}(1+|\xi_2|^2)^{\alpha}(1+|\kappa(\underline{\xi})|)} \\ &\lesssim \int_{|\xi_1|>1} \sum_{\substack{N_2 \\ \text{dyadic}}} \sum_{k=0}^{+\infty} \int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|\xi_2|\sim N_2\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\ll 1\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\sim 2^{-k}\}}d\xi_1 d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}\max(|\xi_1|,N_2)^{2\alpha}(1+N_2^2)^{\alpha}(1+|\xi_1|N_2 2^{-k})}, \end{aligned} \tag{5.2.20}$$

where we haved used the fact that $|\xi_1-\xi_2| \sim \max(|\xi_1|,|\xi_2|)$ to derive the last inequality. Suppose that $d=2$. Then, for every $\xi_1 \in \mathbb{R}^2$, we see that the set of possible ξ_2 with $|\xi_2| \sim N_2$, $\left|\frac{\pi}{2}-\theta\right| \ll 1$ and $\left|\frac{\pi}{2}-\theta\right| \sim 2^{-k}$ is included in an axially symmetric trapezoid $\mathcal{D}$ whose height is $\sim N_2$ and the top and bottom widths are $\sim 2^{-k}N_2$ with an axis of symmetry given by $(\mathbb{R}\xi_1)^{\perp}$. Consequently, $\mathrm{vol}(\mathcal{D}) \sim N_2^2 2^{-k}$ and

$$\int_{\mathbb{R}^2} \mathbb{1}_{\{|\xi_2|\sim N_2\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\ll 1\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\sim 2^{-k}\}}d\xi_2 \lesssim N_2^2 2^{-k}.$$

Suppose now that $d=3$. Then, this time, for every $\xi_1 \in \mathbb{R}^3$, the set of possible ξ_2 with $|\xi_2| \sim N_2$, $\left|\frac{\pi}{2}-\theta\right| \ll 1$ and $\left|\frac{\pi}{2}-\theta\right| \sim 2^{-k}$ is included in a cone $\mathcal{D}$ whose height is $\sim N_2$, whose radius of base disc is $\sim 2^{-k}N_2$ and that presents a symmetry given by the plan $(\mathbb{R}\xi_1)^{\perp}$. Thus, $\mathrm{vol}(\mathcal{D}) \sim N_2^3 2^{-2k}$ and

$$\int_{\mathbb{R}^3} \mathbb{1}_{\{|\xi_2|\sim N_2\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\ll 1\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\sim 2^{-k}\}}d\xi_2 \lesssim N_2^3 2^{-2k}.$$

To sum up,

$$\int_{\mathbb{R}^d} \mathbb{1}_{\{|\xi_2|\sim N_2\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\ll 1\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\sim 2^{-k}\}}d\xi_2 \lesssim N_2^d 2^{-(d-1)k}.$$

Coming back to (5.2.20), we deduce that, for $0<\varepsilon<1$,

$$\begin{aligned}
&\sum_{\substack{N_2\\ \text{dyadic}}}\sum_{k=0}^{+\infty}\int_{\mathbb{R}^d}\frac{\mathbb{1}_{\{|\xi_2|\sim N_2\}}\mathbb{1}_{\{|\frac{\pi}{2}-\theta|\sim 2^{-k}\}}d\xi_2}{(1+|\xi_1|^2)^{2s-\kappa}\max(|\xi_1|,N_2)^{2\alpha}(1+N_2^2)^{\alpha}(1+|\xi_1|N_2 2^{-k})}\\
&\lesssim \sum_{\substack{N_2\\ \text{dyadic}}}\sum_{k=0}^{+\infty}\frac{N_2^d 2^{-(d-1)k}}{(1+|\xi_1|^2)^{2s-\kappa}\max(|\xi_1|,N_2)^{2\alpha}(1+N_2^2)^{\alpha}|\xi_1|^{1-\varepsilon}N_2^{1-\varepsilon}2^{-k+\varepsilon k}}\\
&\lesssim \sum_{\substack{N_2\\ \text{dyadic}}}\frac{N_2^{d-1+\varepsilon}}{(1+|\xi_1|^2)^{2s-\kappa}\max(|\xi_1|,N_2)^{2\alpha}(1+N_2^2)^{\alpha}|\xi_1|^{1-\varepsilon}}.
\end{aligned} \tag{5.2.21}$$

Now, we are dealing with the same sum as in the computations related to $\mathbb{I}_2^{2,1}$. We can mimic the arguments to obtain that

$$\mathbb{I}_2^{2,2}\lesssim 1. \tag{5.2.22}$$

• <u>Come back to $\mathbb{I}$:</u>
Combining the four bounds (5.2.15), (5.2.16), (5.2.19) and (5.2.22), we deduce that the integral $\mathbb{I}$ is finite. That concludes the proof. □

5.3 Deterministic analysis of equation (5.1.1)

Let $1\leq d\leq 3$ be a space dimension and $T>0$ a positive time. Fix α a real number verifying

$$\frac{d}{4}<\alpha<\frac{d}{2}.$$

The aim of this section is to establish the local well-posedness of equation (5.1.1), that is to present the proof of Theorem 5.1.12. Let us remember that our model is understood in the sense of Definition 5.1.11. In other words, we have to prove that there exists v verifying the fixed-point equality

$$\begin{aligned}
v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\rho\mathring{|}_\tau))\,d\tau\\
- i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\rho\mathring{|}_\tau}))\,d\tau+\rho^2\mathring{\mathrm{Y}},\quad t\in[0,T],
\end{aligned}\tag{5.3.1}$$

where the processes $\rho\mathring{|}$ and $\rho^2\mathring{\mathrm{Y}}$ are defined through Proposition 5.1.1 and Proposition 5.1.8. Recall that, for all $p\geq 2$,

$$\rho\mathring{|}\in\mathcal{C}([0,T];\mathcal{W}^{-(\frac{d}{2}-\alpha)-\varepsilon,p}(\mathbb{R}^d)),$$

and that, thanks to multilinear smoothing,

$$\rho^2\mathring{\mathrm{Y}}\in\mathcal{C}([0,T];\mathcal{W}^{-(d-2\alpha)-\varepsilon+\kappa,p}(\mathbb{R}^d)),$$

for every $\varepsilon > 0$ and where

$$\kappa = \begin{cases} 1-\alpha & \text{if } d = 1 \\ \dfrac{3}{2} - \alpha & \text{if } d = 2 \\ 2-\alpha & \text{if } d = 3 \text{ and } \alpha \geq 1 \\ 1 & \text{if } d = 3 \text{ and } \alpha < 1\ . \end{cases}$$

Fix $\varepsilon > 0$ and let $s > 0$ the real number defined by $s = s_{d,\alpha,\varepsilon} = (\frac{d}{2} - \alpha) + \varepsilon$. In the following, we will use a deterministic approach to deal with equation (5.3.1) and we will henceforth consider the pair $(\rho\,\mathsf{I}, \rho^2\,\mathsf{Y})$ as a given element in the space

$$\mathcal{E}_{s,\kappa} := \bigcap_{2 \leq p < \infty} \mathcal{C}([0,T]; \mathcal{W}^{-s,p}(\mathbb{R}^d)) \times \mathcal{C}([0,T]; \mathcal{W}^{-2s+\kappa,p}(\mathbb{R}^d))\,,$$

and then try to solve the more general deterministic equation: for $(\mathbf{\Psi}, \mathbf{\Psi^2}) \in \mathcal{E}_{s,\kappa}$,

$$\begin{aligned} v_t = e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\,d\tau \\ - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\,d\tau + \mathbf{\Psi^2}, \quad t \in [0,T]\,. \end{aligned} \tag{5.3.2}$$

In the next subsection, we present the different tools that will permit us to deal with each term of the above equation.

5.3.1 Technical lemmas

About the Schrödinger operator

Before stating the well-known Strichartz inequalities, let us define the notion of Schrödinger admissible pair.

Definition 5.3.1. *A pair $(p,q) \in [2,+\infty]^2$ is said to be Schrödinger admissible if*

$$(p,q,d) \neq (2,+\infty,2) \ \text{ and } \ \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

Lemma 5.3.2 (Strichartz inequalities, see [1, Paragraph 2.3])**.** *Fix $d \geq 1$ a space dimension and $s \in \mathbb{R}$. Let u stand for the mild solution of equation*

$$\begin{cases} i\partial_t u(t,x) - \Delta u(t,x) = F(t,x)\,, & t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u(0,x) = \phi(x)\,. \end{cases}$$

Then, for all Schrödinger admissible pairs (p,q) and (a,b), it holds that

$$\|u\|_{L^p([0,T];\mathcal{W}^{s,q}(\mathbb{R}^d))} \lesssim \|\phi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^{a'}([0,T];\mathcal{W}^{s,b'}(\mathbb{R}^d))}\,,$$

where the notations a', b' refer to the Hölder conjugates of a, b.

Remark 5.3.3. Strichartz inequality is a fundamental tool to deal with nonlinear Schrödinger equations. It presents the advantage to provide a gain of integrability but the drawback that the solution inherit the (potentially bad) regularity of the initial condition and of the second member F.

The Schrödinger operator can generate a local gain of regularity. This latter is described in [2] where the authors have generalized the so-called Kato smoothing effect that offers locally a gain of a half of a derivative (understood in the Sobolev meaning).

Lemma 5.3.4. *Fix $d \geq 1$ a space dimension. Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be of the form $(\mathbf{F}_\rho)$, $0 \leq s, \eta \leq \frac{1}{2}$ and $0 \leq T \leq 1$. Suppose that $\phi \in H^{-s}(\mathbb{R}^d)$ and $F \in L^1([0,T]; H^{-s}(\mathbb{R}^d))$. Let u stand for the mild solution of equation*

$$\begin{cases} i\partial_t u(t,x) - \Delta u(t,x) = F(t,x)\,, \quad t \in [0,T]\,,\, x \in \mathbb{R}^d\,, \\ u(0,x) = \phi(x)\,. \end{cases}$$

Then, it holds that

$$\|u\|_{L^{\frac{1}{\eta}}([0,T];H_\rho^{-s+\eta}(\mathbb{R}^d))} \lesssim \|\phi\|_{H^{-s}(\mathbb{R}^d)} + \|F\|_{L^1([0,T];H^{-s}(\mathbb{R}^d))}\,,$$

where the proportional constant only depends on ρ, s and η.

Let us go back to equation (5.3.2). We see that, compared with [2], the main difference lies in the fact that the worst term between $-i\int_0^t e^{-i(t-\tau)\Delta}(\mathbf{\Psi}_\tau)\,d\tau$ and $\mathbf{\Psi^2}$ is now $-i\int_0^t e^{-i(t-\tau)\Delta}(\mathbf{\Psi}_\tau)\,d\tau$ with its $-s+\eta$ derivatives (according to the previous local smoothing). Consequently, here, v is expected to inherit the regularity of $-i\int_0^t e^{-i(t-\tau)\Delta}(\mathbf{\Psi}_\tau)\,d\tau$ and should be an element of $\mathcal{C}([0,T]; H^{-s+\eta}(\mathbb{R}^d))$. Precisely, by resorting to the local smoothing of the Schrödinger operator (see Lemma 5.3.4), we will be able to prove that, up to multiplication by ρ, $v \in \mathcal{C}([0,T]; \mathcal{W}^{-s+\eta,p}(\mathbb{R}^d))$ (for some $p \geq 2$) with $\eta > 0$ such that $-s+\eta > 0$. Thus, for all $0 \leq t \leq T$, $v(t)$ will be locally a function, allowing us to give a rigorous meaning to $\rho^2|v|^2$.

About the product in Sobolev spaces

The following lemma states that the product of two functions of regularity $s \geq 0$ stays a function of regularity $s \geq 0$ and precises its integrability under an Hölder type condition. See for instance [17, Proposition 1.1, p. 105].

Lemma 5.3.5 (Fractional Leibniz rule)**.** *Let $s \geq 0$, $1 < r < \infty$ and $1 < p_1, p_2, q_1, q_2 < \infty$ satisfying*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}\,.$$

Then, it holds that

$$\|u \cdot v\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \lesssim \|u\|_{\mathcal{W}^{s,p_1}(\mathbb{R}^d)}\|v\|_{L^{p_2}(\mathbb{R}^d)} + \|u\|_{L^{q_1}(\mathbb{R}^d)}\|v\|_{\mathcal{W}^{s,q_2}(\mathbb{R}^d)}\,.$$

Now, we are interested in defining the product $f \cdot g$ when g is a function (of regularity $\beta > 0$) and f is only a distribution of Sobolev regularity $-\alpha < 0$. This is the subject of the lemma below that guarantees that the product makes sense as soon as $\beta > \alpha$ and inherits the worst regularity, namely $-\alpha$. The proof of this result can be found in [15, Section 4.4.3].

Lemma 5.3.6. *Fix $d \geq 1$ a space dimension. Let $\alpha, \beta > 0$ and $1 \leq p, p_1, p_2 < \infty$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad 0 < \alpha < \beta.$$

If $f \in \mathcal{W}^{-\alpha,p_1}(\mathbb{R}^d)$ and $g \in \mathcal{W}^{\beta,p_2}(\mathbb{R}^d)$, then $f \cdot g \in \mathcal{W}^{-\alpha,p}(\mathbb{R}^d)$ and the following bound holds true

$$\|f \cdot g\|_{\mathcal{W}^{-\alpha,p}(\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{W}^{-\alpha,p_1}(\mathbb{R}^d)} \|g\|_{\mathcal{W}^{\beta,p_2}(\mathbb{R}^d)} \,.$$

An interpolation result and a commutator estimate

To end with, we recall a classical interpolation result followed by a commutator estimate proven in [2] that permits to swap a $\mathcal{C}^\infty$ compactly-supported function with the fractional Laplacian.

Lemma 5.3.7. *Fix $d \geq 1$ a space dimension. Let $s, s_1, s_2 \in \mathbb{R}$ and $1 \leq p, p_1, p_2 < \infty$. Suppose that there exists $\theta \in (0,1)$ such that*

$$s = \theta s_1 + (1-\theta) s_2 \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}\,.$$

If $v \in \mathcal{W}^{s_1,p_1}(\mathbb{R}^d) \cap \mathcal{W}^{s_2,p_2}(\mathbb{R}^d)$, then $v \in \mathcal{W}^{s,p}(\mathbb{R}^d)$ and it holds that

$$\|v\|_{\mathcal{W}^{s,p}(\mathbb{R}^d)} \leq \|v\|^{\theta}_{\mathcal{W}^{s_1,p_1}(\mathbb{R}^d)} \|v\|^{1-\theta}_{\mathcal{W}^{s_2,p_2}(\mathbb{R}^d)}\,.$$

Lemma 5.3.8. *For every $s > 0$ and for all $\mathcal{C}^\infty$ compactly-supported functions $\rho, g : \mathbb{R}^d \to \mathbb{R}$, it holds that*

$$\|(\mathrm{Id} - \Delta)^{\frac{s}{2}}(\rho \cdot g) - \rho \cdot (\mathrm{Id} - \Delta)^{\frac{s}{2}}(g)\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|_{H^{s-1}(\mathbb{R}^d)}\,,$$

where the proportional constant only depends on ρ and s.

5.3.2 Statement and proof of our main result

Let us fix once and for all a $\mathcal{C}^\infty$ compactly-supported function $\rho : \mathbb{R}^d \to \mathbb{R}$ of the form $(\mathbf{F}_\rho)$. For all $T \geq 0$, $\eta > 0$, $p, q \geq 2$, let $Y^{s,\eta,(p,q)}_\rho(T)$ be the space defined by

$$Y^{s,\eta,(p,q)}_\rho(T) := \mathcal{C}([0,T]; H^{-s}(\mathbb{R}^d)) \cap L^p([0,T]; \mathcal{W}^{-s,q}(\mathbb{R}^d)) \cap L^{\frac{1}{\eta}}_T H^{-s+\eta}_\rho\,.$$

Theorem 5.3.9. *Let $1 \leq d \leq 3$ be a space dimension and $\rho : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ compactly-supported function of the form $(\mathbf{F}_\rho)$. Besides, assume that $\alpha < \frac{d}{2}$ verifies $\alpha > \alpha_d$, where*

$$\alpha_d = \begin{cases} 1/4 & \text{if } d = 1 \\ 5/6 & \text{if } d = 2 \\ 17/12 & \text{if } d = 3 \end{cases} . \tag{5.3.3}$$

Recall that, for all fixed $\varepsilon > 0$, the real number $s_{d,\alpha,\varepsilon} > 0$ is defined by $s = s_{d,\alpha,\varepsilon} = (\frac{d}{2} - \alpha) + \varepsilon$. Then, if $\varepsilon > 0$ is small enough, one can find parameters $\eta \in [s, 1/2]$ and $p, q \geq 2$ such that, for all $\phi \in H^{-s}(\mathbb{R}^d)$ and $(\mathbf{\Psi}, \mathbf{\Psi^2}) \in \mathcal{E}_{s,\kappa}$, there exists a time $T > 0$ for which equation (5.3.2) admits a unique solution in the above-defined set $Y_\rho^{s,\eta,(p,q)}(T)$.

Our strategy is to establish a fixed-point principle on $\Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}$ the map defined for all $T \geq 0$ and $(\mathbf{\Psi}, \mathbf{\Psi^2}) \in \mathcal{E}_{s,\kappa}$ by

$$\begin{aligned} \Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}(v) := e^{-it\Delta}\phi - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\bar{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\, d\tau \\ - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\, d\tau + \mathbf{\Psi^2}, \quad t \in [0, T]\,. \end{aligned}$$

This will result from the two bounds described in the proposition below.

Proposition 5.3.10. *Assume that $1 \leq d \leq 3$ and that α_d satisfies condition (5.3.3). Then, if $\varepsilon > 0$ is small enough, one can find parameters $\eta > 0$, $p, q \geq 2$ and $\tilde{\varepsilon} > 0$ such that, setting $Y(T) := Y_\rho^{s,\eta,(p,q)}(T)$, the following bounds hold true: for all $0 \leq T \leq 1$, $\phi \in H^{-s}(\mathbb{R}^d)$, $(\mathbf{\Psi}_1, \mathbf{\Psi}_1^\mathbf{2}) \in \mathcal{E}_{s,\kappa}$, $(\mathbf{\Psi}_2, \mathbf{\Psi}_2^\mathbf{2}) \in \mathcal{E}_{s,\kappa}$ and $v, v_1, v_2 \in Y(T)$,*

$$\|\Gamma_{T,\mathbf{\Psi}_1,\mathbf{\Psi}_1^\mathbf{2}}(v)\|_{Y(T)} \lesssim \|\phi\|_{H^{-s}} + T^{\tilde{\varepsilon}}\Big[\|v\|_{Y(T)}^2 + \|\mathbf{\Psi}_1\|_{L_T^\infty \mathcal{W}^{-s,r}}\|v\|_{Y(T)}\Big] + \|\mathbf{\Psi}_1^\mathbf{2}\|_{Y(T)}, \tag{5.3.4}$$

and

$$\begin{aligned} &\|\Gamma_{T,\mathbf{\Psi}_1,\mathbf{\Psi}_1^\mathbf{2}}(v_1) - \Gamma_{T,\mathbf{\Psi}_2,\mathbf{\Psi}_2^\mathbf{2}}(v_2)\|_{Y(T)} \\ &\lesssim T^{\tilde{\varepsilon}}\Big[\|v_1 - v_2\|_{Y(T)}\{\|v_1\|_{Y(T)} + \|v_2\|_{Y(T)}\} + \|\mathbf{\Psi}_1 - \mathbf{\Psi}_2\|_{L_T^\infty \mathcal{W}^{-s,r}}\|v_1\|_{Y(T)} \\ &\qquad + \|\mathbf{\Psi}_2\|_{L_T^\infty \mathcal{W}^{-s,r}}\|v_1 - v_2\|_{Y(T)}\Big] + \|\mathbf{\Psi}_1^\mathbf{2} - \mathbf{\Psi}_2^\mathbf{2}\|_{Y(T)}\,, \end{aligned} \tag{5.3.5}$$

where r depends on s and η and the proportional constants depend only on ρ and s.

The choice of the three parameters η, p, q in the above proposition depends on the space dimension $d \in \{1, 2, 3\}$. For the sake of clarity, we divide our proof into three subcases. As usual, our objective is to bound each term in the expression of $\Gamma_{T,\mathbf{\Psi},\mathbf{\Psi^2}}$ separately. In the following, we will suppose that $0 \leq T \leq 1$.

Proof of Proposition 5.3.10 when $d = 1$

As soon as $\varepsilon > 0$ is small enough, there exists $\eta > 0$ such that

$$2s < \eta < \inf(\frac{1}{2}, \frac{3}{4} - s).$$

Let $(p,q) = (\infty, 2)$. Then,

$$Y(T) = \mathcal{C}([0,T]; H^{-s}(\mathbb{R})) \cap L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}\,.$$

Also, we define $\theta = \frac{s}{\eta} \in (0, \frac{1}{2})$. Notice that a quick computation shows that the pair (p,q) is Schrödinger admissible.

Bound on $e^{-it\Delta}\phi$: As $e^{-it\Delta}$ is a unitary operator on $H^{-s}(\mathbb{R})$, it holds that

$$\|e^{-it\Delta}\phi\|_{L_T^\infty H^{-s}} = \|\phi\|_{H^{-s}(\mathbb{R})}\,.$$

Besides, since $s \leq \frac{1}{2}$ and $\eta \leq \frac{1}{2}$, by local regularization (see Lemma 5.3.4),

$$\|e^{-it\Delta}\phi\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}} \lesssim \|\phi\|_{H^{-s}(\mathbb{R})}\,,$$

and, combining the two previous inequalities, we have established that

$$\|e^{-it\Delta}\phi\|_{Y(T)} \lesssim \|\phi\|_{H^{-s}(\mathbb{R})}.$$

Bound on $-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau$: Since $\eta > 0$ and since ρ is a $\mathcal{C}^\infty$ compactly-supported function, one has

$$\begin{aligned}
&\left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau \right\|_{Y(T)} = \\
&\left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau \right\|_{L_T^\infty H^{-s}} + \left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau \right\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}} \\
&\lesssim \left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau \right\|_{L_T^\infty H^{-s+\eta}} + \left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau \right\|_{L_T^{\frac{1}{\eta}} H^{-s+\eta}} \\
&\lesssim \left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau \right\|_{L_T^\infty H^{-s+\eta}}.
\end{aligned}$$

Resorting to Strichartz inequalities (Lemma 5.3.2), it holds that

$$\left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau \right\|_{Y(T)} \lesssim \|\rho^2|v|^2\|_{L_T^{\frac{4}{3}}\mathcal{W}^{-s+\eta,1}}\,. \tag{5.3.6}$$

Now, according to Leibniz fractional rule (Lemma 5.3.5), for all $t \geq 0$,

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,1}} \lesssim \|\rho v(t,.)\|_{H^{-s+\eta}}\|\rho v(t,.)\|_{L^2}\,,$$

and, by interpolation (Lemma 5.3.7),

$$\|\rho v(t,.)\|_{L^2} \leq \|\rho v(t,.)\|^{\theta}_{H^{-s+\eta}} \|\rho v(t,.)\|^{1-\theta}_{H^{-s}} ,$$

leading to

$$\begin{aligned}\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,1}} &\lesssim \|\rho v(t,.)\|^{1+\theta}_{H^{-s+\eta}} \|\rho v(t,.)\|^{1-\theta}_{H^{-s}} \\ &\lesssim \|v(t,.)\|^{1+\theta}_{H^{-s+\eta}_\rho} \|v(t,.)\|^{1-\theta}_{H^{-s}} + \|v(t,.)\|^2_{H^{-s}} ,\end{aligned}$$

where we have used a commutator estimate (Lemma 5.3.8) to obtain the second inequality. Consequently,

$$\begin{aligned}&\int_0^T dt\, \|\rho^2|v|^2(t,.)\|^{\frac{4}{3}}_{\mathcal{W}^{-s+\eta,1}} \\ &\lesssim \|v\|^{\frac{4}{3}(1-\theta)}_{Y(T)} \int_0^T dt\, \|v(t,.)\|^{\frac{4}{3}(1+\theta)}_{H^{-s+\eta}_\rho} + T\|v\|^{\frac{8}{3}}_{X(T)} \\ &\lesssim T^{1-\frac{4}{3}(1+\theta)\eta} \|v\|^{\frac{4}{3}(1-\theta)}_{Y(T)} \bigg(\int_0^T dt\, \|v(t,.)\|^{\frac{1}{\eta}}_{H^{-s+\eta}_\rho} \bigg)^{\frac{4}{3}(1+\theta)\eta} + T\|v\|^{\frac{8}{3}}_{Y(T)} \\ &\lesssim T^{1-\frac{4}{3}(1+\theta)\eta} \|v\|^{\frac{8}{3}}_{Y(T)} ,\end{aligned}$$

and thus, coming back to (5.3.6), we can conclude that

$$\bigg\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \bigg\|_{Y(T)} \lesssim T^{\frac{3}{4}-(\eta+s)} \|v\|^2_{Y(T)} .$$

Bound on $-i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\, d\tau$**,** $-i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\, d\tau$**:** Since $s \leq \frac{1}{2}$ and $\eta \leq \frac{1}{2}$, by local regularization (Lemma 5.3.4), the following bound holds true

$$\begin{aligned}&\bigg\| -i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\, d\tau \bigg\|_{L^{\frac{1}{\eta}}_T H^{-s+\eta}_\rho} + \bigg\| -i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\, d\tau \bigg\|_{L^{\frac{1}{\eta}}_T H^{-s+\eta}_\rho} \\ &\lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L^1_T H^{-s}} . \qquad (5.3.7)\end{aligned}$$

Let $l > s$ and $2 \leq r, p < \infty$ be such that $1/2 = 1/r + 1/p$. According to Lemma 5.3.6, for all $t \geq 0$,

$$\|(\rho\overline{v}\cdot\mathbf{\Psi})(t,.)\|_{H^{-s}} \lesssim \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{-s,r}} \|\rho v(t,.)\|_{\mathcal{W}^{l,p}} . \qquad (5.3.8)$$

As soon as

$$-s+\eta-l = d(\frac{1}{2}-\frac{1}{p}) = \frac{d}{r} \quad \Leftrightarrow \quad l = -s+\eta-\frac{d}{r} ,$$

we have the Sobolev embedding $H^{-s+\eta}(\mathbb{R}^d) \hookrightarrow \mathcal{W}^{l,p}(\mathbb{R}^d)$. Now, as $-s+\eta > s$, we can pick $r \geq 2$ large enough such that $l = -s+\eta-\frac{d}{r} > s$, and from (5.3.8), we get that

$$\|(\rho\overline{v}\cdot\mathbf{\Psi})(t,.)\|_{H^{-s}} \lesssim \|\mathbf{\Psi}(t,.)\|_{\mathcal{W}^{-s,r}} \|\rho v(t,.)\|_{H^{-s+\eta}} .$$

With the help of a commutator estimate (Lemma 5.3.8), for all $t \geq 0$,

$$\|\rho v(t,.)\|_{H^{-s+\eta}} \lesssim \|v(t,.)\|_{H^{-s+\eta}_\rho} + \|v(t,.)\|_{H^{-s}} ,$$

entailing that

$$\begin{aligned}\|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L^1_T H^{-s}} &\lesssim \|\mathbf{\Psi}\|_{L^\infty_T \mathcal{W}^{-s,r}}\{T^{1-\eta}\|v\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}} + T\,\|v\|_{L^\infty_T H^{-s}}\} \\ &\lesssim T^{1-\eta}\|\mathbf{\Psi}\|_{L^\infty_T \mathcal{W}^{-s,r}}\|v\|_{Y(T)}\,,\end{aligned}$$

which, going back to (5.3.7), leads us to

$$\Big\| - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\,d\tau\Big\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}} + \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\,d\tau\Big\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}}$$
$$\lesssim T^{1-\eta}\|\mathbf{\Psi}\|_{L^\infty_T \mathcal{W}^{-s,r}}\|v\|_{Y(T)}\,. \tag{5.3.9}$$

On the other hand, resorting to Strichartz inequalities (Lemma 5.3.2), it holds that

$$\Big\|-i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\,d\tau\Big\|_{L^\infty_T H^{-s}} + \Big\|-i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\,d\tau\Big\|_{L^\infty_T H^{-s}} \lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L^1_T H^{-s}}\,.$$

We are thus in the same position as in (5.3.7), and we can repeat our arguments to establish that

$$\Big\| - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\,d\tau\Big\|_{L^\infty_T H^{-s}} + \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\,d\tau\Big\|_{L^\infty_T H^{-s}}$$
$$\lesssim T^{1-\eta}\|\mathbf{\Psi}\|_{L^\infty_T \mathcal{W}^{-s,r}}\|v\|_{Y(T)}\,.$$

Combining this estimate with (5.3.9), we have finally obtained that

$$\Big\| - i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\,d\tau\Big\|_{Y(T)} + \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\,d\tau\Big\|_{Y(T)}$$
$$\lesssim T^{1-\eta}\|\mathbf{\Psi}\|_{L^\infty_T \mathcal{W}^{-s,r}}\|v\|_{Y(T)}\,.$$

Bound on $\mathbf{\Psi_1^2}$: Keeping in mind that $\kappa = 1-\alpha$ and $\eta < \frac{1}{2}$, $-s+\eta \leq -2s+\kappa$ if $\varepsilon > 0$ is small enough and we immediately have that

$$\|\mathbf{\Psi_1^2}\|_{Y(T)} \leq \|\mathbf{\Psi_1^2}\|_{L^\infty_T H^{-2s+\kappa}}.$$

Combining the above bounds provides us with (5.3.4). (5.3.5) can easily be obtained in an analogous manner.

Proof of Proposition 5.3.10 when $d = 2$

As soon as $\varepsilon > 0$ is small enough, there exists $\eta > 0$ such that

$$2s < \eta < \frac{1}{2} - s.$$

Let $(p,q) = (4,4)$. Then,

$$Y(T) = \mathcal{C}([0,T]; H^{-s}(\mathbb{R}^2)) \cap L^4([0,T]; \mathcal{W}^{-s,4}(\mathbb{R}^2)) \cap L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}\,.$$

Also, we define $\theta = \frac{s}{\eta} \in (0, \frac{1}{2})$. Notice that a quick computation shows that the pair (p, q) is Schrödinger admissible.
Bound on $e^{-it\Delta}\phi$: Exactly as for $d = 1$ (see Section 5.3.2), we show that

$$\|e^{-it\Delta}\phi\|_{Y(T)} \lesssim \|\phi\|_{H^{-s}(\mathbb{R}^2)}.$$

Bound on $-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau$: Since $\eta > 0$ and since ρ is a $\mathcal{C}^\infty$ compactly-supported function, one has

$$\begin{aligned}
\Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{Y(T)} &= \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L^\infty_T H^{-s}} \\
&+ \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L^4_T \mathcal{W}^{-s,4}} + \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}} \\
&\lesssim \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L^\infty_T H^{-s+\eta}} + \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L^4_T \mathcal{W}^{-s+\eta,4}} \\
&\quad + \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L_T^{\frac{1}{\eta}} H^{-s+\eta}} \\
&\lesssim \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L^\infty_T H^{-s+\eta}} + \Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{L^4_T \mathcal{W}^{-s+\eta,4}},
\end{aligned}$$

and, resorting to Strichartz inequalities (Lemma 5.3.2), we deduce that

$$\Big\| - i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{Y(T)} \lesssim \|\rho^2|v|^2\|_{L_T^{r'}\mathcal{W}^{-s+\eta,l'}} \tag{5.3.10}$$

where

$$(r, l) = (\frac{4}{1+\theta}, \frac{4}{1-\theta})$$

is a Schrödinger admissible pair. Now, according to Leibniz fractional rule (Lemma 5.3.5), for all $t \geq 0$,

$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,l'}} \lesssim \|\rho v(t,.)\|_{H^{-s+\eta}}\|\rho v(t,.)\|_{L^{\frac{4}{1+\theta}}}.$$

Then, by interpolation (see Lemma 5.3.7), the bound below holds true

$$\|\rho v(t,.)\|_{L^{\frac{4}{1+\theta}}} \leq \|\rho v(t,.)\|^\theta_{H^{-s+\eta}}\|\rho v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,4}}$$

and, consequently,

$$\begin{aligned}
\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,l'}} &\lesssim \|\rho v(t,.)\|^{1+\theta}_{H^{-s+\eta}}\|\rho v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,4}} \\
&\lesssim \|v(t,.)\|^{1+\theta}_{H_\rho^{-s+\eta}}\|v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,4}} + \|v(t,.)\|^{1+\theta}_{H^{-s}}\|v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,4}},
\end{aligned}$$

where we have used a commutator estimate (Lemma 5.3.8) to obtain the second inequality. Hölder's inequality with $\lambda = \frac{3-\theta}{2} > 1$ yields

$$\int_0^T dt\, \|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,l'}}^{r'}$$
$$\lesssim \left(\int_0^T dt\, \|v(t,.)\|_{H_\rho^{-s+\eta}}^{(1+\theta)r'\lambda}\right)^{\frac{1}{\lambda}} \left(\int_0^T dt\, \|v(t,.)\|_{\mathcal{W}^{-s,4}}^{(1-\theta)r'\lambda'}\right)^{\frac{1}{\lambda'}} + \|v\|_{Y(T)}^{r'(1+\theta)} \int_0^T dt\, \|v_t\|_{\mathcal{W}^{-s,4}}^{r'(1-\theta)}.$$

According to the assumptions on our parameters (recall that $\eta < \frac{1}{2} - s$ and $\theta = \frac{s}{\eta}$),

$$(1+\theta)r'\lambda = 2(1+\theta) < \frac{1}{\eta} \quad \text{and} \quad (1-\theta)r'\lambda' = 4\,,$$

yielding

$$\int_0^T dt\, \|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,l'}}^{r'}$$
$$\lesssim T^{\frac{1-2\eta(1+\theta)}{\lambda}} \left(\int_0^T dt\, \|v(t,.)\|_{H_\rho^{-s+\eta}}^{\frac{1}{\eta}}\right)^{(1+\theta)r'\eta} \|v\|_{Y(T)}^{\frac{4}{\lambda'}} + T^{1-\frac{r'(1-\theta)}{4}} \|v\|_{Y(T)}^{r'(1+\theta)} \|v\|_{Y(T)}^{r'(1-\theta)}$$
$$\lesssim T^{\frac{1-2\eta-2s}{\lambda}} \|v\|_{Y(T)}^{(1+\theta)r'+\frac{4}{\lambda'}} + T^{1-\frac{r'(1-\theta)}{4}} \|v\|_{Y(T)}^{2r'}.$$

The previous estimate can be reformulated as

$$\|\rho^2|v|^2\|_{L_T^{r'}\mathcal{W}^{-s+\eta,l'}} \lesssim \{T^{\tilde{\varepsilon}} + T^{\frac{1}{2}}\}\|v\|_{Y(T)}^2\,,$$

with $\tilde{\varepsilon} = \frac{1}{2}(1-2\eta-2s)$. Coming back to (5.3.10), we have finally obtained that

$$\left\| -i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{Y(T)} \lesssim \{T^{\tilde{\varepsilon}} + T^{\frac{1}{2}}\}\|v\|_{Y(T)}^2.$$

Bound on $-i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\, d\tau$**,** $-i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\, d\tau$**:** Repeating the arguments used for $d=1$, we first establish that

$$\left\|-i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\, d\tau\right\|_{L_T^{\frac{1}{\eta}}H_\rho^{-s+\eta}} + \left\|-i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\, d\tau\right\|_{L_T^{\frac{1}{\eta}}H_\rho^{-s+\eta}} \lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L_T^1H^{-s}}\,,$$

and then

$$\|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L_T^1H^{-s}} \lesssim T^{1-\eta}\|\mathbf{\Psi}\|_{L_T^\infty\mathcal{W}^{-s,r}}\|v\|_{Y(T)}.$$

On the other hand, resorting to Strichartz inequalities (Lemma 5.3.2), we can write

$$\left(\left\| -i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\, d\tau\right\|_{L_T^\infty H^{-s}} + \left\| -i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\, d\tau\right\|_{L_T^\infty H^{-s}}\right)$$
$$+\left(\left\| -i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\, d\tau\right\|_{L_T^4\mathcal{W}^{-s,4}} + \left\| -i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\, d\tau\right\|_{L_T^4\mathcal{W}^{-s,4}}\right)$$
$$\lesssim \|\rho\overline{v}\cdot\mathbf{\Psi}\|_{L_T^1H^{-s}}.$$

Combining the two previous bounds, we obtain that

$$\left\| - i \int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau) \cdot (\mathbf{\Psi}_\tau))\, d\tau \right\|_{Y(T)} + \left\| - i \int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau) \cdot (\overline{\mathbf{\Psi}_\tau}))\, d\tau \right\|_{Y(T)}$$
$$\lesssim T^{1-\eta} \|\mathbf{\Psi}\|_{L_T^\infty \mathcal{W}^{-s,r}} \|v\|_{Y(T)} \,.$$

Bound on $\mathbf{\Psi}_1^{\mathbf{2}}$: Keeping in mind that $\kappa = \frac{3}{2} - \alpha$ and $\eta < \frac{1}{2}$, $-s+\eta \leq -2s+\kappa$ if $\varepsilon > 0$ is small enough and we immediately have that

$$\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{Y(T)} \leq \|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L_T^\infty H^{-2s+\kappa}} + \|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L_T^\infty \mathcal{W}^{-2s+\kappa,4}} .$$

Proof of Proposition 5.3.10 when $d = 3$

As soon as $\varepsilon > 0$ is small enough, there exists $\eta > 0$ such that

$$2s < \eta < \frac{1}{4} - s.$$

Let $(p, q) = (2, 6)$. Then,

$$Y(T) = \mathcal{C}([0,T]; H^{-s}(\mathbb{R}^3)) \cap L^2([0,T]; \mathcal{W}^{-s,6}(\mathbb{R}^3)) \cap L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta} \,.$$

Also, we define $\theta = \frac{s}{\eta} \in (0, \frac{1}{2})$. Notice that a quick computation shows that the pair (p, q) is Schrödinger admissible.
Bound on $e^{-it\Delta}\phi$: Again, the arguments are those used for $d = 1$ and $d = 2$ yielding

$$\|e^{-it\Delta}\phi\|_{Y(T)} \lesssim \|\phi\|_{H^{-s}(\mathbb{R}^3)}.$$

Bound on $-i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau$: By definition,

$$\begin{aligned}
\left\| - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{Y(T)} &= \left\| - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{L_T^\infty H^{-s}} \\
&+ \left\| - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{L_T^2 \mathcal{W}^{-s,6}} + \left\| - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{L_T^{\frac{1}{\eta}} H_\rho^{-s+\eta}} \\
&\lesssim \left\| - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{L_T^\infty H^{-s+\eta}} + \left\| - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{L_T^2 \mathcal{W}^{-s+\eta,6}},
\end{aligned}$$

since $\eta > 0$ and since ρ is a $\mathcal{C}^\infty$ compactly-supported function. Now, resorting to Strichartz inequalities (Lemma 5.3.2), we deduce that

$$\left\| - i \int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\, d\tau \right\|_{Y(T)} \lesssim \|\rho^2|v|^2\|_{L_T^{r'} \mathcal{W}^{-s+\eta,l'}} \tag{5.3.11}$$

where
$$(r,l)=\Big(\frac{4}{1+2\theta},\frac{3}{1-\theta}\Big)$$
is a Schrödinger admissible pair. Now, according to Leibniz fractional rule (Lemma 5.3.5), for all $t\geq 0$,
$$\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,l'}}\lesssim \|\rho v(t,.)\|_{H^{-s+\eta}}\|\rho v(t,.)\|_{L^{\frac{6}{1+2\theta}}}\,.$$
Then, by interpolation (see Lemma 5.3.7), the bound below holds true
$$\|\rho v(t,.)\|_{L^{\frac{6}{1+2\theta}}}\leq \|\rho v(t,.)\|^{\theta}_{H^{-s+\eta}}\|\rho v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,6}}$$
and, consequently,
$$\begin{aligned}\|\rho^2|v|^2(t,.)\|_{\mathcal{W}^{-s+\eta,l'}}&\lesssim \|\rho v(t,.)\|^{1+\theta}_{H^{-s+\eta}}\|\rho v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,6}}\\&\lesssim \|v(t,.)\|^{1+\theta}_{H^{-s+\eta}_\rho}\|v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,6}}+\|v(t,.)\|^{1+\theta}_{H^{-s}}\|v(t,.)\|^{1-\theta}_{\mathcal{W}^{-s,6}}\,,\end{aligned}$$
where we have used a commutator estimate (Lemma 5.3.8) to obtain the second inequality. Hölder's inequality with $\lambda=3-2\theta>1$ yields
$$\begin{aligned}&\int_0^T dt\,\|\rho^2|v|^2(t,.)\|^{r'}_{\mathcal{W}^{-s+\eta,l'}}\\&\lesssim\Big(\int_0^T dt\,\|v(t,.)\|^{(1+\theta)r'\lambda}_{H^{-s+\eta}_\rho}\Big)^{\frac{1}{\lambda}}\Big(\int_0^T dt\,\|v(t,.)\|^{(1-\theta)r'\lambda'}_{\mathcal{W}^{-s,6}}\Big)^{\frac{1}{\lambda'}}+\|v\|^{r'(1+\theta)}_{Y(T)}\int_0^T dt\,\|v_t\|^{r'(1-\theta)}_{\mathcal{W}^{-s,6}}\,.\end{aligned}$$
According to the assumptions on our parameters (recall that $\eta<\frac14-s$ and $\theta=\frac{s}{\eta}$),
$$(1+\theta)r'\lambda=4(1+\theta)<\frac{1}{\eta}\quad\text{and}\quad(1-\theta)r'\lambda'=2\,,$$
yielding
$$\begin{aligned}&\int_0^T dt\,\|\rho^2|v|^2(t,.)\|^{r'}_{\mathcal{W}^{-s+\eta,l'}}\\&\lesssim T^{\frac{1-4\eta(1+\theta)}{\lambda}}\Big(\int_0^T dt\,\|v(t,.)\|^{\frac{1}{\eta}}_{H^{-s+\eta}_\rho}\Big)^{(1+\theta)r'\eta}\|v\|^{\frac{2}{\lambda'}}_{Y(T)}+T^{1-\frac{r'(1-\theta)}{2}}\|v\|^{r'(1+\theta)}_{Y(T)}\|v\|^{r'(1-\theta)}_{Y(T)}\\&\lesssim T^{\frac{1-4\eta-4s}{\lambda}}\|v\|^{(1+\theta)r'+\frac{2}{\lambda'}}_{Y(T)}+T^{1-\frac{r'(1-\theta)}{2}}\|v\|^{2r'}_{Y(T)}\,.\end{aligned}$$
The previous estimate can be reformulated as
$$\|\rho^2|v|^2\|_{L^{r'}_T\mathcal{W}^{-s+\eta,l'}}\lesssim\{T^{\tilde\varepsilon}+T^{\frac14}\}\|v\|^2_{Y(T)}\,,$$
with $\tilde\varepsilon=\frac14(1-4\eta-4s)$. Coming back to (5.3.11), we have finally obtained that
$$\Big\|-i\int_0^t e^{-i(t-\tau)\Delta}(\rho^2|v_\tau|^2)\,d\tau\Big\|_{Y(T)}\lesssim\{T^{\tilde\varepsilon}+T^{\frac14}\}\|v\|^2_{Y(T)}\,.$$

Bound on $-i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\,d\tau$, $-i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\,d\tau$: Repeating the arguments used for $d=2$ (remark indeed that $-s+\eta>s$), we establish that

$$\begin{aligned}&\left\|-i\int_0^t e^{-i(t-\tau)\Delta}((\rho\overline{v}_\tau)\cdot(\mathbf{\Psi}_\tau))\,d\tau\right\|_{Y(T)}+\left\|-i\int_0^t e^{-i(t-\tau)\Delta}((\rho v_\tau)\cdot(\overline{\mathbf{\Psi}_\tau}))\,d\tau\right\|_{Y(T)}\\&\lesssim T^{1-\eta}\|\mathbf{\Psi}\|_{L^\infty_T\mathcal{W}^{-s,r}}\|v\|_{Y(T)}\,.\end{aligned}$$

Bound on $\mathbf{\Psi}_1^{\mathbf{2}}$: Keeping in mind that $\kappa=2-\alpha$ and $\eta<\frac{1}{2}$, $-s+\eta\leq-2s+\kappa$ if $\varepsilon>0$ is small enough and we immediately have that

$$\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{Y(T)}\leq\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L^\infty_T H^{-2s+\kappa}}+\|\mathbf{\Psi}_1^{\mathbf{2}}\|_{L^\infty_T\mathcal{W}^{-2s+\kappa,6}}\,.$$

That concludes the proof.

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