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Existence et multiplicité de solutions pour des problèmes elliptiques avec croissance critique dans le gradient

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Résumé

Dans cette thèse, nous donnons des résultats d'existence, de non-existence, d'unicité et de multiplicité de solutions pour des équation aux dérivées partielles avec croissance critique dans le gradient. Les principales méthodes utilisées dans nos preuves sont des arguments variationnels, la théorie des sous et sur-solutions, des estimations à priori et la théorie de la bifurcation.

La thèse se compose de six chapitres. Dans le chapitre 0 nous introduisons le sujet de thèse et nous présentons les résultats principaux. Le chapitre 1 porte sur l'étude d'une équation du type p-Laplacien avec croissance critique dans le gradient et dépendant d'un paramètre. En fonction de l'intervalle où se trouve le paramètre, nous obtenons l'existence et l'unicité d'une solution ou nous montrons l'existence et la multiplicité de solutions. Dans les chapitres 2 et 3, nous poursuivons notre étude dans le cas où l'opérateur utilisé est le Laplacien mais, contrairement au chapitre 1, nous étudions le cas où les coefficients changent de signe. Nous obtenons à nouveau des résultats d'existence et de multiplicité de solutions. Dans le chapitre 4, nous étudions des problèmes non-locaux du type Laplacien fractionnaire avec différents termes de gradient non-local. Nous montrons des résultats d'existence et de non-existence de solutions pour différentes équations de ce type. Finalement, dans le chapitre 5 nous présentons quelques problèmes ouverts liés au contenu de la thèse et des perspectives de recherche.

Mots clés: *Equations elliptiques non-linéaires, gradient carré, croissance critique, p-Laplacien, Laplacien, Laplacien fractionnaire, gradient non-local.*

Abstract

In this thesis, we provide existence, non-existence, uniqueness and multiplicity results for partial differential equations with critical growth in the gradient. The principal techniques employed in our proofs are variational techniques, lower and upper solution theory, a priori estimates and bifurcation theory.

The thesis consists of six chapters. In chapter 0, we introduce the topic of the thesis and we present the main results. Chapter 1 deals with a p-Laplacian type equation with critical growth in the gradient. This equation will depend on a real parameter. Depending on the interval where this parameter lives, we obtain the existence and uniqueness of one solution or we prove the existence and multiplicity of solutions. In chapters 2 and 3, we continue our study in the case where the operator is the Laplacian. However, unlike chapter 1, we study the case where the coefficient functions may change sign. We obtain again existence and multiplicity results. In chapter 4, we study non-local problems of fractional Laplacian type with different non-local gradient terms. We prove existence and non-existence results for different equations of this type. Finally, in chapter 5, we present some open problems related to the content of the thesis and some research perspectives.

Key words: *Non-linear elliptic equations, gradient square, critical growth, p-Laplacian, Laplacian, franctional Laplacian, non-local gradient.*

Contents

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Introduction

This thesis is devoted to the study of elliptic partial differential equations involving nonlinear gradient terms. More precisely, we investigate the existence, non-existence and multiplicity of solutions to several problems with critical growth in the gradient. As a model case, we can think of the boundary value problem

$$\begin{cases} -\Delta u = c(x)u + \mu(x)|\nabla u|^2 + h(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, c and h belong to suitable L^p-spaces, μ belongs to $L^\infty(\Omega)$ and the solutions are searched in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

There exist several mathematical reasons that make the study of nonlinear elliptic PDE with quadratic growth in the gradient interesting. For instance, J. L. Kazdan and R. J. Kramer observed in 1978 that second order PDE with quadratic growth in the gradient are invariant under changes of variable of type $v = F(u)$. This took them to claim on [78, page 619] that *"In the long run, the class of semilinear equations should be less important than some more general class of equations that is invariant under changes of variables"*. From a pure mathematical point of view, it is also worth noting that, in Riemannian geometry, this type of equations naturally appears in the study of gradient Ricci solitons, see for instance [89, Section 1]. On the other hand, concerning more practical reasons, we would like to mention that problem (1) with $c \equiv 0$ corresponds to the stationary case of the Kardar-Parisi-Zhang model of growing interfaces introduced in [77].

Now, before going further in our introduction, we would like to clarify two points. If one thinks about the classical theory of elliptic PDEs, it seems natural to look for solutions to (1) in $H_0^1(\Omega)$. Hence, a normal question is:

Why are we looking for solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$?

For $N \geq 3$, let us consider the boundary boundary problem

$$-\Delta u = |\nabla u|^2, \quad u \in H_0^1(B_1(0)), \tag{2}$$

where $B_1(0) := \{x \in \mathbb{R}^N : |x| < 1\}$. On the one hand, on [4, page 21], it was observed that, for any $0 \le m < 1$,

$$u_m(x) = \log\left(\frac{|x|^{2-N} - m}{1-m}\right),$$

is a solution to (2). On the other hand, by [17, Theorem 1.1] we know that the unique solution to (2) belonging to $L^\infty(\Omega)$ is the trivial one, namely $u \equiv 0$. Hence, we easily observe that the set of solutions to (2) is completely different when imposing or not that the solutions belong to $L^\infty(\Omega)$.

Another question we should address is:

Why do we say that (1) has critical growth in the gradient?

For $N \ge 3$ and $q > 1$, let us consider the boundary value problem

$$-\Delta u = |\nabla u|^q, \quad u \in H_0^1(B_1(0)) \cap L^\infty(B_1(0)), \tag{3}$$

and distinguish two cases:

a) $1 < q \le 2$: It follows from [17, Theorem 1.1] that the unique solution to (3) is $u \equiv 0$.

b) $q > 2$: Let us introduce

$$\beta = \frac{q-2}{q-1} \quad \text{and} \quad \alpha = \frac{(N-2+\beta)^{\frac{1}{q-1}}}{\beta},$$

and define $v(x) := \alpha(1-|x|^\beta)$. By direct computations we observe that $u \equiv 0$ and v are both solutions to (3). Hence, for $q > 2$, problem (3) has at least two different solutions.

We observe that the set of solutions to (3) is different for $1 < q \le 2$ and $q > 2$ and so, that the problems behave very differently. We conclude then that $q = 2$ is a turning point in the behaviour of the problem. This justifies somehow the terminology of *critical growth in the gradient* when we refer to the power $q = 2$.

For $1 < p < \infty$, let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denote the p-Laplacian operator (observe that for $p = 2$ it is precisely the Laplacian). In the first part of the thesis, Chapters 1, 2 and 3, we deal with the existence and multiplicity of solutions to boundary value problems of the form

$$-\Delta_p u = c_\lambda(x)|u|^{p-2}u + \mu(x)|\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{4}$$

with c_λ depending on a real parameter λ. Here, $\Omega \subset \mathbb{R}^N$, $N \ge 2$, is a bounded domain with smooth boundary, c_λ and h belong to $L^q(\Omega)$ for some $q > \max\{N/p, 1\}$ and μ belongs to $L^\infty(\Omega)$. Let us specify that in Chapters 2 and 3 we consider $p = 2$. In that case, problem (4) reduces to

$$-\Delta u = c_\lambda(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{5}$$

which is precisely the model case (1) with $c \equiv c_\lambda$.

First of all, we are going to present the state of the art concerning problems (4) and (5) when we began the thesis in October 2016.

The study of nonlinear elliptic PDEs with a gradient dependence up to the critical growth was essentially initiated by L. Boccardo, F. Murat and J.-P. Puel in the 80's [23–25]. This type of problems have generated since then a large literature. In addition to several works strictly related to the content of this thesis (that we will detail next), several directions have been investigated. For instance, let us mention here some recent works concerning subcritical growth in the gradient [68, 69, 97], supercritical growth in the gradient [94], low regularity coefficients and regularizing effects [15, 16] and pointwise estimates via symmetrization methods [70].

Now, we focus precisely on the literature concerning existence and multiplicity of solutions to problems (4) and (5).

In the case where $c_\lambda(x) \leq -\alpha_0 < 0$ a.e. in Ω for some $\alpha_0 > 0$, now referred to as the *coercive case*, the existence of a solution to (4) is a particular case of the results of [23–25]. Nevertheless, it is still an open question if the uniqueness of such solution holds. Some partial results are in order: for $p = 2$, the uniqueness follows from the results of [19, 20]; for $1 < p < 2$, it seems possible to prove the uniqueness of solutions modifying the proof of [17, Theorem 1.1] and, for $2 < p < \infty$ and more regular coefficients ($c_\lambda, \mu, h \in L^\infty(\Omega)$), it seems reachable adapting the proof of [85, Theorem 1.2]. The *weakly coercive case* $c_\lambda \equiv 0$ was studied in [62] where, for $\|\mu h\|_{N/p}$ small enough, the existence of a solution to (4) is obtained. For $p = 2$, the uniqueness of solution was also obtained in [19] requiring h small enough in an appropriate sense. The *limit coercive case* where one only requires $c_\lambda \leq 0$ a.e. in Ω (i.e. allowing parts of the domain where $c_\lambda \equiv 0$ and parts of it where $c_\lambda < 0$) proved to be more complex to treat. Until this thesis, just the situation where $p = 2$ had been treated. In the recent paper [18], the authors observed that the existence of a solution to (5) is not guaranteed and gave sufficient conditions to ensure such existence. In that paper, the uniqueness of solutions is also proved in this framework. We refer likewise to [17] where more general uniqueness results were obtained. Nevertheless, as already mentioned, the papers concerning the *limit coercive case* $c_\lambda \leq 0$ a.e. in Ω dealt with (5), i.e. $p = 2$. Finally, let us stress that all these results were obtained without requiring any sign conditions on μ and h.

As we will detail later (see Section 0.1), if μ is a constant, the change of variable $v = \frac{p-1}{\mu}(e^{\frac{\mu}{p-1}u} - 1)$ allows to transform problem (4) into a new one which admits a variational formulation (i.e. a problem whose solutions can be obtained as critical points of an associated functional). When $c_\lambda \leq -\alpha_0 < 0$ a.e. in Ω, the associated functional, defined in $W_0^{1,p}(\Omega)$, is coercive. If $c_\lambda \leq 0$ a.e. in Ω, the coerciveness may be lost but it is possible to find sufficient conditions to ensure it. Nevertheless, as soon as $c_\lambda^+ \not\equiv 0$, the functional becomes unbounded from below. Hence, if $c_\lambda \not\leq 0$ a.e. in Ω, i.e. $c_\lambda \gneqq 0$ or c_λ changes sign, problem (4) behaves very differently and becomes much richer than for $c_\lambda \leq 0$ a.e. in Ω. For $c_\lambda \not\leq 0$ a.e. in Ω, up to Chapter 1 of the present thesis, just the particular choice $p = 2$ had been addressed. Furthermore, even for $p = 2$, this case remained unexplored until very recently.

Let us fix for the moment $p = 2$. The first paper dealing with $c_\lambda \not\leq 0$ a.e. in Ω was [76] where the authors considered $c_\lambda = \lambda c \gneqq 0$. Following [104], which addressed a particular case, the authors studied (5) with $\mu(x) \equiv \mu > 0$ but without sign conditions on h. For $\lambda > 0$ and $\|\mu h\|_{N/2}$ small enough it was proved the existence of at least two solutions to (5) using the previously mentioned variational formulation. The main issue to derive the existence of solutions to (5) was then to prove the boundedness of the Palais-Smale sequences to the variational problem and to ensure that the change of variable could be undone (i.e. to verify that the solutions v to the variational problem satisfied $v > -1/\mu$). This result has now been complemented in several ways. The restriction μ constant was first removed in [18] and extended to $\mu(x) \geq \mu_1 > 0$ a.e. in Ω, at the expense of adding the hypothesis $h \gneqq 0$. In this direction we refer also to [50], where, assuming stronger regularity on c and h, the authors removed the condition $h \gneqq 0$. In [50], under different sets of assumptions, the authors clarified the structure of the set of solutions to (5) in the case $\lambda c \gneqq 0$. Note that, in the frame of viscosity solutions and fully nonlinear equations and under corresponding assumptions, similar conclusions were obtained very recently in [92]. In [18, 50, 92] where μ is non constant, the variational approach used in [76] was no longer available. Topological arguments, relying on the derivation of a priori bounds for certain classes of solutions, were used.

Before we developed Chapters 2 and 3 of the present thesis, the only known work where c_λ may change sign was [75] (see also [64] for related problems). Assuming $\mu > 0$ constant, $h \gneqq 0$ and μh and c_λ^+ small in an appropriate sense, the existence of at least two non-negative solutions was proved. Since μ is constant in [75], the problem fitted in the variational framework set up in [76]. The main issue was then to prove that the Palais-Smale sequences were bounded. Due to the indefinite sign of c_λ, several difficulties had to be faced.

When $c_\lambda \not\leq 0$ a.e. in Ω (i.e. the *non-coercive case*), all the above mentioned results require either μ to be constant or to be uniformly bounded from below by a positive constant. In [108], assuming that c_λ, μ and h were non-negative, a first attempt to remove these restrictions on μ was presented. Following the approach of [18], the proofs of the existence results reduce to obtaining a priori bounds on the non negative solutions to (5). The approach developed in [108] is based on interpolation and elliptic estimates in weighted Lebesgue spaces. It works well in low dimension but the possibility to extend it to an arbitrary dimension is not apparent.

In the rest of the introduction we are going to present the main contributions of this thesis. We split it into four sections. Each section corresponds to one chapter of the thesis. In Section 0.1, which corresponds to Chapter 1, we deal with (4) assuming that c_λ has a sign. Sections 0.2 and 0.3, which correspond to Chapters 2 and 3 respectively, are devoted to the study of (5) in the case where c_λ may change sign. Finally, in Section 0.4, which corresponds to Chapter 4, we address a different topic. We present there our main contributions to the study of nonlocal problems involving nonlocal "gradient terms". Section 0.4 is self-contained and independent of the others three sections.

Each chapter of the present thesis (except for Chapter 5) corresponds to a research article (already published or preprint) and can be read independently. Chapter 1 corresponds to the paper [46] in collaboration with C. De Coster. Chapter 2 corresponds to [48] and it is done in collaboration with C. De Coster and L. Jeanjean. Chapter 3 is based on [47] which is a joint work with C. De Coster. Finally, Chapter 4 corresponds to [5] and it is a joint work with B. Abdellaoui. Finally, in Chapter 5, we present some open problems and describe some future projects.

Notation.

1) We denote $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

2) In $\mathbb{R}^N$, we use the notations $|x| = (x_1^2 + \ldots + x_N^2)^{1/2}$ and $B_R(y) = \{x \in \mathbb{R}^N : |x - y| < R\}$.

3) For $v \in L^1(\Omega)$ we define $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$.

4) For $h_1, h_2 \in L^1(\Omega)$ we write:
- $h_1 \leq h_2$ if $h_1(x) \leq h_2(x)$ for a.e. $x \in \Omega$,
- $h_1 \lneqq h_2$ if $h_1 \leq h_2$ and $\text{meas}(\{x \in \Omega : h_1(x) < h_2(x)\}) > 0$.

5) For $p \in [1, +\infty[$, the norm $(\int_\Omega |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. We denote by p' the conjugate exponent of p, namely $p' = p/(p-1)$ and by p^* the Sobolev critical exponent i.e. $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ in case $p \geq N$. The norm in $L^\infty(\Omega)$ is $\|u\|_\infty = \text{esssup}_{x\in\Omega} |u(x)|$.

6) The space $W_0^{1,p}(\Omega)$ is equipped with the norm $\|u\|_{W_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p}$.

0.1 c_λ has a sign

In the first chapter of the thesis, Chapter 1, we assume that c_λ has a sign and we address the more general case problem (4). For any $1 < p < \infty$, we consider the boundary value problem

$$-\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu(x)|\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{6}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2,\ \text{is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ c \gneqq 0 \text{ and } \mu \in L^\infty(\Omega). \end{cases} \tag{7}$$

We prove existence and uniqueness results in the *limit coercive case* $\lambda \leq 0$ and existence and multiplicity results in the *non-coercive case* $\lambda > 0$. Moreover, considering stronger regularity assumptions on the coefficient functions, we clarify the structure of the set of solutions in the *non-coercive case* $\lambda > 0$. We provide now some details on the main results of Chapter 1.

To state the first main result of this thesis let us define

$$m_{p,\lambda}^+ := \begin{cases} \inf_{u \in W_\lambda} \displaystyle\int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^+\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx, & \text{if } W_\lambda \neq \emptyset, \\ +\infty, & \text{if } W_\lambda = \emptyset, \end{cases}$$

and

$$m_{p,\lambda}^- := \begin{cases} \inf_{u \in W_\lambda} \displaystyle\int_\Omega \Big(|\nabla u|^p + \Big(\frac{\|\mu^-\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx, & \text{if } W_\lambda \neq \emptyset, \\ +\infty, & \text{if } W_\lambda = \emptyset, \end{cases}$$

where

$$W_\lambda := \Big\{ w \in W_0^{1,p}(\Omega) : \lambda c(x) w(x) = 0 \text{ a.e. } x \in \Omega,\ \|w\|_{W_0^{1,p}(\Omega)} = 1 \Big\}.$$

The following result, whose proof combines lower and upper solution and variational techniques, generalizes those obtained in [18, Section 3].

Theorem 0.1.1. [Theorem 1.1.1, Chapter 1] *Assume that* (7) *holds and that* $\lambda \leq 0$. *Then if* $m_{p,\lambda}^+ > 0$ *and* $m_{p,\lambda}^- > 0$, *the problem* (6) *has at least one solution.*

Remark 0.1.1.

a) The space W_λ is independent of the size of λ. It depends only on the fact that $\lambda = 0$ or $\lambda \neq 0$.

b) The assumption $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$ connects the cases $\lambda c(x) \leq -\alpha_0 < 0$ and $\lambda c(x) \equiv 0$. On one hand, if $\lambda c(x) < 0$ a.e. in Ω, we have $W_\lambda = \emptyset$ and hence $m_{p,\lambda}^+ = m_{p,\lambda}^- = +\infty$. On the other hand, if $\lambda c(x) \equiv 0$, then $W_\lambda = W_0^{1,p}(\Omega)$ and $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$ holds for example under a suitable smallness condition on $\mu^+ h^+$ and $\mu^- h^-$. In particular observe that, if μ and h have opposite signs, then $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$.

c) If h is either non-negative or non-positive, our hypotheses correspond to the ones introduced in [18, Section 3] for $p=2$. However, observe that for all $u \in W_0^{1,p}(\Omega)$, it follows that

$$\int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^+\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx \geq \int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^+\|_\infty}{p-1}\Big)^{p-1} h^+(x)|u|^p \Big) dx$$

and

$$\int_\Omega \Big(|\nabla u|^p + \Big(\frac{\|\mu^-\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx \geq \int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^-\|_\infty}{p-1}\Big)^{p-1} h^-(x)|u|^p \Big) dx.$$

Hence, if h does not have a sign, our hypotheses are weaker than the ones introduced in [18] even for $p=2$.

In the rest of Chapter 1 we assume that μ is constant, namely we replace (7) by

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ c \gneqq 0 \text{ and } \mu > 0. \end{cases} \tag{8}$$

Observe that there is no loss of generality in assuming $\mu > 0$ since, if u is a solution to (6) with $\mu < 0$, then $w = -u$ is a solution to

$$-\Delta_p w = \lambda c(x)|w|^{p-2}w - \mu|\nabla w|^p - h(x), \quad w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

In [17], for $p=2$ but assuming only (7), the uniqueness of solutions when $\lambda \leq 0$ was obtained as a direct consequence of a suitable comparison principle, see [17, Corollary 3.1]. Nevertheless, as we show in Remark 1.3.3, such kind of principle does not hold in general when $p \neq 2$. Actually, the issue of uniqueness for equations of the form of (6) appears widely open. If partial results seem reachable adapting existing techniques (see for instance [44, 85, 96, 98]), a result covering the full generality of (6) seems, so far, out of reach. Theorem 0.1.2 below, whose proof makes use of some ideas from [6], crucially relies on the assumption that μ is constant. It permits however to treat the *limit coercive case* where c may vanish in some parts of Ω.

Theorem 0.1.2. [Theorem 1.1.2, Chapter 1] *Assume that* (8) *holds and suppose* $\lambda \leq 0$. *Then* (6) *has at most one solution.*

Actually, Theorem 0.1.2 above is a particular case of the more general comparison principle that we state next. Let us consider the boundary value problem

$$-\Delta_p u = \mu|\nabla u|^p + f(x,u), \quad u \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega}), \tag{9}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is } L^1\text{- Carathéodory and } f(x,s) \leq f(x,t) \text{ for a.e. } x \in \Omega \text{ and all } t \leq s, \\ \mu > 0. \end{cases} \tag{10}$$

Remark 0.1.2.

a) We refer to [49, Definition I-3.1] for the definition of L^p-Carathéodory we use here.

b) The assumption $\mu > 0$ is not a restriction. If $u \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a solution to (9) with $\mu < 0$ then $w = -u \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a solution to

$$-\Delta_p w = -\mu|\nabla w|^p - f(x,-w), \quad w \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega}),$$

with $-f(x,-s)$ satisfying the assumption (10).

Theorem 0.1.3. [Theorem 1.3.1, Chapter 1] *Assume that* (10) *holds. If* u_1, $u_2 \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ *are respectively a lower and an upper solution to* (9)*, then* $u_1 \leq u_2$.

Remark 0.1.3. The definition of lower and upper solution to (9) that we use is stated in Definition 1.2.1.

Let us highlight a particular case of (6) that will play an important role in the subsequent results. Namely, we introduce the problem

$$-\Delta_p u = \mu|\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{11}$$

First of all, using the notation

$$m_p := \inf\Big\{ \int_\Omega \Big(|\nabla w|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1} h(x)|w|^p\Big) dx : w \in W_0^{1,p}(\Omega), \|w\|_{W_0^{1,p}(\Omega)} = 1 \Big\}, \tag{12}$$

we completely characterise the existence of solution to (11).

Theorem 0.1.4. [Theorem 1.1.3, Chapter 1] *Assume that* (8) *holds. Then* (11) *has a solution if, and only if,* $m_p > 0$.

Remark 0.1.4. This result again improves, for μ constant, [18] and it allows to observe that, in case $h \leq 0$, (11) has always a solution while the case $h \gneqq 0$ is the "worse" case for the existence of a solution. In case h changes sign, the negative part of h "helps" in order to have a solution to (11).

Gluing together Theorems 0.1.1, 0.1.2 and 0.1.4 we obtain the following existence and uniqueness result for $\lambda \leq 0$.

Corollary 0.1.5. [Corollary 1.1.4, Chapter 1] *Assume that* (8) *holds and suppose that* (11) *has a solution. Then, for all* $\lambda \leq 0$, (6) *has an unique solution.*

Having at hand this information for the *limit coercive case* $\lambda \leq 0$, we turn to the study of the *non-coercive case*, namely when $\lambda > 0$. First, using mainly variational techniques as in [76], we prove the following result.

Theorem 0.1.6. [Theorem 1.1.5, Chapter 1] *Assume that* (8) *holds and suppose that* (11) *has a solution. Then there exists* $\Lambda > 0$ *such that, for any* $0 < \lambda < \Lambda$, (6) *has at least two solutions.*

Next, considering stronger regularity assumptions and combining the theory of lower and upper solution with variational techniques, we derive a more precise information on the structure of the set of solutions in the *non-coercive case*. These informations complement Theorem 0.1.6. Let us denote by $\gamma_1 > 0$ the first eigenvalue of the problem

$$-\Delta_p u = \gamma c(x)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega). \tag{13}$$

Under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ c \text{ and } h \text{ belong to } L^\infty(\Omega), \\ c \gneqq 0 \text{ and } \mu > 0, \end{cases} \tag{14}$$

we prove the following result.

Theorem 0.1.7. [Theorem 1.1.6, Chapter 1] *Assume that (14) holds and suppose that (11) has a solution u_0. Then:*

- *If $h \lneqq 0$, for every $\lambda > 0$, (6) has at least two solutions $u_{\lambda,1}$, $u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ with $u_{\lambda,1} \ll 0$.*
- *If $h \gneqq 0$, then $u_0 \gg 0$ and there exists $\overline{\lambda} \in (0, \gamma_1)$ such that:*
  - *for every $0 < \lambda < \overline{\lambda}$, (6) has at least two solutions $u_{\lambda,1}$, $u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ satisfying $u_{\lambda,i} \geq u_0$ for $i = 1, 2$;*
  - *for $\lambda = \overline{\lambda}$, (6) has at least one solution $u_{\overline{\lambda}} \in \mathcal{C}_0^1(\overline{\Omega})$ satisfying $u_{\overline{\lambda}} \geq u_0$;*
  - *for any $\lambda > \overline{\lambda}$, (6) has no non-negative solution.*

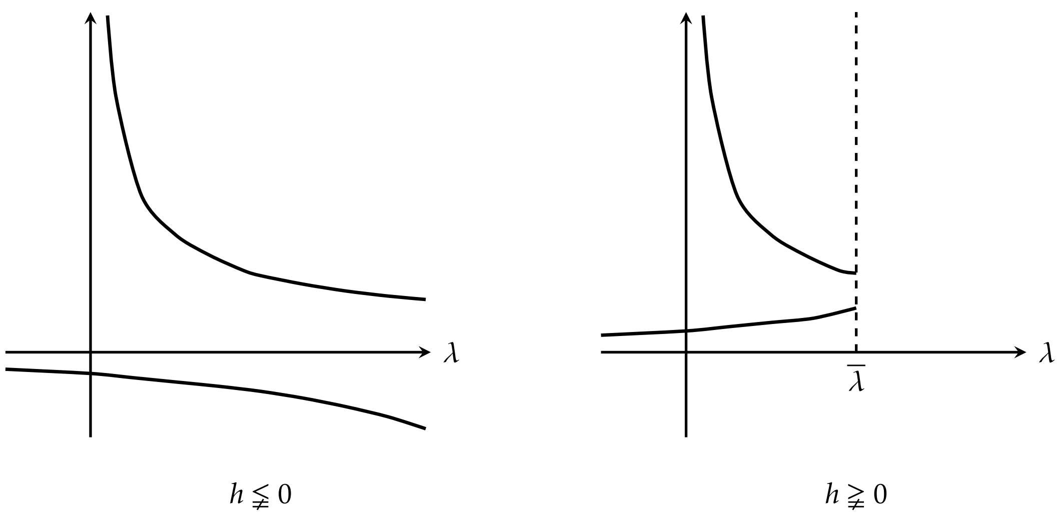

Figure 1: Illustration of Theorem 0.1.7

Remark 0.1.5.

a) The order notation $u \ll v$ for arbitrary functions $u, v \in \mathcal{C}^1(\overline{\Omega})$ is given in Definition 1.1.1.

b) As observed above (see for instance Theorem 0.1.4), in the case $h \lneqq 0$, the assumption that (11) has a solution is automatically satisfied.

c) In the case $\mu < 0$, we have the opposite result i.e., two solutions for every $\lambda > 0$ in case $h \gneqq 0$ and, in case $h \lneqq 0$, the existence of $\overline{\lambda} > 0$ such that (6) has at least two negative solutions, at least one negative solution or no non-positive solution according to $0 < \lambda < \overline{\lambda}$, $\lambda = \overline{\lambda}$ or $\lambda > \overline{\lambda}$.

In case $h \gneqq 0$, we know that for $\lambda > \overline{\lambda}$ (given in the previous theorem), (6) has no non-negative solutions but this does not exclude the possibility of having negative or sign changing solutions. Changing the problem a little we are able to prove again the existence of at least two solutions. We consider the boundary value problem

$$-\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu|\nabla u|^p + kh(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{15}$$

with a dependence in the size of h and we obtain the following result.

Theorem 0.1.8. [Theorem 1.1.7, Chapter 1] *Assume that* (14) *holds and that* $h \gneqq 0$. *Let*

$$k_0 = \sup\left\{k \in [0,+\infty) : \forall\, w \in W_0^{1,p}(\Omega),\ \int_\Omega \left(|\nabla w|^p - \left(\frac{\mu}{p-1}\right)^{p-1} k\, h(x)|w|^p\right) dx > 0\right\}.$$

Then:

- *For all* $\lambda \in (0,\gamma_1)$, *there exists* $\overline{k} = \overline{k}(\lambda) \in (0,k_0)$ *such that, for all* $k \in (0,\overline{k})$, *the problem* (15) *has at least two solutions* $u_{\lambda,1}$, $u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ *with* $u_{\lambda,i} \gg 0$ *and for all* $k > \overline{k}$, *the problem* (15) *has no solution. Moreover, the function* $\overline{k}(\lambda)$ *is non-increasing.*
- *For* $\lambda = \gamma_1$, *the problem* (15) *has a solution if and only if* $k = 0$. *In that case, the solution is unique and it is equal to* 0.
- *For all* $\lambda > \gamma_1$, *there exist* $0 < \tilde{k}_1 \leq \tilde{k}_2 < +\infty$ *such that, for all* $k \in (0,\tilde{k}_1)$, *the problem* (15) *has at least two solutions* $u_{\lambda,1}$, $u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ *with* $u_{\lambda,1} \ll 0$ *and* $\min u_{\lambda,2} < 0$, *for all* $k > \tilde{k}_2$, *the problem* (15) *has no solution and, in case* $\tilde{k}_1 < \tilde{k}_2$, *for all* $k \in (\tilde{k}_1,\tilde{k}_2)$, *the problem* (15) *has at least one solution* u_λ *with* $u_\lambda \not\ll 0$ *and* $\min u_\lambda < 0$. *Moreover, the function* $\widetilde{k_1}(\lambda)$ *is non-decreasing.*

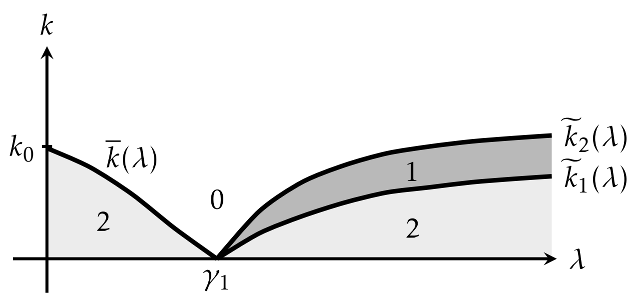

Figure 2: Existence regions of Theorem 0.1.8

Some comments about the proofs of Theorems 0.1.6, 0.1.7 and 0.1.8 are in order. As already said, when $\mu > 0$ is a constant, it is possible to perform a change of variable and to reduce (6) to a semilinear problem. More precisely, introducing

$$v = \frac{p-1}{\mu}\left(e^{\frac{\mu}{p-1}u} - 1\right),$$

one can easily check that u is a solution to (6) if, and only if, $v > -\frac{p-1}{\mu}$ is a solution to

$$-\Delta_p v = \lambda c(x) g(v) + \left(1 + \frac{\mu}{p-1}v\right)^{p-1} h(x), \quad v \in W_0^{1,p}(\Omega), \tag{16}$$

where g satisfies

$$g(s) = \left|\frac{p-1}{\mu}\Big(1+\frac{\mu}{p-1}s\Big)\ln\Big(1+\frac{\mu}{p-1}s\Big)\right|^{p-2}\frac{p-1}{\mu}\Big(1+\frac{\mu}{p-1}s\Big)\ln\Big(1+\frac{\mu}{p-1}s\Big), \quad \text{if } s > -\frac{p-1}{\mu}. \tag{17}$$

Working with problem (16) has the advantage that, with a suitable choice of g when $s \le -\frac{p-1}{\mu}$, the problem has a variational structure. Nevertheless, we have to overcome several difficulties.

First we have to verify that the solutions to (16) satisfy $v > -\frac{p-1}{\mu}$. In order to do that, we prove the existence of a non-positive lower solution $\underline{u}_\lambda$ to (6) such that every upper solution β to (6) satisfies $\beta \ge \underline{u}_\lambda$. This allows us to transform problem (16) into a new one, which, unlike the variational problems considered in [75,76], is completely equivalent to (6). More precisely, let us define

$$\alpha_\lambda = \frac{p-1}{\mu}\Big(e^{\frac{\mu}{p-1}\underline{u}_\lambda} - 1\Big) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{18}$$

and observe that, since $\underline{u}_\lambda$ is non-positive and belongs to $L^\infty(\Omega)$, $0 \ge \alpha_\lambda \ge -\frac{p-1}{\mu} + \varepsilon$ for some $\varepsilon > 0$. Then, for any $\lambda \in \mathbb{R}$, we consider the auxiliary problem

$$-\Delta_p v = f_\lambda(x, v), \quad v \in W_0^{1,p}(\Omega), \tag{19}$$

where

$$f_\lambda(x,s) = \begin{cases} \lambda c(x) g(s) + \Big(1+\dfrac{\mu}{p-1}s\Big)^{p-1} h(x), & \text{if } s \ge \alpha_\lambda(x), \\ \lambda c(x) g(\alpha_\lambda(x)) + \Big(1+\dfrac{\mu}{p-1}\alpha_\lambda(x)\Big)^{p-1} h(x), & \text{if } s \le \alpha_\lambda(x), \end{cases} \tag{20}$$

and we prove in Proposition 1.5.2 that $v \in W_0^{1,p}(\Omega)$ is a solution to (19) if, and only if,

$$u = \frac{p-1}{\mu}\ln\Big(1+\frac{\mu}{p-1}v\Big) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

is a solution to (6).

The main advantages of problem (19) are that it admits a variational formulation and it is completely equivalent to (6). We shall then look for solutions to (19) as critical points of an associated functional I_λ. When $\lambda > 0$, the functional I_λ is unbounded from below and presents a concave-convex type geometry. Then, in trying to obtain critical points of I_λ, the fact that g is only slightly superlinear at infinity is a difficulty. It implies that I_λ does not satisfies an Ambrosetti-Rabinowitz-type condition and proving that the Palais-Smale (Cerami in our case) sequences are bounded becomes challenging. Actually, the proof of Lemma 1.8.1 (boundedness of the Cerami sequences) is one of the main difficulties of Chapter 1 and make use of some ideas that are new in the literature and may be useful in other settings.

Once we have proved the Cerami condition (see Definition 1.2.3) for I_λ with $\lambda > 0$, in order to prove Theorems 0.1.6, 0.1.7 and 0.1.8, we look for critical points of I_λ which are either local-minimum or of a mountain-pass type. In Theorem 0.1.6 the geometry of I_λ is “simple” and permits to use only variational arguments. In Theorems 0.1.7 and 0.1.8 however it is not so clear, looking directly to I_λ, where to search for critical points. We shall then combine variational techniques and lower and upper solution arguments. In both theorems a first solution is obtained through the existence of well-ordered lower and upper solutions. This solution is further proved to be a local minimum of I_λ and it is then possible to obtain a second solution by a mountain pass argument.

0.2 Problem (5) with the three coefficient functions changing sign

Let us fix from now on $p = 2$, i.e. we are going to deal with (5). In Chapter 2 we pursue the study of (5) in the case where $c_\lambda \not\leq 0$ a.e. in Ω. More precisely, we address the situation where the three coefficients functions c_λ, μ and h may change sign. As observed in [75], the structure of the set of solutions to (5) depends on the size of c_λ^+ but it is not affected by the size of c_λ^-. Let us then write c_λ under the form $c_\lambda = \lambda c_+ - c_-$ where we recall λ is a real parameter. We consider the boundary value problem

$$-\Delta u = (\lambda c_+(x) - c_-(x))u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{21}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2, \text{ is a bounded domain with boundary } \partial\Omega \text{ of class } \mathcal{C}^{1,1}, \\ c_+,\, c_-,\, h^+ \text{ belong to } L^q(\Omega) \text{ for some } q > N/2 \text{ and } \mu, h^- \text{ belong to } L^\infty(\Omega), \\ c_+(x) \geq 0,\; c_-(x) \geq 0 \text{ and } c_-(x)c_+(x) = 0 \text{ a.e. in } \Omega, \\ |\Omega_+| > 0, \text{ where } \Omega_+ := \mathrm{Supp}(c_+), \\ \text{there exists a } \varepsilon > 0 \text{ such that } \mu(x) \geq \mu_1 > 0 \text{ and } c_- \equiv 0 \text{ in } \Omega_1 := \{x \in \Omega : d(x, \Omega_+) < \varepsilon\}. \end{cases} \tag{22}$$

Remark 0.2.1.

a) For a definition of $\mathrm{Supp}(f)$ with $f \in L^p(\Omega)$, for some $p \geq 1$, we refer to [26, Proposition 4.17].

b) The condition $c_- \equiv 0$ in $\{x \in \Omega : d(x, \Omega_+) < \varepsilon\}$ for some $\varepsilon > 0$ is reminiscent of the so-called "thick zero set" condition first introduced in [10].

c) Under the regularity assumptions of (22), any solution to (21) belongs to $\mathcal{C}^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. This can be deduced from [82, Theorem IX-2.2] (see also [17, Proposition 2.1]).

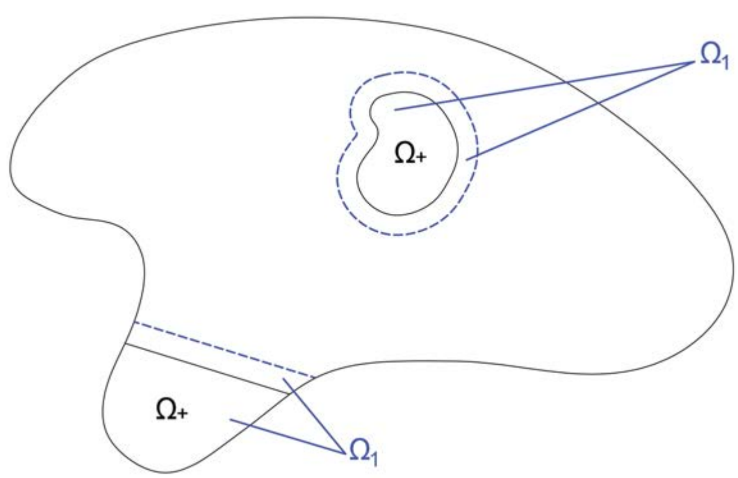

Figure 3: Illustration of assumption (22)

As in [18,50,108] we obtain our existence results using a topological approach, relying thus on the derivation of an a priori bound. In that direction our main result is the following.

Theorem 0.2.1. [Theorem 2.1.1, Chapter 2] *Assume* (22)*. Then, for any* $\Lambda_2 > \Lambda_1 > 0$*, there exists a constant* $M > 0$ *such that, for each* $\lambda \in [\Lambda_1, \Lambda_2]$*, any solution to* (21) *satisfies* $\sup_\Omega u \le M$*.*

Having at hand this a priori bound and following the strategy of [18], we show the existence of a continuum of solutions to (21). More precisely, defining

$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (21)\}, \tag{23}$$

and considering

$$-\Delta u = -c_-(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{24}$$

which corresponds to (21) with $\lambda = 0$, we prove the following theorem.

Theorem 0.2.2. [Theorem 2.1.2, Chapter 2] *Assume* (22) *and suppose that* (24) *has a solution* u_0 *with* $c_+ u_0 \gneqq 0$*. Then, there exists a continuum* $\mathscr{C} \subset \Sigma$ *such that the projection of* $\mathscr{C}$ *on the* λ*-axis is an unbounded interval* $(-\infty, \overline{\lambda}]$ *for some* $\overline{\lambda} \in (0, +\infty)$ *and* $\mathscr{C}$ *bifurcates from infinity to the right of the axis* $\lambda = 0$*. Moreover:*

1) *for all* $\lambda \le 0$*, the problem* (21) *has an unique solution* u_λ *and this solution satisfies* $u_0 - \|u_0\|_\infty \le u_\lambda \le u_0$*;*
2) *there exists* $\lambda_0 \in (0, \overline{\lambda}]$ *such that, for all* $\lambda \in (0, \lambda_0)$*, the problem* (21) *has at least two solutions with* $u_{\lambda,i} \ge u_0$ *for* $i = 1, 2$*.*

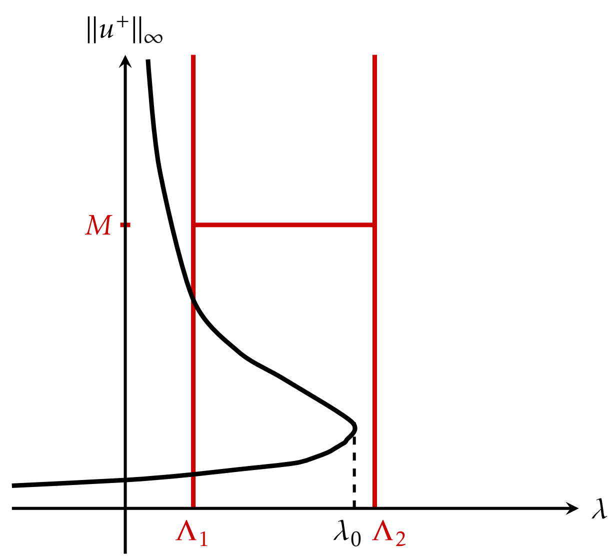

Figure 4: Illustration of Theorems 0.2.1 and 0.2.2

Remark 0.2.2.

a) Theorem 0.2.2, *1)* generalizes [18, Theorem 1.3].
b) If $h \ge 0$ in Ω, [17, Lemma 2.2] implies that the solution to (24) is non-negative.

c) In Theorem 0.1.1 we give sufficient conditions to ensure the existence of a solution to (24).

Let us now give some ideas about the proofs. As already mentioned, our main existence result, Theorem 0.2.2, relies on the derivation of an a priori bound on the solutions to (21). Since we do not have global sign conditions, the approaches used in [18, 50, 108] to obtain the a priori bound do not apply and a new strategy is required. To this aim, we further develop some techniques first sketched in the unpublished work [105]. These techniques, in the framework of viscosity solutions and fully nonlinear equations, now lies at the heart of the paper [92]. We also make use of some ideas from [64].

First of all, we show that it is sufficient to control the behaviour of the solutions to (21) in Ω_+. This can be proved under a weaker assumption than (22). More precisely, under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with boundary } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c_+,\ c_- \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > N/2,\ \mu \text{ belong to } L^\infty(\Omega), \\ c_+(x) \geq 0,\ c_-(x) \geq 0 \text{ and } c_-(x)c_+(x) = 0 \text{ a.e. in } \Omega, \\ |\Omega_+| > 0, \text{ where } \Omega_+ := \operatorname{Supp}(c_+). \end{cases} \tag{25}$$

Lemma 0.2.3. [Lemma 2.4.1, Chapter 2] *Assume that* (25) *holds. Then, there exists* $M > 0$ *such that, for any* $\lambda \in \mathbb{R}$, *any solution* u *to* (21) *satisfies*

$$-\sup_{\Omega_+} u^- - M \leq u \leq \sup_{\Omega_+} u^+ + M.$$

Once we have this extra information, by compactness, we are lead to study what happens around a (unknown) point $\overline{x} \in \overline{\Omega}_+$. We shall consider separately the cases $\overline{x} \in \overline{\Omega}_+ \cap \Omega$ and $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$. A local analysis is then made respectively in a ball or a semiball centered at $\overline{x}$.

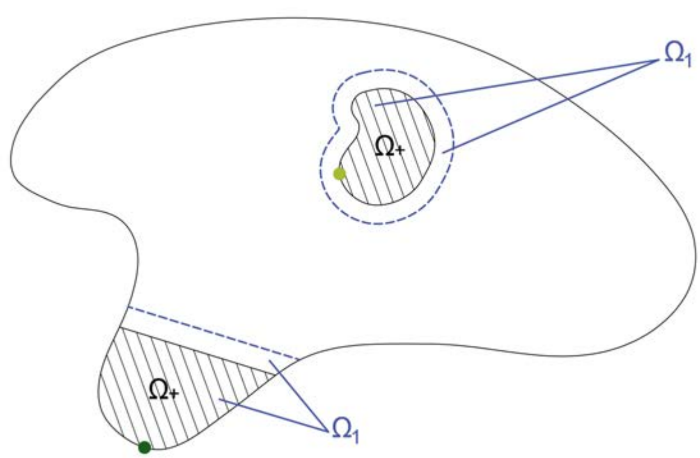

Figure 5: Localization process

Figure 6: Zoom around $\overline{x}$

If similar analysis had previously been performed in other contexts when $\overline{x} \in \Omega$ we believe it is not the case when $\overline{x} \in \partial\Omega$. For $\overline{x} \in \overline{\Omega_+} \cap \Omega$ we have the following result.

Lemma 0.2.4. [Lemma 2.4.2, Chapter 2] *Assume that* (22) *holds and that* $\overline{x} \in \overline{\Omega}_+ \cap \Omega$. *For each* $\Lambda_2 > \Lambda_1 > 0$, *there exist* $M_I > 0$ *and* $R > 0$ *such that, for any* $\lambda \in [\Lambda_1, \Lambda_2]$, *any solution* u *to* (21) *satisfies* $\sup_{B_R(\overline{x})} u \leq M_I$.

The proof of this result is based on the use of a *local maximum principle* that we borrow from [66, 90] (see Lemma 2.2.1 for the precise statement) and the classical *weak Harnack inequality* (see Lemma 2.2.3).

For $\overline{x} \in \Omega_+ \cap \partial\Omega$, the classical local estimates do not apply. The key to our approach is the use of a new *boundary weak Harnack inequality*. Actually, a major part of Chapter 2 is devoted to establish this inequality. This is done in a more general context than the one we need to treat (21). In particular, it also cover the case of the p-Laplacian with a zero order term. More precisely, for $1 < p < \infty$, we consider the boundary value problem

$$-\Delta_p u + a(x)|u|^{p-2}u = 0, \quad u \in W_0^{1,p}(\omega). \tag{26}$$

and we prove the following *boundary weak Harnack inequality*.

Theorem 0.2.5. [Theorem 2.3.1, Chapter 2] *Let* $\omega \subset \mathbb{R}^N$, $N \geq 2$, *be a bounded domain with boundary* $\partial\omega$ *of class* $C^{1,1}$ *and let* $a \in L^\infty(\omega)$ *be a non-negative function. Assume that* u *is a non-negative upper solution to* (26) *and let* $x_0 \in \partial\omega$. *Then, there exist* $\overline{R} > 0$, $\varepsilon = \varepsilon(p, \overline{R}, \|a\|_\infty, \omega) > 0$ *and* $C = C(p, \overline{R}, \varepsilon, \|a\|_\infty, \omega) > 0$ *such that, for all* $R \in (0, \overline{R}]$,

$$\inf_{B_R(x_0)\cap\omega} \frac{u(x)}{d(x,\partial\omega)} \geq C \left(\int_{B_R(x_0)\cap\omega} \left(\frac{u(x)}{d(x,\partial\omega)} \right)^{\varepsilon} dx \right)^{1/\varepsilon}.$$

We believe this *boundary weak Harnack inequality* is of independent interest and will proved to be useful in other settings. Its proof uses ideas introduced by B. Sirakov [106]. In [106] such type of inequalities is established for an uniformly elliptic operator and viscosity solutions. However, since our context is quite different, the result of [106] does not apply to our situation and we need to work out an adapted proof. In particular, we would like to mention Lemmas 2.3.6 and 2.3.7. The proofs we provide of these results present an alternative approach which is shorter and somehow simpler than the one developed in [106] to prove the corresponding results.

Having at hand Theorem 0.2.5, we obtain the counterpart of Lemma 0.2.4 for $\overline{x} \in \Omega_+ \cap \partial\Omega$. Namely, we prove the following result.

Lemma 0.2.6. [Lemma 2.4.3, Chapter 2] *Assume that* (22) *holds and that* $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$. *For each* $\Lambda_2 > \Lambda_1 > 0$, *there exist* $R > 0$ *and* $M_B > 0$ *such that, for any* $\lambda \in [\Lambda_1, \Lambda_2]$, *any solution to* (21) *satisfies* $\sup_{B_R(\overline{x})\cap\Omega} u \leq M_B$.

The combination of Lemmas 0.2.3, 0.2.4 and 0.2.6 gives us the proof of Theorem 0.2.1. Once we have proved Theorem 0.2.1, in order to prove Theorem 0.2.2, we follow the approach first used in [18]. Nevertheless, as we do not have global sign conditions (neither on the coefficients functions nor on the solutions), several difficulties appear. We refer to Chapter 2, Section 2.5 for more details.

We would like to end this section presenting a corollary of Theorem 0.2.5 that we hope will be useful in other settings. Consider the boundary value problem

$$-\Delta u + a(x)u = b(x), \qquad u \in H_0^1(\omega), \tag{27}$$

under the assumption

$$\begin{cases} \omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with boundary } \partial\omega \text{ of class } \mathcal{C}^{1,1}, \\ a \in L^\infty(\omega),\ b^- \in L^p(\omega) \text{ for some } p > N \text{ and } b^+ \in L^1(\omega), \\ a \geq 0 \text{ a.e. in } \omega. \end{cases} \tag{28}$$

Corollary 0.2.7. [Corollary 2.3.9, Chapter 2] *Under the assumption* (28), *assume that* $u \in H^1(\omega)$ *is a non-negative upper solution to* (27) *and let* $x_0 \in \partial\omega$. *Then, there exist* $\overline{R} > 0$, $\varepsilon = \varepsilon(\overline{R}, \|a\|_\infty, \omega) > 0$, $C_1 = C_1(\overline{R}, \varepsilon, \|a\|_\infty, \omega) > 0$ *and* $C_2 = C_2(\omega, \|a\|_\infty) > 0$ *such that, for all* $R \in (0, \overline{R}]$,

$$\inf_{B_R(x_0)\cap\omega} \frac{u(x)}{d(x,\partial\omega)} \geq C_1 \left(\int_{B_R(x_0)\cap\omega} \left(\frac{u(x)}{d(x,\partial\omega)} \right)^{\varepsilon} dx \right)^{1/\varepsilon} - C_2 \|b^-\|_{L^p(\omega)}.$$

0.3 Removing the "thick zero set" condition

In Chapter 3 we continue the study of (5) in the case where $c_\lambda \not\leq 0$ a.e. in Ω. At the expense of considering μ constant, we remove the "thick zero set" condition on c_λ considered in Chapter 2. We write again c_λ under the form $c_\lambda = \lambda c_+ - c_-$ where λ is a real parameter and we consider the boundary value problem

$$-\Delta u = (\lambda c_+(x) - c_-(x))u + \mu|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{29}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c_+,\ c_-, \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > N/2, \\ \mu > 0,\ c_+ \gneqq 0,\ c_- \geq 0 \text{ and } c_+(x)c_-(x) = 0 \text{ a.e. in } \Omega. \end{cases} \tag{30}$$

Remark 0.3.1. Since h does not have a sign, there is no loss of generality in assuming $\mu > 0$.

Before we address the study of (29) in the case where $c_\lambda = \lambda c_+ - c_-$ may change sign, we completely characterize the *limit coercive case* $c_\lambda \leq 0$. More generally, let us consider

$$-\Delta u = -d(x)u + \mu|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{31}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ d \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > N/2, \\ \mu > 0 \text{ and } d \geq 0, \end{cases} \tag{32}$$

and define

$$m_d := \begin{cases} \inf_{u \in W_d} \int_\Omega (|\nabla u|^2 - \mu h(x) u^2)\,dx, & \text{if} \quad W_d \neq \emptyset, \\ +\infty, & \text{if} \quad W_d = \emptyset, \end{cases} \tag{33}$$

where

$$W_d := \{w \in H_0^1(\Omega) : d(x)w(x) = 0 \text{ a.e. in } \Omega,\ \|w\|_{H_0^1(\Omega)} = 1\}.$$

We prove the following sharp result.

Theorem 0.3.1. [Theorem 3.1.1, Chapter 3] *Assume that* (32) *holds. Then* (31) *has a solution if, and only if,* $m_d > 0$.

Remark 0.3.2.

a) This theorem generalizes [18, Proposition 3.1 and Remark 3.2] and, for $p = 2$, Theorem 0.1.4.

b) By [17, Theorem 1.1] we know that the solution obtained is unique.

Due to its importance in the rest of the chapter, let us make explicit an immediate corollary of the previous theorem. We consider the boundary value problem

$$-\Delta u = -c_-(x)u + \mu|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{34}$$

which corresponds to (29) with $\lambda = 0$, and we have the following result.

Corollary 0.3.2. [Corollary 3.1.2, Chapter 3] *Assume that* (32) *holds with* $d \equiv c_-$. *Then* (34) *has a solution if, and only if,* $m_{c_-} > 0$.

Now, having at hand this information about the *limit coercive case* $c_\lambda \leq 0$, we turn to the study of the *non-coercive case* $\lambda > 0$. First, using mainly variational techniques, we prove the following theorem.

Theorem 0.3.3. [Theorem 3.1.3, Chapter 3] *Assume* (30) *and suppose that* (34) *has a solution. Then, there exists* $\Lambda > 0$ *such that, for all* $0 < \lambda < \Lambda$, (29) *has at least two solutions.*

Remark 0.3.3. This result improves and generalizes the main result obtained in [75].

Next, considering stronger regularity assumptions on the coefficient functions and combining lower and upper solution and variational techniques, we improve the conclusions of Theorem 0.3.3. We derive a more precise information on the structure of the set of solutions to (29) when $\lambda > 0$. Under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{1,1}, \\ c_+,\, c_-, \text{ and } h \text{ belong to } L^p(\Omega) \text{ for some } p > N, \\ \mu > 0,\ c_+ \gneqq 0,\ c_- \geq 0 \text{ and } c_+(x)c_-(x) = 0 \text{ a.e. in } \Omega, \end{cases} \tag{35}$$

we prove the following theorems.

Theorem 0.3.4. [Theorem 3.1.4, Chapter 3] *Assume (35) and suppose that (34) has a solution u_0 with $c_+u_0 \gneqq 0$. Then, every u solution to (29) with $\lambda > 0$ and $c_+u \geq 0$ satisfies $u \gg u_0$. Moreover, there exists $\overline{\lambda} \in]0, +\infty[$, such that:*

- *for every $\lambda \in]0, \overline{\lambda}[$, (29) has at least two solutions $u_{\lambda,1}, u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ such that $u_{\lambda,1} \gg u_0$;*
- *(29) with $\lambda = \overline{\lambda}$ has at least one solution $u_{\overline{\lambda}} \in \mathcal{C}_0^1(\overline{\Omega})$ such that $u_{\overline{\lambda}} \geq u_0$;*
- *for $\lambda > \overline{\lambda}$ the problem (29) has no solutions u such that $c_+u \geq 0$.*

Theorem 0.3.5. [Theorem 3.1.5, Chapter 3] *Assume (35) and suppose that (34) has a solution u_0 with $c_+u_0 \lneqq 0$. Then, for every $\lambda > 0$, the problem (29) has at least two solutions $u_{\lambda,1}, u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ such that $u_{\lambda,1} \ll u_0$.*

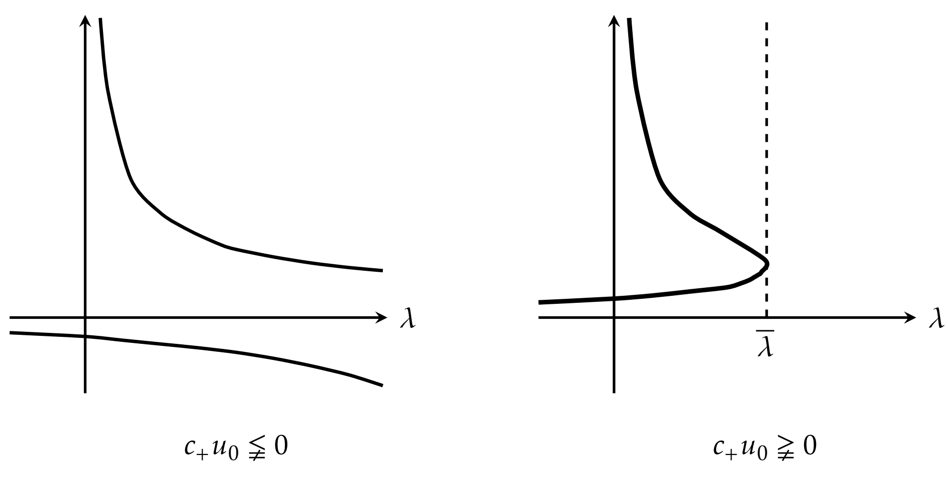

Figure 7: Illustration of Theorems 0.3.4 and 0.3.5

Remark 0.3.4.

a) We recall that the order notion "$\gg$" is given in Definition 1.1.1.

b) Under the assumption (35), every solution to (29) belongs to $\mathcal{C}_0^1(\overline{\Omega})$. This was proved in [50, Theorem 2.2].

c) At the expense of considering $\mu > 0$ constant instead of $\mu \in L^\infty(\Omega)$ with $\mu(x) \geq \mu_1 > 0$ in Ω, Theorems 0.3.4 and 0.3.5 extend the main existence results of [50] to the case where c may change sign. Moreover, unlike [50], we do not assume global sign conditions on u_0 (solution to (34)). Hence, even in the case where $c_- \equiv 0$, i.e. c has a sign, our hypotheses are weaker than the corresponding ones in [50].

d) Theorem 0.3.4 removes the "thick zero set" condition on the support of c_λ considered in Theorem 0.2.2 and gives somehow a more precise information. In turn, here μ is constant and we require a stronger regularity on c_λ and h^+.

Finally, we give sufficient conditions in terms of h ensuring that the hypotheses of Theorem 0.3.4 or of Theorem 0.3.5 are satisfied.

Corollary 0.3.6. [Corollary 3.1.6, Chapter 3] *Assume* (35) *and suppose that* (34) *has a solution:*

- *If $h \gneqq 0$, then the conclusions of Theorem 0.3.4 hold.*
- *If $h \lneqq 0$, then the conclusions of Theorem 0.3.5 hold.*

Remark 0.3.5. In the case where $h \lneqq 0$, problem (34) has always a solution.

We provide now some ideas about the proofs of Theorems 0.3.3, 0.3.4 and 0.3.5. As in Chapter 1 we exploit here the fact that μ is a constant. Nevertheless, due to the indefinite sign of $c_\lambda = \lambda c_+ - c_-$, we have to overcome new difficulties.

First, using the fact that μ is constant, we perform a change of variable and reduce (29) to an equivalent semilinear problem which presents a variational formulation. To that end, the key is the construction of a lower solution to (29) below every upper solution to this problem. The fact that c_λ has no sign causes several difficulties in this construction. See Chapter 3, Section 3.4 for more details.

Once we have this equivalent problem, we shall look for solutions as critical points of an associated functional I_λ. When λ is positive the functional is unbounded from below and presents a concave-convex-type geometry. Furthermore, it does not satisfy an Ambrosetti-Rabinowitz-type condition and c_λ and h have no sign. In this context, to prove that the Palais-Smale sequences are bounded may be challenging. Due to the indefinite sign of c_λ, the approach introduced in Chapter 1 cannot be adapted. Our proof here is inspired by [75]. However, since we do not impose $h \gneqq 0$, the proof is more involved. The role of the lower solution previously discussed is again crucial. We refer to Chapter 3, Section 3.5 for more details.

Having at hand the boundedness of the Palais-Smale sequences, we argue as in Chapter 1. Theorem 0.3.3 is proved using mainly variational techniques as in [75,76]. Nevertheless, since our hypotheses are weaker than the corresponding ones in [75,76], to prove that we have a mountain-pass geometry becomes much more involved. In Theorems 0.3.4 and 0.3.5 we combine lower and upper solution arguments and variational techniques. In both theorems a first solution is obtained throughout the existence of well-ordered lower and upper solutions. This solution is further proved to be a local minimum. Then, we obtain a second solution by a mountain-pass type argument.

Another key ingredient in the proofs of Theorems 0.3.4 and 0.3.5 is the following estimate that can be seen as a combination of the Strong maximum principle and the Hopf's Lemma with unbounded lower order coefficients. Under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{1,1}, \\ c \text{ belongs to } L^p(\Omega) \text{ and } B = (B^1, \ldots, B^N) \text{ belongs to } (L^p(\Omega))^N \text{ for some } p > N, \\ c \geq 0, \end{cases} \tag{36}$$

we obtain the following result.

Theorem 0.3.7. *Assume (36) and let $u \in \mathcal{C}^1(\overline{\Omega})$ be an upper solution to*

$$-\Delta u + \langle B(x), \nabla u\rangle + c(x)u = 0, \quad u \in H_0^1(\Omega). \tag{37}$$

Then, either $u \equiv 0$ or $u \gg 0$.

Remark 0.3.6.

a) The case where $B \in (L^\infty(\Omega))^N$ and $c \in L^\infty(\Omega)$ is nowadays classical and can be founded for instance in [112, Theorem 3.27].

b) Theorem 0.3.7 can be obtained as a corollary from [100, Theorem 4.1]. Nevertheless, for the benefit of the reader, we provide a self-contained simplified proof in Chapter 3, Appendix 3.7.

0.4 Nonlocal "gradient terms"

In Chapter 4 we address a different topic. We turn to study existence and non-existence results for several nonlocal problems involving nonlocal "gradient terms".

In the last fifteen years, there has been an increasing interest in the study of partial differential equations involving integro-differential operators. In particular, the case of the fractional Laplacian has been widely studied and is nowadays a very active field of research. This is due not only to its mathematical richness but also to the fact that the fractional Laplacian has appeared in a great number of equations modeling real world phenomena, especially those which take into account nonlocal effects. Among others, let us mention applications in quasi-geostrophic flows [32], quantum mechanic [83], mathematical finances [14, 37], obstacle problems [21, 22, 31] and crystal dislocation [58, 59, 111].

The first aim of Chapter 4 is to discuss, depending on the real parameter $\lambda > 0$, the existence and non-existence of solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x)\mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{38}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ s \in (1/2, 1), \\ f \in L^m(\Omega) \text{ for some } m > N/2s \text{ and } \mu \in L^\infty(\Omega). \end{cases} \tag{39}$$

Throughout the chapter, $(-\Delta)^s$ stands for the, by know classical, fractional Laplacian operator. For a smooth function u and $s \in (0,1)$, it can be defined as

$$(-\Delta)^s u(x) := a_{N,s} \text{ p.v.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where

$$a_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1} = -\frac{2^{2s}\Gamma\left(\frac{N}{2}+s\right)}{\pi^{\frac{N}{2}}\Gamma(-s)},$$

is a normalization constant and "p.v." is an abbreviation for "in the principal value sense". In (38), $\mathbb{D}_s^2$ is a nonlocal diffusion term. It plays the role of the "gradient square" in the nonlocal case and is given by

$$\mathbb{D}_s^2(u) = \frac{a_{N,s}}{2} \text{ p.v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy. \tag{40}$$

Since they will not play a role in this thesis, we normalize the constants appearing in the definitions of $(-\Delta)^s$ and $\mathbb{D}_s^2$ and we omit the p.v. sense. However, let us stress that these constants guarantee that

$$\lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x), \quad \forall\, u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{41}$$

and

$$\lim_{s\to 1^-} \mathbb{D}_s^2(u(x)) = |\nabla u(x)|^2, \quad \forall\, u \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{42}$$

We refer to [55] and [30] respectively for a proof of (41) and (42). Hence, at least formally, if $s \to 1^-$ in (38), we recover the local problem

$$\begin{cases} -\Delta u = \mu(x)|\nabla u|^2 + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{43}$$

Observe that this equation has played an essential role in Chapters 1, 2 and 3. It is precisely our model problem (1) with $c \equiv 0$. In these chapters, the existence of solutions is proved using either a priori estimates or, when it is possible, a suitable change of variable to obtain an equivalent semilinear problem. However, neither of these techniques seem to be appropriate to deal with the nonlocal problem (38).

In the spirit of the existing results for the local case, see for instance Theorem 0.1.1, our first main result shows the existence of a weak solution to (38) under a smallness condition on λf.

Theorem 0.4.1. [Theorem 4.1.1, Chapter 4] *Assume that* (39) *holds. Then there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \leq \lambda^*$, (38) *has a weak solution* $u \in W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ *for some* $\alpha > 0$.

Remark 0.4.1.

a) The definition of weak solutions to (38) is given in Definition 4.1.1 and the definition of $W_0^{s,2}(\Omega)$ will be introduced in Section 4.2.

b) For $\lambda f \equiv 0$, $u \equiv 0$ is a solution to (38) that obviously belongs to $W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$. Hence, there is no loss of generality to assume that $\lambda > 0$.

The counterpart of $|\nabla u|^2$ in (43) is played in (38) by $\mathbb{D}_s^2(u)$. This term appears in several applications. For instance, let us mention [30, 91, 101] where it naturally appears as the equivalent of $|\nabla u|^2$ when considering fractional harmonic maps into the sphere.

Let us now give some ideas of the proof of Theorem 0.4.1. The existence of solutions to (38) can be related to the regularity of the solutions to a linear equation of the form

$$\begin{cases} (-\Delta)^s v = h(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{44}$$

In Chapter 4, Section 4.3, we obtain sharp Calderón-Zygmund type regularity results for the fractional Poisson equation (44) with low integrability data. We believe these results are of independent interest and will be useful in other settings. Actually, Section 4.3 can be read as an independent part of Chapter 4. We refer the interested reader to Propositions 4.3.1, 4.3.3 and 4.3.4.

Having at hand suitable regularity results for (44) and inspired by [94, Section 6], we develop a fixed point argument to obtain a solution to (38). Note that, due to the nonlocality of the operator and of the "gradient term", the approach of [94] has to be adapted significantly. In particular, the form of the set where we apply the fixed point argument seems to be new in the literature. We consider a subset of $W_0^{s,1}(\Omega)$ where, in some sense, we require more "differentiability" and more integrability. This extra "differentiability" is a purely nonlocal phenomenon and it is related with our regularity results for (44). See Section 4.4 for more details.

Let us also stress that the restriction $s \in (1/2, 1)$ comes from the regularity results of Section 4.3. If suitable regularity results for (44) with $s \in (0, 1/2]$ were available, our fixed point argument would provide the same existence result Theorem 0.4.1 also for $s \in (0, 1/2]$.

Next, we prove that the smallness condition imposed in Theorem 0.4.1 is somehow necessary.

Theorem 0.4.2. [Theorem 4.1.2, Chapter 4] *Assume (39) and suppose that $\mu(x) \geq \mu_1 > 0$ and $f^+ \not\equiv 0$. Then there exists $\lambda^{**} > 0$ such that, for all $\lambda > \lambda^{**}$, (38) has no weak solutions in $W_0^{s,2}(\Omega)$.*

Remark 0.4.2.

a) Observe that, if v is a solution to

$$\begin{cases} (-\Delta)^s v = -\mu(x)\,\mathbb{D}_s^2(v) - \lambda f(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $u = -v$ is a solution to (38). Hence, if $\mu(x) \leq -\mu_1 < 0$ and $f^- \not\equiv 0$ we recover the same kind of non-existence result and the smallness condition is also required.

b) Since we do not use the regularity results of Section 4.3, the restriction $s \in (1/2, 1)$ is not necessary in the proof of Theorem 0.4.2. The result holds for all $s \in (0, 1)$.

Also, in order to show that the regularity imposed on the data f is almost optimal, we provide a counterexample to our existence result when the regularity condition on f is not satisfied. The proof makes use of the Hardy potential.

Theorem 0.4.3. [Theorem 4.1.3, Chapter 4] *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$, let $s \in (0,1)$ and let $\mu \in L^\infty(\Omega)$ such that $\mu(x) \geq \mu_1 > 0$. Then, for all $1 \leq p < \frac{N}{2s}$, there exists $f \in L^p(\Omega)$ such that (38) has no weak solutions in $W_0^{s,2}(\Omega)$ for any $\lambda > 0$.*

Using the same kind of approach than in Theorem 0.4.1, i.e. regularity results for (44) and our fixed point argument, one can obtain existence results for related problems involving different nonlocal diffusion terms and different nonlinearities.

First, we deal with the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x)\, u\, \mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{45}$$

For $\mu(x) \equiv 1$, this problem can be seen as a particular case of the fractional harmonic maps problem considered in [30, 91]. We derive the following existence result for λf small enough.

Theorem 0.4.4. [Theorem 4.1.4, Chapter 4] *Assume that* (39) *holds. Then, there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \le \lambda^*$, (45) *has a weak solution* $u \in W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ *for some* $\alpha > 0$.

Next, motivated by some other results on fractional harmonic maps into the sphere [41, 42] and some classical results of harmonic analysis [109, Chapter V], we consider a different diffusion term. Depending on the real parameter $\lambda > 0$, we study the existence of solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x) |(-\Delta)^{\frac{s}{2}} u|^q + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{46}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \ge 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ f \in L^m(\Omega) \text{ for some } m \ge 1 \text{ and } \mu \in L^\infty(\Omega), \\ s \in (1/2, 1) \text{ and } 1 < q < \dfrac{N}{(N - ms)^+}. \end{cases} \tag{47}$$

Theorem 0.4.5. [Theorem 4.1.5, Chapter 4] *Assume that* (47) *holds. Then there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \le \lambda^*$, (46) *has a weak solution* $u \in W_0^{s,1}(\Omega)$.

Remark 0.4.3.

a) The notion of weak solution to (46) will be given in Definition 4.1.2.

b) The regularity results for (44) that we need to prove Theorem 0.4.5 are different from the ones used in Theorems 0.4.1 and 0.4.4. Nevertheless, the restriction $s \in (1/2, 1)$ still arises out of these regularity results. See Proposition 4.3.5 for more details.

Finally, for $s \in (0,1)$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, following [95, 103], we define the (distributional Riesz) *fractional gradient of order s* as the vector field $\nabla^s : \mathbb{R}^N \to \mathbb{R}$ given by

$$\nabla^s \phi(x) := \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^s} \frac{x-y}{|x-y|} \frac{dy}{|x-y|^N}, \quad \forall\, x \in \mathbb{R}^N. \tag{48}$$

Then we deal with the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x) |\nabla^s u|^q + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{49}$$

Theorem 0.4.6. [Theorem 4.1.6, Chapter 4] *Assume that* (47) *holds. Then there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \le \lambda^*$, (49) *has a weak solution* $u \in W_0^{s,1}(\Omega)$.

1

Existence and multiplicity for elliptic p-Laplacian problems with critical growth in the gradient

1.1 Introduction and main results

Let $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ denote the p-Laplacian operator. We consider, for any $1 < p < \infty$, the boundary value problem

$$-\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu(x)|\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{P_λ}$$

under the assumptions

$$\begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ c \gneqq 0 \text{ and } \mu \in L^\infty(\Omega). \end{cases} \tag{A_0}$$

The study of quasilinear elliptic equations with a gradient dependence up to the critical growth $|\nabla u|^p$ was initiated by L. Boccardo, F. Murat and J.P. Puel in the 80's and it has been an active field of research until now. Under the condition $\lambda c(x) \leq -\alpha_0 < 0$ for some $\alpha_0 > 0$, which is now referred to as the *coercive case*, the existence of solutions is a particular case of the results of [23, 25, 43]. The *weakly coercive case* ($\lambda = 0$) was studied in [62] where, for $\|\mu h\|_{N/p}$ small enough, the existence of an unique solution is obtained, see also [1]. The *limit coercive case*, where one just require that $\lambda c(x) \leq 0$ and hence c may vanish only on some parts of Ω, is more complex and was left open until [18]. In that paper, for the case $p = 2$, it was observed that, under the assumption (A_0), the existence of solutions to (P_λ) is not guaranteed. Sufficient conditions in order to ensure the existence of solution were given.

The case $\lambda c(x) \gneqq 0$ also remained unexplored until very recently. First, in [76] the authors studied problem (P_λ) with $p = 2$. Assuming $\lambda > 0$ and μh small enough, in an appropriate sense, they proved the existence of at least two solutions. This result has now be complemented in several ways. In [75] the existence of two solutions is obtained, allowing the function c to change sign with $c^+ \not\equiv 0$ but assuming $h \gneqq 0$. In both [75,76] $\mu > 0$ is assumed constant. In [18] the restriction μ constant was removed but assuming that $h \gneqq 0$. Finally, in [50], under stronger regularity on the coefficients, cases where μ is non constant and h is non-positive or has no sign were treated. Actually in [50], under different sets of assumptions, the authors clarify the structure of the set of solutions to (P_λ) in the non-coercive case. Now, concerning (P_λ) with $p \neq 2$, the only results in the case $\lambda c \gneqq 0$ are, up to our knowledge, presented in [1,72]. In [72] the case c constant and $h \equiv 0$ is covered and in [1], the model equation is $-\Delta_p u = |\nabla u|^p + \lambda f(x)(1+u)^b$, $b \geq p-1$ and $f \gneqq 0$.

To state our first main result let us define

$$m_{p,\lambda}^+ := \begin{cases} \inf\limits_{u \in W_\lambda} \displaystyle\int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^+\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx, & \text{if } W_\lambda \neq \emptyset, \\ +\infty, & \text{if } W_\lambda = \emptyset, \end{cases}$$

and

$$m_{p,\lambda}^- := \begin{cases} \inf\limits_{u \in W_\lambda} \displaystyle\int_\Omega \Big(|\nabla u|^p + \Big(\frac{\|\mu^-\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx, & \text{if } W_\lambda \neq \emptyset, \\ +\infty, & \text{if } W_\lambda = \emptyset, \end{cases}$$

where

$$W_\lambda := \{ w \in W_0^{1,p}(\Omega) : \lambda c(x) w(x) = 0 \text{ a.e. } x \in \Omega, \ \|w\| = 1 \}.$$

Using these notations, we state the following result which generalizes the results obtained in [18, Section 3].

Theorem 1.1.1. *Assume that (A_0) holds and that $\lambda \leq 0$. Then if $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$, the problem (P_λ) has at least one solution.*

Remark 1.1.1.

a) The space W_λ depends only on the fact that $\lambda = 0$ or $\lambda \neq 0$.

b) The assumption $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$ connects the cases $\lambda c(x) \leq -\alpha_0 < 0$ and $\lambda c(x) \equiv 0$. In fact, in case $\lambda c(x) < 0$ a.e. in Ω we have $W_\lambda = \emptyset$ and hence $m_{p,\lambda}^+ = m_{p,\lambda}^- = +\infty$. On the other hand, if $\lambda c(x) \equiv 0$, then $W_0 = W_0^{1,p}(\Omega)$ and $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$ holds for example under a smallness condition on $\|\mu^+\|_\infty^{p-1} h^+$ and $\|\mu^-\|_\infty^{p-1} h^-$ as in Appendix 1.10. In particular observe that $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$ in case $\mu \geq 0$ and $h \leq 0$ as well as in case $\mu \leq 0$ and $h \geq 0$.

c) If h is either non-negative or non-positive our hypotheses correspond to the ones introduced in [18] for $p = 2$. However, observe that for all $u \in W_0^{1,p}(\Omega)$, it follows that

$$\int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^+\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx \geq \int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^+\|_\infty}{p-1}\Big)^{p-1} h^+(x)|u|^p \Big) dx$$

and

$$\int_\Omega \Big(|\nabla u|^p + \Big(\frac{\|\mu^-\|_\infty}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx \geq \int_\Omega \Big(|\nabla u|^p - \Big(\frac{\|\mu^-\|_\infty}{p-1}\Big)^{p-1} h^-(x)|u|^p \Big) dx.$$

Hence, if h does not have a sign, our hypotheses improve the ones introduced in [18] even for $p = 2$.

In the rest of the chapter we assume that μ is constant. Namely, we replace (A_0) by

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2\,, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ c \gneqq 0 \text{ and } \mu > 0\,. \end{cases} \tag{A_1}$$

Observe that there is no loss of generality in assuming $\mu > 0$ since, if u is a solution to (P_λ) with $\mu < 0$, then $w = -u$ satisfies

$$-\Delta_p w = \lambda c(x)|w|^{p-2}w - \mu|\nabla w|^p - h(x).$$

In [17], for $p = 2$ but assuming only (A_0), the uniqueness of solutions when $\lambda \leq 0$ was obtained as a direct consequence of a comparison principle, see [17, Corollary 3.1]. As we show in Remark 1.3.3, such kind of principle does not hold in general when $p \neq 2$. Actually the issue of uniqueness for equations of the form of (P_λ) appears widely open. If partial results, assuming for example $1 < p \leq 2$ or $\lambda c(x) \leq -\alpha_0 < 0$, seem reachable adapting existing techniques, see in particular [85,96,98], a result covering the full generality of (P_λ) seems, so far, out of reach. Theorem 1.1.2 below, whose proof makes use of some ideas from [6], crucially relies on the assumption that μ is constant. It permits however to treat the limit case (P_0) which plays an important role in the chapter.

Theorem 1.1.2. *Assume that (A_1) holds and suppose $\lambda \leq 0$. Then (P_λ) has at most one solution.*

Let us now introduce

$$m_p := \inf\Big\{ \int_\Omega \Big(|\nabla w|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1} h(x)|w|^p \Big) dx : w \in W_0^{1,p}(\Omega),\, \|w\| = 1 \Big\}. \tag{1.1.1}$$

We can state the following result.

Theorem 1.1.3. *Assume that (A_1) holds. Then (P_0) has a solution if, and only if, $m_p > 0$.*

Theorem 1.1.3 provides, so to say, a characterization in term of a first eigenvalue of the existence of solution to (P_0). This result again improves, for μ constant, [18] and it allows to observe that, in case $h \lneqq 0$, (P_0) has always a solution while the case $h \gneqq 0$ is the "worse" case for the existence of a solution. In case h changes sign, the negative part of h "helps" in order to have a solution to (P_0) as explained in Remark 1.1.1. We give in Appendix 1.10, sufficient conditions on h^+ in order to ensure $m_p > 0$.

Remark 1.1.2. Observe that the sufficient part of Theorem 1.1.3 is direct. Indeed, if $m_p > 0$ then $m_{p,0}^+ > 0$ and $m_{p,0}^- > 0$ and Theorem 1.1.1 implies that (P_0) has a solution.

Gluing together the previous results we obtain the following existence and uniqueness result for $\lambda \leq 0$.

Corollary 1.1.4. *Assume that (A_1) holds and suppose that (P_0) has a solution. Then, for all $\lambda \leq 0$, (P_λ) has an unique solution.*

Now, we turn to the study the non-coercive case, namely when $\lambda > 0$. First, using mainly variational techniques we prove the following result.

Theorem 1.1.5. *Assume that (A_1) holds and suppose that (P_0) has a solution. Then there exists $\Lambda > 0$ such that, for any $0 < \lambda < \Lambda$, (P_λ) has at least two solutions.*

As we shall see in Corollary 1.9.4, the existence of a solution to (P_0) is, in some sense, necessary for the existence of a solution when $\lambda > 0$.

Next, considering stronger regularity assumptions, we derive informations on the structure of the set of solutions in the non-coercive case. These informations complement Theorem 1.1.5. We denote by $\gamma_1 > 0$ the first eigenvalue of the problem

$$-\Delta_p u = \gamma c(x)|u|^{p-2}u\,, \quad u \in W_0^{1,p}(\Omega)\,, \tag{1.1.2}$$

and we introduce the following order notions.

Definition 1.1.1. For $h_1, h_2 \in L^1(\Omega)$ we write

- $h_1 \leq h_2$ if $h_1(x) \leq h_2(x)$ for a.e. $x \in \Omega$,
- $h_1 \lneqq h_2$ if $h_1 \leq h_2$ and $\mathrm{meas}(\{x \in \Omega : h_1(x) < h_2(x)\}) > 0$.

For $u, v \in \mathcal{C}^1(\overline{\Omega})$ we write

- $u < v$ if, for all $x \in \Omega$, $u(x) < v(x)$,
- $u \ll v$ if $u < v$ and, for all $x \in \partial\Omega$, either $u(x) < v(x)$, or, $u(x) = v(x)$ and $\frac{\partial u}{\partial \nu}(x) > \frac{\partial v}{\partial \nu}(x)$, where ν denotes the exterior unit normal.

Under the assumptions

$$\begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ c \text{ and } h \text{ belong to } L^\infty(\Omega), \\ c \gneqq 0 \text{ and } \mu > 0, \end{cases} \tag{A_2}$$

we state the following theorem.

Theorem 1.1.6. *Assume that (A_2) holds and suppose that (P_0) has a solution u_0. Then:*

- *If $h \lneqq 0$, for every $\lambda > 0$, (P_λ) has at least two solutions u_1, u_2 with $u_1 \ll 0$.*
- *If $h \gneqq 0$, then $u_0 \gg 0$ and there exists $\overline{\lambda} \in (0, \gamma_1)$ such that:*
  - *for every $0 < \lambda < \overline{\lambda}$, ($P_\lambda$) has at least two solutions satisfying $u_i \geq u_0$;*
  - *for $\lambda = \overline{\lambda}$, ($P_\lambda$) has at least one solution satisfying $u \geq u_0$;*
- – ∘] *for any $\lambda > \overline{\lambda}$, ($P_\lambda$) has no non-negative solution.*

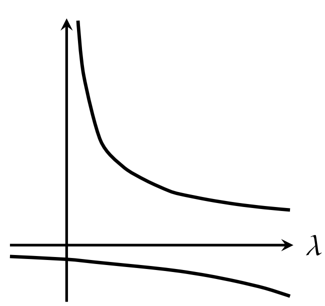

Figure 1.1: Illustration of Theorem 1.1.6 with $h \lneqq 0$

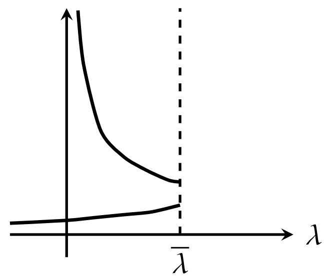

Figure 1.2: Illustration of Theorem 1.1.6 with $h \gneqq 0$

Remark 1.1.3.

a) As observed above, in the case $h \lneqq 0$, the assumption that (P_0) has a solution is automatically satisfied.

b) In the case $\mu < 0$, we have the opposite result i.e., two solutions for every $\lambda > 0$ in case $h \gneqq 0$ and, in case $h \lneqq 0$, the existence of $\overline{\lambda} > 0$ such that (P_λ) has at least two negative solutions, at least one negative solution or no non-positive solution according to $0 < \lambda < \overline{\lambda}$, $\lambda = \overline{\lambda}$ or $\lambda > \overline{\lambda}$.

In case $h \gneqq 0$, we know that for $\lambda > \overline{\lambda}$, (P_λ) has no non-negative solution but this does not exclude the possibility of having negative or sign changing solutions. Actually, we are able to prove the following result changing a little the point of view. We consider the boundary value problem

$$-\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu|\nabla u|^p + kh(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{$P_{\lambda,k}$}$$

with a dependence in the size of h and we obtain the following result.

Theorem 1.1.7. *Assume that (A_2) holds and that $h \gneqq 0$. Let*

$$k_0 = \sup\Big\{k \in [0,+\infty) : \forall\, w \in W_0^{1,p}(\Omega),\ \int_\Omega \Big(|\nabla w|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1} k\, h(x)|w|^p\Big)dx > 0\Big\}.$$

Then:

- *For all $\lambda \in (0,\gamma_1)$, there exists $\overline{k} = \overline{k}(\lambda) \in (0,k_0)$ such that, for all $k \in (0,\overline{k})$, the problem $(P_{\lambda,k})$ has at least two solutions u_1, u_2 with $u_i \gg 0$ and for all $k > \overline{k}$, the problem $(P_{\lambda,k})$ has no solution. Moreover, the function $\overline{k}(\lambda)$ is non-increasing.*

- *For $\lambda = \gamma_1$, the problem $(P_{\lambda,k})$ has a solution if and only if $k = 0$. In that case, the solution is unique and it is equal to 0.*

- *For all $\lambda > \gamma_1$, there exist $0 < \tilde{k}_1 \leq \tilde{k}_2 < +\infty$ such that, for all $k \in (0,\tilde{k}_1)$, the problem $(P_{\lambda,k})$ has at least two solutions with $u_{\lambda,1} \ll 0$ and $\min u_{\lambda,2} < 0$, for all $k > \tilde{k}_2$, the problem $(P_{\lambda,k})$ has no solution and, in case $\tilde{k}_1 < \tilde{k}_2$, for all $k \in (\tilde{k}_1,\tilde{k}_2)$, the problem $(P_{\lambda,k})$ has at least one solution u with $u \not\ll 0$ and $\min u < 0$. Moreover, the function $\widetilde{k_1}(\lambda)$ is non-decreasing.*

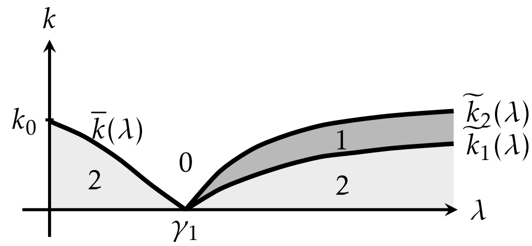

Figure 1.3: Existence regions of Theorem 1.1.7

Let us now say some words about our proofs. First note that when μ is assumed constant it is possible to perform a Hopf-Cole change of variable. Introducing

$$v = \frac{p-1}{\mu}\Big(e^{\frac{\mu}{p-1}u} - 1\Big),$$

we can check that u is a solution to (P_λ) if, and only if, $v > -\frac{p-1}{\mu}$ is a solution to

$$-\Delta_p v = \lambda c(x) g(v) + \Big(1 + \frac{\mu}{p-1} v\Big)^{p-1} h(x), \quad v \in W_0^{1,p}(\Omega), \tag{1.1.3}$$

where g is an arbitrary function satisfying

$$g(s) = \Big|\frac{p-1}{\mu}\Big(1+\frac{\mu}{p-1}s\Big)\ln\Big(1+\frac{\mu}{p-1}s\Big)\Big|^{p-2}\frac{p-1}{\mu}\Big(1+\frac{\mu}{p-1}s\Big)\ln\Big(1+\frac{\mu}{p-1}s\Big), \quad \text{if } s > -\frac{p-1}{\mu}.$$

Working with problem (1.1.3) presents the advantage that one may assume, with a suitable choice of g when $s \le -\frac{p-1}{\mu}$, that it has a variational structure. Nevertheless, from this point we face several difficulties.

First, we need a control from below on the solutions to (1.1.3), i.e. having found a solution to (1.1.3) one needs to check that it satisfies $v > -\frac{p-1}{\mu}$, in order to perform the opposite change of variable and obtain a solution to (P_λ). To that end, in Section 1.4, we prove the existence of a lower solution $\underline{u}_\lambda$ to (P_λ) such that every upper solution β of (P_λ) satisfies $\beta \ge \underline{u}_\lambda$. This allows us to transform the problem (1.1.3) in a new one, which has the advantage of being completely equivalent to (P_λ). Note that the existence of the lower solution ultimately relies on the existence of an a priori lower bound. See Lemma 1.4.1 for a more general result.

We denote by I_λ the functional associated to the new problem, see (1.5.5) for a precise definition. The "geometry" of I_λ crucially depends on the sign of λ. When $\lambda \le 0$ is it essentially coercive and one may search for a critical point as a global minimum. When $\lambda > 0$ the functional I_λ becomes unbounded from below and presents something like a concave-convex geometry. Then, in trying to obtain a critical point, the fact that g is only slightly superlinear at infinity is a difficulty. It implies that I_λ does not satisfies an Ambrosetti-Rabinowitz-type condition and proving that Palais-Smale or Cerami sequences are bounded may be challenging. In the case of the Laplacian, when $p = 2$, dealing with this issue is now relatively standard but for elliptic problems with a p-Laplacian things are more complex and we refer to [51, 72, 73, 86] in that direction. Note however that in these last works, it is always assumed a kind of homogeneity condition which is not available here. Consequently, some new ideas are required, see Section 1.8.

Having at hand the Cerami condition for I_λ with $\lambda > 0$, in order to prove Theorems 1.1.5, 1.1.6 and 1.1.7, we shall look for critical points which are either local-minimum or of mountain-pass type. In Theorem 1.1.5 the geometry of I_λ is "simple" and permits to use only variational arguments. In Theorems 1.1.6 and 1.1.7 however it is not so clear, looking directly to I_λ, where to search for critical points. We shall then make uses of lower and upper solutions arguments. In both theorems a first solution is obtained through the existence of well-ordered lower and upper solutions. This solution is further proved to be a local minimum of I_λ and it is then possible to obtain a second solution by a mountain pass argument. Our approach here follows the strategy presented in [33, 34, 53]. See also [12].

Finally, concerning Theorem 1.1.1, where μ is not assumed to be constant, we obtain our solution through the existence of lower and an upper solution which correspond to solutions to (P_λ) where $\mu = -\|\mu^-\|_\infty$ and $\mu = \|\mu^+\|_\infty$ respectively, see Section 1.6.

The chapter is organized as follows. In Section 1.2, we recall preliminary general results that are used in the rest of the chapter. In Section 1.3, we give a comparison principle and prove the uniqueness result for $\lambda \le 0$. Section 1.4 is devoted to the existence of the lower solution. In Section 1.5, we construct the modified problem that we use to obtain the existence results. The coercive and limit-coercive cases, corresponding to $\lambda \le 0$ are studied in Section 1.6 where we prove Theorem 1.1.1. Theorem 1.1.3 which gives a necessary and sufficient condition to the existence of a solution to (P_0) is established in Section 1.7. In Section 1.8 we show that I_λ has, for $\lambda > 0$ small, a mountain pass geometry and that the Cerami compactness condition holds. This permits to give the proof of Theorem 1.1.5. Section 1.9 contains the proofs of Theorems 1.1.6 and 1.1.7. Finally in an Appendix we give conditions on h^+ that ensure that $m_p > 0$.

Notation.

1) For $p \in [1, +\infty[$, the norm $(\int_\Omega |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. We denote by p' the conjugate exponent of p, namely $p' = p/(p-1)$ and by p^* the Sobolev critical exponent i.e. $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ in case $p \ge N$. The norm in $L^\infty(\Omega)$ is $\|u\|_\infty = \operatorname{esssup}_{x\in\Omega}|u(x)|$.
2) For $v \in L^1(\Omega)$ we define $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$.
3) The space $W_0^{1,p}(\Omega)$ is equipped with the norm $\|u\| := \left(\int_\Omega |\nabla u|^p\, dx\right)^{1/p}$.
4) We denote $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0)$.
5) For a, $b \in L^1(\Omega)$ we denote $\{a \le b\} = \{x \in \Omega : a(x) \le b(x)\}$.

1.2 Preliminaries

In this section we present some definitions and known results which are going to play an important role throughout all the chapter. First of all, we present some results on lower and upper solutions adapted to our setting. Let us consider the problem

$$-\Delta_p u + H(x, u, \nabla u) = f(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{1.2.1}$$

where f belongs to $L^1(\Omega)$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function.

Definition 1.2.1. We say that $\alpha \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a *lower solution to* (1.2.1) if $\alpha^+ \in W_0^{1,p}(\Omega)$ and, for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$, if follows that

$$\int_\Omega |\nabla \alpha|^{p-2} \nabla \alpha \nabla \varphi \, dx + \int_\Omega H(x, \alpha, \nabla \alpha) \varphi \, dx \leq \int_\Omega f(x) \varphi \, dx.$$

Similarly, $\beta \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is an *upper solution to* (1.2.1) if $\beta^- \in W_0^{1,p}(\Omega)$ and, for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$, if follows that

$$\int_\Omega |\nabla \beta|^{p-2} \nabla \beta \nabla \varphi \, dx + \int_\Omega H(x, \beta, \nabla \beta) \varphi \, dx \geq \int_\Omega f(x) \varphi \, dx.$$

Theorem 1.2.1. [24, Theorems 3.1 and 4.2] *Assume the existence of a non-decreasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $k \in L^1(\Omega)$ such that*

$$|H(x, s, \xi)| \leq b(|s|)[k(x) + |\xi|^p], \quad \text{a.e. } x \in \Omega, \ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

If there exist a lower solution α and an upper solution β of (1.2.1) *with $\alpha \leq \beta$, then there exists a solution u of* (1.2.1) *with $\alpha \leq u \leq \beta$. Moreover, there exists u_{min} (resp. u_{max}) minimum (resp. maximum) solution to* (1.2.1) *with $\alpha \leq u_{min} \leq u_{max} \leq \beta$ and such that, every solution u of* (1.2.1) *with $\alpha \leq u \leq \beta$ satisfies $u_{min} \leq u \leq u_{max}$.*

Next, we state the strong comparison principle for the p-Laplacian.

Theorem 1.2.2. [88, Theorem 1.3] [40, Proposition 2.4] *Assume that $\partial\Omega$ is of class $\mathcal{C}^2$ and let f_1, $f_2 \in L^\infty(\Omega)$ with $f_2 \gneqq f_1 \geq 0$. If u_1, $u_2 \in \mathcal{C}_0^{1,\tau}(\overline{\Omega})$, $0 < \tau \leq 1$, are respectively solution to*

$$-\Delta_p u_i = f_i, \quad \text{in } \Omega, \ \text{ for } i = 1, 2, \tag{P_i}$$

such that $u_2 = u_1 = 0$ on $\partial\Omega$. Then $u_2 \gg u_1$.

We need also the following anti-maximum principle.

Proposition 1.2.3. [67, Theorem 5.1] *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded domain with $\partial\Omega$ of class $\mathcal{C}^{1,1}$, c, $\bar{h} \in L^\infty(\Omega)$, γ_1 the first eigenvalue of* (1.1.2). *If $\bar{h} \gneqq 0$, then there exists $\delta_0 > 0$ such that, for all $\lambda \in (\gamma_1, \gamma_1 + \delta_0)$, every solution w of*

$$-\Delta_p w = \lambda c(x) |w|^{p-2} w + \bar{h}(x), \quad u \in W_0^{1,p}(\Omega) \tag{1.2.2}$$

satisfies $w \ll 0$.

The following result is the well known Picone's inequality for the p-Laplacian. We state it for completeness.

Proposition 1.2.4. [11, Theorem 1.1] *Let u, $v \in W^{1,p}(\Omega)$ with $u \geq 0$, $v > 0$ in Ω and $\frac{u}{v} \in L^\infty(\Omega)$. Denote*

$$L(u,v) = |\nabla u|^p + (p-1)\left(\frac{u}{v}\right)^p |\nabla v|^p - p\left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2} \nabla v \nabla u,$$
$$R(u,v) = |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right) |\nabla v|^{p-2} \nabla v.$$

Then, it follows that

- *$L(u,v) = R(u,v) \geq 0$ a.e. in Ω.*
- *$L(u,v) = 0$ a.e. in Ω if, and only if, $u = kv$ for some constant $k \in \mathbb{R}$.*

Now, we consider the boundary value problem

$$-\Delta_p v = g(x,v), \qquad v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{1.2.3}$$

being $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that, for all $s_0 > 0$, there exists $A > 0$, with

$$|g(x,s)| \leq A, \quad \text{a.e. } x \in \Omega, \ \forall\, s \in [-s_0, s_0]. \tag{1.2.4}$$

This problem can be handled variationally. Let us consider the associated functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(v) := \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \int_\Omega G(x,v)\,dx, \quad \text{where} \quad G(x,s) := \int_0^s g(x,t)\,dt.$$

We can state the following result.

Proposition 1.2.5. [52, Proposition 3.1] *Under the assumption (1.2.4), assume that α and β are respectively a lower and an upper solution to (1.2.3) with $\alpha \leq \beta$ and consider*

$$M := \left\{v \in W_0^{1,p}(\Omega) : \alpha \leq v \leq \beta\right\}.$$

Then the infimum of Φ on M is achieved at some v, and such v is a solution to (1.2.3).

Definition 1.2.2. A *lower solution* $\alpha \in \mathcal{C}^1(\overline{\Omega})$ is said to be *strict* if every solution u of (1.2.1) with $u \geq \alpha$ satisfies $u \gg \alpha$.

Similarly, an *upper solution* $\beta \in \mathcal{C}^1(\overline{\Omega})$ is said to be *strict* if every solution u of (1.2.1) such that $u \leq \beta$ satisfies $u \ll \beta$.

Corollary 1.2.6. *Assume that (1.2.4) is valid and that α and β are strict lower and upper solutions to (1.2.3) belonging to $\mathcal{C}^1(\overline{\Omega})$ and satisfying $\alpha \ll \beta$. Then there exists a local minimizer v of the functional Φ in the $\mathcal{C}_0^1$-topology. Furthermore, this minimizer is a solution to (1.2.3) with $\alpha \ll v \ll \beta$.*

Proof. First of all observe that Proposition 1.2.5 implies the existence of $v \in W_0^{1,p}(\Omega)$ solution to (1.2.3), which minimizes Φ on $M := \{v \in W_0^{1,p}(\Omega) : \alpha \leq v \leq \beta\}$. Moreover, since g is an L^∞-Carathéodory function, the classical regularity results (see [56, 87]) imply that $v \in \mathcal{C}^{1,\tau}(\overline{\Omega})$ for some $0 < \tau < 1$. Since the lower and the upper solutions are strict, it follows that $\alpha \ll v \ll \beta$ and so, there is a $\mathcal{C}_0^1$-neighbourhood of v in M. Hence, it follows that v minimizes locally Φ in the $\mathcal{C}_0^1$-topology. □

Proposition 1.2.7. [52, Proposition 3.9] *Assume that g satisfies the following growth condition*

$$|g(x,s)| \leq d\,(1 + |s|^\sigma), \quad \text{a.e. } x \in \Omega,\ \textit{all } s \in \mathbb{R},$$

for some $\sigma \leq p^* - 1$ and some positive constant d. Let $v \in W_0^{1,p}(\Omega)$ be a local minimizer of Φ for the $\mathcal{C}_0^1$-topology. Then $v \in \mathcal{C}_0^{1,\tau}(\overline{\Omega})$ for some $0 < \tau < 1$ and v is a local minimizer of Φ in the $W_0^{1,p}$-topology.

We now recall abstract results in order to find critical points of Φ other than local minima.

Definition 1.2.3. Let $(X, \|\cdot\|)$ be a real Banach space with dual space $(X^*, \|\cdot\|_*)$ and let $\Phi : X \to \mathbb{R}$ be a $\mathcal{C}^1$ functional. The functional Φ satisfies the *Cerami condition at level* $c \in \mathbb{R}$ if, for any *Cerami sequence* at level $c \in \mathbb{R}$, i.e. for any sequence $\{x_n\} \subset X$ with

$$\Phi(x_n) \to c \quad \text{ and } \quad \|\Phi'(x_n)\|_*(1 + \|x_n\|) \to 0,$$

there exists a subsequence $\{x_{n_k}\}$ strongly convergent in X.

Theorem 1.2.8. [61, Corollary 9, Section 1, Chapter IV] *Let $(X, \|\cdot\|)$ be a real Banach space. Suppose that $\Phi : X \to \mathbb{R}$ is a $\mathcal{C}^1$ functional. Take two points $e_1, e_2 \in X$ and define*

$$\Gamma := \{\varphi \in \mathcal{C}([0,1], X) : \varphi(0) = e_1, \varphi(1) = e_2\},$$

and

$$c := \inf_{\varphi \in \Gamma} \max_{t \in [0,1]} \Phi(\varphi(t)).$$

Assume that Φ satisfies the Cerami condition at level c and that

$$c > \max\{\Phi(e_1), \Phi(e_2)\}.$$

Then, there is a critical point of Φ at level c, i.e. there exists $x_0 \in X$ such that $\Phi(x_0) = c$ and $\Phi'(x_0) = 0$.

Theorem 1.2.9. [65, Corollary 1.6] *Let $(X, \|\cdot\|)$ be a real Banach space and let $\Phi : X \to \mathbb{R}$ be a $\mathcal{C}^1$ functional. Suppose that $u_0 \in X$ is a local minimum, i.e. there exists $\varepsilon > 0$ such that*

$$\Phi(u_0) \le \Phi(u), \quad \text{ for } \|u - u_0\| \le \varepsilon,$$

and assume that Φ satisfies the Cerami condition at any level $d \in \mathbb{R}$. Then, the following alternative holds:

i) either there exists $0 < \gamma < \varepsilon$ such that $\inf\{\Phi(u) : \|u - u_0\| = \gamma\} > \Phi(u_0)$,

ii) or, for each $0 < \gamma < \varepsilon$, Φ has a local minimum at a point u_γ with $\|u_\gamma - u_0\| = \gamma$ and $\Phi(u_\gamma) = \Phi(u_0)$.

Remark 1.2.1. In [65], Theorem 1.2.9 is proved assuming the Palais-Smale condition which is stronger than our Cerami condition. Nevertheless, modifying slightly the proof, it is possible to obtain the same result with the Cerami condition.

1.3 Comparison principle and uniqueness results

In this section, we state a comparison principle and, as a consequence, we obtain uniqueness result for (P_λ) with $\lambda \le 0$, proving Theorem 1.1.2. Consider the boundary value problem

$$-\Delta_p u = \mu |\nabla u|^p + f(x,u), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{1.3.1}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \ge 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a } L^1\text{-Carathéodory function with } f(x,s) \le f(x,t) \text{ for a.e. } x \in \Omega \text{ and all } t \le s, \\ \mu > 0. \end{cases} \tag{1.3.2}$$

Remark 1.3.1. As above, the assumption $\mu > 0$ is not a restriction. If $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution to (1.3.1) with $\mu < 0$ then $w = -u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution to

$$-\Delta_p w = -\mu|\nabla w|^p - f(x,-w), \quad w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

with $-f(x,-s)$ satisfying the assumption (1.3.2).

Under a stronger regularity on the solutions, we can prove a comparison principle for (1.3.1). The proof relies on the Picone's inequality (Proposition 1.2.4) and is inspired by some ideas of [6].

Theorem 1.3.1. *Assume that* (1.3.2) *holds. If* $u_1, u_2 \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ *are respectively a lower and an upper solution to* (1.3.1), *then* $u_1 \le u_2$.

Proof. Suppose that u_1, u_2 are respectively a lower and an upper solution to (1.3.1). For simplicity denote $t = \frac{p\mu}{p-1}$ and consider as test function

$$\varphi = \left[e^{tu_1} - e^{tu_2}\right]^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

First of all, observe that

$$\nabla\varphi = t\left[\nabla u_1 e^{tu_1} - \nabla u_2 e^{tu_2}\right]\chi_{\{u_1>u_2\}},$$

with χ_A the characteristic function of the set A. Hence, using assumptions (1.3.2), it follows that

$$\begin{aligned}\int_{\{u_1>u_2\}}\Big(\Big[|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\Big]\Big(t\nabla u_1 e^{tu_1} - t\nabla u_2 e^{tu_2}\Big) - \mu\Big[|\nabla u_1|^p - |\nabla u_2|^p\Big]\Big(e^{tu_1} - e^{tu_2}\Big)\Big)dx \\ \le \int_{\{u_1>u_2\}}\Big(f(x,u_1) - f(x,u_2)\Big)\Big(e^{tu_1} - e^{tu_2}\Big)dx \le 0.\end{aligned}$$

Observe that

$$\begin{aligned}\int_{\{u_1>u_2\}}&\Big[|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\Big]\Big(t\nabla u_1 e^{tu_1} - t\nabla u_2 e^{tu_2}\Big)dx \\ &\qquad - \mu\int_{\{u_1>u_2\}}\Big[|\nabla u_1|^p - |\nabla u_2|^p\Big]\Big(e^{tu_1} - e^{tu_2}\Big)dx \\ &= \int_{\{u_1>u_2\}} e^{tu_1}\Big[|\nabla u_1|^p(t-\mu) + \mu|\nabla u_2|^p - t|\nabla u_2|^{p-2}\nabla u_2\nabla u_1\Big]dx \\ &\qquad + \int_{\{u_1>u_2\}} e^{tu_2}\Big[|\nabla u_2|^p(t-\mu) + \mu|\nabla u_1|^p - t|\nabla u_1|^{p-2}\nabla u_1\nabla u_2\Big]dx.\end{aligned} \tag{1.3.3}$$

Next, as $\nabla e^{tu_i} = t\nabla u_i e^{tu_i}$, $i = 1,2$, we have

$$|\nabla u_i|^p = \frac{|\nabla e^{tu_i}|^p}{t^p e^{tpu_i}} \qquad i = 1,2.$$

Hence, using the above identities, and as $\frac{\mu}{t-\mu} = p-1$ and $\frac{t}{t-\mu} = p$, it follows that,

$$\begin{aligned}&e^{tu_1}\Big[|\nabla u_1|^p(t-\mu) + \mu|\nabla u_2|^p - t|\nabla u_2|^{p-2}\nabla u_2\nabla u_1\Big] \\ &\qquad = \frac{t-\mu}{t^p e^{t(p-1)u_1}}\Big[|\nabla e^{tu_1}|^p + (p-1)\Big(\frac{e^{tu_1}}{e^{tu_2}}\Big)^p|\nabla e^{tu_2}|^p - p\Big(\frac{e^{tu_1}}{e^{tu_2}}\Big)^{p-1}|\nabla e^{tu_2}|^{p-2}\nabla e^{tu_2}\nabla e^{tu_1}\Big], \\ &e^{tu_2}\Big[|\nabla u_2|^p(t-\mu) + \mu|\nabla u_1|^p - t|\nabla u_1|^{p-2}\nabla u_1\nabla u_2\Big] \\ &\qquad = \frac{t-\mu}{t^p e^{t(p-1)u_2}}\Big[|\nabla e^{tu_2}|^p + (p-1)\Big(\frac{e^{tu_2}}{e^{tu_1}}\Big)^p|\nabla e^{tu_1}|^p - p\Big(\frac{e^{tu_2}}{e^{tu_1}}\Big)^{p-1}|\nabla e^{tu_1}|^{p-2}\nabla e^{tu_1}\nabla e^{tu_2}\Big].\end{aligned}$$

Then, by (1.3.3), we have

$$\begin{aligned}&\int_{\{u_1>u_2\}} \frac{t-\mu}{t^p e^{t(p-1)u_1}}\Big[|\nabla e^{tu_1}|^p+(p-1)\Big(\frac{e^{tu_1}}{e^{tu_2}}\Big)^p|\nabla e^{tu_2}|^p-p\Big(\frac{e^{tu_1}}{e^{tu_2}}\Big)^{p-1}|\nabla e^{tu_2}|^{p-2}\nabla e^{tu_2}\nabla e^{tu_1}\Big]dx\\&+\int_{\{u_1>u_2\}} \frac{t-\mu}{t^p e^{t(p-1)u_2}}\Big[|\nabla e^{tu_2}|^p+(p-1)\Big(\frac{e^{tu_2}}{e^{tu_1}}\Big)^p|\nabla e^{tu_1}|^p-p\Big(\frac{e^{tu_2}}{e^{tu_1}}\Big)^{p-1}|\nabla e^{tu_1}|^{p-2}\nabla e^{tu_1}\nabla e^{tu_2}\Big]dx\le 0.\end{aligned} \tag{1.3.4}$$

By Picone's inequality (Proposition 1.2.4), we know that both brackets in (1.3.4) are positive and are equal to zero if and only if $e^{tu_1}=ke^{tu_2}$ for some $k\in\mathbb{R}$. As $t-\mu>0$, thanks to (1.3.4), we deduce the existence of $k\in\mathbb{R}$ such that

$$e^{tu_1}=ke^{tu_2} \quad \text{ in } \{u_1>u_2\}. \tag{1.3.5}$$

Since u_1 and u_2 are continuous on $\overline{\Omega}$ and satisfy $u_1-u_2\le 0$ on $\partial\Omega$, we deduce that $u_1=u_2$ on $\partial\{u_1>u_2\}$. Hence, (1.3.5) applied to $x\in\partial\{u_1>u_2\}$, implies $k=1$. This implies that $u_1=u_2$ in $\{u_1>u_2\}$, which proves $u_1\le u_2$, as desired. □

Corollary 1.3.2. *Assume that (A_1) holds and suppose $\lambda\le 0$. If u_1, $u_2\in W^{1,p}(\Omega)\cap\mathcal{C}(\overline{\Omega})$ are respectively a lower and an upper solution to (P_λ), then $u_1\le u_2$.*

Proof. Define the function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ given by

$$f(x,s)=\lambda c(x)|s|^{p-2}s+h(x).$$

Since (A_1) holds and $\lambda\le 0$, f is a L^1-Carathéodory function which satisfies (1.3.2). Consequently, the proposition follows from Theorem 1.3.1 □

The following result guarantees the regularity that we need to apply the previous comparison principle.

Lemma 1.3.3. *Assume that (A_1) holds and suppose $\lambda\le 0$. Then, any solution to (P_λ) belongs to $\mathcal{C}^{0,\tau}(\overline{\Omega})$.*

Proof. This follows directly from [82, Theorem IX-2.2]. □

Proof of Theorem 1.1.2. The proof is just the combination of Corollary 1.3.2 and Lemma 1.3.3. □

Remark 1.3.2. It is important to note that this comparison and uniqueness results do not hold in general for solution belonging only to $W_0^{1,p}(\Omega)$. See [96, Example 1.1]

Remark 1.3.3. Under the assumption (1.3.2), a comparison principle for the problem

$$-\Delta u=\mu|\nabla u|^p+f(x,u),\quad u\in H_0^1(\Omega)\cap L^\infty(\Omega),$$

with $1\le p\le 2$ is proved in [17, Corollary 3.1]. The following counter-example (see [98, p.7]) shows that there is no hope to obtain a similar result when $p>2$. For $N=2$ and $R>0$, consider the problem on the ball

$$\begin{cases}-\Delta_4 u=|\nabla u|^2 & \text{in } B(0,R),\\ \quad u=0 & \text{on } \partial B(0,R).\end{cases}$$

We easily see that $u_1=0$ and $u_2=\frac{1}{8}(R^2-|x|^2)$ are both solutions to the above problem belonging to $W_0^{1,4}(B(0,R))\cap L^\infty(B(0,R))$.

1.4 A priori lower bound and existence of a lower solution

As explained in the introduction, the aim of this section is to find a lower solution below every upper solution to problem (P_λ). First of all, we show that under a rather mild assumption (in particular no sign on c is required) the solutions to (P_λ) admit a lower bound. Precisely we consider problem (P_λ) assuming now

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}. \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ \mu \in L^\infty(\Omega) \text{ satisfies } 0 < \mu_1 \leq \mu(x) \leq \mu_2. \end{cases} \tag{1.4.1}$$

Adapting the proof of [50, Lemma 3.1], based in turn on ideas of [8], we obtain

Lemma 1.4.1. *Under the assumptions (1.4.1), for any $\lambda \geq 0$, there exists a constant $M_\lambda > 0$ with $M_\lambda := M(N, p, q, |\Omega|, \lambda, \mu_1, \|c^+\|_q, \|h^-\|_q) > 0$ such that, every $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ upper solution to (P_λ) satisfies*

$$\min_{\Omega} u > -M_\lambda.$$

Proof. Let us split the proof in two steps.

Step 1: *There exists a positive constant $M_1 = M_1(p, q, N, |\Omega|, \lambda, \mu_1, \|c^+\|_q, \|h^-\|_q) > 0$ such that $\|u^-\| \leq M_1$.*

First of all, observe that for every function $u \in W^{1,p}(\Omega)$, it follows that

$$\nabla\Big((u^-)^{\frac{p+1}{p}}\Big) = \frac{p+1}{p}(u^-)^{1/p}\nabla u^-, \quad \text{and so,} \quad |\nabla u^-|^p u^- = \Big(\frac{p}{p+1}\Big)^p \Big|\nabla (u^-)^{\frac{p+1}{p}}\Big|^p. \tag{1.4.2}$$

Suppose that $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is an upper solution to (P_λ) and let us consider $\varphi = u^-$ as a test function. Under the assumptions (1.4.1), it follows that

$$\begin{aligned} -\int_\Omega |\nabla u^-|^p dx &\geq -\lambda \int_\Omega c(x)|u^-|^p dx + \int_\Omega \mu(x)|\nabla u^-|^p u^- dx + \int_\Omega h(x) u^- dx \\ &\geq -\lambda \int_\Omega c^+(x)|u^-|^p dx + \mu_1 \int_\Omega |\nabla u^-|^p u^- dx - \int_\Omega h^-(x) u^- dx. \end{aligned} \tag{1.4.3}$$

By (1.4.2) and (1.4.3), we have that

$$\mu_1 \Big(\frac{p}{p+1}\Big)^p \int_\Omega \Big|\nabla (u^-)^{\frac{p+1}{p}}\Big|^p dx + \int_\Omega |\nabla u^-|^p dx \leq \lambda \int_\Omega c^+(x)|u^-|^p dx + \int_\Omega h^-(x) u^- dx. \tag{1.4.4}$$

Firstly, we apply Young's inequality and, for every $\varepsilon > 0$, it follows that

$$\begin{aligned} \int_\Omega c^+(x)|u^-|^p dx &= \int_\Omega (c^+(x))^{1/p}|u^-|^{1/p}(c^+(x))^{\frac{p-1}{p}}|u^-|^{\frac{(p+1)(p-1)}{p}} dx \\ &\leq C(\varepsilon)\int_\Omega c^+(x) u^- dx + \varepsilon \int_\Omega c^+(x)\Big((u^-)^{\frac{p+1}{p}}\Big)^p dx \end{aligned}$$

Moreover, applying Hölder and Sobolev inequalities, observe that

$$\int_\Omega c^+(x)\Big((u^-)^{\frac{p+1}{p}}\Big)^p dx \leq \|c^+\|_q \|(u^-)^{\frac{p+1}{p}}\|^p_{\frac{qp}{q-1}} \leq S\,\|c^+\|_q \|\nabla (u^-)^{\frac{p+1}{p}}\|_p^p$$

with S the constant from the embedding from $W_0^{1,p}(\Omega)$ into $L^{\frac{qp}{q-1}}(\Omega)$. Hence, choosing ε small enough to ensure that $\varepsilon\, S\, \lambda\|c^+\|_q \le \frac{\mu_1}{2}\left(\frac{p}{p+1}\right)^p$ and substituting in (1.4.4), we apply again Hölder and Sobolev inequalities and we find a constant $C = C(\mu_1, \lambda, \|c^+\|_q, p, q, |\Omega|, N)$ such that

$$\frac{\mu_1}{2}\left(\frac{p}{p+1}\right)^p \|\nabla(u^-)^{\frac{p+1}{p}}\|_p^p + \|\nabla u^-\|_p^p \le \Big(\|h^-\|_q + C(\varepsilon)\|c^+\|_q\Big)\|u^-\|_{\frac{q}{q-1}} \le C(\|h^-\|_q + \|c^+\|_q)\|\nabla u^-\|_p .$$

This allows to conclude that

$$\|u^-\| \le \Big(C\Big(\|h^-\|_q + \|c^+\|_q\Big)\Big)^{\frac{1}{p-1}} =: M_1 .$$

Step 2: *Conclusion.*

Since (1.4.1) holds, every $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ upper solution to (P_λ) satisfies

$$-\Delta_p u \ge \lambda c(x)|u|^{p-2}u - h^-(x), \quad \text{in } \Omega . \tag{1.4.5}$$

Moreover, observe that 0 is also an upper solution to (1.4.5). Hence, since the minimum of two upper solution is an upper solution (see [38, Corollary 3.3]), it follows that $\min(u,0)$ is an upper solution to (1.4.5). Furthermore, observe that $\min(u,0)$ is an upper solution to

$$-\Delta_p u \ge \lambda c^+(x)|u|^{p-2}u - h^-(x), \quad \text{in } \Omega .$$

Hence, applying [98, Theorem 6.1.2], we have the existence of $M_2 = M_2(N, p, \lambda, |\Omega|, \|c^+\|_q) > 0$ and $M_3 = M_3(N, p, \lambda, |\Omega|, \|c^+\|_q) > 0$ such that

$$\sup_\Omega u^- \le M_2\Big[\|u^-\|_p + \|h^-\|_q\Big] \le M_3\Big[\|u^-\| + \|h^-\|_q\Big].$$

Finally, the result follows by Step 1. □

Remark 1.4.1.

a) Observe that the lower bound does not depend on h^+ and c^-. In particular, we have the same lower bound for all $h \ge 0$ and all $c \le 0$.

b) Since c does not have a sign, there is no loss of generality in assuming $\lambda \ge 0$. If we consider $\lambda \le 0$, we recover the same result with M_λ depending on $\|c^-\|_q$ instead of $\|c^+\|_q$.

Proposition 1.4.2. *Under the assumptions* (A_1)*, for any* $\lambda \in \mathbb{R}$*, there exists* $\underline{u}_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *lower solution to* (P_λ) *such that, for every* β *upper solution to* (P_λ)*, we have* $\underline{u}_\lambda \le \min\{0, \beta\}$.

Proof. We need to distinguish in our proof the cases $\lambda \le 0$ and $\lambda \ge 0$. First we assume that $\lambda \le 0$. By Lemma 1.4.1, we have a constant $M > 0$ such that every upper solution β of (P_λ) satisfies $\beta \ge -M$. Let α be the solution to

$$-\Delta_p u = -h^-(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

It is then easy to prove that $\underline{u} = \alpha - M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a lower solution to (P_λ) with $\underline{u} \le -M$. By the choice of M, this implies that $\underline{u} \le \overline{u}$ for every upper solution $\overline{u}$ of (P_λ).

Now, when $\lambda \geq 0$ we first introduce the auxiliary problem

$$\begin{cases} -\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu|\nabla u|^p - h^-(x) - 1, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4.6}$$

Thanks to the previous lemma, there exists $M_\lambda > 0$ such that, for every $\beta_1 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ upper solution to (1.4.6), we have $\beta_1 \geq -M_\lambda$. Now, for $k > M_\lambda$, we introduce the problem

$$\begin{cases} -\Delta_p u = -\lambda c(x)k^{p-1} - h^-(x) - 1, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4.7}$$

and denote by α_λ its solution. Since $-\lambda c(x)k^{p-1} - h^-(x) - 1 < 0$, the comparison principle (see for instance [93, Lemma A.0.7]) implies that $\alpha_\lambda \leq 0$. Observe that, for every β_1 upper solution to (P_λ), we have that

$$-\Delta_p \beta_1 \geq \lambda c(x)|\beta_1|^{p-2}\beta_1 + \mu|\nabla\beta_1|^p - h^-(x) - 1 \geq -\lambda c(x)k^{p-1} - h^-(x) - 1 = -\Delta_p \alpha_\lambda.$$

Consequently, it follows that

$$\begin{cases} -\Delta_p \beta_1 \geq -\Delta_p \alpha_\lambda, & \text{in } \Omega, \\ \quad \beta_1 \geq \alpha_\lambda = 0, & \text{on } \partial\Omega, \end{cases}$$

and, applying again the comparison principle, that $\beta_1 \geq \alpha_\lambda$.

Now, we introduce the problem

$$\begin{cases} -\Delta_p u = \lambda c(x)|\widetilde{T}_k(u)|^{p-2}\widetilde{T}_k(u) + \mu|\nabla u|^p - h^-(x) - 1, & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4.8}$$

where

$$\widetilde{T}_k(s) = \begin{cases} -k, & \text{if } s \leq -k, \\ s, & \text{if } s > -k. \end{cases}$$

Observe that β_1 and 0 are upper solutions to (1.4.8). Recalling that the minimum of two upper solution is an upper solution (see [38, Corollary 3.3]), it follows that $\overline{\beta} = \min\{0, \beta_1\}$ is an upper solution to (1.4.8). As α_λ is a lower solution to (1.4.8) with $\alpha_\lambda \leq \overline{\beta}$, applying Theorem 1.2.1, we conclude the existence of $\underline{u}_\lambda$ minimum solution to (1.4.8) with $\alpha_\lambda \leq \underline{u}_\lambda \leq \overline{\beta} = \min\{0, \beta_1\}$.

As, for every upper solution β of (P_λ), β is an upper solution to (1.4.8), we have $\alpha_\lambda \leq \beta$. Recalling that $\underline{u}_\lambda$ is the minimum solution to (1.4.8) with $\alpha_\lambda \leq \underline{u}_\lambda \leq 0$, we deduce that $\underline{u}_\lambda \leq \beta$.

It remains to prove that $\underline{u}_\lambda$ is a lower solution to (P_λ). First, observe that $\underline{u}_\lambda$ is an upper solution to (1.4.6). By construction, this implies that $\underline{u}_\lambda \geq -M_\lambda > -k$. Consequently, $\underline{u}_\lambda$ is a solution to (1.4.6) and so, a lower solution to (P_λ). □

1.5 The Functional setting

Let us introduce some auxiliary functions which are going to play an important role in the rest of the chapter. Define

$$g(s) = \begin{cases} \left|\frac{p-1}{\mu}\left(1 + \frac{\mu}{p-1}s\right)\ln\left(1 + \frac{\mu}{p-1}s\right)\right|^{p-2} \frac{p-1}{\mu}\left(1 + \frac{\mu}{p-1}s\right)\ln\left(1 + \frac{\mu}{p-1}s\right), & s > -\frac{p-1}{\mu}, \\ 0, & s \leq -\frac{p-1}{\mu}, \end{cases} \tag{1.5.1}$$

$$G(s)=\int_0^s g(t)\,dt \qquad \text{and} \qquad H(s)=\frac{1}{p}g(s)s-G(s)\,.$$

In the following lemma we prove some properties of these functions.

Lemma 1.5.1.

i) The function g is continuous on $\mathbb{R}$, satisfies $g>0$ on $\mathbb{R}^+$ and there exists $D>0$ with $-D\le g\le 0$ on $\mathbb{R}^-$. Moreover, $G\ge 0$ on $\mathbb{R}$.

ii) For any $\delta>0$, there exists $\overline{c}=\overline{c}(\delta,\mu,p)>0$ such that, for any $s>\frac{p-1}{\mu}$, $g(s)\le \overline{c}\,s^{p-1+\delta}$.

iii) $\lim_{s\to+\infty} g(s)/s^{p-1}=+\infty$ and $\lim_{s\to+\infty} G(s)/s^p=+\infty$.

iv) There exists $R>0$ such that the function H satisfies $H(s)\le\left(\frac{s}{t}\right)^{p-1}H(t)$, for $R\le s\le t$.

v) The function H is bounded on $\mathbb{R}^-$.

Proof. i) By definition, it is obvious that g is continuous, $g>0$ on $\mathbb{R}^+$ and g is bounded and $g\le 0$ on $\mathbb{R}^-$. This implies also that $G\ge 0$ by integration.

ii) First of all, recall that for any $\varepsilon>0$ there exists $c=c(\varepsilon)>0$ such that $\ln(s)\le c(\varepsilon)s^{\varepsilon}$ for all $s\in(1,\infty)$. This implies that, for any $\delta>0$,

$$\lim_{s\to+\infty}\frac{g(s)}{s^{p-1+\delta}}=\lim_{s\to+\infty}\left(\frac{(p-1)(1+\frac{\mu}{p-1}s)}{\mu s}\right)^{p-1}\frac{\left(\ln(1+\frac{\mu}{p-1}s)\right)^{p-1}}{s^{\delta}}=0\,.$$

Hence, there exists $R>\frac{p-1}{\mu}$ such that, for all $s>R$,

$$\frac{g(s)}{s^{p-1+\delta}}\le 1\,.$$

As the function $\frac{g(s)}{s^{p-1+\delta}}$ is continuous on the compact set $[\frac{p-1}{\mu},R]$, we have a constant $C>0$ with

$$\frac{g(s)}{s^{p-1+\delta}}\le C \quad \text{on } [\tfrac{p-1}{\mu},R].$$

The result follows for $\overline{C}=\max(C,1)$.

iii) As

$$\lim_{s\to+\infty}\frac{\frac{p-1}{\mu}\left(1+\frac{\mu}{p-1}s\right)\ln\left(1+\frac{\mu}{p-1}s\right)}{s}=+\infty\,,$$

and $p>1$, we easily deduce that

$$\lim_{s\to+\infty}\frac{g(s)}{s^{p-1}}=+\infty$$

and, by L'Hospital's rule

$$\lim_{s\to+\infty}\frac{G(s)}{s^{p}}=+\infty\,.$$

iv) First of all, integrating by parts, we observe that, for any $s\ge 0$,

$$G(s)=\left(\frac{p-1}{\mu}\right)^p\left[\frac{1}{p}\left(1+\frac{\mu}{p-1}s\right)^p\left(\ln\left(1+\frac{\mu}{p-1}s\right)\right)^{p-1}-\frac{\mu}{p}\int_0^s\left(1+\frac{\mu}{p-1}t\right)^{p-1}\left(\ln\left(1+\frac{\mu}{p-1}t\right)\right)^{p-2}dt\right],$$

and so, for any $s \geq 0$, it follows that

$$H(s) = \frac{1}{p}\Big(\frac{p-1}{\mu}\Big)^p\Big[\mu\int_0^s\Big(1+\frac{\mu}{p-1}t\Big)^{p-1}\Big(\ln\Big(1+\frac{\mu}{p-1}t\Big)\Big)^{p-2}dt-\Big(1+\frac{\mu}{p-1}s\Big)^{p-1}\Big(\ln\Big(1+\frac{\mu}{p-1}s\Big)\Big)^{p-1}\Big].$$

To prove *iv)*, we show that the function $\varphi(s) := \frac{H(s)}{s^{p-1}}$ is non-decreasing on $[R,+\infty)$ for some $R>0$. Observe that

$$\varphi'(s) = \frac{1}{s^p}[H'(s)s-(p-1)H(s)].$$

Hence, we just need to prove that $H'(s)s-(p-1)H(s) \geq 0$ for $s \geq R$. After some simple computations, we see that it is enough to prove the existence of $R>0$ such that, for all $s \geq R$, $\kappa(s) \geq 0$ where

$$\begin{aligned}\kappa(s) =&\Big(1+\frac{\mu}{p-1}s\Big)^{p-2}\Big(\ln\Big(1+\frac{\mu}{p-1}s\Big)\Big)^{p-2}\Big(\Big(\frac{\mu s}{p-1}\Big)^2+\ln\Big(1+\frac{\mu}{p-1}s\Big)\Big)\\ &-\mu\int_0^s\Big(1+\frac{\mu}{p-1}t\Big)^{p-1}\Big(\ln\Big(1+\frac{\mu}{p-1}t\Big)\Big)^{p-2}dt.\end{aligned}$$

Observe that

$$\begin{aligned}\kappa'(s) =& \frac{\mu}{p-1}\Big(1+\frac{\mu}{p-1}s\Big)^{p-3}\Big(\ln\Big(1+\frac{\mu}{p-1}s\Big)\Big)^{p-3}\\ &\Big[(p-2)\Big(\frac{\mu s}{p-1}-\ln\Big(1+\frac{\mu}{p-1}s\Big)\Big)^2+\Big(\frac{\mu s}{p-1}\Big)^2\ln\Big(1+\frac{\mu}{p-1}s\Big)\Big].\end{aligned}$$

Hence, we distinguish two cases:

i) In case $p \geq 2$, it is obvious that $\kappa'(s) > 0$, for any $s > 0$. This implies that κ is increasing and so, that $\kappa(s) > 0$ for $s > 0$, since $\kappa(0) = 0$.

ii) If $1 < p < 2$, as $\lim_{s\to\infty}\kappa'(s) = +\infty$, there exists $R_1 > 0$ such that, for any $s \geq R_1$, we have $\kappa'(s) > 1$ and hence, there exists $R_2 \geq R_1$ such that $\kappa(s) > 0$, for any $s \geq R_2$.

In any case, we can conclude the existence of $R \geq 0$ such that $\kappa(s) > 0$ for any $s \geq R$. Consequently, there exists $R > 0$ such that $\varphi'(s) > 0$, for $s \geq R$, which means that φ is non-decreasing for $s \geq R$ and hence H satisfies $H(s) \leq \big(\frac{s}{t}\big)^{p-1}H(t)$, for $R \leq s \leq t$.

v) This follows directly from the definition of the functions g and G. □

Next, we define the function

$$\alpha_\lambda = \frac{p-1}{\mu}\Big(e^{\frac{\mu}{p-1}\underline{u}_\lambda}-1\Big) \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \tag{1.5.2}$$

where $\underline{u}_\lambda \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ is the lower solution to (P_λ) obtained in Proposition 1.4.2. Before going further, since $\underline{u}_\lambda \leq 0$, observe that $0 \geq \alpha_\lambda \geq -\frac{p-1}{\mu}+\varepsilon$ for some $\varepsilon > 0$.

Now, for any $\lambda \in \mathbb{R}$, let us consider the auxiliary problem

$$-\Delta_p v = f_\lambda(x,v), \quad v \in W_0^{1,p}(\Omega), \tag{Q_λ}$$

where

$$f_\lambda(x,s) = \begin{cases} \lambda c(x) g(s) + \Big(1 + \dfrac{\mu}{p-1} s\Big)^{p-1} h(x), & \text{if } s \geq \alpha_\lambda(x), \\ \lambda c(x) g(\alpha_\lambda(x)) + \Big(1 + \dfrac{\mu}{p-1} \alpha_\lambda(x)\Big)^{p-1} h(x), & \text{if } s \leq \alpha_\lambda(x), \end{cases} \tag{1.5.3}$$

where g is defined by (1.5.1). In the following lemma, we prove some properties of the solutions to (Q_λ).

Lemma 1.5.2. *Assume that (A_1) holds. Then, it follows that:*

i) Every solution to (Q_λ) belongs to $L^\infty(\Omega)$.

ii) Every solution v of (Q_λ) satisfies $v \geq \alpha_\lambda$.

iii) A function $v \in W_0^{1,p}(\Omega)$ is a solution to (Q_λ) if, and only if, the function

$$u = \frac{p-1}{\mu} \ln\Big(1 + \frac{\mu}{p-1} v\Big) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

is a solution to (P_λ).

Proof. i) This follows directly from [82, Theorem IV-7.1].

ii) First of all, observe that α_λ is a lower solution to (Q_λ). For a solution $v \in W_0^{1,p}(\Omega)$ of (Q_λ), we have $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by the previous step and, for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$,

$$\int_\Omega \Big[|\nabla v|^{p-2} \nabla v - |\nabla \alpha_\lambda|^{p-2} \nabla \alpha_\lambda\Big] \nabla \varphi \, dx \geq \int_\Omega \Big[f_\lambda(x,v) - f_\lambda(x,\alpha_\lambda)\Big] \varphi \, dx.$$

Now, since there exist constants d_1, $d_2 > 0$ such that for all ξ, $\eta \in \mathbb{R}^N$,

$$\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \geq \begin{cases} d_1 (|\xi| + |\eta|)^{p-2} |\xi - \eta|^2, \text{ if } 1 < p < 2, \\ d_2 |\xi - \eta|^p, \text{ if } p \geq 2, \end{cases} \tag{1.5.4}$$

(see for instance [93, Lemma A.0.5]), we choose $\varphi = (\alpha_\lambda - v)^+$ and obtain that

$$0 \geq \int_{\{\alpha_\lambda \geq v\}} \Big[|\nabla v|^{p-2} \nabla v - |\nabla \alpha_\lambda|^{p-2} \nabla \alpha_\lambda\Big] \nabla(\alpha_\lambda - v) \, dx \geq \int_{\{\alpha_\lambda \geq v\}} \Big[f_\lambda(x,v) - f_\lambda(x,\alpha_\lambda)\Big](\alpha_\lambda - v) \, dx = 0.$$

Consequently, using again (1.5.4), we deduce that $\alpha_\lambda = v$ in $\{\alpha_\lambda \geq v\}$ and so, that $v \geq \alpha_\lambda$.

iii) Suppose that $v \in W_0^{1,p}(\Omega)$ is a solution to (Q_λ). The first parts, $i)$, $ii)$ imply that $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is such that $v \geq \alpha_\lambda \geq -\frac{p-1}{\mu} + \varepsilon$ with $\varepsilon > 0$ and hence $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Let us prove that u is a (weak) solution to (P_λ). Let ϕ be an arbitrary function belonging to $\mathcal{C}_0^\infty(\Omega)$ and define $\varphi = \phi/(1 + \frac{\mu}{p-1} v)^{p-1}$. It follows that $\varphi \in W_0^{1,p}(\Omega)$. As $e^{\frac{\mu u}{p-1}} = 1 + \frac{\mu}{p-1} v$, we have the following identity

$$\begin{aligned} \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi \, dx &= \int_\Omega e^{\mu u} |\nabla u|^{p-2} \nabla u \Bigg(\frac{\nabla \phi}{\big(1 + \frac{\mu}{p-1} v\big)^{p-1}} - \frac{\mu \phi \nabla v}{\big(1 + \frac{\mu}{p-1} v\big)^p} \Bigg) dx \\ &= \int_\Omega \frac{e^{\mu u}}{\big(1 + \frac{\mu}{p-1} v\big)^{p-1}} |\nabla u|^{p-2} \nabla u \Bigg(\nabla \phi - \frac{\mu \phi \nabla \Big(\frac{p-1}{\mu}\big(e^{\frac{\mu}{p-1} u} - 1\big)\Big)}{1 + \frac{\mu}{p-1} v} \Bigg) dx \\ &= \int_\Omega |\nabla u|^{p-2} \nabla u \big(\nabla \phi - \mu \phi \nabla u\big) \, dx = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx - \mu \int_\Omega |\nabla u|^p \phi \, dx. \end{aligned}$$

On the other hand, by definition of g, observe that

$$\begin{aligned}\int_\Omega \Big[\lambda c(x)g(v)+&\Big(1+\frac{\mu}{p-1}v\Big)^{p-1}h(x)\Big]\varphi\,dx\\ &=\int_\Omega\Big[\lambda c(x)\Big|\frac{p-1}{\mu}\ln\Big(1+\frac{\mu}{p-1}v\Big)\Big|^{p-2}\Big(\frac{p-1}{\mu}\ln\Big(1+\frac{\mu}{p-1}v\Big)\Big)+h(x)\Big]\phi\,dx\\ &=\int_\Omega\Big[\lambda c(x)|u|^{p-2}u+h(x)\Big]\phi\,dx.\end{aligned}$$

As v is a solution to (Q_λ) we deduce from these two identities that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi\;dx=\int_\Omega\Big[\lambda c(x)|u|^{p-2}u+\mu|\nabla u|^p+h(x)\Big]\phi\,dx,$$

and so, u is a solution to (P_λ), as desired.

On the same way, assume that $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ is a solution to (P_λ). By Proposition 1.4.2 we know that $u\geq \underline{u}_\lambda$. Hence, it follows that $v=\frac{p-1}{\mu}\big(e^{\frac{\mu u}{p-1}}-1\big)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and satisfies $v\geq\alpha_\lambda\geq-\frac{p-1}{\mu}+\varepsilon$ for some $\varepsilon>0$. Arguing exactly as before, we deduce that v is a solution to (Q_λ). □

Remark 1.5.1. Arguing exactly as in the proof of Lemma 1.5.2, iii), we can show that $v_1\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ (respectively $v_2\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$) is a lower solution (respectively an upper solution) of (Q_λ) if, and only if, the function

$$u_1=\frac{p-1}{\mu}\ln\Big(1+\frac{\mu}{p-1}v_1\Big)\qquad\Big(\text{respectively } u_2=\frac{p-1}{\mu}\ln\Big(1+\frac{\mu}{p-1}v_2\Big)\Big)$$

is a lower solution (respectively an upper solution) of (P_λ).

The interest of problem (Q_λ) comes from the fact that it has a variational formulation. We can obtain the solutions to (Q_λ) as critical points of the functional $I_\lambda: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined as

$$I_\lambda(v)=\frac{1}{p}\int_\Omega|\nabla v|^p\,dx-\int_\Omega F_\lambda(x,v)dx, \tag{1.5.5}$$

where we define

$$F_\lambda(x,s)=\lambda c(x)G(s)+\frac{p-1}{\mu p}\Big(1+\frac{\mu}{p-1}s\Big)^p h(x),\quad \text{if } s\geq\alpha_\lambda(x), \tag{1.5.6}$$

and

$$\begin{aligned}F_\lambda(x,s)=&\Big[\lambda c(x)g(\alpha_\lambda(x))+\Big(1+\frac{\mu}{p-1}\alpha_\lambda(x)\Big)^{p-1}h(x)\Big](s-\alpha_\lambda)\\ &+\lambda c(x)G(\alpha_\lambda(x))+\frac{p-1}{\mu p}\Big(1+\frac{\mu}{p-1}\alpha_\lambda(x)\Big)^p h(x),\quad\text{if } s\leq\alpha_\lambda(x).\end{aligned}\tag{1.5.7}$$

Observe that under the assumptions (A_1), since g has subcritical growth (see Lemma 1.5.1), $I\in\mathcal{C}^1(W_0^{1,p}(\Omega),\mathbb{R})$ (see for example [57] page 356).

Lemma 1.5.3. *Assume that* (A_1) *holds and let* $\lambda\in\mathbb{R}$ *be arbitrary, Then, any bounded Cerami sequence for* I_λ *admits a convergent subsequence.*

Proof. Let $\{v_n\}\subset W_0^{1,p}(\Omega)$ be a bounded Cerami sequence for I_λ at level $d\in\mathbb{R}$. We are going to show that, up to a subsequence, $v_n\to v\in W_0^{1,p}(\Omega)$ for a $v\in W_0^{1,p}(\Omega)$.

Since $\{v_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$, up to a subsequence, we can assume that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, $v_n \to v$ in $L^r(\Omega)$, for $1 \le r < p^*$, and $v_n \to v$ a.e. in Ω. First of all, recall that $\langle I'_\lambda(v_n), v_n - v\rangle \to 0$ with

$$\langle I'_\lambda(v_n), v_n - v\rangle = \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx - \int_{\{v_n \ge \alpha_\lambda\}} \lambda c(x) g(v_n)(v_n - v)\,dx$$
$$- \int_{\{v_n \ge \alpha_\lambda\}} \Big(1 + \frac{\mu}{p-1} v_n\Big)^{p-1} (v_n - v)\,h(x)\,dx - \int_{\{v_n \le \alpha_\lambda\}} f_\lambda(x, \alpha_\lambda(x))(v_n - v)\,dx.$$

Let $0 < \delta < (\frac{p}{N} - \frac{1}{q})p^*$, $r < p^*$ and $s < \frac{p^*}{p-1+\delta}$ such that $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1$. Using Lemma 1.5.1 ii), and the Sobolev embedding as well as Hölder inequality, we have that

$$\begin{aligned}\Big|\lambda \int_{\{v_n \ge \alpha_\lambda\}} c(x) g(v_n)(v_n - v)\,dx\Big| &\le |\lambda| \int_\Omega |c(x)||g(v_n)||v_n - v|\,dx \le |\lambda|\|c\|_q \|g(v_n)\|_s \|v_n - v\|_r \\ &\le D|\lambda|\|c\|_q \Big(1 + \|v_n\|^{p-1+\delta}_{(p-1+\delta)s}\Big)\|v_n - v\|_r \\ &\le DS|\lambda|\|c\|_q \Big(1 + \|v_n\|^{p-1+\delta}\Big)\|v_n - v\|_r\,.\end{aligned}$$

Since $\|v_n\|$ is bounded and $v_n \to v$ in $L^r(\Omega)$, for $1 \le r < p^*$, we obtain

$$\lambda \int_{\{v_n \ge \alpha_\lambda\}} c(x) g(v_n)(v_n - v)\,dx \to 0\,.$$

Arguing in the same way, we have

$$\int_{\{v_n \ge \alpha_\lambda\}} \Big(1 + \frac{\mu}{p-1} v_n\Big)^{p-1} (v_n - v)\,h(x)\,dx + \int_{\{v_n \le \alpha_\lambda\}} f_\lambda(x, \alpha_\lambda(x))(v_n - v)\,dx \to 0\,.$$

So, we deduce that

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx \to 0\,. \tag{1.5.8}$$

Hence, applying [57, Theorem 10], we conclude that $v_n \to v$ in $W_0^{1,p}(\Omega)$, as desired. □

1.6 Sharp existence results on the limit coercive case

In this section, following ideas from [18, Section 3], we prove Theorem 1.1.1. As a preliminary step, considering $\mu > 0$ constant, we introduce

$$m_{p,\lambda} := \begin{cases} \displaystyle\inf_{u \in W_\lambda} \int_\Omega \Big(|\nabla u|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1} h(x)|u|^p \Big) dx\,, & \text{if} \quad W_\lambda \ne \emptyset, \\ +\infty\,, & \text{if} \quad W_\lambda = \emptyset. \end{cases}$$

where

$$W_\lambda := \{w \in W_0^{1,p}(\Omega) : \lambda c(x) w(x) = 0 \text{ a.e. } x \in \Omega\,,\ \|w\| = 1\}$$

and we define

$$m := \inf_{u \in W_0^{1,p}(\Omega)} I_\lambda(u) \in \mathbb{R} \cup \{-\infty\}.$$

Proposition 1.6.1. *Assume that* (A_1) *holds,* $\lambda \leq 0$ *and that* $m_{p,\lambda} > 0$. *Then* m *is finite and it is reached by a function* $v \in W_0^{1,p}(\Omega)$. *Consequently the problem* (P_λ) *has a solution.*

Proof. To prove that I_λ has a global minimum since, by Lemma 1.5.3, any bounded Cerami sequence has a convergent subsequence it suffices to show that I_λ is coercive. Having found a global minimum $v \in W_0^{1,p}(\Omega)$ we deduce, by Lemma 1.5.2, that $u = \frac{p-1}{\mu}\ln\left(1 + \frac{\mu}{p-1}v\right)$ is a solution to (P_λ). To show that I_λ is coercive we consider an arbitrary sequence $\{v_n\} \subset W_0^{1,p}(\Omega)$ such that $\|v_n\| \to \infty$ and we prove that

$$\lim_{n\to\infty} I_\lambda(v_n) = +\infty.$$

Assume by contradiction that, along a subsequence, $I_\lambda(v_n)$ is bounded from above and hence

$$\limsup_{n\to\infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} \leq 0. \tag{1.6.1}$$

We introduce the sequence $w_n = \frac{v_n}{\|v_n\|}$, for all $n \in \mathbb{N}$ and observe that, up to a subsequence $w_n \rightharpoonup w$ weakly in $W_0^{1,p}(\Omega)$, $w_n \to w$ in $L^r(\Omega)$, for $1 \leq r < p^*$, and $w_n \to w$ a.e. in Ω. We consider two cases:

Case 1): $w^+ \notin W_\lambda$. In that case, the set $\Omega_0 = \{x \in \Omega : \lambda c(x) w^+(x) \neq 0\} \subset \Omega$ has non-zero measure and so, it follows that $v_n(x) = w_n(x)\|v_n\| \to \infty$ a.e. in Ω_0. Hence, taking into account that $G \geq 0$ and $\lim_{s\to+\infty} G(s)/s^p = +\infty$ (see Lemma 1.5.1) and using Fatou's Lemma, we have

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \frac{\lambda c(x) G(v_n)}{|v_n|^p}|w_n|^p\,dx &\leq \limsup_{n\to\infty} \int_{\Omega_0} \frac{\lambda c(x) G(v_n)}{|v_n|^p}|w_n|^p\,dx \\ &\leq \int_{\Omega_0} \limsup_{n\to\infty} \frac{\lambda c(x) G(v_n)}{|v_n|^p}|w_n|^p\,dx = -\infty.\end{aligned} \tag{1.6.2}$$

On the other hand, observe that for any $v \in W_0^{1,p}(\Omega)$, we can rewrite

$$\begin{aligned}I_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \int_\Omega \lambda c(x) G(v)\,dx + \int_{\{v\leq\alpha_\lambda\}} \lambda c(x) G(v)\,dx \\ - \frac{p-1}{p\mu}\int_{\{v\geq\alpha_\lambda\}}\left(1 + \frac{\mu}{p-1}v\right)^p h(x)\,dx - \int_{\{v\leq\alpha_\lambda\}} F_\lambda(x,v)\,dx.\end{aligned}$$

Hence, considering together (1.6.1) and (1.6.2), we obtain

$$0 \geq \limsup_{n\to\infty}\frac{I_\lambda(v_n)}{\|v_n\|^p} \geq \liminf_{n\to\infty}\frac{I_\lambda(v_n)}{\|v_n\|^p} \geq -C - \limsup_{n\to\infty}\int_\Omega \frac{\lambda c(x) G(v_n)}{\|v_n\|^p}\,dx = +\infty,$$

and so, *Case 1)* cannot occur.

Case 2): $w^+ \in W_\lambda$. First of all, since $\lambda c \leq 0$ and $G \geq 0$ (see Lemma 1.5.1), observe that for any $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}I_\lambda(v) \geq \frac{1}{p}\int_\Omega \left(|\nabla v|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x)(v^+)^p\right)dx - \frac{1}{p}\left(\frac{\mu}{p-1}\right)^{p-1}\int_{\{v\geq\alpha_\lambda\}} h(x)(v^-)^p\,dx \\ - \frac{p-1}{\mu p}\int_{\{v\geq\alpha_\lambda\}}\left[\left(1 + \frac{\mu}{p-1}v\right)^p - \left(\frac{\mu}{p-1}\right)^p |v|^p\right] h(x)\,dx - \int_{\{v\leq\alpha_\lambda\}} F_\lambda(x,v)\,dx.\end{aligned}$$

Moreover, observe that

$$\begin{aligned}\frac{1}{p}\Big|\int_{\{v\geq\alpha_\lambda\}}\Big[\Big(1+\frac{\mu}{p-1}v\Big)^p-\Big(\frac{\mu}{p-1}\Big)^p|v|^p\Big]h(x)\,dx\Big|&\\ &=\Big|\int_{\{v\geq\alpha_\lambda\}}\Big(\int_0^1\Big|s+\frac{\mu}{p-1}v\Big|^{p-2}\Big(s+\frac{\mu}{p-1}v\Big)\,ds\Big)h(x)\,dx\Big| \qquad (1.6.3)\\ &\leq\int_\Omega\Big(1+\frac{\mu}{p-1}|v|\Big)^{p-1}|h(x)|\,dx\leq D\|h\|_q\Big(1+\|v\|^{p-1}\Big),\end{aligned}$$

for some constant $D>0$. Thus, for any $v\in W_0^{1,p}(\Omega)$, it follows that

$$\begin{aligned}I_\lambda(v)\geq\frac{1}{p}\int_\Omega\Big(|\nabla v|^p-\Big(\frac{\mu}{p-1}\Big)^{p-1}h(x)(v^+)^p\Big)dx-\frac{1}{p}\Big(\frac{\mu}{p-1}\Big)^{p-1}\int_{\{v\geq\alpha_\lambda\}}h(x)(v^-)^p\,dx&\\ -D\|h\|_q\Big(1+\|v\|^{p-1}\Big)-\int_{\{v\leq\alpha_\lambda\}}F_\lambda(x,v)\,dx.&\end{aligned}\tag{1.6.4}$$

Hence, using that by the definition of F_λ (see (1.5.6) and (1.5.7)) there exists $m\in L^q(\Omega)$, $q>\max\{N/p,1\}$, such that, for a.e. $x\in\Omega$ and all $s\leq 0$,

$$|F_\lambda(x,s)|\leq m(x)(1+|s|),\tag{1.6.5}$$

and applying (1.6.1) and (1.6.4), we deduce, as $w^+\in W_\lambda$, that

$$0\geq\limsup_{n\to\infty}\frac{I_\lambda(v_n)}{\|v_n\|^p}\geq\liminf_{n\to\infty}\frac{I_\lambda(v_n)}{\|v_n\|^p}\geq\frac{1}{p}\int_\Omega\Big(|\nabla w|^p-\Big(\frac{\mu}{p-1}\Big)^{p-1}h(x)(w^+)^p\Big)dx\geq\frac{1}{p}\min\{1,m_{p,\lambda}\}\|w\|^p\geq 0,$$

and so, that

$$\lim_{n\to\infty}\frac{I_\lambda(v_n)}{\|v_n\|^p}=0\qquad\text{and}\qquad w\equiv 0.$$

Finally, taking into account that $w_n\to 0$ in $L^r(\Omega)$, for $1\leq r<p^*$, we obtain the contradiction

$$0=\lim_{n\to\infty}\frac{I_\lambda(v_n)}{\|v_n\|^p}\geq\frac{1}{p}.$$

Hence, *Case 2)* cannot occur. □

Proof of Theorem 1.1.1. To prove this result, we look for a couple of lower and upper solutions (α,β) of (P_λ) with $\alpha\leq\beta$ and then we apply Theorem 1.2.1. First, assume that both $\|\mu^+\|_\infty>0$ and $\|\mu^-\|_\infty>0$. Observe that any solution to

$$-\Delta_p u=\lambda c(x)|u|^{p-2}u+\|\mu^+\|_\infty|\nabla u|^p+h(x),\quad u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\tag{1.6.6}$$

is an upper solution to (P_λ) and, any solution to

$$-\Delta_p u=\lambda c(x)|u|^{p-2}u-\|\mu^-\|_\infty|\nabla u|^p+h(x),\quad u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\tag{1.6.7}$$

is a lower solution to (P_λ). Now, since $m_{p,\lambda}^+>0$, Proposition 1.6.1 ensures the existence of $\beta\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution to (1.6.6). In the same way, $m_{p,\lambda}^->0$ implies the existence of $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution to

$$-\Delta_p v=\lambda c(x)|v|^{p-2}v+\|\mu^-\|_\infty|\nabla v|^p-h(x),\quad v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),$$

and hence $\alpha = -v$ is a solution to (1.6.7). Moreover, Lemma 1.3.3 implies $\alpha, \beta \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Hence, since α is a lower solution to (1.6.6), it follows that $\alpha \le \beta$, thanks to Theorem 1.3.1. Thus, we can apply Theorem 1.2.1 to conclude the proof. Now note that if $\|\mu^+\|_\infty = 0$, (1.6.6) reduces to

$$-\Delta_p u = \lambda c(x)|u|^{p-2}u + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{1.6.8}$$

which has a solution by [57, Theorem 13]. This solution corresponds again to an upper solution to (P_λ). Similarly, we can justify the existence of the lower solution when $\|\mu^-\|_\infty = 0$. □

1.7 A necessary and sufficient condition

In this section we prove Theorem 1.1.3. First of all, following the ideas of [18], inspired in turn in ideas of [4], we find a necessary condition for the existence of a solution to (P_0). Recall that the problem (P_0) is given by

$$-\Delta_p u = \mu|\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{P_0}$$

Proposition 1.7.1. *Assume that (A_1) holds and suppose that (P_0) has a solution. Then m_p defined by (1.1.1) satisfies $m_p > 0$.*

Proof. Assume that (P_0) has a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, for any $\phi \in \mathcal{C}_0^\infty(\Omega)$, it follows that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(|\phi|^p)\,dx - \mu \int_\Omega |\nabla u|^p |\phi|^p\,dx - \int_\Omega h(x)|\phi|^p\,dx = 0. \tag{1.7.1}$$

Now, applying Young's inequality, observe that

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(|\phi|^p)\,dx &= p\int_\Omega |\phi|^{p-2}\phi|\nabla u|^{p-2}\nabla u\nabla\phi\,dx \le p\int_\Omega |\phi|^{p-1}|\nabla u|^{p-1}|\nabla\phi|\,dx \\ &\le \mu\int_\Omega |\phi|^p|\nabla u|^p\,dx + \left(\frac{p-1}{\mu}\right)^{p-1}\int_\Omega |\nabla\phi|^p\,dx.\end{aligned}$$

Hence, substituting in (1.7.1), multiplying by $\left(\frac{\mu}{p-1}\right)^{p-1}$ and using the density of $\mathcal{C}_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, we obtain

$$\int_\Omega \left(|\nabla\phi|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x)|\phi|^p\right)dx \ge 0, \quad \forall\, \phi \in W_0^{1,p}(\Omega). \tag{1.7.2}$$

Arguing by contradiction, assume that

$$\inf\left\{\int_\Omega \left(|\nabla\phi|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x)|\phi|^p\right)dx : \phi \in W_0^{1,p}(\Omega),\ \|\phi\| = 1\right\} = 0.$$

By standard arguments there exists $\phi_0 \in \mathcal{C}^{0,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$, with $\phi_0 > 0$ in Ω such that

$$\int_\Omega |\nabla\phi_0|^p\,dx = \left(\frac{\mu}{p-1}\right)^{p-1}\int_\Omega h(x)|\phi_0|^p\,dx. \tag{1.7.3}$$

Now, substituting the above identity in (1.7.1) with $\phi = \phi_0$, we have that

$$\int_\Omega \left(|\nabla\phi_0|^p + (p-1)\left(\frac{\mu}{p-1}\right)^p \phi_0^p|\nabla u|^p - p\left(\frac{\mu}{p-1}\right)^{p-1}\phi_0^{p-1}|\nabla u|^{p-2}\nabla u\nabla\phi_0\right)dx = 0. \tag{1.7.4}$$

Finally, observe that

$$\frac{\mu}{p-1}\nabla u = \frac{1}{e^{\frac{\mu}{p-1}u}}\nabla e^{\frac{\mu}{p-1}u}.$$

Hence, by substituting in (1.7.4), we deduce that

$$\int_\Omega \Big(|\nabla\phi_0|^p + (p-1)\Big(\frac{\phi_0}{e^{\frac{\mu}{p-1}u}}\Big)^p|\nabla e^{\frac{\mu}{p-1}u}|^p - p\Big(\frac{\phi_0}{e^{\frac{\mu}{p-1}u}}\Big)^{p-1}|\nabla e^{\frac{\mu}{p-1}u}|^{p-2}\nabla e^{\frac{\mu}{p-1}u}\nabla\phi_0\Big)dx = 0. \tag{1.7.5}$$

Applying Proposition 1.2.4, this proves the existence of $k \in \mathbb{R}$ such that

$$\phi_0 = k e^{\frac{\mu}{p-1}u}.$$

As $\phi_0 = 0$ and $e^{\frac{\mu}{p-1}u} = 1$ on $\partial\Omega$, this implies that $k = 0$ which contradicts the fact that $\phi_0 > 0$ in Ω. □

Proof of Theorem 1.1.3. The proof is just the combination of Proposition 1.7.1 and of Remark 1.1.2. □

Proof of Corollary 1.1.4. We see, combining Theorems 1.1.1 and 1.1.3, that if (P_0) has a solution then (P_λ) has a solution for any $\lambda \le 0$. Moreover this solution is unique by Theorem 1.1.2. □

1.8 On the Cerami conditon and the Mountain-Pass Geometry

We are going to show that, for any $\lambda > 0$, the Cerami sequences for I_λ at any level are bounded. The proof is inspired by [76], see also [74]. Nevertheless it requires to develop some new ideas. In view of Lemma 1.5.3, this will imply that I_λ satisfies the Cerami condition at any level $d \in \mathbb{R}$.

Lemma 1.8.1. *Fixed $\lambda > 0$ arbitrary, assume that (A_1) holds and suppose that $m_p > 0$ with m_p defined by (1.1.1). Then, the Cerami sequences for I_λ at any level $d \in \mathbb{R}$ are bounded.*

Proof. Let $\{v_n\} \subset W_0^{1,p}(\Omega)$ be a Cerami sequence for I_λ at level $d \in \mathbb{R}$. First we claim that $\{v_n^-\}$ is bounded. Indeed since $\{v_n\}$ is a Cerami sequence, we have that

$$\langle I_\lambda'(v_n), v_n^-\rangle = -\int_\Omega |\nabla v_n^-|^p dx - \int_\Omega f_\lambda(x, v_n)v_n^- dx \to 0 \tag{1.8.1}$$

from which, since $f_\lambda(x,s)$ is bounded on $\Omega \times \mathbb{R}^-$, the claim follows. To prove that $\{v_n^+\}$ is also bounded we assume by contradiction that $\|v_n\| \to \infty$. We define

$$\Omega_n^+ = \{x \in \Omega : v_n(x) \ge 0\} \qquad \text{and} \qquad \Omega_n^- = \Omega \setminus \Omega_n^+,$$

and introduce the sequence $\{w_n\} \subset W_0^{1,p}(\Omega)$ given by $w_n = v_n/\|v_n\|$. Observe that $\{w_n\} \subset W_0^{1,p}(\Omega)$ is bounded in $W_0^{1,p}(\Omega)$. Hence, up to a subsequence, it follows that $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$, $w_n \to w$ strongly in $L^r(\Omega)$ for $1 \le r < p^*$, and $w_n \to w$ a.e. in Ω. We split the proof in several steps.

Step 1: $cw \equiv 0$.

As $\|v_n^-\|$ is bounded and by assumption $\|v_n\| \to \infty$, clearly $w^- \equiv 0$. It remains to show that $cw^+ \equiv 0$. Assume by contradiction that $cw^+ \not\equiv 0$ i.e., defining $\Omega^+ := \{x \in \Omega : c(x)w(x) > 0\}$, we assume $|\Omega^+| > 0$. Since $\|v_n\| \to \infty$ and $\langle I'_\lambda(v_n), v_n\rangle \to 0$, it follows that

$$\frac{\langle I'_\lambda(v_n), v_n\rangle}{\|v_n\|^p} \to 0. \tag{1.8.2}$$

First of all, observe that

$$\begin{aligned}\langle I'_\lambda(v_n), v_n\rangle = \|v_n\|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1}\int_{\Omega_n^+} h(x)|v_n|^p\,dx - \int_{\Omega_n^+} \lambda c(x) g(v_n) v_n\,dx - \int_{\Omega_n^-} f_\lambda(x,v_n)\,v_n\,dx\\ - \int_{\Omega_n^+}\Big[\Big(1+\frac{\mu}{p-1}v_n\Big)^{p-1} - \Big(\frac{\mu}{p-1}\Big)^{p-1}|v_n|^{p-2}v_n\Big]v_n h(x)\,dx.\end{aligned} \tag{1.8.3}$$

Now, since $f_\lambda(x,s)$ is bounded on $\Omega\times\mathbb{R}^-$, we deduce that

$$\frac{1}{\|v_n\|^p}\int_{\Omega_n^-} f_\lambda(x,v_n)\,v_n\,dx \to 0. \tag{1.8.4}$$

Moreover, using that $w_n \to w$ in $L^r(\Omega)$, $1 \le r < p^*$, with $w^- \equiv 0$, we have

$$\frac{1}{\|v_n\|^p}\int_{\Omega_n^+} |v_n|^p h(x)\,dx = \int_{\Omega_n^+} |w_n|^p h(x)\,dx \to \int_\Omega w^p h(x)\,dx, \tag{1.8.5}$$

Next, we are going to show that

$$\frac{1}{\|v_n\|^p}\int_{\Omega_n^+}\Big[\Big(1+\frac{\mu}{p-1}v_n\Big)^{p-1} - \Big(\frac{\mu}{p-1}\Big)^{p-1}|v_n|^{p-2}v_n\Big]v_n h(x)\,dx \to 0. \tag{1.8.6}$$

Observe that

$$\int_{\Omega_n^+}\Big[\Big(1+\frac{\mu}{p-1}v_n\Big)^{p-1} - \Big(\frac{\mu}{p-1}\Big)^{p-1}v_n^{p-1}\Big]v_n h(x)\,dx = (p-1)\int_{\Omega_n^+}\Big[\int_0^1\Big(s+\frac{\mu}{p-1}v_n\Big)^{p-2}ds\Big]v_n h(x)\,dx.$$

We consider separately the case $p \ge 2$ and the case $1 < p < 2$. In case $p \ge 2$, there exists $D > 0$ such that

$$\Big|\int_{\Omega_n^+}\Big[\int_0^1\Big(s+\frac{\mu}{p-1}v_n\Big)^{p-2}ds\Big]v_n h(x)\,dx\Big| \le \frac{p-1}{\mu}\int_{\Omega_n^+}\Big(1+\frac{\mu}{p-1}v_n\Big)^{p-1}|h(x)|\,dx \le D\|h\|_q(1+\|v_n\|^{p-1}).$$

On the other hand, in case $1 < p < 2$, we have a constant $D > 0$ with

$$\Big|\int_{\Omega_n^+}\Big[\int_0^1\Big(s+\frac{\mu}{p-1}v_n\Big)^{p-2}ds\Big]v_n h(x)\,dx\Big| \le \frac{p-1}{\mu}\int_{\Omega_n^+}\Big(\frac{\mu}{p-1}v_n\Big)^{p-1}|h(x)|\,dx \le D\|h\|_q\|v_n\|^{p-1}.$$

The claim (1.8.6) follows then directly from the above inequalities. So, substituting (1.8.3), (1.8.4), (1.8.5) and (1.8.6) in (1.8.2) and using that g is bounded on $\mathbb{R}^-$, we deduce that

$$\lambda\int_\Omega c(x)\frac{g(v_n)}{v_n^{p-1}}w_n^p\,dx \to 1 - \Big(\frac{\mu}{p-1}\Big)^{p-1}\int_\Omega w^p h(x)\,dx. \tag{1.8.7}$$

Let us prove that this is a contradiction. By Lemma 1.5.1, we know that $\lim_{s\to+\infty} g(s)/s^{p-1} = +\infty$ and as $w_n \to w > 0$ a.e. in Ω^+, it follows that

$$c(x)\frac{g(v_n)}{v_n^{p-1}} w_n^p \to +\infty \text{ a.e. in } \Omega^+.$$

Since $|\Omega^+| > 0$, we have

$$\int_{\Omega^+} c(x)\frac{g(v_n)}{v_n^{p-1}} w_n^p dx \to +\infty. \tag{1.8.8}$$

On the other hand, as $g \geq 0$ on $\mathbb{R}^+$, $\frac{g(s)}{s^{p-1}}$ is bounded on $\mathbb{R}^-$ and $\|w_n\|_{pq'}$ is bounded, we have

$$\int_{\Omega\setminus\Omega^+} c(x)\frac{g(v_n)}{v_n^{p-1}} w_n^p \, dx \geq -D. \tag{1.8.9}$$

So (1.8.8) and (1.8.9) together give a contradiction with (1.8.7). Consequently, we conclude that $cw \equiv 0$.

Step 2: *Let us introduce a new functional $J_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as*

$$J_\lambda(v) = I_\lambda(v) - \frac{p-1}{\mu p}\int_{\{v\geq\alpha_\lambda\}}\Big[\Big(1+\frac{\mu}{p-1}v\Big)^p - \Big(\frac{\mu}{p-1}\Big)^p |v|^p\Big] h^-(x)\, dx$$

and let us introduce the sequence $\{z_n\} \subset W_0^{1,p}(\Omega)$ defined by $z_n = t_n v_n$, where $t_n \in [0,1]$ satisfies

$$J_\lambda(z_n) = \max_{t\in[0,1]} J_\lambda(t v_n),$$

(if t_n is not unique we choose its smallest possible value). We claim that

$$\lim_{n\to\infty} J_\lambda(z_n) = +\infty.$$

We argue again by contradiction. Suppose the existence of $M < +\infty$ such that

$$\liminf_{n\to\infty} J_\lambda(z_n) \leq M, \tag{1.8.10}$$

and introduce a sequence $\{k_n\} \subset W_0^{1,p}(\Omega)$, defined as

$$k_n = \Big(\frac{2pM}{m_p}\Big)^{\frac{1}{p}} w_n = \Big(\frac{2pM}{m_p}\Big)^{\frac{1}{p}} \frac{v_n}{\|v_n\|}.$$

Let us prove, taking M bigger if necessary, that for n large enough we have

$$J_\lambda(k_n) > \frac{3}{2}M. \tag{1.8.11}$$

As $\Big(\frac{2pM}{m_p}\Big)^{\frac{1}{p}} \frac{1}{\|v_n\|} \in [0,1]$ for n large enough, this will give the contradiction

$$\frac{3}{2}M \leq \liminf_{n\to\infty} J_\lambda(k_n) \leq \liminf_{n\to\infty} J_\lambda(z_n) \leq M.$$

First of all, observe that $k_n \rightharpoonup k := \left(\frac{2pM}{m_p}\right)^{\frac{1}{p}} w$ in $W_0^{1,p}(\Omega)$, $k_n \to k$ in $L^r(\Omega)$, for $1 \le r < p^*$, and $k_n \to k$ a.e. in Ω. By the properties of G (see Lemma 1.5.1) together with $k \ge 0$ and $ck \equiv 0$, it is easy to prove that

$$\int_{\{k_n \ge \alpha_\lambda\}} c(x) G(k_n)\, dx \to \int_\Omega c(x) G(k)\, dx = 0. \tag{1.8.12}$$

As $w^- \equiv 0$, we have $\chi_n^- \to 0$ a.e. in Ω where χ_n^- is the characteristic function of Ω_n^-. Recall (see (1.6.5)) that we have $m \in L^q(\Omega)$, $q > \max\{N/p, 1\}$, such that, for a.e. $x \in \Omega$ and all $s \le 0$,

$$|F_\lambda(x,s)| \le m(x)(1 + |s|).$$

This implies that

$$\int_{\{k_n \le \alpha_\lambda\}} F_\lambda(x, k_n)\, dx \to 0, \qquad \text{as well as} \qquad \int_{\{k_n \le \alpha_\lambda\}} |k_n|^p h(x)\, dx \to 0. \tag{1.8.13}$$

Taking into account (1.8.12) and (1.8.13) we obtain that

$$\begin{aligned} J_\lambda(k_n) = \frac{1}{p} \int_\Omega & \left(|\nabla k_n|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x) |k_n|^p \right) dx \\ & - \frac{p-1}{p\mu} \int_{\{k_n \ge \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1} k_n\right)^p - \left(\frac{\mu}{p-1}\right)^p |k_n|^p \right] h^+(x)\, dx + o(1). \end{aligned} \tag{1.8.14}$$

Now, observe that, by definition of m_p,

$$\frac{1}{p} \int_\Omega \left(|\nabla k_n|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x) |k_n|^p \right) dx \ge \frac{1}{p} m_p \|k_n\|^p = 2M. \tag{1.8.15}$$

Furthermore, arguing as in (1.6.3), observe that

$$\frac{1}{p} \left| \int_{\{k_n \ge \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1} k_n\right)^p - \left(\frac{\mu}{p-1}\right)^p |k_n|^p \right] h^+(x)\, dx \right| \le C \|h^+\|_q \left(1 + \left(\frac{2pM}{m_p}\right)^{\frac{p-1}{p}} \right),$$

where C is independent of M. This implies that

$$J_\lambda(k_n) \ge 2M - C \|h^+\|_q \left(1 + \left(\frac{2pM}{m_p}\right)^{\frac{p-1}{p}} \right) + o(1),$$

and, taking M bigger if necessary, for any $n \in \mathbb{N}$ large enough, (1.8.11) follows.

Step 3: For $n \in \mathbb{N}$ large enough, $t_n \in (0,1)$.

By the definition of J_λ and using that

$$\left(1 + \frac{\mu}{p-1} s\right)^p - \left(\frac{\mu}{p-1}\right)^p s^p \ge 0, \quad \forall\, s \ge 0,$$

observe that

$$\begin{aligned} J_\lambda(v_n) &\le I_\lambda(v_n) - \frac{p-1}{p\mu} \int_{\{\alpha_\lambda \le v_n \le 0\}} \left[\left(1 + \frac{\mu}{p-1} v_n\right)^p - \left(\frac{\mu}{p-1}\right)^p |v_n|^p \right] h^-(x)\, dx \\ &\le I_\lambda(v_n) + \frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\{\alpha_\lambda \le v_n \le 0\}} |v_n|^p\, h^-(x) dx \\ &\le I_\lambda(v_n) + \frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \|\alpha_\lambda\|_\infty^p \|h^-\|_1 . \end{aligned}$$

Consequently, since $I_\lambda(v_n) \to d$, there exists $D > 0$ such that, for all $n \in \mathbb{N}$, $J_\lambda(v_n) \le D$. Thus, taking into account that $J_\lambda(0) = -\frac{p-1}{p\mu}\|h^+\|_1$ and $J_\lambda(t_n v_n) \to +\infty$, we conclude that $t_n \in (0,1)$ for n large enough.

Step 4: *Conclusion.*

First of all, as $t_n \in (0,1)$ for n large enough, by the definition of z_n, observe that $\langle J_\lambda'(z_n), z_n\rangle = 0$, for those n. Thus, it follows that

$$\begin{aligned} J_\lambda(z_n) &= J_\lambda(z_n) - \frac{1}{p}\langle J_\lambda'(z_n), z_n\rangle \\ &= \lambda \int_{\{z_n \ge \alpha_\lambda\}} c(x) H(z_n)\, dx - \frac{p-1}{p\mu} \int_{\{z_n \ge \alpha_\lambda\}} \Big(1 + \frac{\mu}{p-1} z_n\Big)^{p-1} h^+(x)\, dx \\ &\qquad - \int_{\{z_n \le \alpha_\lambda\}} \Big[F_\lambda(x, z_n) - \frac{1}{p} f_\lambda(x, z_n) z_n\Big] dx. \end{aligned}$$

Using the definition of $f_\lambda(x,s)$ for $s \le \alpha_\lambda(x)$ and the fact that $\|z_n^-\|$ is bounded, we easily deduce the existence of $D_1 > 0$ such that, for all n large enough,

$$\frac{p-1}{p\mu} \int_\Omega \Big|1 + \frac{\mu}{p-1} z_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} z_n\Big) h^+(x)\, dx \le -J_\lambda(z_n) + \lambda \int_\Omega c(x) H(z_n)\, dx + D_1 . \tag{1.8.16}$$

Now, since $\{v_n\}$ is a Cerami sequence, observe that (again for n large enough)

$$\begin{aligned} d + 1 &\ge I_\lambda(v_n) - \frac{1}{p}\langle I_\lambda'(v_n), v_n\rangle \\ &= \lambda \int_{\{v_n \ge \alpha_\lambda\}} c(x) H(v_n)\, dx - \frac{p-1}{p\mu} \int_{\{v_n \ge \alpha_\lambda\}} \Big(1 + \frac{\mu}{p-1} v_n\Big)^{p-1} h(x)\, dx \\ &\qquad - \int_{\{v_n \le \alpha_\lambda\}} \Big[F_\lambda(x, v_n) - \frac{1}{p} f_\lambda(x, v_n) v_n\Big] dx \end{aligned}$$

and, as above, there exists a constant $D_2 > 0$ such that

$$\lambda \int_\Omega c(x) H(v_n)\, dx \le \frac{p-1}{p\mu} \int_\Omega \Big|1 + \frac{\mu}{p-1} v_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} v_n\Big) h(x)\, dx + D_2. \tag{1.8.17}$$

Moreover, observe that

$$\begin{aligned} &\int_\Omega \Big|1 + \frac{\mu}{p-1} v_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} v_n\Big) h(x)\, dx \\ &\quad = \int_\Omega \Big|1 + \frac{\mu}{p-1} v_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} v_n\Big) h^+(x)\, dx - \int_\Omega \Big|1 + \frac{\mu}{p-1} v_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} v_n\Big) h^-(x)\, dx \\ &\quad = \frac{1}{t_n^{p-1}} \int_\Omega \Big|t_n + \frac{\mu}{p-1} z_n\Big|^{p-2} \Big(t_n + \frac{\mu}{p-1} z_n\Big) h^+(x)\, dx - \int_\Omega \Big|1 + \frac{\mu}{p-1} v_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} v_n\Big) h^-(x)\, dx \\ &\quad \le \frac{1}{t_n^{p-1}} \int_\Omega \Big|1 + \frac{\mu}{p-1} z_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} z_n\Big) h^+(x)\, dx - \int_\Omega \Big|1 + \frac{\mu}{p-1} v_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} v_n\Big) h^-(x)\, dx. \end{aligned}$$

Considering together this inequality with (1.8.16) and (1.8.17), we obtain that

$$\begin{aligned} \lambda \int_\Omega c(x) H(v_n)\, dx \le{}& D_2 - \frac{J_\lambda(z_n)}{t_n^{p-1}} + \frac{\lambda}{t_n^{p-1}} \int_\Omega c(x) H(z_n)\, dx + \frac{D_1}{t_n^{p-1}} \\ &- \frac{p-1}{p\mu} \int_\Omega \Big|1 + \frac{\mu}{p-1} v_n\Big|^{p-2} \Big(1 + \frac{\mu}{p-1} v_n\Big) h^-(x)\, dx. \end{aligned} \tag{1.8.18}$$

Now, since H is bounded on $\mathbb{R}^-$, there exists $D_3 > 0$ such that, for all $n \in \mathbb{N}$,

$$\int_{\Omega_n^-} c(x)H(z_n)\,dx \le D_3. \tag{1.8.19}$$

On the other hand, using $iv)$ of Lemma 1.5.1, it follows that

$$\int_{\Omega_n^+} c(x)H(z_n)\,dx \le t_n^{p-1} \int_{\Omega_n^+} c(x)H(v_n)\,dx + D_4, \tag{1.8.20}$$

for some positive constant D_4. Hence, substituting (1.8.19) and (1.8.20) in (1.8.18), it follows that

$$\lambda \int_{\Omega_n^-} c(x)H(v_n)\,dx \le D_5 - \frac{J_\lambda(z_n) - D_6}{t_n^{p-1}} - \frac{p-1}{p\mu} \int_\Omega \Big|1 + \frac{\mu}{p-1}v_n\Big|^{p-2}\Big(1 + \frac{\mu}{p-1}v_n\Big)h^-(x)\,dx.$$

Arguing as in the previous steps, observe that

$$\int_\Omega \Big|1 + \frac{\mu}{p-1}v_n\Big|^{p-2}\Big(1 + \frac{\mu}{p-1}v_n\Big)h^-(x)\,dx \ge -\int_{\Omega_n^-} \Big|1 + \frac{\mu}{p-1}v_n\Big|^{p-1} h^-(x)\,dx \ge -D_7\|h^-\|_q\Big(1 + \|v_n^-\|^{p-1}\Big),$$

and so, we have that

$$\lambda \int_{\Omega_n^-} c(x)H(v_n)\,dx \le D_5 - \frac{J_\lambda(z_n) - D_6}{t_n^{p-1}} + D_7\|h^-\|_q\Big(1 + \|v_n^-\|^{p-1}\Big).$$

By Step 1, we know that $\|v_n^-\|$ is bounded and Step 4 shows that $J_\lambda(z_n) \to \infty$. Recall also that, by Step 4, $t_n \in (0,1)$. This implies that

$$\lambda \int_{\Omega_n^-} c(x)H(v_n)\,dx \to -\infty \tag{1.8.21}$$

which contradicts the fact that H is bounded on $\mathbb{R}^-$. This allows to conclude that the Cerami sequences for I_λ at level $d \in \mathbb{R}$ are bounded. $\square$

Now, we turn to the verification of the mountain pass geometry when $\lambda \ge 0$ is small.

Lemma 1.8.2. *Assume that* (A_1) *holds and suppose that $m_p > 0$. For $\lambda \ge 0$ small enough, there exists $r > 0$ such that $I_\lambda(v) > I_\lambda(0)$ for $\|v\| = r$.*

Proof. For an arbitrary fixed $r > 0$, let $v \in W_0^{1,p}(\Omega)$ be such that $\|v\| = r$. We can write

$$\begin{aligned}
I_\lambda(v) = &\frac{1}{p}\int_\Omega \Big(|\nabla v|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1}(v^+)^p h(x)\Big)dx - \frac{1}{p}\Big(\frac{\mu}{p-1}\Big)^{p-1}\int_{\{\alpha_\lambda \le v \le 0\}} |v|^p h(x)\,dx \\
&- \frac{p-1}{p\mu}\int_{\{v \ge \alpha_\lambda\}} \Big[\Big(1 + \frac{\mu}{p-1}v\Big)^p - \Big(\frac{\mu}{p-1}\Big)^p |v|^p\Big]h(x)\,dx \\
&- \int_{\{v \le \alpha_\lambda\}} h(x)\Big[\Big(1 + \frac{\mu}{p-1}\alpha_\lambda\Big)^{p-1}(v - \alpha_\lambda) + \frac{p-1}{p\mu}\Big(1 + \frac{\mu}{p-1}\alpha_\lambda\Big)^p\Big]dx \\
&- \lambda\Big(\int_{\{v \ge \alpha_\lambda\}} c(x)G(v)\,dx + \int_{\{v \le \alpha_\lambda\}} c(x)\big[g(\alpha_\lambda)(v - \alpha_\lambda) + G(\alpha_\lambda)\big]dx\Big).
\end{aligned}$$

Now, observe that, as above,

$$\left|\int_{\{v\geq\alpha_\lambda\}}\Big[\Big(1+\frac{\mu}{p-1}v\Big)^p-\Big(\frac{\mu}{p-1}\Big)^p|v|^p\Big]h(x)\,dx\right|\leq p\int_\Omega\Big(1+\frac{\mu}{p-1}|v|\Big)^{p-1}|h(x)|\,dx\leq D_1(1+r^{p-1}),$$

with D_1 independent of λ. In the same way, using the fact that $\alpha_\lambda\in[-\frac{p-1}{\mu},0]$, we deduce that

$$\Big|-\frac{1}{p}\Big(\frac{\mu}{p-1}\Big)^{p-1}\int_{\{\alpha_\lambda\leq v\leq 0\}}|v|^p h(x)\,dx\Big|\leq D_2,$$
$$\Big|-\int_{\{v\leq\alpha_\lambda\}}h(x)\Big[\Big(1+\frac{\mu}{p-1}\alpha_\lambda\Big)^{p-1}(v-\alpha_\lambda)+\frac{p-1}{p\mu}\Big(1+\frac{\mu}{p-1}\alpha_\lambda\Big)^p\Big]\,dx\Big|\leq D_3+D_4 r,$$

with D_2, D_3 and D_4 independent of λ. Finally, observe that

$$\begin{aligned}\int_\Omega\Big(|\nabla v|^p-\Big(\frac{\mu}{p-1}\Big)^{p-1}(v^+)^p h(x)\Big)\,dx&=\int_\Omega\Big(|\nabla v^+|^p-\Big(\frac{\mu}{p-1}\Big)^{p-1}(v^+)^p h(x)\Big)\,dx+\int_\Omega|\nabla v^-|^p\,dx\\&\geq m_p\|v^+\|^p+\|v^-\|^p\geq\min\{1,m_p\}\|v\|^p=\min\{1,m_p\}\,r^p.\end{aligned}$$

So, we obtain that

$$\begin{aligned}I_\lambda(v)\geq{}&\frac{1}{p}\min\{1,m_p\}\,r^p-D_1r^{p-1}-D_4r-D_5,\\&-\lambda\Big(\int_{\{v\geq\alpha_\lambda\}}c(x)G(v)\,dx+\int_{\{v\leq\alpha_\lambda\}}c(x)\big[g(\alpha_\lambda)(v-\alpha_\lambda)+G(\alpha_\lambda)\big]\,dx\Big),\end{aligned}\tag{1.8.22}$$

where the constants D_i are independent of λ. Moreover, observe that for r large enough,

$$\frac{1}{p}\min\{1,m_p\}\,r^p-D_1r^{p-1}-D_4r-D_5\geq\frac{1}{2p}\min\{1,m_p\}\,r^p+I_\lambda(0).\tag{1.8.23}$$

On the other hand, by Lemma 1.5.1, for every $\delta>0$,

$$\Big|\Big(\int_{\{v\geq\alpha_\lambda\}}c(x)G(v)\,dx+\int_{\{v\leq\alpha_\lambda\}}c(x)\big[g(\alpha_\lambda)(v-\alpha_\lambda)+G(\alpha_\lambda)\big]\,dx\Big)\Big|\leq D_6r^{p+\delta}+D_7r+D_8,\tag{1.8.24}$$

for some constant D_6, D_7, D_8 independent of λ. Hence, for λ small enough, we have

$$\lambda\Big(\int_{\{v\geq\alpha_\lambda\}}c(x)G(v)\,dx+\int_{\{v\leq\alpha_\lambda\}}c(x)\big[g(\alpha_\lambda)(v-\alpha_\lambda)+G(\alpha_\lambda)\big]\,dx\Big)\leq\frac{1}{4p}\min\{1,m_p\}\,r^p,\tag{1.8.25}$$

and so, gathering (1.8.22), (1.8.23) and (1.8.25), we conclude that

$$I_\lambda(v)\geq\frac{1}{4p}\min\{1,m_p\}\,r^p+I_\lambda(0)>I_\lambda(0).$$

□

Lemma 1.8.3. *Assume that (A_1) holds and that $m_p>0$. For any $\lambda>0$, $M>0$, and $r>0$, there exists $w\in W_0^{1,p}(\Omega)$ such that $\|w\|>r$ and $I_\lambda(w)\leq -M$.*

Proof. Consider $v \in C_0^\infty(\Omega)$ such that $v \geq 0$ and $cv \not\equiv 0$ and let us take $t \in \mathbb{R}^+$, $t \geq 1$. First of all, as $\alpha_\lambda \leq 0$, observe that

$$I_\lambda(tv) \leq \frac{1}{p} t^p \int_\Omega \Big(|\nabla v|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1} |v|^p h(x) \Big) dx - \lambda t^p \int_\Omega c(x) v^p \frac{G(tv)}{t^p v^p} dx$$
$$+ \frac{p-1}{p\mu} \int_\Omega \Big[\Big(1 + \frac{\mu}{p-1} tv\Big)^p - \Big(\frac{\mu}{p-1}\Big)^p (tv)^p \Big] h^-(x)\, dx.$$

As above, we have

$$\frac{1}{p} \int_\Omega \Big[\Big(1 + \frac{\mu}{p-1} tv\Big)^p - \Big(\frac{\mu}{p-1}\Big)^p (tv)^p \Big] h^-(x)\, dx \leq t^{p-1} \int_\Omega \Big(1 + \frac{\mu}{p-1} v\Big)^{p-1} h^-(x)\, dx.$$

Hence, we obtain

$$I_\lambda(tv) \leq t^p \Big[\frac{1}{p} \int_\Omega \Big(|\nabla v|^p - \Big(\frac{\mu}{p-1}\Big)^{p-1} |v|^p h(x) \Big) dx - \lambda \int_\Omega c(x) v^p \frac{G(tv)}{t^p v^p} dx + \frac{1}{t} \frac{p-1}{\mu} \Big\| 1 + \frac{\mu}{p-1} v \Big\|_\infty^{p-1} \|h^-\|_1 \Big].$$

Now, since by Lemma 1.5.1, we have

$$\lim_{t\to\infty} \lambda \int_\Omega c(x) v^p \frac{G(tv)}{(tv)^p} dx = \infty,$$

we deduce that $\lim\limits_{t\to\infty} I_\lambda(tv) = -\infty$ from which the lemma follows. □

Proposition 1.8.4. *Assume that* (A_1) *holds and suppose that* $m_p > 0$. *Moreover, suppose that* $\lambda \geq 0$ *is small enough in order to ensure that the conclusion of Lemma 1.8.2 holds. Then,* I_λ *possesses a critical point* $v \in B(0,r)$ *with* $I_\lambda(v) \leq I_\lambda(0)$, *which is a local minimum of* I_λ.

Proof. From Lemma 1.8.2, we see that there exists $r > 0$ such that

$$m := \inf_{v \in B(0,r)} I_\lambda(v) \leq I_\lambda(0) \qquad \text{and} \qquad I_\lambda(v) > I_\lambda(0) \ \text{ if } \ \|v\| = r.$$

Let $\{v_n\} \subset B(0,r)$ be such that $I_\lambda(v_n) \to m$. Since $\{v_n\}$ is bounded, up to a subsequence, it follows that $v_n \rightharpoonup v \in W_0^{1,p}(\Omega)$. By the weak lower semicontinuity of the norm and of the functional I_λ, we have

$$\|v\| \leq \liminf_{n\to\infty} \|v_n\| \leq r \quad \text{ and } \quad I_\lambda(v) \leq \liminf_{n\to\infty} I_\lambda(v_n) = m \leq I_\lambda(0).$$

Finally, as $I_\lambda(v) > I_\lambda(0)$ if $\|v\| = r$, we deduce that $v \in B(0,r)$ is a local minimum of I_λ. □

Proof of Theorem 1.1.5. Assume that $\lambda > 0$ is small enough in order to ensure that the conclusion of Lemma 1.8.2 holds. By Proposition 1.8.4 we have a first critical point, which is a local minimum of I_λ. On the other hand, since the Cerami condition holds, in view of Lemmata 1.8.2. and 1.8.3, we can apply Theorem 1.2.8 and obtain a second critical point of I_λ at the mountain-pass level. This gives two different solutions to (Q_λ). Finally, by Lemma 1.5.2, we obtain two solutions to (P_λ). □

1.9 Proof of Theorems 1.1.6 and 1.1.7

In this section, we assume the stronger assumption (A_2). In that case, we are able to improve our results on the non-coercive case.

Proposition 1.9.1. *Assume that (A_2) holds with $h \lneqq 0$. Then, for every $\lambda > 0$, there exists $v \in \mathcal{C}_0^{1,\tau}(\overline{\Omega})$, for some $0 < \tau < 1$, with $v \ll 0$, which is a local minimum of I_λ in the $W_0^{1,p}$-topology and a solution to (Q_λ) with $v \geq \alpha_\lambda$ (with α_λ defined by (1.5.2)).*

Proof. First of all, observe that, as $h \lneqq 0$, we have $m_p > 0$ and hence, by Theorem 1.1.3, (P_0) has a solution $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. By Lemma 1.5.2,

$$v_0 = \frac{p-1}{\mu}\left(e^{\frac{\mu}{p-1}u_0} - 1\right) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

is then a weak solution to

$$\begin{cases} -\Delta_p v_0 = \left(1 + \dfrac{\mu}{p-1} v_0\right)^{p-1} h(x) \lneqq 0, & \text{in } \Omega, \\ \quad v_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

As moreover, $\left(1+\frac{\mu}{p-1}v_0\right)^{p-1} h(x) \in L^\infty(\Omega)$, it follows from [56,87] that $v_0 \in \mathcal{C}_0^{1,\tau}(\overline{\Omega})$, for some $\tau \in (0,1)$ and, by the strong maximum principle (see [114]), that $v_0 \ll 0$. Now, we split the rest of the proof in three steps.

Step 1: 0 *is a strict upper solution to* (Q_λ).

Observe that 0 is an upper solution to (Q_λ). In order to prove that 0 is strict, let $v \leq 0$ be a solution to (Q_λ). As $g \leq 0$ on $\mathbb{R}^-$ (see Lemma 1.5.1), it follows that v is a lower solution to (Q_0) and so, thanks to the comparison principle, see Corollary 1.3.2, $v \leq v_0 \ll 0$. Hence, 0 is a strict upper solution to (Q_λ).

Step 2: (Q_λ) *has a strict lower solution* $\underline{\alpha} \ll 0$.

By construction $\underline{\alpha} = \alpha_\lambda - 1$ is a lower solution to (Q_λ). Moreover, as every solution v of (Q_λ) satisfies $v \geq \alpha_\lambda \gg \underline{\alpha}$, we conclude that $\underline{\alpha}$ is a strict lower solution to (Q_λ).

Step 3: *Conclusion.*

By Corollary 1.2.6, Proposition 1.2.7, and Lemma 1.5.2, we have the existence of $v \in W_0^{1,p}(\Omega) \cap \mathcal{C}_0^{1,\tau}(\overline{\Omega})$, local minimum of I_λ and solution to (Q_λ) such that $\alpha_\lambda \leq v \ll 0$ as desired. □

Proof of the first part of Theorem 1.1.6. By Proposition 1.9.1, there exists a first critical point, which is a local minimum of I_λ. By Theorem 1.2.9 and since the Cerami condition holds, we have two options. If we are in the first case, then together with Lemma 1.8.3, we see that I_λ has the mountain-pass geometry and by Theorem 1.2.8, we have the existence of a second solution. In the second case, we have directly the existence of a second solution to (Q_λ). Then by Lemma 1.5.2 we conclude to the existence of two solutions to (P_λ). □

Now, we consider the case $h \gneqq 0$.

Lemma 1.9.2. *Assume that (A_2) holds and suppose that $h \gneqq 0$. Recall that γ_1 denotes the first eigenvalue of (1.1.2). It follows that:*

i) For any $0 \le \lambda < \gamma_1$, any solution u of the problem (P_λ) satisfies $u \gg 0$.

ii) For $\lambda = \gamma_1$, the problem (P_λ) has no solution.

iii) For $\lambda > \gamma_1$, the problem (P_λ) has no non-negative solution.

Proof. Observe first that, taking u^- as test function in (P_λ), we obtain

$$-\int_\Omega \Big(|\nabla u^-|^p - \lambda\, c(x)|u^-|^p\Big)\,dx = \int_\Omega \Big(\mu|\nabla u|^p u^- + h(x)u^-\Big)\,dx. \tag{1.9.1}$$

i) For $\lambda < \gamma_1$, there exists $\varepsilon > 0$ such that, for every $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \Big(|\nabla u|^p - \lambda c(x)|u|^p\Big)\,dx \ge \varepsilon \|u\|^p\,.$$

Consequently, as $h \gneqq 0$ and $\mu > 0$, we have that

$$0 \ge -\varepsilon\|u^-\|^p \ge -\int_\Omega \Big(|\nabla u^-|^p - \lambda c(x)|u^-|^p\Big)\,dx = \int_\Omega \Big(\mu|\nabla u|^p u^- + h(x)u^-\Big)\,dx \ge 0,$$

which implies that $u^- = 0$ and so that $u \ge 0$. Hence $-\Delta_p u \gneqq 0$ and by the strong maximum principle (see [114]), we have $u \gg 0$.

ii) In case $\lambda = \gamma_1$ we have, for every $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \Big(|\nabla u|^p - \gamma_1 c(x)|u|^p\Big)\,dx \ge 0. \tag{1.9.2}$$

Assume by contradiction that (P_λ) has a solution u. By (1.9.1) and (1.9.2), and using that $h \gneqq 0$ and $\mu > 0$, we have in particular

$$\int_\Omega \Big(|\nabla u^-|^p - \gamma_1 c(x)|u^-|^p\Big)\,dx = 0.$$

This implies that $u^- = k\varphi_1$ for some $k \in \mathbb{R}$ and φ_1 the first eigenfunction of (1.1.2) and hence, either $u \equiv 0$ or $u \ll 0$. As $h \not\equiv 0$, the first case cannot occur as 0 is not a solution to (P_λ). In the second case, as $h \gneqq 0$, we have

$$\int_\Omega h(x)u^-\,dx > 0$$

which contradicts (1.9.1), (1.9.2) and $\mu > 0$.

iii) Suppose by contradiction that u is a non-negative solution to (P_λ). As in the proof of i), we prove $u \gg 0$ and hence, there exists $D_1 > 0$ such that $u \ge D_1 d$ with $d(x) = \mathrm{dist}(x, \partial\Omega)$. Let $\varphi_1 > 0$ be the first eigenfunction of (1.1.2). As $\varphi_1 \in \mathcal{C}^1(\overline{\Omega})$, we have $D_2 > 0$ such that $\varphi_1 \le D_2 d$. This implies that $\frac{\varphi_1}{u} \in L^\infty(\Omega)$ and $\frac{\varphi_1^p}{u^{p-1}} \in W_0^{1,p}(\Omega)$ with

$$\nabla\Big(\frac{\varphi_1^p}{u^{p-1}}\Big) = p\Big(\frac{\varphi_1}{u}\Big)^{p-1}\nabla\varphi_1 - (p-1)\Big(\frac{\varphi_1}{u}\Big)^p \nabla u.$$

Hence we can take $\frac{\varphi_1^p}{u^{p-1}}$ as test function in (P_λ) and we have that

$$\lambda \int_\Omega c(x)\varphi_1^p\,dx + \int_\Omega \Big[\mu|\nabla u|^p + h(x)\Big]\frac{\varphi_1^p}{u^{p-1}}\,dx = \int_\Omega \nabla\Big(\frac{\varphi_1^p}{u^{p-1}}\Big)|\nabla u|^{p-2}\nabla u\,dx.$$

On the other hand, applying Proposition 1.2.4, we obtain

$$\gamma_1 \int_\Omega c(x)\varphi_1^p\,dx = \int_\Omega |\nabla\varphi_1|^p\,dx \geq \int_\Omega \nabla\Big(\frac{\varphi_1^p}{u^{p-1}}\Big)|\nabla u|^{p-2}\nabla u\,dx.$$

Consequently, gathering together both inequalities, we have the contradiction

$$0 \geq (\gamma_1-\lambda)\int_\Omega c(x)\varphi_1^p\,dx \geq \int_\Omega [\mu|\nabla u|^p + h(x)]\frac{\varphi_1^p}{u^{p-1}}\,dx > 0. \tag{1.9.3}$$

□

Corollary 1.9.3. *Assume that* (A_2) *holds. If, for some* $\lambda > 0$, (P_λ) *has a solution* $u_\lambda \geq 0$ *then* (P_0) *has a solution.*

Proof. Observe that u_λ is an upper solution to (P_0). By Proposition 1.4.2, we know that (P_0) has a lower solution α with $\alpha \leq u_\lambda$. The conclusion follows from Theorem 1.2.1. □

Corollary 1.9.4. *Assume that* (A_2) *holds with* $h \gneqq 0$. *If* (P_λ) *has a solution for some* $\lambda \in (0,\gamma_1)$, *then* (P_0) *has a solution.*

Proof. If (P_λ) has a solution u, by Lemma 1.9.2, we have $u \gg 0$. The result follows from Corollary 1.9.3. □

Proposition 1.9.5. *Assume that* (P_0) *has a solution* $u_0 \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *and suppose that* (A_2) *holds with* $h \gneqq 0$. *Then there exists* $\overline{\lambda} < \gamma_1$ *such that:*

- *i) For every* $0 < \lambda < \overline{\lambda}$, *there exists* $v \in \mathcal{C}_0^{1,\tau}(\overline{\Omega})$, *for some* $0 < \tau < 1$, *with* $v \gg 0$, *which is a local minimum of* I_λ *in the* $W_0^{1,p}$*-topology and a solution to* (Q_λ).
- *ii) For* $\lambda = \overline{\lambda}$, *there exists* $u \in \mathcal{C}_0^{1,\tau}(\overline{\Omega})$, *for some* $0 < \tau < 1$, *with* $u \geq u_0$, *which is a solution to* (P_λ).
- *iii) For* $\lambda > \overline{\lambda}$, *the problem* (P_λ) *has no non-negative solution.*

Proof. Defining

$$\overline{\lambda} = \sup\{\lambda : (P_\lambda) \text{ has a non-negative solution } u_\lambda\},$$

we directly obtain that, for $\lambda > \overline{\lambda}$, the problem (P_λ) has no non-negative solution and, by Lemma 1.9.2 *ii*), we see that $\overline{\lambda} \leq \gamma_1$. Moreover, arguing exactly as in the first part of Proposition 1.9.1, we deduce that

$$v_0 = \frac{p-1}{\mu}\Big(e^{\frac{\mu}{p-1}u_0} - 1\Big) \in W_0^{1,p}(\Omega)\cap\mathcal{C}_0^1(\overline{\Omega})$$

satisfies $v_0 \gg 0$. Now, fix $\lambda \in (0,\overline{\lambda})$.

Step 1: 0 *is a strict lower solution to* (Q_λ).

The proof of this step follows the corresponding one of Proposition 1.9.1.

Step 2: (Q_λ) *has a strict upper solution.*

By the definition of $\overline{\lambda}$ we can find $\delta \in (\lambda,\overline{\lambda})$ and a non-negative solution u_δ of (P_δ). As above, we easily see that

$$v_\delta = \frac{p-1}{\mu}\Big(e^{\frac{\mu}{p-1}u_\delta} - 1\Big) \in W_0^{1,p}(\Omega)\cap\mathcal{C}_0^1(\overline{\Omega})$$

is a non-negative upper solution to (Q_λ) and $v_\delta \gg 0$. Moreover, if v is a solution to (Q_λ) with $v \leq v_\delta$, Theorem 1.2.2 implies that $v \ll v_\delta$. Hence, v_δ is a strict upper solution to (P_λ).

Step 3: *Proof of i).*

The conclusion follows as in Proposition 1.9.1.

Step 4: *Existence of a solution for $\lambda = \overline{\lambda}$.*

Let $\{\lambda_n\}$ be a sequence with $\lambda_n < \overline{\lambda}$ and $\lambda_n \to \overline{\lambda}$ and $\{v_n\}$ be the corresponding sequence of minimum of I_{λ_n} obtained in i). This implies that $\langle I'_{\lambda_n}(v_n), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p}(\Omega)$. By the above construction, we also have

$$I_{\lambda_n}(v_n) \le I_{\lambda_n}(0) = -\frac{p-1}{p\mu}\int_\Omega h(x)\,dx.$$

Arguing exactly as in Lemmata 1.8.1 and 1.5.3, we prove easily the existence of $v \in W_0^{1,p}(\Omega)$ such that $v_n \to v$ in $W_0^{1,p}(\Omega)$ with v a solution to (Q_λ) for $\lambda = \overline{\lambda}$. As $v_n \ge 0$ we obtain also $v \ge 0$, and, by Lemma 1.5.2, we have the existence of a solution u of (P_λ) with $u \ge 0$. As u is then an upper solution to (P_0), we conclude that $u \ge u_0$.

Step 5: $\overline{\lambda} < \gamma_1$.

As by Lemma 1.9.2, the problem (P_λ) has no solution for $\lambda = \gamma_1$, this follows from Step 4. □

Proof of the second part of Theorem 1.1.6. By Lemma 1.9.2, we have $u_0 \gg 0$. Let us consider $\overline{\lambda} \in (0, \gamma_1)$ given by Proposition 1.9.5. Hence, for $\lambda < \overline{\lambda}$, there exists a first critical point u_1, which is a local minimum of I_λ. We then argue as in the proof of the first part to obtain the second solution u_2 of (P_λ). By Lemma 1.9.2, these two solutions satisfy $u_i \gg 0$ and, by Theorem 1.3.1, we conclude that $u_i \ge u_0$. Now, for $\lambda = \overline{\lambda}$, respectively $\lambda > \overline{\lambda}$, the result follows respectively from Proposition 1.9.5 *ii)* and *iii)*. □

Proof of Theorem 1.1.7. **Part 1: Case $\lambda \in (0, \gamma_1)$.**

Step 1: *There exists $k > 0$ such that $(P_{\lambda,k})$ has at least one solution.*

Let $\lambda_0 \in (\lambda, \gamma_1)$ and δ small enough such that

$$\lambda_0\, s^{p-1} \ge \lambda\Big(\frac{p-1}{\mu}\Big(1 + \frac{\mu}{p-1}s\Big)\ln\Big(1 + \frac{\mu}{p-1}s\Big)\Big)^{p-1}, \qquad \forall\, s \in [0, \delta].$$

Define w as a solution to

$$-\Delta_p w = \lambda_0\, c(x)|w|^{p-2}w + h(x), \quad w \in W_0^{1,p}(\Omega).$$

As $\lambda_0 < \gamma_1$, we have $w \gg 0$.

For l small enough, $\tilde{\beta} = lw$ satisfies $0 \le \tilde{\beta} \le \delta$ and, for k such that $l^{p-1} \ge \big(1 + \frac{\mu}{p-1}\delta\big)^{p-1}k$, it is easy to prove that $\beta = \frac{p-1}{\mu}\ln\big(1 + \frac{\mu}{p-1}\tilde{\beta}\big)$ is an upper solution to $(P_{\lambda,k})$ with $\beta \ge 0$. As 0 is a lower solution to $(P_{\lambda,k})$, the claim follows from Theorem 1.2.1.

Step 2: *For $k \ge k_0$, the problem $(P_{\lambda,k})$ has no solution.*

Let u be a solution to $(P_{\lambda,k})$. By Lemma 1.9.2, we have $u \gg 0$. This implies that u is an upper solution to $(P_{0,k})$. As 0 is a lower solution to $(P_{0,k})$, by Theorem 1.2.1, the problem $(P_{0,k})$ has a solution and hence, by Proposition 1.7.1, $m_p > 0$ which means that $k < k_0$. This implies that, for $k \ge k_0$, the problem $(P_{\lambda,k})$ has no solution.

Step 3: $\overline{k} = \sup\{k \in (0,k_0) : (P_{\lambda,k})$ *has at least one solution*$\} < k_0$.

Assume by contradiction that $\overline{k} = k_0$. Let $\{k_n\}$ be an increasing sequence such that $k_n \to \overline{k}$, $k_n \geq \frac{1}{2}\overline{k}$ and there exists $\{u_n\}$ a sequence of solutions to (P_{λ,k_n}). As in the previous step we have that u_n is an upper solution to $(P_{0,\frac{1}{2}\overline{k}})$. By Theorem 1.3.1, we know that $u_n \geq u_0$ with $u_0 \gg 0$ the solution to $(P_{0,\frac{1}{2}\overline{k}})$. Now, let $\phi \in W_0^{1,p}(\Omega) \cap \mathcal{C}_0^1(\overline{\Omega})$ with $\phi \gg 0$ and

$$\left(\frac{p-1}{\mu}\right)^{p-1} \int_\Omega |\nabla \phi|^p \, dx = k_0 \int_\Omega h(x)\phi^p \, dx.$$

Using ϕ^p as test function and applying Young inequality as in the proof of Proposition 1.7.1, it follows that

$$\begin{aligned}\left(\frac{p-1}{\mu}\right)^{p-1} \int_\Omega |\nabla \phi|^p \, dx &\geq \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(|\phi|^p)\, dx - \mu \int_\Omega |\phi|^p |\nabla u_n|^p \, dx \\ &= \lambda \int_\Omega c(x)|u_n|^{p-2}u_n\phi^p \, dx + k_n \int_\Omega h(x)\phi^p \, dx \\ &\geq \lambda \int_\Omega c(x)|u_0|^{p-2}u_0\phi^p \, dx + k_n \int_\Omega h(x)\phi^p \, dx.\end{aligned}$$

Passing to the limit, we have the contradiction

$$\left(\frac{p-1}{\mu}\right)^{p-1} \int_\Omega |\nabla \phi|^p \, dx \geq \lambda \int_\Omega c(x)|u_0|^{p-2}u_0\phi^p \, dx + \left(\frac{p-1}{\mu}\right)^{p-1} \int_\Omega |\nabla \phi|^p \, dx.$$

Step 4: *For $k > \overline{k}$, the problem $(P_{\lambda,k})$ has no solution and for $k < \overline{k}$, the problem $(P_{\lambda,k})$ has at least two solutions u_1, u_2 with $u_i \gg 0$.*

The first statement is obvious by definition of $\overline{k}$. Now, for $k < \overline{k}$, let $\tilde{k} \in (k, \overline{k})$ such that $(P_{\lambda,\tilde{k}})$ has a solution $\tilde{u}$. By Lemma 1.9.2, we have $\tilde{u} \gg 0$. Then, it is easy to observe that $\beta_1 = \left(\frac{k}{\tilde{k}}\right)^{\frac{1}{p-1}} \tilde{u}$ and $\beta_2 = \tilde{u}$ are both upper solutions to $(P_{\lambda,k})$ with $0 \ll \beta_1 \ll \beta_2$.

Observe that 0 is a strict lower solution to $(P_{\lambda,k})$. As $\beta_1 \gg 0$ is an upper solution to $(P_{\lambda,k})$, by Theorem 1.2.1, the problem $(P_{\lambda,k})$ has a minimum solution u_1 with $0 \ll u_1 \leq \beta_1$.

In order to prove the existence of the second solution, observe that if β_2 is not strict, it means that $(P_{\lambda,k})$ has a solution u_2 with $u_2 \leq \beta_2$ but $u_2 \not\ll \beta_2$. Then $u_2 \neq u_1$ and we have our two solutions. If β_2 is strict, we argue as in the proof of Theorem 1.1.6.

Step 5: *The function $\overline{k}(\lambda)$ is non-increasing.*

Let us consider $\lambda_1 < \lambda_2$, $\tilde{k} < \overline{k}(\lambda_2)$ and $\tilde{u} \gg 0$ a solution to $(P_{\lambda_2,\tilde{k}})$. It is easy to prove that $\tilde{u}$ is an upper solution to $(P_{\lambda_1,\tilde{k}})$. As 0 is a lower solution to $(P_{\lambda_1,\tilde{k}})$ with $0 \leq \tilde{u}$, by Theorem 1.2.1, the problem $(P_{\lambda_1,\tilde{k}})$ has a solution. This implies that $\overline{k}(\lambda_1) \geq \overline{k}(\lambda_2)$.

Part 2: Case $\lambda = \gamma_1$.

By Lemma 1.9.2, we know that the problem (P_{γ_1}) has no solution for $k > 0$. Moreover, by (1.9.1), we see that if (P_{γ_1}) with $h \equiv 0$ has a non-trivial solution, then $u \gneqq 0$ and hence, by the strong maximum principle $u \gg 0$. Arguing as in the proof of iii) of Lemma 1.9.2, we obtain the same contradiction (1.9.3).

Part 3: Case $\lambda > \gamma_1$.

Step 1: *There exists $k > 0$ such that $(P_{\lambda,k})$ has at least one solution $u \ll 0$.*

By Proposition 1.2.3 with $\bar{h} = h$, there exists $\delta_0 > 0$ such that, for $\lambda \in (\gamma_1, \gamma_1 + \delta_0)$, the solution to

$$-\Delta_p w = \lambda\, c(x)|w|^{p-2}w + h(x), \quad w \in W_0^{1,p}(\Omega), \tag{1.9.4}$$

satisfies $w \ll 0$. Let us fix $\lambda_0 \in (\gamma_1, \min(\gamma_1 + \delta_0, \lambda))$ and δ small enough such that

$$\lambda_0 |s|^{p-2}s \geq \lambda \left| \frac{p-1}{\mu}\Big(1 + \frac{\mu}{p-1}s\Big) \ln\Big(1 + \frac{\mu}{p-1}s\Big)\right|^{p-2} \frac{p-1}{\mu}\Big(1 + \frac{\mu}{p-1}s\Big) \ln\Big(1 + \frac{\mu}{p-1}s\Big), \quad \forall\, s \in [-\delta, 0].$$

Define w as a solution to

$$-\Delta_p w = \lambda_0\, c(x)|w|^{p-2}w + h(x), \quad u \in W_0^{1,p}(\Omega).$$

As $\gamma_1 < \lambda_0 < \gamma_1 + \delta_0$, we have $w \ll 0$.

For l small enough, $\tilde{\beta} = lw$ satisfies $-\min(\delta, \frac{p-1}{\mu}) < \tilde{\beta} \leq 0$ and, for $k \leq l^{p-1}$, it is easy to prove that $\beta = \frac{p-1}{\mu} \ln\Big(1 + \frac{\mu}{p-1}\tilde{\beta}\Big)$ is an upper solution to $(P_{\lambda,k})$ with $\beta \ll 0$. By Proposition 1.4.2, $(P_{\lambda,k})$ has a lower solution α with $\alpha \leq \beta$ and the claim follows from Theorem 1.2.1.

Step 2: *For k large enough, the problem $(P_{\lambda,k})$ has no solution.*

Otherwise, let u be a solution to $(P_{\lambda,k})$. By Lemma 1.4.1 and Remark 1.4.1, we have $M_\lambda > 0$ such that, for all $k > 0$, the corresponding solution u satisfies $u \geq -M_\lambda$. Let $\phi \in \mathcal{C}_0^1(\overline{\Omega})$ with $\phi \gg 0$. Using ϕ^p as test function, by Young inequality as in the proof of Proposition 1.7.1, it follows that

$$\begin{aligned}
\Big(\frac{p-1}{\mu}\Big)^{p-1} \int_\Omega |\nabla \phi|^p\, dx &\geq \int_\Omega |\nabla u|^{p-2} \nabla u \nabla(\phi^p)\, dx - \mu \int_\Omega \phi^p |\nabla u|^p\, dx \\
&= \lambda \int_\Omega c(x)|u|^{p-2}u\phi^p\, dx + k \int_\Omega h(x)\phi^p\, dx \\
&\geq -\lambda M^{p-1} \int_\Omega c(x)\phi^p\, dx + k \int_\Omega h(x)\phi^p\, dx.
\end{aligned}$$

which is a contradiction for k large enough.

Step 3: *Define $\tilde{k}_1 = \sup\{k > 0 : (P_{\lambda,k})$ has at least one solution $u \ll 0\}$. For $k < \tilde{k}_1$, the problem $(P_{\lambda,k})$ has at least two solutions with $u_1 \ll 0$ and* $\min u_2 < 0$.

For $k < \tilde{k}_1$, let $\tilde{k} \in (k, \tilde{k}_1)$ such that $(P_{\lambda,\tilde{k}})$ has a solution $\tilde{u} \ll 0$. It is then easy to observe that $\beta_1 = \tilde{u}$ and $\beta_2 = \Big(\frac{k}{\tilde{k}}\Big)^{\frac{1}{p-1}} \tilde{u}$ are both upper solutions to $(P_{\lambda,k})$ with $\beta_1 \ll \beta_2 \ll 0$.

By Proposition 1.4.2, $(P_{\lambda,k})$ has a lower solution α with $\alpha \leq \beta_1$ and hence, by Theorem 1.2.1, the problem $(P_{\lambda,k})$ has a minimum solution u_1 with $\alpha \leq u_1 \leq \beta_1$.

In order to prove the existence of the second solution, observe that if β_2 is not strict, it means that $(P_{\lambda,k})$ has a solution u_2 with $u_2 \leq \beta_2$ but $u_2 \not\ll \beta_2$. Then $u_2 \neq u_1$ and we have our two solutions. If β_2 is strict, we argue as in the proof of Theorem 1.1.6.

Step 4: *Define $\tilde{k}_2 = \sup\{k > 0 : (P_{\lambda,k})$ has at least one solution$\}$. For $k > \tilde{k}_2$, the problem $(P_{\lambda,k})$ has no solution and, in case $\tilde{k}_1 < \tilde{k}_2$, for all $k \in (\tilde{k}_1, \tilde{k}_2)$, the problem $(P_{\lambda,k})$ has at least one solution u with $u \not\ll 0$ and* $\min u < 0$.

The first statement follows directly from the definition of $\tilde{k}_2$. In case $\tilde{k}_1 < \tilde{k}_2$, for $k \in (\tilde{k}_1, \tilde{k}_2)$, let $\tilde{k} \in (k, \tilde{k}_2)$ such that $(P_{\lambda,\tilde{k}})$ has a solution $\tilde{u}$. Observe that $\tilde{u}$ is an upper solution to $(P_{\lambda,k})$. Again, Proposition 1.4.2 gives us a lower solution α of $(P_{\lambda,k})$ with $\alpha \le \tilde{u}$ and hence, by Theorem 1.2.1, the problem $(P_{\lambda,k})$ has a solution u. By definition of $\tilde{k}_1$, we have that $u \not\ll 0$ and by Lemma 1.9.2, we know that $\min u < 0$.

Step 5: *The function $\tilde{k}_1(\lambda)$ is non-decreasing.*

Let us consider $\lambda_1 < \lambda_2$, $k < \tilde{k}_1(\lambda_1)$ and $u \ll 0$ a solution to $(P_{\lambda_1,k})$. It is easy to prove that u is an upper solution to $(P_{\lambda_2,k})$. Again, applying Proposition 1.4.2 and Theorem 1.2.1, we prove that the problem $(P_{\lambda_2,k})$ has a solution $u \ll 0$. This implies that $\tilde{k}_1(\lambda_1) \le \tilde{k}_1(\lambda_2)$. □

1.10 Appendix. Sufficient conditions

Lemma 1.10.1. *Given $f \in L^r(\Omega)$, $r > \max\{N/p, 1\}$ if $p \neq N$ and $1 < r < \infty$ if $p = N$, let us consider*

$$E_f(u) = \left(\int_\Omega \left(|\nabla u|^p - f(x)|u|^p \right) dx \right)^{\frac{1}{p}}$$

for an arbitrary $u \in W_0^{1,p}(\Omega)$. It follows that:

i) If $1 < p < N$ and $\|f^+\|_{N/p} < S_N$, $E_f(u)$ is an equivalent norm in $W_0^{1,p}(\Omega)$.

ii) If $p = N$ and $\|f^+\|_r < S_{N,r}$, $E_f(u)$ is an equivalent norm in $W_0^{1,p}(\Omega)$.

iii) If $p > N$ and $\|f^+\|_1 < S_N$, $E_f(u)$ is an equivalent norm in $W_0^{1,p}(\Omega)$.

where, for $p \neq N$, S_N denotes the optimal constant in the Sobolev inequality, i.e.

$$S_N = \inf\left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \|u\|_{p^*} = 1 \right\},$$

and, for $p = N$,

$$S_{N,r} = \inf\left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \|u\|_{\frac{Nr}{r-1}} = 1 \right\}.$$

Proof. We give the proof for $1 < p < N$. The other cases can be done in the same way. First of all, by applying Hölder and Sobolev's inequalities, observe that, for any $h \in L^{\frac{N}{p}}(\Omega)$, it follows that

$$\int_\Omega h(x)|u|^p dx \le \|h\|_{\frac{N}{p}} \|u\|_{p^*}^p \le \frac{1}{S_N} \|h\|_{\frac{N}{p}} \|\nabla u\|_p^p.$$

On the one hand, by using this inequality, observe that

$$\int_\Omega \left(|\nabla u|^p - f(x)|u|^p \right) dx \le \|u\|^p \left(1 + \frac{\|f\|_{\frac{N}{p}}}{S_N} \right).$$

On the other hand, following the same argument, we obtain that

$$\int_\Omega \left(|\nabla u|^p - f(x)|u|^p \right) dx \ge \int_\Omega \left(|\nabla u|^p - f^+(x)|u|^p \right) dx \ge \|u\|^p \left(1 - \frac{\|f^+\|_{\frac{N}{p}}}{S_N} \right) = A\|u\|^p$$

with $A > 0$ since $\|f^+\|_{\frac{N}{p}} < S_N$. The result follows. □

As an immediate Corollary, we have a sufficient condition to ensure that $m_p > 0$.

Corollary 1.10.2. *Recall that m_p is defined by (1.1.1). Under the assumptions (A_1), it follows that:*

i) If $1 < p < N$, then $\|h^+\|_{N/p} < \left(\frac{p-1}{\mu}\right)^{p-1} S_N$ implies $m_p > 0$.

ii) If $p = N$, then $\|h^+\|_q < \left(\frac{p-1}{\mu}\right)^{p-1} S_{N,q}$ implies $m_p > 0$.

iii) If $p > N$, then $\|h^+\|_1 < \left(\frac{p-1}{\mu}\right)^{p-1} S_N$ implies $m_p > 0$.

2

A priori bounds and multiplicity of solutions for an indefinite elliptic problem with critical growth in the gradient

2.1 Introduction and main results

The chapter deals with the existence and multiplicity of solutions for boundary value problems of the form

$$-\Delta u = c_\lambda(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{Q_λ}$$

with c_λ depending on a real parameter λ. Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$, c_λ and h belong to $L^q(\Omega)$ for some $q > N/2$ and μ belongs to $L^\infty(\Omega)$.

This type of problem, which started to be studied by L. Boccardo, F. Murat and J.P. Puel in the 80's, has attracted a new attention these last years. Under the condition $c_\lambda \leq -\alpha_0 < 0$ a.e. in Ω for some $\alpha_0 > 0$, the existence of a solution to (Q_λ) is a particular case of the results of [23, 25] and its uniqueness follows from [19, 20]. The case $c_\lambda \equiv 0$ was studied in [4, 62] and the existence requires some smallness condition on $\|\mu h\|_{N/2}$. The situation where one only requires $c_\lambda \leq 0$ a.e. in Ω (i.e. allowing parts of the domain where $c_\lambda \equiv 0$ and parts of it where $c_\lambda < 0$) proved to be more complex to treat. In the recent papers [18, 46], the authors explicit sufficient conditions for the existence of solutions to (Q_λ). Moreover, in [18], the uniqueness of solution is established (see also [17] in that direction). All these results were obtained without requiring any sign conditions on μ and h.

In case $c_\lambda = \lambda c \gneqq 0$, as we shall discuss later, problem (Q_λ) behaves very differently and becomes much richer. Following [104], which considers a particular case, [76] studied (Q_λ) with $\mu(x) \equiv \mu > 0$ and $\lambda c \gneqq 0$ but without a sign condition on h. The authors proved the existence of at least two solutions when $\lambda > 0$ and $\|(\mu h)^+\|_{N/2}$ are small enough. The restriction μ constant was removed in [18] and extended to $\mu(x) \geq \mu_1 > 0$ a.e. in Ω, at the expense of adding the hypothesis $h \gneqq 0$. Next, in [50], assuming stronger regularity on c and h, the authors removed the condition $h \gneqq 0$. In this paper, it is also lightened that the structure of the set of solutions when $\lambda > 0$, crucially depends on the sign of the (unique) solution to (Q_0). Note that, in [46], the above results are extended to the

p-Laplacian case. Also, in the frame of viscosity solutions and fully nonlinear equations, under corresponding assumptions, similar conclusions have been obtained very recently in [92].

We refer to [76] for an heuristic explanation on how the behavior of (Q_λ) is affected by the change of sign in front of the linear term. Actually, in the case where $\mu(x) \equiv \mu$ is a constant, it is possible to transform problem (Q_λ) into a new one which admits a variational formulation. When $c_\lambda \leq -\alpha_0 < 0$, the associated functional, defined on $H_0^1(\Omega)$, is coercive. If $c_\lambda \lneqq 0$, the coerciveness may be lost and when $c_\lambda \gneqq 0$, in fact as soon as $c_\lambda^+ \gneqq 0$, the functional is unbounded from below. In [76] this variational formulation was directly used to obtain the solutions. In [18, 50] where μ is non constant, topological arguments, relying on the derivation of a priori bounds for certain classes of solutions, were used.

The only known results where c_λ may change sign are [47, 75] (see also [64] for related problems). They both concern the case where μ is a positive constant. In [75], assuming $h \gneqq 0$, μh and c_λ^+ small in an appropriate sense, the existence of at least two non-negative solutions was proved. In [47], the authors show that the loss of positivity of the coefficient of u does not affect the structure of the set of solutions to (Q_λ) observed in [50] when $c_\lambda = \lambda c \gneqq 0$. Since μ is constant in [47, 75], it is possible to treat the problem variationally. The main issue, to derive the existence of solutions, is then to show the boundedness of the Palais-Smale sequences.

When $c_\lambda \gneqq 0$, all the above mentioned results require either μ to be constant or to be uniformly bounded from below by a positive constant (or similarly bounded from above by a negative constant). In [108], assuming that the three coefficients functions are non-negative, a first attempt to remove these restrictions on μ is presented. Following the approach of [18], the proofs of the existence results reduce to obtaining a priori bounds on the non negative solutions to (Q_λ). First it is observed in [108] that a necessary condition is the existence of a ball $B(x_0, \rho) \subset \Omega$ and $\nu > 0$ such that $\mu \geq \nu$ and $c \geq \nu$ on $B(x_0, \rho)$. When $N = 2$ this condition also proves to be sufficient. If $N = 3$ or 4 the condition $\mu \geq \mu_0 > 0$ on a set $\omega \subset \Omega$ such that $\operatorname{supp}(c) \subset \overline{\omega}$ permits to obtain the a priori bounds. Other sets of conditions are presented when $N = 3$ and $N = 5$. However, if the approach developed in [108], which relies on interpolation and elliptic estimates in weighted Lebesgue spaces, works well in low dimension, the possibility to extend it to dimension $N \geq 6$ is not apparent.

In this chapter we pursue the study of (Q_λ) and consider situations where the three coefficients functions c_λ, μ and h may change sign. We define for $v \in L^1(\Omega)$, $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$. As observed already in [47], the structure of the solution set depends on the size of the positive hump (i.e. c_λ^+) but it is not affect by the size of the negative hump (i.e. c_λ^-). Hoping to clarify this, we now write c_λ under the form $c_\lambda = \lambda c_+ - c_-$ and consider the problem

$$-\Delta u = (\lambda c_+(x) - c_-(x))u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{P_λ}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2, \text{ is a bounded domain with boundary } \partial\Omega \text{ of class } \mathcal{C}^{1,1}, \\ c_+, c_-, h^+ \in L^q(\Omega) \text{ for some } q > N/2,\ \mu, h^- \in L^\infty(\Omega), \\ c_+(x) \geq 0,\ c_-(x) \geq 0 \text{ and } c_-(x)c_+(x) = 0 \text{ a.e. in } \Omega, \\ |\Omega_+| > 0, \text{ where } \Omega_+ := \operatorname{Supp}(c_+) \\ \text{there exists a } \varepsilon > 0 \text{ such that } \mu(x) \geq \mu_1 > 0 \text{ and } c_- = 0 \text{ in } \{x \in \Omega : d(x, \Omega_+) < \varepsilon\}. \end{cases} \tag{A_1}$$

For a definition of Supp(f) with $f \in L^p(\Omega)$, for some $p \geq 1$, we refer to [26, Proposition 4.17]. Note also that the condition that $c_- = 0$ on $\{x \in \Omega : d(x, \Omega_+) < \varepsilon\}$ for some $\varepsilon > 0$, is reminiscent of the so-called "thick zero set" condition first introduced in [10].

We also observe that, under the regularity assumptions of condition (A_1), any solution to (P_λ) belongs to $\mathcal{C}^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. This can be deduce from [82, Theorem IX-2.2], see also [17, Proposition 2.1].

As in [18, 50, 108] we obtain our results using a topological approach, relying thus on the derivation of a priori bounds. In that direction our main result is the following.

Theorem 2.1.1. *Assume (A_1). Then, for any $\Lambda_2 > \Lambda_1 > 0$, there exists a constant $M > 0$ such that, for each $\lambda \in [\Lambda_1, \Lambda_2]$, any solution to ($P_\lambda$) satisfies $\sup_\Omega u \leq M$.*

Having at hand this a priori bound, following the strategy of [18], we show the existence of a continuum of solutions to (P_λ). More precisely, defining

$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (P_\lambda)\}, \tag{2.1.1}$$

we prove the following theorem.

Theorem 2.1.2. *Assume (A_1) and suppose that (P_0) has a solution u_0 with $c_+ u_0 \gneqq 0$. Then, there exists a continuum $\mathscr{C} \subset \Sigma$ such that the projection of $\mathscr{C}$ on the λ-axis is an unbounded interval $(-\infty, \overline{\lambda}]$ for some $\overline{\lambda} \in (0, +\infty)$ and $\mathscr{C}$ bifurcates from infinity to the right of the axis $\lambda = 0$. Moreover:*

1) *for all $\lambda \leq 0$, the problem (P_λ) has an unique solution u_λ and this solution satisfies $u_0 - \|u_0\|_\infty \leq u_\lambda \leq u_0$.*
2) *there exists $\lambda_0 \in (0, \overline{\lambda}]$ such that, for all $\lambda \in (0, \lambda_0)$, the problem ($P_\lambda$) has at least two solutions with $u_i \geq u_0$ for $i = 1, 2$.*

Remark 2.1.1.

a) Theorem 2.1.2, 1) generalizes [18, Theorem 1.2].

b) Note that problem (P_0) is given by

$$-\Delta u = -c_-(x) u + \mu(x) |\nabla u|^2 + h(x), \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

In [18, 46] the authors give sufficient conditions to ensure the existence of a solution to (P_0). Moreover, if $h \geq 0$ in Ω, [17, Lemma 2.2] implies that the solution to (P_0) is non-negative.

Let us give some ideas of the proofs. As we do not have global sign conditions, the approaches used in [18, 50, 108] to obtain the a priori bounds do not apply anymore and another strategy is required. To this aim, we further develop some techniques first sketched in the unpublished work [105]. These techniques, in the framework of viscosity solutions to fully nonlinear equations, now lies at the heart of the paper [92]. We also make use of some ideas from [64]. First we show, in Lemma 2.4.1, that it is sufficient to control the behavior of the solutions on $\overline{\Omega}_+$. By compactness, we are then reduced to study what happens around an (unknown) point $\overline{x} \in \overline{\Omega}_+$. We shall consider separately the alternative cases $\overline{x} \in \overline{\Omega}_+ \cap \Omega$ and $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$. A local analysis is made respectively in a ball or a semiball centered at $\overline{x}$. If similar analysis, based on the use of Harnack type inequalities, had previously been performed in other contexts when $\overline{x} \in \Omega$, we believe it is not the case when

$\overline{x} \in \partial\Omega$. For $\overline{x} \in \partial\Omega$, the key to our approach is the use of boundary weak Harnack inequality. Actually a major part of the chapter is devoted to establishing this inequality. This is done in a more general context than needed for (P_λ). In particular it also cover the case of the p-Laplacian with a zero order term. We believe that this "boundary weak Harnack inequality", see Theorem 2.3.1, is of independent interest and will proved to be useful in other settings. Its proof uses ideas introduced by B. Sirakov [106]. In [106] such type of inequalities is established for an uniformly elliptic operator and viscosity solutions. However, since our context is quite different, the result of [106] does not apply to our situation and we need to work out an adapted proof.

We now describe the organization of the chapter. In Section 2.2, we present some preliminary results which are needed in the development of our proofs. In Section 2.3, we prove the boundary weak Harnack inequality for the p-Laplacian. The a priori bound, namely Theorem 2.1.1, is proved in Section 2.4. Finally Section 2.5 is devoted to the proof of Theorem 2.1.2.

Notation.

1) In $\mathbb{R}^N$, we use the notations $|x| = \sqrt{x_1^2 + \ldots + x_N^2}$ and $B_R(y) = \{x \in \mathbb{R}^N : |x-y| < R\}$.

2) We denote $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

3) For $h_1, h_2 \in L^1(\Omega)$ we write

- $h_1 \le h_2$ if $h_1(x) \le h_2(x)$ for a.e. $x \in \Omega$,
- $h_1 \lneqq h_2$ if $h_1 \le h_2$ and $\text{meas}(\{x \in \Omega : h_1(x) < h_2(x)\}) > 0$.

2.2 Preliminary results

In this section, we collect some results which will play an important role throughout the chapter. First of all, let us consider the boundary value problem

$$-\Delta u + H(x, u, \nabla u) = f, \qquad u \in H_0^1(\omega) \cap L^\infty(\omega). \tag{2.2.1}$$

Here $\omega \subset \mathbb{R}^N$ is a bounded domain, $f \in L^1(\omega)$ and $H : \omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function.

Definition 2.2.1. We say that $\alpha \in H^1(\omega) \cap L^\infty(\omega)$ is a *lower solution* to (2.2.1) if $\alpha^+ \in H_0^1(\omega)$ and, for all $\varphi \in H_0^1(\omega) \cap L^\infty(\omega)$ with $\varphi \ge 0$, we have

$$\int_\omega \nabla\alpha \nabla\varphi \, dx + \int_\omega H(x, \alpha, \nabla\alpha)\varphi \, dx \le \int_\omega f(x)\varphi \, dx.$$

Similarly, $\beta \in H^1(\omega) \cap L^\infty(\omega)$ is an *upper solution* to (2.2.1) if $\beta^- \in H_0^1(\omega)$ and, for all $\varphi \in H_0^1(\omega) \cap L^\infty(\omega)$ with $\varphi \ge 0$, we have

$$\int_\omega \nabla\beta \nabla\varphi \, dx + \int_\omega H(x, \beta, \nabla\beta)\varphi \, dx \ge \int_\omega f(x)\varphi \, dx.$$

Next, we consider the boundary value problem

$$-\Delta u + a(x)u = b(x), \qquad u \in H_0^1(\omega), \tag{2.2.2}$$

under the assumption

$$\begin{cases} \omega \subset \mathbb{R}^N, \ N \ge 2, \text{ is a bounded domain,} \\ a, \ b \in L^r(\omega) \ \text{for some } r > N/2. \end{cases} \tag{2.2.3}$$

Remark 2.2.1. With the regularity imposed in the following lemmas and in the absence of a gradient term in the equation, we do not need the lower and upper solutions to be bounded. The full Definition 2.2.1 will however be needed in other parts of the chapter.

Lemma 2.2.1. **(Local Maximum Principle)** *Under the assumption (2.2.3), assume that $u \in H^1(\omega)$ is a lower solution to (2.2.2). For any ball $B_{2R}(y) \subset \omega$ and any $s > 0$, there exists $C = C(s, r, \|a\|_{L^r(B_{2R}(y))}, R) > 0$ such that*

$$\sup_{B_R(y)} u^+ \leq C\Big[\Big(\fint_{B_{2R}(y)} (u^+)^s dx\Big)^{1/s} + \|b^+\|_{L^r(B_{2R}(y))}\Big].$$

Proof. See for instance [66, Theorem 8.17] and [90, Corollary 3.10]. □

Lemma 2.2.2. **(Boundary Local Maximum Principle)** *Under the assumption (2.2.3), assume that $u \in H^1(\omega)$ is a lower solution to (2.2.2) and let $x_0 \in \partial\omega$. For any $R > 0$ and any $s > 0$, there exists $C = C(s, r, \|a\|_{L^r(B_{2R}(x_0)\cap\omega)}, R) > 0$ such that*

$$\sup_{B_R(x_0)\cap\omega} u^+ \leq C\Big[\Big(\fint_{B_{2R}(x_0)\cap\omega} (u^+)^s dx\Big)^{1/s} + \|b^+\|_{L^r(B_{2R}(x_0)\cap\omega)}\Big].$$

Proof. See for instance [66, Theorem 8.25] and [90, Corollary 3.10 and Theorem 3.11]. □

Remark 2.2.2. Lemmas 2.2.1 and 2.2.2 proof's are done in [66] for $a \in L^\infty(\omega)$ and $s > 1$. Nevertheless, as it is remarked on page 193 of that book, the proof is valid for $a \in L^r(\omega)$ with $r > N/2$ and, following closely the proof of [90, Corollary 3.10], the proofs can be extended for any $s > 0$.

Lemma 2.2.3. **(Weak Harnack Inequality)** *Under the assumption (2.2.3), assume that $u \in H^1(\omega)$ is a non-negative upper solution to (2.2.2). Then, for any ball $B_{4R}(y) \subset \omega$ and any $1 \leq s < \frac{N}{N-2}$ there exists $C = C(s, r, \|a\|_{L^r(B_{4R}(y))}, R) > 0$ such that*

$$\inf_{B_R(y)} u \geq C\Big[\Big(\fint_{B_{2R}(y)} u^s dx\Big)^{1/s} - \|b^-\|_{L^r(B_{4R}(y))}\Big].$$

Proof. See for instance [66, Theorem 8.18] and [90, Theorem 3.13]. □

Now, inspired by [27, Lemma 3.2] (see also [60, Appendix A]), we establish the following version of the Brezis-Cabré Lemma.

Lemma 2.2.4. *Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\omega$ of class $\mathcal{C}^{1,1}$ and let $a \in L^\infty(\omega)$ and $f \in L^1(\omega)$ be non-negative functions. Assume that $u \in H^1(\omega)$ is an upper solution to*

$$-\Delta u + a(x)u = f(x), \quad u \in H_0^1(\omega).$$

Then, for every $B_{2R}(y) \subset \omega$, there exists $C = C(R, y, \omega, \|a\|_\infty) > 0$ such that

$$\inf_\omega \frac{u(x)}{d(x,\partial\omega)} \geq C \int_{B_R(y)} f(x)\, dx.$$

Proof. First of all, as a and f are non-negative, by the weak maximum principle, it follows that

$$\inf_\omega \frac{u(x)}{d(x,\partial\omega)} \geq 0.$$

Now let $B_{2R}(y) \subset \omega$. By the above inequality, we can assume without loss of generality that

$$\int_{B_R(y)} f(x)\, dx > 0.$$

We split the proof into three steps.

Step 1: *There exists* $c_1 = c_1(R, y, \omega, \|a\|_\infty) > 0$ *such that*

$$\frac{u(x)}{d(x,\partial\omega)} \geq c_1 \int_{B_R(y)} f(x)\,dx, \quad \forall\, x \in \overline{B_{R/2}(y)}. \tag{2.2.4}$$

Since f is non-negative, observe that u is a non-negative upper solution to

$$-\Delta u + a(x)u = 0, \quad u \in H_0^1(\omega).$$

Hence, by Lemma 2.2.3, there exists a constant $c_2 = c_2(R, \|a\|_\infty) > 0$ such that

$$u(x) \geq c_2 \int_{B_R(y)} u\,dx, \quad \forall\, x \in \overline{B_{R/2}(y)}. \tag{2.2.5}$$

Now, let us denote by ξ the solution to

$$\begin{cases} -\Delta\xi + \|a\|_\infty \xi = \chi_{B_R(y)}, & \text{in } \omega, \\ \xi = 0, & \text{on } \partial\omega. \end{cases} \tag{2.2.6}$$

By [29, Theorem 3], there exists a constant $c_3 = c_3(R, y, \omega, \|a\|_\infty) > 0$ such that, for all $x \in \omega$, $\xi(x) \geq c_3 d(x, \partial\omega)$. Thus, since $B_{2R}(y) \subset \omega$, f is non-negative and $d(x,\partial\omega) \geq R$ for $x \in B_R(y)$, it follows that

$$\int_{B_R(y)} u\,dx = \int_\omega u\Big(-\Delta\xi + \|a\|_\infty \xi\Big)\,dx \geq \int_\omega f(x)\,\xi\,dx \geq c_3 \int_\omega f(x)\,d(x,\partial\omega)\,dx \geq c_3 R \int_{B_R(y)} f(x)\,dx.$$

Hence, substituting the above information in (2.2.5) we obtain for $c_4 = c_2 c_3 R$

$$u(x) \geq c_4 \int_{B_R(y)} f(x)\,dx, \quad \forall\, x \in \overline{B_{R/2}(y)}, \tag{2.2.7}$$

from which, since $\omega \subset \mathbb{R}^N$ is bounded, (2.2.4) follows.

Step 2: *There exists* $c_5 = c_5(R, y, \omega, \|a\|_\infty) > 0$ *such that*

$$\frac{u(x)}{d(x,\partial\omega)} \geq c_5 \int_{B_R(y)} f(x)\,dx, \quad \forall\, x \in \omega \setminus \overline{B_{R/2}(y)}. \tag{2.2.8}$$

Let w be the unique solution to

$$\begin{cases} -\Delta w + \|a\|_\infty w = 0, & \text{in } \omega \setminus \overline{B_{R/2}(y)}, \\ w = 0, & \text{on } \partial\omega, \\ w = 1, & \text{on } \partial B_{R/2}(y). \end{cases} \tag{2.2.9}$$

Still by [29, Theorem 3], there exists $c_6 = c_6(R, y, \omega, \|a\|_\infty) > 0$ such that $w(x) \geq c_6 d(x, \partial\omega)$ for all $x \in \omega \setminus \overline{B_{R/2}(y)}$. On the other hand, let us introduce

$$v(x) = \frac{u(x)}{c_4 \int_{B_R(y)} f(x)\,dx},$$

with c_4 given in (2.2.7). Observe that v is an upper solution to (2.2.9). Hence, by the standard comparison principle, it follows that $v(x) \geq w(x)$ for all $x \in \omega \setminus \overline{B_{R/2}(y)}$ and (2.2.8) follows.

Step 3: *Conclusion.*

The result follows from (2.2.4) and (2.2.8). $\square$

2.3 Boundary weak Harnack inequality

In this section we present a *boundary weak Harnack inequality* that will be central in the proof of Theorem 2.1.1. As we believe this type of inequality has its own interest, we establish it in the more general framework of the p-Laplacian. Recalling that $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$, we introduce the boundary value problem

$$-\Delta_p u + a(x)|u|^{p-2}u = 0, \quad u \in W_0^{1,p}(\omega). \tag{2.3.1}$$

Let us also recall that $u \in W^{1,p}(\omega)$ is an *upper solution to* (2.3.1) if $u^- \in W_0^{1,p}(\omega)$ and, for all $\varphi \in W_0^{1,p}(\omega)$ with $\varphi \geq 0$, it follows that

$$\int_\omega |\nabla u|^{p-2}\nabla u \,\nabla\varphi \, dx + \int_\omega a(x)|u|^{p-2}u\,\varphi\, dx \geq 0.$$

We then prove the following result.

Theorem 2.3.1. (Boundary Weak Harnack Inequality) Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\omega$ of class $\mathcal{C}^{1,1}$ and let $a \in L^\infty(\omega)$ be a non-negative function. Assume that u is a non-negative upper solution to (2.3.1) and let $x_0 \in \partial\omega$. Then, there exist $\overline{R} > 0$, $\varepsilon = \varepsilon(p, \overline{R}, \|a\|_\infty, \omega) > 0$ and $C = C(p, \overline{R}, \varepsilon, \|a\|_\infty, \omega) > 0$ such that, for all $R \in (0, \overline{R}]$,

$$\inf_{B_R(x_0)\cap\omega} \frac{u(x)}{d(x,\partial\omega)} \geq C\left(\int_{B_R(x_0)\cap\omega}\left(\frac{u(x)}{d(x,\partial\omega)}\right)^\varepsilon dx\right)^{1/\varepsilon}.$$

As already indicated, in the proof of Theorem 2.3.1 we shall make use of some ideas from [106].

Before going further, let us introduce some notation that we will be used throughout the section. We define

$$r := r(N,p) = \begin{cases} \dfrac{N(p-1)}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases}$$

and denote by $Q_\rho(y)$ the cube of center y and side of length ρ, i.e.

$$Q_\rho(y) = \{x \in \mathbb{R}^N : |x_i - y_i| < \rho/2 \text{ for } i = 1, \dots, N\}.$$

In case the center of the cube is ρe with $e = (0, 0, \dots, 1/2)$, we use the notation $Q_\rho = Q_\rho(\rho e)$.

Let us now introduce several auxiliary results that we shall need to prove Theorem 2.3.1. We begin recalling the following comparison principle for the p-Laplacian.

Lemma 2.3.2. [112, Lemma 3.1] *Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $a \in L^\infty(\omega)$ be a non-negative function. Assume that $u, v \in W^{1,p}(\omega)$ satisfy (in a weak sense)*

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u \leq -\Delta_p v + a(x)|v|^{p-2}v, & \text{in } \omega, \\ u \leq v, & \text{on } \partial\omega. \end{cases}$$

Then, it follows that $u \leq v$.

As a second ingredient, we need the weak Harnack inequality.

Theorem 2.3.3. [90, Theorem 3.13] *Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $a \in L^\infty(\omega)$ be a non-negative function. Assume that $u \in W^{1,p}(\omega)$ is a non-negative upper solution to*

$$-\Delta_p u + a(x)|u|^{p-2}u = 0, \quad u \in W_0^{1,p}(\omega),$$

and let $Q_\rho(x_0) \subset \omega$. Then, for any $\sigma, \tau \in (0,1)$ and $\gamma \in (0,r)$, there exists $C = C(p,\gamma,\sigma,\tau,\rho,\|a\|_\infty) > 0$ such that

$$\inf_{Q_{\tau\rho}(x_0)} u \geq C\Big(\int_{Q_{\sigma\rho}(x_0)} u^\gamma \, dx \Big)^{1/\gamma}.$$

In the next result, we deduce a more precise information on the dependence of C with respect to ρ. This is closely related to [113, Theorem 1.2] where however the constant still depends on ρ.

Corollary 2.3.4. *Let a be a non-negative constant and $\gamma \in (0,r)$. There exists $C = C(p,\gamma,a) > 0$ such that, for all $0 < \tilde{\rho} \leq 1$, any $u \in W^{1,p}(Q_{\frac{3\tilde{\rho}}{2}}(e))$ non-negative upper solution to*

$$-\Delta_p u + a|u|^{p-2}u = 0, \quad u \in W_0^{1,p}(Q_{\frac{3\tilde{\rho}}{2}}(e)), \tag{2.3.2}$$

satisfies

$$\inf_{Q_{\tilde{\rho}}(e)} u \geq C\,\tilde{\rho}^{-N/\gamma}\Big(\int_{Q_{\tilde{\rho}}(e)} u^\gamma \, dx \Big)^{1/\gamma}.$$

Proof. Let $C = C(p,a,\gamma) > 0$ be the constant given by Theorem 2.3.3 applied with $\rho = \frac{3}{2}$ and $\sigma = \tau = \frac{2}{3}$. This means that if $v \in W^{1,p}(Q_{\frac{3}{2}}(e))$ is a non-negative upper solution to

$$-\Delta_p v + a|v|^{p-2}v = 0, \qquad v \in W_0^{1,p}(Q_{\frac{3}{2}}(e)), \tag{2.3.3}$$

then

$$\inf_{Q_1(e)} v(y) \geq C\Big(\int_{Q_1(e)} v^\gamma \, dy \Big)^{1/\gamma}.$$

As $0 < \tilde{\rho} \leq 1$, observe that if u is a non-negative upper solution to (2.3.2), then v defined by $v(y) = u(\tilde{\rho}y', \tilde{\rho}(y_N - \frac{1}{2}) + \frac{1}{2})$, where $y = (y', y_N)$ with $y' \in \mathbb{R}^{N-1}$, is a non-negative upper solution to (2.3.3). Thus, we can conclude that

$$\inf_{Q_{\tilde{\rho}}(e)} u(x) = \inf_{Q_1(e)} v(y) \geq C\Big(\int_{Q_1(e)} v^\gamma \, dy \Big)^{1/\gamma} = C\tilde{\rho}^{-N/\gamma}\Big(\int_{Q_{\tilde{\rho}}(e)} u^\gamma \, dx \Big)^{1/\gamma}.$$

□

Finally, we introduce a technical result of measure theory.

Lemma 2.3.5. [71, Lemma 2.1] *Let $E \subset F \subset Q_1$ be two open sets. Assume there exists $\alpha \in (0,1)$ such that:*

- $|E| \leq (1-\alpha)|Q_1|$.
- *For any cube $Q \subset Q_1$, $|Q \cap E| \geq (1-\alpha)|Q|$ implies $Q \subset F$.*

Then, it follows that $|E| \leq (1 - c\alpha)|F|$ for some constant $c = c(N) \in (0,1)$.

Now, we can perform the proof of the main result. We prove the boundary weak Harnack inequality for cubes and as consequence we obtain the desired result.

Lemma 2.3.6 (**Growth lemma**). *Let a be a non-negative constant. Given $\nu > 0$, there exists $k = k(p,\nu,a) > 0$ such that, if $u \in W^{1,p}(Q_{\frac{3}{2}})$ is a non-negative upper solution to*

$$-\Delta_p u + a|u|^{p-2}u = 0, \quad u \in W_0^{1,p}(Q_{\frac{3}{2}}),$$

and the following inequality holds

$$|\{x \in Q_1 : u(x) > x_N\}| \geq \nu. \tag{2.3.4}$$

Then $u(x) > kx_N$ in Q_1.

Remark 2.3.1. Before we prove the Lemma, observe that there is no loss of generality in considering a a non-negative constant instead of $a \in L^\infty(Q_{\frac{3}{2}})$ non-negative. If $u \geq 0$ satisfies

$$-\Delta_p u + a(x)|u|^{p-2}u \geq 0, \quad \text{in } Q_{\frac{3}{2}},$$

then u satisfies also

$$-\Delta_p u + \|a\|_\infty |u|^{p-2}u \geq 0, \quad \text{in } Q_{\frac{3}{2}}.$$

Proof. Let us define $S_\delta = Q_{\frac{3}{2}} \setminus Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)$ and fix $c_1 = c_1(\nu) \in (0, \frac{1}{2})$ small enough in order to ensure that $|S_\delta| \leq \frac{\nu}{2}$ for any $0 < \delta \leq c_1$.

Step 1: *For all $\delta \in (0, c_1]$, it follows that $|\{x \in Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right) : u(x) > x_N\}| \geq \frac{\nu}{2}$.*

Directly observe that

$$\{x \in Q_1 : u(x) > x_N\} \subset \{x \in Q_{\frac{3}{2}} : u(x) > x_N\} \subset \{x \in Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right) : u(x) > x_N\} \cup S_\delta.$$

Hence, Step 1 follows from (2.3.4) and the choice of c_1.

Step 2: *For any $\varepsilon > 0$ and all $\delta \in (0, c_1]$, the following inequality holds*

$$\left(\int_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} u^\varepsilon \, dx\right)^{1/\varepsilon} \geq \frac{\delta}{2}\left(\frac{\nu}{2}\right)^{1/\varepsilon}. \tag{2.3.5}$$

Since $u \geq 0$ and, for any $x \in Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)$ we have $x_N \geq \frac{\delta}{2}$, it follows that

$$\begin{aligned}\int_{Q_{\frac{3}{2}-\delta}(\frac{3}{2}e)} u^\varepsilon \, dx &\geq \int_{\{x \in Q_{\frac{3}{2}-\delta}(\frac{3}{2}e):\ u(x) \geq x_N\}} u^\varepsilon \, dx \\ &\geq \int_{\{x \in Q_{\frac{3}{2}-\delta}(\frac{3}{2}e): u(x) \geq x_N\}} \left(\frac{\delta}{2}\right)^\varepsilon dx = \left(\frac{\delta}{2}\right)^\varepsilon \left|\left\{x \in Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right) : u(x) \geq x_N\right\}\right|.\end{aligned}$$

Step 2 follows then from Step 1.

Step 3: *For any $\varepsilon \in (0,r)$ and all $\delta \in (0,c_1]$, there exists $C_\delta = C_\delta(p,\varepsilon,\delta,a) > 0$ such that*

$$\inf_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} \frac{u(x)}{x_N} \geq \frac{\delta C_\delta}{3}\Big(\frac{\nu}{2}\Big)^{1/\varepsilon}.$$

By Theorem 2.3.3 applied with $\rho = \frac{3}{2}$, $x_0 = \frac{3}{2}e$ and $\tau = \sigma = 1 - \frac{2}{3}\delta$, there exists a constant $C_\delta = C_\delta(p,\varepsilon,\delta,a) > 0$ such that

$$\inf_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} u(x) \geq C_\delta\Big(\int_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} u^\varepsilon\,dx\Big)^{1/\varepsilon}.$$

Since for all $x \in Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)$ we have $x_N \leq \frac{3}{2}$, Step 3 follows from the above inequality and Step 2.

Step 4: *Conclusion.*

We fix $\varepsilon \in (0,r)$, define $k_\delta = \frac{\delta C_\delta}{3}\left(\frac{\nu}{2}\right)^{1/\varepsilon}$ and introduce $\eta : [-\frac{3-2c_1}{4}, \frac{3-2c_1}{4}]^{N-1} \to \mathbb{R}$ a $\mathcal{C}^\infty$ function satisfying

$$\eta(x_1,\ldots,x_{N-1}) = \begin{cases} 0, & \text{if } (x_1,\ldots,x_{N-1}) \in \Big[-\frac{1}{2},\frac{1}{2}\Big]^{N-1}, \\ \frac{c_1}{2}, & \text{if } (x_1,\ldots,x_{N-1}) \in \partial_{\mathbb{R}^{N-1}}\Big(\Big[-\frac{3-2c_1}{4},\frac{3-2c_1}{4}\Big]^{N-1}\Big), \end{cases}$$

and

$$0 \leq \eta(x_1,\ldots,x_{N-1}) \leq \frac{c_1}{2} \qquad \text{for } (x_1,\ldots,x_{N-1}) \in \Big[-\frac{3-2c_1}{4},\frac{3-2c_1}{4}\Big]^{N-1}.$$

Moreover, we consider the auxiliary function

$$v_\delta(x_1,\ldots,x_N) = \frac{1}{\delta}\Big(x_N - \eta(x_1,\ldots,x_{N-1})\Big)^2 + \Big(x_N - \eta(x_1,\ldots,x_{N-1})\Big)$$

defined in

$$\omega_\delta = \Big\{(x_1,\ldots,x_N) \in \Big[-\frac{3-2c_1}{4},\frac{3-2c_1}{4}\Big]^{N-1} \times \Big[0,\frac{c_1}{2}\Big] : \eta(x_1,\ldots,x_{N-1}) \leq x_N \leq \frac{\delta}{2}\Big\}.$$

Observe that, in ω_δ, we have $0 \leq x_N - \eta(x_1,\ldots,x_{N-1}) \leq \frac{\delta}{2}$. Hence, there exists $c_2 = c_2(p,\nu,a) \in (0,c_1]$ such that, for all $0 < \delta \leq c_2$,

$$-\Delta_p v_\delta + a|v_\delta|^{p-2}v_\delta \leq -\frac{2}{\delta}(p-1) + 2^{p-1}\Big|\sum_{i=1}^{N-1}\frac{\partial}{\partial x_i}\Big[\Big(\sum_{i=1}^{N-1}(\frac{\partial\eta}{\partial x_i})^2 + 1\Big)^{\frac{p-2}{2}}\frac{\partial\eta}{\partial x_i}\Big]\Big| + \frac{3a}{4}\delta \leq 0, \quad \text{in } \omega_\delta.$$

On the other hand, we define $u_\delta = \frac{2u}{k_\delta}$ and immediately observe that

$$-\Delta_p u_\delta + a|u_\delta|^{p-2}u_\delta \geq 0, \quad \text{in } \omega_\delta.$$

Now, since by Step 3, we have

$$u_\delta \geq \frac{2k_\delta}{k_\delta}\frac{\delta}{2} = \delta \geq v_\delta, \quad \text{for } x_N = \frac{\delta}{2},$$

it follows that

$$u_\delta \geq v_\delta \quad \text{on } \partial\omega_\delta.$$

Then, applying Lemma 2.3.2, it follows that, for any $\delta \in (0, c_2]$, $v_\delta \le u_\delta$ in ω_δ. For $\delta = c_2/2$, we obtain in particular

$$u(x) \ge \frac{1}{2} k_{\frac{c_2}{2}} v_{\frac{c_2}{2}}(x) = \frac{1}{2} k_{\frac{c_2}{2}} \Big(\frac{2}{c_2} x_N^2 + x_N \Big) \ge \frac{1}{2} k_{\frac{c_2}{2}} x_N , \quad \text{in } \omega_{\frac{c_2}{2}} \cap Q_1 .$$

The result then follows from the above inequality and Step 3. □

Lemma 2.3.7. *Let $a \in L^\infty(Q_4)$ be a non-negative function. Assume that $u \in W^{1,p}(Q_4)$ is a non-negative upper solution to*

$$-\Delta_p u + a(x)|u|^{p-2}u = 0 , \quad u \in W_0^{1,p}(Q_4), \tag{2.3.6}$$

satisfying

$$\inf_{Q_1} \frac{u(x)}{x_N} \le 1 .$$

Then, there exist $M = M(p, \|a\|_\infty) > 1$ and $\mu \in (0,1)$ such that

$$\Big| \{x \in Q_1 : u(x)/x_N > M^j\} \Big| < (1-\mu)^j , \quad \forall\, j \in \mathbb{N}. \tag{2.3.7}$$

Proof. Let us fix some notation that we use throughout the proof. We fix $\gamma \in (0, r)$ and consider $C_1 = C_1(p, \|a\|_\infty) > 0$ the constant given by Corollary 2.3.4. We introduce $\alpha \in (0,1)$ and fix $C_2 \in (0,1)$ the constant given by Lemma 2.3.5. Moreover, we choose $\nu = (1-\alpha)\big(\frac{1}{4}\big)^N$ and denote by $k = k(\nu, p, \|a\|_\infty) \in (0,1)$ the constant given by Lemma 2.3.6 applied to an upper solution to

$$-\Delta_p u + 2^p \|a\|_\infty |u|^{p-2}u = 0 , \quad u \in W_0^{1,p}(Q_{\frac{3}{2}}), \tag{2.3.8}$$

with the chosen ν. Let us point out that, if u is a non-negative upper solution to (2.3.6), then u is a non-negative upper solution to (2.3.8). Finally, we consider

$$M \ge \max\Big\{ \frac{1}{k}, \frac{4}{C_1} (1-\alpha)^{-1/\gamma} \Big\},$$

and we are going to show that (2.3.7) holds with $\mu = \alpha C_2$.

First of all, observe that $\{x \in Q_1 : u(x)/x_N > M\} \subset \{x \in Q_1 : k u(x) > x_N\}$. Hence, since $\inf_{Q_1} k u(x)/x_N \le k$, Lemma 2.3.6 implies that

$$|\{x \in Q_1 : u(x)/x_N > M\}| \le |\{x \in Q_1 : k u(x) > x_N\}| < \nu < 1 - \alpha < 1 - C_2 \alpha \tag{2.3.9}$$

and, in particular, (2.3.7) holds for $j = 1$. Now, let us introduce, for $j \in \mathbb{N} \setminus \{1\}$,

$$E = \{x \in Q_1 : u(x)/x_N > M^j\} \quad \text{and} \quad F = \{x \in Q_1 : u(x)/x_N > M^{j-1}\}.$$

Since $M > 1$ and $j \in \mathbb{N} \setminus \{1\}$, observe that (2.3.9) implies that

$$|E| = |\{x \in Q_1 : u(x)/x_N > M^j\}| \le |\{x \in Q_1 : u(x)/x_N > M\}| \le 1 - \alpha , \tag{2.3.10}$$

and the first assumption of Lemma 2.3.5 is satisfied.

Claim: *For every cube $Q_\rho(x_0) \subset Q_1$ such that*

$$|E \cap Q_\rho(x_0)| \ge (1-\alpha)|Q_\rho(x_0)| = (1-\alpha)\rho^N . \tag{2.3.11}$$

we have $Q_\rho(x_0) \subset F$.

Let us denote $x_0 = (x_0', x_{0_N})$ with $x_0' \in \mathbb{R}^{N-1}$. We define the new variable $y = \left(\frac{x'-x_0'}{\rho'}, \frac{x_N}{\rho'}\right)$, where $\rho' = 2x_{0_N}$, and the rescaled function $v(y) = \frac{1}{\rho'} u(\rho' y' + x_0', \rho' y_N)$. Then v is a non-negative upper solution to

$$-\Delta_p v + 2^p \|a\|_\infty |v|^{p-2} v = 0\,, \quad \text{in } Q_{4/\rho'}\Big(-x_0'/\rho', 2/\rho'\Big). \tag{2.3.12}$$

Moreover, observe that

$$x \in E \cap Q_\rho(x_0) \quad \text{if and only if} \quad y \in \{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\}\,,$$

and so, that (2.3.11) is equivalent to

$$|\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\}| \geq (1-\alpha)|Q_{\rho/\rho'}(e)| = (1-\alpha)\Big(\frac{\rho}{\rho'}\Big)^N. \tag{2.3.13}$$

Observe also that the embedding $Q_\rho(x_0) \subset Q_1$ implies that $\rho \leq \rho' \leq 2-\rho$ and $|x_{0,i}| \leq \frac{1-\rho}{2}$ for $i \in \{1,\ldots,N-1\}$. In particular, we have $Q_{\frac{3}{2}} \subset Q_{4/\rho'}\Big(-x_0'/\rho', 2/\rho'\Big)$. Hence v is an upper solution to (2.3.8).

Now, we distinguish two cases:

Case 1: $\rho \geq \rho'/4$. Observe that v/M^j is a non-negative upper solution to (2.3.8). Moreover, as $\rho \leq \rho'$, (2.3.13) implies that

$$|\{y \in Q_1 : v(y)/M^j > y_N\}| \geq |\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\}| \geq \nu\,.$$

Hence, by Lemma 2.3.6, $v(y)/M^j > k y_N$ in Q_1 and so, by the definition of k, $v(y)/y_N > M^{j-1}$ in $Q_{\rho/\rho'}(e)$. This implies that $u(x)/x_N > M^{j-1}$ in $Q_\rho(x_0)$.

Case 2: $\rho < \rho'/4$. Recall that v/M^j is a non-negative upper solution to (2.3.8). Hence, v/M^j is also a non-negative upper solution to

$$-\Delta_p u + 2^p \|a\|_\infty |u|^{p-2} u = 0\,, \quad \text{in } Q_{\frac{3\rho}{2\rho'}}(e) \subset Q_{\frac{3}{2}}\,,$$

Thus, by Corollary 2.3.4, we deduce that

$$\inf_{Q_{\rho/\rho'}(e)} \frac{v(y)}{M^j} \geq C_1\bigg(\Big(\frac{\rho}{\rho'}\Big)^{-N} \int_{Q_{\rho/\rho'}(e)} \Big(\frac{v}{M^j}\Big)^\gamma dy\bigg)^{1/\gamma}. \tag{2.3.14}$$

Now, let us introduce

$$G = \{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > 1/4\}\,,$$

and, as $y_N > 1/4$ for all $y \in Q_{\rho/\rho'}(e)$, observe that (2.3.13) implies the following inequality

$$|G| \geq |\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\}| \geq (1-\alpha)\Big(\frac{\rho}{\rho'}\Big)^N.$$

Hence, we deduce that

$$\int_{Q_{\rho/\rho'}(e)} \Big(\frac{v}{M^j}\Big)^\gamma dy \geq \int_G \Big(\frac{v}{M^j}\Big)^\gamma dy > \Big(\frac{1}{4}\Big)^\gamma |G| \geq \Big(\frac{1}{4}\Big)^\gamma (1-\alpha)\Big(\frac{\rho}{\rho'}\Big)^N,$$

and so, by (2.3.14), that

$$\inf_{Q_{\rho/\rho'}(e)} \frac{v}{M^j} > \frac{C_1}{4}(1-\alpha)^{1/\gamma}.$$

Finally, using that $M \geq \frac{4}{C_1}(1-\alpha)^{-1/\gamma}$ and that $y_N \leq 1$ in $Q_{\rho/\rho'}(e)$, we deduce that $v(y) > M^{j-1} y_N$ in $Q_{\rho/\rho'}(e)$. Thus, we can conclude that $u(x)/x_N > M^{j-1}$ in $Q_\rho(x_0)$.

In both cases we prove that $u(x)/x_N > M^{j-1}$ in $Q_\rho(x_0)$. This means that $Q_\rho(x_0) \subset F$ and so, the Claim is proved.

Since (2.3.10) and the Claim hold, we can apply Lemma 2.3.5 and we obtain that $|E| \leq (1 - C_2\alpha)|F|$, i.e.

$$|\{x \in Q_1 : u(x)/x_N > M^j\}| \leq (1 - C_2\alpha)|\{x \in Q_1 : u(x)/x_N > M^{j-1}\}|, \quad \forall\, j \in \mathbb{N} \setminus \{1\}.$$

Iterating in j and using (2.3.9), the result follows with $\mu = C_2\alpha \in (0,1)$ depending only on N. $\square$

Theorem 2.3.8 (Boundary weak Harnack inequality for cubes). *Let $a \in L^\infty(Q_4)$ be a non-negative function. Assume that $u \in W^{1,p}(Q_4)$ is a non-negative upper solution to*

$$-\Delta_p u + a(x)|u|^{p-2}u = 0, \quad u \in W_0^{1,p}(Q_4),$$

Then, there exist $\varepsilon = \varepsilon(p, \|a\|_\infty) > 0$ and $C = C(p, \varepsilon, \|a\|_\infty) > 0$ such that

$$\inf_{Q_1} \frac{u(x)}{x_N} \geq C\Big(\int_{Q_1} \Big(\frac{u(x)}{x_N}\Big)^\varepsilon dx\Big)^{1/\varepsilon}.$$

Proof. Let us split the proof into three steps.

Step 1: *Assume that $\inf_{Q_1} \frac{u(x)}{x_N} \leq 1$. Then, there exist $\varepsilon = \varepsilon(p, \|a\|_\infty) > 0$ and $C = C(p, \varepsilon, \|a\|_\infty) > 0$ such that, for all $t \geq 0$,*

$$|\{x \in Q_1 : u(x)/x_N > t\}| \leq C \min\{1, t^{-2\varepsilon}\}.$$

Let us define the real valued function

$$f(t) = |\{x \in Q_1 : u(x)/x_N > t\}|,$$

and let M and μ be the constants obtained in Lemma 2.3.7. We define

$$C = \max\{(1-\mu)^{-1}, M^{2\varepsilon}\} > 1 \quad \text{and} \quad \varepsilon = -\frac{1}{2}\frac{\ln(1-\mu)}{\ln M} > 0.$$

If $t \in [0, M]$, we easily get

$$|\{x \in Q_1 : u(x)/x_N > t\}| \leq 1 \leq CM^{-2\varepsilon} \leq C\min\{1, t^{-2\varepsilon}\}.$$

Hence, let us assume $t > M > 1$. Without loss of generality, we assume $t \in [M^j, M^{j+1}]$ for some $j \in \mathbb{N}$, and it follows that

$$\frac{\ln t}{\ln M} - 1 \leq j \leq \frac{\ln t}{\ln M}.$$

Since f is non-increasing and $1 - \mu \in (0,1)$, the above inequality and Lemma 2.3.7 imply

$$f(t) \leq f(M^j) \leq (1-\mu)^j \leq (1-\mu)^{\frac{\ln t}{\ln M} - 1}. \tag{2.3.15}$$

Finally, observe that

$$\ln\Big((1-\mu)^{\frac{\ln t}{\ln M}-1}\Big) = \Big(\frac{\ln t}{\ln M} - 1\Big)\ln(1-\mu) = \ln t \frac{\ln(1-\mu)}{\ln M} - \ln(1-\mu) \leq -2\varepsilon \ln t + \ln C = \ln(Ct^{-2\varepsilon}). \tag{2.3.16}$$

The Step 1 then follows from (2.3.15), (2.3.16) and the fact that $\min\{1, t^{-2\varepsilon}\} = t^{-2\varepsilon}$ for $t \geq 1$.

Step 2: *Assume that* $\inf_{Q_1}\frac{u(x)}{x_N}\leq 1$. *Then, there exists* $C=C(p,\varepsilon,\|a\|_\infty)>0$ such that

$$\int_{Q_1}\left(\frac{u(x)}{x_N}\right)^{\varepsilon}dx\leq C<+\infty. \tag{2.3.17}$$

Directly, applying [66, Lemma 9.7], we obtain that

$$\int_{Q_1}\left(\frac{u(x)}{x_N}\right)^{\varepsilon}dx=\varepsilon\int_0^{\infty}t^{\varepsilon-1}|\{x\in Q_1: u(x)/x_N>t\}|\,dt.$$

Hence, (2.3.17) follows from Step 1.

Step 3: *Conclusion.*

Let us introduce the function

$$v=\frac{u}{\inf_{y\in Q_1}\frac{u(y)}{y_N}+\beta},$$

where $\beta>0$ is an arbitrary positive constant. Obviously, v satisfies the hypothesis of Step 2. Hence, applying Step 2, we obtain that

$$\int_{Q_1}\left(\frac{u(x)}{x_N}\right)^{\varepsilon}\left(\frac{1}{\inf_{y\in Q_1}\frac{u(y)}{y_N}+\beta}\right)^{\varepsilon}dx\leq C,$$

or equivalently that

$$\frac{1}{C^{1/\varepsilon}}\left(\int_{Q_1}\left(\frac{u(x)}{x_N}\right)^{\varepsilon}dx\right)^{1/\varepsilon}\leq\inf_{Q_1}\frac{u(x)}{x_N}+\beta.$$

Letting $\beta\to 0$ we obtain the desired result. □

Proof of Theorem 2.3.1. Thanks to the regularity of the boundary, there exists $\overline{R}>0$ and a diffeomorphism φ such that $\varphi(B_{\overline{R}}(x_0)\cap\omega)\subset Q_1$ and $\varphi(B_{\overline{R}}(x_0)\cap\partial\omega)\subset\{x\in\partial Q_1: x_N=0\}$. The result then follows from Theorem 2.3.8. □

We end this section by presenting a corollary of Theorem 2.3.1. Consider the equation

$$-\Delta u+a(x)u=b(x),\qquad u\in H_0^1(\omega), \tag{2.3.18}$$

under the assumption

$$\begin{cases}\omega\subset\mathbb{R}^N,\ N\geq 2,\ \text{is a bounded domain with boundary } \partial\omega \text{ of class } \mathcal{C}^{1,1},\\ a\in L^{\infty}(\omega),\ b^-\in L^p(\omega) \text{ for some } p>N \text{ and } b^+\in L^1(\omega),\\ a\geq 0 \text{ a.e. in } \omega.\end{cases} \tag{2.3.19}$$

Corollary 2.3.9. *Under the assumption (2.3.19), assume that $u\in H^1(\omega)$ is a non-negative upper solution to (2.3.18) and let $x_0\in\partial\omega$. Then, there exist $\overline{R}>0$, $\varepsilon=\varepsilon(\overline{R},\|a\|_\infty,\omega)>0$, $C_1=C_1(\overline{R},\varepsilon,\|a\|_\infty,\omega)>0$ and $C_2=C_2(\omega,\|a\|_\infty)>0$ such that, for all $R\in(0,\overline{R}]$,*

$$\inf_{B_R(x_0)\cap\omega}\frac{u(x)}{d(x,\partial\omega)}\geq C_1\left(\int_{B_R(x_0)\cap\omega}\left(\frac{u(x)}{d(x,\partial\omega)}\right)^{\varepsilon}dx\right)^{1/\varepsilon}-C_2\|b^-\|_{L^p(\omega)}.$$

In order to prove Corollary 2.3.9 we need the following lemma

Lemma 2.3.10. *Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\omega$ of class $\mathcal{C}^{1,1}$ and let $a \in L^\infty(\omega)$ and $g \in L^p(\omega)$, $p > N$, be non-negative functions. Assume that $u \in H^1(\omega)$ is a lower solution to*

$$-\Delta u + a(x)u = g(x), \quad u \in H_0^1(\omega).$$

Then there exists $C = C(\omega, \|a\|_\infty) > 0$ such that

$$\sup_{\omega} \frac{u(x)}{d(x,\partial\omega)} \leq C\|g\|_{L^p(\omega)}.$$

Proof. First of all, observe that it is enough to prove the result for v solution to

$$\begin{cases} -\Delta v + a(x)v = g(x), & \text{in } \omega, \\ v = 0, & \text{on } \partial\omega. \end{cases}$$

as, by the standard comparison principle it follows that $u \leq v$. Applying [66, Theorem 9.15 and Lemma 9.17] we deduce that $v \in W_0^{2,p}(\omega)$ and there exists $C_1 = C_1(\omega, \|a\|_\infty) > 0$ such that

$$\|v\|_{W^{2,p}(\omega)} \leq C_1\|g\|_{L^p(\omega)}.$$

Moreover, as $p > N$, by Sobolev's inequality, we have $C_2 = C_2(\omega, \|a\|_\infty)$ with

$$\|v\|_{\mathcal{C}^1(\overline{\omega})} \leq C_2\|g\|_{L^p(\omega)},$$

and so, we easily deduce that

$$v(x) \leq C_3\|g\|_{L^p(\omega)} d(x,\partial\omega), \quad \forall\, x \in \omega.$$

Hence, since $u \leq v$, the result follows from the above inequality. □

Proof of Corollary 2.3.9. Let $w \geq 0$ be the solution to

$$\begin{cases} -\Delta w + a(x)w = b^-(x), & \text{in } \omega, \\ w = 0, & \text{on } \partial\omega. \end{cases} \tag{2.3.20}$$

Observe that $v = u + w$ satisfies

$$\begin{cases} -\Delta v + a(x)v \geq 0, & \text{in } \omega, \\ v \geq 0, & \text{on } \partial\omega. \end{cases} \tag{2.3.21}$$

Hence, by Theorem 2.3.1, there exist $\overline{R} > 0$, $\varepsilon = \varepsilon(p, \overline{R}, \|a\|_\infty, \omega) > 0$ and $C = C(p, \overline{R}, \varepsilon, \|a\|_\infty, \omega) > 0$ such that, for all $R \in (0, \overline{R}]$,

$$\inf_{B_R(x_0)\cap\omega} \frac{v(x)}{d(x,\partial\omega)} \geq C\left(\int_{B_R(x_0)\cap\omega} \left(\frac{v(x)}{d(x,\partial\omega)}\right)^\varepsilon dx\right)^{1/\varepsilon}. \tag{2.3.22}$$

On the other hand, by Lemma 2.3.10, there exists $C_2 = C_2(\omega, \|a\|_\infty) > 0$ such that

$$\sup_{\omega} \frac{w(x)}{d(x,\partial\omega)} \leq C_2\|b^-\|_{L^p(\omega)}. \tag{2.3.23}$$

From (2.3.22), (2.3.23) and using that $u = v - w$, the corollary follows observing that $w \geq 0$ and hence $v \geq u$. □

2.4 A priori bound

This section is devoted to the proof of Theorem 2.1.1. As a first step we observe that, to obtain our a priori upper bound on the solutions to (P_λ), we only need to control the solutions on Ω^+. This can be proved under a weaker assumption than (A_1). More precisely, we assume

$$\begin{cases} \Omega \subset \mathbb{R}^N,\, N \geq 2, \text{ is a bounded domain with boundary } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c_+,\ c_- \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > N/2,\ \mu \text{ belongs to } L^\infty(\Omega), \\ c_+(x) \geq 0,\ c_-(x) \geq 0 \text{ and } c_-(x)c_+(x) = 0 \text{ a.e. in } \Omega, \\ |\Omega_+| > 0, \text{ where } \Omega_+ := \mathrm{Supp}(c_+), \end{cases} \tag{B}$$

and we prove the next result.

Lemma 2.4.1. *Assume that (B) holds. Then, there exists $M > 0$ such that, for any $\lambda \in \mathbb{R}$, any solution u of (P_λ) satisfies*

$$-\sup_{\Omega_+} u^- - M \leq u \leq \sup_{\Omega_+} u^+ + M.$$

Remark 2.4.1. Let us point out that if $c_+ \equiv 0$, i.e. $|\Omega_+| = 0$, the problem (P_λ) reduces to

$$-\Delta u = -c_-(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{2.4.1}$$

which is independent of λ. If (2.4.1) has a solution, by [18, Proposition 4.1] it is unique and so, we have an a priori bound.

Proof. In case problem (P_λ) has no solution for any $\lambda \in \mathbb{R}$, there is nothing to prove. Hence, we assume the existence of $\tilde{\lambda} \in \mathbb{R}$ such that ($P_{\tilde{\lambda}}$) has a solution $\tilde{u}$. We shall prove the result with $M := 2\|\tilde{u}\|_\infty$. Let u be an arbitrary solution to (P_λ).

Step 1: $u \leq \sup_{\Omega_+} u^+ + M$.

Setting $D := \Omega \backslash \overline{\Omega}_+$ we define $v = u - \sup_{\partial D} u^+$. We then obtain

$$-\Delta v = -c_-(x)v + \mu(x)|\nabla v|^2 + h(x) - c_-(x)\sup_{\partial D} u^+ \leq -c_-(x)v + \mu(x)|\nabla v|^2 + h(x), \quad \text{in } D.$$

As $v \leq 0$ on ∂D, the function v is a lower solution to

$$-\Delta z = -c_-(x)z + \mu(x)|\nabla z|^2 + h(x), \qquad u \in H_0^1(D) \cap L^\infty(D). \tag{2.4.2}$$

Setting $\tilde{v} = \tilde{u} + \|\tilde{u}\|_\infty$ we observe that

$$-\Delta \tilde{v} = -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x) + c_-(x)\|\tilde{u}\|_\infty \geq -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x), \quad \text{in } D,$$

and thus, as $\tilde{v} \geq 0$ on ∂D, the function $\tilde{v}$ is an upper solution to (2.4.2). By [17, Lemma 2.1], we know that $u, \tilde{u} \in H^1(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and hence, $v, \tilde{v} \in H^1(D) \cap W^{1,N}_{loc}(D) \cap \mathcal{C}(\overline{D})$. Applying [17, Lemma 2.2] we conclude that $v \leq \tilde{v}$ in D namely, that

$$u - \sup_{\partial D} u^+ \leq \tilde{u} + \|\tilde{u}\|_\infty, \quad \text{in } D.$$

This gives that

$$u \le \tilde{u} + \|\tilde{u}\|_\infty + \sup_{\partial D} u^+, \quad \text{in } D,$$

and hence

$$u \le M + \sup_{\Omega_+} u^+, \quad \text{in } \Omega.$$

Step 2: $u \ge -\sup_{\Omega_+} u^- - M$.

We now define $v = u + \sup_{\partial D} u^-$ and obtain $v \ge 0$ on ∂D as well as

$$-\Delta v = -c_-(x)v + \mu(x)|\nabla v|^2 + h(x) + c_-(x)\sup_{\partial D} u^- \ge -c_-(x)v + \mu(x)|\nabla v|^2 + h(x), \quad \text{in } D.$$

Thus v is an upper solution to (2.4.2). Now defining $\tilde{v} = \tilde{u} - \|\tilde{u}\|_\infty$, again, we have $\tilde{v} \le 0$ on ∂D as well as

$$-\Delta \tilde{v} = -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x) - c_-(x)\|\tilde{u}\|_\infty \le -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x), \quad \text{in } D.$$

Thus $\tilde{v}$ is a lower solution to (2.4.2). As previously we have that $v, \tilde{v} \in H^1(D) \cap W^{1,N}_{loc}(D) \cap \mathcal{C}(\overline{D})$ and applying [17, Lemma 2.2] we obtain that $\tilde{v} \le v$ in D. Namely

$$\tilde{u} - \|\tilde{u}\|_\infty \le u + \sup_{\partial D} u^-, \quad \text{in } D.$$

Thus

$$u \ge \tilde{u} - \|\tilde{u}\|_\infty - \sup_{\partial D} u^-, \quad \text{in } D,$$

and without restriction we get that

$$u \ge -\sup_{\Omega_+} u^- - M, \quad \text{in } \Omega,$$

ending the proof. □

Now, let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution to (P_λ). Following [18, Proposition 6.1], we introduce

$$w_i(x) = \frac{1}{\mu_i}\left(e^{\mu_i u(x)} - 1\right) \quad \text{and} \quad g_i(s) = \frac{1}{\mu_i}\ln(1 + \mu_i s), \qquad i = 1, 2, \tag{2.4.3}$$

where μ_1 is given in (A_1) and $\mu_2 = \operatorname{esssup} \mu(x)$. Observe that

$$u = g_i(w_i) \quad \text{and} \quad 1 + \mu_i w_i = e^{\mu_i u}, \qquad i = 1, 2,$$

and that, by standard computations,

$$-\Delta w_i = (1 + \mu_i w_i)\Big[(\lambda c_+(x) - c_-(x))g_i(w_i) + h(x)\Big] + e^{\mu_i u}|\nabla u|^2(\mu(x) - \mu_i). \tag{2.4.4}$$

Using (2.4.4) we shall obtain a uniform a priori upper bound on u in a neighborhood of any fixed point $\overline{x} \in \overline{\Omega}_+$. We consider the two cases $\overline{x} \in \overline{\Omega}_+ \cap \Omega$ and $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$ separately.

Lemma 2.4.2. *Assume that (A_1) holds and that $\overline{x} \in \overline{\Omega}_+ \cap \Omega$. For each $\Lambda_2 > \Lambda_1 > 0$, there exist $M_I > 0$ and $R > 0$ such that, for any $\lambda \in [\Lambda_1, \Lambda_2]$, any solution u of (P_λ) satisfies $\sup_{B_R(\overline{x})} u \le M_I$.*

Proof. Under the assumption (A_1) we can find a $R > 0$ such that $\mu(x) \ge \mu_1 > 0$, $c_- \equiv 0$ in $B_{4R}(\overline{x}) \subset \Omega$ and $c_+ \gneqq 0$ in $B_R(\overline{x})$. For simplicity, in this proof, we denote $B_{mR} = B_{mR}(\overline{x})$, for $m \in \mathbb{N}$.

Since $c_- \equiv 0$ and $\mu(x) \geq \mu_1$ in B_{4R}, observe that (2.4.4) reduces to

$$-\Delta w_1 + \mu_1 h^-(x) w_1 \geq \lambda(1+\mu_1 w_1)c_+(x)g_1(w_1) + h^+(x)(1+\mu_1 w_1) - h^-(x), \quad \text{in } B_{4R}. \tag{2.4.5}$$

Let z_2 be the solution to

$$-\Delta z_2 + \mu_1 h^-(x) z_2 = -\Lambda_2 c_+(x)\frac{e^{-1}}{\mu_1}, \qquad z_2 \in H_0^1(B_{4R}). \tag{2.4.6}$$

By classical regularity arguments (see for instance [82, Theorem III-14.1]), $z_2 \in \mathcal{C}(\overline{B_{4R}})$. Hence, there exists $D = D(\overline{x}, \mu_1, \Lambda_2, \|h^-\|_{L^q(B_{4R})}, \|c_+\|_{L^q(B_{4R})}, q, R) > 0$ such that

$$z_2 \geq -D \text{ in } B_{4R}. \tag{2.4.7}$$

Moreover, by the weak maximum principle [66, Theorem 8.1], we have that $z_2 \leq 0$. Now defining $v_1 = w_1 - z_2 + \frac{1}{\mu_1}$, and since $\min_{[-1/\mu_i, +\infty[}(1+\mu_i s)g_i(s) = -\frac{e^{-1}}{\mu_i}$, we observe that v_1 satisfies

$$-\Delta v_1 + \mu_1 h^-(x) v_1 \geq \Lambda_1 c_+(x)(1+\mu_1 w_1)g_1(w_1)^+, \quad \text{in } B_{4R}. \tag{2.4.8}$$

Also, since $w_1 > -1/\mu_1$, we have $v_1 > 0$ in $\overline{B_{4R}}$. Note also that $0 < 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2$ in $\overline{B_{4R}}$. Now, we split the rest of the proof into four steps.

Step 1: *There exist $C_1 = C_1(\overline{x}, \Lambda_1, \Lambda_2, R, \mu_1, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that*

$$k := \inf_{B_R} v_1(x) \leq C_1. \tag{2.4.9}$$

In case $\mu_1 \inf_{B_R} v_1(x) \leq 1 + \mu_1 D$, where D is given by (2.4.7), the Step 1 is proved. Hence, we assume that

$$\mu_1 v_1(x) \geq 1 + \mu_1 D, \qquad \forall\, x \in B_R. \tag{2.4.10}$$

In particular, $\mu_1 v_1 + \mu_1 z_2 \geq 1$ on B_R. Now, by Lemma 2.2.4 applied on (2.4.8) with $\omega = B_{4R}$, there exists $C = C(R, \|h^-\|_{L^\infty(B_{4R})}, \mu_1, \Lambda_1, \overline{x}) > 0$ such that,

$$\begin{aligned} k &\geq C \int_{B_R} c_+(y)\Big(\mu_1 v_1(y) + \mu_1 z_2(y)\Big)\ln\Big(\mu_1 v_1(y) + \mu_1 z_2(y)\Big)dy \\ &\geq C \int_{B_R} c_+(y)(\mu_1 k - \mu_1 D)\ln\Big(\mu_1 k - \mu_1 D\Big)dy \\ &= C(\mu_1 k - \mu_1 D)\ln\Big(\mu_1 k - \mu_1 D\Big)\|c_+\|_{L^1(B_R)}. \end{aligned}$$

As $c_+ \gneqq 0$ in B_R, comparing the growth in k of the various terms, we deduce that k must remain bounded and thus the existence of $C_1 = (\overline{x}, \Lambda_1, \Lambda_2, R, \mu_1, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that (2.4.9) holds.

Step 2: *For any $1 \leq s < \frac{N}{N-2}$, there exists $C_2 = C_2(\overline{x}, \mu_1, R, s, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that*

$$\int_{B_{2R}} (1+\mu_1 w_1)^s\, dx \leq C_2.$$

Applying Lemma 2.2.3 to (2.4.8), we deduce the existence of $C = C(s, \mu_1, R, \|h^-\|_{L^q(B_{4R})}) > 0$ such that

$$\left(\int_{B_{2R}} v_1^s\, dx\right)^{1/s} \leq C \inf_{B_R} v_1\,.$$

The Step 2 follows from Step 1 observing that $0 \leq 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2 \leq \mu_1 v_1$.

Step 3: *For any* $1 \le s < \frac{N}{N-2}$*, we have, for the constant* $C_2 > 0$ *introduced in Step 2, that*

$$\int_{B_{2R}} \Big(1+\mu_2 w_2\Big)^{\frac{\mu_1 s}{\mu_2}} dx \le C_2.$$

This directly follows from Step 2 since, by the definition of w_i, we have

$$(1+\mu_2 w_2)^{\frac{\mu_1}{\mu_2}} = (e^{\mu_2 u})^{\frac{\mu_1}{\mu_2}} = e^{\mu_1 u} = (1+\mu_1 w_1).$$

Step 4: *Conclusion.*

We will show the existence of $C_3 = C_3(\overline{x}, \mu_1, \mu_2, R, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that

$$\sup_{B_R} w_2 \le C_3\,.$$

Thus, thanks to the definition of w_2, we can conclude the proof. Let us fix $s \in [1, \frac{N}{N-2})$, $r \in (\frac{N}{2}, q)$ and $\alpha = \frac{(q-r)\mu_1 s}{\mu_2 q r}$ and let $c_\alpha > 0$ such that

$$\ln(1+x) \le (1+x)^\alpha + c_\alpha, \quad \forall\ x \ge 0.$$

We introduce the auxiliary functions

$$\begin{aligned} a(x) &= \Lambda_2 c_+(x)(1+\mu_2 w_2)^\alpha + c_\alpha \Lambda_2 c_+(x) + \mu_2 h^+(x),\\ b(x) &= \frac{\Lambda_2}{\mu_2} c_+(x)(1+\mu_2 w_2)^\alpha + c_\alpha \frac{\Lambda_2}{\mu_2} c_+(x) + h^+(x) + c_-(x)\frac{e^{-1}}{\mu_2}, \end{aligned}$$

and, as $\mu(x) \le \mu_2$, we deduce from (2.4.4) that w_2 satisfies

$$\begin{cases} -\Delta w_2 \le a(x) w_2 + b(x) & \text{in } \Omega\,,\\ \quad w_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, as $q/r > 1$, by Step 3 and Hölder inequality, it follows that

$$\begin{aligned} \int_{B_{2R}} (c_+(x)(1+\mu_2 w_2)^\alpha)^r dx &\le \|c_+\|^r_{L^q(B_{2R})} \Big(\int_{B_{2R}} (1+\mu_2 w_2)^{\frac{\alpha q r}{q-r}} dx \Big)^{\frac{q-r}{q}} \\ &\le \|c_+\|^r_{L^q(B_{2R})} \Big(\int_{B_{2R}} (1+\mu_2 w_2)^{\frac{\mu_1 s}{\mu_2}} dx \Big)^{\frac{q-r}{q}} \le C_2^{\frac{q-r}{q}} \|c_+\|^r_{L^q(B_{2R})}. \end{aligned}$$

Hence, there exists $D(\overline{x}, \mu_1, \mu_2, s, R, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}, r, \|h^+\|_{L^q(B_{2R})}) > 0$ such that

$$\max\{\|a\|_{L^r(B_{2R})}, \|b\|_{L^r(B_{2R})}\} \le D\,. \tag{2.4.11}$$

Applying then Lemma 2.2.1, there exists $C(\overline{x}, \mu_1, \mu_2, s, R, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^q(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that

$$\sup_{B_R} w_2^+ \le C\Big[\Big(\int_{B_{2R}} (w_2^+)^{\frac{\mu_1}{\mu_2}s} dx\Big)^{\frac{\mu_2}{\mu_1 s}} + \|b\|_{L^r(B_{2R})}\Big] \le C\Big[\Big(\int_{B_{2R}} (w_2^+)^{\frac{\mu_1}{\mu_2}s} dx\Big)^{\frac{\mu_2}{\mu_1 s}} + D\Big].$$

On the other hand, by Step 3, we get

$$\int_{B_{2R}} (w_2^+)^{\frac{\mu_1}{\mu_2}s} dx \le C(\mu_1, \mu_2, s) \int_{B_{2R}} (1+\mu_2 w_2)^{\frac{\mu_1}{\mu_2}s} dx \le C(\mu_1, \mu_2, s)\, C_2\,,$$

and the result follows. □

Lemma 2.4.3. *Assume that (A_1) holds and that $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$. For each $\Lambda_2 > \Lambda_1 > 0$, there exist $R > 0$ and $M_B > 0$ such that, for any $\lambda \in [\Lambda_1, \Lambda_2]$, any solution to ($P_\lambda$) satisfies $\sup_{B_R(\overline{x})\cap\Omega} u \leq M_B$.*

Proof. Let $\overline{R} > 0$ given by Theorem 2.3.1. Under the assumption (A_1), we can find $R \in (0, \overline{R}/2]$ and $\Omega_1 \subset \Omega$ with $\partial\Omega_1$ of class $\mathcal{C}^{1,1}$ such that $B_{2R}(\overline{x}) \cap \Omega \subset \Omega_1$ and $\mu(x) \geq \mu_1 > 0$, $c_- \equiv 0$ and $c_+ \gneqq 0$ in Ω_1.

Since $c_- \equiv 0$ and $\mu(x) \geq \mu_1$ in Ω_1, observe that (2.4.4) reduces to

$$-\Delta w_1 + \mu_1 h^-(x) w_1 \geq \lambda(1 + \mu_1 w_1) c_+(x) g_1(w_1) + h^+(x)(1 + \mu_1 w_1) - h^-(x), \quad \text{in } \Omega_1 \tag{2.4.12}$$

Let z_2 be the solution to

$$-\Delta z_2 + \mu_1 h^-(x) z_2 = -\Lambda_2 c_+(x) \frac{e^{-1}}{\mu_1}, \qquad z_2 \in H_0^1(\Omega_1). \tag{2.4.13}$$

As in Lemma 2.4.2, $z_2 \in \mathcal{C}(\overline{\Omega_1})$ and there exists a $D = D(\mu_1, \Lambda_2, \|h^-\|_{L^q(B_{4R})}, \|c_+\|_{L^q(B_{4R})}, q, \Omega_1) > 0$ such that $-D \leq z_2 \leq 0$ on Ω_1. Now defining $v_1 = w_1 - z_2 + \frac{1}{\mu_1}$ we observe that v_1 satisfies

$$-\Delta v_1 + \mu_1 h^-(x) v_1 \geq \Lambda_1 c_+(x)(1 + \mu_1 w_1) g_1(w_1)^+, \quad \text{in } \Omega_1. \tag{2.4.14}$$

and $v_1 > 0$ on $\overline{\Omega_1}$. Note also that $0 < 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2$ on $\overline{\Omega_1}$. Next, we split the rest of the proof into three steps.

Step 1: *There exists $C_1 = C_1(\Omega_1, \overline{x}, \Lambda_1, \Lambda_2, R, \mu_1, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ such that*

$$\inf_{B_{2R}(\overline{x})\cap\Omega_1} \frac{v_1(x)}{d(x, \partial\Omega_1)} \leq C_1.$$

Choose $R_2 > 0$ and $y \in \Omega$ such that $B_{4R_2}(y) \subset B_{2R}(\overline{x}) \cap \Omega$ and $c_+ \gneqq 0$ in $B_{R_2}(y)$. As in Step 1 of Lemma 2.4.2, there exists $C = C(\Omega_1, y, \Lambda_1, \Lambda_2, R_2, \mu_1, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ such that

$$\inf_{B_{R_2}(y)} v_1(x) \leq C.$$

We conclude by observing, since $B_{4R_2}(y) \subset B_{2R}(\overline{x}) \cap \Omega_1$, that

$$\inf_{B_{2R}(\overline{x})\cap\Omega_1} \frac{v_1(x)}{d(x, \partial\Omega_1)} \leq \inf_{B_{R_2}(y)} \frac{v_1(x)}{d(x, \partial\Omega_1)} \leq \frac{1}{3R_2} \inf_{B_{R_2}(y)} v_1(x).$$

Step 2: *There exist $\varepsilon = \varepsilon(\overline{R}, \mu_1, \|h^-\|_{L^\infty(\Omega_1)}, \Omega_1) > 0$ and $C_2 = C_2(\overline{x}, \mu_1, R, \overline{R}, s, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ such that*

$$\left(\int_{B_{2R}(\overline{x})\cap\Omega} (1 + \mu_1 w_1)^\varepsilon \, dx \right)^{1/\varepsilon} \leq C_2.$$

By Theorem 2.3.1 applied on (2.4.14) and Step 1, we obtain constants $\varepsilon = \varepsilon(\overline{R}, \mu_1, \|h^-\|_{L^\infty(\Omega_1)}, \Omega_1) > 0$ and $C = C(\Omega_1, \overline{x}, \mu_1, \varepsilon, \overline{R}, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ such that

$$\left(\int_{B_{2R}(\overline{x})\cap\Omega_1} \left(\frac{v_1(x)}{d(x, \partial\Omega_1)} \right)^\varepsilon dx \right)^{1/\varepsilon} \leq C.$$

This clearly implies, since $\Omega_1 \subset \Omega$, that

$$\left(\int_{B_{2R}(\overline{x})\cap\Omega_1} v_1(x)^\varepsilon dx \right)^{1/\varepsilon} \leq C \operatorname{diam}(\Omega).$$

The Step 2 then follows observing that $0 \leq 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2 \leq \mu_1 v_1$ and taking into account that $B_{2R}(\overline{x}) \cap \Omega = B_{2R}(\overline{x}) \cap \Omega_1$.

Step 3: *Conclusion.*

Arguing exactly as in Step 3 and 4 of Lemma 2.4.2, using Lemma 2.2.2 and Step 2, we show the existence of $C_3 = C_3(\overline{x}, \mu_1, \mu_2, R, \Lambda_1, \Lambda_2, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(B_{2R}(\Omega_1))}) > 0$ such that

$$\sup_{B_R(\overline{x})\cap\Omega} w_2 \le C_3 .$$

Hence, the proof of the lemma follows by the definition of w_2. □

Proof of Theorem 2.1.1. Arguing by contradiction we assume the existence of sequences $\{\lambda_n\} \subset [\Lambda_1, \Lambda_2]$, $\{u_n\}$ solutions to (P_λ) for $\lambda = \lambda_n$ and of points $\{x_n\} \subset \Omega$ such that

$$u_n(x_n) = \max\{u_n(x) : x \in \overline{\Omega}\} \to \infty, \quad \text{as } n \to \infty. \tag{2.4.15}$$

Observe that Lemma 2.4.1 and (2.4.15) together imply the existence of a sequence of points $y_n \in \overline{\Omega}_+$ such that

$$u_n(y_n) = \max\{u_n(y) : y \in \overline{\Omega}_+\} \to \infty, \quad \text{as } n \to \infty. \tag{2.4.16}$$

Passing to a subsequence if necessary, we may assume that $\lambda_n \to \overline{\lambda} \in [\Lambda_1, \Lambda_2]$ and $y_n \to \overline{y} \in \overline{\Omega}_+$. Now, let us distinguish two cases:

- If $\overline{y} \in \overline{\Omega}_+ \cap \Omega$, Lemma 2.4.2 shows that we can find $R_I > 0$ and $M_I > 0$ such that, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution to (P_λ), then $\sup_{B_{R_I}(\overline{y})} u \le M_I$. This contradicts (2.4.16).

- If $\overline{y} \in \overline{\Omega}_+ \cap \partial\Omega$, Lemma 2.4.3 shows that we can find $R_B > 0$ and $M_B > 0$ such that, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution to (P_λ), then $\sup_{B_{R_B}(\overline{y})\cap\Omega} u \le M_B$. Again, this contradicts (2.4.16).

As (2.4.16) cannot happen, the result follows. □

2.5 Proof of Theorem 2.1.2

Let us begin with a preliminary result.

Lemma 2.5.1. *Under the assumption (A_1), assume that (P_0) has a solution u_0 for which there exist $\overline{x} \in \Omega$ and $R > 0$ such that $c_+ u_0 \gneqq 0$, $c_- \equiv 0$ and $\mu \ge 0$ in $B_R(\overline{x})$. Then there exists $\overline{\Lambda} \in (0, \infty)$ such that, for $\lambda \ge \overline{\Lambda}$, the problem (P_λ) has no solution u with $u \ge u_0$ in $B_R(\overline{x})$.*

Proof. Let us introduce $\overline{c}(x) := \min\{c_+(x), 1\}$. Observe that $0 \lneqq \overline{c} \le c_+$ and define $\gamma_1^1 > 0$ as the first eigenvalue of the problem

$$\begin{cases} -\Delta\varphi = \gamma\overline{c}(x)\varphi & \text{in } B_R(\overline{x}), \\ \varphi = 0 & \text{on } \partial B_R(\overline{x}). \end{cases} \tag{2.5.1}$$

By standard arguments, there exists $\varphi_1^1 \in C_0^1(\overline{B_R(\overline{x})})$ an associated first eigenfunction such that $\varphi_1^1(x) > 0$ for all $x \in B_R(\overline{x})$ and, denoting by n the outward normal to $\partial B_R(\overline{x})$, we also have

$$\frac{\partial \varphi_1^1(x)}{\partial n} < 0, \quad \text{on } \partial B_R(\overline{x}). \tag{2.5.2}$$

Now, let us introduce the function $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, defined as

$$\phi(x) = \begin{cases} \varphi_1^1(x), & x \in B_R(\overline{x}), \\ 0 & x \in \Omega \setminus B_R(\overline{x}), \end{cases}$$

and suppose that u is a solution to (P_λ) such that $u \geq u_0$ in $B_R(\overline{x})$. First observe that, in view of (2.5.2) and as $u \geq u_0$ on $\overline{B_R(\overline{x})}$, there exists a constant $C > 0$ independent of u such that

$$\int_{\partial B_R(\overline{x})} u \frac{\partial \varphi_1^1}{\partial n} \, dS \leq C. \tag{2.5.3}$$

Thus on one hand, using (2.5.1) and (2.5.3), we obtain

$$\begin{aligned} \int_\Omega \Big(\nabla\phi \nabla u + c_-(x)\,\phi\, u \Big) dx &= \int_{B_R(\overline{x})} \nabla\varphi_1^1 \nabla u \, dx = - \int_{B_R(\overline{x})} u \Delta \varphi_1^1 \, dx + \int_{\partial B_R(\overline{x})} u \frac{\partial \varphi_1^1}{\partial n} \, dS \\ &\leq - \int_{B_R(\overline{x})} u \Delta\varphi_1^1 \, dx + C = \gamma_1^1 \int_{B_R(\overline{x})} \overline{c}(x)\, \varphi_1^1 \, u \, dx + C \leq \gamma_1^1 \int_\Omega c_+(x)\, \phi\, u \, dx + C. \end{aligned} \tag{2.5.4}$$

On the other hand, considering ϕ as test function in (P_λ) we observe that

$$\int_\Omega \Big(\nabla\phi\nabla u + c_-(x)\,\phi\, u \Big) dx = \lambda \int_\Omega c_+(x)\, u\, \phi \, dx + \int_\Omega \Big(\mu(x)|\nabla u|^2 + h(x) \Big) \phi \, dx. \tag{2.5.5}$$

From (2.5.4) and (2.5.5), we then deduce that, for a $D > 0$ independent of u.

$$\begin{aligned} (\gamma_1^1 - \lambda) \int_\Omega c_+(x)\,\phi\, u \, dx &\geq \int_\Omega \Big(\mu(x)|\nabla u|^2 + h(x) \Big) \phi \, dx - C \\ &= \int_{B_R(\overline{x})} \Big(\mu(x)|\nabla u|^2 + h(x) \Big) \varphi_1^1 \, dx - C \geq -D. \end{aligned} \tag{2.5.6}$$

As $c_+ u_0 \gneqq 0$ in $B_R(\overline{x})$, we have that

$$\int_\Omega c_+(x)\,\phi\, u \, dx \geq \int_\Omega c_+(x)\, \phi\, u_0 \, dx > 0.$$

Hence, for $\lambda > \gamma_1^1$ large enough, we obtain a contradiction with (2.5.6). □

Proof of Theorem 2.1.2. We treat separately the cases $\lambda \leq 0$ and $\lambda > 0$.

Part 1: $\lambda \leq 0$.

Observe that for $\lambda \leq 0$ we have $\lambda c_+ - c_- \leq -c_-$ and hence the result follows from [18, Lemma 5.1, Proposition 4.1, Proposition 5.1, Theorem 2.2] as in the proof of [18, Theorem 1.2]. Moreover, observe that u_0 is an upper solution to (P_λ). Hence we conclude that $u_\lambda \leq u_0$ by [17, Lemmas 2.1 and 2.2].

Part 2: $\lambda > 0$.

Consider, for $\lambda \geq 0$ the modified problem

$$-\Delta u + u = (\lambda c_+(x) - c_-(x) + 1)\,((u-u_0)^+ + u_0) + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{$\overline{P}_\lambda$}$$

As in the case of (P_λ), any solution to ($\overline{P}_\lambda$) belongs to $\mathcal{C}^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. Moreover, observe that u is a solution to ($\overline{P}_\lambda$) if and only if it is a fixed point of the operator $\overline{T}_\lambda$ defined by $\overline{T}_\lambda : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega}) : v \mapsto u$ with u the solution to

$$-\Delta u + u - \mu(x)|\nabla u|^2 = (\lambda c_+(x) - c_-(x) + 1)\,((v - u_0)^+ + u_0) + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Applying [18, Lemma 5.2], we see that $\overline{T}_\lambda$ is completely continuous. Now, we denote

$$\overline{\Sigma} := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (\overline{P}_\lambda)\}$$

and we split the rest of the proof into three steps.

Step 1: *If u is a solution to $(\overline{P}_\lambda)$ then $u \geq u_0$ and hence it is a solution to (P_λ).*

Observe that $(u-u_0)^+ + u_0 - u \geq 0$. Also we have that $\lambda c_+(x)((u-u_0)^+ + u_0) \geq \lambda c_+(x)u_0 \geq 0$. Hence, we deduce that a solution u of $(\overline{P}_\lambda)$ is an upper solution to

$$-\Delta u = -c_-(x)\left((u-u_0)^+ + u_0\right) + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{2.5.7}$$

Then the result follows from [17, Lemmas 2.1 and 2.2] noting that u_0 is a solution to (2.5.7).

Step 2: *u_0 is the unique solution to $(\overline{P}_0)$ and $i(I - \overline{T}_0, u_0) = 1$.*

Again the uniqueness of the solution to $(\overline{P}_0)$ can be deduced from [17, Lemmas 2.1 and 2.2]. Now, in order to prove that $i(I - \overline{T}_0, u_0) = 1$, we consider the operator S_t defined by $S_t : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega}) : v \mapsto u$ with u the solution to

$$-\Delta u + u - \mu(x)|\nabla u|^2 = t[(-c_-(x)+1)\left(u_0 + (v-u_0)^+ - (v-u_0-1)^+\right) + h(x)], \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

First, observe that there exists $R > 0$ such that, for all $t \in [0,1]$ and all $v \in \mathcal{C}(\overline{\Omega})$,

$$\|S_t v\|_\infty < R.$$

This implies that

$$\deg(I - S_1, B(0,R)) = \deg(I, B(0,R)) = 1.$$

By [17, Lemmas 2.1 and 2.2], we see that u_0 is the only fixed point of S_1. Hence, by the excision property of the degree, for all $\varepsilon > 0$ small enough, it follows that

$$\deg(I - S_1, B(u_0, \varepsilon)) = \deg(I - S_1, B(0,R)) = 1.$$

Thus, as for $\varepsilon < 1$, $S_1 = \overline{T}_0$, we conclude that

$$i(I - \overline{T}_0, u_0) = \lim_{\varepsilon \to 0} \deg(I - \overline{T}_0, B(u_0, \varepsilon)) = \lim_{\varepsilon \to 0} \deg(I - S_1, B(u_0, \varepsilon)) = 1.$$

Step 3: *Existence and behavior of the continuum.*

By [99, Theorem 3.2] (see also [18, Theorem 2.2]), there exists a continuum $\mathscr{C} \subset \overline{\Sigma}$ such that $\mathscr{C} \cap ([0,\infty) \times \mathcal{C}(\overline{\Omega}))$ is unbounded. By Step 1, we know that if $u \in \mathscr{C}$ then $u \geq u_0$ and is a solution to (P_λ). Thus applying Lemma 2.5.1, we deduce that $\mathrm{Proj}_{\mathbb{R}} \mathscr{C} \cap [0,\infty) \subset [0, \overline{\Lambda}]$. By Theorem 2.1.1 and Step 1, we deduce that for every $\Lambda_1 \in (0, \overline{\Lambda})$, there is an a priori bound on the solutions to $(\overline{P}_\lambda)$ for $\lambda \in [\Lambda_1, \overline{\Lambda}]$. Hence, the projection of $\mathscr{C} \cap ([\Lambda_1, \overline{\Lambda}) \times \mathcal{C}(\overline{\Omega}))$ on $\mathcal{C}(\overline{\Omega})$ is bounded, and so, we deduce that $\mathscr{C}$ emanates from infinity to the right of $\lambda = 0$. Finally, since $\mathscr{C}$ contains $(0, u_0)$ with u_0 the unique solution to (P_0), we conclude that there exists $\lambda_0 \in (0, \overline{\Lambda})$ such that problem $(\overline{P}_\lambda)$, and thus problem (P_λ), has at least two solutions satisfying $u \geq u_0$ for $\lambda \in (0, \lambda_0)$. $\square$

3

Two solutions for an indefinite elliptic problem with critical growth in the gradient

3.1 Introduction and main results

In this chapter we will study the existence and multiplicity of solutions to boundary value problems of the form

$$\begin{cases} -\Delta u = c(x)u + \mu(x)|\nabla u|^2 + h(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary, c and h belong to $L^q(\Omega)$ for some $q > N/2$, μ belongs to $L^\infty(\Omega)$ and the solutions are searched in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

There exist several mathematical reasons that make the study of nonlinear elliptic PDEs with quadratic growth in the gradient interesting. For instance, J. L. Kazdan and R. J. Kramer observed in 1978 that second order PDEs with quadratic growth in the gradient are invariant under changes of variable of type $v = F(u)$. This took them to claim on [78, page 619] that *"In the long run, the class of semilinear equations should be less important than some more general class of equations that is invariant under changes of variables"*. From a pure mathematical point of view, it is also worth noting that, in Riemannian geometry, this type of equations naturally appears in the study of gradient Ricci solitons, see for instance [89, Section 1]. On the other hand, concerning more practical reasons, we would like to mention that problem (P) with $c \equiv 0$ corresponds to the stationary case of the Kardar-Parisi-Zhang model of growing interfaces introduced in [77].

The study of nonlinear elliptic PDEs with a gradient dependence up to the critical growth was essentially initiated by L. Boccardo, F. Murat and J.-P. Puel in the 80's. In the case where $c(x) \leq \alpha_0 < 0$ a.e. in Ω for some $\alpha_0 < 0$, now referred to as the *coercive case*, the existence of a solution to (P) is a particular case of the results of [23–25] and its uniqueness follows from [19,20]. The *weakly coercive case* $c \equiv 0$ was first studied in [62] where, for $\|\mu h\|_{N/2}$ small enough, the authors proved the existence of a solution to (P). For $\mu(x) \equiv \mu > 0$ constant and $h \gneqq 0$ theses results were then improved in [4]. Finally, in the recent work [46] we completely characterised the existence of solutions to (P) in the *weakly coercive case* $c \equiv 0$. The *limit coercive case* where one only requires $c(x) \le 0$ a.e. in Ω (i.e. allowing parts of the domain where $c \equiv 0$ and parts of it where $c < 0$) proved to be more complex to treat. In [18], the authors observed that the existence of a solution to (P) is not guaranteed and gave sufficient conditions to ensure such existence. In case h does not have a sign, weaker sufficient conditions can be found in [46]. The fact that the uniqueness also holds in the *limit coercive case* $c \le 0$ was proved in [18]. We refer likewise to [17] for more general uniqueness results in this framework. Finally, let us point out that, except for [4], all these results were obtained without requiring any sign conditions on μ and h.

If $c(x) \not\le 0$ a.e. in Ω, i.e. $c \gneqq 0$ or c changes sign, problem (P) behaves very differently and becomes much more richer than for $c \le 0$. The first paper which addressed this situation was [76]. Following [104], which considered a particular case, the authors studied (P) with $c \gneqq 0$ and $\mu(x) \equiv \mu > 0$ constant. For $\|c\|_q$ and $\|\mu h\|_{N/2}$ small enough the existence of two solutions to (P) was obtained. This result has now been improved in several ways. The restriction μ constant was first removed in [18]. In that paper the authors imposed on c a dependence on a real parameter λ and considered $\lambda c \gneqq 0$. For $\mu(x) \ge \mu_1 > 0$ a.e. in Ω and $h \gneqq 0$, they proved the existence of at least two solutions for $\lambda > 0$ small enough. In this direction we refer also to [50] where, imposing stronger regularity on c and h, the authors removed the condition $h \gneqq 0$. Under different sets of assumptions, the authors clarified the structure of the set of solutions to (P) for $\lambda c \gneqq 0$. Note that in [46] the above results were extended to the more general p-Laplacian case at the expense of considering μ constant. Also, in the frame of viscosity solutions and fully nonlinear equations, similar conclusions have been obtained in [92] under corresponding assumptions. All the above mentioned results require either μ to be constant or to be uniformly bounded from below by a positive constant (or similarly bounded from above by a negative constant). In [108], assuming that λc, μ and h were non-negative, a first attempt to remove these restriction was presented. Under suitable assumptions on the support of the coefficient functions and for $N \le 5$, the existence of at least two solutions for $\lambda > 0$ small enough was obtained. Finally, let us point out that the only papers dealing with c which may change sign are [48, 75]. In [75], the authors dealt with $\mu(x) \equiv \mu > 0$ constant and $h \gneqq 0$ and they proved the existence of two solutions to (P) for $\|c^+\|_q$ and $\|\mu h\|_{N/2}$ small enough. The restrictions $\mu > 0$ constant and $h \gneqq 0$ were removed in [48] at the expense of considering a "thick zero set" condition on the support of c and suitable assumptions on μ. Let us stress that [48] is the unique paper dealing with the *non-coercive case* $c \not\le 0$ where μ may change sign.

In this chapter we pursue the study of (P) and consider several situations where c and h may change sign. At the expense of considering μ constant we remove the "thick zero set" condition on c considered in [48]. Moreover, we extend in several directions the previously known results and we clarify the structure of the set of solutions in the case where $c^+ \not\equiv 0$.

As a first main result, we completely characterize the *limit coercive case*. Let us consider the boundary value problem

$$-\Delta u = -d(x)u + \mu|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{3.1.1}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \ge 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ d \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > N/2, \\ \mu > 0 \text{ and } d \ge 0, \end{cases} \tag{3.1.2}$$

and define

$$m_d := \begin{cases} \inf\limits_{u\in W_d} \displaystyle\int_\Omega (|\nabla u|^2 - \mu h(x) u^2) dx, & \text{if} \quad W_d \neq \emptyset, \\ +\infty, & \text{if} \quad W_d = \emptyset, \end{cases} \tag{3.1.3}$$

where

$$W_d := \{w \in H_0^1(\Omega) : d(x)w(x) = 0 \text{ a.e. in } \Omega,\ \|w\| = 1\}.$$

We prove the following sharp result.

Theorem 3.1.1. *Assume that (3.1.2) holds. Then (3.1.1) has a solution if, and only if,* $m_d > 0$.

Remark 3.1.1.

a) This result generalizes [18, Proposition 3.1 and Remark 3.2] and, for $p = 2$, [46, Theorem 1.3].

b) By [17, Theorem 1.1] we know that the solution obtained is unique.

As observed in [48], the structure of the set of solutions to (P) depends on the size of c^+ but it is not affected by the size of c^-. In order to clarify this, we replace c by a function $c_\lambda := \lambda c_+ - c_-$ with λ a real parameter. More precisely, we consider the boundary value problem

$$-\Delta u = c_\lambda(x) u + \mu |\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{P_λ}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c_+,\ c_- \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > N/2, \\ \mu > 0,\ c_+ \gneqq 0,\ c_- \geq 0 \text{ and } c_+(x)c_-(x) = 0 \text{ a.e. } x \in \Omega. \end{cases} \tag{A_1}$$

Remark 3.1.2. Since h does not have a sign, there is no loss of generality in assuming $\mu > 0$. If u is a solution to (P_λ) with $\mu < 0$ then $w = -u$ solves

$$-\Delta w = c_\lambda(x) w - \mu |\nabla w|^2 - h(x), \quad w \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Before going further and due to its importance on the rest of the chapter, let us stress that for $\lambda = 0$ the problem (P_λ) reduces to

$$-\Delta u = -c_-(x) u + \mu |\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{P_0}$$

Then, as an immediate corollary of Theorem 3.1.1, we have the following result.

Corollary 3.1.2. *Assume that (A_1) holds. Then (P_0) has a solution if, and only if,* $m_{c_-} > 0$.

Now, having at hand this satisfactory information about the *limit coercive case*, we turn to the study of the *non-coercive case* $\lambda > 0$. First, using mainly variational techniques, we prove the following theorem.

Theorem 3.1.3. *Assume (A_1) and suppose that (P_0) has a solution. Then, there exists* $\Lambda > 0$ *such that, for all* $0 < \lambda < \Lambda$, *(P_λ) has at least two solutions.*

Remark 3.1.3. This result improves and generalizes the main result obtained in [75].

Next, considering stronger regularity assumptions on the coefficient functions and combining lower and upper solution and variational techniques, we improve the conclusions of Theorem 3.1.3. We derive a more precise information on the structure of the set of solutions to (P_λ) when $\lambda > 0$. Let us first introduce the following order notions.

Definition 3.1.1. For $h_1, h_2 \in L^1(\Omega)$ we write

- $h_1 \le h_2$ if $h_1(x) \le h_2(x)$ for a.e. $x \in \Omega$,
- $h_1 \lneqq h_2$ if $h_1 \le h_2$ and $\text{meas}(\{x \in \Omega : h_1(x) < h_2(x)\}) > 0$.

For $u, v \in \mathcal{C}^1(\overline{\Omega})$ we write

- $u < v$ if, for all $x \in \Omega$, $u(x) < v(x)$,
- $u \ll v$ if $u < v$ and, for all $x \in \partial\Omega$, either $u(x) < v(x)$, or, $u(x) = v(x)$ and $\frac{\partial u}{\partial \nu}(x) > \frac{\partial v}{\partial \nu}(x)$, where ν denotes the exterior unit normal.

Under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \ge 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{1,1}, \\ c_+, c_-, \text{ and } h \text{ belong to } L^p(\Omega) \text{ for some } p > N, \\ \mu > 0,\ c_+ \gneqq 0,\ c_- \ge 0 \text{ and } c_+(x)c_-(x) = 0 \text{ a.e. in } \Omega, \end{cases} \tag{A_2}$$

we prove the following theorems.

Theorem 3.1.4. *Assume (A_2) and suppose (P_0) has a solution u_0 with $c_+u_0 \gneqq 0$. Then, every u solution to (P_λ) with $\lambda > 0$ and $c_+u \ge 0$ satisfies $u \gg u_0$. Moreover, there exists $\overline{\lambda} \in]0, +\infty[$, such that:*

- *for every $\lambda \in]0, \overline{\lambda}[$, ($P_\lambda$) has at least two solutions $u_{\lambda,1}, u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ such that $u_{\lambda,1} \gg u_0$;*
- *(P_λ) with $\lambda = \overline{\lambda}$ has at least one solution $u_{\overline{\lambda}} \in \mathcal{C}_0^1(\overline{\Omega})$ such that $u_{\overline{\lambda}} \ge u_0$;*
- *for $\lambda > \overline{\lambda}$ the problem (P_λ) has no solution u such that $c_+u \ge 0$.*

Theorem 3.1.5. *Assume (A_2) and suppose (P_0) has a solution u_0 with $c_+u_0 \lneqq 0$. Then, for every $\lambda > 0$, the problem (P_λ) has at least two solutions $u_{\lambda,1}, u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ such that $u_{\lambda,1} \ll u_0$.*

Remark 3.1.4.

a) Under the assumption (A_2), every solution to (P_λ) belongs to $\mathcal{C}_0^1(\overline{\Omega})$. This was proved in [50, Theorem 2.2].

b) At the expense of considering $\mu > 0$ constant instead of $\mu \in L^\infty(\Omega)$ with $\mu(x) \ge \mu_1 > 0$ in Ω, Theorems 3.1.4 and 3.1.5 extend the main existence results of [50] to the case where c may change sign. Moreover, unlike [50], we do not assume global sign conditions on u_0 (solution to (P_0)). Hence, even in the case where $c_- \equiv 0$, i.e. c has a sign, our hypotheses are weaker than the corresponding ones in [50].

c) Theorem 3.1.4 removes the "thick zero set" condition on the support of c_λ considered in [48, Theorem 1.2] and gives somehow a more precise information. In turn, here μ is constant and we require a stronger regularity on the coefficient functions c_λ and h^+.

Finally, we give sufficient conditions in terms of h ensuring that the hypotheses of Theorem 3.1.4 or of Theorem 3.1.5 are satisfied.

Corollary 3.1.6. *Assume that* (A_2) *holds and suppose that* (P_0) *has a solution:*

- *If* $h \gneqq 0$*, then the conclusions of Theorem 3.1.4 hold.*
- *If* $h \lneqq 0$*, then the conclusions of Theorem 3.1.5 hold.*

Remark 3.1.5. In case $h \lneqq 0$, the problem (P_0) has always a solution.

We provide now some ideas of the proofs of Theorems 3.1.3, 3.1.4 and 3.1.5. First of all we should notice that, as μ is assumed to be a constant, we can perform a Hopf-Cole change of variable and reduce (P_λ) to a semilinear problem. Considering

$$v = \frac{1}{\mu}\left(e^{\mu u} - 1\right), \tag{3.1.4}$$

one can check that u is a solution to (P_λ) if, and only if, $v > -1/\mu$ is a solution to

$$-\Delta v = c_\lambda(x) g(v) + (1 + \mu v) h(x), \quad v \in H_0^1(\Omega), \tag{3.1.5}$$

where g is given by

$$g(s) = \frac{1}{\mu}(1 + \mu s)\ln(1 + \mu s), \quad \text{for } s > -1/\mu.$$

Hence, we need a control from below on the solutions to (3.1.5). More precisely, if v is a solution to (3.1.5), we need to verify that $v > -1/\mu$. To this aim, in Section 3.4, we construct a lower solution $\underline{u}_\lambda$ to (P_λ) below every upper solution to this problem. The fact that c_λ has no sign causes several difficulties in this construction. We refer to Proposition 3.4.2 for more details. This lower solution allows us to introduce a new problem, which is completely equivalent to (P_λ). We define

$$\alpha_\lambda = \frac{1}{\mu}\left(e^{\mu \underline{u}_\lambda} - 1\right)$$

and introduce the problem

$$-\Delta v = f_\lambda(x, v), \quad v \in H_0^1(\Omega), \tag{Q_λ}$$

where

$$f_\lambda(x,s) = \begin{cases} c_\lambda(x) g(s) + (1 + \mu s) h(x), & \text{if } s \geq \alpha_\lambda(x), \\ c_\lambda(x) g(\alpha_\lambda(x)) + (1 + \mu \alpha_\lambda(x)) h(x), & \text{if } s \leq \alpha_\lambda(x). \end{cases} \tag{3.1.6}$$

Then, we show that u is a solution to (P_λ) if, and only if, v defined by (3.1.4) is a solution to (Q_λ).

The main advantage of the problem (Q_λ) which respect to (P_λ) is that it admits a variational formulation. We shall then look for solutions to (Q_λ) as critical points of an associated functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$I_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F_\lambda(x, v) dx,$$

where $G(s) = \int_0^s g(t)dt$,

$$F_\lambda(x,s) = c_\lambda(x) G(s) + \frac{1}{2\mu}(1 + \mu s)^2 h(x), \quad \text{if } s \geq \alpha_\lambda(x),$$

and

$$F_\lambda(x,s) = \Big[c_\lambda(x)g(\alpha_\lambda(x)) + (1+\mu\alpha_\lambda(x))h(x)\Big](s-\alpha_\lambda(x)) + c_\lambda(x)\,G(\alpha_\lambda(x)) + \frac{1}{2\mu}(1+\mu\alpha_\lambda(x))^2 h(x), \quad \text{if } s \le \alpha_\lambda(x).$$

When λ is positive this functional becomes unbounded from below and presents a concave-convex type geometry. Then, in trying to obtain critical points, we have to overcome several difficulties. First, we shall notice that g is only slightly superlinear at infinity. Hence, I_λ does not satisfies an Ambrosetti-Rabinowitz type condition. Moreover, the coefficient functions c_λ and h have no sign. In this context, to prove that the Palais-Smale sequences are bounded may be challenging. Our proof is inspired by [75]. However, since we do not impose $h \gneqq 0$, the proof becomes more involved. The role of the lower solution that we will construct in Proposition 3.4.2 is again crucial. See Section 3.5 for more details.

Having at hand the Palais-Smale condition for I_λ with $\lambda > 0$, we shall look for critical points which are either local minimum or of a mountain-pass type. In Theorem 3.1.3, we work mainly with variational techniques as in [75, 76]. Nevertheless, since our hypotheses are weaker than the corresponding ones in [75,76], to prove that the mountain-pass geometry holds becomes much more involved. In Theorems 3.1.4 and 3.1.5 we combine lower and upper solution with variational techniques. In both theorems a first solution is obtained throughout the existence of well-ordered lower and upper solution. This solution is further proved to be a local minimum. Then, we obtain a second solution by a mountain-pass type argument.

Another key ingredient in the proofs of Theorems 3.1.4 and 3.1.5 is the following estimate that can be seen as a combination of the Strong maximum principle and the Hopf's Lemma with unbounded lower order coefficients. Under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \ge 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{1,1}, \\ c \text{ belongs to } L^p(\Omega) \text{ and } B = (B^1,\dots,B^N) \text{ belongs to } (L^p(\Omega))^N \text{ for some } p > N, \\ c \ge 0, \end{cases} \tag{3.1.7}$$

we obtain the following result.

Theorem 3.1.7. *Assume (3.1.7) and let $u \in \mathcal{C}^1(\overline{\Omega})$ be an upper solution to*

$$-\Delta u + \langle B(x), \nabla u\rangle + c(x)u = 0, \quad u \in H_0^1(\Omega). \tag{3.1.8}$$

Then, either $u \equiv 0$ or $u \gg 0$.

Remark 3.1.6.

a) The case where $B \in (L^\infty(\Omega))^N$ and $c \in L^\infty(\Omega)$ is nowadays classical and can be founded for instance in [112, Theorem 3.27].

b) Theorem 3.1.7 can be obtained as a corollary from [100, Theorem 4.1]. Nevertheless, for the benefit of the reader, we provide a self-contained simplified proof in Appendix 3.7.

The rest of the chapter is organized as follows. In Section 3.2 we recall some auxiliary results that will be useful. Section 3.3 is devoted to the proof of Theorem 3.1.1. In Section 3.4 we construct the lower solution α_λ and we present the functional setting to deal with (Q_λ). Section 3.5 is devoted to prove the Palais-Smale condition and to show that, if (P_0) has a solution, then I_λ has a mountain-pass geometry. This allows us to prove Theorem 3.1.3. Section 3.6 is devoted to the proofs of Theorems 3.1.4 and 3.1.5 and Corollary 3.1.6. Finally, in Appendix 3.7, we prove the Hopf's Lemma with unbounded lower order terms. This permits to prove Theorem 3.1.7.

Notation.

1) In $\mathbb{R}^N$, we use the notations $|x| = \sqrt{x_1^2 + \ldots + x_N^2}$ and $B_R(y) = \{x \in \mathbb{R}^N : |x - y| < R\}$.

2) We denote $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0)$.

3) For $v \in L^1(\Omega)$ we define $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$.

4) For a, $b \in L^1(\Omega)$ we denote $\{a \le b\} = \{x \in \Omega : a(x) \le b(x)\}$.

5) The space $H_0^1(\Omega)$ is equipped with the norm $\|u\| := \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2}$.

6) For $p \in [1, +\infty[$, the norm $(\int_\Omega |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. We denote by p' the conjugate exponent of p and by 2^* the Sobolev critical exponent i.e. $2^* = 2N/(N-2)$ if $N \ge 3$ and $2^* = +\infty$ in case $N = 2$. The norm in $L^\infty(\Omega)$ is $\|u\|_\infty = \text{esssup}_{x \in \Omega} |u(x)|$.

3.2 Preliminaries

This section presents some definitions and known results which are going to play an important role throughout the work. Let us start with some theory of lower and upper solution. We consider the boundary value problem

$$-\Delta u + H(x, u, \nabla u) = \xi(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{3.2.1}$$

where ξ belongs to $L^1(\Omega)$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (see [49, Definition I-3.1] for the definition Carathéodory function).

Definition 3.2.1. We say that $\alpha \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a *lower solution to* (3.2.1) if $\alpha^+ \in H_0^1(\Omega)$ and, for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\varphi \ge 0$, if follows that

$$\int_\Omega \nabla \alpha \nabla \varphi \, dx + \int_\Omega H(x, \alpha, \nabla \alpha) \varphi \, dx \le \int_\Omega \xi(x) \varphi \, dx.$$

Similarly, $\beta \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is an *upper solution to* (3.2.1) if $\beta^- \in H_0^1(\Omega)$ and, for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\varphi \ge 0$, if follows that

$$\int_\Omega \nabla \beta \nabla \varphi \, dx + \int_\Omega H(x, \beta, \nabla \beta) \varphi \, dx \ge \int_\Omega \xi(x) \varphi \, dx.$$

Theorem 3.2.1. [24, Theorems 3.1 and 4.2] *Assume the existence of a non-decreasing function $b : \mathbb{R}^+ \to R^+$ and a function $k \in L^1(\Omega)$ such that*

$$|H(x, s, \xi)| \le b(|s|)[k(x) + |\xi|^2], \quad \text{a.e. } x \in \Omega, \ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

If there exist a lower solution α and an upper solution β of (3.2.1) *with $\alpha \le \beta$, then there exists a solution u of* (3.2.1) *with $\alpha \le u \le \beta$. Moreover, there exists u_{min} (resp. u_{max}) minimum (resp. maximum) solution to* (3.2.1) *with $\alpha \le u_{min} \le u_{max} \le \beta$ and such that, every solution u of* (3.2.1) *with $\alpha \le u \le \beta$ satisfies $u_{min} \le u \le u_{max}$.*

Definition 3.2.2. A *lower solution* $\alpha \in \mathcal{C}^1(\overline{\Omega})$ is said to be *strict* if every solution u of (3.2.1) with $u \ge \alpha$ satisfies $u \gg \alpha$. Similarly, an *upper solution* $\beta \in \mathcal{C}^1(\overline{\Omega})$ is said to be *strict* if every solution u of (3.2.1) such that $u \le \beta$ satisfies $u \ll \beta$.

Now, we consider the boundary value problem

$$-\Delta v = f(x,v), \quad v \in H_0^1(\Omega), \tag{3.2.2}$$

being $f : \Omega \times \mathbb{R} \to \mathbb{R}$ an L^p-Carathéodory function (see [49, Definition I-3.1] for the definition of L^p-Carathéodory function) for some $p > N$, such that

$$|f(x,s)| \leq C|s|^{\frac{N+2}{N-2}} + d(x),$$

for some $C > 0$ and $d \in L^{\frac{2N}{N+2}}(\Omega)$. This problem can be handled variationally. We consider the associated functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega F(x,v)\,dx, \quad \text{where} \quad \frac{\partial}{\partial s}F(x,s) = f(x,s),$$

and we recall the following results.

Proposition 3.2.2. [54, Theorem 6] [110, Chapter 1, Theorem 2.4] *Assume that α and β are respectively a lower and an upper solution to (3.2.2) with $\alpha \leq \beta$ and consider*

$$M := \Big\{ v \in H_0^1(\Omega) : \alpha \leq v \leq \beta \Big\}.$$

Then the infimum of J on M is achieved at some v, and such v is a solution to (3.2.2).

Corollary 3.2.3. *Assume that α and β are strict lower and upper solutions to (3.2.2) belonging to $\mathcal{C}^1(\overline{\Omega})$ and satisfying $\alpha \ll \beta$ and let M be defined as in Proposition 3.2.2. Then the minimizer v of J on M is a local minimizer of the functional J in the $\mathcal{C}_0^1$-topology. Furthermore, this minimizer is a solution to (3.2.2) with $\alpha \ll v \ll \beta$.*

Proof. First of all observe that Proposition 3.2.2 implies the existence of $v \in H_0^1(\Omega)$ solution to (3.2.2), which minimizes J on $M = \{v \in H_0^1(\Omega) : \alpha \leq v \leq \beta\}$. Moreover, as f is an L^p-Carathéodory function for some $p > N$, the classical regularity results imply that $v \in \mathcal{C}^1(\overline{\Omega})$. Since the lower and the upper solutions are strict, it follows that $\alpha \ll v \ll \beta$ and so, there is a $\mathcal{C}_0^1$-neighbourhood of v in M. Hence, it follows that v minimizes locally J in the $\mathcal{C}_0^1$-topology. □

Proposition 3.2.4. [54, Theorem 8] [29, Theorem 1] *Assume that there exist $h \in L^p(\Omega)$ for some $p > N$ and $\sigma \leq \frac{2^*(p-1)}{p} - 1$ such that*

$$|f(x,s)| \leq h(x)(1+|s|^\sigma), \quad \text{a.e. } x \in \Omega, \ \forall\, s \in \mathbb{R},$$

and let $v \in H_0^1(\Omega)$ be a local minimizer of J for the $\mathcal{C}_0^1$-topology. Then $v \in \mathcal{C}_0^1(\overline{\Omega})$ and it is a local minimizer of J in the H_0^1-topology.

Remark 3.2.1. If f is an L^∞-Carathéodory function the result holds under the growth condition

$$|f(x,s)| \leq C(1+|s|^\sigma), \quad \text{a.e. } x \in \Omega, \ \forall\, s \in \mathbb{R},$$

for some $C > 0$ and $\alpha \leq 2^* - 1$. In that case, we are exactly in the framework of [29].

Finally, we recall some abstract results in order to find critical points of J other than local minima.

Definition 3.2.3. Let $(X, \|\cdot\|)$ be a real Banach space with dual space $(X^*, \|\cdot\|_*)$ and let $\Phi : X \to \mathbb{R}$ be a $\mathcal{C}^1$ functional. The functional Φ satisfies the *Palais-Smale condition at level* $c \in \mathbb{R}$ if, for any *Palais-Smale sequence* at level $c \in \mathbb{R}$, i.e. for any sequence $\{x_n\} \subset X$ with

$$\Phi(x_n) \to c \quad \text{and} \quad \|\Phi'(x_n)\|_* \to 0\,,$$

there exists a subsequence $\{x_{n_k}\}$ strongly convergent in X.

Theorem 3.2.5. [13, Theorem 2.1] *Let $(X, \|\cdot\|)$ be a real Banach space. Suppose that $\Phi : X \to \mathbb{R}$ is a $\mathcal{C}^1$ functional. Take two points e_1, $e_2 \in X$ and define*

$$\Gamma := \{\varphi \in \mathcal{C}([0,1], X) : \varphi(0) = e_1,\, \varphi(1) = e_2\},$$

and

$$c := \inf_{\varphi\in\Gamma} \max_{t\in[0,1]} \Phi(\varphi(t)).$$

Assume that Φ satisfies the Palais-Smale condition at level c and that

$$c > \max\{\Phi(e_1), \Phi(e_2)\}.$$

Then, there is a critical point of Φ at level c, i.e. there exists $x_0 \in X$ such that $\Phi(x_0) = c$ and $\Phi'(x_0) = 0$.

Theorem 3.2.6. [65, Corollary 1.6] *Let $(X, \|\cdot\|)$ be a real Banach space and let $\Phi : X \to \mathbb{R}$ be a $\mathcal{C}^1$ functional. Suppose that $u_0 \in X$ is a local minimum, i.e. there exists $\varepsilon > 0$ such that*

$$\Phi(u_0) \le \Phi(u), \quad \text{for } \|u - u_0\| \le \varepsilon\,,$$

and assume that Φ satisfies the Palais-Smale condition at any level $d \in \mathbb{R}$. Then:

i) either there exists $0 < \gamma < \varepsilon$ such that $\inf\{\Phi(u) : \|u - u_0\| = \gamma\} > \Phi(u_0)$,

ii) or, for each $0 < \gamma < \varepsilon$, Φ has a local minimum at a point u_γ with $\|u_\gamma - u_0\| = \gamma$ and $\Phi(u_\gamma) = \Phi(u_0)$.

3.3 Solving the limit coercive case

This section is devoted to the proof of Theorem 3.1.1. Let us first recall some of the notation introduced in Section 3.1. For a function $d \in L^q(\Omega)$ for some $q > N/2$ we recall that

$$W_d := \{w \in H_0^1(\Omega) : d(x)w(x) = 0 \text{ a.e. in } \Omega,\ \|w\| = 1\}$$

and

$$m_d := \begin{cases} \displaystyle\inf_{u\in W_d} \int_\Omega (|\nabla u|^2 - \mu h(x)u^2)dx, & \text{if} \quad W_d \neq \emptyset, \\ +\infty, & \text{if} \quad W_d = \emptyset. \end{cases} \tag{3.3.1}$$

Let us emphasize that

$$W_0 = \{w \in H_0^1(\Omega) : \|w\| = 1\}$$

and immediately observe that $W_d \subseteq W_0$.

Remark 3.3.1. Observe that we could have chosen a different normalization in the definition of W_d. In fact, if we define

$$\widetilde{W_d} :=\{w \in H_0^1(\Omega) : d(x)w(x) = 0 \text{ a.e. in } \Omega,\ \|w\|_2 = 1\}$$

and

$$\widetilde{m}_d := \begin{cases} \inf\limits_{u\in\widetilde{W}_d} \displaystyle\int_\Omega (|\nabla u|^2 - \mu h(x)u^2)dx, & \text{if } \quad \widetilde{W}_d \neq \emptyset, \\ +\infty, & \text{if } \quad \widetilde{W}_d = \emptyset, \end{cases} \tag{3.3.2}$$

we can prove that

$$m_d > 0 \iff \widetilde{m}_d > 0.$$

Proof of Theorem 3.1.1. By [46, Theorem 1.1] we know that $m_d > 0$ is a sufficient condition to ensure that (3.1.1) has a solution. Hence, we just have to prove that the existence of a solution to (3.1.1) implies that $m_d > 0$. If $W_d = \emptyset$, the result is obviously true. Hence, we just consider the case where $W_d \neq \emptyset$. In the case where $d \equiv 0$, the result follows from [46, Proposition 7.1]. Thus, we may assume that $d \gneqq 0$.

Assume that (3.1.1) has a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then, it follows that

$$\int_\Omega \Big(\nabla u \nabla(\phi^2) + d(x)u\phi^2 - \mu|\nabla u|^2\phi^2 - h(x)\phi^2\Big)dx = 0, \quad \forall\ \phi \in \mathcal{C}_0^\infty(\Omega). \tag{3.3.3}$$

Now, by Young's inequality, observe that

$$\int_\Omega \nabla u \nabla(\phi^2)dx \leq \int_\Omega \left(\mu|\nabla u|^2\phi^2 + \frac{1}{\mu}|\nabla\phi|^2\right)dx, \quad \forall\ \phi \in \mathcal{C}_0^\infty(\Omega). \tag{3.3.4}$$

Hence, gathering (3.3.3)-(3.3.4) and using the density of $\mathcal{C}_0^\infty(\Omega)$ in $H_0^1(\Omega)$, we have that

$$\int_\Omega \left(\frac{1}{\mu}|\nabla\phi|^2 + d(x)u\phi^2 - h(x)\phi^2\right)dx \geq 0, \quad \forall\ \phi \in H_0^1(\Omega). \tag{3.3.5}$$

Next, since for any $\phi \in W_d$,

$$\int_\Omega d(x)u\phi^2 dx = 0, \tag{3.3.6}$$

and we know that $W_d \subseteq W_0$, we obtain

$$\begin{aligned} \inf_{\phi\in W_d} \int_\Omega \left(\frac{1}{\mu}|\nabla\phi|^2 - h(x)\phi^2\right)dx &= \inf_{\phi\in W_d} \int_\Omega \left(\frac{1}{\mu}|\nabla\phi|^2 + d(x)u\phi^2 - h(x)\phi^2\right)dx \\ &\geq \inf_{\phi\in W_0} \int_\Omega \left(\frac{1}{\mu}|\nabla\phi|^2 + d(x)u\phi^2 - h(x)\phi^2\right)dx \geq 0. \end{aligned} \tag{3.3.7}$$

Assume by contradiction that

$$m_d = \mu \inf_{\phi\in W_d} \int_\Omega \left(\frac{1}{\mu}|\nabla\phi|^2 - h(x)\phi^2\right)dx = 0.$$

Then, by standard arguments there exists $\phi_0 \in W_d \subseteq W_0$ non-negative such that

$$\int_\Omega \left(\frac{1}{\mu}|\nabla\phi_0|^2 - h(x)\phi_0^2\right)dx = 0.$$

Thus, by Remark 3.3.1, (3.3.6) and (3.3.7), we have that

$$\inf_{\phi\in\widetilde{W}_0}\int_\Omega\left(\frac{1}{\mu}|\nabla\phi|^2+d(x)u\phi^2-h(x)\phi^2\right)dx=\inf_{\phi\in W_0}\int_\Omega\left(\frac{1}{\mu}|\nabla\phi|^2+d(x)u\phi^2-h(x)\phi^2\right)dx$$
$$=\int_\Omega\left(\frac{1}{\mu}|\nabla\phi_0|^2+d(x)u\phi_0^2-h(x)\phi_0^2\right)dx=0,$$

and so, that ϕ_0 is an eigenfunction associated to the first eigenvalue (which we are assuming equal to zero) of the eigenvalue problem

$$-\operatorname{div}\left(\frac{\nabla\phi}{\mu}\right)+(d(x)u-h(x))\phi=\lambda\phi,\quad \phi\in H_0^1(\Omega).$$

Applying then [66, Theorem 8.20] and arguing as in [39, Proposition 3.2] we deduce that $\phi_0>0$ in Ω. Since $d\gneqq 0$, this contradicts that $\phi_0\in W_d$ and the result follows. □

3.4 The lower solution and the functional setting

The aim of this section is to introduce a variational problem which is completely equivalent to (P_λ). As explained in the introduction, the key to find this equivalent problem is the construction of a lower solution below every upper solution to (P_λ).

The construction of this lower solution relies on the following a priori lower bound proved by the first author and L. Jeanjean in [50]. Let us consider the boundary value problem

$$-\Delta u=d(x)u+\mu|\nabla u|^2+f(x),\quad u\in H_0^1(\Omega)\cap L^\infty(\Omega),\tag{3.4.1}$$

under the assumption

$$\begin{cases}\Omega\subset\mathbb{R}^N,N\geq 2,\ \text{is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1},\\ d \text{ and } f \text{ belong to } L^q(\Omega) \text{ for some } q>N/2,\\ \mu>0.\end{cases}\tag{3.4.2}$$

Lemma 3.4.1. [50, Lemma 3.1] *Assume* (3.4.2). *Then, there exists a constant $M>0$ with $M:=M(N,q,|\Omega|,\mu_1,\|d^+\|_q,\|f^-\|_q)>0$ such that, every $u\in H^1(\Omega)\cap L^\infty(\Omega)$ upper solution to* (3.4.1) *satisfies*

$$\min_{\overline{\Omega}} u\geq -M.$$

Remark 3.4.1. The lower bound does not depend on f^+ and d^-.

Having at hand this lower bound, we construct the desired lower solution to (P_λ). The proof is inspired by [50, Lemma 4.2] and [46, Proposition 4.2]. Nevertheless, since $c_\lambda=\lambda c_+-c_-$ may change sign, several new ideas are needed.

Proposition 3.4.2. *Under the assumption* (A_1), *for any $\lambda\in\mathbb{R}$, there exists $\underline{u}_\lambda\in H^1(\Omega)\cap L^\infty(\Omega)$ lower solution to (P_λ) such that, for every β upper solution to (P_λ), we have $\underline{u}_\lambda\leq\min\{0,\beta\}$.*

Proof. We shall consider separately the cases $\lambda \le 0$ and $\lambda > 0$. The case $\lambda \le 0$ can be obtained exactly as in [46, Proposition 4.2]. We turn then to study the case where $\lambda > 0$. Fixed $\lambda > 0$ arbitrary, let us denote by $M_{\lambda,1} > 0$ the constant given by Lemma 3.4.1 applied to (P_λ) and let us introduce the auxiliary problem

$$\begin{cases} -\Delta u = \lambda c_+(x)u + \mu|\nabla u|^2 - h^-(x) - \lambda M_{\lambda,1}c_+(x) - 1\,, & \text{in } \Omega, \\ \quad u = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{3.4.3}$$

Thanks to Lemma 3.4.1, there exists $M_{\lambda,2} > 0$ such that, for every $\beta_1 \in H^1(\Omega) \cap L^\infty(\Omega)$ upper solution to (3.4.3), we have $\beta_1 \ge -M_{\lambda,2}$. Now, for $k > M_{\lambda,2}$, we introduce the problem

$$\begin{cases} -\Delta u = -\lambda k c_+(x) - h^-(x) - \lambda M_{\lambda,1}c_+(x) - 1\,, & \text{in } \Omega, \\ \quad u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{3.4.4}$$

and denote by α_k its solution. Since $-\lambda k c_+(x) - h^-(x) - \lambda M_{\lambda,1}c_+(x) - 1 < 0$, the weak maximum principle implies that $\alpha_k \le 0$. Observe that, for every β_1 upper solution to (3.4.3), we have that

$$-\Delta\beta_1 \ge \lambda c_+(x)\beta_1 + \mu|\nabla\beta_1|^2 - h^-(x) - \lambda M_{\lambda,1}c_+(x) - 1 \ge -\lambda k c_+(x) - h^-(x) - \lambda M_{\lambda,1}c_+(x) - 1 = -\Delta\alpha_k\,, \quad \text{in } \Omega\,.$$

Consequently, it follows that

$$\begin{cases} -\Delta\beta_1 \ge -\Delta\alpha_k\,, & \text{in } \Omega, \\ \quad \beta_1 \ge \alpha_k\,, & \text{on } \partial\Omega, \end{cases}$$

and, by the comparison principle, that $\beta_1 \ge \alpha_k$.

Now, we introduce the problem

$$\begin{cases} -\Delta u = \lambda c_+(x)\widetilde{T}_k(u) + \mu|\nabla u|^2 - h^-(x) - \lambda M_{\lambda,1}c_+(x) - 1, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4.5}$$

where

$$\widetilde{T}_k(s) = \begin{cases} -k\,, & \text{if } s \le -k\,, \\ \ s\,, & \text{if } s > -k\,. \end{cases}$$

Observe that β_1 and 0 are upper solutions to (3.4.5). Recalling that the minimum of two upper solutions is an upper solution, it follows that $\overline{\beta} = \min\{0, \beta_1\}$ is an upper solution to (3.4.5). As α_k is a lower solution to (3.4.5) with $\alpha_k \le \overline{\beta}$, applying Theorem 3.2.1, we conclude the existence of $v_{\lambda,k}$ minimal solution to (3.4.5) with $\alpha_k \le v_{\lambda,k} \le \overline{\beta} = \min\{0, \beta_1\}$.

Now, observe that $v_{\lambda,k}$ is an upper solution to (3.4.3). Hence, it follows that $v_{\lambda,k} \ge -M_{\lambda,2} > -k$ and so, that $v_{\lambda,k}$ is a solution to (3.4.3).

Finally, let us introduce $\underline{u}_\lambda = v_{\lambda,k} - M_{\lambda,1}$ and observe that

$$\begin{aligned} -\Delta\underline{u}_\lambda &= \lambda c_+(x)v_{\lambda,k} + \mu|\nabla v_{\lambda,k}|^2 - h^-(x) - \lambda M_{\lambda,1}c_+(x) - 1 \\ &\le (\lambda c_+(x) - c_-(x))\,\underline{u}_\lambda + \mu|\nabla\underline{u}_\lambda|^2 + h(x), \quad \text{in } \Omega. \end{aligned}$$

Hence, we have that $\underline{u}_\lambda$ is a lower solution to (P_λ) with $\underline{u}_\lambda \le -M_{\lambda,1}$. Thus, since every β upper solution to (P_λ) satisfies $\beta \ge -M_{\lambda,1}$, we have that $\underline{u}_\lambda$ is a lower solution to (P_λ) with $\underline{u}_\lambda \le \beta$ for every β upper solution to (P_λ). □

Remark 3.4.2. The constant $\mu > 0$ can be replaced by a function $\mu \in L^\infty(\Omega)$ with $\mu(x) \ge \mu_1 > 0$ a.e. in Ω and the result still holds true.

Corollary 3.4.3. *Under the assumption (A_2), for any $\lambda > 0$, there exists $\underline{u}_\lambda \in \mathcal{C}^1(\overline{\Omega})$ strict lower solution to (P_λ) such that, every $\beta \in \mathcal{C}^1(\overline{\Omega})$ upper solution to (P_λ) satisfies $\beta \gg \underline{u}_\lambda$.*

Proof. Let $\beta \in \mathcal{C}^1(\overline{\Omega})$ be an upper solution to (P_λ). By Proposition 3.4.2, there exists $\underline{u}_\lambda \in H^1(\Omega) \cap L^\infty(\Omega)$ lower solution to (P_λ) such that $\min\{0, \beta\} \ge \underline{u}_\lambda$. Moreover, under the assumption (A_2), this lower solution belongs to $\mathcal{C}^1(\overline{\Omega})$. Now, we introduce $w = \beta - \underline{u}_\lambda \ge 0$ and we are going to show that $w \gg 0$. First of all, by the construction of $\underline{u}_\lambda$, observe that

$$\begin{aligned} -\Delta w &\ge (\lambda c_+(x) - c_-(x))w + \mu\Big(|\nabla \beta|^2 - |\nabla \underline{u}_\lambda|^2\Big) + h^+(x) + 1 \\ &= (\lambda c_+(x) - c^-(x)w + \mu\langle \nabla\beta + \nabla\underline{u}_\lambda, \nabla w\rangle + h^+(x) + 1, \quad \text{in } \Omega. \end{aligned}$$

Equivalently, it follows that

$$-\Delta w - \mu\langle \nabla\beta + \nabla\underline{u}_\lambda, \nabla w\rangle + c_-(x)w \ge \lambda c_+(x)w + h^+(x) + 1 \ge 1, \quad \text{in } \Omega.$$

On the other hand, observe that $w \ge 0$ on $\partial\Omega$. Hence, by Theorem 3.1.7, it follows that $w \gg 0$ and so that $\beta \gg \underline{u}_\lambda$. □

Now, following [46, Section 5] we introduce some auxiliary functions which, together with the lower solution found in Proposition 3.4.2, will let us introduce the desired equivalent problem to (P_λ). These functions will be essential throughout the rest work. We define

$$g(s) = \begin{cases} \dfrac{1}{\mu}(1 + \mu s)\ln(1 + \mu s), & s > -1/\mu, \\ 0, & s \le -1/\mu, \end{cases} \qquad \text{and} \qquad G(s) = \int_0^s g(t)dt. \tag{3.4.6}$$

In the following lemma we recall some properties of these functions.

Lemma 3.4.4.

i) The function g is continuous on $\mathbb{R}$, satisfies $g > 0$ on $\mathbb{R}^+$ and there exists $D > 0$ with $-D \le g \le 0$ on $\mathbb{R}^-$. Moreover, $G \ge 0$ on $\mathbb{R}$.

ii) For any $\delta > 0$, there exists $\overline{c} = \overline{c}(\delta, \mu) > 0$ such that, for any $s > \frac{1}{\mu}$, $g(s) \le \overline{c}\, s^{1+\delta}$.

iii) $\lim_{s\to+\infty} g(s)/s = +\infty$ and $\lim_{s\to+\infty} G(s)/s^2 = +\infty$.

iv) For any $s \in \mathbb{R}$, it follows that $g(s) - s \ge 0$.

Proof. See [46, Lemma 5.1] for $i), ii)$ and $iii)$. See [76, Lemma 7] for $iv)$. □

Next, we define the function

$$\alpha_\lambda = \frac{1}{\mu}(e^{\mu \underline{u}_\lambda} - 1) \in H^1(\Omega) \cap L^\infty(\Omega), \tag{3.4.7}$$

where $\underline{u}_\lambda \in H^1(\Omega) \cap L^\infty(\Omega)$ is the lower solution to (P_λ) obtained in Proposition 3.4.2. Before going further, since $\underline{u}_\lambda \le 0$, observe that $0 \ge \alpha_\lambda \ge -1/\mu + \varepsilon$ for some $\varepsilon > 0$. Having at hand α_λ, for any $\lambda \in \mathbb{R}$, we consider the auxiliary problem

$$-\Delta v = f_\lambda(x, v), \quad v \in H_0^1(\Omega), \tag{Q_λ}$$

where

$$f_\lambda(x,s) = \begin{cases} c_\lambda(x)g(s) + (1+\mu s)h(x), & \text{if } s \geq \alpha_\lambda(x), \\ c_\lambda(x)g(\alpha_\lambda(x)) + (1+\mu\alpha_\lambda(x))h(x), & \text{if } s \leq \alpha_\lambda(x), \end{cases} \tag{3.4.8}$$

with g defined in (3.4.6), and we are going to prove that (Q_λ) is completely equivalent to (P_λ). Following [46, Lemma 5.2] one can obtain the next lemma.

Lemma 3.4.5. *Assume that* (A_1) *holds. Then, it follows that:*

i) Every solution to (Q_λ) *belongs to* $L^\infty(\Omega)$.

ii) Every solution v *to* (Q_λ) *satisfies* $v \geq \alpha_\lambda$.

iii) A function $v \in H_0^1(\Omega)$ *is a solution to* (Q_λ) *if, and only if, the function*

$$u = \frac{1}{\mu}\ln(1+\mu v) \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

is a solution to (P_λ).

Proof. See [46, Lemma 5.2]. □

Remark 3.4.3. On the same way, we can show that $v_1 \in H^1(\Omega) \cap L^\infty(\Omega)$ (respectively $v_2 \in H^1(\Omega) \cap L^\infty(\Omega)$) is a lower solution (respectively an upper solution) to (Q_λ) if, and only if, the function

$$u_1 = \frac{1}{\mu}\ln\Big(1+\mu v_1\Big) \qquad \Big(\text{respectively } u_2 = \frac{1}{\mu}\ln\Big(1+\mu v_2\Big)\Big)$$

is a lower solution (respectively an upper solution) to (P_λ).

The problem (Q_λ) admits a variational formulation. Its solution in $H_0^1(\Omega)$ can be obtained as critical points of the functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$I_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega F_\lambda(x,v)dx, \tag{3.4.9}$$

where

$$F_\lambda(x,s) = c_\lambda(x)\,G(s) + \frac{1}{2\mu}(1+\mu s)^2 h(x), \quad \text{if } s \geq \alpha_\lambda(x), \tag{3.4.10}$$

and

$$\begin{aligned} F_\lambda(x,s) = \Big[c_\lambda(x)g(\alpha_\lambda(x)) + (1+\mu\alpha_\lambda(x))h(x)\Big](s-\alpha_\lambda(x)) \\ + c_\lambda(x)\,G(\alpha_\lambda(x)) + \frac{1}{2\mu}(1+\mu\alpha_\lambda(x))^2 h(x), \quad \text{if } s \leq \alpha_\lambda(x). \end{aligned} \tag{3.4.11}$$

Under the assumption (A_1), since g has subcritical growth (see Lemma 3.4.4), it is standard to show that $I_\lambda \in \mathcal{C}^1(H_0^1(\Omega),\mathbb{R})$.

3.5 On the Palais-Smale condition and the mountain pass geometry

Our first aim in this section is to show that, for any $\lambda > 0$, the functional I_λ previously defined satisfies the Palais-Smale condition (Definition 3.2.3). Later on, we show that, if (P_0) has a solution, then the mountain pass geometry holds for $\lambda > 0$ small enough. As a consequence, we will prove Theorem 3.1.3.

Let us start showing that the Palais-Smale sequences are bounded. The proof is inspired by [75]. However, since we are not considering $h \gneqq 0$, the proof becomes more difficult. The role of the lower solution obtained in Proposition 3.4.2 and the different definition of the functional are both crucial. Let us define

$$m_{c_\lambda} := \begin{cases} \displaystyle\inf_{u \in W_{c_\lambda}} \int_\Omega (|\nabla u|^2 - \mu h(x) u^2) dx, & \text{if} \quad W_{c_\lambda} \neq \emptyset, \\ +\infty, & \text{if} \quad W_{c_\lambda} = \emptyset, \end{cases} \tag{3.5.1}$$

where

$$W_{c_\lambda} = \Big\{ w \in H_0^1(\Omega),\ c_\lambda(x) w(x) = 0 \text{ a.e. } x \in \Omega,\ w \geq 0,\ \|w\| = 1 \Big\}.$$

Remark 3.5.1. Observe that $W_{c_\lambda} \subseteq W_{c_-}$ and so that $m_{c_\lambda} \geq m_{c_-}$.

Lemma 3.5.1. *Fixed $\lambda > 0$ arbitrary, assume (A_1) and suppose that $m_{c_\lambda} > 0$. Then, the Palais-Smale sequences for I_λ at any level $d \in \mathbb{R}$ are bounded.*

Proof. Let $\{v_n\} \subset H_0^1(\Omega)$ be a Palais-Smale sequence for I_λ at level $d \in \mathbb{R}$. First we claim that $\{v_n^-\}$ is bounded. Indeed, since $\{v_n\}$ is a Palais-Smale sequence, there exists a sequence $\varepsilon_n \to 0$ such that

$$-\varepsilon_n \|v_n^-\| \leq \langle I'_\lambda(v_n), v_n^- \rangle \leq \varepsilon_n \|v_n^-\|. \tag{3.5.2}$$

Also, since $f_\lambda(x,s)$ is bounded in $\Omega \times \mathbb{R}^-$, there exist $D_1, D_2 > 0$ such that

$$\langle I'_\lambda(v_n), v_n^- \rangle \leq -\|v_n^-\|^2 + D_1 \|v_n^-\| + D_2. \tag{3.5.3}$$

Gathering (3.5.2) and (3.5.3) we deduce that

$$0 \leq -\|v_n^-\|^2 + (D_1 + \varepsilon_n)\|v_n^-\| + D_2,$$

and the claim follows. To prove that $\{v_n^+\}$ is also bounded we assume by contradiction that $\|v_n\| \to \infty$ and introduce the sequence $\{w_n\} \subset H_0^1(\Omega)$ given by $w_n = \frac{v_n}{\|v_n\|}$. Observe that $\{w_n\}$ is bounded in $H_0^1(\Omega)$. Hence, up to a subsequence, it follows that $w_n \rightharpoonup w$ in $H_0^1(\Omega)$, $w_n \to w$ strongly in $L^r(\Omega)$ for $1 \leq r < 2^*$ and $w_n \to w$ a.e. in Ω. We split the rest of the proof into several steps:

Step 1: $w \equiv 0$.

As $\|v_n^-\|$ is bounded and by assumption $\|v_n\| \to \infty$, clearly $w^- \equiv 0$. It then remains to prove that $w^+ \equiv 0$. We first prove that $c_\lambda w^+ \equiv 0$. Assume by contradiction that $c_\lambda w^+ \not\equiv 0$. Observe that for every $\varphi \in H_0^1(\Omega)$, we can write

$$\begin{aligned} \langle I'_\lambda(v_n), \varphi \rangle = \int_\Omega \nabla v_n \nabla \varphi\, dx - \int_{\{v_n \geq \alpha_\lambda\}} \mu v_n h(x) \varphi\, dx - \int_{\{v_n \geq \alpha_\lambda\}} c_\lambda(x) g(v_n) \varphi\, dx - \int_{\{v_n \geq \alpha_\lambda\}} h(x) \varphi\, dx \\ - \int_{\{v_n \leq \alpha_\lambda\}} f_\lambda(x, v_n) \varphi\, dx \end{aligned} \tag{3.5.4}$$

Hence, using that $f_\lambda(x,s)$ is bounded in $\Omega \times \mathbb{R}^-$ and the convergence of w_n, it follows that

$$\frac{\langle I_\lambda'(v_n),\varphi\rangle}{\|v_n\|} = \int_\Omega \nabla w \nabla \varphi \, dx - \int_{\{w \geq \alpha_\lambda\}} \mu w h(x) \varphi \, dx - \int_{\{v_n \geq \alpha_\lambda\}} c_\lambda(x) g(v_n) \frac{\varphi}{\|v_n\|} dx + o(1), \quad \forall\, \varphi \in H_0^1(\Omega).$$

Actually, using that g is bounded on $\mathbb{R}^-$ and that $w^- \equiv 0$, we obtain that

$$\int_\Omega c_\lambda(x) g(v_n) \frac{\varphi}{\|v_n\|} dx = \int_\Omega \Big(\nabla w \nabla \varphi - \mu w h(x) \varphi \Big) dx + o(1), \quad \forall\, \varphi \in H_0^1(\Omega).$$

Equivalently, we deduce that, for every $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega c_\lambda(x) \frac{g(v_n) - v_n}{\|v_n\|} \varphi \, dx = \int_\Omega \Big(\nabla w \nabla \varphi - (c_\lambda(x) + \mu h(x)) w \varphi \Big) dx + o(1). \tag{3.5.5}$$

Since $c_\lambda w^+ \not\equiv 0$, we may choose $\varphi \in H_0^1(\Omega)$ and a measurable subset $\Omega_\varphi \subset \Omega$ such that

$$|\Omega_\varphi| > 0, \quad c_\lambda w^+ \varphi > 0 \text{ in } \Omega_\varphi \subset \Omega \quad \text{and} \quad c_\lambda w^+ \varphi \equiv 0 \text{ in } \Omega \setminus \Omega_\varphi.$$

As $g(s) - s \geq 0$ on $\mathbb{R}$ (see Lemma 3.4.4), it follows that

$$c_\lambda(x) \frac{g(v_n) - v_n}{\|v_n\|} \varphi \geq 0 \quad \text{a.e. in } \Omega_\varphi.$$

Moreover, observe that

$$\liminf_{n\to\infty} c_\lambda(x) \frac{g(v_n) - v_n}{\|v_n\|} \varphi = \liminf_{n\to\infty} c_\lambda(x) w_n \frac{g(w_n\|v_n\|) - w_n\|v_n\|}{w_n\|v_n\|} \varphi = +\infty \quad \text{a.e. in } \Omega_\varphi.$$

Hence, applying Fatou's Lemma we deduce that

$$\liminf_{n\to\infty} \int_{\Omega_\varphi} c_\lambda(x) \frac{g(v_n) - v_n}{\|v_n\|} \varphi dx = +\infty,$$

which yields a contradiction with (3.5.5). Thus, we conclude that $c_\lambda w \equiv 0$. Now, we take $\varphi = w$ in (3.5.4) and divide by $\|v_n\|$. Using that $c_\lambda w \equiv 0$ and that $\{v_n\}$ is a Palais-Smale sequence, we get that

$$\int_\Omega \nabla w_n \nabla w dx - \int_\Omega \mu w_n w h(x) dx \to 0,$$

and so, since $w_n \rightharpoonup w$ in $H_0^1(\Omega)$ and $w_n \to w$ in $L^r(\Omega)$, for $1 \leq r < 2^*$, that

$$\int_\Omega \Big(|\nabla w|^2 - \mu h(x) w^2 \Big) dx = 0.$$

By this last identity and the facts that $w \geq 0$ and $c_\lambda w \equiv 0$, the condition $m_{c_\lambda} > 0$ implies that $w \equiv 0$.

Step 2: $\displaystyle\int_\Omega c_\lambda(x)\frac{g(v_n^+)}{v_n^+}(w_n^+)^2dx \to 1.$

First of all, observe that

$$\frac{\langle I_\lambda'(v_n), v_n\rangle}{\|v_n\|^2} = 1 - \frac{1}{\|v_n\|^2}\Bigg[\mu\int_{\{v_n\geq\alpha_\lambda\}}(v_n^2)h(x)dx + \int_{\{v_n\geq\alpha_\lambda\}} c_\lambda(x)g(v_n)v_n dx$$
$$+ \int_{\{v_n\geq\alpha_\lambda\}} v_n h(x)dx + \int_{\{v_n\leq\alpha_\lambda\}} f_\lambda(x,v_n)dx\Bigg] \to 0.$$

Hence, using that $w\equiv 0$, we deduce that

$$1 - \frac{1}{\|v_n\|^2}\int_{\{v_n\geq\alpha_\lambda\}} c_\lambda(x)g(v_n)v_n dx \to 0.$$

Moreover, observe that

$$\frac{1}{\|v_n\|^2}\int_{\{v_n\geq\alpha_\lambda\}} c_\lambda(x)g(v_n)v_n dx = \frac{1}{\|v_n\|^2}\int_\Omega c_\lambda(x)g(v_n^+)v_n^+ dx + \frac{1}{\|v_n\|^2}\int_{\{\alpha_\lambda\leq v_n\leq 0\}} c_\lambda(x)g(v_n)v_n dx,$$

and

$$\frac{1}{\|v_n\|^2}\int_{\{\alpha_\lambda\leq v_n\leq 0\}} c_\lambda(x)g(v_n)v_n dx \to 0.$$

Thus, we can conclude that

$$1 - \int_\Omega c_\lambda(x)\frac{g(v_n^+)}{v_n^+}(w_n^+)^2dx = 1 - \frac{1}{\|v_n\|^2}\int_\Omega c_\lambda(x)g(v_n^+)v_n^+dx \to 0.$$

Step 3: $\displaystyle\ln(\|v_n\|)\int_\Omega c_\lambda(x)(w_n^+)^2dx + \int_\Omega c_\lambda(x)(w_n^+)^2\ln\left(\mu w_n^+ + \frac{1}{\|v_n\|}\right)dx \to 1.$

By the definition of g (see (3.4.6)), we can write

$$\frac{g(v_n^+)}{v_n^+} = \frac{1}{\mu v_n^+}\ln(1+\mu v_n^+) + \ln(1+\mu v_n^+) = \frac{\ln(1+\mu\|v_n\|w_n^+)}{\mu\|v_n\|w_n^+} + \ln(1+\mu\|v_n\|w_n^+).$$

Now, observe that

$$\left|c_\lambda(x)\frac{\ln(1+\mu\|v_n\|w_n^+)}{\mu\|v_n\|w_n^+}\right| \leq |c_\lambda(x)|,$$

and so, since $w\equiv 0$, that

$$\frac{1}{\mu}\int_\Omega c_\lambda(x)\frac{\ln(1+\mu\|v_n\|w_n^+)}{\mu\|v_n\|w_n^+}(w_n^+)^2dx \to 0.$$

Applying the previous step, we conclude that

$$\int_\Omega c_\lambda(x)(w_n^+)^2\ln(1+\mu\|v_n\|w_n^+)dx \to 1. \tag{3.5.6}$$

Observe that

$$\ln(1+\mu\|v_n\|w_n^+) = \ln\left(\|v_n\|\left(\mu w_n^+ + \frac{1}{\|v_n\|}\right)\right) = \ln(\|v_n\|) + \ln\left(\mu w_n^+ + \frac{1}{\|v_n\|}\right).$$

Thus, substituting in (3.5.6), we conclude the Step 3.

Step 4: $\ln(\|v_n\|)\displaystyle\int_\Omega c_\lambda(x)(w_n^+)^2dx \to 0.$

First of all, defining

$$H(s)=\frac{1}{2}g(s)s-G(s),$$

observe that

$$\begin{aligned}
I_\lambda(v_n)-\frac{1}{2}\langle I'_\lambda(v_n),v_n\rangle &= \int_{\{v_n\geq\alpha_\lambda\}} c_\lambda(x)H(v_n)dx-\frac{1}{2\mu}\int_{\{v_n\geq\alpha_\lambda\}}(1+\mu v_n)h(x)dx\\
&\qquad -\int_{\{v_n\leq\alpha_\lambda\}}\Big[F_\lambda(x,v_n)-\frac{1}{2}f_\lambda(x,v_n)v_n\Big]dx\\
&=\int_\Omega c_\lambda(x)H(v_n^+)dx+\int_{\{\alpha_\lambda\leq v_n\leq 0\}} c_\lambda(x)H(v_n)dx-\frac{1}{2\mu}\int_{\{v_n\geq\alpha_\lambda\}}(1+\mu v_n)h(x)dx\\
&\qquad -\int_{\{v_n\leq\alpha_\lambda\}}\Big[F_\lambda(x,v_n)-\frac{1}{2}f_\lambda(x,v_n)v_n\Big]dx\\
&=d+\varepsilon_n\|v_n\|+o(1),
\end{aligned}$$

or equivalently

$$\begin{aligned}
\int_\Omega c_\lambda(x)H(v_n^+)dx &= d-\int_{\{\alpha_\lambda\leq v_n\leq 0\}} c_\lambda(x)H(v_n)dx+\frac{1}{2\mu}\int_{\{v_n\geq\alpha_\lambda\}}(1+\mu v_n)h(x)dx\\
&\qquad+\int_{\{v_n\leq\alpha_\lambda\}}\Big[F_\lambda(x,v_n)-\frac{1}{2}f_\lambda(x,v_n)v_n)\Big]dx+\varepsilon_n\|v_n\|+o(1).
\end{aligned}\tag{3.5.7}$$

Now, using that for every $s\geq 0$,

$$H(s)=\frac{s^2}{4}+\frac{s}{2\mu}(1-\ln(1+\mu s))-\frac{1}{2\mu^2}\ln(1+\mu s),$$

and substituting in (3.5.7), we deduce that

$$\begin{aligned}
\frac{1}{4}\int_\Omega c_\lambda(x)(v_n^+)^2dx &= d-\int_{\{\alpha_\lambda\leq v_n\leq 0\}} c_\lambda(x)H(v_n)dx+\frac{1}{2\mu}\int_{\{v_n\geq\alpha_\lambda\}}(1+\mu v_n)h(x)dx\\
&\qquad+\int_{\{v_n\leq\alpha_\lambda\}}\Big[F_\lambda(x,v_n)-\frac{1}{2}f_\lambda(x,v_n)v_n\Big]dx-\frac{1}{2\mu}\int_\Omega c_\lambda(x)v_n^+(1-\ln(1+\mu v_n^+))dx\\
&\qquad+\frac{1}{2\mu^2}\int_\Omega c_\lambda(x)\ln(1+\mu v_n^+)dx+\varepsilon_n\|v_n\|+o(1).
\end{aligned}$$

As a consequence, we obtain that

$$\begin{aligned}
\ln(\|v_n\|)\int_\Omega c_\lambda(x)(w_n^+)^2dx = \frac{4\ln(\|v_n\|)}{\|v_n\|^2}\bigg[& d-\int_{\{\alpha_\lambda\leq v_n\leq 0\}} c_\lambda(x)H(v_n)dx+\frac{1}{2\mu}\int_{\{v_n\geq\alpha_\lambda\}}(1+\mu v_n)h(x)dx\\
&+\int_{\{v_n\leq\alpha_\lambda\}}\Big[F_\lambda(x,v_n)-\frac{1}{2}f_\lambda(x,v_n)v_n\Big]dx\\
&-\frac{1}{2\mu}\int_\Omega c_\lambda(x)v_n^+(1-\ln(1+\mu v_n^+))dx\\
&+\frac{1}{2\mu^2}\int_\Omega c_\lambda(x)\ln(1+\mu v_n^+)dx+\varepsilon_n\|v_n\|\bigg]+o(1).
\end{aligned}$$

We easily deduce that each term of the right hand side goes to zero. Thus, we can conclude that

$$\ln(\|v_n\|)\int_\Omega c_\lambda(x)(w_n^+)^2 dx \to 0 .$$

Step 5: *Conclusion.*

Considering together Steps 3 and 4, we deduce that

$$\int_\Omega c_\lambda(x)(w_n^+)^2 \ln\left(\mu w_n^+ + \frac{1}{\|v_n\|}\right) \to 1 ,$$

which clearly contradicts the fact that $w \equiv 0$. Since we have a contradiction, we conclude that $\|v_n\|$ is bounded, as desired. □

Having at hand the boundedness of the Palais-Smale sequences, it is classical to show that the Palais-Smale condition holds.

Proposition 3.5.2. *Fixed $\lambda > 0$ arbitrary, assume that (A_1) holds and suppose that $m_{c_\lambda} > 0$. Then I_λ satisfies the Palais-Smale condition at any level $d \in \mathbb{R}$.*

Proof. Thanks to Lemma 3.5.1 we know that the Palais-Smale sequences for I_λ at any level $d \in \mathbb{R}$ are bounded. The strong convergence follows in a standard way. See [76, Lemma 11] or [46, Lemma 5.2] for two different approaches adapted to this setting. □

Now, we turn to check that the mountain pass geometry holds for $\lambda > 0$ small enough. We begin with a preliminary estimate.

Lemma 3.5.3. *Fixed $\Lambda_1 > 0$ arbitrary. There exist $D_1 > 0$ and $D_2 > 0$ such that, for every $\lambda \in [0, \Lambda_1]$ and any $v \in H_0^1(\Omega)$, it follows that*

$$I_\lambda(-v^-) \geq \frac{1}{2}\|v^-\|^2 - D_1\|v^-\| - D_2 \tag{3.5.8}$$

Proof. First of all, observe that for all $v \in H_0^1(\Omega)$ we can write

$$\begin{aligned} I_\lambda(-v^-) = \frac{1}{2}\|v^-\|^2 &- \int_{\{0\geq v\geq \alpha_\lambda\}}\left((\lambda c_+(x) - c_-(x))G(-v^-) + \frac{1}{2\mu}(1+2\mu v)^2 h(x)\right)dx \\ &- \int_{\{\alpha_\lambda \geq v\}}\Big[(\lambda c_+(x) - c_-(x))g(\alpha_\lambda) + (1+\mu\alpha_\lambda)h(x)\Big](v-\alpha_\lambda)dx \\ &- \int_{\{\alpha_\lambda\geq v\}}\left((\lambda c_+(x) - c_-(x))G(\alpha_\lambda) + \frac{1}{2\mu}(1+\mu\alpha_\lambda)^2 h(x)\right)dx. \end{aligned} \tag{3.5.9}$$

Hence, using that, for all $\lambda \in \mathbb{R}$, $\alpha_\lambda \in [-1/\mu, 0]$ and Lemma 3.4.4, *i*), we have that, for all $\lambda \in [0, \Lambda_1]$,

$$\begin{aligned} I_\lambda(-v^-) &\geq \frac{1}{2}\|v^-\|^2 - \int_\Omega\left(\lambda c_+(x)\max_{[-1/\mu,0]} G + \frac{1}{2\mu}h^+(x)\right)dx \\ &\qquad - \int_{\{\alpha_\lambda\geq v\}}\Big[\lambda c_+(x)g(\alpha_\lambda) - (1+\mu\alpha_\lambda)h^-(x)\Big](v-\alpha_\lambda)dx \\ &\geq \frac{1}{2}\|v^-\|^2 - \int_\Omega\left(\Lambda_1 c_+(x)\max_{[-1/\mu,0]} G + \frac{1}{2\mu}h^+(x)\right)dx - \int_\Omega\left(\Lambda_1 c_+(x)\max_{[-1/\mu,0]}|g| + h^-(x)\right)v^- dx. \end{aligned}$$

The estimate (3.5.8) follows immediately from the Sobolev inequality. □

Lemma 3.5.4. *Assume (A_1) and suppose that $m_{c_-} > 0$. Then, there exist constants $\Lambda > 0$ and $R > 0$ such that, if $0 \le \lambda \le \Lambda$, then $I_\lambda(v) \ge I_\lambda(0) + \frac{1}{2}$ for all $v \in \partial D$ with $D := \{v \in H_0^1(\Omega) : \|v^+\| < R\}$.*

Proof. Let us begin with some preliminary observations. First of all, we fix $\Lambda_1 > 0$ arbitrary and, by Lemma 3.5.3, we know that, for all $\lambda \in [0, \Lambda_1]$ and all $v \in H_0^1(\Omega)$,

$$I_\lambda(-v^-) \ge \frac{1}{2}\|v^-\|^2 - D_1\|v^-\| - D_2. \tag{3.5.10}$$

This implies the existence of D_3 (independent of λ) such that, for all $\lambda \in [0, \Lambda_1]$ and all $v \in H_0^1(\Omega)$,

$$I_\lambda(-v^-) \ge -D_3. \tag{3.5.11}$$

Now, by the definition of I_λ and Lemma 3.4.4, observe that, for any $\lambda \ge 0$ and any $\delta > 0$, there exists $D_4 > 0$ (independent of λ) such that, for all $v \in H_0^1(\Omega)$,

$$I_\lambda(v^+) \ge I_0(v^+) - \lambda D_4(1 + \|v^+\|^{2+\delta}). \tag{3.5.12}$$

Also, since $m_{c_-} > 0$ by hypothesis, we know that I_0 is coercive (see [46, Proposition 6.1]) and so, that there exists $R > 0$ such that, for all $v \in H_0^1(\Omega)$ with $\|v^+\| = R$,

$$I_0(v^+) \ge -\frac{1}{\mu}\int_\Omega h(x)dx + 1 + D_3. \tag{3.5.13}$$

Gathering (3.5.12) and (3.5.13) we deduce the existence of $0 < \Lambda \le \Lambda_1$ such that, for all $\lambda \in [0, \Lambda]$ and all $v \in H_0^1(\Omega)$ with $\|v^+\| = R$, the following inequality holds

$$I_\lambda(v^+) \ge -\frac{1}{\mu}\int_\Omega h(x)dx + \frac{1}{2} + D_3. \tag{3.5.14}$$

Now, for the constants $\Lambda > 0$ and $R > 0$ previously given, we define $D := \{v \in H_0^1(\Omega) : \|v^+\| < R\}$ and consider an arbitrary $\lambda \in [0, \Lambda]$. In order to finish the proof, we are going to show that

$$I_\lambda(v) \ge I_\lambda(0) + \frac{1}{2}, \quad \forall\, v \in \partial D.$$

Let $v \in \partial D$ fixed but arbitrary. By (3.5.11), (3.5.14) and the fact that $I_\lambda(0) = -\frac{1}{2\mu}\int_\Omega h(x)dx$, we directly obtain that

$$I_\lambda(v) = I_\lambda(v^+) + I_\lambda(-v^-) - I_\lambda(0) \ge -\frac{1}{\mu}\int_\Omega h(x)dx + \frac{1}{2} + D_3 - D_3 + \frac{1}{2\mu}\int_\Omega h(x)dx = I_\lambda(0) + \frac{1}{2}.$$

□

Lemma 3.5.5. *Assume (A_1). For any $\lambda > 0$, $M > 0$ and $R > 0$, there exists $w \in H_0^1(\Omega)$ such that $\|w^+\| > R$ and $I_\lambda(w) \le -M$.*

Proof. Since $c_+ \not\equiv 0$, we can choose $v \in \mathcal{C}_0^\infty(\Omega)$ such that $v \ge 0$, $c_+ v \not\equiv 0$ and $c_- v \equiv 0$. Moreover, let us take $t \in \mathbb{R}^+$, $t \ge 1$. As $\alpha_\lambda \le 0$, observe that

$$\begin{aligned} I_\lambda(tv) &\le \frac{1}{2}t^2\int_\Omega \Big(|\nabla v|^2 - \mu h(x)v^2\Big)dx - \lambda t^2\int_\Omega c_+(x)v^2\frac{G(tv)}{t^2v^2}dx + \frac{t}{\mu}\|1 + \mu v\|_\infty\|h^-\|_1 \\ &= t^2\left[\int_\Omega \Big(|\nabla v|^2 - \mu h(x)v^2\Big)dx - \lambda\int_\Omega c_+(x)v^2\frac{G(tv)}{t^2v^2}dx + \frac{1}{t}\frac{1}{\mu}\|1 + \mu v\|_\infty\|h^-\|_1\right]. \end{aligned}$$

Now, since by Lemma 3.4.4, we have

$$\lim_{t\to\infty} \lambda \int_\Omega c_+(x) v^2 \frac{G(tv)}{t^2 v^2} dx = +\infty,$$

we deduce that $\lim\limits_{t\to\infty} I_\lambda(tv) = -\infty$ and the lemma follows. □

Gathering Lemmas 3.5.4 and 3.5.5 we deduce that, for $\lambda > 0$ small enough, I_λ possess a mountain-pass geometry. Once this is proved, we first show the existence of a local minimum of I_λ and then we prove Theorem 3.1.3.

Proposition 3.5.6. *Assume (A_1) and suppose that $m_{c_-} > 0$ and that $\lambda \geq 0$ is small enough in order to ensure that the conclusion of Lemma 3.5.4 holds. Then, I_λ possesses a critical point v with $I_\lambda(v) \leq I_\lambda(0)$, which is a local minimum.*

Proof. From Lemma 3.5.4, we know that there exist $R > 0$ such that

$$m := \inf_{v\in D} I_\lambda(v) \leq I_\lambda(0) \qquad \text{and} \qquad I_\lambda(v) > I_\lambda(0) \ \text{ if } \ v \in \partial D,$$

where $D := \{v \in H_0^1(\Omega) : \|v^+\| < R\}$. Let $\{v_n\} \subset D$ be such that $I_\lambda(v_n) \to m$. By the definition of D and (3.5.10), we deduce that $\{v_n\}$ is bounded and so, up to a subsequence, it follows that $v_n \rightharpoonup v \in H_0^1(\Omega)$. By the weak lower semicontinuity of the norm and of the functional I_λ, we have

$$\|v^+\| \leq \liminf_{n\to\infty} \|v_n^+\| \leq R \quad \text{and} \quad I_\lambda(v) \leq \liminf_{n\to\infty} I_\lambda(v_n) = m \leq I_\lambda(0).$$

Finally, since, by Lemma 3.5.4, we know that $I_\lambda(v) > I_\lambda(0)$ if $v \in \partial D$, we deduce that $v \in D$ is a local minimum of I_λ. □

Proof of Theorem 3.1.3. Assume that $\lambda > 0$ is small enough in order to ensure that the conclusion of Lemma 3.5.4 holds. By Proposition 3.5.6 we have a first critical point, which is a local minimum of I_λ. On the other hand, since the Palais-Smale condition holds, in view of Lemmas 3.5.4. and 3.5.5, we can apply Theorem 3.2.5 and obtain a second critical point of I_λ at the mountain-pass level. This gives two different solutions to (Q_λ). Finally, by Lemma 3.4.5, we obtain two different solutions to (P_λ). □

3.6 Proof of Theorems 3.1.4 and 3.1.5

This section is devoted to the proofs of Theorems 3.1.4 and 3.1.5 and Corollary 3.1.6. Let us recall that, under the assumption (A_2), every solution to (P_λ) belongs to $\mathcal{C}_0^1(\overline{\Omega})$ (see [50, Theorem 2.2]). We first prove Theorem 3.1.4 and the first part of Corollary 3.1.6.

Lemma 3.6.1. [17, Lemma 2.2] *Assume (A_1). If $u_1, u_2 \in H^1(\Omega) \cap W_{loc}^{1,N}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are respectively a lower and an upper solution to (P_0), then $u_1 \leq u_2$.*

Lemma 3.6.2. *Assume (A_2) and suppose that (P_0) has a solution u_0 with $c_+ u_0 \gneqq 0$. Then, it follows that:*

- *Every u solution to (P_λ) with $\lambda > 0$ and $c_+ u \geq 0$ satisfies $u \gg u_0$.*
- *For all $\lambda > 0$, u_0 is a strict lower solution to (P_λ).*
- *There exists $\overline{\lambda} \in]0, +\infty[$ such that, for all $0 < \lambda < \overline{\lambda}$, ($P_\lambda$) has a strict upper solution u_1 with $u_1 \gg u_0$ and, for all $\lambda > \overline{\lambda}$, ($P_\lambda$) has no solutions u with $c_+ u \geq 0$.*

Proof. Let us split the proof into four steps:

Step 1: *Every u solution to (P_λ) with $\lambda > 0$ and $c_+ u \geq 0$ satisfies $u \gg u_0$.*

Let u be a solution to (P_λ) with $\lambda > 0$ and $c_+ u \geq 0$. We easily observe that $u \in \mathcal{C}_0^1(\overline{\Omega})$ is an upper solution to (P_0) and so, applying Lemma 3.6.1, that $u \geq u_0$ in Ω. Now, arguing as in Corollary 3.4.3, we are going to show that $u \gg u_0$. Let us define $w = u - u_0$ and observe that

$$\begin{cases} -\Delta w - \mu \langle \nabla u + \nabla u_0, \nabla w \rangle + c_-(x) w \geq \lambda c_+(x) u_0, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Applying then Theorem 3.1.7, we deduce that $w \gg 0$ and so that $u \gg u_0$.

Step 2: *For all $\lambda > 0$, u_0 is a strict lower solution to (P_λ).*

As $c_+ u_0 \gneqq 0$, we easily observe that u_0 is a lower solution to (P_λ). Arguing as in Step 1, we deduce that u_0 is a strict lower solution to (P_λ).

Step 3: *The problem (P_λ) has no solution u with $c_+ u \geq 0$ for $\lambda > 0$ large.*

We argue by contradiction. Suppose that u is a solution to (P_λ) with $c_+ u \geq 0$ and let $\gamma_1 > 0$ be the first eigenvalue and $\varphi_1 > 0$ be the first eigenfunction to the eigenvalue problem

$$-\Delta v + c_-(x) v = \gamma c_+(x) v, \quad v \in H_0^1(\Omega).$$

Multiplying (P_λ) by φ_1 and integrating it follows that

$$\begin{aligned} \gamma_1 \int_\Omega c_+(x) u \varphi_1 \, dx &= \int_\Omega \Big(\nabla u \nabla \varphi_1 + c_-(x) u \varphi_1 \Big) dx \\ &= \lambda \int_\Omega c_+(x) u \varphi_1 \, dx + \mu \int_\Omega |\nabla u|^2 \varphi_1 \, dx + \int_\Omega h(x) \varphi_1 \, dx, \end{aligned}$$

or equivalently

$$0 = (\lambda - \gamma_1) \int_\Omega c_+(x) u \varphi_1 \, dx + \mu \int_\Omega |\nabla u|^2 \varphi_1 \, dx + \int_\Omega h(x) \varphi_1 \, dx.$$

Hence, as $u \geq u_0$, if $\lambda > \gamma_1$ it follows that

$$0 \geq (\lambda - \gamma_1) \int_\Omega c_+(x) u_0 \varphi_1 \, dx + \int_\Omega h(x) \varphi_1 \, dx. \tag{3.6.1}$$

Since $c_+ u_0 \gneqq 0$ and $\varphi_1 > 0$, (3.6.1) gives a contradiction for λ large enough.

Step 4: *Let us define $\overline{\lambda} := \sup\{\lambda : (P_\lambda)$ has a solution u with $c_+ u \geq 0\}$. For all $0 < \lambda < \overline{\lambda}$, ($P_\lambda$) has a strict upper solution $u_1 \gg u_0$ and, for all $\lambda > \overline{\lambda}$, ($P_\lambda$) has no solutions with $c_+ u \geq 0$.*

First of all, observe that Step 2 implies $\overline{\lambda} < \infty$. Furthermore, by the definition of $\overline{\lambda}$, it is obvious that, for $\lambda > \overline{\lambda}$, ($P_\lambda$) has no solutions u with $c_+ u \geq 0$.

Let us then consider $0 < \lambda < \overline{\lambda}$. By the definition of $\overline{\lambda}$, we can find $\widetilde{\lambda} \in]\lambda, \overline{\lambda}[$ and $u_{\widetilde{\lambda}}$ solution to ($P_{\widetilde{\lambda}}$) with $c_+ u_{\widetilde{\lambda}} \geq 0$. Then, observe that $u_{\widetilde{\lambda}} \in \mathcal{C}_0^1(\overline{\Omega})$ is an upper solution to (P_0) and, by Step 1, $u_{\widetilde{\lambda}} \gg u_0$.

Finally, we are going to show that $u_{\widetilde{\lambda}}$ is a strict upper solution to (P_λ). Using that $c_+ u_{\widetilde{\lambda}} \geq 0$ and $\widetilde{\lambda} > \lambda$, we deduce that $u_{\widetilde{\lambda}}$ is an upper solution to (P_λ). Finally, let us consider u a solution to (P_λ) with $u \leq u_{\widetilde{\lambda}}$ and introduce $w = u_{\widetilde{\lambda}} - u$. Arguing as in Corollary 3.4.3, we are going to show that $w \gg 0$. Directly observe that

$$\begin{cases} -\Delta w - \mu\langle \nabla u + \nabla u_{\widetilde{\lambda}}, \nabla w\rangle + c_-(x) w \geq (\widetilde{\lambda} - \lambda) c_+(x) u_0\,, & \text{in } \Omega\,, \\ w = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Hence, applying Theorem 3.1.7, we obtain that $w \gg 0$, and so that $u_{\widetilde{\lambda}}$ is strict. $\square$

Proposition 3.6.3. *Assume (A_2), suppose that (P_0) has a solution u_0 with $c_+ u_0 \gneqq 0$ and let $\overline{\lambda} \in]0, +\infty[$ be given by Lemma 3.6.2. Then:*

1) *For every $0 < \lambda < \overline{\lambda}$, there exists $v \in \mathcal{C}_0^1(\overline{\Omega})$ with $v \gg v_0 := \frac{1}{\mu}(e^{\mu u_0} - 1)$, which is a local minimum of I_λ in the H_0^1-topology and a solution to (Q_λ).*

2) *For $\lambda = \overline{\lambda}$, there exists $v \in \mathcal{C}_0^1(\overline{\Omega})$ with $v \geq v_0$, which is a solution to (Q_λ).*

Proof. 1) By Lemma 3.6.2, for all $0 < \lambda < \overline{\lambda}$, u_0 is a strict lower solution to (P_λ) and there exists $u_1 \in \mathcal{C}_0^1(\overline{\Omega})$ strict upper solution to (P_λ). Let us then introduce

$$v_0 = \frac{1}{\mu}\Big(e^{\mu u_0} - 1\Big) \quad \text{and} \quad v_1 = \frac{1}{\mu}\Big(e^{\mu u_1} - 1\Big).$$

By Lemma 3.4.5 and Remark 3.4.3, it follows that $v_0, v_1 \in \mathcal{C}_0^1(\overline{\Omega})$ are a couple of well ordered strict lower and upper solutions to (Q_λ). Hence, applying Corollary 3.2.3 and Proposition 3.2.4 we have the existence of $v \in \mathcal{C}_0^1(\overline{\Omega})$ minimizer of I_λ on $M := \Big\{v \in H_0^1(\Omega) : v_0 \leq v \leq v_1\Big\}$, local minimum of I_λ in the H_0^1-topology and solution to (Q_λ). Moreover, by its construction $v \gg v_0$.

2) Let $\{\lambda_n\}$ be a sequence with $0 < \lambda_n < \overline{\lambda}$ and $\lambda_n \to \overline{\lambda}$ and let $\{v_n\}$ be the corresponding sequence of minimum of I_{λ_n} obtained in 1). This implies that $\langle I'_{\lambda_n}(v_n), \varphi\rangle = 0$ for all $\varphi \in H_0^1(\Omega)$. Moreover, by the construction of the lower and upper solutions, arguing as in Lemma 3.5.3, we obtain that

$$I_{\lambda_n}(v_n) \leq I_{\lambda_n}(v_0) \leq I_{\overline{\lambda}}(v_0) + D_{\overline{\lambda}}$$

for some $D_{\overline{\lambda}} > 0$. Then, arguing as in Lemma 3.5.1 and Proposition 3.5.2, we prove the existence of $v \in H_0^1(\Omega)$ such that $v_n \to v$ in $H_0^1(\Omega)$ with v a solution to (Q_λ) for $\lambda = \overline{\lambda}$. Moreover, as $v_n \geq v_0$ for all $n \in \mathbb{N}$, we deduce that $v \geq v_0$. $\square$

Proof of Theorem 3.1.4. Let us define as in Lemma 3.6.2 and Proposition 3.6.3

$$\overline{\lambda} := \sup\{\lambda : (P_\lambda) \text{ has a solution } u \text{ with } c_+ u \geq 0\}.$$

We split the proof into several steps:

Step 1: *Every u solution to (P_λ) with $c_+ u \geq 0$ satisfies $u \gg u_0$.*

This follows directly from Lemma 3.6.2.

Step 2: *For every $\lambda \in]0, \overline{\lambda}[$, ($P_\lambda$) has at least two solutions $u_{\lambda,1}, u_{\lambda,2} \in \mathcal{C}_0^1(\overline{\Omega})$ such that $u_0 \ll u_{\lambda,1}$.*

By Proposition 3.6.3, for $\lambda < \overline{\lambda}$, there exists a first critical point $v_{\lambda,1} \in \mathcal{C}_0^1(\Omega)$, which is a local minimum of I_λ. Since the Palais-Smale condition at any level $d \in \mathbb{R}$ holds, by Theorem 3.2.6, we have two options. If we are in the first case, then together with Lemma 3.5.5, we see that I_λ has the mountain-pass geometry and by Theorem 3.2.5, we have the existence of a second solution to (Q_λ). In the second case, we have directly the existence of a solution to (Q_λ). Then, by Lemma 3.4.5, we conclude the existence of two solutions to (P_λ). By the construction of $v_{\lambda,1}$, it follows that $u_{\lambda,1} = \frac{1}{\mu}\ln(1+\mu v_{\lambda,1}) \gg u_0$.

Step 3: *Existence of solution to* ($P_{\overline{\lambda}}$)

Let $v_{\overline{\lambda}}$ the solution to ($Q_{\overline{\lambda}}$) obtained in Proposition 3.6.3. Applying Lemma 3.4.5 we conclude that $u_{\overline{\lambda}} = \frac{1}{\mu}\ln(1+\mu v_{\overline{\lambda}})$ is a solution to (P_λ) for $\lambda = \overline{\lambda}$. Moreover, by construction $u_{\overline{\lambda}} \geq u_0$.

Step 4: *For $\lambda > \overline{\lambda}$ the problem (P_λ) has no solutions u with $c_+ u \geq 0$.*

This follows directly from Lemma 3.6.2. □

Proof of the first part of Corollary 3.1.6. Let u_0 be the unique solution to (P_0). Since (A_2) holds, we know that $u_0 \in \mathcal{C}_0^1(\overline{\Omega})$. Now, observe that

$$\begin{cases} -\Delta u_0 + c_-(x) u_0 \geq h(x), & \text{in } \Omega, \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

Hence, since $h \gneqq 0$, by Theorem 3.1.7, it follows that that $u_0 \gg 0$, and so, in particular that $c_+ u_0 \gneqq 0$. The corollary then follows immediately from Theorem 3.1.4. □

Now, we turn to prove Theorem 3.1.5 and the second part of Corollary 3.1.6.

Proposition 3.6.4. *Assume (A_2) and suppose that (P_0) has a solution u_0 with $c_+ u_0 \lneqq 0$. For every $\lambda > 0$ there exists $v \in \mathcal{C}_0^1(\overline{\Omega})$ with $v \ll v_0 := \frac{1}{\mu}(e^{\mu u_0} - 1)$, which is a local minimum of I_λ in the H_0^1-topology and a solution to (Q_λ).*

Proof. By Corollary 3.4.3 we have the existence of a strict lower solution α_λ with $\alpha_\lambda \ll \beta$ for every β upper solution to (P_λ). On the other hand, arguing as in Lemma 3.6.2 we prove that u_0 is a strict upper solution to (P_λ) for all $\lambda > 0$. Arguing exactly as in Proposition 3.6.3 we deduce the existence of $v \in \mathcal{C}_0^1(\overline{\Omega})$ local minimum of I_λ and solution to (Q_λ). Moreover, by its construction $v \ll v_0 = \frac{1}{\mu}\left(e^{\mu u_0} - 1\right)$. □

Proof of Theorem 3.1.5. The result follows from Proposition 3.6.4 arguing exactly as in the proof of Theorem 3.1.4. □

Proof of the second part of Corollary 3.1.6. Since $h \lneqq 0$ and (A_2) holds, the problem (P_0) has always a solution $u_0 \in \mathcal{C}_0^1(\overline{\Omega})$. Moreover, observe that 0 is an upper solution to (P_0). Hence, by Lemma 3.6.1, it follows that $u_0 \leq 0$. Now, as u_0 satisfies

$$-\Delta u_0 - \mu\langle \nabla u_0, \nabla u_0 \rangle + c_-(x) u_0 = h(x) \lneqq 0, \quad \text{in } \Omega,$$

we deduce by Theorem 3.1.7 that $u_0 \ll 0$ and, in particular that $c_+ u_0 \lneqq 0$. Thus, the corollary follows immediately from Theorem 3.1.5. □

3.7 Appendix. Hopf's Lemma ans SMP with unbounded lower order terms

In this section we prove Theorem 3.1.7 which can be seen as a combination of the Strong maximum principle and the Hopf's Lemma. The proof of Theorem 3.1.7 will be obtained as a consequence of the Hopf's Lemma with unbounded lower order term that we prove in Lemma 3.7.6. We give here a simplified proof of [100, Theorem 4.1]. Let us begin with some preliminary results that will be needed to prove Lemma 3.7.6. Throughout the appendix we assume $N \geq 2$.

Lemma 3.7.1. [79, Lemma 4.2] *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\beta \in (L^N(\Omega))^N$ and $\xi \in L^{N/2}(\Omega)$ with $\xi \geq 0$ a.e. in Ω. Then, for every $F \in H^{-1}(\Omega)$, there exists an unique $u \in H_0^1(\Omega)$ solution to*

$$\begin{cases} -\Delta u + \langle \beta(x), \nabla u \rangle + \xi(x) u = F, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.7.1}$$

As a consequence of the previous lemma we obtain an existence result with inhomogeneous boundary conditions.

Corollary 3.7.2. *Let $\omega = B_1(0) \setminus \overline{B_{1/2}(0)}$, $\beta \in (L^p(\omega))^N$ and $\xi \in L^p(\omega)$ for some $p > N$ and assume that $\xi \geq 0$ a.e. in ω. Then, there exists an unique $u \in \mathcal{C}^{1,\tau}(\overline{\omega})$ for some $\tau > 0$ solution to*

$$\begin{cases} -\Delta u + \langle \beta(x), \nabla u \rangle + \xi(x) u = 0, & \text{in } \omega, \\ u = 0, & \text{on } \partial B_1(0), \\ u = 1, & \text{on } \partial B_{1/2}(0), \end{cases} \tag{3.7.2}$$

such that $0 \leq u \leq 1$ in $\overline{\omega}$.

Proof. Let us consider $\varphi \in C^\infty(\mathbb{R}^N)$ given by $\varphi(x) = \frac{4}{3}(1 - |x|^2)$, and observe that $\varphi(x) = 0$ for all $x \in \partial B_1(0)$ and $\varphi(x) = 1$ for all $x \in \partial B_{1/2}(0)$. Moreover, by direct computations it follows that

$$-\Delta\varphi + \langle \beta(x), \nabla\varphi \rangle + \xi(x)\varphi = \frac{8N}{3} - \frac{8}{3}\langle \beta(x), x \rangle + \frac{4}{3}\xi(x)(1 - |x|^2) =: -F \in H^{-1}(\omega).$$

By Lemma 3.7.1 we know that there exists an unique $w \in H_0^1(\omega)$ solution to

$$\begin{cases} -\Delta w + \langle \beta(x), \nabla w \rangle + \xi(x) w = F, & \text{in } \omega, \\ w = 0, & \text{on } \partial\omega. \end{cases} \tag{3.7.3}$$

Then, we define $u = w + \varphi$ and we observe that $u \in H^1(\omega)$ is a solution to (3.7.2). Next, by [79, Proposition 3.10] we deduce that $0 \leq u \leq 1$ in $\overline{\omega}$ and, by [82, Theorem II-15.1] we obtain that $u \in \mathcal{C}^{1,\tau}(\overline{\omega})$ for some $\tau > 0$. Finally, the uniqueness follow from [79, Proposition 3.10]. □

Lemma 3.7.3. *Let $\omega = B_1(0) \setminus \overline{B_{1/2}(0)}$, $\varepsilon \in (0, 1/4)$, $x_0 \in \partial B_1(0)$ and $T : \mathbb{R}^N \to \mathbb{R}^N$ given by $T(x) = \varepsilon^{-1}(x - x_0) + x_0$. Then, it follows that $\omega \subset T(\omega) := \{T(x) : x \in \omega\}$.*

Proof. First of all, observe that

$$T(\omega) = B_{1/\varepsilon}\Big(\Big(1 - \frac{1}{\varepsilon}\Big)x_0\Big) \setminus \overline{B_{1/2\varepsilon}\Big(\Big(1 - \frac{1}{\varepsilon}\Big)x_0\Big)} = \Big\{x \in \mathbb{R}^N : \frac{1}{2\varepsilon} < \Big|x - \Big(1 - \frac{1}{\varepsilon}\Big)x_0\Big| < \frac{1}{\varepsilon}\Big\}. \tag{3.7.4}$$

Now, observe that, for all $x \in \omega$ and all $\varepsilon \in (0, 1/4)$, it follows that

$$\left|x - \left(1 - \frac{1}{\varepsilon}\right)x_0\right| \le |x| + \left|1 - \frac{1}{\varepsilon}\right||x_0| = |x| + \frac{1}{\varepsilon} - 1 < 1 + \frac{1}{\varepsilon} - 1 = \frac{1}{\varepsilon}, \tag{3.7.5}$$

and

$$\left|x - \left(1 - \frac{1}{\varepsilon}\right)x_0\right| \ge \left||x| - \left|1 - \frac{1}{\varepsilon}\right||x_0|\right| = \frac{1}{\varepsilon} - 1 - |x| > \frac{1}{\varepsilon} - 2 > \frac{1}{2\varepsilon}. \tag{3.7.6}$$

Hence, the result follows from (3.7.4)-(3.7.6). □

Lemma 3.7.4. *Let $\omega = B_1(0) \setminus \overline{B_{1/2}(0)}$, $B = (B^1, \ldots, B^N) \in (L^p(\omega))^N$ and $c \in L^p(\omega)$ for some $p > N$, $\varepsilon \in [0, 1/4]$, $B_\varepsilon(y) = (B_\varepsilon^1, \ldots, B_\varepsilon^N) = \varepsilon B(\varepsilon(y - x_0) + x_0)$ and $c_\varepsilon(y) = \varepsilon^2 c(\varepsilon(y - x_0) + x_0)$. Then, it follows that*

- $\|B_\varepsilon^i\|_{L^p(\omega)} \le \varepsilon^{1-\frac{N}{p}} \|B^i\|_{L^p(\omega)}$ *for all $i = 1, \ldots, N$.*
- $\|c_\varepsilon\|_{L^p(\omega)} \le \varepsilon^{2-\frac{N}{p}} \|c\|_{L^p(\omega)}$

Proof. Let $i \in \{1, \ldots, N\}$. We directly observe that

$$\|B_\varepsilon^i\|_{L^p(\omega)}^p = \int_\omega |B_\varepsilon^i(y)|^p dy = \varepsilon^p \int_\omega |B^i(\varepsilon(y - x_0) + x_0)|^p dy = \varepsilon^{p-N} \int_{S(\omega)} |B^i(z)|^p dz, \tag{3.7.7}$$

where $z = S(y) = \varepsilon(y - x_0) + x_0$. Then, arguing as in Lemma 3.7.3, we obtain that $S(\omega) \subset \omega$, and so, taking into account (3.7.7), we deduce that

$$\|B_\varepsilon^i\|_{L^p(\omega)}^p \le \varepsilon^{p-N} \int_\omega |B^i(z)|^p dz = \varepsilon^{p-N} \|B^i\|_{L^p(\omega)}^p.$$

The estimate for c_ε follows arguing on the same way. □

Using the rescaled functions B_ε and c_ε defined in Lemma 3.7.4, we introduce the auxiliary boundary value problem

$$\begin{cases} -\Delta u + \langle B_\varepsilon(x), \nabla u \rangle + c_\varepsilon(x) u = 0, & \text{in } B_1(0) \setminus \overline{B_{1/2}(0)}, \\ u = 0, & \text{on } \partial B_1(0), \\ u = 1, & \text{on } \partial B_{1/2}(0). \end{cases} \tag{P_ε}$$

and we prove the following uniform a priori bound that will be crucial in the proof of Lemma 3.7.6.

Lemma 3.7.5. *Let $\omega = B_1(0) \setminus \overline{B_{1/2}(0)}$, $B = (B_1, \ldots, B_N) \in (L^p(\omega))^N$ and $c \in L^p(\omega)$ for some $p > N$. Then, there exists $M > 0$ such that, for all $\varepsilon \in [0, 1/4]$, any solution to (P_ε) satisfies $\|u\|_{\mathcal{C}^1(\overline{\omega})} \le M$.*

Proof. We argue by contradiction. We assume that there exist sequences $\{\varepsilon_n\} \subset [0, 1/4]$ and $\{u_n\}$ solutions to (P_ε) with $\varepsilon = \varepsilon_n$ such that

$$\|u_n\|_{\mathcal{C}^1(\overline{\omega})} \to +\infty, \quad \text{as } n \to \infty.$$

Without loss of generality (up to a subsequence if necessary) we may assume that

$$1 \le \|u_n\|_{\mathcal{C}^1(\overline{\omega})}, \quad \forall\, n \in \mathbb{N}.$$

We consider then $v_n := \frac{u_n}{\|u_n\|_{\mathcal{C}^1(\overline{\omega})}}$ and we observe that v_n solves

$$\begin{cases} -\Delta v_n + \langle B_{\varepsilon_n}(x), \nabla v_n\rangle + c_{\varepsilon_n}(x) v_n = 0, & \text{in } \omega, \\ v_n = 0, & \text{on } \partial B_1(0), \\ v_n = \dfrac{1}{\|v_n\|_{\mathcal{C}^1(\overline{\omega})}}, & \text{on } \partial B_{1/2}(0). \end{cases} \tag{3.7.8}$$

Now, for all $n \in \mathbb{N}$, let us define

$$\xi_n = \frac{4}{3\|u_n\|_{\mathcal{C}^1(\overline{\omega})}}(1-|x|^2) \in \mathcal{C}^\infty(\mathbb{R}^N) \quad \text{and} \quad w_n = v_n - \xi_n,$$

and observe that w_n solves

$$\begin{cases} -\Delta w_n = -\langle B_{\varepsilon_n}(x), \nabla v_n\rangle - c_{\varepsilon_n}(x) v_n - \dfrac{8N}{3\|u_n\|_{\mathcal{C}^1(\overline{\omega})}}, & \text{in } \omega, \\ w_n = 0, & \text{on } \partial\omega. \end{cases} \tag{3.7.9}$$

By [66, Lemma 9.17], there exists $C(\omega, N) > 0$ such that

$$\|w_n\|_{W^{2,p}(\omega)} \leq C \left\| \langle B_{\varepsilon_n}(x), \nabla v_n\rangle + c_{\varepsilon_n}(x) v_n + \frac{8N}{3\|u_n\|_{\mathcal{C}^1(\overline{\omega})}} \right\|_{L^p(\omega)},$$

and so, since $\|v_n\|_{\mathcal{C}^1(\overline{\omega})} = 1$ for all $n \in \mathbb{N}$, by Lemma 3.7.4, there exists $C_1 = C_1(\omega, N, \|B\|_{(L^p(\omega))^N}, \|c\|_{L^p(\omega)}) > 0$ such that

$$\|w_n\|_{W^{2,p}(\omega)} \leq C_1.$$

From the above inequality, we deduce the existence of $C_2 > 0$ (independent of n) such that

$$\|v_n\|_{W^{2,p}(\omega)} \leq C_2.$$

Since $p > N$, by the Sobolev compact embedding, we deduce that $v_n \to v$ in $\mathcal{C}^1(\overline{\omega})$ for some $v \in \mathcal{C}^1(\overline{\omega})$. This implies that v is a weak solution to

$$\begin{cases} -\Delta v + \langle \overline{B}(x), \nabla v\rangle + \overline{c}(x) v = 0, & \text{in } \omega, \\ v = 0, & \text{on } \partial\omega, \end{cases} \tag{3.7.10}$$

for some $\overline{B} \in (L^p(\omega))^N$ and $\overline{c} \in L^p(\omega)$. Hence, by [79, Proposition 3.10], we deduce that $v \equiv 0$. This contradict the fact that $v_n \to v$ in $\mathcal{C}^1(\overline{\omega})$ and the result follows. □

Having at hand all the needed ingredients, we prove the Hopf's Lemma with unbounded lower order terms.

Lemma 3.7.6. (**Hopf's Lemma**) *Let $B \in (L^p(B_1(0))^N$ and $c \in L^p(B_1(0))$ for some $p > N$ such that $c \geq 0$ a.e. in $B_1(0)$. Let $x_0 \in \partial B_1(0)$ and let $u \in \mathcal{C}^1(\overline{B_1(0)})$ be an upper solution to*

$$\begin{cases} -\Delta u + \langle B(x), \nabla u\rangle + c(x) u = 0, & \text{in } B_1(0), \\ u = 0, & \text{on } \partial B_1(0), \end{cases} \tag{3.7.11}$$

such that $u(x) > u(x_0) = 0$ for all $x \in B_1(0)$. Then $\frac{\partial u}{\partial \nu}(x_0) < 0$, where ν denotes the exterior unit normal.

Proof. Let us fix $\omega := B_1(0) \setminus \overline{B_{1/2}(0)}$ and split the proof into several steps:

Step 1: *Auxiliary regular barrier* φ

Let us consider the problem

$$\begin{cases} -\Delta\varphi = 0, & \text{in } \omega, \\ \varphi = 0, & \text{on } \partial B_1(0), \\ \varphi = 1, & \text{on } \partial B_{1/2}(0). \end{cases} \tag{3.7.12}$$

By [66, Theorem 6.14] we know that there exists $\varphi \in \mathcal{C}^{2,\tau}(\overline{\omega})$ for some $\tau > 0$ solution to (3.7.12). Moreover, by [66, Lemma 3.4 and Theorem 3.5], we know that

$$0 < \varphi(x) < 1, \quad \forall\, x \in \omega, \quad \text{and} \quad \frac{\partial \varphi}{\partial \nu}(x_0) < 0. \tag{3.7.13}$$

Step 2: *Let $M > 0$ given by Lemma 3.7.5. For every $\varepsilon \in (0,1/4)$ there exists $\varphi_\varepsilon \in \mathcal{C}^{1,\tau}(\overline{\omega})$ for some $\tau > 0$ solution to (P_ε) such that $\|\varphi_\varepsilon\|_{\mathcal{C}^1(\overline{\omega})} \leq M$.*

The existence follows from Corollary 3.7.2 and the uniform bound from Lemma 3.7.5.

Step 3: *Let φ_ε the solution to (P_ε) given by Step 2. There exists $\overline{\varepsilon} \in (0,1/4)$ such that, for all $\varepsilon \in (0,\overline{\varepsilon})$, it follows that $\frac{\partial \varphi_\varepsilon}{\partial \nu}(x_0) < 0$.*

Let us define $\psi_\varepsilon := \varphi_\varepsilon - \varphi$ and observe that $\psi_\varepsilon \in \mathcal{C}^{1,\tau}(\overline{\omega})$ for some $\tau > 0$ and solves

$$\begin{cases} -\Delta\psi_\varepsilon = -\langle B_\varepsilon(x), \nabla\varphi_\varepsilon\rangle - c_\varepsilon(x)\varphi_\varepsilon, & \text{in } \omega, \\ \psi_\varepsilon = 0, & \text{on } \partial\omega. \end{cases} \tag{3.7.14}$$

Then, by [66, Lemma 9.17] and Lemma 3.7.4, there exists $C = C(\omega, N) > 0$ such that

$$\begin{aligned} \|\psi_\varepsilon\|_{W^{2,p}(\omega)} &\leq C\, \|\langle B_\varepsilon(x), \nabla\varphi_\varepsilon\rangle + c_\varepsilon(x)\varphi_\varepsilon\|_{L^p(\omega)} \\ &\leq C\|\varphi_\varepsilon\|_{\mathcal{C}^1(\overline{\omega})}\varepsilon^{1-\frac{N}{p}}\left(\sum_{i=1}^N \|B^i\|_{L^p(\omega)} + \varepsilon\|c\|_{L^p(\omega)}\right) \\ &\leq \varepsilon^{1-\frac{N}{p}} CM\left(\sum_{i=1}^N \|B^i\|_{L^p(\omega)} + \|c\|_{L^p(\omega)}\right) =: \varepsilon^{1-\frac{N}{p}} C_2 \end{aligned} \tag{3.7.15}$$

for some C_2 independent of ε. Hence, by the Sobolev continuous embedding, there exists $C_3 > 0$ independent of ε such that

$$\|\psi_\varepsilon\|_{\mathcal{C}^{1,\tau}(\overline{\omega})} \leq \varepsilon^{1-\frac{N}{p}} C_3.$$

We conclude that

$$\lim_{\varepsilon\to 0}\left|\frac{\partial \varphi_\varepsilon}{\partial \nu}(x_0) - \frac{\partial \varphi}{\partial \nu}(x_0)\right| \leq \lim_{\varepsilon\to 0}\|\psi_\varepsilon\|_{\mathcal{C}^{1,\tau}(\overline{\omega})} = 0,$$

and the Step 3 follows.

Step 4: *Conclusion*

Let $u \in \mathcal{C}^1(\overline{B_1(0)})$ be an upper solution to (3.7.11) such that $u(x) > u(x_0) = 0$ for all $x \in B_1(0)$. We fix $\varepsilon > 0$ small enough to ensure that the Step 3 holds and define

$$u_\varepsilon(y) = u(\varepsilon(y - x_0) + x_0).$$

Since we know that $\omega \subset T(\omega)$ by Lemma 3.7.3, we have that u_ε is an upper solution to

$$\begin{cases} -\Delta u_\varepsilon + \langle B_\varepsilon(x), \nabla u_\varepsilon \rangle + c_\varepsilon(x) u_\varepsilon = 0, & \text{in } \omega, \\ u_\varepsilon = 0, & \text{on } \partial\omega. \end{cases} \tag{3.7.16}$$

Then, we define $\overline{u}_\varepsilon = u_\varepsilon - \theta_\varepsilon \varphi_\varepsilon$ with

$$\theta_\varepsilon = \inf_{\partial B_{1/2}(0)} u_\varepsilon > 0,$$

and we have that $\overline{u}_\varepsilon$ is an upper solution to

$$\begin{cases} -\Delta \overline{u}_\varepsilon + \langle B_\varepsilon(x), \nabla \overline{u}_\varepsilon \rangle + c_\varepsilon(x) \overline{u}_\varepsilon = 0, & \text{in } \omega, \\ \overline{u}_\varepsilon = 0, & \text{on } \partial\omega. \end{cases} \tag{3.7.17}$$

Applying then [79, Proposition 3.10] we deduce that $u_\varepsilon - \theta_\varepsilon \varphi_\varepsilon \geq 0$, in $\overline{\omega}$, and so, by Step 3, we conclude that

$$\frac{\partial u}{\partial \nu}(x_0) = \frac{1}{\varepsilon} \frac{\partial u_\varepsilon}{\partial \nu}(x_0) \leq \frac{\theta_\varepsilon}{\varepsilon} \frac{\partial \varphi_\varepsilon}{\partial \nu}(x_0) < 0,$$

as desired.

□

Corollary 3.7.7. *For $z \in \mathbb{R}^N$ and $R > 0$, let $\beta \in (L^p(B_R(z))^N$ and $\xi \in L^p(B_R(z))$ for some $p > N$ such that $\xi \geq 0$ a.e. in $B_R(z)$. Let $x_0 \in \partial B_R(z)$ and let $u \in \mathcal{C}^1(\overline{B_R(z)})$ be an upper solution to*

$$\begin{cases} -\Delta u + \langle \beta(x), \nabla u \rangle + \xi(x) u = 0, & \text{in } B_R(z), \\ u = 0, & \text{on } \partial B_R(z), \end{cases} \tag{3.7.18}$$

such that $u(x) > u(x_0) = 0$ for all $x \in B_R(z)$. Then $\frac{\partial u}{\partial \nu}(x_0) < 0$, where ν denotes the exterior unit normal.

Proof. Let us define $y = T(x) = \frac{1}{R}(x - z)$ and introduce the functions

$$\begin{aligned} v(y) &= u(Ry + z), \\ B(y) &= R\beta(Ry + z), \\ c(y) &= R^2 \xi(Ry + z). \end{aligned} \tag{3.7.19}$$

Observe that if u is an upper solution to (3.7.18) such that $u(x) > 0$ for all $x \in B_R(z)$, then v is an upper solution to

$$\begin{cases} -\Delta v + \langle B(y), \nabla v \rangle + c(y) v = 0, & \text{in } B_1(0), \\ v = 0, & \text{on } \partial B_1(0), \end{cases} \tag{3.7.20}$$

satisfying the hypothesis of Lemma 3.7.6 for some $y_0 = T(x_0) \in \partial B_1(0)$. Thus, we have that

$$\frac{\partial u}{\partial \nu}(x_0) = \frac{1}{R} \frac{\partial v}{\partial \nu}(y_0) < 0,$$

and the result follows. □

Proof of Theorem 3.1.7. The result follows from Corollary 3.7.7 arguing as in [112, Theorem 3.27]. See also [66, Theorem 3.5]. □

4

Nonlinear fractional Laplacian problems with nonlocal "gradient terms"

4.1 Introduction and main results

In the last fifteen years, there has been an increasing interest in the study of partial differential equations involving integro-differential operators. In particular, the case of the fractional Laplacian has been widely studied and is nowadays a very active field of research. This is due not only to its mathematical richness. The fractional Laplacian has appeared in a great number of equations modeling real world phenomena, especially those which take into account nonlocal effects. Among others, let us mention applications in quasi-geostrophic flows [32], quantum mechanic [83], mathematical finances [14, 37], obstacle problems [21, 22, 31] and crystal dislocation [58, 59, 111].

The first aim of the present chapter is to discuss, depending on the real parameter $\lambda > 0$, the existence and non-existence of solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x)\,\mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_λ}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ s \in (1/2, 1), \\ f \in L^m(\Omega) \text{ for some } m > N/2s \text{ and } \mu \in L^\infty(\Omega). \end{cases} \tag{A_1}$$

Throughout the chapter, $(-\Delta)^s$ stands for the, by know classical, fractional Laplacian operator. For a smooth function u and $s \in (0,1)$, it can be defined as

$$(-\Delta)^s u(x) := a_{N,s} \text{ p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

where

$$a_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1} = -\frac{2^{2s}\Gamma\left(\frac{N}{2}+s\right)}{\pi^{\frac{N}{2}}\Gamma(-s)},$$

is a normalization constant and "p.v." is an abbreviation for "in the principal value sense". In (P_λ), $\mathbb{D}_s^2$ is a nonlocal diffusion term. It plays the role of the "gradient square" in the nonlocal case and is given by

$$\mathbb{D}_s^2(u) = \frac{a_{N,s}}{2} \text{ p.v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy. \tag{4.1.1}$$

Since they will not play a role in this thesis, we normalize the constants appearing in the definitions of $(-\Delta)^s$ and $\mathbb{D}_s^2$ and we omit the p.v. sense. However, let us stress that these constants guarantee

$$\lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x), \quad \forall\, u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{4.1.2}$$

and

$$\lim_{s\to 1^-} \mathbb{D}_s^2(u(x)) = |\nabla u(x)|^2, \quad \forall\, u \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{4.1.3}$$

We refer to [55] and [30] respectively for a proof of (4.1.2) and (4.1.3). Hence, at least formally, if $s \to 1^-$ in (P_λ), we recover the local problem

$$\begin{cases} -\Delta u = \mu(x)|\nabla u|^2 + \lambda f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1.4}$$

This equation corresponds to the stationary case of the Kardar-Parisi-Zhang model of growing interfaces introduced in [77]. The existence and multiplicity of solutions to problem (4.1.4) and of its different extensions have been extensively studied and it is still an active field of research. See for instance [4, 18, 25, 46, 62, 68]. In most of these papers, the existence of solutions is proved using either a priori estimates or, when it is possible, a suitable change of variable to obtain an equivalent semilinear problem. However, neither of these techniques seem to be appropriate to deal with the nonlocal case (P_λ).

Let us also point out that in [36], using pointwise estimates on the Green function for the fractional Laplacian, the authors deal with the nonlocal-local problem

$$\begin{cases} (-\Delta)^s u = |\nabla u|^q + \lambda f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.1.5}$$

For $s \in (1/2, 1)$, $1 < q < \frac{N}{N-(2s-1)}$, $f \in L^1(\Omega)$ and $\lambda > 0$ small enough they obtained the existence of a solution to (4.1.5). This existence result was later completed in [7] where, under suitable assumptions on f, the authors showed the existence of a solution to (4.1.5) for all $1 < q < \infty$ and $\lambda > 0$ small enough.

Following [35, 36] we introduce the following notion of weak solution to (P_λ).

Definition 4.1.1. We say that u is a *weak solution* to (P_λ) if u and $\mathbb{D}_s^2(u)$ belong to $L^1(\Omega)$, $u \equiv 0$ in $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$ and

$$\int_\Omega u(-\Delta)^s \phi\, dx = \int_\Omega \Big(\mu(x)\mathbb{D}_s^2(u) + \lambda f(x)\Big)\phi\, dx, \quad \forall\, \phi \in \mathbb{X}_s, \tag{4.1.6}$$

where

$$\mathbb{X}_s := \left\{ \xi \in \mathcal{C}(\mathbb{R}^N) : \operatorname{Supp} \xi \subset \overline{\Omega},\ (-\Delta)^s \xi(x) \text{ exists } \forall\ x \in \Omega \text{ and } |(-\Delta)^s \xi(x)| \le C \text{ for some } C > 0 \right\}. \tag{4.1.7}$$

In the spirit of the existing results for the local case, our first main result shows the existence of a weak solution to (P_λ) under a smallness condition on λf.

Theorem 4.1.1. *Assume that (A_1) holds. Then there exists $\lambda^* > 0$ such that, for all $0 < \lambda \le \lambda^*$, ($P_\lambda$) has a weak solution $u \in W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$.*

Remark 4.1.1.

a) The definition of $W_0^{s,2}(\Omega)$ will be introduced in Section 4.2.

b) In 1983, L. Boccardo, F. Murat and J.P. Puel [23] already pointed out that the existence of solution to (4.1.4) is not guaranteed for every $\lambda f \in L^\infty(\Omega)$. Some extra conditions are needed. Hence, the smallness condition appearing in Theorem 4.1.1 was somehow expected.

c) For $\lambda f \equiv 0$, $u \equiv 0$ is a solution to (P_λ) that obviously belongs to $W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$. Hence, there is no loss of generality to assume that $\lambda > 0$.

The counterpart of $|\nabla u|^2$ in (4.1.4) is played in (P_λ) by $\mathbb{D}_s^2(u)$. This term appears in several applications. For instance, let us mention [30,91,101] where it naturally appears as the equivalent of $|\nabla u|^2$ when considering fractional harmonic maps into the sphere.

Let us now give some ideas of the proof of Theorem 4.1.1. As in the local case, see for instance [94], the existence of solutions to (P_λ) is related to the regularity of the solutions to a linear equation of the form

$$\begin{cases} (-\Delta)^s v = h(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.1.8}$$

In Section 4.3, we obtain sharp Calderón-Zygmund type regularity results for the fractional Poisson equation (4.1.8) with low integrability data. We believe these results are of independent interest and will be useful in other settings. Actually, Section 4.3 can be read as an independent part of the present chapter. In particular, we refer the interested reader to Propositions 4.3.1, 4.3.3 and 4.3.4.

Having at hand suitable regularity results for (4.1.8) and inspired by [94, Section 6], we develop a fixed point argument to obtain a solution to (P_λ). Note that, due to the nonlocality of the operator and of the "gradient term", the approach of [94] has to be adapted significantly. In particular, the form of the set where we apply the fixed point argument seems to be new in the literature. We consider a subset of $W_0^{s,1}(\Omega)$ where, in some sense, we require more "differentiability" and more integrability. This extra "differentiability" is a purely nonlocal phenomena and it is related with our regularity results for (4.1.8). See Section 4.4 for more details.

Let us also stress that the restriction $s \in (1/2, 1)$ comes from the regularity results of Section 4.3. If suitable regularity results for (4.1.8) with $s \in (0, 1/2]$ were available, our fixed point argument would provide the desired existence results to (P_λ). See Section 4.3 for more details.

Next, let us prove that the smallness condition imposed in Theorem 4.1.1 is somehow necessary.

Theorem 4.1.2. *Assume* (A_1) *and suppose that* $\mu(x) \geq \mu_1 > 0$ *and* $f^+ \not\equiv 0$. *Then there exists* $\lambda^{**} > 0$ *such that, for all* $\lambda > \lambda^{**}$, (P_λ) *has no weak solutions in* $W_0^{s,2}(\Omega)$.

Remark 4.1.2.

a) Observe that, if v is a solution to

$$\begin{cases} (-\Delta)^s v = -\mu(x)\,\mathbb{D}_s^2(v) - \lambda f(x), & \text{in } \Omega, \\ \qquad v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $u = -v$ is a solution to (P_λ). Hence, if $\mu(x) \leq -\mu_1 < 0$ and $f^- \not\equiv 0$ we recover the same kind of non-existence result and the smallness condition is also required.

b) Since we do not use the regularity results of Section 4.3, the restriction $s \in (1/2, 1)$ is not necessary in the proof of Theorem 4.1.2. The result holds for all $s \in (0,1)$.

Also, in order to show that the regularity imposed on the data f is almost optimal, we provide a counterexample to our existence result when the regularity condition on f is not satisfied. The proof makes use of the Hardy potential.

Theorem 4.1.3. *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 2$, *be a bounded domain with* $\partial\Omega$ *of class* C^2, *let* $s \in (0,1)$ *and let* $\mu \in L^\infty(\Omega)$ *such that* $\mu(x) \geq \mu_1 > 0$. *Then, for all* $1 \leq p < \frac{N}{2s}$, *there exists* $f \in L^p(\Omega)$ *such that* (P_λ) *has no weak solutions in* $W_0^{s,2}(\Omega)$ *for any* $\lambda > 0$.

Using the same kind of approach than in Theorem 4.1.1, i.e. regularity results for (4.1.8) and our fixed point argument, one can obtain existence results for related problems involving different nonlocal diffusion terms and different nonlinearities.

First, we deal with the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x)\, u\, \mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{$\widetilde{P_\lambda}$}$$

For $\mu(x) \equiv 1$, this problem can be seen as a particular case of the fractional harmonic maps problem considered in [30, 91].

Remark 4.1.3. The notion of weak solution to ($\widetilde{P_\lambda}$) is essentially the same as in Definition 4.1.1. The only difference is that we now require that u and $u\,\mathbb{D}_s^2(u)$ belong to $L^1(\Omega)$.

We derive the following existence result for λf small enough.

Theorem 4.1.4. *Assume that* (A_1) *holds. Then, there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \leq \lambda^*$, ($\widetilde{P_\lambda}$) *has a weak solution* $u \in W_0^{s,2}(\Omega) \cap C^{0,\alpha}(\Omega)$ *for some* $\alpha > 0$.

Next, motivated by some other results on fractional harmonic maps into the sphere [41, 42] and some classical results of harmonic analysis [109, Chapter V], we consider a different diffusion term. Depending on the real parameter $\lambda > 0$, we study the existence of solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{\frac{s}{2}} u|^q + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{Q_λ}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ f \in L^m(\Omega) \text{ for some } m \geq 1 \text{ and } \mu \in L^\infty(\Omega), \\ s \in (1/2, 1) \text{ and } 1 < q < \dfrac{N}{N - ms}. \end{cases} \tag{B_1}$$

Remark 4.1.4. If $m \geq N/s$, we just need to assume $1 < q < \infty$ in (B_1).

Since the diffusion term considered in (Q_λ) is different from the ones in (P_λ) and ($\widetilde{P}_\lambda$), we shall make precise the notion of weak solution to (Q_λ).

Definition 4.1.2. We say that u is a *weak solution* to (Q_λ) if $u \in L^1(\Omega)$, $|(-\Delta)^{\frac{s}{2}} u| \in L^q(\Omega)$, $u \equiv 0$ in $\mathcal{C}\Omega$ and

$$\int_\Omega u(-\Delta)^s \phi \, dx = \int_\Omega \Big(\mu(x)|(-\Delta)^{\frac{s}{2}} u|^q + \lambda f(x)\Big)\phi \, dx, \quad \forall\ \phi \in \mathbb{X}_s, \tag{4.1.9}$$

where $\mathbb{X}_s$ is defined in (4.1.7).

Theorem 4.1.5. *Assume that* (B_1) *holds. Then there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \leq \lambda^*$, ($Q_\lambda$) *has a weak solution* $u \in W_0^{s,1}(\Omega)$.

Remark 4.1.5. The regularity results for (4.1.8) that we need to prove Theorem 4.1.5 are different from the ones used in Theorems 4.1.1 and 4.1.4. Nevertheless, the restriction $s \in (1/2, 1)$ still arises out of these regularity results. See Proposition 4.3.5 for more details.

Finally, for $s \in (0,1)$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, following [95, 103], we define the (distributional Riesz) *fractional gradient of order s* as the vector field $\nabla^s : \mathbb{R}^N \to \mathbb{R}$ given by

$$\nabla^s \phi(x) := \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^s} \frac{x-y}{|x-y|} \frac{dy}{|x-y|^N}, \quad \forall\ x \in \mathbb{R}^N. \tag{4.1.10}$$

Then we deal with the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x)|\nabla^s u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{$\widetilde{Q}_\lambda$}$$

Remark 4.1.6. The notion of weak solution to ($\widetilde{Q}_\lambda$) has to be understood as in Definition 4.1.2.

Theorem 4.1.6. *Assume that* (B_1) *holds. Then there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \leq \lambda^*$, ($\widetilde{Q}_\lambda$) *has a weak solution* $u \in W_0^{s,1}(\Omega)$.

We end this section describing the organization of the chapter. In Section 4.2, we introduce the suitable functional setting to deal with our problems and we also recall some known results that will be useful. In Section 4.3, which is independent of the rest of the chapter, we prove Calderón-Zygmund type regularity results for the fractional Poisson equation (4.1.8). Section 4.4 is devoted to the proofs of Theorems 4.1.1 and 4.1.4. Section 4.5 contains the proofs of Theorems 4.1.2 and 4.1.3. Section 4.6 deals with (Q_λ) and ($\widetilde{Q}_\lambda$), i.e., it is devoted to the proofs of Theorems 4.1.5 and 4.1.6.

Notation.

1) In $\mathbb{R}^N$, we use the notations $|x| = \sqrt{x_1^2 + \ldots + x_N^2}$ and $B_R(y) = \{x \in \mathbb{R}^N : |x-y| < R\}$.

2) For a bounded open set $\Omega \subset \mathbb{R}^N$ we denote its complementary as $\mathcal{C}\Omega$, i.e. $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$.

3) For $p \in (1,\infty)$, we denote by p' the conjugate exponent of p, namely $p' = p/(p-1)$ and by p_s^* the Sobolev critical exponent i.e. $p_s^* = \frac{Np}{N-sp}$ if $sp < N$ and $p_s^* = +\infty$ in case $sp \geq N$.

4) For $u \in L^\infty(\Omega)$ we use the notation $\|u\|_\infty = \|u\|_{L^\infty(\Omega)} = \text{esssup}_{x\in\Omega} |u(x)|$.

4.2 Functional setting and Useful tools

In this section we present the functional setting and some auxiliary results that will play an important role throughout the chapter. We begin recalling the definition of the fractional Sobolev space.

Definition 4.2.1. Let Ω be an open set in $\mathbb{R}^N$ and $s \in (0,1)$. For any $p \in [1,\infty)$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \right\}.$$

It is a Banach space endowed with the usual norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

Having at hand this definition we introduce the suitable space to deal with our problems.

Definition 4.2.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{0,1}$ and $s \in (0,1)$. For any $p \in [1,\infty)$. We define the space $W_0^{s,p}(\Omega)$ as

$$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

It is a Banach space endowed with the norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p},$$

where

$$D_\Omega := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega) = (\Omega \times \mathbb{R}^N) \cup (\mathcal{C}\Omega \times \Omega).$$

The space $W_0^{s,p}(\Omega)$ was first introduced in [102] in the particular case $p = 2$. We refer to [55] for more details on fractional Sobolev spaces. Nevertheless, due to their relevance in this work, we recall here some results involving fractional Sobolev spaces.

We shall make use of the following classical fractional Sobolev inequality. See [95, Proposition 15.5] for a beautiful proof.

Theorem 4.2.1 (Sobolev inequality)**.** *For any $s \in (0,1)$, $p \in [1, \frac{N}{s})$ and $u \in W^{s,p}(\mathbb{R}^N)$, it follows that*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \le S_{N,p} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p},$$

where $S_{N,p} > 0$ is a constant depending only on N and p.

Next, we present a fractional Hardy inequality and some of its consequences. These results will be crucial to show the optimality of the regularity assumptions of Theorem 4.1.1, namely to prove Theorem 4.1.3.

Theorem 4.2.2. [63, Theorem 1.1] *Let $N \ge 2$, $0 < s < 1$ and $p > 1$. Then, for all $u \in C_0^\infty(\mathbb{R}^N)$, it follows that*

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \ge \Lambda_{N,p,s} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx, \tag{4.2.1}$$

where

$$\Lambda_{N,s,p} := 2\int_0^1 \sigma^{ps-1} \left| 1 - \sigma^{\frac{N-ps}{p}} \right| \Phi_{N,s,p}(\sigma) d\sigma > 0, \tag{4.2.2}$$

and

$$\Phi_{N,s,p}(\sigma) := |\mathbb{S}^{N-2}| \int_{-1}^1 \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2\sigma t+\sigma^2)^{\frac{N+ps}{2}}} dt.$$

Proposition 4.2.3. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary $\partial\Omega$ of class C^2 such that $0 \in \Omega$, $0 < s < 1$ and $p > 1$. Then:*

1) [3, Lemma 3.4] *If we set*

$$\Lambda(\Omega) := \inf \left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}} dxdy}{\displaystyle\int_\Omega \frac{|\phi(x)|^p}{|x|^{ps}} dx} : \phi \in C_0^\infty(\Omega) \setminus \{0\} \right\},$$

it follows that $\Lambda(\Omega) = \Lambda_{N,s,p}$ where $\Lambda_{N,s,p} > 0$ is defined in (4.2.2).

2) The weight $|x|^{-ps}$ is optimal in the sense that, for all $\varepsilon > 0$, if follows that

$$\inf \left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}} dxdy}{\displaystyle\int_\Omega \frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}} dx} : \phi \in C_0^\infty(\Omega) \setminus \{0\} \right\} = 0.$$

Proof. Since the proof of 1) can be found in [3, Lemma 3.4], we just provide the proof of 2). Let $\varepsilon > 0$ be fixed but arbitrarily small. We assume by contradiction that there exists a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \Omega$ and

$$\Lambda_\varepsilon(\Omega) := \inf \left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy}{\displaystyle\int_\Omega \frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}}dx} : \phi \in \mathcal{C}_0^\infty(\Omega)\setminus\{0\} \right\} > 0. \tag{4.2.3}$$

Let us then observe that for any $B_r(0) \subset \Omega$, it follows that

$$0 < \Lambda_\varepsilon(\Omega) \leq \Lambda_\varepsilon(B_r(0)). \tag{4.2.4}$$

Moreover, observe that for $\phi \in \mathcal{C}_0^\infty(B_r(0))$ we have that

$$\int_{B_r(0)} \frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}}dx \geq \frac{1}{r^\varepsilon}\int_{B_r(0)} \frac{|\phi(x)|^p}{|x|^{ps}}dx. \tag{4.2.5}$$

Hence, gathering (4.2.4)-(4.2.5), it follows that, for all $\phi \in \mathcal{C}_0^\infty(B_r(0))$,

$$0 < \Lambda_\varepsilon(\Omega) \leq \Lambda_\varepsilon(B_r(0)) \leq \frac{\displaystyle\iint_{D_{B_r(0)}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy}{\displaystyle\int_{B_r(0)} \frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}}dx} \leq r^\varepsilon \frac{\displaystyle\iint_{D_{B_r(0)}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy}{\displaystyle\int_{B_r(0)} \frac{|\phi(x)|^p}{|x|^{ps}}dx}.$$

Thus, by the definition of $\Lambda(B_r(0))$ and 1), we deduce that $0 < \frac{\Lambda_\varepsilon(\Omega)}{r^\varepsilon} \leq \Lambda_{B_r(0)} = \Lambda_{N,s,p}$. Since (by assumption) $\Lambda_\varepsilon(\Omega) > 0$ and $\Lambda_{N,s,p}$ is independent of Ω, letting $r \to 0$, we obtain a contradiction and the result follows. □

In order to prove some of the Calderón-Zygmund type regularity results of Section 4.3, we will use the relation between the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ and the Bessel potential space defined below.

Definition 4.2.3. Let $s \in (0,1)$. For any $p \in [1,\infty)$, the Bessel potential space $L^{s,p}(\mathbb{R}^N)$ is defined as

$$L^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) \text{ such that } u = (I-d)^{-\frac{s}{2}} f \text{ with } f \in L^p(\mathbb{R}^N) \right\}.$$

It is a Banach space endowed with the norm

$$|||u|||_{L^{s,p}(\mathbb{R}^N)} := ||u||_{L^p(\mathbb{R}^N)} + ||f||_{L^p(\mathbb{R}^N)}.$$

Having in mind the *fractional gradient of order s* introduced in (4.1.10), let us point out that in [103, Theorem 1.7] it is proved that

$$L^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) \text{ such that } |\nabla^s u| \in L^p(\mathbb{R}^N) \right\}, \tag{4.2.6}$$

with the equivalent norm

$$||u||_{L^{s,p}(\mathbb{R}^N)} := ||u||_{L^p(\mathbb{R}^N)} + ||\nabla^s u||_{L^p(\mathbb{R}^N)}.$$

Notice also that in the case where s is an integer and $1 < p < \infty$, by [9, Theorem 7.63] we know that $L^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$. Differently, in case $s \in (0,1)$, the two previous spaces does not coincide. However, we have the following result.

Theorem 4.2.4. [9, Theorem 7.63] *Assume that $s \in (0,1)$ and $1 < p < \infty$. For all $0 < \varepsilon < s$, it follows that*

$$L^{s+\varepsilon,p}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \subset L^{s-\varepsilon,p}(\mathbb{R}^N),$$

with continuous inclusions.

Finally, we recall a classical result of harmonic analysis that will be useful in Section 4.3.

Lemma 4.2.5. [109, Theorem I, Section 1.2, Chapter V] *Let $0 < \lambda < N$ and $1 \le p < \ell < \infty$ be such that $\frac{1}{\ell} + 1 = \frac{1}{p} + \frac{\lambda}{N}$. For $g \in L^p(\mathbb{R}^N)$, we define*

$$J_\lambda(g)(x) = \int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^\lambda} dy.$$

Then, it follows that:

a) J_λ is well defined in the sense that the integral converges absolutely for almost all $x \in \mathbb{R}^N$.

b) If $p > 1$, then $\|J_\lambda(g)\|_{L^\ell(\mathbb{R}^N)} \le c_{p,q}\|g\|_{L^p(\mathbb{R}^N)}$.

c) If $p = 1$, then $\left|\{x \in \mathbb{R}^N \,|J_\lambda(g)(x) > \sigma\}\right| \le \left(\dfrac{A\|g\|_{L^1(\mathbb{R}^N)}}{\sigma}\right)^\ell$.

4.3 Regularity results for the fractional Poisson equation

The main goal of this section, which is independent of the rest of the work, is to prove sharp Calderón-Zygmund type regularity results for the fractional Poisson equation

$$\begin{cases} (-\Delta)^s v = h(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.3.1}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \ge 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ s \in (1/2, 1), \\ h \in L^m(\Omega) \text{ for some } m \ge 1. \end{cases} \tag{4.3.2}$$

First of all, let us precise the notion of weak solution to (4.3.1).

Definition 4.3.1. We say that v is a *weak solution* to (4.3.1) if $v \in L^1(\Omega)$, $v \equiv 0$ in $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$ and

$$\int_\Omega v(-\Delta)^s \phi \, dx = \int_\Omega h(x)\phi \, dx, \quad \forall\ \phi \in \mathbb{X}_s,$$

where $\mathbb{X}_s$ is defined in (4.1.7).

Under our assumption (4.3.2), the existence and uniqueness of solutions to (4.3.1) is a particular case of [35, Proposition 2.4] (see also [84, Section 4]). Having this in mind, we prove several regularity results for (4.3.1). Our first main result reads as follows:

Proposition 4.3.1. *Assume* (4.3.2) *and let v be the unique weak solution to* (4.3.1) *and $t \in (0,1)$:*

1) *If $m = 1$, then $v \in W_0^{t,p}(\Omega)$ for all $1 \le p < \frac{N}{N-(2s-t)}$ and there exists $C_1 = C_1(s,t,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{t,p}(\Omega)} \le \|v\|_{W^{t,p}(\mathbb{R}^N)} \le C_1\|h\|_{L^1(\Omega)}.$$

2) *If $1 < m < \frac{N}{2s}$, then $v \in W_0^{t,p}(\Omega)$ for all $1 \le p \le \frac{mN}{N-m(2s-t)}$ and there exists $C_1 = C_1(m,s,t,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{t,p}(\Omega)} \le \|v\|_{W^{t,p}(\mathbb{R}^N)} \le C_1\|h\|_{L^m(\Omega)}.$$

3) *If $\frac{N}{2s} \le m < \frac{N}{2s-1}$, then $v \in W_0^{t,p}(\Omega)$ for all $1 \le p < \frac{mN}{t(N-m(2s-1))}$ and there exists $C_1 = C_1(m,s,t,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{s,p}(\Omega)} \le \|v\|_{W^{t,p}(\mathbb{R}^N)} \le C_1\|h\|_{L^m(\Omega)}.$$

4) *If $m \ge \frac{N}{2s-1}$, then $v \in W_0^{t,p}(\Omega)$ for all $1 \le p < \infty$.*

Remark 4.3.1.

a) The previous results are sharp in the sense that, if "we take $t = s = 1$", we recover the classical sharp regularity results for the local case and those cannot be improved. See for instance [95, Chapter 5].
b) In the particular case of the fractional Laplacian of order $s \in (1/2,1)$ and for $h \in L^1(\Omega)$, we improve the regularity results of [2, 80, 84]. Note however that in the three quoted papers the authors deal with more general operators and cover the full range $s \in (0,1)$. Furthermore, in [80] the authors also deal with measures as data.
c) Since $s \in (1/2,1)$, observe that $t < 2s$ for all $t \in (0,1)$.

As we believe it has its own interest, let us highlight a particular case of the previous result.

Corollary 4.3.2. *Assume* (4.3.2) *and let v be the unique weak solution to* (4.3.1)*:*

1) *If $m = 1$, then $v \in W_0^{s,p}(\Omega)$ for all $1 \le p < \frac{N}{N-s}$ and there exists $C_1 = C_1(s,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{s,p}(\Omega)} \le \|v\|_{W^{s,p}(\mathbb{R}^N)} \le C_1\|h\|_{L^1(\Omega)}.$$

2) *If $1 < m < \frac{N}{2s}$, then $v \in W_0^{s,p}(\Omega)$ for all $1 \le p \le \frac{mN}{N-ms}$ and there exists $C_1 = C_1(m,s,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{s,p}(\Omega)} \le \|v\|_{W^{s,p}(\mathbb{R}^N)} \le C_1\|h\|_{L^m(\Omega)}.$$

3) *If $\frac{N}{2s} \le m < \frac{N}{2s-1}$, then $v \in W_0^{s,p}(\Omega)$ for all $1 \le p < \frac{mN}{s(N-m(2s-1))}$ and there exists $C_1 = C_1(m,s,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{s,p}(\Omega)} \le \|v\|_{W^{s,p}(\mathbb{R}^N)} \le C_1\|h\|_{L^m(\Omega)}.$$

4) *If $m \ge \frac{N}{2s-1}$, then $v \in W_0^{s,p}(\Omega)$ for all $1 \le p < \infty$.*

In the following two results we complete the information obtained in Proposition 4.3.1 when $h \in L^m(\Omega)$ for some $m > N/2s$.

Proposition 4.3.3. *Assume* (4.3.2) *and let v be the unique weak solution to* (4.3.1) *and $t \in (0,s)$:*

1) *If $\frac{N}{2s} \le m < \frac{N}{2s-t}$ then $v \in W_0^{t,p}(\Omega)$ for all $1 \le p < \frac{mN}{N-m(2s-t)}$ and there exists $C_2 = C_2(m,s,t,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{t,p}(\Omega)} \le \|v\|_{W^{t,p}(\mathbb{R}^N)} \le C_2\|h\|_{L^m(\Omega)}.$$

2) *If $m \ge \frac{N}{2s-t}$ then $v \in W_0^{t,p}(\Omega)$ for all $1 \le p < \infty$ and there exists $C_2 = C_2(m,s,t,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{t,p}(\Omega)} \le \|v\|_{W^{t,p}(\mathbb{R}^N)} \le C_2\|h\|_{L^m(\Omega)}.$$

Proposition 4.3.4. *Assume* (4.3.2) *and let v be the unique weak solution to* (4.3.1) *and $t \in (s,1)$. If $\frac{N}{2s} \le m < \frac{N}{s}$ then $v \in W_0^{t,p}(\Omega)$ for all $1 \le p < \frac{mN}{N-m(2s-t)}$ and there exists $C_2 = C_2(m,s,t,p,\Omega) > 0$ such that*
$$\|v\|_{W_0^{t,p}(\Omega)} \le \|v\|_{W^{t,p}(\mathbb{R}^N)} \le C_2\|h\|_{L^m(\Omega)}.$$

Remark 4.3.2. Notice that, in the case where $t \in (s,1)$, Propositions 4.3.1 and 4.3.4 complete and somehow give a more precise information than the result obtained in [81].

Remark 4.3.3. The proofs of Propositions 4.3.1, 4.3.3 and 4.3.4 are postponed to Subsection 4.3.1

Due to the nonlocality of the fractional Laplacian, several notions of regularity can be studied. The following results, which generalize the fractional regularity proved in [84, Theorem 24] with a different approach, can be seen as the counterpart of Proposition 4.3.1 to deal with (Q_λ) and ($\widetilde{Q}_\lambda$).

Proposition 4.3.5. *Assume* (4.3.2) *and let v be the unique weak solution to* (4.3.1) *and $t \in (0,s]$:*

1) *If $m = 1$, then $(-\Delta)^{\frac{t}{2}} v \in L^p(\Omega)$ for all $1 \le p < \frac{N}{N-(2s-t)}$ and there exists $C_3 = C_3(s,t,p,\Omega) > 0$ such that*
$$\|(-\Delta)^{\frac{t}{2}} v\|_{L^p(\Omega)} \le C_3\|h\|_{L^1(\Omega)}.$$

2) *If $1 < m < \frac{N}{2s-t}$, then $(-\Delta)^{\frac{t}{2}} v \in L^p(\Omega)$ for all $1 \le p \le \frac{mN}{N-m(2s-t)}$ and there exists $C_3 = C_3(s,t,m,p,\Omega) > 0$ such that*
$$\|(-\Delta)^{\frac{t}{2}} v\|_{L^p(\Omega)} \le C_3\|h\|_{L^m(\Omega)}.$$

3) *If $m \ge \frac{N}{2s-t}$ then $(-\Delta)^{\frac{t}{2}} v \in L^p(\Omega)$ for all $1 \le p < \infty$ and there exists $C_3 = C_3(s,t,m,p,\Omega) > 0$ such that*
$$\|(-\Delta)^{\frac{t}{2}} v\|_{L^p(\Omega)} \le C_3\|h\|_{L^m(\Omega)}.$$

Corollary 4.3.6. *Assume* (4.3.2) *and let v be the unique weak solution to* (4.3.1)*:*

1) *If $m = 1$, then $|\nabla^s v| \in L^p(\Omega)$ for all $1 \le p < \frac{N}{N-s}$ and there exists $C_4 = C_4(s,p,\Omega) > 0$ such that*
$$\|\nabla^s v\|_{L^p(\Omega)} \le C_4\|h\|_{L^1(\Omega)}.$$

2) *If $1 < m < \frac{N}{s}$, then $|\nabla^s v| \in L^p(\Omega)$ for all $1 \le p \le \frac{mN}{N-ms}$ and there exists $C_4 = C_4(s,m,p,\Omega) > 0$ such that*
$$\|\nabla^s v\|_{L^p(\Omega)} \le C_4\|h\|_{L^m(\Omega)}.$$

3) If $m \geq \frac{N}{s}$ then $|\nabla^s v| \in L^p(\Omega)$ for all $1 \leq p < \infty$ and there exists $C_4 = C_4(s,m,p,\Omega) > 0$ such that

$$\|\nabla^s v\|_{L^p(\Omega)} \leq C_4 \|h\|_{L^m(\Omega)}.$$

Remark 4.3.4. The proofs of Proposition 4.3.5 and Corollary 4.3.6 will be given in Subsection 4.3.2

Now, before proving Propositions 4.3.1, 4.3.3, 4.3.4, 4.3.5 and Corollary 4.3.6, we state some known results that will be useful in our proofs. First of all, we gather in the following lemma several results of [84].

Lemma 4.3.7. [84, Theorems 13, 15, 16, 23, 24] *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{0,1}$, let $s \in (0,1)$ and assume that $h \in L^m(\Omega)$ for some $m \geq 1$. Then problem* (4.3.1) *has an unique weak solution. Moreover:*

1) If $m = 1$, then $v \in L^p(\Omega)$ for all $1 \leq p < \frac{N}{N-2s}$ and there exists $C_5 = C_5(s,p,\Omega) > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq C_5 \|h\|_{L^1(\Omega)}.$$

2) If $1 < m < \frac{N}{2s}$, then $v \in L^p(\Omega)$ for all $1 \leq p \leq \frac{mN}{N-2ms}$ and there exists $C_5 = C_5(s,m,p,\Omega) > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq C_5 \|h\|_{L^m(\Omega)}.$$

3) If $m \geq \frac{N}{2s}$, then $v \in L^p(\Omega)$ for all $1 \leq p < \infty$ and there exists $C_5 = C_5(s,m,p,\Omega) > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq C_5 \|h\|_{L^m(\Omega)}.$$

Considering stronger assumptions, the first author and I. Peral proved in [7] that the unique weak solution to (4.3.1) belongs to a suitable local Sobolev space. More precisely, under the assumption (4.3.2) the authors obtained the following result.

Lemma 4.3.8. [7, Lemma 2.15] *Assume* (4.3.2) *and let v be the unique weak solution to* (4.3.1)*:*

1) If $m = 1$, then $v \in W^{1,p}(\mathbb{R}^N)$ for all $1 \leq p < \frac{N}{N-(2s-1)}$ and there exists $C_6 = C_6(s,p,\Omega) > 0$ such that

$$\|v\|_{W^{1,p}(\mathbb{R}^N)} \leq \overline{C}_6 \|\nabla v\|_{L^p(\Omega)} \leq C_6 \|h\|_{L^1(\Omega)}.$$

2) If $1 < m < \frac{N}{2s-1}$, then $v \in W^{1,p}(\mathbb{R}^N)$ for all $1 \leq p \leq \frac{mN}{N-m(2s-1)}$ and there exists $C_6 = C_6(m,s,p,\Omega) > 0$ such that

$$\|v\|_{W^{1,p}(\mathbb{R}^N)} \leq \overline{C}_6 \|\nabla v\|_{L^p(\Omega)} \leq C_6 \|h\|_{L^m(\Omega)}.$$

3) If $m \geq \frac{N}{2s-1}$, then $v \in W^{1,p}(\mathbb{R}^N)$ for all $1 \leq p < \infty$ and there exists $C_6 = C_6(m,s,p,\Omega) > 0$ such that

$$\|v\|_{W^{1,p}(\mathbb{R}^N)} \leq \overline{C}_6 \|\nabla v\|_{L^p(\Omega)} \leq C_6 \|h\|_{L^m(\Omega)}.$$

As last ingredient to prove our regularity results we need an interpolation result that we borrow from [28]. Let us introduce the real numbers $0 \leq s_1 \leq \eta \leq s_2 \leq 1$ and $1 \leq p_1, p_2, p \leq \infty$ and assume that they satisfy the relations

$$\eta = \theta s_1 + (1-\theta)s_2 \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{with} \quad 0 < \theta < 1. \tag{4.3.3}$$

Moreover, let us introduce the condition

$$s_2 = p_2 = 1 \quad \text{and} \quad \frac{1}{p_1} \leq s_1. \tag{4.3.4}$$

Lemma 4.3.9. [28, Theorem 1] *Assume that* (4.3.3) *holds and* (4.3.4) *fails. Then, for every* $\theta \in (0,1)$, *there exists a constant* $C = C(s_1, s_2, p_1, p_2, \theta) > 0$ *such that*

$$\|w\|_{W^{\eta,p}(\mathbb{R}^N)} \le C\|w\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^N)}\|w\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R}^N)}, \quad \forall\, w \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,p_2}(\mathbb{R}^N).$$

4.3.1 Proofs of Propositions 4.3.1, 4.3.3 and 4.3.4.

Having at hand all the needed ingredients, we prove our first regularity result.

Proof of Proposition 4.3.1. 1) Let v be the unique weak solution to (4.3.1). On the one hand, by Lemma 4.3.7, 1), we know that

$$\|v\|_{L^{p_1}(\mathbb{R}^N)} \le C_5\|h\|_{L^1(\Omega)}, \qquad \forall\, 1 \le p_1 < \frac{N}{N-2s}. \tag{4.3.5}$$

On the other hand, by Lemma 4.3.8, 1), we know that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \le C_6\|h\|_{L^1(\Omega)}, \qquad \forall\, 1 \le p_2 < \frac{N}{N-(2s-1)}. \tag{4.3.6}$$

Also, by Lemma 4.3.9 applied with $\eta = t$, $s_1 = 0$ and $s_2 = 1$, we have that

$$\|v\|_{W^{t,p}(\mathbb{R}^N)} \le C\|v\|^{1-t}_{L^{p_1}(\mathbb{R}^N)}\|v\|^{t}_{W^{1,p_2}(\mathbb{R}^N)}. \tag{4.3.7}$$

The result follows from (4.3.5)-(4.3.7) using that

$$1 \ge \frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} > \frac{(1-t)(N-2s)+t(N-(2s-1))}{N} = \frac{N-(2s-t)}{N}.$$

2) Let v be the unique weak solution to (4.3.1). By Lemma 4.3.7, 2), we know that

$$\|v\|_{L^{p_1}(\mathbb{R}^N)} \le C_5\|h\|_{L^m(\Omega)}, \qquad \forall\, 1 \le p_1 \le \frac{mN}{N-2ms}. \tag{4.3.8}$$

Also, by Lemma 4.3.8, 2), we know that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \le C_6\|h\|_{L^m(\Omega)}, \qquad \forall\, 1 \le p_2 \le \frac{mN}{N-m(2s-1)}. \tag{4.3.9}$$

Finally, by Lemma 4.3.9 applied with $\eta = t$, $s_1 = 0$ and $s_2 = 1$, we know that (4.3.7) holds. The result follows from (4.3.7), (4.3.8) and (4.3.9) using that

$$1 \ge \frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} \ge \frac{(1-t)(N-2ms)+t(N-m(2s-1))}{mN} = \frac{N-m(2s-t)}{mN}.$$

3) Let v be the unique weak solution to (4.3.1). By Lemma 4.3.7, 3), we know that

$$\|v\|_{L^{p_1}(\mathbb{R}^N)} \le C_5\|h\|_{L^m(\Omega)}, \qquad \forall\, 1 \le p_1 < \infty. \tag{4.3.10}$$

By Lemma 4.3.8, 2), we know that (4.3.9) holds. Moreover, by Lemma 4.3.9 applied with $\eta = t$, $s_1 = 0$ and $s_2 = 1$, it follows that (4.3.7) holds. The result follows from (4.3.7), (4.3.9) and (4.3.10) using that

$$1 \ge \frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} > \frac{t(N-m(2s-1))}{mN}.$$

4) Let v be the unique weak solution to (4.3.1). By Lemma 4.3.7, 3), we know that (4.3.10) holds. On the other hand, by Lemma 4.3.8, 3), we have that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \leq C_6 \|h\|_{L^m(\Omega)}, \qquad \forall\ 1 \leq p_2 < \infty. \tag{4.3.11}$$

The result follows from Lemma 4.3.9 applied with $\eta = t, s_1 = 0$ and $s_2 = 1$. □

Remark 4.3.5. The restriction $s \in (1/2, 1)$ comes from Lemma 4.3.8. It is not expected that Lemma 4.3.8 holds true for $s \in (0, 1/2]$. Hence, with this approach we are limited to deal with $s \in (1/2, 1)$.

Now, using the Bessel potential space $L^{s,p}(\mathbb{R}^N)$ (see Definition 4.2.3), Theorem 4.2.4 and Lemma 4.3.10 below, we prove Propositions 4.3.3 and 4.3.4. We begin proving a regularity result in the Bessel potential space.

Lemma 4.3.10. [9, Theorem 7.58] *Let $0 < t < s < 1$, $1 < p < \frac{N}{s-t}$ and $q = \frac{Np}{N-p(s-t)}$. Then $W^{s,p}(\mathbb{R}^N) \subset W^{t,q}(\mathbb{R}^N)$ with continuous inclusion.*

Proposition 4.3.11. *Assume (4.3.2) and let v be the unique weak solution to (4.3.1):*

1) *If $m = 1$, then $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leq p < \frac{N}{N-s}$ and there exists $C_7 = C_7(s, p, \Omega) > 0$ such that*

$$\|v\|_{L^{s,p}(\mathbb{R}^N)} \leq C_7 \|h\|_{L^1(\Omega)}.$$

2) *If $1 < m < \frac{N}{s}$, then $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leq p \leq \frac{mN}{N-ms}$ and there exists $C_7 = C_7(m, s, p, \Omega) > 0$ such that*

$$\|v\|_{L^{s,p}(\mathbb{R}^N)} \leq C_7 \|h\|_{L^m(\Omega)}.$$

3) *If $m \geq \frac{N}{s}$, then $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leq p < \infty$ and there exists $C_7 = C_7(m, s, p, \Omega) > 0$ such that*

$$\|v\|_{L^{s,p}(\mathbb{R}^N)} \leq C_7 \|h\|_{L^m(\Omega)}.$$

Proof. Assume (4.3.2) and let v be the unique weak solution to (4.3.1). Taking into account (4.2.6), we have just to show the regularity of $|\nabla^s v|$ where ∇^s is defined in (4.1.10). By a density argument and [95, Lemma 15.9] we have that

$$|\nabla^s v(x)| \leq \frac{1}{N-(1-s)} \int_{\mathbb{R}^N} \frac{|\nabla v(y)|}{|x-y|^{N-(1-s)}} dy \qquad \text{a.e. in } \mathbb{R}^N. \tag{4.3.12}$$

Let us then split into three cases:

1) $m = 1$.

By Lemma 4.3.8, 1), we get $v \in W^{1,q}(\mathbb{R}^N)$ for all $1 \leq q < \frac{N}{N-(2s-1)}$. Thus, by Lemma 4.2.5, we conclude that $|\nabla^s v(x)| \in L^p(\mathbb{R}^N)$ for all $1 \leq p < \frac{N}{N-s}$.

2) $1 < m < \frac{N}{s}$.

The result follows arguing on the same way, using now (4.3.12), Lemma 4.3.8, 2) and Lemma 4.2.5.

3) $m \geq \frac{N}{s}$.

In this case, since $m \geq \frac{N}{s}$ and Ω is a bounded domain then $f \in L^{\bar{m}}(\Omega)$ for all $\bar{m} < \frac{N}{s}$. In particular, using the second point it follows that $v \in L^{s,\bar{p}}(\mathbb{R}^N)$ for all $1 \leq \bar{p} \leq \frac{\bar{m}N}{N-s\bar{m}}$. Letting $\bar{m} \uparrow \frac{N}{s}$, we reach that $\bar{p} \uparrow \infty$. Hence $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leq p < \infty$. $\square$

Now, using the above regularity result in the Bessel potential space, we obtain the following lemma.

Lemma 4.3.12. *Assume (4.3.2) and let v be the unique weak solution to (4.3.1):*

1) If $\frac{N}{2s} \leq m < \frac{N}{s}$ then $v \in W_0^{s',p}(\Omega)$ for all $0 < s' < s$ and all $1 \leq p \leq \frac{mN}{N-ms}$. Moreover, there exists $C_8 = C_8(m, s', p, \Omega) > 0$ such that

$$\|v\|_{W_0^{s',p}(\Omega)} \leq \|v\|_{W^{s',p}(\mathbb{R}^N)} \leq C_8 \|h\|_{L^m(\Omega)}.$$

2) If $m \geq \frac{N}{s}$, then $v \in W_0^{s',p}(\Omega)$ for all $0 < s' < s$ and all $1 \leq p < \infty$. Moreover, there exists $C_8 = C_8(m, s', p, \Omega) > 0$ such that

$$\|v\|_{W_0^{s',p}(\Omega)} \leq \|v\|_{W^{s',p}(\mathbb{R}^N)} \leq C_8 \|h\|_{L^m(\Omega)}.$$

Proof. First observe that without loss of generality we can assume that $p > 1$. For $p = 1$ the result follows from Proposition 4.3.1 and the continuous embedding $W^{s,p}(\mathbb{R}^N) \subset W^{s',p}(\mathbb{R}^N)$. We consider then two cases:

1) $\frac{N}{2s} < m < \frac{N}{s}$

Let v be the unique weak solution to (4.3.1). By Proposition 4.3.11, 2) we know that $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leq p \leq \frac{mN}{N-ms}$. Thus, by Theorem 4.2.4 we conclude that $v \in W_0^{s',p}(\Omega)$ for all $0 < s' < s$ and

$$\|v\|_{W_0^{s',p}(\Omega)} \leq C \|v\|_{L^{s,p}(\mathbb{R}^N)} \leq C_8 \|h\|_{L^m(\Omega)}.$$

2) $m \geq \frac{N}{s}$

Let v be the unique weak solution to (4.3.1). In this case, by Proposition 4.3.11, we know that $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leq p < \infty$. Hence, by Theorem 4.2.4, we conclude. $\square$

Having at hand Lemmas 4.3.10 and 4.3.12, we prove Propositions 4.3.3 and 4.3.4.

Proof of Proposition 4.3.3. First observe that, since $t \in (0, s)$ it follows that $\frac{N}{2s-t} < \frac{N}{s}$. Then we split into two cases:

1) $\frac{N}{2s} \leq m < \frac{N}{2s-t}$.

Let v be the unique weak solution to (4.3.1). On the one hand, by Lemma 4.3.12, 1), we have that $v \in W^{s',p_1}(\mathbb{R}^N)$ for all $0 < s' < s$ and all $1 \leq p_1 \leq \frac{mN}{N-ms}$ and that there exists $C_8 > 0$ such that

$$\|v\|_{W_0^{s',p_1}(\Omega)} \leq \|v\|_{W^{s',p_1}(\mathbb{R}^N)} \leq C_8 \|h\|_{L^m(\Omega)}. \tag{4.3.13}$$

On the other hand, by Lemma 4.3.10, we have that $v \in W^{\eta,q_1}(\mathbb{R}^N)$ for all $0 < \eta < s' < s$ and all $1 \leq q_1 \leq \frac{Np_1}{N-p_1(s'-\eta)} \leq \frac{mN}{N-m(s+s'-\eta)}$. Moreover, there exists $C > 0$ such that

$$\|v\|_{W^{\eta,q_1}(\mathbb{R}^N)} \leq C \|v\|_{W^{s',p_1}(\mathbb{R}^N)} \leq C\, C_8 \|h\|_{L^m(\Omega)}. \tag{4.3.14}$$

We fix then $p \in [1, \frac{mN}{N-m(2s-t)})$ and observe that we can find $\overline{\eta} \in (t, s')$ such that

$$1 \leq p \leq \frac{mN}{N - m(s + s' - \overline{\eta})}.$$

The result follows from (4.3.14) using the continuous embedding $W^{\overline{\eta},p}(\mathbb{R}^N) \subset W^{t,p}(\mathbb{R}^N)$.

2) $m \geq \frac{N}{2s-t}$.

The result follows arguing as in the proof of Proposition 4.3.11, 3) using Proposition 4.3.3, 1). □

Proof of Proposition 4.3.4. On the one hand, by Lemma 4.3.12, 1), we have that

$$\|v\|_{W_0^{s',p_1}(\Omega)} \leq \|v\|_{W^{s',p_1}(\mathbb{R}^N)} \leq C_8 \|h\|_{L^m(\Omega)}, \quad \forall\, 0 < s' < s,\ \forall\, 1 \leq p_1 \leq \frac{mN}{N - ms}. \tag{4.3.15}$$

On the other hand, by Lemma 4.3.8, 2), it follows that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \leq C_6 \|h\|_{L^m(\Omega)}, \quad \forall\, 1 \leq p_2 \leq \frac{mN}{N - m(2s-1)}. \tag{4.3.16}$$

Also, by Lemma 4.3.9 applied with $\eta = t$, $s_1 = s'$ and $s_2 = 1$, we know that

$$\|v\|_{W^{t,p'}(\mathbb{R}^N)} \leq C \|v\|_{W^{s',p_1}(\mathbb{R}^N)}^{\frac{1-t}{1-s'}} \|v\|_{W^{1,p_2}(\mathbb{R}^N)}^{\frac{t-s'}{1-s'}}, \tag{4.3.17}$$

with

$$\frac{1}{p'} = \frac{1}{1-s'}\left(\frac{1-t}{p_1} + \frac{t-s'}{p_2}\right).$$

We fix then an arbitrary $1 \leq p < \frac{mN}{N-m(2s-t)}$ and observe that we can choose $s' < s$ such that $p' = p$. Hence, the result follows from (4.3.15)-(4.3.17). □

4.3.2 Proofs of Proposition 4.3.5 and Corollary 4.3.6

Next, using again Lemma 4.3.8 but with a different approach, we prove Proposition 4.3.5. As a consequence we will obtain Corollary 4.3.6.

Proof of Proposition 4.3.5. Let v be the unique weak solution to (4.3.1) and define, for $x \in \mathbb{R}^N$ arbitrary,

$$S_1 := \{y \in \mathbb{R}^N : \operatorname{dist}(y, \Omega) > 2\} \quad \text{and} \quad S_2 := \{y \in \mathbb{R}^N : \operatorname{dist}(y, \Omega) \leq 2 \text{ and } |x-y| \geq 1\}.$$

Then, observe that, for all $x \in \Omega$,

$$\begin{aligned} |(-\Delta)^{\frac{t}{2}} v(x)| &\leq \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|}{|x-y|^{N+t}} dy \\ &\leq \int_{S_1} \frac{|v(x)-v(y)|}{|x-y|^{N+t}} dy + \int_{S_2} \frac{|v(x)-v(y)|}{|x-y|^{N+t}} dy + \int_{B_1(x)} \frac{|v(x)-v(y)|}{|x-y|^{N+t}} dy \\ &=: I_1(x) + I_2(x) + I_3(x). \end{aligned} \tag{4.3.18}$$

Now, let us estimate each one of the three terms. First observe that

$$\begin{aligned} I_1(x) &= \int_{S_1} \frac{|v(x)|}{|x-y|^{N+t}} dy \leq \int_{S_1} \frac{|v(x)|}{\operatorname{dist}(y,\Omega)^{N+t}} dy \\ &= |v(x)| \int_{S_1} \frac{dy}{\operatorname{dist}(y,\Omega)^{N+t}} dy = c_1(N,t,\Omega)|v(x)|, \qquad \forall\, x \in \Omega. \end{aligned} \tag{4.3.19}$$

Next, using that Ω is a bounded domain and the triangular inequality, we deduce that

$$I_2(x) \leq \int_{S_2} |v(x)-v(y)| dy \leq c_2(\Omega)|v(x)| + \|v\|_{L^1(\Omega)}, \qquad \forall\, x \in \Omega. \tag{4.3.20}$$

Finally, following the arguments of [55, Proposition 2.2], we deduce that

$$\begin{aligned} I_3(x) &= \int_{B_1(0)} \frac{|v(x)-v(x+z)|}{|z|} \frac{1}{|z|^{N+t-1}} dz = \int_{B_1(0)} \int_0^1 \frac{|\nabla v(x+\tau z)|}{|z|^{N+t-1}} d\tau dz \\ &\leq \int_0^1 \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} \tau^{t-1} dw d\tau = \left(\int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} dw \right) \left(\int_0^1 \tau^{t-1} d\tau \right) \\ &= \frac{1}{t} \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} dw, \qquad \forall\, x \in \Omega. \end{aligned} \tag{4.3.21}$$

From (4.3.18)-(4.3.21), we deduce that

$$|(-\Delta)^{\frac{t}{2}} v(x)| \leq c(s,t,\Omega) \left(|v(x)| + \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} dw + \|v\|_{L^1(\Omega)} \right), \qquad \forall\, x \in \Omega, \tag{4.3.22}$$

and so, exploiting again the fact that Ω is a bounded domain and using the Hölder and triangular inequalities, for all $1 \leq p < \infty$, we obtain that

$$\|(-\Delta)^{\frac{t}{2}} v\|_{L^p(\Omega)} \leq c_2(s,t,\Omega) \left(\|v\|_{L^p(\Omega)} + \left\| \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} dw \right\|_{L^p(\mathbb{R}^N)} \right). \tag{4.3.23}$$

Now, let us split the proof into three parts:

1) $m = 1$.

By Lemma 4.3.8, 1), we know that $v \in W_0^{1,\sigma}(\Omega)$ for all $1 \leq \sigma < \frac{N}{N-(2s-1)}$ and there exists $C_6 = C_6(s,\sigma,\Omega) > 0$ such that

$$\|\nabla v\|_{L^\sigma(\mathbb{R}^N)} \leq C_6 \|h\|_{L^1(\Omega)}, \qquad \forall\, 1 \leq \sigma < \frac{N}{N-(2s-1)}.$$

Thus, applying Lemma 4.2.5, we deduce that

$$\left\| \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} dw \right\|_{L^\ell(\mathbb{R}^N)} \leq C\, C_6 \|h\|_{L^1(\Omega)}, \quad \forall\, 1 \leq \ell < \frac{N}{N-(2s-t)}. \tag{4.3.24}$$

Also, by Lemma 4.3.7, 1), we know that

$$\|v\|_{L^\gamma(\Omega)} \leq C_5 \|h\|_{L^1(\Omega)}, \quad \forall\, 1 \leq \gamma < \frac{N}{N-2s}. \tag{4.3.25}$$

Taking into account (4.3.24)-(4.3.25), the result follows from (4.3.23).

2) $1 < m < \frac{N}{2s-t}$.

Observe that, by Lemma 4.3.8, 2), it follows that $v \in W_0^{1,\sigma}(\Omega)$ for all $1 \le \sigma \le \frac{mN}{N-m(2s-1)}$ and there exists $C_6 = C_6(m,s,\sigma,\Omega) > 0$ such that

$$\|\nabla v\|_{L^\sigma(\mathbb{R}^N)} \le C_6 \|h\|_{L^m(\Omega)}, \qquad \forall\, 1 \le \sigma \le \frac{mN}{N-m(2s-1)}.$$

Hence, by Lemma 4.2.5, we deduce that

$$\left\| \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} dw \right\|_{L^\ell(\mathbb{R}^N)} \le C\, C_6 \|h\|_{L^m(\Omega)}, \quad \forall\, 1 \le \ell \le \frac{mN}{N-m(2s-t)}. \tag{4.3.26}$$

Also, by Lemma 4.3.7, 2) and 3) we know that

$$\|v\|_{L^\gamma(\Omega)} \le C_5 \|h\|_{L^m(\Omega)}, \quad \forall\, 1 \le \gamma < \frac{mN}{N-2ms}, \quad \text{if} \quad 1 \le m < \frac{N}{2s} \tag{4.3.27}$$

and

$$\|v\|_{L^\gamma(\Omega)} \le C_5 \|h\|_{L^m(\Omega)}, \quad \forall\, 1 \le \gamma < \infty, \quad \text{if} \quad \frac{N}{2s} \le m < \frac{N}{2s-1}. \tag{4.3.28}$$

Taking into account (4.3.26)-(4.3.28), the result follows from (4.3.23).

3) $m \ge \frac{N}{2s-t}$.

The result follows arguing as in the proof of Proposition 4.3.11, 3) using Proposition 4.3.5, 2). □

Proof of Corollary 4.3.6. By [95, Lemma 15.9] we know that

$$\nabla^s u(x) = \frac{1}{N-(1-s)} \int_{\mathbb{R}^N} \frac{\nabla u(y)}{|x-y|^{N+s-1)}} dy. \tag{4.3.29}$$

Hence, we have that

$$|\nabla^s u(x)| \le C \int_{\mathbb{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N+s-1}} dy. \tag{4.3.30}$$

The result then follows arguing as in the proof of Proposition 4.3.5. □

4.3.3 Convergence and compactness

We end this section presenting a result of convergence and one of compactness for the fractional Poisson equation (4.3.1). They will be used in the proofs of Theorems 4.1.1, 4.1.4, 4.1.5 and 4.1.6.

Proposition 4.3.13. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$, let $s \in (1/2,1)$, let $\{h_n\} \subset L^1(\Omega)$ be a sequence such that $h_n \to h$ in $L^1(\Omega)$ and let v_n be the unique weak solution to*

$$\begin{cases} (-\Delta)^s v_n = h_n(x), & \text{in } \Omega, \\ v_n = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

for all $n \in \mathbb{N}$, and v be the unique weak solution to

$$\begin{cases} (-\Delta)^s v = h(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then $v_n \to v$ in $W_0^{s,p}(\Omega)$ for all $1 \le p < \frac{N}{N-s}$.

Proof. First of all observe that the existence of v_n and v are insured by Lemma 4.3.7. Now, let us define $w_n = v_n - v$ and observe that w_n satisfies

$$\begin{cases}(-\Delta)^s w_n = h_n(x) - h(x), & \text{in } \Omega,\\ w_n = 0, & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases}$$

Applying Proposition 4.3.1 with $m = 1$, it follows that

$$\|w_n\|_{W_0^{s,p}(\Omega)} \leq C_3 \|h_n - h\|_{L^1(\Omega)}, \quad \forall\, 1 \leq p < \frac{N}{N-s}.$$

Hence, since $h_n \to h$ in $L^1(\Omega)$, it follows that $w_n \to 0$ in $W_0^{s,p}(\Omega)$ for all $1 \leq p < \frac{N}{N-s}$ and so, that $v_n \to v$ in $W_0^{s,p}(\Omega)$ for all $1 \leq p < \frac{N}{N-s}$, as desired. □

Proposition 4.3.14. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$, let $s \in (1/2, 1)$ and let $h \in L^1(\Omega)$. Then the operator $\mathcal{S} : L^1(\Omega) \to W_0^{s,p}(\Omega)$ given by $\mathcal{S}(h) = v$ with v the unique weak solution to (4.3.1) is compact for all $1 \leq p < \frac{N}{N-s}$.*

Proof. Let $\{f_n\} \subset L^1(\Omega)$ be a bounded sequence. By [36, Proposition 2.4] we know that $\mathcal{S}$ is a compact operator from $L^1(\Omega)$ to $W_0^{1,p_1}(\Omega)$ for all $1 \leq \theta < \frac{N}{N-(2s-1)}$. Hence, for all $1 \leq \theta < \frac{N}{N-(2s-1)}$, up to a subsequence we have that $\mathcal{S}(f_n) \to v$ for some $v \in W_0^{1,\theta}(\Omega)$. By Sobolev inequality, this implies, for all $1 \leq \sigma < \frac{N}{N-2s}$, that $\mathcal{S}(f_n) \to v$ in $L^\sigma(\Omega)$ and $v \in L^\sigma(\Omega)$.

Now, applying Lemma 4.3.9 with $\eta = s$, $s_1 = 0$ and $s_2 = 1$, we obtain that

$$\begin{aligned}\|\mathcal{S}(f_n) - v\|_{W_0^{s,p}(\Omega)} &\leq C\|\mathcal{S}(f_n) - v\|_{L^\sigma(\mathbb{R}^N)}^{1-s} \|\mathcal{S}(f_n) - v\|_{W^{1,\theta}(\mathbb{R}^N)}^{s}\\ &= C\|\mathcal{S}(f_n) - v\|_{L^\sigma(\Omega)}^{1-s} \|\mathcal{S}(f_n) - v\|_{W_0^{1,\theta}(\Omega)}^{s},\end{aligned} \tag{4.3.31}$$

for p satisfying

$$\frac{1}{p} = \frac{1-s}{\sigma} + \frac{s}{\theta}. \tag{4.3.32}$$

Hence, the result follows from (4.3.31) using that

$$1 \geq \frac{1}{p} > \frac{N-s}{N}.$$

□

4.4 Proofs of Theorems 4.1.1 and 4.1.4

This section is devoted to prove Theorems 4.1.1 and 4.1.4. As indicated in the introduction, once we have the regularity results of Section 4.3, we follow the approach first develop in [94, Section 6]. Let us begin with two elementary technical lemmas that will be useful in the proofs of both theorems.

Lemma 4.4.1. *Let $a, b > 0$, $p > 1$ and $c^* := \frac{p-1}{p}\left(\frac{1}{pa^p b}\right)^{\frac{1}{p-1}}$. Then, the function $g : [0,\infty) \to \mathbb{R}$ given by*

$$g(t) = a^p(bt + c^*)^p - t,$$

has exactly one root $t^* \in (0, \infty)$.

Proof. First observe that, $g'(t)=0$ if and only if

$$t=t^*:=\frac{1}{b}\left(\frac{1}{pa^pb}\right)^{\frac{1}{p-1}}-\frac{c^*}{b}=\frac{1}{pb}\left(\frac{1}{pa^pb}\right)^{\frac{1}{p-1}}\in(0,\infty).$$

Moreover, observe that

$$g''(t^*)=(p-1)pa^pb^2\left(\frac{1}{pa^pb}\right)^{\frac{p-2}{p-1}}>0.$$

Thus, we deduce that g has an strict global minimum on $t=t^*$. Finally, observe that

$$g(t^*)=a^p\left(\frac{1}{pa^pb}\right)^{\frac{p}{p-1}}-\frac{1}{b}\left(\frac{1}{pa^pb}\right)^{\frac{1}{p-1}}+\frac{p-1}{pb}\left(\frac{1}{pa^pb}\right)^{\frac{1}{p-1}}=0,\quad g(0)>0 \text{ and } \lim_{t\to\infty}g(t)=\infty.$$

Hence, we conclude that g has exactly one root $t^*\in(0,\infty)$. □

Lemma 4.4.2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{0,1}$ and let $s\in(0,1)$. For all $\varepsilon>0$ satisfying $0<s-\varepsilon<s+\varepsilon<1$ and all $1\le\sigma<r$ there exists $C_9=C_9(s,\varepsilon,\sigma,r,\Omega)>0$ such that*

$$\left\|\left(\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}}dy\right)^{\frac{1}{\sigma}}\right\|_{L^r(\Omega)}\le C_9\|u\|_{W_0^{s+\varepsilon,r}(\Omega)},\quad \forall\ u\in W_0^{s+\varepsilon,r}(\Omega). \tag{4.4.1}$$

Proof. First of all, observe that

$$\begin{aligned}
&\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}}dy\right)^{\frac{r}{\sigma}}dx\\
&=\int_\Omega\left(\int_{\mathbb{R}^N\cap\{|x-y|<1\}}\frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}}dy+\int_{\mathbb{R}^N\cap\{|x-y|\ge1\}}\frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}}dy\right)^{\frac{r}{\sigma}}dx\\
&\le c_{r,\sigma}\left[\int_\Omega\left(\int_{\mathbb{R}^N\cap\{|x-y|<1\}}\frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}}dy\right)^{\frac{r}{\sigma}}dx+\int_\Omega\left(\int_{\mathbb{R}^N\cap\{|x-y|\ge1\}}\frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}}dy\right)^{\frac{r}{\sigma}}dx\right]\\
&=:c_{r,\sigma}(J_1+J_2).
\end{aligned}\tag{4.4.2}$$

Let us then estimate J_1. Applying Hölder inequality, we have that

$$\begin{aligned}
J_1&=\int_\Omega\left(\int_{\mathbb{R}^N\cap\{|x-y|<1\}}\frac{|u(x)-u(y)|^\sigma}{|x-y|^{\frac{N\sigma}{r}+(s+\varepsilon)\sigma}}\frac{|x-y|^{\varepsilon\sigma}}{|x-y|^{N-\frac{N\sigma}{r}}}dy\right)^{\frac{r}{\sigma}}dx\\
&\le\int_\Omega\left(\int_{\mathbb{R}^N\cap\{|x-y|<1\}}\frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}}dy\right)\left(\int_{\mathbb{R}^N\cap\{|x-y|<1\}}\frac{dy}{|x-y|^{N-\frac{\varepsilon\sigma r}{r-\sigma}}}\right)^{\frac{r-\sigma}{\sigma}}dx.
\end{aligned}$$

Furthermore, since

$$\int_{\mathbb{R}^N\cap\{|x-y|<1\}}\frac{dy}{|x-y|^{N-\frac{\varepsilon\sigma r}{r-\sigma}}}=\int_{B_1(0)}\frac{dz}{|z|^{N-\frac{\varepsilon\sigma r}{r-\sigma}}}=C_{J_1}(\varepsilon,\sigma,r)<\infty,$$

we deduce that

$$J_1\le\widetilde{C}_{J_1}\int_\Omega\left(\int_{\mathbb{R}^N\cap\{|x-y|<1\}}\frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}}dy\right)dx\le\widetilde{C}_{J_1}\|u\|^r_{W_0^{s+\varepsilon,r}(\Omega)}. \tag{4.4.3}$$

Now, arguing as with J_1, we obtain that

$$J_2 = \int_{\Omega}\left(\int_{\mathbb{R}^N\cap\{|x-y|\geq 1\}} \frac{|u(x)-u(y)|^{\sigma}}{|x-y|^{\frac{N\sigma}{r}+(s-\varepsilon)\sigma}} \frac{dy}{|x-y|^{N-\frac{N\sigma}{r}+\varepsilon\sigma}}\right)^{\frac{r}{\sigma}} dx$$
$$\leq \int_{\Omega}\left(\int_{\mathbb{R}^N\cap\{|x-y|\geq 1\}} \frac{|u(x)-u(y)|^{r}}{|x-y|^{N+(s-\varepsilon)r}} dy\right)\left(\int_{\mathbb{R}^N\cap\{|x-y|\geq 1\}} \frac{dy}{|x-y|^{N+\frac{\varepsilon\sigma r}{r-\sigma}}}\right)^{\frac{r-\sigma}{\sigma}} dx.$$

Hence, since

$$\int_{\mathbb{R}^N\cap\{|x-y|\geq 1\}} \frac{dy}{|x-y|^{N+\frac{\varepsilon\sigma r}{r-\sigma}}} = \int_{\mathbb{R}^N\setminus B_1(0)} \frac{dz}{|z|^{N+\frac{\varepsilon\sigma r}{r-\sigma}}} = C_{J_2}(\varepsilon,\sigma,r) < \infty,$$

and $W_0^{s+\varepsilon,r}(\Omega) \subset W_0^{s-\varepsilon,r}(\Omega)$, it follows that

$$J_2 \leq \widetilde{C}_{J_2} \int_{\Omega}\left(\int_{\mathbb{R}^N\cap\{|x-y|\geq 1\}} \frac{|u(x)-u(y)|^{r}}{|x-y|^{N+(s-\varepsilon)r}} dy\right) dx \leq \widetilde{C}_{J_2}\|u\|^r_{W_0^{s-\varepsilon,r}(\Omega)} \leq \overline{C}_{J_2}\|u\|^r_{W_0^{s+\varepsilon,r}(\Omega)}. \tag{4.4.4}$$

The result follows from (4.4.2), (4.4.3) and (4.4.4). □

4.4.1 Proof of Theorem 4.1.1

Let us begin recalling that, under the assumption (A_1), $f \in L^m(\Omega)$ for some $m > \frac{N}{2s}$. Hence, since we are working in a bounded domain, without loss of generality, we can assume that $m \in (\frac{N}{2s}, \frac{N}{2s-1})$. Moreover, observe that, for $\lambda f \equiv 0$, $u \equiv 0$ is a solution to (P_λ) and, for $\mu \equiv 0$, (P_λ) reduces to (4.3.1). Hence, we may assume that $\|\mu\|_\infty \neq 0$ and $\|f\|_{L^m(\Omega)} \neq 0$.

Next, we fix some notation that we use throughout this subsection. First, we fix $r = r(m,s) > 0$ such that

$$1 < 2m < r < \frac{mN}{s(N-m(2s-1))},$$

and $\varepsilon = \varepsilon(r,m,s) > 0$ such that

$$1 < r < \frac{mN}{(s+\varepsilon)(N-m(2s-1))} < \frac{mN}{s(N-m(2s-1))}, \quad s+\varepsilon < 1, \quad \text{and } s-\varepsilon > \frac{1}{2}.$$

Also, we introduce and fix the constants C_1, given by Proposition 4.3.1, 3) applied with $t = s+\varepsilon$ and $p = r$, $C_{10} := C_9^2|\Omega|^{\frac{r-2m}{rm}}$, where C_9 is the constant given by Lemma 4.4.2, and

$$\lambda^* := \frac{1}{4\|f\|_{L^m(\Omega)}C_1^2 C_{10}\|\mu\|_\infty}.$$

By the definition of λ^* and Lemma 4.4.1, we know there exists and unique $l \in (0,\infty)$ such that

$$C_1(C_{10}\|\mu\|_\infty l + \lambda^*\|f\|_{L^m(\Omega)}) = l^{\frac{1}{2}}. \tag{4.4.5}$$

Having fixed the above constants, we introduce

$$E := \left\{v \in W_0^{s,1}(\Omega) : \iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} dxdy \leq l^{\frac{r}{2}}\right\},$$

which is a closed convex set of $W_0^{s,1}(\Omega)$. Then, we define $T : E \to W_0^{s,1}(\Omega)$ by $T(\varphi) = u$, where u is the weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)\,\mathbb{D}_s^2(\varphi) + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.4.6}$$

and observe that problem (P_λ) is equivalent to the fixed point problem $u = T(u)$. Hence, to prove Theorem 4.1.1, we shall show that T has fixed point belonging to $W_0^{s,2}(\Omega)\cap\mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$.

Lemma 4.4.3. *Assume that (A_1) holds. Then T is well defined.*

Proof. First of all, by Hölder inequality and Lemma 4.4.2, observe that for all $\varphi \in E$,

$$\int_\Omega \mathbb{D}_s^2(\varphi)dx \le c(r,\Omega)\left(\int_\Omega (\mathbb{D}_s^2(\varphi))^{\frac{r}{2}}dx\right)^{\frac{2}{r}} \le c\,C_9^2\|\varphi\|^2_{W_0^{s+\varepsilon,r}(\Omega)} = c\,C_9^2\,l. \tag{4.4.7}$$

Hence, for all $\varphi \in E$, it follows that

$$\|\mu(x)\,\mathbb{D}_s^2(\varphi) + \lambda f(x)\|_{L^1(\Omega)} \le c\,C_9^2\|\mu\|_\infty l + |\lambda|\|f\|_{L^1(\Omega)} = C < \infty. \tag{4.4.8}$$

Thanks to Lemma 4.3.7 and Proposition 4.3.1, if the right hand side in (4.4.6) belongs to $L^1(\Omega)$, problem (4.4.6) has an unique weak solution and it belongs to $W_0^{s,1}(\Omega)$. Thus, the result follows from (4.4.8). □

Lemma 4.4.4. *Assume (A_1) and let $0 < \lambda \le \lambda^*$. Then $T(E) \subset E$.*

Proof. For an arbitrary $\varphi \in E$, we define $u = T(\varphi)$. Now, by Proposition 4.3.1 and since $0 < \lambda \le \lambda^*$, it follows that

$$\begin{aligned} \left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}}dxdy\right)^{\frac{1}{r}} &\le C_1\left\|\mu(x)\mathbb{D}_s^2(\varphi) + \lambda f(x)\right\|_{L^m(\Omega)} \\ &\le C_1\|\mu\|_\infty\left\|\mathbb{D}_s^2(\varphi)\right\|_{L^m(\Omega)} + C_1\lambda^*\|f\|_{L^m(\Omega)}. \end{aligned} \tag{4.4.9}$$

Also, by Lemma 4.4.2, Hölder inequality and the definition of C_{10}, we obtain that

$$\|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \le |\Omega|^{\frac{r-2m}{rm}}\|(\mathbb{D}_s^2(\varphi))^{\frac{1}{2}}\|^2_{L^r(\Omega)} \le C_9^2|\Omega|^{\frac{r-2m}{rm}}\|\varphi\|^2_{W_0^{s+\varepsilon,r}(\Omega)} = C_{10}\|\varphi\|^2_{W_0^{s+\varepsilon,r}(\Omega)}.$$

Thus, since $\varphi \in E$, we have that

$$\|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \le C_{10}\,l. \tag{4.4.10}$$

From (4.4.5), (4.4.9) and (4.4.10), it follows that

$$\left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}}dxdy\right)^{\frac{1}{r}} \le C_1(C_{10}\|\mu\|_\infty\,l + \lambda^*\|f\|_{L^m(\Omega)}) = l^{\frac{1}{2}}.$$

Hence, since by Proposition 4.3.1 we also know that $u \in W_0^{s,1}(\Omega)$, we conclude that $u \in E$ and so, that $T(E) \subset E$. □

Lemma 4.4.5. *Assume that (A_1) holds. Then T is continuous.*

Proof. Let $\{\varphi_n\} \subset E$ be a sequence such that $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ and define $u_n = T(\varphi_n)$, for all $n \in \mathbb{N}$, and $u = T(\varphi)$. To show that $u_n \to u$ in $W_0^{s,1}(\Omega)$, and so, that T is continuous, we prove that

$$g_n(x) := \mathbb{D}_s^2(\varphi_n) + \lambda f(x) \to g(x) := \mathbb{D}_s^2(\varphi) + \lambda f(x), \quad \text{in } L^1(\Omega). \tag{4.4.11}$$

Indeed, if (4.4.11) holds, the result follows from Proposition 4.3.13.

First of all, using the notation $\psi_n = \varphi_n - \varphi$ and the reverse triangle inequality, we obtain that

$$\begin{aligned}
\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} &= \int_\Omega \left| \int_{\mathbb{R}^N} \frac{|\varphi_n(x)-\varphi_n(y)|^2 - |\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}} dy \right| dx \\
&\le \int_\Omega \left| \int_{\mathbb{R}^N} \frac{\big(|\varphi_n(x)-\varphi_n(y)| + |\varphi(x)-\varphi(y)|\big)|\psi_n(x)-\psi_n(y)|}{|x-y|^{N+2s}} dy \right| dx \\
&= \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\varphi_n(x)-\varphi_n(y)| + |\varphi(x)-\varphi(y)|}{|x-y|^{\frac{N}{2}+s}} \cdot \frac{|\psi_n(x)-\psi_n(y)|}{|x-y|^{\frac{N}{2}+s}} dy \right) dx.
\end{aligned}$$

Applying then Hölder inequality, we deduce that

$$\begin{aligned}
&\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \\
&\quad\le \int_\Omega \left(\int_{\mathbb{R}^N} \frac{(|\varphi_n(x)-\varphi_n(y)| + |\varphi(x)-\varphi(y)|)^2}{|x-y|^{N+2s}} dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}} dy \right)^{\frac{1}{2}} dx \\
&\quad\le \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{(|\varphi_n(x)-\varphi_n(y)| + |\varphi(x)-\varphi(y)|)^2}{|x-y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}} \\
&\quad= \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{(|\varphi_n(x)-\varphi_n(y)| + |\varphi(x)-\varphi(y)|)^2}{|x-y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}} \|\mathbb{D}_s^2(\varphi_n - \varphi)\|_{L^1(\Omega)}^{\frac{1}{2}} \\
&\quad=: I_1 \cdot I_2 .
\end{aligned}$$

Taking into account the above inequality, if we show that I_1 is bounded and I_2 goes to zero, we deduce that $\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \to 0$.

Claim 1: I_1 *is bounded.*

Directly observe that

$$\begin{aligned}
I_1 &\le 2\left[\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\varphi_n(x)-\varphi_n(y)|^2}{|x-y|^{N+2s}} dy \right) dx + \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}} dy \right) dx \right] \\
&= 2\Big[\|\mathbb{D}_s^2(\varphi_n)\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \Big].
\end{aligned} \tag{4.4.12}$$

Since $\varphi_n,\ \varphi \in E$ for all $n \in \mathbb{N}$, by (4.4.7), we have that

$$\Big[\|\mathbb{D}_s^2(\varphi_n)\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \Big] \le 2c\, C_9^2\, l < \infty,$$

and so, that I_1 is bounded.

Claim 2: I_2 *goes to zero.*

Let $\theta \in (0,1)$ be small enough to ensure that $\frac{2-\theta}{1-\theta} < r$. By Hölder inequality, it follows that

$$\begin{aligned}
\|\mathbb{D}_s^2(\varphi_n - \varphi)\|_{L^1(\Omega)} &= \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^2}{|x-y|^{N+2s}} dy \right) dx \\
&= \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^\theta}{|x-y|^{(N+s)\theta}} \frac{|\psi_n(x)-\psi_n(y)|^{2-\theta}}{|x-y|^{N(1-\theta)+s(2-\theta)}} dy \right) dx = (*)
\end{aligned}$$

$$(*) \leq \int_\Omega \left[\left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|}{|x-y|^{N+s}} dy \right)^\theta \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{\frac{2-\theta}{1-\theta}}}{|x-y|^{N+s\frac{2-\theta}{1-\theta}}} dy \right)^{1-\theta} \right] dx$$
$$\leq \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|}{|x-y|^{N+s}} dy \right) dx \right)^\theta \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{\frac{2-\theta}{1-\theta}}}{|x-y|^{N+s\frac{2-\theta}{1-\theta}}} dy \right) dx \right)^{1-\theta}.$$

Hence, since $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ implies that

$$\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|}{|x-y|^{N+s}} dy \right) dx \to 0, \tag{4.4.13}$$

if we prove that

$$\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{\frac{2-\theta}{1-\theta}}}{|x-y|^{N+s\frac{2-\theta}{1-\theta}}} dy \right) dx \tag{4.4.14}$$

is bounded, we can conclude that I_2 goes to zero, as desired. Since we have chosen $\theta \in (0,1)$ small enough in order to ensure that $\frac{2-\theta}{1-\theta} < r$ and Ω is a bounded domain, it follows that

$$\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{\frac{2-\theta}{1-\theta}}}{|x-y|^{N+s\frac{2-\theta}{1-\theta}}} dy \right) dx \leq C(r,\Omega) \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{\frac{2-\theta}{1-\theta}}}{|x-y|^{N+s\frac{2-\theta}{1-\theta}}} dy \right)^{\frac{r}{\frac{2-\theta}{1-\theta}}} dx \right)^{\frac{\frac{2-\theta}{1-\theta}}{r}}. \tag{4.4.15}$$

Applying then Lemma 4.4.2 and the triangular inequality we have that

$$\begin{aligned} \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{\frac{2-\theta}{1-\theta}}}{|x-y|^{N+s\frac{2-\theta}{1-\theta}}} dy \right) dx &\leq \overline{C} \|\psi_n\|_{W_0^{s+\varepsilon,r}(\Omega)} \\ &\leq \widetilde{C} \Big[\|\varphi_n\|_{W_0^{s+\varepsilon,r}(\Omega)} + \|\varphi\|_{W_0^{s+\varepsilon,r}(\Omega)} \Big] \\ &\leq 2\widetilde{C} l^{\frac{1}{2}} = \widehat{C}, \end{aligned} \tag{4.4.16}$$

where $\overline{C}, \widetilde{C}$ and $\widehat{C}$ are positive constants independent of n. Thus, we conclude that (4.4.14) is indeed bounded.

From Claim 1 and 2 we deduce that $\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \to 0$. This implies that $g_n \to g$ in $L^1(\Omega)$, as desired, and the result follows. □

Lemma 4.4.6. *Assume that (A_1) holds. Then T is compact.*

Proof. Let $\{\varphi_n\} \subset E$ be a bounded sequence in $W_0^{s,1}(\Omega)$ and define $u_n = T(\varphi_n)$ for all $n \in \mathbb{N}$. We have to show that $u_n \to u$ in $W_0^{s,1}(\Omega)$ for some $u \in W_0^{s,1}(\Omega)$.

Since $\{\varphi_n\} \subset E$ for all $n \in \mathbb{N}$, arguing as in Lemma 4.4.3, we deduce that $\{\mathbb{D}_s^2(\varphi_n)\}$ is a bounded sequence in $L^1(\Omega)$. Hence, if we define

$$g_n(x) := \mathbb{D}_s^2(\varphi_n) + \lambda f(x), \quad \forall\, n \in \mathbb{N},$$

we have that $\{g_n\}$ is a bounded sequence in $L^1(\Omega)$. The result then follows from Proposition 4.3.14. □

Proof of Theorem 4.1.1. Since E is a closed convex set of $W_0^{s,1}(\Omega)$ and, by Lemmas 4.4.3, 4.4.4, 4.4.5, and 4.4.6, we know that T is continuous, compact and satisfies $T(E) \subset E$, we can apply the Schauder fixed point Theorem to obtain $u \in E$ such that $T(u) = u$. Thus, we conclude that (P_λ) has a weak solution for all $0 < \lambda \leq \lambda^*$. Finally, since $u \in W_0^{s,1}(\Omega) \cap W_0^{s,r}(\Omega)$ for some $1 < 2 < r$, by Lemma 4.3.9 applied with $s_1 = s_2 = s$, we deduce that $u \in W_0^{s,2}(\Omega)$. Moreover, since $r > N/s$, by [55, Theorem 8.2], we know that every $\varphi \in E$ belongs to $\mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$. □

4.4.2 Proof of Theorem 4.1.4

First observe that, as before, without loss of generality we can assume $m \in (\frac{N}{2s}, \frac{N}{s})$, $\|\mu\|_\infty \neq 0$ and $\|f\|_{L^m(\Omega)} \neq 0$. Next, let us fix some notation. We fix

$$r = \frac{3mN}{N + ms} \tag{4.4.17}$$

and $\varepsilon := \varepsilon(r, m, s) > 0$ such that

$$1 < r < \frac{mN}{N - m(s-\varepsilon)} < \frac{mN}{N - ms}, \quad s + \varepsilon < 1 \quad \text{and } s - \varepsilon > \frac{1}{2}.$$

Also, we introduce and fix the constants C_2, given by Corollary 4.3.4 applied with $t = s + \varepsilon$ and $p = r$, $C_{11} := S_{N,r} C C_9^{2m}$, where $S_{N,r}$ is the optimal constant in the Sobolev inequality (Theorem 4.2.1), C is the smallest constant guaranteeing the continuous embedding $W_0^{s+\varepsilon,r}(\Omega) \subset W_0^{s,r}(\Omega)$ and C_9 is the constant given by Lemma 4.4.2, and

$$\lambda^* := \frac{2}{3\|f\|_{L^m(\Omega),}} \left(\frac{1}{3C_2^3 C_{11} \|\mu\|_\infty} \right)^{\frac{1}{2}}.$$

Then, by Lemma 4.4.1 we know that there exists and unique $l \in (0, \infty)$ such that

$$C_2(C_{11}\|\mu\|_\infty l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{\frac{1}{3}}. \tag{4.4.18}$$

Having fixed all these constants, we define

$$E_1 := \left\{ v \in W_0^{s,1}(\Omega) : \iint_{D_\Omega} \frac{|u(x) - u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} dxdy \leq l^{\frac{r}{3}} \right\},$$

which is a closed convex set of $W_0^{s,1}(\Omega)$, and $T_1 : E_1 \to W_0^{s,1}(\Omega)$ by $T_1(\varphi) = u$, with u the unique weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)\varphi \mathbb{D}_s^2(\varphi) + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.4.19}$$

Observe that $(\widetilde{P_\lambda})$ is equivalent to the fixed point problem $u = T_1(u)$. Hence, we shall prove that T_1 has a fixed point belonging to $W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$.

Lemma 4.4.7. *For all $\varphi \in W_0^{s+\varepsilon,r}(\Omega)$, it follows that*

$$\|\varphi \mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \leq C_{11} \|\varphi\|^3_{W_0^{s+\varepsilon,r}(\Omega)}. \tag{4.4.20}$$

Proof. First observe that, with the above notation, we have that

$$2 < \frac{2mr_s^*}{r_s^* - m} = r.$$

Hence, by Hölder and Sobolev inequalities and using that $W_0^{s+\varepsilon,r}(\Omega) \subset W_0^{s,r}(\Omega)$ with continuous inclusion, we obtain that

$$\|\varphi\,\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)}^m \leq S_{N,r} C \|\varphi\|_{W_0^{s+\varepsilon,r}(\Omega)}^m \left\| \left(\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}} dy \right)^{\frac{1}{2}} \right\|_{L^r(\Omega)}^{2m}.$$

Since $r > 2$, the result follows from Lemma 4.4.2. ☐

Corollary 4.4.8. *Assume that* (A_1) *holds. Then* T_1 *is well defined.*

Proof. Since Ω is a bounded domain and $m > \frac{N}{2s} > 1$ the result follows from Lemma 4.4.7 arguing as in the proof of Lemma 4.4.3. ☐

Lemma 4.4.9. *Assume* (A_1) *and let* $0 < \lambda \leq \lambda^*$. *Then* $T_1(E_1) \subset E_1$.

Proof. Let us consider an arbitrary $\varphi \in E$ and define $u = T_1(\varphi)$. By Corollary 4.3.4, since that $0 < \lambda \leq \lambda^*$, we have that

$$\left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} dx dy \right)^{\frac{1}{r}} \leq C_2 \|\mu\|_\infty \left\| \varphi\,\mathbb{D}_s^2(\varphi) \right\|_{L^m(\Omega)} + C_2 \lambda^* \|f\|_{L^m(\Omega)}. \tag{4.4.21}$$

Hence, since $\varphi \in E$, by Lemma 4.4.7 and (4.4.18), it follows that

$$\left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} dx dy \right)^{\frac{1}{r}} \leq C_2 (C_{11} \|\mu\|_\infty l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{\frac{1}{3}}.$$

Thus, as by Proposition 4.3.1 we also know that $u \in W_0^{s,1}(\Omega)$, we conclude that $u \in E_1$ and so, that $T_1(E_1) \subset E_1$. ☐

Lemma 4.4.10. *Assume* (A_1). *Then* T_1 *is continuous.*

Proof. Let $\{\varphi_n\} \subset E$ be a sequence such that $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ and define $u_n = T_1(\varphi_n)$, for all $n \in \mathbb{N}$, and $u = T_1(\varphi)$. Arguing as in the proof of Lemma 4.4.5, we just have to prove that

$$\varphi_n \mathbb{D}_s^2(\varphi_n) \to \varphi\,\mathbb{D}_s^2(\varphi), \quad \text{in } L^1(\Omega). \tag{4.4.22}$$

First observe that, since $r > \frac{N}{s} > \frac{N}{s+\varepsilon}$, for all $\varphi \in E_1$, it follows that

$$\|\varphi\|_{L^\infty(\Omega)} \leq C \iint_{D_\Omega} \frac{|\varphi(x)-\varphi(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} dx dy \leq l^{\frac{r}{3}}. \tag{4.4.23}$$

Hence, since $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$, by Vitali's Convergence Theorem we deduce that $\varphi_n \to \varphi$ in $L^\alpha(\Omega)$ for all $1 \leq \alpha < \infty$.

Next, observe that

$$\begin{aligned}\|\varphi_n \mathbb{D}_s^2(\varphi_n) - \varphi \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} &= \|\varphi_n(\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)) + \mathbb{D}_s^2(\varphi)(\varphi_n - \varphi)\|_{L^1(\Omega)} \\ &\leq \|\varphi_n(\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi))\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)(\varphi_n - \varphi)\|_{L^1(\Omega)} \\ &\leq \|\varphi_n\|_\infty \|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \|\varphi_n - \varphi\|_{L^{m'}(\Omega)} \\ &=: I_1 + I_2\end{aligned} \tag{4.4.24}$$

Then, arguing exactly as in Lemma 4.4.5 and using that $\|\varphi_n\|_\infty \leq C$ (independent of n) we deduce that $I_1 \to 0$. On the other hand, we know that $\|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} < \infty$. Hence, since $\varphi_n \to \varphi$ in $L^\alpha(\Omega)$ for all $1 \leq \alpha < \infty$, we also obtain that $I_2 \to 0$. We then conclude that (4.4.22) holds, as desired. □

Proof of Theorem 4.1.4. Observe that the compactness of T_1 follows arguing exactly as in Lemma 4.4.6. Hence, since E_1 is a closed convex set of $W_0^{s,1}(\Omega)$ and, by Lemmas 4.4.9, 4.4.8 and 4.4.10 we know that T_1 is well defined, continuous and satisfies $T_1(E_1) \subset E_1$, we can apply the Schauder fixed point Theorem to obtain $u \in E_1$ such that $T_1(u) = u$. Thus, we conclude that $(\widetilde{P_\lambda})$ has a weak solution for all $0 < \lambda \leq \lambda^*$. Finally, since $u \in W_0^{s,1}(\Omega) \cap W_0^{s,r}(\Omega)$ for some $1 < 2 < r$, by Lemma 4.3.9 we deduce that $u \in W_0^{s,2}(\Omega)$. Moreover, since $r > N/s$, by [55, Theorem 8.2], we know that every $\varphi \in E_1$ belongs to $\mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$. □

4.5 Proofs of Theorems 4.1.2 and 4.1.3

In this section we prove Theorems 4.1.2 and 4.1.3. The aim of these theorems is to justify the hypotheses considered in Theorem 4.1.1. First we prove that (P_λ) has no solutions for λ large and so, that the smallness condition is somehow necessary to have existence of solution.

Proof of Theorem 4.1.2. Assume that (P_λ) has a solution $u \in W_0^{s,2}(\Omega)$ and let $\phi \in \mathcal{C}_0^\infty(\Omega)$ be an arbitrary function such that

$$\int_\Omega f(x)\phi^2(x)dx > 0,$$

Considering ϕ^2 as test function in (P_λ) we observe that

$$\int_\Omega (-\Delta)^s u\, \phi^2(x)dx = \int_\Omega \mu(x) \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)dydx + \lambda \int_\Omega f(x)\phi^2(x)dx. \tag{4.5.1}$$

Now, on one hand, since $\mu(x) \geq \mu_1 > 0$ and $\mathbb{D}_s^2$ is symmetric in x, y, it follows that

$$\begin{aligned}\int_\Omega \mu(x) \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)dydx &= \iint_{D_\Omega} \mu(x) \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)dydx \\ &\geq \mu_1 \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)dydx \\ &= \frac{\mu_1}{2} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)dydx + \frac{\mu_1}{2} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(y)dydx \\ &\geq \frac{\mu_1}{4} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} (\phi(x)+\phi(y))^2 dydx.\end{aligned} \tag{4.5.2}$$

On the other hand, by Young's inequality, it follows that

$$
\begin{aligned}
\int_\Omega (-\Delta)^s u\, \phi^2(x) dx &= \iint_{D_\Omega} \frac{(u(x)-u(y))(\phi^2(x)-\phi^2(y))}{|x-y|^{N+2s}} dy dx \\
&= \iint_{D_\Omega} \frac{(u(x)-u(y))(\phi(x)-\phi(y))(\phi(x)+\phi(y))}{|x-y|^{N+2s}} dy dx \\
&\leq \iint_{D_\Omega} \frac{|u(x)-u(y)||\phi(x)+\phi(y)|}{|x-y|^{\frac{N}{2}+s}} \cdot \frac{|\phi(x)-\phi(y)|}{|x-y|^{\frac{N}{2}+s}} dy dx \\
&\leq \frac{\mu_1}{4} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} (\phi(x)+\phi(y))^2 dy dx + \frac{1}{\mu_1} \iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s'}} dy dx
\end{aligned}
\tag{4.5.3}
$$

Hence, substituting (4.5.2) and (4.5.3) to (4.5.1), we deduce that, if (P_λ) has a solution, then

$$
\frac{1}{\mu_1} \iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy dx \geq \lambda \int_\Omega f(x)\phi^2(x) dx, \tag{4.5.4}
$$

which gives a contradiction for λ large enough. $\square$

Now, we prove Theorem 4.1.3. This theorem shows that the regularity considered on f is almost optimal. Just the limit case $f \in L^{\frac{N}{2s}}(\Omega)$ remains open.

Proof of Theorem 4.1.3. Without loss of generality we choose a bounded domain Ω with boundary $\partial\Omega$ of class $\mathcal{C}^2$ such that $0 \in \Omega$. Consider then

$$
f(x) = \frac{1}{|x|^{\frac{N-\varepsilon}{m}}}, \tag{4.5.5}
$$

for some $\varepsilon \in (0,1)$ to be chosen later and observe that, since Ω is bounded, $f \in L^m(\Omega)$.

We assume by contradiction that, for all $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that (P_λ) has a solution $u \in W_0^{s,2}(\Omega)$. Arguing as in the proof of Theorem 4.1.2, we conclude that, for all $\phi \in \mathcal{C}_0^\infty(\Omega) \setminus \{0\}$,

$$
\frac{1}{\mu_1} \iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy dx \geq \lambda_\varepsilon \int_\Omega f(x)\phi^2(x) dx = \lambda_\varepsilon \int_\Omega \frac{\phi^2(x)}{|x|^{\frac{N-\varepsilon}{m}}} dx. \tag{4.5.6}
$$

Thus, we deduce that

$$
0 < \mu_1 \lambda_\varepsilon \inf \left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy dx}{\displaystyle\int_\Omega \frac{\phi^2(x)}{|x|^{\frac{N-\varepsilon}{m}}} dx} : \phi \in \mathcal{C}_0^\infty(\Omega) \setminus \{0\} \right\}. \tag{4.5.7}
$$

Nevertheless, since $m < \frac{N}{2s}$, we can choose $\varepsilon > 0$ small enough to ensure that $\frac{N-\varepsilon}{m} > 2s$. In that case, by Proposition 4.2.3, 2), we have that

$$
\inf \left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}} dy dx}{\displaystyle\int_\Omega \frac{|\phi(x)|^2}{|x|^{\frac{N-\varepsilon}{m}}} dx} : \phi \in \mathcal{C}_0^\infty(\Omega) \setminus \{0\} \right\} = 0,
$$

which contradicts (4.5.7). Hence, the result follows. $\square$

4.6 Proofs of Theorems 4.1.5 and 4.1.6

This section is devoted to the proofs of Theorems 4.1.5 and 4.1.6. First, having at hand Proposition 4.3.5, we prove Theorem 4.1.5 using again a fixed point argument. The proof is similar to the ones performed in Section 4.4. Hence, we skip some details.

Since Ω is bounded, without loss of generality, we assume that $1 \le m < \frac{N}{s}$. Also, if $\lambda f \equiv 0$, it follows that $u \equiv 0$ is a solution to (Q_λ) and, if $\mu \equiv 0$, (Q_λ) reduces to (4.3.1). Hence, we may also assume that $\|\mu\|_\infty \neq 0$ and $\|f\|_{L^m(\Omega)} \neq 0$.

Next, we fix some notation that will be used throughout the section. First, we fix $r = r(m,s,q) > 0$ such that

$$1 < qm < r < \frac{mN}{N-ms},$$

C_3 the constant given by Proposition 4.3.5 with $p = r$ and

$$\lambda^* = \frac{q-1}{q\|f\|_{L^m(\Omega)}} \left(\frac{1}{qC_3^q |\Omega|^{\frac{r-qm}{r}} \|\mu\|_\infty} \right)^{\frac{1}{q-1}}.$$

Then, by the definition of λ^* and Lemma 4.4.1, we know that there exists an unique $l \in (0,\infty)$ such that

$$C_3(\|\mu\|_{L^\infty(\Omega)}|\Omega|^{\frac{r-qm}{mr}} l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{\frac{1}{q}}. \tag{4.6.1}$$

With the above constants fixed, we introduce

$$E_2 := \left\{ v \in W_0^{s,1}(\Omega) : \|(-\Delta)^{\frac{s}{2}} v\|_{L^r(\Omega)} \le l^{\frac{1}{q}} \right\},$$

and observe that E_2 is a closed convex set of $W_0^{s,1}(\Omega)$. Then, we define $T_2 : E_2 \to W_0^{s,1}(\Omega)$ by $T_2(\varphi) = u$ with u the unique weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{\frac{s}{2}}\varphi|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.6.2}$$

and observe that (Q_λ) is equivalent to the fixed point problem $u = T_2(u)$. Hence, we shall show that T_2 has a fixed point.

Lemma 4.6.1. *Assume (B_1) and let $0 < \lambda \le \lambda^*$. Then T_2 is well defined, $T_2(E_2) \subset E_2$ and T_2 is compact.*

Proof. The proof of this lemma follows as in Lemmas 4.4.3, 4.4.4 and 4.4.6 using Proposition 4.3.5 instead of Proposition 4.3.1. □

Remark 4.6.1. The only point in the proof of the previous lemma where we use $0 < \lambda \le \lambda^*$ is to show that $T_2(E_2) \subset E_2$. The rest holds for every $\lambda \in \mathbb{R}$.

Lemma 4.6.2. *Assume that (B_1) holds. Then T_2 is continuous.*

Proof. Let $\{\varphi_n\} \subset E_2$ be a sequence such that $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ and define $u_n = T_2(\varphi_n)$, for all $n \in \mathbb{N}$, and $u = T_2(\varphi)$. We shall show that $u_n \to u$ in $W_0^{s,1}(\Omega)$. Observe that $w_n = u_n - u$ satisfies

$$\begin{cases} (-\Delta)^s w_n = \mu(x)\left(|(-\Delta)^{\frac{s}{2}}\varphi_n|^q - |(-\Delta)^{\frac{s}{2}}\varphi|^q\right), & \text{in } \Omega, \\ w_n = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.6.3}$$

Hence, if we show that

$$\mu(x)\left(|(-\Delta)^{\frac{s}{2}}\varphi_n|^q - |(-\Delta)^{\frac{s}{2}}\varphi|^q\right) \to 0, \quad \text{in } L^1(\Omega), \tag{4.6.4}$$

the result follows from Proposition 4.3.13. Directly, since $\varphi_n, \varphi \in E_2$ and $\mu \in L^\infty(\Omega)$, applying the Mean Value Theorem and Hölder inequality, we deduce that

$$\left\|\mu(x)\left(|(-\Delta)^{\frac{s}{2}}\varphi_n|^q - |(-\Delta)^{\frac{s}{2}}\varphi|^q\right)\right\|_{L^1(\Omega)} \leq C\left(\int_\Omega |(-\Delta)^{\frac{s}{2}}(\varphi_n-\varphi)|^q dx\right)^{\frac{1}{q}}, \tag{4.6.5}$$

where C is a positive constant depending only on $\|\mu\|_{L^\infty(\Omega)}$, l, q and Ω. By (4.6.5), if we show that

$$\int_\Omega |(-\Delta)^{\frac{s}{2}}(\varphi_n-\varphi)|^q dx \to 0, \tag{4.6.6}$$

the continuity of the operator follows from Proposition 4.3.13.

Since $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$, it follows that $\varphi_n - \varphi \to 0$ almost everywhere in Ω. Furthermore, observe that, for all measurable subset $\omega \subset \Omega$, we have that

$$\int_\omega |(-\Delta)^{\frac{s}{2}}(\varphi_n-\varphi)|^q dx \leq 2l|\omega|^{\frac{r-q}{q}}.$$

Hence, by Vitali's convergence Theorem, (4.6.6) holds and the result follows. □

Proof of Theorem 4.1.5. Since E_2 is a closed convex set of $W_0^{s,1}(\Omega)$ and, by Lemmas 4.6.1 and 4.6.2, we know that T_2 is continuous, compact and satisfies $T_2(E_2) \subset E_2$, we can apply the Schauder fixed point Theorem to obtain $u \in E_2$ such that $T_2(u) = u$. Thus, we conclude that (Q_λ) has a weak solution for all $0 < \lambda \leq \lambda^*$. □

Proof of Theorem 4.1.6. Having at hand Corollary 4.3.6, the result follows arguing as in Theorem 4.1.5. □

5 Open problems and perspectives

5.1 High multiplicity results for elliptic PDEs with critical growth in the gradient

Chapters 1, 2 and 3 are devoted to the study of elliptic partial differential equations with critical growth in the gradient. As a model case, we can consider the problem

$$-\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{P_λ}$$

with λ a real parameter. Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary, c and h belong to $L^q(\Omega)$ for some $q > N/2$ and μ belongs to $L^\infty(\Omega)$.

Our work in this direction is mainly devoted to the study of the *non-coercive cases* where $\lambda c \not\leq 0$ a.e. in Ω, i.e. $\lambda c \gneqq 0$ or λc change sign. Nevertheless, despite the results obtained and the significant number of recent papers by other authors [18, 50, 75, 76, 92, 108], the structure of the set of solutions is far from being well understood. So far, concerning multiplicity results, just the existence of two solutions (under suitable hypotheses) has been established. The main question we would like to address in this direction is the existence of more than two solutions for $c \gneqq 0$ and $\lambda > 0$ large enough.

The phase plane analysis obtained in [50, Section 6] for $N = 1$ and μ, c and h positive constants seems to indicate that, for $\lambda > 0$ large enough, the problem (P_λ) has more than two solutions. Furthermore, it seems that there exists a link with the spectrum of the boundary value problem

$$-\Delta u = \gamma c(x)u, \quad u \in H_0^1(\Omega).$$

For the particular case μ constant and $h \equiv 0$ we hope to obtain our first higher multiplicity results adapting the lower and upper solutions techniques introduced by C. De Coster in [45]. More precisely, for μ constant, $h \equiv 0$ and $\lambda > 0$ large enough we hope to obtain three non-trivial solutions and the trivial one. Nevertheless, as soon as $h \not\equiv 0$, we cannot use this approach and new ideas are needed. In particular, we wish to determine which assumptions on h give rise to more than two solutions to (P_λ). This is certainly a challenging open problem, but we believe that it can be tackled within the transformed variational framework set up in Chapter 1 and by using topological-variational methods in the presence of invariant sets.

The first preliminary step we plan to do is the study of the radial case (already interesting in itself). More precisely, assuming that c, μ and h are radial functions, we will address the problem

$$\begin{cases} -\Delta u = \lambda c(|x|)u + \mu(|x|)|\nabla u|^2 + h(|x|), & \text{in } B_R(0), \\ u = 0, & \text{on } \partial B_R(0), \end{cases} \tag{P_λ^R}$$

where $B_R(0) := \{x \in \mathbb{R}^N : |x| < R\}$. Our aim is to obtain a more precise information on the structure of the set of solutions to (P_λ^R) for the *non-coercive case* $\lambda c \gneqq 0$. In particular, we wish to characterize branches of radial solutions (depending on λ) and to study whether bifurcation of nonradial solutions may occur along the branch. In such way, we expect to have a bigger intuition with regard to the general problem (P_λ) which should allow us to address the desired question of higher multiplicity.

Another challenging question is the existence of possible limiting and concentration behaviour of solutions to (P_λ) as λ tends to $+\infty$. We expect that local extrema of the data functions c, μ should play a significant role with regard to the determination of possible concentration points. This expectation is based on the fact that a rescaling transformation of the form $w(x) = u(x_0 + \frac{x}{\sqrt{\lambda}})$ leads to the equation

$$\begin{cases} -\Delta w = c(x_0 + \dfrac{x}{\sqrt{\lambda}})w + \mu(x_0 + \dfrac{x}{\sqrt{\lambda}})|\nabla w|^2 + \dfrac{h(x_0 + \frac{x}{\sqrt{\lambda}})}{\lambda}, & \text{in } \Omega_\lambda, \\ w = 0, & \text{on } \partial\Omega_\lambda, \end{cases}$$

on a rescaled domain Ω_λ. In the limit $\lambda \to +\infty$, an equation with constant coefficients $c(x_0)$ and $\mu(x_0)$ is obtained. This equation then again fits in the variational framework set up in Chapter 1.

Using solutions to the limiting equation as building blocks, we wish to set of a Lyapunov-Schmidt type reduction framework to derive the existence of solutions to (P_λ) for large λ with one or more concentration points. In various cases, we expect new results on the multiplicity of solutions to (P_λ) for large λ.

5.2 Boundary weak Harnack inequality: the optimal ε

In Theorem 2.3.1, Chapter 2, we prove a new boundary weak Harnack inequality. More precisely, for $1 < p < \infty$, we consider the boundary value problem

$$-\Delta_p u + a(x)|u|^{p-2}u = 0, \quad u \in W_0^{1,p}(\Omega). \tag{P_{BHP}}$$

and we prove the following result.

Theorem 5.2.1. [Theorem 2.3.1, Chapter 2] *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class $C^{1,1}$ and let $a \in L^\infty(\Omega)$ be a non-negative function. Assume that u is a non-negative upper solution to (26) and let $x_0 \in \partial\Omega$. Then, there exist $\overline{R} > 0$, $\varepsilon = \varepsilon(p, \overline{R}, \|a\|_\infty, \Omega) > 0$ and $C = C(p, \overline{R}, \varepsilon, \|a\|_\infty, \Omega) > 0$ such that, for all $R \in (0, \overline{R}]$,*

$$\inf_{B_R(x_0)\cap\Omega} \frac{u(x)}{d(x,\partial\Omega)} \geq C\left(\int_{B_R(x_0)\cap\Omega} \left(\frac{u(x)}{d(x,\partial\Omega)}\right)^\varepsilon dx \right)^{1/\varepsilon}.$$

In the very recent preprint [107], B. Sirakov found the optimal ε for uniformly elliptic PDE in divergence form. More precisely, he deals with equations of the form

$$-\operatorname{div}(A(x)\nabla u) + b(x)|\nabla u| + f(x) = 0, \quad u \in H_0^1(\Omega), \tag{P_{BH-2}}$$

where A is a symmetric matrix, b and f are non-negative functions and, for some $\lambda > 0$ and $p > N$,

$$A \geq \lambda I, \qquad A \in W^{1,p}(\Omega), \qquad b, f \in L^p(\Omega).$$

He proves the following result.

Theorem 5.2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$. Assume that u is a non-negative upper solution to (P_{BH-2}) and let $x_0 \in \partial\Omega$. Then, there exist $\overline{R} > 0$ and $C = C(\|A\|_{W^{1,p}(\Omega)}, \lambda, p, \|b\|_{L^p(\Omega)}) > 0$ such that, for all $R \in (0, \overline{R}]$,*

$$\inf_{B_R(x_0)\cap\Omega} \frac{u(x)}{d(x,\partial\Omega)} \geq C\left(\left(\int_{B_R(x_0)\cap\Omega}\left(\frac{u(x)}{d(x,\partial\Omega)}\right)^s dx\right)^{1/s} - \|f\|_{L^p(B_2R(x_0)\cap\Omega)}\right),$$

for each $s < 1$.

Having in mind both results, we would like to obtain the corresponding optimal ε for the non-uniformly elliptic case of the p-Laplacian, namely problem (P_{BHP}). The proof of B. Sirakov is based on a not at all trivial Moser-type iteration. We hope to be able to perform this kind of iteration in the more general case of the p-Laplacian. Nevertheless, it is worth to point out that, since the p-Laplacian is a non-linear and non-uniformly elliptic operator, several difficulties are expected to appear and some new ideas will be needed. Even more generally, we would like to deal with a boundary value problem of the form

$$-\operatorname{div}(\boldsymbol{A}(x,u,\nabla u)) + \boldsymbol{B}(x,u,\nabla u) = 0, \quad u \in W_0^{1,p}(\Omega),$$

with $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfying suitable regularity and growth assumption on the line of the considered in [90, Chapter 3].

5.3 Nonlocal elliptic PDEs

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain and $s \in (1/2, 1)$. In Section 4.3, Chapter 4, we obtain sharp Calderón-Zygmund type regularity results for the fractional Poisson equation

$$\begin{cases} (-\Delta)^s u = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_f}$$

with $f \in L^m(\Omega)$ for some $m \geq 1$. Nevertheless, we shall observe that our results do not cover the full range $s \in (0,1)$. It will be our aim here to extend this results to the full range $s \in (0,1)$ and more general operators. We hope here to perform a purely non-local proof and cover the full range $s \in (0,1)$.

In Chapter 4, these regularity results are then used to obtain existence and non-existence results to a problem of the form

$$\begin{cases} (-\Delta)^s u = \mu(x)\,\mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$P_{N\lambda}$}$$

where $\lambda > 0$ is a real parameter, f belongs to a suitable Lebesgue space, $\mu \in L^\infty(\Omega)$ and $\mathbb{D}_s^2$ is a nonlocal "gradient square" term given by

$$\mathbb{D}_s^2(u) = \frac{a_{N,s}}{2}\ \text{p.v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dy\,.$$

This problem can be seen as the nonlocal counterpart of (P_λ) in the case where $c \equiv 0$. In comparison with the local problem (P_λ), very little is known about the nonlocal case. To be more precise, let us introduce the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = c(x)u + \mu(x)\,\mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda > 0$ is a real parameter, c and f belongs to a suitable Lebesgue space and $\mu \in L^\infty(\Omega)$. Having in mind the known results for the local case (P_λ), it seems interesting to address the following questions:

1) Does the uniqueness of (smooth) solutions hold for $c(x) \le 0$?

2) Under the assumption $c(x) \le \alpha_0 < 0$ a.e. in Ω. Is it possible to remove the smallness condition on λ imposed on λ?

3) Is it possible to prove the existence of more than one solution for $c(x) \gneqq 0$, $\mu(x) \ge \mu_1 > 0$ and $\lambda > 0$ small enough?

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