École doctorale IAEM Lorraine

Scattering Theory for Mathematical Models of the Weak Interaction

THÈSE

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par

Benjamin ALVAREZ

Composition du jury

Rapporteurs :	Volker Bach	Professeur à l'Université de Braunschweig
	Christian Gérard	Professeur à l'Université Paris-Sud
Examinateurs :	Sébastien Breteaux	Maître de conférences à l'Université de Lorraine
	Renata Bunoiu	Maîtresse de conférences à l'Université de Lorraine
	Jérémy Faupin	Professeur à l'Université de Lorraine (Directeur de Thèse)
	Dietrich Häfner	Professeur à l'Université Grenoble Alpes
	Annalisa Panati	Maîtresse de conférences à l'Université de Toulon

Institut Élie Cartan de Lorraine — UMR 7503

Mis en page avec la classe thesul.

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شكراً لـ"هشام" و"سعاد" على تبادلاتنا الثرية لثقافتينا اللتان اجتمعتا

Ευχαριστώ, φίλε. Μην ξεχνάτε:
"Ἄλλὰ μὴ βιάζῃς τὸ ταξείδι διόλου.
Καλλίτερα χρόνια πολλὰ νὰ διαρκέσει.
Καὶ γέρος πιὰ ν᾽ ἀράξῃς στὸ νησί,
πλούσιος μὲ ὅσα κέρδισες στὸν δρόμο,
μὴ προσδοκώντας πλούτη νὰ σὲ δώσῃ ἡ Ἰθάκη.
Ἡ Ἰθάκη σ᾽ ἔδωσε τ᾽ ὡραῖο ταξίδι.
Χωρὶς αὐτὴν δὲν θἄβγαινες στὸν δρόμο.
Ἄλλα δὲν ἔχει νὰ σὲ δώσει πιά.
Κι ἂν πτωχικὴ τὴν βρῇς, ἡ Ἰθάκη δὲν σὲ γέλασε.
Ἔτσι σοφὸς ποὺ ἔγινες, μὲ τόση πείρα,
ἤδη θὰ τὸ κατάλαβες ᾑ Ἰθάκες τί σημαίνουν."

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[1]dont la volonté de cueillir le sens dans un R.E.V d'occident n'aura conduit, après cogitation, qu'à une heureuse désillusion.

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________________________________ (Non, pas toi)

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[2] Crédit à Matthieu

"Everything we have done so far is mathematically respectable, although some of the results have been phrased in informal language. To make further progress, however, it is necessary to make a bargain with the devil."

Quantum Field Theory, A Tourist Guide for Mathematicians, Gerald B.Folland.

"C'est en de semblables aventures que j'ai moi-même prodigué mes forces, usé mes ans. Il est certain que dans quelque étagère de l'univers ce livre total doit exister ; je supplie les dieux ignorés qu'un homme – ne fût-ce qu'un seul, il y a des milliers d'années – l'ait eu entre les mains, l'ait lu. Si l'honneur, la sagesse et la joie ne sont pas pour moi, qu'ils soient pour d'autres. Que le ciel existe, même si ma place est l'enfer. Que je sois outragé et anéanti, pourvu qu'en un être, en un instant, Ton énorme Bibliothèque se justifie."

La bibliothèque de Babel, Jorge Luis Borges.

A Zélie, Flore, Luc et Eve.

Contents

Introduction générale

Dans une conférence célèbre donnée à l'Institut Royal de Grande Bretagne en Avril 1900, intitulée *deux nuages du XIXeme siècle sur la théorie de la chaleur et de la lumière*, Lord Kelvin présentait les deux grandes difficultés auxquelles étaient soumis les scientifiques de son temps : le problème de l'expérience de Michelson et Morley et celui du rayonnement du corps noir. Ce discours marqua les esprits puisque ces deux petits nuages constitueraient, quelques années plus tard le point de départ de la physique moderne. Le vingtième siècle a connu une succession de révolutions conceptuelles que les meilleurs esprits du dix-neuvième ne pouvaient pas anticiper et il nous semble pertinent de rappeler succinctement quelques évènements marquants qui ont conduit à la création du modèle standard de la physique des particules.

Derrière le premier nuage : la relativité restreinte

Le problème de la lumière

En 1687 Newton publie *Principia mathematica* qui servira de fondement à la mécanique classique ainsi qu'à sa formalisation mathématique. De plus, il y expose l'idée que la lumière serait constituée de petites billes ou de petits corpuscules se déplaçant instantanément. Cette dernière conception sera contredite par les expériences de diffraction de Young et de Fresnel au dix-neuvième siècle, mettant en évidence le comportement ondulatoire de la lumière. Ce dernier point de vue s'impose progressivement et la lumière finira par être décrite comme un phénomène électromagnétique par les célèbres équations de Maxwell.

Au dix-neuvième siècle, tous les phénomènes ondulatoires connus ont besoin d'un support pour se propager. Le son, par exemple, ne peut pas se propager sans air ni les vagues sans eau. L'enjeu est donc de trouver le support sur lequel la lumière se propage et que l'on baptise éther luminifère. De toute évidence la lumière d'étoiles très lointaines arrive jusqu'à nous. L'onde ne s'atténue donc que très peu et l'éther doit être ainsi très rigide. Cependant, cet objet ne semble pas perturber le mouvement des astres et doit donc être d'une viscosité très faible. De plus l'éther semble devoir être présent dans tout l'univers et être relativement homogène et isotrope. On observe par ailleurs que les équations de Maxwell ne sont pas invariantes par les transformations Galiléennes ce qui implique que les phénomènes électromagnétiques pourraient ne pas être semblables dans tous les référentiels inertiels. Le principe de relativité de la mécanique classique, serait-il erroné ? L'éther pourrait-il constituer un référentiel privilégié pour décrire le mouvement des astres ? Calculer la vitesse relative de la Terre par rapport à l'éther ne permettrait-il pas de démontrer son existence ?

Si la Terre est en mouvement par rapport à l'éther alors en appliquant une simple composition des vitesses on s'attendrait à ce que la célérité de la lumière soit différente suivant la direction de propagation choisie. Or, le déplacement de la Terre peut être très faible par rapport à la vitesse de la lumière. L'idée serait donc de comparer deux rayons lumineux, se propageant dans des directions différentes, à l'aide d'une méthode d'interférométrie capable de mettre en évidence de petits écarts. C'est entre 1881 et 1887 que Michelson et Morley tentent de mettre en application cette idée dans une série d'expériences qui porte maintenant leurs noms. Tous les résultats sont négatifs. La vitesse de la lumière semble constante dans toutes les directions. Le mouvement de la Terre par rapport à l'éther doit être extrêmement faible.

Face à ce résultat Ernst March est le premier à suggérer que l'éther n'existe pas mais la majorité des scientifiques ne soutiennent pas encore cette opinion. En 1889 Fitzgerald avance l'hypothèse de la contraction des longueurs. Cette dernière sera reprise de façon indépendante en 1892 par Lorentz dans un article où il actualise la formulation des futures transformées de Lorentz et introduit la notion de temps local qui n'est encore vu que comme un artifice mathématique. Poincaré modifie encore ces transformées et les baptise "équations de Lorentz" ou "transformations de Lorentz" en hommage aux travaux de ce dernier. Le modèle présenté par Poincaré part du principe de relativité de la mécanique classique, suppose que la vitesse de la lumière est constante dans tous les référentiels et utilise les transformations de Lorentz pour les changements de référentiels. En 1898, Poincaré aurait même suggéré qu'il n'y avait pas de bonne raison de ne pas voir le temps local identifié par Lorentz comme un temps physique. Cependant, Poincaré conserve l'idée d'éther ainsi que de temps et d'espace absolus. Une autre interprétation physique, considérée comme plus satisfaisante, sera présentée en 1905 par Einstein dans son célèbre article *De l'électrodynamique des corps en mouvement* et la formalisation mathématique sera perfectionnée en 1907 par Minkowski. On abandonne alors définitivement l'éther luminifère, le temps n'est plus une grandeur absolue mais dépend de l'observateur et il est considéré au même titre qu'une variable d'espace. La physique s'exprime maintenant dans un espace de dimension quatre muni de la métrique de Minkowski.

Vérifications expérimentales

La relativité restreinte a pu être testée à de nombreuses reprises. Nous citerons ici quelques expériences célèbres.

1. (1932) L'expérience Kennedy-Thorndike : l'expérience de Michelson et Morley a été modifiée afin de vérifier que la vitesse de la lumière ne dépendait pas de la vitesse du dispositif expérimental. La dilatation du temps a été ainsi vérifiée indirectement.

2. (1938 - 1941) Les expériences d'Ives–Stilwell : première preuve directe de la dilatation du temps au travers d'un effet Doppler lumineux.

3. (1971) L'expérience de Hafele-Keating : une horloge atomique a été chargée dans un avion commercial pour un tour du monde de 41 h. L'expérience fut répétée dans des conditions analogues pour un trajet de 49 h. Après ce périple, une différence significative a été observée entre les temps qu'affichaient les horloges embarquées et celles qui étaient restées au sol.

4. (1975) L'expérience de Alley : dans l'étude précédente, la trajectoire précise de l'horloge était difficile à estimer. Pour en avoir un contrôle plus précis, un avion entier a été réservé pour un vol de 15h. Le résultat a confirmé, encore une fois, la théorie.

Derrière le deuxième nuage : la mécanique quantique

La catastrophe ultraviolette et l'effet photoélectrique : un nouveau regard sur la lumière

Lorsque l'on chauffe un objet, il émet un rayonnement. C'est une expérience assez commune et que l'on peut observer avec des plaques électriques par exemple. Trop chaudes, elles deviennent rouges. Ce phénomène ne semble pas fondamentalement dépendre de la nature de l'objet étudié. Il est donc naturel de considérer un cas idéal : le corps noir. Par définition, un corps noir est un objet théorique absorbant tout rayonnement incident. Plus précisément et d'après la loi de Kirchhoff, la proportion d'un rayonnement incident absorbé par un objet quelconque est égale au rapport du rayonnement émis et du rayonnement qu'aurait un corps noir à la même température. L'étude du rayonnement d'un corps noir permet ainsi de comprendre le rayonnement de n'importe quel système.

En 1879, Stefan propose une formule empirique reliant la puissance moyenne émise par unité de surface $R(T)$ à la température du corps noir :

$$R(T) = \sigma T^4$$

avec σ la constante de Stefan. Cette relation a été déduite des lois de la thermodynamique par Boltzmann en 1884. Cependant, elle ne répond que partiellement à la question puisque l'on ne connaît toujours pas la puissance émise associée à une longueur d'onde λ. Nous recherchons donc la puissance spectrale émise $R(\lambda, T)$ telle que :

$$R(T) = \int R(\lambda, T) d\lambda = \sigma T^4.$$

Cette puissance peut être mesurée expérimentalement.

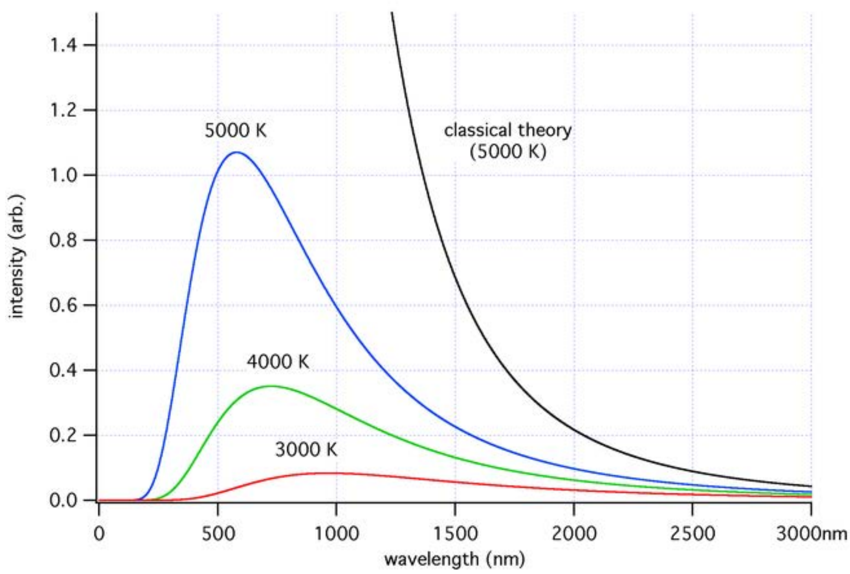

Figure 1: puissance spectrale émise [3]

[3]image issue de https://fr.wikiversity.org/wiki/Capteur/Capteur_de_temp

Rayleigh et Jeans ont déterminé une expression de $R(\lambda, T)$ à partir des lois de la mécanique classique. Comme illustré sur la figure 1, ce résultat diverge à mesure que la longueur d'onde s'approche de zéro et l'on s'attend à une puissance dissipée infinie. C'est en 1911 qu'Ehrenfest baptise cette contradiction : la catastrophe ultraviolette.

En 1900, Planck remarque qu'en supposant que l'émission ne puisse se faire que de façon discrète et non de façon continue, il est possible d'obtenir une nouvelle expression qui décrit parfaitement les résultats observés. L'énergie d'un quanta est :

$$E = nh\nu.$$

avec n un nombre entier et h la constante de Planck. Cette hypothèse était révolutionnaire pour l'époque et il semblerait que Planck la considérait lui-même comme un artifice mathématique. Elle sera pourtant reprise en 1905 par Einstein pour interpréter le problème de l'effet photoélectrique.

L'effet photoélectrique consiste en une émission de photons par un matériau soumis à un rayonnement électromagnétique. En 1887 une célèbre expérience est réalisée par Hertz sur cet effet. Des électrodes sont soumises à un rayonnement ultraviolet et un effet photoélectrique est observé au-dessus d'une certaine fréquence mais, en dessous de cette dernière, rien ne se passe. Ce phénomène de seuil ne peut être compris qu'en utilisant l'idée des quantas de Planck. Cependant, ce que suggère Einstein c'est que l'énergie n'est pas seulement quantifiée lors de l'émission mais en tout point de l'espace. La lumière peut ainsi être vue comme un ensemble de corpuscules qui seront baptisés photons en 1926 par Lewis.

Le problème de l'atome

En 1911, Rutherford présente son modèle planétaire de l'atome. Les électrons chargés négativement gravitent autour d'un noyau chargé positivement à l'image des planètes autour du soleil. Cependant, ce modèle contredit l'électromagnétisme. En effet, une charge accélérée émet un rayonnement électromagnétique et donc perd de l'énergie. L'atome doit donc être instable avec une durée de vie très faible, ce qui n'est évidemment pas le cas.

Un autre problème se pose aux physiciens. Le spectre d'émission des éléments chimiques est discret, ce qui correspond aux célèbres raies spectrales. Le cas le plus simple est celui de l'hydrogène. Bien qu'aucune explication ne soit connue, une certaine structure est empiriquement constatée. Citons par exemple la série de Balmer proposée en 1885 et sa généralisation proposée par Rydberg en 1888. On constate alors que les raies spectrales de l'hydrogène sont liées, dans ces formules, à des nombres entiers.

En 1913, Bohr propose un nouveau modèle de l'atome permettant de résoudre ces deux problèmes. Trois hypothèses sont alors nécessaires. La première est qu'il existe des orbites stables pour l'électron sur lesquelles il ne rayonne pas. La seconde hypothèse est qu'il ne peut quitter une orbite pour une autre que par absorption ou émission d'un quanta d'énergie $nh\nu$. Ces deux hypothèses expliquent très bien les raies spectrales mais ne permettent pas de retrouver les formules précédemment découvertes par Blamer et Rydberg. Il faut en effet choisir, parmi une infinité d'orbites potentiellement stables, lesquelles sont physiquement recevables. C'est l'objet de la troisième hypothèse : le moment cinétique de l'électron est quantifié. A l'aide de ces trois postulats, la formule de Rydberg peut être prédite théoriquement.

Le modèle de Bohr est probablement l'un des premiers modèles théoriques utilisant l'idée de la quantification. Cependant un certain nombre de problèmes demeurent. D'abord, ces trois hypothèses ne semblent pas trouver de véritable justification physique. Le modèle semble juste correspondre à l'observation. De plus, une telle approche ne semble pas pouvoir expliquer le comportement d'atomes plus complexes comme l'hélium.

Dualité onde-corpuscule

Les expériences de Young et Fresnel mettent en évidence le caractère ondulatoire de la lumière. D'autre part, l'effet photoélectrique suggère un comportement corpusculaire de la lumière. Le photon semble donc avoir une double nature. De Broglie soupçonne alors que cette propriété est commune à toutes les particules. Il propose ainsi, en 1924, une généralisation de la formule des quantas de Planck aux particules massives qui permet de déterminer la longueur d'onde d'une particule à partir de son impulsion p :

$$\lambda = \frac{h}{p}.$$

En particulier, la nature ondulatoire de l'électron est mise en évidence en 1927 par Davisson et Germer dans une expérience de double fentes d'Young. C'est également à cette époque que la mécanique quantique se formalise suite à l'établissement de l'équation de Schrödinger en 1925 et au principe d'indétermination (aussi appelé d'incertitude) de Heisenberg en 1927.

Vérifications expérimentales

Nous donnons quelques exemples d'expériences ayant validé la mécanique quantique.

1. L'effet Ehrenberg-Siday-Aharonov-Bohm est un phénomène prédit par la mécanique quantique : une particule chargée est affectée par un potentiel électromagnétique dans une région de l'espace où le champ est nul, ce qui est impossible à expliquer en mécanique classique. Cet effet a été prédit par Ehrenberg et Siday en 1949 puis par Aharonov et Bohm en 1959. Il a été observé par de nombreuses expériences réalisées par Chambers en 1960, par Möllenstedt et Bayh en 1962 ou encore par Tonomura et al. en 1982. Le potentiel était jusque là vu comme un artifice mathématique.

2. (1972) Les expériences de Freedman et Clauser : premières preuves que les inégalités de Bell sont transgresser comme le prédit la mécanique quantique. Ces inégalités étaient les conséquences d'une théorie déterministe locale à variables cachées, qui aurait pu expliquer les états intriqués avec une approche classique.

3. (1981-1982) Les expériences d'Aspect affinent les résultats de 1972. Les inégalités de Bell semblent systématiquement transgresser. La théorie des variables cachées est rejetée.

Une unification des théories quantiques et relativistes : la théorie des champs

Les tentatives de rassembler la mécanique quantique et la relativité restreinte dans un même cadre formel apparaissent très tôt. En 1926 Klein et Gordon proposent une équation inspirée par celle de Schrödinger. Puisque l'énergie libre d'une particule classique :

$$E = \frac{p^2}{2m}$$

se quantifie ainsi :

$$i\hbar\frac{\partial}{\partial_t} = \frac{-\hbar^2\nabla^2}{2m}$$

alors il est naturel de quantifier l'énergie relativiste :

$$E^2 = p^2c^2 + m^2c^2$$

de la même façon :

$$\left(i\hbar\frac{\partial}{\partial_t}\right)^2 = (-i\hbar\nabla)^2 + m^2c^4.$$

Ici, $\hbar$ est la constante de Planck réduite :

$$\hbar = \frac{h}{2\pi}.$$

En définissant :

$$\Box = -\left(i\hbar\frac{\partial}{\partial_t}\right)^2 + (-i\hbar\nabla)^2,$$

nous obtenons l'équation de Klein-Gordon pour une fonction d'onde libre ψ :

$$(\Box + m^2)\psi = 0.$$

Malheureusement, cette équation soulève de nombreuses questions. D'abord, il est possible de démontrer qu'il existe des états ayant une énergie négative. Retirer ces solutions reviendrait à se restreindre à un sous-ensemble non dense de l'espace de Hilbert. Il n'est donc pas possible de les ignorer. Les particules devraient ainsi passer indéfiniment à des états d'énergie toujours plus basse en émettant des photons, ce qui est contraire à ce que nous observons. Par ailleurs, l'équation de Klein-Gordon est une équation d'ordre deux en temps alors que la densité de probabilité relativiste n'est que d'ordre un. Ainsi peut-elle prendre des valeurs négatives ce qui est absurde. En 1928, Dirac propose une nouvelle approche permettant d'éviter ces contradictions. L'objectif est d'avoir une dérivée d'ordre un dans l'équation de Klein-Gordon. L'idée est donc d'en prendre la racine carrée. Notons qu'à partir de maintenant nous nous placerons dans le système d'unités naturelles où :

$$c = \hbar = 1.$$

Soient α et β deux constantes telles que :

$$i\frac{\partial\psi}{\partial t} = (-i\alpha\nabla + m\beta)\psi$$

et

$$\begin{aligned}\left(i\frac{\partial\psi}{\partial t}\right)^2 &= (-i\alpha\nabla + m\beta)(-i\alpha\nabla + m\beta)\psi \\ &= \left((-i\alpha\nabla)^2 - im(\beta\alpha + \alpha\beta)\nabla + m^2\beta^2\right)\psi \\ &= (-i\nabla)^2 + m^2.\end{aligned}$$

Ces conditions sont impossibles à vérifier à moins de considérer α et β comme des matrices. Introduisons alors les matrices γ^μ pour $\mu = 0, 1, 2, 3$. Nous utiliserons la notation d'Einstein :

$$\gamma \cdot \nabla = \gamma^i \cdot \nabla_i$$

En prenant le carré de l'expression suivante :

$$(i\gamma^0\partial_t + i\gamma^\mu\nabla_\mu - m)\psi \quad = \quad 0$$

nous obtenons

$$\begin{aligned} \gamma^0\gamma^0 &= 1 \\ \gamma^i\gamma^j &= -\delta_{i,j} \ \text{ pour i,j = 1,2,3} \\ \gamma^i\gamma^0 + \gamma^0\gamma^i &= 0 \ \text{ pour i=1,2,3.} \end{aligned}$$

Le choix le plus simple de matrice est :

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

où les σ_i sont les matrices de Pauli:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On en déduit l'équation de Dirac :

$$i\frac{\partial\psi}{\partial t} = (-i\gamma^0\gamma^i\nabla_i + m\gamma^0)\psi.$$

Désormais la densité de probabilité de la particule est positive. Néanmoins, l'énergie du système physique peut toujours être négative. Pour résoudre ce dernier problème, Dirac remarque que les solutions de son équation ne peuvent être que des fermions, c'est-à-dire des particules ne pouvant pas cohabiter dans le même état quantique (c'est le principe d'exclusion de Pauli). Dirac imagine alors un vide constitué d'une infinité de particules, occupant tous les niveaux d'énergies négatives et empêchant donc la matière de s'effondrer.

Si le vide est plein de particules, nous pourrions imaginer qu'il serait possible d'exciter un de ces états pour l'amener à un état d'énergie positive. Autrement dit, de créer une particule. Le vide possèderait ainsi un trou qui aurait la même masse que la particule créée mais avec des nombres quantiques internes différents. La rencontre d'un trou et d'une particule conduirait à la destruction de ces deux entités. C'est la découverte théorique des antiparticules (1931). Cette prédiction sera confirmée expérimentalement en 1932 par Anderson.

Cette interprétation, communément appelée Mer de Dirac, n'est toutefois pas très satisfaisante. Notons, par exemple, qu'il est impossible de la généraliser dans le cas bosonique, c'est-à-dire, lorsque les particules ne vérifient pas le principe d'exclusion de Pauli et peuvent alors occuper simultanément n'importe quel état quantique. Cependant, on constate qu'il est très difficile d'établir une théorie relativiste et quantique pour une seule particule. Le point de vue le plus pertinent serait donc de voir les équations au sens des champs.

Les bases de la théorie des champs posées par Dirac, en particulier par son travail de 1927 sur une théorie de l'électrodynamique quantique, seront affinées par Jordan qui introduira la notion d'opérateur de création et d'annihilation. En 1929, une première théorie des champs est proposée pour l'électromagnétisme. Parmi les scientifiques qui y ont contribué, citons entre autre, Heisenberg, Pauli et Born. Cependant, de nombreux problèmes doivent encore être résolus. En particulier, des termes divergents apparaissent dans les calculs et il faut attendre 1949 pour qu'une piste de traitement systématique de ces singularités soit avancée. Citons, entre autre, les contributions de Dyson, Schwinger, Tomonaga et de Feynman. Le succès de l'électrodynamique quantique (QED) incite les scientifiques à introduire un cadre rassemblant les différentes interactions fondamentales de la physique. A ce jour, le modèle standard de la physique des particules rassemble trois de ces quatres interactions : l'électromagnétisme, l'interaction faible et l'interaction forte.

Motivation physique et plan de la thèse

Les particules du modèle Standard

Les particules du modèle standard sont divisées en deux grandes familles : les fermions et les bosons. Les fermions sont décrits par une fonction d'onde antisymétrique et obéissent au principe d'exclusion de Pauli. Il existe deux grandes familles de fermions élémentaires. D'abord il y a les leptons dont l'électron est le représentant le plus emblématique. Il est possible d'observer d'autres entités semblables à l'électron mais avec des masses plus importantes : le muon et le tau. Enfin, il existe un neutrino associé à chacune des particules précédemment citées. Les quarks constituent l'autre grande famille de fermions. Ce sont les constituants fondamentaux de structures composites, comme les protons et les neutrons, et l'on peut en dénombrer trois paires, à l'instar des leptons. D'abord la paire up et down, puis charm et strange et enfin top et bottom.

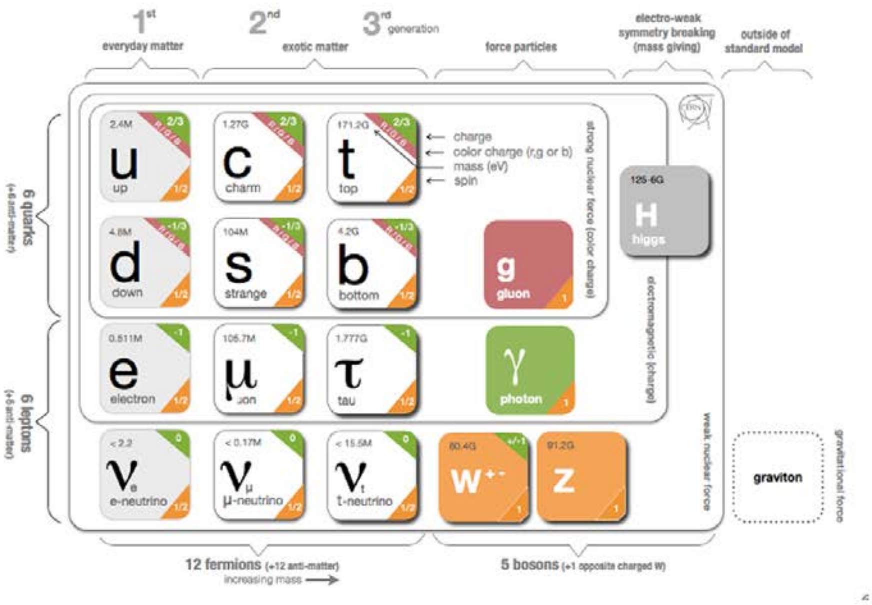

Figure 2: Les particules du modèle standard. [4]

[4]image issue de http://davidgalbraith.org/portfolio/ux-standard-model-of-the-standard-model/

Ces particules peuvent interagir en s'échangeant des quantas d'interaction qui sont des bosons. Contrairement aux fermions, ces derniers sont modélisés par des fonctions d'onde symétriques et peuvent occuper un même état quantique. Par exemple, le boson de l'interaction électromagnétique est le photon, ceux de l'interaction faible sont W^+, W^- et Z^0 et il existe finalement huit bosons pour l'interaction forte que l'on nomme gluons. Nous ne développerons pas plus la présentation du modèle standard et nous nous contenterons de cette description sommaire des objets qui seront manipulés dans la thèse. Nous renvoyons par exemple à [72] pour plus de détails.

L'existence de telles entités est mise en évidence lors d'expériences effectuées dans des accélérateurs de particules. Le principe est d'amener des états physiques (comme des protons pour le LHC) à une vitesse très proche de celle de la lumière avant de les envoyer sur une cible telle que d'autres particules accélérées. Lors d'une collision, des états sont susceptibles de se diffuser pour atteindre le détecteur. Il est alors assez naturel d'introduire un opérateur permettant de retrouver l'état de sortie à partir de l'état d'entrée avant l'expérience. Un tel opérateur est appelé matrice de diffusion. Par ailleurs, il est tout aussi naturel de vouloir étudier le comportement en temps long de tels états puisque, s'éloignant du point de collision et donc d'interactions possibles, on pourrait s'attendre à ce qu'ils soient asymptotiquement libres. Enfin, la matrice de diffusion intervient dans les calculs de sections efficaces qui sont des objets clés dans les mesures expérimentales. Dans cette thèse, nous nous proposons d'explorer la théorie de la diffusion pour quelques modèles simplifiés issus du modèle standard.

Plan de la thèse

Chapitre I Dans le premier chapitre nous introduisons le formalisme mathématique utilisé tout au long de ce travail. Nous commençons par quelques rappels sur le cas quantique avant d'introduire les outils fondamentaux nécessaires à l'étude des champs : espace de Fock, opérateur de seconde quantification, opérateurs de création et d'annihilation, des estimations classiques ainsi que la notion de diffusion et de complétude asymptotique. Nous terminons par un résumé très succinct de la littérature concernant le problème de diffusion. Ce premier état de l'art sera régulièrement complété tout au long des autres chapitres.

Chapitre II Le deuxième chapitre a été publié dans [2] et nous en présentons ici une version plus détaillée. Le résultat principal de ce chapitre est la preuve de la complétude asymptotique pour un modèle de désintégration des bosons de l'interaction faible $W^\pm$ en leptons. Les neutrinos seront considérés comme des particules massives. Notre preuve repose sur l'établissement d'estimations de propagation ainsi que sur deux théories de Mourre. Le spectre de l'hamiltonien ne contient qu'une seule valeur propre, E, qui est la borne inférieure du spectre ainsi qu'une branche absolument continue qui s'étend après un trou spectral entre E et $E + m$ où m est la masse de la plus légère des particules.

Chapitre III Dans ce chapitre, publié dans [2], nous reprenons le problème du chapitre II pour l'adapter dans le cas plus compliqué où l'on suppose que les neutrinos n'ont pas de masse. Le modèle est simplifié afin de faire apparaître une quantité liée au nombre de particules qui sera conservée. Une des principales nouveautés de ce travail réside dans l'établissement d'estimations de propagation adaptées au cas non-massif. Ces dernières jouent un rôle fondamental dans notre preuve de la complétude asymptotique qui constitue l'un des résultats majeurs de cette thèse.

Chapitre IV Dans le chapitre quatre, qui est une version détaillée de [3], nous nous proposons de démontrer que nous pouvons définir un hamiltonien pour un nombre fini mais quelconque de fermions. L'idée principale est d'effectuer une interpolation entre deux N-estimations. Nous démontrons par ailleurs l'existence d'un état fondamental en suivant les méthodes développées par [56].

Chaptire V Le dernier chapitre présente un travail en cours où nous considérons un modèle jouet invariant par translation. Son spectre est étudié à l'aide d'une théorie de Mourre faiblement conjuguée.

Introduction

In April 1900, Lord Kelvin gave a famous talk in front of the Royal Institute of Great Britain entitled *Nineteenth-Century Clouds over the Dynamical Theory of Heat and Light*. He presented two major difficulties faced by scientists at that time: the Michelson and Morley experiment and the black body radiation. However, these two tiny clouds were the birth of modern physics. Major conceptual breakthroughs, which were impossible to anticipate, have been done in the twentieth century and it might be of great interest to sum up quickly some of the key events which have led to the standard model of particle physics.

Behind the first cloud : Special Relativity

The question of light

In 1687, *Principia mathematica* is published by Newton. Together with the foundation of classical mechanics, works about light are exposed. It was seen as small particles with an infinite propagation speed. This thesis is contradicted in the ninetieth century by Young and Fresnel experiment. Light seems actually to behave like a wave. This point of view becomes more and more popular and the famous Maxwell's equations give an accurate description of light and electromagnetic phenomena.

At that time, it is believed that waves need a support to propagate, like the sound in the air or the waves in water. It is then natural to introduce the support of light waves called Luminiferous aether. First, the light of distant stars reach the earth, which implies that the aether has to be rigid. Moreover, Newton's theory gives a satisfying description of the motion of planets which means that the aether does not impact massive objects. It seems to be present everywhere in the universe with homogeneous and isotropic properties. Together with the fact that Maxwell's equations are not invariant upon Galilean transformation, some natural questions arise. Is the relativity principle from classical mechanics wrong? Can the aether be seen as an universal reference frame? Can we prove its existence by measuring the speed of Earth in this referential?

If Earth moves through the aether then, applying the classical velocity composition laws, the speed of light should depend on the direction of propagation. However, the speed of the planet may be small compared to the one of light. To overcome this issue, the idea is to compare different light beams through interference in order to highlight velocity differences. Between 1881 and 1887, Michelson and Morley applied this concept in famous experiments which are now named after them, but they failed to show any significant differences. The Earth might have a really small motion with respect to the aether, which is an unexpected result.

To overcome this difficulty, Ernst March proposes to remove the aether from the theory. However this approach is not very popular. In 1889, the hypothesis of length contraction is proposed by Fitzgerald. Independently, Lorentz proposed the same idea in an article where transformations, which will be named after him, model this concept and where the notion of proper time is introduced. This last concept is, however, seen as a mathematical artefact. Poincaré improves the formulation of these transformations and calls them "Lorentz transformations". His model starts form the relativity principle of classical mechanics, the fact that the speed of light is constant in any reference frame and uses Lorentz transformations to obtain correct velocity composition laws. It is sometimes said that he even questioned the fact that the proper time cannot be seen as a physical time. Nonetheless, the idea of aether is conserved together with the idea of absolute space and time. Another interpretation, which is considered as more satisfying, is presented in 1905 by Einstein in one of the most famous articles in the history : *On the electrodynamics of moving bodies.* Finally, the mathematical framework of special relativity will be improved by Minkowski in 1907. The aether is removed from the theory, time is no longer seen as an absolute concept but rather as quantity depending on the frame. Physical equations have to be studied in a four degrees space with Minkowski metric.

Experimental texts

Special relativity has been tested in many experiments. We present some of them:

1. (1932) Kennedy-Thorndike experiment: the Michelson and Morley experiment has been modified to test that the speed of light does not depend on the velocity of the interferometer. It is an indirect proof of time dilation.

2. (1938 - 1941) Ives–Stilwell experiments: time dilation is observed through a Doppler effect on light beams.

3. (1971) Hafele-Keating experiment: a commercial plane is equipped with an atomic clock to travel around the world in 41 h. The experiment has been repeated on another trip of 49 h. Significant delays have been observed on clocks on the planes with respect to clocks which were sitting on the ground.

4. (1975) Alley expirement: In the Hafele-Keating experiment, the trajectories of the clocks were not easy to estimate. To increase the precision of the test, an entire plane was reserved for a 15 hours experiment. The results was in agreement with the theory.

Behind the second cloud : Quantum Mechanics

The ultraviolet catastrophe and the photoelectric effect : a new point of view about light

When the temperature of a body increases, it starts to emit some radiation. This phenomenon can be observed in every day life. For example, a hotplate may become red if it is too hot. This behaviour does not seem directly related to the type of matter which constitutes the object. It is then natural to consider an ideal situation called the black body. As a definition, a black body is a theoretical model which is supposed to absorb any incident radiation. Moreover, according to the Kirchhoff's law, the proportion of the incident radiation which is absorbed is equal to its emission divided by the theoretical emission it would have if it were a black body. Therefore, understanding the properties of black bodies allowed us to understand the behaviour of any type of physical objects.

In 1879, Stefan relates the average power emitted per surface unit $R(T)$ with the black body temperature through the following empirical equation:

$$R(T) = \sigma T^4$$

with σ the Stefan constant. In 1884 Boltzmann derives this relation from thermodynamic principals. However, the answer is not yet complete because it is not possible to compute the power emitted at a specific wavelength λ. We would like to describe the spectral power $R(\lambda, T)$, which can be experimentally measured, verifying:

$$R(T) = \int R(\lambda, T) d\lambda = \sigma T^4.$$

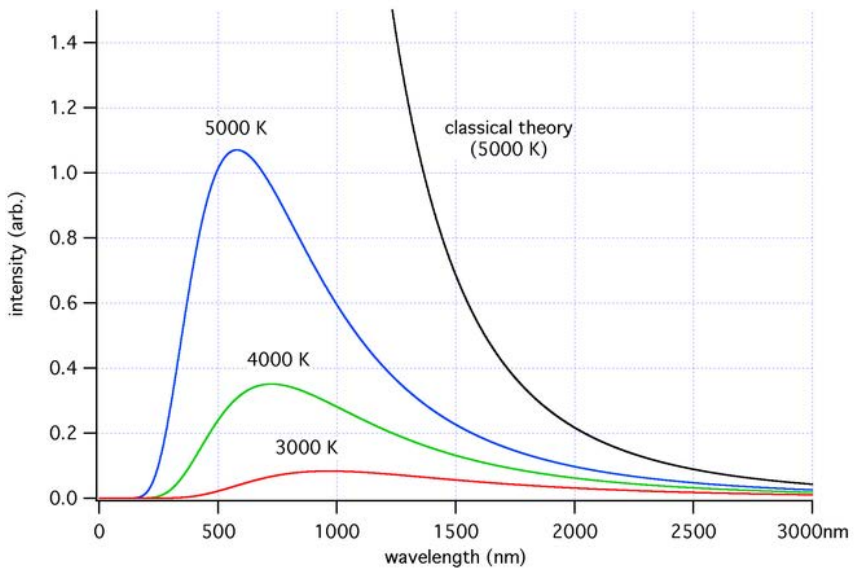

Figure 3: Emitted spectral power [5]

Rayleigh and Jeans derived an expression $R(\lambda, T)$ from classical mechanics but, as illustrated in Figure 3, the formula were not in agreement with the experiments. The lower the wavelength is, the higher the emitted power is. Ehrenfest called this paradox the ultraviolet catastrophe in 1911.

In 1900, Planck shows that assuming that the emission is quantised, it is possible to derive an accurate theoretical description of the black body radiation. The quantised behaviour is for example illustrated in the following famous formula:

$$E = nh\nu.$$

E is the energy of a quanta, n is a natural number and h is the so called Planck constant. This hypothesis was a revolution at that time and it seems that Planck himself saw it as a mathematical trick rather than a physical postulate. In 1905, Einstein uses this hypothesis to explain the photoelectric effect.

The photoelectric effect may be presented as an emission of electrons by some material excited by an electromagnetic wave. In a famous experiment realised by Hertz in 1887, two

[5] Figure from https://fr.wikiversity.org/wiki/Capteur/Capteur_de_temp

electrodes are exposed to an ultraviolet radiation and a photoelectric effect is observed over a certain frequency. The threshold cannot be explained classically but using the idea of quanta introduced by Planck. However, Einstein suggests that the energy is quantised everywhere and not only at emission. Light might then be seen as a particle, which will be called photon in 1926 by Lewis.

The problem of the atom

In 1911, Rutherford establishes his model of the atom, built like a solar system. The electrons with negative charge turn around the nuclei which have a positive charge. However, this point of view is not consistent with electromagnetism because an accelerated charge is supposed to emit radiation. Therefore it would be expected that the electron would loose energy and that the atom would be unstable.

Another important issue is to understand the emission spectrum of chemical elements which are discrete. Focusing on the simplest element, the hydrogen atom, empirical relations between spectral lines are established, for example by Balmer in 1885 and its generalisation by Rydberg in 1888. Their structure seems to depend strongly on natural numbers.

Bohr proposes a solution to these two problems in 1913 with his famous model. Three hypotheses are needed. First, stable trajectories, where the electron do not have to emit electromagnetic waves, exist. Next, an electron can move from one orbit to another only by emission or absorption of a energy quanta $nh\nu$. These two first hypotheses explain why spectral lines are observed but cannot predict them. The correct orbits, among infinite possibilities, have to be chosen. This is the purpose of the third hypothesis: the angular momentum of the electron is quantised. Finally, the Rydberg formula can be predicted theoretically by this model which is probably one of the first ones to use the idea of quantisation.

Unfortunately, a lot of problems remain. Is there any physical motivation to the the previous three hypotheses? Is this model only a phenomenological one built to match with experiment? Why is it so hard to generalised this approach to more complicated structures like the Helium atom?

Wave-particle duality

On the one hand, Young and Fresnel experiments suggest that light is a wave. On the other hand, the photoelectric effect shows that light might be seen as a beam of particles. Therefore, the photon seems to have a double nature and De Broglie imagines that this behaviour has to be a general property of particles. In 1924, he proposes a generalisation of Planck quanta formula to compute the wavelength λ of a massive particle from its momentum p:

$$\lambda = \frac{h}{p}.$$

The wave behaviour of the electron is, in particular, experimentally discovered in 1927 by Davisson and Germer in a famous double-slit experiment. Moreover, quantum mechanics starts to find a better description and formalism with the Schrödinger equation (1925) and the Heisenberg uncertainty principle (1927).

Experimental texts

We give some experiments which have confirmed Quantum Mechanics.

1. The Ehrenberg-Siday-Aharonov-Bohm effect is a phenomenon predicted by quantum mechanics: a charged particle is perturbed by a electromagnetic potential in a region of space with null field, which contradicts classical mechanics. Predicted by Ehrenberg and Siday in 1949 and next by Aharonoy and Bohm in 1959, the effect has been tested many times: by Chambers in 1960, Möllenstedt and Bayh in 1962 or by Tonomura et al. in 1962. These experiments are of great interest as they show that the potential has some physical meaning and cannot be reduced to a mathematical trick.

2. (1972) Freedman and Clauser experiments: first evidence of Bell's inequalities violation. These inequalities are consequences of the hypothesis of hidden variables which could have explained classically entangled states.

3. (1981-1982) The Aspect experiments refine the results of 1972. Bell's inequalities seem to be violated in any situation. The hypothesis of hidden variables is refuted.

An unification of Special Relativity and Quantum Mechanics: Quantum Field Theory

In 1926 Klein and Gordon propose a first attempt to build a relativistic quantum model based on Schrödinger equation. Since the classical free energy of a particle described by:

$$E = \frac{p^2}{2m}$$

is quantised in the following way:

$$i\hbar \frac{\partial}{\partial_t} = \frac{-\hbar^2 \nabla^2}{2m}$$

it seems then natural to quantised the relativistic free energy

$$E^2 = p^2 c^2 + m^2 c^2$$

in the same way:

$$\left(i\hbar \frac{\partial}{\partial_t} \right)^2 = (-i\hbar\nabla)^2 + m^2 c^4.$$

$\hbar$ stands for the reduced Palnck's constant:

$$\hbar = \frac{h}{2\pi}.$$

Defining:

$$\square = -\left(i\hbar \frac{\partial}{\partial_t} \right)^2 + (-i\hbar\nabla)^2,$$

Klien-Gordon equation is obtained for a free wave function ψ:

$$(\square + m^2)\psi = 0.$$

Unfortunately, many problems arise while studying this equation. First, states with negative energy are predicted and it is not possible to remove them directly as it would restrain the physical domain on a non-dense subset of the Hilbert space. Particles should move indefinitely to lower and lower energy states emitting photons which is not what is observed. Moreover, the Klein-Gordon equation is of order two in time whereas the probability density is of order one and consequently may take negative values. In 1928, Dirac proposes a solution to these difficulties. The first goal is to obtain an equation of first degree in time. The idea is then to consider the square root of Klein-Gordon equation. From now on, natural units will be used:

$$c = \hbar = 1.$$

Let α and β be two constants such that:

$$i\frac{\partial\psi}{\partial t} = (-i\alpha\nabla + m\beta)\psi$$

and

$$\begin{aligned}\left(i\frac{\partial\psi}{\partial t}\right)^2 &= (-i\alpha\nabla + m\beta)(-i\alpha\nabla + m\beta)\psi \\ &= \left((-i\alpha\nabla)^2 - im(\beta\alpha + \alpha\beta)\nabla + m^2\beta^2\right)\psi \\ &= (-i\nabla)^2 + m^2.\end{aligned}$$

It is not possible to fulfil these conditions except if α et β are assumed to be matrices. Let us introduce the following matrices: γ^μ for $\mu = 0, 1, 2, 3$. The Einstein notation will be used:

$$\gamma \cdot \nabla = \gamma^i \cdot \nabla_i.$$

The square root of Klein-Gordon equation is:

$$(i\gamma^0\partial_t + i\gamma^\mu\nabla_\mu - m)\psi = 0$$

Therefore:

$$\begin{aligned}\gamma^0\gamma^0 &= 1 \\ \gamma^i\gamma^j &= -\delta_{i,j} \text{ for i,j = 1,2,3} \\ \gamma^i\gamma^0 + \gamma^0\gamma^i &= 0 \text{ for i=1,2,3.}\end{aligned}$$

The simplest choice seems to be:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where σ_i stand for the so-called Pauli's matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dirac's equation becomes:

$$i\frac{\partial\psi}{\partial t} = (-i\gamma^0\gamma^i\nabla_i + m\gamma^0)\psi.$$

The probability density is now positive. Nevertheless, negative energy states are still possible. Dirac suggests then that the vacuum is full of particles. Indeed, his equation models fermions, which means that they cannot be in the same quantum state according to Pauli exclusion principle. Consequently, all the negative energy levels are full and matter cannot collapse any more.

If the vacuum contains particles, it should be possible to excite one of these states to bring it to a positive energy level or, in other words, to create a particle together with a hole in the vacuum. The hole would have the same mass but opposite intern quantum numbers. Moreover, a collision with a particle would lead to the destruction of both structures together with an emission of photons. This is the theoretical discovery of antiparticles (1931) which will be experimentally confirmed in 1932 by Anderson.

This interpretation, called Dirac's sea, is however not entirely satisfactory. For example, it is hard to generalise this theory for bosons which can occupy the same quantum state. Nonetheless, an important idea have been formulated: it seems impossible to build a relativistic quantum theory for one isolated particle. The best point view is to understand Dirac's equation in terms of fields.

The basis of Quantum Field Theory proposed by Dirac, in particular in his work about quantum electrodynamics in 1927, will be completed by Jordan who will introduce, for example, the notion of creation and annihilation operator. A first quantum field theory for electromagnetism is proposed in 1929. Many scientists have been involved in this process such as Heisenberg, Pauli and Born. However, many difficulties have to be overcome such as divergences which appear during computations. A standard procedure to treat them will be proposed in 1949 thanks to the works of Dyson, Schwinger, Tomonaga and Feynman. The accuracy of quantum electrodynamics (QED) encourages scientists to gather all the fundamental interactions in a single formal framework. Today, the standard model of particle physics models the electromagnetic, the weak and the strong interactions.

Physical Motivation and Thesis content

A brief overview of the Standard Model

There are two families of particle in the standard model: fermions and bosons. Fermions are described by antisymmetric wave functions and fulfil the Pauli exclusion principle. Among them, leptons and quarks can be found. The most famous leptons is the electron and it is possible to observe particles with the same characteristics but heavier: the muon and the tau. Finally, there exists a neutrino associated to each of them. On the other hand, quarks are the main constituents of complex structures, such as protons and neutrons, and can be dived, like fermions, in three pairs: up and down, charm and strange, top and bottom.

All of these particles can interact exchanging bosons which are modelled by symmetric wave functions and may occupy the same quantum state. For example, the boson of electromagnetism is the photon. There are three weak bosons: W^+, W^- and Z^0 and finally, there exist eight bosons for the strong interaction called gluons. We will not propose a complete description of the standard model and we refer to [72] for more details.

Existence of such entities are experimentally proved in accelerators of particles. The idea is to accelerate a state (such as protons at the LHC) until it reaches the speed of light and then to send it on a target, like other particles going the other way around. After collision, some particles

[6] Figure from http://davidgalbraith.org/portfolio/ux-standard-model-of-the-standard-model/

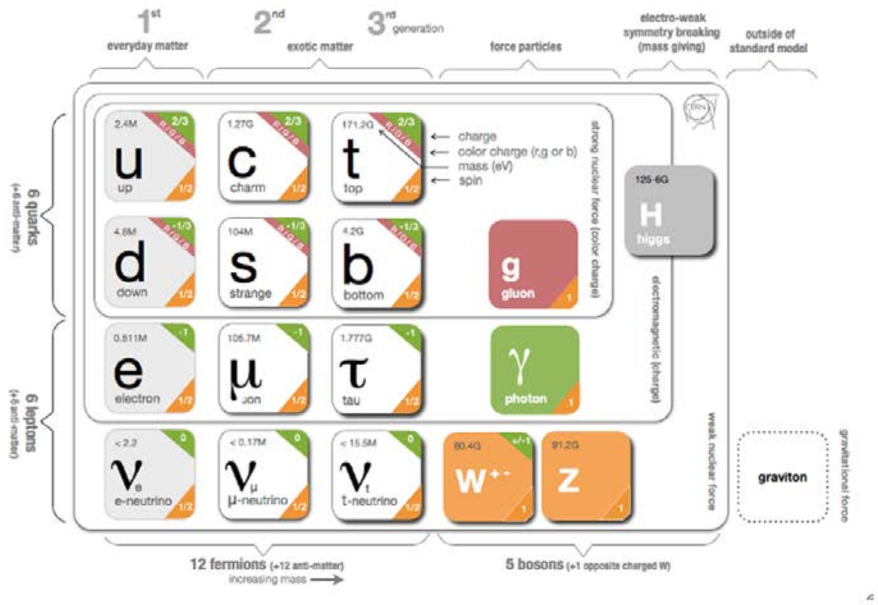

Figure 4: The particles of the Standard Model [6]

may scattered to reach the detector and it is natural to introduce a mathematical tool which build the out-coming states knowing the in-coming one. Such an operator is called scattering matrix. Moreover, it is tempted to study the asymptotic behaviour as time goes to infinity of such states since they are supposed to behave as free particles while escaping from the collision and then the interaction. Finally, the scattering matrix is a key element to compute scattering cross sections which are involved in measurements. In this work we will study scattering theories for simplified mathematical model from the standard model.

Thesis content

Chapter I In the first chapter the mathematical formalism used along this work will be presented. Starting from quantum mechanics remainders, fundamental tools for quantum fields will be introduced: Fock spaces, second quantisation operator, creation and annihilation operator, some classical estimates together with the notion of scattering and asymptotic completeness. Finally, a short overview of the literature will be done which will be completed in the other chapters.

Chapter II The second chapter has been published in [2] and we present a detailed version of our work. The main result of this chapter is the proof of asymptotic completeness for a model of the weak decay of the $W^{\pm}$ bosons into the full family of leptons is done. Neutrinos will be assumed to be massive. The proof rely on propagation estimates together with two versions of Mourre estimate. The spectrum of the Hamiltonian has only one eigenvalue E which is the infimum of the spectrum together with an absolutely continuous branch after a spectral gap between E and $E + m$ where m is the mass of the lightest particle.

Chapter III In this chapter, published in [2], the problem presented in chapter II is adapted to the more difficult massless neutrinos case. The model is simplified in order to introduce a conserved quantity related to the number of particles. An important result is the establishment of propagation estimates adapted to the massless case. They constitute a key tool in our proof of asymptotic completeness which is one of our main contribution.

Chapitre IV This chapter corresponds to [3] where we prove that and Hamiltonian with an abitrarly finitely large number of fermions can be definied. The main idea is to use an interpolation argument on two N-estimates. The existence of a ground state is shown following a strategy presented in [56].

Chaptire V This last chapter gathers works in progress about a translation invariant toy model. Its spectrum is studied thanks to a weakly conjugate Mourre theory.

Chapter 1

Quantum field formalism

1.1 Quantum Mechanics framework

Quantum field theory inherited a lot from quantum mechanics. This section is then devoted to some basic reminders about quantum framework and it may be skipped by the discerning reader. We fix some notations while recalling the definition of Hilbert spaces and of self-adjoint operators. We moreover present the different types of spectrum that can be associated to a self-adjoint operator together with their physical interpretations. Finally, a simple introduction to scattering theory is presented. For more details, we refer to [7, 81, 82, 83].

1.1.1 Self-adjoint and symmetric operators

The physical states are described as elements of a **separable Hilbert space** $\mathscr{H}$ which is defined as follow:

Definition 1.1.1 (Separable Hilbert space)**.** *A separable Hilbert space $\mathscr{H}$ is a real or complex vector space such that the following three properties are true.*

1. *It is equipped with a scalar product $\langle . | . \rangle$ and the following norm can then be defined:*
$$\|f\| = \langle f | f \rangle^{\frac{1}{2}} .$$
2. *It is complete under the norm $\|.\|$.*
3. *It is separable, which means that $\mathscr{H}$ has a countable orthonormal basis.*

Only separable Hilbert spaces will be considered and they will be referred simply as Hilbert spaces.

Example 1.1.2. *Let λ be the Lebesgue measure. Let $\mathscr{L}^2(\mathbb{R}^n)$ be the set of measurable functions f from $\mathbb{R}^n$ to $\mathbb{C}$ such that*
$$\int_{\mathbb{R}^n} |f(x)|^2 \lambda(dx)$$
is finite and equipped with the following scalar product:
$$\forall f, g \in \mathscr{L}^2(\mathbb{R}^n), \langle f | g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x) \lambda(dx).$$
Defining $L^2(\mathbb{R}^n)$ as the quotient of $\mathscr{L}^2(\mathbb{R}^n)$ with the set of functions with zero norm, one obtains a Hilbert space. Note that $\lambda(dx)$ will be often noted dx.

$\mathfrak{h} = L^2(\mathbb{R}^3)$ is often used as the one particle Hilbert space. More precisely, describing a physical particle by a wave function $\Psi \in \mathfrak{h} = L^2(\mathbb{R}^3)$ means that the probability of finding the particle in a region $\mathscr{S} \subset \mathbb{R}^3$ of space is equal to $\displaystyle\int_{\mathscr{S}} |\Psi(x)|^2 d^3x$.

The physical observables are modeled by self-adjoint operators. Therefore, some basic reminders about linear maps are listed here.

Definition 1.1.3. *1. (linear maps) A map from $\mathscr{H}$ to $\mathscr{H}$ is said to be linear if:*

$$\forall f, g \in \mathscr{H}, \alpha \in \mathbb{C}, A(\alpha f + g) = \alpha Af + Ag.$$

2. (Bounded operators) A bounded operator is a linear operator from $\mathscr{H}$ to $\mathscr{H}$ such that:

$$\exists M \in \mathbb{R}^+, \forall f \in \mathscr{H}, \|Af\| \leqslant M\|f\|.$$

The set of all bounded operators is denoted by $\mathscr{B}(\mathscr{H})$.

For a bounded operator A, we can define its norm by:

$$\begin{aligned} \|A\| &= \inf\{M | \forall f \in \mathscr{H}, \|Af\| \leqslant M\|f\|\} \\ &= \inf_{f\neq 0} \frac{\|Af\|}{\|f\|}. \end{aligned}$$

For each operator $A \in \mathscr{B}(\mathscr{H})$ there exists $A^* \in \mathscr{B}(\mathscr{H})$, called the adjoint operator, such that:

$$\forall f, g \in \mathscr{H}, \langle A^* f | g \rangle = \langle f | Ag \rangle .$$

Definition 1.1.4. *An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be self-adjoint if $A^* = A$.*

Different notions of convergence can be defined both for vectors and operators. We recall them here to fix notation.

Definition 1.1.5. *Let $\{f_n\}$ be a sequence of vectors in $\mathscr{H}$, f a vector in $\mathscr{H}$, $\{A_n\}$ a sequence of bounded operators and $A \in \mathscr{B}(\mathscr{H})$.*

1. *We say that $\{f_n\}$ converges strongly to f and we note $s - \lim_{n\to\infty} f_n = f$ if $\|f_n - f\| \underset{n\to\infty}{\to} 0$.*
2. *$\{f_n\}$ is said to converge weakly to f, noted $w - \lim_{n\to\infty} f_n = f$ if:*

$$\forall g \in \mathscr{H}, \langle g|f_n\rangle \underset{n\to\infty}{\to} \langle g|f\rangle .$$

3. *$\{A_n\}$ is said to converge in norm to A, noted $\lim_{n\to\infty} A_n = A$, if*

$$\|A_n - A\| \underset{n\to\infty}{\to} 0.$$

4. *We say that $\{A_n\}$ converges strongly to A, noted $s - \lim_{n\to\infty} A_n = A$ if*

$$\forall f \in \mathscr{H}, \|(A_n - A)f\| \underset{n\to\infty}{\to} 0.$$

5. *$\{A_n\}$ is said to converge weakly to A, noted $w - \lim_{n\to\infty} A_n = A$ if*

$$\forall h, g \in \mathscr{H}, \langle g|A_n h\rangle \underset{n\to\infty}{\to} \langle g|Ah\rangle .$$

Unfortunately, physical observables can rarely be described by bounded operators and it is required to use unbounded ones.

Definition 1.1.6 (Unbounded operators)**.** *Let $\mathscr{D}(A)$ be a linear subspace of $\mathscr{H}$. A is an (unbounded) operator on $\mathscr{H}$ if:*

$$\forall f, g \in \mathscr{D}(A), \alpha \in \mathbb{C}, A(\alpha f + g) = \alpha A f + A g.$$

Let A and B be two unbounded operators. The following notations will be used:

1. $A = B$ stands for $\mathscr{D}(A) = \mathscr{D}(B)$ and $\forall f \in \mathscr{D}(A), Af = Bf$,
2. $A \subseteq B$ stands for $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and $\forall f \in \mathscr{D}(A), Af = Bf$.

Example 1.1.7 (Position operator)**.** *We define the operator Q on $L^2(\mathbb{R})$ as:*

$$(Qf)(x) = xf(x).$$

This operator is clearly unbounded and its domain is:

$$\mathscr{D}(Q) = \left\{ f : \mathbb{R} \to \mathbb{C} | \int_{\mathbb{R}^3} (1 + x^2)|f(x)|^2 dx < \infty \right\}.$$

Example 1.1.8 (Momentum operator)**.** *We define the operator P in the sense of distribution on $L^2(\mathbb{R})$ as:*

$$(Pf)(x) = -i\frac{df}{dx}(x),$$

with domain:

$$\mathscr{D}(P) = \left\{ f : \mathbb{R} \to \mathbb{C} | \int_{\mathbb{R}^3} (1 + k^2)|\hat{f}(k)|^2 dk < \infty \right\},$$

where $\hat{f}$ is the Fourier transform of f.

Definition 1.1.9 (adjoint)**.** *Let $(A, \mathscr{D}(A))$ be an unbounded operator with dense domain. The adjoint A^* of A is defined in the following way:*

1. *for all $f \in \mathscr{H}$, f belongs to the domain $\mathscr{D}(A^*)$ of A^* if there exists $f^* \in \mathscr{H}$ such that:*

$$\forall g \in \mathscr{D}(A), \langle f^* | g \rangle = \langle f | Ag \rangle$$

2. *the mapping A^* is then defined as $A^* f = f^*$.*

Definition 1.1.10 (Self-adjoint operators)**.** *An operator A is said to be self-adjoint if $A = A^*$ meaning that $\mathscr{D}(A^*) = \mathscr{D}(A)$ and for any $f \in \mathscr{D}(A)$, $A^* f = Af$.*

Example 1.1.11. *The position operator Q and the momentum operator P on $L^2(\mathbb{R})$ are self-adjoint.*

The equality requirement of the domains is important. If for all f and g in the domain of A the following equality holds:

$$\langle Af | g \rangle = \langle f | Ag \rangle,$$

then we only have that $\mathscr{D}(A) \subset \mathscr{D}(A^*)$ and the operator is said to be **symmetric**.

Many difficulties arise when manipulating unbounded operators. For example, if $\{f_n\}$ is a sequence in the domain of an operator A converging to f, the sequence $\{Af_n\}$ might not converge to Af. Actually, f might not even be in the domain of A. This motivates the following definitions of closability, closure and of closed operator.

Definition 1.1.12. *Let $(A, \mathscr{D}(A))$ be an unbounded operator.*

1. ***Closable:*** *A is said to be closable if the following properties*

   (a) $\forall n \in \mathbb{N}, f_n \in \mathscr{D}(A)$,

   (b) $s - \lim_{n\to\infty} f_n = 0$,

   (c) $\{Af_n\}$ *is strongly Cauchy*

   imply that $s - \lim_{n\to\infty} Af_n = 0$.

2. ***Closure of an operator:*** *if A is closable then there exists an operator $\overline{A}$, called the closure, such that for any Cauchy sequences $\{f_n\}$ in the domain of A which converges to f and such that $\{Af_n\}$ is also Cauchy, we have $\overline{A}f = s - \lim_{n\to\infty} Af_n$.*

3. ***Closed:*** *A is closed if the following properties*

   (a) $\forall n \in \mathbb{N}, f_n \in \mathscr{D}(A)$,

   (b) $s - \lim_{n\to\infty} f_n = f$,

   (c) $\{Af_n\}$ *is strongly Cauchy*

   imply that $f \in \mathscr{D}(A)$ *and* $s - \lim_{n\to\infty} Af_n = Af$.

In general, unbounded operators may have many closed extensions. The closure is the smallest one as shown by the following proposition (see [7]):

Proposition 1.1.13. *If $A \subseteq B$ and B is closed, then A is closable and $\overline{A} \subseteq B$.*

A symmetric operator with a self-adjoint closure is said to be **essentially self-adjoint**. A complete description of symmetric operators and their extensions can be found in [81]. We introduce the notion of maximal symmetric operators:

Definition 1.1.14 (Deficiency indices)**.** *Let A be a symmetric operator. The deficiency indices are defined as:*

1. $n_+(A) = \dim(\mathrm{Ker}(i - A^*)) = \dim(\mathrm{Ran}(i + A)^\perp)$.

2. $n_-(A) = \dim(\mathrm{Ker}(i + A^*)) = \dim(\mathrm{Ran}(-i + A)^\perp)$.

These numbers constitute an interesting criteria to classify closed symmetric operators.

Proposition 1.1.15. *Let A be a closed symmetric operator.*

1. *A is self-adjoint if and only if $n_+(A) = n_-(A) = 0$.*

2. *A has self-adjoint extensions if and only if $n_+(A) = n_-(A)$.*

3. *Assume $n_+(A) = 0$, $n_-(A) \neq 0$ or $n_+(A) \neq 0$, $n_-(A) = 0$, then A has no nontrivial symmetric extensions.*

Definition 1.1.16 (Maximal symmetric operator)**.** *A closed symmetric operator is said to be maximal symmetric if it has no nontrivial symmetric extensions.*

To build self-adjoint operators a strategy is to perturb an operator A, which is known to be self-adjoint, with some other symmetric operator B. It is indeed natural to think that $A+B$ is self-adjoint if the perturbation is not too large, in a sense that has to be defined.

Definition 1.1.17. *Let A and B be two unbounded operators. B is said to be A-**bounded** if:*

1. $\mathscr{D}(A) \subseteq \mathscr{D}(B)$,
2. *for some real numbers a and b and all $\phi \in \mathscr{D}(A)$ we have:*

$$\|B\phi\| \leqslant a\|A\phi\| + b\|\phi\|. \tag{1.1}$$

Note that the infimum of the set of constants a such that (1.1) is true, for some b which may depend on a, is called the relative bound of B with respect to A.

We can now present the following important theorem:

Theorem 1.1 (Kato-Rellich theorem)**.** *Suppose that A is a self-adjoint operator and B is a symmetric operator which is A-bounded with a relative bound strictly smaller than 1. Then $A+B$ is self-adjoint on $\mathscr{D}(A)$. Moreover, if A is bounded below by m, then $A+B$ is also bounded below by:*

$$m - \max\left\{\frac{b}{(1-a)}, a|m| + b\right\},$$

where a and b are defined in (1.1)*.*

Quadratic forms are also natural objects to study:

Definition 1.1.18 (Quadratic forms)**.** *Let $\mathscr{Q}(q)$ be a dense linear subset of $\mathscr{H}$. $q : \mathscr{Q}(q) \times \mathscr{Q}(q) \to \mathbb{C}$ is a quadratic form if for any $\phi, \psi \in \mathscr{Q}(q)$, $q(., \psi)$ is conjugate linear and $q(\phi, .)$ is linear. $\mathscr{Q}(q)$ is called the form domain of q. Moreover, if $q(\phi,\psi) = \overline{q(\psi,\phi)}$, the quadratic form is said to be symmetric.*

Note that one can canonically associates a quadratic form q to any self-adjoint operator A (see [81, Example 2, Page 277]) and the form domain of A is defined as the domain of this quadratic form : $\mathscr{Q}(A) = \mathscr{Q}(q)$. A result analogous to the Kato-Rellich theorem exists for quadratic forms.

Theorem 1.2 (the KLMN theorem)**.** *Let A be a positive self-adjoint operator. Let β be a symmetric quadratic form on $\mathscr{Q}(A)$, the form domain of A, such that:*

$$\forall \phi \in \mathscr{Q}(A), \quad |\beta(\phi,\phi)| \leqslant a\,\langle\phi|A\phi\rangle + b\,\langle\phi|\phi\rangle\,,$$

for some $a < 1$ and b a real number. Then, there exists an unique self-adjoint operator C with $\mathscr{Q}(C) = \mathscr{Q}(A)$ and:

$$\forall \phi,\psi \in \mathscr{Q}(C), \quad \langle\phi|C\psi\rangle = \langle\phi|A\psi\rangle + \beta(\phi,\psi).$$

Moreover, C is bounded below by $-b$.

Remark 1.1.19. *If we replace $\beta(\phi,\psi)$ by $\langle\phi|B\psi\rangle$ where B is a self-adjoint operator, we see that the KLMN theorem gives a sense to $A+B$ which may differ from the usual operator addition.*

Now that the notion of self-adjoint operators have been clarified we will recall the notion of spectrum and see how it is related to scattered particles.

1.1.2 Spectral theorem and spectrum of an operator

Throughout this section, A will be an operator acting on $\mathscr{H}$.

Definition 1.1.20 (The resolvent set)**.** *The resolvent set, $\rho(A)$, is the set of all complex numbers, z, such that $(A-zI)$ is invertible with bounded inverse:*

$$\rho(A) = \left\{z \in \mathbb{C} | (A-zI)^{-1} \in \mathscr{B}(\mathscr{H})\right\}.$$

Note that $(A-zI)^{-1}$ will be written as $(A-z)^{-1}$.

The resolvent set is a way to answer the very natural question of invertibility of an operator. Another natural object to study is then the complementary set of the resolvent one.

Definition 1.1.21 (spectrum)**.** *Let A be an operator. The set of complex number z such that $A-z$ does not have any bounded inverse is called the spectrum, $\sigma(A)$, and it is defined as:*

$$\sigma(A) = \mathbb{C} \backslash \rho(A).$$

We recall some very well known results that will be used later on.

Proposition 1.1.22. *1. **First resolvent equation**: if $z_1, z_2 \in \rho(A)$, then:*

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1-z_2)(A-z_1)^{-1}(A-z_2)^{-1}$$

2. The spectrum of a self-adjoint operator A is real. Moreover, if $z = x+iy$ with x and y some real numbers such that y is different from zero, then $z \in \rho(A)$ and:

$$\|(A-z)^{-1}\| \leqslant \frac{1}{|y|}.$$

Physically, the spectrum $\sigma(A)$ can be seen as the set of all possible values that can be obtained by measuring the observable A. The characterization of this spectrum is then an important task.

Definition 1.1.23 (Spectral family)**.** *We consider a projection-valued function E defined from $\mathbb{R}$ to the subset of self-adjoint projections in $\mathscr{B}(\mathscr{H})$. It is a right continuous spectral family, that we will be simply called spectral family, if:*

1. $\forall \lambda, \mu \in \mathbb{R}, \quad E_\lambda E_\mu = E_{min\{\lambda,\mu\}}$,
2. $s - \lim_{\lambda \to -\infty} E_\lambda = 0$ *and* $s - \lim_{\lambda \to +\infty} E_\lambda = I$,
3. $\forall \lambda \in \mathbb{R}, E_\lambda = s - \lim_{\epsilon \to 0^+} E_{\lambda+\epsilon}$.

The measure of some half open subset $(a,b] \subset \mathbb{R}$ can be defined as:

$$E((a,b]) = E_b - E_a$$

and it is not hard to see that $E((a,b])$ is a projection. A remarkable property is that for any two half open sets $(a,b], (c,d]$ of $\mathbb{R}$ the following equality holds:

$$E((a,b])E((c,d]) = E((a,b] \cap (c,d]). \tag{1.2}$$

We can now extend these operators to the full Borel σ-algebra. This leads to the concept of spectral measures and the so-called spectral theorem associates uniquely self-adjoint operators to spectral measures.

Theorem 1.3 (Spectral theorem). *1. If an operator A is self-adjoint then there exists a unique spectral measure $\{E_A\}$ such that:*

$$A = \int \lambda E_A(d\lambda). \tag{1.3}$$

2. The converse is also true. Let A be an operator. Assume that there exists a spectral measure $\{E_A\}$ such that (1.3) *holds, then A is self-adjoint.*

Example 1.1.24 (Spectral family of the position operator). *Let $\mathscr{H} = L^2(\mathbb{R})$. The spectral family of the position operator Q is defined as:*

$$(E_\lambda f)(x) = \begin{cases} 0 & \text{if } x > \lambda \\ f(x) & \text{if } x \leqslant \lambda \end{cases}.$$

The spectral theorem is a fundamental tool to study the spectrum of an Hamiltonian. Let A be a self-adjoint operator. For any $f \in \mathscr{H}$ we can define a unique measure in the usual sense through the spectral measure. Let B be a Borel set. We consider:

$$m_f(B) = \langle f | E_A(B) f \rangle = \| E_A(B) f \|^2.$$

Let A be a self-adjoint operator defined on a Hilbert space $\mathscr{H}$. Let ϕ be a continuous function from $\mathbb{R}$ to $\mathbb{C}$. One way to define the operator $\phi(A)$ is to use the spectral theorem. Let $\{E_\lambda\}$ be the spectral family associated to A. Then $\phi(A)$ is an operator defined as:

$$\begin{aligned} \phi(A) &= \int \phi(\lambda) E_A(d\lambda), \\ \mathscr{D}(A) &= \left\{ f \in \mathscr{H} \middle| \int |\phi(\lambda)|^2 m_f(d\lambda) < \infty \right\}. \end{aligned}$$

The measure m_f can be decomposed uniquely through the Lebesgue decomposition theorem into an absolutely continuous part with respect to the Lebesgue measure, m_{ac}, and a singular part, m_s. We moreover introduce m_p defined on each Borel set B as:

$$m_p(B) = \sum_{\lambda \in B} m_f(\{\lambda\}).$$

This is clearly the non-continuous part of the singular measure. We then define the singular continuous measure:

$$m_{sc} = m_s - m_p.$$

Note that for any $f \in \mathscr{H}$, there exist three unique vectors f_p, f_{sc} and f_{ac} such that:

$$\begin{aligned} f &= f_p + f_{sc} + f_{ac}, \\ m_{f_p} &= m_p, \\ m_{f_{sc}} &= m_{sc}, \\ m_{f_{ac}} &= m_{ac}. \end{aligned}$$

This justifies the following definitions:

$$\begin{aligned} \mathscr{H}_{ac}(A) &= \{ f \in \mathscr{H} | m_f \text{ is absolutely continuous} \}, \\ \mathscr{H}_{sc}(A) &= \{ f \in \mathscr{H} | m_f \text{ is singular continuous} \} \end{aligned}$$

and $\mathscr{H}_{pp}(A)$ is the subset spanned by the eigenvectors of A. It is moreover a well-know fact that:

$$\mathscr{H} = \mathscr{H}_p(A) \oplus \mathscr{H}_{sc}(A) \oplus \mathscr{H}_{ac}(A).$$

Note that $\mathscr{H}_p(A)$, $\mathscr{H}_{ac}(A)$ and $\mathscr{H}_{sc}(A)$ are A-invariant Hilbert spaces. Let A_p, A_{sc} and A_{ac} be the restriction of A on $\mathscr{H}_p(A)$, $\mathscr{H}_{sc}(A)$ and $\mathscr{H}_{ac}(A)$ we then have:

$$\sigma(A) = \sigma(A_p) \cup \sigma(A_{sc}) \cup \sigma(A_{ac}).$$

This leads to a first characterisation of the spectrum.

1. The point spectrum, $\sigma_{pp}(A)$ is the set of eigenvalues of A and $\overline{\sigma_{pp}(A)} = \sigma(A_p)$.
2. The absolutely continuous spectrum is $\sigma_{ac}(A) = \sigma(A_{ac})$.
3. The singular continuous spectrum is $\sigma_{sc}(A) = \sigma(A_{sc})$.

This point of view finds a physical interpretation that will be made precise in the next section. Another standard characterization of the spectrum will be used in this document. We define:

1. The set of isolated eigenvalues with finite multiplicity : $\sigma_d(A)$.
2. The essential spectrum : $\sigma_{ess}(A) = \mathbb{C} \backslash \sigma_d(A)$.

Example 1.1.25 (Spectrum of the position operator)**.** *Let B be a Borel set and $f \in \mathscr{H} = L^2(\mathbb{R})$. We then have:*

$$m_f(B) = \|E_Q(B)f\|^2 = \langle f|E_Q(B)f\rangle = \int_B |f(x)|^2 dx.$$

According to the Radon-Nikodym theorem, m_f is absolutely continuous with respect to the Lebesgue measure, which means that Q has only absolutely continuous spectrum.

1.1.3 Time evolution

Let H, a self-adjoint operator on a Hilbert space, be the Hamiltonian of a physical system. The propagation of a quantum state ψ is described by the Schrödinger equation:

$$i\hbar \frac{d\psi}{dt} = H\psi.$$

The wave function can be described explicitly by:

$$\psi(t) = e^{-\frac{i}{\hbar}Ht}\psi(t=0).$$

The family $\{e^{-\frac{i}{\hbar}Ht}\}_{t\in\mathbb{R}}$ is an evolution group. We recall that $\{U_t\}_{t\in\mathbb{R}}$ is said to be an evolution group if

- for any $t \in \mathbb{R}$, U_t is unitary,
- for any $s, t \in \mathbb{R}$, $U_s U_t = U_{s+t}$,
- for any $t \in \mathbb{R}$, $s - \lim\limits_{s\to 0} (U_{t+s} - U_t) = 0$.

A self-adjoint operator always defines a unitary group. The converse is also true and we recall the following well known proposition:

Proposition 1.1.26 (Stone's Theorem)**.** *Let $\{U_t\}$ be an evolution group on a Hilbert space $\mathscr{H}$. Let us define the following unbounded operator:*

$$\begin{aligned} \mathscr{D}(A) &= \left\{ f \in \mathscr{H} \,\middle|\, s - \lim_{t \to 0} \frac{1}{t}\left(U_t - I\right) f \text{ exists.} \right\}, \\ \forall f \in \mathscr{H}, \quad Af &= s - \lim_{t \to 0} \frac{i}{t}\left(U_t - I\right) f. \end{aligned}$$

A is then a self-adjoint operator and $U_t = e^{-iAt}$. We moreover have, for any f in $\mathscr{D}(A)$:

$$\frac{d}{dt} U_t f = -iAU_t f = -iU_t A f.$$

From now on, we suppose that two self-adjoint operators are given, H generating the full dynamics and H_0 the free one. The free evolution group is then defined by the free Hamiltonian H_0, which is the energy observable of a system without interaction. It will be noted:

$$U_t^0 = e^{-\frac{i}{\hbar} H_0 t}$$

and the total evolution group defined by the total Hamiltonian H will be noted:

$$U_t = e^{-\frac{i}{\hbar} H t}.$$

Example 1.1.27. *A typical example of free Hamiltonian is $H_0 = P^2$. Its classical analogue would be the derivative of the position to the power two, i.e the velocity to the power two. Therefore it is homogeneous, up to some mass terms, to the free kinetic energy.*

An example of interactive Hamiltonian would be $H = P^2 + V(Q)$, where V is some function and Q the position operator, such that $V(Q)$ is P^2-bounded of bound lower than 1.

1.1.4 Scattered particles, bounded particles and scattering theory

In this section we present a physical interpretation of the absolutely continuous spectrum and eigenvalues together with the vectors associated to them in the context of Example 1.1.27. In other words, $\mathscr{H} = L^2(\mathbb{R}^3)$, $H_0 = P^2$ and $H = H_0 + V(Q)$. As said in the introduction, the goal of scattering theory is to understand the asymptotic behaviour of physical systems. In particular we are interested in defining scattered states and bound states.

Scattered states are supposed to escape from interaction with time, which means that they disperse to infinity.

Let Q be the position operator defined previously. We define χ_{B_R} a function equal to one inside the ball of radius R and zero outside so that:

$$(\chi_{B_R} f)(x) = \begin{cases} f(x) & \text{if} \quad x \in B_R \\ 0 & \text{if} \quad x \notin B_R \end{cases}. \tag{1.4}$$

As time goes to infinity, a scattered state has to escape from any ball of radius R.

Definition 1.1.28 (Set of scattered states)**.** *We set*

$$\begin{aligned} \mathcal{M}_\infty &= \left\{ f \in \mathscr{H} \,\middle|\, \forall R > 0, \lim_{t \to \infty} \|\chi_{B_R} U_t f\| = 0 \right\}, \\ \mathcal{M}_{-\infty} &= \left\{ f \in \mathscr{H} \,\middle|\, \forall R > 0, \lim_{t \to -\infty} \|\chi_{B_R} U_t f\| = 0 \right\}. \end{aligned}$$

In the same way, a bound state has to be confined in some ball of radius R. $I - \chi_{B_R}$ is the position operator restrained outside a ball of radius R which leads to the following definition:

Definition 1.1.29 (Set of bound states)**.** *We set*

$$\begin{aligned} \mathcal{M}_0 &= \left\{ f \in \mathscr{H} \,\middle|\, \lim_{R\to\infty} \sup_{t\in[0,\infty)} \|(I - \chi_{B_R})U_t f\| = 0 \right\} , \\ \mathcal{M}_{-0} &= \left\{ f \in \mathscr{H} \,\middle|\, \lim_{R\to\infty} \sup_{t\in(-\infty,0]} \|(I - \chi_{B_R})U_t f\| = 0 \right\} . \end{aligned}$$

The two previous sets do not cover all the mathematical cases. In particular, to treat singular states, we have to introduce the following sets:

Definition 1.1.30. *We set*

$$\begin{aligned} \overline{\mathcal{M}}_\infty &= \left\{ f \in \mathscr{H} \,\middle|\, \forall R > 0, \lim_{T\to\infty} \int_0^T \frac{1}{T} \|\chi_{B_R} U_t f\|^2 dt = 0 \right\} , \\ \overline{\mathcal{M}}_{-\infty} &= \left\{ f \in \mathscr{H} \,\middle|\, \forall R > 0, \lim_{T\to\infty} \int_{-T}^0 \frac{1}{T} \|\chi_{B_R} U_t f\|^2 dt = 0 \right\} . \end{aligned}$$

The following proposition gives a first intuition of the physical interpretation of the spectrum of an operator.

Proposition 1.1.31. *The following inclusion holds:*

1. $\mathcal{M}_\infty \subset \overline{\mathcal{M}}_\infty$ *and* $\mathcal{M}_{-\infty} \subset \overline{\mathcal{M}}_{-\infty}$.
2. $\mathscr{H}_{pp}(H) \subset \mathcal{M}_0 \cap \mathcal{M}_{-0}$.
3. $\overline{\mathcal{M}}_{\pm\infty} \subset \mathscr{H}_c(H)$.

To obtain equalities further assumptions have to be made. We state here some sufficient conditions:

Proposition 1.1.32. *Assume that one of the following Hypotheses holds :*

1. $\chi_{B_R}(H - i)^{-1}$ *is compact for each positive real number* R*,*
2. *For each positive real numbers* R *and* M*,* $\chi_{B_R} E_H([-M, M])$ *is compact,*
3. *There exists a bounded operator* $C \in \mathscr{B}(\mathscr{H})$ *such that* $CU_t = U_t C$ *for any real* t *and* $\chi_{B_R} C$ *is compact for any* R*,*

then:

1. $\mathcal{M}_{-0} \cap \mathcal{M}_0 = \mathscr{H}_{pp}(H)$.
2. $\overline{\mathcal{M}}_{\pm\infty} = \overline{\mathcal{M}}_\infty \cap \overline{\mathcal{M}}_{-\infty} = \mathscr{H}_c(H)$.
3. *If moreover* H *has no singular spectrum then:* $\mathcal{M}_{\pm\infty} = \mathcal{M}_\infty \cap \mathcal{M}_{-\infty} = \mathscr{H}_{ac}(H)$.

This proposition justifies that $\mathscr{H}_{pp}(H)$ can be seen as the set of bound states and $\mathscr{H}_{ac}(H)$ as the set of scattered particles. Moreover $\mathscr{H}_{sc}(H)$ contains neither bound states nor scattered states. The presence of singular continuous spectrum is then problematic.

We can now give a short introduction to scattering theory and to asymptotic completeness. The following description is the simplest one but not the most general. It will be refined later in the context of quantum field theory. For a deeper treatment of scattering theory we refer to [7, 82, 28]. The first intuition we may have is that scattered particles can be seen as free particle as they escape from experiment as time goes to infinity. Mathematically, it means that for an input particle $\phi \in \mathscr{H}_{ac}(H)$, there exists ψ_- so that its free time evolution $U_t^0\psi_-$ looks like the scattered state with its full evolution $U_t\phi$ as time goes to minus infinity. In other words:

$$\|U_t\phi - U_t^0\psi_-\| \underset{t\to-\infty}{\to} 0,$$

which is equivalent to:

$$\phi = s - \lim_{t\to-\infty} U_t^* U_t^0 \psi_-.$$

Therefore, it is tempting to define the following operator:

$$\Omega_- = s - \lim_{t\to-\infty} U_t^* U_t^0.$$

Scattered states can be studied in the same way as time goes to infinity and we have:

$$\Omega_+ = s - \lim_{t\to+\infty} U_t^* U_t^0.$$

Ω_+ and Ω_- are called wave operators. In other words, it is expected that for all scattered states $\phi \in \mathscr{H}_{ac}(H)$, there exists $\psi_-, \psi_+ \in \mathscr{H}$ such that:

$$\phi = \Omega_+\psi_+, \quad (1.5)$$

$$\phi = \Omega_-\psi_-. \quad (1.6)$$

One way to recover the output asymptotic state from the input one is then:

$$\psi_+ = \Omega_+^*\Omega_-\psi_-.$$

Therefore, the definition of the scattering operator is:

$$S = \Omega_+^*\Omega_-. \quad (1.7)$$

Asymptotic completeness is a mathematical formula which translates the fact that S is well defined and that all physical intuitions are true. Assume that the existence of the wave operators has been proved together with the fact that they are isometries. The scattering operator (1.7) is well defined if $\mathrm{Ran}(\Omega_-) \subset \mathscr{D}(\Omega_+) \subset \mathrm{Ran}(\Omega_+)$. A desirable physical property is that S is unitary which leads to the following condition: $\mathrm{Ran}(\Omega_-) = \mathrm{Ran}(\Omega_+)$. Moreover any scattered state has to be asymptotically free. From (1.6) and (1.5) this implies that $\mathscr{H}_{ac}(H) \subset \mathrm{Ran}(\Omega_-) = \mathrm{Ran}(\Omega_+)$. In general the range of the wave operators are contained in $\mathscr{H}_{ac}(H)$ so that: $\mathrm{Ran}(\Omega_-) = \mathrm{Ran}(\Omega_+) = \mathscr{H}_{ac}(H)$. Moreover, the singular spectrum has to be empty. Since:

$$\mathscr{H}_p(H)^\perp = \mathscr{H}_{sc}(H) \oplus \mathscr{H}_{ac}(H),$$

as soon as $\mathscr{H}_{sc}(H) = \emptyset$, $\mathscr{H}_p(H)^\perp = \mathscr{H}_{ac}(H)$. Finally, the asymptotic completeness condition is: $\mathrm{Ran}(\Omega_-) = \mathrm{Ran}(\Omega_+) = \mathscr{H}_p(H)^\perp$. The strategy that has to be followed in this case is then:

1. Prove that the wave operators exist and are isometric.

2. Prove $\mathrm{Ran}(\Omega_-) = \mathrm{Ran}(\Omega_+) = \mathscr{H}_p(H)^\perp$.

1.2 Quantum Field theory framework

In this section we highlight the specificity of quantum field theories and present standard tools that will be used later on. No reminders will be made on special relativity and we refer for example to [52] for a pedagogical mathematical overview of it. Moreover, the following convention will be adopted:

$$c = \hbar = 1.$$

It is the so-called natural units used in particle physics.

1.2.1 From quantum mechanics to quantum field theory

Quantum field theory can be described as a Hilbert space together with a self-adjoint operator. The main difference with usual Quantum Mechanics is that the number of particles is not a conserved quantity and it may be arbitrary large. A typical example of Hilbert space for one particle has been already presented : $\mathfrak{h} = L^2(\mathbb{R}^n)$. Spin and polarisation terms may be added. A natural construction for two particles would then be: $\mathfrak{h} \otimes \mathfrak{h}$ and may be extended to the n-particles case:

$$\bigotimes\nolimits_{\sharp}^{n} \mathfrak{h}.$$

Nevertheless, a bosonic wave function, respectively fermionic wave function, is symmetric, respectively antisymmetric. Therefore, the previous tensor product has to be projected on its symmetric part for bosons and antisymmetric part for fermions. Finally, the number of particles in such Hilbert space is necessarily a fixed conserved quantity. All the possible cases have to be included which leads to the so-called Fock space:

$$\mathscr{F}_s = \bigoplus_{n=0}^{\infty} \bigotimes\nolimits_{s}^{n} \mathfrak{h} \qquad \mathscr{F}_a = \bigoplus_{n=0}^{\infty} \bigotimes\nolimits_{a}^{n} \mathfrak{h},$$

where s stands for the symmetric tensor product and a the antisymmetric tensor product. From now on, in the case where a statement is true in both cases the subscripts a and s will be replaced by $\sharp$. We moreover use the convention that the $n = 0$ case reduces to the complex numbers set $\mathbb{C}$.

The quantum field Hilbert space is built from the quantum one. One might guess that a quantum observable h may give birth to a quantum field one. This is done by second quantisation: $\mathrm{d}\Gamma$. If h is an operator acting on $\mathfrak{h}$ we define $\mathrm{d}\Gamma(h)$ on $\mathscr{F}_\sharp$ by:

$$\mathrm{d}\Gamma(h)|_{\bigotimes_{k=1}^{n}} = \sum_{i=1}^{n} \underbrace{\mathbb{1} \otimes_\sharp \cdots \otimes \mathbb{1}}_{i-1} \otimes_\sharp h \otimes_\sharp \underbrace{\mathbb{1} \otimes_\sharp \cdots \otimes \mathbb{1}}_{n-i}.$$

The domain of $\mathrm{d}\Gamma(h)$ is:

$$\mathrm{d}\Gamma(h) = \left\{ \phi \in \mathscr{F}_\sharp \,\middle|\, \phi^{(n)} \in \otimes_{i=1}^{n} \mathscr{D}(h) \right\},$$

where $\phi^{(n)}$ is the projection of ϕ on $\otimes^n \mathfrak{h}$. It is a closable operator and its closure is denoted by the same symbol. The main idea of the second quantisation is to apply the quantum observable to each particle channel and to sum over all possibilities.

Example 1.2.1 (The number operator). *In the case of one particle, the identity $\mathbb{1}$ is obviously the number operator. In the quantum field case we then have:*

$$N = \mathrm{d}\Gamma(\mathbb{1})$$

defined on:

$$\mathscr{D}(N) = \left\{ \psi \in \mathscr{F}_\sharp \middle| \sum_{n=1}^{\infty} n^2 \|\psi^{(n)}\|_2^2 < \infty \right\},$$

where $\psi^{(n)}$ *is the projection of* ψ *on* $\bigotimes_\sharp^n \mathfrak{h}$.

Example 1.2.2 (Free energy operator). *In the quantum mechanical case, the free non-relativistic energy operator was, up to some mass terms,* P^2. *Adding special relativity, the relativistic free energy of a particle of mass* m *is:* $\sqrt{-\Delta + m^2}$. *Remember that in natural units* $c = 1$. *The free energy operator is then:*

$$H_0 = \mathrm{d}\Gamma(\sqrt{-\Delta + m^2}),$$

defined on:

$$\mathscr{D}(H_0) = \left\{ \psi \in \mathscr{F}_\sharp \middle| \sum_{n=1}^{\infty} \int \left| \sum_{k=1}^{n} \sqrt{|\xi_k|^2 + m^2} \hat{\psi}^{(n)}(\xi_1, \ldots, \xi_n) \right|^2 d\xi_1 \ldots d\xi_n < \infty \right\}.$$

We recall that by applying a Fourier transform, a differential operator with constant coefficients can be seen as a multiplication operator.

With new physical processes, new mathematical tools are required in modelling. We introduce some of them in the next two sections.

1.2.2 Creation and annihilation operators

In this section we present a fundamental tool to treat quantum field problems: the creation and annihilation operators. The bosonic case is first treated in great details, the fermionic case having no major differences. Let $\mathfrak{h}$ be $L^2(\mathbb{R}^n)$ and $\mathscr{F}_s$ be the symmetric Fock space over $\mathfrak{h}$.

Definition 1.2.3 (Bosonic creation and annihilation operators). *Let* $\{f_i\}_{i\in[0,n]}$ *and* h *elements of* $\mathfrak{h}$. *We define the bosonic creation operator* $a^*(h)$, *respectively annihilation operator* $a(h)$, *as:*

$$a^*(h) f_1 \otimes_s \cdots \otimes_s f_n = \sqrt{n+1} h \otimes_s f_1 \otimes_s \cdots \otimes_s f_n \quad (1.8)$$

$$a(h) f_0 \otimes_s \cdots \otimes_s f_n = \sqrt{n+1} \langle h | f_0 \rangle f_1 \otimes_s \cdots \otimes_s f_n \quad (1.9)$$

$$a(h)\Omega = 0, \quad (1.10)$$

where $\Omega = (1, 0, 0, \ldots)$ *is the vacuum state. They are extended by linearity. Moreover, they are closable and their closure are denoted by the same symbol.*

Moreover the operator a^* and a verify the usual canonical commutation relations:

Proposition 1.2.4. *Let* h_1 *and* h_2 *vectors of* $\mathfrak{h}$. *We then have:*

$$[a(h_1), a^*(h_2)] = \langle h_1 | h_2 \rangle \mathbb{1}$$

$$[a(h_1), a(h_2)] = [a^*(h_1), a^*(h_2)] = 0$$

It is important to note that the operators a^* and a are controlled by $N^{\frac{1}{2}}$.

Proposition 1.2.5. *Let* $h \in \mathfrak{h}$ *and* $\psi \in \mathscr{D}(N^{\frac{1}{2}})$. *Then* $\psi \in \mathscr{D}(a^*(h)) \cap \mathscr{D}(a(h))$ *and:*

$$\|a^*(h)\psi\|_2 \leqslant \|h\|_2 \|(N+1)^{\frac{1}{2}} \psi\|_2 \qquad \|a(h)\psi\|_2 \leqslant \|h\|_2 \|N^{\frac{1}{2}} \psi\|_2$$

The creation and annihilation can then be applied on wave functions with finite number of particles. Let $\phi^{(n+1)}$, respectively $\phi^{(n)}$, be a square integrable wave function with $n+1$, respectively n, particles. For any $\xi_1, \ldots, \xi_{n+1}$ and ξ in $\mathbb{R}^n$ we have:

$$\begin{aligned}
(a(h)\phi^{(n+1)})(\xi_1, \ldots, \xi_n) &= \sqrt{n+1} \int \overline{h(\xi)} \phi^{(n+1)}(\xi, \xi_1, \ldots, \xi_n) d\xi, \\
(a^*(h)\phi^{(n)})(\xi_1, \ldots, \xi_{n+1}) &= \frac{1}{\sqrt{n+1}} \sum_{k=1}^{n+1} h(\xi_k) \phi^{(n)}(\xi_1, \ldots, \hat{\xi}_k, \ldots, \xi_{n+1}),
\end{aligned}$$

where the notation $\phi^{(n)}(\xi_1, \ldots, \hat{\xi}_k, \ldots, \xi_{n+1})$ means that $\phi^{(n)}$ is applied on all the variables $\xi_1, \ldots, \xi_{n+1}$ but ξ_k.

From these expressions, it might be tempting to introduce the following operators used in physics text books:

$$(a(\xi)\phi^{(n+1)})(\xi_1, \ldots, \xi_n) = \sqrt{n+1}\phi^{(n+1)}(\xi, \xi_1, \ldots, \xi_n) \tag{1.11}$$

$$(a^*(\xi)\phi^{(n)})(\xi_1, \ldots, \xi_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{k=1}^{n+1} \delta(\xi_k - \xi) \phi^{(n)}(\xi_1, \ldots, \hat{\xi}_k, \ldots, \xi_{n+1}). \tag{1.12}$$

The idea is to have a "function version" of the creation and annihilation operators such that:

$$\begin{aligned}
a(h) &= \int \overline{h(\xi)} a(\xi) d\xi, \\
a^*(h) &= \int h(\xi) a^*(\xi) d\xi.
\end{aligned}$$

To get a better understanding of these expressions let us introduce the Schwartz space.

Definition 1.2.6 (Schwartz space)**.** *The Schwartz space is defined as:*

$$\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \forall \alpha, \beta \in \mathbb{N}^n, |f|_{\alpha,\beta} < \infty\}$$

where

$$|f|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

We moreover define $\mathscr{D}_{\mathscr{S}}$ as the set of wave functions such that its projection onto $\otimes_{k=1,s}^n \mathfrak{h}$ vanish for all but finitely many naturals numbers n and belongs to the Schwartz space.

The expressions $a^*(\xi)$ and $a(\xi)$ are perfectly well defined as quadratic forms defined on $\mathscr{D}_{\mathscr{S}}$. Hamiltonians expressed in terms of (1.11) and (1.12) have then to be understood as operators associated to a quadratic form.

Example 1.2.7. *The number operator may be written as:*

$$N = \int a^*(\xi) a(\xi) d\xi.$$

Furthermore, let $\omega(\xi) = \sqrt{|\xi|^2 + m^2}$ where m is the mass of the particle. We have:

$$H_0 = \int \omega(\xi) a^*(\xi) a(\xi) d\xi.$$

Canonical commutation relations can be expressed in terms of (1.12) and (1.11). Let ξ_1 and ξ_2 be vectors of $\mathbb{R}^n$,

$$\begin{aligned} [a(\xi_1), a^*(\xi_2)] &= \delta(\xi_1 - \xi_2)\mathbb{1} \\ [a(\xi_1), a(\xi_2)] = [a^*(\xi_1), a^*(\xi_2)] &= 0. \end{aligned}$$

The definition of the creation b^* and annihilation operators b in the fermionic case is the same but theirs properties differ. We then give a short description of them.

Proposition 1.2.8. *Let h_1 and h_2 be vectors of $\mathfrak{h}$, N be the number operator on the fermionic Fock space, ξ_1 and ξ_2 be vectors of $\mathbb{R}^n$.*

1. *The operators b^* and b fulfil the canonical anti-commutation relations:*

$$\begin{aligned} \{b^*(h_1), b(h_2)\} &= \langle h_1 | h_2 \rangle \mathbb{1}, \\ \{b(h_1), b(h_2)\} = \{b^*(h_1), b^*(h_2)\} &= 0, \\ \{b^*(\xi_1), b(\xi_2)\} &= \delta(\xi_1 - \xi_2), \\ \{b(\xi_1), b(\xi_2)\} = \{b^*(\xi_1), b^*(\xi_2)\} &= 0. \end{aligned}$$

where the notation $\{\cdot, \cdot\}$ is defined for two operators A, B as:

$$\{A, B\} = AB + BA.$$

2. *A direct consequence of the anti-commutation relations is the boundedness of the fermionic creation and annihilation operators.*

$$\begin{aligned} \|b^*(h_1)\| &= \|h_1\|, \\ \|b(h_2)\| &= \|h_2\|. \end{aligned}$$

1.2.3 A classical strategy to build Quantum field models and N_τ esimates

Typical quantum field models are perturbations of the free Hamiltonian H_0 with an interaction term. The interaction terms will be of the form:

$$H_I = \int G(\xi_1, \dots, \xi_p, \xi_{p+1} \dots, \xi_n) d^*(\xi_1) \dots d^*(\xi_p) d(\xi_{p+1}) \dots d(\xi_n) + h.c, \tag{1.13}$$

where d refers to either bosonic or fermionic operators and where h.c stands for hermitian conjugate which is the adjoint of the previous expression. Such operators are defined on a tensor product of Fock spaces associated to each particle species: $\mathscr{H} = \bigotimes_{k=1}^{n} \mathscr{F}_{\sharp,k}$.

Let $B = (B_1, B_2, \dots, B_n)$ be operators acting on the Hilbert spaces $(\mathfrak{h}_1, \dots, \mathfrak{h}_n)$. We define $\overline{\mathrm{d}\Gamma}(B)$ on $\mathscr{H} = \bigotimes_{k=1}^{n} \mathscr{F}_{\sharp,k}$ by:

$$\begin{aligned} \overline{\mathrm{d}\Gamma}(B) : \mathscr{H} &\to \mathscr{H} \\ \overline{\mathrm{d}\Gamma}(B) &= \sum_{j=1}^{p} \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{j-1} \otimes \mathrm{d}\Gamma(B_j) \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{p-j}. \end{aligned}$$

The definition of the particle number operator N can be extended as follow:

$$\begin{array}{rcl} N_i : \mathscr{H} & \rightarrow & \mathscr{H} \\ N_i & = & \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \mathrm{d}\Gamma(\mathbb{1}) \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{p-i} \\ N : \mathscr{H} & \rightarrow & \mathscr{H} \\ N & = & \sum_{i=1}^{p} N_i = \overline{\mathrm{d}\Gamma}(\mathbb{1}). \end{array}$$

In the same way, H_0 is the sum of the free energy of each particle. Note that $\overline{\mathrm{d}\Gamma}$ will be denoted as $\mathrm{d}\Gamma$.

Next, regularity properties have to be made on the kernel G. For example, if the kernel is square integrable, H_I is a quadratic form defined on the set of wave functions with a finite number of particles. The strategy is then to apply Kato-Rellich theorem to have a self-adjoint operator.

Example 1.2.9. *Let us consider a boson of mass $m > 0$. The Hilbert space is a symmetric Fock space. Let $h \in \mathfrak{h}$. The following Van Hove Hamiltonian:*

$$H = H_0 + a(h) + a^*(h)$$

defines a symmetric operator. Moreover, for $\psi \in \mathscr{D}(H_0)$:

$$\begin{array}{rcl} \|(a(h) + a^*(h))\psi\| & \leqslant & \|a(h)\psi\| + \|a^*(h)\psi\|, \\ & \leqslant & 2\|h\| \|(N+1)^{\frac{1}{2}}\psi\|. \end{array}$$

Therefore:

$$\begin{array}{rcl} \|(a(h) + a^*(h))\psi\|^2 & \leqslant & 4\|h\|^2 \langle \psi | (N+1)\psi \rangle, \\ & \leqslant & 4\|h\|^2 \|\psi\| \, \|(N+1)\psi\|. \end{array}$$

Note that for any positive numbers a, b, μ we have:

$$\begin{array}{rcl} (\mu a - \frac{1}{\mu} b)^2 & = & \mu^2 a^2 + \frac{1}{\mu^2} b^2 - 2ab \\ & \geqslant & 0. \end{array}$$

Let then $\mu > 0$

$$\begin{array}{rcl} \|(a(h) + a^*(h))\psi\|^2 & \leqslant & 2\|h\|^2 \left(\frac{2\mu}{\mu} \|\psi\| \, \|(N+1)\psi\| \right), \\ & \leqslant & 2\|h\|^2 \left(\mu^2 \|(N+1)\psi\|^2 + \frac{1}{\mu^2} \|\psi\|^2 \right). \end{array}$$

Let us note that:

$$\begin{array}{rcl} H_0 & = & d\Gamma(\sqrt{k^2 + m^2}) \\ & \geqslant & d\Gamma(\sqrt{m^2}) \\ & \geqslant & m d\Gamma(\mathbb{1}) \\ & \geqslant & mN. \end{array}$$

Finally:

$$\|(a(h) + a^*(h))\psi\|^2 \quad \leqslant \quad 2\|h\|^2 \frac{\mu^2}{m^2} \|H_0\psi\|^2 + 2\|h\|^2 \frac{1}{\mu^2} \|\psi\|^2 .$$

Choosing:

$$\mu < \frac{m^2}{\sqrt{2}\|h\|}$$

the Kato-Rellich theorem can be applied.

From this example, we see that a critical point is to estimate H_I with respect to N or H_0. This operation may be technical and the so-called N_τ estimates, presented in [51], are often useful tools. We quickly recall them here.

Let us first introduce the N_τ operators for $0 \leqslant \tau \leqslant 1$:

$$N_{\tau,i} = \int \omega_i(k_i)^\tau d^*(k_i) d(k_i) dk_i,$$

where

$$\omega_i(k_i) = \sqrt{k_i^2 + m_i^2}$$

and m_i is the mass of the i^{th} particle. Note that:

$$\begin{aligned} N_{0,i} &= N_i, \\ N_{1,i} &= H_{0,i}, \end{aligned}$$

where $H_{0,i}$ stands for the free energy of the i^{th} particle. Let A, B and C be such that:

$$\begin{aligned} \{1, \ldots, n\} &\subset A \cup B \cup C \\ C &\subset \{1, \ldots, p\} \\ A &\subset \{p+1, \ldots, n\} \\ B \cap C = B \cap A &= \varnothing. \end{aligned}$$

The cardinal of A, B and C will be denoted by $|A|, |B|$ and $|C|$. We will moreover use the following notation $a \lesssim b$, for positive numbers a and b, meaning that $a \leqslant Cb$ where C is a positive constant independent of the parameters involved. The following N_τ estimates hold:

Proposition 1.2.10. *Assume that the kernel G of* (1.13) *is in $\mathscr{S}'(\mathbb{R}^n)$. Therefore* (1.13) *is a densely defined bilinear form on $\mathscr{D}_{\mathscr{S}}$ and:*

1. $$\|N_\tau^{-\frac{|C|}{2}} H_I N_\tau^{-\frac{|A|}{2}} (N_\tau + 1)^{-\frac{|B|}{2}}\| \lesssim \left\| \Pi_{i \in A \cup C}\ \omega_i(k_i)^{-\frac{\tau}{2}} G \right\| .$$

2. *If B contains at least one fermion:*
$$\|N_\tau^{-\frac{|C|}{2}} H_I N_\tau^{-\frac{|A|}{2}} (N_\tau + 1)^{-\frac{(|B|-1)}{2}}\| \lesssim \left\| \Pi_{i \in A \cup C}\ \omega_i(k_i)^{-\frac{\tau}{2}} G \right\| .$$

3. *If there are exactly j bosons in the model:*
$$\|H_I (N+1)^{-\frac{j}{2}}\| \leqslant |G|_s,$$

where $|.|_s$ is some Schwartz norm.

To conclude this section, we mention the following useful inequality [29]:

$$\|N^{-\frac{1}{2}} \mathrm{d}\Gamma(B) u\| \leqslant \|\mathrm{d}\Gamma(B^* B)^{\frac{1}{2}} u\|. \tag{1.14}$$

1.2.4 Scattering theory for Quantum fields [29, 67]

We already have seen a version of scattering theory which can be used for one quantum particle systems. Unfortunately, it cannot be applied directly in the context of quantum fields. Starting from the one particle case, some physical and mathematical intuitions to justify the new definitions of the wave operators and of asymptotic completeness are given.

Note that, as before, H labels the full Hamiltonian whereas H_0 stands for the free Hamiltonian. Let us assume that only one species of particle, for example a boson species, is involved. The Hilbert space reduces then to a symmetric Fock space. Let ϕ be our physical initial state. Asymptotically, it has to look like, in some sense, a bound state, u, and a scattered state of asymptotically free bosons. Suppose that the later one is of the form $\prod_{i=1}^{n} a^*(h_i)\Omega$ where the functions h_i are elements of $\mathfrak{h}$ and Ω is the vacuum of the Fock space $\mathscr{F}_s$. Both u and $\prod_{i=1}^{n} a^*(h_i)\Omega$ are elements of the same Hilbert space but will be treated differently. It is then natural to describe the asymptotic behaviour as an element of $\mathscr{H}\otimes\mathscr{H}$: $u \otimes \prod_{i=1}^{n} a^*(h_i)\Omega$. The time evolution of u is given by $U_t = e^{-itH}$ and the time evolution of $\prod_{i=1}^{n} a^*(h_i)\Omega$ is $U_t^0 = e^{-itH_0}$. This suggests to define the following extended Hamiltonian on $\mathscr{H}\otimes\mathscr{H}$:

$$H^{\mathrm{ext}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H_0,$$

so that:

$$e^{-itH^{\mathrm{ext}}} u \otimes \prod_{i=1}^{n} a^*(h_i)\Omega = U_t u \otimes U_t^0 \prod_{i=1}^{n} a^*(h_i)\Omega.$$

Now we have to find a way to compare this state to $U_t\phi$. The main problem is that they are not elements of the same Hilbert space. A mathematical procedure has then to be found to send $\mathscr{H}\otimes\mathscr{H}$ onto $\mathscr{H}$. For a detector, a non scattered particle is a particle which cannot be detected. In other words, the bound wave function u may be seen as an asymptotic vacuum. Therefore, it is tempting to define:

$$I : \left\{ \begin{array}{ccc} \mathscr{H}\otimes\mathscr{H} & \to & \mathscr{H} \\ u \otimes \prod_{i=1}^{n} a^*(h_i)\Omega & \to & \prod_{i=1}^{n} a^*(h_i)u \end{array} \right. .$$

It is called the scattering identification operator. It is defined on $\mathscr{D} = \bigcup_{n=0}^{\infty} \left\{ \mathscr{D}((N+1)^{\frac{n}{2}}) \otimes \bigotimes_{k=1,s}^{n} \mathfrak{h} \right\}$ and it can be extended by linearity. The wave operators become:

$$\begin{aligned} \Omega^{\pm}\left(u \otimes \prod_{i=1}^{n} a^*(h_i)\Omega\right) &= \lim_{t\to\pm\infty} U_t^* I \left(e^{-itH^{\mathrm{ext}}} u \otimes \prod_{i=1}^{n} a^*(h_i)\Omega\right) \\ &= \lim_{t\to\pm\infty} U_t^* I \left(U_t u \otimes U_t^0 \prod_{i=1}^{n} a^*(h_i)\Omega\right) \\ &= \lim_{t\to\pm\infty} U_t^* I \left(U_t u \otimes \prod_{i=1}^{n} \left(U_t^0 a^*(h_i) U_t^{0*}\right)\Omega\right) \\ &= \lim_{t\to\pm\infty} U_t^* \prod_{i=1}^{n} \left(U_t^0 a^*(h_i) U_t^{0*}\right) U_t u \\ &= \lim_{t\to\pm\infty} \prod_{i=1}^{n} \left(U_t^* U_t^0 a^*(h_i) U_t^{0*} U_t\right) u. \end{aligned}$$

Consequently, a natural way to define the wave operators is:

$$\Omega^{\pm} : \left\{ \begin{array}{ccc} \mathscr{K}^{\pm} \otimes \mathscr{H} & \to & \mathscr{H} \\ u \otimes \prod\limits_{i=1}^{n} a^*(h_i)\Omega & \to & \prod\limits_{i=1}^{n} a^{\pm *}(h_i)u \end{array} \right. ,$$

where $a^{\pm\sharp}$ are the so-called asymptotic creation and annihilation operators formally defined by:

$$\begin{aligned} a^{\pm *}(h) &= \lim_{t\to\pm\infty} U_t^* U_t^0 a^*(h) U_t^{0*} U_t \\ a^{\pm}(h) &= \lim_{t\to\pm\infty} U_t^* U_t^0 a(h) U_t^{0*} U_t \end{aligned}$$

and $\mathscr{K}^{\pm}$ is the space of asymptotic vacua defined by :

$$\mathscr{K}^{\pm} = \left\{ u \in \mathscr{F}_s | \forall h \in \mathfrak{h}, a^{\pm}(h)u = 0 \right\}.$$

The precise definition can be found for example in [29] and is recalled in Section 2.4.1.

Non scattered states are expected to be gathered in the subspace spanned by the eigenvalues of H:

$$\mathscr{K}^{\pm} = \mathscr{H}_{pp}(H).$$

The wave operators still have to be isometric. They moreover are expected to be onto and hence unitary, as bound states are taken into account. This implies the unitarity of the scattering operator. These two conditions form the asymptotic completeness.

Let us give some precisions on these new objects, defining them in the context of many particle species models.

To define the scattering identification operator, it is necessary to introduce first new tools about Fock spaces. The domains will not be specified here and some cares will be required when they will be applied. From now on, $\mathfrak{h}_i$ will be a Hilbert space.

Definition 1.2.11. *Let B be an operator on $\mathfrak{h}$. We define $\Gamma(B)$, an operator on a (symmetric or antisymetric) Fock space $\mathfrak{F}_\sharp(\mathfrak{h})$ by:*

$$\begin{aligned} \Gamma(B) : \mathfrak{F}_\sharp(\mathfrak{h}) &\to \mathfrak{F}_\sharp(\mathfrak{h}) \\ \Gamma(B)_{|\overset{n}{\otimes}_\sharp \mathfrak{h}} &= \underbrace{B \otimes \cdots \otimes B}_{n} \\ \Gamma(B)\Omega &= \Omega. \end{aligned}$$

The previous definition may be extended to $\mathscr{H} = \mathfrak{F}_\sharp(\mathfrak{h}_1) \otimes \cdots \otimes \mathfrak{F}_\sharp(\mathfrak{h}_p)$. Where p is the number of particles, and hence, of Fock spaces.

Definition 1.2.12. *Let $B = (B_1, B_2, \ldots, B_p)$ be operators acting on the Hilbert spaces $(\mathfrak{h}_1, \ldots, \mathfrak{h}_p)$. We define $\overline{\Gamma}(B)$ on $\mathscr{H} = \mathfrak{F}_\sharp(\mathfrak{h}_1) \otimes \cdots \otimes \mathfrak{F}_\sharp(\mathfrak{h}_p)$ by:*

$$\begin{aligned} \overline{\Gamma}(B) : \mathscr{H} &\to \mathscr{H} \\ \overline{\Gamma}(B) &= \Gamma(B_1) \otimes \cdots \otimes \Gamma(B_p). \end{aligned}$$

$\overline{\Gamma}$ *may be noted* Γ.

Let:

$$\mathscr{H}(\mathfrak{h}_1,\ldots,\mathfrak{h}_n)=\bigotimes_{k=1}^{n}\mathscr{F}_\sharp(\mathfrak{h}_k).$$

The splitting operator U is supposed to send $\mathscr{H}(\mathfrak{h}_1\oplus\mathfrak{h}_{n+1},\ldots,\mathfrak{h}_n\oplus\mathfrak{h}_{2n})$ onto $\mathscr{H}(\mathfrak{h}_1,\ldots,\mathfrak{h}_n)\otimes\mathscr{H}(\mathfrak{h}_{n+1},\ldots,\mathfrak{h}_{2n})$. Let us first define the operator U_1 for the first Fock space. If it is a symmetric one:

$$\begin{aligned}
U_1:\mathscr{F}_s(\mathfrak{h}_1\oplus\mathfrak{h}_{n+1})\otimes\mathscr{F}_\sharp(\mathfrak{h}_2)\otimes\cdots\otimes\mathscr{F}_\sharp(\mathfrak{h}_n) &\rightarrow \mathscr{F}_s(\mathfrak{h}_1)\otimes\mathscr{F}_\sharp(\mathfrak{h}_2)\otimes\cdots\otimes\mathscr{F}_\sharp(\mathfrak{h}_n)\otimes\mathscr{F}_s(\mathfrak{h}_{n+1})\\
U_1\Omega\otimes\Omega\otimes\cdots\otimes\Omega &= \Omega\otimes\Omega\otimes\cdots\otimes\Omega\otimes\Omega\\
U_1\left(a^\sharp(h_1+h_{n+1})\otimes\psi\right) &= \left(a^\sharp(h_1)\otimes\psi\otimes\mathbb{1}+\mathbb{1}\otimes\psi\otimes a^\sharp(h_{n+1})\right)U_{s_1},
\end{aligned}$$

where ψ is a $\Pi_i d_i^\sharp(h_i)$-type operator acting on $\mathscr{F}_\sharp(\mathfrak{h}_2)\otimes\cdots\otimes\mathscr{F}_\sharp(\mathfrak{h}_n)$. In the fermionic case, [5] shows that it is possible to define two different splitting operators. They are, however, equal up to the left multiplication of an operator of the type: $(-\mathbb{1})^B$ where B is a finite product of N-type operators so, in practice, only one of them will be used. We define the left and right splitting operator for the first fermionic Fock space, labelled by l and r, as follow:

$$\begin{aligned}
U_{1,l/r}:\mathscr{F}_a(\mathfrak{h}_1\oplus\mathfrak{h}_{n+1})\otimes\mathscr{F}_\sharp(\mathfrak{h}_2)\otimes\cdots\otimes\mathscr{F}_\sharp(\mathfrak{h}_n) &\rightarrow \mathscr{F}_a(\mathfrak{h}_1)\otimes\mathscr{F}_\sharp(\mathfrak{h}_2)\otimes\cdots\otimes\mathscr{F}_\sharp(\mathfrak{h}_n)\otimes\mathscr{F}_s(\mathfrak{h}_{n+1})\\
U_{1,l/r}\Omega\otimes\Omega\otimes\cdots\otimes\Omega &= \Omega\otimes\Omega\otimes\cdots\otimes\Omega\otimes\Omega\\
U_{1,l}\left(b^\sharp(h_1+h_{n+1})\otimes\psi_1\right) &= \left(b^\sharp(h_1)\otimes\psi_1\otimes\mathbb{1}+(-\mathbb{1})^N\otimes\psi_1\otimes b^\sharp(h_{n+1})\right)U_{1,l}\\
U_{1,r}\left(b^\sharp(h_1+h_{n+1})\otimes\psi_1\right) &= \left(b^\sharp(h_1)\otimes\psi_1\otimes(-\mathbb{1})^N+\mathbb{1}\otimes\psi_1\otimes b^\sharp(h_{n+1})\right)U_{1,r}.
\end{aligned}$$

All of these operators extend to unitary operators. Going on with this procedure, a family of splitting operators $\mathfrak{U}=(U_1,\ U_2,\ U_3,\ldots,U_n)$ can be defined for the entire Hilbert space.

Proposition 1.2.13. *Let $U_i\in\mathfrak{U}$. Then, U_i is an unitary operator from $\mathscr{H}(\mathfrak{h}_1,\ldots,\mathfrak{h}_i\oplus\mathfrak{h}_{i+n},\ldots,\mathfrak{h}_n)\otimes\mathscr{F}_\sharp(h_{n+1})\otimes\cdots\otimes\mathscr{F}_\sharp(h_{i+n-1})$ to $\mathscr{H}(\mathfrak{h}_1,\ldots,\mathfrak{h}_i,\ldots,\mathfrak{h}_n)\otimes\mathscr{F}_\sharp(h_{n+1})\otimes\cdots\otimes\mathscr{F}_\sharp(h_{i+n-1})\otimes\mathscr{F}_\sharp(h_{i+n})$ and:*

$$U_i\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_1)\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{i-1})}\otimes d\Gamma\left(\begin{bmatrix} b_1 & 0\\ 0 & b_2\end{bmatrix}\right)\otimes\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_{i+1})\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{i+n-1})}$$

$$=\left(\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_1)\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{i-1})}\otimes d\Gamma(b_1)\otimes\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_{i+1})\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{i+n-1})}+\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_1)\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{i+n-1})}\otimes d\Gamma(b_2)\right)U_i$$

for any b_1 acting on $\mathfrak{h}_i$ and b_2 acting on $\mathfrak{h}_{i+n}$. If U_i is a fermionic splitting operator, we introduce the following operators:

$$N_{i,1}=\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_1)\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{i-1})}\otimes N\otimes\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_{i+1})\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{n+i-1})}\otimes\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_{n+i})}$$

$$N_{i,2}=\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_1)\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{i-1})}\otimes\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_i)}\otimes\mathbb{1}_{\mathfrak{F}_\sharp(\mathfrak{h}_{i+1})\otimes\cdots\otimes\mathfrak{F}_\sharp(\mathfrak{h}_{2n})}\otimes N,$$

and we have:

$$U_{i,r}=(-1)^{N_{i,1}N_{i,2}}U_{i,l}.$$

Finally, two general splitting operators can be defined:

$$\begin{aligned} U_{L,R} : \mathscr{H}(\mathfrak{h}_1 \oplus \mathfrak{h}_{n+1}, \ldots, \mathfrak{h}_n \oplus \mathfrak{h}_{2n}) &\rightarrow \mathscr{H}(\mathfrak{h}_1, \ldots, \mathfrak{h}_n) \otimes \mathscr{H}(\mathfrak{h}_{n+1}, \ldots, \mathfrak{h}_{2n}) \\ U_L &= U_{n,l} \ldots U_{1,l} \\ U_R &= U_{n,r} \ldots U_{1,r}. \end{aligned}$$

Let $j_{i,0}, j_{i,\infty}$ be operators on $\mathfrak{h}_i$. We define the operator j_i as:

$$\begin{aligned} j_i \;:\; \mathfrak{h}_i &\rightarrow \mathfrak{h}_i \oplus \mathfrak{h}_i \\ h &\rightarrow (j_{i,0}h, j_{i,\infty}h), \end{aligned}$$

we then have:

$$\begin{aligned} j_i^* \;:\; \mathfrak{h}_i \oplus \mathfrak{h}_i &\rightarrow \mathfrak{h}_i \\ (h_0, h_\infty) &\rightarrow j_{i,0}^* h_0 + j_{i,\infty}^* h_\infty. \end{aligned}$$

Considering $J = (j_1, \ldots, j_n)$ a family of such maps, we define the following operator:

$$\Gamma(J) \;:\; \mathscr{H}(\mathfrak{h}_1, \ldots, \mathfrak{h}_n) \rightarrow \mathscr{H}(\mathfrak{h}_1 \oplus \mathfrak{h}_1, \ldots, \mathfrak{h}_n \oplus \mathfrak{h}_n).$$

Using now the left splitting operator we define:

$$\begin{aligned} \check{\Gamma}(J) \;:\; \mathscr{H}(\mathfrak{h}_1, \ldots, \mathfrak{h}_n) &\rightarrow \mathscr{H}(\mathfrak{h}_1, \ldots, \mathfrak{h}_n) \otimes \mathscr{H}(\mathfrak{h}_1, \ldots, \mathfrak{h}_n) \\ \check{\Gamma}(J) &= U_L \Gamma(J). \end{aligned}$$

Now, let

$$\begin{aligned} i_j \;:\; \mathfrak{h}_j \oplus \mathfrak{h}_j &\rightarrow \mathfrak{h}_j \\ (h_0, h_\infty) &\rightarrow h_0 + h_\infty. \end{aligned}$$

Considering $i = (i_1, \ldots, i_n)$ a family of such map, we define the scattering identification operator by:

$$I = \Gamma(i) U^* = \check{\Gamma}(i^*)^*.$$

The asymptotic creation and annihilation operators are defined in the same way for bosons and fermions. The asymptotic vacua and wave operators are then easy to generalised:

$$\mathscr{K}^\pm = \{f \in \mathscr{H} | \forall i \in \{1, \ldots, n\}, \forall h_i \in \mathfrak{h}_i, d(h_i) f = 0\},$$

where $d(h_i)$ stands for the annihilation operator, either for fermions or bosons, acting on $\mathfrak{F}_\sharp(\mathfrak{h}_i)$.

$$\Omega^\pm = \begin{cases} \mathscr{K}^\pm \otimes \mathscr{H} & \rightarrow \quad \mathscr{H} \\ u \otimes \prod_{i=1}^n d^*(h_i)\Omega & \rightarrow \quad \prod_{i=1}^n d^{\pm *}(h_i) u \end{cases},$$

which is extended by linearity. The main goals of scattering theory is first to prove the existence of the asymptotic creation and annihilation operators which implies the existence of the wave operators. Next, the latter must be shown to be unitary. Finally $\mathscr{K}^\pm$ has to be equal to $\mathscr{H}_{pp}(H)$ for asymptotic completeness to hold.

1.3 An introduction to Mourre's theory

As shown before, the characterisation of the spectrum of a self-adjoint operator is an important task. Conjugate operator methods will be presented in this section. They will be used later to prove the absence of singular continuous spectrum and to establish propagation estimates which are key arguments to show asymptotic completeness. Let us first recall the following well known result (see for example [7]):

Proposition 1.3.1 (Putnam's Theorem)**.** *Let A and H be bounded self-adjoint operators. Let C be a bounded operator and assume:*

$$[H, iA] \geqslant CC^*.$$

Then, for all $\lambda \in \mathbb{R}$, $\epsilon \neq 0$ and $f \in \mathscr{H}$:

$$\left| \mathrm{Im} \left\langle Cf \left| (H - \lambda - i\epsilon)^{-1} Cf \right.\right\rangle \right| \leqslant 4\|A\| \|f\|^2.$$

Moreover, $\mathrm{Ran}(C) \subset \mathscr{H}_{ac}(H)$ *and if, in addition,* $\mathrm{Ker}(C^*) = \{0\}$*, then the spectrum of H is purely absolutely continuous.*

An interesting point to highlight is that if λ is in the spectrum of the self-adjoint operator H, the resolvent $(H - \lambda)^{-1}$ is ill-defined as an operator. The Putnam's theorem gives sense to $\mathrm{Im} \langle f \left| (H - \lambda \pm i0^+)^{-1} f \right.\rangle$. Mourre's theory may be seen as a generalisation of this result that can be applied for unbounded operators. We will first present the original version of Mourre's theory with self-adjoint conjugate operators [77, 8]. This last assumption may be relaxed first to maximal symmetric operators and finally to the so-called singular Mourre's theory [47].

1.3.1 Mourre's theory with self-adjoint conjugate operator

Let H and A be two unbounded self-adjoint operators on a Hilbert space $\mathscr{H}$. The main idea is to find an inequality to control $[H, A]$ as in the Putnam's theorem. Next, some regularity properties on H with respect to A have to be assumed in order to obtain information on the spectrum.

The first difficulty is to define precisely the commutator $[H, A]$. If $\mathscr{D}(A) \cap \mathscr{D}(H)$ is dense, $i[H, A]$ denotes the symmetric form defined on $\mathscr{D}(A) \cap \mathscr{D}(H)$ as:

$$\forall \psi, \phi \in \mathscr{D}(A) \cap \mathscr{D}(H), \langle \psi \left| i[H, A] \phi \right.\rangle = i \left(\langle H\psi \left| A\phi \right.\rangle - \langle A\psi \left| H\phi \right.\rangle \right).$$

Furthermore, if it is closed and bounded below, the quadratic form is associated to a unique self-adjoint operator which will be denoted in the same way: $i[H, A]$.

Next, the regularity conditions with respect to the conjugate operator A are defined as follow:

Definition 1.3.2. *H is said to be $C^n(A)$ if for any $\phi \in \mathscr{H}$, there exists $z \in \rho(H)$ such that the map*

$$s \to e^{-isA}(H - z)^{-1} e^{isA} \phi$$

is $C^n(\mathbb{R})$.

We moreover introduce the Mourre inequalities:

Definition 1.3.3. *Let c_0 be a positive constant and K a compact operator. The spectral measure of an operator H applied on $J \subset \mathbb{R}$ will be denoted $\mathbb{1}_J(H)$. We define the following Mourre inequalities on a segment $[a, b]$:*

1. ***Strict Mourre inequality:***

$$\mathbb{1}_{[a,b]}(H)[H,iA]\mathbb{1}_{[a,b]}(H) \geqslant c_0\mathbb{1}_{[a,b]}(H).$$

2. ***Mourre inequality:***

$$\mathbb{1}_{[a,b]}(H)[H,iA]\mathbb{1}_{[a,b]}(H) \geqslant c_0\mathbb{1}_{[a,b]}(H) + K.$$

The main elements of Mourre's theory have been introduced and we present now the so-called virial theorem.

Theorem 1.4 (Virial theorem)**.** *Assume that H is $C^1(A)$ and that a strict Mourre estimate holds on a segment I. We then have:*

$$\sigma_{pp}(H) \cap I = \varnothing.$$

Another virial theorem holds with a Mourre estimate.

Theorem 1.5 (Virial theorem)**.** *Assume that H is $C^1(A)$ and that a Mourre estimate holds on a segment I, then H has only a finite number of eigenvalues with finite multiplicity in I.*

More information on the spectrum can be obtained if more regularity is assumed.

Theorem 1.6 (Limiting absorption principle)**.** *Let us assume that:*

1. *H is $C^2(A)$,*

2. *H verify a Mourre estimate on $I \subset \mathbb{R}$,*

then $\sigma_{sc}(H) \cap I = \varnothing$ and for any closed interval $J \subset I\backslash\sigma_{pp}(H)$ and $s > \frac{1}{2}$

$$\sup_{z\in\mathscr{J}^\pm} \left\| \langle A\rangle^{-s} (H-z)^{-1} \langle A\rangle^{-s} \right\| < \infty, \tag{1.15}$$

where:

$$\mathscr{J}^\pm = \{z \in \mathbb{C} | Re(z) \in J, \pm Im(z) > 0\}.$$

The inequality (1.15) *is called limiting absorption principle. With a strict Mourre estimate,* (1.15) *holds for all closed interval $J \subset I$ and in particular:*

$$\sigma(H) \cap I = \sigma_{ac}(H) \cap I.$$

1.3.2 Mourre's theory with non-self-adjoint operators

Mourre's theory can be extended in the context of non-self-adjoint conjugate operators. We summarized here the main concepts and ideas behind.

Definition 1.3.4 (C_0-semigroup)**.** *A C_0-semigroup is a map $t \to W_t$ from $\mathbb{R}^+$ to $\mathscr{B}(\mathscr{H})$ such that:*

1. *$W_0 = \mathbb{1}$,*

2. *$\forall s,t \geqslant 0, W_tW_s = W_{t+s}$,*

3. *$s - \lim_{t\to 0^+} W_t = \mathbb{1}$.*

From any C_0-semigroup a generator A, which will be used as a non-self-adjoint conjugate operator in an extended Mourre's theory, can be defined. Its domain is:

$$\mathscr{D}(A) = \left\{ f \in \mathscr{H} | \lim_{t \to 0^+} (it)^{-1}(W_t f - f) \text{ exists} \right\}$$

or equivalently:

$$\mathscr{D}(A) = \{ f \in \mathscr{H} | \exists C \in \mathbb{R}, \forall t \in [0,1], \|W_t f - f\| \leqslant Ct \} .$$

The operator A is then defined on $\mathscr{D}(A)$ as:

$$\forall f \in \mathscr{D}(A), Af = \lim_{t \to 0^+} (it)^{-1}(W_t f - f).$$

This generator can be shown to be closed and densely defined. We will present a simplified version of Mourre's theory and we refer to [47] for more details.

Let $\mathscr{D}(|H|^{\frac{1}{2}})$ be equipped with the norm $\|u\|^2_{\mathscr{G}(H)} = \langle u|(|H|+1)u\rangle$. Another norm which is used in the literature is $\|u\|^2_{\mathscr{G}(H)} = \langle u| \langle H \rangle u\rangle$ where $\langle H \rangle = (|H|^2+1)^{\frac{1}{2}}$. It is not hard to see that these two norms are equivalent. The completion of $(\mathscr{D}(|H|^{\frac{1}{2}}), \|.\|_{\mathscr{G}(H)})$ is noted $\mathscr{G}(H)$ and it is not hard to see that $\mathscr{G}(H) \subset \mathscr{H}$. The dual space of $\mathscr{G}(H)$ will be noted $\mathscr{G}^*(H)$. We moreover have that $\mathscr{G}(H) \subset \mathscr{H} \subset \mathscr{G}^*(H)$ and if $x \in \mathscr{H}$, then $\|x\|_{\mathscr{G}^*(H)} = \|(|H|+1)^{-\frac{1}{2}} x\|$ (or equivalently $\|x\|_{\mathscr{G}^*(H)} = \| \langle H \rangle^{-\frac{1}{2}} x\|$).

Clearly $H \in \mathscr{L}(\mathscr{G}(H), \mathscr{G}^*(H))$. Let us assume that:

$$\begin{aligned} W_t\mathscr{G}(H) &\subseteq \mathscr{G}(H), && (1.16)\\ W_t^*\mathscr{G}(H) &\subseteq \mathscr{G}(H), && (1.17)\\ \sup_{0<t<1} \|W_t f\|_{\mathscr{G}(H)} &< \infty && (1.18)\\ \sup_{0<t<1} \|W_t^* f\|_{\mathscr{G}(H)} &< \infty && (1.19) \end{aligned}$$

then the restriction of $\{W_t\}$ and $\{W_t^*\}$ on $\mathscr{G}(H)$ are C_0-semigroups. Moreover, $\{W_t\}$ and $\{W_t^*\}$ extend to a C_0-semigroups on $\mathscr{G}^*(H)$. All of them will be denoted $\{W_t\}$ and $\{W_t^*\}$.

Definition 1.3.5. *Let $\{W_t\}$ be a C_0-semigroup which verifies* (1.16)*,* (1.17)*,* (1.18) *and* (1.19)*. Let A be its generator. H is said to be $C^1(A, \mathscr{G}(H), \mathscr{G}^*(H))$ if there exists $c > 0$ such that for any $t \in [0,1]$:*

$$\|W_t H - H W_t\|_{\mathscr{L}(\mathscr{G}(H), \mathscr{G}^*(H))} \leqslant ct. \tag{1.20}$$

In this case there exists $H' \in \mathscr{L}(\mathscr{G}(H), \mathscr{G}^*(H))$ such that:

$$\forall \psi, \phi \in \mathscr{D}(H), \lim_{t \to 0^+} \frac{1}{t} \left(\langle \psi | W_t H \phi \rangle - \langle H\psi | W_t \phi \rangle \right) = \langle \psi | H' \phi \rangle .$$

H' identifies with the quadratic form $[H, iA]$. Moreover, if $H' \in C^1(A, \mathscr{G}(H), \mathscr{G}^*(H))$, H is said to be $C^2(A, \mathscr{G}(H), \mathscr{G}^*(H))$. Let us note that $H \in C^1(A, \mathscr{G}(H), \mathscr{G}^*(H))$ implies that if f is an eigenvector of H, $\langle f|H' f\rangle = 0$.

Finally, H verifies a strict Mourre estimate with respect to A on an open interval J if there exist $a, b > 0$ such that:

$$H' \geqslant [a\mathbb{1}_J(H) - b\mathbb{1}_{\mathbb{R}\setminus J}(H)] \langle H \rangle .$$

Together with the assumption that $H \in C^1(A, \mathscr{G}(H), \mathscr{G}^*(H))$, it implies the virial theorem which states that on any compact set $I \subset J$, $\sigma_{pp}(H) \cap I = \varnothing$. Assuming that $H \in C^2(A, \mathscr{G}(H), \mathscr{G}^*(H))$, a limiting absorption principle holds.

The previous framework requires that the commutator $[H, iA]$ belongs to $\mathscr{L}(\mathscr{G}(H), \mathscr{G}^*(H))$ which might be not true in application. To overcome this difficulty we present briefly the so-called singular Mourre's theory [47]. Let H, M be two self-adjoint operators and R a symmetric operator on $\mathscr{H}$. Let us assume that $H \in C^1(M)$, $M \geqslant 0$, $[H, M]$ is relatively H-bounded and $\mathscr{D}(H) \subset \mathscr{D}(R)$. Let $\mathscr{G}(H) = \mathscr{D}(|H|^{\frac{1}{2}}) \cap \mathscr{D}(M^{\frac{1}{2}})$ be equipped with $\|f\|^2_{\mathscr{G}(H)} = \langle f | (|H| + M + 1) f \rangle$. We moreover define $\mathscr{G}^*(H)$ as the dual space of $\mathscr{G}(H)$. If $f \in \mathscr{H}$, then $\|f\|_{\mathscr{G}^*(H)} = \|(|H| + M + 1)^{-\frac{1}{2}} f\|$. Therefore H and M can be seen as elements of $\mathscr{L}(\mathscr{G}(H), \mathscr{G}^*(H))$. As before, let us assume that $\{W_t\}$ and $\{W_t^*\}$ are C_0-semigroups verifying (1.16), (1.17), (1.18) and (1.19) so that they can be extended to $\mathscr{G}^*(H)$ and restricted to $\mathscr{G}(H)$ as C_0-semigroups. If A is the generator of W_t we may define in the same way as before a notion of regularity with respect to A. H is said to be in $C^1(A, \mathscr{G}(H), \mathscr{G}^*(H))$ if (1.20) holds and it implies that there exits $H' \in \mathscr{L}(\mathscr{G}(H), \mathscr{G}^*(H))$ which identifies to $[H, iA]$ and that is assumed to verify $H' = M + R$. In this case, a virial theorem and a limiting absorption principle can be obtained under the same conditions as before.

1.4 Propagation Estimates

The methodology used in the next two chapters is strongly inspired by previous work done in [28], [29] and [5] for example. We will start studying the spectrum of the model using different versions of Mourre's theory. These results give then important tools to prove the so-called propagation estimates which are fundamental results in our approach. We introduce them briefly in this section.

Scattering theory requires to understand the asymptotic behaviour of physical systems. The very first idea that might come in mind is that particles cannot go faster than the speed of light. Therefore, some regions of space and time are forbidden to them. More precisely if R is big enough and as time goes to infinity, it is expected that no particle can be found in a region such that $Rt \leqslant |x| \leqslant R't$ where $R' > R$ and x is the position of the particle. A natural operator to consider is then $\mathbb{1}_{[R,R']}\left(\frac{|x|}{t}\right)$. Moreover, the number of particles is undetermined in a quantum field theory so we may focus on $\mathrm{d}\Gamma\left(\mathbb{1}_{[R,R']}\left(\frac{|x|}{t}\right)\right)$. Finally, the Heisenberg picture might be a better point of view to study the asymptotic behaviour of a chosen state. Let $\chi \in C_0^\infty(\mathbb{R})$, the energy of an initial state u may be controlled with $\chi(H)$. A natural requirement would then be:

$$\forall u \in \mathscr{H}, \forall R' > R > 1, \lim_{t \to \infty} \left\langle \chi(H) u \middle| e^{itH} \mathrm{d}\Gamma\left(\mathbb{1}_{[R,R']}\left(\frac{|x|}{t}\right)\right) e^{-itH} \chi(H) u \right\rangle = 0.$$

Intuitively, this requirement means that the number of particles located in a region such that $Rt \leqslant |x| \leqslant R't$ is vanishingly small. However, this statement is too strong in general. The propagation estimate that is usually found in the literature states that there exists $C > 0$ such that:

$$\int_1^\infty \left\langle \chi(H) u \middle| e^{itH} \mathrm{d}\Gamma\left(\mathbb{1}_{[R,R']}\left(\frac{|x|}{t}\right)\right) e^{-itH} \chi(H) u \right\rangle \frac{dt}{t} \quad \leqslant \quad C\|u\|^2$$

or equivalently

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(\mathbb{1}_{[R,R']}\left(\frac{|x|}{t}\right)\right)^{\frac{1}{2}} e^{-itH}\chi(H)u \right\|^2 \frac{dt}{t} \quad \leqslant \quad C\|u\|^2.$$

Establishing a maximal propagation velocity estimate leads to another natural question: is there any minimal propagation velocity? Assuming that u is a bound state, it is not clear that such an estimate can be established because it is supposed to remain localised. For scattered particles, however, such a velocity is expected and, as time goes to infinity, scattered states are supposed to escape from any balls. Therefore, assuming $\chi \in C_0^\infty(\mathbb{R})$ supported in $\mathbb{R}\backslash(\sigma(H)\backslash\sigma_{ac}(H))$ and with the same type of ideas developed for the previous propagation estimate, there should exist some $\epsilon > 0$ and $C > 0$ such that for any $u \in \mathscr{H}$

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(\mathbb{1}_{[0,\epsilon]}\left(\frac{|x|}{t}\right)\right)^{\frac{1}{2}} e^{-itH}\chi(H)u \right\|^2 \frac{dt}{t} \leqslant C\|u\|^2.$$

This expression translate correctly the fact that a particle related to the absolutely continuous spectrum behave asymptotically like a scattered particle. However, such an expression is difficult to obtain and a weaker property is usually proved:

$$\int_1^\infty \left\| \Gamma\left(\mathbb{1}_{[0,\epsilon]}\left(\frac{|x|}{t}\right)\right) e^{-itH}\chi(H)u \right\|^2 \frac{dt}{t} \leqslant C\|u\|^2.$$

This property is enough to prove asymptotic completeness even if its consequences are weaker as only one particle may be deduced to escape at infinity. In particular, it is hard to recover the description of scattered particles as vectors related to the absolutely continuous spectrum from this new propagation estimates.

So far, we have seen that the asymptotic velocity as the limit as t goes to infinity of $\frac{|x|}{t}$ (at least for one particle) and it might seem natural to see $\frac{|x|}{t}$ as an average velocity. Let us describe another point of view that might be adopted. The free energy of a particle E verify the following relation:

$$E = \sqrt{p^2 + m^2}$$

where p is the momentum and m the mass of the particle. Therefore:

$$\frac{\partial E}{\partial p} = \frac{p}{E}.$$

Together with the fact that:

$$\begin{aligned} p &= \gamma m_0 v, \\ E &= \gamma m_0, \\ \gamma &= \frac{1}{\sqrt{1-v^2}}, \end{aligned}$$

v being the velocity of the particle, we then have:

$$\frac{\partial E}{\partial p} = v.$$

As a conclusion, $\nabla\omega(k)$ can be seen as an instantaneous velocity of the particle. It is moreover expected that the instantaneous and the average velocities converge as time goes to infinity. Let $0 < c_0 < c_1$, we will focus on particles such that $c_0 t \leqslant |x| \leqslant c_1 t$. We then expect that

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(\left| \mathbb{1}_{[c_0,c_1]}\left(\frac{|x|}{t}\right)\left(\frac{x_i}{t} - \partial_i\omega(k)\right) + h.c \right| \right)^{\frac{1}{2}} e^{-itH}\chi(H)u \right\|^2 \frac{dt}{t} \leqslant C\|u\|^2.$$

Another weaker version may be found in the literature:

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(\left\langle \left(\frac{x}{t} - \nabla\omega\right) \left| \mathbb{1}_{[\mathrm{v}_0,\mathrm{v}_1]}(\frac{x}{t}) \left(\frac{x}{t} - \nabla\omega\right) \right\rangle \right)^{\frac{1}{2}} e^{-itH}\chi(H)u \right\|^2 \frac{dt}{t} \leqslant C\|u\|^2.$$

These different results provide a good description of the asymptotic behaviour of physical systems and are also essential tools to prove the existence of an inverse operator to the wave operator. This last object allows us to show that the propagation of a state associated to the absolutely continuous spectrum does not reduce to the asymptotic vacuum. The asymptotic completeness will follow from this fact (see Section 2.4 and 3.5).

1.5 A brief overview of the literature

Scattering theory is an important area of mathematics. The main ideas may be found in [82, 28, 95, 96] where a complete introduction can be found. We refer to [7] for a less complete but more pedagogical introduction. [28] explores in details the case of scattering theory of classical and quantum N-particle systems. We also refer to [26, 53, 88]. Several concepts of the Scattering theory for Quantum Field models have been developped in [63, 64, 65] and [40, 41]. First results regarding asymptotic completeness for systems with matter coupled to non-relativistic radiation have been obtained in [9, 67, 89]. The strategy and technics used in the case of N-particles quantum mechanics can be generalised in the context of quantum field theory. For example, [29], respectively [5], prove asymptotic completeness for massive Pauli-Fierz Hamiltonians including one boson family, respectively one fermion family. See also [43, 42] for Pauli-Fierz Hamiltonians with an infrared cut-off. Spatially cut-off $P(\phi)_2$ Hamiltonians have been treated in [30] and in particular an elegant argument is used to prove the unitarity of the wave operators. Moreover, abstract Quantum Field Hamiltonians have been studied in [50]. Many difficulties may occur such as infrared or ultraviolet divergences which have been studied for Van Hove Hamiltonians [27], Pauli-Fierz models [31] or scalar electrons and massless scalar bosons [40]. Asymptotic completeness is especially hard to prove for massless models. We refer to [49] for a very detailed discussion about the massless Nelson models, [33] for massless spin-boson models and [38] for general massless Pauli-Fierz models, including the standard model of non-relativistic Quantum elctrodynamics. We highlight also the work of [21] on maximal velocity of photons in non-relativistic quantum electrodynamic and [39] for minimal velocity estimates. Finally, translation invariant models present further difficulties to overcome and we refer to [40, 41] for the first study of one-electron coupled to the photon field (compton Scattering), [45, 80] for more recent results and [75, 76, 36] where a translation-invariant Nelson model is treated.

The spectral theory of various models arising from the Fermi weak interaction has been studied in [10, 16, 17, 18, 19, 58] (see also [15, 90] for models related to Quantum electrodynamics). In particular, the existence and uniqueness of a ground state, as well as the absolute continuity of the essential spectrum in intervals above the ground state energy, have been obtained under different reasonable conditions, both for massive and massless fields. However, to our knowledge, the scattering theory for such models has never been studied before. This is one of the main purpose of this PhD thesis.

Chapter 2

Scattering Theory for massive models of the $W^\pm$ decay

2.1 Introduction

As said in the introduction, this chapter is published in [2] and we present a detailed version of it. We are interested in the weak interaction between the vector bosons $W^\pm$ and the full family of leptons. In what follows, the mass of a particle p will be denoted by m_p. It is equal to the mass of the corresponding antiparticle. Physically, the following inequalities hold:

$$m_{\nu_e} < m_{\nu_\mu} < m_\tau < m_e < m_\mu < m_\tau < m_W.$$

In this chapter neutrinos will be assumed to be massive as recent experiments have provided evidences for nonzero neutrino masses (see, e.g., [78] and references therein). However, the standard model with massless neutrinos remains widely studied in theoretical physics and it is conceptually interesting to consider models where neutrinos are supposed to be massless, which will be done in the next chapter. The interaction term is given, in the Lagrangian formalism and for each lepton channel l, by (see, e.g., [54, 55] and references therein)

$$I = g \int \left[\Psi_l(x)^\dagger \gamma^0 \gamma^\alpha (1-\gamma_5) \Psi_{\nu_l}(x) W_\alpha(x) + \Psi_{\nu_l}(x)^\dagger \gamma^0 \gamma^\alpha (1-\gamma_5) \Psi_l(x) W_\alpha(x)^*\right] d^3x, \tag{2.1}$$

with

$$\Psi_l(x) = (2\pi)^{-\frac{3}{2}} \sum_{s_1=\pm\frac{1}{2}} \int \left[\frac{u(p_1,s_1)e^{\mathrm{i}p_1\cdot x}}{(2(|p_1|^2+m_l^2)^{\frac{1}{2}})^{\frac{1}{2}}} b_{l,+}(p_1,s_1) + \frac{v(p_1,s_1)e^{-\mathrm{i}p_1\cdot x}}{(2(|p_1|^2+m_l^2)^{\frac{1}{2}})^{\frac{1}{2}}} b_{l,-}^*(p_1,s_1)\right] d^3p_1, \tag{2.2}$$

$$\Psi_{\nu_l}(x) = (2\pi)^{-\frac{3}{2}} \sum_{s_2=\pm\frac{1}{2}} \int \left[\frac{u(p_2,s_2)e^{\mathrm{i}p_2\cdot x}}{(2(|p_2|^2+m_{\nu_l}^2)^{\frac{1}{2}})^{\frac{1}{2}}} c_{l,+}(p_2,s_2) + \frac{v(p_2,s_2)e^{-\mathrm{i}p_2\cdot x}}{(2(|p_2|^2+m_{\nu_l}^2)^{\frac{1}{2}})^{\frac{1}{2}}} c_{l,-}^*(p_2,s_2)\right] d^3p_2, \tag{2.3}$$

$$W_\alpha(x) = (2\pi)^{-\frac{3}{2}} \sum_{\lambda=-1,0,1} \int \left[\frac{\epsilon_\alpha(p_3,\lambda)e^{\mathrm{i}p_3\cdot x}}{(2(|p_3|^2+m_W^2)^{\frac{1}{2}})^{\frac{1}{2}}} a_+(p_3,\lambda) + \frac{\epsilon_\alpha^*(p_3,\lambda)e^{-\mathrm{i}p_3\cdot x}}{(2(|p_3|^2+m_W^2)^{\frac{1}{2}})^{\frac{1}{2}}} a_-^*(p_3,\lambda)\right] d^3p_3. \tag{2.4}$$

Here, u and v are the solutions to the Dirac equation (normalized as in [55, (2.13)]), ϵ_α is a polarisation vector, γ^α, $\alpha = 0, \dots, 3$ and γ_5 are the usual gamma matrices and g is a coupling constant. Moreover, the index $l \in \{e, \mu, \tau\}$ labels the lepton families, $p_1, p_2, p_3 \in \mathbb{R}^3$ stand for the momentum variables of fermions and bosons, $s_i \in \{-\frac{1}{2}, \frac{1}{2}\}$ denotes the spin of fermions and $\lambda \in \{-1, 0, 1\}$ the spin of bosons. The operators $b_{l,+}(p_1, s_1)$ and $b^*_{l,+}(p_1, s_1)$ are annihilation and creation operators for the electron if $l = e$, muon if $l = \mu$ and tau if $l = \tau$. The operators $b_{l,-}(p_1, s_1)$ and $b^*_{l,-}(p_1, s_1)$ are annihilation and creation operators for the associated antiparticles. Likewise, $c_{l,+}(p_2, s_2)$ and $c^*_{l,+}(p_2, s_2)$ (respectively $c_{l,-}(p_2, s_2)$ and $c^*_{l,-}(p_2, s_2)$) stand for annihilation and creation operators for the neutrinos of the l-family (respectively antineutrinos) and the operators $a_+(p_3, \lambda)$ and $a^*_+(p_3, \lambda)$ (respectively $a_-(p_3, \lambda)$ and $a^*_-(p_3, \lambda)$) are annihilation and creation operators for the boson W^- (respectively W^+). For shortness, we denote by $\xi_i = (p_i, s_i)$, $i = 1, 2$, the quantum variables for fermions, and $\xi_3 = (p_3, \lambda)$ for bosons.

It should be mentioned that, when neutrinos are supposed to be massive, a slightly different interaction term can be found in the literature (see, e.g., [94]). More precisely, massive neutrinos fields $(\tilde{\Psi}_{\nu_1}, \tilde{\Psi}_{\nu_2}, \tilde{\Psi}_{\nu_3})$ may be defined by applying a 3×3 unitary matrix transformation to the fields $(\Psi_{\nu_1}, \Psi_{\nu_2}, \Psi_{\nu_3})$ in (2.3). Our results can be proven without any noticeable change if one considers such interaction terms. We will not do so in the present work.

Inserting (2.2)–(2.4) into (2.1), integrating with respect to x, and using the convention

$$\int d\xi_1 d\xi_2 d\xi_3 = \sum_{s_1=\pm\frac{1}{2}} \sum_{s_2=\pm\frac{1}{2}} \sum_{\lambda=-1,0,1} \int d^3p_1 d^3p_2 d^3p_3,$$

we arrive at the formal expression

$$\begin{aligned} gH_I := g\sum_{j=1}^{4} H_I^{(j)} := g \sum_{l\in\{e,\mu,\tau\}} \sum_{\epsilon=\pm} \int \Big\{ &\Big[G^{(1)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) b^*_{l,\epsilon}(\xi_1) c^*_{l,-\epsilon}(\xi_2) a_\epsilon(\xi_3) + \text{h.c.} \Big] \\ &+ \Big[G^{(2)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) b^*_{l,-\epsilon}(\xi_1) c^*_{l,\epsilon}(\xi_2) a^*_\epsilon(\xi_3) + \text{h.c.} \Big] \\ &+ \Big[G^{(3)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) b^*_{l,-\epsilon}(\xi_1) c_{l,-\epsilon}(\xi_2) a^*_\epsilon(\xi_3) + \text{h.c.} \Big] \\ &+ \Big[G^{(4)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) b^*_{l,\epsilon}(\xi_1) c_{l,\epsilon}(\xi_2) a_\epsilon(\xi_3) + \text{h.c.} \Big] \Big\} d\xi_1 d\xi_2 d\xi_3, \end{aligned} \tag{2.5}$$

where we set $-\epsilon = \mp$ if $\epsilon = \pm$. The kernels $G^{(j)}_{l,\epsilon}$, $j = 1, \dots, 4$, are formally given by:

$$G^{(j)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) = \int G^{(j)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3,x)dx,$$

with:

$$G^{(1)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3,x) = (2\pi)^{-\frac{9}{2}} \begin{cases} \dfrac{\overline{u(p_1,s_1)}\gamma^\alpha(1-\gamma_5)v(p_2,s_2)\epsilon_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{\mathrm{i}(-p_1-p_2+p_3)\cdot x} & \text{if } \epsilon = +, \\ \dfrac{\overline{u(p_2,s_2)}\gamma^\alpha(1-\gamma_5)v(p_1,s_1)\epsilon_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{\mathrm{i}(-p_1-p_2+p_3)\cdot x} & \text{if } \epsilon = -, \end{cases}$$

$$G^{(2)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3,x) = (2\pi)^{-\frac{9}{2}} \begin{cases} \dfrac{\overline{u(p_1,s_1)}\gamma^\alpha(1-\gamma_5)v(p_2,s_2)\epsilon^*_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{-\mathrm{i}(p_1+p_2+p_3)\cdot x} & \text{if } \epsilon = -, \\ \dfrac{\overline{u(p_2,s_2)}\gamma^\alpha(1-\gamma_5)v(p_1,s_1)\epsilon^*_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{-\mathrm{i}(p_1+p_2+p_3)\cdot x} & \text{if } \epsilon = +, \end{cases}$$

$$G^{(3)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3,x) = (2\pi)^{-\frac{9}{2}} \begin{cases} \frac{\overline{u(p_1,s_1)}\gamma^\alpha(1-\gamma_5)u(p_2,s_2)\epsilon^*_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{\mathrm{i}(-p_1+p_2-p_3)\cdot x} & \text{if } \epsilon = -, \\ \\ \frac{\overline{v(p_2,s_2)}\gamma^\alpha(1-\gamma_5)v(p_1,s_1)\epsilon^*_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{\mathrm{i}(-p_1+p_2-p_3)\cdot x} & \text{if } \epsilon = +, \end{cases}$$

$$G^{(4)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3,x) = (2\pi)^{-\frac{9}{2}} \begin{cases} \frac{\overline{u(p_1,s_1)}\gamma^\alpha(1-\gamma_5)u(p_2,s_2)\epsilon_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{\mathrm{i}(-p_1+p_2+p_3)\cdot x} & \text{if } \epsilon = +, \\ \\ \frac{\overline{v(p_2,s_2)}\gamma^\alpha(1-\gamma_5)v(p_1,s_1)\epsilon_\alpha(p_3,\lambda)}{(2(|p_2|^2+m^2_{\nu_l})^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_3|^2+m^2_W)^{\frac{1}{2}})^{\frac{1}{2}}(2(|p_1|^2+m^2_l)^{\frac{1}{2}})^{\frac{1}{2}}} e^{\mathrm{i}(-p_1+p_2+p_3)\cdot x} & \text{if } \epsilon = -. \end{cases}$$

The kernels may be then written in the following way:

$$G^{(1)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) = f^{(1)}_{l,\epsilon,1}(\xi_1)f^{(1)}_{l,\epsilon,2}(\xi_2)f^{(1)}_{l,\epsilon,3}(\xi_3)\delta(-p_1-p_2+p_3), \tag{2.6}$$
$$G^{(2)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) = f^{(2)}_{l,\epsilon,1}(\xi_1)f^{(2)}_{l,\epsilon,2}(\xi_2)f^{(2)}_{l,\epsilon,3}(\xi_3)\delta(p_1+p_2+p_3), \tag{2.7}$$
$$G^{(3)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) = f^{(3)}_{l,\epsilon,1}(\xi_1)f^{(3)}_{l,\epsilon,2}(\xi_2)f^{(3)}_{l,\epsilon,3}(\xi_3)\delta(-p_1+p_2-p_3), \tag{2.8}$$
$$G^{(4)}_{l,\epsilon}(\xi_1,\xi_2,\xi_3) = f^{(4)}_{l,\epsilon,1}(\xi_1)f^{(4)}_{l,\epsilon,2}(\xi_2)f^{(4)}_{l,\epsilon,3}(\xi_3)\delta(-p_1+p_2+p_3), \tag{2.9}$$

where the maps $p_i \mapsto f^{(j)}_{l,\epsilon,i}(\xi_i)$ are bounded in any compact set of $\mathbb{R}^3$. Now, the free Hamiltonian is given by:

$$\begin{aligned} H_0 = & \sum_{l\in\{e,\mu,\tau\}}\sum_{\epsilon=\pm}\int \omega^{(1)}_l(\xi_1)b^*_{l,\epsilon}(\xi_1)b_{l,\epsilon}(\xi_1)d\xi_1 + \sum_{l\in\{e,\mu,\tau\}}\sum_{\epsilon=\pm}\int \omega^{(2)}_l(\xi_2)c^*_{l,\epsilon}(\xi_2)c_{l,\epsilon}(\xi_2)d\xi_2 \\ & + \sum_{\epsilon=\pm}\int \omega^{(3)}(\xi_3)a^*_\epsilon(\xi_3)a_\epsilon(\xi_3)d\xi_3, \end{aligned} \tag{2.10}$$

with the dispersion relations

$$\omega^{(1)}_l(\xi_1) = \sqrt{(p_1^2+m_l^2)}, \quad \omega^{(2)}_l(\xi_2) = \sqrt{(p_2^2+m^2_{\nu_l})}, \quad \omega^{(3)}(\xi_3) = \sqrt{(p_3^2+m^2_{W^\pm})}. \tag{2.11}$$

The total Hamiltonian is defined by

$$H = H_0 + gH_I. \tag{2.12}$$

Since the kernels $G^{(j)}$ are singular, the formal expressions (2.5)–(2.12) do not define a self-adjoint operator in Fock space (see the next section for the precise definition of the Hilbert space that we consider). In order to obtain such a self-adjoint operator, following a standard procedure in constructive QFT (see e.g. [51] and references therein), we introduce ultraviolet and spatial cut-offs in the interaction Hamiltonian. Of course, eventually, it would be desirable to find a renormalization procedure allowing one to remove those cut-offs. This constitutes, however, an important open problem which will be quickly discussed, for spacial cut-offs, on a toy model in the last chapter. Let $\Lambda > 0$ be a fixed ultraviolet parameter and let $B(0,\Lambda)$ denotes the ball centered at 0 and of radius Λ in $\mathbb{R}^3$. In the formal expression (2.5), we introduce ultraviolet cut-offs, i.e., we replace $f^{(j)}_{l,\epsilon,i}(\xi_i)$ by $\chi_{B(0,\Lambda)}(p_i)f^{(j)}_{l,\epsilon,i}(\xi_i)$, for some smooth function $\chi_{B(0,\Lambda)}$ supported in $B(0,\Lambda)$, and we replace the Dirac delta function $\delta(p)$ by an approximation, $\delta_n(p) = n^3\delta_1(np)$, for some smooth and compactly supported function δ_1. The resulting kernels are still denoted by the same symbols $G^{(j)}_{l,\epsilon}$. In particular, $G^{(j)}_{l,\epsilon}$ are now square integrable. As will be shown in the next section, square integrability of the kernels is actually sufficient to prove that $H = H_0 + H_I$ defines a self-adjoint operator in Fock space. Finally, some of our results will be proven in the weak coupling regime. We will therefore keep the coupling constant to highlight such phenomena.

The kernel will always be supposed to be square integrable and, in some cases, stronger regularity assumptions on $G^{(j)}_{l,\epsilon}$ will be required. We emphasize that the function $p_i \to f^{(j)}_{l,\epsilon,i}(\xi_i)$ in the definition of kernels $G^{(j)}_{l,\epsilon}$ are bounded near the origin. In particular, the weak interaction Hamiltonian is more regular in the infrared region than the QED one (where the infrared singularity is of order $|k|^{-\frac{1}{2}}$ near the origin, with k the momentum of photons). On the other hand, contrary to the Pauli-Fierz Hamiltonian of the standard model of non-relativistic QED, in the case of weak interactions considered here, there is no unitary Pauli-Fierz transformation on which one can rely to improve the infrared singularity. In our main result (see Theorem 2.1.1 below), if the masses of the neutrinos are supposed to be strictly positive, the physical infrared singularity will be covered by our assumptions. However, if the neutrinos are supposed to be massless (see the next chapter), we will have to impose regularisation in the model in order to establish our main result. This will be made more precise below.

The spectral theory of such models of the weak interaction has been studied in particular in [6, 10, 16, 17, 18, 19] (see also [15, 90] for related models of QED). Without entering into details, the results established in these references show that, for weak coupling, and under suitable assumptions on the kernels, H is self-adjoint and has a ground state (i.e. $E := \inf \sigma(H)$ is an eigenvalue of H), and the essential spectrum of H coincides with the semi axis $\sigma_{\text{ess}}(H) = [E + m_\nu, \infty)$, with $m_\nu = \min(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau})$. In particular, if the masses of the neutrinos vanish, the ground state energy is an eigenvalue of H embedded into its essential spectrum. Moreover, except for the ground state energy, the spectrum of H below the electron mass is purely absolutely continuous.

In this chapter, together with the following one, we complement the previous spectral results by studying the structure of the essential spectrum in the whole semi-axis $[E + m_\nu, \infty)$ (not only below the electron mass) and by relaxing the weak coupling assumption in the case where the masses of the neutrinos do not vanish.

Our main purpose is then to study scattering theory for models of the form (2.5)–(2.12) and, in particular, to prove asymptotic completeness. Scattering theory for models of non-relativistic matter coupled to a massive, bosonic quantum field – massive Pauli-Fierz Hamiltonians – has been considered by many authors. See, among others, [9, 27, 29, 31, 41, 42, 43, 44, 45, 67, 63, 64, 89]; see also [5] for fermionic Pauli-Fierz systems, [30] for spatially cut-off $P(\varphi)_2$ Hamiltonians, [50] for abstract QFT Hamiltonians, and [21, 34, 35, 33, 38, 39, 49] for massless Pauli-Fierz Hamiltonians. A large part of the techniques we used are adapted from the ones developed in these references. In this chapter we will study the massive neutrinos case. The massless one will be treated in chapter three.

As said in the previous chapter, $\mathscr{H}$, the Hilbert space of the model, is defined as a tensor product of Fock spaces (see the next section for precise definitions). The definition of the space of asymptotic vacua is adapted in the following way, in the context of many particle species:

$$\mathscr{K}^{\pm} := \{u,\, d^{\pm}(h)u = 0 \text{ for all asymptotic annihilation operator } d^{\pm}(h)\},$$

where $d^{\pm}(h)$ stands for either $a^{\pm}_{\epsilon}(h)$, with $h \in L^2(\mathbb{R}^3 \times \{-1, 0, 1\})$, or $b^{\pm}_{l,\epsilon}(h)$ or $c^{\pm}_{l,\epsilon}(h)$, with $h \in L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\})$.

Recall that the parameter $j \in \{1, \ldots, 4\}$ labels the different interaction terms in (2.5) and that the index $l \in \{e, \mu, \tau\}$ labels the lepton families. Moreover $\epsilon = \pm$. In what follows, for shortness, we say that

$$\text{“}G \in L^2\text{” if, for all } j,\ l \text{ and } \epsilon,\ G^{(j)}_{l,\epsilon} \text{ is square integrable.}$$

Recall that s_1, s_2 denote the spin variables for fermions and that λ denotes the spin variable for bosons. We say that

$$\text{“}G \in \mathbb{H}^\mu\text{” if, for all } j,\ l,\ \epsilon,\ s_1,\ s_2 \text{ and } \lambda,\ G^{(j)}_{l,\epsilon}(s_1,\cdot,s_2,\cdot,\lambda,\cdot) \text{ belongs to the Sobolev space } \mathbb{H}^\mu(\mathbb{R}^9).$$

Remembering that the dispersion relations $\omega^{(i)}_l$, $i = 1, 2$, $l \in \{e, \mu, \tau\}$ and $\omega^{(3)}$ are defined in (2.11), we set

$$a_{(i),l} := \frac{\mathrm{i}}{2}\big(\nabla_{p_i} \cdot \nabla\omega^{(i)}_l(p_i) + \nabla\omega^{(i)}_l(p_i) \cdot \nabla_{p_i}\big), \quad i = 1, 2, \quad l \in \{e, \mu, \tau\}, \tag{2.13}$$

$$a_{(3)} := \frac{\mathrm{i}}{2}\big(\nabla_{p_3} \cdot \nabla\omega^{(3)}(p_3) + \nabla\omega^{(3)}(p_3) \cdot \nabla_{p_3}\big), \tag{2.14}$$

$$b_{(i),l} := \frac{\mathrm{i}}{2}\big((p_i \cdot \nabla\omega^{(i)}_l(p_i))^{-1} p_i \cdot \nabla_{p_i} + \nabla_{p_i} \cdot p_i(p_i \cdot \nabla\omega^{(i)}_l(p_i))^{-1}\big), \quad i = 1, 2, \quad l \in \{e, \mu, \tau\}, \tag{2.15}$$

$$b_{(3)} := \frac{\mathrm{i}}{2}\big((p_3 \cdot \nabla\omega^{(3)}(p_3))^{-1} p_3 \cdot \nabla_{p_3} + \nabla_{p_3} \cdot p_3(p_3 \cdot \nabla\omega^{(3)}(p_3))^{-1}\big), \tag{2.16}$$

as partial differential operators acting on $L^2(d\xi_1 d\xi_2 d\xi_3)$. To shorten the statement of some of our results below, we introduce the notation “$a_{(i),\cdot}G \in L^2$” with the following meaning: for $i = 1, 2$, we say that

$$\text{“}a_{(i),\cdot}G \in L^2\text{” if, for all } j,\ l \text{ and } \epsilon,\ a_{(i),l}G^{(j)}_{l,\epsilon} \text{ is square integrable,}$$

and, for $i = 3$, we say that

$$\text{“}a_{(3),\cdot}G \in L^2\text{” if, for all } j,\ l \text{ and } \epsilon,\ a_{(3)}G^{(j)}_{l,\epsilon} \text{ is square integrable.}$$

The notation $a_{(i),\cdot}a_{(i'),\cdot}G \in L^2$ is defined analogously, and likewise for $b_{(i),\cdot}G \in L^2$ and $b_{(i),\cdot}b_{(i'),\cdot}G \in L^2$.

The main results of this chapter can be stated as follow.

Theorem 2.1.1. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5)*. Assume that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3,$$

and that $G \in \mathbb{H}^{1+\mu}$ for some $\mu > 0$. Then the wave operators $\Omega^\pm$ exist and are asymptotically complete. Suppose in addition that

$$b_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3, \quad b_{(i),\cdot}b_{(i'),\cdot}G \in L^2, \quad i, i' = 1, 2, 3.$$

Then there exists $g_0 > 0$, which does not depend on m_{ν_e}, m_{ν_μ}, m_{ν_τ}, such that, for all $|g| \leqslant g_0$, $H - E$ is unitarily equivalent to H_0.

We will follow the strategy presented in [29, 30, 5]. In Section 2.1.1, we show that the Hamiltonian (2.12) defines a self-adjoint operator on a Hilbert space given as a tensor product of antisymmetric and symmetric Fock spaces. The result holds without any restriction on the coupling constant g. Section 2.1.2 also contains the proof of some technical estimates that are used in the subsequent sections.

In Section 2.2, we recall results giving the existence of a ground state and the location of the essential spectrum of H, and we study the essential spectrum by means of suitable versions of Mourre's conjugate operator method.

Section 2.3 is devoted to the proof of several propagation estimates.

Finally, in Section 2.4, we prove some properties of the asymptotic fields and wave operators, and we use the results of Sections 2.2 and 2.3 to prove Theorem 2.1.1.

Let us recall that throughout the entire document, the notation $a \lesssim b$, for positive numbers a and b, stands for $a \leqslant Cb$ where C is a positive constant independent of the parameters involved.

2.1.1 The Model

In this section we show that the Hamiltonian of the model, formally defined in (2.12), identifies with a self-adjoint operator in a tensor product of Fock spaces for fermions and bosons. For fermions, we define $\Sigma_1 := \mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}$ and, for bosons, $\Sigma_2 := \mathbb{R}^3 \times \{-1, 0, 1\}$. The one-particle Hilbert space for fermions is $\mathfrak{h}_1 := L^2(\Sigma_1)$ and for bosons $\mathfrak{h}_2 := L^2(\Sigma_2)$. The anti-symmetric Fock space for fermions is denoted by $\mathfrak{F}_a := \oplus_{n=0}^{\infty} \otimes_a^n \mathfrak{h}_1$, where $\otimes_a$ stands for the antisymmetric tensor product and where we use the usual convention $\otimes_a^0 \mathfrak{h}_1 = \mathbb{C}$. The symmetric Fock space for bosons is $\mathfrak{F}_s := \oplus_{n=0}^{\infty} \otimes_s^n \mathfrak{h}_2$, where $\otimes_s$ stands for the symmetric tensor product and $\otimes_s^0 \mathfrak{h}_2 = \mathbb{C}$. Every family l of leptons contains either an electron, a muon or a tau, the associated antiparticle, and a neutrino and its antineutrino. Consequently, the Hilbert space for each lepton family is

$$\mathfrak{F}_l := \bigotimes^{4} \mathfrak{F}_a,$$

and we denote the full leptonic Hilbert space by

$$\mathfrak{F}_L := \bigotimes^{3} \mathfrak{F}_l.$$

Analogously, the bosonic Hilbert space is given by

$$\mathfrak{F}_W := \bigotimes^{2} \mathfrak{F}_s.$$

The total Hilbert space is

$$\mathscr{H} := \mathfrak{F}_W \otimes \mathfrak{F}_L.$$

In other words, $\mathscr{H}$ is the tensor product of 14 Fock spaces, 2 symmetric Fock spaces for the bosons $W^{\pm}$ and 12 anti-symmetric Fock spaces for the fermions.

The number operators for neutrinos and antineutrinos are defined by

$$N_{\nu_l} := \sum_{\epsilon=\pm} \int_{\Sigma_1} c_{l,\epsilon}^*(\xi_2) c_{l,\epsilon}(\xi_2) d\xi_2, \quad N_{\text{neut}} := \sum_{l \in \{e,\mu,\tau\}} N_{\nu_l},$$

and likewise

$$N_l := \sum_{\epsilon=\pm} \int_{\Sigma_1} b_{l,\epsilon}^*(\xi_1) b_{l,\epsilon}(\xi_1) d\xi_1, \quad N_{\text{lept}} := \sum_{l \in \{e,\mu,\tau\}} N_l, \quad N_W := \sum_{\epsilon=\pm} \int_{\Sigma_2} a_{\epsilon}^*(\xi_3) a_{\epsilon}(\xi_3) d\xi_3.$$

The total number operator is

$$N = N_{\text{lept}} + N_{\text{neut}} + N_W.$$

Using the Kato-Rellich theorem together with N_τ estimates [51], it is proven in [19] that the total Hamiltonian H defined in (2.12) is a self-adjoint operator in $\mathscr{H}$, with domain $\mathscr{D}(H) = \mathscr{D}(H_0)$, provided that the kernels $G^{(j)}_{l,\epsilon}$ are square integrable and $g \ll 1$. In this section, we extend this result to any value of g.

Recall that the notation "$G \in L^2$" means that, for all j, l and ϵ, $G^{(j)}_{l,\epsilon}$ is square integrable. We denote by $\|G\|_2$ the sum over j, l and ϵ of the L^2-norms of $G^{(j)}_{l,\epsilon}$,

$$\|G\|_2 := \sum_{j,l,\epsilon} \|G^{(j)}_{l,\epsilon}\|_2.$$

Theorem 2.1.2. *Suppose that $G \in L^2$. Then, for all $g \in \mathbb{R}$, the Hamiltonian H in* (2.12) *is self-adjoint with domain $\mathscr{D}(H) = \mathscr{D}(H_0)$.*

The proof of Theorem 2.1.2 will be a consequence of the following two lemmas.

Lemma 2.1.3. *Suppose that $G \in L^2$. Then*

$$\|H_I(N_{\text{lept}} + N_W + 1)^{-1}\| \lesssim \|G\|_2.$$

Proof. It is a consequence of N_τ estimates. A detailed proof is given in Annex A. □

The second lemma is a slight generalization of [51, Proposition 1.2.3(c)].

Lemma 2.1.4. *Suppose that for all j, l, ϵ, s_1, s_2 and λ, $G^{(j)}_{l,\epsilon}(s_1, \cdot, s_2, \cdot, \lambda, \cdot)$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^9)$. Then*

$$\|H_I(N_W + 1)^{-\frac{1}{2}}\| \leqslant \mathrm{C}(G),$$

where $\mathrm{C}(G)$ is a positive constant depending on $G^{(j)}_{l,\epsilon}$.

Proof. Let $\{e_i\}$ be an orthonormal basis of $L^2(\mathbb{R})$ composed of eigenvectors, corresponding to the eigenvalues $\lambda_i = (2i+1)$, of the one-dimensional harmonic oscillator $h_{\text{ho}} := -\frac{d^2}{dx^2} + x^2$. We consider the orthonormal basis $\{e_{i_1} \otimes \cdots \otimes e_{i_9}\}$ in $L^2(\mathbb{R}^9)$. Below we use the notation $\mathrm{i}_1 = (i_1, i_2, i_3)$, $\mathrm{i}_2 = (i_3, i_4, i_5)$, $\mathrm{i}_3 = (i_6, i_7, i_9)$ and a sum over $(\mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3)$ corresponds to a sum over $(i_1, \ldots, i_9) \in \mathbb{N}^9$. Moreover $e_{\mathrm{i}_1} := e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$ and likewise for e_{i_2} and e_{i_3}.

As in the proof of the previous lemma, we consider for instance the term $H^{(1)}_{I,1,+}$ occurring in H_I, see (A.1). Let

$$H^{(1)}_{I,1,+}(s_1, s_2, \lambda) := \int G^{(1)}_{1,+}(s_1, p_1, s_2, p_2, \lambda, p_3) b^*_{1,+}(p_1, s_1) c^*_{1,-}(p_2, s_2) a_+(p_3, \lambda) dp_1 dp_2 dp_3,$$

so that

$$H^{(1)}_{I,1,+} = \sum_{s_1, s_2, \lambda} H^{(1)}_{I,1,+}(s_1, s_2, \lambda).$$

To prove the lemma, it suffices to verify that, for any fixed s_1, s_2 and λ,

$$\|H^{(1)}_{I,1,+}(s_1, s_2, \lambda)(N_W + 1)^{-\frac{1}{2}}\| \leqslant \mathrm{C}(G).$$

Decomposing $G^{(1)}_{1,+}(\cdot, s_1, \cdot, s_2, \cdot, s_3)$ into the orthonormal basis $\{e_{i_1} \otimes \cdots \otimes e_{i_9}\}$, we see that

$$H^{(1)}_{I,1,+}(s_1, s_2, \lambda) = \sum_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} \alpha_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} b^*_{1,+,s_1}(e_{\mathrm{i}_1}) \otimes c^*_{1,-,s_2}(e_{\mathrm{i}_2}) \otimes a_{+,\lambda}(e_{\mathrm{i}_3}) + h.c,$$

where we have set $\alpha_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} := \langle e_{\mathrm{i}_1} \otimes e_{\mathrm{i}_2} \otimes e_{\mathrm{i}_3}, G^{(1)}_{1,+}(\cdot, s_1, \cdot, s_2, \cdot, s_3)\rangle$, $b^*_{1,+,s_1}(h) = \int_{\mathbb{R}^3} h(p_1) b^*_{1,+}(s_1, p_1) dp_1$, for $h \in L^2(\mathbb{R}^3)$, and likewise for $c^*_{1,-,s_2}(h)$ and $a_{+,\lambda}(h)$. This yields

$$\begin{aligned}&\left\| H^{(1)}_{I,1,+}(s_1, s_2, \lambda)(N_W + 1)^{-\frac{1}{2}} \right\| \\ &= \sum_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} \left\| \alpha_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} b^*_{1,+,s_1}(e_{\mathrm{i}_1}) \otimes c^*_{1,-,s_2}(e_{\mathrm{i}_2}) \otimes a_{+,\lambda}(e_{\mathrm{i}_3})(N_W + 1)^{-\frac{1}{2}} \right\| \leqslant \sum_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} |\alpha_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3}|.\end{aligned}$$

Observe that

$$\alpha_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} = \left(\prod_{\ell=1}^{9} \frac{1}{(2i_\ell + 1)} \right) \left\langle e_{\mathrm{i}_1} \otimes e_{\mathrm{i}_2} \otimes e_{\mathrm{i}_3} \,\middle|\, (\otimes_{\ell=1}^{9} h_{\mathrm{ho}}) G^{(1)}_{1,+}(\cdot, s_1, \cdot, s_2, \cdot, s_3) \right\rangle .$$

The Cauchy-Schwarz inequality then gives

$$\sum_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} |\alpha_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3}| \leqslant \Big(\sum_{\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3} \Big(\prod_{\ell=1}^{9} \frac{1}{(2i_\ell + 1)^2} \Big) \Big)^{\frac{1}{2}} \left\| (\otimes_{\ell=1}^{9} h_{\mathrm{ho}}) G^{(1)}_{1,+}(\cdot, s_1, \cdot, s_2, \cdot, s_3) \right\|_{L^2(\mathbb{R}^9)} ,$$

which concludes the proof. □

Now we are ready to prove Theorem 2.1.2.

Proof of Theorem 2.1.2. In this proof, we underline the dependence of the interaction Hamiltonian on the kernels $G^{(j)}_{l,\epsilon}$ by writing $H_I = H_I(G)$. According to Lemma 2.1.3, for $\Psi \in \mathscr{D}(N_{\mathrm{lept}} + N_W)$, there exists $a > 0$ such that, for all $G \in L^2$,

$$\|H_I(G)\Psi\| \leqslant a\|G\|_2 \left(\|(N_{\mathrm{lept}} + N_W)\Psi\| + \|\Psi\|\right).$$

Let $\delta > 0$. There exists $G_\delta \in L^2$ such that for all j, l, ϵ, s_1, s_2 and λ, $G^{(j)}_{\delta,l,\epsilon}(s_1, \cdot, s_2, \cdot, \lambda, \cdot)$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^9)$ and

$$\|G - G_\delta\|_2 \leqslant \frac{\delta}{a}.$$

Hence

$$\|(H_I(G) - H_I(G_\delta))\Psi\| \leqslant \varepsilon \left(\|(N_{\mathrm{lept}} + N_W)\Psi\| + \|\Psi\|\right). \tag{2.17}$$

Moreover, by Lemma 2.1.4,

$$\|H_I(G_\delta)\Psi\| \leqslant \mathrm{C}(G_\delta)\|(N_W + 1)^{\frac{1}{2}}\Psi\|, \tag{2.18}$$

with $\mathrm{C}(G_\delta) > 0$. For $\mu > 0$, we have that

$$\|(N_W + 1)^{\frac{1}{2}}\Psi\|^2 \leqslant \mu\|N_W\Psi\|^2 + ((4\mu)^{-1} + 1)\|\Psi\|^2 \leqslant \left(\mu^{\frac{1}{2}}\|N_W\Psi\| + ((4\mu)^{-1} + 1)^{\frac{1}{2}}\|\Psi\|\right)^2.$$

Inserting this into (2.18) and choosing $\mu^{\frac{1}{2}} = \varepsilon\mathrm{C}(G_\delta)^{-1}$, this implies that

$$\|H_I(G_\delta)\Psi\| \leqslant \delta\|N_W\Psi\| + \mathrm{c}_\delta\|\Psi\|, \tag{2.19}$$

for some positive constant c_δ.

Equations (2.17) and (2.19) show that $H_I(G)$ is relatively $(N_{\mathrm{lept}}+N_W)$-bounded with relative bound 0. Since the masses m_l and m_W are positive, it is not difficult to deduce that $N_{\mathrm{lept}} + N_W$ is relatively H_0-bounded. Therefore $H_I(G)$ is relatively H_0-bounded with relative bound 0. Applying the Kato-Rellich theorem concludes the proof. □

2.1.2 Technical estimates

This section is devoted to some technical lemmas and properties which will be used later. Proofs and notations in this section are close to those of [29].

Number-energy estimates

Lemma 2.1.5. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *Assume that $G \in L^2$.*

(i) For all $m \in \mathbb{Z}$, uniformly for z in a compact set of $\{z \in \mathbb{C}, \pm|\Im z| > 0\}$, the operator $(N+1)^{-m}(H-z)^{-1}(N+1)^{m+1}$ extends to a bounded operator satisfying

$$\left\|(N+1)^{-m}(H-z)^{-1}(N+1)^{m+1}\right\| = \mathcal{O}(|\Im(z)|^{-\alpha_m}),$$

where α_m denotes an integer depending on m.

(ii) Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. Then, for all $m, p \in \mathbb{N}$, $N^m \chi(H) N^p$ extends to a bounded operator.

Proof. As in the proof of Theorem 2.1.2, we underline the dependence of the interaction Hamiltonian on the kernels $G_{l,\epsilon}^{(j)}$ by writing $H_I = H_I(G)$. First, observe that

$$[H, N] = [H_I(G), N].$$

A direct computation gives

$$[H_I^{(1)}(G), N_{\mathrm{lept}}] = [H_I^{(1)}(G), N_{\mathrm{neut}}] = -[H_I^{(1)}(G), N_W] = \mathrm{i}H_I^{(1)}(\mathrm{i}G), \tag{2.20}$$

$$[H_I^{(2)}(G), N_{\mathrm{lept}}] = [H_I^{(2)}(G), N_{\mathrm{neut}}] = [H_I^{(2)}(G), N_W] = \mathrm{i}H_I^{(2)}(\mathrm{i}G), \tag{2.21}$$

$$[H_I^{(3)}(G), N_{\mathrm{lept}}] = -[H_I^{(3)}(G), N_{\mathrm{neut}}] = [H_I^{(3)}(G), N_W] = \mathrm{i}H_I^{(3)}(\mathrm{i}G), \tag{2.22}$$

$$[H_I^{(4)}(G), N_{\mathrm{lept}}] = -[H_I^{(4)}(G), N_{\mathrm{neut}}] = -[H_I^{(4)}(G), N_W] = \mathrm{i}H_I^{(4)}(\mathrm{i}G). \tag{2.23}$$

In particular, since $G \in L^2$, Lemma 2.1.3 together with the fact that $N_{\mathrm{lept}} + N_W$ is relatively H-bounded show that $\|[H, N](H+z)^{-1}\| = \mathcal{O}(|\Im z|^{-1})$. Likewise,

$$\|\mathrm{ad}_N^j(H)(H-z)^{-1}\| = \mathcal{O}(|\Im z|^{-1}), \quad j \in \mathbb{N}. \tag{2.24}$$

Next, for $m \in \mathbb{N}$, commuting N^m through $(H-z)^{-1}$, we obtain that

$$(H-z)^{-1}N^m = N^m(H-z)^{-1} + \sum_{k=1}^{m} N^{m-l}(H-z)^{-1}B_{m,k}(z),$$

where, by (2.24), the operator $B_{m,k}(z)$ satisfies $\|B_{m,k}(z)\| = \mathcal{O}(|\Im z|^{-c_{m,k}})$, with $c_{m,k}$ a positive integer. Therefore

$$\begin{aligned}(N+1)^{-m}(H-z)^{-1}(N+1)^{m+1} &= (N+1)(H-z)^{-1} + (N+1)^{-m}\sum_{i=1}^{m} N^{m-l}(H-z)^{-1}B_{m,k}(z) \\ &= \mathcal{O}(|\Im z|^{-\alpha_m}),\end{aligned}$$

where in the second equality we used that N is H-bounded. This proves (i).

To prove (ii), let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ and $m, p \in \mathbb{N}$. Then

$$N^m\chi(H)N^p = \Big(\prod_{k=0}^{m-1} N^{m-k}(H+\mathrm{i})^{-1}N^{k-m+1}\Big)(H+\mathrm{i})^m\chi(H)(H+\mathrm{i})^p\Big(\prod_{k=0}^{p-1} N^{k-p+1}(H+\mathrm{i})^{-1}N^{p-k}\Big),$$

where we used the convention $\prod_{k=0}^{m-1} A_k := A_0A_1\dots A_{m-1}$ for any operators $A_0, \dots, A_{m-1}$. Hence (i) implies (ii). $\square$

Number energy estimates in the "extended" setting

In the remainder of this section, we give results that concern an auxiliary "extended Hamiltonian", defined in a same way as in [29]. The latter has already been introduced in section 1.2.4, together with some basics objects. We recall that the "extended Hilbert space" is

$$\mathscr{H}^{\mathrm{ext}} := \mathscr{H} \otimes \mathscr{H}$$

and the extended Hamiltonian, acting on $\mathscr{H}^{\mathrm{ext}}$, is defined by

$$H^{\mathrm{ext}} := H \otimes \mathbb{1} + \mathbb{1} \otimes H_0.$$

The idea is that the first component of $\mathscr{H}^{\mathrm{ext}}$ corresponds to bound states, while the second component corresponds to states localized near infinity. Let us first introduce new tools. The number operators in the extended setting are defined by

$$N_{\mathrm{lept},0} := N_{\mathrm{lept}} \otimes \mathbb{1}, \quad N_{\mathrm{lept},\infty} := \mathbb{1} \otimes N_{\mathrm{lept}},$$

and likewise for N_{neut}, N_W and the total number operator N.

The operators $\mathrm{d}\Gamma(Q,R)$ **and** $\mathrm{d}\check{\Gamma}(Q,R)$ Let q, r be two operators on $\mathfrak{h}_i$. The operator $\mathrm{d}\Gamma(q,r) : \mathfrak{F}_\sharp(\mathfrak{h}_i) \to \mathfrak{F}_\sharp(\mathfrak{h}_i)$ considered in [29, 5] is defined by

$$\mathrm{d}\Gamma(q,r)|_{\otimes^n \mathfrak{h}_i} = \sum_{j=1}^{n} \underbrace{q \otimes \cdots \otimes q}_{j-1} \otimes r \otimes \underbrace{q \otimes \cdots \otimes q}_{n-j}.$$

Given q, r, s three operators in $\mathfrak{h}_i$, with $\|q\| \leqslant 1$, the following estimates are proven in [29, 5]:

$$\left|\left\langle \mathrm{d}\Gamma(q, rs)u, v\right\rangle\right| \leqslant \left\|\mathrm{d}\Gamma(r^*r)^{\frac{1}{2}} v\right\|\left\|\mathrm{d}\Gamma(s^*s)^{\frac{1}{2}} u\right\|, \tag{2.25}$$

for all $u \in \mathscr{D}(\mathrm{d}\Gamma(r^*r)^{1/2})$ and $v \in \mathscr{D}(\mathrm{d}\Gamma(s^*s)^{1/2})$, and

$$\left\|N^{-\frac{1}{2}} \mathrm{d}\Gamma(q,r)u\right\| \leqslant \left\|\mathrm{d}\Gamma(r^*r)^{\frac{1}{2}} u\right\|, \tag{2.26}$$

for all $u \in \mathscr{D}(\mathrm{d}\Gamma(r^*r)^{1/2})$.

Let now $Q = (q_1, \dots, q_{14})$ and $R = (r_1, \dots, r_{14})$ be two sequences of operators such that q_i and r_i act on $\mathfrak{h}_i$. The operator $\mathrm{d}\Gamma(Q,R) : \mathscr{H} \to \mathscr{H}$ is defined by

$$\mathrm{d}\Gamma(Q,R) = \sum_{i=1}^{14} \Gamma\big((q_1, \dots, q_{i-1})\big) \otimes \mathrm{d}\Gamma(q_i, r_i) \otimes \Gamma\big((q_{i+1}, \dots, q_{14})\big).$$

Moreover, similarly as in Section 1.2.4, we define

$$\begin{aligned} \mathrm{d}\check{\Gamma}(Q,R) \ &: \ \mathscr{H}(\mathfrak{h}_1, \mathfrak{h}_2) \to \mathscr{H}(\mathfrak{h}_1, \mathfrak{h}_2) \otimes \mathscr{H}(\mathfrak{h}_1, \mathfrak{h}_2) \\ \mathrm{d}\check{\Gamma}(Q,R) &= U_L \mathrm{d}\Gamma(Q,R), \end{aligned}$$

where U_L is the unitary operator of Section 1.2.4.

With these definitions, the estimate recalled in (2.26) easily generalizes to the following lemma.

Lemma 2.1.6. *Let $Q = (q_1, \dots, q_{14})$ and $R = (r_1, \dots, r_{14})$ be finite sequences of operators, with $\|q_i\| \leqslant 1$. We have that*

$$\left\|N^{-\frac{1}{2}}\mathrm{d}\Gamma(Q,R)u\right\| \leqslant \left\|\mathrm{d}\Gamma(R^*R)^{\frac{1}{2}}u\right\|,$$

for all $u \in \mathscr{D}(\mathrm{d}\Gamma(R^*R)^{1/2})$, where $N = \mathrm{d}\Gamma((\mathbb{1}, \dots, \mathbb{1}))$ is the total number operator in $\mathscr{H}$. Moreover,

$$\left\|(N_0 + N_\infty))^{-\frac{1}{2}}\mathrm{d}\check{\Gamma}(Q,R)u\right\| \leqslant \left\|\mathrm{d}\check{\Gamma}(R^*R)^{\frac{1}{2}}u\right\|$$

for all $u \in \mathscr{D}(\mathrm{d}\check{\Gamma}(R^*R)^{1/2})$.

Intertwining property We state some intertwining properties that will be used later on. In the next lemma, $a_i^\sharp$ stands for a bosonic creation or annihilation operator acting on the i^{th} Fock space (note that $\mathscr{H}$ is the tensor product of 14 Fock spaces, and hence $\mathscr{H} \otimes \mathscr{H}$ is the tensor product of 28 Fock spaces). Likewise, $d_{i,b}^\sharp$ stands for a fermionic creation or annihilation operator acting on the i^{th} Fock space.

Lemma 2.1.7. *Let $J = \{(j_{1,0}, j_{1,\infty}), \dots, (j_{14,0}, j_{14,\infty})\}$ be a family of operators defined as in Section 1.2.4. For $i = 1, 2$, and $h_2 \in \mathfrak{h}_2$, we have that*

$$\check{\Gamma}(J)\int h_2(\xi)a_i^\sharp(\xi)d\xi = \int \left\{(j_{i,0}h_2)(\xi)a_i^\sharp(\xi) + (j_{i,\infty}h_2)(\xi)a_{i+14}^\sharp(\xi)\right\} d\xi\, \check{\Gamma}(J).$$

Likewise, for $i = 3, \dots, 14$ and $h_1 \in \mathfrak{h}_1$, we have that

$$\check{\Gamma}(J)\int h_1(\xi)d_i^\sharp(\xi)d\xi = \int \left\{(j_{i,0}h_1)(\xi)d_{i,b}^\sharp(\xi) + (j_{i,\infty}h_1)(\xi)\,(-1)^{N_i}\, d_{i+14,b}^\sharp(\xi)\right\} d\xi\, \check{\Gamma}(J),$$

where we have set $N_i = \displaystyle\int d_{i,b}^*(\xi)d_{i,b}(\xi)d\xi$.

Proof. We prove for instance the first intertwining property with $i = 1$ and $a^\sharp = a^*$, the proof of the other statements is analogous. Recall the notation $j_i = (j_{i,0}, j_{i,\infty})$. We have that

$$\Gamma\left(j_1\right) a^\sharp(h_2) = a^\sharp(j_1h_2)\Gamma\left(j_1\right),$$

(see [29]). Therefore,

$$\begin{aligned}
\check{\Gamma}(J)a^*(h_2) &= U_L\Gamma(J)\left\{a^*(h_2)\otimes\mathbb{1}_{\mathfrak{F}_s}\otimes\mathbb{1}_{\mathfrak{F}_L}\right\}\\
&= U_L\left\{\Gamma(j_1)a^*(h_2)\otimes\Gamma(j_2)\otimes\Gamma\left(\{j_3,\dots,j_{14}\}\right)\right\}\\
&= U_L\left\{a^*(j_1h_2)\Gamma(j_1)\otimes\Gamma(j_2)\otimes\Gamma\left(\{j_3,\dots,j_{14}\}\right)\right\}\\
&= U_L\left\{a^*(j_1h_2)\otimes\mathbb{1}_{\mathfrak{F}_s}\otimes\mathbb{1}_{\mathfrak{F}_L}\right\}\Gamma(J)\\
&= \left\{a^*(j_{1,0}h_2)\otimes\mathbb{1}_{\mathfrak{F}_s}\otimes\mathbb{1}_{\mathfrak{F}_L}\otimes\mathbb{1}_{\mathscr{H}}+\mathbb{1}_{\mathscr{H}}\otimes a^*(j_{1,\infty}h_2)\otimes\mathbb{1}_{\mathfrak{F}_s}\otimes\mathbb{1}_{\mathfrak{F}_L}\right\}\check{\Gamma}(J).
\end{aligned}$$

This corresponds to the first equality in the statement of the lemma, for $i = 1$ and $a^\sharp = a^*$. □

We conclude these remainders with another useful intertwining property.

Lemma 2.1.8. *Let $B = (b_1, \dots, b_{14})$ and $J = \{(j_{1,0}, j_{1,\infty}), \dots, (j_{14,0}, j_{14,\infty})\}$ be families of operators defined as before. We have that*

$$\left(\mathrm{d}\Gamma(B)\otimes\mathbb{1}_{\mathscr{H}}+\mathbb{1}_{\mathscr{H}}\otimes\mathrm{d}\Gamma(B)\right)\check{\Gamma}(J)-\check{\Gamma}(J)\mathrm{d}\Gamma(B) = \mathrm{d}\check{\Gamma}(J,[B,J]).$$

Proof. It suffices to write

$$\begin{aligned}
&(\mathrm{d}\Gamma(B)\otimes \mathbb{1}_{\mathscr{H}} + \mathbb{1}_{\mathscr{H}}\otimes \mathrm{d}\Gamma(B))\,\check{\Gamma}(J) - \check{\Gamma}(J)\mathrm{d}\Gamma(B)\\
&= (\mathrm{d}\Gamma(B)\otimes \mathbb{1}_{\mathscr{H}} + \mathbb{1}_{\mathscr{H}}\otimes \mathrm{d}\Gamma(B))\,U_L\Gamma(J) - U_L\Gamma(J)\mathrm{d}\Gamma(B)\\
&= U_L\left\{\mathrm{d}\Gamma\left(\begin{bmatrix} B & 0\\ 0 & B\end{bmatrix}\right)\Gamma(J) - \Gamma(J)\mathrm{d}\Gamma(B)\right\}\\
&= U_L\sum_{i=1}^{n}\left\{\Gamma((j_1,\dots,j_{i-1}))\otimes\left(\mathrm{d}\Gamma\left(\begin{bmatrix} b_i & 0\\ 0 & b_i\end{bmatrix}\right)\Gamma(j_i) - \Gamma(j_i)\mathrm{d}\Gamma(b_i)\right)\otimes\Gamma((j_{i+1},\dots,j_n))\right\}\\
&= U_L\sum_{i=1}^{n}\left\{\Gamma((j_1,\dots,j_{i-1}))\otimes \mathrm{d}\Gamma(j_i,[b_i,j_i])\otimes\overline{\Gamma}((j_{i+1},\dots,j_n))\right\}\\
&= \mathrm{d}\check{\Gamma}(J,[B,J]).
\end{aligned}$$

This proves the lemma. $\square$

The proof of Lemma 2.1.5 can be adapted in a straightforward way to obtain the following results.

Lemma 2.1.9. *Under the conditions of Lemma 2.1.5, for all $m \in \mathbb{Z}$, we have that*

$$\left\|(N_0 + N_\infty + 1)^{-m}(H^{\text{ext}} - z)^{-1}(N_0 + N_\infty + 1)^{m+1}\right\| = \mathcal{O}(|\Im(z)|^{-\alpha_m}),$$

uniformly for z in a compact set of $\{z \in \mathbb{C}, \pm|\Im z| > 0\}$, where α_m denotes an integer depending on m. Moreover, for all $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ and $m, p \in \mathbb{N}$, $(N_0 + N_\infty)^m\chi(H^{\text{ext}})(N_0 + N_\infty)^p$ extends to a bounded operator.

Let $(j_{1,0},\dots,j_{14,0}) \in \mathrm{C}_0^\infty(\mathbb{R}^3)$ and $(j_{1,\infty},\dots,j_{14,\infty}) \in \mathrm{C}^\infty(\mathbb{R}^3)$ be families of functions satisfying $j_{\ell,0} \geqslant 0$, $j_{\ell,\infty} \geqslant 0$, $j_{\ell,0}^2 + j_{\ell,\infty}^2 = 1$ and $j_{\ell,0} = 1$ near 0. Recall that the total Hilbert space $\mathscr{H}$ is a tensor product of 14 Fock spaces. We set $j = ((j_{1,0}, j_{1,\infty}),\dots,(j_{14,0}, j_{14,\infty}))$ and, for any $R \geqslant 1$, $j^R = ((j^R_{1,0}, j^R_{1,\infty}),\dots,(j^R_{14,0}, j^R_{14,\infty}))$, with $j^R_{\ell,\sharp} = j_{\ell,\sharp}(\frac{x}{R})$, $x = \mathrm{i}\nabla_p$, and p is the momentum of the particle labeled by $\ell \in \{1,\dots,14\}$.

Recall that we write "$G \in \mathbb{H}^\mu$" if, for all j, l, ϵ, s_1, s_2 and λ, $G^{(j)}_{l,\epsilon}(s_1,\cdot,s_2,\cdot,\lambda,\cdot)$ belongs to the Sobolev space $\mathbb{H}^\mu(\mathbb{R}^9)$.

Lemma 2.1.10. *(i) $G \in \mathbb{H}^\mu$ implies that, for all j, l, ϵ, s_1, s_2 and λ,*

$$\forall R > 1,\ \left\|\mathbb{1}_{[R,\infty)}(|x_i|)G^{(j)}_{l,\epsilon}(s_1,\cdot,s_2,\cdot,\lambda,\cdot)\right\|_2 \lesssim R^{-\mu}, \tag{2.27}$$

where $x_i = \mathrm{i}\nabla_{p_i}$, $i = 1, 2, 3$.

(ii) Conversely, if for all j, l, ϵ, s_1, s_2 and λ, (2.27) *holds, then $G \in \mathbb{H}^\alpha$ for any $\alpha < \mu$.*

Proof. For simplicity, the case of a single variable L^2 function v is considered. The result can be naturally generalised to our case. Let us first prove that the hypothesis $v \in \mathbb{H}^\mu$ implies:

$$\forall R > 1,\ \left\|\mathbb{1}_{[R,\infty)}(|x|)v\right\|_2 \lesssim R^{-\mu}.$$

We have that

$$\|\mathbb{1}_{[R,\infty]}(|x|)v\|^2 = \int \mathbb{1}_{[R,\infty]}(|x|)|\hat{v}(x)|^2 dx \tag{2.28}$$

$$\leqslant \int \mathbb{1}_{[R,\infty]}(|x|)\frac{(|x|^2)^\mu}{R^{2\mu}}|\hat{v}(x)|^2 dx \tag{2.29}$$

$$\leqslant CR^{-2\mu}. \tag{2.30}$$

To prove (ii), we observe that:

$$|x|^{\alpha'} v = |x|^{\alpha'} \mathbb{1}_{[0,1]}(|x|)v + |x|^{\alpha'} \mathbb{1}_{[1,+\infty)}(|x|)v.$$

First, $\left\| |x|^{\alpha'} \mathbb{1}_{[0,1]}(|x|)v \right\| \leqslant \|v\|$. Moreover:

$$\begin{aligned}
\left\| |x|^{\alpha'} \mathbb{1}_{[1,+\infty)}(|x|)v \right\| &= \left\| \sum_{n=0}^{\infty} |x|^{\alpha'} \mathbb{1}_{[2^n,2^{n+1}]}(|x|)v \right\| \\
&\leqslant \sum_{n=0}^{\infty} 2^{(n+1)\alpha'} \| \mathbb{1}_{[2^n,2^{n+1}]}(|x|)v \| \\
&\leqslant C \sum_{n=0}^{\infty} 2^{(n+1)\alpha'} 2^{-\mu n} \\
&\leqslant C 2^{\alpha'} \sum_{n=0}^{\infty} 2^{n(\alpha'-\mu)}.
\end{aligned}$$

If $\mu > \alpha'$ then (ii) is proven. □

Lemma 2.1.11. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian (2.12) with H_I given by (2.5). Assume that $G \in L^2$. Let j^R be defined as above. Then*

$$(H^{\mathrm{ext}} + \mathrm{i})^{-1} \check{\Gamma}(j^R) - \check{\Gamma}(j^R)(H + \mathrm{i})^{-1} = o(R^0), \quad R \to \infty. \tag{2.31}$$

In particular, for any $\chi, \chi' \in \mathrm{C}_0^\infty(\mathbb{R})$, we have that

$$\big\{\chi(H^{\mathrm{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\chi(H)\big\}\chi'(H) = o(R^0), \quad R \to \infty. \tag{2.32}$$

If $G \in \mathbb{H}^\mu$ with $\mu > 0$, then

$$(H^{\mathrm{ext}} + \mathrm{i})^{-1} \check{\Gamma}(j^R) - \check{\Gamma}(j^R)(H + \mathrm{i})^{-1} = \mathcal{O}(R^{-\min(1,\mu)}), \quad R \to \infty, \tag{2.33}$$

and in particular, for any $\chi, \chi' \in \mathrm{C}_0^\infty(\mathbb{R})$ and $\mu \geqslant 1$, we have that

$$\big\{\chi(H^{\mathrm{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\chi(H)\big\}\chi'(H) \in \mathcal{O}(R^{-1}), \quad R \to \infty. \tag{2.34}$$

Proof. We first note that

$$(H^{\mathrm{ext}} + \mathrm{i})^{-1} \check{\Gamma}(j^R) - \check{\Gamma}(j^R)(H + \mathrm{i})^{-1} = (H^{\mathrm{ext}} + \mathrm{i})^{-1} \big(\check{\Gamma}(j^R)H - H^{\mathrm{ext}}\check{\Gamma}(j^R)\big)(H + \mathrm{i})^{-1},$$

and a direct computation gives

$$H_0^{\mathrm{ext}} \check{\Gamma}(j^R) - \check{\Gamma}(j^R)H_0 = \mathrm{d}\check{\Gamma}(j^R, [\omega, j^R]).$$

It follows from Lemma 2.1.6 that

$$\begin{aligned}
\left\| \mathrm{d}\check{\Gamma}(j^R, [\omega, j^R])(N_0 + N_\infty + 1)^{-1} \right\| &= \left\| (N_0 + N_\infty + 1)^{-\frac{1}{2}} \mathrm{d}\check{\Gamma}(j^R, [\omega, j^R])(N + 1)^{-\frac{1}{2}} \right\| \\
&\leqslant \left\| [\omega, j^R]^* [\omega, j^R] \right\|^{\frac{1}{2}} = \mathcal{O}(R^{-1}).
\end{aligned}$$

Moreover, we have that

$$(H_I(G)\otimes\mathbb{1})\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H_I(G)=\sum_{j=1}^{4}(H_I^{(j)}(G)\otimes\mathbb{1})\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H_I^{(j)}(G).$$

Considering for instance the term

$$H^{(1)}_{I,1,+}(G)=\int G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)b^*_{1,+}(\xi_1)c^*_{1,-}(\xi_2)a_+(\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

occurring in $H_I^{(1)}(G)$. Using the intertwining properties of Lemma 2.1.7, one verifies that

$$(H^{(1)}_{I,1,+}(G)\otimes\mathbb{1})\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H^{(1)}_{I,1,+}(G)$$

can be expressed as a sum of operators of the form

$$\int j^R(x_1,x_2,x_3)G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)b^{*,\sharp}_{1,+}(\xi_1)c^{*,\sharp}_{1,-}(\xi_2)a^{\sharp}_+(\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

where $j^R(x_1,x_2,x_3)$ stands for either $1-j^R_{\ell_1,0}(x_1)j^R_{\ell_2,0}(x_2)j^R_{\ell_3,0}(x_3)$ or $j^R_{\ell_1,\sharp}(x_1)j^R_{\ell_2,\sharp}(x_2)j^R_{\ell_3,\sharp}(x_3)$, with at least one of the $j^R_{\ell,\sharp}$ equal to $j^R_{\ell,\infty}$. Moreover $a^{\sharp}_+(\xi_3)$ stands for either $a_+(\xi_3)\otimes\mathbb{1}$ or $\mathbb{1}\otimes a_+(\xi_3)$ and likewise for $b^{*,\sharp}_{1,+}(\xi_1)$ and $c^{*,\sharp}_{1,-}(\xi_2)$.

Therefore, proceeding as in Lemma 2.1.3, one deduces that if $G\in L^2$, then

$$\left\|\left\{(H_I(G)\otimes\mathbb{1})\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H_I(G)\right\}(N_0+N_\infty+1)^{-1}\right\|=o(R^0),\quad R\to\infty.$$

Similarly, if $G\in\mathbb{H}^\mu$, we obtain that

$$\left\|\left\{(H_I(G)\otimes\mathbb{1})\check{\Gamma}(j^R)-\check{\Gamma}(j^R)H_I(G)\right\}(N_0+N_\infty+1)^{-1}\right\|=\mathcal{O}(R^{-\mu}),\quad R\to\infty.$$

Putting together the previous estimates proves (2.31) and (2.33).

To prove (2.32) and (2.34), let $\tilde{\chi}\in \mathrm{C}_0^\infty(\mathbb{C})$ be an almost analytic extension of χ satisfying:

$$\tilde{\chi}|_{\mathbb{R}}=\chi,\quad |\partial_{\bar z}\tilde{\chi}(z)|\leqslant \mathrm{C}_n|\Im(z)|^n,\quad n\in\mathbb{N}.$$

Using the Helffer-Sjöstrand functional calculus, we have that

$$\begin{aligned}&\left(\chi(H^{\mathrm{ext}})\check{\Gamma}(j^R)-\check{\Gamma}(j^R)\chi(H)\right)\chi'(H)\\&=\int\partial_{\bar z}\tilde{\chi}(z)\left((z-H^{\mathrm{ext}})^{-1}\check{\Gamma}(j^R)-\check{\Gamma}(j^R)(z-H)^{-1}\right)\chi'(H)dz\wedge d\bar z.\end{aligned}$$

Combining (2.31), (2.33) and Lemma 2.1.5, we obtain (2.32) and (2.34). □

2.2 Spectral Theory

In this section, we do the spectral analysis of the Hamiltonian H. We begin with recalling results giving the existence of a ground state, next we study the structure of the essential spectrum by means of suitable versions of Mourre's conjugate operator method.

2.2.1 Existence of a ground state and location of the essential spectrum

Recall the notation $E = \inf \sigma(H)$. The next theorem shows, in particular, that H has a ground state, i.e., that E is an eigenvalue of H.

Theorem 2.2.1. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5)*. Assume that $G \in L^2$. Then*

$$\sigma_{\text{ess}}(H) = [E + m_\nu, \infty).$$

In particular, E is a discrete eigenvalue of H.

This result is known as the HVZ theorem and its proof may be found in [29] in the bosonic context or [5] in the fermionic case. Let j^R be defined as in the lemma (2.1.11) adding the assumption that for all i, $j_{i,0}^2 + j_{i,\infty}^2 = 1$. We define $q^R = ((j_{1,0}^R)^2, \dots, (j_{14,0}^R)^2)$. We start by proving the following lemma:

Lemma 2.2.2. *Assume $G \in L^2$. Then the operator $\overline{\Gamma}(q^R)(H + i)^{-1}$ is compact on $\mathscr{H}$.*

Proof. First, we note that:

$$\overline{\Gamma}(q^R)(H_0 + i)^{-1}(H_0 + i)(H + i)^{-1}$$

and as $(H_0 + i)(H + i)^{-1}$ is bounded it is enough to show that $\overline{\Gamma}(q^R)(H_0 + i)^{-1}$ is compact. Since $\mathbb{1}_{[n,\infty)}(N)(H_0 + i)^{-1}$ tends to 0 in norm when n tends to infinity, it is enough to prove the compactness of $\overline{\Gamma}(q^R)(H_0 + i)^{-1}$ on every n-particle sector. The conclusion follows from Proposition 2.34 in [7]. □

Proof of Theorem 2.2.1. The fact that

$$\sigma_{\text{ess}}(H) \subset [E + m_\nu, \infty) \tag{2.35}$$

is a consequence of (2.32) in Lemma 2.1.11. Indeed, if χ belongs to $\mathrm{C}_0^\infty((-\infty, E + m_\nu))$, using that the operator $\check{\Gamma}(j^R)$ defined in the previous section is isometric, we have that

$$\chi(H) = \check{\Gamma}(j^R)^* \check{\Gamma}(j^R)\chi(H) = \check{\Gamma}(j^R)^* \chi(H^{\text{ext}})\check{\Gamma}(j^R) + o(R^0), \quad R \to \infty.$$

Since $N_\infty = \mathbb{1} \otimes N$ commutes with H^{ext} in $\mathscr{H} = \mathscr{H}^{\text{ext}} \otimes \mathscr{H}^{\text{ext}}$, and since

$$H^{\text{ext}}\mathbb{1}_{[1,\infty)}(N_\infty) \geqslant (E + m_\nu)\mathbb{1}_{[1,\infty)}(N_\infty),$$

this yields

$$\begin{aligned}\chi(H) = \check{\Gamma}(j^R)^*(\mathbb{1} \otimes \Pi_\Omega)\chi(H^{\text{ext}})\check{\Gamma}(j^R) + o(R^0) &= \check{\Gamma}(j^R)^*(\mathbb{1} \otimes \Pi_\Omega)\check{\Gamma}(j^R)\chi(H) + o(R^0), \quad R \to \infty,\\ &= \check{\Gamma}(q^R)\chi(H) + o(R^0), \quad R \to \infty,\end{aligned}$$

where Π_Ω denotes the projection onto the vacuum in $\mathscr{H}$. The second equality in the previous equation is another consequence of (2.32). The inclusion (2.35) then follows from Lemma 2.2.2.

The converse inclusion can be proven by constructing a Weyl sequence associated to λ for any $\lambda \in [E + m_\nu, \infty)$ in the same way as in [29, Theorem 4.1] or [5, Theorem 4.3]. □

2.2.2 Spectral analysis for any value of the coupling constant

In this section, we study the structure of the essential spectrum of H using Mourre's conjugate operator theory [77, 8]. Remembering that the dispersion relation $\omega_l^{(i)}$, $i = 1, 2$, $l \in \{e, \mu, \tau\}$ and $\omega^{(3)}$ are defined in Introduction, we set

$$a_{(i),l} := \frac{\mathrm{i}}{2}\big(\nabla_{p_i} \cdot \nabla\omega_l^{(i)}(p_i) + \nabla\omega_l^{(i)}(p_i) \cdot \nabla_{p_i}\big), \quad i = 1, 2, \quad l \in \{e, \mu, \tau\}, \tag{2.36}$$

$$a_{(3)} := \frac{\mathrm{i}}{2}\big(\nabla_{p_3} \cdot \nabla\omega^{(3)}(p_3) + \nabla\omega^{(3)}(p_3) \cdot \nabla_{p_3}\big). \tag{2.37}$$

In particular, the operators $a_{(i),l}$ are self-adjoint and their domains are given by $\mathscr{D}(a_{(i),l}) = \{h \in \mathfrak{h}_1, a_{(i),l}h \in \mathfrak{h}_1\}$, where $a_{(i),l}h$ should be understood in the sense of distributions. Likewise, $a_{(3)}$ is self-adjoint with domain $\mathscr{D}(a_{(3)}) = \{h \in \mathfrak{h}_2, a_{(3)}h \in \mathfrak{h}_2\}$. Using the notation

$$\mathrm{d}\Gamma(q) = \sum_{i=1}^{14} \mathrm{d}\Gamma(q_i)$$

as operators on $\mathscr{H}$, with $q = (q_1, \dots, q_{14})$, q_1, q_2 operators on $\mathfrak{h}_2$ and $(q_3, \dots, q_{14})$ operators on $\mathfrak{h}_1$, we set

$$\begin{aligned} &A := \mathrm{d}\Gamma(a),\\ &a := (a_{(3)}, a_{(3)}, a_{(1),1}, a_{(1),1}, a_{(2),1}, a_{(2),1}, a_{(1),2}, a_{(1),2}, a_{(2),2}, a_{(2),2}, a_{(1),3}, a_{(1),3}, a_{(2),3}, a_{(2),3}). \end{aligned} \tag{2.38}$$

Hence the following notations will be used: $a_1 = a_{(3)}$, $a_2 = a_{(3)}$, $a_3 = a_{(1),1}$, $\dots$, $a_{14} = a_{(2),3}$. A direct computation, done in the first chapter, gives

$$[H_0, \mathrm{i}A] = \mathrm{d}\Gamma(|\nabla\omega|^2), \tag{2.39}$$

in the sense of quadratic forms, with:

$$\nabla\omega := (\nabla\omega^{(3)}, \nabla\omega^{(3)}, \nabla\omega_1^{(1)}, \nabla\omega_1^{(1)}, \nabla\omega_1^{(2)}, \nabla\omega_1^{(2)}, \nabla\omega_2^{(1)}, \nabla\omega_2^{(1)}, \nabla\omega_2^{(2)}, \nabla\omega_2^{(2)}, \nabla\omega_3^{(1)}, \nabla\omega_3^{(1)}, \nabla\omega_3^{(2)}, \nabla\omega_3^{(2)}).$$

Moreover, writing $H_I = H_I(G)$, we have that

$$[H_I(G), \mathrm{i}A] = [H_I^{(1)}(G) + H_I^{(2)}(G) + H_I^{(3)}(G) + H_I^{(4)}(G), \mathrm{i}A], \tag{2.40}$$

where

$$[H_I^{(1)}(G), \mathrm{i}A] = \sum_{i=1}^{2} \mathrm{i}H_I^{(1)}(\mathrm{i}a_iG) - \sum_{i=3}^{14} \mathrm{i}H_I^{(1)}(\mathrm{i}a_iG), \tag{2.41}$$

$$[H_I^{(2)}(G), \mathrm{i}A] = -\sum_{i=1}^{14} \mathrm{i}H_I^{(2)}(\mathrm{i}a_iG), \tag{2.42}$$

$$[H_I^{(3)}(G), \mathrm{i}A] = -\sum_{i\in\{1,2,5,6,9,10,13,14\}}^{14} \mathrm{i}H_I^{(3)}(\mathrm{i}a_iG) + \sum_{i\in\{3,4,7,8,11,12\}}^{14} \mathrm{i}H_I^{(3)}(\mathrm{i}a_iG), \tag{2.43}$$

$$[H_I^{(4)}(G), \mathrm{i}A] = -\sum_{i\in\{5,6,9,10,13,14\}}^{14} \mathrm{i}H_I^{(4)}(\mathrm{i}a_iG) + \sum_{i\in\{1,2,3,4,7,8,11,12\}}^{14} \mathrm{i}H_I^{(4)}(\mathrm{i}a_iG). \tag{2.44}$$

The main result of this section is Theorem 2.2.4 below, which proves that H satisfies a Mourre estimate with respect to A in any interval that does not intersect the set of thresholds (see (2.47) below for the definition of thresholds in our context). As recalled before, in order to be able to deduce useful spectral properties of H, in addition to the Mourre estimate, one needs to establish that H is regular enough w.r.t. A. This is the purpose of the following lemma.

Recall that, for $i = 1, 2$, the notation "$a_{(i),\cdot}G \in L^2$" means that, for all j, l and ϵ, $a_{(i),l}G^{(j)}_{l,\epsilon}$ is square integrable, and that "$a_{(3),\cdot}G \in L^2$" means that for all j, l and ϵ, $a_{(3)}G^{(j)}_{l,\epsilon}$ is square integrable.

Lemma 2.2.3. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *Assume that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3. \tag{2.45}$$

Then H is of class $\mathrm{C}^1(A)$. If in addition

$$a_{(i),\cdot}a_{(i'),\cdot}G \in L^2, \quad i, i' = 1, 2, 3, \tag{2.46}$$

then H is of class $\mathrm{C}^2(A)$.

Proof. Proceeding as in [46, Section 4], using that $\mathscr{D}(H) = \mathscr{D}(H_0)$, it is not difficult to verify that, for all $s \in \mathbb{R}$, $e^{-\mathrm{i}sA}\mathscr{D}(H) \subset \mathscr{D}(H)$. In order to prove that H is of class $\mathrm{C}^1(A)$ if (2.45) holds, it then suffices to verify that $[H, \mathrm{i}A]$ extends to an H-bounded operator. From (2.39) and the fact that $|\nabla\omega|^2$ is bounded, it follows that $[H_0, \mathrm{i}A]$ is relatively N-bounded, and therefore relatively H_0-bounded since the masses of all the particles are positive. Moreover, it follows from Lemma 2.1.3 and (2.40)–(2.44) that $[H_I(G), \mathrm{i}A]$ is also H_0-bounded under assumption (2.45).

Likewise, to prove that H is of class $\mathrm{C}^2(A)$ if assumption (2.46) holds, it suffices to verify that the second commutator $[[H, \mathrm{i}A], \mathrm{i}A]$ extends to a relatively H-bounded operator. The result then follows from computing $[[H, \mathrm{i}A], \mathrm{i}A]$ in the same way as in (2.39)–(2.44) and using the same arguments as before. □

The set of thresholds is defined by

$$\tau = \sigma_{\mathrm{pp}}(H) + \Big\{ \sum_{i=1}^{14} m_i n_i,\ n_i \in \mathbb{N},\ (n_1, \dots, n_{14}) \neq (0, \dots, 0) \Big\}, \tag{2.47}$$

where m_i denotes the mass of the particle i (i.e. $m_1 = m_2 = m_W$, $m_3 = m_4 = m_e$, $m_5 = m_6 = m_{\nu_e}$, *etc*).

Theorem 2.2.4. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *Assume that* (2.45) *holds. Let $\lambda \in \mathbb{R}\backslash\tau$. There exist $\varepsilon > 0$, $\mathrm{c}_0 > 0$ and a compact operator K such that*

$$\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H)[H, \mathrm{i}A]\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H) \geqslant \mathrm{c}_0\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H) + K. \tag{2.48}$$

In particular, for all interval $[\lambda_1, \lambda_2]$ such that $[\lambda_1, \lambda_2] \cap \tau = \varnothing$, H has at most finitely many eigenvalues with finite multiplicities in $[\lambda_1, \lambda_2]$ and, as a consequence, $\sigma_{\mathrm{pp}}(H)$ can accumulate only at τ, which is a countable set.

If in addition (2.46) *holds, then $\sigma_{\mathrm{sc}}(H) = \varnothing$.*

Proof. The proof of (2.48) uses arguments developed in [29] and [5] (see also [28]). Let $E = \inf \sigma(H)$. We introduce the following notations:

$$d(\lambda) = \overset{\Omega^{\perp}}{\inf_{\sigma_{pp}(H)+\overline{d\Gamma}(\omega(k))}} \left(\overline{d\Gamma}(|\nabla\omega(k)|^2)\right),$$

which can be rewritten as:

$$\begin{cases} 0 & \text{if} \lambda \in [E, E + m_{\nu_e}) \\ \inf \left\{ \sum_{j=1}^{N} \sum_{i=1}^{n_j} |\nabla \omega^{(j)}(k_i)|^2 \;\middle|\; \lambda_1 + \sum_{j=1}^{N} \sum_{i=1}^{n_j} \omega^{(j)}(k_i) = \lambda,\ \forall j \leqslant N, n_j \in \mathbb{N},\ \exists j,\ n_j \neq 0,\ \lambda_1 \in \sigma_{pp}(H) \right\} & \text{otherwise} \end{cases}$$

We define also:

$$\tilde{d}(\lambda) = \inf \left\{ \sum_{j=1}^{N} \sum_{i=1}^{n_j} |\nabla \omega^{(j)}(k_i)|^2 \;\middle|\; \lambda_1 + \sum_{j=1}^{N} \sum_{i=1}^{n_j} \omega^{(j)}(k_i) = \lambda,\ \forall j \leqslant N, n_j \in \mathbb{N},\ \lambda_1 \in \sigma_{pp}(H) \right\}.$$

Note that:

$$\tilde{d}(\lambda) = \begin{cases} d(\lambda), \lambda \notin \sigma_{pp}(H) \\ 0, \lambda \in \sigma_{pp}(H) \end{cases}.$$

We also introduce

$$\Delta_{\lambda}^{\kappa} = [\lambda - \kappa, \lambda + \kappa]$$

$$d^{\kappa}(\lambda) = \inf_{\mu \in \Delta_{\lambda}^{\kappa}} d(\mu)$$

$$\tilde{d}^{\kappa}(\lambda) = \inf_{\mu \in \Delta_{\lambda}^{\kappa}} \tilde{d}(\mu).$$

We will use an induction and the statements that will be proved are:

- $H_1(n)$: Let $\epsilon > 0$ and $\lambda \in [E, E + nm_{\nu_e})$. Then there exists a compact operator K_0, an interval Δ which contains λ such that:

$$\mathbb{1}_{\Delta}(H)[H, iA]\mathbb{1}_{\Delta}(H) \geqslant (d(\lambda) - \epsilon)\mathbb{1}_{\Delta}(H) + K_0.$$

- $H_2(n)$: Let $\epsilon > 0$ and $\lambda \in [E, E + nm_{\nu_e})$. Then there exists an interval Δ containing λ such that:

$$\mathbb{1}_{\Delta}(H)[H, iA]\mathbb{1}_{\Delta}(H) \geqslant (\tilde{d}(\lambda) - \epsilon)\mathbb{1}_{\Delta}(H).$$

- $H_3(n)$: Let $\kappa > 0$, $\epsilon_0 > 0$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that for all $\lambda \in [E, E + nm_{\nu_e} - \epsilon_0]$, one has:

$$\mathbb{1}_{\Delta_{\lambda}^{\kappa}}(H)[H, iA]\mathbb{1}_{\Delta_{\lambda}^{\kappa}}(H) \geqslant (\tilde{d}^{\kappa}(\lambda) - \epsilon)\mathbb{1}_{\Delta_{\lambda}^{\kappa}}(H).$$

- $S_1(n)$: τ is a closed countable set in $[E, E + nm_{\nu_e}]$.

- $S_2(n)$: for all $\lambda_1 < \lambda_2 \leqslant E + nm_{\nu_e}$ with $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, we have $dim\mathbb{1}^{pp}_{[\lambda_1, \lambda_2]}(H) < \infty$.

The following assertions will be proved:

$$H_1(n) \Rightarrow H_2(n) \tag{2.49}$$

$$H_2(n) \Rightarrow H_3(n) \tag{2.50}$$

$$H_1(n) \Rightarrow S_2(n) \tag{2.51}$$

$$S_2(n-1) \Rightarrow S_1(n) \tag{2.52}$$

$$S_1(n) \text{ and } H_3(n-1) \Rightarrow H_1(n). \tag{2.53}$$

First, (2.52) is obvious since property $S_2(n-1)$ means that $\sigma_{pp}(H) \cap [E, E+(n-1)m_{\nu_e}]$ is a closed countable set, which implies $S_1(n-1)$.

Part I: To prove (2.49), let us first fix $\lambda \notin \sigma_{pp}(H)$. By $H_1(n)$ we can find Δ_1 containing λ and a compact operator K_1 such that:

$$\mathbb{1}_{\Delta_1}(H)[H, iA]\mathbb{1}_{\Delta_1}(H) \geqslant (d(\lambda) - \frac{\epsilon}{2})\mathbb{1}_{\Delta_1}(H) + K_1.$$

Since $s - \lim_{\Delta \to \{\lambda\}} \mathbb{1}_\Delta(H) = 0$ and using the compactness of K_1, we can find an open set Δ such that:

$$\mathbb{1}_\Delta(H) K_1 \mathbb{1}_\Delta(H) \geqslant -\frac{\epsilon}{2}\mathbb{1}_\Delta(H).$$

As $\tilde{d}(\lambda) = d(\lambda)$ we have:

$$\mathbb{1}_\Delta(H)[H, iA]\mathbb{1}_\Delta(H) \geqslant (\tilde{d}(\lambda) - \epsilon)\mathbb{1}_\Delta(H).$$

If $\lambda \in \sigma_{pp}(H)$ then $P = \mathbb{1}_{\{\lambda\}}(H) \neq 0$. Moreover $(1-P)K_1(1-P)$ is again compact and there exists a finite rank projection F onto a subspace of the range of P such that:

$$\|(1-P)K_1(1-P) - (1-F)K_1(1-F)\| \leqslant \frac{\epsilon}{6}. \tag{2.54}$$

Note that F commutes with H. We now have:

$$\begin{aligned}
\mathbb{1}_{\Delta_1}(H)[H, iA]\mathbb{1}_{\Delta_1}(H) &= \mathbb{1}_{\Delta_1}(H)(1-F)[H, iA](1-F)\mathbb{1}_{\Delta_1}(H) \\
&+ \mathbb{1}_{\Delta_1}(H)P[H, iA]P\mathbb{1}_{\Delta_1}(H) \\
&- \mathbb{1}_{\Delta_1}(H)(1-F)P[H, iA]P(1-F)\mathbb{1}_{\Delta_1}(H) \\
&+ \mathbb{1}_{\Delta_1}(H)F[H, iA](1-P)\mathbb{1}_{\Delta_1}(H) + h.c \\
&= R_1 + R_2 + R_3 + R_4 + R_4^*.
\end{aligned}$$

Using assertion H_1 we have:

$$\begin{aligned}
\mathbb{1}_{\Delta_1}(H)(1-F)[H, iA](1-F)\mathbb{1}_{\Delta_1}(H) &= (1-F)\mathbb{1}_{\Delta_1}(H)[H, iA]\mathbb{1}_{\Delta_1}(H)(1-F) \\
&\geqslant (d(\lambda) - \frac{\epsilon}{3})\mathbb{1}_{\Delta_1}(H)(1-F) + (1-F)K_0(1-F).
\end{aligned}$$

Using (2.54) we deduce that:

$$\mathbb{1}_{\Delta_1}(H)(1-F)[H,iA](1-F)\mathbb{1}_{\Delta_1}(H) \quad \geqslant \quad -\frac{\epsilon}{3}\mathbb{1}_{\Delta_1}(H)(1-F)+(1-P)K_0(1-P)-\frac{\epsilon}{6}.$$

From Lemma 2.2.3, the virial theorem can be applied and:

$$R_2 = R_3 = 0.$$

Finally:

$$R_4 + R_4^* \quad = \quad F^* R_4 + R_4^* F,$$

and the Cauchy-Schwarz inequality leads to:

$$\begin{aligned} R_4 + R_4^* &\geqslant -(F^*F)^{\frac{1}{2}}(R_4^* R_4)^{\frac{1}{2}} \\ &\geqslant -\frac{\epsilon}{6}(F^*F) - \frac{3}{2\epsilon}(R_4^* R_4) \\ &\geqslant -\frac{\epsilon}{6} - \frac{3}{2\epsilon}(1-P)\mathbb{1}_{\Delta_1}(H)\left(\mathbb{1}_{\Delta_1}(H)[H,iA]F[H,iA]\mathbb{1}_{\Delta_1}(H)\right)\mathbb{1}_{\Delta_1}(H)(1-P). \end{aligned}$$

Defining:

$$K_2 = -\frac{3}{2\epsilon}\mathbb{1}_{\Delta_1}(H)[H,iA]F[H,iA]\mathbb{1}_{\Delta_1}(H).$$

We then have :

$$\mathbb{1}_{\Delta_1}(H)[H,iA]\mathbb{1}_{\Delta_1}(H) \quad \geqslant \quad -\frac{\epsilon}{3}\mathbb{1}_{\Delta_1}(H)(1-F)+(1-P)(K_0+K_2)(1-P)-\frac{\epsilon}{3}.$$

Since $K_0 + K_2$ is compact and $\mathbb{1}_{\Delta_1}(H)(1-P)$ tends strongly to zero as Δ tends to $\{\lambda\}$, we then have, for a sufficiently small Δ:

$$\mathbb{1}_{\Delta}(H)[H,iA]\mathbb{1}_{\Delta}(H) \geqslant -\epsilon\mathbb{1}_{\Delta}(H).$$

Part II: Let us now prove (2.50). We fix $\kappa > 0$, $\epsilon_0 > 0$ and for all $\lambda \in [E, E + nm_{\nu_e} - \epsilon_0]$, we can find $\delta(\lambda) < \kappa$ such that:

$$\mathbb{1}_{\Delta_\lambda^{\delta(\lambda)}}(H)[H,iA]\mathbb{1}_{\Delta_\lambda^{\delta(\lambda)}}(H) \geqslant (\tilde{d}(\lambda)-\epsilon)\mathbb{1}_{\Delta_\lambda^{\delta(\lambda)}}(H).$$

A finite sequence of λ_i, $i = 1,\ldots,N$ can be found to cover $[E, E + nm_{\nu_e} - \epsilon_0]$ with $\Delta_{\lambda_i}^{\delta(\lambda_i)}$. We then have:

$$\mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H)[H,iA]\mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H) \geqslant (\tilde{d}(\lambda_i)-\epsilon)\mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H).$$

We set

$$\delta = \min_{i=1,\ldots,N}(\delta(\lambda_i))$$

and, then, there exists $i \in \{1,\ldots,N\}$, depending on λ, such that $\Delta_\lambda^\delta \subset \Delta_{\lambda_i}^{\delta(\lambda_i)}$ and $\lambda_i \in \Delta_\lambda^\delta$. Moreover $\tilde{d}^\kappa(\lambda) \leqslant \tilde{d}(\lambda)$, so:

$$\begin{aligned} \mathbb{1}_{\Delta_\lambda^\delta}(H)[H,iA]\mathbb{1}_{\Delta_\lambda^\delta}(H) &\geqslant (\tilde{d}(\lambda_i)-\epsilon)\mathbb{1}_{\Delta_\lambda^\delta}(H) \\ &\geqslant (\tilde{d}^\kappa(\lambda)-\epsilon)\mathbb{1}_{\Delta_\lambda^\delta}(H). \end{aligned}$$

Part III: (2.51) is a direct consequence of the virial theorem. Let $\lambda_1, \lambda_2 \in [E, E + nm_{\nu_e})$ such that $[\lambda_1, \lambda_2] \cap \tau = \varnothing$. We can again find a finite sequence $\{\lambda_i\}_{i=1,\ldots,n}$ such that $\delta_{\lambda_i}^{\delta(\lambda_i)}$ cover $[\lambda_1, \lambda_2]$ and :

$$\mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H)[H, iA]\mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H) \geqslant (d(\lambda_i) - \epsilon)\mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H) + K_i \tag{2.55}$$

with $(d(\lambda_i) - \epsilon) > 0$. For each $\delta_{\lambda_i}^{\delta(\lambda_i)}$ let us take an orthonormal sequence of eigenvectors $\{\Psi_i\}_n$ with eigenvalues in $\delta_{\lambda_i}^{\delta(\lambda_i)}$. Using virial theorem and (2.55) :

$$\begin{aligned}
0 &= \left\langle \Psi_{i,n} \middle| \mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H)[H, iA]\mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H)\Psi_{i,n} \right\rangle \\
&\geqslant (d(\lambda_i) - \epsilon) \left\langle \Psi_{i,n} \middle| \mathbb{1}_{\Delta_{\lambda_i}^{\delta(\lambda_i)}}(H)\Psi_{i,n} \right\rangle + \langle \Psi_{i,n} | K_i \Psi_{i,n} \rangle \\
&\geqslant (d(\lambda_i) - \epsilon) \|\Psi_{i,n}\|^2 + \langle \Psi_{i,n} | K_i \Psi_{i,n} \rangle
\end{aligned}$$

If there is an infinite number of such eigenvectors they converge weakly to zero. Since K_i is compact then $\langle \Psi_{i,n} | K_i \Psi_{i,n} \rangle$ converges to zero too. We would have then

$$0 \geqslant (d(\lambda_i) - \epsilon) \|\Psi_{i,n}\|^2$$

which is a contradiction.

Part IV: We now prove statement (2.53). Let $\chi \in C_0^\infty([E, E + nm_{\nu_e}))$. Then:

$$\begin{aligned}
\chi(H)[H, iA]\chi(H) &= \check{\Gamma}(j^R)^*\check{\Gamma}(j^R)\chi(H)[H, iA]\chi(H) \\
&= \check{\Gamma}(j^R)^* \left(\mathbb{1}_{\{0\}}(N_\infty) + \mathbb{1}_{[1,\infty[}(N_\infty)\right) \check{\Gamma}(j^R)\chi(H)[H, iA]\chi(H) \\
&= \check{\Gamma}(j^R)^* \left(\mathbb{1}_{\{0\}}(N_\infty)\right) \check{\Gamma}(j^R)\chi(H)[H, iA]\chi(H) \\
&+ \check{\Gamma}(j^R)^* \left(\mathbb{1}_{[1,\infty[}(N_\infty)\right) \check{\Gamma}(j^R)\chi(H)[H, iA]\chi(H) \\
&= \Gamma(q^R)\chi(H)[H, iA]\chi(H) \\
&+ \check{\Gamma}(j^R)^* \left(\mathbb{1}_{[1,\infty[}(N_\infty)\right) \chi(H^{ext})\check{\Gamma}(j^R)[H, iA]\chi(H) + O(R^{-1}).
\end{aligned}$$

Moreover, $[H, iA]$ is almost of the same shape as H. Using the computation that have been done before, and Lemma 2.1.11 we have that:

$$([H, iA] \otimes \mathbb{1}_{\mathscr{H}} + \mathbb{1}_{\mathscr{H}} \otimes d\Gamma(|\nabla\omega|^2))\check{\Gamma}(j^R) - \check{\Gamma}(j^R)[H, iA] = O(R^{-1}),$$

which leads to:

$$\begin{aligned}
\chi(H)[H, iA]\chi(H) &= \Gamma(q^R)\chi(H)[H, iA]\chi(H) \\
&+ \check{\Gamma}(j^R)^* \left(\mathbb{1}_{[1,\infty[}(N_\infty)\right) \chi(H^{ext})([H, iA] \otimes \mathbb{1}_{\mathscr{H}} \\
&+ \mathbb{1}_{\mathscr{H}} \otimes d\Gamma(|\nabla\omega|^2))\chi(H^{ext})\check{\Gamma}(j^R) + O(R^{-1}).
\end{aligned}$$

$\Gamma(q^R)\chi(H)[H, iA]\chi(H)$ is compact. We will call it $K_1(R)$. Assuming now $\lambda \in \tau$ then $\sup_{\kappa>0} d^\kappa(\lambda) = d(\lambda) = 0$. If $\lambda \notin \tau$ then, by closedness of τ in $[E, E + nm_{\nu_e})$, there is not any sequence of τ elements converging to λ. So, there exists $\kappa_0 > 0$ small enough such that for all $\kappa < \kappa_0$, $d^\kappa(\lambda) \neq 0$. As a trivial consequence $\sup_{\kappa>0} d^\kappa(\lambda) = d(\lambda)$. So, there exists κ small enough so that $d^\kappa(\lambda) \geqslant d(\lambda) + \frac{\epsilon}{3}$.

Let $\lambda_1 \in [E, E+(n-1)m_{\nu_e})$ then by hypothesis H_3 we have:

$$\mathbb{1}_{\Delta^\kappa_{\lambda_1}}(H)[H, iA]\mathbb{1}_{\Delta^\kappa_{\lambda_1}}(H) \geqslant (\tilde{d}^\kappa(\lambda_1) - \frac{\epsilon}{3})\mathbb{1}_{\Delta^\kappa_{\lambda_1}}(H).$$

Replacing λ_1 by $\lambda - \overline{d\Gamma}(\Omega_j)$, where $(\Omega_j)_i = \omega_i$ if $m_i \leqslant m_j$ and $(\Omega_j)_i = 0$ otherwise, multiplying to the right by $\mathbb{1}_{[1,\infty[}(N_\infty)$ to be sure that we can at least withdraw one particle we obtain:

$$\begin{aligned} & \left(\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j)) \otimes \mathbb{1}_{\mathscr{H}}\right)[H, iA] \otimes \mathbb{1}_{\mathscr{H}} \left(\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j)) \otimes \mathbb{1}_{\mathscr{H}}\right)\mathbb{1}_{[1,\infty[}(N_\infty) \\ \geqslant \quad & (\tilde{d}^\kappa(\lambda - d\Gamma(\Omega_j)) - \frac{\epsilon}{3})\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\mathbb{1}_{[1,\infty[}(N_\infty). \end{aligned}$$

Therefore:

$$\begin{aligned} & \left(\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\right)\{[H, iA] \otimes \mathbb{1}_{\mathscr{H}} + \mathbb{1}_{\mathscr{H}} \otimes d\Gamma(|\nabla\omega|)\}\left(\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\right)\mathbb{1}_{[1,\infty[}(N_\infty) \\ \geqslant \quad & \left(\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\right)(\tilde{d}^\kappa(\lambda - d\Gamma(\Omega_j)) + \mathbb{1}_{\mathscr{H}} \otimes d\Gamma(|\nabla\omega|) - \frac{\epsilon}{3})\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\mathbb{1}_{[1,\infty[}(N_\infty) \\ \geqslant \quad & \left(d^\kappa(\lambda) - \frac{\epsilon}{3}\right)\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\mathbb{1}_{[1,\infty[}(N_\infty) \\ \geqslant \quad & \left(d(\lambda) - \frac{2\epsilon}{3}\right)\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\mathbb{1}_{[1,\infty[}(N_\infty). \end{aligned}$$

This shows that:

$$\chi(H^{ext})([H, iA]\otimes\mathbb{1}_{\mathscr{H}} + \mathbb{1}_{\mathscr{H}}\otimes d\Gamma(|\nabla\omega|^2))\chi(H^{ext})\left(\mathbb{1}_{[1,\infty[}(N_\infty)\right) \geqslant \left(d(\lambda) - \frac{2\epsilon}{3}\right)\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\mathbb{1}_{[1,\infty[}(N_\infty),$$

and consequently that:

$$\chi(H^{ext})\left(\mathbb{1}_{[1,\infty[}(N_\infty)\right)([H, iA]\otimes\mathbb{1}_{\mathscr{H}} + \mathbb{1}_{\mathscr{H}}\otimes d\Gamma(|\nabla\omega|^2))\chi(H^{ext}) \geqslant \left(d(\lambda) - \frac{2\epsilon}{3}\right)\mathbb{1}_{\Delta^\kappa_\lambda}(H + d\Gamma(\Omega_j))\mathbb{1}_{[1,\infty[}(N_\infty).$$

Therefore:

$$\begin{aligned} \chi(H)[H, iA]\chi(H) &= \Gamma(q^R)\chi(H)[H, iA]\chi(H) \\ &+ \check{\Gamma}(j^R)^*\left(\mathbb{1}_{[1,\infty[}(N_\infty)\right)\chi(H^{ext})([H, iA] \otimes \mathbb{1}_{\mathscr{H}} \\ &+ \mathbb{1}_{\mathscr{H}} \otimes d\Gamma(|\nabla\omega|^2))\chi(H^{ext})\check{\Gamma}(j^R) + O(R^{-1}) \\ &\geqslant (d(\lambda) - \frac{2\epsilon}{3})\chi^2(H) + K_1(R) + O(R^{-1}). \end{aligned}$$

Recalling that $K_1(R)$ is compact and choosing R large enough proves H_1 as long as we stay in the subset $[E, E + nm_{\nu_e})$ of the spectrum. We have just shown that the hypothesis $H_3(n-1)$, together with $S_1(n)$ implies $H_1(n)$. Note that for only one electron neutrino, H_1 is obvious. Therefore, an induction argument is enough to conclude the proof. □

2.2.3 Spectral analysis for small coupling constant and regularized kernels

In this section, we improve the results of Theorem 2.2.4 by imposing stronger conditions on the kernels and treating the coupling constant g as a small parameter. The main idea consists in considering a different, non-self-adjoint conjugate operator, in order to obtain a global Mourre estimate without compact remainder. We introduce the following operators:

$$b_{(i),l} := \frac{\mathrm{i}}{2}\big((p_i \cdot \nabla\omega_l^{(i)}(p_i))^{-1} p_i \cdot \nabla_{p_i} + \nabla_{p_i} \cdot p_i (p_i \cdot \nabla\omega_l^{(i)}(p_i))^{-1}\big), \quad i = 1,2, \quad l \in \{e,\mu,\tau\}, \tag{2.56}$$

$$b_{(3)} := \frac{\mathrm{i}}{2}\big((p_3 \cdot \nabla\omega^{(3)}(p_3))^{-1} p_3 \cdot \nabla_{p_3} + \nabla_{p_3} \cdot p_3 (p_3 \cdot \nabla\omega^{(3)}(p_3))^{-1}\big), \tag{2.57}$$

acting on $\mathfrak{h}_1$, and $b_{(3)}$ acting on $\mathfrak{h}_2$. In particular, the operators $b_{(i),l}$ with domains $\mathrm{C}_0^\infty(\mathbb{R}^3\backslash\{0\}) \times \{-\frac{1}{2},\frac{1}{2}\}$ are symmetric, and likewise $b_{(3)}$ with domain $\mathrm{C}_0^\infty(\mathbb{R}^3\backslash\{0\}) \times \{-1,0,1\}$ is symmetric. Their closures are denoted by the same symbols. Moreover, we set

$$\begin{aligned} &B := \mathrm{d}\Gamma(b), \\ &b := (b_{(3)}, b_{(3)}, b_{(1),1}, b_{(1),1}, b_{(2),1}, b_{(2),1}, b_{(1),2}, b_{(1),2}, b_{(2),2}, b_{(2),2}, b_{(1),3}, b_{(1),3}, b_{(2),3}, b_{(2),3}), \end{aligned} \tag{2.58}$$

(hence $b_1 = b_{(3)}$, $b_2 = b_{(3)}$, $b_3 = b_{(1),1}$, $\dots$, $b_{14} = b_{(2),3}$.). Then $\dim(\mathrm{Ker}(B^* - \mathrm{i})) = 0$, B generates a C_0-semigroup of isometries $\{W_t\}_{t\geqslant 0}$ and a direct computation gives

$$[H_0, \mathrm{i}B] = N, \tag{2.59}$$

in the sense of quadratic forms. Moreover, the commutators $[H_I^{(j)}(G), \mathrm{i}B]$, $j = 1,\dots,4$ are given by (2.41)–(2.44) with b_i instead of a_i.

A straightforward modification of [16, Section 5.1] shows that W_t and W_t^* preserve $\mathscr{D}(|H|^{\frac{1}{2}})$, and that for all $\phi \in \mathscr{D}(|H|^{\frac{1}{2}})$,

$$\sup_{0<t<1} \|W_t\phi\|_{\mathscr{D}(|H|^{\frac{1}{2}})} < \infty, \quad \sup_{0<t<1} \|W_t^*\phi\|_{\mathscr{D}(|H|^{\frac{1}{2}})} < \infty. \tag{2.60}$$

Before proving a Mourre estimate and deducing from it spectral properties of H, we must show, as in the previous section, that H is regular enough with respect to the conjugate operator.

Recall again that, for $i = 1,2$, the notation "$b_{(i),\cdot}G \in L^2$" means that, for all j, l and ϵ, $b_{(i),l}G_{l,\epsilon}^{(j)}$ is square integrable, and that "$b_{(3),\cdot}G \in L^2$" means that, for all j, l and ϵ, $b_{(3)}G_{l,\epsilon}^{(j)}$ is square integrable.

Lemma 2.2.5. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5)*. Assume that*

$$G \in L^2, \quad b_{(i),\cdot}G \in L^2, \quad i = 1,2,3. \tag{2.61}$$

Then H is of class $\mathrm{C}^1(B; \mathscr{D}(|H|^{\frac{1}{2}}); \mathscr{D}(|H|^{\frac{1}{2}})^*)$. If in addition

$$b_{(i),\cdot}b_{(i'),\cdot}G \in L^2, \quad i,i' = 1,2,3, \tag{2.62}$$

then H is of class $\mathrm{C}^2(B; \mathscr{D}(|H|^{\frac{1}{2}}); \mathscr{D}(|H|^{\frac{1}{2}})^*)$.

Proof. Suppose that (2.61) holds. In order to verify that H is of class $\mathrm{C}^1(B;\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$, since (2.60) holds, it suffices to prove (see [47, Proposition 5.2]) that the quadratic form $[H,B]$ defined on $\mathscr{D}(|H|^{\frac{1}{2}})\cap\mathscr{D}(B)$ by $\langle\phi,[H,B]\phi\rangle=\langle\phi,HB\phi\rangle-\langle B^*\phi,H\phi\rangle$ extends to an element of $\mathscr{L}(\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$. By (2.59) and the fact that N is relatively H-bounded, it is clear that $[H_0,B]$ extends to an element of $\mathscr{L}(\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$. That $[H_I,B]$ also extends to an element of $\mathscr{L}(\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$ follows from the expression of the commutator (given by (2.41)–(2.44) with b_i instead of a_i, as mentioned above) together with Lemma 2.1.3, which can be applied under the hypothesis (2.61).

To prove that H is of class $\mathrm{C}^2(B;\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$ under the further assumption (2.62), it suffices to verify similarly that $[[H,B],B]$ extends to an element of $\mathscr{L}(\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$. But we have that $[[H_0,B],B]=0$ and a similar computation as before shows that $[[H_I,B],B]$ extends to an element of $\mathscr{L}(\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$ by Lemma 2.1.3. □

The next theorem establishes a global Mourre estimate for H, from which, using the regularity properties proven in the previous lemma, we can deduce the desired spectral properties of H.

Theorem 2.2.6. *Consider the Hamiltonian* (2.12) *with H_I given by* (2.5) *and assume that* (2.61) *holds. There exist $g_0>0$, $\mathrm{c}>0$ and $\mathrm{d}>0$ such that, for all values of the masses of the neutrinos $m_{\nu_e}>0$, $m_{\nu_\mu}>0$, $m_{\nu_\tau}>0$,*

$$[H,\mathrm{i}B]\geqslant \mathrm{c}\mathbb{1}-\mathrm{d}\Pi_\Omega, \tag{2.63}$$

where Π_Ω denotes the projection onto the vacuum in $\mathscr{H}$. In particular, $E=\inf\sigma(H)$ is the only eigenvalue of H, and E is non-degenerate. If, in addition, (2.62) *holds, then the spectrum of H in $[E+m_\nu,\infty)$ is purely absolutely continuous.*

Proof. Recall that, in the sense of quadratic forms on $\mathscr{D}(B)\cap\mathscr{D}(H_0)$ we have that $[H_0,\mathrm{i}B]=N$ and that $[H_I,\mathrm{i}B]$ is relatively N-bounded by Lemma 2.1.3. Therefore there exists $\mathrm{c}_1>0$ and $\mathrm{c}_2>0$, which do not depend on the masses of the neutrinos, such that

$$\langle\psi,[H_I,\mathrm{i}B]\psi\rangle\leqslant \mathrm{c}_1\langle\psi,N\psi\rangle+\mathrm{c}_2\|\psi\|^2.$$

This yields

$$\begin{aligned}[H,\mathrm{i}B]=[H_0,\mathrm{i}B]+g[H_I,\mathrm{i}B]&\geqslant N-\mathrm{c}_1gN-\mathrm{c}_2g\\&\geqslant(1-\mathrm{c}_1g)(N+\Pi_\Omega-\Pi_\Omega)-\mathrm{c}_2g\\&\geqslant(1-\mathrm{c}_1g)\mathbb{1}-(1-\mathrm{c}_1g)\Pi_\Omega-\mathrm{c}_2g\\&\geqslant(1-(\mathrm{c}_1+\mathrm{c}_2)g)\mathbb{1}-(1-\mathrm{c}_1g)\Pi_\Omega,\end{aligned}$$

which proves (2.63).

The fact that (2.63) implies that H has at most one eigenvalue is a consequence of the virial theorem (see Lemma 10 of [68]), which holds since $H\in\mathrm{C}^1(B;\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$ by Lemma 2.2.5, together with the fact that $\dim(\mathrm{Ran}(\Pi_\Omega))=1$. See, e.g., [67, Lemma 10].

That the spectrum of H in $[E+m_\nu,\infty)$ is purely absolutely continuous if (2.62) is satisfied is a consequence of the abstract results recalled at the beginning of this section together with the fact that $H\in\mathrm{C}^2(B;\mathscr{D}(|H|^{\frac{1}{2}});\mathscr{D}(|H|^{\frac{1}{2}})^*)$, by Lemma 2.2.5. □

2.3 Propagation Estimates

In this section, we use the method of propagation observables that was developed in N-body scattering theory (see e.g. [88, 69, 53, 26, 28] and references therein). This method was adapted to the context of Pauli-Fierz or $P(\varphi)_2$ Hamiltonians in several papers (see, in particular, [29, 30, 43, 44, 5, 45, 21, 38]). In this section only the case where all particles are massive will be considered and the propagation estimates that we prove are straightforwardly adapted from [53, 29].

The basic approach that we follow to prove our propagation estimates is the following: let H be a self-adjoint operator on a Hilbert space $\mathscr{H}$ and let $\Phi(t)$ be a time-dependent family of self-adjoint operators. Suppose that for some $u \in \mathcal{H}$,

$$\big\langle e^{-\mathrm{i}tH}u, \Phi(t)e^{-\mathrm{i}tH}u\big\rangle \leqslant \mathrm{C}_u, \quad \text{uniformly in } t \geqslant 1,$$

and that one of the following two conditions holds,

$$\partial_t\big\langle e^{-\mathrm{i}tH}u, \Phi(t)e^{-\mathrm{i}tH}u\big\rangle \geqslant \big\langle e^{-\mathrm{i}tH}u, \Psi(t)e^{-\mathrm{i}tH}u\big\rangle - \sum_{j=1}^{n}\big\langle e^{-\mathrm{i}tH}u, B_j^*(t)B_j(t)e^{-\mathrm{i}tH}u\big\rangle, \tag{2.64}$$

$$\partial_t\big\langle e^{-\mathrm{i}tH}u, \Phi(t)e^{-\mathrm{i}tH}u\big\rangle \leqslant -\big\langle e^{-\mathrm{i}tH}u, \Psi(t)e^{-\mathrm{i}tH}u\big\rangle + \sum_{j=1}^{n}\big\langle e^{-\mathrm{i}tH}u, B_j^*(t)B_j(t)e^{-\mathrm{i}tH}u\big\rangle, \tag{2.65}$$

where $\Psi(t)$ are positive operators and $B_i(t)$ are families of time-dependent operators such that

$$\int_1^\infty \big\|B_i(t)e^{-\mathrm{i}tH}u\big\|^2 dt \leqslant \mathrm{C}_u. \tag{2.66}$$

Then, integrating with respect to t, one obtains that

$$\int_1^\infty \big\langle e^{-\mathrm{i}tH}u, \Psi(t)e^{-\mathrm{i}tH}u\big\rangle dt \lesssim \mathrm{C}_u, \tag{2.67}$$

which is sometimes called a weak propagation estimate for the family of observables $\Psi(t)$. Observe that the left-hand-sides of (2.64)–(2.65) can be rewritten as

$$\big\langle e^{-\mathrm{i}tH}u, \mathbf{D}\Phi(t)e^{-\mathrm{i}tH}u\big\rangle,$$

where $\mathbf{D}$ stands for the Heisenberg derivative $\mathbf{D}\Phi(t) = \partial_t\Phi(t) + [H, \mathrm{i}\Phi(t)]$. Therefore, to prove the propagation observable (2.67), it suffices to find a family of operators $\Phi(t)$ whose Heisenberg derivative “dominates” $\Psi(t)$, in the sense that $\mathbf{D}\Phi(t) \geqslant \Psi(t)$ or $\mathbf{D}\Phi(t) \leqslant -\Psi(t)$, up to remainder terms that are integrable in the sense of (2.66). The strategy usually consists in comparing the time derivative $\partial_t\Phi(t)$ and the commutator $[H, \mathrm{i}\Phi(t)]$, possibly by means of a “commutator expansion” of $[H, \mathrm{i}\Phi(t)]$. We refer the reader to e.g. [28] for details on the method of propagation observables, see also [38, Section 2] for a description of the method of propagation observables for Pauli-Fierz Hamiltonians, closely related to the approach that we will follow here.

It is useful to introduce the following notations

$$\mathbf{d}_{0l}^{(i)}b(t) = \frac{\partial b}{\partial t}(t) + [\omega_l^{(i)}(p_i), \mathrm{i}b(t)], \quad i = 1, 2, \quad \mathbf{d}_0^{(3)}b(t) = \frac{\partial b}{\partial t}(t) + [\omega^{(3)}(p_i), \mathrm{i}b(t)],$$

if $b(t)$ is a family of operators, acting on $\mathfrak{h}_1$ if $i = 1, 2$, or acting on $\mathfrak{h}_2$ if $i = 3$. Likewise, we set

$$\mathbf{D}_0B(t) = \frac{\partial B}{\partial t}(t) + [H_0, \mathrm{i}B(t)], \quad \mathbf{D}B(t) = \frac{\partial B}{\partial t}(t) + [H, \mathrm{i}B(t)],$$

if $B(t)$ is a family of operators acting on $\mathscr{H}$. Note that if $B(t) = (b_1(t), \dots, b_{14}(t))$, remembering that the total Hilbert space $\mathscr{H}$ is the tensor product of 14 Fock spaces, then, as functions of t,

$$\begin{aligned}\mathbf{D}_0 \mathrm{d}\Gamma(B) =: \mathrm{d}\Gamma(\mathbf{d}_0 b) &:= \mathrm{d}\Gamma(\mathbf{d}_{0,1} b_1, \dots, \mathbf{d}_{0,14} b_{14})\\ &:= \mathrm{d}\Gamma\big(\mathbf{d}_0^{(3)} b_1, \mathbf{d}_0^{(3)} b_2, \mathbf{d}_{01}^{(1)} b_3, \mathbf{d}_{01}^{(1)} b_4, \mathbf{d}_{01}^{(2)} b_5, \mathbf{d}_{01}^{(2)} b_6, \mathbf{d}_{02}^{(1)} b_7,\\ &\qquad \mathbf{d}_{02}^{(1)} b_8, \mathbf{d}_{02}^{(2)} b_9, \mathbf{d}_{02}^{(2)} b_{10}, \mathbf{d}_{03}^{(1)} b_{11}, \mathbf{d}_{03}^{(1)} b_{12}, \mathbf{d}_{01}^{(2)} b_{13}, \mathbf{d}_{01}^{(2)} b_{14}\big).\end{aligned}$$

In the remainder of this section, we prove the propagation estimates that will be used in Section 2.4.

As mentioned before, the proofs of the propagation estimates of this section are almost straightforward adaptations of the ones in [29, 30]. However the proofs will be recalled for consistency.

To shorten expressions below, we set $x_i = (x_i^{(1)}, x_i^{(2)}, x_i^{(3)}) = \mathrm{i}\nabla$, $i = 1, \dots, 14$, and, for $R = (R_1, \dots, R_{14})$ and $R' = (R'_1, \dots, R'_{14})$,

$$\mathbb{1}_{[R,R']}(|x|) = \big(\mathbb{1}_{[R_1,R'_1]}(|x_1|), \dots, \mathbb{1}_{[R_{14},R'_{14}]}(|x_{14}|)\big) = \big(\mathbb{1}_{[R_1,R'_1]}(\sqrt{-\Delta}), \dots, \mathbb{1}_{[R_{14},R'_{14}]}(\sqrt{-\Delta})\big), \tag{2.68}$$

and likewise for other functions of x. The operators $\mathrm{d}\Gamma(\mathbb{1}_{[R,R']}(|x|))$, $\Gamma(\mathbb{1}_{[R,R']}(|x|))$ are then defined, as in the previous sections, following the conventions of Chapter 1. We also set

$$\omega := (\omega^{(3)}, \omega^{(3)}, \omega_1^{(1)}, \omega_1^{(1)}, \omega_1^{(2)}, \omega_1^{(2)}, \omega_2^{(1)}, \omega_2^{(1)}, \omega_2^{(2)}, \omega_2^{(2)}, \omega_3^{(1)}, \omega_3^{(1)}, \omega_3^{(2)}, \omega_3^{(2)}). \tag{2.69}$$

Recall that, for $i = 1, 2$, the notation "$a_{(i),\cdot} G \in L^2$" means that, for all j, l and ϵ, $a_{(i),l} G_{l,\epsilon}^{(j)}$ is square integrable, and that "$a_{(3),\cdot} G \in L^2$" means that for all j, l and ϵ, $a_{(3)} G_{l,\epsilon}^{(j)}$ is square integrable.

Theorem 2.3.1. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *Assume that*

$$G \in L^2, \quad a_{(i),\cdot} G \in L^2, \quad i = 1, 2, 3.$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

(i) Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$, $R = (R_1, \dots, R_{14})$ and $R' = (R'_1, \dots, R'_{14})$ be such that $R'_i > R_i > 1$. There exists $\mathrm{C} > 0$ such that, for all $u \in \mathscr{H}$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\mathbb{1}_{[R,R']}\Big(\frac{|x|}{t}\Big)\Big)^{\frac{1}{2}} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C} \|u\|^2.$$

(ii) Let $0 < \mathrm{v}_0 < \mathrm{v}_1$ and $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. There exists $\mathrm{C} > 0$ such that, for all $l \in \{e, \mu, \tau\}$ and $u \in \mathscr{H}$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\Big\langle \big(\frac{x}{t} - \nabla\omega\big), \mathbb{1}_{[\mathrm{v}_0,\mathrm{v}_1]}\big(\frac{x}{t}\big)\big(\frac{x}{t} - \nabla\omega\big)\Big\rangle\Big)^{\frac{1}{2}} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C} \|u\|^2.$$

(iii) Let $0 < \mathrm{v}_0 < \mathrm{v}_1$, $J \in \mathrm{C}_0^\infty(\{x \in \mathbb{R}^3, \mathrm{v}_0 < |x| < \mathrm{v}_1\})$ and $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. There exists $\mathrm{C} > 0$ such that, for all $l \in \{e, \mu, \tau\}$ and $u \in \mathscr{H}$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\Big| J\Big(\frac{x}{t}\Big)\Big(\frac{x^{(\ell)}}{t} - \partial_{(\ell)}\omega\Big) + \mathrm{h.c}\Big|\Big)^{\frac{1}{2}} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C} \|u\|^2.$$

(iv) *Let* $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ *be supported in* $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$. *There exist* $\epsilon > 0$ *and* $\mathrm{C} > 0$ *such that, for all* $u \in \mathscr{H}$,

$$\int_1^\infty \Big\| \Gamma\Big(\mathbb{1}_{[0,\epsilon]}\Big(\frac{|x|}{t}\Big)\Big) \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C}\|u\|^2.$$

Proof. (i) By definition of the operator $\mathrm{d}\Gamma$, we have that

$$\begin{aligned}\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\mathbb{1}_{[R,R']}\Big(\frac{|x|}{t}\Big)\Big)^{\frac12} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} &= \int_1^\infty \Big\| \sum_{i=1}^{14} \mathrm{d}\Gamma_i\Big(\mathbb{1}_{[R_i,R_i']}\Big(\frac{\sqrt{-\Delta}}{t}\Big)\Big)^{\frac12} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \\ &\lesssim \sum_{i=1}^{14} \int_1^\infty \Big\| \mathrm{d}\Gamma_i\Big(\mathbb{1}_{[R_i,R_i']}\Big(\frac{\sqrt{-\Delta}}{t}\Big)\Big)^{\frac12} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t},\end{aligned} \tag{2.70}$$

where we have set

$$\mathrm{d}\Gamma_i(B) = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \mathrm{d}\Gamma(B) \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{14-i}. \tag{2.71}$$

Next we proceed as in [29, 5]. Let $F \in \mathrm{C}^\infty(\mathbb{R})$ be equal to 1 near ∞, to 0 on a compact set near the origin, and such that $F'(s) \geqslant \mathbb{1}_{[R_i,R_i']}(s)$. We define in addition

$$\Phi(t) = \chi(H)\mathrm{d}\Gamma_i\Big(F\Big(\frac{|x_i|}{t}\Big)\Big)\chi(H), \qquad b(t) = \mathbf{d}_{0,i}F\left(\frac{|x_i|}{t}\right).$$

A direct computation then shows that

$$\mathbf{D}\Phi(t) = \chi(H)b(t)\chi(H) + \chi(H)\,\mathrm{i}\Big[H_I, \mathrm{d}\Gamma_i\Big(F\Big(\frac{|x_i|}{t}\Big)\Big)\Big]\chi(H).$$

Using pseudo-differential calculus (or a commutator expansion at second order, see, e.g., [28]) gives:

$$\begin{aligned} b(t) &= \frac{\partial}{\partial t}F\left(\frac{|x_j|}{t}\right) + \left[\omega_j(k_j), iF\left(\frac{|x_j|}{t}\right)\right] \\ &= F'\left(\frac{|x_j|}{t}\right)\frac{-|x_j|}{t^2} + \frac{x_j}{|x_j|t}\nabla\omega_j(k_j)F'\left(\frac{|x_j|}{t}\right) + O(t^{-2}) \\ &= F'\left(\frac{|x_j|}{t}\right)\left(\frac{-|x_j|}{t^2} + \frac{x_j}{|x_j|t}\nabla\omega_j(k_j)\right) + O(t^{-2}) \\ &\leqslant \frac{-C_0}{t}\mathbb{1}_{[R_j,R'j]}(s) + O(t^{-2}), \end{aligned}$$

where C_0 is a positive constant. Besides, since F_i vanishes near 0 and since $G \in \mathbb{H}^{1+\mu}$, we have that

$$\Big\| F_i\Big(\frac{|x_i|}{t}\Big)G\Big\|_2 = \mathcal{O}(t^{-1-\mu}).$$

Therefore one can deduce from Lemma 2.1.3 together with the fact that N is relatively H-bounded that

$$\chi(H)\Big[H_I, \mathrm{d}\Gamma_i\Big(F\Big(\frac{|x_i|}{t}\Big)\Big)\Big]\chi(H) = \mathcal{O}(t^{-1-\mu}).$$

Hence the condition (2.64) is satisfied and it suffices to apply the abstract method recalled at the beginning of Section 2.3.

(ii) The problem reduces again to the one particle case noticing that:

$$\begin{aligned}\int_1^\infty \left\| \Theta_{[v_0,v_1]}(t)^{\frac{1}{2}} \chi(H) e^{-itH} u \right\|^2 \frac{dt}{t} &\leqslant \int_1^\infty \left\| \sum_{i=1}^p \Theta_{i,[v_0,v_1]}(t)^{\frac{1}{2}} \chi(H) e^{-itH} u \right\|^2 \frac{dt}{t} \\ &\leqslant \sum_{i=1}^p \int_1^\infty \left\| \Theta_{i,[v_0,v_1]}(t)^{\frac{1}{2}} \chi(H) e^{-itH} u \right\|^2 \frac{dt}{t}.\end{aligned}$$

Were the following notations have been used:

$$\Theta_{[v_0,v_1]}(t) = \mathrm{d}\Gamma\Big(\Big\langle \frac{x}{t} - \nabla\omega \Big| \mathbb{1}_{[\mathrm{v}_0,\mathrm{v}_1]}\left(\frac{x}{t}\right)\left(\frac{x}{t} - \nabla\omega\right)\Big\rangle\Big), \tag{2.72}$$

$$\Theta_{i,[v_0,v_1]}(t) = d\Gamma_i\left(\left\langle \frac{x_i}{t} - \nabla\omega(k_i) \middle| \mathbb{1}_{[v_0,v_1]}\left(\frac{x_i}{t}\right)\left(\frac{x_i}{t} - \nabla\omega_i(k_i)\right)\right\rangle\right). \tag{2.73}$$

We can now proceed as in [29, 5]. Let $R_0(x)$ be an increasing function such that:

$$\begin{aligned}&R_0(x_j) = 0, \ \text{for } |x_j| \leqslant \frac{v_0}{2} \\ &R_0(x_j) = \frac{1}{2}x_j^2 + c, \ \text{for } |x_j| \geqslant 2v_1 \\ &R_0(x_j) \geqslant \mathbb{1}_{[v_{0j},v_{1j}]}(|x_j|).\end{aligned}$$

Moreover, fixing $v_{2j} > v_{1j} + 1$, the function R can be defined as:

$$R(x_j) = F(|x_j|)R_0(x_j),$$

where F is chosen so that, $F(s) = 1, \ s \leqslant v_{1j}, \ F(s) = 0, \ s \geqslant v_{2j}$. A simple computation gives:

$$\begin{aligned}\nabla^2_{x_j} R(x_j) &= \nabla^2_{x_j} F(|x_j|)R_0(x_j) + 2\nabla_{x_j} F(|x_j|)\nabla_{x_j} R_0(x_j) + F(|x_j|)\nabla^2_{x_j} R_0(x_j) \\ &\geqslant -C\mathbb{1}_{[v_{1j},v_{2j}]}(|x_j|) + F(|x_j|)\nabla^2_{x_j} R_0(x_j) \\ &\geqslant -C\mathbb{1}_{[v_{1j},v_{2j}]}(x_j) + \mathbb{1}_{[v_{0j},v_{1j}]}(|x_j|)\nabla^2_{x_j} R_0(x_j) \\ &\geqslant \mathbb{1}_{[v_{0j},v_{1j}]}(x_j) - C\mathbb{1}_{[v_{1j},v_{2j}]}(x_j).\end{aligned}$$

Let

$$b_i(t) = R\left(\frac{x_j}{t}\right) - \frac{1}{2}\left(\left\langle \nabla R\left(\frac{x_j}{t}\right) \middle| \frac{x_j}{t} - \nabla\omega_j(k_j)\right\rangle\right).$$

Considering:

$$\Phi(t) = \chi(H) d\Gamma_i(b(t))\chi(H).$$

Note that:

$$\begin{aligned}D\Phi(t) &= \chi(H)\frac{\partial}{\partial t} d\Gamma_j\left(b(t)\right)\chi(H) + \chi(H) i \left[H, d\Gamma_j\left(b(t)\right)\right]\chi(H) \\ &= \chi(H) D_0 d\Gamma_j\left(b(t)\right)\chi(H) + \chi(H) i \left[H_I, d\Gamma_j\left(b(t)\right)\right]\chi(H) \\ &= \chi(H) d\Gamma_j\left(d_0 b(t)\right)\chi(H) + \chi(H) i \left[H_I, d\Gamma_j\left(b(t)\right)\right]\chi(H).\end{aligned}$$

On the one hand we have:

$$\begin{aligned} d_0 b(t) &= \frac{-x_j}{t^2}\nabla R\left(\frac{x_j}{t}\right) - \frac{1}{2}\left(\left\langle \frac{-x_j}{t^2}\nabla^2 R\left(\frac{x_j}{t}\right)\middle| \frac{x_j}{t} - \omega_j(k_j)\right\rangle + \left\langle \nabla R\left(\frac{x_j}{t}\right)\middle|\frac{-x_j}{t^2}\right\rangle + h.c\right) \\ &+ \frac{1}{t}\nabla\omega_j(k_j)\nabla R\left(\frac{x_j}{t}\right) + \frac{1}{2t}\nabla\omega_j(k_j)\left(\left\langle \nabla^2 R\left(\frac{x_j}{t}\right)\middle|\frac{x_j}{t} - \omega_j(k_j)\right\rangle\right. \\ &+ \left.\left\langle \nabla R\left(\frac{x_j}{t}\right)\middle| \mathbb{1}\right\rangle + h.c\right) + O(t^{-2}) \\ &= \frac{1}{t}\left\langle \frac{x_j}{t} - \nabla\omega_j(k_j)\middle|\nabla^2 R\left(\frac{x}{t}\right)\left(\frac{x}{t} - \nabla\omega_j(k_j)\right)\right\rangle + O(t^{-2}) \\ &\geqslant \frac{1}{t}\left\langle \frac{x_j}{t} - \nabla\omega_j(k_j)\middle| \mathbb{1}_{[v_{0j},v_{1j}]}\left(\frac{|x_j|}{t}\right)\left(\frac{x_j}{t} - \nabla\omega_j(k_j)\right)\right\rangle \\ &- \frac{C}{t}\left\langle \frac{x_j}{t}\nabla\omega(k_j)\middle| \mathbb{1}_{[v_{1j},v_{2j}]}\left(\frac{|x_j|}{t}\right)\left(\frac{x_j}{t} - \nabla\omega_j(k_j)\right)\right\rangle + O(t^{-2}). \end{aligned}$$

So as a consequence:

$$D_0\Phi(t) \geqslant \frac{1}{t}\Theta_{i,[v_0,v_1]}(t) - \frac{C}{t}d\Gamma_i\left(\mathbb{1}_{[v_{1i},v_{2i}]}\left(\frac{|x_i|}{t}\right)\right) + O(t^{-2}).$$

On the other hand and as before:

$$\chi(H)\left[H_I, id\Gamma_j\left(b(t)\right)\right]\chi(H) = O(t^{-1-\mu}).$$

Therefore, the abstract method recalled at the beginning of Section 2.3 can be used again to prove (ii).

(iii) Let us first recall the following lemma, which holds in the one particle case (see [29]):

Lemma 2.3.2. *Let $A = (\frac{x}{t} - \nabla\omega(k))^2 + t^{-\delta}$, $\delta > 0$. Let $J, J_1, J_2 \in C_0^\infty(\mathbb{R}^d)$ with $J_1 = 1$ near $suppJ$, $J_2 = 1$ near $suppJ_1$, $0 \leqslant J_2 \leqslant 1$. Then*

i) $J(\frac{x}{t})A^{\frac{1}{2}} = O(1)$

ii) $[A^{1/2}, J(\frac{x}{t})] = O(t^{\frac{\delta}{2}-1})$

iii) $d_0 A^{\frac{1}{2}} = -\frac{1}{t}A^{\frac{1}{2}} + O(t^{-1-\frac{\delta}{2}})$

For $1 \leqslant i \leqslant d$ and for $\epsilon = inf(\delta, 1 - \frac{\delta}{2})$.

iv) $|J(\frac{x}{t})(\frac{x(i)}{t} - \partial_i\omega(k)) + h.c| \leqslant CJ_2(\frac{x}{t})A^{\frac{1}{2}}J_2(\frac{x}{t}) + Ct^{-\frac{\epsilon}{2}}$

v) $J(\frac{x}{t})(\frac{x_i}{t} - \partial_i\omega(k))A^{\frac{1}{2}}J_1(\frac{x}{t}) + h.c \leqslant C\left\langle(\frac{x}{t} - \nabla\omega(k))\,\middle|\, J_2^2(\frac{x}{t})(\frac{x}{t} - \nabla\omega(k))\right\rangle + Ct^{-\epsilon}$

Now, (iii) can be proved. As usual, it is enough to prove the proposition for one particle sector. Let

$$b(t) = J_1(\frac{x_j}{t})A_j^{\frac{1}{2}}J_1(\frac{x_j}{t}),$$

and

$$\Phi(t) = -\chi(H)d\Gamma_i(b(t))\chi(H),$$

where

$$A_j = (\frac{x_j}{t} - \nabla\omega(k_j))^2 + t^{-\delta}.$$

According to Lemma 2.3.2 (i) and Lemma 2.1.5, $\Phi(t) = O(1)$.

$$\begin{aligned} D\Phi(t) &= \chi(H) D_0 d\Gamma_j\left(b(t)\right)\chi(H) + \chi(H) i\left[H_I, d\Gamma_j\left(b(t)\right)\right]\chi(H) \\ &= \chi(H) d\Gamma_j\left(d_0 b(t)\right)\chi(H) + \chi(H) i\left[H_I, d\Gamma_j\left(b(t)\right)\right]\chi(H). \end{aligned}$$

The commutator $[H_I, id\Gamma_j(b(t))]$ is treated as usual and:

$$\chi(H)\left[H_I, id\Gamma_j\left(b(t)\right)\right]\chi(H) = O(t^{-1-\mu}).$$

Moreover,

$$d_0 b(t) = \left(d_0 J_1\left(\frac{x_j}{t}\right)\right) A_j^{\frac{1}{2}} J_1\left(\frac{x_j}{t}\right) + h.c + J_1\left(\frac{x_j}{t}\right)\left(d_0 A_j^{\frac{1}{2}}\right) J_1\left(\frac{x_j}{t}\right).$$

On the one hand, Lemma 2.3.2 (iii) implies

$$d_0 A_j^{\frac{1}{2}} = -\frac{A_j^{\frac{1}{2}}}{t} + O(t^{-1-\frac{\delta}{2}}).$$

Consequently:

$$\begin{aligned} J_1\left(\frac{x_j}{t}\right)\left(d_0 A_j^{\frac{1}{2}}\right) J_1\left(\frac{x_j}{t}\right) &= J_1\left(\frac{x_j}{t}\right)\left(-\frac{A_j^{\frac{1}{2}}}{t} + O(t^{-1-\frac{\delta}{2}})\right) J_1\left(\frac{x_j}{t}\right) \\ &= J_1\left(\frac{x_j}{t}\right)\left(-\frac{A_j^{\frac{1}{2}}}{t}\right) J_1\left(\frac{x_j}{t}\right) + O(t^{-1-\frac{\delta}{2}}), \end{aligned}$$

and according to Lemma 2.3.2 (iv):

$$\begin{aligned} C J_1\left(\frac{x_j}{t}\right) A_j J_1\left(\frac{x_j}{t}\right) + Ct^{-\frac{\epsilon}{2}} &\geqslant \left| J\left(\frac{x_j}{t}\right)\left(\frac{x_j(i)}{t} - \partial_i\omega(k_j)\right) + h.c\right| \\ J_1\left(\frac{x_j}{t}\right) A_j J_1\left(\frac{x_j}{t}\right) &\geqslant C_0\left| J\left(\frac{x_j}{t}\right)\left(\frac{x_j(i)}{t} - \partial_i\omega(k_j)\right) + h.c\right| - t^{-\frac{\epsilon}{2}}. \end{aligned}$$

Then:

$$-J_1\left(\frac{x_j}{t}\right)\left(d_0 A_j^{\frac{1}{2}}\right) J_1\left(\frac{x_j}{t}\right) \geqslant \frac{C_0}{t}\left| J\left(\frac{x_j}{t}\right)\left(\frac{x_j(i)}{t} - \partial_i\omega(k_j)\right) + h.c\right| - Ct^{-1-\epsilon}. \quad (2.74)$$

On the other hand,

$$\begin{aligned} d_0 J_1\left(\frac{x_j}{t}\right) &= \frac{-x_j}{t^2}\nabla J_1\left(\frac{x_j}{t}\right) + i\left[\omega(k_j), J_1\left(\frac{x_j}{t}\right)\right] \\ &= \frac{-x_j}{t^2}\nabla J_1\left(\frac{x_j}{t}\right) + \frac{1}{t}\nabla\omega(k_j)\nabla J_1\left(\frac{x_j}{t}\right) + O(t^{-2}). \end{aligned}$$

Applying now Lemma 2.3.2 (v) gives:

$$-\left(d_0 J_1\left(\frac{x_j}{t}\right)\right) A^{\frac{1}{2}} J_1\left(\frac{x_j}{t}\right) + h.c \geqslant -\frac{C}{t}\left\langle \frac{x_j}{t} - \nabla\omega(k_j)\right| J_2\left(\frac{x_j}{t}\right)\left(\frac{x_j}{t} - \nabla\omega(k_j)\right)\right\rangle + O(t^{-1-\epsilon}). \quad (2.75)$$

The result follows from (2.74), (2.75).

(iv) Let us first consider χ supported near $\lambda \in \mathbb{R}\backslash(\tau \cup \sigma_{pp}(H))$. Using the first version of the Mourre estimate presented in Theorem 2.2.4, the function χ can be chosen equal to one near λ and *supp*χ small enough so that:

$$\chi(H)[H, iA]\chi(H) \geqslant C_0\chi^2(H).$$

Let $\epsilon > 0$, $\{q_i\}_{i\in\{1,\ldots,14\}} \in C_0^\infty\left(\{|x_i| \leqslant 2\epsilon\}\right)$ such that $0 \leqslant q_i \leqslant 1$, $q_i = 1$ near on $\{|x_i| \leqslant \epsilon\}$ and let $q^t = q\left(\frac{x}{t}\right)$. Let:

$$\Phi(t) = \chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H).$$

A is $(N+1)$ bounded and because of the support of q^t:

$$\pm\Gamma(q^t)\frac{A}{t}\Gamma(q^t) \leqslant C\epsilon(N+1) \quad \left\|\frac{A}{t}\Gamma(q^t)(N+1)^{-1}\right\| \leqslant C\epsilon. \tag{2.76}$$

To prove the second estimates it has to be noted that:

$$\left\|\frac{A}{t}\Gamma(q^t)(N+1)^{-1}\right\| \leqslant \left\|\Gamma(q^t)\frac{A}{t}(N+1)^{-1}\right\| + \left\|\left[\frac{A}{t}, \Gamma(q^t)\right](N+1)^{-1}\right\|$$

and using Helffer-Sjöstrand formula:

$$\begin{aligned}
\left[\frac{A}{t}, \Gamma(q^t)\right] &= \int \partial_{\bar{z}}\tilde{q}(z)\left(z - \frac{x}{t}\right)^{-1}\left[\frac{A}{t}, \frac{x}{t}\right]\left(z - \frac{x}{t}\right)^{-1} \\
&= \int \partial_{\bar{z}}\tilde{q}(z)\left(z - \frac{x}{t}\right)^{-1}\frac{x}{t^2}\nabla^2\omega(p)\left(z - \frac{x}{t}\right)^{-1}.
\end{aligned}$$

Using (2.76) and Lemma 2.1.5, it can be shown that $\Phi(t)$ is uniformly bounded. Furthermore:

$$\begin{aligned}
D\Phi(t) &= D\chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H) \\
&= \chi(H)\left(\frac{\partial}{\partial t}\left\{\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\right\} + \left[H, \Gamma(q^t)\frac{A}{t}\Gamma(q^t)\right]\right)\chi(H) \\
&= \chi(H)\left(\frac{\partial}{\partial t}\left\{\Gamma(q^t)\right\}\frac{A}{t}\Gamma(q^t) + \Gamma(q^t)\frac{\partial}{\partial t}\left\{\frac{A}{t}\right\}\Gamma(q^t)\right. \\
&+ \left.\Gamma(q^t)\frac{A}{t}\frac{\partial}{\partial t}\left\{\Gamma(q^t)\right\} + \left[H, \Gamma(q^t)\frac{A}{t}\Gamma(q^t)\right]\right)\chi(H) \\
&= \chi(H)d\Gamma(q^t, d_0q^t)\frac{A}{t}\Gamma(q^t)\chi(H) + h.c \\
&+ \chi(H)\left[H_I, \Gamma(q^t)\right]\frac{A}{t}\Gamma(q^t)\chi(H) + h.c \\
&+ t^{-1}\chi(H)\Gamma(q^t)\left[H, A\right]\Gamma(q^t)\chi(H) \\
&- t^{-1}\chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H) \\
&= R_1(t) + R_2(t) + R_3(t) + R_4(t).
\end{aligned}$$

A straightforward computation leads to:

$$\|R_2(t)\| = O(t^{-1-\mu}).$$

To evaluate $R_1(t)$ we need to calculate $d_0 q^t$:

$$\begin{aligned} d_0 q^t(x) &= \partial_t q\left(\frac{x}{t}\right) + i\left[\omega(k), q\left(\frac{x}{t}\right)\right] \\ &= -\frac{1}{2t}\left\langle \frac{x}{t} - \nabla\omega(k) \middle| \nabla q\left(\frac{x}{t}\right)\right\rangle + h.c + O(t^{-2}) \\ &= \frac{1}{t} g_t + r_t, \end{aligned}$$

where $r_t = O(t^{-2})$ and according to Lemma 2.1.6 and (2.76):

$$\left\| \chi(H) d\Gamma(q^t, r_t)\frac{A}{t}\Gamma(q^t)\chi(H)\right\| = O(t^{-2}).$$

Considering:

$$\begin{aligned} B_1 &= \chi(H) d\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}} \\ B_2^* &= (N+1)^{\frac{1}{2}}\frac{A}{t}\Gamma(q^t)\chi(H), \end{aligned}$$

we have:

$$\begin{aligned} \frac{1}{t}\chi(H) d\Gamma(q^t, g_t)\frac{A}{t}\Gamma(q^t)\chi(H) + h.c &= t^{-1}B_1 B_2^* + t^{-1} B_2 B_1^* \\ &\geqslant -(\epsilon_0 t)^{-1} B_1 B_1^* - \epsilon_0 t^{-1} B_2 B_2^*. \end{aligned}$$

Note that:

$$B_2 B_2^* = \chi(H)\Gamma(q^t)\frac{A^2}{t^2}(N+1)\Gamma(q^t)\chi(H).$$

Moreover:

$$\begin{aligned} \Gamma(q^t)\frac{A^2}{t^2}\Gamma(q^t) &= \Gamma(q^t)\frac{A^2}{t^2}(N+1)^{-2}(N+1)^2\Gamma(q^t) \\ &\leqslant (N+1)\Gamma^2(q^t)(N+1) \\ \chi(H)(N+1)^p\chi(H) &\leqslant C\chi^2(H). \end{aligned}$$

Therefore and by analogy to Lemma 2.1.5:

$$\begin{aligned} -B_2 B_2^* &\geqslant -C\chi(H)(N+1)^{\frac{3}{2}}\Gamma^2(q^t)(N+1)^{\frac{3}{2}}\chi(H) \\ &\geqslant -C\Gamma(q^t)\chi(H)(N+1)^3\chi(H)\Gamma(q^t) + O(t^{-1}) \\ &\geqslant -C\Gamma(q^t)\chi(H)^2\Gamma(q^t) - Ct^{-1} \\ &\geqslant -C\chi(H)\Gamma^2(q^t)\chi(H) \ - Ct^{-1}. \end{aligned}$$

Next,

$$B_1 B_1^* = \chi(H) d\Gamma(q^t, g^t)(N+1)^{-1} d\Gamma(q^t, g^t)\chi(H)$$

and using Lemma 2.1.6 again:

$$\begin{aligned} |\langle u\, | B_1 B_1^* u\rangle| &= |\langle B_1^* u\, | B_1^* u\rangle| \\ &= \left|\left\langle (N+1)^{-\frac{1}{2}} d\Gamma(q^t, g^t)\chi(H)u \,\middle|\, (N+1)^{-\frac{1}{2}} d\Gamma(q^t, g^t)\chi(H)u\right\rangle\right| \\ &= \left\|(N+1)^{-\frac{1}{2}} d\Gamma(q^t, g^t)\chi(H)u\right\|^2 \\ &\leqslant \left\| d\Gamma(g^{t*}g^t)^{\frac{1}{2}}\chi(H)u\right\|^2. \end{aligned}$$

Applying (ii):

$$\int_1^\infty \|B_1 e^{-itH} u\|^2 \frac{dt}{t} \leqslant C\|u\|^2.$$

Next:

$$\begin{aligned} R_3(t) &= t^{-1}\chi(H)\Gamma(q^t)\,[H,A]\,\Gamma(q^t)\chi(H) \\ &= t^{-1}\Gamma(q^t)\chi(H)\,[H,A]\,\chi(H)\Gamma(q^t) + O(t^{-2}) \\ &\geqslant C_0 t^{-1}\Gamma(q^t)\chi^2(H)\Gamma(q^t) - Ct^{-2} \\ &\geqslant C_0 t^{-1}\chi(H)\Gamma^2(q^t)\chi(H) - Ct^{-2} \end{aligned}$$

and finally:

$$\begin{aligned} -R_4(t) &= t^{-1}\chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H) \\ &= t^{-1}\chi(H)\Gamma(q^t)\frac{A}{t}(N+1)^{-1}(N+1)\Gamma(q^t)\chi(H) \\ &\leqslant \frac{C\epsilon}{t}\chi(H)\Gamma(q^t)(N+1)\Gamma(q^t)\chi(H) \\ &\leqslant \frac{C\epsilon}{t}\chi(H)(N+1)^{\frac{1}{2}}\Gamma^2(q^t)(N+1)^{\frac{1}{2}}\chi(H) \\ &\leqslant \frac{C\epsilon}{t}\Gamma(q^t)\chi(H)(N+1)\chi(H)\Gamma(q^t) + Ct^{-2} \\ &\leqslant \frac{C_2\epsilon}{t}\Gamma(q^t)\chi(H)^2\Gamma(q^t) + Ct^{-2} \\ &\leqslant \frac{C_2\epsilon}{t}\chi(H)\Gamma^2(q^t)\chi(H) + Ct^{-2}. \end{aligned}$$

We now sum up every thing:

$$\begin{aligned} D\Phi(t) &= R_1(t) + R_2(t) + R_3(t) + R_4(t) \\ &\geqslant R_1(t) - (\epsilon_0 t)^{-1}B_1B_1^* - \epsilon_0 t^{-1}B_2B_2^* + R_3(t) + R_4(t) \\ &\geqslant (-\epsilon_0 C_1 + C - \epsilon C_2)t^{-1}\chi(H)\Gamma(q^t)^2\chi(H) - (\epsilon_0 t)^{-1}B_1B_1^* - Ct^{-1-\mu}. \end{aligned}$$

For ϵ and ϵ_0 small enough we have $\tilde{C}_0 = (-\epsilon_0 C_1 + C - \epsilon C_2) > 0$ and

$$D\Phi(t) \geqslant \frac{\tilde{C}_0}{t}\chi(H)\Gamma(q^t)^2\chi(H) - R(t) - Ct^{-1-\mu},$$

where $R(t)$ is integrable. Finally, we conclude using the method presented at the begining of this section. To extend this result to any χ with compact support contained in $\mathbb{R}$, we refer to Proposition 4.4.7 in [28]. □

2.4 Asymptotic Completeness

In this section, we prove Theorem 2.1.1. We begin by recalling the definitions and basic properties of the asymptotic spaces and of the wave operators in Subsection 2.4.1. Subsection 2.4.2 is devoted to the proof of an important ingredient of Theorem 2.1.1, namely the existence of inverse wave operators. Finally, in Subsection 2.4.3, we establish asymptotic completeness of the wave operators.

2.4.1 The asymptotic space and the wave operators

Most of the results of this section are straightforward adaptations of corresponding results established in [29, 5] for Pauli-Fierz Hamiltonians. They are recalled for consistency. However, it should be noted that the unitarity of the wave operators cannot be proved in the same way as in [29, 5] if the masses of the neutrinos vanish. We rely instead on an elegant argument due to [30].

We first define the so-called asymptotic creation and annihilation operators. Note that the fermionic asymptotic creation or annihilation operators $b^{\pm,\sharp}_{l,\epsilon}(h)$ (for leptons) and $c^{\pm,\sharp}_{l,\epsilon}(h)$ (for neutrinos), with $h \in \mathfrak{h}_1$, are bounded. The bosonic operators $a^{\pm,\sharp}_{\epsilon}(h)$, with $h \in \mathfrak{h}_2$, are closed but unbounded. For $h_2 \in \mathfrak{h}_2$ we define $h^{(3)}_{2,t} := e^{-\mathrm{i}t\omega^{(3)}}h_2$. Likewise, for $h_1 \in \mathfrak{h}_1$, we set $h^{(2)}_{1,t} := e^{-\mathrm{i}t\omega^{(2)}}h_1$ and $h^{(1)}_{1,t} := e^{-\mathrm{i}t\omega^{(1)}}h_1$. For all $h_1 \in \mathfrak{h}_1$, $h_2 \in \mathfrak{h}_2$ and $\epsilon = \pm$, we introduce the following notations

$$
\begin{aligned}
\phi^{(a)}_{\epsilon}(h_2) &= \frac{1}{\sqrt{2}}\left(a^*_{\epsilon}(h_2) + a_{\epsilon}(h_2)\right),\\
\phi^{(b)}_{l,\epsilon}(h_1) &= \frac{1}{\sqrt{2}}\left(b^*_{l,\epsilon}(h_1) + b_{l,\epsilon}(h_1)\right),\\
\phi^{(c)}_{l,\epsilon}(h_1) &= \frac{1}{\sqrt{2}}\left(c^*_{l,\epsilon}(h_1) + c_{l,\epsilon}(h_1)\right).
\end{aligned}
$$

Assuming that $G \in \mathbb{H}^{1+\mu}$ for some $\mu > 0$, the asymptotic bosonic fields can be defined in the same way as in [29, Section 5.2], as generators of the asymptotic Weyl operators. The latter are defined as the strong limits

$$
W^{(a),+}_{\epsilon}(h_2) = \operatorname*{s-lim}_{t\to+\infty} e^{\mathrm{i}tH} e^{\mathrm{i}t\phi^{(a)}_{\epsilon}(h_{2,t})} e^{-\mathrm{i}tH}
$$

and we have

$$
W^{(a),+}_{\epsilon}(h_2) = e^{\mathrm{i}\phi^{(a),+}_{\epsilon}(h_2)}.
$$

The asymptotic fermionic fields can be defined similarly, or, equivalently, as the strong limits

$$
\phi^{(b),+}_{l,\epsilon}(h_1) = \operatorname*{s-lim}_{t\to+\infty} e^{\mathrm{i}tH}\phi^{(b)}_{l,\epsilon}(h_{1,t})e^{-\mathrm{i}tH}, \quad \phi^{(c),+}_{l,\epsilon}(h_1) = \operatorname*{s-lim}_{t\to+\infty} e^{\mathrm{i}tH}\phi^{(c)}_{l,\epsilon}(h_{1,t})e^{-\mathrm{i}tH}.
$$

The results stated in the next theorem are straightforward adaptations of corresponding results established in [29, 5].

Theorem 2.4.1. *Suppose that $G \in \mathbb{H}^{1+\mu}$ for some $\mu > 0$.*

i) For any $h_2 \in \mathfrak{h}_2$, the asymptotic bosonic creation and annihilation operators $a^{+\sharp}_{\epsilon}(h_2)$ defined on $\mathscr{D}(a^{+\sharp}_{\epsilon}(h_2)) = \mathscr{D}(\phi^{(a),+}_{\epsilon}(h_2)) \cap \mathscr{D}(\phi^{(a),+}_{\epsilon}(\mathrm{i}h_2))$ by

$$
a^{+*}_{\epsilon}(h_2) = \frac{1}{\sqrt{2}}(\phi^{(a),+}_{\epsilon}(h_1) - \mathrm{i}\phi^{(a),+}_{\epsilon}(\mathrm{i}h_2)), \qquad a^{+}_{\epsilon}(h_2) = \frac{1}{\sqrt{2}}(\phi^{(a),+}_{\epsilon}(h_2) + \mathrm{i}\phi^{(a),+}_{\epsilon}(\mathrm{i}h_2)),
$$

are closed operators. Moreover, we have that $\mathscr{D}((|H|+1)^{\frac{1}{2}}) \subset \mathscr{D}(a_\epsilon^{+\sharp}(h_2))$ *and*

$$\|a^{+\sharp}(h_2)u\| \leqslant C\|h_2\|\|(|H|+1)^{\frac{1}{2}}u\|.$$

ii) For any $h_1 \in \mathfrak{h}_1$, *the asymptotic fermionic creation and annihilation operators defined by*

$$b_{l,\epsilon}^{+*}(h_1) = \frac{1}{\sqrt{2}}(\phi_{l,\epsilon}^{(b),+}(h_1) - \mathrm{i}\phi_{l,\epsilon}^{(b),+}(\mathrm{i}h_1)), \qquad b_{l,\epsilon}^{+}(h_1) = \frac{1}{\sqrt{2}}(\phi_{l,\epsilon}^{(b),+}(h_1) + \mathrm{i}\phi_{l,\epsilon}^{(b),+}(\mathrm{i}h_1)),$$
$$c_{l,\epsilon}^{+*}(h_1) = \frac{1}{\sqrt{2}}(\phi_{l,\epsilon}^{(c),+}(h_1) - \mathrm{i}\phi_{l,\epsilon}^{(c),+}(\mathrm{i}h_1)), \qquad c_{l,\epsilon}^{+}(h_1) = \frac{1}{\sqrt{2}}(\phi_{l,\epsilon}^{(c),+}(h_1) + \mathrm{i}\phi_{l,\epsilon}^{(c),+}(\mathrm{i}h_1)),$$

are bounded operators.

iii) The following commutation relations hold (in the sense of quadratic forms)

$$\begin{aligned}
&[a^+(h_2), a^{+*}(g_2)] = \langle h_2, g_2\rangle \mathbb{1},\\
&[a^+(h_2), a^+(g_2)] = [a^{+*}(h_2), a^{+*}(g_2)] = 0,\\
&\{b^+(h_1), b^{+*}(g_1)\} = \langle h_1, g_1\rangle \mathbb{1},\\
&\{b^+(h_1), b^+(g_1)\} = \{b^{+*}(h_1), b^{+*}(g_1)\} = 0,\\
&\{c^+(h_1), c^{+*}(g_1)\} = \langle h_1, g_1\rangle \mathbb{1},\\
&\{c^+(h_1), c^+(g_1)\} = \{c^{+*}(h_1), c^{+*}(g_1)\} = 0,\\
&\{b^+(h_1), c^+(g_1)\} = \{b^+(h_1), c^{+*}(g_1)\} = 0,\\
&\{b^{+*}(h_1), c^+(g_1)\} = \{b^{+*}(h_1), c^{+*}(g_1)\} = 0,\\
&[b^{+\sharp}(h_1), a^{+\sharp}(h_2)] = [c^{+\sharp}(h_1), a^{+\sharp}(h_2)] = 0.
\end{aligned}$$

iv) We have that

$$e^{\mathrm{i}tH}a^{+\sharp}(h_2)e^{-\mathrm{i}tH} = a^{+\sharp}(h_{2,-t}), \quad e^{\mathrm{i}tH}b^{+\sharp}(h_1)e^{-\mathrm{i}tH} = b^{+\sharp}(h_{1,-t}), \quad e^{\mathrm{i}tH}c^{+\sharp}(h_1)e^{-\mathrm{i}tH} = c^{+\sharp}(h_{1,-t}),$$

and the following "pulltrough formulae" are satisfied

$$\begin{aligned}
a^{+*}(h_2)H &= Ha^{+*}(h_2) - a^{+*}(\omega^{(3)}h_2), \quad & a^+(h_2)H &= Ha^+(h_2) + a^+(\omega^{(3)}h_2),\\
b^{+*}(h_1)H &= Hb^{+*}(h_1) - b^{+*}(\omega^{(1)}h_1), \quad & b^+(h_1)H &= Hb^+(h_1) + b^+(\omega^{(1)}h_1),\\
c^{+*}(h_1)H &= Hc^{+*}(h_1) - c^{+*}(\omega^{(2)}h_1), \quad & c^+(h_1)H &= Hc^+(h_1) + c^+(\omega^{(2)}h_1).
\end{aligned}$$

Proof. We focus on the proof of the bosonic case. To prove (i), let $h \in \mathfrak{h}_{20} = C_0^\infty(\Sigma_2 \; \{0\})$ and let us notice that

$$\begin{aligned}
[a_\epsilon^*(h), iH_0] &= a_\epsilon^*(-i\omega^{(3)}h)\\
[a_\epsilon(h), iH_0] &= a_\epsilon(-i\omega^{(3)}h).
\end{aligned}$$

As a consequence:

$$\begin{aligned}
\partial_t\left\{e^{-iH_0t}a_\epsilon^*(h)e^{iH_0t}\right\} &= e^{-iH_0t}[a_\epsilon^*(h), iH_0]e^{iH_0t}\\
&= e^{-iH_0t}a_\epsilon^*(-i\omega^{(3)}h)e^{iH_0t}.
\end{aligned}$$

Moreover:

$$\phi_\epsilon^{(a)}(h_t) = e^{-iH_0t}\phi_\epsilon^{(a)}(h)e^{iH_0t}.$$

Therefore:

$$\begin{aligned}\partial_t \phi_\epsilon^{(a)}(h_t) &= [\phi_\epsilon^{(a)}(h_t), iH_0]\\ \partial_t \left\{e^{itH}\phi_\epsilon^{(a)}(h_t)e^{-itH}\right\} &= e^{itH}\left\{iH\phi_\epsilon^{(a)}(h_t) + [\phi_\epsilon^{(a)}(h_t), iH_0] - \phi_\epsilon^{(a)}(h_t)iH\right\}e^{-itH}\\ &= e^{itH}\left\{[igH_I, \phi_\epsilon^{(a)}(h_t)]\right\}e^{-itH}.\end{aligned}$$

By defining:

$$B(\xi) = \frac{1}{\sqrt{2}}\sum_{l=1}^{3}\sum_{\epsilon=\pm,\epsilon'=\pm}\sum_{\epsilon''}\int d^3\xi_1 d^3\xi_2 \left\{G^{(1)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi)b^*_{l,\epsilon}(\xi_1)c^*_{l,\epsilon'}(\xi_2) + \overline{G^{(2)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi)}b_{l,\epsilon'}(\xi_1)c_{l,\epsilon}(\xi_2)\right\},$$

the commutator $[H_I, \phi_\epsilon^{(a)}(h_t)]$ can be written as:

$$[H_I, \phi_\epsilon^{(a)}(h_t)] = \langle h_t|B\rangle - \overline{\langle h_t|B\rangle} = 2i\Im\langle h_t|B\rangle,$$

where:

$$\langle h_t\,|B\,\rangle = \int \overline{h_t(\xi)}B(\xi)d\xi.$$

Therefore

$$\partial_t \left\{e^{itH}\phi_\epsilon^{(a)}(h_t)e^{-itH}\right\} = e^{itH}\left\{-2g\Im\langle h_t|B\rangle\right\}e^{-itH}$$

and we have:

$$e^{itH}\phi_\epsilon^{(a)}(h_t)e^{-itH} - \phi_\epsilon^{(a)}(h) = -2g\int_0^t e^{isH}\Im\langle h_s|B\rangle e^{-isH}ds.$$

We note that:

$$\begin{aligned}\left\|\langle h_t|B\rangle\,(N+1)^{-1}\right\| &\lesssim \left\|\int \overline{h_t(\xi)}G(\xi,.,.)d\xi\right\|,\\ &\lesssim \left\|\int e^{i\omega(\xi)t}\overline{h(\xi)}G(\xi,.,.)d\xi\right\|,\end{aligned}$$

using a stationary phase argument, it is not difficult to see that:

$$\left\|\langle h_t|B\rangle\,(N+1)^{-1}\right\| \lesssim t^{-1-\mu}.$$

For $h \in \mathfrak{h}_2$, there exists a sequence $\{h_n\}$ of elements belonging to $\mathfrak{h}_{20}$ such that $h = \lim_{n\to+\infty} h_n$. Since $\|(N+1)^{\frac{1}{2}}(H+i)^{-\frac{1}{2}}\| < \infty$, the following relations are true for all u:

$$\begin{aligned}\|(\phi(h_{n,t}-h_t)(H+i)^{-\frac{1}{2}}u\| &\leqslant \|h_{n,t}-h_t\|\ \|(N+1)^{\frac{1}{2}}(H+i)^{-\frac{1}{2}}u\|\\ &\leqslant C\|h_{n,t}-h_t\|\|u\|.\end{aligned}$$

This proves that $\phi(h_{n,t}-h_t)(H+i)^{-\frac{1}{2}}$ converges in norm to zero which implies (i). We prove now the pulltrhrough formulas (iv), the remaining parts being obvious. Starting from:

$$a^{+\sharp}(h)e^{-isH} = e^{-isH}a^{+\sharp}(h_{-s}),$$

we have:

$$\begin{aligned} a^{+\sharp}(h)\left(\frac{e^{-isH}-\mathbb{1}}{-is}\right) &= \frac{\mathbb{1}}{is}a^{+\sharp}(h)+\frac{e^{-isH}}{-is}a^{+\sharp}(h_{-s}) \\ &= \frac{a^{+\sharp}(h_{-s})-a^{+\sharp}(h)}{-is}+\frac{e^{-isH}-\mathbb{1}}{-is}a^{+\sharp}(h_{-s}). \end{aligned}$$

And as s goes to 0 the results hold.

□

The space of asymptotic vacua is defined by

$$\mathscr{K}^{\pm}=\left\{u\in\mathscr{D}(N_W^{\frac{1}{2}}),\ \forall h_1\in\mathfrak{h}_2, h_2,h_3\in\mathfrak{h}_1, l,\epsilon, a_\epsilon^{\pm}(h_1)u=b_{l,\epsilon}^{\pm}(h_2)u=c_{l,\epsilon}^{\pm}(h_3)u=0\right\}.$$

The asymptotic space is

$$\mathscr{H}^{\pm}=\mathscr{K}^{\pm}\otimes\mathscr{H}.$$

The following proposition can be proven in the same way as [29, Proposition 5.5].

Proposition 2.4.2. *Consider the Hamiltonian* (2.12) *with H_I given by* (2.5) *and suppose that*

$$G\in\mathbb{H}^{1+\mu}\ \textit{for some}\ \mu>0.$$

Then

(i) $\mathscr{K}^{\pm}$ *is closed and H-invariant.*

(ii) For all $n\in\mathbb{N}$ and $h_1,\dots,h_n$, $\mathscr{K}^{\pm}$ is contained in the domain of $d^{\pm,*}(h_1)\dots d^{\pm,*}(h_n)$, where $d^{\pm,*}(h_i)$ stands for any of the operators $a_\epsilon^{\pm,*}(h_i)$, with $h_i\in\mathfrak{h}_1$, or $b_{l,\epsilon}^{\pm,*}(h_i)$ or $c_{l,\epsilon}^{\pm,*}(h_i)$, with $h_i\in\mathfrak{h}_2$.

(iii) $\mathscr{H}_{\mathrm{pp}}(H)\subset\mathscr{K}^{\pm}$.

Proof. (i) is obvious. To prove (ii) we will use an induction argument. If there is one particle, since $\mathscr{D}(d^{+*}(h_1))=\mathscr{D}(d^{+}(h_1))$, the proposition is true. Let us assume (ii) for $n-1$ particles. Then, for $u\in\mathscr{K}^{+}$ we have, using commutation relations, $d^{+*}(h_1)\dots d^{+*}(h_{n-1})u\in\mathscr{D}(d^{+}(h_n))$. This implies (ii) for n bosons.

Finally, to prove (iii) let $u\in\mathscr{H}_{pp}(H)$. Then

$$Hu=\lambda_u u$$

and

$$\begin{aligned} e^{itH}a(h_t)e^{-itH}u &= e^{itH-it\lambda_u}a(h_t)u \\ &= (E+i)e^{itH-it\lambda_u}a(h_t)(H+i)^{-1}u. \end{aligned}$$

Since $w-\lim_{t\to+\infty}h_t=0$, we have that $s-\lim_{t\to+\infty}a(h_t)(H+i)^{-1}=0$, which implies $a^{+}(h)u=0$. The same methods would show that $b^{+}(h)u=c^{+}(h)u=0$. □

Recall that the wave operators $\Omega^{\pm} : \mathscr{H}^{\pm} \to \mathscr{H}$ are defined in Section 2.1 and that $H^{\text{ext}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H_0$. The following properties are standard consequences of the definitions.

Proposition 2.4.3. *Consider the Hamiltonian* (2.12) *with H_I given by* (2.5) *and suppose that*

$$G \in \mathbb{H}^{1+\mu} \textit{ for some } \mu > 0.$$

Then $\Omega^{\pm}$ are isometric and we have that

$$H\Omega^{\pm} = \Omega^{\pm} H^{\text{ext}}.$$

Moreover,

$$d^{\pm,\sharp}(h)\Omega^{\pm} = \Omega^{\pm}(\mathbb{1} \otimes d^{\sharp}(h)),$$

where $d^{\pm,\sharp}(h)$ stands for any of the operators $a_{\epsilon}^{\pm,\sharp}(h)$, with $h \in \mathfrak{h}_1$, or $b_{l,\epsilon}^{\pm,\sharp}(h)$ or $c_{l,\epsilon}^{\pm,\sharp}(h)$, with $h \in \mathfrak{h}_2$.

Theorem 2.4.4. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5)*. Suppose that*

$$G \in \mathbb{H}^{1+\mu} \textit{ for some } \mu > 0.$$

Then $\Omega^{\pm}$ are unitary maps form $\mathscr{H}^{\pm}$ to $\mathscr{H}$.

The proof is a straightforward adaptation of the one presented in [29].

In the remainder of this subsection we recall the notion of "extended wave operators", defined as in [29], together with some of their properties. Recall that $\mathscr{H}^{\text{ext}} = \mathscr{H} \otimes \mathscr{H}$ and $H^{\text{ext}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H_0$. We set

$$\mathscr{D}(\Omega^{\text{ext},\pm}) = \bigotimes_{n=0}^{\infty} \mathscr{D}((|H|^{\frac{n}{2}}) \otimes \bigoplus_{n_1+\cdots+n_{14}=n} \otimes_s^{n_1} \mathfrak{h}_2 \otimes \otimes_s^{n_2} \mathfrak{h}_2 \otimes \otimes_a^{n_3} \mathfrak{h}_1 \otimes \cdots \otimes \otimes_a^{n_{14}} \mathfrak{h}_1,$$

and

$$\begin{aligned} &\Omega^{\text{ext},\pm} : \mathscr{D}(\Omega^{\text{ext},\pm}) \to \mathscr{H} \\ &\Omega^{\text{ext},\pm} \psi \otimes d^*(h_1) \dots d^*(h_n)\Omega = d^{\pm,*}(h_1) \dots d^{\pm,*}(h_n)\psi, \end{aligned} \tag{2.77}$$

where, as above, $d^{\pm,*}(h_i)$ stands for any of the operators $a_{\epsilon}^{\pm,*}(h_i)$, with $h_i \in \mathfrak{h}_1$, or $b_{l,\epsilon}^{\pm,*}(h_i)$ or $c_{l,\epsilon}^{\pm,*}(h_i)$, with $h_i \in \mathfrak{h}_2$, and likewise for $d^*(h_i)$. The definition (2.77) extends to any vector in $\mathscr{D}(\Omega^{\text{ext},+})$ by linearity. The fact that $\Omega^{\text{ext},\pm}$ are well-defined follows from the properties of the asymptotic creation operators.

It follows directly from the definitions of $\Omega^{\pm}$ and $\Omega^{\text{ext},\pm}$ that

$$\Omega^{\text{ext},\pm}|_{\mathscr{H}^{\pm}} = \Omega^{\pm}.$$

Moreover, in the same way as in [29, Theorem 5.7], it is not difficult to verify that, for all $u \in \mathscr{D}(\Omega^{\text{ext},\pm})$,

$$\underset{t\to\pm\infty}{\text{s-lim}}\, e^{\mathrm{i}tH} I e^{-\mathrm{i}tH^{\text{ext}}} u = \Omega^{\text{ext},\pm} u.$$

where the scattering identification operator I is defined in the first chapter.

2.4.2 The geometric inverse wave operators

In this section we establish the existence of two asymptotic observables using the propagation estimates of Section 2.3. The methods we used may be found in [29, 43, 5]. We begin with the following important proposition.

Proposition 2.4.5. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *We moreover assume that:*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3,$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

Let $\delta > 0$ and q_i, $i \in \{1, \ldots, 14\}$ be functions in $\mathrm{C}_0^\infty(\{x \in \mathbb{R}^3, |x| \leqslant 2\delta\})$ such that $0 \leqslant q_i \leqslant 1$, $q_i = 1$ on $\{x \in \mathbb{R}^3, |x| \leqslant \delta\}$ and let $q^t = (q_1(\frac{x}{t}), \ldots, q_{14}(\frac{x}{t}))$. The following limits exist

$$\Gamma^{\pm}(q) := \underset{t\to\pm\infty}{\text{s-lim}}\, e^{\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{-\mathrm{i}tH}.$$

Moreover, for all $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$, there exists $\delta > 0$ such that

$$\Gamma^{\pm}(q)\chi(H) = 0.$$

Proof. Using a density argument and by analogy to Lemma 2.1.11, it is enough to show the existence of the following limit:

$$s - \lim_{t\to\pm\infty} e^{itH}\chi(H)\Gamma(q^t)\chi(H)e^{-itH},$$

where $\chi \in C_0^\infty(\mathbb{R})$.

$$\begin{aligned}
\partial_t\left\{e^{itH}\chi(H)\Gamma(q^t)\chi(H)e^{-itH}\right\} &= e^{itH}\chi(H)\left\{iH\Gamma(q^t) + \partial_t\Gamma(q^t) - \Gamma(q^t)iH\right\}\chi(H)e^{-itH} \\
&= e^{itH}\chi(H)\left\{D_0\Gamma(q^t) + i[H_I, \Gamma(q^t)]\right\}\chi(H)e^{-itH} \\
&= e^{itH}\chi(H)\left\{d\Gamma(q^t, d_0q^t) + i[H_I, \Gamma(q^t)]\right\}\chi(H)e^{-itH}.
\end{aligned} \tag{2.78}$$

On the one hand:

$$\chi(H)[H_I, \Gamma(q^t)]\chi(H) = O(t^{-1-\mu}),$$

and on the other hand:

$$d_0q^t = \frac{1}{t}g^t + r^t,$$

where $r^t = O(t^{-2})$ and

$$g^t = -\frac{1}{2}\left(\left\langle \frac{x}{t} - \nabla\omega(k)\middle|\nabla q\left(\frac{x}{t}\right)\right\rangle + h.c\right).$$

First:

$$\left\|\chi(H)d\Gamma(q^t, r^t)\chi(H)\right\| = O(t^{-2}).$$

Next:

$$\left|\left\langle e^{-itH}\chi(H)u\,\middle|d\Gamma(q^t, g^t)\chi(H)e^{-itH}u\right\rangle\right| \quad \leqslant \quad \left\|d\Gamma(|g^t|)^{\frac{1}{2}}u\right\|^2$$

and the propagation estimates of Proposition 2.3.1(iii) can be applied to the right-hand side, which proves the integrability of (2.78).

The last statement is a consequence of 2.3.1(iv). For $\chi \in C_0^\infty(\mathbb{R}\backslash\tau \cup \sigma_{pp}(H))$ there exist $\epsilon > 0$ and a quattuordecuple q, $q_i \in C_0^\infty([-\epsilon, \epsilon])$ and $q_i = 1$ for $|x_i| \leqslant \frac{\epsilon}{2}$ such that:

$$\int_1^\infty \|\Gamma(q^t)\chi(H)e^{-itH}u\|^2 \frac{dt}{t} \leqslant C\|u\|^2,$$

and by Theorem 2.3.1(iv):

$$\|\Gamma(q^t)\chi(H)e^{-itH}u\| \underset{t\to+\infty}{\to} \|\Gamma^{\pm}(q)\chi(H)u\|,$$

therefore $\Gamma^{\pm}(q)\chi(H) = 0$.

□

We introduce the following notations that will be used in the proof of the next theorem:

$$\begin{aligned}
\check{\mathbf{d}}_{0l}^{(i)} b(t) &= \frac{\partial b}{\partial t}(t) + \mathrm{i}\big(\omega_l^{(i)}(p_i) \oplus \omega_l^{(i)}(p_i) b(t) - b(t)\omega_l^{(i)}(p_i)\big), \quad i = 1, 2, \\
\check{\mathbf{d}}_{0}^{(3)} b(t) &= \frac{\partial b}{\partial t}(t) + \mathrm{i}\big(\omega^{(3)}(p_3) \oplus \omega^{(3)}(p_3) b(t) - b(t)\omega^{(3)}(p_3)\big),
\end{aligned}$$

if $b(t)$ is a family of operators from $\mathfrak{h}_1$ to $\mathfrak{h}_1 \oplus \mathfrak{h}_1$, if $i = 1, 2$, or from $\mathfrak{h}_2$ to $\mathfrak{h}_2 \oplus \mathfrak{h}_2$ if $j = 3$. Likewise we set

$$\check{\mathbf{D}}_0 B(t) = \frac{\partial B}{\partial t}(t) + \mathrm{i}(H_0 \otimes \mathbb{1} + \mathbb{1} \otimes H_0)B(t) - \mathrm{i}B(t)H_0,$$

for any family of operators $B(t) : \mathscr{H} \to \mathscr{H}^{\mathrm{ext}}$. Note that if $B(t) = (b_1(t), \ldots, b_{14}(t))$ then, as functions of t,

$$\begin{aligned}
\check{\mathbf{D}}_0 \mathrm{d}\Gamma(B) =: \mathrm{d}\Gamma(\check{\mathbf{d}}_0 b) &:= \mathrm{d}\Gamma(\check{\mathbf{d}}_{0,1} b_1, \ldots, \check{\mathbf{d}}_{0,14} b_{14}) \\
&:= \mathrm{d}\Gamma\big(\check{\mathbf{d}}_0^{(3)} b_1, \check{\mathbf{d}}_0^{(3)} b_2, \check{\mathbf{d}}_{01}^{(1)} b_3, \check{\mathbf{d}}_{01}^{(1)} b_4, \check{\mathbf{d}}_{01}^{(2)} b_5, \check{\mathbf{d}}_{01}^{(2)} b_6, \check{\mathbf{d}}_{02}^{(1)} b_7, \\
&\qquad \check{\mathbf{d}}_{02}^{(1)} b_8, \check{\mathbf{d}}_{02}^{(2)} b_9, \check{\mathbf{d}}_{02}^{(2)} b_{10}, \check{\mathbf{d}}_{03}^{(1)} b_{11}, \check{\mathbf{d}}_{03}^{(1)} b_{12}, \check{\mathbf{d}}_{01}^{(2)} b_{13}, \check{\mathbf{d}}_{01}^{(2)} b_{14}\big).
\end{aligned}$$

The main result of this subsection is stated in the following theorem. It shows the existence of inverse wave operators.

Theorem 2.4.6. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *We moreover assume that:*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3,$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

Let $\delta > 0$ and $j_{0,i}$, $i \in \{1, \ldots, 14\}$ be functions in $\mathrm{C}_0^\infty(\{x \in \mathbb{R}^3, |x| \leqslant 2\delta\})$ such that $0 \leqslant j_{0,i} \leqslant 1$, $j_{0,i} = 1$ on $\{x \in \mathbb{R}^3, |x| \leqslant \delta\}$ and let $j_{\infty,i} = 1 - j_{0,i}$, $j_i = (j_{0,i}, j_{\infty,i})$. Let $J^t = (j_1(\frac{x}{t}), \ldots, j_{14}(\frac{x}{t}))$.

(i) The following limits exist

$$W^{\pm}(J) := \underset{t\to+\infty}{\text{s-lim}}\, e^{\pm\mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}\big(J^t\big)e^{\mp\mathrm{i}tH}.$$

(ii) For all $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$*, we have that*

$$W^{\pm}(J)\chi(H) = \chi(H^{\mathrm{ext}})W^{\pm}(J).$$

(iii) Let $q = (q_1, \ldots, q_{14})$ *be such that* $q_i j_{i,0} = j_{i,0}$*. Then*

$$\big(\Gamma^{\pm}(q) \otimes \mathbb{1}\big)W^{\pm}(J) = W^{\pm}(J).$$

(iv) For all $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$*, we have that*

$$\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^{\pm}(J) = \chi(H).$$

Proof. (i) A density argument and Lemma 2.1.11 show that it is enough to prove the existence of the following limit for some $\chi \in C_0^\infty(\mathbb{R}^d)$:

$$s - \lim_{t\to\pm\infty} e^{itH^{ext}}\chi(H^{ext})\check{\Gamma}(J^t)\chi(H)e^{-itH}.$$

As usual, we compute:

$$\begin{aligned}
\partial_t\left\{e^{itH^{ext}}\chi(H^{ext})\check{\Gamma}(J^t)\chi(H)e^{-itH}\right\} &= e^{itH^{ext}}\chi(H^{ext})\left\{iH^{ext}\check{\Gamma}(J^t) + \partial_t\check{\Gamma}(J^t) - \check{\Gamma}(J^t)iH\right\}\chi(H)e^{-itH}\\
&= e^{itH^{ext}}\chi(H^{ext})\left\{\check{D}_0(\Gamma)(J^t) + iH_I^{ext}\check{\Gamma}(J^t)\right.\\
&- \left. i\check{\Gamma}(J^t)H_I\right\}\chi(H)e^{-itH}.
\end{aligned}$$

Using Lemma 2.1.11:

$$\chi(H^{ext})\left\{iH_I^{ext}\check{\Gamma}(J^t) - i\check{\Gamma}(J^t)H_I\right\}\chi(H) = O(t^{-1-\mu}).$$

Since $\check{D}_0(\Gamma)(J^t)) = d\check{\Gamma}(J^t, d_0J^t)$ and:

$$\begin{aligned}
d_0J^t &= \frac{1}{t}g^t + r^t,\\
g^t &= (g_0^t, g_\infty^t),\\
g_\epsilon^t &= -\frac{1}{2}\left(\left(\frac{x}{t} - \nabla\omega(k)\right)\nabla J_\epsilon\left(\frac{x}{t}\right) + h.c\right),\\
r^t &= O(t^{-2}),
\end{aligned}$$

we may use Lemma 2.1.6 to prove that:

$$\|\chi(H^{ext})d\check{\Gamma}(J^t, r^t)\chi(H)\| = O(t^{-2}).$$

Let $u_i^t = e^{itH}u_i$, then:

$$\begin{aligned}
\left|\left\langle u_1^t \left|\chi(H^{ext})d\check{\Gamma}(J^t, g^t)\chi(H)u_2^t\right.\right\rangle\right| &\leqslant \left\|d\Gamma(|g_0^t|)^{\frac{1}{2}} \otimes \mathbb{1}\ \chi(H^{ext})u_2^t\right\| \left\|d\Gamma(|g_0^t|)^{\frac{1}{2}}\ \chi(H)u_1^t\right\|\\
&+ \left\|\mathbb{1} \otimes d\Gamma(|g_\infty^t|)^{\frac{1}{2}}\ \chi(H^{ext})u_2^t\right\| \left\|d\Gamma(|g_\infty^t|)^{\frac{1}{2}}\ \chi(H)u_1^t\right\|.
\end{aligned}$$

Next, we use Proposition 2.3.1(iii) to conclude.

(ii) This is a standard intertwining property.

(iii) It suffices to write

$$\begin{aligned}
\big(\Gamma^\pm(q)\otimes\mathbb{1}\big)W^\pm(J)u &= \big(e^{\pm itH}\Gamma(q^t)e^{\mp itH}\otimes\mathbb{1}\big)e^{\pm itH^{\mathrm{ext}}}\check{\Gamma}\big(J^t\big)e^{\mp itH}u+o(1)\\
&= \big(e^{\pm itH^{\mathrm{ext}}}(\Gamma(q^t)\otimes\mathbb{1}\big)\check{\Gamma}\big(J^t\big)e^{\mp itH}u+o(1)\\
&= \big(e^{\pm itH^{\mathrm{ext}}}\check{\Gamma}\big(J^t\big)e^{\mp itH}u+o(1)\\
&= W^\pm(J)u+o(1),
\end{aligned}$$

where we used that $(\Gamma(q^t)\otimes\mathbb{1}\big)\check{\Gamma}\big(J^t\big)=\check{\Gamma}\big(J^t\big)$ because $q_i j_{i,0}=j_{i,0}$ in the third equality.

(iv) This is again standard intertwining properties. □

2.4.3 Asymptotic completeness

We are now ready to conclude the proof of Theorem 2.1.1. We begin by showing that the pure point spectral subspace of H, $\mathscr{H}_{\mathrm{pp}}(H)$, and the spaces of asymptotic vacua $\mathscr{K}^\pm$ coincide. Note that our reasoning process is slightly different from that of [29, 5].

Theorem 2.4.7. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5)*. We moreover assume that:*

$$G\in L^2,\quad a_{(i),\cdot}G\in L^2,\quad i=1,2,3,$$

and that

$$G\in\mathbb{H}^{1+\mu}\text{ for some }\mu>0.$$

Therefore, for any coupling constant g,

$$\mathscr{H}_{\mathrm{pp}}(H)=\mathscr{K}^\pm.$$

Proof. By Proposition 2.4.2, we know that

$$\mathscr{H}_{\mathrm{pp}}(H)\subset\mathscr{K}^\pm.$$

Since in addition $\mathscr{H}_{\mathrm{pp}}(H)$ and $\mathscr{K}^\pm$ are closed, it remains to establish that $\mathscr{H}_{\mathrm{pp}}(H)^\perp\subset(\mathscr{K}^\pm)^\perp$. In turn, since $\sigma_{\mathrm{pp}}(H)$ can only accumulate at the closed countable set τ, it suffices to prove that $\mathrm{Ran}(\chi(H))\subset(\mathscr{K}^\pm)^\perp$ for all $\chi\in\mathrm{C}_0^\infty(\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H)))$.

Let $\chi\in\mathrm{C}_0^\infty(\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H)))$ and let $u=\chi(H)v$. Let J be defined as in the statement of Theorem 2.4.6. By Theorem 2.4.6 (iv), we have that

$$\chi(H)=\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^\pm(J)=\Omega^{\mathrm{ext},\pm}(\mathbb{1}\otimes\Pi_\Omega)\chi(H^{\mathrm{ext}})W^\pm(J)+\Omega^{\mathrm{ext},\pm}(\mathbb{1}\otimes\Pi_\Omega^\perp)\chi(H^{\mathrm{ext}})W^\pm(J),$$

where Π_Ω denotes the projection onto the Fock vacuum and $\Pi_\Omega^\perp$ the projection onto its orthogonal complement. We claim that the first term vanishes. Indeed, we can write

$$\begin{aligned}
\Omega^{\mathrm{ext},\pm}(\mathbb{1}\otimes\Pi_\Omega)\chi(H^{\mathrm{ext}})W^\pm(J) &= \Omega^{\mathrm{ext},\pm}(\chi(H)\otimes\Pi_\Omega)W^\pm(J)\\
&= \Omega^{\mathrm{ext},\pm}(\chi(H)\otimes\Pi_\Omega)(\Gamma^\pm(q)\otimes\mathbb{1})W^\pm(J),
\end{aligned}$$

by Theorem 2.4.6 (iii), where q is as in the statement of this result. Since $\chi(H)\Gamma^{\pm}(q) = 0$ by Proposition 2.4.5, we see that this term indeed vanishes. Hence we have proven that

$$\chi(H)v = \Omega^{\mathrm{ext},\pm}(\mathbb{1} \otimes \Pi_{\Omega}^{\perp})\chi(H^{\mathrm{ext}})W^{\pm}(J)v.$$

Since $\Omega^{\mathrm{ext},\pm}(\mathbb{1} \otimes d^*(h)) = d^{\pm,*}(h)\Omega^{\mathrm{ext},\pm}$ for any kind of creation operator $d^*(h)$, the last equality clearly shows that $\chi(H)v \in (\mathscr{K}^{\pm})^{\perp}$. This concludes the proof of the theorem. □

Finally, as a consequence of Theorem 2.4.7, we deduce that $H - E$ and H_0 are unitary equivalent if the conditions on G are strengthened.

Corollary 2.4.8. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} are positive and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *Under the conditions of Theorem 2.4.7, and assuming in addition that*

$$b_{(i),\cdot}G \in L^2, \quad i = 1,2,3, \quad b_{(i),\cdot}b_{(i'),\cdot}G \in L^2, \quad i, i' = 1,2,3, \tag{2.79}$$

then there exists $g_0 > 0$, which does not depend on m_{ν_e}, m_{ν_μ}, m_{ν_τ}, such that, for all $|g| \leqslant g_0$, $H - E$ and H_0 are unitarily equivalent.

Proof. It follows from Proposition 2.4.3, Theorem 2.4.4 and Theorem 2.4.7 that H is unitary equivalent to $H|_{\mathscr{H}_{\mathrm{pp}}(H)} \otimes \mathbb{1} + \mathbb{1} \otimes H_0$. In the case where neutrinos are massive, Theorems 2.2.1 and 2.2.6 imply that, for $|g| \leqslant g_0$, $\mathscr{H}_{\mathrm{pp}}(H) = \{E\}$ and E is a simple eigenvalue of H. This shows that $H - E$ and H_0 are unitarily equivalent. Moreover, by Theorem 2.2.6, g_0 can be chosen independently of the values of m_{ν_e}, m_{ν_μ}, m_{ν_τ}.

□

Chapter 3

Scattering Theory for massless models of the $W^{\pm}$ decay

3.1 Introduction

This chapter, published in [2], is dedicated to a simplified model of the weak decay of the $W^{\pm}$ bosons with massless neutrinos. As said in the previous chapter, physical evidences that the neutrinos are massive are now available but theirs masses seem to be really small and massless neutrinos hypothesis is still widely considered. Moreover, many other particles, for example the photon, are massless and it is then interesting to consider massless fields. Let us recall the different interactive terms presented in the introduction of chapter 2.

$$\begin{aligned}
H_I^{(1)} &:= \sum_{l\in\{l,\nu,\tau\}} \sum_{\epsilon=\pm} \int \Big[G_{l,\epsilon}^{(1)}(\xi_1,\xi_2,\xi_3) b_{l,\epsilon}^*(\xi_1) c_{l,-\epsilon}^*(\xi_2) a_\epsilon(\xi_3) + \text{h.c.} \Big] \\
H_I^{(2)} &:= \sum_{l\in\{l,\nu,\tau\}} \sum_{\epsilon=\pm} \int \Big[G_{l,\epsilon}^{(2)}(\xi_1,\xi_2,\xi_3) b_{l,-\epsilon}^*(\xi_1) c_{l,\epsilon}^*(\xi_2) a_\epsilon^*(\xi_3) + \text{h.c.} \Big] \\
H_I^{(3)} &:= \sum_{l\in\{l,\nu,\tau\}} \sum_{\epsilon=\pm} \int \Big[G_{l,\epsilon}^{(3)}(\xi_1,\xi_2,\xi_3) b_{l,-\epsilon}^*(\xi_1) c_{l,-\epsilon}(\xi_2) a_\epsilon^*(\xi_3) + \text{h.c.} \Big] \\
H_I^{(4)} &:= \sum_{l\in\{l,\nu,\tau\}} \sum_{\epsilon=\pm} \int \Big[G_{l,\epsilon}^{(1)}(\xi_1,\xi_2,\xi_3) b_{l,\epsilon}^*(\xi_1) c_{l,\epsilon}(\xi_2) a_\epsilon(\xi_3) + \text{h.c.} \Big] .
\end{aligned}$$

We recall that the Dirac distributions in kernels are approximated by some smooth and compactly supported functions. An important property of the interaction Hamiltonian (2.5) is that it preserves the lepton number, in the sense that $N_{l^-} + N_{\nu_l} - N_{l^+} - N_{\bar{\nu}_l}$ commutes with H_I. Here, N_p stands for the number operator corresponding to a particle p. We moreover observe that the first interactive term in (2.5), $H_I^{(1)}$, describes explicitly processes like

$$W^+ \to l^+ + \nu_l, \qquad W^- \to l^- + \overline{\nu}_l, \tag{3.1}$$

while $H_I^{(2)}$ prevents the bare vacuum from being a bound state, as expected from physics. To study a process like (3.1), it is reasonable, in a first approximation, to keep only these first two terms, thus considering the simpler interaction Hamiltonian given by

$$H = H_0 + g(H_I^{(1)} + H_I^{(2)}) \tag{3.2}$$

where H_0 has been defined in (2.10) and g is again a real coupling constant highlighting the strength of the interaction. Under the assumption that the masses of the neutrinos vanish, we will make this approximation. The advantage is that the differences of number operators $N_{l^-} - N_{\bar{\nu}_l}$ and $N_{l^+} - N_{\nu_l}$ are preserved by the Hamiltonian. This property will be essential in some of our arguments.

As in the previous chapter, our main goal is to prove asymptotic completeness which can be stated in the following way:

Theorem 3.1.1. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by* (2.5)*. Assume that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad |p_3|^{-1}a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3,$$

and that $G \in \mathbb{H}^{1+\mu}$ for some $\mu > 0$. Then there exists $g_0 > 0$ such that, for all $|g| \leqslant g_0$, the wave operators $\Omega^\pm$ exist and are asymptotically complete. Suppose in addition that

$$b_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3, \quad b_{(i),\cdot}b_{(i'),\cdot}G \in L^2, \quad i, i' = 1, 2, 3.$$

Then there exists $g_0' > 0$ such that, for all $|g| \leqslant g_0'$, $H - E$ is unitarily equivalent to H_0.

Remark 3.1.2. *The assumption that $|p_3|^{-1}a_{(i),\cdot}G \in L^2$ in Theorem 3.1.1 can be replaced by $|p_1|^{-1}a_{(i),\cdot}G \in L^2$.*

As mentioned above, physically, the masses of the neutrinos are extremely small. We emphasize that Theorem 2.1.1 holds for $|g|$ small enough, uniformly with respect to the masses of the neutrinos. As a consequence, from the observation that the Hamiltonian with massive neutrinos converges to that with massless neutrinos, in the norm resolvent sense, as the masses of the neutrinos go to 0, one can deduce that, in the massless case, $H - E$ is approximately unitarily equivalent to H_0. This means that there exists a sequence of unitary operators (U_n) such that $U_n H_0 U_n^* \to H - E$, as $n \to \infty$, in the norm resolvent sense. The conclusion of Theorem 3.1.1, which concerns also the massless case, is, of course, significantly stronger since it shows that $H - E$ and H_0 are unitarily equivalent (not only approximately unitarily equivalent). However, Theorem 3.1.1 only holds for the simplified Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$.

For Pauli-Fierz Hamiltonians, contributions to scattering theory involving massless particles include [49, 34, 38, 39, 35, 33]. In particular, for the massless spin-boson model, asymptotic completeness has been established in [38, 33], using the uniform bound on the number of emitted particles proven in [34]. This property, however, has not been proven for more general massless Pauli-Fierz Hamiltonians, yet. In our setting, controlling the number of emitted particles is made possible thanks to the fact that $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, with N_{lept} the number of leptons and antileptons and N_{neut} the number of neutrinos and antineutrinos.

The proof of Theorem 3.1.1 constitutes one of the main novelty of this work compared to previous results in the literature. Here we cannot follow directly the approach of [29, 30, 5] because of the presence of massless particles. To obtain a useful Mourre estimate, we combine singular Mourre's Theory [47] together with an induction argument of [29] and smallness of the coupling constant g. Using this Mourre estimate and the fact that $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, we then establish propagation estimates. Our propagation estimates resemble those proven in [29], but with a different time-dependent propagation observable. Our choice of the (one-particle) propagation observable is inspired in part by that used in [38, 39]: it is a time-dependent modification of the usual "position operator", especially fitted to handle singularities due to the presence of massless particles. We do not take the same propagation observable as in [38, 39] because we follow a different approach, closer to that of [29], to prove asymptotic completeness.

As in [21, 38, 39], an important ingredient to prove the propagation estimates is the control of the observable $\mathrm{d}\Gamma(|p_2|^{-1})$ along the time evolution (where p_2 is the momentum of a neutrino). More precisely, we prove that, for suitable initial states, the expectation of this observable along the evolution grows slower than linearly in time t, which is crucial to estimate some remainder terms in the propagation estimates.

We will follow the same organisation as in the previous chapter. However, the massless case is harder and more technical than the massive one. We strongly recommend to read Chapter 2 to get some good intuition of our strategy. The model was defined in the introduction of Chapter 2. Therefore, we start with the proof of some Technical Estimates which are strongly inspired by those of Section 2.1.2.

In Section 3.3.2 the spectrum of H and a Mourre estimates, adapted to the massless case and which will be a key argument to prove propagation estimates in Section 3.4, is established.

Finally, Theorem 3.1.1 is proved in Section 3.5.

Let us recall again that throughout the entire document, the notation $a \lesssim b$, for positive numbers a and b, stands for $a \leqslant \mathrm{C}b$ where C is a positive constant independent of the parameters involved.

3.2 The Model and Technical Estimates

The estimates presented in Section 2.1.2 are adapted in the context of massless fields.

Lemma 3.2.1. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian* (2.12) *with H_I given by* (2.5)*. Assume that $G \in L^2$.*

(i) For all $m \in \mathbb{Z}$, uniformly for z in a compact set of $\{z \in \mathbb{C}, \pm|\Im z| > 0\}$, the operator $(N+1)^{-m}(H-z)^{-1}(N+1)^m$ extends to a bounded operator satisfying

$$\left\|(N+1)^{-m}(H-z)^{-1}(N+1)^m\right\| = \mathcal{O}(|\Im(z)|^{-\alpha_m}),$$

where α_m denotes an integer depending on m. Moreover, the operator $(N_{\text{lept}} + N_W + 1)^{-m}(H-z)^{-1}(N_{\text{lept}} + N_W + 1)^{m+1}$ extends to a bounded operator satisfying

$$\left\|(N_{\text{lept}} + N_W + 1)^{-m}(H-z)^{-1}(N_{\text{lept}} + N_W + 1)^{m+1}\right\| = \mathcal{O}(|\Im(z)|^{-\beta_m}),$$

where β_m denotes an integer depending on m.

(ii) Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. Then, for all $m, p \in \mathbb{N}$, $(N_{\text{lept}} + N_W)^m \chi(H)(N_{\text{lept}} + N_W)^p$ extends to a bounded operator.

Proof. The proof is analogous to that of Lemma 2.1.5. The only difference is that N_{lept} and N_W are still relatively H-bounded, but N_{neut} is not anymore. □

Lemma 3.2.2. *Under the conditions of Lemma 3.2.1, for all $m \in \mathbb{Z}$, we have that*

$$\left\|(N_0 + N_\infty + 1)^{-m}(H^{\text{ext}} - z)^{-1}(N_0 + N_\infty + 1)^m\right\| = \mathcal{O}(|\Im(z)|^{-\alpha_m}),$$

uniformly for z in a compact set of $\{z \in \mathbb{C}, \pm|\Im z| > 0\}$, where α_m denotes an integer depending on m. Moreover,

$$\begin{aligned}\|(N_{\mathrm{lept},0} + N_{W,0} + N_{\mathrm{lept},\infty} + N_{W,\infty} + 1)^{-m}(H^{\mathrm{ext}} - z)^{-1} \\ (N_{\mathrm{lept},0} + N_{W,0} + N_{\mathrm{lept},\infty} + N_{W,\infty} + 1)^{m+1}\| = \mathcal{O}(|\Im(z)|^{-\alpha_m}),\end{aligned}$$

and for all $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ and $m, p \in \mathbb{N}$, $(N_{\mathrm{lept},0} + N_{W,0} + N_{\mathrm{lept},\infty} + N_{W,\infty})^m \chi(H^{\mathrm{ext}})(N_{\mathrm{lept},0} + N_{W,0} + N_{\mathrm{lept},\infty} + N_{W,\infty})^p$ extends to a bounded operator.

Lemma 3.2.3. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian (2.12) with H_I given by (2.5). Assume that $G \in L^2$. Let j^R be defined as above. Then*

$$(H^{\mathrm{ext}} + \mathrm{i})^{-1}\check{\Gamma}(j^R) - \check{\Gamma}(j^R)(H + \mathrm{i})^{-1}(N_{\mathrm{neut}} + 1)^{-1} \in o(R^0). \tag{3.3}$$

If $G \in \mathbb{H}^\mu$ with $\mu > 0$, then

$$(H^{\mathrm{ext}} + \mathrm{i})^{-1}\check{\Gamma}(j^R) - \check{\Gamma}(j^R)(H + \mathrm{i})^{-1}(N_{\mathrm{neut}} + 1)^{-1} \in \mathcal{O}(R^{-\min(1,\mu)}). \tag{3.4}$$

In particular, for any $\chi, \chi' \in \mathrm{C}_0^\infty(\mathbb{R})$ and $\mu \geqslant 1$, we have that

$$\Big(\chi(H^{\mathrm{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\chi(H)\Big)\,\chi'(H)(N_{\mathrm{neut}} + 1)^{-1} \in \mathcal{O}(R^{-1}). \tag{3.5}$$

Proof. It suffices to adapt the proof of Lemma 2.1.11, using Lemma 3.2.1. □

3.3 Spectral Theory

As in the massive case, we first recall suitable assumptions implying existence of a ground state for H, next we study the structure of the essential spectrum using a suitable version of Mourre's theory.

3.3.1 Existence of a ground state and location of the essential spectrum

We recall here, without proof, a result due to [19, 10].

Theorem 3.3.1. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian (2.12) with H_I given by (2.5). Assume that $G \in L^2$ and that $|p_2|^{-1}G \in L^2$. Then, there exists $g_0 > 0$ such that, for all $|g| \leqslant g_0$, H has a unique ground state, i.e., $E = \inf \sigma(H)$ is a non-degenerate eigenvalue of H. Moreover,*

$$\sigma(H) = \sigma_{\mathrm{ac}}(H) = [E, \infty).$$

Let us mention that the conclusion of Theorem 3.3.1 does not exclude the presence of eigenvalues or singular continuous spectrum in the interval $[E, \infty)$. Proving absence of embedded eigenvalues and of singular continuous spectrum will be one of the main purposes of the next section. Theorem 3.3.1 is proven in [19, 10] for $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$, but the proof goes through without any substantial modification for the full Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)} + H_I^{(3)} + H_I^{(4)})$.

3.3.2 Spectral analysis

Now, we turn to the study of the essential spectrum $[E, \infty)$ of H. Let us mention that, in [10], it is proven that the spectrum of H in $[E, m_e)$ is purely absolutely continuous (except for the ground state energy, which is an eigenvalue as recalled in Theorem 3.3.1). This result is proven using Mourre's theory, with a conjugate operator given as the generator of dilatations "restricted to low-energies". The idea of employing such a conjugate operator originated in [46], see also [23]. The method of [46] is particularly efficient in that it only requires that the kernels of the interaction Hamiltonian belong to the domain of the generator of dilatations, which does not require much regularity of the kernels in the low-energy regime. However, it is presently not known how to extend this approach to prove the absence of singular continuous spectrum in the whole interval $[E, \infty)$ and not only in $[E, m_e)$.

On the other hand, if one assumes that the kernels belong to the domain of the operator b_i in (2.56)–(2.57), one can proceed as in Section 2.2.3. The conjugate operator is still given by (2.58). Note that $b_{(2),l} = a_{(2),l}$ when the masses of the neutrinos vanish. As in Section 2.2.3, one verifies that B is the generator of a C_0-semigroup of isometries and that $[H_0, \mathrm{i}B] = N$. In particular, $[H_0, \mathrm{i}B]$ is not relatively H-bounded anymore. Nevertheless, one can use singular Mourre's theory developed in [47] (see also [37]) and which framework is recalled in Section 1.3.2. Therefore, considering the following hypotheses

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3. \tag{3.6}$$

$$a_{(i),\cdot}a_{(i'),\cdot}G \in L^2, \quad i, i' = 1, 2, 3, \tag{3.7}$$

$$G \in L^2, \quad b_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3. \tag{3.8}$$

$$b_{(i),\cdot}b_{(i'),\cdot}G \in L^2, \quad i, i' = 1, 2, 3, \tag{3.9}$$

and choosing $M = [H_0, \mathrm{i}B] = N$ and $R = [H_I, \mathrm{i}B]$ and proceeding exactly as in the proofs of Lemma 2.2.5 and Theorem 2.2.6, we obtain the following result.

Theorem 3.3.2. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian* (2.12) *with H_I given by* (2.5)*. Assume that* (3.8) *holds. There exist $g_0 > 0$, $\mathrm{c} > 0$ and $\mathrm{d} > 0$ such that,*

$$[H, \mathrm{i}B] \geqslant \mathrm{c}\mathbb{1} - \mathrm{d}\Pi_\Omega,$$

where Π_Ω denotes the projection onto the vacuum in $\mathscr{H}$. In particular, H has at most one eigenvalue, which is non-degenerate. If, in addition, (3.9) *holds, then, except for the ground state energy E, the spectrum of H in $[E, \infty)$ is purely absolutely continuous.*

Theorem 3.3.2 provides a complete description of the spectrum of H for small enough values of g and under strong assumptions on the kernels G. However, in view of applications to scattering theory in Section 3.5 – more precisely, in order to prove the propagation estimates in Section 3.4 that will be subsequently used in Section 3.5 – the conjugate operator B chosen in Theorem 3.3.2 is too singular. The singularity here comes from the presence of massive particles. Indeed, for massive particles, the operators b_i are strongly singular near the origin $p_i = 0$ because of the factor $(p_i \cdot \nabla\omega_l^{(i)}(p_i))^{-1}$ in the definitions (2.56)–(2.57).

For this reason, we need to prove a Mourre estimate with another conjugate operator, namely the operator A defined in (2.38). Notice that this operator is not self-adjoint when neutrinos are supposed to be massless, because in this case the operators $a_{(2),l}$ are not self-adjoint, only maximal symmetric. The domains of the operators $a_{(2),l}$ are explicitly given as follows: let

$T : L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}) \to L^2(\mathbb{R}_+) \otimes L^2(\mathrm{S}^2) \otimes \mathbb{C}^2$ be the unitary operator, going from cartesian coordinate to polar coordinate, defined by $(Tu)(r, \theta, s) := ru(r\theta, s)$. Then

$$\mathscr{D}(a_{(2),l}) = T^{-1}\big(\mathbb{H}_0^1(\mathbb{R}_+) \otimes L^2(\mathrm{S}^2) \otimes \mathbb{C}^2\big), \quad a_{(2),l} = T^{-1}(\mathrm{i}\partial_r)T, \tag{3.10}$$

where $\mathbb{H}_0^1(\mathbb{R}_+)$ is the usual Sobolev space with Dirichlet boundary condition at 0.

As in Section 2.2.3, one verifies that $\dim(\mathrm{Ker}(A^* - \mathrm{i})) = 0$, and that A generates a C_0-semigroup of isometries $\{\tilde{W}_t\}_{t \geqslant 0}$ such that $\tilde{W}_t$ and $\tilde{W}_t^*$ preserve $\mathscr{G} := \mathscr{D}(|H|^{\frac{1}{2}}) \cap \mathscr{D}(N_{\mathrm{neut}}^{\frac{1}{2}})$. Moreover, for all $\phi \in \mathscr{G}$,

$$\sup_{0<t<1} \|\tilde{W}_t \phi\|_{\mathscr{G}} < \infty, \quad \sup_{0<t<1} \|\tilde{W}_t^* \phi\|_{\mathscr{G}} < \infty.$$

A direct computation gives

$$[H_0, \mathrm{i}A] = N_{\mathrm{neut}} + \sum_{l \in \{l, \nu, \tau\}} \mathrm{d}\Gamma(|\nabla \omega_l^{(1)}(p_1)|^2) + \mathrm{d}\Gamma(|\nabla \omega^{(3)}(p_3)|^2), \tag{3.11}$$

in the sense of quadratic forms, where

$$\mathrm{d}\Gamma(|\nabla \omega_l^{(1)}(p_1)|^2) = \sum_{\epsilon=\pm} \int |\nabla \omega_l^{(1)}(p_1)|^2 b_{l,\epsilon}^*(\xi_1) b_{l,\epsilon}(\xi_1) d\xi_1,$$

$$\mathrm{d}\Gamma(|\nabla \omega^{(3)}(p_3)|^2) = \sum_{\epsilon=\pm} \int |\nabla \omega^{(3)}(p_3)|^2 a_\epsilon^*(\xi_3) a_\epsilon(\xi_3) d\xi_3.$$

Moreover, the commutators $[H_I^{(j)}(G), \mathrm{i}A]$, $j = 1, \dots, 4$, are given by (2.41)–(2.44). In particular, the commutator $[H, \mathrm{i}A]$ is not relatively H-bounded. For this reason, we work in the setting of singular Mourre's theory.

The following lemma can be proven in the same way as Lemma 2.2.5.

Lemma 3.3.3. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *Assume that* (3.6) *holds. Then H is of class $\mathrm{C}^1(A; \mathscr{G}; \mathscr{G}^*)$. If in addition* (3.7) *holds, then H is of class $\mathrm{C}^2(A; \mathscr{G}; \mathscr{G}^*)$.*

Recall that the set of thresholds, τ, is defined in (2.47). We are now ready to prove the main result of this section.

Theorem 3.3.4. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian* (2.12) *with H_I given by* (2.5). *Assume that*

$$G \in L^2, \quad a_{(i),\cdot} G \in L^2, \quad |p_3|^{-1} a_{(i),\cdot} G \in L^2, \quad i = 1, 2, 3. \tag{3.12}$$

There exists $g_0 > 0$ such that, for all $|g| \leqslant g_0$ and $\lambda \in \mathbb{R} \backslash \tau$, there exist $\varepsilon > 0$, $\mathrm{c}_0 > 0$, $\mathrm{d} > 0$ and a compact operator K such that

$$[H, \mathrm{i}A] \geqslant \mathrm{c}_0(N_{\mathrm{neut}} + \mathbb{1}) - \mathrm{d}(\mathbb{1} - \mathbb{1}_{[\lambda-\varepsilon, \lambda+\varepsilon]}(H))(1 + H^2)^{\frac{1}{2}} + K. \tag{3.13}$$

In particular, for all interval $[\lambda_1, \lambda_2]$ such that $[\lambda_1, \lambda_2] \cap \tau = \varnothing$, H has at most finitely many eigenvalues with finite multiplicities in $[\lambda_1, \lambda_2]$ and, as a consequence, $\sigma_{\mathrm{pp}}(H)$ can accumulate only at τ, which is a countable set. If in addition

$$a_{(i),\cdot} a_{(i'),\cdot} G \in L^2, \quad i, i' = 1, 2, 3, \tag{3.14}$$

then $\sigma_{\mathrm{sc}}(H) = \varnothing$.

Remark 3.3.5. *The following weaker version of the Mourre estimate,*

$$[H, \mathrm{i}A] \geqslant \mathrm{c}_0 \mathbb{1} - \mathrm{d}(\mathbb{1} - \mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H))(1 + H^2)^{\frac{1}{2}} + K,$$

would be sufficient for the conclusions of Theorem 3.3.4 to hold. But the Mourre estimate with the operator N_{neut} in right-hand-side of (3.13) *will be important in our proof of some propagation estimates in Section 3.4.*

Proof of Theorem 3.3.4. In this proof, for any interval Δ, χ_Δ will refer to a function in $\mathrm{C}_0^\infty(\mathbb{R})$ such that $\Delta \subset \mathrm{supp}\,(\chi_\Delta)$. For all $\lambda \in \mathbb{R}$, we set

$$\begin{aligned} d(\lambda) = \inf \Big\{ \mu \in \mathbb{R} \text{ such that } \mu = &\sum_{\substack{l \in \{l,\nu,\tau\} \\ \epsilon = \pm}} \sum_{i=1}^{n_{l,\epsilon}} |\nabla \omega_l^{(1)}(p_{1,i,l,\epsilon})|^2 + \sum_{\epsilon=\pm} \sum_{i=1}^{n_\epsilon} |\nabla \omega^{(3)}(p_{3,i,\epsilon})|^2, \\ &\lambda_1 + \sum_{\substack{l \in \{l,\nu,\tau\} \\ \epsilon = \pm}} \sum_{i=1}^{n_{l,\epsilon}} \omega_l^{(1)}(p_{1,i,l,\epsilon}) + \sum_{\epsilon=\pm} \sum_{i=1}^{n_\epsilon} \omega^{(3)}(p_{3,i,\epsilon}) = \lambda,\ \lambda_1 \in \sigma_{\mathrm{pp}}(H), \\ &n_{l,\epsilon}, n_\epsilon \in \mathbb{N} \text{ and at least one of the } n_{l,\epsilon} \text{ or } n_\epsilon \text{ is} \neq 0,\ p_{1,i,l,\epsilon}, p_{3,i,\epsilon} \in \mathbb{R}^3 \Big\}, \end{aligned}$$

with the convention that $\inf \varnothing = 0$. The definition of $\tilde{d}(\lambda)$ is the same, except that we do not impose the restriction that at least one of the $n_{l,\epsilon}$ or n_ϵ is different from 0. One can then verify that $\tilde{d}(\lambda) = d(\lambda)$, if $\lambda \notin \sigma_{\mathrm{pp}}(H)$, and $\tilde{d}(\lambda) = 0$ if $\lambda \in \sigma_{\mathrm{pp}}(H)$. We also introduce, for $\kappa > 0$,

$$\Delta_\lambda^\kappa = [\lambda - \kappa, \lambda + \kappa], \quad d^\kappa(\lambda) = \inf_{\mu \in \Delta_\lambda^\kappa} d(\mu), \quad \tilde{d}^\kappa(\lambda) = \inf_{\mu \in \Delta_\lambda^\kappa} \tilde{d}(\mu).$$

Recalling (3.11), we set

$$H_0' := \frac{1}{2} N_{\mathrm{neut}} + \sum_{l \in \{l,\nu,\tau\}} \mathrm{d}\Gamma(|\nabla \omega_l^{(1)}(p_1)|^2) + \mathrm{d}\Gamma(|\nabla \omega^{(3)}(p_3)|^2),$$

so that

$$[H_0, \mathrm{i}A] = \frac{1}{2} N_{\mathrm{neut}} + H_0'.$$

We also set $H_I' := [H_I, \mathrm{i}A]$.

We follow the general strategy of the proof of [29, Theorem 4.3]. Let $m_1 := \inf(m_e, m_\mu, m_\tau, m_W) > 0$. We will prove by induction that the following properties hold for any $n \in \mathbb{N}^*$.

$H_1(n)$: Let $\varepsilon > 0$ and $\lambda \in [E, E + nm_1)$. There exist a constant c, a compact operator K_0 and an interval Δ containing λ such that

$$H_0' + gH_I' \geqslant \Big(\min\big(\frac{1}{4}, d(\lambda)\big) - \varepsilon \Big) \mathbb{1} - \mathrm{c}\big(\mathbb{1} - \chi_\Delta(H)\big)(1 + H^2)^{\frac{1}{2}} + K_0.$$

$H_2(n)$: Let $\varepsilon > 0$ and $\lambda \in [E, E + nm_1)$. There exist a constant c and an interval Δ containing λ such that

$$H_0' + gH_I' \geqslant \Big(\min\big(\frac{1}{4}, \tilde{d}(\lambda)\big) - \varepsilon \Big) \mathbb{1} - \mathrm{c}\big(\mathbb{1} - \chi_\Delta(H)\big)(1 + H^2)^{\frac{1}{2}}.$$

$H_3(n)$: Let $\kappa > 0$, $\varepsilon_0 > 0$ and $\varepsilon > 0$. There exist a constant c and $\delta > 0$ such that, for all $\lambda \in [E, E + nm_1 - \varepsilon_0]$, one has

$$H_0' + gH_I' \geqslant \Big(\min\big(\frac{1}{4}, \tilde{d}^{\kappa}(\lambda)\big) - \varepsilon \Big) \mathbb{1} - \mathrm{c}\big(\mathbb{1} - \chi_{\Delta_\lambda^\delta}(H)\big)(1 + H^2)^{\frac{1}{2}}.$$

$S_1(n)$: τ is a closed countable set in $[E, E + nm_1]$.

$S_2(n)$: for all λ_1, λ_2 such that $\lambda_1 < \lambda_2 \leqslant E + nm_1$ and $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, H has finitely many eigenvalues, with finite multiplicities, in $[\lambda_1, \lambda_2]$.

We claim that, for all $n \in \mathbb{N}^*$,

$$S_2(n-1) \Rightarrow S_1(n) \tag{3.15}$$

$$S_1(n) \ \text{ and } \ H_3(n-1) \Rightarrow H_1(n). \tag{3.16}$$

$$H_1(n) \Rightarrow H_2(n) \tag{3.17}$$

$$H_2(n) \Rightarrow H_3(n) \tag{3.18}$$

$$H_1(n) \Rightarrow S_2(n). \tag{3.19}$$

By definition, $\tau \cap [E, E + m_1) = \emptyset$ and hence $S_1(1)$ is obviously satisfied. We refer to the proof of Theorem 2.2.4 and to [77, 8, 28] for the proofs of (3.15), (3.17), (3.18) and (3.19).

Using that the commutators $[H_I^{(j)}(G), \mathrm{i}A]$, $j = 1, \dots, 4$, are given by (2.41)–(2.44), a direct application of the N_τ estimates of Glimm and Jaffe (see [51, Proposition 1.2.3(b)]) shows that there exist $\mathrm{c}_1 > 0$ and $\mathrm{c}_2 > 0$ such that

$$\big|\langle \Psi, H_I' \Psi \rangle\big| \leqslant \mathrm{c}_1 \langle \Psi, H_0' \Psi \rangle + \mathrm{c}_2 \|\Psi\|^2, \tag{3.20}$$

for all $\Psi \in \mathscr{D}(H_0')$. Here it should be noticed that $|\nabla \omega^{(3)}(p_3)|^2 = p_3^2(p_3^2 + m_W^2)^{-1}$. Hence, according to [51, Proposition 1.2.3(b)], the constant c_1 can be chosen to be proportional to $\||p_3|^{-1}G\|_2 + \|G\|_2$, which is finite by (3.12).

Recall that the operator $\check{\Gamma}(j^R) : \mathscr{H} \to \mathscr{H}^{\mathrm{ext}}$ has been defined in Section 1.2.4 (see also Section 2.1.2). Using that $\check{\Gamma}(j^R)$ is an isometry, we can write

$$\begin{aligned} H_0' + gH_I' &= \check{\Gamma}(j^R)^* \check{\Gamma}(j^R)(H_0' + gH_I') \\ &= \check{\Gamma}(j^R)^*(H_0' \otimes \mathbb{1} + \mathbb{1} \otimes H_0' + gH_I' \otimes \mathbb{1} + \mathrm{Rem}_R)\check{\Gamma}(j^R), \end{aligned}$$

with $\mathrm{Rem}_R(N_0 + N_\infty + 1)^{-1} = o(R^0)$. The last equality can be proven in the same way as in the proofs of Lemmas 2.1.11–3.2.3. We decompose

$$\begin{aligned} [H, \mathrm{i}A] = &\ \check{\Gamma}(j^R)^*(H_0' \otimes \mathbb{1} + \mathbb{1} \otimes H_0' + gH_I' \otimes \mathbb{1} + \mathrm{Rem}_R)\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R) \\ &+ \check{\Gamma}(j^R)^*(H_0' \otimes \mathbb{1} + \mathbb{1} \otimes H_0' + gH_I' \otimes \mathbb{1} + \mathrm{Rem}_R)\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R), \end{aligned} \tag{3.21}$$

and estimate the two terms separately. For the second one, we notice that

$$(\mathbb{1} \otimes H_0')\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty}) \geqslant \frac{1}{2}(\mathbb{1} \otimes N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty}) \geqslant \frac{1}{2}\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty}). \tag{3.22}$$

By (3.20), we have that

$$\begin{aligned} &\check{\Gamma}(j^R)^*(H_0' \otimes \mathbb{1} + \mathbb{1} \otimes H_0' + gH_I' \otimes \mathbb{1} + \mathrm{Rem}_R)\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R) \\ &\geqslant \check{\Gamma}(j^R)^*((1 - \mathrm{c}_1 g)H_0' \otimes \mathbb{1} - \mathrm{c}_2 g + \mathbb{1} \otimes H_0' + \mathrm{Rem}_R)\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R) \\ &\geqslant \check{\Gamma}(j^R)^*((1 - \mathrm{c}_1 g)H_0' \otimes \mathbb{1} - \mathrm{c}_2 g + \mathbb{1} \otimes H_0' \\ &\quad + \mathrm{Rem}_R(N_0 + N_\infty + 1)^{-1}(N_0 + N_\infty + 1))\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R) \\ &\geqslant \check{\Gamma}(j^R)^*((1 - \mathrm{c}_1 g)H_0' \otimes \mathbb{1} - \mathrm{c}_2 g + \mathbb{1} \otimes H_0' + o(R^0)(N_0 + N_\infty + 1))\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R). \end{aligned}$$

For g small enough and R large enough, since N_{lept} and N_W are relatively H-bounded, this yields

$$\begin{aligned}
&\check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+\mathbb{1}\otimes H_0'+gH_I'\otimes\mathbb{1}+\mathrm{Rem}_R)\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)\\
&\geqslant \check{\Gamma}(j^R)^*((1-\mathrm{c}_1g)H_0'\otimes\mathbb{1}-\mathrm{c}_2g+\mathbb{1}\otimes H_0'+o(R^0)(N_{\mathrm{neut},0}+N_{\mathrm{neut},\infty}))\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)\\
&+o(R^0)\check{\Gamma}(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)(1+H^2)^{\frac12}\\
&\geqslant \check{\Gamma}(j^R)^*(-\mathrm{c}_2g+\frac12\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})+o(R^0)(N_{\mathrm{neut},0}+N_{\mathrm{neut},\infty}))\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)\\
&+o(R^0)\check{\Gamma}(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)(1+H^2)^{\frac12}\\
&\geqslant \frac14\check{\Gamma}(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)+o(R^0)\check{\Gamma}(j^R)^*(N_{\mathrm{neut},0}+N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)\\
&+o(R^0)\check{\Gamma}(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)(1+H^2)^{\frac12}\\
&\geqslant \frac14\check{\Gamma}(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)+o(R^0)\check{\Gamma}(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\mathrm{neut},\infty})\check{\Gamma}(j^R)(1+H^2)^{\frac12},
\end{aligned}\tag{3.23}$$

where (3.22) and the fact that $H_0'\geqslant 0$ have been used.

Now, we consider the first term in (3.21). As in the proof of Theorem 2.2.1, we use the fact that, for all bounded interval I and $R>0$, $\check{\Gamma}(j^R)^*(\mathbb{1}\otimes\Pi_\Omega)\check{\Gamma}(j^R)\chi_I(H)$ is compact. Thus, using that $N_{\mathrm{neut}}\leqslant H_0'$, that H_I' is relatively $(N_{\mathrm{lept}}+N_W)$-bounded and that N_{lept} and N_W are relatively H-bounded, we can write

$$\begin{aligned}
&\check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+\mathbb{1}\otimes H_0'+gH_I'\otimes\mathbb{1}+\mathrm{Rem}_R)\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{\{0\}}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\\
&=\check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+gH_I'\otimes\mathbb{1}+\mathrm{Rem}_R)(\mathbb{1}\otimes\Pi_\Omega))\check{\Gamma}(j^R)\\
&\geqslant \check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+gH_I'\otimes\mathbb{1}+o(R^0)(N_{\mathrm{neut}}\otimes\mathbb{1}+N_{\mathrm{lept}}\otimes\mathbb{1}+N_W\otimes\mathbb{1}))(\mathbb{1}\otimes\Pi_\Omega))\check{\Gamma}(j^R)\\
&\geqslant \check{\Gamma}(j^R)^*((1-o(R^0))H_0'\otimes\mathbb{1}+gH_I'\otimes\mathbb{1}+o(R^0)(N_{\mathrm{lept}}\otimes\mathbb{1}+N_W\otimes\mathbb{1}))(\mathbb{1}\otimes\Pi_\Omega))\check{\Gamma}(j^R)\\
&\geqslant \check{\Gamma}(j^R)^*(gH_I'\otimes\mathbb{1}+o(R^0)(N_{\mathrm{lept}}\otimes\mathbb{1}+N_W\otimes\mathbb{1}))(\mathbb{1}\otimes\Pi_\Omega))\check{\Gamma}(j^R)\\
&\geqslant K_{R,I}-\mathrm{c}_3\check{\Gamma}(j^R)^*(\mathbb{1}\otimes\Pi_\Omega)\check{\Gamma}(j^R)(\mathbb{1}-\chi_I(H))(1+H^2)^{\frac12},
\end{aligned}\tag{3.24}$$

for any bounded interval I and $R>0$ large enough, where $K_{R,I}$ is compact and c_3 is a positive constant.

It remains to consider

$$\begin{aligned}
&\check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+\mathbb{1}\otimes H_0'+gH_I'\otimes\mathbb{1}+\mathrm{Rem}_R)\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\\
&\geqslant \check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+gH_I'\otimes\mathbb{1}+o(R^0)(N_0+N_\infty+1))\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R).
\end{aligned}$$

Here we used that $\mathbb{1}\otimes H_0'\geqslant 0$. We introduce $\mathbb{1}=\chi_I(H)+(\mathbb{1}-\chi_I(H))$ on the right. Using that $H_0'\otimes\mathbb{1}\geqslant 0$, we can write

$$\begin{aligned}
&\check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+\mathbb{1}\otimes H_0'+gH_I'\otimes\mathbb{1}+\mathrm{Rem}_R)\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\\
&\geqslant \check{\Gamma}(j^R)^*((1-o(R^0))(H_0'+gH_I')\otimes\mathbb{1})\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\chi_I(H)\\
&\quad+\check{\Gamma}(j^R)^*(o(R^0)gH_I'\otimes\mathbb{1}+o(R^0)(N_0+N_\infty+1))\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\chi_I(H)\\
&\quad+\check{\Gamma}(j^R)^*(H_0'\otimes\mathbb{1}+gH_I'\otimes\mathbb{1}+o(R^0)(N_0+N_\infty+1))\\
&\qquad\qquad\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)(\mathbb{1}-\chi_I(H)).
\end{aligned}$$

Hence, using again that H_I' is relatively $(N_{\text{lept}} + N_W)$-bounded and that N_{lept} and N_W are relatively H-bounded, we obtain that

$$\begin{aligned}
&\check{\Gamma}(j^R)^*(H_0' \otimes \mathbb{1} + \mathbb{1} \otimes H_0' + gH_I' \otimes \mathbb{1} + \mathrm{Rem}_R)\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R) \\
&\geqslant \check{\Gamma}(j^R)^*((1 - o(R^0))(H_0' + gH_I') \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_I(H) \\
&\quad + o(R^0)\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)(1 + H^2)^{\frac{1}{2}}\chi_I(H) \\
&\quad - \mathrm{c}_4\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)(\mathbb{1} - \chi_I(H))(1 + H^2)^{\frac{1}{2}} \\
&\geqslant \check{\Gamma}(j^R)^*((1 - o(R^0))(H_0' + gH_I') \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_I(H) \\
&\quad + o(R^0)\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_I(H) \\
&\quad - \mathrm{c}_4\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)(\mathbb{1} - \chi_I(H))(1 + H^2)^{\frac{1}{2}},
\end{aligned} \tag{3.25}$$

for any bounded interval I and $R > 0$, where c_4 is a positive constant.

Now, we can prove $H_1(1)$ and (3.16). To prove $H_1(1)$, let $\varepsilon > 0$, $\lambda \in [E, E + m_1)$ and Δ be an interval containing λ and supported in $[E, E + m_1)$. The function $\chi_\Delta \in \mathrm{C}_0^\infty(\mathbb{R})$ is chosen such that $\mathrm{supp}(\chi_\Delta) \subset (-\infty, E + m_1)$ and we consider in addition $\tilde{\chi}_\Delta \in \mathrm{C}_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(\tilde{\chi}_\Delta) \subset (-\infty, E + m_1)$ and $\tilde{\chi}_\Delta\chi_\Delta = \chi_\Delta$. Then, using that $H_0' \geqslant 0$, the fact that H_I' is $(N_{\text{lept}} + N_W)$-bounded as before, and Lemma 3.2.3, the first term in the right-hand-side of (3.25) (with $I = \Delta$) can be estimated, for R large enough, in the following way:

$$\begin{aligned}
&\check{\Gamma}(j^R)^*((1 - o(R^0))(H_0' + gH_I') \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&\geqslant \check{\Gamma}(j^R)^*((1 - o(R^0))H_0' \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&\quad - \mathrm{c}_5\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H), \\
&\geqslant \check{\Gamma}(j^R)^*((1 - o(R^0))H_0' \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&- \mathrm{c}_5\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\tilde{\chi}_\Delta(H^{\text{ext}})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&+ \mathrm{c}_5\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\left(\tilde{\chi}_\Delta(H^{\text{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\tilde{\chi}_\Delta(H)\right)\chi_\Delta(H), \\
&\geqslant \check{\Gamma}(j^R)^*((1 - o(R^0))H_0' \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&+ \mathrm{c}_5\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\left(\tilde{\chi}_\Delta(H^{\text{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\tilde{\chi}_\Delta(H)\right)\chi_\Delta(H).
\end{aligned}$$

The last equality comes from the fact that $H^{\text{ext}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H_0 \geqslant E + m_1$ on the range of the operator $\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})$, and hence that $\tilde{\chi}_\Delta(H^{\text{ext}})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty}) = 0$, since $\tilde{\chi}_\Delta$ is supported in $(-\infty, E + m_1)$. As said before, the idea is now to apply Lemma 3.2.3.

$$\begin{aligned}
&\check{\Gamma}(j^R)^*((1 - o(R^0))(H_0' + gH_I') \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&\geqslant \check{\Gamma}(j^R)^*((1 - o(R^0))H_0' \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&+ \mathrm{c}_5\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty}) \\
&\left(\tilde{\chi}_\Delta(H^{\text{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\tilde{\chi}_\Delta(H)\right)\chi_\Delta(H)(N_{neut} + 1)(N_{neut} + 1)^{-1}, \\
&\geqslant \check{\Gamma}(j^R)^*((1 - o(R^0))H_0' \otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H) \\
&+ \mathrm{c}_5\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty}) \\
&\left(\tilde{\chi}_\Delta(H^{\text{ext}})\check{\Gamma}(j^R) - \check{\Gamma}(j^R)\tilde{\chi}_\Delta(H)\right)(N_{neut} + 1)\chi_\Delta(H)(N_{neut} + 1)^{-1} \\
&+ \mathrm{o}(R^{-1})\mathrm{c}_5\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty} + N_{W,\infty})\check{\Gamma}(j^R)\chi_\Delta(H),
\end{aligned}$$

applying now Lemma 2.1.8 we have:

$$\begin{aligned}
&\geqslant \check\Gamma(j^R)^*((1-o(R^0))H_0'\otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_\Delta(H)\\
&+\mathrm{c}_5\check\Gamma(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})(N_{neut}\otimes\mathbb{1}+\mathbb{1}\otimes N_{neut}+1)\\
&\Big(\tilde\chi_\Delta(H^{\text{ext}})\check\Gamma(j^R)-\check\Gamma(j^R)\tilde\chi_\Delta(H)\Big)\,\chi_\Delta(H)(N_{neut}+1)^{-1}\\
&+\mathrm{o}(R^{-1})\mathrm{c}_5\check\Gamma(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_\Delta(H),\\
&\geqslant \check\Gamma(j^R)^*((1-o(R^0))H_0'\otimes \mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_\Delta(H)\\
&+\mathrm{o}(R^{-1})\mathrm{c}_5\check\Gamma(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})(N_{neut}\otimes\mathbb{1}+\mathbb{1}\otimes N_{neut}+1)\check\Gamma(j^R)\chi_\Delta(H)\\
&+\mathrm{o}(R^{-1})\mathrm{c}_5\check\Gamma(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_\Delta(H),\\
&\geqslant 0. \qquad (3.26)
\end{aligned}$$

Since $d(\lambda)=0$ for $\lambda\in(-\infty,E+m_1)$, fixing R large enough, $I=\Delta$ in (3.24) and (3.25), and combining Equations (3.21)–(3.26), we obtain:

$$\begin{aligned}
[H,\mathrm{i}A] \;\geqslant\;& \frac14\check\Gamma(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\text{neut},\infty})\check\Gamma(j^R)+o(R^0)\Big(\check\Gamma(j^R)^*\mathbb{1}_{[1,\infty)}(N_{\text{neut},\infty})\check\Gamma(j^R)(1+H^2)^{\frac12}\\
+\;&\check\Gamma(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_I(H)\Big)\\
-\;&\mathrm{c}_3\check\Gamma(j^R)^*(\mathbb{1}\otimes\Pi_\Omega)\check\Gamma(j^R)(\mathbb{1}-\chi_I(H))(1+H^2)^{\frac12}\\
-\;&\mathrm{c}_4\check\Gamma(j^R)^*\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)(\mathbb{1}-\chi_I(H))(1+H^2)^{\frac12}+K_{R,I}
\end{aligned}$$

which is $H_1(1)$.

To prove (3.16), let $\varepsilon>0$ and $\lambda\in[E,E+nm_1)$. We go back to (3.25), with $I=\Delta_\mu^\delta$, and consider again the first term in the right-hand-side. By $S_1(n)$, τ, where d vanishes, is a closed countable set in $[E,E+nm_1]$, which implies that there exists κ such that $\tau\cap\Delta_\lambda^\kappa\neq\varnothing$ and then $d(\lambda)=\sup_{\kappa>0}d^\kappa(\lambda)$. Hence there exists $\kappa>0$ such that $d^\kappa(\lambda)>d(\lambda)-\varepsilon/3$. Let $\varepsilon_0>0$ be such that $\lambda\in[E,E+nm_1-\varepsilon_0]$. By $H_3(n-1)$, we know that there exist $\mathrm{c}_6\in\mathbb{R}$ and $\delta>0$ such that

$$H_0'+gH_I'\geqslant(\min(\frac14,\tilde d^\kappa(\mu))-\frac{\varepsilon}{3})\mathbb{1}-\mathrm{c}_6(\mathbb{1}-\tilde\chi_{\Delta_\mu^\delta}(H))(1+H^2)^{\frac12}, \qquad (3.27)$$

for all $\mu\in[E,E+(n-1)m_1-\varepsilon_0]$. Here $\tilde\chi_{\Delta_\lambda^\delta}$ is chosen such that $\tilde\chi_{\Delta_\lambda^\delta}\chi_{\Delta_\lambda^\delta}=\tilde\chi_{\Delta_\lambda^\delta}$, where $\chi_{\Delta_\lambda^\delta}$ is the function appearing in (3.25). We begin by estimating the first term in the right-hand-side of (3.25) as

$$\begin{aligned}
&\check\Gamma(j^R)^*((1-o(R^0))(H_0'+gH_I')\otimes\mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_{\Delta_\lambda^\delta}(H)\\
&\geqslant\check\Gamma(j^R)^*((1-o(R^0))(\min(\frac14,\tilde d^\kappa(\mu))-\frac{\varepsilon}{3})\mathbb{1}\otimes\mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_{\Delta_\lambda^\delta}(H)\\
&-\mathrm{c}_6\check\Gamma(j^R)^*((1-o(R^0))((\mathbb{1}-\tilde\chi_{\Delta_\mu^\delta}(H))(1+H^2)^{\frac12}\otimes\mathbb{1})\mathbb{1}_{\{0\}}(N_{\text{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})\check\Gamma(j^R)\chi_{\Delta_\lambda^\delta}(H).
\end{aligned}$$

On the range of $\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})$, we have that $\mathbb{1}\otimes H_0\geqslant m_1$. Therefore, by (3.27) and the functional calculus, with $\mu=\lambda-1\otimes H_0$, we obtain that

$$\tilde d(\lambda-\mathbb{1}\otimes H_0)=\int\tilde d(\lambda-\Omega(p))E(dp)$$

where E is the spectral measure of the momentum operator and Ω the sum of all the free energies of the particles. Therefore, on the range of $\mathbb{1}_{[1,\infty)}(N_{\text{lept},\infty}+N_{W,\infty})$ the following equality holds:

$$\tilde d(\lambda-\mathbb{1}\otimes H_0)=d(\lambda)$$

and

$$\begin{aligned}
&(1-o(R^0))\check{\Gamma}(j^R)^*(\min(\frac{1}{4},\tilde{d}^\kappa(\lambda-\mathbb{1}\otimes H_0))-\frac{\varepsilon}{3})\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\chi_{\Delta_\lambda^\delta}(H)\\
&=(1-o(R^0))\check{\Gamma}(j^R)^*(\min(\frac{1}{4},d^\kappa(\lambda))-\frac{\varepsilon}{3})\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\chi_{\Delta_\lambda^\delta}(H)\\
&\geqslant(1-o(R^0))(\min(\frac{1}{4},d(\lambda))-\frac{2\varepsilon}{3})\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\chi_{\Delta_\lambda^\delta}(H)\\
&\geqslant(\min(\frac{1}{4},d(\lambda))-\varepsilon)\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\chi_{\Delta_\lambda^\delta}(H).
\end{aligned}\tag{3.28}$$

Moreover, let $\tilde{\tilde{\chi}}_{\Delta_\lambda^\delta}\in \mathrm{C}_0^\infty(\mathbb{R})$ be such that $\tilde{\tilde{\chi}}_{\Delta_\lambda^\delta}\chi_{\Delta_\lambda^\delta}=\chi_{\Delta_\lambda^\delta}$. Using Lemma 2.1.11, we write

$$\begin{aligned}
&\mathrm{c}_6\check{\Gamma}(j^R)^*((1-o(R^0))((\mathbb{1}-\tilde{\chi}_{\Delta_\mu^\delta}(H))(1+H^2)^{\frac{1}{2}}\otimes\mathbb{1})\\
&\qquad\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\check{\Gamma}(j^R)\chi_{\Delta_\lambda^\delta}(H)\tilde{\tilde{\chi}}_{\Delta_\lambda^\delta}(H)\\
&=\mathrm{c}_6((1-o(R^0))\check{\Gamma}(j^R)^*\mathbb{1}_{\{0\}}(N_{\mathrm{neut},\infty})\mathbb{1}_{[1,\infty)}(N_{\mathrm{lept},\infty}+N_{W,\infty})\\
&\qquad(((\mathbb{1}-\tilde{\chi}_{\Delta_\mu^\delta}(H))(1+H^2)^{\frac{1}{2}}\otimes\mathbb{1})\chi_{\Delta_\lambda^\delta}(H^{\mathrm{ext}})\check{\Gamma}(j^R)\tilde{\tilde{\chi}}_{\Delta_\lambda^\delta}(H)+o(R^0)\\
&=o(R^0).
\end{aligned}\tag{3.29}$$

The last equality comes from $\mu=\lambda-\mathbb{1}\otimes H_0$ and $\tilde{\chi}_{\Delta_\lambda^\delta}\chi_{\Delta_\lambda^\delta}=\tilde{\chi}_{\Delta_\lambda^\delta}$. Fixing R large enough and $I=\Delta_\lambda^\delta$, Equations (3.21), (3.23), (3.24), (3.25), (3.28) and (3.29) prove (3.16). This concludes the proof of (3.13).

The fact that $\sigma_{\mathrm{sc}}(H)=\varnothing$ if (3.14) holds follows from Lemma 3.3.3 and the abstract results recalled above. □

3.4 Propagation Estimates

In this section we prove the counterpart of Theorem 2.3.1 in the case where the masses of the neutrinos vanish. As mentioned at the beginning of Section 2.3, for massless neutrinos we cannot directly adapt [29, 30], hence some parts of the proof require substantial modifications. A first issue comes from the fact that, in order to control remainder terms in some commutator expansions, one needs to control the second quantization of expressions of the form $[\mathrm{i}\nabla_{p_2},[\mathrm{i}\nabla_{p_2},\omega_l^{(2)}(p_2)]]$, where $\omega_l^{(2)}$ are the dispersion relations of neutrinos. In the massive case, such commutators are bounded. Their second quantizations are therefore relatively N-bounded and, hence, relatively H-bounded. In the massless case, however, $[\mathrm{i}\nabla_{p_2},[\mathrm{i}\nabla_{p_2},\omega_l^{(2)}(p_2)]]$ is of order $\mathcal{O}(|p_2|^{-1})$ near 0. To overcome this difficulty, the idea is to show that $\langle e^{-\mathrm{i}tH}u,\mathrm{d}\Gamma(|p_2|^{-1})e^{-\mathrm{i}tH}u\rangle\lesssim t^\gamma$ for some $\gamma<1$ and for suitable states $u\in\mathscr{H}$. This idea was used in [21, 38, 39] for Pauli-Fierz Hamiltonians, modifying an argument of [49]. In Lemma 3.4.1, we adapt [21, 38, 39] to our context.

A second issue that has to be dealt with, in the massless case, to prove a suitable minimal velocity estimate, is that we have to use the Mourre estimate stated in Theorem 3.3.4, with a positive term proportional to the number operator N_{neut}. This is again required in order to be able to control some remainder terms. Some care must also be taken because of the fact that the considered conjugate operator (see (2.38)) is not self-adjoint. Propagation estimates analogous to those of Theorem 2.3.1 are established in Theorem 3.4.2. An important difference with the massive case is that the propagation estimates hold only for states in a dense subset of the total Hilbert space.

In Theorem 3.4.3, we prove a second version of propagation estimates involving a time-dependent modified one-particle "position" operator. Theorem 3.4.3 will be a crucial input in our proof of asymptotic completeness of the wave operators in Section 3.5.

For shortness, we set $\mathrm{d}\Gamma(|p_2|^{-\alpha}) = \mathrm{d}\Gamma(h^\alpha)$ where $h^\alpha = (h_1^\alpha, \dots, h_{14}^\alpha)$ with $h_i^\alpha = 0$ if $i \in \{1,2,3,4,7,8,11,12\}$ (corresponding to the label of a massive particle), and h_i^α is the operator of multiplication by $|p_2|^{-\alpha}$ if $i \in \{5,6,9,10,13,14\}$ (corresponding to the label of a neutrino). The first part of the following lemma (see (3.30)) is adapted from [21, Lemma 4.1]. For technical reasons, that will appear in the proofs of Theorems 3.4.2 and 3.4.3 below, we also need a new, related estimate, see (3.31).

Lemma 3.4.1. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian* (3.2) *with H_I given by* (2.5). *Let $0 \leqslant \alpha \leqslant 1$ and assume that*

$$G \in L^2, \quad |p_2|^{-1-\mu} G \in L^2 \text{ for some } \mu > 0.$$

Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. There exists $\mathrm{C} > 0$ such that, for all $u \in \mathscr{D}(\mathrm{d}\Gamma(|p_2|^{-\alpha})^{\frac12})$,

$$\begin{aligned}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma(|p_2|^{-\alpha}) e^{-\mathrm{i}tH}\chi(H)u\Big\rangle \ &\leqslant\ \mathrm{C}t^{\frac{\alpha}{1+\mu}}\big(\|\mathrm{d}\Gamma(|p_2|^{-\alpha})^{\frac12}u\|^2 + \|N_{\mathrm{neut}}^{\frac12}u\|^2 + \|u\|^2\big)\\ &+\ \|\mathrm{d}\Gamma(|p_2|^{-\alpha})^{\frac12}u\|^2 + \|u\|^2, \end{aligned} \tag{3.30}$$

and for all $u \in \mathscr{D}(\mathrm{d}\Gamma(|p_2|^{-\alpha}))$,

$$\Big\|\mathrm{d}\Gamma(|p_2|^{-\alpha}) e^{-\mathrm{i}tH}\chi(H)u\Big\| \leqslant \mathrm{C}t^{\frac{\alpha}{1+\mu}}\big(\|\mathrm{d}\Gamma(|p_2|^{-\alpha})u\| + \|N_{\mathrm{neut}}u\| + \|u\|\big) + \|\mathrm{d}\Gamma(|p_2|^{-\alpha})^{\frac12}u\|^2 + \|u\|^2. \tag{3.31}$$

Proof. To prove (3.30), we consider a function $f_0 \in \mathrm{C}^\infty([0,\infty);\mathbb{R})$ such that f_0 is decreasing, $f_0(r) = 1$ on $[0,1]$ and $f_0(r) = 0$ on $[2,\infty)$. Let $f_\infty = 1 - f_0$. For $\nu > 0$, we decompose

$$\mathrm{d}\Gamma(|p_2|^{-\alpha}) = \mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu |p_2|)\big) + \mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_\infty(t^\nu |p_2|)\big), \tag{3.32}$$

and insert this into the left-hand-side of (3.30). Since f_∞ is supported in $[1,\infty)$, $N_{\mathrm{neut}} - N_{\mathrm{lept}}$ commutes with H and N_{lept} is relatively H-bounded the second term can be estimated as

$$\begin{aligned}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_\infty(t^\nu|p_2|)\big) e^{-\mathrm{i}tH}\chi(H)u\Big\rangle &\leqslant t^{\alpha\nu}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\big(f_\infty(t^\nu|p_2|)\big) e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\leqslant t^{\alpha\nu}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, N_{\mathrm{neut}} e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\leqslant t^{\alpha\nu}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, (N_{\mathrm{neut}} - N_{\mathrm{lept}}) e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &+ t^{\alpha\nu}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, N_{\mathrm{lept}} e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\leqslant t^{\alpha\nu}\Big\|\chi(H)|N_{\mathrm{neut}} - N_{\mathrm{lept}}|^{\frac12}u\Big\|^2\\ &+ t^{\alpha\nu}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, N_{\mathrm{lept}} e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\leqslant t^{\alpha\nu}\Big\|\chi(H)(N_{\mathrm{neut}}^{\frac12} + N_{\mathrm{lept}}^{\frac12})u\Big\|^2\\ &+ t^{\alpha\nu}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, N_{\mathrm{lept}} e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\lesssim t^{\alpha\nu}\big(\|N_{\mathrm{neut}}^{\frac12}u\|^2 + \|u\|^2\big). \end{aligned} \tag{3.33}$$

To estimate the evolution of the first term in (3.32), we differentiate

$$
\begin{aligned}
&\partial_t\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle \\
&= \nu t^{\nu-1}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\big(|p_2|^{1-\alpha} f_0'(t^\nu|p_2|)\big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle \\
&\quad + \Big\langle e^{-\mathrm{i}tH}\chi(H)u, \big[H, \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)\big]e^{-\mathrm{i}tH}\chi(H)u\Big\rangle.
\end{aligned}
$$

Since $f_0' \leqslant 0$ and since H_0 commutes with $\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)$, this implies that

$$
\begin{aligned}
&\partial_t\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle \\
&\leqslant \Big\langle e^{-\mathrm{i}tH}\chi(H)u, \big[H_I(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)\big]e^{-\mathrm{i}tH}\chi(H)u\Big\rangle.
\end{aligned} \tag{3.34}
$$

Similarly as in (2.20)–(2.23), a direct computation gives

$$
\begin{aligned}
\big[H_I^{(1)}(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)\big] &= -H_I^{(1)}(\mathrm{i}|p_2|^{-\alpha} f_0(t^\nu|p_2|)G), \\
\big[H_I^{(2)}(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)\big] &= -H_I^{(2)}(\mathrm{i}|p_2|^{-\alpha} f_0(t^\nu|p_2|)G), \\
\big[H_I^{(3)}(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)\big] &= H_I^{(3)}(\mathrm{i}|p_2|^{-\alpha} f_0(t^\nu|p_2|)G), \\
\big[H_I^{(4)}(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)\big] &= H_I^{(4)}(\mathrm{i}|p_2|^{-\alpha} f_0(t^\nu|p_2|)G).
\end{aligned}
$$

By Lemma 2.1.3, the support of f_0 and the assumption that $|p_2|^{-1-\mu}G \in L^2$, we obtain that

$$
\begin{aligned}
\big\|\big[H_I(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)\big](N_{\mathrm{lept}} + N_W + 1)^{-1}\big\| &\lesssim \|(|p_2|^{-\alpha} f_0(t^\nu|p_2|))G\|_2 \\
&\lesssim \|(|p_2|^{1+\mu-\alpha} f_0(t^\nu|p_2|)|p_2|^{-1-\mu})G\|_2 \\
&\lesssim t^{-(1+\mu-\alpha)\nu}\|(|p_2|^{-(1+\mu)})G\|_2.
\end{aligned} \tag{3.35}
$$

Since $N_{\mathrm{lept}} + N_W$ is relatively H-bounded, integrating (3.34) shows that

$$
\begin{aligned}
&\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(t^\nu|p_2|)\big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle - \Big\langle \chi(H)u, \mathrm{d}\Gamma\big(|p_2|^{-\alpha}\big)\chi(H)u\Big\rangle \\
&\leqslant \int_0^t \Big\langle e^{-\mathrm{i}sH}\chi(H)u, \big[H_I(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(s^\nu|p_2|)\big)\big]e^{-\mathrm{i}sH}\chi(H)u\Big\rangle ds, \\
&\leqslant \int_0^t \|e^{-\mathrm{i}sH}\chi(H)u\|\,\big\|\big[H_I(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(s^\nu|p_2|)\big)\big]e^{-\mathrm{i}sH}\chi(H)u\big\|\, ds, \\
&\leqslant \|\chi(H)u\|\,\|(N_{lept} + N_W + 1)\chi(H)u\| \int_0^t \big\|\big[H_I(G), \mathrm{i}\mathrm{d}\Gamma\big(|p_2|^{-\alpha} f_0(s^\nu|p_2|)\big)\big](N_{lept} + N_W + 1)^{-1}\big\|\, ds, \\
&\lesssim \|u\|^2 \int_0^t s^{-(1+\mu-\alpha)\nu}\|(|p_2|^{-(1+\mu)})G\|_2 ds, \\
&\lesssim \|u\|^2\, t^{1-(1+\mu-\alpha)\nu}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\langle \chi(H)u, d\Gamma(|p_2|^{-\alpha})\chi(H)u\rangle &= \Big\|d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}\chi(H)u\Big\|^2, \\
&= \Big\|\chi(H)d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}u\Big\|^2 + \Big\|[d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}, \chi(H)]u\Big\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|[d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}, \chi(H)]u\right\|^2 &= \left\|[d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}, \chi(H)]\left(d\Gamma(|p_2|^{-\alpha})+1\right)^{-1}\left(d\Gamma(|p_2|^{-\alpha})+1\right)u\right\|^2, \\
&\lesssim \left(\left\|d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}\chi(H)\left(d\Gamma(|p_2|^{-\alpha})+1\right)^{-1}\right\|^2 + \left\|\chi(H)\left(d\Gamma(|p_2|^{-\alpha})+1\right)^{-\frac{1}{2}}\right\|^2\right) \\
&\quad \left\|\left(d\Gamma(|p_2|^{-\alpha})+1\right)u\right\|^2, \\
&\lesssim \left(\left\|\left(d\Gamma(|p_2|^{-\alpha})+1\right)^{-\frac{1}{2}}\chi(H)\right\|^2\right. \\
&+ \left\|\left(d\Gamma(|p_2|^{-\alpha})+1\right)^{-\frac{1}{2}}[\chi(H), d\Gamma(|p_2|^{-\alpha})]\left(d\Gamma(|p_2|^{-\alpha})+1\right)^{-1}\right\|^2 \\
&+ \left.\left\|\chi(H)\left(d\Gamma(|p_2|^{-\alpha})+1\right)^{-\frac{1}{2}}\right\|^2\right)\left\|\left(d\Gamma(|p_2|^{-\alpha})+1\right)u\right\|^2.
\end{aligned}
$$

Therefore:

$$
\langle\chi(H)u, d\Gamma(|p_2|^{-\alpha})\chi(H)u\rangle \lesssim \left\|d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}u\right\|^2 + \|u\|^2
$$

and

$$
\begin{aligned}
\left\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_0(t^\nu|p_2|)\right)e^{-\mathrm{i}tH}\chi(H)u\right\rangle &\lesssim t^{1-(1+\mu-\alpha)\nu}\left(\|\mathrm{d}\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}u\|^2 + \|u\|^2\right) \\
&+ \left\|d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}u\right\|^2 + \|u\|^2. \qquad (3.36)
\end{aligned}
$$

Equations (3.33) and (3.36) yield

$$
\begin{aligned}
\left\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma\left(|p_2|^{-\alpha}\right)e^{-\mathrm{i}tH}\chi(H)u\right\rangle &\lesssim \max(t^{1-(1+\mu-\alpha)\nu}, t^{\alpha\nu})\left(\|\mathrm{d}\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}u\|^2 + \|N_{\mathrm{neut}}^{\frac{1}{2}}u\|^2 + \|u\|^2\right) \\
&+ \left\|d\Gamma(|p_2|^{-\alpha})^{\frac{1}{2}}u\right\|^2 + \|u\|^2.
\end{aligned}
$$

Choosing $\nu = (1+\mu)^{-1}$ concludes the proof of (3.30).

To establish (3.31), we modify the proof as follows. On the one hand, we have that

$$
\left\|\mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_\infty(t^\nu|p_2|)\right)e^{-\mathrm{i}tH}\chi(H)u\right\| \leqslant t^{-\nu\alpha}\left\|N_{\mathrm{neut}}e^{-\mathrm{i}tH}\chi(H)u\right\| \lesssim t^{-\nu\alpha}\left(\left\|N_{\mathrm{neut}}u\right\| + \|u\|\right), \quad (3.37)
$$

and on the other hand

$$
\begin{aligned}
&\left\|\mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_0(t^\nu|p_2|)\right)e^{-\mathrm{i}tH}\chi(H)u\right\| \\
&= \left\|e^{\mathrm{i}tH}\mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_0(t^\nu|p_2|)\right)e^{-\mathrm{i}tH}\chi(H)u\right\| \\
&\leqslant \left\|\mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_0(|p_2|)\right)e^{-\mathrm{i}H}\chi(H)u\right\| + \int_1^t \left\|e^{\mathrm{i}sH}\left[\mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_0(s^\nu|p_2|)\right), \mathrm{i}H\right]e^{-\mathrm{i}sH}\chi(H)u\right\|ds \\
&\lesssim \left\|\mathrm{d}\Gamma\left(|p_2|^{-\alpha}\right)u\right\| + \|[\mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_0(|p_2|)\right), e^{-\mathrm{i}H}\chi(H)]u\| + \int_1^t s^{-(1+\mu-\alpha)\nu}\left\|(|p_2|^{-(1+\mu)})G\right\|_2\|u\|ds, \\
&\lesssim \left\|\mathrm{d}\Gamma\left(|p_2|^{-\alpha}\right)u\right\| + \|[\mathrm{d}\Gamma\left(|p_2|^{-\alpha}f_0(|p_2|)\right), e^{-\mathrm{i}H}\chi(H)]u\| + \int_1^t s^{-(1+\mu-\alpha)\nu}\left\|(|p_2|^{-(1+\mu)})G\right\|_2\|u\|ds,
\end{aligned}
\qquad (3.38)
$$

where we used (3.35) and the fact that $N_{\mathrm{lept}} + N_W$ is relatively H-bounded in the last inequality. Combining (3.37) and (3.38), we deduce (3.31) similarly as above. □

Now we are ready to prove a first version of the propagation estimates in the massless case. The general strategy is the same as in Theorem 2.3.1, with some important technical modifications. In particular, we have to use Lemma 3.4.1, the particular form of the Mourre estimate stated in Theorem 3.3.4 and the fact that $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H.

Recall that the notation $\mathbb{1}_{[R,R']}(|x|)$ has been introduced in (2.68).

Theorem 3.4.2. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by (2.5). Assume that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad |p_3|^{-1}a_{(i),\cdot}G \in L^2, \quad i = 1,2,3,$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

(i) Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$, $R = (R_1, \dots, R_{14})$ and $R' = (R'_1, \dots, R'_{14})$ be such that $R'_i > R_i > 1$. There exists $\mathrm{C} > 0$ such that, for all $u \in \mathscr{D}(N_{\mathrm{neut}}^{\frac12}) \cap \mathscr{D}(\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12})$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\mathbb{1}_{[R,R']}\Big(\frac{|x|}{t}\Big)\Big)^{\frac12} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C}\big(\|N_{\mathrm{neut}}^{\frac12} u\|^2 + \|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12} u\|^2 + \|u\|^2\big).$$

(ii) Let $0 < \mathrm{v}_0 < \mathrm{v}_1$ and $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. There exists $\mathrm{C} > 0$ such that, for all $u \in \mathscr{D}(N_{\mathrm{neut}}^{\frac12}) \cap \mathscr{D}(\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12})$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\Big\langle \big(\frac{x}{t} - \nabla\omega\big), \mathbb{1}_{[\mathrm{v}_0,\mathrm{v}_1]}\big(\frac{x}{t}\big)\big(\frac{x}{t} - \nabla\omega\big)\Big\rangle\Big)^{\frac12} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t}$$
$$\leqslant \mathrm{C}\big(\|N_{\mathrm{neut}}^{\frac12} u\|^2 + \|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12} u\|^2 + \|u\|^2\big).$$

(iii) Let $0 < \mathrm{v}_0 < \mathrm{v}_1$, $J \in \mathrm{C}_0^\infty(\{x \in \mathbb{R}^3, \mathrm{v}_0 < |x| < \mathrm{v}_1\})$ and $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. There exists $\mathrm{C} > 0$ such that, for all $l \in \{e, \mu, \tau\}$ and $u \in \mathscr{D}(N_{\mathrm{neut}}^{\frac12}) \cap \mathscr{D}(\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12})$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\Big| J\Big(\frac{x}{t}\Big)\Big(\frac{x^{(\ell)}}{t} - \partial_{(\ell)}\omega\Big) + \mathrm{h.c}\Big|\Big)^{\frac12} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t}$$
$$\leqslant \mathrm{C}\big(\|N_{\mathrm{neut}}^{\frac12} u\|^2 + \|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12} u\|^2 + \|u\|^2\big).$$

(iv) There exists $g_0 > 0$ such that, for all $|g| \leqslant g_0$, the following holds: let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ be supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$. There exist $\delta > 0$ and $\mathrm{C} > 0$ such that

$$\int_1^\infty \Big\| \Gamma\Big(\mathbb{1}_{[0,\delta]}\Big(\frac{|x|}{t}\Big)\Big) \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C}\big\|(\mathrm{d}\Gamma(|p_2|^{-1}) + N_{\mathrm{neut}} + 1)^{\frac12} (N_{\mathrm{neut}} + 1)^{\frac32} u\big\|^2,$$

for all $u \in \mathscr{D}((\mathrm{d}\Gamma(|p_2|^{-1}) + N_{\mathrm{neut}} + 1)^{\frac12}(N_{\mathrm{neut}} + 1)^{\frac32})$.

Proof. We prove (i) and (iv), underlying the differences with [29]. The proofs of (ii) and (iii) can be deduced by adapting [29, Section 6], using furthermore arguments similar to those used to prove (i).

(i) Let $F_i \in \mathrm{C}^\infty(\mathbb{R};\mathbb{R})$ be a non-decreasing function such that $F_i = 0$ near 0, $F_i = \text{const}$ near $+\infty$ and $F'_i \geqslant \mathbb{1}_{[R_i,R'_i]}$. As in the proof of Theorem 2.3.1, (2.70) holds and therefore it suffices to prove that

$$\int_1^\infty \Big\| \mathrm{d}\Gamma_i\Big(\mathbb{1}_{[R_i,R'_i]}\Big(\frac{|x_i|}{t}\Big)\Big)^{\frac12} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \lesssim \big(\|N_{\mathrm{neut}}^{\frac12} u\|^2 + \|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12} u\|^2 + \|u\|^2\big)$$

for all $u \in \mathscr{D}(N_{\mathrm{neut}}^{\frac{1}{2}}) \cap \mathscr{D}(\mathrm{d}\Gamma(|p_2|^{-1})^{\frac{1}{2}})$ and $i \in \{1, \ldots, 14\}$, where $\mathrm{d}\Gamma_i$ is defined in (2.71). Let

$$f_i(t) := \Big\langle e^{-\mathrm{it}H}\chi(H)u, \mathrm{d}\Gamma_i\Big(F_i\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{it}H}\chi(H)u\Big\rangle. \tag{3.39}$$

Note that, since $\mathrm{d}\Gamma_i(F_i(\frac{|x_i|}{t})) \leqslant \mathrm{C}N$, for some positive constant C, since N_{lept} and N_W are relatively H-bounded and since $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, we can write

$$\begin{aligned}
f_i(t) &\leqslant \Big\langle e^{-\mathrm{it}H}\chi(H)u, \mathrm{C}Ne^{-\mathrm{it}H}\chi(H)u\Big\rangle \\
&= \mathrm{C}\Big\langle e^{-\mathrm{it}H}\chi(H)u, \big(2N_{\mathrm{lept}} + N_W + N_{\mathrm{neut}} - N_{\mathrm{lept}}\big)e^{-\mathrm{it}H}\chi(H)u\Big\rangle \\
&\lesssim \|u\|^2 + \Big\langle \big(N_{\mathrm{neut}} - N_{\mathrm{lept}}\big)^{\frac{1}{2}}u, \chi(H)^2\big(N_{\mathrm{neut}} - N_{\mathrm{lept}}\big)^{\frac{1}{2}}u\Big\rangle \\
&\lesssim \|N_{\mathrm{neut}}^{\frac{1}{2}}u\|^2 + \|u\|^2.
\end{aligned}$$

Hence $f_i(t)$ is uniformly bounded.

Differentiating f_i gives

$$\begin{aligned}
\partial_t f_i(t) &= \Big\langle e^{-\mathrm{it}H}\chi(H)u, \mathbf{D}\mathrm{d}\Gamma_i\Big(F_i\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{it}H}\chi(H)u\Big\rangle \\
&= \Big\langle e^{-\mathrm{it}H}\chi(H)u, \mathrm{d}\Gamma_i\Big(\mathbf{d}_0F_i\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{it}H}\chi(H)u\Big\rangle \\
&\quad + \Big\langle e^{-\mathrm{it}H}\chi(H)u, \Big[H_I^{(1)}(G) + H_I^{(2)}(G), \mathrm{id}\Gamma_i\Big(F_i\Big(\frac{|x_i|}{t}\Big)\Big)\Big]e^{-\mathrm{it}H}\chi(H)u\Big\rangle,
\end{aligned} \tag{3.40}$$

where $\mathbf{D}$ and $\mathbf{d}_0$ denotes the Heisenberg derivatives defined at the beginning of Section 2.3.

Using that $F_i' \geqslant \mathbb{1}_{[R_i,R_i']}$ and $R_i > 1$, a commutator expansion at second order (see, e.g., [28], [21, Lemma 5.2] and [43]) gives

$$\mathbf{d}_0F_i\Big(\frac{|x_i|}{t}\Big) \leqslant \frac{-\mathrm{C}_0}{t}\mathbb{1}_{[R_i,R_i']}\Big(\frac{|x_i|}{t}\Big) + \mathcal{O}(t^{-2}),$$

if $i \in \{1, 2, 3, 4, 7, 8, 11, 12\}$ (corresponding to the label of a massive particle), and

$$\mathbf{d}_0F_i\Big(\frac{|x_i|}{t}\Big) \leqslant \frac{-\mathrm{C}_0}{t}\mathbb{1}_{[R_i,R_i']}\Big(\frac{|x_i|}{t}\Big) + \mathcal{O}(t^{-2})|p_2|^{-1},$$

if $i \in \{5, 6, 9, 10, 13, 14\}$ (corresponding to the label of a neutrino). To prove this last point, the critical part is to evaluate the commutator $[F(\frac{|x|}{t}), |p_2|]$. We note $L(x) = F(\sqrt{x})$ and using Helffer-Sjöstrand formula we have:

$$\begin{aligned}
\left[F\left(\frac{|x|}{t}\right), |p_2|\right] &= \frac{i}{2\pi}\int \partial_{\bar z}\tilde{L}(z)\left(z - \frac{x^2}{t^2}\right)^{-1}\left[\frac{x^2}{t^2}, |p_2|\right]\left(z - \frac{x^2}{t^2}\right)^{-1} dz \wedge d\bar z, \\
&= \frac{i}{2\pi t^2}\int \partial_{\bar z}\tilde{L}(z)\left(z - \frac{x^2}{t^2}\right)^{-1}\left(x\frac{p_2}{|p_2|} + \frac{p_2}{|p_2|}x\right)\left(z - \frac{x^2}{t^2}\right)^{-1} dz \wedge d\bar z, \\
&= \frac{i}{2\pi t^2}\int \partial_{\bar z}\tilde{L}(z)\left(z - \frac{x^2}{t^2}\right)^{-2}\left(x\frac{p_2}{|p_2|} + \frac{p_2}{|p_2|}x\right) dz \wedge d\bar z \\
&+ \frac{i}{2\pi t^2}\int \partial_{\bar z}\tilde{L}(z)\left(z - \frac{x^2}{t^2}\right)^{-2}\left[\left(x\frac{p_2}{|p_2|} + \frac{p_2}{|p_2|}x\right), \frac{x^2}{t^2}\right]\left(z - \frac{x^2}{t^2}\right)^{-1} dz \wedge d\bar z,
\end{aligned} \tag{3.41}$$

where $\tilde{L}$ is an almost analytic extension of L such that:

$$\begin{aligned}\tilde{L}|_{\mathbb{R}} &= L\\ |\partial_{\bar{z}}\tilde{L}| &\leqslant C_n|\mathrm{Im}(z)|^n,\ n\in\mathbb{N}.\end{aligned}$$

We note that:

$$\begin{aligned}\frac{1}{t^2}\int\partial_{\bar{z}}\tilde{L}(z)\left(z-\frac{x^2}{t^2}\right)^{-2}\left(x\frac{p_2}{|p_2|}+\frac{p_2}{|p_2|}x\right)dz\wedge d\bar{z} &= \frac{1}{t^2}\nabla L\left(\frac{x^2}{t^2}\right)\left(x\frac{p_2}{|p_2|}+\frac{p_2}{|p_2|}x\right),\\ &= \frac{1}{t}\nabla F\left(\frac{x}{t}\right)\frac{p_2}{|p_2|}.\end{aligned}$$

and

$$\begin{aligned}\left[\left(x\frac{p_2}{|p_2|}+\frac{p_2}{|p_2|}x\right),\frac{x^2}{t^2}\right] &= \frac{x^2}{t^2}\left(|p_2|^{-1}-p_2^2|p_2|^{-3}\right)\\ &+ 2\frac{x}{t}\left(|p_2|^{-1}-p_2^2|p_2|^{-3}\right)\frac{x}{t}\\ &+ \left(|p_2|^{-1}-p_2^2|p_2|^{-3}\right)\frac{x^2}{t^2}\end{aligned}\tag{3.42}$$

so

$$\left[F\left(\frac{|x|}{t}\right),|p_2|\right] = \frac{i}{2\pi t}\nabla F\left(\frac{x}{t}\right)\frac{p_2}{|p_2|}+O(t^{-2})|p_2|^{-1}.$$

In the first case, using as above that $N_{\mathrm{lept}}-N_{\mathrm{neut}}$ commutes with H, this gives

$$\begin{aligned}&\Big\langle e^{-\mathrm{i}tH}\chi(H)u,\mathrm{d}\Gamma_i\Big(\mathbf{d}_0F_i\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\leqslant-\frac{\mathrm{C}_0}{t}\Big\langle e^{-\mathrm{i}tH}\chi(H)u,\mathrm{d}\Gamma_i\Big(\mathbb{1}_{[R_i,R_i']}\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle+\mathcal{O}(t^{-2})\Big\langle e^{-\mathrm{i}tH}\chi(H)u,Ne^{-\mathrm{i}tH}\chi(H)u\Big\rangle,\\ &\leqslant-\frac{\mathrm{C}_0}{t}\Big\langle e^{-\mathrm{i}tH}\chi(H)u,\mathrm{d}\Gamma_i\Big(\mathbb{1}_{[R_i,R_i']}\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle+\mathcal{O}(t^{-2})\big(\|N_{\mathrm{neut}}^{\frac12}u\|^2+\|u\|^2\big),\end{aligned}\tag{3.43}$$

if $i\in\{1,2,3,4,7,8,11,12\}$. In the second case (if i corresponds to the label of a neutrino), we use in addition Lemma 3.4.1, yielding

$$\begin{aligned}&\Big\langle e^{-\mathrm{i}tH}\chi(H)u,\mathrm{d}\Gamma_i\Big(\mathbf{d}_0F_i\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\leqslant-\frac{\mathrm{C}_0}{t}\Big\langle e^{-\mathrm{i}tH}\chi(H)u,\mathrm{d}\Gamma_i\Big(\mathbb{1}_{[R_i,R_i']}\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle+\mathcal{O}(t^{-2})\Big\langle e^{-\mathrm{i}tH}\chi(H)u,\mathrm{d}\Gamma(|p_2|^{-1})e^{-\mathrm{i}tH}\chi(H)u\Big\rangle,\\ &\leqslant-\frac{\mathrm{C}_0}{t}\Big\langle e^{-\mathrm{i}tH}\chi(H)u,\mathrm{d}\Gamma_i\Big(\mathbb{1}_{[R_i,R_i']}\Big(\frac{|x_i|}{t}\Big)\Big)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle+\mathcal{O}(t^{-2+\frac{2}{2+\mu}})\big(\|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12}u\|^2+\|u\|^2\big)\\ &+\mathcal{O}(t^{-2})\big(\|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac12}u\|^2+\|u\|^2\big).\end{aligned}\tag{3.44}$$

Note that Hardy's inequality implies that, if $G\in\mathbb{H}^{1+\mu}$ for some $\mu>0$, then $|p_2|^{-1-\mu}G\in L^2$ (if $\mu<1/2$), and hence Lemma 3.4.1 can indeed be applied.

The commutators $[H_I^{(j)}(G),\mathrm{d}\Gamma_i(F_i(\frac{|x_i|}{t}))]$, $j=1,2$, are given by expressions similar to (2.41)–(2.42), with the operator $F_i(\frac{|x_i|}{t})$ instead of a_i. Since F_i vanishes near 0 and $G\in\mathbb{H}^{1+\mu}$, we have that

$$\Big\|F_i\Big(\frac{|x_i|}{t}\Big)G\Big\|_2=\mathcal{O}(t^{-1-\mu}).$$

Therefore we deduce from Lemma 2.1.3 that

$$\begin{aligned}&\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Big[H_I^{(1)}(G)+H_I^{(2)}(G), \mathrm{id}\Gamma_i\Big(F_i\Big(\frac{|x_i|}{t}\Big)\Big)\Big]e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\\ &\leqslant \mathrm{C}t^{-1-\mu}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, (N_W+N_{\mathrm{lept}})e^{-\mathrm{i}tH}\chi(H)u\Big\rangle = \mathcal{O}(t^{-1-\mu})\|u\|^2. \end{aligned}\tag{3.45}$$

Integrating (3.40) over $[1,\infty)$, using that $t^{-2+\frac{2}{2+\mu}}$ and $t^{-1-\mu}$ are integrable, we obtain the statement of (i) by combining (3.43)–(3.44) and (3.45).

(iv) Let $\lambda \in \mathbb{R}\backslash\{\tau\cup\sigma_{\mathrm{pp}}(H)\}$. Clearly, since $N_{\mathrm{lept}}-N_{\mathrm{neut}}$ commutes with H, it suffices to prove (iv) for all $u\in \mathrm{Ran}(\mathbb{1}_{\{n\}}(N_{\mathrm{lept}}-N_{\mathrm{neut}}))$, $n\in\mathbb{Z}$. Hence we fix $n\in\mathbb{Z}$ and $u\in \mathrm{Ran}(\mathbb{1}_{\{n\}}(N_{\mathrm{lept}}-N_{\mathrm{neut}}))$. Recall that, by Theorem 3.3.4, there exist $\varepsilon>0$, $\mathrm{c}_0>0$ and $\mathrm{C}_\lambda>0$ such that

$$[H,\mathrm{i}A]\geqslant \mathrm{c}_0(N_{\mathrm{neut}}+\mathbb{1})-\mathrm{C}_\lambda(\mathbb{1}-\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H))(1+H^2)^{\frac{1}{2}}.$$

For $\tilde{\chi}\in \mathrm{C}_0^\infty((\lambda-\varepsilon,\lambda+\varepsilon))$, this implies that

$$[H,\mathrm{i}A]\geqslant \mathrm{c}_0(N_{\mathrm{neut}}+\mathbb{1})-\mathrm{C}_\lambda(\mathbb{1}-\tilde{\chi}^2(H))(1+H^2)^{\frac{1}{2}}. \tag{3.46}$$

Let $\chi\in \mathrm{C}_0^\infty((\lambda-\varepsilon,\lambda+\varepsilon))$ be such that $\chi\tilde{\chi}=\chi$. Let $\delta>0$ and q_i, $i\in\{1,\dots,14\}$ be functions in $\mathrm{C}_0^\infty(\{x\in\mathbb{R}^3, |x|\leqslant 2\delta\})$ such that $0\leqslant q_i\leqslant 1$, $q_i=1$ on $\{x\in\mathbb{R}^3,|x|\leqslant\delta\}$ and let $q^t=(q_1(\frac{\mathrm{i}\nabla}{t}),\dots,q_{14}(\frac{\mathrm{i}\nabla}{t}))$. Let

$$h(t):=\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(q^t)\frac{A}{t}\Gamma(q^t)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle.$$

Let us first recall that

$$\begin{aligned} A &= d\Gamma(a)\\ a_k &= \frac{i}{2}\left(\nabla_k\cdot\nabla\omega_k(p)+h.c\right).\end{aligned}$$

Since $\nabla\omega_l^{(i)}$, $i=1,2$, $l\in\{l,\nu,\tau\}$ and $\nabla\omega^{(3)}$ are bounded, it is not difficult to observe, considering the support of q, that

$$\pm\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\leqslant \mathrm{C}\delta(N+1) \quad\text{ and }\quad \Big\|\frac{A}{t}\Gamma(q^t)(N+1)^{-1}\Big\|\leqslant \mathrm{C}\delta. \tag{3.47}$$

Here it should be noticed that $\Gamma(q^t)$ maps $\mathscr{H}$ to $\mathscr{D}(A)$. Indeed a vector in $\mathrm{Ran}(\Gamma(q^t))$ belongs to the Sobolev space $\mathbb{H}^s(\mathbb{R}^3)$, for any $s>0$, as a function of any of the momentum variable p_i. It is then regular in a neighborhood of zero and using, in particular, (3.10), it is not difficult to verify that $\mathrm{Ran}(\Gamma(q^t))\subset\mathscr{D}(A)$. In the second estimate of (3.47), we refer to the proof of (2.76). Note that Hardy's inequality in $\mathbb{R}^3$, $\||p_2|^{-1}u\|_2\lesssim\|\nabla_{p_2}u\|_2$, has to be used.

From (3.47) and using as before that $N_{\mathrm{lept}}-N_{\mathrm{neut}}$ commutes with H, we obtain that

$$|h(t)|\leqslant\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(q^t)\frac{A}{t}\Gamma(q^t)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\lesssim\|N_{\mathrm{neut}}^{\frac{1}{2}}u\|^2+\|u\|^2. \tag{3.48}$$

In particular, $h(t)$ is uniformly bounded. Furthermore, one can compute

$$\begin{aligned}
\partial_t h(t) &= \Big\langle e^{-\mathrm{i}tH}u, \Big(\chi(H)\mathrm{d}\Gamma(q^t, \mathbf{d}_0 q^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{h.c.}\Big)e^{-\mathrm{i}tH}u\Big\rangle \\
&\quad + \Big\langle e^{-\mathrm{i}tH}u, \Big(\chi(H)[H_I^{(1)}(G) + H_I^{(2)}(G), \mathrm{i}\Gamma(q^t)]\frac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{h.c.}\Big)e^{-\mathrm{i}tH}u\Big\rangle \\
&\quad + t^{-1}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(q^t)[H, \mathrm{i}A]\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle \\
&\quad - t^{-1}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle \\
&=: R_1(t) + R_2(t) + R_3(t) + R_4(t). \qquad (3.49)
\end{aligned}$$

Observe that A being symmetric, A^* is an extension of A so that the second terms in $R_1(t)$ and $R_2(t)$ are indeed the hermitian conjugates of the first ones. In what follows we consider each term $R_\sharp(t)$ separately.

We begin with $R_2(t)$. From the assumption that $G \in \mathbb{H}^{1+\mu}$, the commutation relations of Section 2.1.2 and Lemma 2.1.3, one can show that

$$\big\|[H_I^{(1)}(G) + H_I^{(2)}(G), \Gamma(q^t)](N_{\mathrm{lept}} + N_W)^{-1}\big\| = \mathcal{O}(t^{-1-\mu}).$$

Together with (3.47) this implies that

$$|R_2(t)| \lesssim \mathcal{O}(t^{-1-\mu})\big\|(N+1)\chi(H)e^{-\mathrm{i}tH}u\big\|^2,$$

since $N_{\mathrm{lept}} + N_W$ is relatively H-bounded. Since $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, we deduce as before that

$$|R_2(t)| \lesssim \mathcal{O}(t^{-1-\mu})\|(N_{\mathrm{neut}} + 1)u\|^2. \qquad (3.50)$$

To evaluate $R_1(t)$, we compute, by means of a commutator expansion (see [28] and [21, Section 5]),

$$\mathbf{d}_0 q_i^t(x_i) = -\frac{1}{2t}\Big\langle \frac{x_i}{t} - \nabla\omega_i, \nabla q_i\big(\frac{x_i}{t}\big)\Big\rangle + \mathrm{h.c.} + \mathcal{O}(t^{-2}), \qquad (3.51)$$

if $i \in \{1,2,3,4,7,8,11,12\}$ (corresponding to the label of a massive particle), and

$$\mathbf{d}_0 q_i^t(x_i) = -\frac{1}{2t}\Big\langle \frac{x_i}{t} - \nabla\omega_i, \nabla q_i\big(\frac{x_i}{t}\big)\Big\rangle + \mathrm{h.c.} + \mathcal{O}(t^{-\frac{3}{2}})|p_2|^{-\frac{1}{2}}, \qquad (3.52)$$

if $i \in \{5,6,9,10,13,14\}$ (corresponding to the label of a neutrino). Here it should be noted that that the remainder term in (3.52) can be computed to be of order $|p_2|^{-1+\gamma}\mathcal{O}(t^{-2+\gamma})$ for any $0 \leqslant \gamma \leqslant 1$. The same calculation as (3.41) may be done and Hardy's inequality may be used in (3.42). Which leads to:

$$(z - \frac{x^2}{t^2})^{-1}\left[\left(x\frac{p_2}{|p_2|} + \frac{p_2}{|p_2|}x\right), \frac{x^2}{t^2}\right](z - \frac{x^2}{t^2})^{-1} = O(t^{-1}). \qquad (3.53)$$

An interpolation argument, using Hadamard's three lines lemma for example, would lead to

$$(z - \frac{x^2}{t^2})^{-1}\left[\left(x\frac{p_2}{|p_2|} + \frac{p_2}{|p_2|}x\right), \frac{x^2}{t^2}\right](z - \frac{x^2}{t^2})^{-1} = O(t^{-2+\gamma})|p_2|^{-1+\gamma}. \qquad (3.54)$$

For more details, see [21, Lemma 5.2]. Here we choose $\gamma = \frac{1}{2}$ for convenience, see below.

Let $g_i^t := -\frac{1}{2}\langle \frac{x_i}{t} - \nabla\omega_i, \nabla q_i\big(\frac{x_i}{t}\big)\rangle + \text{h.c.}$ and let r_i^t be the remainder in (3.51) or (3.52). For $i \in \{1,2,3,4,7,8,11,12\}$, by (3.47), we deduce that

$$\begin{aligned}\Big|\Big\langle e^{-\mathrm{it}H}u, \Big(\chi(H)\mathrm{d}\Gamma(q_i^t, r_i^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\rangle\Big| &\lesssim \|\mathrm{d}\Gamma(q_i^t, r_i^t)\chi(H)e^{-\mathrm{it}H}u\| \\ &\quad \Big\|\Big(\frac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\| \\ &\lesssim \mathcal{O}(t^{-2})\big\|N\chi(H)e^{-\mathrm{it}H}u\big\|^2 \\ &\lesssim \mathcal{O}(t^{-2})\|(N_{\text{neut}}+1)u\|^2, \qquad (3.55)\end{aligned}$$

where we used again that $N_{\text{lept}} - N_{\text{neut}}$ commutes with H in the second inequality. The case $i \in \{5,6,9,10,13,14\}$ corresponding to massless particles is more difficult. Using (3.52) and Lemma 2.1.6, we write

$$\begin{aligned}&\Big|\Big\langle e^{-\mathrm{it}H}u, \Big(\chi(H)\mathrm{d}\Gamma(q_i^t, r_i^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\rangle\Big| \\ &\Big|\Big\langle e^{-\mathrm{it}H}u, \Big(\chi(H)\mathrm{d}\Gamma(q_i^t, r_i^t)(N+1)^{-\frac{1}{2}}(N+1)^{\frac{1}{2}}\frac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\rangle\Big| \\ &\lesssim \mathcal{O}(t^{-\frac{3}{2}})\Big\|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac{1}{2}}\chi(H)e^{-\mathrm{it}H}u\Big\|\big\|(N+1)^{\frac{3}{2}}\chi(H)e^{-\mathrm{it}H}u\big\|.\end{aligned}$$

Applying Lemma 3.4.1, and using that $N_{\text{lept}} - N_{\text{neut}}$ commutes with H, it thus follows that

$$\begin{aligned}&\Big|\Big\langle e^{-\mathrm{it}H}u, \Big(\chi(H)\mathrm{d}\Gamma(q_i^t, r_i^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\rangle\Big| \\ &\lesssim \mathcal{O}(t^{-\frac{3}{2}+\frac{1}{2+\mu}})\big(\|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac{1}{2}}u\| + \|N_{\text{neut}}^{\frac{1}{2}}u\| + \|u\|\big)\big(\big\|(N+1)^{\frac{3}{2}}u\big\| + \|u\|\big). \qquad (3.56)\end{aligned}$$

Summing over i, we obtain from (3.55) and (3.56) that

$$\begin{aligned}&\Big|\Big\langle e^{-\mathrm{it}H}u, \Big(\chi(H)\mathrm{d}\Gamma(q^t, r^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\rangle\Big| \\ &\lesssim \mathcal{O}(t^{-2})\|(N_{\text{neut}}+1)u\|^2 + \mathcal{O}(t^{-\frac{3}{2}+\frac{1}{2+\mu}})\big(\|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac{1}{2}}u\| + \|u\|\big)\big(\big\|(N+1)^{\frac{3}{2}}u\big\| + \|u\|\big). \quad (3.57)\end{aligned}$$

To estimate the term corresponding to $\frac{1}{t}\mathrm{d}\Gamma(q^t, g^t)$, we proceed similarly but use (ii) instead of Lemma 3.4.1. We introduce $\tilde{q}^t$, defined as q^t, such that $\tilde{q}^t q^t = q^t$ and hence $\Gamma(q^t) = \Gamma(\tilde{q}^t)\Gamma(q^t)$. This yields

$$\begin{aligned}&\frac{1}{t}\Big|\Big\langle e^{-\mathrm{it}H}u, \Big(\chi(H)\mathrm{d}\Gamma(q_i^t, g_i^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\rangle\Big| \\ &= \frac{1}{t}\Big|\Big\langle e^{-\mathrm{it}H}u, \Big(\chi(H)\mathrm{d}\Gamma(q_i^t, g_i^t)(N+1)^{-\frac{1}{2}}(N+1)^{\frac{1}{2}}\frac{A}{t}\Gamma(\tilde{q}^t)\Gamma(q^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{it}H}u\Big\rangle\Big| \\ &\lesssim \frac{1}{t}\Big\|\mathrm{d}\Gamma((g_i^t)^*g_i^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{it}H}u\Big\|\big\|(N+1)^{\frac{3}{2}}\Gamma(q^t)\chi(H)e^{-\mathrm{it}H}u\big\|.\end{aligned}$$

Let us recall that the goal is to use the framework presented in the introduction of Section 2.3. We then have to isolate the term we want to study, $\Gamma(q^t)\chi(H)e^{-\mathrm{it}H}$:

$$\lesssim \frac{\alpha(|n|+1)^3}{t}\Big\|\mathrm{d}\Gamma((g_i^t)^*g_i^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{it}H}u\Big\|^2 + \frac{1}{\alpha(|n|+1)^3 t}\big\|(N+1)^{\frac{3}{2}}\Gamma(q^t)\chi(H)e^{-\mathrm{it}H}u\big\|^2,$$

where we recall that $n \in \mathbb{Z}$ has been fixed such that $u \in \mathrm{Ran}(\mathbb{1}_{\{n\}}(N_{\text{lept}} - N_{\text{neut}}))$. This parameter has been introduced in order to control the $(N+1)^{\frac{3}{2}}$ which appears together with $\Gamma(q^t)\chi(H)e^{-\mathrm{it}H}$.

The parameter $\alpha > 0$ will be determined later. Summing over i, since $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, this implies that

$$\begin{aligned}
&\frac{1}{t}\Big|\Big\langle e^{-\mathrm{i}tH}u, \Big(\chi(H)\mathrm{d}\Gamma(q^t, g^t)\frac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{h.c.}\Big)e^{-\mathrm{i}tH}u\Big\rangle\Big| \\
&\lesssim \frac{\alpha}{t}\Big\|\mathrm{d}\Gamma((g^t)^* g^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}(|N_{\mathrm{neut}} - N_{\mathrm{lept}}| + 1)^{\frac{3}{2}}u\Big\|^2 + \frac{1}{\alpha t}\big\|\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \\
&\quad + \frac{1}{\alpha t}\big\|(N_{\mathrm{lept}} + N_W + 1)^{\frac{3}{2}}\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2. \qquad (3.58)
\end{aligned}$$

Remembering that $\tilde{\chi}\chi = \chi$, the last term of (3.58) can be decomposed into

$$\begin{aligned}
&\frac{1}{\alpha t}\big\|(N_{\mathrm{lept}} + N_W + 1)^{\frac{3}{2}}\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \\
&= \frac{1}{\alpha t}\big\|(N_{\mathrm{lept}} + N_W + 1)^{\frac{3}{2}}\big[\Gamma(q^t), \tilde{\chi}(H)\big]\chi(H)e^{-\mathrm{i}tH}u\big\|^2 + \frac{1}{\alpha t}\big\|(N_{\mathrm{lept}} + N_W + 1)^{\frac{3}{2}}\tilde{\chi}(H)\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2.
\end{aligned}$$

Using the Helffer-Sjöstrand functional calculus and similar arguments as before, one verifies that

$$\big\|\big[\Gamma(q^t), \chi(H)\big](N_{\mathrm{neut}} + 1)^{-1}\big\| = \mathcal{O}(t^{-1}). \qquad (3.59)$$

By (ii) in Lemma 3.2.1, it is not difficult to deduce from (3.59) and the previous equality that

$$\frac{1}{\alpha t}\big\|(N_{\mathrm{lept}} + N_W + 1)^{\frac{3}{2}}\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \lesssim \frac{1}{\alpha t^2}\big\|(N_{\mathrm{neut}}^2 + 1)u\big\|^2 + \frac{1}{\alpha t}\big\|\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2. \qquad (3.60)$$

Combining (3.56), (3.58) and (3.60) gives

$$\begin{aligned}
|R_1(t)| &\lesssim \frac{\alpha}{t}\Big\|\mathrm{d}\Gamma((g^t)^* g^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}(|N_{\mathrm{neut}} - N_{\mathrm{lept}}| + 1)^{\frac{3}{2}}u\Big\|^2 + \frac{1}{\alpha t}\big\|\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \\
&\quad + \mathcal{O}(t^{-2})\|(N_{\mathrm{neut}}^2 + N_{\mathrm{neut}} + 1)u\|^2 \\
&\quad + \mathcal{O}(t^{-\frac{3}{2}+\frac{1}{2+\mu}})\big(\|\mathrm{d}\Gamma(|p_2|^{-1})^{\frac{1}{2}}u\| + \|N_{\mathrm{neut}}^{\frac{1}{2}}u\| + \|u\|\big)\big(\|(N_{\mathrm{neut}} + 1)^{\frac{3}{2}}u\| + \|u\|\big). \qquad (3.61)
\end{aligned}$$

Remember that the first term of this expression will be controlled using (ii). We move on to the study of $R_3(t)$. It follows from (3.46) that

$$R_3(t) \geqslant \frac{1}{t}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(q^t)\big(\mathrm{c}_0(N_{\mathrm{neut}} + \mathbb{1}) - \mathrm{C}_\lambda(\mathbb{1} - \tilde{\chi}^2(H))\big)\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle.$$

Using (3.59) together with the previous equation, the facts that $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, that N_W and N_{lept} are relatively H-bounded and that $\chi\tilde{\chi} = \chi$, this gives

$$R_3(t) \geqslant \frac{\mathrm{c}_0}{t}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(q^t)(N_{\mathrm{neut}} + \mathbb{1})\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle + \mathcal{O}(t^{-2})\|(N_{\mathrm{neut}} + 1)u\|^2. \qquad (3.62)$$

To estimate $R_4(t)$, it suffices to apply (2.76) together with the fact that N_W and N_{lept} are relatively H-bounded. This yields

$$|R_4(t)| \lesssim \frac{\delta}{t}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(q^t)(N_{\mathrm{neut}} + 1)\Gamma(q^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle. \qquad (3.63)$$

Putting together (3.50), (3.61), (3.62) and (3.63), we finally arrive at

$$\begin{aligned}\partial_t h(t) \geqslant & \Big(\frac{c_0}{t} - \frac{1}{\alpha t} - \frac{\delta}{t}\Big)\Big\langle e^{-itH}u, \chi(H)\Gamma(q^t)(N_{\rm neut}+\mathbb{1})\Gamma(q^t)\chi(H)e^{-itH}u\Big\rangle \\ & + \frac{\alpha}{t}\Big\| d\Gamma(g_t^* g_t)^{\frac{1}{2}}\chi(H)e^{-itH}(|N_{\rm neut}-N_{\rm lept}|+1)^{\frac{3}{2}}u\Big\|^2 + \mathcal{O}(t^{-\min(2,1+\mu)})\|(N_{\rm neut}^2+N_{\rm neut}+1)u\|^2 \\ & + \mathcal{O}(t^{-\frac{3}{2}+\frac{1}{2+\mu}})\big(\|d\Gamma(|p_2|^{-1})^{\frac{1}{2}}u\|+\|u\|\big)\big(\|(N_{\rm neut}+1)^{\frac{3}{2}}u\|+\|u\|\big).\end{aligned}$$

Fixing α large enough and δ small enough and integrating over t from 1 to ∞, we obtain from (3.48) and (ii) that

$$\begin{aligned}&\int_1^\infty \Big\langle e^{-itH}u, \chi(H)\Gamma(q^t)(N_{\rm neut}+1)\Gamma(q^t)\chi(H)e^{-itH}u\Big\rangle\frac{dt}{t} \\ &\lesssim \big\|(d\Gamma(|p_2|^{-1})+N_{\rm neut}+1)^{\frac{1}{2}}(N_{\rm neut}+1)^{\frac{3}{2}}u\big\|^2.\end{aligned}$$

This proves (iv) for any $\chi \in C_0^\infty((\lambda-\varepsilon,\lambda+\varepsilon))$. The extension of the result to any $\chi \in C_0^\infty(\mathbb{R}\backslash(\sigma_{\rm pp}(H)\cup\tau))$ follows from standard arguments (see, e.g., [28, Proposition 4.4.7]). □

These estimates are of interest as they describe the propagation of particles in the massless case. Unfortunately they are too singular to be used in the proof of asymptotic completeness. We sum up here the critical part of the proof. As in the massive case, inverse operators will be built considering the asymptotic behaviour of some operator $W(t)$. The following strategy will be used:

$$\|(W(t_1)-W(t_2))u\| = \left\|\int_t^{t'}\partial_s W(s)\right\| = \sup_v \int_t^{t'}\langle v\,|\partial_s W(s)u\rangle\ .$$

The estimation of $\langle v\,|\partial_s W(s)u\rangle$ can lead to the existence of the inverse wave operators. The third propagation estimates is then required and the term $d\Gamma(|p|^{-\alpha})$ would constraint v on a dense domain whereas bounds has to be established for any v. We then have to find another formulation of the propagation estimates which absorb this singularity. The idea is to introduce a time-dependent modified position operator which behaves asymptotically as the usual position operator. In accordance with the notations previously introduced in this section, we set

$$x_{t,\rho} := \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}} x \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}},$$

meaning that $x_{t,\rho,i} = x_i = i\nabla$ if $i \in \{1,2,3,4,7,8,11,12\}$ (corresponding to the label of a massive particle), and

$$x_{t,\rho,i} = \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}} x_i \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}} = \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}}(i\nabla_{p_2})\Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}},$$

if $i \in \{5,6,9,10,13,14\}$ (corresponding to the label of a neutrino). Likewise,

$$\omega_{t,\rho} := \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}} \omega \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}},$$

where the notation ω has been introduced in (2.69). Similarly as above, for $R = (R_1,\dots,R_{14})$ and $R' = (R_1',\dots,R_{14}')$, we have that

$$\mathbb{1}_{[R,R']}(|x_{t,\rho}|) = \big(\mathbb{1}_{[R_1,R_1']}(|x_{t,\rho,1}|),\dots,\mathbb{1}_{[R_{14},R_{14}']}(|x_{t,\rho,14}|)\big).$$

We now turn to the proof of the following theorem.

Theorem 3.4.3. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by* (2.5). *Assume that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad |p_3|^{-1}a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3,$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

(i) Let $\rho > 0$ be such that $(1+\mu)^{-1} < \rho < 1$ and let $c > \rho^{-1}$. Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$, $R = (R_1, \dots, R_{14})$ and $R' = (R'_1, \dots, R'_{14})$ be such that $R'_i > R_i > c$. There exists $\mathrm{C} > 0$ such that, for all $u \in \mathscr{D}(N_{\mathrm{neut}}^{\frac{1}{2}})$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\mathbb{1}_{[R,R']}\Big(\frac{|x_{t,\rho}|}{t}\Big)\Big)^{\frac{1}{2}} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C}\big(\|N_{\mathrm{neut}}^{\frac{1}{2}} u\|^2 + \|u\|^2\big).$$

(ii) Let $\rho > 0$ be such that $(1+\mu)^{-1} < \rho < 1$. Let $0 < \mathrm{v}_0 < \mathrm{v}_1$ and $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. There exists $\mathrm{C} > 0$ such that, for all $u \in \mathscr{D}(N_{\mathrm{neut}}^{\frac{1}{2}})$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\Big\langle \big(\frac{x_{t,\rho}}{t} - \nabla\omega_{t,\rho}\big), \mathbb{1}_{[\mathrm{v}_0,\mathrm{v}_1]}\big(\frac{x_{t,\rho}}{t}\big)\big(\frac{x_{t,\rho}}{t} - \nabla\omega_{t,\rho}\big)\Big\rangle\Big)^{\frac{1}{2}} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t}$$
$$\leqslant \mathrm{C}\big(\|N_{\mathrm{neut}}^{\frac{1}{2}} u\|^2 + \|u\|^2\big).$$

(iii) Let $\rho > 0$ be such that $(1+\mu)^{-1} < \rho < 1$. Let $0 < \mathrm{v}_0 < \mathrm{v}_1$, $J \in \mathrm{C}_0^\infty(\{x \in \mathbb{R}^3, \mathrm{v}_0 < |x| < \mathrm{v}_1\})$ and $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. There exists $\mathrm{C} > 0$ such that, for all $l \in \{l, \nu, \tau\}$ and $u \in \mathscr{D}(N_{\mathrm{neut}}^{\frac{1}{2}})$,

$$\int_1^\infty \Big\| \mathrm{d}\Gamma\Big(\Big| J\Big(\frac{x_{t,\rho}}{t}\Big)\Big(\frac{x_{t,\rho}^{(\ell)}}{t} - \partial_{(\ell)}\omega_{t,\rho}\Big) + \mathrm{h.c}\Big|\Big)^{\frac{1}{2}} \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C}\big(\|N_{\mathrm{neut}}^{\frac{1}{2}} u\|^2 + \|u\|^2\big).$$

(iv) There exists $g_0 > 0$ such that, for all $|g| \leqslant g_0$, the following holds: let $\rho > 0$ be such that $(1+\mu)^{-1} < \rho < 1$. Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ be supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$. There exist $\delta > 0$ and $\mathrm{C} > 0$ such that, for all $u \in \mathscr{D}((\mathrm{d}\Gamma(|p_2|^{-1}) + N_{\mathrm{neut}} + 1)(N_{\mathrm{neut}} + 1)^{\frac{3}{2}})$,

$$\int_1^\infty \Big\| \Gamma\Big(\mathbb{1}_{[0,\delta]}\Big(\frac{|x_{t,\rho}|}{t}\Big)\Big) \chi(H) e^{-\mathrm{i}tH} u \Big\|^2 \frac{dt}{t} \leqslant \mathrm{C}\big\| (\mathrm{d}\Gamma(|p_2|^{-1}) + N_{\mathrm{neut}} + 1)(N_{\mathrm{neut}} + 1)^{\frac{3}{2}} u \big\|^2.$$

Proof. (i) The proof is similar to that of Theorem 3.4.2 (i), with the following differences. Let $F_i : \mathbb{R} \to \mathbb{R}$ be a bounded non-decreasing function supported in $(1, \infty)$. Instead of the function f_i in (3.39), we consider

$$\tilde{f}_i(t) := \Big\langle e^{-\mathrm{i}tH}\chi(H)u, \mathrm{d}\Gamma_i\Big(F_i\Big(\frac{|x_{t,\rho,i}|}{ct}\Big)\Big) e^{-\mathrm{i}tH}\chi(H)u \Big\rangle,$$

with $c > 1$. Obviously, in the massive case ($i \in \{1, 2, 3, 4, 7, 8, 11, 12\}$) the proof is the same than in Theorem 3.4.2. Let $i \in \{5, 6, 9, 10, 13, 14\}$. In the same way as for f_i, since F_i is bounded, we have that

$$\tilde{f}_i(t) \lesssim \|N_{\mathrm{neut}}^{\frac{1}{2}} u\|^2 + \|u\|^2. \tag{3.64}$$

Write $F_i(r) = \tilde{F}_i(r^2)$. Note that

$$x_{t,\rho,i}^2 = \sum_{\ell=1}^{3} \Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}} x_i^{(\ell)} \Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big) x_i^{(\ell)} \Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}}.$$

Proceeding as in [21, Lemma 5.2], we compute

$$\begin{aligned}
\mathbf{d}_0 \tilde{F}_i\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big) &= \Big\{ -\frac{2}{t}\frac{x_{t,\rho,i}^2}{c^2t^2} + \frac{\rho}{t}\Big(\frac{t^{-\rho}}{|p_2| + t^{-\rho}}\frac{x_{t,\rho,i}^2}{c^2t^2} + \text{h.c.}\Big) + \frac{1}{t}\Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\frac{p_2}{|p_2|}\cdot\frac{x_{t,\rho,i}}{ct} + \text{h.c.}\Big)\Big\} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big) \\
&\quad + \mathcal{O}(t^{-2+\rho}) \\
&= \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \Big\{ -\frac{2}{t}\frac{x_{t,\rho,i}^2}{c^2t^2} + \frac{\rho}{t}\Big(\frac{t^{-\rho}}{|p_2| + t^{-\rho}}\frac{x_{t,\rho,i}^2}{c^2t^2} + \text{h.c.}\Big) \\
&\qquad + \frac{1}{t}\Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\frac{p_2}{|p_2|}\cdot\frac{x_{t,\rho,i}}{ct} + \text{h.c.}\Big)\Big\} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} + \mathcal{O}(t^{-2+\rho}).
\end{aligned}$$

Using that $t^{-\rho}(|p_2| + t^{-\rho})^{-1} = 1 - |p_2|(|p_2| + t^{-\rho})^{-1}$ and commuting $|p_2|^{1/2}(|p_2| + t^{-\rho})^{-1/2}$ with $\tilde{F}_i'(x_{t,\rho,i}^2/c^2t^2)^{1/2}$ (using again [21, Lemma 5.2]), we obtain that

$$\begin{aligned}
&= \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \Big\{ -\frac{2-2\rho}{t}\frac{x_{t,\rho,i}^2}{c^2t^2}\Big\} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \\
&+ \Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \Big\{ -\frac{2\rho}{t}\frac{x_{t,\rho,i}^2}{c^2t^2} + \frac{1}{ct}\Big(\frac{p_2}{|p_2|}\cdot\frac{x_{t,\rho,i}}{ct} + \text{h.c.}\Big)\Big\} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}} \\
&+ \mathcal{O}(t^{-2+\rho}).
\end{aligned}$$

From the properties of the support of F_i', we then deduce that

$$\begin{aligned}
\mathbf{d}_0 \tilde{F}_i\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big) &\leqslant -2\tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \Big\{\frac{1-\rho}{t}\frac{x_{t,\rho,i}^2}{c^2t^2}\Big\} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \\
&- 2\Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \Big\{\frac{\rho - c^{-1}}{t}\frac{x_{t,\rho,i}^2}{c^2t^2}\Big\} \tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big)^{\frac{1}{2}} \Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}} + \mathcal{O}(t^{-2+\rho}).
\end{aligned}$$

Using again the properties of the support of F_i' we have:

$$\mathbf{d}_0 \tilde{F}_i\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big) \leqslant -2\frac{1-\rho}{t}\tilde{F}_i'\Big(\frac{x_{t,\rho,i}^2}{c^2t^2}\Big) - 2\frac{\rho - c^{-1}}{t}\Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}} \tilde{F}_i'\Big(\frac{x_{t,\rho}^2}{c^2t^2}\Big)\Big(\frac{|p_2|}{|p_2| + t^{-\rho}}\Big)^{\frac{1}{2}} + \mathcal{O}(t^{-2+\rho}). \tag{3.65}$$

The first two terms are non-positive since $\rho < 1$ and $c > \rho^{-1}$. Moreover, the commutators

$$\Big[H_I^{(j)}(G), \mathrm{d}\Gamma_i\Big(F_i\Big(\frac{|x_{t,\rho,i}|}{ct}\Big)\Big)\Big], \quad j = 1, 2,$$

are given by expressions similar to (2.41)–(2.42), with the operator $F_i(\frac{|x_{t,\rho,i}|}{ct})$ instead of a_i. Since F_i vanishes near 0 and $G \in \mathbb{H}^{1+\mu}$, one can verify using interpolation that G belongs to the domain of $|x_{t,\rho,i}|^{1+\mu}$, and hence

$$\Big\|F_i\Big(\frac{|x_{t,\rho,i}|}{ct}\Big)G\Big\|_2 = \mathcal{O}(t^{-1-\mu}).$$

Therefore, in the same way as for Equation (3.45), Lemma 2.1.3 yields

$$\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Big[H_I^{(1)}(G)+H_I^{(2)}(G), \mathrm{id}\Gamma_j\Big(F_i\Big(\frac{|x_{t,\rho,i}|}{ct}\Big)\Big)\Big]e^{-\mathrm{i}tH}\chi(H)u\Big\rangle = \mathcal{O}(t^{-1-\mu})\|u\|^2. \quad (3.66)$$

Using (3.64), (3.65) and (3.66), one can conclude that (i) holds by arguing in the same way as in the proof of Theorem 3.4.2 (i).

(iv) Again, the proof resembles that of Theorem 3.4.2 (iv), we focus on the differences. We consider $\chi \in \mathrm{C}_0^\infty((\lambda-\varepsilon, \lambda+\varepsilon))$, $\delta > 0$ and q_i, $i \in \{1, \dots, 14\}$ as in the proof of Theorem 3.4.2. Let $u \in \mathrm{Ran}(\mathbb{1}_{\{n\}}(N_{\mathrm{lept}} - N_{\mathrm{neut}}))$ for some $n \in \mathbb{Z}$. Let $\tilde{q}^t = (q_1(\frac{x_{t,\rho,1}}{t}), \dots, q_{14}(\frac{x_{t,\rho,14}}{t}))$ and

$$\tilde{h}(t) := \Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(\tilde{q}^t)\frac{A}{t}\Gamma(\tilde{q}^t)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle.$$

To verify that $\tilde{h}(t)$ is uniformly bounded, we modify (2.76) as follows. We can decompose

$$A = \sum_{i=1}^{14} \mathrm{d}\Gamma_i(a_i).$$

Clearly the massive cases ($i \in \{1, 2, 3, 4, 7, 8, 11, 12\}$) can be handled as in (2.76). For $i \in \{5, 6, 9, 10, 13, 14\}$, we recall that

$$a_i = \frac{\mathrm{i}}{2}\Big(\frac{p_2}{|p_2|}\cdot\nabla + \nabla\cdot\frac{p_2}{|p_2|}\Big)$$

and we have:

$$\begin{aligned}\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(\tilde{q}^t)\frac{\mathrm{d}\Gamma_i(a_i)}{t}\Gamma(\tilde{q}^t)e^{-\mathrm{i}tH}\chi(H)u\Big\rangle &= \Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(\tilde{q}^t)\Big[\frac{\mathrm{d}\Gamma_i(a_i)}{t}, \Gamma(\tilde{q}^t)\Big]e^{-\mathrm{i}tH}\chi(H)u\Big\rangle \\ &\quad + \Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(\tilde{q}^t)^2\frac{\mathrm{d}\Gamma_i(a_i)}{t}e^{-\mathrm{i}tH}\chi(H)u\Big\rangle. \end{aligned} \quad (3.67)$$

We estimate each term separately. First, we compute, using [21, Lemma 5.2],

$$\Big[\frac{a_i}{t}, q_i\Big(\frac{x_{t,\rho,i}}{t}\Big)\Big] = \mathcal{O}(t^{-1})|p_2|^{-1}.$$

This implies that

$$\begin{aligned}\Big|\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(\tilde{q}^t)\Big[\frac{\mathrm{d}\Gamma_i(a_i)}{t}, \Gamma(\tilde{q}^t)\Big]e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\Big| &\lesssim t^{-1}\|u\|\|\mathrm{d}\Gamma(|p_2|^{-1})e^{-\mathrm{i}tH}\chi(H)u\| \\ &\lesssim \|u\|\big(\|u\| + \|N_{\mathrm{neut}}u\| + \big\|\mathrm{d}\Gamma(|p_2|^{-1})u\big\|\big), \end{aligned} \quad (3.68)$$

where we used Lemma 3.4.1 in the second inequality. Next we have that

$$
\begin{aligned}
a_i &= \frac{1}{2}\Big(\frac{p_2}{|p_2|}\cdot x_i + x_i \cdot \frac{p_2}{|p_2|}\Big) \\
&= \sum_{l\in\{1,2,3\}} \Big(x_i^{(\ell)}\frac{p_2^{(\ell)}}{|p_2|} + \frac{1}{2|p_2|} - \frac{(p_2^{(\ell)})^2}{2|p_2|^3}\Big) \\
&= \sum_{l\in\{1,2,3\}} \Big(x_i^{(\ell)}\frac{|p_2|}{|p_2|+t^{-\rho}}\Big(1+\frac{t^{-\rho}}{|p_2|}\Big)\frac{p_2^{(\ell)}}{|p_2|} + \frac{1}{2|p_2|} - \frac{(p_2^{(\ell)})^2}{2|p_2|^3}\Big) \\
&= \sum_{l\in\{1,2,3\}} \Bigg(\Bigg(\Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}}x_i^{(\ell)} + \Big[x_i^{(\ell)}, \Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}}\Big]\Bigg)\Big(\frac{|p_2|}{|p_2|+t^{-\rho}}\Big)^{\frac{1}{2}}\Big(1+\frac{t^{-\rho}}{|p_2|}\Big)\frac{p_2^{(\ell)}}{|p_2|} + \frac{1}{2|p_2|} - \frac{(p_2^{(\ell)})^2}{2|p_2|^3}\Bigg) \\
&= \sum_{l\in\{1,2,3\}} \Big(\Big(x_{t,\rho,j}^{(\ell)} + \mathcal{O}(\frac{1}{|p_2|+t^{-\rho}})\Big)\Big(1+\frac{t^{-\rho}}{|p_2|}\Big)\frac{p_2^{(\ell)}}{|p_2|} + \frac{1}{2|p_2|} - \frac{(p_2^{(\ell)})^2}{2|p_2|^3}\Big) \\
&= \sum_{l\in\{1,2,3\}} x_{t,\rho,j}^{(\ell)}\big(1+\mathcal{O}(t^{-\rho})|p_2|^{-1}\big) + \mathcal{O}(1)|p_2|^{-1}.
\end{aligned}
$$

This proves

$$
a_i = \sum_{l\in\{1,2,3\}} x_{t,\rho,i}^{(\ell)}\big(1+\mathcal{O}(t^{-\rho})|p_2|^{-1}\big) + \mathcal{O}(1)|p_2|^{-1}. \tag{3.69}
$$

Using that $|x_{t,\rho,i}| \leqslant \delta t$ on the support of q_i^t, we deduce from the previous equation that

$$
\begin{aligned}
&\Big|\Big\langle e^{-\mathrm{i}tH}\chi(H)u, \Gamma(\tilde{q}^t)^2\frac{\mathrm{d}\Gamma_i(a_i)}{t}e^{-\mathrm{i}tH}\chi(H)u\Big\rangle\Big| \\
&\lesssim \delta\|u\|\big(\|N_{\mathrm{neut}}e^{-\mathrm{i}tH}\chi(H)u\| + \mathcal{O}(t^{-1-\rho})\|\mathrm{d}\Gamma(|p_2|^{-1})e^{-\mathrm{i}tH}\chi(H)u\|\big) + \mathcal{O}(t^{-1})\|u\|\|\mathrm{d}\Gamma(|p_2|^{-1})e^{-\mathrm{i}tH}\chi(H)u\| \\
&\lesssim \|u\|\big(\|N_{\mathrm{neut}}u\| + \|\mathrm{d}\Gamma(|p_2|^{-1})u\| + \|u\|\big),
\end{aligned} \tag{3.70}
$$

where we used Lemma 3.4.1 and the fact that $\rho > (1+\mu)^{-1}$ in the second inequality. We deduce from (3.68) and (3.70) that (3.67), and therefore $\tilde{h}(t)$, are uniformly bounded in t.

Next, we can decompose the derivative $\partial_t\tilde{h}(t) = \tilde{R}_1(t) + \tilde{R}_2(t) + \tilde{R}_3(t) + \tilde{R}_4(t)$ analogously as $h(t)$ in (3.49). The term $\tilde{R}_2(t)$ can be estimated in the same way as $R_2(t)$, using estimates similar to (3.68) and (3.70) instead of (2.76). This yields

$$
|\tilde{R}_2(t)| \lesssim \mathcal{O}(t^{-1-\mu})\|u\|\big(\|(N_{\mathrm{neut}}u\| + \|\mathrm{d}\Gamma(|p_2|^{-1})u\| + \|u\|\big). \tag{3.71}
$$

The estimate for $\tilde{R}_3(t)$ is identical to that of $R_3(t)$, yielding

$$
\tilde{R}_3(t) \geqslant \frac{\mathrm{c}_0}{t}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(\tilde{q}^t)(N_{\mathrm{neut}}+\mathbb{1})\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle + \mathcal{O}(t^{-2})\|(N_{\mathrm{neut}}+1)u\|^2. \tag{3.72}
$$

To estimate $\tilde{R}_4(t)$, we proceed as in (3.68)–(3.70). This gives

$$
\begin{aligned}
|\tilde{R}_4(t)| = \frac{1}{t}|\tilde{h}(t)| &\lesssim \frac{\delta}{t}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(\tilde{q}^t)(N_{\mathrm{neut}}+1)\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle \\
&\quad + t^{-1-\rho}\|u\|\|\mathrm{d}\Gamma(|p_2|^{-1})e^{-\mathrm{i}tH}\chi(H)u\| \\
&\lesssim \frac{\delta}{t}\Big\langle e^{-\mathrm{i}tH}u, \chi(H)\Gamma(\tilde{q}^t)(N_{\mathrm{neut}}+1)\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\Big\rangle \\
&\quad + t^{-1-\rho+\alpha(1+\mu)^{-1}}\|u\|\big(\|\mathrm{d}\Gamma(|p_2|^{-1})u\| + \|u\|\big),
\end{aligned} \tag{3.73}
$$

the second inequality being a consequence of Lemma 3.4.1.

Next we consider $\tilde{R}_1(t)$. As in the proof of Theorem 3.4.2, we compute

$$\mathbf{d}_0\tilde{q}_i^t(x_{t,\rho,i}) = -\frac{1}{2t}\Big\langle\frac{x_{t,\rho,i}}{t} - \nabla\omega_{t,\rho,i}, \nabla\tilde{q}_t\big(\frac{x_{t,\rho,i}}{t}\big)\Big\rangle + \text{h.c.} + \mathcal{O}(t^{-2+\rho}),$$

and set $\tilde{g}_i^t := -\frac{1}{2}\langle\frac{x_{t,\rho,i}}{t} - \nabla\omega_{t,\rho,i}, \nabla\tilde{q}_j\big(\frac{x_{t,\rho,i}}{t}\big)\rangle + \text{h.c.}$. Let also $\tilde{r}_i^t$ be the remainder term in the previous equality. We have that

$$\begin{aligned}
&\Big|\Big\langle e^{-\mathrm{i}tH}u, \Big(\chi(H)\mathrm{d}\Gamma(\tilde{q}_i^t,\tilde{r}_i^t)\frac{\mathrm{d}\Gamma_i(a_i)}{t}\Gamma(\tilde{q}^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{i}tH}u\Big\rangle\Big| \\
&\lesssim \mathcal{O}(t^{-2+\rho})\big\|N_{\text{neut}}\chi(H)e^{-\mathrm{i}tH}u\big\|\Big\|\frac{\mathrm{d}\Gamma_i(a_i)}{t}\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\Big\| \\
&\lesssim \mathcal{O}(t^{-2+\rho})\big\|N_{\text{neut}}\chi(H)e^{-\mathrm{i}tH}u\big\|\Big(\big\|N_{\text{neut}}\chi(H)e^{-\mathrm{i}tH}u\big\| + \mathcal{O}(t^{-\rho})\big\|\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\|\Big),
\end{aligned}$$

where we used (3.68)–(3.70) in the second inequality. By Lemma 3.4.1, and since $N_{\text{neut}} - N_{\text{lept}}$ commutes with H and N_{lept} is relatively H-bounded, we obtain that

$$\begin{aligned}
&\Big|\Big\langle e^{-\mathrm{i}tH}u, \Big(\chi(H)\mathrm{d}\Gamma_i(\tilde{q}_i^t,\tilde{r}_i^t)\frac{\mathrm{d}\Gamma_i(a_i)}{t}\Gamma(\tilde{q}^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{i}tH}u\Big\rangle\Big| \\
&\lesssim \mathcal{O}(t^{-2+\rho})\big(\|N_{\text{neut}}u\| + \|u\|\big)\big(\|N_{\text{neut}}u\| + \|\mathrm{d}\Gamma(|p_2|^{-1})u\| + \|u\|\big).
\end{aligned}$$

To estimate the term corresponding to $\frac{1}{t}\mathrm{d}\Gamma(\tilde{q}_i^t,\tilde{g}_i^t)$, we write

$$\begin{aligned}
&\frac{1}{t}\Big|\Big\langle e^{-\mathrm{i}tH}u, \Big(\chi(H)\mathrm{d}\Gamma(\tilde{q}_i^t,\tilde{g}_i^t)\frac{\mathrm{d}\Gamma_i(a_i)}{t}\Gamma(\tilde{q}^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{i}tH}u\Big\rangle\Big| \\
&= \frac{1}{t}\Big|\Big\langle e^{-\mathrm{i}tH}u, \Big(\chi(H)\mathrm{d}\Gamma(\tilde{q}_i^t,\tilde{g}_i^t)(N_{\text{neut}}+1)^{-\frac{1}{2}}(N_{\text{neut}}+1)^{\frac{1}{2}}\frac{\mathrm{d}\Gamma_i(a_i)}{t}\Gamma(\tilde{q}^t)\chi(H) + \text{h.c.}\Big)e^{-\mathrm{i}tH}u\Big\rangle\Big| \\
&\lesssim \frac{1}{t}\Big\|\mathrm{d}\Gamma((\tilde{g}_i^t)^*\tilde{g}_i^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}u\Big\|\Big(\big\|(N_{\text{neut}}+1)^{\frac{3}{2}}\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\big\| \\
&\qquad + \mathcal{O}(t^{-\rho})\big\|(N_{\text{neut}}+1)^{\frac{1}{2}}\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\|\Big),
\end{aligned}$$

where we used again (3.68)–(3.70). We expand the last expression and estimate the two terms separately. The first one is estimated exactly as in the proof of Theorem 3.4.2, yielding

$$\begin{aligned}
&\frac{1}{t}\Big\|\mathrm{d}\Gamma((\tilde{g}_i^t)^*\tilde{g}_i^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}u\Big\|\big\|(N_{\text{neut}}+1)^{\frac{3}{2}}\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\big\| \\
&\lesssim \frac{\alpha}{t}\Big\|\mathrm{d}\Gamma((\tilde{g}^t)^*\tilde{g}^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}(|N_{\text{neut}}-N_{\text{lept}}|+1)^{\frac{3}{2}}u\Big\|^2 + \frac{1}{\alpha t}\big\|\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2 + \frac{1}{\alpha t^2}\big\|(N_{\text{neut}}^2+1)u\big\|^2,
\end{aligned}$$

with $\alpha > 0$. For the second term, we have that

$$\begin{aligned}
&\frac{1}{t^{1+\rho}}\Big\|\mathrm{d}\Gamma((\tilde{g}_i^t)^*\tilde{g}_i^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}u\Big\|\big\|(N_{\text{neut}}+1)^{\frac{1}{2}}\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\| \\
&\lesssim \frac{1}{t^{1+\rho}}\Big\|\mathrm{d}\Gamma((\tilde{g}_i^t)^*\tilde{g}_i^t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}u\Big\|^2 + \frac{1}{t^{1+\rho}}\big\|(N_{\text{neut}}+1)^{\frac{1}{2}}\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\|^2,
\end{aligned}$$

and, since N_{lept} is relatively H-bounded,

$$\begin{aligned}
&\big\|(N_{\text{neut}}+1)^{\frac{1}{2}}\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \\
&= \big\|\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}(|N_{\text{neut}}-N_{\text{lept}}|+1)^{\frac{1}{2}}u\big\|^2 + \big\|N_{\text{lept}}^{\frac{1}{2}}\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \\
&\leqslant \big\|\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}(|N_{\text{neut}}-N_{\text{lept}}|+1)^{\frac{1}{2}}u\big\|^2 + \big\|(c_1H+c_2)^{\frac{1}{2}}\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\|^2,
\end{aligned}$$

where c_1 and c_2 are real numbers. Since one easily verifies that $[H, \mathrm{d}\Gamma(|p_2|^{-1})]$ is relatively H-bounded, Lemma 3.4.1 implies that

$$\big\|(N_{\mathrm{neut}}+1)^{\frac{1}{2}}\mathrm{d}\Gamma(|p_2|^{-1})\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \lesssim t^{(1+\mu)^{-1}}\big\|(\mathrm{d}\Gamma(|p_2|^{-1})+1)(N_{\mathrm{neut}}+1)^{\frac{1}{2}}u\big\|^2.$$

Putting together the previous estimates, we obtain that

$$\begin{aligned}|\tilde{R}_1(t)| &\lesssim \frac{\alpha+1}{t}\Big\|\mathrm{d}\Gamma(\tilde{g}_t^*\tilde{g}_t)^{\frac{1}{2}}\chi(H)e^{-\mathrm{i}tH}(|N_{\mathrm{neut}}-N_{\mathrm{lept}}|+1)^{\frac{3}{2}}u\Big\|^2 + \frac{1}{\alpha t}\big\|\Gamma(\tilde{q}^t)\chi(H)e^{-\mathrm{i}tH}u\big\|^2 \\ &+ \mathcal{O}(t^{-2+\rho})\big(\|N_{\mathrm{neut}}u\|+\|u\|\big)\big(\|N_{\mathrm{neut}}u\|+\|\mathrm{d}\Gamma(|p_2|^{-1})u\|+\|u\|\big) \\ &+ \mathcal{O}(t^{-1-\rho+(1+\mu)^{-1}})\big\|(\mathrm{d}\Gamma(|p_2|^{-1})+1)(N_{\mathrm{neut}}+1)^{\frac{1}{2}}u\big\|^2. \end{aligned} \tag{3.74}$$

From (3.71)–(3.74), we conclude the proof in the same way as for Theorem 3.4.2 (iv). □

3.5 Asymptotic completeness

In this section, we prove Theorem 3.1.1. We start from the strategy used for Theorem 2.1.1 but we need the propagation estimates of Theorem 3.4.3 instead of the ones of Section 2.3.

3.5.1 The asymptotic space and the wave operators

We start recalling some well-known results which can be proved in the same way as in Section 2.4.

Proposition 3.5.1. *Consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by* (2.5) *and suppose that*

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

Then

(i) $\mathscr{K}^{\pm}$ is closed and H-invariant.

(ii) For all $n \in \mathbb{N}$ and $h_1, \dots, h_n$, $\mathscr{K}^{\pm}$ is contained in the domain of $d^{\pm,*}(h_1)\dots d^{\pm,*}(h_n)$, where $d^{\pm,*}(h_i)$ stands for any of the operators $a_\epsilon^{\pm,*}(h_i)$, with $h_i \in \mathfrak{h}_1$, or $b_{l,\epsilon}^{\pm,*}(h_i)$ or $c_{l,\epsilon}^{\pm,*}(h_i)$, with $h_i \in \mathfrak{h}_2$.

(iii) $\mathscr{H}_{\mathrm{pp}}(H) \subset \mathscr{K}^{\pm}$.

Proposition 3.5.2. *Consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by* (2.5) *and suppose that*

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

Then $\Omega^{\pm}$ are isometric and we have that

$$H\Omega^{\pm} = \Omega^{\pm}H^{\mathrm{ext}}.$$

Moreover,

$$d^{\pm,\sharp}(h)\Omega^{\pm} = \Omega^{\pm}(\mathbb{1} \otimes d^{\sharp}(h)),$$

where $d^{\pm,\sharp}(h)$ stands for any of the operators $a_\epsilon^{\pm,\sharp}(h)$, with $h \in \mathfrak{h}_1$, or $b_{l,\epsilon}^{\pm,\sharp}(h)$ or $c_{l,\epsilon}^{\pm,\sharp}(h)$, with $h \in \mathfrak{h}_2$.

Theorem 3.5.3. *Suppose that the masses of the neutrinos m_{ν_e}, m_{ν_μ}, m_{ν_τ} vanish and consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by (2.5). Suppose that*

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

Then $\Omega^{\pm}$ are unitary maps form $\mathscr{H}^{\pm}$ to $\mathscr{H}$.

Proof. We will highlight an argument presented in [30].

Let $\mathfrak{f}_1 \subset \mathfrak{h}_1, \mathfrak{f}_2 \subset \mathfrak{h}_2$ be two subspaces of finite dimensions. For $u \in \mathscr{H}$, let

$$n^{\pm}_{\mathfrak{f}_1,\mathfrak{f}_2}(u) = \sum_{\epsilon=\pm} \sum_{i=1}^{\dim \mathfrak{f}_1} \|a^{\pm}_{\epsilon}(h_i)u\|^2 + \sum_{\epsilon=\pm, l\in\{e,\mu,\tau\}} \sum_{j=1}^{\dim \mathfrak{f}_2} \left(\|b^{\pm}_{l,\epsilon}(g_j)u\|^2 + \|c^{\pm}_{j,\epsilon}(g_k)u\|^2\right),$$

where $\{h_i\}$ and $\{g_j\}$ are orthonormal bases of $\mathfrak{f}_1$ and $\mathfrak{f}_2$, respectively. Note that if $u \notin \mathscr{D}(a^{\pm}_{l,\epsilon}(h_i))$ for some i, then $n^{\pm}_{\mathfrak{f}_1,\mathfrak{f}_2}(u) = \infty$. Clearly,

$$\begin{aligned} n^{\pm}_{\mathfrak{f}_1,\mathfrak{f}_2}(u) &= \lim_{t\to\infty} \sum_{\epsilon=\pm} \sum_{i=1}^{\dim \mathfrak{f}_1} \|a_{\epsilon}(h_{i,\pm t})e^{\mp \mathrm{i}tH}u\|^2 + \sum_{\epsilon=\pm, l\in\{e,\mu,\tau\}} \sum_{j=1}^{\dim \mathfrak{f}_2} \left(\|b_{l,\epsilon}(g_{j,\pm t})e^{\mp \mathrm{i}tH}u\|^2 + \|c_{l,\epsilon}(g_{j,\pm t})e^{\mp \mathrm{i}tH}u\|^2\right) \\ &\leqslant \left\langle e^{\mp \mathrm{i}tH}u, N e^{\mp \mathrm{i}tH}u \right\rangle. \end{aligned}$$

Decomposing $N = N_W + 2N_{\mathrm{lept}} + (N_{\mathrm{neut}} - N_{\mathrm{lept}})$ and using that N_W and N_{lept} are relatively H-bounded and that $N_{\mathrm{neut}} - N_{\mathrm{lept}}$ commutes with H, we deduce that

$$n^{\pm}_{\mathfrak{f}_1,\mathfrak{f}_2}(u) \lesssim \langle u, |H|u\rangle + \langle u, N_{\mathrm{neut}}u\rangle + \|u\|^2. \tag{3.75}$$

Now, as in [30, Theorem 4.3], one can verify that

$$\operatorname{Ran}\Omega^{\pm} = \overline{\mathscr{D}(n^{\pm})}, \tag{3.76}$$

where

$$n^{\pm}(u) = \sup_{\mathfrak{f}_1\subset\mathfrak{h}_1, \mathfrak{f}_2\subset\mathfrak{h}_2} n^{\pm}_{\mathfrak{f}_1,\mathfrak{f}_2}(u)$$

and $\mathscr{D}(n^{\pm}) = \{u \in \mathscr{H}, n^{\pm}(u) < \infty\}$. In the previous equation, the supremum is taken over all finite dimensional subspaces $\mathfrak{f}_1$, $\mathfrak{f}_2$. From (3.75) we deduce that $\mathscr{D}(n^{\pm})$ contains the dense subset $\mathscr{D}(|H|^{\frac{1}{2}}) \cap \mathscr{D}(N_{\mathrm{neut}}^{\frac{1}{2}})$. By (3.76), this shows that $\Omega^{\pm}$ are onto. □

3.5.2 The geometric inverse wave operators

In this section we establish the existence of two asymptotic observables using the propagation estimates of Section 3.4. Compared to similar results proven in [29, 43, 5], the main difficulty we encounter comes from the fact that the propagation observables of Section 3.4 only hold for a dense set of states, and for suitable norms. For this reason, the results of this section are not straightforward modifications of previous papers.

We begin with the following important proposition.

Proposition 3.5.4. *Suppose that the masses of the neutrinos vanish and consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by (2.5). Suppose that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad |p_3|^{-1}a_{(i),\cdot}G \in L^2, \quad i = 1, 2, 3,$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

Let $\delta > 0$ *and* q_i, $i \in \{1, \dots, 14\}$ *be functions in* $\mathrm{C}_0^\infty(\{x \in \mathbb{R}^3, |x| \leqslant 2\delta\})$ *such that* $0 \leqslant q_i \leqslant 1$, $q_i = 1$ *on* $\{x \in \mathbb{R}^3, |x| \leqslant \delta\}$ *and let* $\tilde{q}^t = (q_1(\frac{x_{t,\rho}}{t}), \dots, q_{14}(\frac{x_{t,\rho}}{t}))$. *The following limits exist*

$$\Gamma^{\pm}(q) := \operatorname*{s-lim}_{t\to\pm\infty} e^{\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{-\mathrm{i}tH}.$$

Moreover, for all $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ *supported in* $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$, *there exists* $\delta > 0$ *such that*

$$\Gamma^{\pm}(q)\chi(H) = 0.$$

Proof. It suffices to prove the existence of

$$\lim_{t\to\pm\infty} e^{\pm\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{\mp\mathrm{i}tH}u,$$

for u in a dense subset of $\mathscr{H}$. We consider

$$\mathscr{E} := \big\{u \in \mathscr{H}, \exists\chi \in \mathrm{C}_0^\infty(\mathbb{R}), n \in \mathbb{N}, u = \chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u\big\}.$$

Let $u \in \mathscr{E}$ and let $\tilde{\chi} \in \mathrm{C}_0^\infty(\mathbb{R})$ be such that $\tilde{\chi}\chi = \chi$. As in the proof of Theorem 3.4.3 (iv), the Helffer-Sjöstrand functional shows that

$$\big\|\big[\Gamma(\tilde{q}^t), \tilde{\chi}(H)\big](N_{\mathrm{neut}} + 1)^{-1}\big\| = \mathcal{O}(t^{-1}).$$

Since $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H and $\Gamma(\tilde{q}^t)$ and since N_{lept} is relatively H-bounded, we deduce that

$$\begin{aligned}
&e^{\pm\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{\mp\mathrm{i}tH}\chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u\\
&= \tilde{\chi}(H)e^{\pm\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{\mp\mathrm{i}tH}\chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u + \mathcal{O}(t^{-1})\\
&= \mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})\tilde{\chi}(H)e^{\pm\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{\mp\mathrm{i}tH}\chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u + \mathcal{O}(t^{-1}).
\end{aligned}$$

For the last equality, we recall that $\Gamma(\tilde{q}^t)$ commutes with N. To shorten notations, let $\chi_{(n)}(H) := \chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})$, $\tilde{\chi}_{(n)}(H) := \tilde{\chi}(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})$. By the previous equality, it now suffices to prove the existence of

$$\lim_{t\to\pm\infty} \tilde{\chi}_{(n)}(H)e^{\pm\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u.$$

Set $W(t) := \tilde{\chi}_{(n)}(H)e^{\pm\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)$ and write, for $t' > t \geqslant 1$,

$$\big\|W(t')u - W(t)u\big\| = \Big\|\int_t^{t'} \partial_s W(s)uds\Big\| \leqslant \sup_{v\in\mathscr{H},\|v\|=1} \int_t^{t'} |\langle v, \partial_s W(s)u\rangle| ds. \tag{3.77}$$

We compute

$$\begin{aligned}
\langle v, \partial_t W(t)u\rangle =& \langle v, \partial_t \left(\tilde{\chi}_{(n)}(H)e^{\pm\mathrm{i}tH}\Gamma(\tilde{q}^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)\right) u\rangle\\
&= \pm\langle v, \tilde{\chi}_{(n)}(H)e^{\pm\mathrm{i}tH}\big(\mathbf{D}_0\Gamma(\tilde{q}^t) + \mathrm{i}g[H_I^{(1)} + H_I^{(2)}, \Gamma(\tilde{q}^t)]\big)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\rangle\\
&= \pm\langle v, \tilde{\chi}_{(n)}(H)e^{\pm\mathrm{i}tH}\big(\mathrm{d}\Gamma(\tilde{q}^t, \mathbf{d}_0\tilde{q}^t) + \mathrm{i}g[H_I^{(1)} + H_I^{(2)}, \Gamma(\tilde{q}^t)]\big)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\rangle.
\end{aligned} \tag{3.78}$$

We will show that the right-hand-side is integrable in t on $[1,\infty)$.

We invoke arguments closely related to those used in the proof of Theorem 3.4.3 (iv). First, the assumption that $G \in \mathbb{H}^{1+\mu}$, the commutation relations of Section 2.1.2 and Lemma 2.1.3 imply that

$$\big\|[H_I^{(1)}(G) + H_I^{(2)}(G), \Gamma(\tilde{q}^t)](N_{\mathrm{lept}} + N_W)^{-1}\big\| = \mathcal{O}(t^{-1-\mu}).$$

Since N_W and N_{lept} are relatively H-bounded, this yields

$$\big\|[H_I^{(1)} + H_I^{(2)}, \Gamma(\tilde{q}^t)]\chi(H)\big\| \lesssim t^{-1-\mu}.$$

Now we consider the term involving $\mathrm{d}\Gamma(\tilde{q}^t, \mathbf{d}_0\tilde{q}^t)$ in (3.78). As in the proof of Theorem 3.4.3 (iv), we have that

$$\mathbf{d}_0\tilde{q}_i^t(x_{t,\rho,i}) = -\frac{1}{2t}\Big\langle \frac{x_{t,\rho,i}}{t} - \nabla\omega_{t,\rho,i}, \nabla q_i\big(\frac{x_{t,\rho,i}}{t}\big)\Big\rangle + \text{h.c.} + \mathcal{O}(t^{-2}),$$

if $i \in \{1,2,3,4,7,8,11,12\}$ (corresponding to the label of a massive particle), and

$$\mathbf{d}_0\tilde{q}_i^t(x_{t,\rho,i}) = -\frac{1}{2t}\Big\langle \frac{x_{t,\rho,i}}{t} - \nabla\omega_{t,\rho,i}, \nabla q_i\big(\frac{x_{t,\rho,i}}{t}\big)\Big\rangle + \text{h.c.} + \mathcal{O}(t^{-2+\rho}),$$

if $i \in \{5,6,9,10,13,14\}$ (corresponding to the label of a neutrino). We treat the second case, namely $i \in \{5,6,9,10,13,14\}$, the case of $i \in \{1,2,3,4,7,8,11,12\}$ being easier.

Let $\tilde{g}_i^t := -\frac{1}{2}\langle \frac{x_{t,\rho,i}}{t} - \nabla\omega_{t,\rho,i}(k), \nabla q_i\big(\frac{x_{t,\rho,i}}{t}\big)\rangle + \text{h.c.}$ and let $\tilde{r}_i^t = \mathbf{d}_0\tilde{q}_i^t(x_{t,\rho,i}) - \frac{1}{t}\tilde{g}_i^t = \mathcal{O}(t^{-2+\rho})$. For the term corresponding to $\tilde{r}_i^t$, we have that

$$\big\|e^{\pm\mathrm{i}tH}\tilde{\chi}_{(n)}(H)\mathrm{d}\Gamma(\tilde{q}_i^t, \tilde{r}_i^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\| \lesssim \mathcal{O}(t^{-2+\rho})\big\|Ne^{-\mathrm{i}tH}\chi_{(n)}(H)u\big\| = \mathcal{O}(t^{-2+\rho}).$$

The equality comes from the facts that $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, N_W and N_{lept} are relatively H-bounded and that $(N+1)\tilde{\chi}_{(n)}(H)$ is bounded.

To estimate the term corresponding to $\frac{1}{t}\mathrm{d}\Gamma(\tilde{q}_i^t, \tilde{g}_i^t)$, we use (2.25), yielding

$$\begin{aligned}&\frac{1}{t}\big|\big\langle e^{\mp\mathrm{i}tH}\tilde{\chi}_{(n)}(H)v, \mathrm{d}\Gamma(\tilde{q}_i^t, \tilde{g}_i^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\rangle\big|\\ &\leqslant \frac{1}{t}\big\|\mathrm{d}\Gamma(|\tilde{g}_i^t|)^{\frac{1}{2}}e^{\mp\mathrm{i}tH}\tilde{\chi}_{(n)}(H)v\big\|\big\|\mathrm{d}\Gamma(|\tilde{g}_i^t|)^{\frac{1}{2}}e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\|.\end{aligned}$$

By (iii) of Theorem 3.4.3,

$$\int_1^\infty \frac{1}{t}\big\|\mathrm{d}\Gamma(|\tilde{g}_i^t|)^{\frac{1}{2}}e^{\mp\mathrm{i}tH}\tilde{\chi}_{(n)}(H)v\big\|\big\|\mathrm{d}\Gamma(|\tilde{g}_i^t|)^{\frac{1}{2}}e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\|dt \lesssim \|u\|\|v\|.$$

From (3.77) and the previous computations, we easily deduce that for any $\varepsilon > 0$,

$$\big\|W(t')u - W(t)u\big\| \leqslant \varepsilon,$$

for t and t' large enough. This proves that the limits $\Gamma^{\pm}(\tilde{q})$ exist.

The fact that $\Gamma^{\pm}(\tilde{q})\chi(H) = 0$ for all $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$ is a consequence of Theorem 3.4.3 (iv). Indeed, Theorem 3.4.3 (iv) shows that $\Gamma^{\pm}(\tilde{q})\chi(H)u = 0$ for all u in a dense subset of $\mathscr{H}$. Since $\Gamma^{\pm}(\tilde{q})\chi(H)$ is bounded, the statement follows. □

We turn now to the proof of the main result of this subsection.

Theorem 3.5.5. *Suppose that the masses of the neutrinos vanish and consider the Hamiltonian* $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ *with* $H_I^{(1)}$ *and* $H_I^{(2)}$ *given by* (2.5)*. Suppose that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad |p_3|^{-1}a_{(i),\cdot}G \in L^2, \quad i = 1,2,3,$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

Let $\delta > 0$ *and* $j_{0,i}$*,* $i \in \{1,\dots,14\}$ *be functions in* $\mathrm{C}_0^\infty(\{x \in \mathbb{R}^3, |x| \leqslant 2\delta\})$ *such that* $0 \leqslant j_{0,i} \leqslant 1$*,* $j_{0,i} = 1$ *on* $\{x \in \mathbb{R}^3, |x| \leqslant \delta\}$ *and let* $j_{\infty,i} = 1 - j_{0,i}$*,* $j_i = (j_{0,i}, j_{\infty,i})$*. Let* $\tilde{J}^t = (j_1(\frac{x_{t,\rho,1}}{t}),\dots,j_{14}(\frac{x_{t,\rho,14}}{t}))$*.*

(i) *The following limits exist*

$$W^\pm(J) := \underset{t\to+\infty}{\text{s-lim}}\, e^{\pm\mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}\big(\tilde{J}^t\big)e^{\mp\mathrm{i}tH}.$$

(ii) *For all* $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$*, we have that*

$$W^\pm(J)\chi(H) = \chi(H^{\mathrm{ext}})W^\pm(J).$$

(iii) *Let* $q = (q_1,\dots,q_{14})$ *be such that* $q_i j_{i,0} = j_{i,0}$*. Then*

$$\big(\Gamma^\pm(q)\otimes\mathbb{1}\big)W^\pm(J) = W^\pm(J).$$

(iv) *For all* $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$*, we have that*

$$\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^\pm(J) = \chi(H).$$

The same holds if the masses of the neutrinos m_{ν_e}*,* m_{ν_μ}*,* m_{ν_τ} *are positive and if one considers the Hamiltonian* (2.12) *with* H_I *given by* (2.5)*.*

Proof. (i) As in the proof of Proposition 3.5.4, it suffices to prove the existence of

$$\lim_{t\to\pm\infty} e^{\pm\mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}(\tilde{J}^t)e^{\mp\mathrm{i}tH}u,$$

for u in a dense subset of $\mathscr{H}$. We consider again

$$\mathscr{E} = \big\{u \in \mathscr{H}, \exists\chi \in \mathrm{C}_0^\infty(\mathbb{R}), n \in \mathbb{N}, u = \chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u\big\},$$

and fix $u \in \mathscr{E}$. Let $\tilde{\chi} \in \mathrm{C}_0^\infty(\mathbb{R})$ be such that $\tilde{\chi}\chi = \chi$. In the same way as in Lemma 2.1.11, one verifies that

$$\big\|\big(\check{\Gamma}(\tilde{J}^t)\tilde{\chi}(H) - \chi(H^{\mathrm{ext}})\check{\Gamma}(\tilde{J}^t)\big)(N_{\mathrm{neut}}+1)^{-1}\big\| = \mathcal{O}(t^{-1}).$$

Using that $N_{\mathrm{lept}} - N_{\mathrm{neut}}$ commutes with H, $\check{\Gamma}(\tilde{J}^t)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}}) = \mathbb{1}_{[-n,n]}(N^{\mathrm{ext}}_{\mathrm{neut}} - N^{\mathrm{ext}}_{\mathrm{lept}})\check{\Gamma}(\tilde{J}^t)$, and that N_{lept} is relatively H-bounded, we deduce that

$$\begin{aligned}
&e^{\pm\mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}(\tilde{J}^t)e^{\mp\mathrm{i}tH}\chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u\\
&= \tilde{\chi}(H^{\mathrm{ext}})e^{\pm\mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}(\tilde{J}^t)e^{\mp\mathrm{i}tH}\chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u + \mathcal{O}(t^{-1})\\
&= \mathbb{1}_{[-n,n]}(N^{\mathrm{ext}}_{\mathrm{neut}} - N^{\mathrm{ext}}_{\mathrm{lept}})\tilde{\chi}(H^{\mathrm{ext}})e^{\pm\mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}(\tilde{J}^t)e^{\mp\mathrm{i}tH}\chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}} - N_{\mathrm{lept}})u + \mathcal{O}(t^{-1}).
\end{aligned}$$

Similarly as in the proof of Proposition 3.5.4, to shorten notation, we set $\chi_{(n)}(H) := \chi(H)\mathbb{1}_{[-n,n]}(N_{\mathrm{neut}}-N_{\mathrm{lept}})$ and $\tilde\chi_{(n)}(H^{\mathrm{ext}}) := \tilde\chi(H^{\mathrm{ext}})\mathbb{1}_{[-n,n]}(N^{\mathrm{ext}}_{\mathrm{neut}}-N^{\mathrm{ext}}_{\mathrm{lept}})$. By the previous equality, it now suffices to prove the existence of

$$\lim_{t\to\pm\infty}\tilde\chi_{(n)}(H^{\mathrm{ext}})e^{\pm \mathrm{i}tH^{\mathrm{ext}}}\check\Gamma(\tilde J^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u.$$

Set $\check W(t) := \tilde\chi_{(n)}(H^{\mathrm{ext}})e^{\pm \mathrm{i}tH^{\mathrm{ext}}}\check\Gamma(\tilde J^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)$ and write, for $t'>t\geqslant 1$,

$$\big\|\check W(t')u-\check W(t)u\big\| = \Big\|\int_t^{t'}\partial_s\check W(s)uds\Big\| \leqslant \sup_{v\in\mathscr{H}^{\mathrm{ext}},\|v\|=1}\int_t^{t'}|\langle v,\partial_s W(s)u\rangle|ds. \tag{3.79}$$

We compute

$$\begin{aligned}
&\langle v,\partial_t\check W(t)u\rangle = \langle v,\partial_t\Big(\tilde\chi_{(n)}(H^{\mathrm{ext}})e^{\pm \mathrm{i}tH^{\mathrm{ext}}}\check\Gamma(\tilde J^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)\Big)u\rangle\\
&= \pm\langle v,\tilde\chi_{(n)}(H^{\mathrm{ext}})e^{\pm \mathrm{i}tH^{\mathrm{ext}}}\big\{\check{\mathbf{D}}_0\check\Gamma(\tilde J^t)+\mathrm{i}g\big(((H_I^{(1)}+H_I^{(2)})\otimes\mathbb{1})\check\Gamma(\tilde J^t)-\check\Gamma(\tilde J^t)(H_I^{(1)}+H_I^{(2)})\big)\big\}e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\rangle\\
&= \pm\langle v,\tilde\chi_{(n)}(H^{\mathrm{ext}})e^{\pm \mathrm{i}tH^{\mathrm{ext}}}\big\{\mathrm{d}\check\Gamma(\tilde J^t,\check{\mathbf{d}}_0\tilde J^t)\\
&\qquad\qquad +\mathrm{i}g\big(((H_I^{(1)}+H_I^{(2)})\otimes\mathbb{1})\check\Gamma(\tilde J^t)-\check\Gamma(\tilde J^t)(H_I^{(1)}+H_I^{(2)})\big)\big\}e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\rangle.
\end{aligned}$$

As in the proof of Lemma 2.1.11, we have that

$$\tilde\chi_{(n)}(H^{\mathrm{ext}})\big(((H_I^{(1)}+H_I^{(2)})\otimes\mathbb{1})\check\Gamma(\tilde J^t)-\check\Gamma(\tilde J^t)(H_I^{(1)}+H_I^{(2)})\big)\chi_{(n)}(H)=\mathcal{O}(t^{-1-\mu}).$$

Moreover, similarly as in the proof of Proposition 3.5.4, we decompose

$$\check{\mathbf{d}}_0\tilde J^t=\frac1t\tilde G^t+\tilde R^t,\quad \tilde G^t=(\tilde g_0^t,\tilde g_\infty^t),\quad \tilde g_\sharp^t=-\frac12\Big(\Big(\frac{x_{t,\rho}}{t}-\nabla\omega_{t,\rho}\Big)\nabla\tilde J_\sharp\Big(\frac{x_{t,\rho}}{t}\Big)+\mathrm{h.c.}\Big),\quad \tilde R^t=\mathcal{O}(t^{-2+\rho}),$$

and, for all $i\in\{1,\dots,14\}$, we have that

$$\big\|e^{\pm\mathrm{i}tH^{\mathrm{ext}}}\tilde\chi_{(n)}(H^{\mathrm{ext}})\mathrm{d}\check\Gamma(\tilde J_i^t,\tilde R_i^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\|\lesssim\mathcal{O}(t^{-2+\rho})\big\|Ne^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\|=\mathcal{O}(t^{-2+\rho}).$$

The term corresponding to $\frac1t\mathrm{d}\check\Gamma(\tilde J_i^t,\tilde G_i^t)$ is estimated as

$$\begin{aligned}
&\frac1t\big|\big\langle e^{\mp\mathrm{i}tH^{\mathrm{ext}}}\tilde\chi_{(n)}(H^{\mathrm{ext}})v,\mathrm{d}\check\Gamma(\tilde J_i^t,\tilde G_i^t)e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\rangle\big|\\
&\leqslant\frac1t\big\|(\mathrm{d}\Gamma(|\tilde g_{0,i}^t|)^{\frac12}\otimes\mathbb{1})e^{\mp\mathrm{i}tH^{\mathrm{ext}}}\tilde\chi_{(n)}(H^{\mathrm{ext}})v\big\|\big\|\mathrm{d}\Gamma(|\tilde g_{0,i}^t|)^{\frac12}e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\|\\
&\quad+\frac1t\big\|(\mathbb{1}\otimes\mathrm{d}\Gamma(|\tilde g_{\infty,i}^t|)^{\frac12})e^{\mp\mathrm{i}tH^{\mathrm{ext}}}\tilde\chi_{(n)}(H^{\mathrm{ext}})v\big\|\big\|\mathrm{d}\Gamma(|\tilde g_{\infty,i}^t|)^{\frac12}e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\|.
\end{aligned}$$

By (iii) of Theorem 3.4.3,

$$\int_1^\infty\frac1t\big\|(\mathrm{d}\Gamma(|\tilde g_{0,i}^t|)^{\frac12}\otimes\mathbb{1})e^{\mp\mathrm{i}tH^{\mathrm{ext}}}\tilde\chi_{(n)}(H^{\mathrm{ext}})v\big\|\big\|\mathrm{d}\Gamma(|\tilde g_{0,i}^t|)^{\frac12}e^{\mp\mathrm{i}tH}\chi_{(n)}(H)u\big\|dt\lesssim\|u\|\|v\|,$$

and likewise for the second term in the right-hand-side of the previous inequality. Eq. (3.79) and the previous estimates imply that, for any $\varepsilon>0$,

$$\big\|W(t')u-W(t)u\big\|\leqslant\varepsilon,$$

for t and t' large enough, which proves that the limits $W^{\pm}(\tilde J)$ exist.

(ii) This is a standard intertwining property.

(iii) It suffices to write

$$\begin{aligned}\big(\Gamma^{\pm}(q)\otimes\mathbb{1}\big)W^{\pm}(J)u &= \big(e^{\pm \mathrm{i}tH}\Gamma(q^t)e^{\mp \mathrm{i}tH}\otimes\mathbb{1}\big)e^{\pm \mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}\big(\tilde{J}^t\big)e^{\mp \mathrm{i}tH}u + o(1)\\ &= \big(e^{\pm \mathrm{i}tH^{\mathrm{ext}}}(\Gamma(q^t)\otimes\mathbb{1}\big)\check{\Gamma}\big(\tilde{J}^t\big)e^{\mp \mathrm{i}tH}u + o(1)\\ &= \big(e^{\pm \mathrm{i}tH^{\mathrm{ext}}}\check{\Gamma}\big(\tilde{J}^t\big)e^{\mp \mathrm{i}tH}u + o(1)\\ &= W^{\pm}(J)u + o(1),\end{aligned}$$

where we used that $(\Gamma(q^t)\otimes\mathbb{1}\big)\check{\Gamma}\big(\tilde{J}^t\big) = \check{\Gamma}\big(\tilde{J}^t\big)$ because $q_i j_{i,0} = j_{i,0}$ in the third equality.

(iv) This is again standard intertwining property. □

3.5.3 Asymptotic completeness

Now that the propagation estimates have been established in the massless case, together with some important consequences presented in 3.5.5 and 3.5.4, the proof of Theorem 3.1.1 is a straightforward modification of the one of Theorem 2.1.1.

Theorem 3.5.6. *Suppose that the masses of the neutrinos vanish and consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by (2.5). Suppose that*

$$G \in L^2, \quad a_{(i),\cdot}G \in L^2, \quad |p_3|^{-1}a_{(i),\cdot}G \in L^2, \quad i = 1,2,3,$$

and that

$$G \in \mathbb{H}^{1+\mu} \text{ for some } \mu > 0.$$

There exists $g_0 > 0$ such that, for all $|g| \leqslant g_0$,

$$\mathscr{H}_{\mathrm{pp}}(H) = \mathscr{K}^{\pm}.$$

Corollary 3.5.7. *Suppose that the masses of the neutrinos vanish and consider the Hamiltonian $H = H_0 + g(H_I^{(1)} + H_I^{(2)})$ with $H_I^{(1)}$ and $H_I^{(2)}$ given by (2.5). Under the conditions of Theorem 3.5.6, and assuming in addition that*

$$b_{(i),\cdot}G \in L^2, \quad i = 1,2,3, \quad b_{(i),\cdot}b_{(i'),\cdot}G \in L^2, \quad i,i' = 1,2,3, \tag{3.80}$$

the operators $H - E$ and H_0 are unitarily equivalent.

Chapter 4

Fermionic Hamiltonians

4.1 Introduction

This chapter corresponds to [3] and explores a topic which is not directly related to scattering theories and it is rather independent of the previous section. While studying Hamiltonians from the weak interaction a very natural question arised. How much particle species can be included in a physical process? In particular, in the fermionic case, where the creation and annihilation operators are bounded, it would be expected to be possible to include as many particle species as we want. Let us then consider a finite number of interacting fermion fields of spin 1/2. Each field is associated to a different species of particles. The free energy is

$$H_f = \sum_{i=1}^{n} \int \omega_i(k_i) b_i^*(\xi_i) b_i(\xi_i) d\xi_i, \tag{4.1}$$

where $\xi_i = (k_i, \lambda_i) \in \mathbb{R}^3 \times \{-1/2, 1/2\}$, with k_i the momentum variable and λ_i the spin variable for the i^{th} particle. Moreover, $b_i^\sharp(\xi_i)$, where $b_i^\sharp$ stands for b_i^* or b_i, are the usual fermionic creation and annihilation operators for the i^{th} particle. The relativistic dispersion relations $\omega_i(k_i)$ are defined by

$$\omega_i(k_i) = \sqrt{k_i^2 + m_i^2},$$

with $m_i \geqslant 0$ the mass of the i^{th} field. The interaction terms are of the form

$$\int G(\xi_1, \ldots, \xi_n) b_1^*(\xi_1) \ldots b_p^*(\xi_p) b_{p+1}(\xi_{p+1}) \ldots b_n(\xi_n) d\xi_1 \ldots d\xi_n + \text{h.c.}, \tag{4.2}$$

the first term representing a process where p particles are created while $n-p$ particles are annihilated, with $0 \leqslant p \leqslant n$. The second term, which is the hermitian conjugate of the first one, represents the inverse process. In the previous equation, the interaction form factor G is supposed to be square integrable, which makes (4.2) a well-defined quadratic form. The interaction Hamiltonian that we consider in this chapter, H_I, is given by the sum over all possible processes of interaction terms of the form (4.2); see the next section. Formally, the total Hamiltonian is then defined by $H = H_f + H_I$.

Important physical examples of processes that can be described by such a model arise from the Fermi theory of weak interactions. For instance, the weak decay of a muon into an electron, a muon neutrino and an electron antineutrino, or the scattering of an electron with an electron neutrino are well-described at low energy by the Fermi theory (see e.g. [55, Chapter 2]).

Considering the formal Hamiltonian associated to the Lagrangian of the theory, and introducing ultraviolet and spatial cut-offs, one obtains an expression of the form above. Note that, in such processes, four spin 1/2 fermion fields interact. More details on the Fermi theory of weak interactions, as well as other examples, will be given in Section 4.1.3.

We emphasize that, in the model we consider, the masses of the fields may vanish. Physically, while in the classical version of the Standard Model of Quantum Filed Theory, neutrinos were supposed to be massless, it is now experimentally established that they have a small positive mass. Nevertheless, massless neutrinos are still considered in the physics literature as they constitute relevant approximations if one computes the first terms of the scattering cross-sections corresponding to weak decay processes. Moreover, models involving massless quasi-fermions, such as Weyl fermions (see e.g. [59, Section 2.7] or [84, Chapter 2]) are frequently studied, and we think it is mathematically interesting to study a general model covering both the massive and the massless cases.

Another remark is concerning the spin of the particles. The spinor space for a field of massive Dirac fermions is 4-dimensional. Splitting into particles and anti-particles, this corresponds to a state space given as a direct sum of two anti-symmetric Fock spaces over $L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$ (hence two fermion fields of spin 1/2). Proceeding analogously for a massless Dirac field with a 2-dimensional spinor space should lead to a direct sum of two anti-symmetric Fock spaces over $L^2(\mathbb{R}^3)$. To keep uniform notations, however, we consider only spin 1/2 fermionic fields, regardless of the masses of the associated particles. The spin degrees of freedom do not play any relevant role in our analysis, anyway.

Our first main concern will be to show that the formal Hamiltonian H defines a self-adjoint operator on Fock space. An obvious obstruction is that the kinetic energy is only quadratic in the creation and annihilation operators, while the interaction Hamiltonian is of order n. Nevertheless, using that all particles involved are fermions, an argument due to Glimm and Jaffe [51] shows that, if all interaction form factors G belong to the Schwartz space of rapidly decreasing functions, then H_I is a bounded operator. In particular, H identifies to a self-adjoint operator. In fact, inspecting the proof of [51, Proposition 1.2.3], one can see that, to apply the argument, it is actually sufficient that all G belong to the domain of some power of the harmonic oscillator. Still, this condition imposes a constraint on the infrared behavior of the form factors which is much too strong to cover physically realistic cases. To circumvent this problem, our strategy will consist in applying suitable interpolation arguments in order to obtain refined N_τ estimates. The latter can then be applied to the abstract model studied here, with only *mild* regularity assumptions on the infrared behavior of the form factors. Note that, as may be expected, the regularity assumptions that we will have to impose will be slightly stronger in the case of massless fields than in the massive case. The refined N_τ estimates, combined with a perturbation argument, will allow us to obtain the self-adjointness of H.

Once the self-adjointness of H is established, our next concern will be to prove the existence of a ground state. In the case where all fields are massive, we will adapt an argument of [29], while in the more difficult case where some fields are supposed to be massless, we will employ an induction argument and follow the approach of [56]. In both cases, the proof will have to rely on the refined N_τ estimates established previously, instead of the classical ones.

The question of asymptotic completeness has not been treated in this case. We believe that the massive case does not require a lot of modifications with respect to chapter two. The massless case would be of course an open problem. However, it might be tempted to use the methods developed in chapter 3 to prove asymptotic completeness for simplified models. The difficulty would then be to find a linear combination of interactive terms which conserved an interesting number quantity. Intuitively, for N particles, 2^{N-1} interaction terms have to be considered and each massless particles that are introduced reduce the number of terms that can be treated. For example, for p massless species only 2^{N-p-1} terms can be involved.

The remainder of this section is organized as follows. In Section 4.1.1, we define precisely the model considered in this chapter, next we state our main results in Section 4.1.2. Section 4.1.3 is concerned with examples arising from the Fermi theory of weak interactions.

4.1.1 The model

As mentioned above, we consider in this chapter an abstract class of models representing a finite number, n, of interacting fermion fields. The total Hilbert space is the tensor product of n antisymmetric Fock spaces,

$$\mathscr{H} := \bigotimes_{i=1}^{n} \mathscr{F}, \qquad \mathscr{F} := \mathbb{C} \oplus \bigoplus_{l=1}^{\infty} \otimes_a^l \Big(L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}) \Big). \tag{4.3}$$

Here $\otimes_a^l$ stands for the anti-symmetric tensor product. Throughout this chapter, we use the notation $\xi = (k, \lambda) \in \mathbb{R}^3 \times \{-1/2, 1/2\}$, i.e. k stands for the momentum variable and λ the spin variable. Moreover, to distinguish between the n different species of particles, the variable corresponding to the i^{th} Fock space will be denoted by $\xi_i = (k_i, \lambda_i)$.

The free Hamiltonian, acting on $\mathscr{H}$, is the sum of the second quantizations of the free relativistic energy of n particles of masses $m_i \geqslant 0$,

$$H_f := \sum_{i=1}^{n} H_{f,i},$$

where $H_{f,i}$ acts on the i^{th} Fock space and is given by

$$H_{f,i} := \mathrm{d}\Gamma(\omega_i(k_i)), \quad \omega_i(k_i) := \sqrt{k_i^2 + m_i^2}, \quad m_i \geqslant 0.$$

Let $\mathscr{F}_{\text{fin}}(\mathcal{S})$ be the subset of $\mathscr{F}$ consisting of vectors $(\varphi_0, \varphi_1, \dots) \in \mathscr{F}$ such that, for all $l \in \mathbb{N}^*$, $\varphi_l \in \otimes_a^l L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$ identifies to a function in the Schwartz space $\mathcal{S}(\mathbb{R}^{3l}; \mathbb{C}^{2^l})$ and $\varphi_l = 0$ for all but finitely many l's. We recall that, in the sense of quadratic forms on $\mathscr{F}_{\text{fin}}(\mathcal{S}) \times \mathscr{F}_{\text{fin}}(\mathcal{S})$, $H_{f,i}$ is given by Equation (4.1), where $b_i^*(\xi_i)$, respectively $b_i(\xi_i)$, stands for the fermionic creation operator-valued distribution, respectively annihilation operator-valued distribution, acting on the i^{th} Fock space. The following anti-commutation relations are supposed to hold:

$$\begin{aligned}
&\{b_i(\xi_i), b_i^*(\xi_i')\} = \delta(\xi_i - \xi_i'), \\
&\{b_i(\xi_i), b_i(\xi_i')\} = \{b_i^*(\xi_i), b_i^*(\xi_i')\} = 0, \\
&\{b_i^\sharp(\xi_i), b_j^\sharp(\xi_j)\} = 0, \quad i < j,
\end{aligned}$$

for all $i, j \in \{1, \dots, n\}$. Note that the third equation means that, following the convention described in [92, Sections 4.1 and 4.2], we suppose that creation or annihilation operators associated to different species of particles anti-commute. Our analysis can be straightforwardly modified if one adopts the convention that they commute instead.

In what follows, we use the notation

$$\int f(\xi_i) d\xi_i = \sum_{\lambda_i \in \{-\frac{1}{2}, \frac{1}{2}\}} \int_{\mathbb{R}^3} f(k_i, \lambda_i) dk_i.$$

The interaction Hamiltonian is given by

$$H_I := \sum_{p=0}^{n} \sum_{\{i_1,\dots,i_n\}\in\mathfrak{I}_p} \big(H^{(p)}_{I,i_1,\dots,i_n}(G^{(p)}_{i_1,\dots,i_n}) + \text{h.c.} \big), \tag{4.4}$$

where we have set

$$\mathfrak{I}_p := \big\{ \{i_1,\dots,i_n\} = \{1,\dots,n\},\, i_1 < \dots < i_p,\, i_{p+1} < \dots < i_n,\, i_1 < i_{p+1} \big\}, \tag{4.5}$$

and

$$H^{(p)}_{I,i_1,\dots,i_n}(G^{(p)}_{i_1,\dots,i_n}) := \int G^{(p)}_{i_1,\dots,i_n}(\xi_1,\dots,\xi_n) b^*_{i_1}(\xi_{i_1}) \dots b^*_{i_p}(\xi_{i_p}) b_{i_{p+1}}(\xi_{i_{p+1}}) \dots b_{i_n}(\xi_{i_n}) d\xi_1 \dots d\xi_n.$$

If $p = 0$ in (4.5), it should be understood that the conditions $i_1 < \dots < i_p$ and $i_1 < i_{p+1}$ are empty, and likewise if $p = 1$, $p = n-1$ or $p = n$. As mentioned above, physically, the expression of $H^{(p)}_{I,i_1,\dots,i_n}(G^{(p)}_{i_1,\dots,i_n})$ represents an interaction process where a particle of each species labeled by $i_1,\dots,i_p$ is created, while a particle of each species labeled by $i_{p+1},\dots,i_n$ is annihilated. Summing over $\mathfrak{I}_p$ insures that all possible creation or annihilation processes are considered (in particular, the condition $i_1 < i_{p+1}$ appearing in the definition of $\mathfrak{I}_p$ ensures that each process is not considered twice due to the presence of the hermitian conjugate term).

Assuming that G is square integrable, it is not difficult to verify (see [81, Theorem X.44]) that $H^{(p)}_{I,i_1,\dots,i_n}(G^{(p)}_{i_1,\dots,i_n})$ defines a quadratic form on

$$\Big(\hat{\otimes}_{i=1}^n \mathscr{F}_{\text{fin}}(\mathcal{S})\Big) \times \Big(\hat{\otimes}_{i=1}^n \mathscr{F}_{\text{fin}}(\mathcal{S})\Big),$$

where $\hat{\otimes}$ stands for the algebraic tensor product.

Formally, the total Hamiltonian that we will study is given by

$$H = H_f + H_I. \tag{4.6}$$

It is a well-defined quadratic form on $\big(\hat{\otimes}_{i=1}^n \mathscr{F}_{\text{fin}}(\mathcal{S})\big) \times \big(\hat{\otimes}_{i=1}^n \mathscr{F}_{\text{fin}}(\mathcal{S})\big)$.

4.1.2 Results

Before stating our main results, we introduce some notations. Let

$$[\![n_1,n_2]\!] := \{n_1, n_1+1, \dots, n_2\}, \qquad [\![n_1,n_2]\!]_{i_0} := \{n_1, n_1+1, \dots, n_2\} \backslash \{i_0\},$$

for any integers $n_1 < n_2$ and i_0. We distinguish massive and massless particles,

$$[\![1,n]\!] = [\![1,n]\!]^{>} \cup [\![1,n]\!]^{0},$$

with

$$[\![1,n]\!]^{>} := \{i \in [\![1,n]\!], m_i > 0\}, \quad [\![1,n]\!]^{0} := \{i \in [\![1,n]\!], m_i = 0\}.$$

We set in addition

$$[\![1,n]\!]^{>}_{i_0} := [\![1,n]\!]^{>} \backslash \{i_0\} \quad \text{if } i_0 \in [\![1,n]\!]^{>}, \qquad [\![1,n]\!]^{>}_{i_0} := [\![1,n]\!]^{>} \quad \text{if } i_0 \notin [\![1,n]\!]^{>},$$

and likewise for $[\![1,n]\!]^{0}_{i_0}$.

Given operators $A_1, \ldots, A_n$, we adopt the convention that

$$\prod_{i \in [\![1,n]\!]} A_i u := A_1 \cdots A_n u,$$

for any $u \in \mathcal{D}(A_1 \cdots A_n) = \{v \in \mathcal{D}(A_n), A_n v \in \mathcal{D}(A_1 \cdots A_{n-1})\}$, where $\mathcal{D}(A)$ stands for the domain of an operator A.

Recalling the notation $\xi = (k, \lambda) \in \mathbb{R}^3 \times \{-1/2, 1/2\}$, we let

$$\mathrm{h}_i^j := -\frac{d^2}{dk_{i,j}^2} + k_{i,j}^2 \tag{4.7}$$

be the harmonic oscillator in one-dimension corresponding to the variable $k_{i,j}$, with

$$k_i = (k_{i,1}, k_{i,2}, k_{i,3}).$$

The corresponding operator acting on the variable $k_{i,j}$ in $\otimes_{i=1}^n L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$ is denoted by the same symbol. In other words, for $G \in \otimes_{i=1}^n L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$,

$$(\mathrm{h}_i^j G)(\xi_1, \ldots, \xi_n) = -\frac{\partial^2 G}{\partial k_{i,j}^2}(\xi_1, \ldots, \xi_n) + k_{i,j}^2 G(\xi_1, \ldots, \xi_n),$$

with $\xi_i = (k_{i,1}, k_{i,2}, k_{i,3}, \lambda_i)$.

Our main results are summarized in the following two theorems. The first one shows that H identifies to a self-adjoint operator on $\mathscr{H}$. The second one establishes the existence of a ground state for H, under stronger assumptions on the kernels $G^{(p)}_{i_1,\ldots,i_n}$.

Theorem 4.1 (Self-adjointness)**.** *Let $i_0 \in [\![1, n]\!]$ and $\varepsilon > 0$. Suppose that, for all $p \in [\![0, n]\!]$ and all set of integers $\{i_1, \ldots, i_n\} \in \mathfrak{I}_p$,*

$$G^{(p)}_{i_1,\ldots,i_n} \in \mathcal{D}\Big(\Big(\prod_{\substack{i \in [\![1,n]\!]^{>}_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2} - \frac{1}{n-1} + \varepsilon}\Big)\Big(\prod_{\substack{i \in [\![1,n]\!]^{0}_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2} - \frac{5}{6}\frac{1}{n-1} + \varepsilon}\Big)\Big).$$

Then the quadratic form H defined in (4.6) *extends to a self-adjoint operator on $\mathscr{H}$ with domain*

$$\mathcal{D}(H) = \mathcal{D}(H_f).$$

Moreover, H is semi-bounded from below and any core for H_f is a core for H.

For any $\Lambda > 0$, we set $B_\Lambda := \{\xi = (k, \lambda) \in \mathbb{R}^3 \times \{-1/2, 1/2\}, |k| \leqslant \Lambda\}$.

Theorem 4.2 (Existence of a ground state)**.** *Under the conditions of Theorem 4.1, suppose in addition that there exists $1 \leqslant r < 2$ such that, for all $i' \in [\![1, n]\!]^0$, p and $i_1, \ldots, i_n$ as in the statement of Theorem 4.1, $1 \leqslant r < 2$ and $\Lambda > 0$,*

$$\int_{B_\Lambda} |k_{i'}|^{-2r} \Big\| \big(\mathrm{S} G^{(p)}_{i_1,i_2,\ldots,i_n}\big)(\cdots, \xi_{i'}, \cdots) \Big\|_2^r d\xi_{i'} < \infty, \tag{4.8}$$

and

$$\int_{B_\Lambda} |k_{i'}|^{-r} \Big\| \big(\mathrm{S}(\nabla_{k_{i'}} G^{(p)}_{i_1,i_2,\ldots,i_n})\big)(\cdots, \xi_{i'}, \cdots) \Big\|_2^r d\xi_{i'} < \infty, \tag{4.9}$$

where S *stands for the operator*

$$\mathrm{S} := \Big(\prod_{\substack{i \in [\![1,n]\!]^{>}_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2} - \frac{1}{n-1} + \varepsilon} \Big) \Big(\prod_{\substack{i \in [\![1,n]\!]^{0}_{i_0} \setminus \{i'\} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2} - \frac{5}{6}\frac{1}{n-1} + \varepsilon} \Big),$$

on $L^2((\mathbb{R}^3 \times \{-1/2, 1/2\})^n)$. *Then* H *has a ground state, i.e.,* $E := \inf \sigma(H)$ *is an eigenvalue of* H.

We emphasize that our results hold without any restriction on the strength of the interaction. This means that, if one introduces a coupling constant g into the model and study the Hamiltonian $H_g = H_f + gH_I$, then H_g is self-adjoint and has a ground state *for all* values of $g \in \mathbb{R}$. In the next section, we give concrete examples insuring that the conditions on the kernels $G^{(p)}_{i_1,i_2,\dots,i_n}$ stated in the previous theorems are satisfied.

4.1.3 Applications to mathematical models of the weak interactions

As mentioned above, the main examples we have in mind of physical models of the form studied in this chapter come from the Fermi theory of weak interactions.

Fermions involved in the Lagrangian of the Standard Model are quarks and leptons. Each fermion is a Dirac particle with spin 1/2 and is distinct from its antiparticle. Recall that there are six quarks, up(u), down(d), strange(s), charm(c), bottom(b), top(t) and six leptons e_-, ν_e, μ_-, ν_μ, τ_-, ν_τ where ν_e (respectively ν_μ, ν_τ) is the electron neutrino (respectively the muon neutrino, the tau neutrino).

We are mainly interested in the quark and lepton Lagrangian, which arises by taking the normal Dirac kinetic energy and replacing the ordinary derivative by the covariant one. See [92, 93], [59] and [72, Section 6.2, (6.11)]. This Lagrangian is used to calculate quark and lepton interactions. The full Lagrangian of the fermions is the sum of the lepton electroweak Lagrangian and the quark QCD Lagrangian. It is a finite sum of terms involving two fermions with spin 1/2 together with gauge bosons. See [72]. Hamiltonian models of the weak interaction involving two fermion fields and one boson field have been studied in the three previous chapters and in [10, 17, 18, 19].

Well known physical examples of interacting fermion fields with spin 1/2 are given by the Fermi Theory of weak interactions with $V-A$ (Vector-Axial vector) coupling. In these examples, the Fermi Theory is associated to an *effective* low-energy electroweak Lagrangian obtained by the contraction of the propagators of the gauge bosons $W^\pm$ and Z in low energy electroweak processes. See e.g. [59, Chapter 5]. For instance, the weak decay of a muon into an electron, a muon neutrino and an electron antineutrino,

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e,$$

the scattering of an electron with an electron neutrino,

$$e^- + \nu_e \to \nu_e + e^-,$$

or the muon production in the scattering of an electron with a muon neutrino,

$$e^- + \nu_\mu \to \mu^- + \nu_e,$$

are well-described at low energy by the Fermi Theory of weak interactions. See [55, Chapter 2].

Another fundamental example is the β decay, i.e., the weak decay of the neutron. In the Fermi model, the β decay is the weak decay of a neutron into a proton, an electron and an electron antineutrino,

$$n \to p + e^- + \bar{\nu}_e.$$

Note that protons and neutrons are baryons, i.e., composite particles. Baryons are composed of three quarks: A neutron is composed of an up quark (u) and two down quarks (d), a proton is composed of two up quarks (u) and one down quark (d). Protons and neutrons may be approximately regarded as bound states of three quarks. In the quark model, the β decay is the weak decay of a down quark (d) into a up quark (u), an electron and an electron antineutrino,

$$d \to u + e^- + \bar{\nu}_e.$$

In this model also, four fermions of spin 1/2 interact.

Recently, the decays of mesons B into mesons D have also been experimentally studied. Mesons are bosons composed of a quark and an antiquark. A meson B is composed of a bottom antiquark ($\bar{b}$) and a quark. A meson D is composed of a charm quark (c) and an antiquark. The following decays have been observed (see [78, 71, 24]):

$$B^- \to D^0 + l^- + \bar{\nu}_l, \qquad \bar{B}^0 \to D^+ + l^- + \bar{\nu}_l, \tag{4.10}$$

and

$$B^+ \to \bar{D}^0 + l^+ + \nu_l. \tag{4.11}$$

Here $B^+ = (u\bar{b})$, $B^- = (\bar{u}b)$, $\bar{B}^0 = (\bar{d}b)$, $D^0 = (c\bar{u})$, $D^+ = (c\bar{d})$, $\bar{D}^0 = (u\bar{c})$, l^- (respectively l^+) is a lepton of negative electric charge (respectively positive electric charge) and ν_l, $\bar{\nu}_l$ are lepton neutrino and antineutrino. The decays (4.10) correspond to the transition of a quark b into a quark c,

$$b \to c + l^- + \bar{\nu}_l. \tag{4.12}$$

The decay (4.11) is the transition of an antiquark $\bar{b}$ to an antiquark $\bar{c}$,

$$\bar{b} \to \bar{c} + l^+ + \nu_l. \tag{4.13}$$

The decay (4.13) is the charge conjugation of the decay (4.12). Both involve four fermions of spin 1/2.

For an example of computation about the six-fermion process $b \to dq\bar{q}l^+l^-$, with $q \in (u, d, s)$, see e.g. [66]. More generally, all physical processes we have in mind involve an even number of fermions. Nevertheless, for the sake of mathematical generality, we will consider in this chapter an arbitrary number n of fermions, with n either even or odd. Some modifications of the proof in the odd case will be required.

In all the previous examples involving four Dirac particles, the formal Hamiltonian obtained from the corresponding Lagrangian is of the form (4.6), with one or two massless fields (assuming that neutrinos are treated as massless, in accordance with the classical form of the Standard Model). After introduction of a (smooth) high-energy cut-off of parameter Λ and a spatial cut-off, the kernels obtained from physics can be supposed to be of the form

$$G(\xi_1, \ldots, \xi_4) = f_1(k_1) \cdots f_4(k_4) \delta_{\mathrm{reg}}(k_1, \ldots, k_4),$$

as suggested in the previous chapters (see also [17, 18]) and where we disregarded the dependence on the spin variables for simplicity, $\delta_{\mathrm{reg}}(k_1, \ldots, k_4)$ is a regularization of the Dirac distributions appearing due to momentum conservation, and the f_i are C^∞ in $\mathbb{R}^3 \backslash \{0\}$, satisfying the estimates

$$|\partial^\alpha_{k_{i,j}} f_i(k_i)| \leqslant C_\alpha |k_i|^{\nu_i - \alpha} \mathbb{1}_{\cdot \leqslant \Lambda}(|k_i|), \quad i = 1, \ldots, 4, \quad j = 1, \ldots, 3, \quad \alpha \in \mathbb{N}. \tag{4.14}$$

Physically, we have that $\nu_i = 0$ for any i.

For $n = 4$, Theorem 4.1 shows that if all interaction form factors G satisfy

$$G \in \mathcal{D}\Big(\Big(\prod_{\substack{i\in[\![1,n]\!]^{>}_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{6}+\varepsilon}\Big)\Big(\prod_{\substack{i\in[\![1,n]\!]^{0}_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{2}{9}+\varepsilon}\Big)\Big), \tag{4.15}$$

for some $\varepsilon > 0$, then H is self-adjoint. Due to the presence of smooth ultraviolet cut-offs, and assuming that $\delta_{\mathrm{reg}}(k_1,\dots,k_n)$ is also smooth, the condition (4.15) is implied by the requirement that, for massive particles, each function f_i belongs to the Sobolev space $H^{1+\varepsilon}(\mathbb{R}^3)$ for some $\varepsilon > 0$ (except possibly the one labelled by i_0 that must only belong to $L^2(\mathbb{R}^3)$), while for massless particles, each function f_i belongs to $H^{4/3+\varepsilon}(\mathbb{R}^3)$ (again, except possibly the one labelled by i_0). Hence we need $\nu_i > -1/2$ in (4.14) for massive particles (i.e. $i \in [\![1,n]\!]^{>}_{i_0}$), and $\nu_i > -1/6$ for massless particles (i.e. $i \in [\![1,n]\!]^{0}_{i_0}$), and $\nu_{i_0} > -3/2$. In particular, the physical case $\nu_i = 0$ is covered by our assumptions.

As for the conditions (4.8) and (4.9) in Theorem 4.2, they concern only the massless fields. One can check that they are satisfied provided that $\nu_i \geqslant 1/2$ in (4.14), for any $i \in [\![1,n]\!]^0$. Hence, to prove the existence of a ground state, if massless fields are involved, we need to impose an infrared regularization compared to the physical case. This is due to the method employed [56] whose advantage is to allow us to establish the existence of a ground state without any restriction on the strength of the interaction. If one introduces a coupling parameter g into the model and use perturbative methods [13, 79], it is likely that one can rely on our refined N_τ estimates to prove the existence of a ground state for $H = H_f + gH_I$ for small enough values of g, without imposing any infrared regularization.

4.2 Self-adjointness

In this section we prove that the total Hamiltonian H identifies to a self-adjoint operator, i.e., we prove Theorem 4.1. The strategy consists in establishing relative bounds of H_I with respect to H_f. We begin with relative bounds in the sense of forms, next we turn to operator bounds.

In the following two subsections, we concentrate on a particular term of the interaction Hamiltonian H_I (see (4.4)) that, for simplicity, we write as

$$H_I(G) = \int G(\xi_1,\dots,\xi_n) b_1^*(\xi_1)\dots b_p^*(\xi_p) b_{p+1}(\xi_{p+1})\dots b_n(\xi_n) d\xi_1\dots d\xi_n, \tag{4.16}$$

for some $0 \leqslant p \leqslant n$.

4.2.1 Form bounds

As will be recalled more precisely in the proof of the next lemma, the usual N_τ estimates of Glimm and Jaffe [51] show that $H_I(G)$ is relatively form-bounded with respect to $H_f^{n/2}$. Our first aim is to find suitable conditions on the kernel G such that H_I is relatively form-bounded with respect to lower powers of H_f.

We recall that, for any function $f \in L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$,

$$\|b^\sharp(f)\| = \|f\|_2, \tag{4.17}$$

with the usual notations

$$b^*(f) = \int f(\xi)b^*(\xi)d\xi = \sum_{\lambda\in\{-\frac{1}{2},\frac{1}{2}\}} \int_{\mathbb{R}^3} f(k,\lambda)b^*_\lambda(k)dk,$$
$$b(f) = \int \bar{f}(\xi)b(\xi)d\xi = \sum_{\lambda\in\{-\frac{1}{2},\frac{1}{2}\}} \int_{\mathbb{R}^3} \bar{f}(k,\lambda)b_\lambda(k)dk.$$

Here $b^\sharp(f)$, $b^\sharp_\lambda(f)$ denote the usual fermionic creation and annihilation operators in $\mathscr{F}$. For $g \in L^2(\mathbb{R}^3)$ and $\lambda \in \{-1/2, 1/2\}$, the notation $b^\sharp_\lambda(g)$ stands for

$$b^*_\lambda(g) = \int_{\mathbb{R}^3} g(k)b^*_\lambda(k)dk, \qquad b_\lambda(g) = \int_{\mathbb{R}^3} \bar{g}(k)b_\lambda(k)dk,$$

and hence $\|b^\sharp_\lambda(g)\| = \|g\|_2$.

We begin with a lemma which is close to Proposition 1.2.3 (b) in [51]. We give a short proof for the convenience of the reader.

Lemma 4.2.1. *For all* $i_0 \in [\![1,n]\!]$, $G \in \mathcal{D}(\prod_{i\in[\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}})$ *and* $\varphi \in \mathcal{D}((\sum_{i\in[\![1,n]\!]_{i_0}} H_{f,i})^{(n-1)/2})$, *we have that*

$$\big|\langle\varphi, H_I(G)\varphi\rangle\big| \leqslant \Big\| \prod_{i\in[\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}} G\Big\|_2 \Big\|\Big(\sum_{i\in[\![1,n]\!]_{i_0}} H_{f,i} + 1\Big)^{\frac{n-1}{2}} \varphi\Big\|^2. \tag{4.18}$$

Proof. Assume that $i_0 \in [\![1,p]\!]$. We write

$$\begin{aligned}&\big|\langle\varphi, H_I(G)\varphi\rangle\big| \\ &= \Big| \int \Big\langle \prod_{i\in[\![1,p]\!]_{i_0}} b_i(\xi_i)\varphi, \Big(\int G(\xi_1,\dots\xi_n)b^*_{i_0}(\xi_{i_0})d\xi_{i_0}\Big) \prod_{i\in[\![p+1,n]\!]} b_i(\xi_i)\varphi\Big\rangle \prod_{i\in[\![1,n]\!]_{i_0}} d\xi_i\Big|.\end{aligned}$$

Using the Cauchy-Schwarz inequality and the fact that, for a.e. ξ_i, $i \in [\![1,n]\!]_{i_0}$,

$$\Big\|\int G(\xi_1,\dots\xi_n)b^*_{i_0}(\xi_{i_0})d\xi_{i_0}\Big\| = \|G(\xi_1,\dots,\xi_{i_0-1},\cdot,\xi_{i_0+1},\dots,\xi_n)\|_2,$$

(see (4.17)), we obtain that

$$\begin{aligned}&\big|\langle\varphi, H_I(G)\varphi\rangle\big| \\ &\leqslant \int \Big\|\prod_{i\in[\![1,p]\!]_{i_0}} b_i(\xi_i)\varphi\Big\| \|G(\xi_1,\dots,\xi_{i_0-1},\cdot,\xi_{i_0+1},\dots,\xi_n)\|_2 \Big\|\prod_{i\in[\![p+1,n]\!]} b_i(\xi_i)\varphi\Big\| \prod_{i\in[\![1,n]\!]_{i_0}} d\xi_i.\end{aligned} \tag{4.19}$$

Now, we observe that

$$\begin{aligned}\int \prod_{i\in[\![1,p]\!]_{i_0}} \omega_i(k_i)\Big\|\prod_{i\in[\![1,p]\!]_{i_0}} b_i(\xi_i)\varphi\Big\|^2 \prod_{i\in[\![1,p]\!]_{i_0}} d\xi_i &= \Big\langle\varphi, \prod_{i\in[\![1,p]\!]_{i_0}} H_{f,i}\varphi\Big\rangle \\ &\leqslant \Big\langle\varphi, \Big(\sum_{i\in[\![1,p]\!]_{i_0}} H_{f,i}\Big)^{p-1}\varphi\Big\rangle,\end{aligned}$$

and likewise,

$$\int \prod_{i\in[\![p+1,n]\!]} \omega_i(k_i) \Big\| \prod_{i\in[\![p+1,n]\!]} b_i(\xi_i)\varphi \Big\|^2 \prod_{i\in[\![p+1,n]\!]} d\xi_i \leqslant \Big\langle \varphi, \Big(\sum_{i\in[\![p+1,n]\!]} H_{f,i} \Big)^{n-p} \varphi \Big\rangle.$$

Combining this with (4.19) and the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} &\big|\langle \varphi, H_I(G)\varphi\rangle\big| \\ &\leqslant \Big\| \prod_{i\in[\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}} G \Big\|_2 \Big\langle \varphi, \Big(\sum_{i\in[\![1,p]\!]_{i_0}} H_{f,i} \Big)^{p-1} \varphi \Big\rangle^{\frac{1}{2}} \Big\langle \varphi, \Big(\sum_{i\in[\![p+1,n]\!]} H_{f,i} \Big)^{n-p} \varphi \Big\rangle^{\frac{1}{2}}. \end{aligned} \tag{4.20}$$

The estimate (4.18) follows directly from (4.20). The argument is similar in the case where $i_0 \in [\![p+1, n]\!]$. □

Remark 4.2.2. *The previous proof shows that the following more precise estimate holds:*

$$\big|\langle \varphi, H_I(G)\psi\rangle\big| \leqslant \Big\| \prod_{i\in[\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}} G \Big\|_2 \Big\| \prod_{i\in[\![1,p]\!]_{i_0}} H_{f,i}^{\frac{1}{2}} \varphi \Big\| \Big\| \prod_{i\in[\![p+1,n]\!]_{i_0}} H_{f,i}^{\frac{1}{2}} \psi \Big\|.$$

This refined estimate will be useful in the next section.

Remark 4.2.3. *Note that the previous result is similar to Lemma 2.1.3.*

Next, we prove another lemma which, in our setting, is a slight improvement of Proposition 1.2.3 (c) in [51] (see also [5]). The idea is that, if G is sufficiently regular (i.e. belongs to a suitable Schwartz space), then $H_I(G)$ extends to a bounded quadratic form. Our improvement compared to [51] consists in showing that regularity in all variables *but one* is sufficient to obtain boundedness of $H_I(G)$. This will be important in applications.

Recall that the one-dimensional harmonic oscillator h_i^j has been defined in (4.7). A basis of normalized eigenvectors of h_i^j is denoted by $(e_l)_{l\in\mathbb{N}}$, so that $\mathrm{h}_i^j e_l = (2l+1)e_l$.

Lemma 4.2.4. *For all $s > 1/2$, there exists $\mathrm{C}_s > 0$ such that, for all $i_0 \in [\![1, n]\!]$, $G \in \mathcal{D}(\prod_{\substack{i\in[\![1,n]\!]_{i_0} \\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^s)$ and $\varphi \in \mathscr{H}$,*

$$\big|\langle \varphi, H_I(G)\varphi\rangle\big| \leqslant \mathrm{C}_s \Big\| \prod_{\substack{i\in[\![1,n]\!]_{i_0} \\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^s G \Big\|_2 \|\varphi\|^2. \tag{4.21}$$

Proof. Assume that $i_0 \in [\![1, p]\!]$. We have that

$$\begin{aligned} &\big|\langle \varphi, H_I(G)\varphi\rangle\big| \\ &= \Big| \int \Big\langle \prod_{i\in[\![1,p]\!]} b_i(\xi_i)\varphi, G(\xi_1,\dots,\xi_n) \prod_{i\in[\![p+1,n]\!]} b_i(\xi_i)\varphi \Big\rangle \prod_{i\in[\![1,n]\!]} d\xi_i \Big| \\ &= \Big| \sum_{\substack{\lambda_i\in\{-\frac{1}{2},\frac{1}{2}\}: \\ i\in[\![1,n]\!]_{i_0}}} \int \Big\langle \varphi, \sum_{\substack{l_i^j\in\mathbb{N}: \\ i\in[\![1,n]\!]_{i_0}, j\in[\![1,3]\!]}} \big\langle e_{l_1^1} \otimes \cdots \otimes e_{l_i^j} \otimes \cdots \otimes e_{l_n^3}, G\big((\cdot,\lambda_1),\cdots,\xi_{i_0},\cdots,(\cdot,\lambda_n)\big)\big\rangle_2 \\ &\qquad b^*(\xi_{i_0}) \prod_{i\in[\![1,p]\!]_{i_0}} b^*_{i,\lambda_i}(\otimes_{j=1}^3 e_{l_i^j}) \prod_{i\in[\![p+1,n]\!]} b_{i,\lambda_i}(\otimes_{j=1}^3 e_{l_i^j})\varphi \Big\rangle d\xi_{i_0} \Big|, \end{aligned}$$

where the subscript in the first sum above means that for each $i \in [\![1,n]\!]_{i_0}$, we sum over $\lambda_i \in \{-1/2, 1/2\}$, and likewise, in the second sum, for each couple $(i,j) \in [\![1,n]\!]_{i_0} \times [\![1,3]\!]$, we sum over $l_i^j \in \mathbb{N}$. The scalar product $\langle \cdot, \cdot \rangle_2$ appearing in the right side of the last equality stands for the scalar product in $L^2(\mathbb{R}^{3(n-1)})$.

Using that h_i^j is self-adjoint and that $\mathrm{h}_i^j e_{l_1^1} \otimes \cdots \otimes e_{l_n^3} = (2l_i^j + 1) e_{l_1^1} \otimes \cdots \otimes e_{l_n^3}$ for all i, j, we obtain that

$$
\begin{aligned}
\big|\langle \varphi, H_I(G)\varphi \rangle\big| = \Big| \sum_{\substack{\lambda_i \in \{-\frac12, \frac12\}: \\ i \in [\![1,n]\!]_{i_0}}} \int \Big\langle \varphi, & \sum_{\substack{l_i^j \in \mathbb{N}: \\ i \in [\![1,n]\!]_{i_0}, j \in [\![1,3]\!]}} \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} \frac{1}{(2l_i^j+1)^s} \big\langle e_{l_1^1} \otimes \cdots \otimes e_{l_n^3}, \\
& \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^s G\big((\cdot, \lambda_1), \cdots, \xi_{i_0}, \cdots, (\cdot, \lambda_n)\big) \big\rangle_2 \\
& b^*(\xi_{i_0}) \prod_{i \in [\![1,p]\!]_{i_0}} b^*_{i,\lambda_i}(\otimes_{j=1}^3 e_{l_i^j}) \prod_{i \in [\![p+1,n]\!]} b_{i,\lambda_i}(\otimes_{j=1}^3 e_{l_i^j}) \varphi \Big\rangle d\xi_{i_0} \Big| \\
\leqslant \sum_{\substack{\lambda_i \in \{-\frac12, \frac12\}: \\ i \in [\![1,n]\!]_{i_0}}} \sum_{\substack{l_i^j \in \mathbb{N}: \\ i \in [\![1,n]\!]_{i_0}, j \in [\![1,3]\!]}} \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} & \frac{1}{(2l_i^j+1)^s} \Big| \Big\langle \varphi, \Big(\int \big\langle e_{l_1^1} \otimes \cdots \otimes e_{l_n^3}, \\
& \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^s G\big((\cdot, \lambda_1), \cdots, \xi_{i_0}, \cdots, (\cdot, \lambda_n)\big) \big\rangle_2 b^*(\xi_{i_0}) d\xi_{i_0} \Big) \\
& \prod_{i \in [\![1,p]\!]_{i_0}} b^*_{i,\lambda_i}(\otimes_{j=1}^3 e_{l_i^j}) \prod_{i \in [\![p+1,n]\!]} b_{i,\lambda_i}(\otimes_{j=1}^3 e_{l_i^j}) \varphi \Big\rangle \Big|.
\end{aligned}
$$

Next, by (4.17), we see that the operator into parentheses in the last equation is bounded and satisfies

$$
\begin{aligned}
& \Big\| \int \big\langle e_{l_1^1} \otimes \cdots \otimes e_{l_n^3}, \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^s G\big((\cdot, \lambda_1), \cdots, \xi_{i_0}, \cdots, (\cdot, \lambda_n)\big) \big\rangle_{L^2(\mathbb{R}^{3(n-1)})} b^*(\xi_{i_0}) d\xi_{i_0} \Big\| \\
& = \Big(\int \big| \big\langle e_{l_1^1} \otimes \cdots \otimes e_{l_n^3}, \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^s G\big((\cdot, \lambda_1), \cdots, \xi_{i_0}, \cdots, (\cdot, \lambda_n)\big) \big\rangle_2 \big|^2 d\xi_{i_0} \Big)^{\frac12}.
\end{aligned}
$$

Combining this with the fact that $\|b^\sharp_{i,\lambda_i}(\otimes_{j=1}^3 e_{l_i^j})\| = \| \otimes_{j=1}^3 e_{l_i^j} \|_2 = 1$, we deduce that

$$
\begin{aligned}
\big|\langle \varphi, H_I(G)\varphi \rangle\big| \leqslant \|\varphi\|^2 & \sum_{\substack{\lambda_i \in \{-\frac12, \frac12\}: \\ i \in [\![1,n]\!]_{i_0}}} \sum_{\substack{l_i^j \in \mathbb{N}: \\ i \in [\![1,n]\!]_{i_0}, j \in [\![1,3]\!]}} \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} \frac{1}{(2l_i^j+1)^s} \\
& \Big(\int \big| \big\langle e_{l_1^1} \otimes \cdots \otimes e_{l_n^3}, \prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^s G\big((\cdot, \lambda_1), \cdots, \xi_{i_0}, \cdots, (\cdot, \lambda_n)\big) \big\rangle_2 \big|^2 d\xi_{i_0} \Big)^{\frac12}.
\end{aligned}
$$

Applying again the Cauchy-Schwarz inequality, using that $\sum_{l_i^j \in \mathbb{N}} (2l_i^j + 1)^{-2s} < \infty$ since $s > 1/2$, and that the sum over the λ_i's is finite, we obtain

$$\begin{aligned}\big|\langle\varphi, H_I(G)\varphi\rangle\big| &\leqslant \mathrm{C}_s\|\varphi\|^2\Big(\sum_{\substack{\lambda_i\in\{-\frac12,\frac12\}:\\ i\in[\![1,n]\!]_{i_0}}}\ \sum_{\substack{l_i^j\in\mathbb{N}:\\ i\in[\![1,n]\!]_{i_0},j\in[\![1,3]\!]}}\int\Big|\Big\langle e_{l_1^1}\otimes\cdots\otimes e_{l_n^3},\\ &\qquad\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^s G\big((\cdot,\lambda_1),\cdots,\xi_{i_0},\cdots,(\cdot,\lambda_n)\big)\Big\rangle_2\Big|^2 d\xi_{i_0}\Big)^{\frac12}\\ &= \mathrm{C}_s\|\varphi\|^2\Big\|\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^s G\Big\|_2,\end{aligned}$$

for some positive constant C_s. This concludes the proof. □

Remark 4.2.5. *This lemma together with its proof may be seen as a generalisation of Lemma 2.1.4.*

Now we interpolate the estimates given by Lemmas 4.2.1 and 4.2.4. This gives the following proposition.

Proposition 4.2.6. *Let $s > 1/2$ and $0 \leqslant \theta \leqslant 1$. There exists a positive constant $\mathrm{C}_{s,\theta}$ such that, for all $i_0 \in [\![1, n]\!]$,*

$$G \in \mathcal{D}\Big(\prod_{\substack{i\in[\![1,n]\!]^>_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\theta s}\prod_{\substack{i\in[\![1,n]\!]^0_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{12}+\theta(s-\frac{1}{12})}\Big),$$

and $\varphi \in \mathcal{D}((\sum_{i\in[\![1,n]\!]_{i_0}} H_{f,i})^{\frac{n-1}{2}(1-\theta)})$, we have that

$$\begin{aligned}&\big|\langle\varphi, H_I(G)\varphi\rangle\big|\\ &\leqslant \mathrm{C}_{s,\theta}\Big\|\Big(\prod_{\substack{i\in[\![1,n]\!]^>_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\theta s}\Big)\Big(\prod_{\substack{i\in[\![1,n]\!]^0_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{12}+\theta(s-\frac{1}{12})}\Big)G\Big\|_2\Big\|\Big(\sum_{i\in[\![1,n]\!]_{i_0}}H_{f,i}+1\Big)^{\frac{n-1}{2}(1-\theta)}\varphi\Big\|^2.\end{aligned}\tag{4.22}$$

Proof. To interpolate the estimates given by Lemmas 4.2.1 and 4.2.4, we rewrite (4.18) in a weaker version that will be more convenient. For massive particles, $i \in [\![1, n]\!]^>$, we have that $\|\omega_i(k_i)^{-\frac12}u\|_2 \lesssim \|u\|_2$, while in the case of massless particles, $i \in [\![1, n]\!]^0$, we write

$$\begin{aligned}\|\omega_i(k_i)^{-\frac12}u\|_2 &= \|(k_{i,1}^2+k_{i,2}^2+k_{i,3}^2)^{-\frac14}u\|_2\\ &\lesssim \||k_{i,1}|^{-\frac16}|k_{i,2}|^{-\frac16}|k_{i,3}|^{-\frac16}u\|_2\\ &\lesssim \|(|\partial_{k_{i,1}}|^{\frac16}|\partial_{k_{i,2}}|^{\frac16}|\partial_{k_{i,3}}|^{\frac16})u\|_2\\ &\lesssim \Big\|\prod_{j\in[\![1,3]\!]}(\mathrm{h}_i^j)^{\frac{1}{12}}u\Big\|_2, \quad i\in[\![1,n]\!]^0.\end{aligned}$$

In the first inequality, we used that $abc \lesssim a^3 + b^3 + c^3$ for any positive numbers a, b, c, and in the second inequality we used Hardy's inequality in $\mathbb{R}$. Hence (4.18) implies that

$$\big|\langle\varphi, H_I(G)\varphi\rangle\big| \lesssim \Big\|\prod_{\substack{i\in[\![1,n]\!]^0_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{12}}G\Big\|_2\Big\|\Big(\sum_{i\in[\![1,n]\!]_{i_0}}H_{f,i}+1\Big)^{\frac{n-1}{2}}\varphi\Big\|^2.\tag{4.23}$$

Now we proceed to interpolation. Let $\tilde{G} \in L^2$. For $\tilde{\varphi} \in \mathscr{H}$, consider the map

$$f : z \mapsto \Big\langle \Big(\sum_{i \in [\![1,n]\!]_{i_0}} H_{f,i} + 1 \Big)^{-\frac{n-1}{2}(1-z)} \tilde{\varphi},$$
$$H_I\Big(\Big(\prod_{\substack{i \in [\![1,n]\!]_{i_0}^{>} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{-zs} \Big) \Big(\prod_{\substack{i \in [\![1,n]\!]_{i_0}^{0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{-\frac{1}{12} - z(s - \frac{1}{12})} \Big) \tilde{G} \Big) \Big(\sum_{i \in [\![1,n]\!]_{i_0}} H_{f,i} + 1 \Big)^{-\frac{n-1}{2}(1-z)} \tilde{\varphi} \Big\rangle.$$

Since the operators h_i^j and $\sum H_{f,i} + 1$ are positive and invertible, one verifies that f is analytic in $\{z \in \mathbb{C}, 0 < \mathrm{Re}(z) < 1\}$, and bounded and continuous in $\{z \in \mathbb{C}, 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$. Moreover, Equations (4.18) and (4.21) show that

$$\sup_{\mathrm{Re}(z)=0} |f(z)| \lesssim \|\tilde{G}\|_2 \|\tilde{\varphi}\|^2, \qquad \sup_{\mathrm{Re}(z)=1} |f(z)| \lesssim \|\tilde{G}\|_2 \|\tilde{\varphi}\|^2.$$

Applying Hadamard's three lines lemma, we deduce that

$$\sup_{0 \leqslant \mathrm{Re}(z) \leqslant 1} |f(z)| \lesssim \|\tilde{G}\|_2 \|\tilde{\varphi}\|^2.$$

Taking $\mathrm{Im}(z) = 0$, we obtain that

$$\Big| \Big\langle \Big(\sum_{i \in [\![1,n]\!]_{i_0}} H_{f,i} + 1 \Big)^{-\frac{n-1}{2}(1-\theta)} \tilde{\varphi} \,, H_I\Big(\Big(\prod_{\substack{i \in [\![1,n]\!]_{i_0}^{>} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{-\theta s} \Big) \Big(\prod_{\substack{i \in [\![1,n]\!]_{i_0}^{0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{-\frac{1}{12} - \theta(s - \frac{1}{12})} \Big) \tilde{G} \Big)$$
$$\Big(\sum_{i \in [\![1,n]\!]_{i_0}} H_{f,i} + 1 \Big)^{-\frac{n-1}{2}(1-\theta)} \tilde{\varphi} \Big\rangle \Big| \lesssim \|\tilde{G}\|_2 \|\tilde{\varphi}\|^2,$$

for any $0 \leqslant \theta \leqslant 1$, $\tilde{G} \in L^2$ and $\tilde{\varphi} \in \mathscr{H}$. Applying this to

$$\tilde{G} = \prod_{\substack{i \in [\![1,n]\!]_{i_0}^{>} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\theta s} \prod_{\substack{i \in [\![1,n]\!]_{i_0}^{0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{12} + \theta(s - \frac{1}{12})} G \qquad \text{and} \qquad \tilde{\varphi} = \Big(\sum_{i \in [\![1,n]\!]_{i_0}} H_{f,i} + 1 \Big)^{\frac{n-1}{2}(1-\theta)} \varphi,$$

this implies the statement of the lemma. □

4.2.2 Operator bounds

In this section we improve the results of Section 4.2.1 by establishing relative bounds of $H_I(G)$ with respect to H_f. The first step is to prove the following lemma, using Remark 4.2.2.

Lemma 4.2.7. *For all* $i_0 \in [\![1,n]\!]$, $G \in \mathcal{D}(\prod_{i \in [\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}})$ *and* $\varphi \in \mathcal{D}((\sum_{i \in [\![1,n]\!]_{i_0}} H_{f,i})^{(n-1)/2})$, *we have that*

$$\big\| H_I(G) \varphi \big\| \leqslant \Big\| \prod_{i \in [\![1,n]\!]_{i_0}} (1 + \omega_i(k_i)^{-\frac{1}{2}}) G \Big\|_2 \Big\| \Big(\sum_{i \in [\![1,n]\!]_{i_0}} H_{f,i} + 1 \Big)^{\frac{n-1}{2}} \varphi \Big\|. \tag{4.24}$$

Proof. Suppose that $i_0 \in [\![1,p]\!]$. It follows from Remark 4.2.2 that, for all $G \in \mathcal{D}(\prod_{i \in [\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}})$,

$$\Big(\prod_{i \in [\![1,p]\!]_{i_0}} H_{f,i}^{\frac{1}{2}} \Big)^{-1} H_I(G) \Big(\prod_{i \in [\![p+1,n]\!]} H_{f,i}^{\frac{1}{2}} \Big)^{-1} \leqslant \Big\| \prod_{i \in [\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}} G \Big\|_2. \tag{4.25}$$

In the previous equation and in the remainder of this proof, even if the operators $H_{f,i}$ are not invertible, the notation $H_{f,i}^{-1}$ should be understood as the operator acting on the orthogonal complement of the Fock vacuum and whose restriction to any n-particle subspace of the Fock space is given by the corresponding multiplication operator (hence, for instance, the restriction of $H_{f,i}^{-1}$ to the 1-particle space is the operator of multiplication by $\omega_i(k_i)^{-1}$).

Let $G \in \mathcal{D}(\prod_{i\in[\![1,n]\!]_{i_0}} \omega_i(k_i)^{-\frac{1}{2}})$. We claim that

$$H_I(G)\Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac{1}{2}}\Big)^{-1} \in \mathcal{L}(\mathcal{H}). \tag{4.26}$$

To prove (4.26), we proceed similarly as in the proof of Lemma 4.2.1 and use in addition the pull-through formula. For all $\varphi, \psi \in \mathcal{H}$, we write

$$\begin{aligned}
&\Big|\Big\langle\varphi, H_I(G)\Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac{1}{2}}\Big)^{-1}\psi\Big\rangle\Big| \\
&= \Big|\int\Big\langle \prod_{i\in[\![1,p]\!]_{i_0}} b_i(\xi_i)\varphi, \\
&\qquad \Big(\int G(\xi_1,\dots\xi_n)b_{i_0}^*(\xi_{i_0})d\xi_{i_0}\Big)\prod_{i\in[\![p+1,n]\!]} b_i(\xi_i)\Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac{1}{2}}\Big)^{-1}\varphi\Big\rangle \prod_{i\in[\![1,n]\!]_{i_0}} d\xi_i\Big| \\
&= \Big|\int\Big\langle \prod_{i\in[\![1,p]\!]_{i_0}} b_i(\xi_i)\Big(\prod_{i\in[\![1,p]\!]_{i_0}} H_{f,i}^{\frac{1}{2}}\Big)u, \\
&\qquad \Big(\int G(\xi_1,\dots\xi_n)b_{i_0}^*(\xi_{i_0})d\xi_{i_0}\Big)\prod_{i\in[\![p+1,n]\!]} b_i(\xi_i)v\Big\rangle \prod_{i\in[\![1,n]\!]_{i_0}} d\xi_i\Big|,
\end{aligned}$$

where we have set

$$u := \Big(\prod_{i\in[\![1,p]\!]_{i_0}} H_{f,i}^{\frac{1}{2}}\Big)^{-1}\varphi, \quad v := \Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac{1}{2}}\Big)^{-1}\psi, \tag{4.27}$$

to shorten notations.

Using the pull-through formula $b(\xi_i)f(H_{f,i}) = f(H_{f,i}+\omega_i(k_i))b(\xi_i)$ for any measurable function f (which easily follows from the anti-commutation relation together with the spectral theorem), we obtain that

$$\begin{aligned}
&\Big|\Big\langle\varphi, H_I(G)\Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac{1}{2}}\Big)^{-1}\psi\Big\rangle\Big| \\
&= \Big|\int\Big\langle \prod_{i\in[\![1,p]\!]_{i_0}} b_i(\xi_i)u, \\
&\qquad \Big(\prod_{i\in[\![1,p]\!]_{i_0}} \big(H_{f,i}+\omega_i(k_i)\big)^{\frac{1}{2}}\Big)\Big(\int G(\xi_1,\dots\xi_n)b_{i_0}^*(\xi_{i_0})d\xi_{i_0}\Big)\prod_{i\in[\![p+1,n]\!]} b_i(\xi_i)v\Big\rangle \prod_{i\in[\![1,n]\!]_{i_0}} d\xi_i\Big|.
\end{aligned}$$

Next, we observe that

$$\big\|(\omega_i(k_i)+1)^{-\frac{1}{2}}(H_{f,i}+\omega_i(k_i))^{\frac{1}{2}}(H_{f,i}+1)^{-\frac{1}{2}}\big\| \leqslant 1. \tag{4.28}$$

Applying (4.28), the Cauchy-Schwarz inequality and the fact that, for a.e. ξ_i, $i \in [\![1,n]\!]_{i_0}$,

$$\Big\| \int G(\xi_1,\dots\xi_n) b^*_{i_0}(\xi_{i_0}) d\xi_{i_0} \Big\| = \| G(\xi_1,\dots,\xi_{i_0-1},\cdot,\xi_{i_0+1},\dots,\xi_n) \|_2 ,$$

we deduce that

$$\begin{aligned}
&\Big| \langle \varphi, H_I(G) \Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac12} \Big)^{-1} \psi \rangle \Big| \\
&= \int \Big\| \prod_{i\in[\![1,p]\!]_{i_0}} b_i(\xi_i) u \Big\| \prod_{i\in[\![1,p]\!]_{i_0}} (\omega_i(k_i)+1)^{\frac12} \| G(\xi_1,\dots,\xi_{i_0-1},\cdot,\xi_{i_0+1},\dots,\xi_n)\|_2 \\
&\quad \times \Big\| \prod_{i\in[\![p+1,n]\!]} b_i(\xi_i) \Big(\prod_{i\in[\![1,p]\!]_{i_0}} (H_{f,i}+1)^{\frac12} \Big) v \Big\| \prod_{i\in[\![1,n]\!]_{i_0}} d\xi_i .
\end{aligned}$$

In the same way as in the proof of Lemma 4.2.1, this yields

$$\begin{aligned}
&\Big| \langle \varphi, H_I(G) \Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac12} \Big)^{-1} \psi \rangle \Big| \\
&\leqslant \Big\| \Big(\prod_{i\in[\![1,p]\!]_{i_0}} (1+\omega_i(k_i))^{-\frac12} \Big) \Big(\prod_{i\in[\![p+1,n]\!]} \omega_i(k_i)^{-\frac12} \Big) G \Big\|_2 \\
&\quad \times \Big\| \Big(\prod_{i\in[\![1,p]\!]_{i_0}} H_{f,i}^{\frac12} \Big) u \Big\| \times \Big\| \Big(\prod_{i\in[\![p+1,n]\!]} H_{f,i}^{\frac12} \Big) \Big(\prod_{i\in[\![1,p]\!]_{i_0}} (H_{f,i}+1)^{\frac12} \Big) v \Big\| .
\end{aligned}$$

Remembering the definition of u and v in (4.27), we conclude that

$$\begin{aligned}
&\Big| \langle \varphi, H_I(G) \Big(\prod_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}+1)^{\frac12} \Big)^{-1} \psi \rangle \Big| \\
&\leqslant \Big\| \Big(\prod_{i\in[\![1,p]\!]_{i_0}} (1+\omega_i(k_i))^{-\frac12} \Big) \Big(\prod_{i\in[\![p+1,n]\!]} \omega_i(k_i)^{-\frac12} \Big) G \Big\|_2 \|\varphi\| \|\psi\| .
\end{aligned}$$

This implies the statement of the lemma in the case where $i_0 \in [\![1,p]\!]$. The case where $i_0 \in [\![p+1,n]\!]$ can be treated in the same way. □

Now, as in the previous section, we use an interpolation argument to obtain the following relative bound.

Proposition 4.2.8. *Let $s > 1/2$ and $0 \leqslant \theta \leqslant 1$. There exists a positive constant $\mathrm{C}_{s,\theta}$ such that, for all $i_0 \in [\![1,n]\!]$,*

$$G \in \mathcal{D}\Big(\prod_{\substack{i\in[\![1,n]\!]^{>}_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\theta s} \prod_{\substack{i\in[\![1,n]\!]^{0}_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{12}+\theta(s-\frac{1}{12})} \Big),$$

and $\varphi \in \mathcal{D}((\sum_{i\in[\![1,n]\!]_{i_0}} H_{f,i})^{\frac{n-1}{2}(1-\theta)})$, we have that

$$\begin{aligned}
&\| H_I(G)\varphi \| \\
&\leqslant \mathrm{C}_{s,\theta} \Big\| \Big(\prod_{\substack{i\in[\![1,n]\!]^{>}_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\theta s} \Big) \Big(\prod_{\substack{i\in[\![1,n]\!]^{0}_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{12}+\theta(s-\frac{1}{12})} \Big) G \Big\|_2 \Big\| \Big(\sum_{i\in[\![1,n]\!]_{i_0}} H_{f,i} + 1 \Big)^{\frac{n-1}{2}(1-\theta)} \varphi \Big\| . \qquad (4.29)
\end{aligned}$$

Proof. It follows from Lemma 4.2.4 that, for all $s > 1/2$, there exists $\mathrm{C}_s > 0$ such that, for all $i_0 \in [\![1,n]\!]$, $G \in \mathcal{D}(\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^s)$ and $\varphi, \psi \in \mathscr{H}$,

$$\big|\big\langle \varphi, H_I(G)\psi \big\rangle\big| \leqslant \mathrm{C}_s \Big\| \prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^s G \Big\|_2 \|\varphi\| \|\psi\|. \tag{4.30}$$

Considering the map

$$f : z \mapsto \Big\langle \varphi \Big| H_I\Big(\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}^{>}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{-zs}\Big)\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}^{0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{-\frac{1}{12}-z(s-\frac{1}{12})}\Big)\tilde{G}\Big)\Big(\sum_{i\in[\![1,n]\!]_{i_0}} H_{f,i} + 1\Big)^{-\frac{n-1}{2}(1-z)} \psi \Big\rangle,$$

on $\{z \in \mathbb{C}, 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$, it suffices to proceed in the same way as in Lemma 4.2.6, using Hadamard's three lines lemma together with (4.24) and (4.30). ☐

Remark 4.2.9. *The constants $\mathrm{C}_{s,\theta}$ in Propositions 4.2.6 and 4.2.8 depend on the positive masses m_i, $i \in [\![1,n]\!]^{>}$. More precisely, inspecting the proof, one can write*

$$\mathrm{C}_{s,\theta} = \tilde{\mathrm{C}}_{s,\theta}\Big(\prod_{i\in[\![1,n]\!]^{>}} m_i^{-\theta/2}\Big),$$

where $\tilde{\mathrm{C}}_{s,\theta}$ is independent of m_i. In the next section, to establish the existence of a ground state for the Hamiltonian H, it will be important to have relative bounds that hold uniformly in the masses m_i, for i in some subset $\mathrm{I} \subset [\![1,n]\!]^{>}$. To obtain such bounds, it suffices to modify the proof of Proposition 4.2.6 by replacing the estimate $\|\omega_i(k_i)^{-1/2}u\|_2 \leqslant m_i^{-1/2}\|u\|_2$, for $i \in \mathrm{I}$, by $\|\omega_i(k_i)^{-1/2}u\|_2 \leqslant \||k_i|^{-1/2}u\|_2$. This leads to the following more precise relative bounds

$$\begin{aligned}\|H_I(G)\varphi\| \leqslant \tilde{\mathrm{C}}_{s,\theta} \min_{\mathrm{I}\subset[\![1,n]\!]_{i_0}^{>}} \Big\| \Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}^{>}\setminus \mathrm{I}\\ j\in[\![1,3]\!]}} m_i^{-\frac{\theta}{2}} (\mathrm{h}_i^j)^{\theta s}\Big)\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}^{0}\cup \mathrm{I}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{12}+\theta(s-\frac{1}{12})}\Big) G \Big\|_2 \\ \times \Big\| \Big(\sum_{i\in[\![1,n]\!]_{i_0}} H_{f,i} + 1\Big)^{\frac{n-1}{2}(1-\theta)} \varphi \Big\|,\end{aligned} \tag{4.31}$$

where $\tilde{\mathrm{C}}_{s,\theta}$ is independent of the masses m_i's.

4.2.3 Self-adjointness of H

Using Proposition 4.2.8, we are now able to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\varepsilon > 0$ and consider a term $H^{(p)}_{I,i_1,\dots,i_n}(G^{(p)}_{i_1,\dots,i_n})$ occurring in the sum defining H_I, see (4.4). Possibly changing variables, we can assume without loss of generality that $H^{(p)}_{I,i_1,\dots,i_n}(G^{(p)}_{i_1,\dots,i_n})$ is given by an expression of the form (4.16), hence, to shorten notations, we write $H^{(p)}_{I,i_1,\dots,i_n}(G^{(p)}_{i_1,\dots,i_n}) = H_I(G)$, with

$$G \in \mathcal{D}\Big(\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}^{>}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}^{0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{5}{6}\frac{1}{n-1}+\varepsilon}\Big)\Big).$$

Applying Proposition 4.2.8 with $\theta = 1 - \frac{2(1-\varepsilon)}{n-1}$ and $s = 1/2 + \kappa$, with κ small enough, we obtain that

$$\|H_I(G)\varphi\| \lesssim \Big\| \Big(\prod_{\substack{i \in [\![1,n]\!]_{i_0}^{>} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2} - \frac{1}{n-1} + \varepsilon} \Big) \Big(\prod_{\substack{i \in [\![1,n]\!]_{i_0}^{0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2} - \frac{5}{6}\frac{1}{n-1} + \varepsilon} \Big) G \Big\|_2 \big\|(H_f + 1)^{1-\varepsilon}\varphi\big\|, \tag{4.32}$$

for all $\varphi \in \mathcal{D}(H_f^{1-\varepsilon})$. Next, we observe that

$$\begin{aligned} \big\|(H_f + 1)^{1-\varepsilon}\varphi\big\| &= \big\langle \varphi, (H_f + 1)^{2-2\varepsilon}\varphi \big\rangle^{\frac{1}{2}} \\ &\leqslant \big(\mu^2 \big\langle \varphi, (H_f + 1)^2 \varphi \big\rangle + \mathrm{C}_\mu \|\varphi\|^2 \big)^{\frac{1}{2}} \\ &\leqslant \mu \big\| H_f \varphi \big\| + \mathrm{C}_\mu \|\varphi\|, \end{aligned} \tag{4.33}$$

for any $\mu > 0$, the first inequality being a consequence of Young's inequality.

Combining (4.32) and (4.33), since $\mu > 0$ can be fixed arbitrarily small, we deduce that $H_I(G)$ is relatively H_f-bounded with relative bound 0. Since the other terms in the sum occurring in (4.4) can be treated in the same way, we deduce from the Kato-Rellich theorem that, indeed, H extends to a self-adjoint operator satisfying $\mathcal{D}(H) = \mathcal{D}(H_f)$. Semi-boundedness of H and the fact that any core for H_f is a core for H are other consequences of the Kato-Rellich theorem. □

4.3 Existence of a ground state

In this section, we prove the existence of a ground state for the Hamiltonian H defined by Theorem 4.1, i.e., we prove Theorem 4.2. In Section 4.3.1, we begin by studying the simplest case where all fermion fields are supposed to be massive. Next, in Section 4.3.2, we consider the case where excatly one field is massless, and, in Subsection 4.3.3, we consider the general case, using an induction in $l \in [\![1, n]\!]$, where l represents the number of massless fields involved.

4.3.1 Models with only massive fields

In this section, we suppose that the masses of all the particles are positive. We set

$$m := \min_{i \in [\![1,n]\!]} m_i > 0.$$

We prove the existence of a ground state for H by adapting a method due to [29] (see also [90]). The proofs follows closely Section 4.2 of [17], the main difference being that we have to use the relative bounds of Proposition 4.2.8 instead of the usual N_τ estimates.

We begin by introducing the operators U and $\check{\Gamma}_R$ adapted from [29] (see also [5, 67]) to our context. Recall that the antisymmetric Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$ has been defined in (4.3). We denote by $\tilde{\mathcal{F}}$ the antisymmetric Fock space over $L^2(\mathbb{R}^3 \times \{-1/2, 1/2\}) \oplus L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$. Let

$$U : \mathscr{F} \to \mathscr{F} \otimes \mathscr{F},$$

be defined by

$$U\Omega = \Omega \otimes \Omega, \quad Ub^*(f \oplus g) = (b^*(f) \otimes \mathbb{1} + (-1)^N \otimes b^*(g))U,$$

for any $f, g \in L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$, where $\Omega_{\tilde{\mathcal{F}}}$, respectively $\Omega_{\mathcal{F}}$, denotes the vacuum in $\tilde{\mathcal{F}}$, respectively in $\mathcal{F}$, and $N = \mathrm{d}\Gamma(\mathbb{1})$ denotes the number operator in $\mathscr{F}$. Let $j_0 \in \mathrm{C}^\infty([0, \infty); [0, 1])$

be such that $j_0 \equiv 1$ on $[0, 1/2]$ and $j_0 \equiv 0$ on $[1, \infty)$, and let $j_\infty = \sqrt{1 - j_0^2}$. For $R > 0$, let $j_\sharp^R := j_\sharp(y^2/R^2)$ on $L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$, where $j_\sharp$ stands for j_0 or j_∞ and $y = \mathrm{i}\nabla_k$. Let

$$j^R : L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}) \to L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}) \oplus L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}), \quad j^R(f) := (j_0^R(f), j_\infty^R(f)).$$

We recall that

$$\check{\Gamma}(j^R) : \mathscr{F} \to \mathscr{F} \otimes \mathscr{F}, \quad \check{\Gamma}(j^R) := U\Gamma(j^R),$$

where, as usual, for an operator a on $L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$, the operator $\Gamma(a)$ on $\mathscr{F}$ is defined by its restriction to $\otimes_a^l L^2(\mathbb{R}^3 \times \{-1/2, 1/2\})$ as $\Gamma(a) = a \otimes \cdots \otimes a$, and $\Gamma(a)\Omega = 1$.

The "extended" Hilbert space is

$$\mathscr{H}^{\mathrm{ext}} := \mathscr{H} \otimes \mathscr{H},$$

where, recall, $\mathscr{H} = \otimes_{i=1}^n \mathscr{F}$. The operator $\check{\Gamma}_R : \mathscr{H} \to \mathscr{H}^{\mathrm{ext}}$ is defined by

$$\check{\Gamma}_R := \otimes_{i=1}^n \check{\Gamma}(j^R),$$

and the extended Hamiltonian, acting on $\mathscr{H}^{\mathrm{ext}}$, is

$$H^{\mathrm{ext}} := H \otimes \mathbb{1} + \mathbb{1} \otimes H_f.$$

Under the conditions of Theorem 4.1, one verifies, by adapting the proof of that theorem in a straightforward way, that $H_I \otimes \mathbb{1}$ is relatively H^{ext} bounded with relative bound 0.

Recall that, for $p \in [\![0, n]\!]$, the set $\mathfrak{J}_p$ has been defined in (4.5).

Lemma 4.3.1. *Let $i_0 \in [\![1, n]\!]$ and $\varepsilon > 0$. Suppose that, for all $p \in [\![0, n]\!]$ and all set of integers $\{i_1, \ldots, i_n\} \in \mathfrak{J}_p$,*

$$G_{i_1,\ldots,i_n}^{(p)} \in \mathcal{D}\Big(\prod_{\substack{i \in [\![1,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2} - \frac{1}{n-1} + \varepsilon} \Big).$$

Then, for any $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ we have that

$$\chi(H^{\mathrm{ext}})\check{\Gamma}_R - \check{\Gamma}_R \chi(H) \to 0, \quad R \to \infty. \tag{4.34}$$

Proof. The proof can be adapted from that of [17, Lemma 4.3] and previous results of chapter two. The main difference is that we have to use the relative bound of Proposition 4.2.8 instead of the usual N_τ estimates. Namely, considering, as in the proof of Theorem 4.1, a particular term of H_I of the form $H_{I,i_1,\ldots,i_n}^{(p)}(G_{i_1,\ldots,i_n}^{(p)}) = H_I(G)$, with $H_I(G)$ given by (4.16), one can compute

$$\begin{aligned}
&(H_I(G) \otimes \mathbb{1})\check{\Gamma}_R - \check{\Gamma}_R H_I(G) \\
&= \int \Big(1 - \prod_{i=1}^n j_0(\frac{x_i^2}{R^2})\Big) G(\xi_1, \ldots, \xi_n) b_1^{*,0}(\xi_1) \ldots b_p^{*,0}(\xi_p) b_{p+1}^0(\xi_{p+1}) \ldots b_n^0(\xi_n) d\xi_1 \cdots d\xi_n \\
&+ \sum_{\substack{\{\alpha_i\} \in \{0,\infty\}^n \\ \exists j, \alpha_j \neq 0}} \int \Big(\prod_{i=1}^n j_{\alpha_i}(\frac{x_i^2}{R^2}) \Big) G(\xi_1, \ldots, \xi_n) b_1^{*,\alpha_1}(\xi_1) \ldots b_p^{*,\alpha_p}(\xi_p) b_{p+1}^{\alpha_{p+1}}(\xi_{p+1}) \ldots b_n^{\alpha_n}(\xi_n) \\
&\qquad d\xi_1 \cdots d\xi_n.
\end{aligned}$$

Here we have set $x_i = \mathrm{i}\nabla_{k_i}$, $b_i^{\sharp,0} := b_i^\sharp \otimes \mathbb{1}$ and $b_i^{\sharp,\infty} = (-1)^{N_i} \otimes b^\sharp$, where $b^\sharp$ stands for b or b^* and $N_i = \displaystyle\int b^*(\xi_i)b(\xi_i)d\xi_i$ is the number operator in the i^{th} Fock space. The subscript $\{\alpha_i\} \in \{0,\infty\}^n$, $\exists j, \alpha_j \neq 0$ means that, for each term of the sum, at least one of the creation of annihilation operator $b_i^{\sharp,\alpha_i}$ is equal to b_i^{∞,α_i}.

Proceeding as in the proof of Proposition 4.2.8, one then verifies that

$$\begin{aligned}
&\big\|\big((H_I(G)\otimes\mathbb{1})\check{\Gamma}_R - \check{\Gamma}_R H_I(G)\big)\varphi\big\| \\
&\leqslant \mathrm{C}_{s,\theta}\Big\{\Big\|\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\theta s}\Big)\Big(1-\prod_{i=1}^n j_0(\frac{x_i^2}{R^2})\Big)G\Big\|_2 \\
&\qquad + \sum_{\substack{\{\alpha_i\}\in\{0,\infty\}^n\\ \exists j,\alpha_j\neq 0}} \Big\|\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\theta s}\Big)\Big(\prod_{i=1}^n j_{\alpha_i}(\frac{x_i^2}{R^2})\Big)G\Big\|_2\Big\} \\
&\qquad \times \Big\|\Big(\sum_{i\in[\![1,n]\!]_{i_0}} (H_{f,i}\otimes\mathbb{1} + \mathbb{1}\otimes H_{f,i}) + 1\Big)^{(n-1)(1-\theta)}\varphi\Big\|,
\end{aligned} \tag{4.35}$$

for any $s > 1/2$ and $0 \leqslant \theta \leqslant 1$. Fixing θ and s as in the proof of Theorem 4.1, and using that $H_{f,i}\otimes\mathbb{1} + \mathbb{1}\otimes H_{f,i}$ is relatively H^{ext} bounded, one deduces from the previous estimate that

$$\begin{aligned}
&\big\|\big((H_I(G)\otimes\mathbb{1})\check{\Gamma}_R - \check{\Gamma}_R H_I(G)\big)(H^{\mathrm{ext}}+i)^{-1}\big\| \\
&\leqslant \mathrm{C}_{s,\theta}\Big\{\Big\|\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)\Big(1-\prod_{i=1}^n j_0(\frac{x_i^2}{R^2})\Big)G\Big\|_2 \\
&\qquad + \sum_{\substack{\{\alpha_i\}\in\{0,\infty\}^n\\ \exists j,\alpha_j\neq 0}} \Big\|\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)\Big(\prod_{i=1}^n j_{\alpha_i}(\frac{x_i^2}{R^2})\Big)G\Big\|_2\Big\}.
\end{aligned} \tag{4.36}$$

Using pseudo-differential calculus together with the fact that G belongs to the domain of $\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}$, it is not difficult to see that the right-hand-side of the previous equation goes to 0, as $R\to\infty$. Since the other terms occurring in the definition of H_I can be treated in the same way, the rest of the proof is a straightforward adaptation of [17, Lemma 4.3]. □

Given Lemma 4.3.1, one deduces the location of the essential spectrum of H as stated in the following proposition. Again, the proof is a straightforward adaptation of a corresponding result in [17] (see [17, Theorem 3.5] and Chapter two). Details are left to the reader.

Proposition 4.3.2. *Let $i_0 \in [\![1,n]\!]$ and $\varepsilon > 0$. Suppose that, for all $p \in [\![0,n]\!]$ and all set of integers $\{i_1,\dots,i_n\} \in \mathfrak{I}_p$,*

$$G^{(p)}_{i_1,\dots,i_n} \in \mathcal{D}\Big(\prod_{\substack{i\in[\![1,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big).$$

The essential spectrum of H is given by

$$\sigma_{\mathrm{ess}}(H) = [E+m,\infty),$$

where $E = \inf\sigma(H)$ and $m = \min_{i\in[\![1,n]\!]}(m_i) > 0$. In particular, E is a (discrete) eigenvalue of H.

We mention that, as in Proposition 4.3.2, the location of the essential spectrum for quantum models involving massive fermionic fields has also been established in [5, 90]. The proof of [5, Theorem 4.3] is similar to ours, while in [90, Theorem 1.1], it it proven that $[E+m,\infty)\subset\sigma_{\mathrm{ess}}(H)$ by different arguments. It is likely that the abstract result of [90] can be applied to our context.

4.3.2 Models with one massless field

In this section, we suppose that one field is massless, while all the other ones are massive. To fix ideas, we suppose that $m_1=0$ and $m=\min_{i\in[\![2,n]\!]} m_i>0$. It will be convenient to write the total Hamiltonian H in (4.6) as $H=H_{m_1=0}$, in other words,

$$H_{m_1=0}=H_{f,m_1=0}+H_I=\mathrm{d}\Gamma(|k_1|)+\sum_{i=2}^{n}\mathrm{d}\Gamma(\sqrt{k_i^2+m_i^2})+H_I,\quad m_i>0,\quad 2\leqslant i\leqslant n,$$

where H_I is given by (4.4). Obviously the situation is identical if $m_i=0$ for some $i\in[\![1,n]\!]$ while all the other m_i's are positive.

In order to prove that the Hamiltonian $H_{m_1=0}$ has a ground state, we follow the strategy of [56]. First, we approximate $H_{m_1=0}$ by a family of operators H_{m_1}, where

$$H_{m_1}=H_{f,m_1}+H_I=\sum_{i=1}^{n}\mathrm{d}\Gamma(\sqrt{k_i^2+m_i^2})+H_I,\quad m_i>0,\quad 1\leqslant i\leqslant n,$$

then we let $m_1\to 0$. Proposition 4.3.2 shows that H_{m_1} has a ground state Φ_{m_1}. The strategy then consists in showing that Φ_{m_1} converges strongly, as $m_1\to 0$, to a (non-vanishing) ground state of $H_{m_1=0}$.

We set $E_{m_1}:=\inf\sigma(H_{m_1})$ and $E_{m_1=0}:=\inf\sigma(H_{m_1=0})$.

Proposition 4.3.3. *Let $i_0\in[\![1,n]\!]$ and $\varepsilon>0$. Suppose that, for all $p\in[\![0,n]\!]$ and all set of integers $\{i_1,\dots,i_n\}\in\mathfrak{I}_p$,*

$$G^{(p)}_{i_1,\dots,i_n}\in\mathcal{D}\Big(\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac12-\frac{1}{n-1}+\varepsilon}\Big)\Big(\prod_{\substack{i\in[\![1,1]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac12-\frac56\frac{1}{n-1}+\varepsilon}\Big)\Big).$$

Then $E_{m_1}\to E_{m_1=0}$ as $m_1\downarrow 0$. Moreover,

$$E_{m_1=0}=\lim_{m_1\downarrow 0}\langle\Phi_{m_1},H_{m_1=0}\Phi_{m_1}\rangle. \tag{4.37}$$

Proof. For $m_1>m_1'>0$, we have that $H_{m_1}\geqslant H_{m_1'}\geqslant H$. This implies that the map $m_1\mapsto E_{m_1}$ is non-decreasing and bounded below by $E_{m_1=0}$. Hence there exists E^* such that

$$\lim_{m_1\downarrow 0}E_{m_1}=:E^*\geqslant E_{m_1=0}.$$

We prove that $E^*\leqslant E_{m_1=0}$. Let $\delta>0$ and let $\varphi\in\mathcal{D}(H_{m_1=0})$ be a normalized vector such that $\langle\varphi,H_{m_1=0}\varphi\rangle\leqslant E_{m_1=0}+\delta$. Note that

$$H_{m_1}\leqslant\mathrm{d}\Gamma(|k_1|+m_1)+\sum_{i\in[\![2,n]\!]}\mathrm{d}\Gamma(\sqrt{k_i^2+m_i^2})+H_I\leqslant H_{m_1=0}+m_1N_1,$$

where $N_1=\int b_1^*(\xi_1)b_1(\xi_1)d\xi_1$ is the number operator in the first Fock space.

Let $\tilde{\varphi}_\ell = \mathbb{1}_{[0,\ell]}(N_1)\varphi$. Clearly, since N_1 commutes with $\mathrm{d}\Gamma((k_i^2+m_i^2)^{1/2})$, $i \in [\![1,n]\!]$, we have that

$$\lim_{\ell\to\infty} \big\langle \tilde{\varphi}_\ell, H_{f,m_1=0}\tilde{\varphi}_\ell \big\rangle = \big\langle \varphi, H_{f,m_1=0}\varphi \big\rangle.$$

Moreover, applying Proposition 4.2.6, we see that there exist $a>0$ and $b>0$ such that

$$\big|\big\langle \tilde{\varphi}_\ell - \varphi, H_I(\tilde{\varphi}_\ell-\varphi)\big\rangle\big| \leqslant a\big\langle \tilde{\varphi}_\ell-\varphi, H_{f,m_1=0}(\tilde{\varphi}_\ell-\varphi)\big\rangle + b\|\tilde{\varphi}_\ell-\varphi\|^2,$$

which tends to zero as $\ell\to\infty$. This shows that, fixing ℓ large enough, we have

$$\langle \tilde{\varphi}_\ell-\varphi, H_{m_1=0}(\tilde{\varphi}_\ell-\varphi)\rangle \leqslant \delta,$$

and therefore

$$\begin{aligned}
E^* &\leqslant \langle \tilde{\varphi}_\ell, H_{m_1}\tilde{\varphi}_\ell\rangle \\
&\leqslant \langle \tilde{\varphi}_\ell, H_{m_1=0}\tilde{\varphi}_\ell\rangle + m_1\langle \tilde{\varphi}_\ell, N_1\tilde{\varphi}_\ell\rangle \\
&\leqslant \langle \varphi, H_{m_1=0}\varphi\rangle + \delta + m_1\langle \tilde{\varphi}_\ell, N_1\tilde{\varphi}_\ell\rangle \\
&\leqslant E + 2\delta + m_1\langle \tilde{\varphi}_\ell, N_1\tilde{\varphi}_\ell\rangle.
\end{aligned}$$

Letting $m_1\to 0$, we obtain that

$$E^* \leqslant E + 2\delta.$$

Hence $E^*\leqslant E$ since $\delta>0$ is arbitrary.

To prove (4.37), it suffices to observe that

$$E_{m_1=0} \leqslant \langle \Phi_{m_1}, H_{m_1=0}\Phi_{m_1}\rangle \leqslant \langle \Phi_{m_1}, H_{m_1}\Phi_{m_1}\rangle = E_{m_1}.$$

Letting $m_1\to 0$ concludes the proof. □

The next step is to prove that Φ_{m_1} converges strongly, as $m_1\to 0$, along some subsequence, to a non-vanishing vector of the Hilbert space which will turn out to be a ground state of $H_{m_1=0}$. An important ingredient is to control the expectation of the number operator N_1 in the approximate ground state Φ_{m_1}, uniformly in $m_1>0$. This is the purpose of the following proposition.

Proposition 4.3.4. *Let $i_0 \in [\![1,n]\!]$ and $\varepsilon>0$. Suppose that, for all $p\in[\![0,n]\!]$ and all set of integers $\{i_1,\dots,i_n\}\in\mathfrak{I}_p$,*

$$G^{(p)}_{i_1,\dots,i_n} \in \mathcal{D}\Big(\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac12-\frac{1}{n-1}+\varepsilon}\Big)\Big(\prod_{\substack{i\in[\![1,1]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac12-\frac56\frac{1}{n-1}+\varepsilon}\Big)\Big).$$

For a.e. $\xi_1=(k_1,\lambda_1)\in\mathbb{R}^3\times\{-1/2,1/2\}$, $k_1\neq 0$, we have that

$$\big\|b_1(\xi_1)\Phi_{m_1}\big\| \leqslant \mathrm{C}_0|k_1|^{-1}\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac12-\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2, \tag{4.38}$$

where C_0 is a positive constant independent of m_1 and the sum runs over all integers $i_2,\dots,i_n$ such that $\{i_2,\dots,i_n\}=\{2,\dots,n\}$, $i_2<\dots<i_p$ and $i_{p+1}<\dots<i_n$.

Proof. We have to distinguish the case where the number of different fields, n, is even from that where it is odd.

Suppose first that n is even. Since Φ_{m_1} is a ground state of H_{m_1}, we have that

$$b_1(\xi_1)(H_{m_1} - E_{m_1})\Phi_{m_1} = 0,$$

and the pull-through formula then yields

$$(H_{m_1} - E_{m_1} + (k_1^2 + m_1^2)^{\frac{1}{2}})b_1(\xi_1)\Phi_{m_1} + [b_1(\xi_1), H_I]\Phi_{m_1} = 0. \tag{4.39}$$

Recalling the expression (4.4) of H_I, a direct computation shows that

$$\begin{aligned}&[b_1(\xi_1), H_I]\\ &= \sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\int G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\dots,\xi_n)b^*_{i_2}(\xi_{i_2})\dots b^*_{i_p}(\xi_{i_p})b_{i_{p+1}}(\xi_{i_{p+1}})\dots b_{i_n}(\xi_{i_n})d\xi_2\dots d\xi_n,\end{aligned} \tag{4.40}$$

where the second sum runs over all integers $i_2,\dots,i_n$ such that $\{i_2,\dots,i_n\} = \{2,\dots,n\}$, $i_2 < \dots < i_p$ and $i_{p+1} < \dots < i_n$. Applying Proposition 4.2.8, we obtain that

$$\begin{aligned}&\big\|[b_1(\xi_1), H_I]\Phi_{m_1}\big\|\\ &\leqslant \mathrm{C}_{s,\theta,n}\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\theta s}\Big)G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2\Big\|\Big(\sum_{i\in[\![2,n]\!]_{i_0}}H_{f,i}+1\Big)^{\frac{n-2}{2}(1-\theta)}\Phi_{m_1}\Big\|,\end{aligned}$$

for any $s > 1/2$ and $0 \leqslant \theta \leqslant 1$, where $\mathrm{C}_{s,\theta,n}$ is a positive constant independent of $m_1 > 0$. Fixing s and θ as in the proof of Theorem 4.1, it is then not difficult to deduce from the previous estimate that

$$\begin{aligned}&\big\|[b_1(\xi_1), H_I]\Phi_{m_1}\big\|\\ &\leqslant \mathrm{C}\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2\Big\|\Big(\sum_{i\in[\![2,n]\!]_{i_0}}H_{f,i}+1\Big)\Phi_{m_1}\Big\|,\end{aligned}$$

where $\mathrm{C} > 0$ does not depend on m_1. Together with (4.39), since

$$\big\|(H_{m_1} - E_{m_1} + (k_1^2 + m_1^2)^{\frac{1}{2}})^{-1}\big\| \leqslant |k_1|^{-1},$$

we obtain that

$$\begin{aligned}&\big\|b_1(\xi_1)\Phi_{m_1}\big\|\\ &\leqslant \mathrm{C}|k_1|^{-1}\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2\Big\|\Big(\sum_{i\in[\![2,n]\!]_{i_0}}H_{f,i}+1\Big)\Phi_{m_1}\Big\|.\end{aligned}$$

Moreover, applying (4.31) in Remark 4.2.9, with $I = \{1\}$, shows that

$$\big\|H_I\Phi_{m_1}\big\| \leqslant \mathrm{C}'\big\|\big(H_{f,m_1}+1\big)\Phi_{m_1}\big\|,$$

for some positive constant C' independent of m_1. Combined with the previous equation and the fact that $\|H_{m_1}\Phi_{m_1}\| = |E_{m_1}|$ is uniformly bounded in $m_1 \geqslant 0$ in a compact set (since $m_1 \mapsto E_{m_1}$ is non-decreasing), this yields that

$$\big\|b_1(\xi_1)\Phi_{m_1}\big\| \leqslant \mathrm{C}''|k_1|^{-1}\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2,$$

with $\mathrm{C}'' > 0$ independent of m_1. This proves (4.38) in the case where n is even.

If n is odd, the proof has to be modified as follows. Using anti-commutation relations, we now find that

$$[b_1(\xi_1), H_I] = -2H_I b_1(\xi_1) + H_I'(\xi_1), \tag{4.41}$$

where

$$H_I'(\xi_1) := \sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\int G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\dots,\xi_n)b^*_{i_2}(\xi_{i_2})\dots b^*_{i_p}(\xi_{i_p})b_{i_{p+1}}(\xi_{i_{p+1}})\dots b_{i_n}(\xi_{i_n})d\xi_2\dots d\xi_n.$$

The identity (4.39) is thus replaced by

$$\big(H_{m_1} - 2H_I - E_{m_1} + (k_1^2+m_1^2)^{\frac{1}{2}}\big)b_1(\xi_1)\Phi_{m_1} + H_I'(\xi_1)\Phi_{m_1} = 0.$$

Now, using that n is odd, a direct computation gives

$$(-1)^N H_{m_1}(-1)^N = H_{m_1} - 2H_I,$$

where $N = \sum_{i=1}^{n}\int b_i^*(\xi_i)b_i(\xi_i)d\xi_i$ is the total number operator. Therefore, H_{m_1} and $H_{m_1} - 2H_I$ are unitarily equivalent and hence we have that $\inf\sigma(H_{m_1} - 2H_I) = E_{m_1}$. This shows that $(H_{m_1} - 2H_I - E_{m_1} + (k_1^2+m_1^2)^{1/2})$ is invertible with inverse bounded by $|k_1|^{-1}$. The rest of the proof is identical to that of the previous case. □

We mention that, to treat the case where n is odd in the preceding proof, another possibility is to use an argument of [4], commuting the operators $(-1)^N b_1(\xi_1)$ instead of $b_1(\xi_1)$ in (4.41).

The following further technical estimate will be used in the proof of the existence of a ground state for $H_{m_1=0}$.

Proposition 4.3.5. *Under the conditions of Proposition 4.3.4, for a.e. $\xi_1 = (k_1,\lambda_1) \in \mathbb{R}^3 \times \{-1/2, 1/2\}$, $k_1 \neq 0$, we have that*

$$\begin{aligned}\big\|\nabla_{k_1}(b_1(\xi_1)\Phi_{m_1})\big\| \leqslant{}& \mathrm{C}_0|k_1|^{-2}\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2\\ &+ \mathrm{C}_0|k_1|^{-1}\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)(\nabla_{k_1}G^{(p)}_{1,i_2,\dots,i_n})(\xi_1,\cdot,\dots,\cdot)\Big\|_2,\end{aligned} \tag{4.42}$$

where C_0 is a positive constant independent of m_1.

Proof. Consider for instance the case where n is even. We rewrite (4.39) as

$$b_1(\xi_1)\Phi_{m_1} = -\big(H_{m_1} - E_{m_1} + (k_1^2 + m_1^2)^{\frac{1}{2}}\big)^{-1}[b_1(\xi_1), H_I]\Phi_{m_1}. \tag{4.43}$$

Differentiating w.r.t. k_1 and using (4.40), we obtain that

$$\nabla_{k_1}(b_1(\xi_1)\Phi_{m_1}) = -\nabla_{k_1}\Big(\big(H_{m_1} - E_{m_1} + (k_1^2 + m_1^2)^{\frac{1}{2}}\big)^{-1}\Big)$$
$$\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\int G^{(p)}_{1,i_2,\dots,i_n}(\xi_1,\dots,\xi_n)b^*_{i_2}(\xi_{i_2})\dots b^*_{i_p}(\xi_{i_p})b_{i_{p+1}}(\xi_{i_{p+1}})\dots b_{i_n}(\xi_{i_n})\Phi_{m_1}d\xi_2\dots d\xi_n$$
$$-\big(H_{m_1} - E_{m_1} + (k_1^2 + m_1^2)^{\frac{1}{2}}\big)^{-1}$$
$$\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\int(\nabla_{k_1}G^{(p)}_{1,i_2,\dots,i_n})(\xi_1,\dots,\xi_n)b^*_{i_2}(\xi_{i_2})\dots b^*_{i_p}(\xi_{i_p})b_{i_{p+1}}(\xi_{i_{p+1}})\dots b_{i_n}(\xi_{i_n})\Phi_{m_1}d\xi_2\dots d\xi_n.$$

Proceeding as in the proof of Proposition 4.3.4, it is not difficult to deduce from the previous equality that (4.42) holds.

The proof of (4.42) in the case where n is odd is analogous, using (4.41) instead of (4.39). □

Remark 4.3.6. *Proceeding in the same way as in Propositions 4.3.4 and 4.3.5, one can estimate the norms of $b(\xi_l)\Phi_{m_1}$ and $\nabla_{k_l}(b(\xi_l)\Phi_{m_1})$, $l \in [\![2,n]\!]$, as*

$$\big\|b_l(\xi_l)\Phi_{m_1}\big\| \leqslant \mathrm{C}_0\omega_l(k_l)^{-1}$$
$$\times\sum_{p=0}^{n}\sum_{i_1,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)\Big(\prod_{\substack{i\in[\![1,1]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_1^j)^{\frac{1}{2}-\frac{5}{6}\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{i_1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2,$$

and

$$\big\|\nabla_{k_l}(b_l(\xi_l))\Phi_{m_1}\big\| \leqslant \mathrm{C}_0\omega_l(k_l)^{-2}$$
$$\times\Big\{\sum_{p=0}^{n}\sum_{i_1,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)\Big(\prod_{\substack{i\in[\![1,1]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_1^j)^{\frac{1}{2}-\frac{5}{6}\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{i_1,i_2,\dots,i_n}(\xi_1,\cdot,\dots,\cdot)\Big\|_2$$
$$+\,\omega_l(k_l)\sum_{p=0}^{n}\sum_{i_1,\dots,i_n}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon}\Big)\Big(\prod_{\substack{i\in[\![1,1]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_1^j)^{\frac{1}{2}-\frac{5}{6}\frac{1}{n-1}+\varepsilon}\Big)(\nabla_{k_l}G^{(p)}_{i_1,i_2,\dots,i_n})(\xi_1,\cdot,\dots,\cdot)\Big\|_2\Big\},$$

where $\omega_l(k_l) = (k_l^2 + m_l^2)^{1/2}$ and C_0 is a positive constant independent of m_1. Theses estimates are not optimal, but they are sufficient for our purpose.

We can finally prove the main theorem of this section.

Theorem 4.3. *Under the conditions of Proposition 4.3.4, the operator $H_{m_1=0}$ has a ground state, i.e., there exists $\Phi_{m_1=0} \in \mathcal{D}(H_{m_1=0})$, $\Phi_{m_1=0} \neq 0$, such that*

$$H_{m_1=0}\Phi_{m_1=0} = E_{m_1=0}\Phi_{m_1=0}.$$

Proof. Let $(m_1^{(j)})_{j\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $m_1^{(j)} \to 0$ as $j \to \infty$. To shorten notations, we denote by $\Phi_j \in \mathcal{D}(H_{m_1^{(j)}})$ a normalized ground state of $H_{m_1^{(j)}}$, which exists according to Proposition 4.3.2. By Proposition 4.3.3, we have that

$$\lim_{j\to\infty}\big\|(H_{m_1=0} - E_{m_1=0})^{\frac{1}{2}}\Phi_j\big\| = 0.$$

To prove the theorem, we claim that it suffices to show that (Φ_j) converges strongly, as $j \to \infty$. Indeed, if we prove that there exists $\Phi_{m_1=0}$ such that $\|\Phi_j - \Phi_{m_1=0}\|$, as $j \to \infty$, we can deduce from the previous equality that $\Phi_{m_1=0} \in \mathcal{D}((H_{m_1=0} - E_{m_1=0})^{\frac{1}{2}})$, $\|\Phi_{m_1=0}\| = 1$ and $(H_{m_1=0} - E_{m_1=0})^{\frac{1}{2}}\Phi_{m_1=0} = 0$. The statement of the theorem then follows.

Now, we prove that (Φ_j) converges strongly as $j \to \infty$. We decompose

$$\Phi_j = \sum_{l_1,\dots,l_n \in \mathbb{N}} \Phi_j^{(l_1,\dots,l_n)} \in \otimes_a^{l_1} L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}) \otimes \cdots \otimes_a^{l_n} L^2(\mathbb{R}^3 \times \{-\frac{1}{2}, \frac{1}{2}\}).$$

Recall that N stands for the total number operator in $\mathscr{H}$. By Proposition 4.3.4 and Remark 4.3.6, using Hypothesis (4.8), one can see that $\langle \Phi_j, N\Phi_j \rangle$ is uniformly bounded in $j \in \mathbb{N}$. From this property, one can deduce that the strong convergence of $\Phi_j^{(l_1,\dots,l_n)}$ for any $l_1,\dots,l_n \in \mathbb{N}$ implies the strong convergence of Φ_j.

Moreover, we claim that it suffices to prove the L^2 convergence of $\Phi_j^{(l_1,\dots,l_n)}$ on any compact subset of $(\mathbb{R}^3 \times \{-1/2, 1/2\})^{l_1+\cdots+l_n}$. Indeed, we observe that

$$\begin{aligned} &\Phi_j^{(l_1,\dots,l_n)}(\xi_1^{(1)},\dots,\xi_1^{(l_1)},\dots,\xi_n^{(1)},\dots,\xi_n^{(l_n)}) \\ &= \frac{1}{\sqrt{l_1}} b_1(\xi_1^{(1)}) \Phi_j^{(l_1-1,l_2,\dots,l_n)}(\xi_1^{(2)},\dots,\xi_1^{(l_1)},\dots,\xi_n^{(1)},\dots,\xi_n^{(l_n)}). \end{aligned}$$

Recall that $B_\Lambda = \{(k,\lambda) \in \mathbb{R}^3 \times \{-1/2, 1/2\}, |k| \leqslant \Lambda\}$. From Proposition 4.3.4, it follows that

$$\left\|\mathbb{1}_{B_\Lambda^c}(\xi_1^{(1)}) b_1(\xi_1)\Phi_j\right\| \leqslant \frac{\mathrm{C}_0}{\Lambda} \sum_{p=0}^{n} \sum_{i_2,\dots,i_n} \Big\| \Big(\prod_{\substack{i \in [\![2,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon} \Big) G_{1,i_2,\dots,i_n}^{(p)}(\xi_1,\cdot,\dots,\cdot) \Big\|_2,$$

uniformly in $j \in \mathbb{N}$. This implies that

$$\left\|\mathbb{1}_{B_\Lambda^c}(\cdot)\Phi_j^{(l_1,\dots,l_n)}\right\|_2^2 \lesssim \frac{1}{\Lambda^2} \sum_{p=0}^{n} \sum_{i_2,\dots,i_n} \Big\| \Big(\prod_{\substack{i \in [\![2,n]\!]_{i_0} \\ j \in [\![1,3]\!]}} (\mathrm{h}_i^j)^{\frac{1}{2}-\frac{1}{n-1}+\varepsilon} \Big) G_{1,i_2,\dots,i_n}^{(p)} \Big\|_2^2. \tag{4.44}$$

Here $\mathbb{1}_{B_\Lambda^c}(\cdot)\Phi_j^{(l_1,\dots,l_n)}$ should be understood as the map

$$\big(\xi_1^{(1)},\dots,\xi_1^{(l_1)},\dots,\xi_n^{(1)},\dots,\xi_n^{(l_n)}\big) \mapsto \mathbb{1}_{B_\Lambda^c}(\xi_1^{(1)})\Phi_j^{(l_1,\dots,l_n)}\big(\xi_1^{(1)},\dots,\xi_1^{(l_1)},\dots,\xi_n^{(1)},\dots,\xi_n^{(l_n)}\big).$$

Clearly, the right hand side of (4.44) can be made arbitrarily small, uniformly in $j \in \mathbb{N}$, for Λ large enough. An analogous estimate holds if $\xi_1^{(1)}$ is replaced by any of the other variable $\xi_i^{(\ell)}$. This shows that it suffices to establish the L^2 convergence of $\Phi_j^{(l_1,\dots,l_n)}$ on any compact subset of $(\mathbb{R}^3 \times \{-1/2, 1/2\})^{l_1+\cdots+l_n}$.

Now we prove that $(\Phi_j^{(l_1,\dots,l_n)})_{j\in\mathbb{N}}$ converges strongly on

$$B_\Lambda^{l_1+\cdots+l_n} \subset (\mathbb{R}^3 \times \{-1/2, 1/2\})^{l_1+\cdots+l_n}.$$

We note that, since $\|\Phi_j\| = 1$ for all $j \in \mathbb{N}$, $(\Phi_j^{(l_1,\dots,l_n)})_{j\in\mathbb{N}}$ is a bounded sequence in L^2, hence, by the Banach-Alaoglu theorem, it converges weakly in L^2, along some subsequence. Consider a weakly convergent subsequence which, to simplify, is denoted by the same symbol $(\Phi_j^{(l_1,\dots,l_n)})_{j\in\mathbb{N}}$.

We define the space $W^{1,r}(B_\Lambda^{l_1+\cdots+l_n})$ as the set of all measurable maps f from $B_\Lambda^{l_1+\cdots+l_n}$ to $\mathbb{C}$ such that, for all values of the spin variables $(\lambda_1^{(1)},\dots,\lambda_n^{(l_n)})$, the map

$$f\big(\cdot,\lambda_1^{(1)},\cdot,\lambda_1^{(2)},\dots,\cdot,\lambda_n^{(l_n)}\big)$$

belongs to the Sobolev space $W^{1,r}(\{|k|\leqslant\Lambda\}^{l_1+\cdots+l_n})$. We claim that, for all $j\in\mathbb{N}$, $\Phi_j^{(l_1,\dots,l_n)}$ belongs to $W^{1,r}(B_\Lambda^{l_1+\cdots+l_n})$ provided that $1\leqslant r<2$. Indeed, since $r<2$ and $\Phi_j^{(l_1,\dots,l_n)}\in L^2(B_\Lambda^{l_1+\cdots+l_n})$, we see that $\Phi_j^{(l_1,\dots,l_n)}\in L^r(B_\Lambda^{l_1+\cdots+l_n})$. Moreover, similarly as in [56], we can compute

$$\begin{aligned}
&\int_{B_\Lambda^{l_1+\cdots+l_n}}\Big|(\nabla_{k_1^{(1)}}\Phi_j^{(l_1,\dots,l_n)})\big(\xi_1^{(1)},\dots,\xi_n^{(l_n)}\big)\Big|^r d\xi_1^{(1)}\dots d\xi_n^{(l_n)}\\
&=\frac{1}{\sqrt{l_1}^r}\int_{B_\Lambda^{l_1+\cdots+l_n}}\Big|(\nabla_{k_1^{(1)}}b_1(\xi_1^{(1)})\Phi_j^{(l_1-1,l_2,\dots,l_n)})\big(\xi_1^{(2)},\dots,\xi_n^{(l_n)}\big)\Big|^r d\xi_1^{(1)}\dots d\xi_n^{(l_n)}\\
&\lesssim\int_{B_\Lambda}\Big(\int_{B_\Lambda^{(l_1-1)+\cdots+l_n}}\Big|(\nabla_{k_1^{(1)}}b_1(\xi_1^{(1)})\Phi_j^{(l_1-1,l_2,\dots,l_n)})\big(\xi_1^{(2)},\dots,\xi_n^{(l_n)}\big)\Big|^2 d\xi_1^{(2)}\dots d\xi_n^{(l_n)}\Big)^{\frac{r}{2}}d\xi_1^{(1)}\\
&\leqslant\int_{B_\Lambda}\|\nabla_{k_1^{(1)}}(b_1(\xi_1^{(1)})\Phi_j)\|^r d\xi_1^{(1)},
\end{aligned}$$

the first inequality being a consequence of Hölder's inequality. Applying Proposition 4.3.5, we obtain that

$$\begin{aligned}
&\int_{B_\Lambda^{l_1+\cdots+l_n}}\Big|(\nabla_{k_1^{(1)}}\Phi_j^{(l_1,\dots,l_n)})\big(\xi_1^{(1)},\dots,\xi_n^{(l_n)}\big)\Big|^r d\xi_1^{(1)}\dots d\xi_n^{(l_n)}\\
&\lesssim\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\int_{B_\Lambda}|k_1^{(1)}|^{-2r}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac12-\frac{1}{n-1}+\varepsilon}\Big)G^{(p)}_{1,i_2,\dots,i_n}(\xi_1^{(1)},\cdot,\dots,\cdot)\Big\|_2^r d\xi_1^{(1)}\\
&\quad+\sum_{p=0}^{n}\sum_{i_2,\dots,i_n}\int_{B_\Lambda}|k_1^{(1)}|^{-r}\Big\|\Big(\prod_{\substack{i\in[\![2,n]\!]_{i_0}\\ j\in[\![1,3]\!]}}(\mathrm{h}_i^j)^{\frac12-\frac{1}{n-1}+\varepsilon}\Big)(\nabla_{k_1^{(1)}}G^{(p)}_{1,i_2,\dots,i_n}(\xi_1^{(1)},\cdot,\dots,\cdot))\Big\|_2^r d\xi_1^{(1)},
\end{aligned}$$

which is finite by assumption. In the same way, one can verify that the other derivatives $\nabla_{k_i^{(\ell)}}\Phi_j^{(l_1,\dots,l_n)}$ belong to L^r. Hence $\Phi_j^{(l_1,\dots,l_n)}\in W^{1,r}(B_\Lambda^{l_1+\cdots+l_n})$.

Finally, since $(\Phi_j^{(l_1,\dots,l_n)})_{j\in\mathbb{N}}$ converges weakly in $L^2(B_\Lambda^{l_1+\cdots+l_n})$, it also converges weakly in $W^{1,r}(B_\Lambda^{l_1+\cdots+l_n})$, because $r<2$. As in [56], applying the Rellich-Kondrachov theorem, one then obtain that $(\Phi_j^{(l_1,\dots,l_n)})_{j\in\mathbb{N}}$ converges strongly in $L^2(B_\Lambda^{l_1+\cdots+l_n})$. This concludes the proof of the theorem. □

4.3.3 Proof of Theorem 4.2

In order to conclude the proof of Theorem 4.2, it suffices to proceed by induction in the following way. The induction hypothesis (H_l) is that, if l particles are massless and $n-l$ are massive, then H has a ground state. It has been shown in Section 4.3.1 that (H_0) holds. Assuming that (H_l) holds true, we proceed to prove that (H_{l+1}) holds exactly in the same way as in Section 4.3.2. More precisely, assuming that $m_1=m_2=\cdots=m_{l+1}=0$ and $\min_{i\in[\![l+2,n]\!]}m_i>0$, the total Hamiltonian

$$H\equiv H_{m_1=0,\dots,m_l=0,m_{l+1}=0}$$

is approximated by the family of operators

$$H_{m_1=0,\ldots,m_l=0,m_{l+1}}, \quad m_{l+1} > 0,$$

where the free energy for the $(l+1)^{\mathrm{th}}$ massless field, $\mathrm{d}\Gamma(|k_{l+1}|)$, is replaced by $\mathrm{d}\Gamma(k_{l+1}^2 + m_{l+1}^2)^{1/2}$. By the induction hypothesis, $H_{m_1=0,\ldots,m_l=0,m_{l+1}}$ has a normalized ground state $\Psi_{m_1=0,\ldots,m_l=0,m_{l+1}}$. One shows that $\Psi_{m_1=0,\ldots,m_l=0,m_{l+1}}$ converges strongly, as $m_{l+1} \to 0$, to a ground state of $H_{m_1=0,\ldots,m_l=0,m_{l+1}=0}$ by adapting the proof given in Section 4.3.2 in a straightforward way. Details are left to the reader.

Chapter 5

Translation invariant toy model

5.1 Introduction

In chapters two and three, mathematical models of the weak interaction have been considered. For example, the decay of the W^- into an electron and its antineutrino is described by an Hamiltonian of the from:

$$H = H_0 + \lambda \int_{\mathbb{R}^3} g_1(p_1)g_2(p_2)g_2(p_3)e^{-i(p_1+p_2-p_3)x}b^*(p_1)c^*(p_2)a(p_3)dxdp_1dp_2dp_3 + \text{h.c}, \tag{5.1}$$

where the free Hamiltonian H_0 has been already defined in (2.10) and the physical expressions of g_1, g_2 and g_3 may be found in (2.6). Note that the polarisation and spin terms have been dropped here for simplicity. The problematic term is:

$$\int_{\mathbb{R}^3} e^{-i(p_1+p_2-p_3)x}dx, \tag{5.2}$$

which is usually regularized through a spacial cutoff. As an example, let f be a function defined from $\mathbb{R}^3$ to $\mathbb{R}$, integrable and square integrable. Let moreover χ be a compactly bounded function equal to one on $[0, 1]$. Since the kernels g_1, g_2 and g_3 are bounded

$$G_n(p_1, p_2, p_3) = g_1(p_1)g_2(p_2)g_3(p_3)n^3 f(n(p_1 + p_2 - p_3))\chi\left(\frac{|p_1|}{n}\right)\chi\left(\frac{|p_2|}{n}\right)\chi\left(\frac{|p_3|}{n}\right) \tag{5.3}$$

is clearly square integrable and has already been treated before. A natural question which arises is to remove such cut-off. Our aim, in this chapter, is to show that one can define a translation invariant Hamiltonian H as in (5.1) and that H coincides with the limit, in the strong resolvent sense, of a sequence of non translation invariant Hamiltonians $\{H_n\}$ with interaction kernels formally given by (5.3). As a first step, an abstract class of translation invariant interactive fields will be considered. Both bosons and fermions are considered. For simplicity, we will study the case of two particles. The Hilbert space is a tensor product of two Fock spaces:

$$\mathscr{H} = \mathscr{F}_\sharp \otimes \mathscr{F}_\sharp$$

where $\mathscr{F}_\sharp$ stands for either the symmetric (for bosons) or antisymmetric (for fermions) Fock space over $L^2(\mathbb{R}^3)$. The free energy is defined as:

$$H_0 = \sum_{i=1}^{2} \int \omega_i(p_i)d_i^*(p_i)d_i(p_i)dp_i,$$

where p_i stands for the momentum of the i^{th} particle, d_i and d_i^* are the usual creation and annihilation operators of the i^{th} particle satisfying the canonical commutation relations in the bosonic case and the canonical anticommutation relations in the fermionic one. The relativistic dispersion relation for a particle of mass $m_i \geqslant 0$ is given by:

$$\omega_i(p_i) = \sqrt{p_i^2 + m_i^2}.$$

The spin components have been dropped to simplify mathematical expressions. Their treatment do not introduce any major difficulties. The interaction term is formally given by:

$$H_I = \int g(p) d_1^*(p) d_2(p) + \text{h.c.} \tag{5.4}$$

To give a better understanding of the interaction term, this formal expression may be applied to a continuous square integrable test function $\psi^{(m,n)}$ with m particles of type one and n of type two. For example, in the bosonic case we have:

$$\begin{aligned}\left(\int g(p) d_1^*(p) d_2(p) dp \psi^{(m,n)}\right)(k_1, \dots, k_{m+1}, q_1, \dots, q_{n-1}) \quad = \\ \sqrt{\frac{n}{m+1}} g(k_i) \sum_{i=1}^{m+1} \psi^{(m,n)}(k_1, \dots, \hat{k_i}, \dots, k_{m+1}, k_i, q_1, \dots, q_{n-1}).\end{aligned}$$

The full Hamiltonian is then given by $H = H_0 + H_I$. It can be approximated by non translation invariant Hamiltonians whose interaction terms are defined as:

$$H_{I,n} = \int \tilde{G}_n(p_1, p_2) b_1^*(p_1) b_2(p_2) + \text{h.c.} \tag{5.5}$$

The strength of the interactions will be highlighted by a coupling constant λ:

$$H = H_0 + \lambda H_I \tag{5.6}$$

$$H_n = H_0 + \lambda H_{I,n} \tag{5.7}$$

The study of translation invariant models remains a difficult problem which has been explored for example in [40, 41, 44, 80, 75, 76, 36, 62, 11, 61, 60, 74, 57, 12, 22, 14] and references therein. Let us highlight one of the issues that can be encountered. For example, we may focus on another natural object to consider:

$$H_I^{(2)} = \int g(p) b_1^*(p) b_2^*(p) + \text{h.c.}$$

It might be tempting, as before, to apply this formal expression to a test function. But $H_I^{(2)}$ acts on the vacuum as:

$$(H_I^{(2)}\Omega)(k_1, k_2) = \delta(k_1 - k_2) g(k_1).$$

This expression cannot be seen as a square integrable function. We believe that a renormalisation procedure may be required to define such terms. Such attempts have been successfully done for example in [51] where a broad description of $(\phi_2)^{2n}$ is provided. The ultraviolet cut-off is removed adding a divergent constant which is the infimum of the approximated model. Moreover, finite propagation speed principle is used to show that the Heisenberg dynamics of any bounded operator is independent of the spacial cutoff which makes possible, in some cases, to absorb the divergences into some mass terms. The case $n = 2$ is also treated in dimension 2 and 3 and is closer to strategies used in physics textbooks for the ϕ^4 theory. A divergent constant is added to obtain a Wick ordered expression and a multiplicative renormalisation, which might recall the standard procedure to remove the bubble diagrams, is done. Such attempts have not been tested in our context yet and it is beyond the scope of this chapter. We will focus on the interaction term defined in (5.4) and the following theorem will be proved:

Theorem 5.1. *1. Suppose that we have $l \in \{0, 1, 2\}$ massless particles and that $p \to |p|^{-l} g(p)$ is a bounded function, then there exists λ_0 such that for any positive $\lambda < \lambda_0$ the Hamiltonian H defined in (5.6) with the interaction term given by (5.4) is a self-adjoint operator such that $\mathscr{D}(H) = \mathscr{D}(H_0)$.*

2. *Assume moreover that g is a bounded continuous function. There exists $\tilde{G}_n \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ such that the Hamiltonians defined in (5.7) with interaction terms given by (5.5) are self-adjoint for any value of the coupling constant and $\mathscr{D}(H_n) = \mathscr{D}(H_0)$. Moreover, for $\lambda < \lambda_0$ H_n converges strongly in the resolvent sense to H.*

3. *Let us assume that g is bounded and:*

$$\||p|^{-l} \partial^\alpha g\|_\infty < \infty \quad l = 0, 1, 2 \text{ and } |\alpha| = 0, 1, 2, \tag{5.8}$$

then for any natural number n, there exists a constant λ_n such that for any $\lambda < \lambda_n$, the spectrum of the Hamiltonians defined in (5.7) with an interaction term given by (5.5) is:

$$\sigma(H_n) = \{0\} \cup [m, \infty). \tag{5.9}$$

m is the mass of the lightest particle, 0 is the only eigenvalue, it is non degenerate, and the spectra on $[m, \infty)$ are purely absolutely continuous.

There moreover exists a constant λ_2 such that for any $\lambda < \lambda_2$ the same results holds for the Hamiltonian defined in (5.6) with an interaction term given by (5.4).

The third point is of interest as it provides an example of an application of the weakly conjugate Mourre's theory.

Remark 5.1.1. *In the case where both particles are bosons or fermions, the total Hamiltonian H can be rewritten as follows. Let U be the unitary operator defined as in Section 1.2.4 identifying $\mathscr{H}$ with $\mathscr{F}_\sharp(h \oplus h)$,*

$$U : \mathscr{F}_\sharp(h \oplus h) \to \mathscr{H}.$$

A direct computation then shows that

$$U^{-1} H U = \mathrm{d}\Gamma(a),$$

with

$$a = \begin{pmatrix} \omega_1 & g \\ \bar{g} & \omega_2 \end{pmatrix}.$$

One can then study the operator $\mathrm{d}\Gamma(a)$ in $\mathscr{F}_\sharp(h \oplus h)$, which significantly simplifies the analysis. However, as mentioned at the beginning of this chapter, the toy model involving two particles that we consider serves as a preparation to analyse the model related to the weak interaction *theory involving one boson and two fermions. The methods used in the present chapter extends to such a model, details will be given elsewhere (see [1]).*

In the case where the two particles involved are one boson and one fermion, we mention that it might be possible to use the notion of super Fock spaces (see [32, Section 3.3.9]) in order to rewrite the total Hamiltonian in a simple second-quantized form as above. Super Fock spaces may also be useful to analyse the weak interaction model, we believe it would interesting to study this in future works.

5.2 The model

The model is defined on a tensor product of Fock spaces. Defining $\mathfrak{h} = L^2(\mathbb{R}^3)$ we have for bosons:

$$\mathscr{F}_s = \bigoplus_{n=0}^{\infty} \bigotimes_{s,k=0}^{n} \mathfrak{h}$$

and for fermions:

$$\mathscr{F}_a = \bigoplus_{n=0}^{\infty} \bigotimes_{a,k=0}^{n} \mathfrak{h}.$$

The Hilbert space is:

$$\mathscr{H} = \mathscr{F}_\sharp \otimes \mathscr{F}_\sharp,$$

where $\mathscr{F}_\sharp$ stands for either $\mathscr{F}_a$ or $\mathscr{F}_s$. We consider the following model:

$$H_I = \int g(p) b_1^*(p) b_2(p) dp + \text{h.c.}$$

$b_1^\sharp$, respectively $b_2^\sharp$, stands for the usual creation and annihilation operators acting on the first, respectively second, Fock space.

Proposition 5.2.1. *Assume g is bounded and $m_1, m_2 > 0$ then, H_I is a symmetric operator. Moreover there exists λ_0 such that for any positive $\lambda < \lambda_0$, the Hamiltonian H defined in* (5.6) *with the interaction term given by* (5.4) *is self-adjoint and any core of H_0 is a core for H.*

Proof. Let $\psi^{(m,n)} \in \mathscr{H}$ be a wave function with a number m of the first particle species and n of the second one. We define:

$$\tilde{\psi}_p(k_1, \ldots, k_{m+1}, q_1, \ldots, q_{n-1}) = \left(b_1^*(p) b_2(p) \psi^{(m,n)}\right)(k_1, \ldots, k_{m+1}, q_1, \ldots, q_{n-1})$$

which leads to:

$$\tilde{\psi}_p(k_1, \ldots, k_{m+1}, q_1, \ldots, q_{n-1}) = \sqrt{\frac{n}{m+1}} \sum_{i=1}^{m+1} \delta_{p-k_i} \psi^{(m,n)}(k_1, \ldots, \hat{k}_i, \ldots, k_{m+1}, p, q_1, \ldots, q_{n-1}).$$

For simplicity we introduce the following notation:

$$\psi^{(m,n)}(k_1, \ldots, \hat{k}_i, \ldots, k_{m+1}, k_i, q_1, \ldots, q_{n-1}) = \psi^{(m,n)}(\hat{k}_i, k_i).$$

We then have:

$$\int g(p) \tilde{\psi}_p dp = \sqrt{\frac{n}{m+1}} \sum_{i=1}^{m+1} g(k_i) \psi^{(m,n)}(\hat{k}_i, k_i).$$

Therefore:

$$\begin{aligned}\left\|\int g(p)\tilde{\psi}_p dp\right\|_2^2 &\leqslant \frac{n}{m+1}\left\|\sum_{i=1}^{m+1} g(k_i)\psi^{(m,n)}(\hat{k}_i,k_i)\right\|_2^2 \\ &\leqslant \frac{n}{m+1}\left(\sum_{i=1}^{m+1}\left\|g(k_i)\psi^{(m,n)}(\hat{k}_i,k_i)\right\|_2\right)^2\end{aligned}$$

Moreover using the boundedness of the kernel g we have:

$$\left(\left\|g(k_i)\psi^{(m,n)}(\hat{k}_i,k_i)\right\|_2\right) \lesssim \|\psi\|_2.$$

The adjoint term may be treated in the same way which implies:

$$\|H_I\psi\| \lesssim \sqrt{n(m+1)}\|\psi\|_2.$$

Consequently,

$$\|H_I(N+1)^{-1}\psi\| \lesssim \|\psi\|_2$$

therefore, for coupling constants small enough, the Kato-Rellich theorem can be applied which implies self-adjointness.

□

Adapting the proof in the case where massless particles are involved, we obtain:

Proposition 5.2.2. *Let us assume that there are $l \in \{0,1,2\}$ massless particle species in the model. Assume the boundedness of $p \to |p|^{-l}g(p)$, then the Hamiltonian H defined in* (5.6) *with the interaction term given by* (5.4) *is self-adjoint and any core of H_0 is a core for H.*

Remark 5.2.3. 1. *Proposition 5.2.1 can be proved also in the same way as what has been done in Lemma 4.2.7. Considering two wave functions $\Psi, \Phi \in \mathscr{D}(N^{\frac{1}{2}})$ it is not hard to prove that:*

$$|\langle\Psi\,|H\Phi\rangle\,| \leqslant \left\|N^{\frac{1}{2}}\Phi\right\|\left\|N^{\frac{1}{2}}\Psi\right\|$$

and consequently that $(N+1)^{-\frac{1}{2}}H(N+1)^{-\frac{1}{2}}$ is a bounded operator. The conclusion follows from commutation properties and the Kato Rellich theorem.

2. *The operator defined in Proposition 5.2.1 is translation invariant in that it commutes with the momentum operator:*

$$[H,P_i] \;=\; 0 \text{ for } i=1,2,3.$$

Here

$$P_i = \mathrm{d}\Gamma(p_{1,i}) + \mathrm{d}\Gamma(p_{2,i}),$$

where

$$\begin{aligned}p_1 &= (p_{1,1},p_{1,2},p_{1,3})\\ p_2 &= (p_{2,1},p_{2,2},p_{2,3})\end{aligned}$$

are the usual momentum operators acting on $L^2(\mathbb{R}^3)$.

3. *Let us assume that the particles have the same masses. Then it is not hard to see that H commutes with H_0. Therefore it is impossible to establish a scattering theory in the sense presented in the previous chapters.*

5.3 Non translation invariant approximation

In this part the previous Hamiltonian is approximated by a non translation invariant one. We define:

$$G_n(p_1,p_2) = \frac{1}{2}g(p_1)\chi(\frac{|p_1|}{n})\chi(\frac{|p_2|}{n}) + \frac{1}{2}g(p_2)\chi(\frac{|p_2|}{n})\chi(\frac{|p_1|}{n})$$

where χ is a positive function bounded by 1 which is equal to zero outside $[0,1]$. We then have:

$$\int |G_n(p_1,p_2)|^2 dp_1 dp_2 \quad \lesssim \quad \|g\|_\infty^2 n^2.$$

Let f be a positive compactly supported function of integral one. We may then define the following operator:

$$H_{I,n} = \int G_n(p_1,p_2)n^3 f(n(p_1-p_2))b_1^*(p_1)b_2(p_2)dp_1dp_2 + \text{h.c.}$$

Using the same type of arguments as in chapter two it can be proved that $H_n = H_0 + \lambda H_{I,n}$ is self-adjoint for any λ. We may define:

$$\tilde{G}_n(p_1,p_2) = G_n(p_1,p_2)n^3 f(n(p_1-p_2)).$$

Proposition 5.3.1. *If g is continuous and bounded then H_n defined by* (5.7) *and* (5.5) *converges strongly in the resolvent sense to H defined by* (5.6) *and* (5.4) *for any coupling constant $\lambda < \lambda_0$.*

Proof. Let $\psi^{(l,m)}$ be a continuous and compactly supported wave function with m particles of the first species and l of the second one. We have that:

$$\begin{aligned}
&\left\|(H-H_n)\psi^{(l,m)}\right\| = \lambda\left\|(H_I - H_{I,n})\psi^{(l,m)}\right\| \\
= &\ \lambda\left\|\left(\int g(p)b_1^*(p)b_2(p)dp \ - \int nG_n(p_1,p_2)f(n(p_1-p_2))b_1^*(p_1)b_2(p_2)dp_1dp_2\right)\psi^{(l,m)}\right\| \\
= &\ \lambda\left\|\left(\int g(p)f(y)b_1^*(p)b_2(p)dpdy \ - \int G_n(p,-\frac{y}{n}+p)f(y)b_1^*(p)b_2(-\frac{y}{n}+p)dpdy\right)\psi^{(l,m)}\right\| \\
= &\ \lambda\left\|\sqrt{\frac{l}{m+1}}\sum_{j=1}^{m+1}\int\left(g(k_i)\psi^{(l,m)}(\hat{k}_i,k_i) \ - \ G_n(k_i,k_i-\frac{y}{n})\psi^{(l,m)}(\hat{k}_i,-\frac{y}{n}+k_i)\right)f(y)dy\right\|,
\end{aligned}$$

where the notation:

$$\psi^{(m,n)}(k_1,\ldots,\hat{k}_i,\ldots,k_{m+1},k_i,q_2,\ldots,q_n) = \psi^{(m,n)}(\hat{k}_i,k_i)$$

has been used. By compactness of the support of the test function the integrand can be dominated by f up to some constant. We then have:

$$\lim_{n\to\infty}\int\left(g(k_i)\psi^{(l,m)}(\hat{k}_i,k_i) - G_n(k_i,k_i-\frac{y}{n})\psi^{(l,m)}(\hat{k}_i,-\frac{y}{n}+k_i)\right)f(y)dy = 0.$$

Defining:

$$\begin{gathered}
F_n(k_1,\ldots,\hat{k}_i,\ldots,k_{m+1},k_i,q_2,\ldots,q_n) = \\
\int\left(g(k_i)\psi^{(l,m)}(\hat{k}_i,k_i) - G_n(k_i,k_i-\frac{y}{n})\psi^{(l,m)}(\hat{k}_i,-\frac{y}{n}+k_i)\right)f(y)dy.
\end{gathered}$$

Dominating F_n using again the compactness of the support of the test function, we can conclude using Theorem VIII.25a of [81]. □

Remark 5.3.2. *So far we have seen that a translation invariant model can be approximated a by non translation one. We believe that the methods developed here may also be used to remove spacial cutoff of models and it would be interesting to apply them on models presented in Chapter two and three.*

5.4 Spectral property

This section is devoted to the study of the spectra of H_n and H. They can be shown to be purely absolutely continuous, except at 0 which is the only eigenvalue. The proof is based on a Mourre estimate in the so-called weakly conjugate operator framework.

Let us first prove the following theorem:

Theorem 5.2. *Let H_n be an operator defined by* (5.7) *and* (5.5). *Let us assume that:*

$$\tilde{G}_n \in L^2 \tag{5.10}$$

$$\| |p_i|^{-l} |p_j|^{-l'} \partial^\alpha G_n \|_\infty < \infty \quad l + l' = 0,1,2 \quad |\alpha| = 0,1,2, \text{uniformly in n} \tag{5.11}$$

and $\tilde{G}_n$ is differentiable, then for any n there exists a constant λ_n such that for any positive λ such that $\lambda < \lambda_n$ we have:

$$\sigma(H_n) = \{0\} \cup [m, \infty),$$

where m is the mass of the lightest particle. Moreover, 0 is the only eigenvalue of H_n and its spectrum in $[m, \infty)$ is purely absolutely continuous.

To prove this theorem we use a Mourre theory with a weakly conjugate operator. We refer to [47, 86, 73] and references therein. We define:

$$a_j = \frac{i}{2}(-p_j \cdot \nabla_{p_j} + \text{h.c}) \quad \text{with} \quad j = 1, 2, \tag{5.12}$$

which are self-adjoint on $\mathscr{D}(a_j) = \{h \in \mathfrak{h} | a_j h \in \mathfrak{h}\}$. The conjugate operator is defined as:

$$A = \mathrm{d}\Gamma(a) \mathbb{1}_{\Omega^\perp}. \tag{5.13}$$

It generates an evolution group which will be called U_t.

Let us consider any self-adjoint operator H on $\mathscr{H}$. Its domain $\mathscr{D}(H)$ may be considered together with the norm: $\|x\|^2_{\mathscr{G}} = \langle x \,|\langle H \rangle\, x\rangle$ and its completion $\mathscr{G}$ is naturally included in $\mathscr{H}$. Note that $\langle H \rangle$ stands for $(H^2 + \mathbb{1})^{\frac{1}{2}}$. Its dual space is denoted $\mathscr{G}^*$. Identifying $\mathscr{H}$ and $\mathscr{H}^*$, we have that $\mathscr{H} \subset \mathscr{G}^*$. Note that if $x \in \mathscr{H}$, $\|x\|^2_{\mathscr{G}^*} = \left\langle x \middle| \langle H \rangle^{-1} x \right\rangle$. $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ and $H \in \mathscr{L}(\mathscr{G}, \mathscr{G}^*)$. We first impose that for any $t \in \mathbb{R}$

$$U_t \mathscr{G} \subset \mathscr{G}$$

which implies that U_t is also an evolution group on $\mathscr{G}$ together with the fact that $U_{-t} H U_t \in \mathscr{L}(\mathscr{G}, \mathscr{G}^*)$. We then say that $H \in C^1(A, \mathscr{G}, \mathscr{G}^*)$ if the application $t \to U_{-t} H U_t$ is differentiable. Its differential at 0 identifies with $[H, iA]$, which extends to an element of $\mathscr{L}(\mathscr{G}, \mathscr{G}^*)$. The operator A will be said to be weakly conjugate to H if and only if $H \in C^1(A, \mathscr{G}, \mathscr{G}^*)$ and $[H, iA] > 0$, which means that $[H, iA] \geqslant 0$ and $[H, iA]$ is injective.

In the same way the completion of $\mathscr{G}$ equipped with the norm: $\|x\|_{\mathcal{B}} = \langle x \,|[H, iA]x\rangle^{\frac{1}{2}}$ may be defined and will be referred as $\mathcal{B}$ together with its dual space $\mathcal{B}^*$. Considering:

$$\mathcal{E} = \left\{ g \in \mathcal{B}^* \,\middle|\, \|g\|_{\mathcal{E}} = \int t^{-\frac{3}{2}} \|U_t g - g\|_{\mathcal{B}^*} dt < \infty \right\} \tag{5.14}$$

the following results holds [86, 73]:

Theorem 5.3. *Assume that A is weakly conjugate to H and that $[H, iA] \in C^1(A, \mathcal{B}, \mathcal{B}^*)$. Then there exists a constant $C > 0$ such that:*

$$\left| \left\langle f | (H - \lambda \mp \mu)^{-1} f \right\rangle \right| \leqslant C \|f\|_{\mathcal{E}}^2 \tag{5.15}$$

for all $\lambda \in \mathbb{R}$, $\mu > 0$ and $f \in \mathcal{E}$. In particular the spectrum of H is purely absolutely continuous.

Before applying this framework to our model, let us note that for any natural number n, H_n preserves $\mathscr{H}_\Omega = \mathrm{vect}(\Omega)$ and we have:

$$\mathscr{H} = \mathscr{H}_\Omega \oplus \mathscr{H}_{\Omega^\perp}.$$

It is then enough to study:

$$H_{n,\Omega} = H_n|_{\mathscr{H}_\Omega} = 0|_{\mathscr{H}_\Omega} \tag{5.16}$$

$$H_{n,\Omega^\perp} = H_n|_{\mathscr{H}_{\Omega^\perp}}. \tag{5.17}$$

Therefore we will focus on (5.17).

Proposition 5.4.1. *Assume Hypotheses* (5.11)*. A is weakly conjugate to $H_{n,\Omega^\perp}$.*

Proof. We first prove that $[H_{n,\Omega^\perp}, iA] > 0$. In the massive case we have:

$$[H_0, iA] = \mathrm{d}\Gamma\left(\frac{p_1^2}{\sqrt{p_1^2 + m_1^2}}\right) + \mathrm{d}\Gamma\left(\frac{p_2^2}{\sqrt{p_2^2 + m_2^2}}\right) \tag{5.18}$$

and in the massless case:

$$[H_0, iA] = \mathrm{d}\Gamma\left(|p_1|\right) + \mathrm{d}\Gamma\left(|p_2|\right). \tag{5.19}$$

The massless case being easier, we will focus on the massive one. Let us note that:

$$\begin{aligned}
[H_{I,n}, iA] &= \lambda \int (a_2 - a_1) G_n(p_1,p_2) n^3 f(n(p_1 - p_2)) b_1^*(p_1) b_2(p_2) dp_1 dp_2 + h.c. \\
&= \lambda \int (p_2 \cdot \nabla_{p_2} G_n(p_1,p_2) - p_1 \cdot \nabla_{p_1} G_n(p_1,p_2)) n^3 f(n(p_1 - p_2)) b_1^*(p_1) b_2(p_2) dp_1 dp_2 \\
&+ \lambda \int n^3 G_n(p_1,p_2)(p_2 \cdot \nabla_{p_2} f(n(p_1 - p_2)) - p_1 \cdot \nabla_{p_1} f(n(p_1 - p_2))) b_1^*(p_1) b_2(p_2) dp_1 dp_2 + h.c.
\end{aligned}$$

We have :

$$\begin{aligned}
[H_{I,n}, iA] &= \lambda \int (p_2 \cdot \nabla_{p_2} G_n(p_1,p_2) - p_1 \cdot \nabla_{p_1} G_n(p_1,p_2)) n^3 f(n(p_1 - p_2)) b_1^*(p_1) b_2(p_2) dp_1 dp_2 \\
&- \lambda \int n^4 G_n(p_1,p_2)(p_2 + p_1) \cdot \nabla f(n(p_1 - p_2)) b_1^*(p_1) b_2(p_2) dp_1 dp_2 + h.c.
\end{aligned}$$

Let us consider a wave function Φ in the domain of $\mathrm{d}\Gamma\left(\frac{p_2^2}{\sqrt{p_2^2+m_2^2}}\right)^{\frac{1}{2}}$ and of $\mathrm{d}\Gamma\left(\frac{p_1^2}{\sqrt{p_1^2+m_1^2}}\right)^{\frac{1}{2}}$. We have:

$$|A| = \left|\left\langle \Phi \left| \int p_2 \cdot \nabla_{p_2} G_n(p_1,p_2) n^3 f(n(p_1-p_2)) b_1^*(p_1) b_2(p_2) dp_1 dp_2 \Phi \right.\right\rangle\right| .$$

For any $j \in \{1,2,3\}$ we have:

$$\begin{aligned}
|A_j| &= \left|\iint \left\langle b_1(p_2+\frac{u}{n})\Phi \left| \partial_{2,j} G_n(p_2+\frac{u}{n},p_2) f(u) p_{2,j} b_2(p_2)\Phi \right.\right\rangle dp_2 du\right| \\
&\leqslant \iint \left|\left\langle \omega_2^{\frac{1}{2}}(p_2)\overline{f(u)\partial_{2,j}G_n(p_2+\frac{u}{n},p_2)} b_1(p_2+\frac{u}{n})\Phi \left| \omega_2^{-\frac{1}{2}}(p_2) p_{2,j} b_2(p_2)\Phi \right.\right\rangle\right| dp_2 du,
\end{aligned}$$

applying the Cauchy-Schwarz inequality we have:

$$\begin{aligned}
|A_j| &\leqslant \iint \left\|\omega_2^{\frac{1}{2}}(p_2)\overline{f(u)\partial_{2,j}G_n(p_2+\frac{u}{n},p_2)} b_1(p_2+\frac{u}{n})\Phi\right\| \left\|\omega_2^{-\frac{1}{2}}(p_2) p_{2,j} b_2(p_2)\Phi\right\| dp_2 du \\
&\leqslant \int\sqrt{\int \left\|\omega_2^{\frac{1}{2}}(p_2)\overline{f(u)\partial_{2,j}G_n(p_2+\frac{u}{n},p_2)} b_1(p_2+\frac{u}{n})\Phi\right\|^2 dp_2 \int \left\|\omega_2^{-\frac{1}{2}}(p_2)|p_{2,j}| b_2(p_2)\Phi\right\|^2 dp_2 du} \\
&\leqslant \int\sqrt{\int \left\|\omega_1^{-\frac{1}{2}}(p_2)|p_2|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\omega_2^{\frac{1}{2}}(p_2-\frac{u}{n})\overline{f(u)\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})} b_1(p_2)\Phi\right\|^2 dp_2} \\
&\quad \sqrt{\int \left\|\omega_2^{-\frac{1}{2}}(p_2)|p_{2,j}| b_2(p_2)\Phi\right\|^2 dp_2 du}.
\end{aligned}$$

Let us note that $p_2 \to \omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\omega_2^{\frac{1}{2}}(p_2-\frac{u}{n})\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}$ is uniformly bounded with respect to u. Indeed:

$$\begin{aligned}
\left\|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\omega_2^{\frac{1}{2}}(p_2-\frac{u}{n})\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}\right\|_\infty &\leqslant \left\|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\omega_2^{\frac{1}{2}}(p_2)\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}\right\|_\infty \\
&+ \left\|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\left|\frac{u}{n}\right|^{\frac{1}{4}}\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}\right\|_\infty
\end{aligned}$$

For the first term, we have for $|p_2| \geqslant \Lambda$:

$$\left\|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\omega_2^{\frac{1}{2}}(p_2)\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}\right\|_\infty \leqslant \left(1+\frac{m_1^2}{\Lambda^2}\right)^{\frac{1}{4}}\left(1+\frac{m_2^2}{\Lambda^2}\right)^{\frac{1}{4}}\|\partial_{2,j}G_n\|_\infty,$$

if $|p_2| \leqslant \delta$ we have

$$\left\|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\omega_2^{\frac{1}{2}}(p_2)\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}\right\|_\infty \leqslant \left(\delta^2+m_1^2\right)^{\frac{1}{4}}\left(\delta^2+m_2^2\right)^{\frac{1}{4}}\||p_2|^{-1}\partial_{2,j}G_n\|_\infty$$

which is bounded using and Hypothesis (5.11). Finally, $|p_2| \in [\delta,\Lambda]$, $p_2 \to \omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\omega_2^{\frac{1}{2}}(p_2-\frac{u}{n})\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}$ is a continuous function on a compact set (recall that f is compactly supported). For the second term, using again the support of f we have that:

$$\left\|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\left|\frac{u}{n}\right|^{\frac{1}{4}}\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}\right\|_\infty \lesssim \left\|\omega_1^{\frac{1}{2}}(p_2)|p_2|^{-1}\overline{\partial_{2,j}G_n(p_2,p_2-\frac{u}{n})}\right\|_\infty$$

which is clearly bounded using the same type of argument as before. Finally,

$$\begin{aligned}
|A_j| &\lesssim \int |f(u)| \sqrt{\int \left\| \omega_1^{-\frac{1}{2}}(p_2)|p_2| b_1(p_2)\Phi \right\|^2 dp_2 \int \left\| \omega_2^{-\frac{1}{2}}(p_2)|p_{2,j}| b_2(p_2)\Phi \right\|^2 dp_2} du, \\
&\lesssim \int |f(u)| \sqrt{\int \left\| \omega_1^{-\frac{1}{2}}(p_2)|p_2| b_1(p_2)\Phi \right\|^2 dp_2 \int \left\| \omega_2^{-\frac{1}{2}}(p_2)|p_2| b_2(p_2)\Phi \right\|^2 dp_2} du.
\end{aligned}$$

We conclude that:

$$|A| \leqslant |A_1| + |A_2| + |A_3| \lesssim \int f(u) \sqrt{\left\| d\Gamma\left(\frac{p_1^2}{\sqrt{p_1^2+m_1^2}}\right)^{\frac{1}{2}} \Phi \right\|^2 \left\| d\Gamma\left(\frac{p_2^2}{\sqrt{p_2^2+m_2^2}}\right)^{\frac{1}{2}} \Phi \right\|^2} du.$$

Since f is of integral one:

$$|A| \lesssim \left\| d\Gamma\left(\frac{p_1^2}{\sqrt{p_1^2+m_1^2}}\right)^{\frac{1}{2}} \Phi \right\| \left\| d\Gamma\left(\frac{p_2^2}{\sqrt{p_2^2+m_2^2}}\right)^{\frac{1}{2}} \Phi \right\|.$$

The term
$$\left\langle \Phi \middle| \int p_2 \cdot \nabla_{p_2} G_n(p_1,p_2) n^3 f(n(p_1-p_2)) b_1^*(p_1) b_2(p_2) dp_1 dp_2 \Phi \right\rangle$$
may be treated in the same way. Finally let us consider:

$$B = \left\langle \Phi \middle| \int n^4 G_n(p_1,p_2)(p_2+p_1) \cdot \nabla f(n(p_1-p_2)) b_1^*(p_1) b_2(p_2) dp_1 dp_2 \Phi \right\rangle.$$

The term ∇f is continuous and compactly supported and consequently bounded. The same method as what has been done before leads to:

$$|B| \lesssim n \left\| d\Gamma\left(\frac{p_1^2}{\sqrt{p_1^2+m_1^2}}\right)^{\frac{1}{2}} \Phi \right\| \left\| d\Gamma\left(\frac{p_2^2}{\sqrt{p_2^2+m_2^2}}\right)^{\frac{1}{2}} \Phi \right\|.$$

As a conclusion, we see that there exists a constant c such that:

$$[H_{I,n}, iA] \leqslant \lambda c(1+n) d\Gamma\left(\frac{p^2}{\sqrt{p^2+m^2}}\right)$$

so that:

$$[H_{n,\Omega^\perp}, iA] \geqslant (1-\lambda c(1+n)) d\Gamma\left(\frac{p^2}{\sqrt{p^2+m^2}}\right),$$

which is strictly positive on $\mathscr{H}_{\Omega^\perp}$ if $\lambda < 1/(c(1+n))$. Finally, the fact that $H \in C^1(A, \mathscr{G}, \mathscr{G}^*)$ is a straightforward consequence of the H_0 boundedness of $[H_{\Omega^\perp}, iA]$ together with the fact that $\mathscr{D}(H) = \mathscr{D}(H_0)$ and similar arguments as in [46, Section 4].

□

Proof of Theorem 5.2. From the H_0 boundedness of $[A, [H_{\Omega^\perp}, iA]]$, which is a consequence of Hypotheses (5.11) that $[H_{\Omega^\perp}, iA] \in C^1(A, \mathcal{B}, \mathcal{B}^*)$. We then apply Theorem 5.3 together with Proposition 5.4.1. Finally, to prove that there is no spectrum in $(0, m)$ we may use Hypothesis (5.10) to apply the same method we used to prove Theorem 2.2.1 of Chapter 2. $\square$

Some minor modifications in the previous proof would lead to the following results:

Theorem 5.4. *Let H be the Hamiltonian defined by* (5.6) *and* (5.4)*. Let us assume that g is bounded and:*

$$\||p|^{-l}\partial^\alpha g\|_\infty < \infty \quad l = 0, 1, 2 \text{ and } |\alpha| = 0, 1, 2, \tag{5.20}$$

then there exists a constant λ_2 such that for any positive $\lambda < \lambda_2$ we have:

$$\sigma(H) = \{0\} \cup [m, \infty),$$

where m is the mass of the lightest particle. Moreover, 0 is the only eigenvalue of H and its spectrum in $[m, \infty)$ is purely absolutely continuous.

Note that the hypotheses of Theorem 5.4 are enough to build a kernel G_n verifying the hypotheses of Theorem 5.2.

Appendix A

Proof of Lemma 2.1.3

We give a proof of Lemma 2.1.3 due to [19] and which relies on Proposition 1.2.3(b) of [51]. It is recalled here for consistency. Consider for instance the term

$$H^{(1)}_{I,1,+} := \int G^{(1)}_{1,+}(\xi_1, \xi_2, \xi_3) b^*_{1,+}(\xi_1) c^*_{1,-}(\xi_2) a_+(\xi_3) d\xi_1 d\xi_2 d\xi_3 + h.c, \tag{A.1}$$

occurring in H_I. We begin by verifying that $H^{(1)}_{I,1,+}$ is densely defined. Let us consider the following formal operator:

$$h^{(1)}_I = \int d^3\xi_1 d^3\xi_2 d^3\xi_3 \left[G^{(1)}_{1,+}(\xi_1, \xi_2, \xi_3) b^*_{1,+}(\xi_1) c^*_{1,-}(\xi_2) a_+(\xi_3) \right],$$

so that:

$$H^{(1)}_{I,1,+} = h^{(1)}_I + h^{(1)*}_I.$$

Let $Q_1 = (m, \bar{m}, n_1, \bar{n}_1, o_1, \bar{o}_1) \in \mathbb{N}^6$ and $\psi^{Q_1} \in \mathscr{H}$ any wave function with finite m bosons, $\bar{m}$ antibosons, n_1 electrons, $\bar{n}_1$ positrons, o_1 neutrinos and $\bar{o}_1$ antineutrinos. We then have:

$$h^{(1)}_I \psi^{Q_1} = \sqrt{\frac{m+1}{(\bar{o}_1+1)(n_1+1)}} \sum_{j=1}^{n_1+1} \sum_{i=1}^{\bar{o}_1+1} \int (-1)^{j+i-2} G^{(1)}_{1,+}(q_j, p_i, \xi_3)$$

$$\psi^{Q_1}(\xi_3, k_1, \ldots, k_{(m-1)}, p_1, \ldots, \hat{p_i}, \ldots, p_{\bar{o}_1+1}, q_1, \ldots, \hat{q_j}, \ldots, q_{n_1+1}) d\xi_3. \tag{A.2}$$

We can then conclude that:

$$\begin{aligned} \left\langle h^{(1)}_I \psi^{Q_1} | h^{(1)}_I \psi^{Q_1} \right\rangle \quad &= \quad \left(\frac{m+1}{(\bar{o}_1+1)(n_1+1)} \right) \sum_{j=1}^{n_1+1} \sum_{i=1}^{\bar{o}_1+1} \sum_{j'=1}^{n_1+1} \sum_{i'=1}^{\bar{o}_1+1} \int\int \left\langle G^{(1)}_{1,+}(p_i, q_j, \xi_3) \psi^{Q_1}(\xi_3, \hat{p_i}, \hat{q_j}) \right| \\ &\quad G^{(1)}_{1,+}(p_{i'}, q_{j'}, \xi_4) \psi^{Q_1}(\xi_4, \hat{p_{i'}}, \hat{q_{j'}}) \Big\rangle \, d\xi_3 d\xi_4 \end{aligned}$$

and the Cauchy-Schwartz inequality gives:

$$\begin{aligned}\left\langle h_I^{(1)}\psi^{Q_1}|h_I^{(1)}\psi^{Q_1}\right\rangle &\leqslant \left(\frac{m+1}{(\bar{o}_1+1)(n_1+1)}\right)\sum_{j=1}^{n_1+1}\sum_{i=1}^{\bar{o}_1+1}\sum_{j'=1}^{n_1+1}\sum_{i'=1}^{\bar{o}_1+1}\int\int\left\|G_{1,+}^{(1)}(.,.,\xi_3)\psi^{Q_1}(\xi_3,\hat{p_i},\hat{q_j})\right\|_2 \\ &\quad \left\|G_{1,+}^{(1)}(.,.,\xi_4)\psi^{Q_1}(\xi_4,\hat{p_{i'}},\hat{q_{j'}})\right\|_2 d\xi_3 d\xi_4 \\ &\leqslant \left(\frac{m+1}{(\bar{o}_1+1)(n_1+1)}\right)\sum_{j=1}^{n_1+1}\sum_{i=1}^{\bar{o}_1+1}\sum_{j'=1}^{n_1+1}\sum_{i'=1}^{\bar{o}_1+1}\int\int\left\|G_{1,+}^{(1)}(.,.,\xi_3)\right\|_2\ \left\|\psi^{Q_1}(\xi_3,\hat{p_i},\hat{q_j})\right\|_2 \\ &\quad \left\|G_{1,+}^{(1)}(.,.,\xi_4)\|_2\|\psi^{Q_1}(\xi_4,\hat{p_{i'}},\hat{q_{j'}})\right\|_2 d\xi_3 d\xi_4.\end{aligned}$$

We apply again the Cauchy-Schwartz inequality:

$$\begin{aligned}\left\langle h_I^{(1)}\psi^{Q_1}|h_I^{(1)}\psi^{Q_1}\right\rangle &\leqslant \left(\frac{m+1}{(\bar{o}_1+1)(n_1+1)}\right)\sum_{j=1}^{n_1+1}\sum_{i=1}^{\bar{o}_1+1}\sum_{j'=1}^{n_1+1}\sum_{i'=1}^{\bar{o}_1+1}\|G_{1,+}^{(1)}\|_2^2\ \|\psi^{Q_1}\|_2^2, \\ \|h_I^{(1)}\psi^{Q_1}\|_2^2 &\leqslant \left(\frac{m+1}{(\bar{o}_1+1)(n_1+1)}\right)(n_1+1)^2(\bar{o}_1+1)^2\|G_{1,+}^{(1)}\|_2^2\ \|\psi^{Q_1}\|_2^2, \\ \|h_I^{(1)}\psi^{Q_1}\|_2 &\leqslant \sqrt{(m+1)(\bar{o}_1+1)(n_1+1)}\|G_{1,+}^{(1)}\|_2\ \|\psi^{Q_1}\|_2.\end{aligned}$$

The adjoint operator is densely defined too, which is enough to prove that $h_I^{(1)}$ is closable. These results can be extended to the entire interactive Hamiltonian. This proves that $H_{I,1,+}^{(1)}$ is a well densely defined symmetric operator. It remains to prove that it is H_0-bounded. For almost every $\xi_3 \in \Sigma_2$ consider the closed operators:

$$\begin{aligned}B_{1,+}^{(1)}(\xi_3) &= -\int\overline{G_{1,+}^{(1)}(\xi_1,\xi_2,\xi_3)}b_{1,+}(\xi_1)c_{1,-}(\xi_2)d\xi_1 d\xi_2, \\ \left(B_{1,+}^{(1)}(\xi_3)\right)^* &= \int G_{1,+}^{(1)}(\xi_1,\xi_2,\xi_3)b_{1,+}^*(\xi_1)c_{1,-}^*(\xi_2)d\xi_1 d\xi_2.\end{aligned}$$

Moreover, we claim that for almost every $\xi_3 \in \Sigma_2$, $\mathscr{D}(N_1^{\frac{1}{2}}) \subset \mathscr{D}(B_{1,+}^{(1)}(\xi_3)) \cap \mathscr{D}\left(\left(B_{1,+}^{(1)}(\xi_3)\right)^*\right)$ and for all $\Phi \in \mathscr{D}(N_1^{\frac{1}{2}})$ we have:

$$\|B_{1,+}^{(1)}(\xi_3)\Phi\|_{\mathfrak{F}_L} \leqslant \|G_{1,+}^{(1)}(.,.,\xi_3)\|_{L^2(\Sigma_1\times\Sigma_1)}\|N_1^{\frac{1}{2}}\Phi\|_{\mathfrak{F}_L}, \tag{A.3}$$

$$\left\|\left(B_{1,+}^{(1)}(\xi_3)\right)^*\Phi\right\|_{\mathfrak{F}_L} \leqslant \|G_{1,+}^{(1)}(.,.,\xi_3)\|_{L^2(\Sigma_1\times\Sigma_1)}\|N_1^{\frac{1}{2}}\Phi\|_{\mathfrak{F}_L}. \tag{A.4}$$

Let us prove this for $B_{1,+}^{(1)}(\xi_3)$. Let $Q = (n,\bar{n},q,\bar{q})$ and $Q' = (n',\bar{n}',q',\bar{q}')$ be two elements of $\mathbb{N}^4$. Let $\Psi^{(Q)}$, respectively $\Psi^{(Q')}$, be a wave function with n electrons, $\bar{n}$ positrons, q neutrinos and $\bar{q}$ antineutrinos, respectively n' electrons, $\bar{n'}$ positrons, q' neutrinos and $\bar{q}'$ antineutrinos. We have:

$$\left\langle\Phi^{(Q')}|B_{1,+}^{(1)}(\xi_3)\Phi^{(Q)}\right\rangle = -\int\left\langle\Phi^{(Q')}\mid b_{1,+}(\xi_1)c_{1,-}(\xi_2)\Phi^{(Q)}\right\rangle\overline{G_{1,+}^{(1)}(\xi_1,\xi_2,\xi_3)}d\xi_1 d\xi_2,$$

and hence

$$\begin{aligned}\left|\left\langle \Phi^{(Q')}|B^{(1)}_{1,+}(\xi_3)\Phi^{(Q)}\right\rangle\right| &= \left|\int\left\langle\int G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)c^*_{1,+}(\xi_1)\Phi^{(Q')}d\xi_1 \mid b_{1,-}(\xi_2)\Phi^{(Q)}\right\rangle d\xi_2\right|\\ &= \left|\int\left\langle c^*_{1,+}(G^{(1)}_{1,+}(.,\xi_2,\xi_3))\Phi^{(Q')}\mid b_{1,-}(\xi_2)\Phi^{(Q)}\right\rangle d\xi_2\right|.\end{aligned}$$

The Cauchy-Schwartz inequality gives:

$$\begin{aligned}\left|\left\langle \Phi^{(Q')}|B^{(1)}_{1,+}(\xi_3)\Phi^{(Q)}\right\rangle\right| &\leqslant \left|\int\left\|c^*_{1,+}(G^{(1)}_{1,+}(.,\xi_2,\xi_3))\Phi^{(Q')}\right\|\left\|b_{1,-}(\xi_2)\Phi^{(Q)}\right\|d\xi_2\right|\\ &\leqslant \left|\int\left\|G^{(1)}_{1,+}(.,\xi_2,\xi_3)\right\|\left\|b_{1,-}(\xi_2)\Phi^{(Q)}\right\|d\xi_2\right|\left\|\Phi^{(Q')}\right\|.\end{aligned}$$

Next, using the definition of $c_{1,-}(\xi_2)$ and applying again Cauchy-Schwartz:

$$\begin{aligned}\left|\left\langle \Phi^{(Q')}|B^{(1)}_{1,+}(\xi_3)\Phi^{(Q)}\right\rangle\right| &\leqslant \sqrt{q}\left\|G^{(1)}_{1,+}(.,.,\xi_3)\right\|\left\|\Phi^{(Q)}\right\|\left\|\Phi^{(Q')}\right\|\\ &\leqslant \left\|G^{(1)}_{1,+}(.,.,\xi_3)\right\|\left\|N_1^{\frac{1}{2}}\Phi^{(Q)}\right\|\left\|\Phi^{(Q')}\right\|.\end{aligned}$$

The inequality is true for all wave functions $\Phi^{(Q')}$ with a finite number of particles, which leads to:

$$\|B^{(1)}_{1,+}(\xi_3)\Phi^{(Q)}\| \leqslant \|G^{(1)}_{1,+}(.,.,\xi_3)\|\;\|N_1^{\frac{1}{2}}\Phi^{(Q)}\|.$$

This proves (A.3). The same strategy would prove (A.4). We then introduce the self-adjoint operator:

$$H^{(3)}_{0,+} = \int w^{(3)}(\xi_3)a^*_+(\xi_3)a_+(\xi_3)d\xi_3,$$

and the next step is to prove, for every $\Psi \in \mathscr{D}(H_0)$ and every $\eta > 0$, the following two inequalities:

$$\begin{aligned}\left\|\int\left(B^{(1)}_{1,+}(\xi_3)\right)^*\otimes a_+(\xi_3)d\xi_3\Psi\right\|^2 &\leqslant \left(\int\frac{|G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)}d\xi_1d\xi_2d\xi_3\right)\|(N_1+1)^{\frac{1}{2}}\otimes(H^{(3)}_{0,+})^{\frac{1}{2}}\Psi\|^2\\ \left\|\int\left(B^{(1)}_{1,+}(\xi_3)\right)\otimes a^*_+(\xi_3)d\xi_3\Psi\right\|^2 &\leqslant \left(\int\frac{|G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)}d\xi_1d\xi_2d\xi_3\right)\|(N_1+1)^{\frac{1}{2}}\otimes(H^{(3)}_{0,+})^{\frac{1}{2}}\Psi\|^2\\ &+ \left(\int|G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)|^2d\xi_1d\xi_2d\xi_3\right)\left(\eta\|(N_1+1)^{\frac{1}{2}}\otimes\mathbb{1}\Psi\|^2\right.\\ &+ \left.\frac{1}{4\eta}\|\Psi\|^2\right).\end{aligned}\tag{A.5}$$

To prove these inequalities, let Ψ be in $\mathscr{D}(N_1^{\frac{1}{2}})\hat{\otimes}\mathscr{D}(H^{(3)}_{0,+})$, where $\hat{\otimes}$ is the algebraic tensor product. Considering

$$\Psi_+(\xi_3) = w^{(3)}(\xi_3)^{\frac{1}{2}}((N_1+1)^{\frac{1}{2}}\otimes a(\xi_3))\Psi,$$

we have:

$$\int \|\Psi_+(\xi_3)\|^2 d\xi_3 = \|(N_1+1)^{\frac{1}{2}} \otimes (H_{0,+}^{(3)})^{\frac{1}{2}}\Psi\|^2.$$

Since

$$\int (B_{1,+}^{(1)}(\xi_3))^* \otimes a_+(\xi_3) d\xi_3 \Psi = \int \frac{1}{w^{(3)}(\xi_3)^{\frac{1}{2}}} ((B_{1,+}^{(1)}(\xi_3))^* (N_l+1)^{-\frac{1}{2}} \otimes \mathbb{1})\Psi_+(\xi_3) d\xi_3,$$

then

$$\begin{aligned}
\left\|\int \left(B_{1,+}^{(1)}(\xi_3)\right)^* \otimes a_+(\xi_3) d\xi_3 \Psi\right\|^2 &\leqslant \left(\int \frac{1}{w^{(3)}(\xi_3)^{\frac{1}{2}}} \|(B_{1,+}^{(1)}(\xi_3))^* (N_1+1)^{-\frac{1}{2}}\| \; \|\Psi_+(\xi_3)\| d\xi_3\right)^2 \\
&\leqslant \left(\int \frac{|G_{1,+}^{(1)}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)} d\xi_1 d\xi_2 d\xi_3\right) \|(N_1+1)^{\frac{1}{2}} \otimes (H_{0,+}^{(3)})^{\frac{1}{2}}\Psi\|^2,
\end{aligned}$$

where we have used (A.4). For the next inequality we have:

$$\begin{aligned}
\left\|\int B_{1,+}^{(1)}(\xi_3) \otimes a_+^*(\xi_3) d\xi_3 \Psi\right\|^2 &= \int \left\langle B_{1,+}^{(1)}(\xi_3) \otimes a_+^*(\xi_3)\Psi | B_{1,+}^{(1)}(\xi_3') \otimes a_+^*(\xi_3')\Psi \right\rangle d\xi_3 d\xi_3' \\
&= \int \left\langle B_{1,+}^{(1)}(\xi_3) \otimes a_+(\xi_3') a_+^*(\xi_3)\Psi | B_{1,+}^{(1)}(\xi_3') \otimes \mathbb{1}\Psi \right\rangle d\xi_3 d\xi_3' \\
&= \int \left\langle B_{1,+}^{(1)}(\xi_3) \otimes (a_+^*(\xi_3) a_+(\xi_3') + \delta(\xi_3 - \xi_3'))\Psi \right| \\
&\quad \left| B_{1,+}^{(1)}(\xi_3') \otimes \mathbb{1}\Psi \right\rangle d\xi_3 d\xi_3' \\
&= \int \left\langle B_{1,+}^{(1)}(\xi_3) \otimes a_+(\xi_3')\Psi | B_{1,+}^{(1)}(\xi_3') \otimes a_+(\xi_3)\Psi \right\rangle d\xi_3 d\xi_3' \\
&+ \int \|B_{1,+}^{(1)}(\xi_3) \otimes \mathbb{1}\Psi\|^2 d\xi_3.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int \left\langle B_{1,+}^{(1)}(\xi_3) \otimes a_+(\xi_3')\Psi | B_{1,+}^{(1)}(\xi_3') \otimes a_+(\xi_3)\Psi \right\rangle d\xi_3 d\xi_3' &= \frac{1}{w^{(3)}(\xi_3)^{\frac{1}{2}} w^{(3)}(\xi_3')^{\frac{1}{2}}} \int \left\langle B_{1,+}^{(1)}(\xi_3)(N_1+1)^{-\frac{1}{2}} \otimes \mathbb{1} \right. \\
&\quad \left. \Psi_+(\xi_3') | B_{1,+}^{(1)}(\xi_3')(N_1+1)^{-\frac{1}{2}} \otimes \mathbb{1}\Psi_+(\xi_3) \right\rangle d\xi_3 d\xi_3'.
\end{aligned}$$

The same type of arguments as before are enough to show that:

$$\begin{aligned}
\int \left\langle B_{1,+}^{(1)}(\xi_3) \otimes a_+(\xi_3')\Psi | B_{1,+}^{(1)}(\xi_3') \otimes a_+(\xi_3)\Psi \right\rangle d\xi_3 d\xi_3' &\leqslant \left(\int \frac{|G_{1,+}^{(1)}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)} d\xi_1 d\xi_2 d\xi_3\right) \\
&\quad \|(N_1+1)^{\frac{1}{2}} \otimes (H_{0,+}^{(3)})^{\frac{1}{2}}\Psi\|^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int \|B^{(1)}_{1,+}(\xi_3)\otimes \mathbb{1}\Psi\|^2 d\xi_3 &= \int \left\| \left(B^{(1)}_{1,+}(\xi_3)(N_1+1)^{-\frac{1}{2}}\otimes 1\right)\left((N_1+1)^{\frac{1}{2}}\otimes \mathbb{1}\Psi\right)\right\|^2 d\xi_3\\
&\leqslant \left(\int |G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)|^2 d\xi_1 d\xi_2 d\xi_3\right)\|(N_1+1)^{\frac{1}{2}}\otimes \mathbb{1}\Psi\|^2\\
&\leqslant \left(\int |G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)|^2 d\xi_1 d\xi_2 d\xi_3\right)\langle (N_1+1)\otimes \mathbb{1}\Psi|\Psi\rangle\\
&\leqslant \left(\int |G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)|^2 d\xi_1 d\xi_2 d\xi_3\right)\|(N_1+1)\otimes \mathbb{1}\Psi\|\;\|\Psi\|\\
&\leqslant \left(\int |G^{(1)}_{1,+}(\xi_1,\xi_2,\xi_3)|^2 d\xi_1 d\xi_2 d\xi_3\right)\left(\eta\|(N_1+1)\Psi\|^2+\frac{1}{4\eta}\|\Psi\|^2\right).
\end{aligned}$$

These properties are proved for the $H^{(1)}_{I,+,-}$ and it is easy to see that they extend to the entire interactive term. We now have all the ingredients to prove the lemma using Kato-Rellich theorem:

$$\|H_I\Psi\|^2 \leqslant \sum_{\alpha=1,2}\sum_{l=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\{\left\|\int (B^{(\alpha)}_{l,\epsilon,\epsilon'}(\xi_3))^*\otimes a_\epsilon(\xi_3)\Psi d\xi_3\right\|^2+\left\|\int B^{(\alpha)}_{l,\epsilon,\epsilon'}(\xi_3)\otimes a^*_\epsilon(\xi_3)\Psi d\xi_3\right\|^2\right\}.$$

Using (A.5) we have :

$$\begin{aligned}
\|H_I\Psi\|^2 \leqslant& \sum_{\alpha=1,2}\sum_{l=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\{\left(\int \frac{|G^{(1)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)}d\xi_1 d\xi_2 d\xi_3\right)\|(N_l+1)^{\frac{1}{2}}\otimes (H^{(3)}_{0,\epsilon})^{\frac{1}{2}}\Psi\|^2\right.\\
&+\left(\int \frac{|G^{(1)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)}d\xi_1 d\xi_2 d\xi_3\right)\|(N_l+1)^{\frac{1}{2}}\otimes (H^{(3)}_{0,\epsilon})^{\frac{1}{2}}\Psi\|^2\\
&\left.+\left(\int |G^{(1)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2 d\xi_1 d\xi_2 d\xi_3\right)\left(\eta\|(N_l+1)^{\frac{1}{2}}\otimes \mathbb{1}\Psi\|^2+\frac{1}{4\eta}\|\Psi\|^2\right)\right\}.
\end{aligned}$$

Obviously,

$$w^{(2)}(\xi_3)\geqslant m_W.$$

Therefore,

$$\begin{aligned}
\|H_I\Psi\|^2 \leqslant& \sum_{\alpha=1,2}\sum_{l=1}^{3}\sum_{\epsilon\neq\epsilon'}\left\{\left(\int \frac{|G^{(1)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2}{m_W}d\xi_1 d\xi_2 d\xi_3\right)\|(N_l+1)^{\frac{1}{2}}\otimes (H^{(3)}_{0,\epsilon})^{\frac{1}{2}}\Psi\|^2\right.\\
&+\left(\int \frac{|G^{(1)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2}{m_W}d\xi_1 d\xi_2 d\xi_3\right)\|(N_l+1)^{\frac{1}{2}}\otimes (H^{(3)}_{0,\epsilon})^{\frac{1}{2}}\Psi\|^2\\
&\left.+\left(\int |G^{(1)}_{l,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2 d\xi_1 d\xi_2 d\xi_3\right)\left(\eta\|(N_l+1)^{\frac{1}{2}}\otimes \mathbb{1}\Psi\|^2+\frac{1}{4\eta}\|\Psi\|^2\right)\right\}
\end{aligned}$$

$$\begin{aligned}
\|H_I\Psi\|^2 \leqslant& \sum_{\alpha=1,2}\sum_{l=1}^{3}\sum_{\epsilon\neq\epsilon'}\|G^{(\alpha)}_{l,\epsilon,\epsilon'}\|\left\{\frac{1}{m_W}\left(\|(N_l+1)^{\frac{1}{2}}\otimes (H^{(3)}_{0,\epsilon})^{\frac{1}{2}}\Psi\|^2+\|(N_l+1)^{\frac{1}{2}}\otimes (H^{(3)}_{0,\epsilon})^{\frac{1}{2}}\Psi\|^2\right)\right.\\
&\left.+\left(\eta\|(N_l+1)^{\frac{1}{2}}\otimes \mathbb{1}\Psi\|^2+\frac{1}{4\eta}\|\Psi\|^2\right)\right\}.
\end{aligned}$$

Furthermore, for any positive β and α:

$$\begin{aligned}
\|(N_l+1)^{\frac{1}{2}}\otimes(H_{0,\epsilon}^{(3)})^{\frac{1}{2}}\Psi\|^2 &= \left\langle (N_l+1)^{\frac{1}{2}}\otimes(H_{0,\epsilon}^{(3)})^{\frac{1}{2}}\Psi|(N_l+1)^{\frac{1}{2}}\otimes(H_{0,\epsilon}^{(3)})^{\frac{1}{2}}\Psi\right\rangle \\
&= \left\langle (N_l+1)\otimes\mathbb{1}\Psi|\mathbb{1}\otimes(H_{0,\epsilon}^{(3)})\Psi\right\rangle \\
&\leqslant \|(N_l+1)\otimes\mathbb{1}\Psi\|\ \|\mathbb{1}\otimes(H_{0,\epsilon}^{(3)})\Psi\| \\
&\leqslant (\|(N_l)\otimes\mathbb{1}\Psi\|+\|\Psi\|)\ \|\mathbb{1}\otimes(H_{0,\epsilon}^{(3)})\Psi\| \\
&\leqslant \beta\|(N_l)\otimes\mathbb{1}\Psi\|^2+\frac{1}{4\beta}\|\mathbb{1}\otimes(H_{0,\epsilon}^{(3)})\Psi\|^2+\alpha\|\Psi\|^2+\frac{1}{4\alpha}\|\mathbb{1}\otimes(H_{0,\epsilon}^{(3)})\Psi\|^2 \\
&\leqslant \frac{\beta}{m_1^2}\|H_0\Psi\|^2+\frac{1}{4\beta}\|H_0\Psi\|^2+\alpha\|\Psi\|^2+\frac{1}{4\alpha}\|H_0\Psi\|^2 \\
&\leqslant \frac{1}{2}\left(\frac{1}{m_1^2}+1\right)\|H_0\Psi\|^2+\alpha\|\Psi\|^2+\frac{1}{4\alpha}\|H_0\Psi\|^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\eta\|(N_l+1)\otimes\mathbb{1}\Psi\|^2+\frac{1}{4\eta}\|\psi\|^2 &\leqslant \eta\left(\|N_l\otimes\mathbb{1}\Psi\|^2+\|\Psi\|^2+2\|N_l\otimes\mathbb{1}\Psi\|\ \|\Psi\|\right)+\frac{1}{4\eta}\|\psi\|^2 \\
&\leqslant \frac{\eta}{m_1^2}\|H_0\Psi\|^2+\eta\|\Psi\|^2+2\eta\beta\|N_l\otimes\mathbb{1}\Psi\|^2+\frac{\eta}{2\beta}\|\Psi\|^2+\frac{1}{4\eta}\|\psi\|^2 \\
&\leqslant \left(\frac{\eta}{m_1}+\frac{2\eta\beta}{m_1^2}\right)\|H_0\Psi\|^2+\left(\eta+\frac{\eta}{2\beta}+\frac{1}{4\eta}\right)\|\psi\|^2.
\end{aligned}$$

This leads to:

$$\begin{aligned}
\|H_I\Psi\|^2 \leqslant\ & \sum_{\alpha=1,2}\sum_{l=1}^{3}\sum_{\epsilon\neq\epsilon'}\|G_{l,\epsilon,\epsilon'}^{(\alpha)}\|\left\{\frac{1}{m_W}\left(\frac{1}{m_1^2}+1\right)\|H_0\Psi\|^2+\left(2\alpha+\eta+\frac{\eta}{2\beta}+\frac{1}{4\eta}\right)\|\Psi\|^2+\right. \\
& \left.\left(\frac{1}{2\alpha}+\frac{\eta}{m_1}+\frac{2\eta\beta}{m_1^2}\right)\|H_0\Psi\|^2\right\}.
\end{aligned}$$

Finally, using the Kato-Rellich theorem, this concludes the proved.

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Abstract

In this work, we consider, first, mathematical models of the weak decay of the vector bosons $W^{\pm}$ into leptons. The free quantum field Hamiltonian is perturbed by an interaction term from the standard model of particle physics. After the introduction of high energy and spatial cut-offs, the total quantum Hamiltonian defines a self-adjoint operator on a tensor product of Fock spaces. We study the scattering theory for such models. First, the masses of the neutrinos are supposed to be positive: for all values of the coupling constant, we prove asymptotic completeness of the wave operators. In a second model, neutrinos are treated as massless particles and we consider a simpler interaction Hamiltonian: for small enough values of the coupling constant, we prove again asymptotic completeness, using singular Mourre's theory, suitable propagation estimates and the conservation of the difference of some number operators.

We moreover study Hamiltonian models representing an arbitrary number of spin 1/2 fermion quantum fields interacting through arbitrary processes of creation or annihilation of particles. The fields may be massive or massless. The interaction form factors are supposed to satisfy some regularity conditions in both position and momentum space. Without any restriction on the strength of the interaction, we prove that the Hamiltonian identifies to a self-adjoint operator on a tensor product of anti-symmetric Fock spaces and we establish the existence of a ground state. Our results rely on novel interpolated N_τ estimates. They apply to models arising from the Fermi theory of weak interactions, with ultraviolet and spatial cut-offs.

Finally, the removal of spatial cut-off to define translation invariant toy models we will be quickly discussed in the last chapter.

Résumé

Dans ce travail nous considérons d'abord un modèle mathématique de la désintégration des bosons $W^{\pm}$ en leptons. L'hamiltonien d'énergie libre est perturbé par un terme d'interaction issu du modèle standard de la physique des particules. Après avoir introduit des coupures en hautes énergies ainsi qu'en espace, nous démontrons que l'Hamiltonien est un opérateur auto-adjoint sur un produit tensoriel d'espaces de Fock. Nous en étudions la théorie de la diffusion. D'abord, nous supposons que les neutrinos ont une masse non nulle et la complétude asymptotique est vérifiée pour une valeur quelconque de la constante de couplage. Dans un deuxième temps, nous considérons des neutrinos non massifs dans un modèle simplifié. Nous démontrons alors la complétude asymptotique en supposant que la constante de couplage est suffisamment petite, en utilisant une théorie de Mourre singulière, des estimations de propagation adaptées ainsi que la conservation d'une certaine combinaison linéaire d'opérateurs de nombre de particules.

Nous étudions par ailleurs des modèles de théorie des champs pour un nombre fini mais quelconque de fermions de spin 1/2. Le terme d'interaction est obtenu en considérant toutes les combinaisons possibles pour les opérateurs de création et d'annihilation. Les différents champs peuvent être massifs comme non massifs et le noyau d'interaction doit vérifier des hypothèses de régularité en espace comme en moment. L'hamiltonien est alors un opérateur auto-adjoint, quelque soit l'intensité de l'interaction, sur un produit tensoriel d'espaces de Fock. Nous démontrons par ailleurs l'existence d'un état fondamental. Nos résultats s'appuient sur une interpolation d'estimation en N_τ et peuvent intervenir dans la modélisation de processus d'interaction faible dans la théorie de Fermi.

Nous présenterons enfin une façon de retirer la troncature en espace sur des modèles jouets afin de définir un modèle invariant par translation.