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Grandes déviations précises pour des statistiques de test

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Contents

Introduction

The aim of this dissertation is to study the Sharp Large Deviations of some statistics: the empirical correlation coefficient between 2 random variables and a self-normalized statistic called Moran statistic. Throughout the whole thesis, the study of Laplace's method is presented as a powerful mean of approximating the integral of type $\int_a^b e^{-xp(t)}q(t)dt$ when x goes to infinity.

In the early history of the *Large deviation principle* (LDP), the term "large deviation" is commonly known as refinements of the *Central limit theorem* (CLT), when an expansion is set at some points which are different from the mean. The definition of the LDP is formally introduced at the end of the 1970s. Cramér [22] has first stated the so-called Cramér's theorem for distributions on $\mathbb{R}$ and Chernoff [16] extended this theorem by the following result. Let $\{X_n\}_n$ be a sequence of *independent, identically distributed* (i.i.d.) random variables with law μ on $\mathbb{R}$, $S_n = \sum_{k=1}^n X_k$ and $c > E(X_1)$. Then the following values

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\frac{S_n}{n} \geq c) \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{n} \log P(\frac{S_n}{n} \geq c)$$

are controlled by a function L^*, which is the dual Fenchel-Legendre transform of the log–Laplace function $L(\lambda) = \log E\left(e^{\lambda X_1}\right)$. The extension of Cramér's theorem in dependent case is considered by Plachky and Steinebach [43] (in $\mathbb{R}$) and Gärtner [31] (in $\mathbb{R}^d$). We will mention the LDP and its fundamental properties in Chapter 1 of this thesis. For more details in general cases, we refer to [26] or [27].

The large deviations results only show the comprehensive view of limiting behavior, through their asymptotic upper and lower exponential bounds, of a family of probability measures in terms of a rate function.

In 1960, Bahadur and Rao [6] established an asymptotic expansion of large deviations for the tail probability $P(\frac{S_n}{n} \geq c)$ as follows. Let $\{X_n\}_n$ and S_n be defined as above. Let a be a constant ($a > E(X_1)$). Under some assumptions (which are detailed in Chapter 1), there exists a sequence $(b_n)_n$ of positive numbers such that

$$P(\frac{S_n}{n} \geq a) = \frac{\Lambda^n \, b_n}{(2\pi n)^{1/2}} \left(1 + o(1)\right), \tag{1}$$

and $\log b_n = O(1)$ as $n \to +\infty$. Here, Λ is a constant defined by

$$\Lambda = \inf_{\lambda \in D} \{e^{-a\lambda} E\left(e^{\lambda X_1}\right)\},$$

where the domain $D = \{\lambda : E(e^{\lambda X_1}) < +\infty\}$.

Furthermore, for each $j = 1, 2, \dots,$ there exists a bounded sequence $(c_{j,n})_n$ such that, for any given positive integer k,

$$P(\frac{S_n}{n} \geq a) = \frac{\Lambda^n b_n}{(2\pi n)^{1/2}} \left(1 + \frac{c_{1,n}}{n} + \frac{c_{2,n}}{n^2} + \cdots + \frac{c_{k,n}}{n^k}\right) \left(1 + O\left(\frac{1}{n^{k+1}}\right)\right), \tag{2}$$

as $n \to +\infty$.
The distribution of $\{X_n\}_n$ in paper [6] is considered in three different cases: X_1's distribution is absolutely continuous, X_1 is a lattice variable or X_1 is none of these two cases.

In the spirit of [6], many results on tail probability's asymptotic expansions have been developed and are commonly known as "*sharp large deviation principles*" (SLDP) or "*strong large deviations*" (SLD). The paper of Bahadur and Rao [6] contains the other result of Blackwell and Hodges ([11], 1959) in lattice case. Book studied SLD for weighted sums of i.i.d. random variables ([12], 1972). Chaganty and Sethurama generalized Theorem 1 of [6] to arbitrary sequence of random variables under some conditions on the moment generating function (m.g.f.) of S_n ([14], 1993) and extended their earlier result to multi-dimensional case ([15], 1996). Cho and Joen ([17], 1994) proved SLD theorem for the ratio of the independent random variables. In the statistical field recently, there have also been numerous results. Bercu, Gamboa, and Lavielle ([9], 2000) established SLDP for Gaussian quadratic forms. Bercu and Rouault ([10], 2002) studied SLD for Ornstein–Uhlenbeck processes and later, Bercu, Coutin, and Savy extended the previous results to fractional Ornstein–Uhlenbeck processes ([7], 2011) and non-stationary cases ([8], 2012). Rovira and Tindel studied SLD for a certain class of sets on the Wiener space ([49], 2000) and for the Stochastic Heat Equation ([50], 2001). Joutard obtained SLD results in nonparametric estimation ([32], 2006), for the conditional empirical process ([33], 2008) and for arbitrary sequences of random variables ([35] and [34], 2013). In [35], Joutard illustrated his results with the kernel density estimator, sample variance, Wilcoxon signed-rank statistic and Kendall tau statistic. The large deviations results for each case was proved earlier in [38], [56], [36], respectively. Daouia and Joutard studied SLD properties of the quantile-based frontier estimators ([23], 2009). Zhou and Zhao derived SLD for the log-likelihood ratio of an α-Brownian Bridge ([63], 2013). Zhao, Q. Liu, F. Liu and Yin gave a SLD for the Energy of α-Brownian Bridge ([62], 2013).

In this thesis, we prove SLDP proceeding as in Bercu et al. [9, 10]. Their work is detailed in Chapter 1, where we also briefly mention the work of Joutard [35]. Under the assumption of [35], the SLD result is merely obtained in the first-order expansion. The process in [9, 10] allows us to expand the SLD in higher order depending on the expansion given by Laplace's method. Let us now detail the remaining chapters of this thesis.

Chapter 2 is devoted to the presentation of the powerful so-called Laplace's method (or stationary phase method for the general complex case) which gives the asymptotic behavior –as x goes to infinity– of integrals $I(x) = \int_a^b e^{-xp(t)} q(t)dt$, where the functions p, q and the real numbers a, b, are independent of the parameter x. Such methods appeared in the early 18th century with the work of Laplace ([25], 1820) and the expansion can be given explicitly for several usual functions (see e.g. [37]). Expansions for the Stirling formula and hypergeometric functions are mentioned in Chapter 3. To the best of my knowledge, Laplace's method is often presented in the first order form (in x, see e.g. [28] or [45]) and rarely described in its full expansion as follows (see more details in forthcoming

Theorem 2.3.10)

$$\int_{\mathbb{R}} e^{xp(t)} q(t) dt = e^{xf(t_0)} \left(\frac{c_0(t_0)}{\sqrt{x}} + \frac{c_1(t_0)}{2!\, x^{3/2}} + \cdots + \frac{c_N(t_0)}{(2N)!\, x^{N+1/2}} + O\left(\frac{1}{x^{N+3/2}}\right)\right).$$

The coefficients $c_0, \cdots, c_N$ depend on the values of the k-th derivatives of functions p and q at the minimum t_0 of p.

Chapter 3 presents SLD for the empirical correlation coefficient in two different cases: spherical and Gaussian distributions. In 1895, Karl Pearson introduced an index to measure correlation, which was called *Pearson product-moment correlation coefficient*, *Pearson's correlation coefficient* or more simply *correlation coefficient*. To measure the dependence between two random variables X and Y, the Pearson's correlation coefficient is given by

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}.$$

At the same time, Pearson developed the *empirical Pearson correlation coefficient* between two samples $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ as

$$r_n = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1} (X_i - \bar{X}_n)^2 \sum_{i=1} (Y_i - \bar{Y}_n)^2}}.$$

where $\bar{X}_n = \frac{1}{n}\sum_{k=1}^n X_k$ and $\bar{Y}_n = \frac{1}{n}\sum_{k=1}^n Y_k$ are the empirical means of the samples. In case $E(X)$ and $E(Y)$ are both known, we can consider

$$\tilde{r}_n = \frac{\sum_{i=1}^n (X_i - E(X))(Y_i - E(Y))}{\sqrt{\sum_{i=1} (X_i - E(X))^2 \sum_{i=1} (Y_i - E(Y))^2}}.$$

By using Cauchy–Schwartz inequality, it can be shown that the absolute values of ρ, r_n and $\tilde{r}_n$ are less than or equal to 1. $\rho = \pm 1$ if and only if X and Y are linearly related i.e. there exists a functional relationship between X and Y; and if $\rho = 0$ then we say that X and Y are uncorrelated. The study of the correlation coefficient is detailed in many references (see e.g. [40] or [52]) and it is shown that many “competing” correlation indexes are special cases of Pearson's correlation coefficient ([48]). The SLD results for r_n and $\tilde{r}_n$ when two samples have Gaussian distribution and spherical distribution, respectively, are presented.

Spherical distribution: Muirhead studied the distribution of the sample correlation coefficient in several multivariate cases (see [40]). Under some assumptions detailed later on, we know from [40] that the density function of r_n is

$$\frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})}(1 - r^2)^{(n-4)/2}, \quad (-1 < r < 1)$$

and we can show that the density function of $\tilde{r}_n$ is

$$\frac{\Gamma(\frac{n}{2})}{\pi^{1/2}\Gamma(\frac{n-1}{2})}(1 - r^2)^{(n-3)/2}, \quad (-1 < r < 1).$$

The SLD results of r_n and $\tilde{r}_n$ will be obtained as follows

$$P(r_n \geq c) = \frac{e^{-nL^*(c)-\frac{1}{2}\log(1+4\lambda_c^2)+\frac{3}{2}\log\frac{1+\sqrt{1+4\lambda_c^2}}{2}}}{\lambda_c\sigma_c\sqrt{2\pi n}}(1+o(1))$$

and

$$P(\tilde{r}_n \geq c) = \frac{e^{-nL^*(c)-\frac{1}{4}\log(1+4\lambda_c^2)+\log\frac{1+\sqrt{1+4\lambda_c^2}}{2}}}{\lambda_c\sigma_c\sqrt{2\pi n}}(1+o(1))\,,$$

where $L^*(s) = -\frac{1}{2}\log(1-s^2)$ is the Fenchel–Legendre dual of $L(\lambda)$ which is the limit of the normalized cumulant generating function L_n of r_n

$$L(\lambda) := \lim_{n\to\infty} L_n(\lambda) = \lim_{n\to\infty}\frac{1}{n}\log E(e^{n\lambda r_n})\,.$$

Gaussian distribution: In 2007, Si presented large deviations results for r_n and $\tilde{r}_n$ as follows [55]. Let $(X_i, Y_i), i = 1, 2, \ldots n$ be the i.i.d. sample of $\mathbb{R}^2$-valued Gaussian vector (X, Y). Assume that $\mathrm{Var}(X) = \sigma_1^2 > 0$ and $\mathrm{Var}(Y) = \sigma_2^2 > 0$, $\mathrm{Cov}(X, Y) = \rho\sigma_1\sigma_2$, where $|\rho| < 1$. Then the law of r_n and $\tilde{r}_n$ satisfy the LDP on $\mathbb{R}$ with the same rate function I, where

$$I(s) = \begin{cases} \log\dfrac{1-s\rho}{\sqrt{(1-\rho^2)(1-s^2)}} & , -1 < s < 1, \\ +\infty & , \text{ otherwise.} \end{cases}$$

In Chapter 3, we prove the SLD results for r_n and $\tilde{r}_n$ independently of the work of [55]. Once again, Muirhead [40] gave the density function of r_{n+1} as follows

$$\frac{(n-1)\Gamma(n)}{\Gamma(n+1/2)\sqrt{2\pi}}(1-\rho^2)^{n/2}(1-\rho r)^{-n+1/2}(1-r^2)^{(n-3)/2}$$
$${}_2F_1\left(\frac{1}{2},\frac{1}{2};n+\frac{1}{2};\frac{1}{2}(1+\rho r)\right) \qquad (-1 < r < 1)\,.$$

where ${}_2F_1$ is the hypergeometric function.

We can obtain the SLD for r_n

$$P(r_n \geq c) = \frac{e^{-nL^*(c)+\log\overline{g}_\rho(r_0(\lambda))-\frac{1}{2}\log|\overline{h}''(r_0(\lambda))|}}{\lambda_c\sigma_c\sqrt{2\pi n}}(1+o(1))\,,$$

where for any $-1 < s < 1$,

$$L^*(s) = \log\left(\frac{1-\rho s}{\sqrt{(1-\rho^2)}\sqrt{(1-s^2)}}\right)\,. \tag{3}$$

The explicit form of $L^*(s)$ is obtained and matches $I(s)$. However, the condition $|\rho| \leq \rho_0$, $\rho_0 = \sqrt{3+2\sqrt{3}}/3$, must be added.

The SLD for $\tilde{r}_n$ is given by

$$P(\tilde{r}_n \geq c) = \frac{e^{-nL^*(c)-\frac{1}{4}\log(1-4\lambda_c^2)}}{\lambda_c\sigma_c\sqrt{n}}(1+o(1))\,,$$

where $L^*(s) = -\frac{1}{2}\log(1-s^2)$. Here the function $L(\lambda)$ is similar to one obtained in the spherical case.

Higher-order developments are discussed in both cases and proposed as follows

$$P(r_n \geq c) = \frac{e^{-nL^*(c)+R_0(\lambda_c)}}{\lambda_c \sigma_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{\delta_{c,k}}{n^k} + O\left(\frac{1}{n^{p+1}}\right)\right],$$

where $R_0(\lambda)$ is the function obtained from the expansion of the normalized log–Laplace transform L.

One application of SLD is to study the rejection region of a test using Bahadur exact slope. This slope is studied here for r_n in the Gaussian case to test $(\mathcal{H}_0) : \rho = 0$ against the alternative $(\mathcal{H}_1) : \rho \neq 0$.

Chapter 4 of this thesis studies the SLD result for a self-normalized statistic. A well-defined function of observations, which is a so-called *statistic* (see e.g. [13]), can include many types of property of the sample. There are three commonly used statistics to provide a quick look of the sample: sample mean $\bar{X}_n = \frac{1}{n}\sum_{k=1}^n X_k$, sample variance $\sigma^2 = \frac{1}{n-1}\sum_{k=1}^n (X_k - \bar{X}_n)^2$ and sample standard deviation $\sigma = \sqrt{\sigma^2}$. Among the variety of statistics, self-normalized statistics or scale-free statistics are given by

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{X_i}{\bar{X}_n}\right), \tag{4}$$

which are often used to construct scale-free tests of shape.

In 1997, Shao studied large deviations of such statistics in the special case $f(s) = -s^p$, $p > 1$ [53]. In 2005, Tchirina developed large deviations for a class of scale-free statistics of type (4) under Gamma distribution for various cases of functions f [58]. At the same time, Tchirina considered the statistic $T_n = |\gamma + \frac{1}{n}\sum_{k=1}^n \log \frac{X_i}{\bar{X}_n}|$, where γ is the Euler constant and she obtained large deviations asymptotics under the null exponential hypothesis. She also got results on the Bahadur efficiency of such statistics [57]. These previous quantities are known as Moran statistics. In 2007, Tchirina studied the asymptotic properties of the exponentially tests based on L-statistics $T_n = \frac{1}{n\bar{X}_n}\sum_{k=1}^n w_{i,n} X_{(i)}$, where $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are the order statistics and $w_{i,n}, i = 1, ..., n$, is an array of coefficients [59]. In Chapter 4, we study the SLD for the Moran statistic

$$T_n := \gamma + \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{X_i}{\bar{X}_n}\right),$$

where γ is the Euler constant. The explicit expression of the rate function is not reachable in this case. However, we can present a SLDP.

The Appendix aims to present several definitions used in this thesis, as well as some highly technical computations that helped us to understand the behavior of different coefficients in Laplace development and that can be possibly used for the proofs.

Chapter 1

Large and Sharp Large Deviations

In this first chapter, we recall some elementary definitions and theorems about the LDP and SLDP on which this thesis relies. The work of [6] in continuous case is reformulated in Section 1.2 as the major premise of SLDP. We provide in Section 1.2.2 the framework of the method as in [9] to establish SLD and briefly compare it to the one in [35]. An example is introduced at the beginning of this chapter in order to illustrate the large deviation in the most comprehensive way.

Contents

First of all, let us consider an example, which arises from the Law of Large Numbers and the CLT, in order to have a first view of Large Deviations.

Example

Let μ be the probability measure on $\mathbb{R}$ and $(X_n)_n$ be a sequence of i.i.d. random variables with law μ. Consider $S_n = \sum_{k=1}^n X_k$ and the empirical mean $\bar{X}_n = \frac{1}{n}\sum_{k=1}^n X_k$. It is well-known that if $X_n \in L^2(\mathbb{R})$ with mean $E(X_n) = m$ and variance $\mathrm{Var}(X_n) = \sigma^2$, then

$$\bar{X}_n \xrightarrow[n\to\infty]{} m \qquad \text{a.s}$$

and

$$\sqrt{n}\left(\bar{X}_n - m\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma^2\right).$$

In the particular case of $\mu = \mathcal{N}(0,1)$, remark that $\bar{X}_n$ has a Gaussian distribution $\mathcal{N}(0, n^{-1})$ and for all $x > 0$, $\lim_{n\to\infty} P\left(\left|\bar{X}_n\right| > x\right) = 0$. More precisely, we have

$$P\left(\left|\bar{X}_n\right| > x\right) = \frac{2}{\sqrt{2\pi}} \int_{x\sqrt{n}}^{+\infty} e^{-t^2/2} dt\,.$$

From the change of variable $t = x\sqrt{n} + \frac{s}{\sqrt{n}}$, we obtain

$$P\left(\left|\bar{X}_n\right| > x\right) = \frac{2e^{-nx^2/2}}{\sqrt{2\pi n}} \int_0^{+\infty} e^{-s^2/(2n)-sx} ds\,.$$

From dominated convergence, the preceding integral converges to $\int_0^{+\infty} e^{-sx} ds = x^{-1}$. Therefore, as $n \to +\infty$,

$$P\left(\left|\bar{X}_n\right| > x\right) \sim \frac{2e^{-nx^2/2}}{x\sqrt{2\pi n}}\,,$$

i.e.

$$P\left(\left|\bar{X}_n\right| > x\right) = \frac{2e^{-nx^2/2}}{x\sqrt{2\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{1.1}$$

We can express the previous result in a weak version as

$$\lim_{n\to+\infty} \frac{1}{n} \log P\left(\left|\bar{X}_n\right| > x\right) = -\frac{x^2}{2}\,. \tag{1.2}$$

The equations (1.2) and (1.1) are known as large deviations and precise (or sharp) large deviations, respectively. One can now ask whether the limit (1.2) also holds for non–Gaussian case. The answer of this question is pointed out through Cramér's theorem for i.i.d. random variables: the limit of $n^{-1} \log P\left(\left|\bar{X}_n\right| > x\right)$ depends on μ and always exists. Furthermore, Gärtner–Ellis theorem shows that the preceding result also holds for non-i.i.d. case.

Our motivation here is the following: on the one hand, we give the characterization of the LDP in general cases with Cramér's and Gärtner–Ellis theorems. On the other hand, we study the tail probabilities (SLDP) presenting the results of Bahadur and Rao [6], Bercu et al. [9] and Joutard [35]. This second section is the main part related to our work.

1.1 Large Deviation Principle

We present in this section the LDP and elementary theorems, which are the first steps for SLDP of Section 1.2.

Let $(E, \mathcal{E})$ be a measurable topological space. The LDP characterizes the limiting behavior, as $n \to \infty$, of a family of probability measures $\{\mu_n\}$ on $(E, \mathcal{E})$, in terms of a *rate function*.

Definition 1.1.1 *Let I be a real (or extended-real) function on a topological space E. It is a lower semicontinuous mapping if the level set*

$$\Psi_I(\alpha) := \{x : I(x) \le \alpha\}$$

is closed for every real α. The effective domain of I, denoted by D_I, is defined by

$$D_I := \{x : I(x) < +\infty\}.$$

Definition 1.1.2 *A rate function I is a lower semicontinuous mapping from E to $[0,+\infty]$. A good rate function is a rate function for which all the level sets $\Psi_I(\alpha)$ are compact subsets of E.*

Definition 1.1.3 (The LDP) *The family of probability measures $\{\mu_n\}$ satisfies the LDP with rate function I and speed (or scale) n if, for all $\Gamma \in \mathcal{E}$,*

$$-\inf_{x\in\Gamma^0} I(x) \leq \liminf_{n\to\infty} \frac{1}{n}\log\mu_n(\Gamma) \leq \limsup_{n\to\infty}\frac{1}{n}\log\mu_n(\Gamma) \leq -\inf_{x\in\overline{\Gamma}} I(x). \tag{1.3}$$

Remark 1.1.4 (Uniqueness) *The rate function associated with the LDP of a probability measures $\{\mu_n\}$ family on a metric space (more generally on a regular topological space) is unique.*

Since $\mathcal{E}$ in (1.3) is not necessarily be the Borel σ-field, when the Borel σ-field on E is included in $\mathcal{E}$, the LDP is equivalent to the following inequalities:

i) (Large deviation upper bound) For any closed set F

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(F) \leq -\inf_{x\in F} I(x). \tag{1.4}$$

ii) (Large deviation lower bound) For any open set G

$$\liminf_{n\to\infty}\frac{1}{n}\log\mu_n(G) \geq -\inf_{x\in G} I(x). \tag{1.5}$$

The following theorem shows the transformation of the LDP through continuous mapping.

Theorem 1.1.5 (Contraction Principle) *Let X and Y be Hausdorff spaces and $f : X \to Y$ be a continuous function. Consider a good rate function $I : X \to [0,+\infty]$ and the function $I' : Y \to [0,+\infty]$, defined by $I'(y) := \inf\{I(x) : x \in X, y = f(x)\}$.*

i) I' is a good rate function on Y.

ii) If I controls the LDP associated with $\{\mu_n\}$ on X, then I' controls the LDP associated with $\{\mu_n \circ f^{-1}\}$ on y.

The two following subsections deal with the LD of the empirical mean.

Cramér's Theorem for i.i.d. case

Let μ be a probability measure on $\mathbb{R}$.

Definition 1.1.6 *A* log*–Laplace transform (commonly known as a cumulant generating function or logarithmic m.g.f.) L associated with the law μ is a mapping from $\mathbb{R}$ to $[0,+\infty]$, defined as either*

$$L(\lambda) := \log\int \exp(\lambda x)\ \mu(dx), \tag{1.6}$$

or in case X is a random variable with law μ

$$L(\lambda) := \log E\left(e^{\lambda X}\right). \tag{1.7}$$

Definition 1.1.7 *The dual Fenchel-Legendre transform of L is*

$$L^*(x) := \sup_{\lambda \in \mathbb{R}} \{\lambda x - L(\lambda)\}, \tag{1.8}$$

for $x \in \mathbb{R}$.

Theorem 1.1.8 (Cramér) *Let $(X_n)_n$ be a sequence of i.i.d. random variables with law μ. Define by*

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \, .$$

Then the sequence $(\bar{X}_n)_n$ satisfies a LDP with rate function L^*, namely

i) For any closed set $F \subset \mathbb{R}$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log P\left(\bar{X}_n \in F\right) \leq - \inf_{x \in F} L^*(x). \tag{1.9}$$

ii) For any open set $G \subset \mathbb{R}$,

$$\liminf_{n \to +\infty} \frac{1}{n} \log P\left(\bar{X}_n \in G\right) \geq - \inf_{x \in G} L^*(x). \tag{1.10}$$

The Gärtner–Ellis Theorem

We now extend the Cramér's theorem to the more general Gärtner–Ellis theorem.

Definition 1.1.9 *$y \in \mathbb{R}$ is an exposed point of f if for some $\lambda \in \mathbb{R}$ and all $x \neq y$,*

$$\lambda y - f(y) > \lambda x - f(x). \tag{1.11}$$

λ in (1.11) is called an exposing hyperplane.

Definition 1.1.10 *A convex function $f : \mathbb{R} \to (-\infty, +\infty]$ is essentially smooth if:*

i) The interior of the effective domain D_f^0 is non-empty.

ii) f is differentiable throughout D_f^0.

iii) f is steep, namely, $\lim_{n\to\infty} |f_n'| = +\infty$ whenever $(f_n)_n$ is a sequence in D_f^0 converging to a boundary point of D_f^0.

Let $(Z_n)_n$ be a sequence of random variables, of laws (μ_n). Define the log–Laplace function

$$L_n(\lambda) := \frac{1}{n} \log E\left(e^{n\lambda Z_n}\right). \tag{1.12}$$

Assumption 1.1.11 *For each $\lambda \in \mathbb{R}$, the logarithmic m.g.f., defined as the limit*

$$L(\lambda) := \lim_{n \to +\infty} L_n(\lambda) \tag{1.13}$$

exists as an extended real number. Moreover, $0 \in D_L^0$, where $D_L = \{\lambda \in \mathbb{R} : L(\lambda) < +\infty\}$ is the effective domain of L.

Theorem 1.1.12 (Gärtner–Ellis) *Let Assumption 1.1.11 holds.*

i) For any closed set $F \subset \mathbb{R}$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mu_n(F) \leq -\inf_{x\in F} L^*(x). \tag{1.14}$$

ii) For any open set $G \subset \mathbb{R}$,

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mu_n(G) \geq -\inf_{x\in G\cap\mathcal{F}} L^*(x), \tag{1.15}$$

where $\mathcal{F}$ is the set of exposed points of L^* whose exposing hyperplane belongs to D_L^0.

ii) If L is an essentially smooth, lower semicontinuous function, then the LDP holds with the good rate function L^*.

Remark 1.1.13 *In the particular case where $F = [c, +\infty[$, $c > E(Z_n)$, we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log P(Z_n \geq c) = -\inf_{x\geq c} L^*(x). \tag{1.16}$$

Remark 1.1.14 (Properties of functions L and L^*) *Under Assumption 1.1.11,*

i) L is a convex function and L^* is a convex rate function. Moreover, $L^*(x) \geq 0$ for all $x \in \mathbb{R}$.

ii) Suppose that L is differentiable for some $\lambda \in D_L^0$ and $y = L'(\lambda)$, then

$$L'(\lambda) = y \quad \Longrightarrow \quad L^*(y) = \lambda y - L(\lambda). \tag{1.17}$$

This convexity property above will be used further on.

1.2 Sharp Large Deviation Principle

1.2.1 Motivation and SLDP

The (1.1) and (1.2) not only show the result of large deviations, but also inspire us to develop the tail probability in the asymptotic expansion (1.1). Since the LDP and its properties only give the logarithmic equivalent for $P\left(\left|\bar{X}_n\right| > x\right)$ in term of a rate function, then a "sharper" tool is considered in order to estimate this tail probability.

SLDP has been studied widely and commonly known as a "strong large deviation" in many results. Bahadur and Rao [6] (1960) were one of the first mathematicians establishing such expansions for the sample mean. The sequence of i.i.d. random variables $(X_n)_n$ is considered in three separate cases: X_1's distribution is absolutely continuous, X_1 is a lattice variable (namely, there exists constants x_0 and $d > 0$ such that X_1 is confined to the set $\{x_0 + rd : r = 0, \pm 1, \pm 2 \cdots\}$ with probability one) or X_1 is none of these two cases. The result of Blackwell and Hodges [11] (1959) in the lattice case is contained in [6]. Chaganty and Sethuraman (1993) generalized Theorem 1 of [6] to arbitrary sequences of random variables under some conditions on the m.g.f. of $S_n = X_1 + \cdots + X_n$ [14] and extended this result to multi-dimensional case [15] (1996) (strong LD). Cho and Joen proved a strong LD theorem for the ratio of independent random variables [17] (1996). Joutard obtained SLD results in the nonparametric estimation setting [32] (2006), for the conditional empirical process [33] (2008) and for arbitrary sequences of random variables [35]-[34] (2013). Bercu, Gamboa, and Lavielle established the SLDP and gave the result for Gaussian quadratic forms [9] (2000). Bercu and Rouault (2002) studied the SLD for the Ornstein–Uhlenbeck process [10] and later on, Bercu, Coutin and Savy widened the previous results to non-stationary cases [8] (2012).

We now present briefly the results of Bahadur and Rao [6] (1960):

Let $(X_n)_n$ be a sequence of i.i.d. random variables and a be a constant $(-\infty < a < +\infty)$. Denote by $\varphi(\lambda) = E\left(e^{\lambda X_1}\right)$ the m.g.f. of X_1, where λ is a real variable and $0 < \varphi \leq +\infty$. Define function $\psi(\lambda) = e^{-a\lambda}\varphi(\lambda)$ and let $D_\varphi = \{\lambda : \varphi(\lambda) < +\infty\}$ be the effective domain of φ.

Theorem 1.2.1 ([6]) *Suppose that the distribution of X_1 is absolutely continuous and*

- $P\left(X_1 = a\right) \neq 1$.
- *D_φ is a non-degenerate interval, i.e. D_φ is not a single point.*
- *There exists a positive $\tau \in D_\varphi^0$ such that $\psi(\tau) = \inf_{\lambda \in D_\varphi}\{\psi(\lambda)\} = \Lambda$.*

Then there exists a sequence $(b_n)_n$ of positive numbers such that

$$P\left(\frac{X_1 + \cdots + X_n}{n} \geq a\right) = \frac{\Lambda^n\, b_n}{(2\pi n)^{1/2}}\,(1 + o(1)) \tag{1.18}$$

and

$$\log b_n = O(1) \tag{1.19}$$

as $n \to +\infty$. Furthermore, for each $j = 1, 2, \ldots$ there exists a bounded (possibly constant) sequence $(c_{j,n})_n$ such that, for any given positive integer k,

$$P\left(\frac{X_1 + \cdots + X_n}{n} \geq a\right) = \frac{\Lambda^n b_n}{(2\pi n)^{1/2}}\left(1 + \frac{c_{1,n}}{n} + \frac{c_{2,n}}{n^2} + \cdots + \frac{c_{k,n}}{n^k}\right)\left(1 + O\left(\frac{1}{n^{k+1}}\right)\right) \tag{1.20}$$

as $n \to +\infty$.

Proof:

[Ideas from [6]] We first remark that τ and Λ are uniquely determined by

$$\frac{\varphi'(\tau)}{\varphi(\tau)} = a, \qquad \text{where } \varphi' = \frac{d\varphi}{d\lambda} \tag{1.21}$$

and

$$\Lambda = \psi(\tau), \quad (0 < \Lambda < 1). \tag{1.22}$$

Next, we can decompose $p_n := P\left(\dfrac{X_1 + \cdots + X_n}{n} \geq a\right)$ as $p_n = \Lambda^n I_n$ and expand I_n to finally obtain (1.20).

We now detail I_n as follows

i) Let $Y_1 = X_1 - a$ and $F(y) = P(Y_1 < y)$ be the distribution function (d.f.) of Y_1. Define z_1 a random variable having d.f. $G(z) = \int_{-\infty}^{z} \Lambda^{-1} e^{\tau y}\, dF(y)$.

Remark 1.2.2 *The m.g.f. of z_1 exists in a neighborhood of the origin,*

$$E(z_1) = 0 \tag{1.23}$$

and

$$\sigma^2 := Var\ (z_1) = \frac{\varphi''(\tau)}{\varphi(\tau)} - a^2 < +\infty. \tag{1.24}$$

Define $\alpha = \sigma\tau$, $(0 < \alpha < +\infty)$ and let $z_1, z_2, \ldots$ be i.i.d. random variables. For each n, let

$$u_n = \frac{z_1 + \cdots + z_n}{n^{1/2}\sigma} \tag{1.25}$$

and

$$H_n(x) = P(u_n < x), \quad (-\infty < x < +\infty). \tag{1.26}$$

Then it follows that $p_n = \Lambda^n I_n$, where

$$I_n = n^{1/2}\alpha \int_0^{+\infty} e^{-n^{1/2}\alpha x}\, [H_n(x) - H_n(0)]\, dx. \tag{1.27}$$

The expansion of I_n therefore depends on H_n and we develop it as follows.

ii) Suppose that the d.f. of X_1, denoted by F_1, satisfies

$$\limsup_{|t|\to+\infty} \left| \int_{-\infty}^{+\infty} e^{itx} dF_1(x) \right| < 1. \tag{1.28}$$

Then G also satisfies

$$\limsup_{|t|\to+\infty} \left| \int_{-\infty}^{+\infty} e^{itx} dG(x) \right| < 1. \tag{1.29}$$

and from error estimation in asymptotic expansions (see Cramér, page 81, [21]) we have for each fixed positive integer k,

$$H_n(x) = K_n(x) + R_n(x), \tag{1.30}$$

where

$$K_n(x) = \sum_{j=0}^{k} n^{-j/2} P_j(-\phi) \tag{1.31}$$

where $R_n(x)$ is of order the $n^{-(k+1)/2}$ uniformly in x and we detail below ϕ and P_j

Remark 1.2.3

- *From the CLT, the function ϕ is defined by*

$$\phi(x) = \lim_{n\to+\infty} H_n(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-t^2/2}\, dt \tag{1.32}$$

for every $-\infty < x < +\infty$.

- *P_j are polynomials, obtained by expanding an analytic function $\left[\eta(w/n^{1/2})\right]^n e^{-w^2/2}$ in a domain independent of n as a power series in w as follows*

$$\left[\eta(w/n^{1/2})\right]^n e^{-w^2/2} = \sum_{j=0}^{+\infty} n^{-j/2} P_j(w), \tag{1.33}$$

where $\eta(w)$ is the m.g.f. of Z_1/σ.

From (1.27), we have

$$I_n = n^{1/2}\alpha \int_0^{+\infty} e^{-n^{1/2}\alpha x} \left[K_n(x) - K_n(0)\right] dx + O\left(\frac{1}{n^{k/2+1/2}}\right). \tag{1.34}$$

Next, using integration by parts and Parseval formula, we obtain

$$I_n = \frac{1}{\alpha(2\pi n)^{1/2}} \int_{-\infty}^{+\infty} \left(1 + \frac{it}{n^{1/2}\alpha}\right)^{-1} \left(\sum_{j=0}^{k} n^{-j/2} P_j(it)\right) d\phi(t) + O\left(\frac{1}{n^{k/2+1/2}}\right). \tag{1.35}$$

Define

$$\mu_{r,s} = \int_{-\infty}^{+\infty} (it)^r\, P_s(it)\, d\phi(t) \qquad (r, s = 0, 1, 2, \dots). \tag{1.36}$$

We denote by $\mu_{r,s} = 0$ if $r + s$ is odd and let us define for every n

$$c_{j,n} = \sum_{r+s=2j} \left(-\frac{1}{\alpha}\right)^r \mu_{r,s} \qquad (j = 0, 1, 2, \dots). \tag{1.37}$$

Consequently, it follows that

$$I_n = \frac{1}{\alpha(2\pi n)^{1/2}} \sum_{0\le j<k/2} c_{j,n}\, n^{-j} + O\left(\frac{1}{n^{k/2+1/2}}\right). \tag{1.38}$$

These steps establish the Theorem 1.2.1 with $b_n = \alpha^{-1}$.

$\square$

Through the work of Bahadur and Rao [6], the SLDP is formally known in the following way.

Definition 1.2.4 (SLDP [9]) *Let $(Z_n)_n$ be a sequence of real random variables converging almost surely to some real number v. We say that $(Z_n)_n$ satisfies a Local Sharp Large Deviation Principle of order $p \in \mathbb{N}$ at point $c \in \mathbb{R}$ whenever the following expansion holds*

$$P\left(Z_n \geq c\right) = \frac{a_0 \exp\left(-nb\right)}{\sqrt{n}} \left(1 + \frac{a_1}{n} + \cdots + \frac{a_p}{n^p} + o\left(\frac{1}{n^p}\right)\right), (c > v) \tag{1.39}$$

or

$$P\left(Z_n \leq c\right) = \frac{a_0 \exp\left(-nb\right)}{\sqrt{n}} \left(1 + \frac{a_1}{n} + \cdots + \frac{a_p}{n^p} + o\left(\frac{1}{n^p}\right)\right), (c < v). \tag{1.40}$$

Example 1.2.5 (SLD for the sample variance [35]) *Let X_i have a normal distribution $N(\mu; \sigma^2), \sigma^2 > 0$ and consider the sample variance*

$$Z_n = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

Then for a real a such that $a > \sigma^2$ and n large enough,

$$P(Z_n \geq a) = \frac{e^{-(n-1)I(a)}}{2a\tau_a\sqrt{n\pi}}[1 + o(1)],$$

where $I(a) = \dfrac{1}{2}\left(\dfrac{a}{\sigma^2} - \log(\dfrac{a}{\sigma^2}) - 1\right) > 0$ *and* $\tau_a = \dfrac{a - \sigma^2}{2a\sigma^2}$.

1.2.2 Framework and Main method for establishing SLD

In this section, we propose the framework of [9] for establishing SLDP. This framework is used throughout this thesis. After that, we briefly mention the work of Joutard [35], which gives the SLD results in the first-order expansion under some assumptions.

Framework

Let $(Z_n)_n$ be a sequence of random variables. We now present the outline of the method in four steps:

1. Study functions: $L_n(\lambda) = \dfrac{1}{n} \log E\left(e^{n\lambda Z_n}\right)$ and $L(\lambda) = \lim_{n\to+\infty} L_n(\lambda)$. (Note that L is a convex function).

2. Consider the dual Fenchel-Legendre transform of $L(\lambda)$: $L^*(y) = \sup_{\lambda\in\mathbb{R}}\{\lambda y - L(\lambda)\}$. According to Remark 1.1.14, for each $c \in \mathbb{R}$ if L is differentiable and $c = L'(\lambda_c)$ then $L^*(c) = \lambda_c c - L(\lambda_c)$. Denote by $\sigma_c^2 = L''(\lambda_c) > 0$.

3. Set a new probability Q_n by change of probability:

$$\frac{dQ_n}{dP} = e^{n\lambda_c Z_n - nL_n(\lambda_c)}. \tag{1.41}$$

4. We consider the decomposition $P(Z_n \geq c) = A_n B_n$, where
$$A_n = \exp\left[n(L_n(\lambda_c) - c\lambda_c)\right] \tag{1.42}$$
and
$$B_n = E_n\left(\exp\left[-n\lambda_c(Z_n - c)\right] \mathbb{1}_{Z_n \geq c}\right). \tag{1.43}$$
Here, E_n denotes the expectation under probability Q_n. Decompose L_n as:
$$L_n(\lambda) = L(\lambda) + \frac{1}{n}H(\lambda) + O\left(\frac{1}{n^2}\right). \tag{1.44}$$
We study the expansions of A_n and B_n as follows:
   4a. According to Step 2,
$$\begin{aligned} A_n &= \exp\left[n\left(L(\lambda) + \frac{1}{n}H(\lambda) - c\lambda_c\right) + O\left(\frac{1}{n}\right)\right] \\ &= \exp\left(-nL^*(\lambda_c) + H(\lambda) + O\left(\frac{1}{n}\right)\right). \end{aligned}$$
   4b. Let us denote by
$$U_n = \frac{\sqrt{n}(Z_n - c)}{\sigma_c},$$
   and study $\Phi_n(u)$ be the c.f. of U_n over the probability Q_n, namely $\Phi_n(u) = E_n(e^{iuU_n})$. It follows from Parseval formula that
$$\begin{aligned} B_n &= E_n\left(\exp\left(-\lambda_c\sigma_c\sqrt{n}U_n\right) \mathbb{1}_{Z_n \geq c}\right) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\frac{1}{\lambda_c\sigma_c\sqrt{n} + iu}\right) \Phi_n(u)du = \frac{C_n}{\lambda_c\sigma_c\sqrt{2\pi n}}, \end{aligned}$$
   where
$$C_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(1 + \frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)^{-1} \Phi_n(u)du.$$
   The expansion of $\Phi_n(u)$ gives the SLDP.

Remark the result in [35]

We present here a slightly different method to establish the first-order expansion of SLD as in [35]. This result can apply for the corresponding case 1 and 2 in [6]. We now summarize the assumptions and elementary ideas in [35].

Let $(b_n)_n$ be a sequence of real positive numbers such that $\lim_{n\to\infty} b_n = \infty$. Define
$$\phi_n(t) = E\left(e^{tb_nZ_n}\right)$$
and
$$\varphi_n(t) = \frac{1}{b_n}\log E\left(e^{tb_nZ_n}\right).$$
Assume that there exits the limit $\lim_{n\to+\infty}\varphi_n(t) = \varphi(t)$ for all $t \in (-\alpha, \alpha)$ $(\alpha > 0)$. For constant a such that $|a - \varphi'(0)| > 0$, assume there exits $\tau_a \in \{t \in \mathbb{R} : 0 < |t| < \alpha\}$, such that $\varphi'(\tau_a) = a$.

Assumption 1.2.6 *i) φ_n is an analytic function in $D_C := \{z \in \mathbb{C} : |t| < \alpha\}$ and can be bounded for all $z \in D_C$ and n large enough.*

ii) There exist $\alpha_0 \in (0, \alpha - \tau_a)$ and a function H such that for each $t \in (\tau_a - \alpha_0, \tau_a + \alpha_0)$ and for n large enough,

$$\varphi_n(t) = \varphi(t) + \frac{1}{b_n} H(t) + o\left(\frac{1}{b_n}\right),$$

where the function φ is three times continuously differentiable in $(\tau_a - \alpha_0, \tau_a + \alpha_0)$, $\varphi''(\tau_a) > 0$, and H is continuously differentiable in $(\tau_a - \alpha_0, \tau_a + \alpha_0)$.

iii) There exists $\delta_0 > 0$ such that,

$$\sup_{\delta < |t| < \beta |\tau_a|} \left| \frac{\phi_n(\tau_a + it)}{\phi_n(\tau_a)} \right| = o\left(\frac{1}{\sqrt{b_n}}\right),$$

for any given δ and β such that $0 < \delta < \delta_0 < \beta$.

Then if $a > \varphi'(0)$ and Assumptions (1.2.6) holds, for n large enough,

$$P(Z_n \geq a) = \frac{e^{-b_n I(a) + H(\tau_a)}}{\sigma_a \tau_a \sqrt{2\pi b_n}} [1 + o(1)],$$

where $\tau_a > 0$ is such that $\varphi'(\tau_a) = a$. Further, $I(a) = \tau_a a - \varphi(\tau_a)$ and $\sigma_a^2 = \varphi''(\tau_a)$.

Here, we want to note that the frameworks of [9] and [35] are quite similar. The different step between them is to study the expansion of c.f. Φ_n of U_n in step 4b. Bercu et al. [9] study the expansion by directly bounding Φ_n under assumption of $L_n^{(k)}(\lambda)$ whereas Joutard [35] develops Φ_n based on the results of [14], which only gives the asymptotic behavior in the first-order. The Assumption 1.2.6 is mentioned in order to apply the results of [14].

Résumé

Ce premier chapitre est consacré à la présentation des théorèmes classiques de grandes déviations (PGD) et grandes déviations précises. Dans un premier temps, nous rappelons le résultat de Cramér: soit (X_i) une famille de variables réelles indépendantes et identiquement distribuées, on définit

$$L(\lambda) := \log E\left(e^{\lambda X_1}\right),$$

et la duale de Fenchel–Legendre de L:

$$L^*(x) := \sup_{\lambda \in \mathbb{R}} \{\lambda x - L(\lambda)\},$$

pour $x \in \mathbb{R}$. On a alors

Theorem 1.2.7 (Cramér) *Soit $(X_n)_n$ une suite de v.a. i.i.d. de loi μ. Soit*

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i\,.$$

Alors $(\bar{X}_n)$ satisfait un PGD de fonction de taux L^*, i.e.

i) Pour tout fermé $F \subset \mathbb{R}$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log P\left(\bar{X}_n \in F\right) \leq -\inf_{x\in F} L^*(x). \tag{1.45}$$

ii) Pour tout ouvert $G \subset \mathbb{R}$,

$$\liminf_{n\to+\infty} \frac{1}{n} \log P\left(\bar{X}_n \in G\right) \geq -\inf_{x\in G} L^*(x). \tag{1.46}$$

Une généralisation est donnée par le théorème de Gärtner–Ellis: Soit $(Z_n)_n$ une suite de v.a. de loi (μ_n). On définit la log–Laplace :

$$L_n(\lambda) := \frac{1}{n} \log E\left(e^{n\lambda Z_n}\right). \tag{1.47}$$

On admet que la fonction limite log–Laplace (ou fonction génératrice des cumulants normalisée):

$$L(\lambda) := \lim_{n\to+\infty} L_n(\lambda) \tag{1.48}$$

existe pour tout $\lambda \in \mathbb{R}$, limite éventuellement infinie. On suppose de plus que $0 \in D_L^0$, où $D_L = \{\lambda \in \mathbb{R} : L(\lambda) < +\infty\}$ est le domaine de L. On a alors:

Theorem 1.2.8 (Gärtner–Ellis) *i) Pour tout fermé $F \subset \mathbb{R}$,*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mu_n\left(F\right) \leq -\inf_{x\in F} L^*(x). \tag{1.49}$$

ii) Pour tout ouvert $G \subset \mathbb{R}$,

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mu_n\left(G\right) \geq -\inf_{x\in G\cap\mathcal{F}} L^*(x), \tag{1.50}$$

où $\mathcal{F}$ est l'ensemble des points exposés de L^* dont l'hyperplan exposé est dans D_L^0.

ii) Si L est essentiellement lisse et semi-continue, on a le PGD de fonction de taux L^*.

Dans une deuxième partie, on définit un PGD précises:

Definition 1.2.9 *Soit $(Z_n)_n$ une suite de v.a. réelles qui converge presque sûrement vers un réel v. On dit que $(Z_n)_n$ satisfait un PGD précises d'ordre $p \in \mathbb{N}$ au point $c \in \mathbb{R}$ si on a*

$$P\left(Z_n \geq c\right) = \frac{a_0 \exp\left(-nb\right)}{\sqrt{n}} \left(1 + \frac{a_1}{n} + \cdots + \frac{a_p}{n^p} + o\left(\frac{1}{n^p}\right)\right), (c > v) \tag{1.51}$$

ou

$$P\left(Z_n \leq c\right) = \frac{a_0 \exp\left(-nb\right)}{\sqrt{n}} \left(1 + \frac{a_1}{n} + \cdots + \frac{a_p}{n^p} + o\left(\frac{1}{n^p}\right)\right), (c < v). \tag{1.52}$$

Dans les Chapitres 3 et 4, on prouve un PGD précis en 4 étapes comme suit:
Soit $(Z_n)_n$ une suite de v.a.:

1. On étudie : $L_n(\lambda) = \dfrac{1}{n} \log E\left(e^{n\lambda Z_n}\right)$ et $L(\lambda) = \lim_{n\to+\infty} L_n(\lambda)$. (Remarque: L est convexe).

2. On calcule la duale de Fenchel–Legendre de $L(\lambda)$: $L^*(y) = \sup_{\lambda\in\mathbb{R}}\{\lambda y - L(\lambda)\}$. On a vu que si pour $c \in \mathbb{R}$, L est différentiable et $c = L'(\lambda_c)$ alors $L^*(c) = \lambda_c c - L(\lambda_c)$. On note $\sigma_c^2 = L''(\lambda_c) > 0$.

3. On définit le changement de probabilités Q_n par:
$$\frac{dQ_n}{dP} = e^{n\lambda_c Z_n - nL_n(\lambda_c)}. \tag{1.53}$$

4. On décompose $P(Z_n \geq c) = A_n B_n$, avec
$$A_n = \exp\left[n(L_n(\lambda_c) - c\lambda_c)\right] \tag{1.54}$$
et
$$B_n = E_n\left(\exp\left[-n\lambda_c(Z_n - c)\right] \mathbb{1}_{Z_n\geq c}\right). \tag{1.55}$$
Ici E_n est l'espérance sous Q_n. On décompose L_n comme suit:
$$L_n(\lambda) = L(\lambda) + \frac{1}{n}H(\lambda) + O\left(\frac{1}{n^2}\right). \tag{1.56}$$
Puis on développe A_n and B_n:

4a. D'après l'étape 2,
$$\begin{aligned} A_n &= \exp\left[n\left(L(\lambda) + \frac{1}{n}H(\lambda) - c\lambda_c\right) + O\left(\frac{1}{n}\right)\right] \\ &= \exp\left(-nL^*(\lambda_c) + H(\lambda) + O\left(\frac{1}{n}\right)\right). \end{aligned}$$

4b. On définit
$$U_n = \frac{\sqrt{n}(Z_n - c)}{\sigma_c},$$
et on étudie $\Phi_n(u) = E_n(e^{iuU_n})$. D'après Parseval,
$$\begin{aligned} B_n &= E_n\left(\exp\left(-\lambda_c\sigma_c\sqrt{n}U_n\right) \mathbb{1}_{Z_n\geq c}\right) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\frac{1}{\lambda_c\sigma_c\sqrt{n} + iu}\right)\Phi_n(u)du = \frac{C_n}{\lambda_c\sigma_c\sqrt{2\pi n}}, \end{aligned}$$
où
$$C_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(1 + \frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)^{-1}\Phi_n(u)du.$$
Le développement de $\Phi_n(u)$ donne le PGD précis.

Chapter 2

Laplace transforms and asymptotics

This chapter deals with Laplace's method and the asymptotic expansion of the integral $\int_a^b e^{-xp(t)q(t)dt}$ (see e.g. [45], [42], [28], or [20]). The main work in Section 2.3 gives the explicit form of the integral around the maximum of function $p(t)$ by another approach. The application of Laplace's method is detailed for the incomplete Gamma function, hypergeometric function, Euler integral and Stirling formula.

Contents

2.1 General results

This chapter aims to describe several techniques of integral computations when the integral depends on a parameter tending to infinity. The archetype of integrals studied here will be

$$I(x) = \int_a^b e^{-xp(t)}q(t)dt\,, \tag{2.1}$$

where a, b are real numbers, possibly ∞; p, q are sufficiently smooth real functions and x is a real number. We want to describe the asymptotics of I when $x \to \infty$.

2.1.1 Integration by part

The first idea is to perform integration by parts. It is particularly interesting when we consider a simpler form of (2.40), namely

$$I(x) = \int_0^\infty e^{-xt} q(t) dt\,. \tag{2.2}$$

We assume that q is $\mathcal{C}^\infty$ on $[0, \infty[$ and for any $\sigma \in \mathbb{N}$, independently on N,

$$q^{(N)}(t) = o(e^{\sigma t})\,, \text{ when } t \to \infty\,.$$

An obvious computation gives

Lemma 2.1.1 *We can write for any $N \in \mathbb{N}$*

$$I(x) = \sum_{s=0}^{N} \frac{q^{(s)}(0)}{x^{s+1}} + \epsilon_N(x)\,,$$

where

$$\epsilon_N(x) = \frac{1}{x^n} \int_0^\infty e^{-xt} q^{(N)}(t) dt\,.$$

The main point is now to bound ϵ_N. If q is bounded as follows:

$$\forall t \in [0, \infty), \quad |q^{(N)}(t)| \le K e^{\sigma t}\,, \tag{2.3}$$

where K and σ are real constants independent of N, then for any $N \in \mathbb{N}$ and for $x > \max\{0, \sigma\}$

$$|\epsilon_N(x)| \le \frac{K}{x^N (x - \sigma)}. \tag{2.4}$$

Hence,

$$I(x) \sim \sum_{s=0}^{\infty} \frac{q^{(s)}(0)}{x^{s+1}} \text{ when } x \to \infty. \tag{2.5}$$

When there exist a maximum of $q^{(N)}$ in $[0, \infty[$, say in 0, we can bound

$$|\epsilon_N(x)| \le \frac{|q^{(N)}(0)|}{x^{N+1}}.$$

If we do not have (2.3) then the obvious extension is

$$|\epsilon_N(x)| \le \frac{C_N}{x^{N+1}},$$

where

$$C_N = \sup_{t \in [0,\infty[} |q^{(N)}(t)|$$

and if C_N is big compared to $q^{(N)}(0)$ then we can seek for a bound of type

$$|q^{(N)}(t)| \le |q^{(N)}(0)| e^{\sigma_N t}\,,$$

where σ_N is independent of t then we get similarly to (2.4), when $x > \max(\sigma_N, 0)$

$$|\epsilon_N(x)| \leq \frac{|q^{(N)}(0)|}{x^N(x-\sigma_N)}\,. \tag{2.6}$$

The best value for σ_N is then

$$\sigma_N = \sup_{[0,\infty]} \frac{1}{t} \log \left| \frac{q^{(N)}(t)}{q^{(N)}(0)} \right| \,.$$

Incomplete Gamma function

We give here some computation which is slightly different from the above framework but that will help in the computation of (2.40). The following integral is known as the complementary incomplete Gamma function defined by

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt\,.$$

We have from integration by parts:

$$\Gamma(\alpha, x) = e^{-x}x^{\alpha-1} + (\alpha-1)\Gamma(\alpha-1, x)$$

and therefore

$$\Gamma(\alpha, x) = e^{-x}x^{\alpha-1} \sum_{s=0}^{n-1} \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-s)}{x^s} + \epsilon_n(x), \tag{2.7}$$

where

$$\epsilon_n(x) = (\alpha-1)(\alpha-2)\cdots(\alpha-n) \int_x^\infty e^{-t} t^{\alpha-n-1} dt,$$

which can be bounded by

$$|\epsilon_n(x)| \leq |(\alpha-1)(\alpha-2)\cdots(\alpha-n)| e^{-x} x^{\alpha-n-1}\,.$$

Hence for fixed α and large x,

$$\Gamma(\alpha, x) \sim e^{-x}x^{\alpha-1} \sum_{s=0}^{\infty} \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-s)}{x^s}. \tag{2.8}$$

2.1.2 Watson's Lemma

The idea now is to substitute directly the MacLaurin development in (2.2).

$$q(t) = q(0) + tq'(0) + t^2\frac{q''(0)}{2!} + \cdots + t^N\frac{q^{(N)}(0)}{N!} + R_N(t)\,. \tag{2.9}$$

Of course this development has to be valid throughout the whole range of integration in (2.2). Can a similar expansion can be built if the power of t are non integer? The answer has been given by Watson (see [42]) in the following result

Theorem 2.1.2 *(Watson, 1918) If*

$$q(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu} \text{ when } t \to 0, \tag{2.10}$$

then

$$\int_0^\infty e^{-xt} q(t)dt \sim_{x\to\infty} \sum_{s=0}^{\infty} \Gamma(\frac{s+\lambda}{\mu}) \frac{a_s}{x^{(s+\lambda)/\mu}} .$$

Proof:
For any $N \in \mathbb{N}$ define

$$Q_N(t) = q(t) - \sum_{s=0}^{N-1} a_s t^{(s+\lambda-\mu)/\mu}.$$

Since

$$\int_0^\infty t^{(s+\lambda-\mu)/\mu} e^{-xt} dt = \Gamma\left(\frac{s+\lambda}{\mu}\right),$$

we have

$$I(x) = \sum_{s=0}^{N-1} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} + \int_0^\infty e^{-xt} Q_N(t)dt\,. \tag{2.11}$$

From assumption (2.10), for any N, and for $t \leq k_N$, we have

$$|Q_N(t)| \leq K_N\, t^{(N+\lambda)/\mu-1},$$

and therefore

$$\left|\int_0^{k_N} e^{-xt} Q_N(t)dt\right| \leq \Gamma\left(\frac{N+\lambda}{\mu}\right) \frac{K_N}{x^{(N+\lambda)/\mu}} . \tag{2.12}$$

For the remaining part $\int_{k_N}^\infty e^{-xt} Q_N(t)dt$ we fix X such that $\int_0^\infty e^{-Xt} Q_N(t)dt$ converges and for $x > X$ we have

$$\int_{k_N}^\infty e^{-xt} Q_N(t)dt = \int_{k_N}^\infty e^{-(x-X)t} e^{-Xt} Q_N(t)dt = (x-X) \int_{k_N}^\infty e^{-(x-X)t} \Phi_N(t)dt\,,$$

where

$$\Phi_N(t) = \int_{k_N}^t e^{-xu} Q_N(u)du\,.$$

It is obvious that Φ_N is continuous and bounded on $[k_N, \infty[$. Hence we can define L_N the supremum of Φ_N on $[k_N, \infty[$. Therefore

$$\left|\int_{k_N}^\infty e^{-xt} Q_N(t)dt\right| \leq L_N e^{-(x-X)k_N} . \tag{2.13}$$

Putting together (2.12) and (2.13), we get for some constant $\bar{K}_N$

$$\left| I(x) - \sum_{s=0}^{N-1} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(N+\lambda)/\mu}} \right| \leq \frac{\bar{K}_N}{x^{(s+\lambda)/\mu}}.$$

Application to Hypergeometric function

For any $Re(c) > Re(b) > 0$, we consider the hypergeometric function:

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{(b-1)}(1-t)^{c-b-1}(1-zt)^{-a} dt\,. $$

(see Appendix A for more details on this function). With a suitable change of variable, for any λ we get

$$ {}_2F_1(a,b,c+\lambda;z) = \int_0^\infty u^{b-1} f(u) e^{-\lambda u} du, $$

where

$$ f(u) = \left(\frac{e^u-1}{u}\right)^{b-1} e^{(1-c)u}(1-z+ze^{-u})^{-a}\,. $$

We can develop f in series around 0 as follows:

$$ \begin{aligned} f(u) &= \left(\sum_{k=1}^\infty \frac{u^{k-1}}{k!}\right)^{b-1} \sum_{p=0}^\infty \frac{(1-c)^p u^p}{p!} \left(1-z+z\sum_{r=0}^\infty \frac{(-1)^r u^r}{r!}\right)^{-a} \\ &= \left(1+\sum_{k=1}^\infty \frac{u^k}{(k+1)!}\right)^{b-1} \left(1+\sum_{p=1}^\infty \frac{(1-c)^p u^p}{p!}\right)\left(1+z\sum_{r=1}^\infty \frac{(-1)^r u^r}{r!}\right)^{-a}\,. \end{aligned} $$

Hence we get

$$ u^{b-1} f(u) = \sum_{k=0}^\infty c_k(z) u^{k+b-1}\,, \tag{2.14} $$

where $c_0(z) = 1$, $c_1(z) = \frac{b-1}{2!} + (1-c) + az$ and $c_k(z)$ are polynomials in z.
From Watson's Lemma we have

$$ {}_2F_1(a,b,c+\lambda;z) = \frac{\Gamma(c+\lambda)}{\Gamma(c+\lambda-b)} \sum_{k=0}^\infty c_k(z) \frac{(b)_k}{\lambda^{b+k}}\,. \tag{2.15} $$

2.2 Laplace's method

In this paragraph we look for an expansion of integrals of type (2.40). We assume the following:

Assumption 2.2.1
(i) p has a unique minimum in $[a,b]$ at a.
(ii) p' and q are continuous in a neighborhood of a.
(iii) the integral I converges absolutely in its range for large x.

We assume furthermore the following developments around a:

Assumption 2.2.2

(iv) $p(t) \sim p(a) + \sum_{s=0}^\infty p_s (t-a)^{s+\mu}$.

(v) $q(t) \sim \sum_{s=0}^{\infty} q_s(t-a)^{s+\lambda-1}$.

(vi) We assume furthermore that p *can be differentiated as follows*

$$p'(t) = \sum_{s=0}^{\infty}(s+\mu)p_s(t-a)^{s+\mu-1}.$$

Then we have the following result

Proposition 2.2.3 *Under Assumptions 2.2.1 and 2.2.2, we have*

$$\int_a^b e^{-xp(t)}q(t)dt \sim e^{-xp(a)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}}\,. \tag{2.16}$$

Proof:

Let us define $k \in (a,b)$ such that in $(a,k]$, p' is continuous positive and q is continuous. Since p is strictly increasing on $(a,k]$, we can define

$$v = p(t) - p(a)$$

and

$$f(v) = \frac{q(t)}{p'(t)}.$$

We get

$$e^{xp(a)} \int_a^k e^{-xp(t)}q(t)dt = \int_0^\kappa e^{-xv} f(v)dv\,, \tag{2.17}$$

where $\kappa = p(k) - p(a)$. And we have the expansions

$$t - a \sim \sum_{s=1}^{\infty} c_s v^{s/\mu}$$

and

$$f(v) \sim \sum_{s=1}^{\infty} a_s v^{(s+\lambda-\mu)/\mu}\,.$$

See more details in Olver [42]. We can split f into two terms, a finite sum and the reminder for any $v > 0$,

$$f(v) = \sum_{s=0}^{n-1} a_s v^{(s+\lambda-\mu)/\mu} + v^{(n+\lambda-\mu)/\mu} f_n(v) \tag{2.18}$$

$$\int_0^\kappa e^{-xv} f(v)dv = \sum_{s=0}^{n-1} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} - \epsilon_{n,1}(x) + \epsilon_{n,2}(x),$$

where

$$\epsilon_{n,1}(x) = \sum_{s=0}^{n-1} \Gamma\left(\frac{s+\lambda}{\mu}, \kappa x\right) \frac{a_s}{x^{(s+\lambda)/\mu}},$$

and $\Gamma(u,v)$ is the incomplete Gamma function. Furthermore,

$$\epsilon_{n,2} = \int_0^{\kappa} e^{-xv} v^{(n+\lambda-\mu)\mu} f_n(v)dv\,.$$

From the development of the complementary incomplete Gamma function detailed in 2.1.1, we have

$$\epsilon_{n,1}(x) = O(e^{-\kappa x}/x)$$

Let us bound $\epsilon_{n,2}$. Since κ is finite and f_n is continuous on $[0,\kappa)$,we have

$$\epsilon_{n,2}(x) = O(\frac{1}{x^{(n+\lambda)/\mu}})\,.$$

For the remaining range $[k,b)$, let X be a value for which I is absolutely convergent, and denote by $\eta = \inf_{[k,b)}\{p(t)-p(a)\}$. Hence,

$$\left| e^{xp(a)} \int_{\kappa}^{b} e^{-xp(t)} q(t)dt \right| \le e^{-(x-X)\eta + Xp(a)} \int_{\kappa}^{b} e^{-Xp(t)} |q(t)|dt.$$

And for x large enough the LHS above is bounded by $\varepsilon/x^{-\lambda/\mu}$.

Euler integral and Stirling formula

We will use later on a development of the Euler function and we detail here these computations. We consider Euler's integral in the form

$$\Gamma(x) = x^{-1} \int_0^{\infty} e^{-w} w^x dw \quad (x>0).$$

The integral is zero at $w=0$ and increases to a maximum at $w=x$ then decreases steadily back to zero as $w\to\infty$. Setting $w = x(1+t)$ gives

$$\Gamma(x) = e^{-x}x^x \int_{-1}^{\infty} e^{-xt}(1+t)^x dt = e^{-x}x^x \int_{-1}^{\infty} e^{-xp(t)}dt, \tag{2.19}$$

where $p(t) = t - \log(1+t)$. The minimum occurs at $t=0$.

We get

$$e^x x^{-x}\Gamma(x) = \int_0^{\infty} e^{-xp(t)}dt + \int_0^1 e^{-xp(-t)}dt. \tag{2.20}$$

Since $p'(t) = t/(1+t)$ for $-1<t<1$,

$$p(t) = \sum_{n\ge 2} \frac{(-t)^n}{n} \quad (-1<t<1),$$

then each integral of (2.20) satisfy the conditions of Proposition 2.2.3 and with $v = p(t)$, the reversion of the last expansion yields

$$t = 2^{1/2}v^{1/2} + \frac{2}{3}v + \frac{2^{1/2}}{18}v^{3/2} - \frac{2}{135}v^2 + \frac{2^{1/2}}{1080}v^{5/2} + \dots,$$

which converges for sufficiently small v. Hence,

$$f(v) \equiv \frac{dt}{dv} = a_0 v^{-1/2} + a_1 + a_2 v^{1/2} + \dots. \tag{2.21}$$

For example,

$$a_0 = \frac{2^{1/2}}{2}, \quad a_1 = \frac{2}{3}, \quad a_2 = \frac{2^{1/2}}{12}, \quad a_3 = -\frac{4}{135}, \quad a_4 = \frac{2^{1/2}}{432}.$$

According to Proposition 2.2.3,

$$\int_0^\infty e^{-xp(t)} dt \sim \sum_{s=0}^\infty \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{x^{(s+1)/2}}.$$

Similarly,

$$\int_0^1 e^{-xp(-t)} dt \sim \sum_{s=0}^\infty (-1)^s \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{x^{(s+1)/2}}.$$

We finally obtain

$$\Gamma(x) \sim e^{-x} x^x \left(\frac{2\pi}{x}\right)^{1/2} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots\right) \quad (x \to \infty). \tag{2.22}$$

The leading term in this expansion is known as *Stirling's formula* and no general expression is available for the coefficients.

Remark 2.2.4 *The alternative way of expanding function $\Gamma(z)$ for large z with error bounds is shown in Chapter 8, §4, [42]. It gives*

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{s=1}^{m-1} \frac{\mathbf{B}_{2s}}{2s(2s-1)\, z^{2s-1}} + R_m(z), \tag{2.23}$$

where m is an arbitrary positive integer and

$$R_m(z) = \int_0^\infty \frac{\mathbf{B}_{2s} - \mathbf{B}_{2m}(x - [x])}{2m(x+z)^{2m}} dx = O\left(\frac{1}{z^{2m-1}}\right).$$

Here $\mathbf{B}_s$ and $\mathbf{B}_s(x)$ denote Bernoulli number and Bernoulli polynomial.

2.3 An other approach

In this section, we still consider Laplace's method for integrals

$$\bar{I}(x) = \int_a^b e^{xp(t)} q(t) dt\,, \tag{2.24}$$

as $x \to +\infty$ and the development will be around the maximum of p. We present a method which is slightly different from the previous one, the idea here is to consider a Taylor development of $\sqrt{x} I(x)$ at 0. Some references can be found in [45].

2.3.1 Main results

First, we recall some definitions

Definition 2.3.1 *Partial exponential Bell polynomials are defined for any positive integers $k \leq n$ by*

$$B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum \frac{n!}{c_1! c_2! \cdots c_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}, \tag{2.25}$$

where the sum is taken over all positive integers $c_1, c_2 \cdots, c_{n-k+1}$ such that

$$c_1 + c_2 + \cdots + c_{n-k+1} = k,$$
$$c_1 + 2c_2 + \cdots + (n-k+1)c_{n-k+1} = n.$$

Definition 2.3.2 *The complete exponential Bell polynomials are defined by*

$$B_0 = 1,$$
$$\forall n \geq 1\,, \quad B_n = \sum_{k=1}^{n} B_{n,k}.$$

where $B_{n,k}$ are partial exponential Bell polynomials defined above.

For detailed formulas on Bell polynomials, see Comtet [18, 19]

Proposition 2.3.3 *Let p be a real function of class $\mathcal{C}^\infty(\mathbb{R})$ and q be a real function of class $\mathcal{C}^\infty$ with support on the interval $[-c, c]$ ($c > 0$). We suppose that*

i) $p(0) = p'(0) = 0$,

ii) $p'' < 0$ *on the segment* $[-c, c]$.

Then there exists a function F of class $\mathcal{C}^\infty(\mathbb{R})$ which satisfies, for all $x > 0$,

$$\sqrt{x} \int_{\mathbb{R}} e^{xp(t)} q(t) dt = F(1/\sqrt{x}). \tag{2.26}$$

For all $n \geq 0$, as $x \to +\infty$, we have

$$\int_{\mathbb{R}} e^{xp(t)} q(t) dt = \frac{1}{\sqrt{x}} \left(F(0) + \frac{F''(0)}{2!\, x} + \cdots + \frac{F^{(2n)}(0)}{(2n)!\, x^n} + O\Big(\frac{1}{x^{n+1}}\Big) \right), \tag{2.27}$$

where the coefficients $F^{(k)}(0)$ depend only on the values of the derivatives $p''(0), p^{(3)}(0), \ldots, p^{(k+2)}(0)$ and $q(0), q'(0), \ldots, q^{(k)}(0)$.

In particular,

$$F(0) = q(0)\sqrt{\frac{2\pi}{|p''(0)|}}\,, \quad F^{(2n+1)}(0) = 0 \text{ for all } n \geq 0$$

and for all $n \geq 1$

$$F^{(2n)}(0) = \sqrt{\frac{2\pi}{|p''(0)|}} \sum_{k=0}^{2n} \binom{2n}{k} q^{(2n-k)}(0)$$
$$\cdot \sum_{m=0}^{k} B_{k,m}\left(\frac{p^{(3)}(0)}{2.3}, \ldots, \frac{p^{(k-m+3)}(0)}{(k-m+2)(k-m+3)}\right) \frac{(2m+2n-1)!!}{|p''(0)|^{m+n}},$$

where $(2n+1)!! = 1.3.5\ldots(2n+1)$ and $B_{k,m}(x_1, x_2, \ldots, x_{k-m+1})$ is the partial exponential Bell polynomial.

For an integral on an a-priori non-centered interval $[a, b]$, we have an analogous result

Theorem 2.3.4 *Let (a, b) be a non-empty open interval, possibly non bounded and t_0 be some point in (a, b). Denote by V_{t_0} a neighborhood of t_0 such that $p, q : (a, b) \to \mathbb{R}$ are functions of class $\mathcal{C}^\infty(V_{t_0})$.*

We suppose that

i) p is measurable on (a, b),

ii) The maximum of p is reached at t_0 (i.e. $p'(t_0) = 0$ and $p''(t_0) < 0$),

iii) There exists x_0 such that $\int_a^b e^{x_0 p(t)} |q(t)| dt < +\infty$.

Then there exist coefficients $c_0(t_0), c_1(t_0), \ldots$ depending on derivatives of p and q at t_0, such that for any $N \geq 0$, as $x \to +\infty$ we have

$$\int_a^b e^{xp(t)} q(t) dt = e^{xp(t_0)} \left(\frac{c_0(t_0)}{\sqrt{x}} + \frac{c_1(t_0)}{2!\, x^{3/2}} + \cdots + \frac{c_N(t_0)}{(2N)!\, x^{N+1/2}} + O\left(\frac{1}{x^{N+3/2}}\right)\right). \quad (2.28)$$

Moreover, $(c_N)_N$ can be computed as

$$c_N(t_0) = \sqrt{\frac{2\pi}{|p''(t_0)|}} \sum_{k=0}^{2N} \binom{2N}{k} q^{(2N-k)}(t_0)$$
$$\sum_{m=0}^{k} B_{k,m}\left(\frac{p^{(3)}(t_0)}{2.3}, \ldots, \frac{p^{(k-m+3)}(t_0)}{(k-m+2)(k-m+3)}\right) \frac{(2m+2N-1)!!}{|p''(t_0)|^{m+N}}.$$

2.3.2 Proof of Proposition 2.3.3

Setting $y = t\sqrt{x}$ and $u = 1/\sqrt{x}$, the LHS of (2.26) can be written

$$\sqrt{x} \int_{\mathbb{R}} e^{xp(t)} q(t) dt = \int_{\mathbb{R}} e^{xp(\frac{y}{\sqrt{x}})} q\left(\frac{y}{\sqrt{x}}\right) dy = \int_{\mathbb{R}} e^{p(uy)/u^2} q(uy) dy,$$

which allows us directly write function F with proposal: For any $u \neq 0$

$$F(u) = \int_{\mathbb{R}} e^{p(uy)/u^2} q(uy) dy. \quad (2.29)$$

By changing variable $z = -y$, we obtain

$$F(u) = \int_{\mathbb{R}} e^{p(-uy)/u^2} q(-uy)dy = F(-u),$$

then F is even. We thus get $F^{(2n+1)}(0) = 0$ for all $n \geq 0$ and it remains to extend function F at $u = 0$. It is feasible since the expression $p(uy)/u^2$ can be extended by continuity at $u = 0$.

Indeed, according to Taylor's formula at point u_0 we have

$$p(u) = p(u_0) + p'(u_0)\,(u - u_0) + \int_{u_0}^{u} p''(t)(u - t)dt.$$

For $u \neq 0$, since $p(0) = p'(0) = 0$ then

$$\frac{p(uy)}{u^2} = \frac{1}{u^2}\int_0^{uy} p''(t)(uy - t)dt = y^2 \int_0^1 (1 - s)\,p''(suy)ds.$$

We can therefore define function F by placing for any $u \in \mathbb{R}$

$$F(u) = \int_{\mathbb{R}} e^{r(y,u)} q(uy)dy\,, \tag{2.30}$$

where

$$r(y, u) = y^2 \int_0^1 (1 - s)\,p''(suy)ds, \quad \forall u, y \in \mathbb{R}. \tag{2.31}$$

Due to classical theorems of derivation under the integral, the function $r(y, u)$ is clearly of class $\mathcal{C}^\infty$ with respect to u, for all fixed y.

By using the notation

$$D^k = \frac{\partial^k}{\partial u^k}\,,$$

it is easy to obtain for all $k \geq 0$

$$(D^k r)(y, u) = y^{k+2} \int_0^1 (1 - s)\,s^k\,p^{(k+2)}(suy)ds\,, \tag{2.32}$$

$$(D^k r)(y, 0) = y^{k+2}\,p^{(k+2)}(0) \int_0^1 (s^k - s^{k+1})\,ds = \frac{y^{k+2}}{(k+1)(k+2)} p^{(k+2)}(0), \tag{2.33}$$

and if $|uy| \leq c$,

$$\left|D^k r(y, u)\right| \leq \frac{|y|^{k+2}}{(k+1)(k+2)} M_{k+2}(c)\,, \tag{2.34}$$

where

$$M_{k+2}(c) = \max_{[-c,c]} |p^{(k+2)}(t)|\,.$$

We now prove that function F is of class $\mathcal{C}^\infty(\mathbb{R})$ by inductive method. The first step is detailed in the following lemma.

Lemma 2.3.5 *The function $F(u)$ is well-defined, differentiable and $F'(u)$ is continuous on $\mathbb{R}$.*

Proof:
Setting $E(y,u) = e^{r(y,u)}q(uy)$. We begin to prove that function F in (2.26) is well-defined.

Indeed, we have $p'' < 0$ in $[-c,c]$ then there exists $\epsilon > 0$ such that $p''(t) < -\epsilon$ for all $t \in [-c,c]$. It thus follows from (2.31) that $r(y,u) \leq -\epsilon \, \dfrac{y^2}{2}$ when $|uy| \leq c$.
Besides, function q has support in the interval $[-c,c]$ then $q(uy) = 0$ when $|uy| \geq c$. Therefore, there exists a constant

$$N_0 = \max_{[-c,c]} |q(t)|$$

such that

$$\forall u, y \in \mathbb{R}, \quad |E(y,u)| = e^{r(y,u)} |q(uy)| \leq N_0 \, e^{-\epsilon \, y^2/2}\,, \tag{2.35}$$

hence $E(y,u)$ is bounded independently of u. The function $N_0 \, e^{-\epsilon \, y^2/2}$ is integrable for all y. Therefore F is well-defined and continuous. Let us now show that F is differentiable and F' continuous.
Recall that when $|uy| \leq c$, $Dr(y,u) \leq M_3(c) \, \dfrac{|y|^3}{6}$.
The derivative of $E(y,u)$ with respect to u is

$$DE(y,u) = e^{r(y,u)} \left(Dr(y,u) \, q(uy) + y \, q'(uy) \right) ,$$

which is bounded for all $u, y \in \mathbb{R}$ by

$$\left(M_3(c) \, N_0 \, \frac{|y|^3}{6} + N_1 \, |y| \right) \, e^{-\epsilon \, y^2/2}, \tag{2.36}$$

where

$$N_k = \max_{[-c,c]} |q^{(k)}(t)|$$

and

$$N_1 = \max_{[-c,c]} |q'(t)| \, .$$

Once again, $DE(y,u)$ is bounded independently of u. Besides, the RHS of (2.36) is integrable for all y, then F is differentiable on $\mathbb{R}$ and for $u \in \mathbb{R}$,

$$F'(u) = \int_{\mathbb{R}} \frac{\partial}{\partial u} E(y,u) \, dy = \int_{\mathbb{R}} e^{r(y,u)} \left(Dr(y,u) \, q(uy) + y \, q'(uy) \right) \, dy. \tag{2.37}$$

This leads to the continuity of $F'(u)$.

□

The inductive steps hold according to the following lemma.

Lemma 2.3.6 *Assume that for any $n \geq 1$, $F \in \mathcal{C}^n(\mathbb{R})$ where*

$$\begin{aligned} F^{(n)}(u) &= \int_{\mathbb{R}} \frac{\partial^n}{\partial u^n} \left(e^{r(y,u)} q(uy) \right) \, dy \\ &= \int_{\mathbb{R}} e^{r(y,u)} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} y^{n-k} \, q^{(n-k)}(uy) \, B_k \left(Dr(y,u), \ldots, D^k r(y,u) \right) \, dy. \end{aligned} \tag{2.38}$$

Then $F \in \mathcal{C}^{n+1}(\mathbb{R})$ and $F^{(n+1)}(u)$ is defined as (2.38), where B_k denotes the complete exponential Bell polynomials.

Remark 2.3.7 *The way to obtain the function under the integral in (2.38) is to derive function $e^{r(y,u)}q(uy)$ n-th times. From Leibniz's rule and Faà di Bruno's Formula*

$$\begin{aligned}\frac{\partial^n}{\partial u^n}\left(e^{r(y,u)}q(uy)\right) &= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{\partial^k}{\partial u^k}(e^{r(y,u)})\,\frac{\partial^{n-k}}{\partial u^{n-k}}(q(uy)) \\ &= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{\partial^k}{\partial u^k}(e^{r(y,u)})\,y^{n-k}\,q^{(n-k)}(uy)\,,\end{aligned}$$

in which for $k > 0$,

$$\begin{aligned}\frac{\partial^k}{\partial u^k}(e^{r(y,u)}) &= \sum_{m=1}^{k} e^{r(y,u)} B_{k,m}\left(Dr(y,u),\dots,D^{k-m+1}r(y,u)\right) \\ &= e^{r(y,u)}\,B_k\left(Dr(y,u),\dots,D^k r(y,u)\right).\end{aligned}$$

Here, $B_{k,m}$ and B_k denote the partial and the complete exponential Bell polynomial, respectively (see Definitions 2.3.1 and 2.3.2). $B_0 = 1$, $B_{0,0} = 1$, $B_{k,0} = 0$ for $k \geq 1$ and $B_{0,m} = 0$ for $m \geq 1$.

Hence, for $k \geq 0$

$$\begin{aligned}D^k(e^{r(y,u)}) &= e^{r(y,u)}\sum_{m=0}^{k} B_{k,m}\left(Dr(y,u),\dots,D^{k-m+1}r(y,u)\right) \\ &= e^{r(y,u)}\,B_k\left(Dr(y,u),\dots,D^k r(y,u)\right).\end{aligned}$$

Proof of Lemma 2.3.6:

We prove first that function $F^{(n)}(u)$ in (2.38) is well-defined. Let us first show some bounds on Bell polynomials appearing in (2.38) and shorten notations of the derivative $D^k r(y,u)$ into $D^k r$

$$\left|B_{k,m}\left(Dr,\dots,D^{k-m+1}r\right)\right| \leq \sum_{(*)}\frac{k!}{j_1!j_2!\dots j_{k-m+1}!}\left(\frac{|Dr|}{1!}\right)^{j_1}\cdots\left(\frac{\left|D^{k-m+1}r\right|}{(k-m+1)!}\right)^{j_{k-m+1}}$$

$$\leq \sum_{(*)}\frac{k!}{j_1!j_2!\dots j_{k-m+1}!}\left(\frac{M_3(c)\,|y|^3}{1!\,.2.3}\right)^{j_1}\cdots\left(\frac{M_{k-m+3}(c)\,|y|^{k-m+3}}{(k-m+1)!\,(k-m+2)(k-m+3)}\right)^{j_{k-m+1}},$$

where sequences $j_1, j_2, \dots, j_{k-m+1}$ of non-negative integers satisfy two conditions

$$(*)\begin{cases} j_1 + j_2 + \cdots + j_{k-m+1} = m \\ j_1 + 2j_2 + \cdots + (k-m+1)j_{k-m+1} = k \end{cases}.$$

Defining

$$M_{k,m}(c) = \max\left\{\frac{M_3(c)}{2.3},\dots,\frac{M_{k-m+3}(c)}{(k-m+2)(k-m+3)}\right\},$$

we get

$$\left|B_{k,m}\left(Dr,\ldots,D^{k-m+1}r\right)\right| \leq \sum_{(*)} \frac{k!\,|y|^{k+2m}\,M_{k,m}^m(c)}{j_1!j_2!\ldots j_{k-m+1}!}\left(\frac{1}{1!}\right)^{j_1}\cdots\left(\frac{1}{(k-m+1)!}\right)^{j_{k-m+1}}$$
$$\leq |y|^{k+2m}\,M_{k,m}^m(c)\,B_{k,m}(1,\ldots,1).$$

Back to bounding $F^{(n)}(u)$, we obtain

$$|D^n\left(E(y,u)\right)| \leq e^{r(y,u)}\sum_{k=0}^{n}\binom{n}{k}|q^{(n-k)}(uy)|\sum_{m=0}^{k}|y|^{n+2m}\,M_{k,m}^m(c)\,B_{k,m}(1,\ldots,1)$$
$$\leq e^{-\epsilon y^2/2}\sum_{k=0}^{n}\binom{n}{k}N_{n-k}\sum_{m=0}^{k}|y|^{n+2m}\,M_{k,m}^m(c)\,B_{k,m}(1,\ldots,1). \qquad (2.39)$$

The function bounding $F^{(n)}(u)$ (in the RHS of (2.39)) is once again independent of u and integrable in y. Then $F^{(n)}(u)$ is well-defined and continuous.

We now prove that $F \in \mathcal{C}^{n+1}(\mathbb{R})$.
On the one hand, the function under the integral of $F^{(n)}(u)$ has its derivative

$$D\left(e^{r(y,u)}\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\,q^{(n-k)}(uy)\,B_k\left(Dr(y,u),\ldots,D^kr(y,u)\right)\right)$$
$$= D\left(D^n\left(e^{r(y,u)}q(uy)\right)\right) = D^{n+1}\left(e^{r(y,u)}q(uy)\right)$$
$$= e^{r(y,u)}\sum_{k=0}^{n+1}\binom{n+1}{k}y^{n-k+1}\,q^{(n-k+1)}(uy)\,B_k\left(Dr(y,u),\ldots,D^kr(y,u)\right).$$

By using similar arguments, we not only conclude that $F^{(n)}(u)$ is differentiable on $\mathbb{R}$, but also shows that $F^{(n+1)}(u)$ is continuous and well-defined on $\mathbb{R}$. For $u \in \mathbb{R}$,

$$F^{(n+1)}(u) = \int_{\mathbb{R}} \frac{\partial^{n+1}}{\partial u^{n+1}}\left(e^{r(y,u)}q(uy)\right)\,dy$$
$$= \int_{\mathbb{R}} e^{r(y,u)}\sum_{k=0}^{n+1}\binom{n+1}{k}y^{n-k+1}\,q^{(n-k+1)}(uy)\,B_k\left(Dr(y,u),\ldots,D^kr(y,u)\right)\,dy.$$

□

The remaining work of the proof of Proposition 2.3.3 is to consider the coefficients $F^{(n)}(0)$ on the Taylor's expansion of F at 0.
We showed that $F^{(2n+1)}(0) = 0$ for all $n \geq 0$. Then from (2.38)

$$F^{(2n)}(0) = \int_{\mathbb{R}} e^{r(y,0)}\sum_{k=0}^{2n}\binom{2n}{k}y^{2n-k}\,q^{(2n-k)}(0)\,B_k\left(Dr(y,0),\ldots,D^kr(y,0)\right)\,dy$$
$$= \int_{\mathbb{R}} e^{\frac{1}{2}y^2\,p''(0)}\sum_{k=0}^{2n}\binom{2n}{k}y^{2n-k}\,q^{(2n-k)}(0)\,B_k\left(\frac{p^{(3)}(0)\,y^3}{2.3},\ldots,\frac{p^{(k+2)}(0)\,y^{k+2}}{(k+1)(k+2)}\right)\,dy.$$

Since for $k \geq 0$

$$B_k\left(\frac{p^{(3)}(0)\,y^3}{2.3},\ldots,\frac{p^{(k+2)}(0)\,y^{k+2}}{(k+1)(k+2)}\right)$$

$$= \sum_{m=0}^{k} B_{k,m}\left(\frac{p^{(3)}(0)\,y^3}{2.3},\ldots,\frac{p^{(k-m+3)}(0)\,y^{k-m+3}}{(k-m+2)(k-m+3)}\right)$$

$$= \sum_{m=0}^{k}\sum_{(*)} \frac{k!}{j_1!j_2!\ldots j_{k-m+1}!}\left(\frac{p^{(3)}(0)\,y^3}{1!.2.3}\right)^{j_1}\cdots\left(\frac{p^{(k-m+3)}(0)\,y^{k-m+3}}{(k-m+1)!(k-m+2)(k-m+3)}\right)^{j_{k-m+1}},$$

where sequences $j_1, j_2, \ldots, j_{k-m+1}$ of non-negative integers satisfy two conditions

$$(*)\begin{cases} j_1 + j_2 + \cdots + j_{k-m+1} = m \\ j_1 + 2j_2 + \cdots + (k-m+1)j_{k-m+1} = k \end{cases}.$$

This yields

$$B_k\left(\frac{p^{(3)}(0)y^3}{2.3},\ldots,\frac{p^{(k+2)}(0)\,y^{k+2}}{(k+1)(k+2)}\right)$$
$$= \sum_{m=0}^{k} y^{k+2m} B_{k,m}\left(\frac{p^{(3)}(0)}{2.3},\ldots,\frac{p^{(k-m+3)}(0)}{(k-m+2)(k-m+3)}\right).$$

Then

$$F^{(2n)}(0) = \sum_{k=0}^{2n}\binom{2n}{k} q^{(2n-k)}(0)$$
$$\cdot\sum_{m=0}^{k} B_{k,m}\left(\frac{p^{(3)}(0)}{2.3},\ldots,\frac{p^{(k-m+3)}(0)}{(k-m+2)(k-m+3)}\right)\int_{\mathbb{R}} e^{\frac{1}{2}y^2p''(0)}y^{2n+2m}dy.$$

Remark 2.3.8 (One form of Gaussian integrals) *For* $a > 0$

$$\int_{\mathbb{R}} e^{-\frac{1}{2}a\,x^2}x^{2n}\,dx = \frac{(2n-1)!!}{a^n}\sqrt{\frac{2\pi}{a}},$$

where $(2n-1)!! = 1.3.5\ldots(2n-1) = \dfrac{(2n)!}{2^n\,n!}$ *for* $n \geq 1$.

The assumption of $p'' < 0$ on the segment $[-c, c]$ leads us obtain

$$F^{(2n)}(0) = \sqrt{\frac{2\pi}{|p''(0)|}}\sum_{k=0}^{2n}\binom{2n}{k} q^{(2n-k)}(0)$$
$$\cdot\sum_{m=0}^{k} B_{k,m}\left(\frac{p^{(3)}(0)}{2.3},\ldots,\frac{p^{(k-m+3)}(0)}{(k-m+2)(k-m+3)}\right)\frac{(2m+2n-1)!!}{|p''(0)|^{m+n}},$$

for $n \geq 1$. When $n = 0$, we get

$$F(0) = q(0)\sqrt{\frac{2\pi}{|p''(0)|}}.$$

The remaining of the proof is a Taylor expansion.

2.3.3 Proof of Theorem 2.3.10

We will consider the asymptotic expansion of

$$\bar{I}(x) = \int_a^b e^{xp(t)} q(t) dt,$$

as $x \to +\infty$.

We define a function θ which equals 1 in $V(t_0)$ and such that $0 \leq \theta \leq 1$. Let us consider the function

$$I_2(x) = \int_a^b e^{xp(t)} q(t)(1 - \theta(t)) dt.$$

Lemma 2.3.9 *$I_2(x)$ is negligible with respect to $e^{x\,p(t_0)} x^{-\alpha}$ for any α, as $x \to +\infty$.*

Proof:
Since $1 - \theta(t) = 0$ for all $t \in V_{t_0}$ then

$$I_2(x) = \int_{(a,b)\backslash V_{t_0}} e^{xp(t)} q(t)(1 - \theta(t)) dt.$$

On the one hand, from the assumption of t_0 being the maximum of p on (a, b), we have $p(t_0) > p(t)$, for all $t \in (a, b)\backslash V_{t_0}$. Then there exists $\epsilon > 0$ such that

$$p(t_0) - p(t) \geq \epsilon > 0, \text{ for all } t \in (a, b)\backslash V_{t_0}.$$

On the other hand, for all $x > x_0$

$$|I_2(x)| \leq \int_{(a,b)\backslash V_{t_0}} e^{(x-x_0)p(t)} e^{x_0 p(t)} |q(t)|\, dt \leq \int_a^b e^{(x-x_0)(p(t_0)-\epsilon)} e^{x_0 p(t)} |q(t)|\, dt,$$

then

$$|I_2(x)| \leq e^{xp(t_0)}\, e^{-x\epsilon} \int_a^b e^{x_0 p(t)} |q(t)|\, dt\, e^{-x_0 p(t_0) + \epsilon x_0}.$$

According to the assumption *iii)* of Theorem 2.3.10, we get, for any $x > x_0$,

$$|I_2(x)|\, e^{-xp(t_0)} \leq \delta e^{-x\epsilon} \text{ where constant } \delta = \int_a^b e^{x_0 p(t)} |q(t)|\, dt\, e^{-x_0 p(t_0) + \epsilon x_0}.$$

We know that $\lim_{x \to +\infty} \dfrac{x^\alpha}{a^x} = 0$ for all α and all $a > 1$, this shows that $I_2(x)$ is negligible with respect to $e^{xp(t_0)}\, x^{-\alpha}$ for all α, as $x \to +\infty$.

□

We now only consider the remaining part of integral, namely

$$I_1(x) = \bar{I}(x) - I_2(x) = \int_a^b e^{xp(t)} q(t)\, \theta(t) dt,$$

has formed a development of terms of the scale $e^{xp(t_0)}\, x^{-\alpha}$ (we will see later), which are infinitely large to $I_2(x)$. Thus, the asymptotic expansion of $\bar{I}(x)$ in this scale is reduced to that of $I_1(x)$.

We can find $c > 0$ such that the interval $[t_0 - 2c, t_0 + 2c]$ is contained in (a, b); over $[t_0 - 2c, t_0 + 2c]$, functions p, q are $\mathcal{C}^\infty$ and $p'' < 0$. We assume now that the support of θ is contained in $[t_0 - c, t_0 + c]$.
Putting

$$\psi(y) = q(t_0 + y)\,\theta(t_0 + y)$$

and

$$\varphi(y) = (p(t_0 + y) - p(t_0))\,\chi(y),$$

in which χ is a function of class $\mathcal{C}^\infty$, equals 1 on $[-c, c]$ and has support in $[-2c, 2c]$.
Functions p, q now is extended to φ, ψ on $\mathbb{R}$ and it is easy to see that they are $\mathcal{C}^\infty$ on $\mathbb{R}$ as well. We have

$$\begin{aligned}
e^{-xp(t_0)} I_1(x) &= \int_a^b e^{x(p(t)-p(t_0))} q(t)\,\theta(t)\,dt \\
&= \int_{t_0-c}^{t_0+c} e^{x(p(t)-p(t_0))} q(t)\,\theta(t)\,dt \\
&= \int_{-c}^{c} e^{x(p(t_0+y)-p(t_0))} q(t_0+y)\,\theta(t_0+y)\,dy \\
&= \int_{\mathbb{R}} e^{x\varphi(y)} \psi(y)\,dy.
\end{aligned}$$

We see that function ψ has support in $[-c, c]$ and on this interval, $\chi(y) = 1$, $\varphi''(y) = p''(t_0 + y) < 0$. Moreover, $\varphi(0) = 0$ and $\varphi'(0) = p'(t_0) = 0$.
By using Proposition 2.3.3, there exists function $\bar{F}$ of class $\mathcal{C}^\infty(\mathbb{R})$ and coefficients $c_0, c_1, \ldots$ such that

$$\begin{aligned}
e^{-xp(t_0)} I_1(x) &= \frac{1}{\sqrt{x}} \bar{F}\left(\frac{1}{\sqrt{x}}\right) \\
&= \frac{1}{\sqrt{x}} \left(c_0 + \frac{c_1}{2!\,x} + \cdots + \frac{c_n}{(2n)!\,x^n} + O\left(\frac{1}{x^{n+1}}\right)\right),
\end{aligned}$$

as $x \to +\infty$, in which $c_n = \bar{F}^{(2n)}(0)$.
In particular,

$$\bar{F}(0) = \psi(0)\sqrt{\frac{2\pi}{|\varphi''(0)|}}, \quad \bar{F}^{(2n+1)}(0) = 0 \text{ for all } n \geq 0$$

and for all $n \geq 1$

$$\begin{aligned}
F^{(2n)}(0) = &\sqrt{\frac{2\pi}{|\varphi''(0)|}} \sum_{k=0}^{2n} \binom{2n}{k} \psi^{(2n-k)}(0) \\
&\cdot \sum_{m=0}^{k} B_{k,m}\left(\frac{\varphi^{(3)}(0)}{2.3}, \ldots, \frac{\varphi^{(k-m+3)}(0)}{(k-m+2)(k-m+3)}\right) \frac{(2m+2n-1)!!}{|\varphi''(0)|^{m+n}}.
\end{aligned}$$

This establishes Theorem 2.3.10.

Résumé

Le Chapitre 2 est consacré à la méthode de Laplace et au développement asymptotique d'intégrales. L'archetype des intégrales étudiées ici est

$$I(x) = \int_a^b e^{-xp(t)} q(t)dt\,, \tag{2.40}$$

avec a, b des réels, éventuellement ∞; p, q sont des fonctions réelles suffisamment régulières et x est réel. On cherche l'asymptotique de I quand $x \to \infty$.

Dans un premier temps, des intégrations par parties ainsi qu'un résultat de Watson [42] donnent des décompositions pour la fonction Gamma incomplète, la fonction Hypergéométrique ainsi que l'intégrale d'Euler et la formule de Stirling. Dans une deuxième partie, on donne une décomposition en série:

Theorem 2.3.10 *Soit (a, b) un intervalle ouvert non vide, (a ou b éventuellement ∞) et t_0 un point dans (a, b). Soit V_{t_0} un voisinage de t_0 tel que $p, q : (a, b) \to \mathbb{R}$ sont $\mathcal{C}^\infty(V_{t_0})$.*

On suppose

- *i) p est mesurable sur (a, b),*
- *ii) Le maximum de p est atteint en t_0 (i.e. $p'(t_0) = 0$ et $p''(t_0) < 0$),*
- *iii) Il existe x_0 tel que $\int_a^b e^{x_0 p(t)} |q(t)|dt < +\infty$.*

Alors il existe des coefficients $c_0(t_0), c_1(t_0), \ldots$ dépendant des dérivées de p et q en t_0, tels que pour $N \geq 0$, et $x \to +\infty$ on a:

$$\int_a^b e^{xp(t)} q(t)dt = e^{xp(t_0)} \left(\frac{c_0(t_0)}{\sqrt{x}} + \frac{c_1(t_0)}{2!\, x^{3/2}} + \cdots + \frac{c_N(t_0)}{(2N)!\, x^{N+1/2}} + O\left(\frac{1}{x^{N+3/2}}\right)\right). \tag{2.41}$$

De plus, la suite $(c_N)_N$ est donnée par

$$c_N(t_0) = \sqrt{\frac{2\pi}{|p''(t_0)|}} \sum_{k=0}^{2N} \left(\begin{array}{c} 2N \\ k \end{array} \right) q^{(2N-k)}(t_0)$$
$$\sum_{m=0}^{k} B_{k,m} \left(\frac{p^{(3)}(t_0)}{2.3}, \ldots, \frac{p^{(k-m+3)}(t_0)}{(k-m+2)(k-m+3)} \right) \frac{(2m+2N-1)!!}{|p''(t_0)|^{m+N}}.$$

et $B_{k,m}$ sont les polynômes de Bell.

Chapter 3

Sharp Large deviations for empirical correlation coefficients

Contents

3.1 Introduction

The Pearson correlation coefficient between two random variables X and Y is defined by

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}},$$

whenever this quantity exists. The empirical counterpart is the following. Let us consider two samples $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$. The empirical Pearson correlation coefficient is given by

$$r_n = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}(X_i - \bar{X}_n)^2 \sum_{i=1}(Y_i - \bar{Y}_n)^2}}, \tag{3.1}$$

where $\bar{X}_n = \frac{1}{n}\sum_{k=1}^n X_k$ and $\bar{Y}_n = \frac{1}{n}\sum_{k=1}^n Y_k$ are the empirical means of the samples.

In the Gaussian case, when $E(X)$ and $E(Y)$ are both known, we consider $\tilde{r}_n$:

$$\tilde{r}_n = \frac{\sum_{i=1}^n (X_i - E(X))(Y_i - E(Y))}{\sqrt{\sum_{i=1}(X_i - E(X))^2 \sum_{i=1}(Y_i - E(Y))^2}}. \tag{3.2}$$

When the $(X_i, Y_i)_i$ are a sample from a distribution (X, Y), r_n and $\tilde{r}_n$ converge almost surely to the Pearson correlation coefficient of (X, Y) given above. The coefficients r_n and $\tilde{r}_n$ describe the linear relation between the two random vectors. We study SLD for empirical coefficients r_n and $\tilde{r}_n$ in two general cases: spherical and Gaussian distributions.

This chapter is organized as follows: in Sections 3.2 and 3.3, we present the SLD results in the spherical and Gaussian cases; Section 3.4 is devoted to the proofs and in Section 3.6, we study the Bahadur exact slope of r_n in the Gaussian case.

3.2 Spherical distribution

In this section, we study empirical correlations under the following assumption.

Assumption 3.2.1 *We assume that $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ with $n > 2$ are two independent random vectors where X has a n-variate spherical distribution with $P(X = 0) = 0$ and Y has any distribution with $P(Y \in \{\mathbf{1}\}) = 0$ where $\mathbf{1} = \{k(1, \cdots, 1), k \in \mathbb{R}\}$.*

3.2.1 SLDP for r_n

The strategy is to compute the normalized cumulant generating function of r_n:

$$L_n(\lambda) = \frac{1}{n}\log E(e^{n\lambda r_n}). \tag{3.3}$$

The asymptotics of L_n are given in the following proposition:

Proposition 3.2.2 *For any $\lambda \in \mathbb{R}$, we have*

$$E\left(e^{n\lambda r_n}\right) = \frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})} e^{nh(r_0(\lambda))}\left(\frac{c_0(\lambda)}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right)\right), \tag{3.4}$$

where

- $h(r) = \lambda r + \frac{1}{2}\log(1 - r^2)$,
- $r_0(\lambda)$ *is the unique root in* $]-1, 1[$ *of* $h'(r) = 0$, *i.e.*

$$r_0(\lambda) = \frac{-1 + \sqrt{1 + 4\lambda^2}}{2\lambda}, \tag{3.5}$$

- $g(r) = (1-r^2)^{-2}$ *and* $c_0(\lambda) = \sqrt{\dfrac{2\pi}{|h''(r_0(\lambda))|}}g(r_0(\lambda))\,.$

Therefore

$$L_n(\lambda) = L(\lambda) - \frac{1}{n}\left[\frac{1}{2}\log\sqrt{1+4\lambda^2} - \frac{3}{2}\log\frac{1+\sqrt{1+4\lambda^2}}{2}\right] + O\Big(\frac{1}{n^2}\Big)\,. \tag{3.6}$$

where L is the limit normalized log*–Laplace transform of r_n:*

$$L(\lambda) = h(r_0(\lambda)). \tag{3.7}$$

The proof of this proposition is postponed to Section 3.4. Now we have the following SLDP:

Theorem 3.2.3 *For any $0 < c < 1$, under Assumption (3.2.1), we have*

$$P(r_n \geq c) = \frac{e^{-nL^*(c) - \frac{1}{2}\log(1+4\lambda_c^2) + \frac{3}{2}\log\frac{1+\sqrt{1+4\lambda_c^2}}{2}}}{\lambda_c\sigma_c\sqrt{2\pi n}}(1+o(1)), \tag{3.8}$$

where

- λ_c *is the unique solution of* $L'(\lambda) = c$, *i.e.* $\lambda_c = \dfrac{c}{1-c^2}$,
- $\sigma_c^2 = L''(\lambda_c) = \dfrac{(1-c^2)^2}{1+c^2}$,
- $L^*(y) = -\frac{1}{2}\log(1-y^2)\,.$

Proof:
To prove the SLD on r_n, we proceed as in Bercu et al. [9, 10]. See also Chapter 1 for more details. The following lemma, which proof is given in Section 3.4, gives some basic properties of L:

Lemma 3.2.4 *Let $L(\lambda) = h(r_0(\lambda))$ where h and r_0 are defined in Proposition 3.2.2, we have*

- *L is defined on $\mathbb{R}$, $\mathcal{C}^\infty$ on its domain.*
- *L is a strictly convex function on $\mathbb{R}$, L reaches its minimum at $\lambda = 0$ and $L' \in]-1,1[$.*
- *The Legendre dual of L is defined on $]-1,1[$ and computed as*

$$L^*(y) = \sup_{\lambda\in\mathbb{R}}\{\lambda y - L(\lambda)\} = -\frac{1}{2}\log(1-y^2)\,. \tag{3.9}$$

.

Let $0 < c < 1$ and $\lambda_c > 0$ such that $L'(\lambda_c) = c$. Then

$$L^*(c) = c\lambda_c - L(\lambda_c)\,,$$

We denote by $\sigma_c^2 = L''(\lambda_c)$, and define the following change of probability:

$$\frac{dQ_n}{dP} = e^{\lambda_c n r_n - nL_n(\lambda_c)}\,. \tag{3.10}$$

The expectation under Q_n is denoted by E_n. We write

$$P(r_n \geq c) = A_n B_n\,, \tag{3.11}$$

where

$$A_n = \exp[n(L_n(\lambda_c) - c\lambda_c)]\,,$$
$$B_n = E_n(\exp[-n\lambda_c(r_n - c)]\mathbb{1}_{r_n \geq c})\,.$$

On the first hand, from (3.6)

$$A_n = \exp[-nL^*(c) - \frac{1}{4}\log(1 + 4\lambda_c^2) + \frac{3}{2}\log\frac{1 + \sqrt{1 + 4\lambda^2}}{2}]\left(1 + O\left(\frac{1}{n}\right)\right).$$

On the other hand, let us denote by

$$U_n = \frac{\sqrt{n}(r_n - c)}{\sigma_c},$$

$$\Phi_n(u) = E_n(e^{iuU_n}) = \exp(-\frac{iu\sqrt{n}}{\sigma_c}c + nL_n(\lambda_c + \frac{iu}{\sigma_c\sqrt{n}}) - nL_n(\lambda_c))\,.$$

We have the following technical results on Φ_n, proved in Section 3.4.

Lemma 3.2.5 *For any $K \in \mathbb{N}^*$, $\eta > 0$, for n large enough and any $u \in \mathbb{R}$,*

$$|\Phi_n(u)| \leq \frac{1}{|\lambda_c + \frac{iu}{\sigma_c\sqrt{n}}|^K}\frac{c_0(\lambda)}{c_0^K(\lambda)}(1 + \eta)\,. \tag{3.12}$$

where c_0 and c_0^K are the first coefficients in Laplace's method (see Theorem 2.3.10), respectively and c_0^K corresponds to

$$g^K(r) = (2r)^K(1 - r^2)^{-K-2}\,.$$

From lemma above, choosing $K \geq 2$, we see that Φ_n is in L^2 and by Parseval formula,

$$B_n = E_n[e^{-\lambda_c\sigma_c\sqrt{n}U_n}\mathbb{1}_{U_n \geq 0}] = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\frac{1}{\lambda_c\sigma_c\sqrt{n} + iu}\right)\Phi_n(u)du = \frac{C_n}{\lambda_c\sigma_c\sqrt{2\pi n}},$$

where

$$C_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(1 + \frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)^{-1}\Phi_n(u)du.$$

Lemma 3.2.6 *We have*

$$\lim_{n\to\infty}\Phi_n(u) = e^{-u^2/2} \text{ and } \lim_{n\to\infty}C_n = 1\,.$$

From lemma above, which proof is postponed to Section 3.4, we have equation (3.8).

□

3.2.2 Known expectation

In case $E(X)$ and $E(Y)$ are known, we consider $\tilde{r}_n$ as follows

$$\tilde{r}_n = \frac{(X - E(X))' (Y - E(Y))}{\|X - E(X)\| \, \|Y - E(Y)\|}, \tag{3.13}$$

where X' is the transpose of vector X. We can derive a SLD result similar to the previous one. The following proposition gives the expression of the n.c.g.f. of $\tilde{r}_n$:

Proposition 3.2.7 *For any $\lambda \in \mathbb{R}$, we have*

$$E(e^{n\lambda\tilde{r}_n}) = \frac{\Gamma(\frac{n}{2})}{\pi^{1/2}\Gamma(\frac{n-1}{2})} e^{nh(r_0(\lambda))} \left(\frac{\tilde{c}_0(\lambda)}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right)\right), \tag{3.14}$$

where

- $h(r) = \lambda r + \frac{1}{2}\log(1 - r^2)$,
- $r_0(\lambda)$ *is the unique root in* $]-1, 1[$ *of* $h'(r) = 0$, *i.e.*

$$r_0(\lambda) = \frac{-1 + \sqrt{1 + 4\lambda^2}}{2\lambda},$$

- $\tilde{g}(r) = (1 - r^2)^{-3/2}$ *and* $\tilde{c}_0(\lambda) = \sqrt{\frac{2\pi}{|h''(r_0(\lambda))|}}\tilde{g}(r_0(\lambda))$.

The normalized cumulant generating function of $\tilde{r}_n$ is

$$\tilde{L}_n(\lambda) = h(r_0(\lambda)) - \frac{1}{n}\left[\frac{1}{2}\log\sqrt{1 + 4\lambda^2} - \log\frac{1 + \sqrt{1 + 4\lambda^2}}{2}\right] + O\left(\frac{1}{n^2}\right). \tag{3.15}$$

This proposition is proved in Section 3.4. We have the following SLDP:

Theorem 3.2.8 *For any $0 < c < 1$, under Assumption (3.2.1), we have*

$$P(\tilde{r}_n \geq c) = \frac{\exp^{-nL^*(c) - \frac{1}{4}\log(1+4\lambda_c^2) + \log\frac{1+\sqrt{1+4\lambda_c^2}}{2}}}{\lambda_c\sigma_c\sqrt{2\pi n}}(1 + o(1)). \tag{3.16}$$

Proof:
The proof of Theorem 3.2.8 is exactly similar to the one of Theorem 3.2.3 and formula (3.8) is changed to (3.16) according to the way formula (3.6) is changed to (3.15).

□

3.3 Gaussian case

Assumption 3.3.1 *Let (X, Y) be a $\mathbb{R}^2$-valued Gaussian random vector where $\sigma_1^2 = Var(X)$, $\sigma_2^2 = Var(Y)$ and ρ is the correlation coefficient: $Cov(X, Y) = \rho\sigma_1\sigma_2$. We consider an i.i.d. sample $\{(X_i, Y_i), i = 1, \cdots n\}$ of (X, Y).*

3.3.1 General case

We deal with the Pearson coefficient given in (3.1). Large deviations for (r_n) are detailed in the paper of Si [55]. It can be noted that the contraction principle used by Si is not valid here. The rate function is correct however. We can give an expression of the normalized log–Laplace transform L_n given by (3.3).

Proposition 3.3.2 *Let us define*

$$\rho_0 := \frac{\sqrt{3+2\sqrt{3}}}{3}.$$

For any $\lambda \in \mathbb{R}$ and ρ such that $|\rho| \le \rho_0$, we have the n.c.g.f. of r_n:

$$L_n(\lambda) = \overline{h}(r_0(\lambda)) + \frac{1}{2}\log(1-\rho^2) + \frac{1}{n}\left[\log \overline{g}_\rho(r_0(\lambda)) - \frac{1}{2}\log|\overline{h}''(r_0(\lambda))|\right] + O\left(\frac{1}{n^2}\right), \quad (3.17)$$

in which

- $\overline{h}(r) = \lambda r - \log(1-\rho r) + \frac{1}{2}\log(1-r^2)$,
- $r_0(\lambda)$ *is the unique real root in* $]-1,1[$ *of* $\overline{h}'(r) = 0$,
- $\overline{g}_\rho(r) = (1-\rho^2)^{-1/2}(1-\rho r)^{3/2}(1-r^2)^{-2}$.

The proof of this proposition is postponed to Section 3.4. We prove the following SLDP:

Theorem 3.3.3 *For any $0 \le \rho < c < 1$ and $|\rho| \le \rho_0$ (with the notations of Proposition 3.3.2), we have*

$$P(r_n \ge c) = \frac{e^{-nL^*(c)+\log \overline{g}_\rho(r_0(\lambda_c)) - \frac{1}{2}\log|\overline{h}''(r_0(\lambda_c))|}}{\lambda_c \sigma_c \sqrt{2\pi n}}(1+o(1)), \quad (3.18)$$

where for any $-1 < y < 1$,

$$L^*(y) = \log\left(\frac{1-\rho y}{\sqrt{(1-\rho^2)}\sqrt{(1-y^2)}}\right). \quad (3.19)$$

Proof:
Following the Proof of Theorem 3.2.3, we can easy obtain (3.18). Note that the rate function in Si [55] matches our (3.77).

□

3.3.2 Known expectations

In case $E(X)$ and $E(Y)$ are known; and $\rho = 0$, we have the following result

Proposition 3.3.4 *The normalized cumulant generating function of $\tilde{r}_n$ is given for any $\lambda \in \mathbb{R}$ by*

$$L_n(\lambda) = h(u_0(\lambda)) - \frac{1}{4n}\log(1+4\lambda^2) + O\left(\frac{1}{n^2}\right), \quad (3.20)$$

where

- $h(r) = \lambda r + \frac{1}{2}\log(1-r^2)$,
- $u_0(\lambda)$ *is the unique solution of* $h'(\lambda) = 0$ *in* $]-1,1[$.

The proof is postponed to Section 3.4. The SLDP is therefore:

Theorem 3.3.5 *When* $\rho = 0$ *and under Assumption 3.3.1, for* $0 < c < 1$*, we have*

$$P(\tilde{r}_n \geq c) = \frac{e^{-nL^*(c) - \frac{1}{4}\log(1-4\lambda_c^2)}}{\lambda_c \sigma_c \sqrt{n}}(1+o(1)), \tag{3.21}$$

where L^* *is given in Theorem 3.2.3.*

3.4 Proofs

3.4.1 Proof of Proposition 3.2.2

We know from Muirhead (Theorem 5.1.1, [40]) that

$$(n-2)^{1/2}\frac{r_n}{(1-(r_n)^2)^{1/2}}$$

has a t_{n-2}-distribution. Hence the density function of r_n is

$$f_n(r) = \frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})}(1-r^2)^{(n-4)/2} \quad (-1 < r < 1). \tag{3.22}$$

Applying Theorem 2.3.10, we get

$$\begin{aligned} E\left(e^{n\lambda r_n}\right) &= \int_{-1}^{1} e^{n\lambda r} f_n(r)dr = \int_{-1}^{1} e^{n\lambda r}\frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})}(1-r^2)^{(n-4)/2}dr \\ &= \frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})} e^{nh(r_0(\lambda))}\left(\frac{c_0(\lambda)}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right)\right). \end{aligned}$$

where h, r_0 and c_0 are given in Proposition 3.2.2.

So we have

$$E\left(e^{n\lambda r_n}\right) = \frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})}\sqrt{\frac{2\pi}{n}} e^{nh(r_0(\lambda))}\frac{g(r_0(\lambda))}{\sqrt{|h''(r_0(\lambda))|}}\left(1+O\left(\frac{1}{n}\right)\right) \tag{3.23}$$

$$= \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\sqrt{\frac{2}{n}}\, e^{nh(r_0(\lambda))}\frac{1}{(1-r_0(\lambda)^2)\sqrt{1+r_0(\lambda)^2}}\left(1+O\left(\frac{1}{n}\right)\right) \tag{3.24}$$

From the duplication formula

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z)\,,$$

as well as the Stirling formula

$$\log\Gamma(z) = z\log z - z - \frac{1}{2}\log z + \log\sqrt{2\pi} + O\left(\frac{1}{Re(z)}\right)\,, \text{ as } Re(z) \to \infty\,,$$

formula (3.24) above becomes

$$E(e^{n\lambda r}) = e^{nh(r_0(\lambda))}\frac{1}{(1-r_0(\lambda)^2)\sqrt{1+r_0(\lambda)^2}}\left(1+O\left(\frac{1}{n}\right)\right)$$

With the expression of r_0, we get formula (3.6).

3.4.2 Proof of Lemma 3.2.4

We can explicit the full expression of L:

$$L(\lambda) = \frac{-1+\sqrt{1+4\lambda^2}}{2} - \frac{1}{2}\log(\frac{1+\sqrt{1+4\lambda^2}}{4})\,. \tag{3.25}$$

It is easy to see that L is defined on $\mathbb{R}$, $\mathcal{C}^\infty$ on its domain.

From the definition of L we can deduce

$$L'(\lambda) = r_0(\lambda) + h'(r_0(\lambda)) = r_0(\lambda), \tag{3.26}$$

and by construction of r_0, $L' \in]-1,1[$. Now we can compute

$$L''(\lambda) = r_0'(\lambda) = \frac{1}{2\lambda^2}\left(1 - \frac{1}{\sqrt{1+4\lambda^2}}\right), \tag{3.27}$$

and it is easily seen that $L''(\lambda) > 0$ for any $\lambda \in \mathbb{R}^*$ and $L''(0)$ can be defined by continuity as 1. Hence L is strictly convex on $\mathbb{R}$ and has its minimum at $\lambda = 0$. Moreover, if we have

$$L'(\lambda_c) = r_0(\lambda_c) = c,$$

then $0 < c < 1$ implies $\lambda_c > 0$ and we can obtain

$$4\lambda_c(\lambda_c(1-c^2) - c) = 0.$$

This leads us to the expression

$$\lambda_c = \frac{c}{1-c^2}\,.$$

Hence the preceding expression yields

$$\sigma_c^2 = L''(\lambda_c) = \frac{(1-c^2)^2}{1+c^2}\,.$$

3.4.3 Proof of Lemmas 3.2.5 and 3.2.6

The proof of Lemma 3.2.5 is based on iterated integrations by parts. We detail below the steps.

$$\begin{aligned}\Phi_n(u) = E_n(e^{iuU_n}) &= \int_{\mathbb{R}} e^{iu\frac{\sqrt{n}(r-c)}{\sigma_c}} f_n(r) e^{\lambda_c nr - nL_n(\lambda_c)} dr \\ &= \mathbf{\Gamma}_n\, e^{-iu\frac{\sqrt{n}c}{\sigma_c}} e^{-nL_n(\lambda_c)} \int_{-1}^{1} e^{(iu\frac{\sqrt{n}}{\sigma_c} + \lambda_c n)r} (1-r^2)^{n/2-2} dr,\end{aligned}$$

where for seek of simplicity we denote by

$$\mathbf{\Gamma}_n = \frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})}\,. \tag{3.28}$$

For $K \in \mathbb{N}^*$, performing K integrations by part, since f_n is zero at -1 and 1 when n is large enough, we get:

$$\Phi_n(u) = \mathbf{\Gamma}_n e^{-iu\frac{\sqrt{n}c}{\sigma_c}} e^{-nL_n(\lambda_c)} \cdots$$
$$\cdot \frac{(\frac{n}{2}-2)(\frac{n}{2}-3)\cdots(\frac{n}{2}-K-1)}{\left(iu\frac{\sqrt{n}}{\sigma_c}+\lambda_c n\right)^K} \int_{-1}^{1} e^{(iu\frac{\sqrt{n}}{\sigma_c}+\lambda_c n)r}(-2r)^K(1-r^2)^{n/2-2-K}dr.$$

Hence,

$$|\Phi_n(u)| \leq \mathbf{\Gamma}_n e^{-nL_n(\lambda_c)} \frac{(\frac{n}{2}-2)(\frac{n}{2}-3)\cdots(\frac{n}{2}-K-1)}{\left|iu\frac{\sqrt{n}}{\sigma_c}+\lambda_c n\right|^K} \int_{-1}^{1} e^{\lambda_c nr}(2r)^K(1-r^2)^{n/2-2-K}dr.$$

Using Laplace's method once again (see Chapter 2), for a given $\eta > 0$ we can find N large enough such that for any $n \geq N$,

$$|\Phi_n(u)| \leq \frac{1}{|\lambda_c + \frac{iu}{\sqrt{n}\sigma_c}|^K} \frac{c_0(\lambda)}{c_0^K(\lambda)}(1+\eta)\,. \tag{3.29}$$

$\square$

To prove Lemma 3.2.6, we first split C_n into two terms:

$$C_n = \frac{1}{\sqrt{2\pi}} \int_{|u|\leq n^\alpha} \left(1+\frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)^{-1} \Phi_n(u)du + \frac{1}{\sqrt{2\pi}} \int_{|u|>n^\alpha} \left(1+\frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)^{-1} \Phi_n(u)du. \tag{3.30}$$

For the second term in the RHS of (3.30) we have

$$\begin{aligned}
\left|\int_{|u|>n^\alpha} \frac{1}{\left(1+\frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)}\Phi_n(u)du\right| &\leq \int_{|u|>n^\alpha} \frac{1}{\left|1+\frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right|}\Phi_n(u)du \\
&\leq \int_{|u|>n^\alpha} \frac{1}{|\lambda_c|^K\left|1+\frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right|^{K+1}}du\frac{c_0^K(\lambda_c)}{c_0(\lambda_c)}(1+\eta) \\
&\leq \frac{c_0^K(\lambda_c)}{|\lambda_c|^K c_0(\lambda_c)}(1+\eta)\int_{|u|>n^\alpha} \frac{1}{\left(1+\frac{u^2}{\lambda_c^2\sigma_c^2 n}\right)^{(K+1)/2}}du \\
&\leq \frac{c_0^K(\lambda_c)}{|\lambda_c|^K c_0(\lambda_c)}(1+\eta)(\lambda_c^2\sigma_c^2 n)^{(K+1)/2} 2\frac{n^{-\alpha K}}{K}.
\end{aligned}$$

In order to have a negligible term, it is enough to have $-K\alpha + \frac{K+1}{2} < 0$, i.e. fixing $K=3$, $\alpha = \frac{3}{4}$. Now for the domain $\{|u| \leq n^\alpha\}$, we study more precisely the expression

$$\Phi_n(u) = E_n(e^{iuU_n}) = \exp(-\frac{iu\sqrt{n}}{\sigma_c}c + nL_n(\lambda_c + \frac{iu}{\sigma_c\sqrt{n}}) - nL_n(\lambda_c))\,. \tag{3.31}$$

We first remark that $E(e^{n\lambda r_n})$ is analytic in λ on $\mathbb{R}$, hence it can be expanded by analytic continuation and $L_n(\lambda + iy)$ for $\lambda, y \in \mathbb{R}$ is well defined. From the analyticity,, we can expand in Taylor series the expression (3.31) above.

$$\begin{aligned}
\Phi_n(\lambda_c) &= \exp\{-iu\frac{\sqrt{n}c}{\sigma_c} + n\sum_{k=1}^{\infty}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{L_n^{(k)}(\lambda_c)}{k!}\} \\
&= \exp\{-iu\frac{\sqrt{n}c}{\sigma_c} + n\frac{iu}{\sigma_c\sqrt{n}}L_n'(\lambda_c) + n\sum_{k\geq 2}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{L_n^{(k)}(\lambda_c)}{k!}\}. \qquad (3.32)
\end{aligned}$$

We detail now a development of L_n – and its derivatives – which will be useful in the whole chapter.

Technical Lemma 3.4.1 *For any $\lambda \in \mathbb{R}$, we have*

$$L_n(\lambda) = h(r_0(\lambda)) + \frac{1}{n}\log \boldsymbol{\Gamma}_n - \frac{1}{2n}\log n + \frac{1}{n}R_0(\lambda) + \frac{1}{n}\sum_{k\geq 1}\frac{R_p(\lambda)}{n^p p!}, \qquad (3.33)$$

where $\boldsymbol{\Gamma}_n$ is defined in (3.28) *and*

$$\begin{aligned}
R_0(\lambda) &= \log c_0(\lambda), \qquad (3.34)\\
R_p(\lambda) &= \sum_{1\leq s\leq p}(-1)^{s-1}(s-1)!B_{p,s}(c_1, c_2, \cdots)c_0^{-s}, \qquad (3.35)
\end{aligned}$$

where the coefficients c_i are given by Laplace development (see Section 2.3) and $B_{p,s}$ is the partial exponential Bell polynomials (see (2.25)*).*

Proof of Technical Lemma 3.4.1:
From Chapter 2 we can develop

$$E(e^{n\lambda r_n}) = \frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})}\frac{e^{nh(r_0(\lambda)}}{\sqrt{n}}\sum_{p\geq 0}\frac{c_p(\lambda)}{(2p)!n^p}, \qquad (3.36)$$

where

$$\begin{aligned}
c_p(\lambda) = &\sqrt{\frac{2\pi}{|h''(r_0(\lambda))|}}\sum_{k=0}^{2p}\binom{2p}{k}g^{(2p-k)}(r_0(\lambda)) \\
&\cdot \sum_{m=0}^{k}B_{k,m}\left(\frac{h^{(3)}(r_0(\lambda))}{2.3}, \ldots, \frac{h^{(k-m+3)}(r_0(\lambda))}{(k-m+2)(k-m+3)}\right)\frac{(2m+2p-1)!!}{|h''(t_0)|^{m+p}}. \qquad (3.37)
\end{aligned}$$

From Faà di Bruno formula (see e.g. formula [5c] of Comtet [18]):

$$\log E(e^{n\lambda r_n}) = nh(r_0(\lambda)) + \log\left(\frac{\Gamma(\frac{n-1}{2})}{\sqrt{n}\pi^{1/2}\Gamma(\frac{n-2}{2})}\right) + \log c_0(\lambda) + \sum_{p\geq 1}\frac{R_p(\lambda)}{n^p p!}, \qquad (3.38)$$

where R_p is defined in formula (3.35) above. Hence the formula (3.33) is proven. □

From expressions (3.35) and (3.37), we see that R_p is a polynomial in $g^{(s)}(r_0(\lambda))$ and $h^{(s)}(r_0(\lambda))$ where the derivatives are taken with respect to r. The function $r_0(\lambda)$ is $\mathcal{C}^\infty$ on $\mathbb{R}$. We can therefore express the derivatives of L_n as follows:

$$L_n^{(k)}(\lambda) = L^{(k)}(\lambda) + \frac{R_0^{(k)}(\lambda)}{n} + \frac{1}{n}\sum_{p\geq 1}\frac{R_p^{(k)}(\lambda)}{n^p p!}. \tag{3.39}$$

Back to formula (3.32), and from the choice of λ_c, we have

$$\frac{\partial}{\partial\lambda}h(r_0(\lambda))\Big|_{\lambda=\lambda_c} = L'(\lambda_c) = c$$

and

$$\begin{aligned}
\Phi_n(u) &= \exp\{\frac{iu\sqrt{n}}{\sigma_c}[L_n'(\lambda_c) - c] + n\sum_{k\geq 2}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{L_n^{(k)}(\lambda_c)}{k!}\} \\
&= \exp\{\frac{iu}{\sqrt{n}\sigma_c}[R_0'(\lambda) + \sum_{p\geq 1}\frac{R_p'(\lambda)}{n^p p!}] - \frac{u^2}{2\sigma_c^2}L_n''(\lambda_c) + n\sum_{k\geq 3}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{L_n^{(k)}(\lambda_c)}{k!}\} \\
&= \exp\{-\frac{u^2}{2} + \sum_{k=3}^{2p}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{nL^{(k)}(\lambda_c)}{k!} + \sum_{k=1}^{2p}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{R_0^{(k)}(\lambda_c)}{k!} + \sum_{k\geq 1}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{1}{k!}\sum_{p\geq 1}\frac{R_p^{(k)}(\lambda_c)}{n^p p!}\}.
\end{aligned} \tag{3.40}$$

For p large enough such that $\{u^k/(\sqrt{n})^{k+2p}\}$ is bounded on $\{|u| \leq n^\alpha\}$, we can have a uniform bound on the rest of the sum in the last term on the RHS above. Hence we can write, for a given $m \in \mathbb{N}$ large enough

$$\begin{aligned}
\Phi_n(u) = \exp\{-\frac{u^2}{2} &+ \sum_{k=3}^{2m+3}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{nL^{(k)}(\lambda_c)}{k!} + \sum_{k=1}^{2m+1}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{R_0^{(k)}(\lambda_c)}{k!} \\
&+ \sum_{k=1}^{2m+1}\sum_{p=1}^{s(m)}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k \frac{1}{k!}\frac{R_p^{(k)}(\lambda_c)}{n^p p!}\} + O(\frac{1+|u|^{2m+4}}{n^{m+1}})\}.
\end{aligned} \tag{3.41}$$

We follow the scheme of Cramer [21] Lemma 2, p.72 (see also Bercu and Rouault [10]), and we get the wanted results.

□

Remark 3.4.2 *A thorough study of expressions $L_n^{(k)}$ and $R_p^{(k)}$ are given in the Appendix.*

3.4.4 Proof of Proposition 3.2.7

By symmetry, the mean $EX = 0$ if it exists. Then, $\tilde{r}_n$ from (3.13) becomes

$$\tilde{r}_n = \frac{X'(Y - E(Y))}{\|X\|\,\|Y - E(Y)\|}. \tag{3.42}$$

Applying Theorem 1.5.7 from Muirhead [40], with $\alpha = \dfrac{Y - E(Y)}{\|Y - E(Y)\|} \in \mathbb{R}^n$, then

$$(n-1)^{1/2}\frac{\tilde{r}_n}{(1-\tilde{r}_n^2)^{1/2}}$$

has a t_{n-1}-distribution. Comparing to r_n, the degree of the t-distribution is one degree less than $\tilde{r}_n$.

Hence the density function of $\tilde{r}_n$ is

$$\frac{\Gamma(\frac{n}{2})}{\pi^{1/2}\Gamma(\frac{n-1}{2})}(1-r^2)^{(n-3)/2}, \quad (-1 < r < 1). \tag{3.43}$$

Applying Laplace's method we get

$$\begin{aligned} E\left(e^{n\lambda\tilde{r}_n}\right) &= \int_{-1}^{1} e^{n\lambda r}\frac{\Gamma(\frac{n}{2})}{\pi^{1/2}\Gamma(\frac{n-1}{2})}(1-r^2)^{(n-3)/2}dr \\ &= \frac{\Gamma(\frac{n}{2})}{\pi^{1/2}\Gamma(\frac{n-1}{2})}e^{nh(r_0(\lambda))}\left(\frac{\tilde{c}_0(\lambda)}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right)\right), \end{aligned}$$

where h, r_0 and c_0 are given in Proposition 3.2.7. Then

$$\begin{aligned} E\left(e^{n\lambda\tilde{r}_n}\right) &= \frac{\Gamma(\frac{n}{2})}{\pi^{1/2}\Gamma(\frac{n-1}{2})}\sqrt{\frac{2\pi}{n}}e^{nh(r_0(\lambda))}\frac{\tilde{g}(r_0(\lambda))}{\sqrt{|h''(r_0(\lambda))|}}\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= e^{nh(r_0(\lambda))}\frac{1}{\sqrt{(1-r_0^2(\lambda))(1+r_0^2(\lambda))}}\left(1 + O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{3.44}$$

And we can obtain formula (3.15) from the expression of r_0.

3.4.5 Proof of Proposition 3.3.2

From Muirhead, we know that the density function of a $(n+1)$ sample correlation coefficient r_{n+1} is given by

$$\begin{aligned} &\frac{(n-1)\Gamma(n)}{\Gamma(n+1/2)\sqrt{2\pi}}(1-\rho^2)^{n/2}(1-\rho r)^{-n+1/2}(1-r^2)^{(n-3)/2} \\ &\qquad\qquad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; n + \frac{1}{2}; \frac{1}{2}(1+\rho r)\right) \qquad (-1 < r < 1). \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function (see [42]). Hence Laplace transform is

$$\begin{aligned} E\left(e^{(n+1)\lambda r_{n+1}}\right) &= \frac{(n-1)\Gamma(n)}{\Gamma(n+1/2)\sqrt{2\pi}}(1-\rho^2)^{n/2} \\ &\int_{-1}^{1} e^{(n+1)\lambda r}(1-\rho r)^{-n+1/2}(1-r^2)^{(n-3)/2}{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; n + \frac{1}{2}; \frac{1}{2}(1+\rho r)\right)dr. \end{aligned}$$

Looking for a limit as $n \to \infty$, we can use the following result due to Temme [60, 61] (see also [30] and Section 2.1): the function ${}_2F_1$ has the following Laplace transform representation

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}dt \tag{3.45}$$

and

$$ {}_2F_1(a,b,c+\lambda;z) \sim \frac{\Gamma(c+\lambda)}{\Gamma(c+\lambda-b)} \sum_{s=0}^{\infty} \mathrm{f}_s(z) \frac{(b)_s}{\lambda^{b+s}}\,, \tag{3.46} $$

where the equivalent is for $\lambda \to +\infty$ and

$$ \mathrm{f}(t) = \left(\frac{e^t-1}{t}\right)^{b-1} e^{(1-c)t}(1-z+ze^{-t})^{-a}\,, $$

$$ \mathrm{f}(t) = \sum_{s=0}^{\infty} \mathrm{f}_s(t) t^s. $$

In our case, we get as $n \to \infty$:

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{1}{2}(1+\rho r)\right) \sim \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(n)} \left(\frac{1}{\sqrt{n}} + \frac{2+\rho r}{8n^{3/2}} + o\left(\frac{1}{n^{3/2}}\right)\right)\,. \tag{3.47} $$

Hence we have to deal with the following integral:

$$ \int_{-1}^{1} e^{(n+1)\lambda r}(1-\rho r)^{-n+1/2}(1-r^2)^{(n-3)/2}\left(1+\frac{2+\rho r}{8n}+o\left(\frac{1}{n}\right)\right)dr. \tag{3.48} $$

Neglecting the terms of lower order in n we focus on

$$ \int_{-1}^{1} e^{(n+1)\lambda r}(1-\rho r)^{-n+1/2}(1-r^2)^{(n-3)/2}dr = \int_{-1}^{1} e^{n\bar{h}(r)}\bar{g}(r)dr\,, \tag{3.49} $$

where

$$ \bar{h}(r) = \lambda r - \log(1-\rho r) + \frac{1}{2}\log(1-r^2)\,, \tag{3.50} $$

$$ \bar{g}(r) = e^{\lambda r}\sqrt{(1-\rho r)}(1-r^2)^{-3/2}\,. $$

The following lemma detail the properties of the function $\bar{h}$:

Lemma 3.4.3 *For any $\rho \in]-1,1[$ and $r \in]-1,1[$, the function $\bar{h}$ of formula* (3.50) *is defined for any $\lambda \in \mathbb{R}$. Moreover, $\bar{h}'(r) = 0$ has at least one solution in $]-1,1[$ and $\bar{h}''(r) < 0$ on $]-1,1[$ for any $|\rho| \le \rho_0$ where $\rho_0 = \dfrac{\sqrt{3+2\sqrt{3}}}{3}$.*

Proof:
We compute easily

$$ \bar{h}'(r) = \lambda + \frac{\rho}{1-\rho r} - \frac{r}{1-r^2} $$

and see that $H(r) = \bar{h}'(r)(1-r^2) = 0$ has at least one root in $]-1,1[$ (since $H(-1)H(1) < 0$). Hence there exists at least one solution $r_0 \in]-1,1[$ such that $\bar{h}'(r) = 0$. Next, we compute

$$ \bar{h}''(r) = \frac{\rho^2}{(1-\rho r)^2} - \frac{1+r^2}{(1-r^2)^2} $$

and we have

$$ \bar{h}''(r) < 0 \text{ for any } r \in]-1,1[\iff |\rho| \le \rho_0 := \frac{\sqrt{3+2\sqrt{3}}}{3}\,. \qquad \square $$

We know from Si [55] that the rate function in this case is

$$I_\rho(s) = \log\left(\frac{1-\rho s}{\sqrt{(1-\rho^2)}\sqrt{(1-s^2)}}\right) \text{ for } -1 < s < 1\,. \tag{3.51}$$

However this function is obtained by a contraction principle which is not applicable here (the functions applied in the principle are not continuous, see Dembo and Zeitouni for more details [26]), we claim that the expression above is correct. We prove it below. We have

$$L(\lambda) = \bar{h}(r_0(\lambda)) + \frac{1}{2}\log(1-\rho^2)\,.$$

where r_0 satisfies

$$\bar{h}'(r_0(\lambda)) = 0.$$

Now we compute

$$L'(\lambda) = r_0(\lambda) + r_0'(\lambda)\bar{h}'(r_0(\lambda)) = r_0(\lambda). \tag{3.52}$$

For every $-1 < c < 1$ and λ_c such that $L'(\lambda_c) = c$, we have

$$\begin{aligned}
L^*(c) &= c\lambda_c - L(\lambda_c)\\
&= c\lambda_c - \{\lambda_c r_0(\lambda_c) + \frac{1}{2}\log(1-r_0^2(\lambda_c)) - \log(1-\rho r_0(\lambda_c)) + \frac{1}{2}\log(1-\rho^2)\}\\
&= -\frac{1}{2}\log(1-c^2) + \log(1-\rho c) - \frac{1}{2}\log(1-\rho^2) = \log\frac{1-\rho c}{\sqrt{1-c^2}\sqrt{1-\rho^2}}.
\end{aligned}$$

Because of the dual properties of Legendre transform, the condition of Laplace's method $\overline{h}''(r) < 0$ is compatible to the condition of convexity of I_ρ in $]-1,1[$.

It means that for $\rho_0 < |\rho| < 1$, I_ρ is not convex. We can infer from the fact $I_\rho^* = L$ and $L^* = I_\rho$ that function L does not exist.

From that point, under condition $|\rho| \le \rho_0$, we can get

$$E\left(e^{(n+1)\lambda r_{n+1}}\right) = \frac{n-1}{\sqrt{2n\pi}}(1-\rho^2)^{n/2}\sqrt{\frac{2\pi}{n}}e^{n\bar{h}(r_0(\lambda))}\frac{\overline{g}(r_0(\lambda))}{\sqrt{|\overline{h}''(r_0(\lambda))|}}\left(1+O\left(\frac{1}{n}\right)\right) \tag{3.53}$$

$$= e^{(n+1)\bar{h}(r_0(\lambda))}\frac{(1-\rho^2)^{n/2}(1-\rho r_0(\lambda))^{3/2}}{(1-r_0^2(\lambda))^2\sqrt{|\overline{h}''(r_0(\lambda))|}}\left(1+O\left(\frac{1}{n}\right)\right) \tag{3.54}$$

We can adjust the size of sample into n and obtain

$$E\left(e^{n\lambda r_n}\right) = e^{n\bar{h}(r_0(\lambda))}\frac{(1-\rho^2)^{(n-1)/2}(1-\rho r_0(\lambda))^{3/2}}{(1-r_0^2(\lambda))^2\sqrt{|\overline{h}''(r_0(\lambda))|}}\left(1+O\left(\frac{1}{n}\right)\right), \tag{3.55}$$

which leads us to (3.17).

3.4.6 Proof of Proposition 3.3.4

For the asymptotics of L_n in this case, we follow the steps of Si [55]. Up to considering $X_1 = X - E(X)$ and $Y_1 = Y - E(Y)$, we can boil down to $E(X) = E(Y) = 0$.

If we denote by $\langle , \rangle$ the euclidean scalar product in $\mathbb{R}^2$, and

$$\tilde{X} = \left(\frac{X_1}{\sqrt{\sum_{i=1}^n X_i^2}}, \cdots, \frac{X_n}{\sqrt{\sum_{i=1}^n X_i^2}} \right) \; ; \quad \tilde{Y} = \left(\frac{Y_1}{\sqrt{\sum_{i=1}^n Y_i^2}}, \cdots, \frac{Y_n}{\sqrt{\sum_{i=1}^n Y_i^2}} \right)$$

therefore

$$\tilde{r}_n = \langle \tilde{X}, \tilde{Y} \rangle. \tag{3.56}$$

Large deviations for $\{\tilde{r}_n\}$ are proved in [55]. We derive here the corresponding sharp principle. Since $\tilde{X}$, $\tilde{Y}$ are independent random variables with uniform distribution $\tilde{\sigma}_n$ on the unit sphere $\mathcal{S}^{n-1}$ of $\mathbb{R}^n$, we can compute

$$E\left(e^{\lambda \tilde{r}_n}\right) = \iint_{\mathcal{S}^{n-1} \times \mathcal{S}^{n-1}} e^{\lambda \langle x,y \rangle} \tilde{\sigma}_n(dx) \tilde{\sigma}_n(dy) dx dy \tag{3.57}$$

$$= \frac{a_{n-1}}{a_n} \int_{-1}^{1} e^{\lambda u} \left(\sqrt{1-u^2} \right)^{n-1} du \,, \tag{3.58}$$

where a_n is the area of the unit sphere:

$$a_i = \frac{2\pi^{\frac{i+1}{2}}}{\Gamma(\frac{i+1}{2})} \,.$$

In order to get the SLD, we want to compute the normalized log–Laplace transform: for any $\lambda \in \mathbb{R}$, From Stirling formula (see Chapter 2), we get easily

$$\frac{a_{n-1}}{a_n} = \sqrt{\frac{n}{2\pi}} \left(1 + O\left(\frac{1}{n} \right) \right) .$$

Then we can write

$$\int_{-1}^{1} e^{n\lambda u} \left(\sqrt{1-u^2} \right)^{n-1} du = \int_{-1}^{1} e^{nh(u)} g(u) du \,,$$

where $h(u) = \lambda u + \frac{1}{2} \log(1-u^2)$ and $g(u) = \frac{1}{\sqrt{1-u^2}}$. We apply Laplace's method to get:

$$\int_{-1}^{1} e^{nh(u)} du = e^{nh(u_0(\lambda))} \left(\frac{c_0(\lambda)}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}} \right) \right) , \tag{3.59}$$

where

$$u_0(\lambda) = \frac{-1 + \sqrt{1+4\lambda^2}}{2\lambda} \,, \quad c_0(\lambda) = \sqrt{\frac{2\pi}{|h"(u_0(\lambda))|}} g(u_0(\lambda)) \,.$$

This leads to

$$L_n(\lambda) = h(u_0(\lambda)) - \frac{1}{2n} \log(1+4\lambda^2) + O\left(\frac{1}{n^2} \right) . \tag{3.60}$$

3.5 Any order development

We present in this section a quite general result that can be used to get SLD at any order in our cases, since the so-called Laplace's method, – or stationary phase method – is applied on very smooth functions. Recall that we can write the normalized cumulant generating function as

$$L_n(\lambda) = \frac{1}{n}\log E\left(e^{n\lambda r_n}\right) = L(\lambda) + \frac{1}{n}\log K(n) - \frac{1}{2n}\log n + \frac{R_0(\lambda)}{n} + \frac{1}{n}\sum_{k\geq 1}\frac{R_p(\lambda)}{n^p p!} \quad (3.61)$$

where $K(n)$ is a function of n, R_0 and R_p are given in (3.34), (3.35). For example, the coefficients of $L_n(\lambda)$ and $L_n^{(k)}(\lambda)$, respect to scale n^{-2} are $\frac{F''(r_0)}{2F(r_0)} - \frac{5}{4}$ and $\left[\frac{F''(r_0)}{F(r_0)}\right]'$, respectively.

Note that:

$$\frac{F''(r_0)}{F(r_0)} = \frac{15}{36}|h_2|^{-3}h_3^2 + \frac{1}{4}|h_2|^{-2}h_4 + |h_2|^{-2}h_3\frac{g'(r_0)}{g(r_0)} + |h_2|^{-1}\frac{g''(r_0)}{g(r_0)}, \quad (3.62)$$

$$\frac{g'(r_0)}{g(r_0)} = H'(\lambda) - \frac{1}{2}|h_2|^{-1}h_3, \quad (3.63)$$

and

$$\frac{g''(r_0)}{g(r_0)} = H''(\lambda) + \left(\frac{g'(r_0)}{g(r_0)}\right)^2 - \frac{1}{2}|h_2|^{-2}h_3^2 - \frac{1}{2}|h_2|^{-1}h_4. \quad (3.64)$$

SLD functions can be shown similarly to the method used in both papers of Bercu et al. [9, 10].

Theorem 3.5.1 *In the framework of Sections 3.2 and 3.3, for any* $0 < c < 1$, *there exists a sequence* $\{\delta_{c,k}\}_k$ *such that*

$$P(r_n \geq c) = \frac{e^{-nL^*(c)+R_0(\lambda_c)}}{\lambda_c\sigma_c\sqrt{2\pi n}}\left[1 + \sum_{k=1}^{p}\frac{\delta_{c,k}}{n^k} + O\left(\frac{1}{n^{p+1}}\right)\right]. \quad (3.65)$$

Proof:

Similarly to the proof of Theorem (3.2.3), we can remind briefly the main ideal as follows: From the decomposition $P(r_n \geq c) = A_n B_n$, in which

$$\begin{aligned} A_n &= \exp[n(L_n(\lambda_c) - c\lambda_c)] \\ &= \exp[-nL^*(c) + R_0(\lambda_c) + \sum_{p\geq 1}\frac{R_p(\lambda_c)}{n^p(2p)!}] \\ &= \exp[-nL^*(c) + R_0(\lambda_c))]\left(1 + \sum_{p\geq 1}\frac{\eta_p(\lambda_c)}{n^p(2p)!}\right). \end{aligned}$$

where $\{\eta_p\}_p$ is a sequence of smooth functions of λ. From the development of Φ in (3.40)

$$\left(1 + \frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)^{-1}\Phi_n(u) = e^{-\frac{u^2}{2\sigma_c}}\left(1 + \sum_{k=1}^{2p+1}\frac{P_{p,k}(u)}{n^{k/2}} + \frac{1+u^{6(p+1)}}{n^{p+1}}O(1)\right), \quad (3.66)$$

where $P_{p,k}$ are polynomials in odd powers of u for k odd, and polynomials in even powers of u for k even. From that points, we can complete the proof of Theorem (3.5.1).

□

3.6 Correlation test and Bahadur exact slope

3.6.1 Bahadur slope

Let us recall here some basic facts about Bahadur exact slopes of test statistics. For a reference, see [4] and [41]. Consider a sample $X_1, \cdots, X_n$ having common law μ_θ depending on a parameter $\theta \in \Theta$. To test $(H_0) : \theta \in \Theta_0$ against the alternative $(H_1) : \theta \in \Theta_1 = \Theta \backslash \Theta_0$, we use a test statistic S_n, large values of S_n rejecting the null hypothesis. The p-value of this test is by definition $G_n(S_n)$, where

$$G_n(t) = \sup_{\theta \in \Theta_0} P_\theta(S_n \geq t).$$

The Bahadur exact slope $c(\theta)$ of S_n is then given by the following relation

$$c(\theta) = -2 \liminf_{n \to \infty} \frac{1}{n} \log \left(G_n(S_n) \right). \tag{3.67}$$

Quantitatively, for $\theta \in \Theta_1$, the larger $c(\theta)$ is, the faster S_n rejects H_0.

A theorem of Bahadur (Theorem 7.2 in [5]) gives the following characterization of $c(\theta)$: suppose that $\lim_n n^{-1/2} S_n = b(\theta)$ for any $\theta \in \Theta_1$, and that $\lim_n n^{-1} \log \left(G_n(n^{1/2} t) \right) = -I(t)$ under any $\theta \in \Theta_0$. If I is continuous on an interval containing $b(\Theta_1)$, then $c(\theta)$ is given by:

$$c(\theta) = 2 I(b(\theta)). \tag{3.68}$$

3.6.2 Correlation in the Gaussian case

In the Gaussian case, under Assumption 3.3.1, we have the following strong law of large numbers:

$$r_n \to \rho = \mathrm{cov}(X, Y) \tag{3.69}$$

We wish to test $H_0 : \ \rho = 0$ against the alternative $H_1 : \ \rho \neq 0$. It is obvious that under H_1,

$$\lim_{n \to \infty} r_n = \rho.$$

and this limit is continuous when $\rho \neq 0$.

Besides, we have here

$$G_n(t) = \sup_{\rho \in \Theta_0} P_\rho(\sqrt{n} r_n \geq t)$$

and

$$\frac{1}{n} \log G_n(\sqrt{n} t) \to -\frac{1}{2} \log(1 - t^2).$$

Therefore the Bahadur slope is

$$c(\rho) = \log(1 - \rho^2). \tag{3.70}$$

We show that this statistic is optimal in a certain sense. In the framework above, to test $\theta \in \Theta_0$ against the alternative $\theta \in \Theta_1$ we define the likelihood ratio:

$$\lambda_n = \frac{\sup_{\theta \in \Theta_0} \prod_{i=1}^n \mu_\theta(x_i)}{\sup_{\theta \in \Theta_1} \prod_{i=1}^n \mu_\theta(x_i)}$$

and the related statistic:

$$\hat{S}_n = \frac{1}{n} \log \lambda_n. \tag{3.71}$$

Bahadur showed in [3] that $\hat{S}_n$ is optimal in the following sense: for any $\theta \in \Theta_1$,

$$\lim_{n\to\infty} \frac{1}{n} \log G_n(\hat{S}_n) = -J(\theta), \tag{3.72}$$

where J is the infimum of the Kullback–Leibler information:

$$J(\theta) = \inf\{K(\theta, \theta_0), \theta_0 \in \Theta_0\} \tag{3.73}$$

and

$$K(\theta, \theta_0) = -\int \log[\frac{\mu_{\theta_0}(x)}{\mu_\theta(x)}] d\mu_\theta\,. \tag{3.74}$$

Definition 3.6.1 *Let T_n be a statistic in the parametric framework defined above, then if $c(\theta)$ is the Bahadur slope of T_n, we have*

$$c(\theta) \leq 2J(\theta)$$

and T_n is said to be optimal if the upper bound is reached.

We have the following result on the statistic r_n

Proposition 3.6.2 *The sequence of empirical coefficients $\{r_n\}_n$ is asymptotically optimal in the Bahadur sense ([3]).*

Proof:

We can easily compute the Kullback–Liebler information in this case:

Let $\theta = (\mu, \Sigma)$ corresponds to the distribution of (X, Y) in the case $\theta \in \Theta_1$ and $\theta = (\mu_0, \Sigma_0)$ for $\theta \in \Theta_0$. Since $\rho = 0$ in the case $\theta \in \Theta_0$, the matrix Σ_0 is diagonal.

$$K(\theta, \theta_0) = -\frac{1}{2} \log|\Sigma| + \frac{1}{2} \log|\Sigma_0| - 1 + \frac{1}{2} \mathrm{tr}\Sigma_0^{-1}[\Sigma - (\mu - \mu_0)^t(\mu - \mu_0)], \tag{3.75}$$

where $|\Sigma|$ stands for the determinant of Σ. The infimum in (3.75) is reached when $\mu_0 = \mu$ and the diagonal terms in Σ_0 are the ones of Σ.

Hence,

$$J(\theta) = \inf_{\theta_0 \in \Theta_0} K(\theta, \theta_0) = -\frac{1}{2} \log|\Sigma| + \frac{1}{2} \log \sigma_{11} + \frac{1}{2} \log \sigma_{22} = -\frac{1}{2} \log(1 - \rho^2).$$

□

Résumé

Dans le Chapitre 3, nous étudions les grandes déviations précises pour des coefficients de Pearson empiriques qui sont définis par:

$$r_n = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1} (X_i - \bar{X}_n)^2 \sum_{i=1} (Y_i - \bar{Y}_n)^2}}$$

ou, quand les espérances sont connues,

$$\tilde{r}_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}(X))(Y_i - \mathbb{E}(Y))}{\sqrt{\sum_{i=1} (X_i - \mathbb{E}(X))^2 \sum_{i=1} (Y_i - \mathbb{E}(Y))^2}}\,.$$

Notre cadre est celui d'échantillons (X_i, Y_i) ayant une distribution sphérique ou une distribution gaussienne. Dans chaque cas, le schéma de preuve suit celui de Bercu et al. Dans le cas sphérique, la fonction de taux est donnée par

$$L^*(y) = -\frac{1}{2}\log(1 - y^2)\,. \tag{3.76}$$

Dans le cas Gaussien, les grandes déviations ne sont valides que dans un domaine restreint de corrélation ρ: l'échantillon $\{(X_i, Y_i), i = 1, \cdots n\}$ est issu du vecteur Gaussien (X, Y) avec $\sigma_1^2 = \mathrm{Var}(X)$, $\sigma_2^2 = \mathrm{Var}(Y)$ et ρ est le coefficient de corrélation: $\mathrm{Cov}(X, Y) = \rho\sigma_1\sigma_2$. Soit

$$\rho_0 := \frac{\sqrt{3 + 2\sqrt{3}}}{3}\,.$$

Pour tout $\lambda \in \mathbb{R}$ et ρ tel que $|\rho| \leq \rho_0$, on a alors le PGD précis avec la fonction de taux

$$L^*(y) = \log\left(\frac{1 - \rho y}{\sqrt{(1 - \rho^2)}\sqrt{(1 - y^2)}}\right)\,. \tag{3.77}$$

Chapter 4

Self Normalized statistics

In this chapter, we prove the SLD for a particular case of a self-normalized statistic, which is Moran statistic. We recall a theorem of Darling [24] to study the moment generating function of the statistic. The properties related to the Digamma function and Hurwitz zeta function are mentioned in [42]. The first-order expansion will be shown in Section 4.2 and we also discuss higher-order development.

Contents

4.1 Introduction and model

A self-normalized statistic is formally defined in the following way

Definition 4.1.1 *Let $X_1, X_2, \cdots, X_n$ be a random sample of size n. A self-normalized statistic is defined by*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{X_i}{\bar{X}_n}\right),$$

where f is a real valued function and $\bar{X}_n$ is the empirical mean of $X_1, X_2, \cdots, X_n$:

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k .$$

In this section, we focus on a particular function $f(s) = \log s$. Let $X_1, X_2, \cdots, X_n$ be non-negative random variables and consider the so-called Moran statistic:

$$T_n = \frac{1}{n}\sum_{k=1}^{n} \log\frac{X_i}{\bar{X}_n} + \gamma\,, \tag{4.1}$$

where γ is the Euler constant. This test is known to be the most powerful unbiaised exponential test against the Gamma alternative (see Moran [39] and Plachky and Steinebach [43]). Large deviations for the Moran test has been thoroughly described by Tchirina [57]. We propose here a SLD result. Moreover, our method for computing the normalized cumulant generating function of T_n is completely different from the one of [55] and relies on the results of Darling [24]. For a reference on self-normalized statistics for tests of normality, see the work of Arcones [2].

4.2 Main result

The statistic T_n is used to test H_0 (Exponential distribution) against H_1 (Gamma alternative). As a matter of fact, the random sequence T_n tends to 0 as n tends to infinity. We have the following asymptotics:

Proposition 4.2.1 *a) Under H_0, $T_n \to 0$ a.s., as $n \to \infty$.*

b) Under H_0, $\sqrt{n}T_n \to \mathcal{N}(0, \frac{\pi^2}{6} - 1)$ in distribution, as $n \to \infty$.

The proof of proposition above is given in Tchirina [57], Theorem 1.

We can compute the normalized cumulant generating function of T_n and give its limit as n grows to infinity. It is detailed in the following two propositions which proofs are postponed to Section 4.3.

Proposition 4.2.2 *Under H_0 and for any real $\lambda > -1$, Laplace transform of nT_n is*

$$E[e^{\lambda\, nT_n}] = e^{\lambda n \gamma} \frac{\Gamma(n)\; n^{n\lambda}\; \Gamma^n(\lambda+1)}{\Gamma(n(\lambda+1))}. \tag{4.2}$$

Proposition 4.2.3 *Under H_0, the normalized cumulant generating function of T_n is*

$$L_n(\lambda) = \frac{1}{n} \log E[e^{\lambda\, nT_n}] = L(\lambda) + \frac{1}{2n}\log(\lambda+1) + O\left(\frac{1}{n^2}\right), \tag{4.3}$$

where

$$L(\lambda) = \gamma\lambda - (\lambda+1)\log(\lambda+1) + \lambda + \log\Gamma(\lambda+1)\,. \tag{4.4}$$

We can now present the main result of this chapter:

Theorem 4.2.4 *Under H_0 and for $0 < c < \gamma$,*

$$P(T_n \geq c) = \frac{\exp^{-nL^*(c) + \frac{1}{2}\log(1+\lambda_c)}}{\lambda_c \sigma_c \sqrt{2\pi n}}(1 + o(1)), \tag{4.5}$$

where L^* is the Legendre dual of the function L defined above, namely the limit n.c.g.f. of T_n, and $\lambda_c > 0$ is the unique λ such that $L'(\lambda) = c$.

Proof:
We begin the proof by some results on L:

Proposition 4.2.5 *The function L is strictly convex, $L' \in]-\infty, \gamma[$. Moreover, $L'(0) = 0$ and therefore for any $0 < c < \gamma$ there exists a unique $\lambda_c > -1$ such that $L'(\lambda_c) = c$.*

The proof of this proposition is postponed to the Appendix. To prove the SLD of T_n, we proceed as in [9]. Let us fix $0 < c < \gamma$ and λ_c such that $L'(\lambda_c) = c$. We denote by $\sigma_c^2 = L''(\lambda_c)$ and define the following change of probability:

$$\frac{dQ_n}{dP} = e^{\lambda_c n T_n - n L_n(\lambda_c)}. \tag{4.6}$$

The expectation under Q_n is denoted by E_n. Now

$$P(T_n \geq c) = A_n B_n \tag{4.7}$$

and

$$A_n = \exp[n(L_n(\lambda_c) - c\lambda_c)]\,, \quad B_n = E_n(\exp[-n\lambda_c(T_n - c)]\mathbb{1}_{T_n \geq c})\,.$$

On the first hand,

$$A_n = \exp\left[-nL^*(c) + \frac{1}{2}\log(1+\lambda_c)\right]\left(1 + O\left(\frac{1}{n}\right)\right).$$

On the other hand, let us denote by

$$U_n = \frac{\sqrt{n}(T_n - c)}{\sigma_c},$$
$$\Phi_n(u) = E_n(e^{iuU_n})\,.$$

Lemma 4.2.6 *For n large enough, Φ_n is $L^2(\mathbb{R})$.*

Therefore by Parseval formula,

$$B_n = E_n[e^{-\lambda_c\sigma_c\sqrt{n}U_n}\mathbb{1}_{U_n\geq 0}] = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\frac{1}{\lambda_c\sigma_c\sqrt{n} + iu}\right)\Phi_n(u)du = \frac{C_n}{\lambda_c\sigma_c\sqrt{2\pi n}}$$

where

$$C_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(1 + \frac{iu}{\lambda_c\sigma_c\sqrt{n}}\right)^{-1}\Phi_n(u)du.$$

We have

Lemma 4.2.7 *$C_n \to 1$ as $n \to \infty$.*

From lemma above, we have equation (4.27).

$\square$

4.3 Proofs

4.3.1 Proof of Proposition 4.2.2

We use here a result of Darling [24].

It is well-known (see for instance Proposition 1 of Shorack and Wellner [54] p.335) that the distribution of

$$\left(\frac{X_1}{\sum_{i=1}^n X_i}, \dots, \frac{X_n}{\sum_{i=1}^n X_i}\right)$$

follows a Dirichlet distribution of order n with parameters $(1, \dots, 1)$. In other words, it is the law of n uniform spacing $(D_1, \dots, D_n)$ where $D_i = U_{(i)} - U_{(i-1)}$ for $1 \le i \le n-1$ with $U_{(1)}, \dots, U_{(n-1)}$ are the order statistics of a $(n-1)$-sample of uniform random variables on $[0, 1]$ (with the convention $U_{(0)} = 0$ and $U_{(n)} = 1$).

Remark 4.3.1 *The density distribution of $(D_1, \dots, D_n)$ with respect to the Lebesgue measure of $\mathbb{R}^{n-1}$ (since $D_n = 1 - \sum_{i=1}^{n-1} D_i$) is just the uniform density over the open simplex*

$$\mathcal{S}_{n-1} = \left\{ (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\,,\ x_i > 0\,,\ 1 \le i \le n-1\,,\ \sum_{i=1}^{n-1} x_i < 1 \right\}.$$

We recall that the Lebesgue measure of $\mathcal{S}_{n-1}$ is equal to $\frac{1}{(n-1)!}$. Moreover, the marginal distribution of D_i for $1 \le i \le n-1$ is a Beta distribution Beta$(1, n-1)$ with density $(n-1)(1-x)^{n-2}\mathbb{1}_{]0,1[}(x)$ with respect to the Lebesgue measure on $\mathbb{R}$.

Remark 4.3.2
- *Rao and Sethuraman [46] proved a Central Limit Theorem for the statistic T_n under some alternative assumption that the distribution of the U_i is not uniform (it is assumed that the distribution function is equal to $F(x) = x + \frac{L_n(x)}{n^{1/4}}$).*
- *Rao and Sethuraman [47] have established the weak convergence of the empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{nD_i}$.*
- *Zhou and Jammalamadaka [64] have studied a large deviation result for the Dirichlet distribution for spacings of the form $D_i = U_{[\lambda_i n]} - U_{[\lambda_{i-1} n]}$ with $1 \le i \le k$ and $0 = \lambda_0 < \lambda_1 < \dots < \lambda_k = 1$, and where k is fixed and n goes to infinity.*

Thus, this will allow us to calculate Laplace transform of the statistics T_n since, for any positive real λ,

$$E\big(e^{n\lambda T_n}\big) = e^{n\lambda\gamma} E\Big(e^{\lambda \sum_{i=1}^n \log(nD_i)}\Big) = E\Big(\prod_{i=1}^n e^{\lambda \log(nD_i)}\Big).$$

Then we can apply the following result of Darling (Theorem 2.1, [24]) which is based on the inversion of the Laplace transform of a convolution product:

Theorem 4.3.3 *Let $f_1, \dots, f_n$ be n real-valued functions which the abscissas of convergence of corresponding Laplace transforms are all less than some real c. If $(D_1, \dots, D_n)$ denotes n uniform spacings on $[0, 1]$ then,*

$$E\Big(\prod_{i=1}^n f_i(D_i)\Big) = \frac{(n-1)!}{2i\pi} \int_{\mathcal{B}} \Big(\prod_{i=1}^n \int_0^{+\infty} f_i(x_i)\, e^{-x_i z}\, \mathrm{d}x_i\Big) e^z\, \mathrm{d}z,$$

where $\mathcal{B} = \{c + iy\,,\ y \in \mathbb{R}\}$.

Proof:

The proof of this theorem is based on Laplace transform of a product of convolution. Using the distribution of the $(n-1)$ order statistics $U_{(1)}, \dots, U_{(n)}$, we can calculate as follows:

$$\begin{aligned}
E\Big(\prod_{i=1}^{n} f_i(D_i)\Big) &= (n-1)! \int_{\{0\le x_1 \le \dots \le x_{n-1} \le 1\}} f_1(x_1) f_2(x_2 - x_1) \dots f_n(1 - x_{n-1}) \, \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \\
&= (n-1)! F(1),
\end{aligned}$$

where F is the convolution product of the functions $(f_i)_{1\le i\le n}$ denoted by

$$F(x) = f_1 * \dots * f_n(x),$$

for any positive real x. In order to calculate $F(1)$, we consider the Laplace transform of F and derive that $\mathcal{L}(F) = \mathcal{L}(f_1) * \dots * \mathcal{L}(f_n)$ which is equivalent to

$$\int_0^{+\infty} F(x) e^{-zx} \, \mathrm{d}x = \prod_{i=1}^{n} \int_0^{+\infty} f_i(x_i) e^{-zx_i} \, \mathrm{d}x_i,$$

provided $Rez > c$. Now, we can apply the complex inversion for the Laplace transform which gives,

$$F(x) = \frac{1}{2i\pi} \int_{\mathcal{B}} \Big(\prod_{i=1}^{n} \int_0^{+\infty} f_i(x_i) \, e^{-zx_i} \, \mathrm{d}x_i \Big) e^{zx} \, \mathrm{d}z,$$

and we apply this to $x = 1$ to conclude.

$\square$

Applying this theorem to the functions f_i, $1 \le i \le n$ all equals to the same function $f_i(x) = e^{\lambda \log(nx)}$, this leads to:

$$E\big(e^{n\lambda T_n}\big) = e^{n\lambda\gamma} \frac{(n-1)!}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^z \left(\int_0^{\infty} e^{-rz} e^{\lambda \, \log(nr)} dr \right)^n dz. \tag{4.8}$$

We have

$$\begin{aligned}
\int_0^{\infty} e^{-rz} e^{\lambda \, \log(nr)} dr &= \int_0^{\infty} e^{-rz} (nr)^{\lambda} dr \\
&= \frac{n^{\lambda}}{z^{\lambda+1}} \, \Gamma(\lambda + 1) \qquad\qquad (\text{as } Re\lambda > -1).
\end{aligned}$$

Then we obtain from (4.8) that

$$\begin{aligned}
E[e^{\lambda \, nT_n}] &= e^{n\lambda\gamma} (n-1)! \, n^{n\lambda} \, \Gamma^n(\lambda+1) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^z \, z^{-n(\lambda+1)} dz \\
&= e^{n\lambda\gamma} \frac{\Gamma(n) \, n^{n\lambda} \, \Gamma^n(\lambda+1)}{\Gamma(n(\lambda+1))}.
\end{aligned}$$

4.3.2 Proof of Proposition 4.3.5

From Proposition 4.2.2, we have

$$\frac{1}{n}\log E[e^{\lambda\, nT_n}] = \lambda\gamma + \frac{\log\Gamma(n)}{n} + \lambda\log n + \log\Gamma(\lambda+1) - \frac{\log\Gamma(n(\lambda+1))}{n}. \tag{4.9}$$

The Stirling's formula (see also Remark 2.2.4) gives

$$\begin{aligned}\log\Gamma(n) &= \left(n-\frac{1}{2}\right)\log n - n + \frac{1}{2}\log 2\pi + O\left(\frac{1}{n}\right),\\ \log\Gamma(n(\lambda+1)) &= \left(n(\lambda+1)-\frac{1}{2}\right)(\log n + \log(\lambda+1)) - n(\lambda+1) + \frac{1}{2}\log 2\pi + O\left(\frac{1}{n}\right).\end{aligned}$$

Accordingly,

$$\begin{aligned}\frac{\log\Gamma(n)}{n} - \frac{\log\Gamma(n(\lambda+1))}{n} &= \frac{-n\lambda\log n - \left(n(\lambda+1)-\frac{1}{2}\right)\log(\lambda+1) + n\lambda}{n} + O\left(\frac{1}{n^2}\right)\\ &= -\lambda\log n - (\lambda+1)\log(\lambda+1) + \lambda - \frac{1}{2n}\log(\lambda+1) + O\left(\frac{1}{n^2}\right).\end{aligned}$$

Then we get the limit of the normalized log-Laplace transform of T_n

$$L_n(\lambda) = \frac{1}{n}\log E[e^{\lambda\, nT_n}] \xrightarrow{n\to\infty} L(\lambda) := -(\lambda+1)\log(\lambda+1) + \lambda + \log\Gamma(\lambda+1).$$

4.3.3 Proof of Proposition 4.2.5

Recall that

$$L(\lambda) = \gamma\lambda - (\lambda+1)\log(\lambda+1) + \lambda + \log\Gamma(\lambda+1).$$

For $x > 0$, first and second derivatives of $\log\Gamma(x)$ exist and are known as Digamma function $\psi(x)$ and Hurwitz zeta function $\zeta(x,s)$, respectively (see Chapter 1, [1]). We show that $L(\lambda)$ is a convex function.

Indeed, we can represent $L'(\lambda)$ and $L''(\lambda)$ as

$$\begin{aligned}L'(\lambda) &= \gamma + \psi(\lambda+1) - \log(\lambda+1),\\ L''(\lambda) &= \zeta(\lambda+1,2) - \frac{1}{\lambda+1}.\end{aligned}$$

According to Exercise 42iii and 43b ([1]), we have

$$\psi(x) = \log x - \frac{1}{2x} - \int_0^\infty \frac{2t\,dt}{(x^2+t^2)(e^{2\pi t}-1)}, \tag{4.10}$$

$$\zeta(x,2) = \frac{1}{2x^2} + \frac{1}{x} + \int_0^\infty \frac{4xt\,dt}{(x^2+t^2)^2(e^{2\pi t}-1)}. \tag{4.11}$$

Therefore, for any $\lambda > -1$,

$$L'(\lambda) = \gamma - \frac{1}{2(\lambda+1)} - \int_0^\infty \frac{2t\,dt}{((\lambda+1)^2+t^2)(e^{2\pi t}-1)} < \gamma, \tag{4.12}$$

$$L''(\lambda) = \frac{1}{2(\lambda+1)^2} + \int_0^\infty \frac{4(\lambda+1)t\,dt}{((\lambda+1)^2+t^2)^2(e^{2\pi t}-1)} > 0. \tag{4.13}$$

Moreover, $L'(0) = 0$. The Legendre dual of L exists and is defined by

$$L^*(y) = \begin{cases} \sup_{\lambda>-1}\{\lambda y - L(\lambda)\}, & y < 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.14}$$

Recall that for $x > 0$

$$\psi(x) = \int_0^{+\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt \tag{4.15}$$

is well defined and continuous on $]0, +\infty[$ and besides,

$$\lim_{x\to 0^+} (\psi(x) - \log x) = \lim_{x\to 0^+} \left(\psi(x+1) - \frac{1 + x\log x}{x} \right) = -\infty$$

$$\left(\text{Since } \psi(1) = -\gamma \text{ and } \lim_{x\to 0^+} x\log x = 0 \right).$$

According to [1], Corollary 1.4.5, for $|\arg x| \le \pi - \delta$, $\delta > 0$,

$$\psi(x) = \log x - \frac{1}{2x} - \sum_{j=1}^{m} \frac{B_{2j}}{(2j)} \frac{1}{x^{2j}} + O\left(\frac{1}{x^{2m}}\right). \tag{4.16}$$

Therefore,

$$\lim_{x\to+\infty} (\psi(x) - \log x) = 0. \tag{4.17}$$

After all, one get that for $\lambda > -1$, function $L'(\lambda)$ is continuous, $L'(\lambda) < \gamma$, $L''(\lambda) > 0$, $\lim_{\lambda\to+\infty} L'(\lambda) = \gamma$ and $\lim_{\lambda\to(-1)^+} L'(\lambda) = -\infty$. Even though the explicit form of $L^*(y)$ may not be obtained, we know that for any $0 < c < \gamma$ there exists a unique $\lambda_c > 0$ such that $L'(\lambda_c) = c$, then

$$L^*(c) = c\lambda_c - L(\lambda_c).$$

4.3.4 Proofs of Lemmas 4.2.6 and 4.2.7

As in (3.31) we can write

$$\Phi_n(u) = E_n(e^{iuU_n}) = E[\exp(-\frac{iu\sqrt{n}}{\sigma_c}c + nL_n(\lambda_c + \frac{iu}{\sigma_c\sqrt{n}}) - nL_n(\lambda_c))].$$

To study if Φ is L^2, we can consider $|\Phi(u)|$ and in this expression, there is only the terms depending on u. Therefore we boil down to considering

$$|\exp\{nL_n(\lambda_c + \frac{iu}{\sigma_c\sqrt{n}})\}|. \tag{4.18}$$

From expression (4.2),

$$nL_n(\lambda_c+\frac{iu}{\sigma_c\sqrt{n}}) = \log\Gamma(n) + n(\lambda_c+\frac{iu}{\sigma_c\sqrt{n}})\log n + n\log\Gamma(\lambda_c+\frac{iu}{\sigma_c\sqrt{n}}) - \log\Gamma(n(\lambda_c+\frac{iu}{\sigma_c\sqrt{n}})). \tag{4.19}$$

Hence from (4.18) we consider only

$$|\exp\{n\log\Gamma(\lambda_c+\frac{iu}{\sigma_c\sqrt{n}})-\log\Gamma(n(\lambda_c+\frac{iu}{\sigma_c\sqrt{n}}))\}|. \tag{4.20}$$

We use here an expression for $\log\Gamma$ due to Binet and detailed in Andrews et al. [1] (see Theorem 1.6.3):

$$\log\Gamma(x)=(x-\frac{1}{2})\log x-x+\frac{1}{2}\log(2\pi)+\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tx}}{t}dt\,, \tag{4.21}$$

which leads (4.20) to study

$$|\exp\{(\frac{1}{2}-\frac{n}{2})\log(\lambda_c+\frac{iu}{\sigma_c\sqrt{n}})\}|. \tag{4.22}$$

Obviously, expression (4.22) is L^2, it corresponds to the Fourier transform of a L^2 function. □

Now we can develop as in (3.40):

$$\Phi_n(u)=\exp\{\frac{iu\sqrt{n}}{\sigma_c}[L_n'(\lambda_c)-c]+n\sum_{k\geq 2}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^k\frac{L_n^{(k)}(\lambda_c)}{k!}\}.$$

We see from expression (4.21) that L_n is an analytic function. Proceeding as in the previous chapter, we can develop $L_n^{(k)}$ and get the convergence of C_n.

As a matter of fact, the preceding results of (4.27) can be generalized to any order development. It is based on how far we can develop $L_n(\lambda)$.

The asymptotics of Gamma function with large argument are detailed in [42], as follows

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+\sum_{s=1}^{m-1}\frac{\mathbf{B}_{2s}}{2s(2s-1)z^{2s-1}}+R_m(z), \tag{4.23}$$

where $\mathbf{B}_s$ is Bernoulli number and m is an arbitrary positive integer. Moreover,

$$R_m(z)=\int_0^\infty\frac{\mathbf{B}_{2m}-\mathbf{B}_{2m}(x-[x])}{2m(x+z)^{2m}}dx=O\left(\frac{1}{z^{2m-1}}\right), \tag{4.24}$$

for large $|z|$. From that point, we can proceed as in Chapter 3.

Résumé

Par la suite, nous considérons la statistique de Moran

$$T_n=\frac{1}{n}\sum_{k=1}^n\log\frac{X_i}{\bar{X}_n}+\gamma\,,$$

où γ est la constante d' Euler. La statistique T_n est utilisée pour tester H_0 (distribution exponentielle) contre l'alternative H_1 (Gamma). On a:

Proposition 4.3.4 *a) Sous H_0, $T_n \to 0$ p.s., quand $n \to \infty$.*

b) Sous H_0, $\sqrt{n}T_n \to \mathcal{N}(0, \frac{\pi^2}{6} - 1)$ en distribution, quand $n \to \infty$.

Cette proposition est donnée par Tchirina [57], Theorème 1.

On montre alors :

Proposition 4.3.5 *Sous H_0, pour tout réel $\lambda > -1$, la* log*–Laplace normalisée de T_n est*

$$L_n(\lambda) = \frac{1}{n} \log E[e^{\lambda\, nT_n}] = L(\lambda) + \frac{1}{2n} \log(\lambda + 1) + O\left(\frac{1}{n^2}\right), \tag{4.25}$$

où

$$L(\lambda) = \gamma\lambda - (\lambda + 1) \log(\lambda + 1) + \lambda + \log \Gamma(\lambda + 1)\,. \tag{4.26}$$

Le résultat principal de ce chapitre est le suivant:

Theorem 4.3.6 *Sous H_0 et pour $0 < c < \gamma$,*

$$P(T_n \geq c) = \frac{\exp^{-nL^*(c) + \frac{1}{2}\log(1+\lambda_c)}}{\lambda_c \sigma_c \sqrt{2\pi n}}(1 + o(1)), \tag{4.27}$$

où L^*est la duale de Legendre de L (définie ci-dessus), et $\lambda_c > 0$ est l'unique λ tel que $L'(\lambda) = c$.

Appendix A

Appendix

Contents

A.1 Fundamental definitions and notations

In this section, we deal with the introduction of fundamental notations and definitions used throughout the thesis. Good references could be [42] for paragraph 1; [60], [1] for paragraph 2; [18] or [19] for paragraph 3.

1. We use several notations related to the asymptotic analysis such as $\sim$, o and O. Let us remark the definition of those symbols, as $x \to \infty$, as follows

   i) If $f(x)/g(x) \to 1$, we write

   $$f(x) \sim g(x) \qquad (x \to \infty).$$

   In words, f *is asymptotic to* g, or g *is an asymptotic expansion to* f.

   ii) If $f(x)/g(x) \to 0$, we write

   $$f(x) = o(g(x)) \qquad (x \to \infty).$$

   In words, f *is of order less than* g.

   iii) If $|f(x)/g(x)|$ is bounded, we write

   $$f(x) = O(g(x)) \qquad (x \to \infty).$$

   In words, f *is of order not exceeding* g.

iv) *(Asymptotic expasion)* Let $f(z)$ be a real (or complex) function, $\sum a_s z^{-s}$ be a formal power series (convergent or divergent), and define $R_n(z)$ be a *remainder* as follows

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + + \frac{a_{n-1}}{z^{n-1}} + R_n(z).$$

Then, following Poincaré [44], if $R_n(z) = O(z^{-n})$ for fixed value of n, as $z \to \infty$ in a certain unbounded region $\mathbf{R}$, we say that the series $\sum a_s z^{-s}$ *is an asymptotic expansion of* f and write

$$f(z) \approx a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots \qquad (z \to \infty \text{ in } \mathbf{R}).$$

In collaboration with holomorphic (or analytic) property (see *e.g.* [51], p.198), a complex function f is representable by convergent power series $\sum_{s=0}^{\infty} a_s (z - z_0)^{-s}$ in some open disk centered at z_0, i.e.

$$f(z) = \sum_{s=0}^{\infty} a_s (z - z_0)^{-s}.$$

2. The upcoming paragraphs are to introduce the definition of several special functions.

i) *Gamma function* originated in 1729 and is defined through Euler's integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \qquad (\text{Re } z > 0).$$

Integrating by parts the above integral we get

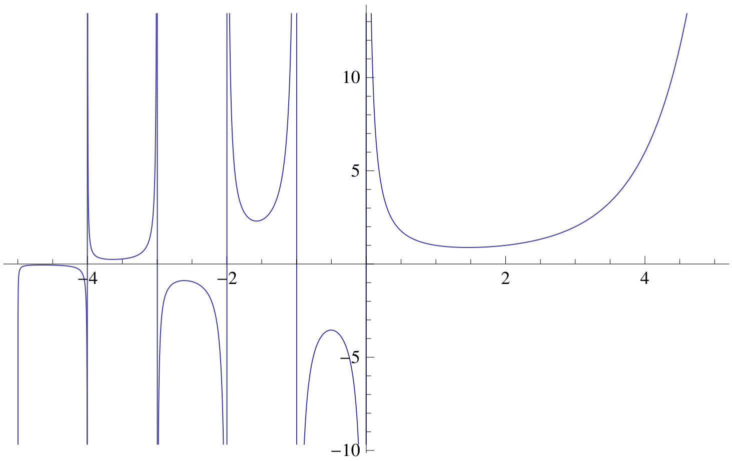

Figure A.1: Gamma function, $y = \Gamma(x)$ for $x \in \mathbb{R}$.

$$\Gamma(z + 1) = z\Gamma(z),$$

and when $z = n$, a positive integer, we have

$$\Gamma(n) = (n - 1)! \qquad (n = 1, 2, \ldots).$$

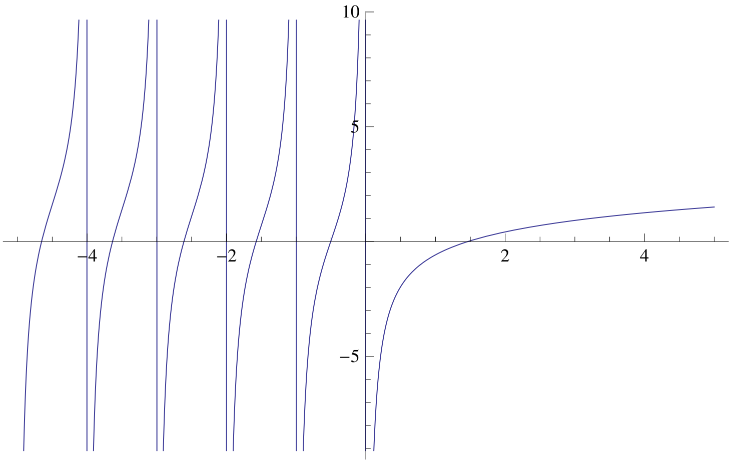

Figure A.2: Digamma function, $y = \psi(x)$ for $x \in \mathbb{R}$.

ii) *Digamma function* or so-called Psi function is defined by

$$\psi(z) = \Gamma'(z)/\Gamma(z).$$

iii) *Hurwitz zeta function* is defined by the series

$$\zeta(z) = \sum_{s=1}^{\infty} \frac{1}{s^z},$$

when Re $z > 1$ and by analytic continuation elsewhere, $\zeta(z)$ is holomorphic in the half-plane Re $z > 1$.

iv) *Hypergeometric function*: The so-called hypergeometric equation is defined, for any real or complex parameters a, b, c by

$$z(1-z)\frac{d^2w}{dz^2} + (c - (a+b+1)z)\frac{dw}{dz} - abw = 0. \tag{A.1}$$

A solution for $|z| < 1$ and $c \neq 0$ is given by the hypergeometric function, which is a converging series

$${}_2F_1(a,b,c;z) = \sum_{s=0}^{\infty} \frac{\Gamma(c)(a)_s(b)_s}{\Gamma(c+s)} \frac{z^s}{s!}, \tag{A.2}$$

where $(\cdot)$ stands for the Pochhammer's notation:
- $(a)_0 = 1$,
- $(a)_s = a(a+1)(a+2)\cdots(a+s-1)$, for $s \geq 1$

For $Re(c) > Re(b) > 0$, we have an integral representation of the hypergeometric function:

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{(b-1)}(1-t)^{c-b-1}(1-zt)^{-a} dt\,.$$

The graphs of $\Gamma(x)$ and $\psi(x)$ for real values of x are in Fig.A.1 and Fig.A.2

3. *The Stirling numbers of the second kind* $S_{p,k}$ count the number of ways to partition a set of p labelled objects into k nonempty unlabelled subsets. $S_{p,k}$ can be computed from the sum

$$S_{p,k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c} k \\ j \end{array}\right) j^p.$$

In this thesis, we use a recurrence relation of the Stirling numbers of the second kind

$$\begin{aligned} S_{p+1,k+1} &= \sum_{j=k}^{p}\left(\begin{array}{c} p \\ j \end{array}\right) S_{j,k} \\ &= \left(\begin{array}{c} p \\ k \end{array}\right) S_{k,k} + \left(\begin{array}{c} p \\ k+1 \end{array}\right) S_{k+1,k} + \cdots + \left(\begin{array}{c} p \\ p \end{array}\right) S_{p,k} \end{aligned}$$

with the initial conditions $S_{p,0} = S_{0,p} = 0$ for $p > 0$ and $S_{0,0} = 1$.

A.2 Some Technical Computations

Although they did not have a direct application in this thesis, we present here two auxiliary technical results that appeared in the process and seemed valuable to us.

We recall here the expression (3.40) of the c.f. $\Phi_n(u)$:

$$\Phi_n(u) = \exp\{\frac{iu}{\sqrt{n}\sigma_c}[L'_n(\lambda_c) - c] - \frac{u^2}{2\sigma_c^2}L''_n(\lambda_c) + n\sum_{p\geq 3}\left(\frac{iu}{\sigma_c\sqrt{n}}\right)^p \frac{L_n^{(p)}(\lambda_c)}{p!}\},$$

where

$$L_n^{(k)}(\lambda) = L^{(k)}(\lambda) + R_0^{(k)}(\lambda) + \frac{1}{n}\sum_{p\geq 1}\frac{R_p^{(k)}(\lambda)}{n^p p!}.$$

We also study the expansion of $nL_n^{(p)}(\lambda_c)$ and the bounding of $R_p^{(k)}(\lambda_c)$. Although such tedious calculus is not applied on the proofs in the main results, it leads us to gain two following theorems. We note that we consider the particular case in spherical distribution to illustrate for the proof in these theorem.

Theorem A.2.1 *For any* $p \geq 3$, $nL_n^{(p)}(\lambda)$ *can be expressed by the following power series*

$$nL_n^{(p)}(\lambda) = (nr_0(\lambda))^p \sum_{s>p} w_s n^{-s}, \tag{A.3}$$

and, namely,

$$w_s = 0, \qquad \textit{for each } s = 0, 1, 2 \ldots, p. \tag{A.4}$$

Theorem A.2.2 *For* $k = 1, 2, \ldots,$ *we have*

$$\left|R_p^{(k)}(\lambda)\right| \leq \delta_{k,p}\,\Delta_p^{p+1}, \tag{A.5}$$

in which constants $\delta_{k,p}$ *and* Δ_p *are computed by* (A.50) *and* (A.51), *respectively.*

The proofs of these results follow.

A.2.1 Proof of Theorem A.2.22

Recall that

$$E\left(e^{n\lambda r_n}\right) = \int_{-1}^{1} e^{n\lambda r} f_n(r)dr = \int_{-1}^{1} e^{n\lambda r} \frac{\Gamma(\frac{n-1}{2})}{\pi^{1/2}\Gamma(\frac{n-2}{2})}(1-r^2)^{(n-4)/2}dr$$

then from the definition of L_n,

$$\begin{aligned} L_n(\lambda) &= \frac{1}{n}\log E\left(e^{n\lambda r_n}\right) \\ &= \frac{1}{n}\log\left\{\int_{-1}^{1} e^{n\lambda r} f_{r_n}(r)dr\right\} = \frac{1}{n}\log\left[\Gamma_n\, F_n(\lambda)\right], \end{aligned}$$

where $F_n(\lambda) := \int_{-1}^{1} e^{nh(\lambda,r)}g(r)dr$, $g(r) = (1-r^2)^{(n-4)/2}$ and $\Gamma_n = \Gamma(\frac{n-1}{2})/(\pi^{1/2}\Gamma(\frac{n-2}{2}))$. It is easy to see that

$$F_n^{(k)}(\lambda) = \int_{-1}^{1} e^{nh(\lambda,r)} g_k(r)\, dr,$$

where $g_k(r) = (nr)^k g(r)$. Then it follows the Faà di Bruno's formula, for every $p = 1, 2, \ldots,$ that

$$nL_n^{(p)}(\lambda) = \sum_{k=1}^{p} \log^{(k)}\left\{F_n(\lambda)\right\} B_{p,k}\left(F_n'(\lambda), F_n''(\lambda), \ldots, F_n^{(p-k+1)}(\lambda)\right),$$

where $B_{p,k}(x_1, \ldots, x_{p-k+1})$ denotes the partial exponential Bell polynomials. According to the Leibniz rule and definition of Bell polynomial, we have

$$nL_n^{(p)}(\lambda) = \sum_{k=1}^{p} \frac{(-1)^{k+1}(k-1)!}{F_n^k(\lambda)} B_{p,k}\left(F_n'(\lambda), F_n''(\lambda), \ldots, F_n^{(p-k+1)}(\lambda)\right).$$

Then

$$\begin{aligned} nL_n^{(p)}(\lambda) = \sum_{k=1}^{p} \frac{(-1)^{k+1}(k-1)!}{F_n^k(\lambda)} \sum_{(j_i)_i,(*)} & \frac{p!}{j_1! j_2! \ldots j_{p-k+1}!} \\ & \cdot \left(\frac{F_n'(\lambda)}{1!}\right)^{j_1} \left(\frac{F_n''(\lambda)}{2!}\right)^{j_2} \cdots \left(\frac{F_n^{(p-k+1)}(\lambda)}{(p-k+1)!}\right)^{j_{p-k+1}} \end{aligned}$$

where $(j_i)_i, (*)$ represents for the meaning that $(j_i)_i$ be the sequences of non-negative integers which satisfy two conditions

$$(*)\begin{cases} j_1 + j_2 + \cdots + j_{p-k+1} = k \\ j_1 + 2j_2 + \cdots + (p-k+1)j_{p-k+1} = p \end{cases}.$$

Remark A.2.3 *The previous definition of the sequences $(j_i)_i$ is still compatible with the original version of the Bell polynomial. Indeed, it follows from the definition that*

$$(*)\begin{cases} j_1 + j_2 + j_3 + \cdots = k \\ j_1 + 2j_2 + 3j_3 + \cdots = p \end{cases}$$

and therefore,

$$j_i = 0, \qquad \textit{for } i = p-k+2, p-k+3, \ldots. \tag{A.6}$$

From now, remind (A.6) and we have the decompositions

$$nL_n^{(p)}(\lambda) = \sum_{k=1}^{p}(-1)^{k+1}(k-1)! \sum_{(j_i)_i,(*)} \frac{p!}{j_1!j_2!\dots}\left(\frac{1}{1!}\right)^{j_1}\left(\frac{1}{2!}\right)^{j_2}\cdots \cdot \left(\frac{F_n'(\lambda)}{F_n(\lambda)}\right)^{j_1}\left(\frac{F_n''(\lambda)}{F_n(\lambda)}\right)^{j_2}\cdots.$$

We now describe

$$P := \sum_{(j_i)_i,(*)} \frac{p!}{j_1!j_2!\dots}\left(\frac{1}{1!}\right)^{j_1}\left(\frac{1}{2!}\right)^{j_2}\cdots\left(\frac{F_n'(\lambda)}{F_n(\lambda)}\right)^{j_1}\left(\frac{F_n''(\lambda)}{F_n(\lambda)}\right)^{j_2}\cdots. \tag{A.7}$$

as a power series. According to Theorem 2.3.10 in Chapter 2, for $i = 1, 2, \dots, p-k+1$, there exists the sequences of numerical coefficients $(c_s)_s$ and $\left(c_{s(i)}\right)_s$ associated with $F_n(\lambda)$ and $F_n^{(i)}(\lambda)$, respectively, namely

$$F_n(\lambda) = e^{nh(\lambda,r_0(\lambda))}\left(\frac{c_0}{\sqrt{n}} + \frac{c_1}{2!\,n^{3/2}} + \dots + \frac{c_p}{(2p)!\,n^{p+1/2}} + O\left(\frac{1}{n^{p+3/2}}\right)\right)$$

and

$$F_n^{(i)}(\lambda) = e^{nh(\lambda,r_0(\lambda))}\left(\frac{c_{0(i)}}{\sqrt{n}} + \frac{c_{1(i)}}{2!\,n^{3/2}} + \dots + \frac{c_{p(i)}}{(2p)!\,n^{p+1/2}} + O\left(\frac{1}{n^{p+3/2}}\right)\right).$$

Here, $r_0(\lambda)$ is from Proposition 3.2.2 and notations r_0, c_s are substituted for $r_0(\lambda), c_s(r_0(\lambda))$, respectively.

Then, it follows

$$\begin{aligned}\frac{F_n^{(i)}(\lambda)}{F_n(\lambda)} &= \frac{c_{0(i)}}{c_0}\,\frac{1 + \dfrac{c_{1(i)}}{n\,c_{0(i)}} + \cdots \dfrac{c_{p(i)}}{(2p)!\,n^p\,c_{0(i)}} + O\left(\dfrac{1}{n^{p+1}}\right)}{1 + \dfrac{c_1}{n\,c_0} + \cdots \dfrac{c_p}{(2p)!\,n^p\,c_0} + O\left(\dfrac{1}{n^{p+1}}\right)}\\ &= \frac{c_{0(i)}}{c_0}\left(1 + \frac{1}{n}\,d_{1(i)} + \frac{1}{n^2}\,d_{2(i)} + \dots + \frac{1}{n^p}\,d_{p(i)}\right)\left(1 + O\left(\frac{1}{n^{p+1}}\right)\right).\end{aligned} \tag{A.8}$$

Substitute (A.8) into (A.7), we obtain

$$\begin{aligned}P = &\sum_{(j_i)_i,(*)} \frac{p!}{j_1!j_2!\dots}\left(\frac{1}{1!}\right)^{j_1}\left(\frac{1}{2!}\right)^{j_2}\cdots\frac{c_{0(1)}^{j_1}\,c_{0(2)}^{j_2}\cdots c_{0(p-k+1)}^{j_{p-k+1}}}{c_0^k}\\ &\cdot\left(1 + \frac{1}{n}\,d_{1(1)} + \dots + \frac{1}{n^p}\,d_{p(1)}\right)^{j_1}\left(1 + \frac{1}{n}\,d_{1(2)} + \dots + \frac{1}{n^p}\,d_{p(2)}\right)^{j_2}\cdots\left(1 + O\left(\frac{1}{n^{p+1}}\right)\right).\end{aligned} \tag{A.9}$$

Lemma A.2.4

$$\frac{c_{0(1)}^{j_1}\,c_{0(2)}^{j_2}\cdots c_{0(p-k+1)}^{j_{p-k+1}}}{c_0^k} = (nr_0)^p. \tag{A.10}$$

Proof:

Indeed, c_0 and $c_{0(i)}$ are the first coefficients in Laplace expansion of F_n and $F_n^{(i)}$, respectively. Hence, for $i = 1, 2, \ldots, p-k+1$,

$$\left(\frac{c_{0(i)}}{c_0^k}\right)^{j_i} = \left(\frac{g_i(r_0)}{g(r_0)}\right)^{j_i} = (nr_0)^{i\,j_i}.$$

By the second condition in $(*)$, (A.10) holds.

□

According to the multinomial theorem and Lemma A.2.4, we obtain

$$\begin{aligned} P = (nr_0)^p & \sum_{(j_i)_i,(*)} \frac{p!}{j_1! j_2! \ldots} \left(\frac{1}{1!}\right)^{j_1} \left(\frac{1}{2!}\right)^{j_2} \cdots \\ & \cdot \sum_{\left(k_{m(1)}\right)_m} \binom{j_1}{k_{0(1)}, k_{1(1)}, \ldots, k_{p(1)}} 1^{k_{0(1)}} d_{1(1)}^{k_{1(1)}} \cdots d_{p(1)}^{k_{p(1)}}\, n^{-\left(k_{1(1)}+2k_{2(1)}+\cdots+pk_{p(1)}\right)} \\ & \cdot \sum_{\left(k_{m(2)}\right)_m} \binom{j_2}{k_{0(2)}, k_{1(2)}, \ldots, k_{p(2)}} 1^{k_{0(2)}} d_{1(2)}^{k_{1(2)}} \cdots d_{p(2)}^{k_{p(2)}}\, n^{-\left(k_{1(2)}+2k_{2(2)}+\cdots+pk_{p(2)}\right)} \cdots \\ & \cdot \left(1 + O\left(\frac{1}{n^{p+1}}\right)\right). \end{aligned} \tag{A.11}$$

Here, for $i = 1, 2, \ldots, p-k+1$, $\left(k_{m(i)}\right)_{m=0,1,\ldots,p}$ be the sequences of all combinations of non-negative integer such that

$$\sum_{m=0}^{p} k_{m(i)} = k_{0(i)} + k_{1(i)} + \cdots + k_{p(i)} = j_i$$

and

$$\binom{j_i}{k_{0(i)}, k_{1(i)}, \ldots, k_{p(i)}} = \frac{j_i!}{k_{0(1)}!\, k_{1(i)}! \cdots k_{p(i)}!}$$

be a multinomial coefficients. Then we can rewrite (A.11) as follows

$$\begin{aligned} P = (nr_0)^p & \sum_{(j_i)_i,(*),\left(k_{m(i)}\right)_{i,m}} \frac{p!}{j_1! j_2! \ldots} \left(\frac{1}{1!}\right)^{j_1} \left(\frac{1}{2!}\right)^{j_2} \cdots \\ & \cdot \frac{j_1!}{\left(j_1 - \sum_{m=1}^{p} k_{m(1)}\right)!} \frac{d_{1(1)}^{k_{1(1)}}}{k_{1(1)}!} \cdots \frac{d_{p(1)}^{k_{p(1)}}}{k_{p(1)}!} \frac{j_2!}{\left(j_2 - \sum_{m=1}^{p} k_{m(2)}\right)!} \frac{d_{1(2)}^{k_{1(2)}}}{k_{1(2)}!} \cdots \frac{d_{p(2)}^{k_{p(2)}}}{k_{p(2)}!} \cdots \\ & \cdot n^{-\left(\sum_{i=1}^{p-k+1} k_{1(i)} + 2\sum_{i=1}^{p-k+1} k_{2(i)} + \cdots + p\sum_{i=1}^{p-k+1} k_{p(i)}\right)} \left(1 + O\left(\frac{1}{n^{p+1}}\right)\right). \end{aligned} \tag{A.12}$$

Define

$$y_i := \sum_{m=1}^{s} k_{m(i)}, \quad 0 \le y_i \le j_i \qquad (i = 1, 2, \ldots, p-k+1). \tag{A.13}$$

$$\tilde{j}_i := j_i - \sum_{m=1}^{s} k_{m(i)} = j_i - y_i \ge 0 \quad (i = 1, 2, \ldots, p-k+1). \tag{A.14}$$

$$x_m := \sum_{i=1}^{p-k+1} k_{m(i)}, \quad x_m \ge 0 \qquad (m = 1, 2, \ldots, p). \tag{A.15}$$

Then (A.12) follows that

$$P = (nr_0)^p \sum_{\left(k_{m(i)}\right)_{i,m}} \frac{d_{1(1)}^{k_{1(1)}}}{k_{1(1)}!} \cdots \frac{d_{p(1)}^{k_{p(1)}}}{k_{p(1)}!} \left(\frac{1}{1!}\right)^{y_1} \frac{d_{1(2)}^{k_{1(2)}}}{k_{1(2)}!} \cdots \frac{d_{p(2)}^{k_{p(2)}}}{k_{p(2)}!} \left(\frac{1}{2!}\right)^{y_2} \cdots$$
$$\cdot \sum_{(\tilde{j}_i)_i, (\tilde{*})} \frac{p!}{\tilde{j}_1! \tilde{j}_2! \ldots} \left(\frac{1}{1!}\right)^{\tilde{j}_1} \left(\frac{1}{2!}\right)^{\tilde{j}_2} \cdots n^{-(x_1 + 2x_2 + \cdots + px_p)} \left(1 + O\left(\frac{1}{n^{p+1}}\right)\right), \tag{A.16}$$

where $(\tilde{j}_i)_i, (\tilde{*})$ represents for the meaning that $(\tilde{j}_i)_i$ be the sequences of non-negative integers which satisfy two conditions

$$(\tilde{*}) \begin{cases} \sum_{i \ge 1} \tilde{j}_i = k - \sum_{i=1}^{p-k+1} y_i \\ \sum_{i \ge 1} i\, \tilde{j}_i = p - \sum_{i=1}^{p-k+1} i\, y_i \end{cases}.$$

Then we obtain from (A.16) that

$$P = (nr_0)^p \sum_{\left(k_{m(i)}\right)_{i,m}} \left(\frac{d_{1(1)}}{1!}\right)^{k_{1(1)}} \frac{1}{k_{1(1)}!} \cdots \left(\frac{d_{p(1)}}{1!}\right)^{k_{p(1)}} \frac{1}{k_{p(1)}!} \cdots$$
$$\cdot \left(\frac{d_{1(2)}}{2!}\right)^{k_{1(2)}} \frac{1}{k_{1(2)}!} \cdots \left(\frac{d_{p(2)}}{2!}\right)^{k_{p(2)}} \frac{1}{k_{p(2)}!} \cdots S_{p - \sum_{i=1}^{p-k+1} iy_i,\, k - \sum_{i=1}^{p-k+1} y_i}$$
$$\cdot \frac{p!}{\left(p - \sum_{i=1}^{p-k+1} iy_i\right)!} n^{-(x_1 + 2x_2 + \cdots + px_p)} \left(1 + O\left(\frac{1}{n^{p+1}}\right)\right), \tag{A.17}$$

where $S_{p,k}$ denotes the Stirling numbers of the second kind. Consequently, it follows that

$$nL_n^{(p)}(\lambda) = (nr_0)^p\, p! \sum_{k=1}^{p} (-1)^{k+1} (k-1)! \sum_{\left(k_{m(i)}\right)_{i,m}} \frac{1}{(p - \sum_i iy_i)!}$$
$$\cdot \prod_{i=1}^{p-k+1} \left(\frac{d_{1(i)}}{i!}\right)^{k_{1(i)}} \frac{1}{k_{1(i)}!} \cdots \prod_{i=1}^{p-k+1} \left(\frac{d_{p(i)}}{i!}\right)^{k_{p(i)}} \frac{1}{k_{p(i)}!}$$
$$\cdot S_{p - \sum_{i=1}^{p-k+1} iy_i,\, k - \sum_{i=1}^{p-k+1} y_i}\, n^{-(x_1 + 2x_2 + \cdots + px_p)} \left(1 + O\left(\frac{1}{n^{p+1}}\right)\right). \tag{A.18}$$

Remark A.2.5 *During the proof, in order to avoid many significant conditions related to the notations of the factorial and the Stirling number of the second kind, we might agree that:*

- $x!$ *is non-zero if* $x \geq 0$,
- $S_{p,k}$ *is non-zero if* $0 < k \leq p$.

Otherwise, they can be eliminated and do not affect to our computations.

The rest of proving Theorem A.2.22 is to show that the sum related to sequence $\left(k_{m(i)}\right)_{i,m}$ in (A.18) can be expanded as a power series $\sum_s h_s n^{-s}$. Moreover, for $s = 1, 2, \ldots, p$, we have

$$\sum_{k=1}^{p}(-1)^{k+1}(k-1)!\, h_s = 0. \tag{A.19}$$

It is shown by induction: firstly in base case by Proposition A.2.6 and in inductive case by Proposition A.2.11.

Proposition A.2.6

$$\sum_{k=1}^{p}(-1)^{k+1}(k-1)!\, h_0 = 0. \tag{A.20}$$

Proof of Proposition A.2.6:
Follow this case, we have $s = 0$. Therefore, $k_{m(i)} = 0$ for all $i = 1, 2, \ldots, p-k+1$ and $m = 1, 2, \ldots, s$. The LHS of (A.20) becomes

$$\sum_{k=1}^{p}(-1)^{k+1}(k-1)!\, S_{p,k} = 0. \tag{A.21}$$

Let us follow the RHS of above equation

$$\begin{aligned}
\sum_{k=1}^{p}(-1)^{k+1}(k-1)!\, S_{p,k} &= \sum_{k=1}^{p}(-1)^{k+1}(k-1)!\, \left(S_{p-1,k-1} - S_{p-1,k}\right) \\
&= \sum_{k=2}^{p}(-1)^{k+1}(k-1)!\, S_{p-1,k-1} + \sum_{k=1}^{p-1}(-1)^{k+1}k!\, S_{p-1,k} \\
&= \sum_{\tilde{k}=1}^{p-1}(-1)^{\tilde{k}}\tilde{k}!\, S_{p-1,\tilde{k}} + \sum_{k=1}^{p-1}(-1)^{k+1}k!\, S_{p-1,k} \\
&= 0.
\end{aligned}$$

$\square$

We begin the very first step of induction of (A.19) by considering the coefficients h_s. We have

$$s = x_1 + 2x_2 + \cdots + s x_s = \sum_{i=1}^{p-k+1} k_{1(i)} + 2\sum_{i=1}^{p-k+1} k_{2(i)} + \cdots + s\sum_{i=1}^{p-k+1} k_{s(i)}. \tag{A.22}$$

For $s > 0$, all combinations of non-negative integer $\left(k_{m(i)}\right)_{i,m}$ set up s- subsets of combinations $x_1, x_2, \ldots, x_s$ which satisfy (A.22) and are defined by $\sum_{i=1}^{p-k+1} k_{1(i)}$, $\sum_{i=1}^{p-k+1} k_{2(i)}$, $\ldots, \sum_{i=1}^{p-k+1} k_{s(i)}$, respectively. For each set of $x_1, x_2, \ldots, x_s$, setting

$$x_0 := x_1 + x_2 + \cdots + x_s = y_1 + y_2 + \cdots + y_{p-k+1}. \tag{A.23}$$

We infer from (A.23) that $0 < x_0 \leq s$. For given $s = 1, 2, \ldots, p$ and for each x_0,

$$\begin{aligned} h_s = & \sum_{\substack{x_0 = \overline{1,s} \\ x_0 = x_1 + \cdots + x_s}} \quad \sum_{\substack{\left(k_{1(i)}\right)_i \\ x_1 = k_{1(1)} + \cdots + k_{1(p-k+1)}}} \prod_{i=1}^{p-k+1} \left(\frac{d_{1(i)}}{i!}\right)^{k_{1(i)}} \frac{1}{k_{1(i)}!} \cdots \\ & \cdot \sum_{\substack{\left(k_{s(i)}\right)_i \\ x_s = k_{s(1)} + \cdots + k_{s(p-k+1)}}} \prod_{i=1}^{p-k+1} \left(\frac{d_{s(i)}}{i!}\right)^{k_{s(i)}} \frac{1}{k_{s(i)}!} \frac{S_{p-\sum_{i=1}^{p-k+1} iy_i, k-x_0}}{\left(p - \sum_{i=1}^{p-k+1} iy_i\right)!}. \end{aligned} \tag{A.24}$$

Since $x_0 > 0$ then there exist at least $k_{m(i)} \neq 0$. It leads to the fact that there exists a positive $m \in [1, s]$ such that $x_m > 0$. Remark that

$$\sum_{i=1}^{p-k+1} iy_i = \sum_{i=1}^{p-k+1} ik_{1(i)} + \sum_{i=1}^{p-k+1} ik_{2(i)} + \cdots + \sum_{i=1}^{p-k+1} ik_{s(i)},$$

and formula (A.24) shows that h_s is not effected by eliminating the case when $x_m = 0$ for some $m \in [1, s]$. Therefore, we can subdivide terms in (A.24) as the following lemma

Lemma A.2.7 *For each $x_0 \leq s$ (s fixed), let M be a positive constant between 1 and x_0 associated with finite non-negative sequence $k_1, k_2, \ldots, k_{p-k+1}$ such that $M = k_1 + k_2 + \cdots + k_{p-k+1}$, then*

$$\begin{aligned} & \sum_{\substack{k_i \geq 0 \\ M = k_1 + \cdots + k_{p-k+1}}} \binom{M}{k_1, k_2, \ldots, k_{p-k+1}} \prod_{i=1}^{p-k+1} \left(\frac{d_{m(i)}}{i!}\right)^{k_i} \frac{S_{q-\sum_{i=1}^{p-k+1} ik_i, k-x_0}}{\left(q - \sum_{i=1}^{p-k+1} ik_i\right)!} \\ & = \sum_{i_1, i_2, \ldots, i_M = 1}^{p-k+1} \frac{d_{m(i_1)}}{i_1!} \frac{d_{m(i_2)}}{i_2!} \cdots \frac{d_{m(i_M)}}{i_M!} \frac{S_{q-i_1-i_2-\cdots-i_M, k-x_0}}{(q - i_1 - i_2 - \cdots - i_M)!} \end{aligned} \tag{A.25}$$

holds for all M from 1 to s.

Proof:
The lemma follows by induction. Indeed, when $M = 1$, we have

$$\begin{aligned} & \sum_{\substack{k_i \geq 0 \\ 1 = k_1 + \cdots + k_{p-k+1}}} \binom{1}{k_1, \ldots, k_{p-k+1}} \prod_{i=1}^{p-k+1} \left(\frac{d_{m(i)}}{i!}\right)^{k_i} \frac{S_{q-\sum_{i=1}^{p-k+1} ik_i, k-x_0}}{\left(q - \sum_{i=1}^{p-k+1} ik_i\right)!} \\ & = \sum_{i=1}^{p-k+1} \frac{d_{m(i)}}{i!} S_{q-i, k-x_0} \frac{1}{(q-i)!} \end{aligned}$$

Assume that (A.25) holds for M, we now prove that

$$\sum_{\substack{\tilde{k}_i \geq 0 \\ M+1=\tilde{k}_1+\cdots+\tilde{k}_{p-k+1}}} \binom{M+1}{\tilde{k}_1,\ldots,\tilde{k}_{p-k+1}} \prod_{i=1}^{p-k+1} \left(\frac{d_{m(i)}}{i!}\right)^{\tilde{k}_i} \frac{S_{q-\sum_{i=1}^{p-k+1} i\tilde{k}_i, k-x_0}}{\left(q-\sum_{i=1}^{p-k+1} i\tilde{k}_i\right)!}$$
$$= \sum_{i_1,\ldots,i_M,i_{M+1}=1}^{p-k+1} \frac{d_{m(i_1)}}{i_1!} \cdots \frac{d_{m(i_M)}}{i_M!} \frac{d_{m(i_{M+1})}}{i_{M+1}!} \frac{S_{q-i_1-\cdots-i_{M+1},k-x_0}}{(q-i_1-\cdots-i_{M+1})!},$$

where $\left(\tilde{k}_i\right)_i$ is a sequence of non-negative numbers associating with $M+1$ such that $M+1=\tilde{k}_1+\cdots+\tilde{k}_{p-k+1}$. Since

$$\binom{M+1}{\tilde{k}_1,\ldots,\tilde{k}_{p-k+1}} = \frac{(M+1)!}{\tilde{k}_1!\ldots\tilde{k}_{p-k+1}!} = \frac{M!\,(\tilde{k}_1+\cdots+\tilde{k}_{p-k+1})}{\tilde{k}_1!\ldots\tilde{k}_{p-k+1}!}$$
$$= \sum_{l=1}^{p-k+1} \binom{M}{\tilde{k}_1,\ldots,\tilde{k}_l-1,\ldots,\tilde{k}_{p-k+1}},$$

then

$$\begin{aligned}\text{LHS} &= \sum_{M+1=\tilde{k}_1+\cdots+\tilde{k}_{p-k+1}} \sum_{l=1}^{p-k+1} \binom{M}{\tilde{k}_1,\ldots,\tilde{k}_l-1,\ldots,\tilde{k}_{p-k+1}} \\ &\quad \cdot \left(\frac{d_{m(1)}}{1!}\right)^{\tilde{k}_1} \cdots \left(\frac{d_{m(l)}}{l!}\right)^{\tilde{k}_l-1} \frac{d_{m(l)}}{l!} \cdots \left(\frac{d_{m(p-k+1)}}{(p-k+1)!}\right)^{\tilde{k}_{p-k+1}} \\ &\quad \cdot \frac{S_{q-\sum_{i\neq l} i\tilde{k}_i - l(\tilde{k}_l-1)-l,k-x_0}}{\left(q-\sum_{i=1}^{p-k+1} i\tilde{k}_i\right)!}.\end{aligned}$$

If we set $k_l=\tilde{k}_l-1$ (for $l=1,2,\ldots p-k+1$) and $k_i=\tilde{k}_i$ (for $i=1,2,\ldots p-k+1$ and $i\neq l$), the sequences of non-negative number $k_1,\ldots,k_{p-k+1}$ associates with M. Therefore,

$$\begin{aligned}\text{LHS} &= \sum_{l=1}^{p-k+1} \frac{d_{m(l)}}{l!} \sum_{M=k_1+\cdots+k_{p-k+1}} \binom{M}{k_1,\ldots,k_l,\ldots,k_{p-k+1}} \\ &\quad \cdot \prod_{i=1}^{p-k+1} \left(\frac{d_{m(i)}}{i!}\right)^{k_i} \frac{S_{q-l-\sum_i ik_i,k-x_0}}{\left(q-l-\sum_{i=1}^{p-k+1} ik_i\right)!} \\ &= \sum_{l=1}^{p-k+1} \frac{d_{m(l)}}{l!} \sum_{i_1,i_2,\ldots,i_M=1}^{p-k+1} \frac{d_{m(i_1)}}{i_1!} \frac{d_{m(i_2)}}{i_2!} \cdots \frac{d_{m(i_M)}}{i_M!} \frac{S_{q-l-i_1-i_2-\cdots-i_{x_m},k-x_0}}{(q-l-i_1-i_2-\cdots-i_{x_m})!}.\end{aligned}$$

It complete the proof where index i_{M+1} replaces for l.

□

For each x_0, we have for all $m = 1, 2, \ldots, s$, x_m can take any value between 1 and x_0. Therefore, we can apply Lemma A.2.7 when $x_m = M$, $q = p - \sum_{i=1}^{p-k+1} iy_i + \sum_{i=1}^{p-k+1} ik_{m(i)}$ and substitute notation $k_{m(i)}$ for k_i

$$\sum_{\substack{\left(k_{m(i)}\right)_i \\ x_m = k_{m(1)} + \cdots + k_{m(p-k+1)}}} \prod_{i=1}^{p-k+1} \left(\frac{d_{m(i)}}{i!}\right)^{k_{m(i)}} \frac{1}{k_{m(i)}!} \frac{S_{p-\sum_{i=1}^{p-k+1} iy_i, k-x_0}}{\left(p - \sum_{i=1}^{p-k+1} iy_i\right)!}$$
$$= \frac{1}{x_m!} \sum_{i_1, i_2, \ldots, i_m = 1}^{p-k+1} \frac{d_{m(i_1)}}{i_1!} \frac{d_{m(i_2)}}{i_2!} \cdots \frac{d_{m(i_{x_m})}}{i_{x_m}!} \frac{S_{p-\sum_{l\neq m, l=\overline{1,s}}^{i=\overline{1,p-k+1}} ik_{l(i)} - (i_1+\cdots+i_{x_m}), k-x_0}}{\left(p - \sum_{l\neq m, l=\overline{1,s}}^{i=\overline{1,p-k+1}} ik_{l(i)} - (i_1 + \cdots + i_{x_m})\right)!}, \tag{A.26}$$

Remark A.2.8 *Next, we substitute notation $\left(i_{l(m)}\right)_l$ for $(i_l)_l$ by meaning the sequence of index generated by the m-th term.*

Accordingly, the entire terms in the form (A.24) of h_s can be computed as

$$h_s = \sum_{\substack{x_0 = \overline{1,s} \\ x_0 = x_1 + \cdots + x_s}} \frac{1}{x_1! x_2! \cdots x_s!} \sum_{(**)} \frac{d_{1(i_{1(1)})} \cdots d_{1(i_{x_1(1)})}}{i_{1(1)}! \cdots i_{x_1(1)}!}$$
$$\cdot \frac{d_{2(i_{1(2)})} \cdots d_{2(i_{x_2(2)})}}{i_{1(2)}! \cdots i_{x_2(2)}!} \cdots \frac{d_{s(i_{1(s)})} \cdots d_{s(i_{x_s(s)})}}{i_{1(s)}! \cdots i_{x_s(s)}!} \frac{S_{p-\sum_m i_{m(1)} - \cdots - \sum_m i_{m(s)}, k-x_0}}{(p - \sum_m i_{m(1)} - \cdots - \sum_m i_{m(s)})!}. \tag{A.27}$$

Here, $(**)$ represent for the meaning that the sequences of index $\left(i_{m(1)}\right)_{m=\overline{1,x_1}}$, $\left(i_{m(2)}\right)_{m=\overline{1,x_2}}$, $\ldots$, $\left(i_{m(s)}\right)_{m=\overline{1,x_s}}$, take their value on $[1, p-k+1]$. Since the number of these indexes is $x_1 + \cdots + x_s = x_0$ and the factors $d_{m(i)}/i!$ respect to m in range $[1, s]$, then we can simplify the expression of h_s as follows

$$h_s = \sum_{x_0=1}^{s} \sum_{x_1+\cdots+x_s=x_0} \frac{1}{x_1! x_2! \cdots x_s!} \sum_{i_1, i_2, \ldots, i_{x_0}=1}^{p-k+1} \frac{d_{m_1(i_1)}}{i_1!} \frac{d_{m_2(i_2)}}{i_2!} \cdots \frac{d_{m_{x_0}(i_{x_0})}}{i_{x_0}!} \cdot$$
$$\cdot \frac{S_{p-i_1-i_2-\cdots-i_{x_0}, k-x_0}}{(p - i_1 - i_2 - \cdots - i_{x_0})!}, \tag{A.28}$$

where $m_1, m_2, \ldots, m_{x_0}$ are the indexes taking values from 1 to s.

The next step is to consider coefficients $d_{m(i)}$. We mainly mention the dependence of results on i and k. We recall that all function respect to λ from now is considered at $\lambda = \lambda_c$, the notations do not change. The following technical lemma will show us that $d_{m(i)}$ can be expressed to a series, which depends on i:

Technical Lemma A.2.9 *We can express*

$$d_{m(i)} = \sum_{\omega=1}^{2m} \binom{i}{\omega} \mathbb{1}_{i \geq \omega} Q_\omega, \tag{A.29}$$

where $\{Q_\omega\}$ is a sequence of constants $Q_\omega(r_0(\lambda_c))$ which is independent of index i.

Proof:

Remark again the formula (A.9), we will proceed to consider the formulas of $g_i^{(k)}$, $\frac{g_i^{(k)}}{g_i} - \frac{g^{(k)}}{g}$, $\frac{c_{m(i)}}{c_{0(i)}} - \frac{c_m}{c_0}$, $d_{\alpha(i)}$.

Firstly, we have the view about $d_{m(i)}$.

$$\sum d_{m(i)} n^{-m} = \frac{\sum_m \frac{c_{m(i)}}{(2m)!\, c_{0(i)}} n^{-m}}{\sum_m \frac{c_m}{(2m)!\, c_0} n^{-m}} = \sum_m C_{m(i)} n^{-m} \left(\sum_m C_m n^{-m} \right)^{-1}$$
$$= \sum_m C_{m(i)} n^{-m} \sum_m B_m n^{-m},$$

where $C_{m(i)} := \frac{c_{m(i)}}{(2m)!\, c_{0(i)}}$, $C_m := \frac{c_m}{(2m)!\, c_0}$ and sequence $(B_m)_m$ can be defined by

$$\begin{cases} B_0\, C_0 = 1 \\ \sum_{l=0}^{m} B_l\, C_{m-l} = 0, \text{ for } m = 1, 2, \ldots \end{cases}.$$

It is easy to see that $C_{0(i)} = C_0 = B_0 = 1$. Since $\sum_{l=0}^{m} B_l\, C_{m-l} = 0$ then

$$d_{m(i)} = \sum_{l=0}^{m} C_{l(i)}\, B_{m-l} = \sum_{l=1}^{m} \left(C_{l(i)} - C_l \right) B_{m-l}$$
$$= \sum_{l=1}^{m} \frac{1}{(2l)!} \left(\frac{c_{l(i)}}{c_{0(i)}} - \frac{c_l}{c_0} \right) B_{m-l}.$$

Secondly, according to Laplace development (Theorem 2.3.10, Chapter 2), we have

$$c_0 = \sqrt{\frac{2\pi}{|h''(r_0)|}}\, g(r_0)$$

and

$$c_l = \sqrt{\frac{2\pi}{|h''(r_0)|}} \sum_{\alpha=0}^{2l} \begin{pmatrix} 2l \\ \alpha \end{pmatrix} g^{(2l-\alpha)}(r_0) \cdot$$
$$\cdot \sum_{\beta=0}^{\alpha} B_{\alpha,\beta} \left(\frac{h^{(3)}(r_0)}{2.3}, \ldots, \frac{h^{(\alpha-\beta+3)}(r_0)}{(\alpha-\beta+2)(\alpha-\beta+3)} \right) \frac{(2\beta+2l-1)!!}{|h''(r_0)|^{\beta+l}}.$$

Therefore,

$$\frac{c_l}{c_0} = \sum_{\alpha=0}^{2l} \frac{g^{(2l-\alpha)}(r_0)}{g(r_0)} D_{\alpha,l}(r_0),$$

where

$$D_{\alpha,l}(r_0) = \begin{pmatrix} 2l \\ \alpha \end{pmatrix} \sum_{\beta=0}^{\alpha} B_{\alpha,\beta} \left(\frac{h^{(3)}(r_0)}{2.3}, \ldots, \frac{h^{(\alpha-\beta+3)}(r_0)}{(\alpha-\beta+2)(\alpha-\beta+3)} \right) \frac{(2\beta+2l-1)!!}{|h''(r_0)|^{\beta+l}}$$

is a polynomial of variable r_0 (namely the derivative of h), depend on l and not depend on i. Similarly, we can obtain

$$\begin{aligned}\frac{c_{l(i)}}{c_{0(i)}}-\frac{c_l}{c_0}&=\sum_{\alpha=0}^{2l-1}\left(\frac{g_i^{(2l-\alpha)}(r_0)}{g_i(r_0)}-\frac{g^{(2l-\alpha)}(r_0)}{g(r_0)}\right)D_{\alpha,l}(r_0)\\&=\sum_{\alpha=1}^{2l}\left(\frac{g_i^{(\alpha)}(r_0)}{g_i(r_0)}-\frac{g^{(\alpha)}(r_0)}{g(r_0)}\right)D_{2l-\alpha,l}(r_0).\end{aligned}\tag{A.30}$$

Thirdly, we have $g(r)=(1-r^2)^{-2}$, then it follows to the Faà di Bruno's formula, for $\alpha=1,2,\ldots,$

$$\begin{aligned}g^{(\alpha)}(r)=\{u\,(v(r))\}^{(\alpha)}&=\sum_{\beta=1}^{\alpha}u^{(\beta)}\,(v(r))\;B_{\alpha,\beta}\left(v'(r),v''(r),\cdots,v^{(\alpha-\beta+1)}(r)\right)\\&=\sum_{\beta=1}^{\alpha}\frac{(-1)^{\beta}\,(\beta+1)!}{(1-r^2)^{\beta+2}}\,B_{\alpha,\beta}\left(-2r,-2,0,\cdots,0\right),\end{aligned}$$

in which, $u(r)=r^{-2}$, $u^{(\beta)}\,(r)=(-1)^{\beta}\,(\beta+1)!\,r^{-(\beta+2)}$ and $v(r)=1-r^2$. Since

$$B_{\alpha,\beta}\left(-2r,-2,0,\cdots,0\right)=\sum_{(j_k)_k,(**)}\frac{\alpha!}{j_1!\,j_2!\cdots}\left(\frac{-2r}{1!}\right)^{j_1}\left(\frac{-2}{2!}\right)^{j_2}\left(\frac{0}{3!}\right)^{j_3}\cdots\left(\frac{0}{(\alpha-\beta+1)!}\right)^{j_{\alpha-\beta+1}},$$

where $(j_k)_k,(**)$ represents for the meaning that $(j_k)_k$ be the sequence of non-negative integers which satisfy two conditions

$$(**)\left\{\begin{array}{l}\sum_{k\geq1}j_k=\beta\\\sum_{k\geq1}k\,j_k=\alpha\end{array}\right..$$

We know that $0^0=1$ and $0^k=0\,(k\geq1)$, then

$$B_{\alpha,\beta}\left(-2r,-2,0,\cdots,0\right)=0\qquad\text{if}\quad j_k\neq0\quad(k=3,4,\cdots,\alpha-\beta+1).$$

Then it follows

$$B_{\alpha,\beta}\left(-2r,-2,0,\cdots,0\right)=\sum_{j_1,j_2,(\tilde{**})}\frac{\alpha!}{j_1!\,j_2!\ldots}\,(-2r)^{j_1}\,(-1)^{j_2},$$

where j_1,j_2 are non-negative integers which satisfy

$$(\tilde{**})\left\{\begin{array}{l}j_1+j_2=\beta\geq0\\j_1+2\,j_2=\alpha\geq0\end{array}\right..$$

Then

$$B_{\alpha,\beta}\left(-2r,-2,0,\cdots,0\right)=\frac{\alpha!}{(2\beta-\alpha)!\,(\alpha-\beta)!}\,(-1)^{\beta}\,(2r)^{2\beta-\alpha}\,\mathbb{1}_{\alpha/2\leq\beta\leq\alpha}.$$

Consequently, we obtain

$$
\begin{aligned}
g^{(\alpha)}(r) &= \sum_{\beta=1}^{\alpha} \frac{(-1)^{\beta}\,(\beta+1)!}{(1-r^2)^{\beta+2}} \, \frac{\alpha!}{(2\beta-\alpha)!\,(\alpha-\beta)!}\,(-1)^{\beta}\,(2r)^{2\beta-\alpha}\,\mathbb{1}_{\alpha/2\le\beta\le\alpha} \\
&= \frac{\alpha!}{r^{\alpha}\,(1-r^2)^2} \sum_{\beta\ge\alpha/2}^{\alpha} \frac{2^{2\beta-\alpha}(\beta+1)!}{(2\beta-\alpha)!\,(\alpha-\beta)!} \left(\frac{r^2}{1-r^2}\right)^{\beta}
\end{aligned}
$$

and therefore

$$
\frac{g^{(\alpha)}(r)}{g(r)} = \frac{\alpha!}{r^{\alpha}} \sum_{\beta\ge\alpha/2}^{\alpha} \frac{2^{2\beta-\alpha}(\beta+1)!}{(2\beta-\alpha)!\,(\alpha-\beta)!} \left(\frac{r^2}{1-r^2}\right)^{\beta}. \tag{A.31}
$$

<u>Fourthly,</u> we know that $g_i(r) = (nr)^i g(r)$ then according to Leibniz's rule, we have

$$
\begin{aligned}
g_i^{(\alpha)}(r) &= n^i \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (r^i)^{(\alpha-\beta)} g^{(\beta)}(r) \\
&= n^i \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left\{ \frac{i!}{(i-\alpha+\beta)!}\, r^{i-\alpha+\beta}\,\mathbb{1}_{\beta\ge\alpha-i} \right\} \cdot \\
&\qquad \cdot \left\{ \frac{\beta!}{r^{\beta}\,(1-r^2)^2} \sum_{\gamma\ge\beta/2}^{\beta} \frac{2^{2\gamma-\beta}(\gamma+1)!}{(2\gamma-\beta)!\,(\beta-\gamma)!} \left(\frac{r^2}{1-r^2}\right)^{\gamma} \right\} \\
&= \frac{n^i r^{i-\alpha}}{(1-r^2)^2} \sum_{\beta=0}^{\alpha} \frac{i!\,\alpha!}{(\alpha-\beta)!\,(i-\alpha+\beta)!}\,\mathbb{1}_{\beta\ge\alpha-i} \sum_{\gamma\ge\beta/2}^{\beta} \frac{2^{2\gamma-\beta}(\gamma+1)!}{(2\gamma-\beta)!\,(\beta-\gamma)!} \left(\frac{r^2}{1-r^2}\right)^{\gamma}.
\end{aligned}
$$

It follows that

$$
\frac{g_i^{(\alpha)}(r)}{g_i(r)} = \frac{\alpha!}{r^{\alpha}} \sum_{\beta=0}^{\alpha} \binom{i}{\alpha-\beta} \mathbb{1}_{\beta\ge\alpha-i} \sum_{\gamma\ge\beta/2}^{\beta} \frac{2^{2\gamma-\beta}(\gamma+1)!}{(2\gamma-\beta)!\,(\beta-\gamma)!} \left(\frac{r^2}{1-r^2}\right)^{\gamma}. \tag{A.32}
$$

From (A.31) and (A.32), we get that

$$
\frac{g_i^{(\alpha)}(r_0)}{g_i(r_0)} - \frac{g^{(\alpha)}(r_0)}{g(r_0)} = \sum_{\beta=0}^{\alpha-1} \binom{i}{\alpha-\beta} \mathbb{1}_{\beta\ge\alpha-i}\, \frac{\alpha!}{r_0^{\alpha}} \sum_{\gamma\ge\beta/2}^{\beta} \frac{2^{2\gamma-\beta}(\gamma+1)!}{(2\gamma-\beta)!\,(\beta-\gamma)!} \left(\frac{r_0^2}{1-r_0^2}\right)^{\gamma}.
$$

Remark A.2.10 *Here, we note the above expression that we can compress the last sum of index γ to the polynomials $A_\beta(r_0)$, which do not depend on i, namely*

$$
\begin{aligned}
A_{2y}(r_0) &= \frac{\alpha!}{r_0^{\alpha}} \sum_{\gamma=y}^{2y} \frac{2^{2\gamma-2y}(\gamma+1)!}{(2\gamma-2y)!\,(2y-\gamma)!} \left(\frac{r_0^2}{1-r_0^2}\right)^{\gamma}, \quad \textit{if } \beta = 2y, \\
A_{2y+1}(r_0) &= \frac{\alpha!}{r_0^{\alpha}} \sum_{\gamma=y+1}^{2y+1} \frac{2^{2\gamma-2y-1}(\gamma+1)!}{(2\gamma-2y-1)!\,(2y+1-\gamma)!} \left(\frac{r_0^2}{1-r_0^2}\right)^{\gamma}, \quad \textit{if } \beta = 2y+1.
\end{aligned}
$$

Then we see that

$$\frac{g_i^{(\alpha)}(r_0)}{g_i(r_0)} - \frac{g^{(\alpha)}(r_0)}{g(r_0)} = \begin{pmatrix} i \\ \alpha \end{pmatrix} \mathbb{1}_{i\geq\alpha} A_0(r_0) + \begin{pmatrix} i \\ \alpha-1 \end{pmatrix} \mathbb{1}_{i\geq\alpha-1} A_1(r_0) + \\ + \cdots + \begin{pmatrix} i \\ 1 \end{pmatrix} \mathbb{1}_{i\geq 1} A_{\alpha-1}(r_0).$$

Accordingly, we can express (A.30) as follows

$$\begin{aligned} \frac{c_{l(i)}}{c_{0(i)}} - \frac{c_l}{c_0} &= \sum_{\alpha=1}^{2l} \Bigg\{ \begin{pmatrix} i \\ \alpha \end{pmatrix} \mathbb{1}_{i\geq\alpha} A_0(r_0) + \begin{pmatrix} i \\ \alpha-1 \end{pmatrix} \mathbb{1}_{i\geq\alpha-1} A_1(r_0) + \\ &\qquad + \cdots + \begin{pmatrix} i \\ 1 \end{pmatrix} \mathbb{1}_{i\geq 1} A_{\alpha-1}(r_0) \Bigg\} D_{2l-\alpha,l}(r_0) \\ &= \begin{pmatrix} i \\ 2l \end{pmatrix} \mathbb{1}_{i\geq 2l} E_{2l}(r_0) + \begin{pmatrix} i \\ 2l-1 \end{pmatrix} \mathbb{1}_{i\geq 2l-1} E_{2l-1}(r_0) + \\ &\qquad + \cdots + \begin{pmatrix} i \\ 1 \end{pmatrix} \mathbb{1}_{i\geq 1} E_1(r_0) \\ &= \sum_{\alpha=1}^{2l} \begin{pmatrix} i \\ \alpha \end{pmatrix} \mathbb{1}_{i\geq\alpha} E_\alpha(r_0), \end{aligned}$$

in which, for $\alpha = 1, 2, \ldots, 2l$

$$E_\alpha(r_0) = \sum_{\beta=0}^{2l-\alpha} A_\beta(r_0)\, D_{2l-\alpha-\beta,l}(r_0).$$

Finally, we imply the last expression

$$\begin{aligned} d_{m(i)} &= \sum_{l=1}^{m} \frac{1}{(2l)!} \sum_{\alpha=1}^{2l} \begin{pmatrix} i \\ \alpha \end{pmatrix} \mathbb{1}_{i\geq\alpha} E_\alpha(r_0)\, B_{m-l}(r_0) \\ &= \sum_{\omega=1}^{2m} \begin{pmatrix} i \\ \omega \end{pmatrix} \mathbb{1}_{i\geq\omega}\, Q_\omega(r_0), \end{aligned}$$

where

$$Q_\omega(r_0) = E_\omega(r_0) \sum_{\alpha\geq\omega/2}^{m} \frac{B_{m-\alpha}(r_0)}{(2\alpha)!}.$$

□

Base on the expression of h_s, we prove the inductive case of (A.19) for any $s \leq p$ by the following proposition

Proposition A.2.11 *Let $x_0(\leq s)$ be given and $m_1, m_2, \ldots, m_{x_0} \leq s$ be arbitrary non-negative numbers such that $1 \leq m_1, m_2, \ldots, m_{x_0} \leq s$, then*

$$\sum_{k=1}^{p}(-1)^{k+1}(k-1)! \sum_{x_0=1}^{s} \sum_{x_1+\cdots+x_s=x_0} \frac{1}{x_1!x_2!\cdots x_s!} \cdot \sum_{i_1,i_2,\ldots,i_{x_0}=1}^{p-k+1} \frac{d_{m_1(i_1)}}{i_1!}\frac{d_{m_2(i_2)}}{i_2!}\cdots\frac{d_{m_{x_0}(i_{x_0})}}{i_{x_0}!} \frac{S_{p-i_1-i_2-\cdots-i_{x_0},k-x_0}}{(p-i_1-i_2-\cdots-i_{x_0})!} = 0. \quad \text{(A.33)}$$

Proof:

Inspired by the result (A.21), we will demonstrate that

$$\sum_{i_1,i_2,\ldots,i_{x_0}=1}^{p-k+1} \frac{d_{m_1(i_1)}}{i_1!}\frac{d_{m_2(i_2)}}{i_2!}\cdots\frac{d_{m_{x_0}(i_{x_0})}}{i_{x_0}!} \frac{S_{p-i_1-i_2-\cdots-i_{x_0},k-x_0}}{(p-i_1-i_2-\cdots-i_{x_0})!}$$

can be expressed by the finite sum formed as $\sum_i Q_i S_{p-i,k}$ where Q_i is a certain constant.

Here again, we remark that the significant condition of notation $S_{p,k}$ is unspoken. However, it will be mentioned specifically on the proof.

Let us consider sequence $(\Sigma_T)_T$, $T = 1, 2 \ldots, x_0$ which satisfies

$$(\Sigma_T)_T : \begin{cases} \Sigma_1 = \displaystyle\sum_{i_1=1}^{p-k+1} \frac{d_{m_1(i_1)}}{i_1!} \frac{S_{p-i_1-i_2-\cdots-i_{x_0},k-x_0}}{(p-i_1-i_2-\cdots-i_{x_0})!} \\ \Sigma_T = \displaystyle\sum_{i_T=1}^{p-k+1} \frac{d_{m_T(i_T)}}{i_T!} \Sigma_{T-1} \end{cases},$$

then the sum we mentioned at the beginning of this proof is expressed by Σ_{x_0}. We now point out the general formula of Σ_T as follows

$$\Sigma_T = \prod_{t=1}^{T} \sum_{j_t=0}^{j_{t-1}+1} (-1)^{j_{t-1}+1-j_t} \binom{j_{t-1}+1}{j_t} \cdot \sum_{\omega_t=1}^{2m_t} \frac{Q_{\omega_t}}{\omega_t!} \frac{S_{p_T^*+j_T+1,k-x_0+T}}{p_T^*!} \mathbb{1}_{p_T^*+j_T+1 \geq k-x_0+T}, \quad \text{(A.34)}$$

in which $j_0 = -1$, $p_T^* = p - \omega_1 - \cdots - \omega_T - i_{T+1} - \cdots - i_{x_0}$ and $T = 1, 2 \ldots, x_0$. The above formula is proved by induction.

When $T = 1$, from Lemma A.2.9 and the recurrence relation of the Stirling numbers of the second kind, for $k - x_0 \geq 0$, we have

$$
\begin{aligned}
\Sigma_1 &= \sum_{i_1=1}^{p-k+1} \frac{d_{m_1(i_1)}}{i_1!} \frac{S_{p-i_1-i_2-\cdots-i_{x_0},k-x_0}}{(p-i_1-i_2-\cdots-i_{x_0})!} \mathbb{1}_{p-i_1-i_2-\cdots-i_{x_0}\geq k-x_0} \\
&= \sum_{i_1=1}^{p-k+1} \left\{ \sum_{\omega_1=1}^{2m_1} \binom{i_1}{\omega_1} \mathbb{1}_{i_1\geq\omega_1} \frac{Q_{\omega_1}}{i_1!} \right\} \frac{S_{p-i_2-\cdots-i_{x_0}-i_1,k-x_0}}{(p-i_2-\cdots-i_{x_0}-i_1)!} \mathbb{1}_{i_1\leq p-i_2-\cdots-i_{x_0}-(k-x_0)} \\
&= \sum_{\omega_1=1}^{2m_1} \frac{Q_{\omega_1}}{\omega_1!} \sum_{i_1=\omega_1}^{p-i_2-\cdots-i_{x_0}-(k-x_0)} \frac{1}{(i_1-\omega_1)!} \frac{S_{p-i_2-\cdots-i_{x_0}-i_1,k-x_0}}{(p-i_2-\cdots-i_{x_0}-i_1)!} \\
&= \sum_{\omega_1=1}^{2m_1} \frac{Q_{\omega_1}}{\omega_1!} \frac{1}{p_1^*!} \sum_{i_1=\omega_1}^{p-i_2-\cdots-i_{x_0}-(k-x_0)} \binom{p_1^*}{p-i_2-\cdots-i_{x_0}-i_1} S_{p-i_2-\cdots-i_{x_0}-i_1,k-x_0} \\
&= \sum_{\omega_1=1}^{2m_1} \frac{Q_{\omega_1}}{\omega_1!} \frac{1}{p_1^*!} \left\{ \binom{p_1^*}{p_1^*} S_{p_1^*,k-x_0} + \binom{p_1^*}{p_1^*-1} S_{p_1^*-1,k-x_0} + \cdots \right. \\
&\qquad\qquad \left. \cdots + \binom{p_1^*}{k-x_0} S_{k-x_0,k-x_0} \right\} \\
&= \sum_{\omega_1=1}^{2m_1} \frac{Q_{\omega_1}}{\omega_1!} \frac{S_{p_1^*+1,k-x_0+1}}{p_1^*!} \mathbb{1}_{p_1^*\geq k-x_0}.
\end{aligned}
$$

So the base case of inductive proof holds. Now, we assume that (A.34) holds for $1 < T < x_0$, $k - x_0 \geq 0$ and then we will show it still holds for $T + 1$. Indeed,

$$
\begin{aligned}
\Sigma_{T+1} &= \sum_{i_{T+1}=1}^{p-k+1} \frac{d_{m_{T+1}(i_{T+1})}}{i_{T+1}!} \Sigma_T \\
&= \sum_{i_{T+1}=1}^{p-k+1} \left\{ \sum_{\omega_{T+1}=1}^{2m_{T+1}} \binom{i_{T+1}}{\omega_{T+1}} \mathbb{1}_{i_{T+1}\geq\omega_{T+1}} Q_{\omega_{T+1}} \frac{1}{i_{T+1}!} \right\} \\
&\cdot \prod_{t=1}^{T} \sum_{j_t=0}^{j_{t-1}+1} (-1)^{j_{t-1}+1-j_t} \binom{j_{t-1}+1}{j_t} \sum_{\omega_t=1}^{2m_t} \frac{Q_{\omega_t}}{\omega_t!} \frac{S_{p_T^*+j_T+1,k-x_0+T}}{p_T^*!} \mathbb{1}_{p_T^*+j_T+1\geq k-x_0+T}.
\end{aligned}
$$

Since $p_T^* + j_T + 1 \geq k - x_0 + T$ is equivalent to $p - \omega_1 - \cdots - \omega_T - i_{T+1} - \cdots - i_{x_0} + j_T + 1 \geq$

$k - x_0 + T$, then

$$\begin{aligned}
\Sigma_{T+1} =& \prod_{t=1}^{T} \sum_{j_t=0}^{j_{t-1}+1} (-1)^{j_{t-1}+1-j_t} \left(\begin{array}{c} j_{t-1}+1 \\ j_t \end{array} \right) \sum_{\omega_t=1}^{2m_t} \frac{Q_{\omega_t}}{\omega_t!} \sum_{\omega_{T+1}=1}^{2m_{T+1}} \frac{Q_{\omega_{T+1}}}{\omega_{T+1}!} \\
& \cdot \sum_{i_{T+1}=1}^{p-k+1} \frac{1}{(i_{T+1}-\omega_{T+1})!} \frac{S_{p_T^*+j_T+1,k-x_0+T}}{p_T^*!} \mathbb{1}_{i_{T+1}\geq \omega_{T+1}} \\
& \qquad \cdot \mathbb{1}_{i_{T+1}\leq p-\omega_1-\cdots-\omega_T-i_{T+2}-\cdots-i_{x_0}+j_T+1-(k-x_0+T)} \\
=& \prod_{t=1}^{T} \sum_{j_t=0}^{j_{t-1}+1} (-1)^{j_{t-1}+1-j_t} \left(\begin{array}{c} j_{t-1}+1 \\ j_t \end{array} \right) \sum_{\omega_t=1}^{2m_t} \frac{Q_{\omega_t}}{\omega_t!} \sum_{\omega_{T+1}=1}^{2m_{T+1}} \frac{Q_{\omega_{T+1}}}{\omega_{T+1}!} \\
& \cdot \frac{1}{p_{T+1}^*!} \sum_{i_{T+1}=\omega_{T+1}}^{\substack{p-\omega_1-\cdots-\omega_T-i_{T+2}-\cdots-i_{x_0}+ \\ +j_T+1-(k-x_0+T)}} \left(\begin{array}{c} p_{T+1}^* \\ p_T^* \end{array} \right) S_{p_T^*+j_T+1,k-x_0+T}.
\end{aligned}$$

Technical Lemma A.2.12 *Let x be given such that $x = 1, 2, \ldots, n-k$, we have*

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \sum_{l=0}^{x} (-1)^{x-l} \left(\begin{array}{c} x \\ l \end{array} \right) \left(\begin{array}{c} n+l \\ k+x \end{array} \right) \mathbb{1}_{x-l\leq n-k}. \tag{A.35}$$

Proof of Technical Lemma A.2.12:
This lemma holds by induction. Indeed, the base step when $x = 1$ is easy to get since

$$\left(\begin{array}{c} n+1 \\ k+1 \end{array} \right) = \left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k+1 \end{array} \right).$$

Now, let us assume (A.35) holds for x, the inductive step also holds for $x+1$ as follows

$$\begin{aligned}
&\sum_{l=0}^{x} (-1)^{x-l} \left(\begin{array}{c} x \\ l \end{array} \right) \left(\begin{array}{c} n+l \\ k+x \end{array} \right) = \sum_{l=0}^{x} (-1)^{x-l} \left(\begin{array}{c} x \\ l \end{array} \right) \left\{ \left(\begin{array}{c} n+l+1 \\ k+x+1 \end{array} \right) - \left(\begin{array}{c} n+l \\ k+x+1 \end{array} \right) \right\} \\
&= \sum_{\tilde{l}=1}^{x+1} (-1)^{x-\tilde{l}+1} \left(\begin{array}{c} x \\ \tilde{l}-1 \end{array} \right) \left(\begin{array}{c} n+\tilde{l} \\ k+x+1 \end{array} \right) + \sum_{l=0}^{x} (-1)^{x-l+1} \left(\begin{array}{c} x \\ l \end{array} \right) \left(\begin{array}{c} n+l \\ k+x+1 \end{array} \right) \\
&= \left(\begin{array}{c} n+x+1 \\ k+x+1 \end{array} \right) + \sum_{l=1}^{x} (-1)^{x-l+1} \left\{ \left(\begin{array}{c} x \\ l-1 \end{array} \right) + \left(\begin{array}{c} x \\ l \end{array} \right) \right\} \left(\begin{array}{c} n+l \\ k+x+1 \end{array} \right) + \\
&\qquad\qquad + (-1)^{x+1} \left(\begin{array}{c} n \\ k+x+1 \end{array} \right) \\
&= \left(\begin{array}{c} n+x+1 \\ k+x+1 \end{array} \right) + \sum_{l=1}^{x} (-1)^{x-l+1} \left(\begin{array}{c} x+1 \\ l \end{array} \right) \left(\begin{array}{c} n+l \\ k+x+1 \end{array} \right) + (-1)^{x+1} \left(\begin{array}{c} n \\ k+x+1 \end{array} \right) \\
&= \sum_{l=0}^{x+1} (-1)^{x-l+1} \left(\begin{array}{c} x+1 \\ l \end{array} \right) \left(\begin{array}{c} n+l \\ k+x+1 \end{array} \right).
\end{aligned}$$

All above transformations are reasonable when $x + 1 - l \leq n - k$.

$\square$

Back to the proof of Proposition A.2.11, by applying formula (A.35), where $x = j_T + 1$ and index l is substituted by j_{T+1}, we can write

$$\binom{p^*_{T+1}}{p^*_T} = \sum_{j_{T+1}=0}^{j_T+1} (-1)^{j_T+1-j_{T+1}} \binom{j_T+1}{j_{T+1}} \binom{p^*_{T+1}+j_{T+1}}{p^*_T+j_T+1} \mathbb{1}_{p^*_{T+1}+j_{T+1}\geq p^*_T+j_T+1},$$

where $p^*_{T+1} + j_{T+1} \geq p^*_T + j_T + 1$ is equivalent to $i_{T+1} \geq \omega_{T+1} + j_T + 1 - j_{T+1}$, then it follows

$$\begin{aligned}
\Sigma_{T+1} &= \prod_{t=1}^{T} \sum_{j_t=0}^{j_{t-1}+1} (-1)^{j_{t-1}+1-j_t} \binom{j_{t-1}+1}{j_t} \sum_{\omega_t=1}^{2m_t} \frac{Q_{\omega_t}}{\omega_t!} \cdot \sum_{\omega_{T+1}=1}^{2m_{T+1}} \frac{Q_{\omega_{T+1}}}{\omega_{T+1}!} \\
&\quad \cdot \sum_{j_{T+1}=0}^{j_T+1} (-1)^{j_T+1-j_{T+1}} \binom{j_T+1}{j_{T+1}} \frac{1}{p^*_{T+1}!} \\
&\quad \cdot \sum_{i_{T+1}=\omega_{T+1}+j_T+1-j_{T+1}}^{\substack{p-\omega_1-\cdots-\omega_T-i_{T+2}-\cdots-i_{x_0}+\\ +j_T+1-(k-x_0+T)}} \binom{p^*_{T+1}+j_{T+1}}{p^*_T+j_T+1} S_{p^*_T+j_T+1,k-x_0+T} \\
&= \prod_{t=1}^{T+1} \sum_{j_t=0}^{j_{t-1}+1} (-1)^{j_{t-1}+1-j_t} \binom{j_{t-1}+1}{j_t} \sum_{\omega_t=1}^{2m_t} \frac{Q_{\omega_t}}{\omega_t!} \frac{1}{p^*_{T+1}!} \\
&\quad \cdot \left\{ \binom{p^*_{T+1}+j_{T+1}}{p^*_{T+1}+j_{T+1}} S_{p^*_{T+1}+j_{T+1},k-x_0+T} + \binom{p^*_{T+1}+j_{T+1}}{p^*_{T+1}+j_{T+1}-1} S_{p^*_{T+1}+j_{T+1}-1,k-x_0+T} + \right. \\
&\qquad \left. + \cdots + \binom{p^*_{T+1}+j_{T+1}}{k-x_0+T} S_{k-x_0+T,k-x_0+T} \right\} \\
&= \prod_{t=1}^{T+1} \sum_{j_t=0}^{j_{t-1}+1} (-1)^{j_{t-1}+1-j_t} \binom{j_{t-1}+1}{j_t} \sum_{\omega_t=1}^{2m_t} \frac{Q_{\omega_t}}{\omega_t!} \frac{1}{p^*_{T+1}!} \\
&\qquad \cdot S_{p^*_{T+1}+j_{T+1}+1,k-x_0+T+1} \mathbb{1}_{p^*_{T+1}+j_{T+1}+1\geq k-x_0+T+1},
\end{aligned}$$

which consequently establishes the proof of formula (A.34).

Remark A.2.13 *For $k - x_0 \geq 0$, we have formula* (A.34). *However, we note that if $x_0 > k$, $S_{p,k-x_0} = 0$ for all $p \geq 0$. Then* (A.34) *still holds for any x_0.*

The entire proof of Proposition A.2.11 is to prove

$$\sum_{k=1}^{p} (-1)^{k+1}(k-1)! \sum_{x_0=1}^{s} \sum_{x_1+\cdots+x_s=x_0} \frac{1}{x_1!x_2!\cdots x_s!} \Sigma_{x_0} = 0. \tag{A.36}$$

It is easy to see that the RHS of above equation, in company with the formula of Σ_{x_0}, implies their equivalence with

$$\sum_{k=1}^{p'} (-1)^{k+1}(k-1)!\, S_{p-\omega_1-\omega_2-\cdots-\omega_{x_0},k} = 0, \tag{A.37}$$

where $p' = p - \omega_1 - \cdots - \omega_{x_0}$. Hence, we finish the proof of Proposition A.2.11.

□

Consequently, Proposition A.2.11 and preceding remark lead us complete the proof of Theorem A.2.22, where $w_s = \sum_{k=1}^{p}(-1)^{k+1}(k-1)!\, h_s$.

A.2.2 Proof of Theorem A.2.23

To study the k-derivatives of $R_p(\lambda)$, according to equation (3.35) of $R_p(\lambda)$, we now study the n-derivatives of the product of functions as follow

Technical Lemma A.2.14 *For given n, we have*

$$\left(\prod_{i=1}^{p+1} f_i(\lambda)\right)^{(n)} = \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_p=0}^{k_{p-1}} \binom{n}{n-k_1, k_1-k_2, \dots k_{p-1}-k_p, k_p} \cdot [f_1(\lambda)]^{(n-k_1)}\,[f_2(\lambda)]^{(k_1-k_2)}\dots[f_p(\lambda)]^{(k_{p-1}-k_p)}\,[f_{p+1}(\lambda)]^{(k_p)}.$$

Proof:
We can prove by induction. Indeed, when $p=1$, according to Leibniz's rule

$$(f_1(\lambda)f_2(\lambda))^{(n)} = \sum_{k_1=0}^{n}\binom{n}{k_1}[f_1(\lambda)]^{(n-k_1)}\,[f_2(\lambda)]^{(k_1)}$$

and the inductive step is proved as follows

$$\begin{aligned}
\left(\prod_{i=1}^{p+2} f_i(\lambda)\right)^{(n)} &= \left(\prod_{i=1}^{p+1} f_i(\lambda)\, f_{p+2}(\lambda)\right)^{(n)}\\
&= \sum_{k=0}^{n}\binom{n}{k}[f_{p+2}(\lambda)]^{(n-k)}\left[\prod_{i=1}^{p+1} f_i(\lambda)\right]^{(k)}\\
&= \sum_{k=0}^{n}\binom{n}{k}[f_{p+2}(\lambda)]^{(n-k)}\sum_{k_1=0}^{k}\cdots\sum_{k_p=0}^{k_{p-1}}\binom{k}{k-k_1,\dots k_{p-1}-k_p, k_p}\\
&\quad\cdot[f_1(\lambda)]^{(k-k_1)}\dots[f_p(\lambda)]^{(k_{p-1}-k_p)}\,[f_{p+1}(\lambda)]^{(k_p)}\\
&= \sum_{k=0}^{n}\sum_{k_1=0}^{k}\cdots\sum_{k_p=0}^{k_{p-1}}\binom{n}{n-k, k-k_1,\dots k_{p-1}-k_p, k_p}\\
&\quad\cdot[f_{p+2}(\lambda)]^{(n-k)}\,[f_1(\lambda)]^{(k-k_1)}\dots[f_p(\lambda)]^{(k_{p-1}-k_p)}\,[f_{p+1}(\lambda)]^{(k_p)}.
\end{aligned}$$

□

We have

$$R_p(\lambda) = \sum_{1\le u\le p}(-1)^{u-1}(u-1)!\sum_{(*)}\frac{p!}{\prod_{i=1}^{p-u+1} j_i!(i!)^{j_i}}\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)^{j_1}\cdots\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)^{j_{p-u+1}}, \tag{A.38}$$

where sequences $j_1, j_2, \ldots, j_{p-u+1}$ of non-negative integers satisfy two conditions

$$(*) \begin{cases} j_1 + j_2 + \cdots + j_{p-u+1} = u \\ j_1 + 2j_2 + \cdots + (p-u+1)j_{p-u+1} = p \end{cases} .$$

then

$$\begin{aligned} R_p^{(k)}(\lambda) &= \sum_{1\le u\le p} (-1)^{u-1}(u-1)! \sum_{(*)} \frac{p!}{\prod_{i=1}^{p-u+1} j_i!(i!)^{j_i}} \frac{d^k}{d\lambda^k} \left\{ \left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)^{j_1} \cdots \left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)^{j_{p-u+1}} \right\} \\ &= \sum_{1\le u\le p} (-1)^{u-1}(u-1)! \sum_{(*)} \frac{p!}{\prod_{i=1}^{p-u+1} j_i!(i!)^{j_i}} \\ &\quad \cdot \sum_{k_1=0}^{k} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{p-u}=0}^{k_{p-u-1}} \binom{k}{k-k_1, k_1-k_2, \ldots, k_{p-u-1}-k_{p-u}, k_{p-u}} \\ &\quad \cdot \left\{ \left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)^{j_1} \right\}^{(k-k_1)} \cdots \left\{ \left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)^{j_{p-u+1}} \right\}^{(k_{p-u})} . \end{aligned}$$

According to Faà di Bruno's formula,

$$\frac{d^k}{d\lambda^k} \{F(G(\lambda))\} = \sum_{s=1}^{k} F^{(s)}(G(\lambda))\ B_{k,s}\left(G'(\lambda), G''(\lambda), \ldots, G^{(k-s+1)}(\lambda)\right).$$

For $F(x) = x^j$ and $G(x) = c_l(\lambda)/c_0(\lambda)$, we have

$$\frac{d^k}{d\lambda^k} \left\{ \left(\frac{c_l(\lambda)}{c_0(\lambda)}\right)^j \right\} = \sum_{s=1}^{k} \mathbb{1}_{j\ge s} \frac{j!}{(j-s)!} \left(\frac{c_l(\lambda)}{c_0(\lambda)}\right)^{j-s} B_{k,s}\left(\left(\frac{c_l(\lambda)}{c_0(\lambda)}\right)', \left(\frac{c_l(\lambda)}{c_0(\lambda)}\right)'', \ldots, \left(\frac{c_l(\lambda)}{c_0(\lambda)}\right)^{(k-s+1)} \right).$$

then apply above formula for $k \ge 0$,

$$\begin{aligned} &\left\{ \left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)^{j_1} \right\}^{(k-k_1)} \cdots \left\{ \left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)^{j_{p-u+1}} \right\}^{(k_{p-u})} \\ &= \sum_{s_1=0}^{k-k_1} \mathbb{1}_{j_1\ge s_1} \frac{j_1!}{(j_1-s_1)!} \left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)^{j_1-s_1} B_{k-k_1,s_1}\left(\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)', \left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)'', \ldots \right) \cdots \\ &\quad \cdot \sum_{s_{p-u+1}=0}^{k_{p-u}} \mathbb{1}_{j_{p-u+1}\ge s_{p-u+1}} \frac{j_{p-u+1}!}{(j_{p-u+1}-s_{p-u+1})!} \left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)^{j_{p-u+1}-s_{p-u+1}} \\ &\quad \cdot B_{k_{p-u},s_{p-u+1}}\left(\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)', \left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)'', \ldots \right). \end{aligned}$$

Hence,

$$
\begin{aligned}
R_p^{(k)}(\lambda) = &\sum_{1\le u\le p}(-1)^{u-1}(u-1)!\sum_{(*)}\frac{p!}{\prod_{i=1}^{p-u+1}(i!)^{j_i}}\\
&\cdot\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{p-u}=0}^{k_{p-u-1}}\binom{k}{k-k_1,k_1-k_2,\ldots,k_{p-u-1}-k_{p-u},k_{p-u}}\\
&\cdot\sum_{s_1=0}^{k-k_1}\sum_{s_2=0}^{k_1-k_2}\cdots\sum_{s_{p-u+1}=0}^{k_{p-u}}\frac{\mathbb{1}_{j_1\ge s_1}\,\mathbb{1}_{j_2\ge s_2}\ldots\mathbb{1}_{j_{p-u+1}\ge s_{p-u+1}}}{(j_1-s_1)!(j_2-s_2)!\ldots(j_{p-u+1}-s_{p-u+1})!}\\
&\cdot\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)^{j_1-s_1}\left(\frac{c_2(\lambda)}{c_0(\lambda)}\right)^{j_2-s_2}\ldots\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)^{j_{p-u+1}-s_{p-u+1}}\\
&\cdot B_{k-k_1,s_1}\left(\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)',\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)'',\ldots\right)B_{k_1-k_2,s_2}\left(\left(\frac{c_2(\lambda)}{c_0(\lambda)}\right)',\left(\frac{c_2(\lambda)}{c_0(\lambda)}\right)'',\ldots\right)\ldots\\
&\cdot B_{k_{p-u},s_{p-u+1}}\left(\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)',\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)'',\ldots\right).
\end{aligned}
$$

The idea of the entire bounding is to define

$$
C_p(\lambda)=\max_{\lambda\in\mathbb{R}}\left\{\left|\frac{c_1(\lambda)}{c_0(\lambda)}\right|,\ldots,\left|\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right|\right\} \tag{A.39}
$$

and

$$
\begin{aligned}
D_p(\lambda)=\max_{\lambda\in\mathbb{R}}&\left\{\left|B_{k-k_1,s_1}\left(\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)',\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)'',\ldots\right)\right|,\ldots,\right.\\
&\left.\left|B_{k_{p-u},s_{p-u+1}}\left(\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)',\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)'',\ldots\right)\right|\right\}.
\end{aligned} \tag{A.40}
$$

The constants to bound $C_p(\lambda)$ and $D_p(\lambda)$ can be found. We will detail constant C_p such that $C_p(\lambda)\le C_p$ in three Technical Lemmas A.2.15, A.2.17, A.2.18. Constant D_p which satisfies $D_p(\lambda)\le D_p$ can be obtained by the similar technique.

We now assume that we have two constant C_p and D_p, then

$$
\begin{aligned}
\left|R_p^{(k)}(\lambda)\right|\le &\sum_{1\le u\le p}(u-1)!\sum_{(*)}\frac{p!}{\prod_{i=1}^{p-u+1}(i!)^{j_i}}\\
&\cdot\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{p-u}=0}^{k_{p-u-1}}\binom{k}{k-k_1,k_1-k_2,\ldots k_{p-1}-k_{p-u-1},k_{p-u-1}}\\
&\cdot\sum_{s_1=0}^{k-k_1}\sum_{s_2=0}^{k_1-k_2}\cdots\sum_{s_{p-u+1}=0}^{k_{p-u}}\frac{\mathbb{1}_{j_1\ge s_1}\,\mathbb{1}_{j_2\ge s_2}\ldots\mathbb{1}_{j_{p-u+1}\ge s_{p-u+1}}}{(j_1-s_1)!(j_2-s_2)!\ldots(j_{p-u+1}-s_{p-u+1})!}\\
&\cdot C_p^{\sum j_i-\sum s_i}\,D_p^{p-u+1}.
\end{aligned}
$$

Since $C_p > 1$,

$$\left|R_p^{(k)}(\lambda)\right| \le \sum_{1\le u\le p} (u-1)!\, C_p^u\, D_p^{p-u+1} \sum_{(*)} \frac{p!}{\prod_{i=1}^{p-u+1} (i!)^{j_i}}$$
$$\cdot \sum_{k_1=0}^{k} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{p-u}=0}^{k_{p-u-1}} \left(\begin{array}{c} k \\ k-k_1, k_1-k_2, \ldots k_{p-1}-k_{p-u-1}, k_{p-u-1} \end{array} \right)$$
$$\cdot \sum_{s_1=0}^{k-k_1} \sum_{s_2=0}^{k_1-k_2} \cdots \sum_{s_{p-u+1}=0}^{k_{p-u}} \frac{\mathbb{1}_{j_1\ge s_1}\, \mathbb{1}_{j_2\ge s_2} \ldots \mathbb{1}_{j_{p-u+1}\ge s_{p-u+1}}}{(j_1-s_1)!(j_2-s_2)!\ldots(j_{p-u+1}-s_{p-u+1})!}.$$

Setting

$$\delta_{k,p} = \sum_{1\le u\le p} (u-1)! \sum_{(*)} \frac{p!}{\prod_{i=1}^{p-u+1} (i!)^{j_i}}$$
$$\cdot \sum_{k_1=0}^{k} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{p-u}=0}^{k_{p-u-1}} \left(\begin{array}{c} k \\ k-k_1, k_1-k_2, \ldots k_{p-1}-k_{p-u-1}, k_{p-u-1} \end{array} \right)$$
$$\cdot \sum_{s_1=0}^{k-k_1} \sum_{s_2=0}^{k_1-k_2} \cdots \sum_{s_{p-u+1}=0}^{k_{p-u}} \frac{\mathbb{1}_{j_1\ge s_1}\, \mathbb{1}_{j_2\ge s_2} \ldots \mathbb{1}_{j_{p-u+1}\ge s_{p-u+1}}}{(j_1-s_1)!(j_2-s_2)!\ldots(j_{p-u+1}-s_{p-u+1})!}, \quad \text{(A.41)}$$

and

$$\Delta_p = \max\{C_p, D_p\}, \quad \text{(A.42)}$$

then we can obtain (A.49) and complete the proof of Theorem A.2.23.

We now present some technical lemmas to prove that there exists constant C_p such that $C_p(\lambda) \le C_p$.

We have $h(r) = \lambda r + \frac{1}{2}\log(1-r^2)$, $h''(r) = -(1+r^2)(1-r^2)^{-2}$. We recall from formula (3.37) that

$$\frac{c_l(\lambda)}{c_0(\lambda)} = \sum_{\alpha=0}^{2l} \left(\begin{array}{c} 2l \\ \alpha \end{array} \right) \frac{g^{(2l-\alpha)}(r_0(\lambda))}{g(r_0(\lambda))}$$
$$\cdot \sum_{\beta=0}^{\alpha} B_{\alpha,\beta} \left(\frac{h^{(3)}(r_0(\lambda))}{2.3.|h''(r_0(\lambda))|}, \ldots, \frac{h^{(\alpha-\beta+3)}(r_0(\lambda))}{(\alpha-\beta+2)(\alpha-\beta+3).|h''(r_0(\lambda))|} \right) \frac{(2\beta+2l-1)!!}{|h''(r_0(\lambda))|^l}. \quad \text{(A.43)}$$

Technical Lemma A.2.15 *For $n \ge 1$*

$$\frac{h^{(n+2)}(r_0(\lambda))}{h''(r_0(\lambda))} = n! \sum_{k\ge(n+1)/2}^{n} 2^{2k-n} \left(\begin{array}{c} k \\ n-k \end{array} \right) \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \left\{ 1 + \frac{2k}{1+r_0^2(\lambda)} \right\} +$$
$$+ n!\, \frac{r_0^2(\lambda)+2s+3}{(1-r_0^2(\lambda))^{s+1}(1+r_0^2(\lambda))}\, \mathbb{1}_{n=2s+2,\, s\ge 0}. \quad \text{(A.44)}$$

Proof of Technical Lemma A.2.15:
Let us express $h''(r) = -(1+r^2)\,g(r)$ then for positive integer n, Leibniz rule gives

$$
\begin{aligned}
& h^{(n+2)}(r) = -\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (1+r^2)^{(n-k)}\, g^{(k)}(r) \\
& = -\left(\begin{array}{c} n \\ n \end{array} \right) (1+r^2)\, g^{(n)}(r) - \left(\begin{array}{c} n \\ n-1 \end{array} \right) (1+r^2)'\, g^{(n-1)}(r) - \left(\begin{array}{c} n \\ n-2 \end{array} \right) (1+r^2)''\, g^{(n-2)}(r)\, \mathbb{1}_{n\geq 2} \\
& = -(1+r^2)\, g^{(n)}(r) - 2nr\, g^{(n-1)}(r) - n(n-1)\, g^{(n-2)}(r)\, \mathbb{1}_{n\geq 2}.
\end{aligned}
$$

Hence,

$$
\frac{h^{(n+2)}(r)}{h''(r)} = \frac{g^{(n)}(r)}{g(r)} + \frac{2nr}{1+r^2}\,\frac{g^{(n-1)}(r)}{g(r)} + \frac{n(n-1)}{1+r^2}\,\frac{g^{(n-2)}(r)}{g(r)}\, \mathbb{1}_{n\geq 2}.
$$

We have

$$
\frac{h^{(3)}(r)}{h''(r)} = \frac{g'(r)}{g(r)} + \frac{2r}{1+r^2} = \frac{4r}{1-r^2} + \frac{2r}{1+r^2}
$$

and we recall formula (A.31) that

$$
\begin{aligned}
\frac{g^{(n)}(r)}{g(r)} &= n! \sum_{k\geq n/2}^{n} \frac{2^{2k-n}(k+1)!}{(2k-n)!\,(n-k)!}\,\frac{r^{2k-n}}{(1-r^2)^k} \\
&= n! \sum_{k\geq n/2}^{n} 2^{2k-n}(k+1) \left(\begin{array}{c} k \\ n-k \end{array} \right) \frac{r^{2k-n}}{(1-r^2)^k}.
\end{aligned}
$$

Then for $n \geq 2$

$$
\begin{aligned}
\frac{h^{(n+2)}(r_0(\lambda))}{h''(r_0(\lambda))} &= n! \sum_{k\geq n/2}^{n} 2^{2k-n}(k+1) \left(\begin{array}{c} k \\ n-k \end{array} \right) \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \\
&\quad + \frac{2nr_0(\lambda)}{1+r_0^2(\lambda)}\,(n-1)! \sum_{k\geq (n-1)/2}^{n-1} 2^{2k-n+1}(k+1) \left(\begin{array}{c} k \\ n-k-1 \end{array} \right) \frac{r_0^{2k-n+1}(\lambda)}{(1-r_0^2(\lambda))^k} \\
&\quad + \frac{n(n-1)}{1+r_0^2(\lambda)}\,(n-2)! \sum_{k\geq (n-2)/2}^{n-2} 2^{2k-n+2}(k+1) \left(\begin{array}{c} k \\ n-k-2 \end{array} \right) \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k}
\end{aligned}
$$

$$
\begin{aligned}
&= n! \sum_{k\geq n/2}^{n} 2^{2k-n}(k+1) \left(\begin{array}{c} k \\ n-k \end{array} \right) \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \\
&\quad + \frac{n!}{1+r_0^2(\lambda)} \sum_{k\geq (n-1)/2}^{n-1} 2^{2k-n+2}(k+1) \left(\begin{array}{c} k \\ n-k-1 \end{array} \right) \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k} \\
&\quad + \frac{n!}{1+r_0^2(\lambda)} \sum_{k\geq (n-2)/2}^{n-2} 2^{2k-n+2}(k+1) \left(\begin{array}{c} k \\ n-k-2 \end{array} \right) \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k}
\end{aligned}
$$

$$= n! \sum_{k \geq n/2}^{n} 2^{2k-n}(k+1) \binom{k}{n-k} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k}$$

$$+ \frac{n!}{1+r_0^2(\lambda)} \Bigg\{ 2^n \, n \binom{n-1}{0} \frac{r_0^n(\lambda)}{(1-r_0^2(\lambda))^{n-1}} + \sum_{k \geq (n-1)/2}^{n-2} 2^{2k-n+2}(k{+}1) \binom{k}{n-k-1} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k}$$

$$+ \sum_{k \geq (n-1)/2}^{n-2} 2^{2k-n+2}(k+1) \binom{k}{n-k-2} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k}$$

$$+ 2^{2k-n+2}(k+1) \binom{k}{n-k-2} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{(n-2)/2 \leq k < (n-1)/2} \Bigg\}$$

$$= n! \sum_{k \geq n/2}^{n} 2^{2k-n}(k+1) \binom{k}{n-k} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k}$$

$$+ \frac{n!}{1+r_0^2(\lambda)} \Bigg\{ 2^n \, n \binom{n-1}{0} \frac{r_0^n(\lambda)}{(1-r_0^2(\lambda))^{n-1}} + \sum_{k \geq (n-1)/2}^{n-2} 2^{2k-n+2}(k{+}1) \binom{k+1}{n-k-1} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k}$$

$$+ 2^{2k-n+2}(k+1) \binom{k}{n-k-2} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{(n-2)/2 \leq k < (n-1)/2} \Bigg\}$$

$$= n! \sum_{k \geq n/2}^{n} 2^{2k-n}(k+1) \binom{k}{n-k} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k}$$

$$+ \frac{n!}{1+r_0^2(\lambda)} \Bigg\{ 2^n \, n \binom{n-1}{0} \frac{r_0^n(\lambda)}{(1-r_0^2(\lambda))^{n-1}} + \sum_{\tilde{k} \geq (n+1)/2}^{n-1} 2^{2\tilde{k}-n} \, \tilde{k} \binom{\tilde{k}}{n-\tilde{k}} \frac{r_0^{2\tilde{k}-n}(\lambda)}{(1-r_0^2(\lambda))^{\tilde{k}-1}}$$

$$+ 2^{2k-n+2}(k+1) \binom{k}{n-k-2} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{(n-2)/2 \leq k < (n-1)/2} \Bigg\}$$

$$= n! \, 2^{2k-n}(k+1) \binom{k}{n-k} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{n/2 \leq k < (n+1)/2}$$

$$+ n! \sum_{k \geq (n+1)/2}^{n} 2^{2k-n}(k+1) \binom{k}{n-k} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} +$$

$$+ \frac{n!}{1+r_0^2(\lambda)} \sum_{\tilde{k} \geq (n+1)/2}^{n} 2^{2\tilde{k}-n} \, \tilde{k} \binom{\tilde{k}}{n-\tilde{k}} \frac{r_0^{2\tilde{k}-n}(\lambda)}{(1-r_0^2(\lambda))^{\tilde{k}-1}} +$$

$$+ \frac{n!}{1+r_0^2(\lambda)} 2^{2k-n+2}(k+1) \binom{k}{n-k-2} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{(n-2)/2 \leq k < (n-1)/2}$$

$$
\begin{aligned}
&= n!\, 2^{2k-n}(k+1) \begin{pmatrix} k \\ n-k \end{pmatrix} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{n/2 \le k < (n+1)/2} \\
&\quad + \frac{n!}{1+r_0^2(\lambda)} 2^{2k-n+2}(k+1) \begin{pmatrix} k \\ n-k-2 \end{pmatrix} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{(n-2)/2 \le k < (n-1)/2} \\
&\qquad + n! \sum_{k \ge (n+1)/2}^{n} 2^{2k-n} \begin{pmatrix} k \\ n-k \end{pmatrix} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \left\{ k+1+k\frac{1-r_0^2(\lambda)}{1+r_0^2(\lambda)} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= n!\, 2^{2k-n}(k+1) \begin{pmatrix} k \\ n-k \end{pmatrix} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{n/2 \le k < (n+1)/2} \\
&\quad + \frac{n!}{1+r_0^2(\lambda)} 2^{2k-n+2}(k+1) \begin{pmatrix} k \\ n-k-2 \end{pmatrix} \frac{r_0^{2k-n+2}(\lambda)}{(1-r_0^2(\lambda))^k} \mathbb{1}_{(n-2)/2 \le k < (n-1)/2} \\
&\qquad + n! \sum_{k \ge (n+1)/2}^{n} 2^{2k-n} \begin{pmatrix} k \\ n-k \end{pmatrix} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \left\{ 1+\frac{2k}{1+r_0^2(\lambda)} \right\}.
\end{aligned}
$$

Remark A.2.16 *We note that if $n = 2s+2$ $(s \ge 0)$, two conditions $n/2 \le k < (n+1)/2$ and $(n-2)/2 \le k < (n-1)/2$ are therefore $k = s+1$ and $k = s$, respectively. If $n = 2s+3$ $(s \ge 0)$, there do not exist any integer k which satisfy these conditions.*

Therefore,

$$
\begin{aligned}
&\frac{h^{(n+2)}(r_0(\lambda))}{h''(r_0(\lambda))} = n! \sum_{k \ge (n+1)/2}^{n} 2^{2k-n} \begin{pmatrix} k \\ n-k \end{pmatrix} \frac{r_0^{2k-n}(\lambda)}{(1-r_0^2(\lambda))^k} \left\{ 1+\frac{2k}{1+r_0^2(\lambda)} \right\} \\
&+n! \left\{ 2^0(s+2) \begin{pmatrix} s+1 \\ s+1 \end{pmatrix} \frac{r_0^0(\lambda)}{(1-r_0^2(\lambda))^{s+1}} + \frac{2^0}{1+r_0^2(\lambda)} (s+1) \begin{pmatrix} s \\ s \end{pmatrix} \frac{r_0^0(\lambda)}{(1-r_0^2(\lambda))^{s}} \right\} \mathbb{1}_{n=2s+2,\, s\ge 0}
\end{aligned}
$$

By simplifying above formula, we get (A.44) and Technical Lemma A.2.15 is proved.

□

Technical Lemma A.2.17 *For $n \ge 1$*

$$
\left| \frac{h^{(n+2)}(r)}{h''(r)} \right| \le n! \frac{2^n}{(1-r^2)^n} M_n, \tag{A.45}
$$

where M_n is the constant defined by $M_n = \sum_{k \ge \frac{n-1}{2}}^{n} \begin{pmatrix} k \\ n-k \end{pmatrix} (2k+1)$. Moreover, $M_{n+1} \ge M_n$ for any $n \ge 1$.

Proof of Technical Lemma A.2.17:
Denote by

$$
\begin{aligned}
H_n(r) := \sum_{k \ge \frac{n+1}{2}}^{n} 2^{2k-n} \begin{pmatrix} k \\ n-k \end{pmatrix} \frac{r^{2k-n}}{(1-r^2)^k} \left\{ \frac{2k}{1+r^2} + 1 \right\} \\
+ \frac{r^2+2s+3}{(1-r^2)^{s+1}(1+r^2)} \mathbb{1}_{n=2s+2,\, s\ge 0},
\end{aligned}
$$

namely, $H_n(r) = n!\,\frac{h^{(n+2)}(r)}{h''(r)}$. Since $|r| \leq 1$ then $0 \leq 1 - r^2 \leq 1$, $|r|^{2k-n} \leq 1$ (if $2k - n \geq 1$) and $(\frac{1}{1-r^2})^k \leq (\frac{1}{1-r^2})^n$ (if $k \leq n$). Therefore,

$$\begin{aligned}
|H_n(r)| &\leq \sum_{k \geq \frac{n+1}{2}}^{n} 2^n \begin{pmatrix} k \\ n-k \end{pmatrix} \frac{1}{(1-r^2)^n}\,(2k+1) + \frac{1}{(1-r^2)^{s+1}}\,\frac{r^2+2s+3}{1+r^2}\,\mathbb{1}_{n=2s+2,\, s\geq 0} \\
&\leq \frac{2^n}{(1-r^2)^n} \sum_{k \geq \frac{n+1}{2}}^{n} \begin{pmatrix} k \\ n-k \end{pmatrix} (2k+1) + \frac{1}{(1-r^2)^{s+1}}\,(2s+3)\,\mathbb{1}_{n=2s+2,\, s\geq 0}.
\end{aligned}$$

Setting

$$\tilde{M}_n := \sum_{k \geq \frac{n+1}{2}}^{n} \begin{pmatrix} k \\ n-k \end{pmatrix} (2k+1),$$

then

$$\begin{aligned}
|H_n(r)| &\leq \frac{2^n}{(1-r^2)^n}\,\tilde{M}_n + \frac{n+1}{(1-r^2)^{[\frac{n}{2}]}}\,\mathbb{1}_{n=2s+2,\, s\geq 0} \\
&\leq \frac{2^n}{(1-r^2)^n} \left\{ \tilde{M}_n + \frac{n+1}{2^n}\,\mathbb{1}_{n=2s+2,\, s\geq 0} \right\} \\
&\leq \frac{2^n}{(1-r^2)^n} \left\{ \tilde{M}_n + (n+1)\,\mathbb{1}_{n=2s+2,\, s\geq 0} \right\}.
\end{aligned}$$

Putting

$$M_n = \sum_{k \geq \frac{n}{2}}^{n} \begin{pmatrix} k \\ n-k \end{pmatrix} (2k+1)$$

then $M_n = \tilde{M}_n + (n+1)\,\mathbb{1}_{n=2s+2,\, s\geq 0}$. Indeed,

$$\begin{aligned}
M_n &= \sum_{k \geq \frac{n+1}{2}}^{n} \begin{pmatrix} k \\ n-k \end{pmatrix} (2k+1) + \begin{pmatrix} k \\ n-k \end{pmatrix} (2k+1)\,\mathbb{1}_{\frac{n+1}{2} > k \geq \frac{n}{2}} \\
&= \tilde{M}_n + \begin{pmatrix} s+1 \\ 0 \end{pmatrix} (2s+3)\,\mathbb{1}_{n=2s+2, s\geq 0}.
\end{aligned}$$

On the other hand, $M_{n+1} \geq M_n$. Indeed, we have

$$M_{n+1} = \sum_{k \geq \frac{n+1}{2}}^{n+1} \begin{pmatrix} k \\ n-k+1 \end{pmatrix} (2k+1) = \sum_{\tilde{k} \geq \frac{n-1}{2}}^{n} \begin{pmatrix} \tilde{k}+1 \\ n-\tilde{k} \end{pmatrix} (2k+3)$$

then

$$M_{n+1} \geq \sum_{\tilde{k} \geq \frac{n}{2}}^{n} \begin{pmatrix} \tilde{k}+1 \\ n-\tilde{k} \end{pmatrix} (2k+3) \geq \sum_{\tilde{k} \geq \frac{n}{2}}^{n} \begin{pmatrix} \tilde{k} \\ n-\tilde{k} \end{pmatrix} (2k+3) \geq M_n.$$

Accordingly, we obtain that

$$|H_n(r)| \leq \frac{2^n}{(1-r^2)^n}\,M_n,$$

which complete the proof of Technical Lemma A.2.17.

$\square$

Technical Lemma A.2.18 *For $l \geq 1$*

$$\left|\frac{c_l(\lambda)}{c_0(\lambda)}\right| \leq N_l, \tag{A.46}$$

where N_l is the constant defined by $N_l = (4l)!\,(2l+1)\,M_{2l}\,(M_{2l+1}+1)^{2l}$. Moreover, $N_{l+1} > N_l$ for any $l \geq 1$.

Proof of Technical Lemma A.2.18:

First of all, let us consider the Bell polynomials in formula (A.43), we have

$$\begin{aligned} &B_{\alpha,\beta}\left(\frac{h^{(3)}(r)}{2.3.|h''(r)|}, \dots, \frac{h^{(\alpha-\beta+3)}(r)}{(\alpha-\beta+2)(\alpha-\beta+3).|h''(r)|}\right) \\ &\qquad = \sum_{(**)} \alpha! \prod_{i=1}^{\alpha-\beta+1} \left(\frac{h^{(i+2)}(r)}{(i+1)(i+2)\,|h''(r)|}\right)^{j_i} \frac{1}{j_i!\,(i!)^{j_i}}, \end{aligned}$$

where sequences $j_1, j_2, \dots, j_{\alpha-\beta+1}$ of non-negative integers satisfy two conditions

$$(**)\left\{\begin{array}{c} j_1 + j_2 + \cdots + j_{\alpha-\beta+1} = \beta \\ j_1 + 2j_2 + \cdots + (\alpha-\beta+1)j_{\alpha-\beta+1} = \alpha \end{array}\right. .$$

Then

$$\begin{aligned} &B_{\alpha,\beta}\left(\frac{h^{(3)}(r)}{2.3.|h''(r)|}, \dots, \frac{h^{(\alpha-\beta+3)}(r)}{(\alpha-\beta+2)(\alpha-\beta+3).|h''(r)|}\right) \\ &\qquad = \alpha! \sum_{(**)} \prod_{i=1}^{\alpha-\beta+1} \left(\frac{(-1)\,i!\,H_i(r)}{(i+1)(i+2)}\right)^{j_i} \frac{1}{j_i!\,(i!)^{j_i}} \\ &\qquad = (-1)^{\beta}\,\alpha! \sum_{(**)} \prod_{i=1}^{\alpha-\beta+1} \left(\frac{H_i(r)}{(i+1)(i+2)}\right)^{j_i} \frac{1}{j_i!}. \end{aligned}$$

Remark A.2.19 (Double factorial or semi-factorial of an odd number) *We recall again the notation !! in Section 2.3 that for $n \geq 1$,*

$$(2n-1)!! = 1.3.5\dots.(2n-1) = \frac{(2n)!}{2^n\,n!},$$

then it is easy to see that $(2n-1)!!$ is increasing in n. Indeed,

$$(2(n+1)-1)!! = \frac{(2n+2)!}{2^{n+1}\,(n+1)!} = \frac{(2n+2)(2n+1)}{2\,(n+1)}\,\frac{(2n)!}{2^n\,n!} \geq (2n-1)!!.$$

Therefore,

$$\begin{aligned}
&\left|\sum_{\beta=0}^{\alpha}(2\beta+2l-1)!!\,B_{\alpha,\beta}\left(\frac{h^{(3)}(r)}{2.3.|h''(r)|},\ldots,\frac{h^{(\alpha-\beta+3)}(r)}{(\alpha-\beta+2)(\alpha-\beta+3).|h''(r)|}\right)\right| \\
&\qquad = \left|\sum_{\beta=0}^{\alpha}(2\beta+2l-1)!!\,(-1)^{\beta}\,\alpha!\sum_{(**)}\prod_{i=1}^{\alpha-\beta+1}\left(\frac{H_i(r)}{(i+1)(i+2)}\right)^{j_i}\frac{1}{j_i!}\right| \\
&\qquad \le \sum_{\beta=0}^{\alpha}(2\alpha+2l-1)!!\,\alpha!\sum_{(**)}\prod_{i=1}^{\alpha-\beta+1}(H_i(r))^{j_i} \\
&\qquad \le (2\alpha+2l-1)!!\,\alpha!\sum_{\beta=0}^{\alpha}\sum_{(**)}\prod_{i=1}^{\alpha-\beta+1}\left(\frac{2^i}{(1-r^2)^i}\,M_i\right)^{j_i} \\
&\qquad \le (2\alpha+2l-1)!!\,\alpha!\,\frac{2^{\alpha}}{(1-r^2)^{\alpha}}\sum_{\beta=0}^{\alpha}\sum_{(**)}M_{\alpha-\beta+1}^{\beta}.
\end{aligned}$$

Remark A.2.20 (The stars and bars method (see more [29])) *For any pair of positive integers n and k, the number of k-tuples of non-negative integers whose sum is n is equal to the number of multi sets of cardinality $k-1$ taken from a set of size $n+1$. Namely, this number is given by the binomial coefficient*

$$\left(\!\!\binom{n+1}{k-1}\!\!\right) := \binom{n+k-1}{n}.$$

Accordingly, when $n := \beta$ and $k := \alpha-\beta+1$, the number of selected sequences $j_1, j_2, \ldots, j_{\alpha-\beta+1}$ which satisfy $(**)$ is equal or less than

$$\binom{\alpha}{\beta}.$$

Hence,

$$\begin{aligned}
&\left|\sum_{\beta=0}^{\alpha}(2\beta+2l-1)!!\,B_{\alpha,\beta}\left(\frac{h^{(3)}(r)}{2.3.|h''(r)|},\ldots,\frac{h^{(\alpha-\beta+3)}(r)}{(\alpha-\beta+2)(\alpha-\beta+3).|h''(r)|}\right)\right| \\
&\qquad \le (2\alpha+2l-1)!!\,\alpha!\,\frac{2^{\alpha}}{(1-r^2)^{\alpha}}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}M_{\alpha-\beta+1}^{\beta} \\
&\qquad \le (2\alpha+2l-1)!!\,\alpha!\,\frac{2^{\alpha}}{(1-r^2)^{\alpha}}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}M_{\alpha+1}^{\beta} \\
&\qquad \le (2\alpha+2l-1)!!\,\alpha!\,\frac{2^{\alpha}}{(1-r^2)^{\alpha}}\,(M_{\alpha+1}+1)^{\alpha}.
\end{aligned}$$

Next, let us bound $\frac{g^{(2l-\alpha)}(r)}{g(r)}$ in formula (A.43). According to formula (A.31), we can obtain

$$
\begin{aligned}
\left|\frac{g^{(2l-\alpha)}(r)}{g(r)}\right| &= \left|(2l-\alpha)! \sum_{k\geq(2l-\alpha)/2}^{2l-\alpha} 2^{2k-(2l-\alpha)}(k+1)\begin{pmatrix} k \\ 2l-\alpha-k\end{pmatrix} \frac{r^{2k-(2l-\alpha)}}{(1-r^2)^k}\right| \\
&\leq \frac{(2l-\alpha)!}{(1-r^2)^{2l-\alpha}} \sum_{k\geq(2l-\alpha)/2}^{2l-\alpha} 2^{2l-\alpha}\,(k+1)\begin{pmatrix} k \\ 2l-\alpha-k\end{pmatrix} \\
&\leq \frac{(2l-\alpha)!}{(1-r^2)^{2l-\alpha}}\, 2^{2l-\alpha} \sum_{k\geq(2l-\alpha)/2}^{2l-\alpha} (2k+1)\begin{pmatrix} k \\ 2l-\alpha-k\end{pmatrix} \\
&= \frac{(2l-\alpha)!}{(1-r^2)^{2l-\alpha}}\, 2^{2l-\alpha}\, M_{2l-\alpha}.
\end{aligned}
$$

We now consider the bounding of $\frac{c_l(\lambda)}{c_0(\lambda)}$, in which, variable r is now substituted by $r_0(\lambda)$. We have

$$
\begin{aligned}
&\left|\frac{c_l(\lambda)}{c_0(\lambda)}\right| \leq \frac{1}{|h''(r_0(\lambda))|^l} \sum_{\alpha=0}^{2l} \frac{(2l)!}{\alpha!\,(2l-\alpha)!} \left|\frac{g^{(2l-\alpha)}(r_0(\lambda))}{g(r_0(\lambda))}\right| \\
&\cdot \left|\sum_{\beta=0}^{\alpha}(2\beta+2l-1)!!\, B_{\alpha,\beta}\left(\frac{h^{(3)}(r_0(\lambda))}{2.3.|h''(r_0(\lambda))|},\dots,\frac{h^{(\alpha-\beta+3)}(r_0(\lambda))}{(\alpha-\beta+2)(\alpha-\beta+3).|h''(r_0(\lambda))|}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq \left(\frac{(1-r_0^2(\lambda))^2}{1+r_0^2(\lambda)}\right)^l \sum_{\alpha=0}^{2l} &\frac{(2l)!}{\alpha!\,(2l-\alpha)!}\frac{(2l-\alpha)!}{(1-r_0^2(\lambda))^{2l-\alpha}}\, 2^{2l-\alpha}\, M_{2l-\alpha} \\
&\cdot (2\alpha+2l-1)!!\,\alpha!\,\frac{2^\alpha}{(1-r_0^2(\lambda))^\alpha}\,\left(M_{\alpha+1}+1\right)^\alpha \\
&\leq \left(\frac{1}{1+r_0^2(\lambda)}\right)^l \sum_{\alpha=0}^{2l}(2l)!\,2^{2l}\,M_{2l-\alpha}\,(2\alpha+2l-1)!!\,\left(M_{\alpha+1}+1\right)^\alpha .
\end{aligned}
$$

Remark again the increasing of double factorial, we have

$$
(4l-1)!! = \frac{(4l)!}{(2l)!\,2^{2l}}\,,
$$

so we can obtain

$$
\begin{aligned}
\left|\frac{c_l(\lambda)}{c_0(\lambda)}\right| &\leq (2l)!\,2^{2l} \sum_{\alpha=0}^{2l} M_{2l}\,(4l-1)!!\,\left(M_{2l+1}+1\right)^\alpha \\
&\leq (4l)! \sum_{\alpha=0}^{2l} M_{2l}\,\left(M_{2l+1}+1\right)^{2l} \\
&\leq (4l)!\,(2l+1)\,M_{2l}\,\left(M_{2l+1}+1\right)^{2l} .
\end{aligned}
$$

Putting $N_l = (4l)!\,(2l{+}1)\,M_{2l}\,(M_{2l+1}{+}1)^{2l}$, we can imply that $N_{l+1} > N_l$ by the increasing of sequence $\{M_n\}_n$.

□

Remark A.2.21 *i) Constant C_p equals to N_{p-u+1}.*

ii) To bound $D_p(\lambda)$, we can compute the k-th derivative of $\frac{g^{(2l-\alpha)}(r)}{g(r)}$ (formula (A.31)*) and $\frac{h^{(n+2)}(r_0(\lambda))}{h''(r_0(\lambda))}$ (formula* (A.44)*) respect to λ.*

Résumé

Le Chapitre 5 est un appendice consacré à deux calculs techniques de combinatoire. Soit L_n la fonction génératrice des cumulants normalisée dans chacun des cas des Chapitres 3 et 4. On montre que

$$L_n^{(k)}(\lambda) = L^{(k)}(\lambda) + R_0^{(k)}(\lambda) + \frac{1}{n}\sum_{p\geq 1}\frac{R_p^{(k)}(\lambda)}{n^p p!}.$$

où les R_k sont donnés par la méthode de Laplace. On montre les deux résultats suivants, dans le cas des variables sphériques:

Theorem A.2.22 *Pour tout $p \geq 3$, $nL_n^{(p)}(\lambda)$ peut être développé comme suit:*

$$nL_n^{(p)}(\lambda) = (nr_0(\lambda))^p \sum_{s>p} w_s n^{-s}, \tag{A.47}$$

avec

$$w_s = 0, \qquad \textit{pour tout } s = 0, 1, 2\ldots, p. \tag{A.48}$$

Theorem A.2.23 *Pour $k = 1, 2, \ldots,$ on a*

$$\left|R_p^{(k)}(\lambda)\right| \leq \delta_{k,p}\,\Delta_p^{p+1}, \tag{A.49}$$

où les constantes $\delta_{k,p}$ and Δ_p sont données par

$$\begin{aligned}\delta_{k,p} = &\sum_{1\leq u\leq p}(u-1)!\sum_{(*)}\frac{p!}{\prod_{i=1}^{p-u+1}(i!)^{j_i}}\\ &\cdot\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{p-u}=0}^{k_{p-u-1}}\binom{k}{k-k_1, k_1-k_2, \ldots k_{p-1}-k_{p-u-1}, k_{p-u-1}}\\ &\cdot\sum_{s_1=0}^{k-k_1}\sum_{s_2=0}^{k_1-k_2}\cdots\sum_{s_{p-u+1}=0}^{k_{p-u}}\frac{\mathbb{1}_{j_1\geq s_1}\mathbb{1}_{j_2\geq s_2}\ldots\mathbb{1}_{j_{p-u+1}\geq s_{p-u+1}}}{(j_1-s_1)!(j_2-s_2)!\ldots(j_{p-u+1}-s_{p-u+1})!},\end{aligned} \tag{A.50}$$

et

$$\Delta_p = \max\{C_p, D_p\}, \tag{A.51}$$

avec

$$C_p(\lambda) = \max_{\lambda\in\mathbb{R}}\left\{\left|\frac{c_1(\lambda)}{c_0(\lambda)}\right|, \ldots, \left|\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right|\right\} \tag{A.52}$$

et

$$\begin{aligned}D_p(\lambda) = \max_{\lambda\in\mathbb{R}}\Bigg\{&\left|B_{k-k_1,s_1}\left(\left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)', \left(\frac{c_1(\lambda)}{c_0(\lambda)}\right)'', \ldots\right)\right|, \ldots,\\ &\left|B_{k_{p-u},s_{p-u+1}}\left(\left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)', \left(\frac{c_{p-u+1}(\lambda)}{c_0(\lambda)}\right)'', \ldots\right)\right|\Bigg\}.\end{aligned} \tag{A.53}$$

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Thi Kim Tien TRUONG

Grandes déviations précises pour des statistiques de test

Résumé :
Cette thèse concerne l'étude de grandes déviations précises pour deux statistiques de test: le coefficient de corrélation empirique de Pearson et la statistique de Moran.
Les deux premiers chapitres sont consacrés à des rappels sur les grandes déviations précises et sur la méthode de Laplace qui seront utilisés par la suite. Par la suite, nous étudions les grandes déviations précises pour des coefficients de Pearson empiriques qui sont définis par: $r_n = \sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)/\sqrt{\sum_{i=1}(X_i - \bar{X}_n)^2 \sum_{i=1}(Y_i - \bar{Y}_n)^2}$ ou, quand les espérances sont connues, $\tilde{r}_n = \sum_{i=1}^n (X_i - \mathbb{E}(X))(Y_i - \mathbb{E}(Y))/\sqrt{\sum_{i=1}(X_i - \mathbb{E}(X))^2 \sum_{i=1}(Y_i - \mathbb{E}(Y))^2}$. Notre cadre est celui d'échantillons (X_i, Y_i) ayant une distribution sphérique ou une distribution gaussienne. Dans chaque cas, le schéma de preuve suit celui de Bercu et al. Par la suite, nous considérons la statistique de Moran $T_n = \frac{1}{n}\sum_{k=1}^n \log \frac{X_i}{\bar{X}_n} + \gamma$, où γ est la constante d' Euler. Enfin l'appendice est consacré aux preuves de résultats techniques.

Mots clés : grandes déviations précises, coefficient de correlation de Pearson, test de Moran, statistiques auto-normalisées, méthode de Laplace.

Sharp Large Deviations for some Test Statistics

Abstract:
This thesis focuses on the study of Sharp large deviations (SLD) for two test statistics: the Pearson's empirical correlation coefficient and the Moran statistic.
The two first chapters aim to recall general results on SLD principles and Laplace's methods used in the sequel. Then we study the SLD of empirical Pearson coefficients, namely $r_n = \sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)/\sqrt{\sum_{i=1}(X_i - \bar{X}_n)^2 \sum_{i=1}(Y_i - \bar{Y}_n)^2}$ and when the means are known, $\tilde{r}_n = \sum_{i=1}^n (X_i - \mathbb{E}(X))(Y_i - \mathbb{E}(Y))/\sqrt{\sum_{i=1}(X_i - \mathbb{E}(X))^2 \sum_{i=1}(Y_i - \mathbb{E}(Y))^2}$. Our framework takes place in two cases of random sample (X_i, Y_i): spherical distribution and Gaussian distribution. In each case, we follow the scheme of Bercu et al. Next, we state SLD for the Moran statistic $T_n = \frac{1}{n}\sum_{k=1}^n \log \frac{X_i}{\bar{X}_n} + \gamma$, where γ is the Euler constant. Finally the appendix is devoted to some technical results.

Keywords : Sharp Large Deviations, Pearson correlation coefficient, Moran test, Self-normalized statistics, Laplace's method.

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