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Sous-variétés spéciales des espaces homogènes

Special subvarieties of homogeneous spaces

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Contents

List of Figures

List of Tables

Introduction

Aim

This thesis deals with the problem of constructing interesting complex algebraic varieties. Our main goal will be to construct and study new special varieties inside homogeneous spaces.

By special varieties we mean either Fano varieties or varieties with trivial canonical bundle. As opposed to varieties of general type, these objects are not so common. On the one hand, for each fixed dimension, it is known that there is only a finite number of families of Fano manifolds; on the other hand, varieties with trivial canonical bundle can be divided essentially in three classes, which are interesting for different reasons: tori, Calabi-Yau (CY) and hyper-Kähler (HK) manifolds.

A great effort has been put in the search and analysis of these objects. Since there exists finitely many families of Fano manifolds, one could ask for their classification: this has been found in dimension 1, 2 and 3 thanks to the works of Del Pezzo, Iskovskikh, Mori and Mukai, among others. The next step, i.e. dimension 4, is being attacked by several research groups (for instance, see the project *Classification, Computation, and Construction: New Methods in Geometry*, or *3C in G*).

The scarcity of the known families of hyper-Kähler manifolds contrasts with their importance among varieties with trivial canonical bundle. Not only do we know essentially only four kinds of such manifolds (Hilbert schemes of points on $K3$ surfaces, generalized Kummer varieties, and the two O'Grady's exceptional examples); but also, we lack explicit descriptions of a general member of these families. Finally, let us mention that for Calabi-Yau manifolds, mirror symmetry is still a quite mysterious phenomenon. Being able to test it on more examples (in particular non toric examples) could help to understand it better.

Motivation

There are interesting connections between special varieties and homogeneous spaces. A striking evidence of this relation is the work of Mukai about $K3$ surfaces and Fano threefolds (as a reference, see [Muk88]). Mukai was able to reinterpret families of Fano threefolds as families of subvarieties of homogeneous spaces. His idea was that on a sufficiently general Fano threefold X of a certain type, one can sometimes prove the existence of a special vector bundle. One gets a morphism, and eventually an embedding, from X to a certain Grassmannian. It turned out that all the families of prime Fano threefolds of degree greater than

10 (for which a classification was already known) admitted a nice description in terms of homogeneous bundles over Grassmannians. This description of the families was helpful to understand better their geometry in relation to the well known geometry of Grassmannians.

Mukai also used the vector bundle method to prove the unirationality of some moduli spaces of polarized $K3$ surfaces with small genus. Indeed, via the same construction, he proved that the general surface is the zero locus of a section of a homogeneous bundle; as a consequence, an open subset of the corresponding moduli space is dominated by an open subset of the space of sections of the bundle ([Muk88], [Muk06], and [Muk92]).

From then, other works have showed the interest of homogeneous spaces in providing examples of special varieties. For instance, it is a difficult problem to provide explicit locally complete families of hyper-Kähler manifolds, and very few are known; among them, two can be seen as varieties of zeroes of a general global section of a homogeneous vector bundle over a Grassmannian; both are families of fourfolds. The first one is the family of varieties of lines in a cubic fourfold, due to Beauville and Donagi ([BD85]). The variety of lines in a cubic fourfold is actually a subvariety of the Grassmannian $\mathrm{Gr}(2,6)$ of projective lines on a projective space of dimension 5. It is the zero locus of a section of the third symmetric power of the dual of the tautological bundle. We denote this family of varieties by X_1. The second one, more recent, is due to Debarre and Voisin ([DV10]). They start from the Grassmannian $\mathrm{Gr}(6,10)$ of 6-dimensional planes in a vector space V of dimension 10, and consider a general skew symmetric 3-form over V. The variety of planes isotropic with respect to this form is the zero locus of a section (which corresponds to the form) of the third anti-symmetric power of the dual of the tautological bundle. They prove that this is a locally complete family, which we denote by X_2, of fourfolds which are hyper-Kähler.

In this thesis we essentially use two methods to obtain special subvarieties of homogeneous spaces. We begin by considering zero loci inside Grassmannians, and then we use the more general construction of orbital degeneracy loci. Finally, the study of bisymplectic Grassmannians will show how we can use the structure of homogeneous spaces in order to study their special subvarieties; this subject would deserve to be investigated further.

Zero loci inside Grassmannians

In Chapter 2 we construct special varieties as zero loci of sections of homogeneous vector bundles. Most of the results in this Chapter can be found in [Ben18].

The two examples X_1 and X_2 were our main motivation. Indeed, we study fourfolds which arise as zero loci of general global sections of homogeneous, completely reducible bundles over ordinary, classical and exceptional Grassmannians (see Definition 1.1.1). We will see that the only hyper-Kähler varieties of this form are those already mentioned; indeed, the following theorem holds:

Theorem 0.0.1. *Suppose* Y *is a hyper-Kähler fourfold which is the zero locus of a general section of a homogeneous, completely reducible, globally generated vector bundle over an ordinary or classical (symplectic or orthogonal) Grassmannian. Then either Y is of type X_1 or of type X_2.*

This theorem will be a direct consequence of the classification theorems contained in Chapter 2. For ordinary Grassmannians, we followed the analogous study done in [Küc95], where the author has classified and then studied the properties of Fano fourfolds with index one obtained in the same way. Already in that case the two constraints for the varieties to be four dimensional and Fano of index one were sufficient to have a classification of the bundles which could give rise to the required varieties.

In Section 2.1, we will generalize this result by giving a classification of fourfolds with trivial canonical bundle (Theorem 2.1.1), following substantially the same ideas and proofs. With the help of the MACAULAY2-package SCHUBERT2 ([GS]) we will determine which subvarieties are Calabi-Yau (CY) and which are irreducible holomorphic symplectic (IHS, which is the same as hyper-Kähler, or HK) among the examples we have found.

The two examples X_1 and X_2 share an additional feature: the homogeneous bundles defining them are both irreducible. In Section 2.1.3 we were able to prove the following result, which holds in any dimension:

Theorem (Theorem 2.1.20). *Let Y be the zero locus of a general section of an irreducible homogeneous bundle over the ordinary Grassmannian. If Y is hyper-Kähler, then it must have dimension 4 and it is of type X_1 or X_2.*

In Section 2.2, we will extend the classification to subvarieties of dimension 4 of the other classical Grassmannians. It should be remarked that, even though the symplectic and orthogonal Grassmannians can already be seen as varieties of zeroes of sections of homogeneous bundles over the ordinary Grassmannian, a new classification needs to be done. In fact, there exist homogeneous bundles over the classical Grassmannians that are not restriction of bundles over the ordinary ones. For instance, the orthogonal of the tautological bundle is not irreducible, and one can quotient it by the tautological bundle. Also, the spin bundles in the orthogonal case do not extend to a bundle on the ordinary Grassmannian.

We will then present the corresponding results for dimension 2 and 3 (Section 2.3). We give the classification for surfaces and threefolds and, for the surfaces, we report also the computation of the degree, which gives the genus of the natural polarization of the surface, and the Euler characteristic. Surprisingly enough, for surfaces, there are many more cases than those considered by Mukai in his work on $K3$ surfaces, and they would be worth being studied thoroughly. We begin doing so in Section 2.3.1 for two threefolds, and in Section 2.3.2 for some maximal families of $K3$ surfaces with Picard number two, but clearly there is much more to investigate. For instance, we identify a locally complete family of $K3$ surfaces with Picard number two (labelled by $(oe9)$ in Section 2.3.2) which admits two quadric projections and whose geometry deserves a more detailed analysis.

We remark that a result similar to Theorem 0.0.1 holds for exceptional Grassmannians, which are quotients of exceptional groups G by parabolic subgroups P_i associated to one simple root α_i. This is a consequence of the analogous exceptional classification given by Theorem 2.4.3. However, in the exceptional situation, for three cases (two subvarieties in E_6/P_5 and one in E_7/P_1) we were not able to determine if the varieties were CY or HK; indeed, the computations with MACAULAY2 were too heavy.

As a final remark, let us point out that the classifications given in this paper give for free the analogous classifications for Fano threefolds, fourfolds and fivefolds (see Remark 2.1.2).

As we were finishing to write down of the article [Ben18], an article by D. Inoue, A. Ito and M. Miura on the same subject was published on arXiv ([IIM16]). In this work the authors prove that, under the same hypothesis as ours, a finite classification is possible for subvarieties of the ordinary Grassmannian with trivial canonical bundle of any fixed dimension (see Theorem 2.1.3). They also study in more detail the case of CY threefolds, giving an explicit classification similar to ours and studying the cases found. On the other hand, they do not deal with the cases of symplectic, orthogonal and exceptional Grassmannians, which is interesting too (for example, see Mukai's articles [Muk88] and [Muk92], in particular the sections on $K3$ surfaces of genus seven and eighteen).

Orbital degeneracy loci

The second method we use to construct varieties is through orbital degeneracy loci (ODL). These are generalizations of the classical degeneracy loci of morphisms between vector bundles, which have already been used to produce examples of special varieties in the literature (see for example [KK10]). They are modelled on a given, fixed, closed stable subvariety Y of a representation V of an algebraic group G.

The study of ODL is a joint project with Sara Angela Filippini, Laurent Manivel and Fabio Tanturri that led to the production of two articles [BFMT17a] and [BFMT17b], in which their fundamental properties are collected. In the first two sections of Chapter 3 we summarize these properties for both classical and orbital degeneracy loci. In this thesis, we decided to illustrate the ODL method by describing three families of hyper-Kähler fourfolds, which is a non published work; a supplementary new contribution to the study of ODL is given by the study of quivers in Section 3.3 and Section 3.4, whose results can be found in [Ben17].

The first example of ODL is given by zero loci of sections (where $Y = \{0\} \subset V$), but this example is not rich enough to understand the subtleties of ODL. When Y is a determinantal variety inside a space of matrices, we recover the classical degeneracy loci of a morphism between vector bundles. In the following we outline an example that can be thought of as a prototype of ODL, which allows us to give an idea of what these objects resemble to. Let us consider the GL_6-orbit closure $Y \subset V \cong \wedge^3\mathbb{C}^6$ of partially decomposable tensors, i.e. tensors that can be written as $v \wedge \sigma$, where $v \in \mathbb{C}^6$ and $\sigma \in \wedge^2\mathbb{C}^6$. The corresponding ODL is the *relativization* of Y inside an ambient variety X. The relativization of V over X is the vector bundle $\wedge^3 E_6$, where E_6 is a vector bundle of rank six over X. Given a section s of $\wedge^3 E_6$, the ODL $D_Y(s)$ is defined as the locus of points $x \in X$ such that $s(x)$ is a partially decomposable tensor. Here we have used the fact that the fiber $(\wedge^3 E_6)_x$ is naturally isomorphic to V.

When the group acting on V is more complicated, one can still define a vector bundle whose fiber is isomorphic to V by using a G-principal bundle over X; the general construction is described in Chapter 3. In order to define an ODL, one needs a section s of a certain vector bundle endowed with some G-structure over an ambient variety X (as we have seen for partially decomposable tensors); if the bundle is globally generated and the section general, an assumption we will always make, one can control the codimension and the singular locus of the ODL through a Bertini type theorem. Indeed, as their *models* are usually singular, ODL are as well singular in general.

In fact, the codimension inside V and the dimension of the singular locus are not the only features of the stable subvariety Y that one is able to relativize to orbital degeneracy loci. The two main ones are the following:

1. a desingularization of Y by a *Kempf collapsing*, when it exists;
2. the minimal G-equivariant free resolution of $\mathcal{O}_Y$.

We will see that Kempf collapsings are a special kind of desingularizations that can be used to obtain free resolutions of $\mathcal{O}_Y$; this is done by using the so called geometric technique, for which a reference is [Wey03] (and which is recalled in Section 1.4). Moreover, for a large class of orbit closures, this kind of desingularizations exists. For instance, they can be constructed for orbit closures inside prehomogeneous spaces (whose definition is recalled in Section 1.4). In [BFMT17b] we analyse thoroughly the case of prehomogeneous parabolic representations, which have the advantage of containing a finite number of orbits. The two properties of Y we mentioned respectively give for ODL:

1. a desingularization of $D_Y(s)$ by the zero locus $\mathscr{Z}(\tilde{s})$ of a section $\tilde{s}$ constructed from s ([BFMT17a, Proposition 2.3]);
2. if Y is Cohen-Macaulay, a locally free resolution of $\mathcal{O}_{D_Y(s)}$ (Theorem 3.2.20).

For our purposes however, this is not enough: in order to construct special varieties, it is essential to be able to control the canonical bundle of ODL. Both constructions 1. and 2. can be used independently to understand what $K_{D_Y(s)}$ is:

Proposition (Proposition 3.2.12). *Suppose that Y has rational singularities and admits a* crepant *Kempf collapsing, i.e. a Kempf collapsing which is a desingularization with trivial relative canonical bundle. Then the resolution $\mathscr{Z}(\tilde{s}) \to D_Y(s)$ is crepant as well, and the canonical bundle of $D_Y(s)$ is the restriction of some line bundle over X.*

Proposition (Corollary 3.2.23). *Suppose that Y has Cohen-Macaulay and Gorenstein ring; then the last term of the induced locally free resolution of $\mathcal{O}_{D_Y(s)}$ is of rank one. As a consequence, $K_{D_Y(s)}$ is the restriction of some line bundle over X.*

In both cases, the line bundle over X whose restriction gives $K_{D_Y(s)}$ can be computed explicitly, and these two tools are effective methods that can be used in a complementary way.

Using these techniques, we have been able to construct many special varieties as ODL. We obtained several dozens Calabi-Yau varieties of dimension 4 from the orbit closure of partially decomposable tensors, from nilpotent orbit closures ([BFMT17a]) and from other orbit closures in parabolic representations ([BFMT17b]), as well as five Calabi-Yau threefolds ([BFMT17a]); in [BFMT17a] we also constructed and analysed several examples of (almost) Fano threefolds and fourfolds. Moreover, we were able to determine the main invariants of the varieties we found through some non trivial computations in cohomology. In Section 3.2.1 we give a sample of those computations by obtaining the Hodge numbers for the Fano fourfolds we found in [BFMT17a].

As a concrete illustration of the theory developed for ODL, and in order to show that they can produce very nice constructions, we study three families of Hilbert schemes of two points on $K3$ surfaces. We start in each case from a $K3$ surface inside a (ordinary, classical) Grassmannian. As we want to construct the Hilbert scheme of length two subschemes, we need to consider two points inside the $K3$ surface, so two points inside the Grassmannian. From such a couple of points seen as linear subspaces, by taking their union (or intersection depending on the geometry of the problem), we obtain a point of another Grassmannian $\mathrm{Gr}(k, n)$. We prove that in the three cases the Hilbert scheme of two points on the $K3$ surface can be obtained by this naive procedure.

Actually, as we always consider a $K3$ surface which is a complete intersection of codimension m, we are able to show that the Hilbert scheme can be seen as a degeneracy locus $D_Y(s)$ of a morphism between vector bundles (see Section 3.1). Its desingularization $\mathscr{Z}(\tilde{s})$ turns out to live inside a product $\mathrm{Gr}(k, n) \times \mathrm{Gr}(k', m)$. We then analyze in detail the second projection towards $\mathrm{Gr}(k', m)$, whose image we prove to be an orbital degeneracy locus constructed from a parabolic representation (see Proposition 3.2.27, 3.2.30 and 3.2.32). We then get the chain of isomorphisms

$$\text{Hilbert scheme of two points on a } K3 \leftrightarrow \text{classical degeneracy locus} \leftrightarrow \text{ODL}$$

which permits to see the Hilbert scheme as an ODL and, in this way, to enlighten some interesting geometric aspects of the HK fourfold considered. Even though these constructions do not give new examples of hyper-Kähler varieties, in our opinion they indicate that there is a chance that searching for HK varieties among ODL may eventually be successful.

In Section 3.3 we study orbit closures inside quiver representations. We chose to analyze the case of quiver orbits because of their nice properties. In particular, when the quivers are of finite type, a complete description of such orbits is known ([Gab72]). Moreover, some aspects of the orbit closures have already been studied, for instance by Sutar in [Sut13] or in her PhD thesis. By using the geometric technique and Reineke's resolutions of singularities, Sutar is able to obtain a locally free resolution of their ideals. She works with a certain quiver of type A_3, and then she extends her results to source-sink quivers of finite type (e.g. of type E). Moreover, from the locally free resolution it is possible to extract further geometric informations, which include whether the orbit closures are normal, Cohen-Macaulay or Gorenstein; in the A_3 case, for example, Sutar determines which orbit closures are Gorenstein.

After recalling basic facts about quiver representations, we explain Reineke's construction of resolutions of singularities of quiver orbit closures. We discuss when those resolutions are crepant for quivers of type A_3, one-way and source-sink quivers of type A_n, and a quiver of type D_4. In the A_3 case we are able to prove that the Gorenstein property studied by Sutar is actually a consequence of the generally stronger condition of admitting a crepant desingularization. Finally, some quiver degeneracy loci fourfolds (and one threefold) with trivial canonical bundle are constructed in Section 3.4 using the results of the previous section. The varieties constructed are just a sample of what is possible to achieve by using these techniques.

To sum up, in Chapter 2 and Chapter 3 we have been able to construct many special varieties inside homogeneous spaces. The methods we have used are very general and leave the door open to searching for other kinds of varieties; for example, one could try to construct curves or surfaces of general type with prescribed invariants.

Bisymplectic Grassmannians

Classical Grassmannians are a special kind of homogeneous spaces for classical groups. Symplectic Grassmannians parametrize subspaces of a given vector space V isotropic with respect to a non degenerate skew-symmetric two-form on

V, while orthogonal Grassmannians parametrize subspaces of V isotropic with respect to a non degenerate symmetric two-form. As these varieties are the “simplest" homogeneous spaces, they have been the object of a lot of attention for more than a century, from different points of view (classical, quantum and equivariant cohomology, derived category, etc.). For G-varieties much less is known when the homogeneity hypothesis is dropped.

In Chapter 4 we focus on some non homogeneous types of Grassmannians, by which we mean varieties that parametrize special classes of subspaces of a given vector space V. By mimicking the definition of symplectic Grassmannians one can define *multisymplectic* Grassmannians: they parametrize subspaces of a given vector space that are isotropic with respect to a finite set of skew-symmetric forms. Similarly, one can define *multiorthogonal* Grassmannians (by considering a finite set of symmetric forms), or *multiorthosymplectic* Grassmannians (by considering symmetric and skew-symmetric forms at the same time).

A remarkable example among multisymplectic Grassmannians is $\mathcal{V}_{22}$; this variety is the prime Fano threefold in Iskovskikh's classification with the largest possible anticanonical degree. Mukai, by using the vector bundle method, interpreted $\mathcal{V}_{22}$ as the *trisymplectic* Grassmannian of 3-dimensional subspaces that are isotropic with respect to three skew-symmetric two-forms on a 7-dimensional vector space (see [Muk02] and Example 1.5.1).

The second remarkable example is given by bisymplectic Grassmannians. We will denote by $\mathrm{I}_2\mathrm{Gr}(k,2n)$ a bisymplectic Grassmannian of k-dimensional subspaces of a $2n$-dimensional vector space V that are isotropic with respect to two general skew-symmetric two-forms. These varieties are always Fano, the canonical bundle being the restriction of $\mathcal{O}(2n-2k+2)$ over $\mathrm{Gr}(k,2n)$ (we suppose that $k \leq n$). In [Kuz15], the case where $k=n$ has been worked out: the bisymplectic Grassmannian $\mathrm{I}_2\mathrm{Gr}(n,2n)$ is isomorphic to $(\mathbf{P}^1)^n$. Moreover, $\mathrm{I}_2\mathrm{Gr}(1,2n) \cong \mathbf{P}^{2n-1}$.

If $2 \leq k \leq n-1$, suppose that the bisymplectic Grassmannian is defined by ω_1 and ω_2; one can prove that if $X = \mathrm{I}_2\mathrm{Gr}(k,2n)$ is smooth, then ω_1 and ω_2 are simultaneously block diagonalizable with 2×2 blocks (Proposition 4.1.6). This has the crucial consequence that the automorphism group of X contains the group $(\mathrm{SL}(2)^n)/\mathbb{Z}^2$, which is the stabilizer of the 2×2 blocks. Even more importantly, a maximal torus $T \cong (\mathbb{C}^*)^n$ inside $\mathrm{SL}(2)^n$ acts on X with a finite number of fixed points. This, as we will see, allows to use all the power of the localization theorems to determine its equivariant cohomology.

Another class of Grassmannians that admit a torus action with a finite number of fixed points are orthosymplectic Grassmannians; again, the existence of the torus action is ensured by the fact that a general skew-symmetric two form and a general symmetric two form are simultaneously (2×2)-block diagonalizable. These varieties would deserve to be studied thoroughly in the future.

Actually one can prove that the automorphism group of bisymplectic Grassmannians is $\mathrm{SL}(2)^n$ up to a finite group. We obtain this result as a consequence of the following theorem:

Theorem (Theorem 4.1.12). *Let $X = \mathrm{I}_2\mathrm{Gr}(k, 2n)$, with $2 \leq k \leq n-1$. If T_X denotes the tangent bundle of X, we have the following isomorphisms:*

$$\mathrm{H}^0(X, T_X) \cong \mathfrak{sl}(2)^n \text{ and } \mathrm{H}^1(X, T_X) \cong \mathbb{C}^{n-3}.$$

This result is obtained by using the fact that X can be seen as the zero locus of a general section of the vector bundle $(\wedge^2\mathcal{U}^*)^{\oplus 2}$ over $\mathrm{Gr}(k, 2n)$. Theorem 4.1.12 implies that bisymplectic Grassmannians form a locally complete family of dimension $n-3$ depending on the choice of the two skew-symmetric forms ω_1 and ω_2. Therefore, in contrast with the case of symplectic Grassmannians, there is not a unique isomorphism class of bisymplectic Grassmannians (even though we refer sometimes to *the* bisymplectic Grassmannian). Moreover, this gives a substantial difference with the cases $k = 1, n$.

The last part of Chapter 4 is devoted to the study of the cohomology of bisymplectic Grassmannians. As we anticipated, the existence of the torus T suggested us to try to understand first the T-equivariant cohomology of X. The torus T is also a maximal torus inside Sp_{2n}, and it acts on the symplectic Grassmannian $\mathrm{IGr}(k, 2n)$ with a finite number of points. Actually, via the natural embedding of $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ inside $\mathrm{IGr}(k, 2n)$, the T-fixed locus is the same for the two varieties. Therefore, we decided to include Section 4.2 to understand better the equivariant cohomology of $\mathrm{IGr}(k, 2n)$, in order to elucidate the equivariant bisymplectic case.

The main results on equivariant cohomology are recalled in Section 4.2.2. Theorem 4.2.12 gives some relations that equivariant classes need to satisfy, and Theorem 4.2.13 asserts that these relations are sufficient to determine the equivariant cohomology as a quotient of a polynomial ring when there exists only a finite number of T-invariant curves. Then, a classical result on torus actions allows to recover easily the classical cohomology from the equivariant one (Theorem 4.2.14).

The equivariant cohomology of symplectic Grassmannians can be understood very well. Not only it is possible to determine it as a subring of the direct sum of copies of a certain polynomial ring (Theorem 4.2.15), but it is also possible to prove that the equivariant classes of Schubert subvarieties (which give an additive basis in cohomology) are completely determined by this description (Proposition 4.2.16); this is similar to what happens, for example, for ordinary Grassmannians (see [KT03]). Even more, one can recover these classes by an inductive method, which consists in computing the product of any Schubert variety with the (unique) hyperplane section:

Theorem (Theorem 4.2.20). *Equation (4.8) and Proposition 4.2.19 determine inductively the equivariant classes of all the Schubert varieties inside $\mathrm{IGr}(k, V)$.*

The analogous results for ordinary Grassmannians are [KT03, Lemma 1 and Proposition 1] (analogous to Proposition 4.2.16) and [KT03, Proposition 2] (analogous to Theorem 4.2.20). The situation is more complicated for bisymplectic

Grassmannians. For these, in general, we were only able to describe the equivariant cohomology as a subring of the direct sum of copies of a certain polynomial ring:

Theorem (Theorem 4.3.15). *The relations in Theorem 4.2.12 are enough to determine the equivariant cohomology of* $\mathrm{I}_2\mathrm{Gr}(k, V)$.

However, we could not prove that this description determines uniquely the equivariant classes of the analog of the Schubert subvarieties inside X; these subvarieties are obtained from the Bialynicki-Birula cell decomposition (see Section 1.2) and provide an additive basis in cohomology. The critical reason for the difference between the behaviour of $\mathrm{IGr}(k, 2n)$ and $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ is that in the homogeneous situation Schubert varieties are actually orbit closures (of a Borel subgroup), which is no longer the case for bisymplectic Grassmannians.

As an application of the results we have found, we were able to compute explicitly the equivariant and classical cohomology of the bisymplectic Grassmannian $\mathrm{I}_2\mathrm{Gr}(2, 6)$, and we obtained equivariant and classical Pieri rules (see Theorem 4.3.19 and Theorem 4.3.21). The variety $\mathrm{I}_2\mathrm{Gr}(2, 6)$ is actually a codimension 2 complete intersection inside $\mathrm{Gr}(2, 6)$. The analysis of this example has allowed us to point out some interesting features that may be generalized to all bisymplectic Grassmannians and may help understanding the (equivariant) cohomology of X in the general case (see the remarks at the end of the section on $\mathrm{I}_2\mathrm{Gr}(2, 6)$).

We would like to underline once more that the work on bisymplectic Grassmannians is far from being over, and we have here just a sample of their basic properties. Apart from completing the study of their cohomology, there are other open questions worthy to be posed about these varieties. For instance, one may be interested in understanding their derived category, or their quantum cohomology (for which one could use the techniques given by the *equivariant quantum cohomology*). We included Section 4.4 at the end of Chapter 4 in order to give some ideas and research directions that one may follow to continue the study of bisymplectic Grassmannians.

1. Generalities

We report in this first chapter the mathematical background that we will need for all the following results. It is divided in two big sections. The first one deals with general facts about homogeneous spaces and the action of algebraic groups on algebraic varieties; we include a section on prehomogeneous spaces, as it will be useful later on. The second part is an introduction to special varieties, where we pointed out the properties that are relevant to our work.

1.1. Homogeneous varieties

In this section we recall some facts about homogeneous varieties and homogeneous bundles; for a more complete exposition see [Ott95]. We introduce some notations that will be used in all chapters, namely those concerning generalized Grassmannians (ordinary, classical and exceptional). One of the aims of this section is also to present Bott's theorem, which is an essential cohomological tool, and that will appear throughout the thesis.

Let G be a reductive complex algebraic group; for instance, one of the classical groups $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{Spin}(2n+1,\mathbb{C})$ or $\mathrm{Spin}(2n,\mathbb{C})$, or one of the exceptional ones G_2, F_4, E_6, E_7, E_8. A variety X is G-homogeneous if it admits a transitive algebraic left action of G. Homogeneous varieties can be seen as quotients G/P of G by a subgroup P. A homogeneous variety G/P is projective if and only if P contains a Borel subgroup B (it is the case of the Grassmannians); in this case the subgroup P is said to be parabolic. Parabolic subgroups can be classified combinatorially by subsets of the simple roots of G. We will be interested mostly in generalized Grassmannians.

Definition 1.1.1. *Generalized Grassmannians are homogeneous varieties G/P where P is a maximal parabolic subgroup of a simple Lie group G. A Grassmannian G/P is said to be ordinary (respectively classical, exceptional) if the group G is $\mathrm{SL}(n,\mathbb{C})$ (respectively a classical group, an exceptional group).*

Example 1.1.2. We will denote by $\mathrm{Gr}(k,n)$ or $\mathrm{Gr}(k,V)$ the ordinary Grassmannian of k-planes in a n-dimensional complex vector space V. It will be the prototype of the ambient variety for the subvarieties we will construct and study in the following chapters.

Example 1.1.3. Classical Grassmannians admit a similar description as that for ordinary ones. Indeed, let V be a vector space of dimension $2n$ (respectively $2n$, $2n+1$) on which a non-degenerate skew-symmetric (resp. symmetric) form is defined. If $k \leq n$, the variety $\mathrm{IGr}(k,2n)$ (resp. $\mathrm{OGr}(k,2n)$, $\mathrm{OGr}(k,2n+1)$) of k-dimensional isotropic subspaces of V with respect to the form is a Grassmannian for $\mathrm{Sp}(2n,\mathbb{C})$ (resp. $\mathrm{SO}(2n,\mathbb{C})$, $\mathrm{Sp}(2n+1,\mathbb{C})$); we will refer to it as the symplectic (resp. even orthogonal, odd orthogonal) Grassmannian.

In the even orthogonal case, the variety of n-dimensional isotropic subspaces of V is the disjoint union of two connected families denoted by $\mathrm{OGr}_+(n,2n)$ and $\mathrm{OGr}_-(n,2n)$. These are isomorphic, and correspond to the quotient of $\mathrm{Spin}(2n,\mathbb{C})$ by the parabolic subgroup associated to one of the last simple roots of D_n. We will denote by $\mathrm{OGr}(n,2n)$ one of these connected families without specifying which one if it is not necessary in the context. Finally, recall that $\mathrm{OGr}(n-1,2n)$ is not a Grassmannian; indeed, it is the quotient of $\mathrm{Spin}(2n,\mathbb{C})$ by the (non maximal!) parabolic subgroup associated to the two last simple roots of D_n (see Section 2.2.4 for more details on this variety).

Example 1.1.4. Exceptional Grassmannians as well admit explicit descriptions which we will not need in the following. More generally, homogeneous spaces G/P admit an interpretation as generalized flag varieties.

We will see in Chapter 2 that exceptional and classical Grassmannians behave similarly to ordinary ones, for instance they have Picard number equal to one. We will use all of them as ambient varieties for several classifications of subvarieties. These will be zero loci of sections of homogeneous bundles (see Section 1.3) over generalized Grassmannians.

Many cohomological properties of homogeneous varieties are known. For instance, an additive basis for their cohomology is given by the so-called Schubert subvarieties, while the multiplicative structure is described by the Borel presentation (see Appendix A.2.1). We will use this description in Chapter 2 and Chapter 3 in order to do some computations of Euler characteristics.

Moreover in Chapter 4, in order to understand better bisymplectic Grassmannians, which are not homogeneous, we will study the (equivariant) cohomology of symplectic Grassmannians. To do so, we will exploit the fact that homogeneous varieties admit an action of a torus $\mathbb{C}^*$ with finitely many fixed points. Such an action defines the so-called Bialynicki-Birula decomposition (see [BB73]). In the next section we recall how to obtain this decomposition, and we will see its relation with the decomposition in Schubert varieties in the homogeneous case.

1.2. The Bialynicki-Birula decomposition

Let us consider a torus $\tau \cong \mathbb{C}^*$ acting on a smooth proper variety X. We will assume that the action has only a finite number of fixed points $\{p_1,\dots,p_r\} = X^\tau$. We recall that the character group $\Xi(\tau)$ of τ is isomorphic to $\mathbb{Z}$. As the point p_i is fixed, the torus acts on its tangent space T_{X,p_i}. As τ is reductive and abelian, its action is diagonalizable on T_{X,p_i}, i.e. we can write

$$T_{X,p_i} = \bigoplus_{\alpha\in\Xi(\tau)} V_\alpha,$$

where V_α is the eigenspace with eigenvalue α. By using the identification $\Xi(\tau) \cong \mathbb{Z}$, we have the following decomposition:

$$T_{X,p_i} = \bigoplus_{\alpha>0} V_\alpha \oplus \bigoplus_{\alpha<0} V_\alpha \oplus V_0 = T_i^+ \oplus T_i^- \oplus T_i^0.$$

Now, the hypothesis that the fixed locus X^τ is finite implies that $T_i^0 = 0$ for $i = 1, \ldots, r$. For each i we define the following varieties:

$$Y_i = \{p \in X \text{ such that, if } t \in \tau, \text{ then } \lim_{t\to\infty} t.p = p_i\}.$$

Clearly from the definition $p_i \in Y_i$ $\forall i$. The following result can be found in [BB73]:

Theorem 1.2.1 (Bialynicki-Birula decomposition). *Let $\tau \cong \mathbb{C}^*$ act on a smooth proper variety X with a finite number of fixed points $\{p_1, \ldots, p_r\} = X^\tau$. Then the subvarieties $Y_i \subset X$ for $i = 1, \ldots, r$ give a cell decomposition of X, meaning that they satisfy the following properties:*

- *Y_i is an affine space smooth at p_i and such that $T_{Y_i,p_i} = T_i^-$;*
- *$Y_i \cap Y_j = \emptyset$ for $i \neq j$, and $X = \bigcup_i Y_i$;*

Remark 1.2.2. Theorem 1.2.1 actually provides two decompositions; the second one is defined by replacing τ with τ^{-1}, i.e. T_i^- with T_i^+.

Remark 1.2.3. Let us denote by Z_i the closure of Y_i. As the Y_i's are isomorphic to an affine space, the classes $[Z_i]$ for $i = 1, \ldots, r$ are a basis for the integer cohomology of X.

In the case when X is a homogeneous space $X = G/P$, things work even better. Indeed, let $T \cong (\mathbb{C}^*)^{\mathrm{rank}(G)}$ be a maximal torus inside G, and B a Borel subgroup of G containing T. Then T acts on G/P with a finite number of fixed points, parametrized by the elements $w \in W/W_P$, where W is the Weyl group of G, and W_P is the Weyl group of the semisimple part of P. We will denote by w^P the element in the class of w of minimal length. Moreover, one can consider the decomposition of X in B-orbits. Indeed, by the Bruhat decomposition for G, it turns out that

$$G/P = \bigcup_{w\in W/W_P} BwP/P,$$

where the union is disjoint. The orbit BwP/P is isomorphic to an affine space of dimension $l(w^P)$. The fixed points of the action of T are the points wP/P for $w \in W/W_P$. The closures of the B-orbits are called Schubert varieties and denoted by σ_w; as it was the case for the subvarieties Z_i, their classes provide a basis for the integer cohomology of X. For the Bruhat decomposition, and in general for flag varieties, the reader can refer to [Bri05].

A general subgroup τ of T still acts with a finite number of fixed points on G/P. Therefore for $X = G/P$ we have two decompositions, the Bruhat one in Schubert varieties and the Bialynicki-Birula one. In fact, the two decompositions are the same. If we choose τ such that it acts positively on the simple roots of B, then it is possible to show (see [BBCM02, Book II, Example 4.2]) that BwP/P is exactly the variety $Y_{wP/P}$ for $w \in W/W_P$.

Remark 1.2.4. This has an interesting consequence for homogeneous spaces. It implies that $Z_w \backslash Y_w$ is the union of $Y_{w'}$ for some $w' \in W/W_P$ (because the Y_w's are orbits!). Moreover if $Z_{w'} \cap Z_w \neq \emptyset$, then either $Z_{w'} \subset Z_w$ or $Z_w \subset Z_{w'}$. Finally, if $w'P/P \in Z_w$, then $Z_{w'} \subset Z_w$. These properties do not hold in general. In Chapter 4 we will show, for instance, that bisymplectic Grassmannians admit the action of a torus τ with finitely many fixed points, and therefore they admit a Bialynicki-Birula decomposition. However, we will see on an explicit example ($\mathrm{I_2Gr}(2,6)$) that it may happen that two distinct Z_i and Z_j have the same dimension, but $Z_i \cap Z_j \neq \emptyset$. This is the principal reason why describing the equivariant cohomology of bisymplectic Grassmannians is more complicated than doing so for symplectic ones.

Even though the Schubert varieties provide a natural additive basis for the integer cohomology of generalized flag varieties, the multiplication rules for them (usually referred to as Littlewood-Richardson rules) can involve very complicated combinatorics. For classical Grassmannians these combinatorics have been worked out, but a lot of work still needs to be done for exceptional Grassmannians and more general flag varieties.

For classical Grassmannians, a multiplicative basis of the cohomology can be described in terms of certain *special* Schubert varieties. For these spaces, it is known how to express Schubert varieties in terms of special ones (Giambelli formulae, see [BKT11] and [BKT17]) and how to multiply any Schubert variety with a special one (Pieri formulae, see [BKT09]). These formulae have been the object of a long and highly non trivial work lasted more than a century.

Special Schubert subvarieties of Grassmannians are given in terms of Chern classes of the so-called *homogeneous* bundles. For instance, for the ordinary Grassmannians, the special Schubert varieties are the Chern classes of the tautological bundle (or, equivalently, of the quotient tautological bundle), which therefore generate multiplicatively the integer cohomology. We introduce in the next section homogeneous bundles, giving the basic properties and definitions which we will use extensively later on.

1.3. Homogeneous bundles

We recall the following important definition:

Definition 1.3.1. *A homogeneous vector bundle $\mathcal{F}$ over a homogeneous variety*

$X = G/P$ is a vector bundle which admits a G-action compatible with the one on the variety X.

If a vector bundle $\mathcal{F}$ is homogeneous, then the fiber $\mathcal{F}_{[P]}$ over the point $[P] \in X$ is stabilized by the subgroup P, i.e. $\mathcal{F}_{[P]}$ is a representation of P; the converse holds as well. More precisely, there is an equivalence of categories between homogeneous vector bundles over G/P and representations of P. Therefore, in this context, one can define irreducible and indecomposable homogeneous bundles, in analogy with the definitions in representation theory.

Notice that P, contrary to G, is not reductive in general. Let P_U be the unipotent factor of P and P_L a Levi factor. The latter is a reductive group. It turns out that a representation $\rho : P \to \mathrm{GL}(V)$, where V is a vector space, is completely reducible if and only if $\rho|_{P_U}$ is trivial. So, completely reducible homogeneous bundles are identified with representations of P_L, and these in turn are identified with their maximal weights. This provides a combinatorial way to classify completely reducible homogeneous bundles which consists in indicating the maximal weights of the irreducible representations to which they correspond. We recall some important homogeneous bundles for classical Grassmannians.

Example 1.3.2. Let $\mathrm{Gr}(k, n)$ be the ordinary Grassmannian. We will denote by $\mathcal{U}$ the tautological bundle of rank k and $\mathcal{U}^*$ its dual, and by $\mathcal{Q}$ the tautological quotient bundle of rank $n-k$; the ample generator of the Picard group of the Grassmannian, which corresponds to $\det(\mathcal{U}^*) = \det(\mathcal{Q})$, will be denoted by $\mathcal{O}(1)$, and $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$.

Example 1.3.3. Let $\mathrm{IGr}(k, 2n)$ ($\mathrm{OGr}(k, m)$, $m = 2n, 2n+1$) be the symplectic (orthogonal) Grassmannian. Let us denote again by $\mathcal{U}$ the tautological bundle of rank k, $\mathcal{U}^*$ its dual, and $\mathcal{Q}$ the tautological quotient bundle of rank $2n-k$; the ample generator of the Picard group of the Grassmannian, which corresponds to $\det(\mathcal{U}^*) = \det(\mathcal{Q})$, will be denoted by $\mathcal{O}(1)$, and $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$.

Moreover, over $\mathrm{IGr}(k, 2n)$, $\mathcal{U}^\perp$ will denote the orthogonal of the tautological bundle of rank $2n-k$, which is not irreducible: in fact, there is an injective homomorphism $\mathcal{U} \to \mathcal{U}^\perp$, and the quotient $\mathcal{U}^\perp/\mathcal{U}$ is irreducible of rank $2n-2k$. The bundle $\mathcal{U}^\perp$ is however indecomposable, and the exact sequence

$$0 \to \mathcal{U} \to \mathcal{U}^\perp \to \mathcal{U}^\perp/\mathcal{U} \to 0$$

is non split; the existence of non split sequences is a consequence of the fact that the parabolic groups are not semisimple.

Over $\mathrm{OGr}(k, 2n+1)$, $\mathcal{T}_{+\frac{1}{2}}$ will denote the spin bundle of rank 2^{n-k} coming from the spin representation, and over $\mathrm{OGr}(k, 2n)$, $\mathcal{T}_{+\frac{1}{2}}$ and $\mathcal{T}_{-\frac{1}{2}}$ will denote the two spin bundles of rank 2^{n-k-1} coming from the two half-spin representations. Finally, over $\mathrm{OGr}(n, 2n)$, the line bundle $\mathcal{O}(1)$ is not a generator of the Picard group. It is actually divisible, and its square root will be denoted by $\mathcal{O}(\frac{1}{2})$ (note that over

$\mathrm{OGr}(n, 2n+1)$, $\mathcal{T}_{+\frac{1}{2}}$ is again a square root of $\mathcal{O}(1)$). The line bundle $\mathcal{O}(\frac{1}{2})$ gives the spinorial embedding of $\mathrm{OGr}(n, 2n)$ in $\mathbb{P}^{2^{n-1}-1}$.

Bott's Theorem

A tool which is very useful when studying homogeneous vector bundles is Bott's theorem (or Borel-Weil-Bott's theorem, see [Bot57]). It gives the cohomology of all irreducible homogeneous vector bundles over G/P in terms of G representations. Depending on the particular group G we are studying, it can be expressed more combinatorially, and we use these reformulations in Chapter 2. Here we give the general statement.

Let $\mathcal{F}$ be an irreducible homogeneous vector bundle over G/P. As irreducible representations of simple groups are parametrized by their maximal weight, we can suppose that $\mathcal{F}$ corresponds to a P_L-representation V_λ with maximal weight λ. We recall that a weight of P_L is a linear combination of the simple roots of P_L, which are the same as those of G; if the coefficients of the linear combination are non negative (respectively nonzero) with respect to the positive simple roots of G, the weight is said to be G-*dominant* (resp. G-regular). Therefore λ can be seen as a weight for G. Let ρ be the sum of the fundamental weights, and W the Weyl group of G. The group W acts on the set of characters of G; moreover, W is a Coxeter group, and as such to each element $w \in W$ we can associate its length $l(w) \in \mathbb{N}$. Finally, let $\lambda^\vee$ denote the maximal weight of V_λ^*.

Theorem 1.3.4 (Bott's theorem)**.** *Let $\mathcal{F}_\lambda$ be an irreducible homogeneous vector bundle over G/P associated to the P_L-representation V_λ.*

If $\lambda^\vee + \rho$ is not G-regular, then

$$\mathrm{H}^i(G/P, \mathcal{F}_\lambda) = 0 \ \ \forall i.$$

If $\lambda^\vee + \rho$ is G-regular and $w(\lambda^\vee + \rho)$ is G-dominant for $w \in W$, then

$$\mathrm{H}^i(G/P, \mathcal{F}_\lambda) = 0 \ \ \forall i \neq l(w) \ \text{ and } \ \mathrm{H}^{l(w)}(G/P, \mathcal{F}_\lambda) = (V^G_{w(\lambda^\vee+\rho)-\rho})^*,$$

where V^G_μ denotes the irreducible G-representation with highest weight μ.

Remark 1.3.5. Notice that the homogeneity condition implies that if an irreducible bundle admits a nonzero section, then it is globally generated. Moreover, an irreducible homogeneous vector bundle associated to V_λ is globally generated if and only if $\lambda^\vee$ is G-dominant. The Borel-Weil theorem usually refers to the result concerning the space of sections $\mathrm{H}^0(G/P, \mathcal{F}_\lambda)$.

We give a useful application of the Borel-Weil Theorem to Grassmannians.

Example 1.3.6. Consider the variety $\mathrm{Gr}(k, 2n) = \mathrm{Gr}(k, V)$, with $k \leq n$. By applying the Borel-Weil theorem, it is easy to see that

$$\mathrm{H}^0(\mathrm{Gr}(k, V), \wedge^2 \mathcal{U}^*) \cong \wedge^2 V^*.$$

A general section ω of $\wedge^2 \mathcal{U}^*$ is a non-degenerate skew-symmetric form over V. The zero locus of ω, which we denote by $\mathscr{Z}(\omega)$, is the set of subspaces $P \in \mathrm{Gr}(k, V)$ such that $\omega(P) = \omega|_P = 0$, i.e. is the set of isotropic subspaces. Therefore there is an isomorphism $\mathscr{Z}(\omega) \cong \mathrm{IGr}(k, V)$, and the symplectic Grassmannian can be seen as a closed subvariety of the ordinary one.

Similarly, the orthogonal Grassmannian $\mathrm{OGr}(k, m)$ is the zero locus of a general section of $S^2 \mathcal{U}^*$ inside $\mathrm{Gr}(k, m)$.

1.4. Prehomogeneous spaces

Even though the subject of “prehomogeneous spaces" is different from those treated in this section, we decided to include it here because they are part of the background material on algebraic groups that will be used later on. More specifically, we will use them to search for *models* for orbital degeneracy loci (Chapter 3, see also [BFMT17b]). These models are nothing more than closed G-stable subvarieties of a given G-representation. We will focus on a certain class among prehomogeneous spaces, called parabolic representations, which are closely related to homogeneous varieties (see Remark 1.4.10). Then, we will introduce Weyman's geometric technique (see [Wey03]) in order to study orbit closures inside such representations.

Prehomogeneous spaces can be thought of as affine generalizations of homogeneous spaces:

Definition 1.4.1. *Let G be an algebraic group. A prehomogeneous space V is a G representation with an open dense G-orbit.*

Example 1.4.2. The group GL_n acts with only two orbits on $\mathbb{C}^n$, which is prehomogeneous. More generally, $\mathrm{GL}_e \times \mathrm{GL}_f$ acts on the space of matrices $M_{e,f}$ with finitely many orbits, parametrized by the rank. Therefore, $M_{e,f}$ is $\mathrm{GL}_e \times \mathrm{GL}_f$-prehomogeneous.

Example 1.4.3. Example 1.4.2 is however particular. In general, prehomogeneous spaces do not need to have finitely many orbits. For instance, by using certain operations called castling transfoms on a given prehomogeneous space (see [Man13]), it is possible to produce another prehomogeneous space. This can be used to show that $\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^5$ is $\mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_5$-prehomogeneous. It turns out that this representation does not have a finite number of orbits.

Among prehomogeneous spaces with finitely many orbits, a large class is provided by parabolic representations. They are parametrized by the choice of a simple root α_i of a simple Lie group (or Dynkin diagram) G. The Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is a Cartan subalgebra, Φ a system of roots with set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$.

Definition 1.4.4. *A $\mathbb{Z}$-grading on $\mathfrak{g}$ is a decomposition*

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$$

such that $[\mathfrak{g}_j, \mathfrak{g}_h] \subset \mathfrak{g}_{j+h}$.

Remark 1.4.5. Given a $\mathbb{Z}$-grading, $\mathfrak{g}_0$ is a Lie subalgebra acting on each $\mathfrak{g}_i$.

Let $H_i \in \mathfrak{h}$ be an element such that $\alpha_j(H_i) = \delta_{ij}$ for $j = 1, \ldots, n$. Then, one obtains a $\mathbb{Z}$-grading by defining

$$\mathfrak{g}_j := \bigoplus_{\alpha\in\phi, \alpha(H_i)=j} \mathfrak{g}_\alpha.$$

With this definition, $\mathfrak{g}_\alpha$ belongs to $\mathfrak{g}_h$ if and only if $\alpha = \sum_j n_j \alpha_j$ with $n_i = h$. Moreover, $\mathfrak{h} \subset \mathfrak{g}_0$. Actually, the semisimple part of the Lie subalgebra $\mathfrak{g}_0$ can be deduced by suppressing the simple root α_i from the Dynkin diagram of $\mathfrak{g}$. Denote by G_0 the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. The following result motivates our interest for this construction (see [Vin79]):

Theorem 1.4.6 (Vinberg)**.** *Consider a $\mathbb{Z}$-grading induced by $H_i \in \mathfrak{h}$. The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is irreducible. Moreover, G_0 acts on $\mathfrak{g}_1$ with a finite number of orbits. In particular, $\mathfrak{g}_1$ is G_0-prehomogeneous.*

Definition 1.4.7. *A parabolic representation is a prehomogeneous space of the form $\mathfrak{g}_1$ described in Theorem 1.4.6.*

Remark 1.4.8. We point out that a parabolic representation $\mathfrak{g}_{-1}$ is contained in the nilpotent cone of the simple Lie algebra $\mathfrak{g}$ of G, and Theorem 1.4.6 is a consequence of the finiteness of nilpotent orbits. These are defined as G-orbits of nilpotent elements inside the Lie algebra $\mathfrak{g}$.

Example 1.4.9. By choosing $G = \mathrm{SL}_{n+1}$ and α_i the ith simple root of the Dynkin diagram A_n, one can see that the parabolic space $\mathfrak{g}_1$ is isomorphic to the space of matrices $M_{k,n+1-k}$. By using the Dynkin diagrams C_n and D_n with the choice of the simple root α_n, one obtains the spaces of symmetric and skew-symmetric matrices. Other examples will be given in Chapter 3.

Remark 1.4.10. There is a nice geometric interpretation of the parabolic representation $\mathfrak{g}_1$ induced by $H_i \in \mathfrak{h}$. Indeed, consider the parabolic group P_i associated to the i-th simple root of G, its Lie algebra $\mathfrak{p}_i$ and the Lie algebra of its Levi factor $\mathfrak{l}_i$. Then

$$\mathfrak{p}_i = \bigoplus_{j\geq 0} \mathfrak{g}_j$$

as $\mathfrak{l}_i$-representations. Over the variety G/P_i, the tangent bundle T_{G/P_i} is the homogeneous bundle associated to the P-representation $\mathfrak{g}/\mathfrak{p}_i$ which, as a $\mathfrak{l}_i$-representation, is $\bigoplus_{j<0}\mathfrak{g}_j$. Moreover $\mathfrak{g}_{-1}$ is naturally the dual of $\mathfrak{g}_1$ (via the Killing form).

Notice that a line passing through the point $[P_i] \in G/P_i$ which is entirely contained inside G/P_i naturally lives in $\mathbf{P}(T_{G/P_i,[P_i]})$. It turns out (see [LM03]) in many cases that the variety of such lines is isomorphic to the unique closed P_i-orbit inside

$$\mathbf{P}(\mathfrak{g}_{-1}) \subset \mathbf{P}(\mathfrak{g}/\mathfrak{p}_i) = T_{G/P_i,[P_i]}.$$

This gives a natural frame to introduce parabolic spaces via generalized Grassmannians.

The geometric method

In Chapter 3 we will be interested in orbit closures inside G-representations. Weyman and his collaborators developed a method (the geometric technique, see [Wey03]) to study the properties of such orbit closures. This method has proved to be very efficient for studying parabolic orbit closures, and it is interesting to us because it provides two fundamental tools, both of which can be relativized to obtain informations on the orbital degeneracy loci (see [BFMT17a], [BFMT17b] or Chapter 3).

The first one is a G-equivariant desingularization of an orbit closure, called a *Kempf collapsing*. Let Y be a closed G-stable subvariety inside a G-representation V. Moreover, suppose Y admits a resolution of singularities of the following type: the desingularization of Y is given by the total space $\mathcal{W}$ of a (homogeneous) vector bundle W over the homogeneous variety G/P. Here $\mathcal{W} \subset \mathcal{V} := V \times G/P$, and the morphism $\mathcal{W} \to V$ is given by the first projection onto V:

$$\begin{tikzcd} \mathcal{V} \arrow[d] & \mathcal{W} \arrow[l, hook'] \arrow[d, "p_W"] \arrow[dr] & \\ V & Y \arrow[l, hook'] & G/P \end{tikzcd} \tag{1.1}$$

The map p_W is therefore proper, surjective and birational; such a desingularization is called a Kempf collapsing (Kempf having used it in [Kem76] to study determinantal varieties).

Almost all orbit closures inside parabolic representations admit a desingularization given by a Kempf collapsing; this may be false for some parabolic representations deduced from E_8.

Remark 1.4.11. Kempf collapsings need not be unique, for a given orbit closure, as illustrated by Example 1.4.12.

Example 1.4.12. Let $M_{e,f}$ be the space of matrices which parametrises morphisms $\mathbb{C}^e \to \mathbb{C}^f$. The determinantal variety of matrices of rank at most $0 \leq r \leq \min\{e,f\}$

is

$$Y_{e,f}^r = \{x \in M_{e,f} |\, \mathrm{rank}(x) \leq r\}$$

It is the closure of the orbit of matrices of rank exactly r. Its singular locus is $Y_{e,f}^{r-1}$; we describe now a Kempf collapsing for $Y_{e,f}^r$. Indeed, consider the Grassmannian $\mathrm{Gr}(e-r,e)$ of $(e-r)$-planes in $\mathbb{C}^e$ (this has to be thought of as parametrising the kernel of the morphisms in $Y_{e,f}^r$). If $W = \mathcal{Q}^* \otimes \mathbb{C}^f$ is the bundle on the Grassmannian, denote by $\mathcal{W} \subset M_{e,f} \times \mathrm{Gr}(e-r,e)$ its total space; then $\mathcal{W}$ is a Kempf collapsing of $Y_{e,f}^r$ via the proper, birational morphism $\mathcal{W} \hookrightarrow M_{e,f} \times \mathrm{Gr}(e-r,e) \to M_{e,f}$. From this description, $Y_{e,f}^r$ is a variety of dimension $r(e+f)-r^2$, singular in codimension $e+f-2r+1$. Another Kempf collapsing for $Y_{e,f}^r$ is given by the total space of $W = (\mathbb{C}^e)^* \otimes \mathcal{U}$ over $\mathrm{Gr}(r,f)$; in this case, the homogeneous variety is parametrizing the images of the morphisms.

Remark 1.4.13. We recall that a desingularization $\pi : Z' \to Z$ is said to be *crepant* if the relative canonical bundle $K_{Z'/Z}$ is trivial. In the case of a Kempf collapsing, the crepancy condition becomes

$$K_{G/P} = \det(W).$$

Using Kempf collapsings and under some additional hypothesis on the cohomology of the bundles involved, it is possible to produce a locally free resolution of the ideal of $Y \subset V$; this is achieved by pushing down a suitable locally free resolution constructed over the variety G/P. Moreover, this method allows to understand some properties of Y, namely if it is normal, with rational singularities, Cohen-Macaulay and Gorenstein. A famous example of such resolutions is given by the Eagon-Northcott complex, or the more general Lascoux complex for determinantal varieties. In [KW12], [KW13] and [KW], the geometric technique is used to study orbit closures in parabolic representations of exceptional type (the classical types having being dealt with by many authors, see [Wey03]).

1.5. Special varieties

This section is devoted to recalling some basic facts about the other protagonists of this work, together with homogeneous spaces: special varieties. By special varieties we will mean Fano varieties and varieties with trivial canonical bundle. Chapter 2 and Chapter 3 are both motivated by the attempt to construct new interesting special varieties, and the subject of Chapter 4 is the study of a particular class of Fano manifolds.

In what follows, we explain the strong relationships between homogeneous spaces and Fano manifolds, which was one of the first motivations for this thesis. This relationship is clearly showed by Mukai's classification of Fano threefolds (see [Muk02] for instance) as subvarieties of certain Grassmannians. In the second part of this section, we introduce Calabi-Yau varieties, hyper-Kähler manifolds, and we recall the Beauville-Bogomolov decomposition for varieties with trivial canonical bundle.

1.5.1. Fano threefolds

A Fano variety is a variety whose anticanonical bundle is ample. Fano varieties are rationally connected and simply connected. Examples of Fano varieties are the projective space, Grassmannians and in general all rational homogeneous spaces.

For each fixed dimension, there is only a finite number of families of Fano manifolds. In dimension 2 Fano surfaces have been classified by Del Pezzo, while in dimension 3 a classification has been given by Iskovskikh when the Picard number is one (see [IP99]), and by Mori and Mukai when it is greater (see [MM03]). In higher dimension the problem of describing all families is probably too hard, even though there is an effort to classify Fano fourfolds (this is one of the aims of the project *3C in G* cited in the Introduction).

The classification given by Iskovskikh has been later reinterpreted by Mukai. Mukai was able to bring to light the strong liaison between Fano varieties and homogeneous spaces, or more precisely Grassmannians.

Indeed, for each family of prime Fano threefold of Picard number one, he could produce a globally generated vector bundle F, say of rank r, living on a general member X of the family. Therefore, one could define the morphism

$$\psi_F : X \to \mathrm{Gr}(r, \mathrm{H}^0(X, F)^*).$$

In fact ψ_F was an embedding of the Fano variety inside the Grassmannian. As a matter of fact, he was able to describe all (families of) Fano threefolds as being zero loci of certain homogeneous bundles over generalized Grassmannians. Most of them turned out to be complete intersections in generalized Grassmannians, as it is the case for the zero locus of a general section of $\mathcal{O}(1)^5$ over $\mathrm{Gr}(2,6)$, or of $\mathcal{O}(\frac{1}{2})^8$ over $\mathrm{Gr}(5,10)$, but not all of them.

Example 1.5.1. A family of Fano threefolds which have been studied from different points of view is the so-called family $\mathcal{V}_{22}$. In Mukai's interpretation, it can be seen as the zero locus of a general section of $(\wedge^2\mathcal{U}^*)^3$ over $\mathrm{Gr}(3,7)$; by the Borel-Weil theorem, it parametrizes 3-dimensional subspaces of $\mathbb{C}^7$ isotropic with respect to three general skew-symmetric forms. We will refer to it as the trisymplectic Grassmannian $\mathrm{I}_3\mathrm{Gr}(3,7)$. In Chapter 4, inspired by this example, we study another similar class of Fano varieties, i.e. bisymplectic Grassmannians.

1.5.2. Varieties with trivial canonical bundle

There are three main categories of varieties which can be thought of as the "building blocks" of Kähler varieties with trivial canonical bundle, as implied by the Beauville-Bogomolov decomposition theorem: complex tori, Calabi-Yau manifolds, and irreducible holomorphic symplectic manifolds. Complex tori of dimension n are just compact quotients of $\mathbb{C}^n$ by a lattice. Let us examine the other two classes in more detail.

For what concerns Calabi-Yau manifolds, in the literature various different definitions can be found; we will use the following one:

Definition 1.5.2. *A manifold X with trivial canonical bundle is of Calabi-Yau type if*

$$\mathrm{H}^0(X, \Omega_X^0) \cong \mathrm{H}^0(X, \Omega_X^{\dim(X)}) \cong \mathbb{C},$$
$$\mathrm{H}^0(X, \Omega_X^k) = 0 \text{ for } 1 \leq k \leq \dim(X),$$

and its dimension is at least 3.

Remark 1.5.3. Calabi-Yau manifolds are manifolds of Calabi-Yau type which are simply connected.

The condition on the dimension is required because $K3$ surfaces, which meet the requirements of Definition 1.5.2, are considered to be irreducible holomorphic symplectic surfaces. This brings us to the last class of varieties we consider:

Definition 1.5.4. *A Kähler manifold Z with trivial canonical bundle and dimension $2n$ is irreducible holomorphic symplectic, or hyper-Kähler, if*

$$\mathrm{H}^0(Z, \Omega_Z^*) = \mathbb{C}[\sigma]/\sigma^{n+1},$$

where $\sigma \in \mathrm{H}^0(Z, \Omega_Z^2)$ is everywhere nondegenerate.

As anticipated before, these definitions gain in importance if we consider the following theorem (see [Bog74]):

Theorem 1.5.5 (Decomposition theorem)**.** *Let Y be a compact Kähler simply connected manifold with $K_Y = \mathcal{O}_Y$. Then*

$$Y = \prod_i X_i \times \prod_j Z_j$$

where
- *X_i are simply connected Calabi-Yau manifolds;*
- *Z_j are simply connected and irreducible holomorphic symplectic.*

This result will be useful later to distinguish which of the varieties that we find are of Calabi-Yau type and which are irreducible holomorphic symplectic.

Example 1.5.6 (Hilbert scheme of points on $K3$ surfaces). Let X be a $K3$ surface. We denote by $X^{[n]}$ the Hilbert scheme of n points on X, or equivalently of length n subschemes of X. $X^{[n]}$ can be constructed as a desingularization of the nth-symmetric product $X^{(n)}$, and has dimension $2n$. Beauville showed in [Bea83] that $X^{[n]}$ is a hyper-Kähler variety, thus providing an example (together with Kummer generalized varieties) of hyper-Kähler varieties in each dimension. The second Betti number of $X^{[n]}$ is $b_2(X^{[n]}) = 23$, therefore these varieties live in a family of dimension 21 (dimension 20 if the family has projective members). As $K3$ surfaces live in a family of dimension 20 (dimension 19 if the $K3$s are projective), the general deformation of $X^{[n]}$ is not an Hilbert scheme of n points on a $K3$ surface.

When $n = 2$, $X^{[2]}$ is the blow-up of $X^{(2)}$ along the diagonal, which is its singular locus. Examples of projective locally complete families of such fourfolds are X_1 and X_2, as explained in the Introduction.

When dealing with fourfolds, it is possible to understand if a variety with trivial canonical bundle is a torus, a Calabi-Yau (CY) or a hyper-Kähler (HK) manifold by computing the Euler characteristic of its trivial bundle:

Proposition 1.5.7. *Suppose Y is a smooth projective fourfold with trivial canonical bundle. If the Euler characteristic of the trivial bundle $\chi(\mathcal{O}_Y)$ is either two or three, then Y is simply connected. Moreover:*
– *If $\chi(\mathcal{O}_Y) = 2$, then Y is CY;*
– *If $\chi(\mathcal{O}_Y) = 3$, then Y is IHS;*

Proof. Let us first suppose to have proven that Y is simply connected. Then we can apply Theorem 1.5.5. Therefore, our variety Y is a product of CY and IHS manifolds, i.e. it is either a product of two $K3$ surfaces, or a CY fourfold, or a IHS one. In the first case $\chi(\mathcal{O}_Y) = 4$, in the second $\chi(\mathcal{O}_Y) = 2$ and in the third $\chi(\mathcal{O}_Y) = 3$, thus proving the last assertion.

Next we turn to the proof of simply connectedness of Y. As a matter of fact, a generalization of the decomposition Theorem 1.5.5 holds ([Bea83]): for any compact Kähler manifold Y with trivial canonical bundle, there exists an étale cover $f : Y' \to Y$ of degree d such that

$$Y' = \prod_i X_i \times \prod_j Z_j \times \prod_k T_k$$

where X_i are simply connected CYs, Z_j are simply connected IHS', and T_k are complex tori. Moreover, a well-known formula says:

$$\chi(Y', \mathcal{O}_{Y'}) = d\ \chi(Y, \mathcal{O}_Y)\ .$$

Recall also that the Euler characteristic of a product is the product of the Euler characteristics of the single factors, and that $\chi(\mathcal{O}_T) = 0$ for T a complex torus.

Therefore, if $\chi(\mathcal{O}_Y)$ is either two or three, then no factor of Y' can be a torus, because $\chi(\mathcal{O}_{Y'}) \neq 0$; as a consequence, for such a cover $f : Y' \to Y$ the degree is

1, f is an isomorphism, and $Y \cong Y'$ is either CY or IHS, and in both cases it is simply connected. $\square$

Remark 1.5.8. Notice that connectedness too is ensured by the hypothesis that $\chi(\mathcal{O}_Y) = 2$ or 3. In fact, the above proof shows that $\chi(\mathcal{O}_Y)$ cannot be equal to one.

2. Zero loci in generalized Grassmannians

As already pointed out in Section 1.5.1, homogeneous spaces can be conveniently used to provide examples of special varieties. The classification of Fano threefolds is not an isolated case. The examples of families of hyper-Kähler fourfolds described in [BD85] and [DV10] are another manifestation of this phenomenon, as explained in the Introduction.

In this chapter, we study fourfolds with trivial canonical bundle which are zero loci of sections of homogeneous, completely reducible bundles over ordinary, classical and exceptional Grassmannians. We give a complete classification of those varieties, and we include also the analogous classification for surfaces and threefolds. In doing so, we study some interesting behaviours of these varieties (Section 2.1.3 and Section 2.4). The chapter includes a result on zero loci of sections of irreducible bundles inside ordinary Grassmannians (see Section 2.1.3). In this framework we were able to prove that, in any dimension, the only hyper-Kähler varieties are those of type X_1 and X_2 (defined in the Introduction).

Some of the results contained in this chapter can be found in [Ben18]. More precisely the article includes the classification theorems in ordinary Grassmannians (Theorem 2.1.1, 2.3.1 and 2.3.4) and classical Grassmannians (Theorem 2.2.1, 2.3.2 and 2.3.5). On the contrary, all the material concerning irreducible bundles (Section 2.1.3), the study of an isomorphism between Calabi-Yau threefolds (Section 2.3.1) and of some families of K3 surfaces (Section 2.3.2), and the classification in exceptional Grassmannians (Section 2.4) is new.

2.1. Fourfolds in ordinary Grassmannians

We will refer to Example 1.3.2 for notations on homogeneous bundles over the ordinary Grassmannian. We will denote by $\Lambda^i(E)$ the i-th exterior power of the bundle E, and $S^i(E)$ the i-th symmetric power of E. Our first main theorem is the following:

Theorem 2.1.1. *Let Y be a fourfold with $K_Y = \mathcal{O}_Y$ which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle $\mathcal{F}$ over $\mathrm{Gr}(k, n)$. Up to the identification of $\mathrm{Gr}(k, n)$ with $\mathrm{Gr}(n - k, n)$, the only possible cases are those appearing in Table B.1 in Appendix B.*

In the classification we have put also the computation of $\chi(\mathcal{O}_Y)$ as it is the quantity that permits to distinguish between CY and IHS manifolds, as the former satisfy $\chi(\mathcal{O}_Y) = 2$ ($\mathrm{H}^0(Y, \Omega^2_Y) = 0$), while the latter satisfy $\chi(\mathcal{O}_Y) = 3$

$(\mathrm{H}^0(Y, \Omega^2_Y) = \mathbb{C})$.

All the subvarieties found are CY manifolds, with the exception of the cases $(d7)$, $(d5)$ and $(b12)$, which we examine now. The case $(b12)$ is the IHS fourfold appearing in [BD85], while the case $(d7)$ is the one appearing in [DV10].

The case $(d5)$ already appears in [Rei72], where the variety of n-planes in the intersection of two quadrics in a space of dimension $2n+2$ is proved to be an abelian variety, the Jacobian variety of an hyperelliptic curve of genus $n+1$. So, for $(d5)$, Y is an abelian variety.

On the other hand, the case $(d6)$ has $\chi(\mathcal{O}_Y) = 4$ because it is not connected. In fact it has two connected components, as if one considers the variety Y_1 of zeroes of a general section of $S^2\mathcal{U}^*$ in $\mathrm{Gr}(4,8)$, it is the set of maximal subspaces isotropic with respect to a general symmetric 2-form. It is well known that Y_1 has two connected components. As a consequence, the connected components of $(d6)$ are two isomorphic CY manifolds, which can be seen as complete intersections in the orthogonal Grassmannian $\mathrm{OGr}(4,8)$. Notice that $\mathrm{OGr}(4,8)$ in turn is isomorphic to $\mathrm{OGr}(1,8)$ because of the symmetries of the Dynkin diagram D_4, and $\mathrm{OGr}(1,8)$ is a quadric in $\mathbb{P}^7$; the isomorphism is given by the embedding of $\mathrm{OGr}(4,8)$ in $\mathbb{P}^7$ defined by the line bundle $\mathcal{O}(\frac{1}{2})$. Therefore each connected component of $(d6)$ is a complete intersection in $\mathbb{P}^7$.

Remark 2.1.2. As already stated in the Introduction, the previous classification gives the analogous one for Fano varieties of dimension 5. Indeed, suppose Y and $\mathcal{F}$ are as in Theorem 2.1.1, and $\mathcal{F} = \mathcal{L} \oplus \mathcal{F}'$, where $\mathcal{L}$ is a homogeneous globally generated line bundle. Then the zero locus of a general section of $\mathcal{F}'$ is a Fano fivefold. Vice versa, consider a zero locus inside a Grassmannian which is a Fano fivefold; an anticanonical section inside of it is a fourfold with trivial canonical bundle, therefore appearing in Table B.1. A similar argument holds for all the other classifications in Grassmannians.

As we mentioned in the Introduction, in [IIM16] zero loci inside ordinary Grassmannians are studied as well; the authors are able to prove the following finiteness result ([IIM16, Corollary 2.3]):

Theorem 2.1.3. *For a fixed positive integer $d > 0$, there are at most finitely many families of d-folds with trivial canonical bundle in Grassmannians which are zero loci of completely reducible globally generated homogeneous vector bundles, up to natural identifications among Grassmannians.*

This indicates that an explicit classification of such subvarieties of Grassmannians in higher dimension is possible.

2.1.1. Classification

The proof of the theorem will be divided into different lemmas and propositions which concern the subvarieties of $\mathrm{Gr}(k,n)$ for different choices of k and n.

Notation 2.1.4. The notations will be similar to those used in [Küc95]. The Grassmannian $\mathrm{Gr}(k,n)$ will be thought of as the quotient G/P_k, where P_k is the maximal parabolic subgroup containing the Borel subgroup of positive standard roots in $G = \mathrm{SL}(n,\mathbb{C})$. Every irreducible homogeneous bundle is represented by its highest weight. A weight is represented by $\beta = (\beta_1,\dots,\beta_n)$ or by $(\beta_1,\dots,\beta_k;\beta_{k+1},\dots,\beta_n)$, where

$$\beta = \beta_1\lambda_1 + \beta_2(\lambda_2-\lambda_1) + \cdots + \beta_{n-1}(\lambda_{n-1}-\lambda_{n-2}) - \beta_n\lambda_{n-1},$$

and the λ_i's are the fundamental weights for $G = \mathrm{SL}(n,\mathbb{C})$. All weights can be renormalized so to have $\beta_n = 0$.

A consequence of the homogeneous condition is that as soon as a homogeneous bundle admits non zero global sections, it is globally generated. Another equivalent condition for a bundle to have global sections is the existence of a G-representation for the *dual* of the weight representing the homogeneous bundle: in this case, the G-representation in question is canonically isomorphic to the space of global sections (see Theorem 1.3.4 and the remark following it).

Remark 2.1.5. As we work with globally generated bundles, from now on the notation will change: to indicate a bundle with highest weight β as before, we will write $(-\beta_k,\dots,-\beta_1;\beta_{k+1},\dots,\beta_n)$, which is equivalent to taking the highest weight of the dual representation. In this way, a bundle $\alpha = (\alpha_1,\dots,\alpha_n)$, according to the new notation, is globally generated when $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$, i.e. when the weight α is dominant under the action of G.

Example 2.1.6. Over the Grassmannian $\mathrm{Gr}(3,7)$ of 3-dimensional spaces in a 7-dimensional space, the dual tautological bundle $\mathcal{U}^*$ of rank 3 will be denoted by its highest weight $(1,0,0;0,0,0,0) = (1,0,\dots,0)$, the tautological quotient bundle $\mathcal{Q}$ by $(1,1,1;1,1,1,0)$, $\mathcal{O}(1) = \Lambda^3\mathcal{U}^*$ by $(1,1,1;0,\dots,0)$.

The rank of a bundle can be calculated explicitly:

$$\mathrm{rank}(\beta_1,\dots,\beta_n) = \dim(\beta_1,\dots,\beta_k) \times \dim(\beta_{k+1},\dots,\beta_n)$$

where

$$\dim(\beta_1,\dots,\beta_r) = \prod_{1\leq i<j\leq r} \frac{j-i+\beta_i-\beta_j}{j-i}$$

is the Weyl character formula (see [FH91, Chapter 24] for this formula and similar ones for classical groups).

The formula comes from the fact that the rank of a homogeneous bundle over G/P is the same as the dimension of the P-module representing it (see Section

1.1). As the P-module is irreducible, it is actually an irreducible module of the Levi factor of P, or of its Lie algebra, which is $\mathfrak{sl}(k) \oplus \mathfrak{sl}(n-k)$ for $G/P = \mathrm{Gr}(k,n)$. Moreover the Weyl character formula gives the dimension of a $\mathfrak{sl}(r)$-module in terms of its highest weight. Putting these facts together, we obtain the formula for the rank in terms of the weight.

Remark 2.1.7. Analogous formulas for classical Grassmannians will be derived in a similar way; it will therefore be necessary to understand what the Levi factor of P is for symplectic and orthogonal Grassmannians.

The first three results have been proved in Küchle's paper ([Küc95, Lemma 3.2, Lemma 3.4, Corollary 3.5]). In what follows, we assume that the subvarieties of $\mathrm{Gr}(k,n)$ we are constructing are zero loci of general sections of the homogeneous bundle $\mathcal{F}$. This bundle being completely reducible, it can be decomposed as a direct sum $\mathcal{F} = \oplus_i \mathcal{E}_i$.

Lemma 2.1.8. *One can assume that the following bundles over* $\mathrm{Gr}(k,n)$ *do not appear as summands in* $\mathcal{F} = \oplus_i \mathcal{E}_i$*:*
(i) the bundles $(1,0,\dots,0)$ *(respectively* $(1,\dots,1,0)$*) corresponding to* $\mathcal{U}^*$ *(resp.* Q *on* $\mathrm{Gr}(n-k,n)$ *); (ii) if* $2k > n$ *the bundles* $(1,1,0,\dots,0)$ *and* $(2,0,\dots,0)$ *(resp.* $(1,\dots,1,0,0)$ *and* $(2,\dots,2,0)$ *for* $2k < n$*).*

One defines:

$$\mathrm{dex}(\beta) = (\frac{|\beta|_1}{k} - \frac{|\beta|_2}{n-k})\,\mathrm{rank}(\beta) \tag{2.1}$$

where $|\beta|_1 = \sum_{i=1}^{k} \beta_i$ and $|\beta|_2 = \sum_{i=k+1}^{n} \beta_i$.

Lemma 2.1.9. *For* $\mathcal{F} = \oplus_i \mathcal{E}_i$ *we have*

$$\mathrm{rank}(\mathcal{F}) = \sum_i \mathrm{rank}(\mathcal{E}_i) = k(n-k) - 4 \ , \tag{2.2}$$

and

$$\sum_i \mathrm{dex}(\mathcal{E}_i) = n \ . \tag{2.3}$$

Proof. The first formula of this lemma is the same as in [Küc95, Lemma 3.4], on the other hand the second is different as in this case one requires $K_Y = \mathcal{O}_Y$.

However, it is worth giving a hint on how to prove (2.3). Let us fix an irreducible P-module V_β given by the weight β. Equation (2.3) is proved if we show that

$$\det(V_\beta) = L^{\mathrm{dex}(V_\beta)},$$

where L is the 1-dimensional representation associated to the highest weight $(1,\dots,1;0,\dots,0)$. Now, the highest weight associated to $\det(V_\beta)$ is the sum of all the weights of V_β. Moreover, the space of weights is invariant under the action of the Weyl group W of P, and for ordinary Grassmannians we have

$$W = \mathfrak{S}_k \times \mathfrak{S}_{n-k},$$

where $\mathfrak{S}_r$ stands for the symmetric group on r; the action on the weights is the expected one. Therefore, it is straightforward to see that the weight associated to $\det(V_\beta)$ is

$$(\dim(V_\beta)\frac{|\beta|_1}{k}, \dots, \dim(V_\beta)\frac{|\beta|_1}{k}; \dim(V_\beta)\frac{|\beta|_2}{n-k}, \dots, \dim(V_\beta)\frac{|\beta|_2}{n-k}).$$

By rescaling, this weight is equal to $(\mathrm{dex}(\beta), \dots, \mathrm{dex}(\beta); 0, \dots, 0)$, thus ending the proof. $\square$

Remark 2.1.10. The same method used to prove (2.3) can be used when dealing with symplectic and orthogonal Grassmannians. The only difference will be that generally one of the factors of the Weyl group of P will be given by a certain *signed* symmetric group: here again what matters is the Weyl group of the Levi factor of P. Therefore, as we will see, the formula for dex depends on the type of Grassmannian we consider; however, the relation

$$\det(V_\beta) = L^{\mathrm{dex}(V_\beta)}$$

which was proved above will hold in general.

Corollary 2.1.11. *Using the correspondence irreducible bundles - weights, in the same hypothesis as before:*
(a) for each bundle $\mathcal{E}_i = (\beta_1^i, \dots, \beta_n^i)$, *we have* $\beta_1^i = \dots = \beta_k^i$ *or* $|\beta^i|_2 = 0$*;*
(b) $\dim(\beta_1, \dots, \beta_r) \geq \binom{r}{i}$ *if* $\beta_i > \beta_{i+1}$.

The strategy of the proof is the same as that of [Küc95]. We recall that the bundle $\mathcal{F}$ that defines the fourfold lives on $\mathrm{Gr}(k, n)$, $2k \leq n$.

Proposition 2.1.12 (Classification for $k \leq 3$)**.** *If* $k \leq 3$*, then for* $\mathcal{F}$ *we have one of the cases labelled by the letters (a), (b), (c) appearing in Table B.1.*

Proof. $k = 1$.
If $|\beta|_2 \neq 0$, then $\mathrm{rank}(\beta) \geq n-1 > n-1-4$, so the only possible case is (a).
$k = 2$.
The variety $\mathrm{Gr}(2,4)$ is a Fano variety, so one can suppose that $n \geq 5$. Calculating the rank, one can see that the only possible bundles $\mathcal{E}_i$ are $(p, q; 0, \dots, 0)$, $(r, r; 1, 0, \dots, 0)$ for $r \geq 1$, $(s, s; 1, \dots, 1, 0)$ for $s \geq 2$. If $(s, s; 1, \dots, 1, 0)$ is present as an addend, the only possibility is (b3) (if $s \geq 3$, then $\mathrm{dex}(s, s; 1, \dots, 1, 0) \geq n$). For the same reason, if $(r, r; 1, 0, \dots, 0)$ is present, $r = 1$ and one has the cases (b10)(.i). Then, one remains only with the bundles $(p, q; 0, \dots, 0)$, for which

$\frac{\text{dex}}{\text{rank}} \geq 1$, which forces $n \leq 8$. These are the remaining (b)-cases.
$k = 3$.

One has $n \geq 6$, and, calculating the rank, for $n \geq 9$ the possible bundles are $(p, q, r; 0, \ldots, 0)$ for $p \geq q \geq r$, $(p, q, r; 0, \ldots, 0) \neq (1, 0, \ldots, 0)$, $\neq (0, \ldots, 0)$; $(r, r, r; 1, 0, \ldots, 0)$ for $r \geq 1$; $(s, s, s; 1, \ldots, 1, 0)$ for $s \geq 2$. An argument similar to the one used when $k = 2$ shows that the only possible bundles are the (c)-cases. □

Lemma 2.1.13 ("Reduction of cases"). *If*

$$\operatorname{rank}(\beta)(|\beta|_1 - 1) \leq k^2 + 4 \tag{2.4}$$

and not considering the cases (d7) and (d8) in Table B.1, then the possible bundles β representing $\mathcal{E}_i$ and appearing as addend of $\mathcal{F}$ for $k, n - k \geq 4$ are the following:

$$(A) = (1, \ldots, 1, 0; 0, \ldots, 0)$$

$$(B) = (1, 1, 0, \ldots, 0)$$
$$(C) = (2, 0, \ldots, 0)$$
$$(Z) = (2, 1, \ldots, 1; 0, \ldots, 0)$$
$$(D) = (p, \ldots, p; \beta_{k+1}, \ldots, \beta_n)$$

Remark 2.1.14. If $\mathcal{F} = \oplus_i \mathcal{E}_i$, the expression $k^2 + 4$ is equal to

$$\sum_i (k(\operatorname{dex}(\mathcal{E}_i)) - \operatorname{rank}(\mathcal{E}_i)) = kn - kn + k^2 + 4.$$

So, if one knows that all the terms in the sum are positive, one can apply the lemma to each bundle appearing in $\mathcal{F}$. This is the case, as we will see, of Proposition 2.1.15.

Proof. Let us suppose that the bundle β is not of type (D). If $\beta_k \geq 2$, then $\operatorname{rank}(\beta) \geq k$ and $|\beta|_1 - 1 \geq 2k$, which gives a contradiction with (2.4). So, suppose $\beta_k = 1$. If $\beta \neq$ (Z), it means that

$$\operatorname{rank}(\beta)(|\beta|_1 - 1) > k(k + 1),$$

which is a contradiction with (2.4). So one can suppose that $\beta_k = 0$. If $|\beta| \geq 5$, as $2 \operatorname{rank}(\beta) \geq k(k - 1)$, we have a contradiction with (2.4). If $|\beta| = 4$, (2.4) implies $k = 4$, but by studying the possible cases one sees that none satisfies (2.4). So $|\beta| \leq 3$. Apart from (B) and (C), there are cases $(3, 0, \ldots, 0)$ not satisfying (2.4), $(2, 1, 0, \ldots, 0)$ not satisfying (2.4), and type (Y) $= (1, 1, 1, 0, \ldots, 0)$. This last bundle satisfies (2.4) only for $k = 4, 5, 6$.

For $k = 4$, (Y)=(A). By studying the possible appearances of (Y) together with the other bundles (A), (B), (C), (Z), (D), the only cases we obtain are (d7) and (d8). □

Proposition 2.1.15 (Classification for $k \geq 4$, $|\beta^i|_2 = 0$)**.** *Suppose $k, n-k \geq 4$. If the bundles appearing as summands in $\mathcal{F} = \oplus_i \mathcal{E}_i$ all satisfy $|\beta^i|_2 = 0$, then the only possible cases are (d1),(d2),(d3),(d4),(d6) of Table B.1.*

Proof. The hypothesis permits to apply Lemma 2.1.13 (see also Remark 2.1.14). Then the only possible bundles are those mentioned: (A), (B), (C), (Z), (D). Let us define, for a bundle β, the quantity

$$\xi = \xi(\beta) = \mathrm{rank}(\beta)(k(\frac{\mathrm{dex}(\beta)}{\mathrm{rank}(\beta)}) - 1).$$

If (Z) is a summand, then

$$k^2 + 4 - \xi(Z) = 4,$$

and among the other bundles the only one which has $\xi \leq 4$ is (D) for $k = 5, p = 1$. But then $n = \mathrm{dex}(Z) + \mathrm{dex}(D) = k + 2$, and this case is in Proposition 2.1.12.

If (A) is a summand, we have

$$k^2 + 4 - \xi(A) = 4 + 2k.$$

Then another (A) cannot be added, otherwise, by computing ξ, $k = 4$ and $n = 7$, and this case is in Proposition 2.1.12. Similarly by adding (C). With (B), it is only possible to have $k = 5$ and $n = 9$, which is prohibited by Lemma 2.1.8, or $k = 4$, which gives cases (d1) and (d9). Then, if there are just line bundles (D) in addition to (A), by studying ξ we have $p \leq 2$, and one can see that all the cases arising have already been studied.

If (B) is a summand, then

$$k^2 + 4 - \xi(B) = 4 + \frac{k^2 - k}{2}.$$

Suppose moreover that there is another (B) ($k^2 + 4 - 2\xi(B) = 4$); then one can have only a bundle of type (D) with $p = 2$ (case (d2)) or two bundles of type (D) with $p = 1$ (case (d3)). If instead one supposes that there is a bundle of type (C), one finds the only possibility to be (d4). Finally, by supposing to add just line bundles, then their number must be $k(n-k) - 4 - \mathrm{rank}(B)$, and the sum of their dex must be equal to $n - \mathrm{dex}(B) = n - k + 1$. By imposing $\frac{\mathrm{dex}}{\mathrm{rank}} \geq 1$, which is true for line bundles, and knowing that $n \geq 2k$ by Lemma 2.1.8, one finds for k the equation $k^2 - k - 10 \leq 0$, which has no solution for $k \geq 4$.

If (C) is a summand, then again

$$k^2 + 4 - \xi(C) = 4 + \frac{k^2 - k}{2};$$

by adding another (C), one gets only the case (d5): $(2, 0, \ldots, 0)^{\oplus 2}$ in $\mathrm{Gr}(4, 10)$.

Therefore, let us suppose the other bundles are only line bundles. One must have $0 \leq \text{rank}(\mathcal{F}) - \text{rank}(C) \leq n - (\text{dex}(C))$ (for line bundles $\frac{\text{dex}}{\text{rank}} \geq 1$). Moreover, by Lemma 2.1.8, one can suppose $n \geq 2$. Putting all together, one finds that the only possible case is (d6).

If there are only bundles of type (D), as

$$\frac{n}{k(n-k)-4} = \frac{\text{dex}}{\text{rank}} \geq 1,$$

and as $k, n-k \geq 4$, no other case arises. □

Before digging in the proof of the classification, let us explain the plan we will follow. Proposition 2.1.12 deals with the classification when $k \leq 3$, and an argument of symmetry allows us to suppose that $2k > n$. This will imply that all remaining bundles $\mathcal{F}$ in the classification satisfy Lemma 2.1.13, which restricts the possible addends of $\mathcal{F}$ to a finite set. The last step of the proof is to ensure that the classification is complete, i.e. that all remaining $\mathcal{F}$ satisfy the hypothesis of Proposition 2.1.15. This is done by a case by case analysis of the possible combinations of addends of the type allowed by Lemma 2.1.13.

Proof of the classification. As a consequence of Proposition 2.1.12, we can suppose $k, n-k \geq 4$. Using the isomorphism of $\text{Gr}(k,n)$ with $\text{Gr}(n-k,n)$, we suppose also $2k \geq n$. When $2k = n$,

$$\xi = \text{rank}(\beta)(k(\text{dex}(\beta)) - 1) = \text{rank}(\beta)((n-k)(\text{dex}(\beta)) - 1),$$

and this symmetry implies that all the bundles satisfy Lemma 2.1.13. Then, dropping the hypothesis $|\beta|_2 = 0$, one only has to "symmetrize" the results found in Proposition 2.1.15; this means that the cases (d2.1), (d3.1) and (d9.1) are to be added to the classification.

So, from now on, $2k > n$. As the expressions of the form $(a, \dots, a; a, \dots, a)$ are not considered, and $\beta_i \geq \beta_{i+1}$, either $(n-k)(|\beta|_1 - 1) \geq k|\beta|_2$ or $(n-k)|\beta|_1 \geq k(|\beta|_2 + 1)$, and in both cases $2k > n$ implies that all the terms of the sum on the right side of $k^2 + 4 = \sum_i \xi(\mathcal{E}^i)$ are positive. Then Lemma 2.1.13 can be applied. As we have Proposition 2.1.15, and using $\text{Gr}(k,n) \longleftrightarrow \text{Gr}(n-k,n)$, it remains to deal just with the following situation: there exists i_0 for which $|\beta^{i_0}|_2 = 0$ (and the corresponding bundle is not of rank one), but this doesn't hold for every i.

By Lemma 2.1.8, this bundle must be either (A) or (Z). As $k \geq 5$, by computing ξ, one cannot have: (A) $\oplus$ (Z), (Z) $\oplus$ (Z), (A) $\oplus$ (A). For the bundles of type (D), let us change notation:

$$(p, \dots, p; \beta_{k+1}, \dots, \beta_n) \longrightarrow (0, \dots, 0; -\delta_1, \dots, -\delta_{n-k}) \longleftrightarrow (\delta_{n-k}, \dots, \delta_1; 0, \dots, 0)$$

where $\longleftrightarrow$ stands for $\text{Gr}(k,n) \longleftrightarrow \text{Gr}(n-k,n)$. Then, if $\beta^{i_0} = \text{(Z)}$, $k^2 + 4 - \xi(Z) = 4$; but the presence of a bundle δ which is of rank $\neq 1$ leads to a contradiction;

indeed

$$\operatorname{rank}(\delta)(\frac{k}{n-k}|\delta|-1) > (n-k)(|\delta|-1) \geq 4(|\delta|-1),$$

where $|\delta^i| = \sum_j \delta^i_j$.

As a result, the bundles present as summands of $\mathcal{F}$ are: one of type (A) and the others of type (D), with at least one which is not a line bundle. One has $k^2+4-\xi(A) = 4+2k$. Then the condition $\sum_{i\neq i_0}(\xi(\beta^i)) + \xi(A) = k^2+4$ becomes

$$\sum_{i\neq i_0}(\frac{k}{n-k}|\delta^i|-1)\operatorname{rank}(\delta^i) = 4+2k.$$

If for the bundle δ which is not of rank one $\delta_1 \geq 1$, then

$$\operatorname{rank}(\delta)(\frac{k}{n-k}|\delta|-1) > k(n-k) \geq 4k,$$

which is a contradiction.

Therefore $\delta_1 = 0$. Define

$$\psi(\delta) = \operatorname{rank}(\delta)(\frac{k}{n-k}|\delta|-1).$$

If $|\delta| \geq 4$, $\psi(\delta) \geq 3k+1$. So, one is led to consider $|\delta| = 2, 3$. With a similar estimate, one can eliminate $\delta = (2, 1, 0, \ldots, 0)$ and $(3, 0, \ldots, 0)$. The bundle $(1, 1, 1, 0, \ldots, 0)$ is possible just for $n-k = 4$; but then

$$n - \operatorname{dex}(A) - \operatorname{dex}(\delta) = 2 \ , \ k(n-k) - 4 - \operatorname{rank}(A) - \operatorname{rank}(\delta) = 3k-8,$$

and one easily verifies that neither line bundles nor the bundles $\beta = (1, \ldots, 1, 0, 0)$, $(2, \ldots, 2, 0)$ can be added to give new cases in the classification. Therefore $\delta = 2$ and coming back to the notation with β, the last cases to study are those with (A)$\oplus(1, \ldots, 1, 0, 0)$ or (A)$\oplus(2, \ldots, 2, 0)$.

If (A)$\oplus(2, ..., 2, 0)$ is a factor of $\mathcal{F}$, then $n - dex(A) - dex(2, ..., 2, 0) = 0$, so there cannot be other bundles. Then equation (2.2) gives

$$0 = 4kn - n^2 - 3k^2 - n - k - 8 = -(n-2k)^2 + k^2 - n - k - 8 = k^2 - a^2 + a - 8 - 3k$$

where $a = 2k-n$. Integer solutions for k are given only if $4a^2-4a+41 = b^2 = c^2+40$, where $c = 2a-1$, and b is an integer. By writing down all the integer solutions for $(b+c)(b-c) = 40$, none gives new cases.

If (A)$\oplus(1, ..., 1, 0, 0)$ is a factor of $\mathcal{F}$, then $n - dex(A) - dex(1, ..., 1, 0, 0) = 2$, so there cannot be summands other than (A), $(1, ..., 1, 0, 0)$ and line bundles. Then $k(n-k) - 4 - rank(A) - rank(1, ..., 1, 0, 0)$ can be only 2 or 1, and equation (2.2) gives

$$0 = 4kn - n^2 - 3k^2 + n - 3k - c = -(n-2k)^2 + k^2 + n - 3k - c = k^2 - a^2 - a - c - k$$

where $a = 2k - n$, and c can be 10 or 12. Integer solutions for k are given only if $4a^2 + 4a + 4c + 1 = b^2 = d^2 + 4c$, where $d = 2a + 1$, and b is an integer. By writing down all the integer solutions for $(b+d)(b-d) = 4c$, for $c = 10, 12$, none gives new cases. □

Remark 2.1.16. Having dealt with the combinatorics of the problem, we turn to the geometry. All the bundles we have considered so far, and appearing in Table B.1, are globally generated (Remark 2.1.5). Therefore, by applying the usual Bertini theorem, our subvarieties with trivial canonical bundle are smooth. The same will hold when dealing with the classical Grassmannians, as in that case too all the bundles considered will be globally generated.

2.1.2. CY vs IHS

We want now to show how to distinguish between CY and IHS manifolds in an efficient way because we have to deal with a great number of cases. Actually for this, by Proposition 1.5.7, it will be enough to compute the Euler characteristic of the trivial bundle of the variety Y with trivial canonical bundle in question.

For the actual computation of $\chi(\mathcal{O}_Y)$ it is possible to use the *Hirzebruch-Riemann-Roch theorem*, which gives:

$$\chi(\mathcal{O}_Y) = \int_Y \mathrm{td}(Y)$$

where $\mathrm{td}(Y) = \mathrm{td}(TY)$ is the todd class of the tangent bundle. SCHUBERT2 allows to compute easily these quantities for subvarieties of Grassmannians.

Another aspect of these varieties that can be studied is their Hodge numbers. A tool which is useful in this sense is the Koszul complex for a variety Y which is the zero locus of a section of a vector bundle $\mathcal{F}$ over another variety G. If the bundle has rank r, and $\mathrm{codim}_G(Y) = r$, then one has the exact sequence:

$$0 \to \Lambda^r \mathcal{F}^* \to \Lambda^{r-1}\mathcal{F}^* \to \cdots \to \Lambda^2 \mathcal{F}^* \to \mathcal{F}^* \to \mathcal{O}_G \to \mathcal{O}_Y \to 0 \ .$$

Using this complex, tensoring it by any other bundle, it is possible to find the cohomology groups of the restriction of the bundle to Y. Moreover, one can use the short exact sequence

$$0 \to \mathcal{F}^*|_Y \to \Omega^1_G|_Y \to \Omega^1_Y \to 0$$

to study the cohomology groups of the cotangent bundle of Y. This is not enough in general; one needs to know the cohomology groups on the variety G. But for

				1				
			0		0			
		0		1		0		
	0		0		0		0	
1		145		628		145		1
	0		0		0		0	
		0		1		0		
			0		0			
				1				

Table 2.1. – Hodge diamond of case (c6) in Table B.1

this it is possible, as $G = \mathrm{Gr}(k, n)$ and $\mathcal{F}$ is homogeneous, to use Bott's theorem ([Bot57], [Küc95, Theorem 2.3] for the version that is needed here).

Example 2.1.17. It is a (lengthy) exercice to compute the Hodge Diamond of case (c6) in Table B.1; the result is displayed in Table 2.1.

However, this method takes some time to be employed, though it is not complicated using the Littlewood-Richardson rule. Therefore we decided not to include computations of cohomology groups apart from when necessary.

2.1.3. Zero loci of Irreducible bundles in ordinary Grassmannians and hyper-Kähler manifolds

The examples of Beauville-Donagi and Debarre-Voisin may suggest that other HK of dimension 4 may exist which are zero loci of homogeneous bundles over ordinary Grassmannians. We proved this is false for completely reducible bundles. It should be noted, however, that the two examples arise by considering homogeneous bundles which are irreducible. One could drop the hypothesis of dimension 4 and keep the irreducibility condition, and try to solve the same problem. This is the motivation for what follows.

Consider a variety $Y \subset Gr(k, n)$ with trivial canonical bundle which is the zero locus of a general section of an irreducible homogeneous bundle F. By counting dimensions, and up to the isomorphism with $Gr(n-k, n)$, we can suppose that

$$F = \mathbb{S}_{\lambda_1,\ldots,\lambda_k}\mathcal{U}^*.$$

In the above expression, $\mathbb{S}_{\lambda_1,\ldots,\lambda_k}(\cdot)$ denotes the Schur functor associated to $\Lambda := (\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \ldots \geq \lambda_k$ are non-negative integers. Let us denote $\lambda :=$

$\sum_i \lambda_i$; the rank of F depends only on Λ. Then, we have the following formulas:

$$k\det(F) = \operatorname{rank}(F)\lambda, \tag{2.5}$$

$$\dim(Y) = k(n-k) - \operatorname{rank}(F).$$

The condition $K_Y = \mathcal{O}_Y$ then, by the adjunction formula, becomes

$$n = \operatorname{rank}(F)\frac{\lambda}{k}.$$

We will use the Koszul complex in order to find when such an Y may be HK. In fact, we can ask ourselves when $\mathrm{H}^2(Y, \mathcal{O}_Y) \cong \mathbb{C}$. By looking at the Koszul complex, this is possible only if, for a certain $u \leq \operatorname{rank}(F)$,

$$\mathrm{H}^{u+2}(\wedge^u F^*) \cong \mathbb{C} \oplus \ \{\text{eventual other terms}\}. \tag{2.6}$$

Therefore, let us study the cohomology of $\wedge^u F^* = \wedge^u(\mathbb{S}_{\lambda_1,\ldots,\lambda_k}\mathcal{U})$. We have that

$$\wedge^u(\mathbb{S}_{\lambda_1,\ldots,\lambda_k}\mathcal{U}) = \oplus\ \mathbb{S}_{\mu_1,\ldots,\mu_k}\mathcal{U}$$

for a finite number of non-increasing sequences $(\mu_1, \ldots, \mu_k)$ such that $\mu := \sum_i \mu_i = \lambda u$. By Bott's theorem, the $(u+2)-$cohomology of $\mathbb{S}_{\mu_1,\ldots,\mu_k}\mathcal{U}$ is isomorphic to $\mathbb{C}$ if and only if there exists an integer $1 \leq j \leq k$ such that

$$j(n-k) = u + 2\ ,$$

$$\mu_i = n - k + j \quad \text{for} \quad i \leq j \quad \text{and}$$

$$\mu_i = j \quad \text{for} \quad i > j\ .$$

Now, as $\mu = \lambda u$, one finds the relation

$$j = \frac{2\lambda}{(\lambda-1)n - k\lambda}.$$

The inequality $j \geq 1$ gives the inequality which bounds the number of cases to study, i.e.

$$n \leq (2+k)\frac{\lambda}{\lambda-1}. \tag{2.7}$$

This is the right bound to study; actually, by imposing condition (2.5), we get:

$$\operatorname{rank}(F) \leq (2+k)\frac{k}{\lambda-1}. \tag{2.8}$$

But $\operatorname{rank}(F)$ grows fast with λ, which can be supposed to be ≥ 3. In what follows we apply this inequality to show that, fixing the "form" of Λ, we can find all the cases in which the variety Y may be HK. Before, let us see some examples

Example 2.1.18 (Beauville-Donagi generalized)*.* We study here the case where

$$F = \mathbb{S}_{\lambda_1,\dots,\lambda_k}\mathcal{U}^* = \mathbb{S}_{\lambda,0,\dots,0}\mathcal{U}^* = S^\lambda\mathcal{U}^*$$

is the λ-th symmetric power of $\mathcal{U}^*$. In this hypothesis,

$$\operatorname{rank}(F) = \binom{k+\lambda-1}{\lambda}.$$

Then one can apply (2.8). For example, in the case $\lambda = 3$, one gets $k \leq 2$, and therefore only the Beauville-Donagi case (type X_1). If $\lambda = 4$, then $k \leq 1$.

Example 2.1.19 (Debarre-Voisin generalized)*.* In this case

$$F = \mathbb{S}_{\lambda_1,\dots,\lambda_k}\mathcal{U}^* = \mathbb{S}_{1,\dots,1,0,\dots,0}\mathcal{U}^* = \wedge^\lambda\mathcal{U}^*$$

is the λ-th exterior power of $\mathcal{U}^*$ ($\lambda \leq k$). In this hypothesis,

$$\operatorname{rank}(F) = \binom{k}{\lambda}.$$

Then, by applying (2.8) for example with $\lambda = 3$, we get $k \leq 6$. Again, the only relevant solution is the Debarre-Voisin example (type X_2). If $\lambda = 4$, then $k \leq 6$; the only possible solution is $k = 6$, but in this case the dimension of Y is odd.

Theorem 2.1.20. *Let Y be the zero locus of a general section of an irreducible homogeneous bundle $F = \mathbb{S}_\Lambda\mathcal{U}^*$ over the ordinary Grassmannian. If Y is hyper-Kähler, then it is of type X_1 or X_2, or it is a quartic inside $\mathbf{P}^3$.*

The proof will be split in several (eleven!) lemmas which deal with different Λ's. We will always suppose that Y is the zero locus of a general section of a irreducible bundle $F = \mathbb{S}_\Lambda\mathcal{U}^*$ over the ordinary Grassmannian $\operatorname{Gr}(k,n)$. For the lemmas, we will assume that F is not a line bundle.

Lemma 2.1.21. *If $\lambda_i \neq \lambda_{i+1}$ with $i \geq 3$ and $k - i \geq 3$, and Y is HK, then Y is of type X_2.*

Proof. The hypothesis implies $k \geq 6$ and $\lambda \geq 3$. Then, equation 2.8 implies $k \leq 6$. For $k = 3$, the only possibility is $\Lambda = (1,1,1,0,0,0)$, i.e. the Debarre-Voisin case. □

Lemma 2.1.22. *If $\lambda_2 \geq \lambda_3 + 2$ (respectively $\lambda_{k-2} \geq \lambda_{k-1} + 2$), $k \geq 4$, and Λ does not satisfy the hypothesis of the previous lemma, then Y is not HK.*

Proof. We have $\lambda \geq 4$ and $\operatorname{rank}(F) \geq \operatorname{rank}(\mathbb{S}_{2,2,0,\dots,0}\mathcal{U}^*)$. This is computable using the Weyl character formula, and with equation 2.8 it implies that $k \leq 2$, which is not possible. □

Lemma 2.1.23. *If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (respectively $\lambda_{k-2} \neq \lambda_{k-1} \neq \lambda_k$), $k \geq 4$, and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. We have $\lambda \geq 3$ and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{2,1,0,\dots,0}\mathcal{U}^*)$. Therefore, using equation 2.8, we have that $k \leq 3$, which is not possible. □

Lemma 2.1.24. *If $\lambda_2 \neq \lambda_3$ and $\lambda_{k-2} \neq \lambda_{k-1}$, $k \geq 4$, and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. We have $\lambda \geq 4$ and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{2,2,1,\dots,1,0,0}\mathcal{U}^*)$. Therefore, using equation 2.8, we have that $k \leq 3$, which is not possible. □

Lemma 2.1.25. *If $\lambda_2 \neq \lambda_3$ and $\lambda_{k-1} \neq \lambda_k$ (respectively $\lambda_1 \neq \lambda_2$ and $\lambda_{k-2} \neq \lambda_{k-1}$), $k \geq 4$, and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. We have $\lambda \geq 3$, and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{2,1,\dots,1,0,0}\mathcal{U}^*)$. Therefore, using equation 2.8, we have that $k \leq 3$, which is not possible. □

Lemma 2.1.26. *If $\lambda_1 \geq \lambda_2 + 3$ (respectively $\lambda_{k-1} \geq \lambda_k + 3$), Λ does not satisfy any of the hypothesis of the previous lemmas and Y is HK, then Y is of type X_1.*

Proof. We have $\lambda \geq 3$, and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{3,0,\dots,0}\mathcal{U}^*)$. Therefore equation 2.8 implies $k \leq 2$. Then, $k = 2$ gives only the Beauville-Donagi case. □

Lemma 2.1.27. *If $\lambda_1 \geq \lambda_2 + 2$ and $\lambda_{k-1} \neq \lambda_k$ (respectively $\lambda_1 \neq \lambda_2$ and $\lambda_{k-1} \geq \lambda_k + 2$), $k \geq 3$, and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. We have $\lambda \geq 4$, and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{3,1,\dots,1,0}\mathcal{U}^*)$. Therefore equation 2.8 implies $k \leq 1$, which is not possible. □

Lemma 2.1.28. *If $\lambda_1 \geq \lambda_2 + 1$ and $\lambda_{k-1} \geq \lambda_k + 1$, $k \geq 3$, and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. We have $\lambda \geq 3$, and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{2,1,\dots,1,0}\mathcal{U}^*)$. Therefore equation 2.8 implies $k \leq 2$, which is not possible. □

Lemma 2.1.29. *If $\lambda_1 \geq \lambda_2 + 2$ (respectively $\lambda_{k-1} \geq \lambda_k + 2$), $k \geq 2$, and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. We have $\lambda \geq 2$, and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{2,0,\dots,0}\mathcal{U}^*)$. If $\lambda \geq 2 + k$ as for $\Lambda = (3, 1, \dots, 1)$, then $k \leq 1$ by equation 2.8.

If $\lambda \geq 2k - 2$ and $k \geq 3$ as for $\Lambda = (2, \dots, 2, 0)$, then $k \leq 3$; the only possibility is $F = \mathbb{S}_{2,2,0}$ in $\mathrm{Gr}(3, 8)$, whose dimension is odd (equal to nine).

Finally, if $\Lambda = (2, 0, \dots, 0)$, then equation 2.5 implies $n = k + 1$. But a section of $S^2\mathcal{U}^*$ over $\mathrm{Gr}(k, k + 1)$ is empty, as there exists no k-plane isotropic with respect to a non-degenerate symmetric form over a space of dimension $k + 1$. □

Lemma 2.1.30. *If $\lambda_2 \geq \lambda_3+1$ (or $\lambda_{k-2} \geq \lambda_{k-1}+1$), $k \geq 4$, and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. We have $\lambda \geq 2$. If $\Lambda = (1,1,0,\dots,0)$, then equation 2.5 implies $n < k$.

If $\Lambda = (1,\dots,1,0,0)$, equation 2.8 implies $k \leq 5$. In the case $k=5$, we have $\wedge^3\mathcal{U}^*$ over $\mathrm{Gr}(5,6)$, for which the zero locus of a general section is empty.

If $\lambda_2 \neq \lambda_3$ and $\lambda_k \geq 1$, then $\lambda \geq 6$ and $\mathrm{rank}(F) \geq \mathrm{rank}(\mathbb{S}_{2,2,1,\dots,1}\mathcal{U}^*)$. Equation 2.8 implies $k \leq 3$, which is not possible. $\square$

Lemma 2.1.31. *If $\lambda_1 \geq \lambda_2+1$ (respectively $\lambda_{k-1} \geq \lambda_k+1$), and Λ does not satisfy any of the hypothesis of the previous lemmas, then Y is not HK.*

Proof. In the hypothesis we made, we have that $\Lambda = (a+1,a,\dots,a)$. Equation 2.5 implies $n=\lambda$, and therefore $\lambda \geq k$. If $\Lambda = (2,1,\dots,1)$, we have $n=k+1$. But in this case Y is 0-dimensional.

If $\lambda \geq 2k-1$ and $k \geq 3$ as for $\Lambda = (2,\dots,2,1)$, equation 2.8 implies $k \leq 3$, i.e. actually $k=3$, $n=5$, $\Lambda = (2,2,1)$. But in this case Y is 5-dimensional, so it cannot be HK (odd dimension). $\square$

Proof of Theorem 2.1.20. The weight Λ either satisfies the hypothesis of the lemmas which go from Lemma 2.1.21 to Lemma 2.1.31, either gives a rank one bundle F. In this last hypothesis, Y is an hypersurface in $\mathrm{Gr}(k,n)$, and therefore it is HK only when it is a $K3$ quartic surface in $\mathbf{P}^3$. $\square$

2.2. Fourfolds in classical Grassmannians

In this section we study subvarieties of classical, i.e. symplectic and orthogonal, Grassmannians, showing that a result similar to Theorem 2.1.1 holds. The geometry of these isotropic Grassmannians is well known (see [FP98, Chapter 6] for some basic properties). They belong to the class of *flag manifolds*, for which a good reference is [Bri05]. Their classical and quantum cohomology has been the subject of several studies (for instance, see [BKT09]).

We will refer to Example 1.3.3 for notations on homogeneous bundles over the classical Grassmannians. The main theorem is the following.

Theorem 2.2.1. *Let Y be a fourfold with $K_Y = \mathcal{O}_Y$ which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle $\mathcal{F}$ over the symplectic Grassmannian $\mathrm{IGr}(k,2n)$ (respectively the odd orthogonal Grassmannian $\mathrm{OGr}(k,2n+1)$, the even orthogonal Grassmannian $\mathrm{OGr}(k,2n)$), and which does not appear in the analogous classification for the ordinary Grassmannian. Up to identifications, the only possible cases are those appearing in Table B.2 (respectively Table B.3, Table B.4).*

Remark 2.2.2. The varieties Y appearing in Theorem 2.2.1 are smooth (see Remark 2.1.16).

Remark 2.2.3. All the cases studied refer to subvarieties of classical Grassmannians which are the quotient of a classical group G by a parabolic subgroup associated to a single simple root (see Definition 1.1.1). So, we have skipped the classification of subvarieties of $\mathrm{OGr}(n-1, 2n)$, because in this case the corresponding parabolic subgroup is associated to the last two simple roots of the Dynkin diagram D_n. However, for the sake of completeness we have also reported the analogous classification for $\mathrm{OGr}(n-1, 2n)$ at the end of this section.

Remark 2.2.4. It is well known that the Grassmannians $\mathrm{OGr}(n-1, 2n-1)$ and $\mathrm{OGr}(n, 2n)$ are isomorphic. But the bundles which are homogeneous in one case may not be homogeneous in the other. For example, consider $\Lambda^2\mathcal{U}^*$ on $\mathrm{OGr}(n, 2n)$, which is the tangent bundle. Pulling back this bundle via the isomorphism gives the tangent bundle T on $\mathrm{OGr}(n-1, 2n-1)$, which is not a priori the second exterior power of a vector bundle homogeneous with respect to $\mathfrak{so}(2n-1)$, and is not irreducible. On the contrary, $\mathcal{O}(1)$ on $\mathrm{OGr}(n, 2n)$ pulls back to the corresponding $\mathcal{O}(1)$ on $\mathrm{OGr}(n-1, 2n-1)$, and the same for $\mathcal{O}(\frac{1}{2})$. So, referring to Table B.3 and Table B.4, one can easily identify cases $(oz3)$ and $(oy1)$, $(oz4)$ and $(oy5)$, $(oz5)$ and $(oy4)$, $(oz7)$ and $(oy1.1)$.

We break the classification given by Theorem 2.2.1 into three parts, which correspond to subvarieties in symplectic, odd and even orthogonal Grassmannians. Furthermore, the method used to understand if the varieties we have found are CY or IHS is the one already used in Section 2.1.2. Indeed, we want to apply Proposition 1.5.7. In order to do so, we need to compute the Euler characteristic of the trivial bundle of the variety. This requires some technical facts about the cohomology of classical Grassmannians; in Appendix A we reported the details of how to do such a computation.

2.2.1. Symplectic Grassmannians

The symplectic Grassmannian $\mathrm{IGr}(k, 2n)$ will be thought of as the quotient G/P_k, where P_k is the maximal parabolic subgroup containing the standard Borel subgroup of positive roots in $G = \mathrm{Sp}(2n, \mathbb{C})$. Every irreducible homogeneous bundle is represented by its highest weight. A weight is represented by $\beta = (\beta_1, \dots, \beta_n)$ or by $(\beta_1, \dots, \beta_k; \beta_{k+1}, \dots, \beta_n)$ when this notation is needed, where

$$\beta = (\beta_1 - \beta_2)\lambda_1 + \lambda_2(\beta_2 - \beta_3) + \cdots + \lambda_{n-1}(\beta_{n-1} - \beta_n) + \lambda_n\beta_n,$$

and the λ_i's are the fundamental weights for $G = \mathrm{Sp}(2n, \mathbb{C})$. Notice that the parabolic algebra $Lie(P_k)$ has Levi factor $\mathfrak{sl}(k) \oplus \mathfrak{sp}(2(n-k))$ for $k \neq 1, n$, which is straightforward by looking at the Dynkin diagram.

Notation 2.2.5. As we work with globally generated bundles, from now on the notation will change: to indicate a bundle with highest weight β as before, we will write $(-\beta_k, \dots, -\beta_1; \beta_{k+1}, \dots, \beta_n)$, which is equivalent to taking the highest weight of the dual representation. In this way, a bundle $\alpha = (\alpha_1, \dots, \alpha_n)$, according to the new notation, is globally generated when $\alpha_1 \geq \dots \geq \alpha_n \geq 0$, i.e. when the weight α is dominant under the action of G. To understand why we use this "dualized" notation, refer to the explication before Remark 2.1.5.

Example 2.2.6. Over the symplectic Grassmannian $\mathrm{IGr}(3, 14)$, the dual tautological bundle $\mathcal{U}^*$ of rank 3 will be denoted by $(1, 0, 0; 0, 0, 0, 0) = (1, 0, \dots, 0)$, and the tautological "orthogonal" bundle $\mathcal{U}^\perp/\mathcal{U}$ of rank 8 by $(0, 0, 0; 1, 0, 0, 0)$.

The dimension of a bundle can be calculated explicitly: suppose $k \neq 1, n$;

$$\mathrm{rank}(\beta_1, \dots, \beta_n) = \dim_{\mathfrak{sl}(k)}(\beta_1, \dots, \beta_k) \times \dim_{\mathfrak{sp}(2(n-k))}(\beta_{k+1}, \dots, \beta_n),$$

where

$$\dim_{\mathfrak{sl}(r)}(\beta_1, \dots, \beta_r) = \prod_{1 \leq i < j \leq r} \frac{j - i + \beta_i - \beta_j}{j - i},$$

and

$$\dim_{\mathfrak{sp}(2r)}(\beta_1, \dots, \beta_r) = \prod_{1 \leq i < j \leq r} \frac{j - i + \beta_i - \beta_j}{j - i} \prod_{1 \leq i \leq j \leq r} \frac{2r + 2 - j - i + \beta_i + \beta_j}{2r + 2 - j - i}$$

are the Weyl character formula relative to the corresponding Lie algebras (see [FH91, Chapter 24, Equation 24.19]). This formula is a consequence of the form of the Levi factor of P and of Remark 2.1.7.

One defines:

$$\mathrm{dex}(\beta) = (\frac{|\beta|_1}{k}) \, \mathrm{rank}(\beta)$$

where $|\beta|_1 = \sum_{i=1}^k \beta_i$. Then, similarly to the case for the ordinary Grassmannian (see Remark 2.1.10),

$$\det(\beta) = \mathcal{O}(\mathrm{dex}(\beta)).$$

Therefore, fourfolds with trivial canonical bundle correspond to homogeneous vector bundles $\mathcal{F} = \oplus_i \mathcal{E}_i$, with

$$\sum_i \mathrm{rank}(\mathcal{E}_i) = k(2n - k) - \frac{k(k-1)}{2} - 4 \ , \tag{2.9}$$

$$\sum_i \mathrm{dex}(\mathcal{E}_i) = 2n - k + 1 \ . \tag{2.10}$$

Classification in symplectic Grassmannian

We recall that $\mathrm{IGr}(k, 2n)$ is embedded naturally in $\mathrm{Gr}(k, 2n)$ as the zero locus of a general section of $\Lambda^2\mathcal{U}^*$ (see Example 1.3.6). The following lemma is useful to avoid repeating cases already considered in Theorem 2.1.1 and will be used throughout the proof of the classification.

Lemma 2.2.7. *Suppose $\mathcal{F} = \oplus_i \mathcal{E}_i$ is as in the hypothesis of Theorem 2.2.1. In order to find new varieties in $\mathrm{IGr}(k, 2n)$ with respect to the case of the ordinary Grassmannian, it is necessary that for at least one bundle $\mathcal{E}_i = \beta_i$, $|\beta_i|_2 \neq 0$.*

Proof. The tautological bundle over $\mathrm{IGr}(k, 2n)$ is the restriction of the tautological bundle over $\mathrm{Gr}(k, 2n)$. Similarly, the bundle represented by the weight $\beta = (\beta_1, \dots, \beta_k, 0, .., 0)$ over the symplectic Grassmannian is the restriction of the bundle represented by the same weight β over the ordinary Grassmannian. Therefore if for all i, $|\beta_i|_2 = 0$, the resulting fourfold is already the zero locus of a homogeneous bundle over $\mathrm{Gr}(k, 2n)$; as a consequence it has already been considered in Theorem 2.1.1 □

So, one has to suppose that $k \neq n$. Moreover, if $k = 1$, $\mathrm{IGr}(1, 2n) = \mathrm{Gr}(1, 2n)$, so one can also suppose $k \neq 1$.

One can assume that $(\mathcal{U}^*)^{\oplus 2}$ over $\mathrm{IGr}(k, 2n)$ does not appear as summand in $\mathcal{F} = \oplus_i \mathcal{E}_i$; in fact, taking a zero section locus of this bundle in $\mathrm{IGr}(k, 2n)$ is equivalent to restricting to the space $\mathrm{IGr}(k, 2(n-1))$.

Finally, remark that for any bundle $\mathcal{E}$ globally generated summand of $\mathcal{F}$ (also for the odd and even orthogonal Grassmannians),

$$\frac{\mathrm{dex}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} = \frac{|\beta|_1}{k} \geq \frac{1}{k} \quad . \tag{2.11}$$

The proof now consists in studying cases with low k (in $\mathrm{IGr}(k, 2n)$), and then eliminating any other possibility.

Proposition 2.2.8 (Classification for $k \leq 3$). *If $k \leq 3$, as suitable $\mathcal{F}$ satisfying the hypothesis of Theorem 2.2.1 one has all and only the bundles appearing in Table B.2.*

Proof. $k = 2$

In this case $\mathrm{rank}(\mathcal{F}) = 4n - 9$, and $\mathrm{dex}(\mathcal{F}) = 2n - 1$. If one has the bundle λ, with $\lambda_1 \neq \lambda_2$ and $|\lambda|_2 \neq 0$, then

$$\mathrm{rank}(\lambda) \geq 2(2(n-2)) = 4n - 8,$$

which is impossible; so, by checking the dimensions of the corresponding modules, and comparing with (2.9) and (2.10), one remains with the bundles $(p, q; 0, \dots, 0)$

for $p \geq q$, $(p, p; 1, 0, \ldots, 0)$ for $n \geq 3$, and finally for $n = 4$ with those mentioned and with $(p, p; 1, 1)$. Then one checks that this gives the bundles (sb) for $n = 3, 4$. For $n > 4$, one knows that there must be one summand of the form $(p, p; 1, 0, \ldots, 0)$, and one sees that (2.10) implies $p = 1$. But then

$$\frac{\operatorname{dex}(\mathcal{F}) - \operatorname{dex}((1,1;1,0,\ldots,0))}{\operatorname{rank}(\mathcal{F}) - \operatorname{rank}((1,1;1,0,\ldots,0))} = \frac{3}{2n-5} \geq \frac{1}{2}$$

by (2.11), which means $n = 5$, for which one can check by hand that there is no other possibility.

$k = 3$

In this hypothesis $\operatorname{rank}(\mathcal{F}) = 6n - 16$, and $\operatorname{dex}(\mathcal{F}) = 2n - 2$. Doing the computation by hand, for $n = 4$ one finds all the cases (sc). For a bundle β such that there exists $1 \leq i \leq k-1$ such that $\beta_i < \beta_{i+1}$ and $|\beta|_2 \neq 0$, the minimal value of dex corresponds to the bundle $(2, 1, 1; 1, 0, \ldots, 0)$. Equation (2.10) then implies that it cannot appear for $n \geq 5$. Then one only has $(p, q, r; 0, \ldots, 0)$ or $(p, p, p; \beta_{k+1}, \ldots, \beta_n)$, and $\beta_{k+1} \neq 0$. As a consequence of (2.11), there is at least once the bundle $(1, 0, \ldots, 0)$, therefore

$$\operatorname{rank}(\mathcal{F}) - \operatorname{rank}((1,0,\ldots,0)) = 6n - 19\ ,\ \operatorname{dex}(\mathcal{F}) - \operatorname{dex}((1,0,\ldots,0)) = 2n - 3.$$

This last equation gives as the only possibility for the second type bundles that $p = 1$, and for $n \geq 6$, $|\beta|_2 = 1$. Studying separately $n = 5$ and $n \geq 6$ one checks that there are no other cases. □

Proof of the classification of Table B.2. As a consequence of the previous proposition, it is sufficient to show that for $k \geq 4$, there is no bundle $\mathcal{F}$ with the properties required in Theorem 2.2.1. As one knows $\operatorname{rank}(\mathcal{F})$ and $\operatorname{dex}(\mathcal{F})$, one finds that, except for the case $k = 4, n = 5$, one of the summands must be $(1, 0, .., 0)$ (otherwise $k \operatorname{dex}(\mathcal{F}) \geq 2 \operatorname{rank}(\mathcal{F})$). As there cannot be two such bundles, one can write

$$\frac{\operatorname{dex}(\mathcal{F}) - \operatorname{dex}((1,0,\ldots,0))}{\operatorname{rank}(\mathcal{F}) - \operatorname{rank}((1,0,\ldots,0))} \geq \frac{2}{k}\ ,$$

which gives $n \leq k + \frac{1}{2} + \frac{4}{k}$. This implies that the only cases that have to be studied are: $(k, n) = (4, 5), (5, 6), (6, 7), (7, 8), (8, 9)$. If $(k, n) = (4, 5)$, as for at least one bundle $\beta_5 \neq 0$, a similar reasoning on $\frac{\text{rank}}{\text{dex}}$ tells us that there must be one bundle $(1, 0, \ldots, 0)$. Then, simple combinatorics prevent any bundle to have the good properties. The remaining cases can be inspected explicitly. □

2.2.2. Odd Orthogonal Grassmannians

The odd orthogonal Grassmannian $\mathrm{OGr}(k, 2n+1)$ will be thought of as the quotient G/P_k, where P_k is the maximal parabolic subgroup containing the standard Borel subgroup of positive roots in $G = \mathrm{SO}(2n+1, \mathbb{C})$. Every irreducible homogeneous bundle is represented by its highest weight. A weight is represented by $\beta = (\beta_1, \dots, \beta_n)$ or by $(\beta_1, \dots, \beta_k; \beta_{k+1}, \dots, \beta_n)$ when this notation is needed, where

$$\beta = (\beta_1 - \beta_2)\lambda_1 + \lambda_2(\beta_2 - \beta_3) + \cdots + \lambda_{n-1}(\beta_{n-1} - \beta_n) + 2\lambda_n\beta_n,$$

the λ_i's are the fundamental weights for $\mathrm{SO}(2n+1, \mathbb{C})$, and the β_i's are all integers or all half integers. Notice that the parabolic algebra $Lie(P_k)$ has Levi factor $\mathfrak{sl}(k) \oplus \mathfrak{so}(2(n-k)+1)$ for $k \neq 1, n$, which is straightforward by looking at the Dynkin diagram.

Notation 2.2.9. As we work with globally generated bundles, from now on the notation will change: to indicate a bundle with highest weight β as before, we will write $(-\beta_k, \dots, -\beta_1; \beta_{k+1}, \dots, \beta_n)$, which is equivalent to taking the highest weight of the dual representation. In this way, a bundle $\alpha = (\alpha_1, \dots, \alpha_n)$, according to the new notation, is globally generated when $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$, i.e. when the weight α is dominant under the action of G. To understand why we use this "dualized" notation, refer to the explication before Remark 2.1.5.

Example 2.2.10. Over the orthogonal Grassmannian $\mathrm{OGr}(3, 15)$, the dual tautological bundle $\mathcal{U}^*$ of rank 3 will be denoted again by $(1,0,0;0,0,0,0)$, the tautological "orthogonal" bundle $\mathcal{U}^\perp/\mathcal{U}$ of rank 9 by $(0,0,0;1,0,0,0)$. With $\mathcal{T}_{+\frac{1}{2}}$ we will denote the bundle coming from the representation $(-\frac{1}{2}, \dots, -\frac{1}{2}; \frac{1}{2}, \dots, \frac{1}{2})$.

The dimension of a bundle can be calculated explicitly: suppose $k \neq 1, n$;

$$\mathrm{rank}(\beta_1, \dots, \beta_n) = \dim_{\mathfrak{sl}(k)}(\beta_1, \dots, \beta_k) \times \dim_{\mathfrak{so}(2(n-k)+1)}(\beta_{k+1}, \dots, \beta_n),$$

where

$$\dim_{\mathfrak{so}(2r+1)}(\beta_1, \dots, \beta_r) = \prod_{1 \leq i < j \leq r} \frac{j - i + \beta_i - \beta_j}{j - i} \prod_{1 \leq i \leq j \leq r} \frac{2r + 1 - j - i + \beta_i + \beta_j}{2r + 1 - j - i}$$

is the Weyl character formula (see [FH91, Chapter 24, Equation 24.29]). This formula is a consequence of the form of the Levi factor of P and of Remark 2.1.7. A similar formula holds when $k = 1, n$.

The definition of the function dex is the same as before (see Remark 2.1.10). Therefore, fourfolds with trivial canonical bundle correspond to homogeneous vector bundles $\mathcal{F} = \sum_i \mathcal{E}_i$, with

$$\sum_i \mathrm{rank}(\mathcal{E}_i) = k(2n + 1 - k) - \frac{k(k+1)}{2} - 4 \ , \tag{2.12}$$

$$\sum_i \mathrm{dex}(\mathcal{E}_i) = 2n - k \ . \tag{2.13}$$

Classification in odd orthogonal Grassmannian

We recall that $\mathrm{OGr}(k, 2n+1)$ is embedded naturally in $\mathrm{Gr}(k, 2n+1)$ as the zero locus of a general section of $S^2\mathcal{U}^*$ (see Example 1.3.6). The following lemma is similar to Lemma 2.2.7.

Lemma 2.2.11. *Suppose $\mathcal{F} = \oplus_i \mathcal{E}_i$ is as in the hypothesis of Theorem 2.2.1. In order to find new varieties in $\mathrm{OGr}(k, 2n+1)$ with respect to the case of the ordinary Grassmannian, it is necessary that for at least one bundle $\mathcal{E}_i = \beta$, $|\beta|_2 \neq 0$ or the β_i's are not integers (they can be half integers).*

Proof. The proof is the same as the one for Lemma 2.2.7. Notice only that half integer weights are associated to spin representations. □

One can assume that $(\mathcal{U}^*)$ over $\mathrm{OGr}(k, 2n+1)$ do not appear as summand in $\mathcal{F} = \oplus_i \mathcal{E}_i$; in fact, taking a zero section locus of this bundle in $\mathrm{OGr}(k, 2n+1)$ is equivalent to restricting to the space $\mathrm{OGr}(k, 2n)$.

The classification will be made in three steps: one has to distinguish the three particular cases: $k = n$, $k = n-1$, $k \leq n-2$. This is a consequence of the difference in these cases of the Dynkin diagram of B_n with the k-th root removed. The classification in Table B.3 is a direct consequence of the following propositions.

Proposition 2.2.12 (Classification for $k = n$)**.** *Over $\mathrm{OGr}(n, 2n+1)$, as suitable $\mathcal{F}$ satisfying the hypothesis of Theorem 2.2.1, one has all and only the cases (oy) appearing in Table B.3.*

Proof. Under the hypothesis of the proposition, $(\frac{1}{2}, \dots, \frac{1}{2})$ is a line bundle, "square root" of $\mathcal{O}(1)$. Moreover, one has $\mathrm{rank}(\mathcal{F}) = \frac{n(n+1)}{2} - 4$, $\mathrm{dex}(\mathcal{F}) = n$. One can study the rate $\frac{\mathrm{dex}}{\mathrm{rank}}$ and obtain constraints. Indeed $\mathrm{dex}(2, 0, \dots, 0) = n+1$, so this bundle cannot appear. On the other hand $\mathrm{dex}(1, 1, 0, \dots, 0) = n-1 = \mathrm{dex}(1, \dots, 1, 0)$, and for all the other bundles which are not line bundles, dex is greater than n. If $(1, 1, 0, \dots, 0)$ appears, $\mathrm{rank}(\mathcal{F}) - \mathrm{rank}(1, 1, 0, \dots, 0) = n-4$. As there must be also at least one bundle with half integers, the only possibility is $n = 6$, i.e. $(oy6)$. Therefore, except for this case, one checks that for all the other possible bundles, and therefore for $\mathcal{F}$, $\frac{\mathrm{dex}}{\mathrm{rank}} \geq \frac{1}{2}$, which implies $n \leq 4$. This gives the other cases (oy). □

Proposition 2.2.13 (Classification for $k = n-1$)**.** *Over $\mathrm{OGr}(n-1, 2n+1)$, as suitable $\mathcal{F}$ satisfying the hypothesis of Theorem 2.2.1, one has all and only the cases (ox) appearing in Table B.3.*

Proof. In this case notice that $(0, \dots, 0; \beta)$ is of rank $2\beta + 1$. If $n \neq 3, 4$, the minimal ratio $\frac{\mathrm{dex}}{\mathrm{rank}}$ is $\frac{2}{n-1}$ given by $(1, 1, 0, \dots, 0)$. But this implies $n \leq 3$. So the only cases to study are $n = 3$, $n = 4$, and this gives the cases (ox). □

Proposition 2.2.14 (Classification for $k \leq n-2$)**.** *If $k \leq n-2$, over $\mathrm{OGr}(k, 2n+1)$, as suitable $\mathcal{F}$ satisfying the hypothesis of Theorem 2.2.1, one has all and only the cases $(ob0)$, $(ob0.1)$ appearing in Table B.3.*

Proof. With this hypothesis, except for $k \leq 3$, we have $\frac{\mathrm{dex}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} \geq \frac{2}{k}$, which implies $k \leq 1$. So three cases have to be considered.

If $k = 1$, one easily sees that no possibility matches the requirements.

If $k = 2$, $\mathrm{rank}(\mathcal{F}) = 4n - 9$, $\mathrm{dex}(\mathcal{F}) = 2n - 2$, and if $n \geq 4$ then $\frac{\mathrm{dex}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} < 1$; therefore there must be at least one bundle $(\frac{1}{2}, \ldots, \frac{1}{2})$, whose rank is 2^{n-2}. This means that $2^{n-2} \leq 4n - 9$, so $3 \leq n \leq 5$. For $n = 4$, one gets the cases ($ob0$), ($ob0.1$), and no case for $n = 5$.

If $k = 3$, the same argument as before gives $5 \leq n \leq 8$, and inspecting case by case one finds that no other variety arises. $\square$

2.2.3. Even Orthogonal Grassmannians

The even orthogonal Grassmannian $\mathrm{OGr}(k, 2n)$ will be thought of as the quotient G/P_k, where P_k is the maximal parabolic subgroup containing the standard Borel subgroup of positive roots in $G = \mathrm{SO}(2n, \mathbb{C})$. Every irreducible homogeneous bundle is represented by its highest weight. A weight is represented by $\beta = (\beta_1, \ldots, \beta_n)$ or by $(\beta_1, \ldots, \beta_k; \beta_{k+1}, \ldots, \beta_n)$ when this notation is needed, where

$$\beta = (\beta_1 - \beta_2)\lambda_1 + \lambda_2(\beta_2 - \beta_3) + \cdots + \lambda_{n-1}(\beta_{n-1} - \beta_n) + \lambda_n(\beta_{n-1} + \beta_n),$$

the λ_i's are the fundamental weights for $\mathrm{SO}(2n, \mathbb{C})$, and the β_i's are all integers or all half integers. Notice that the parabolic algebra $Lie(P_k)$ has Levi factor $\mathfrak{sl}(k) \oplus \mathfrak{so}(2(n-k))$ for $k \neq 1, n, n-1$, which is straightforward by looking at the Dynkin diagram.

Notation 2.2.15. As we work with globally generated bundles, from now on the notation will change: to indicate a bundle with highest weight β as before, we will write $(-\beta_k, \ldots, -\beta_1; \beta_{k+1}, \ldots, \beta_n)$, which is equivalent to taking the highest weight of the dual representation. In this way, a bundle $\alpha = (\alpha_1, \ldots, \alpha_n)$, according to the new notation, is globally generated when $\alpha_1 \geq \cdots \geq \alpha_{n-1} \geq |\alpha_n|$, i.e. when the weight α is dominant under the action of G. To understand why we use this "dualized" notation, refer to the explication before Remark 2.1.5.

Example 2.2.16. Over the orthogonal Grassmannian $\mathrm{OGr}(3, 14)$, with $\mathcal{T}_{\pm\frac{1}{2}}$ we will denote $(-\frac{1}{2}, \ldots, -\frac{1}{2}; \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$ the bundle coming from the spin representations.

The dimension of a bundle can be calculated explicitly: suppose $k \neq 1, n, n-1$; then

$$\mathrm{rank}(\beta_1, \ldots, \beta_n) = \dim_{\mathfrak{sl}(k)}(\beta_1, \ldots, \beta_k) \times \dim_{\mathfrak{so}(2(n-k))}(\beta_{k+1}, \ldots, \beta_n),$$

where

$$\dim_{\mathfrak{so}(2r)}(\beta_1, \ldots, \beta_r) = \prod_{1 \leq i < j \leq r} \frac{j - i + \beta_i - \beta_j}{j - i} \prod_{1 \leq i < j \leq r} \frac{2r - j - i + \beta_i + \beta_j}{2r - j - i}$$

is the Weyl character formula (see [FH91, Chapter 24, Equation 24.41]). This formula is a consequence of the form of the Levi factor of P and of Remark 2.1.7. A similar formula holds when $k = 1, n$.

The definition of the function dex is the same as before (see Remark 2.1.10). Therefore, fourfolds with trivial canonical bundle correspond to homogeneous vector bundles $\mathcal{F} = \sum_i \mathcal{E}_i$, with

$$\sum_i \mathrm{rank}(\mathcal{E}_i) = k(2n-k) - \frac{k(k+1)}{2} - 4 \ , \tag{2.14}$$

$$\sum_i \mathrm{dex}(\mathcal{E}_i) = 2n - k - 1 \ . \tag{2.15}$$

Classification in even orthogonal Grassmannian

We recall that $\mathrm{OGr}(k, 2n)$ is embedded naturally in $\mathrm{Gr}(k, 2n)$ as the zero locus of a general section of $S^2\mathcal{U}^*$ (see Example 1.3.6). The following lemma is similar to Lemma 2.2.7.

Lemma 2.2.17. *Suppose $\mathcal{F} = \oplus_i \mathcal{E}_i$ is as in the hypothesis of Theorem 2.2.1. In order to find new varieties in $\mathrm{OGr}(k, 2n)$ with respect to the case of the ordinary Grassmannian, it is necessary that for at least one bundle $\mathcal{E}_i = \beta$, $|\beta|_2 \neq 0$ or the β_i's are not integers (they can be half integers).*

Proof. The proof is the same as the one for Lemma 2.2.7. Notice only that half integer weights are associated to spin representations. □

One can assume that $\mathcal{U}^*$ over $\mathrm{OGr}(k, 2n)$ does not appear as summand in $\mathcal{F} = \oplus_i \mathcal{E}_i$; in fact, taking a zero section locus of this bundle in $\mathrm{OGr}(k, 2n)$ is equivalent to restricting to the space $\mathrm{OGr}(k, 2n-1)$.

The classification will be made in three steps: one has to distinguish the three particular cases: $k = n$, $k = n-2$, $k \leq n-3$. This is due to the difference in these cases of the Dynkin diagram of D_n with the k-th root removed. The classification of Table B.4 is a direct consequence of the following propositions.

Proposition 2.2.18 (Classification for $k = n$). *Over $\mathrm{OGr}(n, 2n)$, as suitable $\mathcal{F}$ satisfying the hypothesis of Theorem 2.2.1, one has all and only the cases (oz) appearing in Table B.4.*

Proof. The only bundles that do not appear in the ordinary Grassmannian are those with half integer coefficients. One checks that equations (2.14) and (2.15) make it impossible to have twice the bundle $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$. If it appears even once, (2.15) implies that no bundle with integer coefficients which is not a line bundle can appear, therefore $\frac{\mathrm{dex}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} \geq \frac{1}{2}$, which gives $n = 5, 6$. One obtains therefore $(oz1)$ and $(oz2)$. If the bundle $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ is not present, the same argument as before gives that $\frac{\mathrm{dex}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} \geq \frac{1}{2}$, which gives the remaining cases (oz). □

Proposition 2.2.19 (Classification for $k = n-2$)**.** *Over* $\mathrm{OGr}(n-2, 2n)$*, as suitable* $\mathcal{F}$ *satisfying the hypothesis of Theorem 2.2.1, one has all and only the cases* (ow) *appearing in Table B.4.*

Proof. In this case the Levi factor of $Lie(P)$ is given by the Dynkin diagram D_n where the $(n-2)$-th root has been removed. Therefore its semisimple part is equal to $\mathfrak{sl}(n-2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Then, following Remark 2.1.7, we have for a bundle $\beta = (\beta_1, \dots, \beta_n)$,

$$\mathrm{rank}(\beta) = \dim_{\mathfrak{sl}(n-2)}(\beta_1, \dots, \beta_{n-2}) \times \dim_{\mathfrak{sl}(2)}(\beta_{n-1}, \beta_n) \times \dim_{\mathfrak{sl}(2)}(\beta_{n-1}, -\beta_n).$$

If $n \neq 3, 4, 5$, the minimal ratio $\frac{\mathrm{dex}}{\mathrm{rank}}$ is $\frac{2}{n-2}$, given by $(1, 1, 0, \dots, 0)$. But this implies $n \leq 4$. So the only cases to study are $n = 3$, $n = 4$ and $n = 5$, and this gives the cases (ow). $\square$

Proposition 2.2.20 (Classification for $k \leq n-3$)**.** *If* $k \leq n-3$*, over* $\mathrm{OGr}(k, 2n)$*, as suitable* $\mathcal{F}$ *satisfying the hypothesis of Theorem 2.2.1, one has all and only the cases* (ob) *appearing in Table B.4.*

Proof. With this hypothesis, except for $k \leq 3$, we have $\frac{\mathrm{dex}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} \geq \frac{2}{k}$, which implies $k \leq 1$. So three cases have to be considered.

If $k = 1$, one easily sees that no possibility matches the requirements.

If $k = 2$, as in the analogous proposition for the odd orthogonal Grassmannian, there must be at least one bundle $(\frac{1}{2}, \dots, \pm\frac{1}{2})$, whose rank is 2^{n-3}. As $\mathrm{rank}(\mathcal{F}) = 4n - 11$, this gives $5 \leq n \leq 7$. Therefore, studying case by case, one recovers all the cases (ob).

If $k = 3$, the same reason as before gives $6 \leq n \leq 9$. But in all these cases, $\frac{\mathrm{dex}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})}$ is less than $\frac{1}{2}$, therefore no other case arises. $\square$

2.2.4. The case of $OGr(n-1, 2n)$

Let $\mathrm{OGr}(n-1, 2n)$ be the orthogonal Grassmannian of isotropic $(n-1)$-planes in a $2n$-dimensional complex vector space. This variety is different in nature from those considered up to now, as it is not a generalized Grassmannian in the usual sense (it has Picard rank equal to 2).

It is well known that the zero locus of a general section of $S^2\mathcal{U}^*$ in $\mathrm{Gr}(n, 2n)$ consists of two components OGr_+ and OGr_-, each of which is a copy of $\mathrm{OGr}(n, 2n)$. For each point $W_+ \in \mathrm{OGr}_+$, and $W_- \in \mathrm{OGr}_-$ such that $\dim(W_+ \cap W_-) = n-1$, let us denote $W := W_+ \cap W_-$. Then $W \in G := \mathrm{OGr}(n-1, 2n)$. Moreover, given $W \in G$, one can recover W_+ and W_- in a unique way, i.e. there exist two morphisms

$$\pi_{+/-} : G \to \mathrm{OGr}_{+/-} \quad , \quad W \mapsto W_{+/-} \quad .$$

Over OGr_+ (respectively OGr_-) there is a line bundle $\mathcal{O}(\frac{1}{2})_+$ (resp. $\mathcal{O}(\frac{1}{2})_-$) which is the square root of the restriction to OGr_+ (resp. OGr_-) of $\mathcal{O}(1)$ over

$\mathrm{Gr}(n, 2n)$. Therefore there are two line bundles $\mathcal{L}_+ := \pi_+^*\mathcal{O}(\frac{1}{2})_+$ and $\mathcal{L}_- := \pi_-^*\mathcal{O}(\frac{1}{2})_-$ over G. In fact, it can be shown that the Picard group of G is generated by those two line bundles. This description corresponds to the following picture from the point of view of quotients of $\mathrm{SO}(2n, \mathbb{C})$.

Recall that P_i denotes the parabolic subgroup of $\mathrm{SO}(2n, \mathbb{C})$ corresponding to the i-th simple root. The even orthogonal Grassmannian $\mathrm{OGr}(n-1, 2n)$ can be thought of as the quotient $G/P_{n,n-1}$, where $P_{n,n-1}$ is the parabolic subgroup $P_n \cap P_{n-1}$. The reason why G is not considered to be a Grassmannian is exactly because the parabolic subgroup $P_{n,n-1}$ is associated to two, and not one, simple roots. This also explains why $\rho(G) = \mathbb{Z}^2$ (general theory of homogeneous spaces). The two morphisms π_+ and π_- correspond to the two projections

$$G = G/P_{n,n-1} \to G/P_n \cong \mathrm{OGr}_+$$

and

$$G = G/P_{n,n-1} \to G/P_{n-1} \cong \mathrm{OGr}_- \quad .$$

Now we want to understand the relation between $\mathcal{L}_+$, $\mathcal{L}_-$, and $\mathcal{O}(1)$ (which is the restriction to G of the Plücker line bundle $\mathcal{O}(1)$ over $\mathrm{Gr}(n-1, 2n)$). Let $\mathcal{U}$ be the tautological bundle of rank $n-1$, and $\mathcal{U}^*$ its dual, on G. As already mentioned for the classical Grassmannians, there is a vector bundle $\mathcal{U}^\perp/\mathcal{U}$ over G of rank 2. Let us also denote $\mathcal{U}_\pm$ the tautological bundles over $\mathrm{OGr}_\pm$ restricted to G. Then, $\mathcal{U}^\perp = \mathcal{U}_+ + \mathcal{U}_- \subset \mathbb{C}^{2n}$, where the sum is not direct, and, as a consequence,

$$\mathcal{U}^\perp/\mathcal{U} = \mathcal{U}_+/\mathcal{U} \oplus \mathcal{U}_-/\mathcal{U} \quad .$$

In this second equation the sum is actually a direct sum; this comes from the fact that $W = W_+ \cap W_-$ for W an isotropic $(n-1)$-plane in $\mathbb{C}^{2n}$. The quadratic form on $\mathbb{C}^{2n}$ restricts to a form on $\mathcal{U}^\perp$. Since this form descends to a form on $\mathcal{U}^\perp/\mathcal{U}$ which is non degenerate, then $\mathcal{U}^\perp/\mathcal{U} \cong (\mathcal{U}^\perp/\mathcal{U})^*$, which implies $\det(\mathcal{U}^\perp/\mathcal{U}) = 0$. Moreover $\det(\mathcal{U}_\pm) = \pi_\pm^*\mathcal{O}(\frac{1}{2})_\pm^{\otimes 2}$. By taking the determinant of the bundles in the previous equation, we get the important relation

$$\mathcal{L}_+ \otimes \mathcal{L}_- = \mathcal{O}(1) \quad .$$

The following theorem holds:

Theorem 2.2.21. *Let* Y *be a fourfold with $K_Y = \mathcal{O}_Y$ which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle $\mathcal{F}$ over the even orthogonal Grassmannian $\mathrm{OGr}(n-1, 2n)$ (and which does not appear in the analogous classification for the classical Grassmannian). Up to identifications, the only possible cases are those appearing in Table B.5.*

For the proof of the theorem, every irreducible homogeneous bundle is represented by its highest weight. The notations for the bundles on $\mathrm{OGr}(n-1, 2n)$ are the same as those used for $\mathrm{OGr}(k, 2n)$ for general k, as well as the formula for

rank and dex of the bundles (notice that the parabolic algebra $Lie(P_{n-1,n})$ has Levi factor $\mathfrak{sl}(n-1)$). The line bundles $\mathcal{L}_+$ and $\mathcal{L}_-$ will be respectively denoted by $(\frac{1}{2},\ldots,\frac{1}{2})$ and $(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$. Finally, the canonical bundle of $\mathrm{OGr}(n-1,2n)$ is a tensor power of $\mathcal{O}(1)$.

Fourfolds with trivial canonical bundle correspond to homogeneous vector bundles $\mathcal{F}=\sum_i \mathcal{E}_i$, with

$$\sum_i \mathrm{rank}(\mathcal{E}_i) = (n-1)(n+1) - \frac{(n-1)(n)}{2} - 4 \ , \tag{2.16}$$

$$\sum_i \mathrm{dex}(\mathcal{E}_i) = n \ . \tag{2.17}$$

In order to find new varieties with respect to the case of the ordinary Grassmannian, it is necessary that for at least one bundle $\mathcal{E}_i=\beta$, $|\beta|_n \neq 0$ or the β_i's are not integers (they can be half integers).

Proof of Theorem 2.2.21. Suppose the variety is embedded in $\mathrm{OGr}(n-1,2n)$, with $n\geq 4$. Then, for any irreducible homogeneous globally generated bundle $\mathcal{E}_i$ which is a component of $\mathcal{F}$, we have $\frac{\mathrm{dex}(\mathcal{E}_i)}{\mathrm{rank}(\mathcal{E}_i)} \geq \frac{2}{n-1}$. By using equations (2.17) and (2.16), we get the inequality $n\leq 5$. Then, the theorem follows by inspecting all the possible cases. It should be remarked that, if $\mathcal{F}=\oplus_i\mathcal{E}_i=\oplus_i(\beta_1^i,\ldots,\beta_n^i)$ is a suitable vector bundle, then $\sum_i \mathrm{rank}(\mathcal{E}_i)\beta_n^i=0$, which ensures that $\det(\mathcal{F})$ is a multiple of $\mathcal{O}(1)$, as $K_{\mathrm{OGr}(n-1,2n)}$ is. □

2.3. The cases of dimensions 2 and 3

In this section, we will give the analogous results of the previous classifications of varieties with trivial canonical bundle in dimensions 2 and 3. We recall that a pioneering work on $K3$ surfaces (and Fano threefolds) inside Grassmannians has been conducted by Mukai in a more geometrical manner (see [Muk02]). We will recover the families described by Mukai in our classification. The proofs do not present anything new from the previous ones, they follow the exact same strategy, so we omit them. It is perhaps worth remarking that the first problem is to prove the finiteness of the number of such varieties in Grassmannians (for the ordinary Grassmannian, see [IIM16]). The notations are the same used in the previous classifications.

We will begin with the classification in dimension 3.

Theorem 2.3.1. *Let* Y *be a threefold with* $K_Y=\mathcal{O}_Y$ *which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle* $\mathcal{F}$ *over* $\mathrm{Gr}(k,n)$*. Up to the identification of* $\mathrm{Gr}(k,n)$ *with* $\mathrm{Gr}(n-k,n)$*, the only possible cases are those appearing in Table B.6.*

This classification, as already mentioned in the introduction, appears also in [IIM16]. However, we point out the fact that in [IIM16], the cases $(c3)$, $(d3.1)$ and $(d2.1)$ do not appear. The bundles which define them are analogous to the one appearing respectively in the cases $(c2)$, $(d3)$ and $(d2)$, and in fact there are isomorphisms $(c3) \cong (c2)$, $(d3.1) \cong (d3)$ and $(d2.1) \cong (d2)$. These isomorphisms come from the fact that all these cases live in the symplectic Grassmannian $\mathrm{IGr}(n, 2n)$, on which there is a canonical isomorphism of bundles $\mathcal{U}^* \cong \mathcal{Q}$.

Theorem 2.3.2. *Let* Y *be a threefold with* $K_Y = \mathcal{O}_Y$ *which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle* $\mathcal{F}$ *over the symplectic Grassmannian* $\mathrm{IGr}(k, 2n)$ *(respectively the odd orthogonal Grassmannian* $\mathrm{OGr}(k, 2n+1)$*, the even orthogonal Grassmannian* $\mathrm{OGr}(k, 2n)$*) and which does not appear in the analogous classification for the ordinary Grassmannian. Up to identifications, the only possible cases are those appearing in Table B.7 (respectively Table B.8, Table B.9).*

Theorem 2.3.3. *Let* Y *be a threefold with* $K_Y = \mathcal{O}_Y$ *which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle* $\mathcal{F}$ *over the even orthogonal Grassmannian* $\mathrm{OGr}(n-1, 2n)$ *(and which does not appear in the analogous classification for the classical Grassmannian). Up to identifications, the only possible cases are those appearing in Table B.10.*

Now, the classification for dimension 2 (K3 surfaces). In this case we reported the degree of the surface with respect to the bundle $\mathcal{O}(\frac{1}{2})$ for the varieties in $\mathrm{OGr}(n, 2n+1)$ and $\mathrm{OGr}(n, 2n)$, and with respect to the bundle $\mathcal{O}(1)$ in the other cases. For $\mathrm{OGr}(n-1, 2n)$, we reported the degree with respect to $\mathcal{O}(1)$, $\mathcal{L}_+$ and $\mathcal{L}_-$. In this way, one also gets the genus of the $K3$ by the well known formula $\deg = 2 \text{ genus} - 2$.

Theorem 2.3.4. *Let* Y *be a surface with* $K_Y = \mathcal{O}_Y$ *which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle* $\mathcal{F}$ *over* $\mathrm{Gr}(k, n)$*. Up to the identification of* $\mathrm{Gr}(k, n)$ *with* $\mathrm{Gr}(n-k, n)$*, the only possible cases are those appearing in Table B.11.*

Theorem 2.3.5. *Let* Y *be a surface with* $K_Y = \mathcal{O}_Y$ *which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle* $\mathcal{F}$ *over the symplectic Grassmannian* $\mathrm{IGr}(k, 2n)$ *(respectively the odd orthogonal Grassmannian* $\mathrm{OGr}(k, 2n+1)$*, the even orthogonal Grassmannian* $\mathrm{OGr}(k, 2n)$*) and which does not appear in the analogous classification for the ordinary Grassmannian. Up to identifications, the only possible cases are those appearing in Table B.12 (respectively Table B.13, Table B.14).*

Theorem 2.3.6. *Let* Y *be a surface with* $K_Y = \mathcal{O}_Y$ *which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle* $\mathcal{F}$ *over the even orthogonal Grassmannian* $\mathrm{OGr}(n-1, 2n)$ *(and which does not appear in the analogous classification for the classical Grassmannian). Up to identifications, the only possible cases are those appearing in Table B.15.*

2.3.1. An unexpected isomorphism

Among the Calabi-Yau threefolds we have found, we realised that the cases $(c8)$ and $(oz5)$ have the same Hodge numbers (by an explicit computation). In fact, they are the same varieties. This is a consequence of the existence of an isomorphism between the Grassmannian $Gr(3,6)$ and the zero locus of a general section of $\wedge^5\mathcal{U}^*\otimes\mathcal{O}(-\frac{1}{2})$ in the orthogonal Grassmannian $OGr(6,12)$. To prove this, a more explicit description of the bundle $\wedge^5\mathcal{U}^*\otimes\mathcal{O}(-\frac{1}{2})$ is necessary. What follows should convince the reader that among the varieties we found, there could be some isomorphisms which are not easy to detect.

We start by recalling basic facts about the spin representations. Fix a complex vector space V of dimension $2n$ and a non degenerate quadratic form on it. The space of sections $\mathrm{H}^0(OGr(n,V),\wedge^{n-1}\mathcal{U}^*\otimes\mathcal{O}(-\frac{1}{2}))$ of the bundle $\wedge^{n-1}\mathcal{U}^*\otimes\mathcal{O}(-\frac{1}{2})\cong\mathcal{U}\otimes\mathcal{O}(\frac{1}{2})$, by the Borel-Weil Theorem, is isomorphic to the dual of the spin representation $S_-(V)^*$. On the other hand, $\mathrm{H}^0(OGr(n,V),\mathcal{O}(\frac{1}{2}))$ is isomorphic to the dual of the spin representation $S_+(V)^*$. Consider a maximal isotropic subspace F of V. Then, one has the following identifications:

$$S_+(V)\cong\wedge^{even}F=\wedge^+F \quad , \quad S_-(V)\cong\wedge^{odd}F=\wedge^-F.$$

Let us denote by $S(V)$ the exterior algebra $S_+(V)\oplus S_-(V)$; it is isomorphic to $\wedge^\bullet F$. The Clifford algebra $Cl(V)$ acts on $S(V)$. Its action is defined by the action of V, which is as follows: fix an isotropic $F'\subset V$ such that $F'\pitchfork F$; via the duality given by the quadratic form $F'\cong F^*$. Then the action of $Cl(V)$ is induced by the one of V which is the usual wedge product for the vectors in F, while the action of $f^*\in F^*$ on $\wedge^\bullet F$ is defined by:

$$f^*.(v)=f^*(v)\quad \text{for } v\in F,$$

$$f^*.(\xi\wedge\eta)=(f^*.\xi)\wedge\eta+(-1)^{deg(\xi)}\xi\wedge(f^*.\eta).$$

Let us here remark that the choice of another isotropic space G gives another $S(V)$, i.e. $S(V)\cong_{Cl(V)}\wedge^\bullet G$. The isomorphism $\wedge^\bullet F\cong_{Cl(V)}\wedge^\bullet G$ is unique modulo multiplication by a scalar. This isomorphism respects the decomposition $\wedge^-(\cdot)\oplus\wedge^+(\cdot)$ or inverses it depending on the fact that F and G are in the same connected component of isotropic spaces or in different ones.

To every maximal isotropic subspace $W\in V$, we can associate a line $Pf_W\subset S(V)$:

$$W\mapsto Pf_W=\{\phi\in S(V) \text{ such that } w.\phi=0\ \ \forall w\in W\}.$$

In fact, if $W\in\mathrm{OGr}(n,V)_\pm$, then $Pf_W\in S_\pm(V)$. Therefore we get a morphism

$$\eta:\mathrm{OGr}_\pm(n,V)\to\mathbf{P}(S_\pm(V)),\ \ \eta(W)=Pf_W.$$

All vectors $0\neq\phi\in Pf_W$ as W varies among maximal isotropic subspaces $W\subset V$

are called *pure spinors*. From $\phi \in Pf_W \subset S(V)$, it is possible to recover W as

$$W = \{v \in V \text{ such that } v.\phi = 0\}.$$

As a matter of fact, η is an embedding and realizes $\mathrm{OGr}_\pm(k, V)$ as the spinorial variety inside $\mathbf{P}(S_\pm(V))$. From now on we denote by $\mathrm{OGr}(k, V) = \mathrm{OGr}_\pm(k, V)$.

When identifying $S(V)$ with the exterior algebra $\wedge^\bullet W$, we get $Pf_W = \wedge^n W = \det(W)$. The following result can be found in [PS86]:

Proposition 2.3.7. *The line bundle on $OGr(n, V)$ defined by*

$$W \mapsto Pf_W \subset S_+(V) \quad \forall W \in OGr(n, V)$$

is the square root of the bundle $\mathcal{O}(-1)|_{OGr} = \det(\mathcal{U})|_{OGr}$, where $\mathcal{U}$ is the tautological subbundle in $Gr(n, V)$.

Following the same line of ideas, we give a description of the bundle $\wedge^{n-1}\mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2})$. For every maximal isotropic subspace Y, identify $S(V)$ and $\wedge^\bullet Y$. Let us define

$$VPf_Y = \wedge^{n-1} Y \subset \wedge^\bullet Y$$

A more intrinsic way of defining VPf_Y is to say that it is the image $(Y^*. Pf_Y)$ of Pf_Y in $S(V)$ under the action of $Y^* \subset Cl(V)$. The following holds:

Proposition 2.3.8. *The vector bundle on $OGr(n, V)$ defined by*

$$Y \mapsto VPf_Y \subset S_-(V) \quad \forall Y \in OGr(n, V)$$

is the bundle $\wedge^{n-1}\mathcal{U} \otimes \mathcal{O}(\frac{1}{2})$.

Proof. Notice that the two bundles have the same rank. Denote the first bundle in the statement of the proposition by $\mathcal{F}$. Let us rewrite the second one:

$$\wedge^{n-1}\mathcal{U} \otimes \mathcal{O}(\frac{1}{2}) \cong \wedge^1\mathcal{U}^* \otimes \mathcal{O}(-1) \otimes \mathcal{O}(\frac{1}{2}) \cong \mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2})$$

The action of V in the Clifford algebra $Cl(V)$ defines a morphism

$$\psi : V \otimes S_+(V) \to S_-(V)$$

Take $Y \in OGr(n, V)$. The fiber at Y of $\mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2})$ is $Y^* \otimes Pf_Y \subset V \otimes S_+(V)$. By applying ψ, we get that

$$\psi(Y^* \otimes Pf_Y) = (Y^*. Pf_Y) = VPf_Y \subset S_-(V)$$

As a consequence ψ defines a morphism $\tilde{\psi} : \mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2}) \to \mathcal{F}$ which is an isomorphism on the fibers over $OGr(n, V)$; therefore it is an isomorphism of vector bundles. $\square$

Remark 2.3.9. A similar description holds for all spin bundles over orthogonal Grassmannians. Indeed, consider the Grassmannian $\mathrm{OGr}_+(k,V)$, and for simplicity suppose k even. Then the bundle $\mathcal{T}_{+\frac{1}{2}}$ is a subbundle of the trivial bundle $\mathcal{S}_+(V)$ over $\mathrm{OGr}_+(k,V)$; it can be proved to be the image of the morphism

$$\tilde{\psi} : \mathcal{U}^* \otimes \mathcal{S}_+(V) \to \mathcal{S}_+(V),$$

where the morphism $\tilde{\psi}$ is induced on each fiber by ψ.

As a consequence, we have the description of the zero locus of a general section of $\wedge^5\mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2})$ we claimed: fix $n = 6$;

Proposition 2.3.10. *The zero locus $Z(\tilde{s})$ of a general section s of the bundle $\wedge^5\mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2})$ on $OGr(6,V)$ is isomorphic to $Gr(3,6)$.*

Proof. By the Borel-Weil Theorem, the space of sections of $\wedge^5\mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2})$ is given by $S_-(V)^*$ (which is isomorphic to $S_-(V)$ as n is even). So we can suppose $s \in S_-(V)^*$. As s is a generic spinor and V is of dimension 12, it can be written as the sum of two pure spinors ([Igu70]), i.e.

$$s = s_W + s_{W'} \ \ , \ \ s_W \in Pf_W \ \ , s_{W'} \in Pf_{W'}$$

where W, W' are maximal isotropic subspaces. As $s \in S_-(V)^*$, the two subspaces W, W' do not belong to $OGr(6,V)$, but to the other connected component of isotropic maximal subspaces in V. Moreover, by the generality assumption, we can suppose that $W \cap W' = \{0\}$. Therefore we can identify $W' \cong W^*$. Consider the embedding

$$i_{W,W'} : Gr(3,6) \cong Gr(3,W) \to OGr(6,V) \ \ , \ \ P \mapsto \langle P, P^\perp \cap W' \rangle$$

This embedding will give the isomorphism we want. Indeed, let us study $Z(\tilde{s})$.

By proposition 2.3.8, the fiber of the bundle $\wedge^5\mathcal{U}^* \otimes \mathcal{O}(-\frac{1}{2})$ over the point $Y \in OGr(6,V)$ is given by

$$Y \mapsto (VPf_Y)^*,$$

and we have the natural restriction map $S_-(V)^* \to (VPf_Y)^*$. The evaluation of the section s at Y is given by

$$s(Y) = s|_{VPf_Y} \in (VPf_Y)^*.$$

By the description of VPf_Y as $\wedge^5 Y \subset \wedge^{odd} Y \cong S_-(V)$, we get that the section s is zero at Y if and only if $\mu(s) = 0 \ \forall \mu \in \wedge^5 Y$.

If $Y \in Im(i_{W,W'})$, the intersections $Y \cap W$ and $Y \cap W'$ both have dimension 3. We claim that

$$\mu(s) = \mu(s_W + s_{W'}) = 0 \ \forall \mu \in \wedge^5 Y, \tag{2.18}$$

which implies that $s(Y) = 0$.

Indeed, it suffices to verify the equality in (2.18) on a basis of $\wedge^5 Y$. Take $y_1, y_2, y_3 \in Y \cap W$, $y_4, y_5, y_6 \in Y \cap W'$ such that $\langle y_1, y_2, y_3, y_4, y_5, y_6 \rangle = Y$. Then a basis of $\wedge^5 Y$ is given by all the vectors of the form $\mu = y_{i_1} \wedge \ldots \wedge y_{i_5}$. Without losing any generality, we can suppose that $\mu = y_1 \wedge \ldots \wedge y_5$. Then

$$\mu(s_W) = y_2 \wedge \ldots \wedge y_5 \wedge y_1(s_W) = 0$$

because $y_1 \in W$ and $s_W \in Pf_W$. Similarly

$$\mu(s_{W'}) = y_1 \wedge \ldots \wedge y_5(s_{W'}) = 0$$

because $y_5 \in W'$ and $s_{W'} \in Pf_{W'}$. As a consequence $\mu(s) = 0$.

As W and W' do not belong to the same connected component of maximal isotropic subspaces of Y, the intersections $Y \cap W$ and $Y \cap W'$ have odd dimension. If $Y \notin Im(i_{W,W'})$ we can suppose that $\dim(Y \cap W)$ or $\dim(Y \cap W')$ is equal to one; assume $\dim(Y \cap W) = 1$. Let us show that

$$\exists \tilde{\mu} \in \wedge^5 Y \text{ such that } \tilde{\mu}(s_W) \neq 0 \text{ and } \tilde{\mu}(s_{W'}) = 0;$$

this will imply that $s(Y) \neq 0$. Take $0 \neq w \in W \cap Y$, and $0 \neq w' \in W' \cap Y$, and suppose $\langle w, w', y_1, y_2, y_3, y_4 \rangle = Y$. Then $\tilde{\mu} = y_1 \wedge y_2 \wedge y_3 \wedge y_4 \wedge w'$ will do. $\square$

2.3.2. Maximal families of $K3$ surfaces with Picard number two

In Theorem 2.3.4, 2.3.5, 2.3.6 we found all $K3$ surfaces in classical Grassmannians which are zero loci of general sections of homogeneous completely reducible vector bundles. All the varieties we constructed come naturally in families; indeed, by letting the section that defines them vary, we get a family of deformation of those varieties. Mukai studied families of polarized $K3$ surfaces of maximal dimension (i.e. 19) inside Grassmannians (e.g. see [Muk88], [Muk92], [Muk06]). The general element of such families is a $K3$ surface with $\rho = 1$.

In the following we will study families of polarized $K3$ surfaces with Picard number two of maximal dimension (i.e. 18). $K3$ surfaces with $\rho = 2$ have been studied for various reasons. For instance, $K3$ surfaces which admit a double cover over $\mathbf{P}^2$ branched over a sextic are used in [Gal17] to approach the rationality problem of cubic fourfolds.

Ottem focused as well on some $K3$s with $\rho = 2$ in [Ott13], for which he was able to describe the Cox ring. He considered *doubly elliptic* $K3$ surfaces, i.e. $K3$ surfaces which are elliptic bundles over $\mathbf{P}^1$ in two different ways. Doubly elliptic surfaces are similar to some of the surfaces we will analyze more in detail (see Table 2.2, cases $(b12)$, $(oe6)$, $(oe9)$, and the end of section on $(oe9)$).

Finally, studying non general $K3$ surfaces is interesting as one can produce examples of such varieties with non trivial automorphism groups, which in turn can enlighten geometric aspects of these surfaces. The situation where $\rho = 2$ has been investigated in [Bin05]. The analysis of the automorphism group is done in terms of the intersection matrix of rank two surfaces. We will use Bini's results to understand the automorphism group of $K3$ surfaces of type $(oe9)$.

In order to have $\rho = 2$, we will take our examples of $K3$ surfaces inside $\mathrm{OGr}(n-1, 2n)$. The Picard group of this homogeneous space is generated by two line bundles $\mathcal{L}_+$, $\mathcal{L}_-$ such that $\mathcal{L}_+ \otimes \mathcal{L}_- = \mathcal{O}(1)$, where $\mathcal{O}(1)$ is the restriction of $\mathcal{O}(1)$ over $\mathrm{Gr}(n-1, 2n)$ to $\mathrm{OGr}(n-1, 2n)$ (more details on this are given in Section 2.2.4). The fact that the ambient variety of the surfaces we are considering has $\rho = 2$ is a good starting point.

Let us recall some general facts about deformation theory. Let $\mathcal{S} \to B$ be a (smooth proper) deformation of the central fiber $\mathcal{S}_{b_0} = S$. Then there exists a morphism

$$\delta_{KS} : T_{B,b_0} \to \mathrm{H}^1(S, T_S)$$

called the Kodaira-Spencer map (e.g. see [Kod05] or [KS58]). Here T_S denotes the tangent space of S. Elements in $\mathrm{H}^1(S, T_S)$ can be identified with infinitesimal effective deformations

$$\tilde{\mathcal{S}} \to Spec(\mathbb{C}[t]/t^2)$$

modulo isomorphism. Moreover, the dimension of the image of the Kodaira-Spencer map $\delta_{KS}(T_{B,b_0})$ should be thought of as the "effective" dimension of the deformation family $\mathcal{S}$ at b_0.

In our situation S is the zero locus of a general section s of a vector bundle $\mathcal{F}$ over X. The parameter space for the deformation is an open subset B of the space of sections $\mathrm{H}^0(X, \mathcal{F})$ (such that zero loci are smooth, for example). The deformation family can be described as

$$\mathcal{S} = \{(S', s') \text{ s.t. } S' \text{ is the (smooth) zero locus of the section } s' \in \mathrm{H}^0(X, \mathcal{F})\}.$$

The map $\mathcal{S} \to B \subset \mathrm{H}^0(X, \mathcal{F})$ is the natural projection onto the second factor. As B is an open set in an affine space, the Kodaira-Spencer map becomes

$$\delta_{KS} : T_{B,s} \cong \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^1(S, T_S)$$

This morphism is described, for instance, in [Bor83]. On one hand, consider the exact sequence:

$$0 \to \mathcal{I}_S \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{O}_S \otimes \mathcal{F} \to 0,$$

where $\mathcal{I}_S$ is the ideal of S. It induces a map in cohomology $\eta_1 : \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(S, \mathcal{F}|_S)$, whose cokernel is $\mathrm{H}^1(X, \mathcal{I}_S \otimes \mathcal{F})$. On the other hand, the normal exact sequence

$$0 \to T_S \to T_X|_S \to \mathcal{F}|_S \to 0$$

case	bundle $\mathcal{F}$	X= OGr(n-1,2n)	$\mathcal{O}(1)^2$	$\mathcal{L}_+^2$	$\mathcal{L}_-^2$	$\mathcal{L}_+\mathcal{L}_-$
(b12)	$\mathcal{O}(1)^{\oplus 3}$	OGr(2,6)	20	4	4	6
(oe4)	$\mathcal{O}(1) \oplus \mathcal{L}_+(1) \oplus \mathcal{L}_-$	OGr(2,6)	16	4	2	5
(oe5)	$\mathcal{L}_- \oplus \mathcal{L}_- \oplus \mathcal{L}_+^{\otimes 2}(1)$	OGr(2,6)	10	4	0	3
(oe6)	$\mathcal{O}(2) \oplus \mathcal{L}_- \oplus \mathcal{L}_+$	OGr(2,6)	12	2	2	4
(oe9)	$\mathcal{O}(1) \oplus \mathcal{L}_-^{\oplus 3} \oplus \mathcal{L}_+^{\oplus 3}$	OGr(3,8)	28	6	6	8

Table 2.2. – Families of surfaces in OGr(n-1,2n) of dimension 18

induces the map in cohomology $\eta_2 : \mathrm{H}^0(S, \mathcal{F}|_S) \to \mathrm{H}^1(S, T_S)$, whose cokernel is $\mathrm{H}^1(S, T_X|_S)$. Then, the Kodaira-Spencer map is the composition

$$\delta_{KS} = \eta_2 \circ \eta_1.$$

This in turn implies:

Corollary 2.3.11. *If $\mathrm{H}^1(X, \mathcal{I}_S \otimes \mathcal{F}) = 0$, then the codimension of the image of the Kodaira-Spencer map is equal to $\dim \mathrm{H}^1(S, T_X|_S)$. If $\dim \mathrm{H}^1(S, T_X|_S) = 0$, then the family is locally complete.*

In our case, even though $\mathrm{H}^1(X, \mathcal{I}_S \otimes \mathcal{F})$ may be zero, the family cannot be locally complete. Indeed, for us X is a projective variety, and so S is polarized; but a general $K3$ surface is not projective. Actually primitively polarized $K3$ surfaces with a certain degree of the polarization form a hypersurface of the space of $K3$ surfaces. Moreover, we want the $K3$ surface to have $\rho = 2$, so they live in a family of codimension 2 (dimension 18).

In the end, we have the following proposition:

Proposition 2.3.12. *Let S be a $K3$ surface which is the zero locus of a general section of a homogeneous, completely reducible vector bundle $\mathcal{F}$ over $X := \mathrm{OGr}(n-1, 2n)$. Suppose that $\mathrm{H}^1(X, \mathcal{I}_S \otimes \mathcal{F}) = 0$ and $\mathrm{H}^1(S, T_X|_S) \cong \mathbb{C}^2$. Then S lives in a family of dimension 18, and it appears in Table 2.2.*

Proof. To compute the cohomology groups, we used the Koszul complex associated to S, and Bott's theorem for the cohomology of homogeneous bundles over the Grassmannians. □

We analyze in what follows the geometry and the equations defining the $K3$ surfaces in Table 2.2. In particular, we describe cases $(b12)$, $(oe4)$, $(oe5)$ as families of special quartics in $\mathbf{P}^3$; case $(oe6)$ has already been studied in [Ili97]. Case $(oe9)$ is a little bit different from the others.

K3 $(b12)$

The $K3$ surface S lives inside $\mathrm{OGr}(2, V_6)$, where V_6 has dimension 6.

Remark 2.3.13. Actually $\mathrm{OGr}(2, V_6)$ is isomorphic to the flag variety $F(1, 3, V_4)$, where V_4 has dimension 4. The isomorphism is a consequence of the identification of SO_6 and SL_4, but we give a more concrete description. Identify $V_6 \cong \wedge^2 V_4$ for a certain vector space V_4, so that the symmetric form is the wedge product. Then we have a morphism:

$$F(1, 3, V_4) \to \mathrm{Gr}(2, \wedge^2 V_4) \quad , \quad (l \subset P \subset V_4) \mapsto l \wedge P.$$

The image is isotropic because $(l \wedge P) \wedge (l \wedge P) = 0$. Conversely, suppose that $Q \in \mathrm{OGr}(2, \wedge^2 V_4)$; then as Q is isotropic, it is a plane of rank 2 elements in $\wedge^2 V_4$, and therefore it can be written as $l' \wedge P'$.

Moreover $F(1, 3, V_4)$ can be seen as the zero locus of the section $id \in \mathrm{H}^0(\mathbf{P}(V_4) \times \mathbf{P}(V_4^*), \mathcal{O}(1, 1)) \cong \mathrm{End}(V_4)$. Under the isomorphism described above, we have the following identifications:

$$\mathcal{L}_+ \to \mathcal{O}(1, 0) \quad , \quad \mathcal{L}_- \to \mathcal{O}(0, 1) \quad , \quad \mathcal{O}(1) \to \mathcal{O}(1, 1).$$

Therefore, by using the linear system of $\mathcal{L}_+$, $\mathcal{L}_-$, we obtain the two natural projections $\pi_\pm : F(1, 3, 4) \to \mathbf{P}^3$.

The following proposition shows what are the equations of S (we study the projection π_+; the projection π_- admits the same description).

Proposition 2.3.14. *Suppose S is of type $(b12)$. Then π_+ realises S inside $\mathbf{P}(V_4)$ as the zero locus of the determinant of a 4×4 matrix of linear forms. In particular, S is a quartic in $\mathbf{P}^3$.*

Proof. Let V_4, V_4', be two vector spaces of dimension 4. Inside $\mathbf{P}(V_4) \times \mathbf{P}(V_4')$, S is the zero locus of a section s of $\mathbb{C}^4 \otimes \mathcal{O}(1, 1)$. Explicitly, $s \in \mathbb{C}^4 \otimes V_4^* \otimes V_4'^*$, which is also the space of sections of $\mathbb{C}^4 \otimes V_4'^* \otimes \mathcal{O}(1)$ over $\mathbf{P}(V_4)$. A point $l \in \mathbf{P}(V_4)$ belongs to the image of π_+ if and only if there exists $l' \in \mathbf{P}(V_4')$ such that $(l, l') \in S$. This is equivalent to the fact that $s|_l \in l^* \otimes \mathbb{C}^4 \otimes V_4'^* = l^* \otimes \mathrm{Hom}(V_4', \mathbb{C}^4)$ has kernel different from zero. Therefore, $l \in \pi_+(S)$ if and only if $det(s) = 0$, where s is seen as a section of $\mathbb{C}^4 \otimes V_4'^* \otimes \mathcal{O}(1)$ over $\mathbf{P}(V_4)$. Moreover $\pi_+|_S$ is an isomorphism onto its image because the degree of π_+ is 1, and S has trivial canonical bundle. Therefore we get the statement of the proposition. □

Determinantal hypersurfaces have been studied in [Bea00]. More precisely, [Bea00, Corollary 6.6] asserts that a smooth quartic hypersurface in $\mathbf{P}^3$ is determinantal if and only if it contains a non-hyperelliptic curve C of genus three, embedded in $\mathbf{P}^3$ by a linear system of degree 6. Actually, the curve C can be explicitly obtained as a divisor of the cokernel bundle of the morphism:

$$s : \mathcal{O}^4_{\mathbf{P}(V_4)} \to V_4' \otimes \mathcal{O}_{\mathbf{P}(V_4)}(1).$$

K3 $(oe4)$, $(oe5)$, $(oe6)$

The $K3$ surfaces of type $(oe4)$ and $(oe5)$ admit a description similar to those of type $(b12)$. We will denote by $\left(v_1\middle|\dots\middle|v_n\right)$ the matrix whose column vectors are $v_1, \dots, v_n$.

Let S be of type $(oe4)$; then $S \subset \mathbf{P}(V_4) \times \mathbf{P}(V_4')$ is the zero locus of a section s of $\mathcal{F} = 2\mathcal{O}(1,1) \oplus \mathcal{O}(2,1) \oplus \mathcal{O}(0,1)$.

Proposition 2.3.15. *Suppose S is of type $(oe4)$. Then π_+ realises S inside $\mathbf{P}(V_4)$ as the zero locus of the determinant of a square matrix $\left(s_1\middle|\dots\middle|s_4\right)$ of sections such that $s_1, s_2 \in \mathrm{H}^0(\mathbf{P}(V_4), 4\mathcal{O}(1))$, $s_3 \in \mathrm{H}^0(\mathbf{P}(V_4), 4\mathcal{O}(2))$, $s_4 \in \mathrm{H}^0(\mathbf{P}(V_4), 4\mathcal{O})$. In particular, S is a quartic in $\mathbf{P}^3$. The second projection $\pi_-|_S$ is a double covering of $\mathbf{P}^2$ ramified over a sextic.*

Proof. Let V_4, V_4', be two vector spaces of dimension 4. Inside $\mathbf{P}(V_4) \times \mathbf{P}(V_4')$, S is the zero locus of a section s of $(\mathbb{C}^2 \otimes \mathcal{O}(1,1)) \oplus \mathcal{O}(2,1) \oplus \mathcal{O}$. The proof of the first part of the theorem is the same as for the case $(b12)$; the only difference is that

$$s \in (\mathbb{C}^2 \otimes V_4^* \otimes V_4'^*) \oplus (S^2(V_4^*) \otimes V_4'^*) \oplus V_4'^* \cong$$

$$\mathrm{H}^0(\mathbf{P}(V_4), (2\mathcal{O}(1) \otimes V_4'^*) \oplus (\mathcal{O}(2) \otimes V_4'^*) \oplus (\mathcal{O} \otimes V_4'^*))$$

Therefore $\pi_+(S)$ is again the locus where $\det(s) = 0$, but the matrix is of the form showed in the statement of the proposition.

For the second part, as the vector bundle $\mathcal{F}$ contains a copy of $\mathcal{O}(0,1)$, the zero locus of s is contained in $\mathbf{P}(V_4) \times \mathbf{P}^2$. The projection π_- sends S to $\mathbf{P}^2$ and is of degree 2. By the Riemann-Hurwitz formula

$$K_S = \pi^*(\deg(\pi) K_{\mathbf{P}^2} + R),$$

where R is the divisor of the branch locus; the formula implies that R is a curve of degree 6 inside $\mathbf{P}^2$. □

Similarly, let S be of type $(oe5)$; then $S \subset \mathbf{P}(V_4) \times \mathbf{P}(V_4')$ is the zero locus of a section s of $\mathcal{F} = \mathcal{O}(1,1) \oplus \mathcal{O}(3,1) \oplus 2\mathcal{O}(0,1)$.

Proposition 2.3.16. *Suppose S is of type $(oe5)$. Then π_+ realises S inside $\mathbf{P}(V_4)$ as the zero locus of the determinant of a square matrix $\left(s_1\middle|\dots\middle|s_4\right)$ of sections such that $s_1 \in \mathrm{H}^0(\mathbf{P}(V_4), 4\mathcal{O}(1))$, $s_2 \in \mathrm{H}^0(\mathbf{P}(V_4), 4\mathcal{O}(3))$, $s_3, s_4 \in \mathrm{H}^0(\mathbf{P}(V_4), 4\mathcal{O})$. In particular, S is a quartic in $\mathbf{P}^3$. The second projection $\pi_-|_S$ is an elliptic fibration over $\mathbf{P}^1$.*

Proof. Let V_4, V_4', be two vector spaces of dimension 4. Inside $\mathbf{P}(V_4)\times\mathbf{P}(V_4')$, S is the zero locus of a section s of $\mathcal{O}(1,1)\oplus\mathcal{O}(3,1)\oplus(\mathbb{C}^2\otimes\mathcal{O})$. The proof of the first part of the theorem is the same as for the case $(b12)$ and $(oe4)$; the only difference is that

$$s\in(V_4^*\otimes V_4'^*)\oplus(S^3(V_4^*)\otimes V_4'^*)\oplus(\mathbb{C}^2\otimes V_4'^*)\cong$$
$$\mathrm{H}^0(\mathbf{P}(V_4),(\mathcal{O}(1)\otimes V_4'^*)\oplus(\mathcal{O}(3)\otimes V_4'^*)\oplus(2\mathcal{O}\otimes V_4'^*))$$

Therefore $\pi_+(S)$ is again the locus where $\det(s)=0$, but the matrix is of the form showed in the statement of the proposition.

For the second part, as the vector bundle $\mathcal{F}$ contains two copies of $\mathcal{O}(0,1)$, the zero locus of s is contained in $\mathbf{P}(V_4)\times\mathbf{P}^1$. The projection π_- sends S to $\mathbf{P}^1$ and is of degree 3. The preimage of a point $l'\in\mathbf{P}^1\subset\mathbf{P}(V_4')$ is the intersection of the zero locus of $s_1|_{l'}$ and of $s_2|_{l'}$ inside $\mathbf{P}(V_4')\times\{l'\}$. This intersection is a cubic in $\mathbf{P}^2$, i.e. an elliptic curve. □

Finally, for the $K3$ surface S of type $(oe6)$, we will not give the details of the equations, as this variety has already been studied throughly, for instance in [Ili97] (see in particular [Ili97, Corollary 5.3]). We just remark that the two projections $\pi_\pm|_S$ are double coverings of $\mathbf{P}^2$ branched over a sextic.

K3 $(oe9)$

In what follows we just give a brief description of the $K3$ surface S of type $(oe9)$, without stating any result. To our knowledge, this is the only $K3$ surface with $\rho=2$ which is new in the literature.

The variety S lives inside the homogeneous space $\mathrm{OGr}(3,8)$. This variety is the quotient $SO_8/P_{\alpha_3,\alpha_4}$, where P_{α_3,α_4} is the parabolic group associated to the last two simple roots of D_4. It admits two projections

$$\pi_+:\mathrm{OGr}(3,8)\to\mathbb{Q}^6=SO_8/P_{\alpha_3}\quad,\quad\pi_-:\mathrm{OGr}(3,8)\to\mathbb{Q}^6=SO_8/P_{\alpha_4}$$

as explained in Section 2.2.4. The pullback of $\mathcal{O}_{\mathbb{Q}^6}(1)$ via $\pi_\pm$ is $\mathcal{L}_\pm$. As inside $\mathcal{F}=\mathcal{O}(1)\oplus\mathcal{L}_-^{\oplus3}\oplus\mathcal{L}_+^{\oplus3}$ there is a factor $\mathcal{L}_\pm^{\oplus3}$, we have that $\pi_\pm(S)$ is contained inside $\mathbb{Q}^3$. Thus, S admits an embedding in $\mathbb{Q}^3\times\mathbb{Q}^3$.

Remark 2.3.17. S contains two (-2)-curves, i.e. two $\mathbf{P}^1$'s. Indeed, the bundles $\mathcal{L}_+^*\otimes\mathcal{L}_-^{\otimes2}$ and $\mathcal{L}_-^*\otimes\mathcal{L}_+^{\otimes2}$ have degree -2. Moreover, a computation using the Koszul complex and Bott's theorem shows that their space of sections is 1-dimensional. The unique divisors of these two line bundles are therefore two $\mathbf{P}^1$'s contained in S. In the basis given by $M_1=\mathcal{L}_+^*\otimes\mathcal{L}_-^{\otimes2}$ and $M_2=\mathcal{L}_+^{\otimes2}\otimes(\mathcal{L}_-^*)^{\otimes5}$, we have $M_1^2=-2$, $M_2^2=14$, and $M_1M_2=0$; therefore by [Bin05, Theorem 1], the automorphism group of S is trivial.

Finally, we remark that $\pi_\pm$ realizes S as a (possibly singular) intersection of a quadric and a cubic in $\mathbf{P}^4$. This situation is similar to the ones we have for the

$K3$s $(b12)$ and $(oe6)$ and doubly elliptic $K3$s. In each of these cases, we have a general $K3$ surface S with $\rho = 2$ which can be realized in two possible ways S_+ and S_-, which in turn live inside the same locally complete family of $K3$ surfaces with $\rho = 1$.

2.4. Classification in exceptional Grassmannians

In this section we explain how to give an analogous classification as the one in the previous sections for the exceptional Grassmannians: we look for subvarieties with trivial canonical bundle which are zero loci of sections of completely reducible homogeneous bundles. In doing so we will explain once more how one can obtain the quantity dex, in order to generalize it for exceptional Lie groups. As there is a finite number of exceptional groups, the finiteness of the classification is clear; but still, we will show how to find all the possible varieties of low dimension. We will then compute the Euler characteristic in order to distinguish between Calabi-Yau and hyper-Kähler manifolds.

Let us fix some notation: G_2, F_4, E_6, E_7, E_8 will denote the exceptional Lie groups. The exceptional Grassmannians G/P related to them will be denoted by the group G and an index i which represents one of the simple roots α_i of the Dynkin diagram: it will mean that the parabolic subgroup P is the one associated to the i-th simple root. For example $E_6(1)$ will denote the Cayley plane (see [IM]).

The dimension and the index of a homogeneous space $G/P_i = G(i)$ can be recovered from the decomposition of the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is a Cartan subalgebra, Φ a system of roots with set of simple roots Δ. In fact, denote

$$\Phi(i)_{<0} = \{\alpha \in \Phi \text{ s.t. } \alpha = \sum_{\alpha_j \in \Delta} c_j \alpha_j \ , \ c_i < 0\} \ ,$$

and $\mathfrak{p}_i$ the Lie algebra associated to P_i. One has that the tangent bundle of $G(i)$ (which is a homogeneous bundle) is associated to the P_i-module

$$\mathfrak{t} = \mathfrak{g}/\mathfrak{p}_i \cong \bigoplus_{\alpha \in \Phi(i)_{<0}} \mathfrak{g}_\alpha,$$

where the last isomorphism is only as T-modules for a maximal torus T inside P_i. Moreover, from the general theory of homogeneous spaces, the positive generator of the Picard group of $G(i)$ is the line bundle associated to the P_i-module with weight ω_i. Therefore one can compute the dimension and the index of the variety $G(i)$ just by looking at the set Φ. The following picture explains the choice of the labelling of the simple roots in the Dynkin diagram, and Table 2.3 shows the dimensions and indexes of the various exceptional Grassmannians.

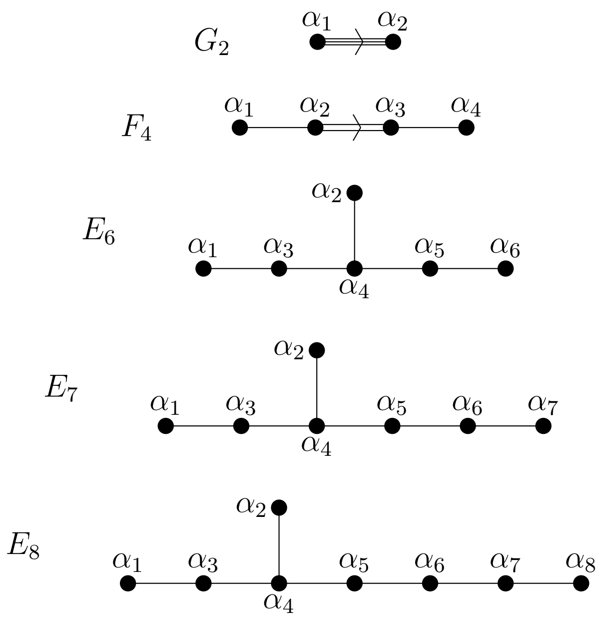


Table 2.3. – Dimension and index of the Fano varieties $G(i) := G/P_i$ for G an exceptional group; see for example [Sno93]

$G(i)$	$\dim(G(i))$	$i(G(i))$	$G(i)$	$\dim(G(i))$	$i(G(i))$	$G(i)$	$\dim(G(i))$	$i(G(i))$
$G_2(1)$	5	3	$G_2(2)$	5	5	$F_4(1)$	15	8
$F_4(2)$	20	5	$F_4(3)$	20	7	$F_4(4)$	15	11
$E_6(1)$	16	12	$E_6(2)$	21	11	$E_6(3)$	25	9
$E_6(4)$	29	7	$E_7(1)$	33	17	$E_7(2)$	42	14
$E_7(3)$	47	11	$E_7(4)$	53	8	$E_7(5)$	50	10
$E_7(6)$	42	13	$E_7(7)$	27	18	$E_8(1)$	78	23
$E_8(2)$	92	17	$E_8(3)$	98	13	$E_8(4)$	106	9
$E_8(5)$	104	11	$E_8(6)$	97	14	$E_8(7)$	83	19
$E_8(8)$	57	29						

The second step in order to be able to classify subvarieties is to classify irreducible homogeneous bundles or, equivalently, irreducible P_i-modules. This can be done by parametrizing them by their highest weights (see Section 1.3), in terms of which we will be able to understand what the determinant of the corresponding homogeneous bundle is. This will be useful in order to apply the adjunction formula, which will tell us when we are dealing with subvarieties with trivial canonical bundle.

2.4.1. Determinant of irreducible bundles

We recall from Section 1.3 that $(P_i)_L$ denotes the Levi factor of P_i. It can be easily read from the Dynkin diagram of G, and the maximal torus in $(P_i)_L$ is the same as the one in G. Denoting by $\mathfrak{l}_i$ the Lie algebra of $(P_i)_L$, this means that $\mathfrak{h} \subset \mathfrak{l}_i \subset \mathfrak{g}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}_i$. As $(P_i)_L$ is reductive, all irreducible $(P_i)_L$-modules V are classified by their maximal weight $\lambda \in \mathfrak{h}^*$.

Denote by $\omega_j \in \mathfrak{h}^*$ the fundamental weight dual to $H_{\alpha_j} \in \mathfrak{h}$. On $\mathfrak{h}$ and $\mathfrak{h}^*$ there is a natural non-degenerate symmetric pairing, given by the Killing form K. Recall that the coroot H_{α_j} is defined by the relations

$$\alpha_k(H_{\alpha_j}) = 2\frac{K(\alpha_j,\alpha_k)}{K(\alpha_j,\alpha_j)} \ \forall k.$$

If E_i denotes the linear space generated by $\{\alpha_j\}_{j\neq i}$, then its orthogonal in $\mathfrak{h}^*$ is $E_i^{\perp} = \mathbb{C}\omega_i$.

Now, if λ corresponds to a completely reducible vector bundle F_λ over G/P_i and therefore to a P_i-module $V = V_\lambda$,

$$\lambda = \sum_j \lambda_j\omega_j \text{ where } \lambda_j \in \mathbb{N} \text{ if } j \neq i \text{ , and } \lambda_i \in \mathbb{Z}.$$

By the Borel-Weil Theorem, the associated bundle is globally generated if and only if $\lambda_i \geq 0$. Actually, λ_i represents just the tensor product (λ_i times) with the positive generator of the Picard group of G/P_i.

The P_i-module V admits a decomposition given by diagonalizing the action of $\mathfrak{h}$: $V = \bigoplus_{\gamma\in\Gamma} U_\gamma$ for a finite set of weights $\Gamma \in \mathfrak{h}^*$. Clearly $\lambda \in \Gamma$. If W_i is the Weyl group of $\mathfrak{l}_i$, by general representation theory the set Γ is contained in the convex hull of $W_i(\lambda)$. Moreover, W_i is generated by the simple reflections with respect to $\{\alpha_j\}_{j\neq i}$, and its action on $\mathbb{C}\omega_i$ is trivial. If $\gamma \in \Gamma$, we use the following notation

$$\gamma = \sum_j \gamma_j\omega_j = \sum_{j\neq i} \tilde{\gamma}_j\alpha_j + c_\gamma\omega_i;$$

what we have said implies that $c_\gamma = c_\lambda$ for every $\gamma \in \Gamma$.

Lemma 2.4.1. *In the notations used so far,*

$$\det(F_\lambda) = H^{\operatorname{rank}(F_\lambda)c_\lambda}$$

where H is the positive generator of the Picard group of G/P_i.

Proof. In order to understand what is the determinant of the bundle F_λ, we try to understand its weight, which we denote by λ_{det}. Clearly, this weight is given by

$$\lambda_{det} = \sum_{\gamma \in \Gamma} \dim(U_\gamma)\gamma.$$

Moreover, λ_{det} will be proportional to ω_i, i.e. $\lambda_{det} = (\lambda_{det})_i \omega_i$ (on the semisimple part of $\mathfrak{l}_i$ the only representation of dimension 1 is the trivial one). As a consequence,

$$\det(F_\lambda) = H^{(\lambda_{det})_i}.$$

Therefore we can compute $\det(F_\lambda)$. First of all, express the (maximal) weight $\lambda = (\lambda_1, ..., \lambda_n)$ in terms of $\alpha_1, ..., \widehat{\alpha}_i, ..., \alpha_n, \omega_i$; this allows to find c_λ. By using the properties of the representations of semisimple algebras , and the fact that $c_\gamma = c_\lambda$ $\forall \gamma \in \Gamma$, one gets

$$\lambda_{det} = \sum_{\gamma \in \Gamma} \dim(U_\gamma)\gamma = \dim(V) c_\lambda \omega_i + x$$

where $x \in E_i$. As we have already remarked, this must be equal to $(\lambda_{det})_i \omega_i$, therefore $x = 0$ and

$$(\lambda_{det})_i = \dim(V) c_\lambda = \mathrm{rank}(F_\lambda) c_\lambda.$$

□

Example 2.4.2 ($\mathrm{Gr}(k, n)$). In this case a Cartan subalgebra is given by diagonal matrices with zero trace. We can identify $\mathfrak{h}^*$ with the vectors $v = (v_1, ..., v_n) = (v_1, ..., v_k; v_{k+1}, ..., v_n)$ in $\mathbb{C}^n$ such that $\sum_j v_j = 0$. A system of simple roots is given by $\alpha_1 = (1, -1, 0, ..., 0), ..., \alpha_{n-1} = (0, ..., 0, 1, -1)$. Here we supposed that the Killing form is the restriction of the standard symmetric form on $\mathbb{C}^n$. With these notations,

$$\omega_k = \frac{k(n-k)}{n}\left(\frac{1}{k}, ..., \frac{1}{k}; -\frac{1}{n-k}, ..., -\frac{1}{n-k}\right)$$

Given a weight $\lambda = (\lambda_1, ..., \lambda_n)$, we get that

$$c_\lambda = \frac{K(\omega_k, \lambda)}{K(\omega_k, \omega_k)} = \frac{\sum_{j=1}^k \lambda_j}{k} - \frac{\sum_{j=k+1}^n \lambda_j}{n-k}$$

and as a consequence, if $dex := \mathrm{rank}(F_\lambda)\left(\frac{\sum_{j=1}^k \lambda_j}{k} - \frac{\sum_{j=k+1}^n \lambda_j}{n-k}\right)$,

$$\det(F_\lambda) = H^{dex}$$

In this way we have found Equation 2.1 once more. The analogous formulas used for the other classical Grassmannians can be recovered in the same way. We will not study in detail these cases, but we will pass directly to the exceptional groups.

In the Appendix C, we reported the value of the coefficient c_λ for all the exceptional Grassmannians $G(i)$.

2.4.2. Exceptional classification

Consider the variety $G(i) = G/P_i$. We will denote by $(\lambda_1, ..., \lambda_n)$ the bundle F_λ associated to the maximal weight $\lambda = \sum_j \lambda_j \omega_j$. Moreover, H will denote the positive generator of the Picard group of $G(i)$. We have the following:

Theorem 2.4.3. *Let G be an exceptional Lie group. Let Y be a surface, a threefold, a fourfold or a sixfold with $K_Y = \mathcal{O}_Y$ which is the variety of zeroes of a general section of a homogeneous, completely reducible, globally generated vector bundle $\mathcal{F}$ over $G(i)$. Then $\mathcal{F}$ appears in one of the Tables B.16, B.17, B.18 in the Appendix.*

Remark 2.4.4. We classified sixfolds too in order to have more examples. For varieties of higher dimension a finite classification still holds, but it is difficult to extract some useful information on the varieties one finds; for instance, the computation of their invariants becomes too heavy for MACAULAY2.

Proof. One can put together Lemma 2.4.1, Lemma C.1 and the Weyl character formulas for the dimension of a representation of a semisimple Lie algebra to compute $\mathrm{rank}(F_\lambda)$ and $\det(F_\lambda)$. By using Table 2.3, one finds all possible globally generated completely reducible vector bundles $F = \oplus_\lambda F_\lambda$ such that the variety of zeroes of a general section has dimension $2, 3, 4$ or 6, and trivial canonical bundle (adjunction formula). This variety will be smooth by a Bertini type theorem, because of the generality of the section.

We just remark that the bundle F_{ω_6} over $E_6(1)$, even if it is globally generated, admits an exact sequence of the following form:

$$0 \to \mathcal{O}_{E_6(1)} \to F_{\omega_6} \to \tilde{F} \to 0$$

for a certain vector bundle $\tilde{F}$ (see for example [FM15, Section 2]). Therefore the zero locus of a general section over it is empty. This means that even though F_{ω_6} would give, numerically, zero loci which have trivial canonical bundle, these loci are in fact empty.

The computation of the Euler characteristic of the trivial bundle, as well as the degree of H for $K3$ surfaces, has been done with the help of [GS] and with the same method used for the subvarieties of the even orthogonal Grassmannian (see Appendix A). Even though it is a long work to do, it is not original nor it presents interesting ideas, so we did not report it. $\square$

3. Orbital degeneracy loci and quiver representations

An effective way to construct subvarieties which are not complete intersections is to use degeneracy loci. Classically, these loci are constructed by starting from a morphism of vector bundles; the locus of points for which the morphism on the fibers has rank bounded by a constant is a (classical) degeneracy locus. This construction can be generalized to any closed G-stable subvariety of a G-representation, e.g. an orbit closure, and they include zero loci as a particular case. In [BFMT17a] and [BFMT17b] the general theory of orbital degeneracy loci is developed.

In this chapter we use orbital degeneracy loci (later on ODL) to continue trying to fulfil our goal of constructing interesting special varieties. We start by collecting some general facts about classical loci; then, we recall the definition and fundamental properties of orbital degeneracy loci. As a proof of the usefulness and richness of degeneracy loci, we construct some Hilbert schemes of two points on a $K3$ surface: the Hilbert schemes will be naturally isomorphic to some classical degeneracy loci. By using such isomorphism, it will then be possible to reinterpret them as orbital degeneracy loci. Thus, on one hand we obtain a new insight on the geometry of the Hilbert schemes, on the other hand, hopefully, the constructions we present may be generalized to study other hyper-Kähler varieties.

The second part of this chapter is devoted to the study of quiver orbit closures and quiver degeneracy loci. Orbit closures inside quiver representations have been studied thoroughly, and admit a desingularization by a Kempf collapsing (a construction due to Reineke). We study the conditions under which those desingularizations permit to control the canonical bundle of the corresponding ODL, in order to be able to construct special varieties. Throughout the chapter, we give applications of the general theory to obtain Fano varieties and varieties with trivial canonical bundle.

The definition of ODL and how to construct a desingularization of such loci can be found in [BFMT17a], while how to obtain a locally free resolution of an ODL is explained in [BFMT17b]. In the latter, moreover, we study ODL which come from Gorenstein orbit closures in parabolic representations, showing the many possibilities ODL offer. We notice however that the techniques developed in the two papers are generalizations of the analogous ones for classical degeneracy loci, which were known before in the literature.

Finally, the content of Section 3.3 on crepant desingularizations of quiver orbit closures, and Section 3.4 on quiver orbital degeneracy loci can be found in [Ben17]. The sections on Hilbert schemes of two points on $K3$ surfaces and on the Hodge numbers of some Fano fourfolds are, on the other hand, new.

3.1. Classical degeneracy loci

Let $M_{e,f}$ be the space of matrices $e \times f$ which parametrises morphisms $\mathbb{C}^e \to \mathbb{C}^f$, and $Y^r_{e,f}$ the determinantal variety defined in Example 1.4.12. As $Y = Y^r_{e,f}$ is a $GL_e \times GL_f$-stable subvariety of an affine $GL_e \times GL_f$-variety, we can consider the relative situation, i.e. degeneracy loci (see [FP98]). Let X be a projective variety, and E, F two bundles on X of respective ranks e, f. Suppose $E^* \otimes F$ is globally generated, and consider a general section $s \in \mathrm{H}^0(X, E^* \otimes F)$. Then

$$D_Y(s) = \{x \in X \,|\, \mathrm{rank}(s(x)) \leq r\}$$

is a variety of dimension $\dim(X) + r(e+f-r) - ef$, singular in codimension $e+f-2r+1$. It has a natural desingularization induced by the Kempf collapsing of $W = \mathcal{Q}^* \otimes \mathbb{C}^f$ over $\mathrm{Gr}(e-r, e)$. Let $\theta : \mathrm{Gr}(e-r, E) \to X$ be the Grassmannian bundle of $(e-r)$-planes in E over X; the section s induces a section of $\theta^*(E^* \otimes F)$, and hence a section $\tilde{s}$ of the bundle

$$\mathcal{Q}_W := \theta^*(E^* \otimes F)/(\mathcal{Q}^* \otimes \theta^* F) = \mathcal{U}^* \otimes \theta^* F$$

over $\mathrm{Gr}(e-r, E)$. It is straightforward to see that a desingularization of $D_Y(s)$ is given by the zero locus $\mathscr{Z}(\tilde{s})$ of the section $\tilde{s}$.

By the adjunction formula, the canonical bundle of $\mathscr{Z}(\tilde{s})$ is

$$K_{\mathscr{Z}(\tilde{s})} = \theta^*(K_X \otimes (\det(E^*) \otimes \det(F))^{e-r}) \otimes \mathcal{O}_{\mathrm{Gr}}(f-e).$$

A way to impose that $\mathscr{Z}(\tilde{s})$ has trivial canonical bundle is to require the two following conditions to be satisfied:

$$f = e, \quad K_X^* = (\det(E^*) \otimes \det(F))^{e-r}.$$

There are two ways to understand why the condition $e = f$ needs to be imposed from the point of view of orbital degeneracy loci (see Section 3.2): firstly, under this assumption, $\det(W) = K_{\mathrm{Gr}(e-r,e)}$ and the Kempf collapsing is crepant; secondly, when $e = f$ (and only in this case), the determinantal orbit closures are Gorenstein.

Application to Hilbert schemes of points

Degeneracy loci have already been successfully used to construct Calabi-Yau threefolds (for instance, in [KK10]). Here, we use them to construct Hilbert schemes of points on $K3$ surfaces, in particular in dimension 4, i.e. $K3^{[2]}$ (see

Example 1.5.6). The classical degeneracy loci we are going to construct will help us build a bridge, in the next section, between ODL and Hilbert schemes. Moreover, in view of the importance of these varieties and in general of hyper-Kähler varieties, we hope that in the future it will be possible to use similar constructions to find more of them. We took inspiration from the paper by Iliev and Manivel ([IM16]), in which a similar situation is analysed.

3.1.1. First HK: $S_{14}^{[2]}$

The construction that follows should remind the reader of the proofs in the paper by Beauville and Donagi. Consider the Grassmannian $\mathrm{Gr}(2, V_6)$, where V_6 is a vector space of dimension 6. The zero locus of a general section t of the vector bundle $6\mathcal{O}(1)$ is a $K3$ surface S_{14} of degree 14 with respect to $\mathcal{O}(1)$; the family thus parametrised is 19-dimensional, and therefore locally complete. We want to construct the Hilbert scheme of points $S_{14}^{[2]}$ associated to a member of this family.

Let us formalise our notation: the section t lives in the space of sections $\mathrm{H}^0(\mathrm{Gr}(2, V_6), 6\mathcal{O}(1))$, which is naturally isomorphic to $V_6'^* \otimes \wedge^2 V_6^*$, where V_6' is another vector space of dimension 6; the reason for considering its dual will be clear in the following. Two general points of S_{14} are represented by two 2-planes P, Q inside V_6, whose intersection is $P \cap Q = 0$. Then $P + Q$ defines a 4-plane in V_6. This suggests to search for $S_{14}^{[2]}$ inside $\mathrm{Gr}(4, V_6)$. Therefore, our choice of the base variety X to construct the degeneracy locus is $X = \mathrm{Gr}(4, V_6)$. Moreover, we want our degeneracy locus to be parametrised by the same space of sections of t, i.e. $V_6'^* \otimes \wedge^2 V_6^*$. Therefore we make the choice $E = V_6' \otimes \mathcal{O}_X$, and $F = \wedge^2 \mathcal{U}_X^*$. Notice that, under this hypothesis, $\mathrm{rank}\, E = \mathrm{rank}\, F = 6$.

Proposition 3.1.1. *Let X, E, F be as before, and let $Y = Y_{6,6}^4$ be the determinantal variety defined in section 3.1. Let $s \in V_6'^* \otimes \wedge^2 V_6^*$ be a general section of $E^* \otimes F$. Then, the Y-degeneracy locus $D_Y(s)$ is the Hilbert scheme of points $S_{14}^{[2]}$, where S_{14} is the $K3$ surface defined by s in $\mathrm{Gr}(2, V_6)$. Moreover, varying s, one obtains a family of dimension 19.*

Proof. As recalled in the previous section, $D_Y(s)$ is a fourfold with trivial canonical bundle ($\mathrm{rank}\, E = \mathrm{rank}\, F$). As it is singular in codimension $e + f - 2r + 1 = 5$, it is in fact smooth. Therefore it is isomorphic to its desingularization $\mathscr{Z}(\tilde{s})$, which is a subvariety of $\mathrm{Gr}(2, V_6') \times \mathrm{Gr}(4, V_6)$. Moreover, as $\tilde{s}$ varies inside

$$\mathrm{H}^0(\mathrm{Gr}(2, V_6') \times \mathrm{Gr}(4, V_6), \mathcal{U}_1^* \otimes \wedge^2 \mathcal{U}_2^*) \cong V_6'^* \otimes \wedge^2 V_6^*,$$

and $H^0(\mathrm{Gr}(a, n), T_{\mathrm{Gr}}) = \mathfrak{sl}_n$, the family we obtain is of dimension

$$\dim(\mathbf{P}(V_6'^* \otimes \wedge^2 V_6^*)) - \dim(SL_6) - \dim(SL_6) = 89 - 35 - 35 = 19.$$

Therefore we only need to show the isomorphism $\eta : \mathscr{Z}(\tilde{s}) \to S_{14}^{[2]}$. The section $\tilde{s}$ defines an homomorphism $V_6' \to \wedge^2 V_6^*$. The image of this map can be seen as a section of $6\mathcal{O}(1)$ inside $\mathrm{Gr}(2, V_6)$, which in turn defines a $K3$ surface S_{14} of degree 14.

Suppose $(P, Q) \in \mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_6') \times \mathrm{Gr}(4, V_6)$. By composing $\tilde{s}$ with the natural projection, we get the morphism

$$\phi : V_6' \to \wedge^2 V_6^* \to \wedge^2 Q^*.$$

The rank of ϕ is at most four; indeed, since $(P, Q) \in \mathscr{Z}(\tilde{s})$, the map

$$P \to V_6' \to \wedge^2 V_6^* \to \wedge^2 Q^*$$

is the zero morphism. As $D_{Y^3_{6,6}}(s)$ is empty, $P = \mathrm{Ker}(\phi)$ and the image of ϕ is of dimension 4. Therefore, the morphism ϕ can be seen as defining a section of $4\mathcal{O}(1)$ over $\mathrm{Gr}(2, Q)$; its zero locus is of dimension 0, degree 2 and contained in S_{14}, i.e. a point in $S_{14}^{[2]}$. This point will be $\eta(P, Q)$.

Conversely, if $R, T \in S_{14} \subset \mathrm{Gr}(2, V_6)$ are general, then $R + T$ is of dimension 4 and defines a point of $D_Y(s) \in \mathrm{Gr}(4, V_6)$. The isomorphism $\theta' : \mathscr{Z}(\tilde{s}) \to D_Y(s)$ then gives a birational inverse to η. As the two varieties have trivial canonical bundle, it extends to a morphism and defines the inverse η^{-1}. □

The above proof also shows that:

Corollary 3.1.2. *The degeneracy locus $D_Y(s)$ is smooth, hence isomorphic to its desingularization $\mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_6') \times \mathrm{Gr}(4, V_6)$; the isomorphism is given by the restriction of the projection to the second factor.*

From now on, we will denote by $\mathscr{Z}_{S_{14}}(\tilde{s})$ the variety $\mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_6') \times \mathrm{Gr}(4, V_6)$.

3.1.2. Second HK: $S_8^{[2]}$

We use the same ideas as in the previous example in order to construct $S_8^{[2]}$, where S_8 is a $K3$ surface of degree 8. A general such surface is obtained as the intersection of three general quadrics in $\mathbf{P}^5 = \mathbf{P}(V_6)$, i.e. a general section t of $3\mathcal{O}(2)$. The space of sections in which t lives is isomorphic to $V_3^* \otimes Sym^2(V_6^*)$, where V_3 is a vector space of dimension 3. Let $l \neq m$ denote two lines in V_6 such that $[l], [m] \in S_8 \subset \mathbf{P}(V_6)$; they define a 2-plane $l + m$ in V_6. This, as before, motivates the choice $X = Gr(2, V_6)$. Moreover as before, as the space of sections is $V_3^* \otimes Sym^2(V_6^*)$, we make the choice $E = V_3 \otimes \mathcal{O}_X$, $F = Sym^2(\mathcal{U}_X^*)$. Notice again that $\mathrm{rank}\, E = \mathrm{rank}\, F = 3$.

Proposition 3.1.3. *Let X, E, F be as before, and let $Y = Y^1_{3,3}$ be the determinantal variety defined in section 3.1. Let $s \in V_3^* \otimes Sym^2 V_6^*$ be a general section of $E^* \otimes F$.*

Then, the Y-degeneracy locus $D_Y(s)$ is the Hilbert scheme of points $S_8^{[2]}$, where S_8 is the $K3$ surface defined by s in $\mathbf{P}(V_6)$. Moreover, varying s, one obtains a family of dimension 19.

Proof. As the proof is similar to the one in Section 3.1.1, we will be more concise. $D_Y(s)$ is a fourfold with trivial canonical bundle; it is smooth, and therefore isomorphic to $\mathscr{Z}(\tilde{s})$. By letting $\tilde{s}$ vary, the family we obtain is of dimension $(63-1)-8-35=19$.

The isomorphism $\eta : \mathscr{Z}(\tilde{s}) \to S_8^{[2]}$ is given as follows. Suppose $(P, Q) \in \mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_3) \times \mathrm{Gr}(2, V_6)$. By using the morphism defined by $\tilde{s}$, we get

$$\phi : V_3 \to Sym^2 V_6^* \to Sym^2 Q^*.$$

As $D_{Y^0_{3,3}}(s)$ is empty and $P \subset \mathrm{Ker}(\phi)$, the image of ϕ is of dimension 1. Therefore ϕ defines a section of $\mathcal{O}(2)$ over $\mathbf{P}(1, Q)$, whose zero locus is of dimension 0, degree 2 and contained in S_8, i.e. a point in $S_8^{[2]}$.

Conversely, if $R, T \in S_8 \subset \mathbf{P}(1, V_6)$ are general, then $R + T$ is of dimension 2 and defines a point of $D_Y(s) \in \mathrm{Gr}(2, V_6)$. The isomorphism $\theta' : \mathscr{Z}(\tilde{s}) \to D_Y(s)$ then gives a (birational) inverse η^{-1} to η. $\square$

In this case too, the above proof shows that:

Corollary 3.1.4. *The degeneracy locus $D_Y(s)$ is smooth, hence isomorphic to its desingularization $\mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_3) \times \mathrm{Gr}(2, V_6)$; the isomorphism is given by the restriction of the projection to the second factor.*

From now on, we will denote by $\mathscr{Z}_{S_8}(\tilde{s})$ the variety $\mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_3) \times \mathrm{Gr}(2, V_6)$.

3.1.3. Third HK: $S_{12}^{[2]}$

The final example is the Hilbert scheme of points of a general $K3$ surface S_{12} of degree 12. These surfaces are embedded in the orthogonal Grassmannian $\mathrm{OGr}(5, V_{10})$ of isotropic 5-planes in the 10-dimensional vector space V_{10} as the intersection of 8 hyperplane sections. In other words, S_{12} is defined as the zero locus of a section t of $8\mathcal{O}(\frac{1}{2})$. In this case, the space of sections is $\mathrm{H}^0(\mathrm{OGr}(5, V_{10}), 8\mathcal{O}(\frac{1}{2})) = V_8^* \otimes S_+(V_{10})$, where V_8 is a vector space of dimension 8, and $S_+(V_{10})$ is one of the two spinor representations of Spin_{10} of dimension 16. Consider two general points of S_{12}, represented by two isotropic 5-planes P, Q in $S_{12} \subset \mathrm{OGr}(5, V_{10})$; as $\mathrm{OGr}(5, V_{10})$ is the connected component of isotropic 5-planes in V_{10} intersecting each other in odd dimension, by generality P and Q will intersect in a 1-dimensional space, i.e. $P \cap Q = l$, where $l \in \mathbf{P}(V_{10})$, or more precisely $l \in \mathrm{OGr}(1, V_{10})$. This guides our choice for $X = \mathrm{OGr}(1, V_{10})$. Moreover, if we denote by $T_{\frac{1}{2}}(1)$ the globally generated rank 8 vector bundle over $\mathrm{OGr}(1, V_{10})$ whose space of sections is $S_+(V_{10})$, we redefine E and F as $E = V_8 \otimes \mathcal{O}_X$, $F = T_{\frac{1}{2}}(1)$. We have again $\operatorname{rank} E = \operatorname{rank} F = 8$.

Proposition 3.1.5. *Let X, E, F be as before, and let $Y = Y^6_{8,8}$ be the determinantal variety defined in section 3.1. Let $s \in V_8^* \otimes T_{\frac{1}{2}}(1)$ be a general section of $E^* \otimes F$. Then, the Y-degeneracy locus $D_Y(s)$ is the Hilbert scheme of points $S_{12}^{[2]}$, where S_{12} is the $K3$ surface defined by s in $\mathrm{OGr}(5, V_{10})$. Moreover, varying s, one obtains a family of dimension 19.*

Proof. The line of the proof for this case is the same as the ones already encountered. $D_Y(s)$ is again a smooth (by dimensional reasons) fourfold with trivial canonical bundle isomorphic to $\mathscr{Z}(\tilde{s})$; varying $\tilde{s}$ the family is of dimension $(128-1)-63-45 = 19$ (recall that $\mathrm{H}^0(\mathrm{OGr}(a, 2n), T_{\mathrm{OGr}}) = \mathfrak{so}_{2n}$).

Let us show the isomorphism $\eta : \mathscr{Z}(\tilde{s}) \to S_{12}^{[2]}$. Suppose $(P, l) \in \mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_8) \times \mathrm{OGr}(1, V_{10})$. Then, as $D_{Y^5_{8,8}}(s)$ is empty,

$$\phi : V_8 \to S^+_{V_{10}} \to S^+_{l^\perp/l}$$

has rank six, with $\mathrm{Ker}(\phi) = P$. Therefore ϕ defines a section of $6\mathcal{O}(1)$ over $\mathrm{OGr}(4, l^\perp/l)$, whose zero locus is of dimension 0 and degree 2, i.e. two points Q_1, Q_2 in $\mathrm{OGr}(4, l^\perp/l)$. Then, $Q_1 + l$, $Q_2 + l$ are two isotropic 5-planes in V_{10} that contain l, and define a point in $S_{12}^{[2]}$. This point will be $\eta(P, l)$.

In order to understand the morphism η^{-1}, we recall some facts about spinors (see [Igu70]). The nullity of a spinor $\psi \in S^+_{V_{10}}$ is the dimension of the maximal isotropic subspace of V_{10} that is annihilated by ψ under the action of the Clifford algebra. Pure spinors have by definition nullity 5, and they parametrise points of $\mathrm{OGr}(5, V_{10})$. The sum of two pure spinors in $S^+_{V_{10}}$ in general has nullity one. The fact that this sum ψ has nullity one reflects the fact that two general isotropic 5-planes in V_{10} (associated to the two pure spinors) intersect in a 1-dimensional space l; with this notation, l is indeed nullified by ψ.

As a consequence, two points $R, T \in S_{12} \subset \mathrm{OGr}(5, V_{10})$ define a point $l = R \cap T \in D_Y(s) \subset \mathrm{OGr}(1, V_{10})$. The isomorphism $\theta' : \mathscr{Z}(\tilde{s}) \to D_Y(s)$ then gives a (birational) inverse η^{-1} to η. □

As in the other cases, we have the following corollary:

Corollary 3.1.6. *The degeneracy locus $D_Y(s)$ is smooth, hence isomorphic to its desingularization $\mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_8) \times \mathrm{OGr}(1, V_{10})$; the isomorphism is given by the restriction of the projection to the second factor.*

From now on, we will denote by $\mathscr{Z}_{S_{12}}(\tilde{s})$ the variety $\mathscr{Z}(\tilde{s}) \subset \mathrm{Gr}(2, V_8) \times \mathrm{OGr}(1, V_{10})$.

3.2. Orbital degeneracy loci

In the following, we refer to [BFMT17a] and [BFMT17b] for general facts about orbital degeneracy loci; we recall just some notation, which was already used in Section 3.1.

Fix an algebraic group G, a finite dimensional G-module V, and a closed G-invariant subvariety $Y \subset V$. A wide class of examples of such Y's is given by orbit closures in prehomogeneous spaces (Section 1.4). More precisely, many of the varieties Y we will use are orbit closures inside parabolic spaces.

Example 3.2.1. $Y^r_{e,f}$ will be the determinantal variety of matrices in $V = M_{e,f}$ of rank at most r. The group acting on V is $G = \mathrm{GL}_e \times \mathrm{GL}_f$. As a parabolic representation, V is obtained from the grading given by the couple (A_{e+f-1}, α_e), where α_e is the e-th simple root of A_{e+f-1}.

Example 3.2.2. Let $G = \mathrm{GL}_6$ and $V = \wedge^3 V_6$ where V_6 is a 6-dimensional vector space; V_6 can be seen as the parabolic representation associated to the couple (E_6, α_2). There are five orbits in V_6: the dense orbit, a hypersurface of degree 4, the orbit of *partially decomposable forms*, the cone over the Grassmannian $\mathrm{Gr}(3,6)$ and the $\{0\}$-orbit. We will be mostly interested in the middle orbit closure, whose description is the following:

$$Y = \{\text{partially decomposable forms}\} =$$

$$= \{\sigma \in \wedge^3 V_6 \text{ s.t. } \sigma = v \wedge \omega \text{ for } v \in V_6 \text{ and } \omega \in \wedge^2 V_6\}.$$

The dimension of Y is 15, and its singular locus is the cone over the Grassmannian $\mathrm{Gr}(3,6)$ (of dimension 10).

Example 3.2.3. One can of course also consider orbit closures which do not live inside parabolic representations; for instance, in the second half of this chapter we will focus on orbit closures inside quiver representations.

The degeneracy locus is a way to relativize Y inside an ambient variety X. The right way to obtain such a relativization is to use the G structure of the *model* $Y \subset V$. For this purpose, one can start by assuming that the variety X is equipped with a G-principal bundle $\mathcal{E}$. For all the following results on ODL, we will also assume that X is smooth and projective.

As a matter of fact, $\mathcal{E}$ defines a functor

$$\mathcal{E}_\bullet : \{G\text{-spaces } Z\} \to \{\text{locally trivial fibrations over } X \text{ with fiber } Z\},$$

$$Z \mapsto \mathcal{E}_Z := \mathcal{E} \times Z/\sim,$$

where for any $e \in \mathcal{E}$, $z \in Z$, $(eg, z) \sim (e, gz)$ for all $g \in G$.

Remark 3.2.4. In particular, if Z is a G-representation, then $\mathcal{E}_Z$ is a vector bundle over X. Therefore, $\mathcal{E}_\bullet$ is a functor that sends any G-module to an $\mathcal{O}_X$-module.

As a consequence, one can construct from the data $(V, X, \mathcal{E})$ a vector bundle $\mathcal{E}_V$ on X, whose fiber over each point is isomorphic to V as a G-module. Being the inclusion $Y \subset V$ G-equivariant, it can be relativized as well over X to the inclusion $\mathcal{E}_Y \subset \mathcal{E}_V$, which are fibrations over X.

Having relativized the model over X, we still need to pull it back *inside* of X. Let s be a section of $\mathcal{E}_V$. Then, one can construct the Y-degeneracy locus associated to s, which is denoted by $D_Y(s)$; indeed, $D_Y(s)$ is defined as the locus of points in X which are sent by s inside $\mathcal{E}_Y$. More intrinsically:

Definition 3.2.5 (ODL, [BFMT17b]). *The Y-degeneracy locus of s, denoted by $D_Y(s)$, is the scheme defined by the Cartesian diagram*

$$\begin{tikzcd} \mathcal{E}_Y \arrow[r] & \mathcal{E}_V \\ D_Y(s) \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\square"] & X \arrow[u, "s"'] \end{tikzcd} \tag{3.1}$$

Its support is $\{x \in X,\ s(x) \in \mathcal{E}_Y \subset \mathcal{E}_V\} = s^{-1}(\mathcal{E}_Y)$.

Example 3.2.6. Let $Y = \{0\}$ inside the standard representation $\mathbb{C}^r$ of GL_r. Then $D_Y(s)$ is the zero locus of the section s.

Example 3.2.7 (Example 3.2.1 continued). In the relative setting, the data of a G-principal bundle $\mathcal{E}$ over a variety X is the same as the data of two vector bundles E and F of respective ranks e and f. With this notation, one obtains that $\mathcal{E}_V \cong \mathrm{Hom}(E, F)$, and a section s of this bundle is a morphism $s : E \to F$. Then $D_{Y^r_{e,f}}(s)$ is the classical degeneracy locus of the morphism s between vector bundles, described in Section 3.1.

If the bundle $\mathcal{E}_V$ is globally generated, and the section s is general, $D_Y(s)$ satisfies nice properties ([BFMT17a, Proposition 2.3]); in particular, $\mathrm{codim}_X(D_Y(s)) = \mathrm{codim}_V(Y)$, its singular locus is $D_{\mathrm{Sing}(Y)}(s)$, and it is normal if Y is normal. These facts are proven by using a Bertini type argument, as it can be done for (the special case of) zero loci.

Example 3.2.8 (Example 3.2.2 continued). In the relative setting, $\mathcal{E}_V$ over X is isomorphic to $\wedge^3 E$, for E a vector bundle over X of rank six. If $\wedge^3 E$ is globally generated and s is a general section, then $D_Y(s)$ has codimension 5 in X and is singular in codimension 5.

In [BFMT17a] we have used the orbit closure of partially decomposable forms as a model to construct many Calabi-Yau and Fano orbital degeneracy loci. It has also been the first example which was not a classical locus; we have studied its geometry in detail, and we have found a Thom-Porteous formula for its class in the cohomology of X. In the next section we will understand why this example is so important.

In the following two sections we will see how to construct desingularizations of ODL and locally free resolutions of their ideals. Both techniques are useful (and crucial!) to control the canonical bundle of the ODL.

3.2.1. Desingularization of ODL

Suppose Y admits a resolution of singularities given by a Kempf collapsing: the desingularization of Y is given by the total space $\mathcal{W}$ of a (homogeneous) vector bundle W over a homogeneous variety G/P (see Section 1.4). Then it is possible to relativize this construction, and obtain a desingularization of $D_Y(s)$.

This resolution of singularities will live inside the variety $\mathcal{E}_{G/P}$, which admits a fibration $\theta : \mathcal{E}_{G/P} \to X$ whose fiber is isomorphic to G/P. The following diagram illustrates the situation we are describing:

$$\begin{tikzcd}
\theta^*\mathcal{E}_V \arrow[d] & \mathcal{E}_W \arrow[l, hook'] \arrow[d] \arrow[dr] & & \\
& & \mathcal{E}_{G/P} \arrow[dd, "\theta"] & \mathscr{Z}(\tilde{s}) \arrow[l, hook'] \arrow[dd, "\theta'"] \\
\mathcal{E}_V & \mathcal{E}_Y \arrow[l, hook'] \arrow[dr] & & \\
& & X & D_Y(s) \arrow[l, hook']
\end{tikzcd}$$

In order to understand better the diagram, we suggest to compare it to its *absolute* version 1.1. Denote by Q_W the quotient of $\theta^*\mathcal{E}_V$ by $\mathcal{E}_W$ (seen as a vector bundle over $\mathcal{E}_{G/P}$). Then s induces a section $\tilde{s}$ of Q_W, and its zero locus $\mathscr{Z}(\tilde{s})$ is the wanted resolution of $D_Y(s)$ (refer to [BFMT17a, Proposition 2.3]).

Such a desingularization becomes very important when trying to understand the canonical bundle of the ODL. We recall the following definition:

Definition 3.2.9. *A variety Z has rational singularities if there exists a desingularization $\pi : Z' \to Z$ of Z such that:*

$\pi_*\mathcal{O}_{Z'} = \mathcal{O}_Z$ *and*

$R^i\pi_*\mathcal{O}_{Z'} = 0$ *for* $i > 0$.

Remark 3.2.10. Having rational singularities implies normality. Moreover, if the properties in the previous definition are satisfied for one desingularization $\pi : Z' \to Z$, they are satisfied for any desingularization of Z.

Remark 3.2.11. For an orbit closure $Y \subset V$, it is possible to verify if it has rational singularities via the geometric method (see Section 1.4).

In this context crepant resolutions (see Remark 1.4.13) are interesting. Morally, with crepant resolutions we are requiring the relative canonical bundle $K_{\mathscr{Z}(\tilde{s})/D_Y(s)}$ to be trivial, which means that the canonical bundle $K_{\mathscr{Z}(\tilde{s})}$ is the pull-back of a line bundle over X. With a further technical hypothesis (rational singularities) one is able to pushforward $K_{\mathscr{Z}(\tilde{s})}$ to recover $K_{D_Y(s)}$ as the restriction of a line bundle over X. More precisely:

Proposition 3.2.12 ([BFMT17a]). *Suppose that Y has rational singularities and the Kempf collapsing $p_W : W \to Y$ satisfies*

$$det(W) = K_{G/P}.$$

If $\mathcal{E}_V$ is globally generated and s is a general section, then the canonical sheaf of $D_Y(s)$ is the restriction of some line bundle on X.

Remark 3.2.13. Rational singularities are needed for two reasons. First of all, they imply the normality of Y, which ensures the normality of $D_Y(s)$ and therefore the existence of a canonical sheaf $K_{D_Y(s)}$. Secondly, it can be proven that $D_Y(s)$ has rational singularities too, and thus the canonical class of the desingularization $\mathscr{Z}(\tilde{s})$ can be pushed forward to the canonical class of $D_Y(s)$.

Notice that the line bundle mentioned in the previous proposition can be computed explicitly, by applying the adjunction formula for zero loci to $\mathscr{Z}(\tilde{s})$.

Example 3.2.14 (Example 3.2.1 continued). The Kempf collapsing of $Y^r_{e,f}$ described in Example 1.4.12 is crepant if and only if $e = f$. This explains the behaviour illustrated at the end of Section 3.1.

Example 3.2.15 (Example 3.2.2 continued). A Kempf collapsing of the orbit closure Y of partially decomposable forms is given by the total space of

$$W = \mathcal{U} \wedge \wedge^2 V_6 = \wedge^2 \mathcal{Q}(-1) \text{ over } \mathbf{P}(V_6).$$

This collapsing turns out to be crepant, and the relative canonical bundle can be computed as:

$$K_{D_Y(s)} = K_X \otimes \det(E)^5.$$

As $D_Y(s)$ is smooth in codimension 4, this model can be used to construct smooth fourfolds which are Fano or with trivial canonical bundle, which was done extensively in [BFMT17a]. The existence of the crepant Kempf collapsing is what brought at first this example to our attention.

Hodge diamond of some Fano fourfolds

In [BFMT17a] we constructed many ODL of Calabi-Yau type and of Fano type in dimension 3 and 4. Essentially, we used the orbit of partially decomposable forms Y in $\wedge^3\mathbb{C}^6$ (Example 3.2.2) and some nilpotent orbit closures (Remark 1.4.8). Most of the latter varieties admit crepant Kempf collapsings. Indeed, a nilpotent orbit closure is contained in a Lie algebra $\mathfrak{g}$. Notice that the total space $\Omega^1_{G/P}$ of the cotangent bundle of a homogeneous variety G/P is a subbundle of the trivial bundle $\mathfrak{g}$ over G/P. Denote by Y_n the image of the natural morphism

$$p : \Omega^1_{G/P} \to G/P \times \mathfrak{g} \to \mathfrak{g}.$$

The tangent bundle of G/P is the homogeneous bundle associated to the P-representation $\mathfrak{g}/\mathfrak{p}$, where $\mathfrak{g}$ (respectively $\mathfrak{p}$) is the Lie algebra of G (resp. P); as a consequence, $\Omega^1_{G/P}$ is associated to the P-module

$$(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}^\perp,$$

which is contained in the nilpotent cone of $\mathfrak{g}$ via the Killing form.

Thus Y_n is a closed, irreducible subvariety of the nilpotent cone, which contains only finitely many orbits, i.e. Y_n is a nilpotent orbit closure. If p is birational onto its image, we obtain that the orbit closure Y_n is desingularized by the crepant Kempf collapsing given by $\Omega^1_{G/P}$.

In this section we want to complement on a result of [BFMT17a]. In this paper, we constructed five families of Fano fourfolds, of which we computed some invariants (see [BFMT17a, Table 8]). Here we report the computation of their Hodge numbers, which could be useful in order to study them more in detail. Moreover, it shows once more why having a (crepant) Kempf collapsing is useful; using the Leray spectral sequence and the Koszul complex for zero loci in the relative situation, it allows to obtain the Hodge diamond. However, applying this method presents some complications, as it involves a strong use of Bott's theorem; in addition, in order to determine the Hodge diamond, one needs to be able to describe completely the maps that appear in the complexes involved, which can become quite difficult.

Theorem 3.2.16. *The Hodge numbers of the Fano fourfolds that appear in [BFMT17a, Table 8] are shown in Table 3.1.*

Table 3.1. – Hodge numbers of Fano degeneracy loci F of dimension 4

	X	E	$h^{1,1}$	$h^{1,2}$	$h^{1,3}$	$h^{2,2}$
$Y \subset \wedge^3\mathbb{C}^6$	$\mathrm{Gr}(3,6)$	$\mathcal{U}_X^* \oplus 3\mathcal{O}_X$	2	1	1	23
$Y \subset \wedge^3\mathbb{C}^6$	$\mathrm{IGr}(2,7)$	$\mathcal{Q}_X \oplus \mathcal{O}_X$	3/4	0/1	1	26/28
$Y \subset \wedge^3\mathbb{C}^6$	$\mathrm{IGr}(2,7)$	$\mathcal{U}_X^* \oplus 4\mathcal{O}_X$	2	0	5	54
(3) $\mathbf{P}^3, SL_4$	$\mathbb{Q}^{13}$	$4\mathcal{O}_X$	2	0	16	114
(7) $\mathbf{P}^4, SL_5$	$\mathbf{P}^{20}$	$5\mathcal{O}_X$	2	0	4	46

Proof. We will explain the computation for the last two varieties inside $\mathbb{Q}^{13}$ and $\mathbf{P}^{20}$. The others are ODL's coming from the orbit of partially decomposable forms;

in [BFMT17a, Appendix A] we explain in detail how to compute their Hodge numbers. Notice just that the indeterminacy in the second fourfold for $h^{1,1}$, $h^{1,2}$, $h^{2,2}$, is the same that was present when computing the Hodge diamond of the Calabi-Yau threefold obtained by cutting it with an anticanonical hyperplane.

For what concerns the last two Fano varieties, they come from nilpotent orbits. However, the bundle E is trivial in the two cases. We will explain here how we computed the Hodge diamond of the Fano F in $X = \mathbf{P}^{20}$; the one in $\mathbb{Q}^{13}$ can be studied similarly. As $F = D_Y(s)$ is smooth, it is isomorphic to its desingularization $\mathscr{Z}(\tilde{s})$. The section s lives in $\mathrm{H}^0(X, \mathfrak{sl}(E) \otimes L)$, where $L = \mathcal{O}_X(1)$. Consider the projective bundle $\theta : \mathbf{P}(E) \to X$; then $\mathscr{Z}(\tilde{s}) \subset \mathbf{P}(E)$ is the zero locus of a section inside

$$\mathrm{H}^0(\mathbf{P}(E), \theta^*(\mathfrak{sl}(E) \otimes L)/\Omega^1_{\mathbf{P}(E)/X} \otimes \theta^* L).$$

This bundle will be denoted by $\mathcal{T} = \tilde{\mathcal{T}} \otimes \theta^* L$.

By using the Koszul complex, to compute the cohomology of $\mathcal{O}_{\mathscr{Z}(\tilde{s})}$, we are reduced to compute $\mathrm{H}^j(\mathbf{P}(E), \wedge^i \mathcal{T}^*)$. This in turn, by the Leray spectral sequence, can be obtained from

$$\mathrm{H}^p(X, R^q\theta_*(\wedge^i \mathcal{T}^*)) = \mathrm{H}^p(X, R^q\theta_*(\theta^* L^{-i} \otimes \wedge^i \tilde{\mathcal{T}}^*)) = \mathrm{H}^p(X, L^{-i} \otimes R^q\theta_*(\wedge^i \tilde{\mathcal{T}}^*)),$$

by the projection formula. Moreover, as E is trivial, $\tilde{\mathcal{T}}$ is too, and

$$R^q\theta_*(\wedge^i \tilde{\mathcal{T}}^*) = R^{q,i} \otimes \mathcal{O}_X$$

for vector spaces $R^{q,i}$ of dimension equal to $\operatorname{rank}(\wedge^i \tilde{\mathcal{T}}^*)$. Therefore we can write

$$\mathrm{H}^p(X, L^{-i} \otimes R^q\theta_*(\wedge^i \tilde{\mathcal{T}}^*)) = \mathrm{H}^p(X, L^{-i}) \otimes R^{q,i},$$

whose computation is an easier task: as i goes from 0 to 20, the only cohomology different from zero of $L^{-i} = \mathcal{O}_X(-i)$ is, for $i = 0$, $\mathrm{H}^0(X, \mathcal{O}_X) \cong \mathbb{C}$. For what concerns the cohomology of $\Omega^1_{\mathscr{Z}(\tilde{s})}$, we can use the exact normal sequence and the exact sequence of the relative tangent bundle of $\mathbf{P}(E)$ over X; by applying again the Koszul complex and the Leray spectral sequence, we are lead to compute the following quantities:

$$\mathrm{H}^p(X, R^q\theta_*(\mathcal{T}^* \otimes \wedge^i \mathcal{T}^*)) = \mathrm{H}^p(X, L^{-i-1} \otimes R^q\theta_*(\tilde{\mathcal{T}}^* \otimes \wedge^i \tilde{\mathcal{T}}^*)) = \mathrm{H}^p(X, L^{-i-1}) \otimes R^{p,i}_N,$$

$$\mathrm{H}^p(X, R^q\theta_*(\Omega^1_{\mathbf{P}(E)/X} \otimes \wedge^i \mathcal{T}^*)) = \mathrm{H}^p(X, L^{-i} \otimes R^q\theta_*(\Omega^1_{\mathbf{P}(E)/X} \otimes \wedge^i \tilde{\mathcal{T}}^*)) =$$
$$= \mathrm{H}^p(X, L^{-i}) \otimes R^{p,i}_{\mathbf{P}(E)/X},$$

$$\mathrm{H}^p(X, R^q\theta_*(\Omega^1_X \otimes \wedge^i \mathcal{T}^*)) = \mathrm{H}^p(X, L^{-i} \otimes \Omega^1_X \otimes R^q\theta_*(\wedge^i \tilde{\mathcal{T}}^*)) = \mathrm{H}^p(X, L^{-i} \otimes \Omega^1_X) \otimes R^{p,i}_{\Omega^1_X},$$

for suitable vector spaces $R^{p,i}_N, R^{p,i}_{\mathbf{P}(E)/X}, R^{p,i}_{\Omega^1_X}$. The dimension of these vector spaces can be computed as described in [BFMT17a], and the rest is just regular cohomology over $\mathbf{P}^{20}$.

Finally, to compute $h^{2,2}$ we used $\chi(\Omega^2_F)$, which is given again in [BFMT17a]. $\square$

3.2.2. Locally free resolution of ODL

One of the aims of [BFMT17b] was to construct a locally free resolution of the ideal of an ODL. We did it by starting from a G-equivariant free resolution of the G-stable subvariety Y of the G-representation V. Furthermore, in the case when Y is Gorenstein, the resolution can be used to understand the canonical bundle of the ODL. In this section we report the main results in this sense. In order to gain in clearness, we will state the results in a less technical situation than the one considered in [BFMT17b].

We begin by recalling some definitions for affine varieties:

Definition 3.2.17. *An affine variety $Z \subset \mathbb{C}^n$ has Cohen-Macaulay ring if there exists a minimal free resolution of $\mathcal{O}_Z$ of length $\operatorname{codim}_{\mathbb{C}^n} Z$. Moreover, Z has Gorenstein ring if and only if the minimal resolution is self-dual.*

Remark 3.2.18. If an affine variety Z has Gorenstein ring, the last term of the minimal resolution is free of rank one (see [Eis95, Corollary 21.16]).

Remark 3.2.19. As already mentioned in Section 1.4, it is concretely possible to verify if an orbit closure inside a parabolic representation is Cohen-Macaulay or Gorenstein by using the geometric technique.

Let us suppose that $Y \subset V$ is Cohen-Macaulay. Let us consider a minimal free resolution $F_\bullet$ of Y of length $\operatorname{codim}_V Y$. As Y is G-stable, we can suppose that the resolution $F_\bullet$ is G-equivariant (as explained in [Wey03]). The terms of the complex $F_\bullet$ are $\mathcal{O}_V$-modules.

Let us consider the relative situation. We fix a variety X and a G-principal bundle $\mathcal{E}$ over it. By applying the functor $\mathcal{E}_\bullet$ defined at the beginning of Section 3.2 to the complex $F_\bullet$, we obtain a locally free complex $\mathcal{E}_{F_\bullet}$, which has the following properties:

- as the F_i's are $\mathcal{O}_V$-modules, in the relative case the $\mathcal{E}_{F_i}$'s become $\mathcal{O}_{\mathcal{E}_V}$-modules. Therefore $\mathcal{E}_{F_\bullet}$ can be seen as a locally free complex over the total space $\mathcal{E}_V$;
- $\mathcal{E}_{F_\bullet}$ can be proved to be an exact complex resolving $\mathcal{O}_{\mathcal{E}_Y}$ (as $\mathcal{E}_Y \subset \mathcal{E}_V$).

As stated here, the complex $\mathcal{E}_{F_\bullet}$ naturally lives on $\mathcal{E}_V$ over X. In order to bring it back to X we can apply the pullback via the section $s : X \to \mathcal{E}_V$. By using the Generic Perfection Theorem (see [EN67]), we obtain:

Theorem 3.2.20 ([BFMT17b]). *Let Y be a G-stable subvariety of the G-representation V with Cohen-Macaulay ring and minimal free resolution*

$$0 \to F_\bullet \to \mathcal{O}_Y \to 0.$$

Moreover, let X be a smooth projective variety and $\mathcal{E}$ a G-principal bundle over it. If $\mathcal{E}_V$ is globally generated and s is a general section of $\mathcal{E}_V$, then the complex

$$0 \to s^*(\mathcal{E}_{F_\bullet}) \to s^*(\mathcal{O}_{\mathcal{E}_Y}) \to 0$$

is a length $\operatorname{codim}_V Y$ *locally free resolution of $\mathcal{O}_{D_Y(s)} = s^*(\mathcal{O}_{\mathcal{E}_Y})$.*

Example 3.2.21 (Example 3.2.1 continued). The determinantal variety $Y^r_{e,f}$ is always Cohen-Macaulay and it is Gorenstein if and only if $e = f$. Suppose $e \geq f$, $r = f - 1$. The resolution of the ideal of the classical degeneracy locus $D_{Y^r_{e,f}}(s)$ given by Theorem 3.2.20 is the well-known Eagon-Northcott complex (e.g. see [Laz04, Theorem B.2.2]).

Example 3.2.22 (Example 3.2.2 continued). A relative version of the resolution of the ideal of the orbit closure Y of partially decomposable forms has been given in [BFMT17b].

The following corollary of Theorem 3.2.20 is what allows to control the canonical bundle.

Corollary 3.2.23. *In the hypothesis of Theorem 3.2.20, assume moreover that Y is normal of codimension c and has Gorenstein ring. Then the last term of the minimal resolution of $\mathcal{O}_Y$ is*

$$F_c = W$$

for W a 1-dimensional representation of G, and

$$K_{D_Y(s)} = K_X \otimes \mathcal{E}^*_W|_{D_Y(s)}$$

Remark 3.2.24. Notice that the bundle $\mathcal{E}_W$ is defined on the whole variety X. This result has therefore the same flavour as Proposition 3.2.12, and it can be used to control $K_{D_Y(s)}$. As a matter of fact, admitting a crepant Kempf collapsing is a condition which is stronger than having Gorenstein ring (see [BFMT17b, Section 2.5]).

Remark 3.2.25. Sometimes we do not have a crepant Kempf collapsing of a given orbit closure, but we know that it is Cohen-Macaulay and Gorenstein. For instance, this is always true for a cone over an ordinary Grassmannian $\mathrm{Gr}(k, n)$: the natural resolution given by $\mathcal{O}(-1)$ over $\mathrm{Gr}(k, n)$ is indeed never crepant. However, it is possible to compute the relative canonical bundle of the ODL knowing that the cone is Gorenstein.

Remark 3.2.26. In [BFMT17b] we were able to compute the last term of the resolution, which gives the (dual of the) relative canonical bundle of the ODL, in many cases. In particular, we did it for all parabolic representations.

Orbital interpretation of Hilbert schemes of points

The examples we have studied in Section 3.1 admit a second interpretation as orbital degeneracy loci. It is interesting to notice that this second interpretation, in the first HK example of $S_{14}^{[2]}$, allows to construct a complete 20-dimensional family of hyper-Kähler fourfolds (it is the same family as the one appearing in [BD85]). Unfortunately, this extension of the family is not possible in the other two cases, as we will show.

More precisely, as a consequence of the description given in Section 3.1, in the cases studied a general member of $S_d^{[2]}$ lives in the product of two varieties which is of the form $\mathrm{Gr}(2, \cdot) \times X$; if before we focused on the second projection, we want to study now the projection onto the first factor. It turns out that this projection realises the Hilbert scheme as another degeneracy locus for examples 3.1.1 and 3.1.3; in example 3.1.2 instead, the first projection realises the Hilbert scheme as a fibration in Jacobians over $\mathrm{Gr}(2, V_3)$ (this situation has been described in [Saw14]). Morally, by "forgetting" the variety X (through the first projection), we get rid of the direct connection with the $K3$ surface S_d, and in our opinion this is a heuristic explanation of the fact that it is possible to extend the family described in Section 3.1.1.

3.2.3. Beauville-Donagi revisited

As we have seen in Section 3.1.1 (and in Corollary 3.1.2), a general member of $S_{14}^{[2]}$ is a subvariety $\mathscr{Z}_{S_{14}}(\tilde{s}) \subset \mathrm{Gr}(2, V_6') \times \mathrm{Gr}(4, V_6)$ (here an important role is played by the fact that the bundle E was chosen to be trivial). The second projection onto $\mathrm{Gr}(4, V_6)$ restricted to $S_{14}^{[2]}$ is an isomorphism with its image; actually, we realised $S_{14}^{[2]}$ starting from the consideration that it should be a subvariety of $\mathrm{Gr}(4, V_6)$. We study now the first projection onto $\mathrm{Gr}(2, V_6')$; we will see that its image can be well understood as orbital degeneracy locus. As a consequence, we have the following result:

Proposition 3.2.27. *Let $\mathscr{Z}_{S_{14}}(\tilde{s}) \subset \mathrm{Gr}(2, V_6') \times \mathrm{Gr}(4, V_6)$ be the Hilbert scheme of points of S_{14} as defined in Section 3.1.1. Then the image of the projection $\mathscr{Z}_{S_{14}}(\tilde{s}) \to \mathrm{Gr}(2, V_6')$ is the zero locus of a section of the bundle $Sym^3\mathcal{U}^*$. By varying this section, one obtains a 20-dimensional family of hyper-Kähler fourfolds.*

Notice that this family is exactly the same as the one that appears in [BD85].

Proof. Consider the prehomogeneous representation V associated to the pair (E_7, α_3), where α_3 is the third simple root (in standard conventions) of the Lie algebra associated to the group E_7. By looking at the Dynkin diagram,

$$V = U_2^* \otimes \wedge^2 U_6^*,$$

where U_2, U_6 are vector spaces of respective dimension 2 and 6 (the duals are taken for later convenience). It is a representation under the action of $G_0 = SL_2 \times SL_6 \times \mathbb{C}^*$. Take the codimension 4 orbit, and denote its closure by Y. Then Y is the closure of the space of morphisms $U_2 \to \wedge^2 U_6^*$ such that the image has zero Pfaffian. A resolution of Y is given by the total space of the bundle

$$W = U_2^* \otimes (\mathcal{O}(-1) \oplus \mathcal{Q}^* \otimes \mathcal{U}^*) \quad \text{over} \quad \mathrm{Gr}(4, U_6).$$

In the relative case, consider a variety X endowed with two vector bundles G, H of respective rank two and six, with $G^* \otimes \wedge^2 H^*$ globally generated. Then, if

$$s \in \mathrm{H}^0(X, G^* \otimes \wedge^2 H^*),$$

we can define $D_Y(s) \subset X$. If s is general, it is a subvariety of codimension 4 resolved by $\mathscr{Z}(\tilde{s})$, where $\tilde{s} \in \mathrm{H}^0(\mathrm{Gr}(4, H), \wedge^2 \mathcal{U}^*)$ (as explained in Section 3.2). By making the choice $X = \mathrm{Gr}(2, V_6')$, $G = \mathcal{U}_X$ and $H = V_6 \otimes \mathcal{O}_X$, it is straightforward to see that $\mathscr{Z}(\tilde{s})$ is exactly the same as $\mathscr{Z}_{S_{14}}(\tilde{s})$ defined after Corollary 3.1.2.

To prove that $D_Y(s)$ lives in a family of dimension 20, notice that $s(x)$ can be seen as a skew-symmetric matrix of forms over the fiber G_x. Since Y is the closure of morphisms with null Pfaffian,

$$D_Y(s) = \{x \in X | Pf(s(x)) = 0\} = \mathscr{Z}(Pf(s)),$$

where $\mathscr{Z}(Pf(s))$ is the zero locus of

$$Pf(s) \in \mathrm{H}^0(X, Sym^3 G^*) = \mathrm{H}^0(\mathrm{Gr}(2, V_6'), Sym^3 \mathcal{U}^*) = Sym^3 V_6'^*.$$

By taking a general section t of $Sym^3 \mathcal{U}^*$ over $\mathrm{Gr}(2, V_6')$, one obtains a family of hyper-Kähler fourfolds $\mathscr{Z}(t)$ deformation equivalent to $S_{14}^{[2]}$ of dimension $(56-1) - 35 = 20$. $\square$

Remark 3.2.28. By fixing an isomorphism $V_6' \cong V_6^*$, our construction and Beauville-Donagi's one are the same. Indeed, while in their construction what is used in order to construct the $K3$ surface is just the image of the morphism $V_6' \to \wedge^2 V_6^*$, in our construction we need to fix the whole morphism. This means that we need to fix also an automorphism of V_6', which is the same as fixing the isomorphism $V_6' \cong V_6^*$.

Remark 3.2.29. Consider the codimension 10 orbit Y in the prehomogeneous representation V associated to the pair (E_7, α_3). Moreover, let $X = \mathrm{Gr}(2, V_9)$ where V_9 has dimension 9, with two bundles $G = \mathcal{U}$, $H = 6\mathcal{O}_X$ on it, of respective ranks 2 and 6. Then, by following exactly the same line of ideas used before, one can prove that the orbital degeneracy locus $D_Y(s)$ (where s is a general section of $G^* \otimes \wedge^2 H$) is again hyper-Kähler and isomorphic to $S_{14}^{[2]}$ (it is smooth as the singularities of the orbital degeneracy locus are in codimension 5).

Indeed, if before we were fixing a subspace of dimension 6 in $\wedge^2 H^*$ in order to construct the $K3$ surface S_{14}, now we are fixing its dual, i.e. a subspace of dimension 9 in $\wedge^2 H$. This gives another embedding of $S_{14}^{[2]}$ in $\mathrm{Gr}(2, V_9)$.

3.2.4. Fibration in Jacobians

The Hilbert scheme of points of a $K3$ surface of degree 8 can be seen as a subvariety $\mathscr{Z}_{S_8}(\tilde{s}) \subset \mathrm{Gr}(2, V_3) \times \mathrm{Gr}(2, V_6)$ by Corollary 3.1.4. In this case, the first projection restricted to $\mathscr{Z}_{S_8}(\tilde{s})$ is surjective. Moreover, it is a fibration in Jacobians, as explained by the following proposition.

Proposition 3.2.30. *Let $\mathscr{Z}_{S_8}(\tilde{s}) \subset \mathrm{Gr}(2, V_3) \times \mathrm{Gr}(2, V_6)$ be the Hilbert scheme of points of S_8 as defined in Section 3.1.2. Then the projection $\mathscr{Z}_{S_8}(\tilde{s}) \to \mathrm{Gr}(2, V_3)$ is a fibration, with fibers isomorphic to the Jacobian of a curve of genus 2.*

Proof. Consider the $GL_2 \times GL_6$-representation

$$V = U_2^* \otimes Sym^2 U_6^* = \mathrm{Hom}(U_2, Sym^2 U_6^*),$$

where U_2, U_6 are vector spaces of respective dimension 2 and 6. A generic element of V defines an element of $\mathrm{Gr}(2, Sym^2 U_6^*)$. This in turn defines the variety $J \subset \mathrm{Gr}(2, U_6)$ of 2-planes isotropic with respect to the symmetric forms generating the 2-plane in $Sym^2 U_6^*$. J can thus be seen as the zero locus of a general section in $\mathrm{H}^0(\mathrm{Gr}(2, U_6), 2(Sym^2 \mathcal{U}^*))$. It has been proved in [Rei72] that such a J is isomorphic to the Jacobian of a curve of genus two.

In the relative setting, take $X = \mathrm{Gr}(2, V_3)$, $G = \mathcal{U}_X$ and $H = V_6 \otimes \mathcal{O}_X$, where the two vector bundles G, H are of respective ranks two and six, such that $G^* \otimes Sym^2 H^*$ is globally generated. Then, on the Grassmannian bundle $\pi : \mathrm{Gr}(2, H) \to X$, consider a generic section $t \in \mathrm{H}^0(\mathrm{Gr}(2, H), \pi^* G^* \otimes \mathrm{Sym}^2 \mathcal{U}^*)$. From what we have said before, its zero locus $\mathscr{Z}(t)$ is a variety which admits a surjective fibration in Jacobians $\pi' : \mathscr{Z}(t) \to X$ over its image. But $\mathscr{Z}(t)$ is actually the same as $\mathscr{Z}_{S_8}(\tilde{s})$ for $t = \tilde{s}$, and we get the assertion of the proposition. □

Remark 3.2.31. As in Remark 3.2.29, instead of constructing the Hilbert scheme $S_8^{[2]}$ from a subspace of dimension 3 inside $Sym^2 H^*$ (the image of the morphism defined by s), we can consider its dual space, i.e. a subspace of dimension 18 inside $Sym^2 H$. By doing so, we obtain an embedding from $S_8^{[2]}$ into $\mathrm{Gr}(2, V_{18})$, where V_{18} has dimension 18 (we do not give further details, as the method to prove this statement is similar to the one already used).

3.2.5. A Spinor orbit

The case of the Hilbert scheme of points of a $K3$ surface of degree 12 is similar to the Beauville-Donagi case: the first projection onto $\mathrm{Gr}(2, V_8)$ is an isomorphism that realises it as an orbital degeneracy locus; however, in this case, the family doesn't extend inside $\mathrm{Gr}(2, V_8)$ to a 20-dimensional family. Moreover, the image of $\mathscr{Z}_{S_{12}}(\tilde{s})$ cannot be seen as the zero locus of a general section of a vector bundle.

Proposition 3.2.32. *Let $\mathscr{Z}_{S_{12}}(\tilde{s}) \subset \mathrm{Gr}(2, V_8) \times \mathrm{OGr}(1, V_{10})$ be the Hilbert scheme of points of S_{12} as defined in section 3.1.3. Then the projection $\mathscr{Z}_{S_{12}}(\tilde{s}) \to \mathrm{Gr}(2, V_8)$ is an isomorphism onto its image.*

Proof. Consider in this case the prehomogeneous representation V associated to the pair (E_7, α_6), where α_6 is the sixth simple root of E_7. In this hypothesis

$$V = U_2^* \otimes S_+(U_{10}),$$

where U_2, U_{10} are vector spaces of respective dimension 2 and 10. It is a representation under the action of $G_0 = SL_2 \times \mathrm{Spin}_{10} \times \mathbb{C}^*$. There is only one codimension 8 orbit, whose closure we denote by Y: it is the closure of the space of morphisms $U_2 \to S_+(U_{10})$ such that the image is a 2-plane of spinors which all annihilate the same isotropic line in U_{10}. Indeed, a resolution of Y is given by the total space $\mathcal{W}$ of the bundle

$$W = U_2^* \otimes T_{\frac{1}{2}}(1) \quad \text{over} \quad \mathrm{OGr}(1, U_{10}).$$

This orbit closure is singular at the points representing morphisms $\phi : U_2 \to S_+(U_{10})$ whose image is a line inside $\mathrm{OGr}(5, U_{10})$. This is a consequence of the fact that a spinor in $S_+(U_{10})$ that is not pure, has nullity equal to one; therefore if the image of the morphism ϕ is not contained in $\mathrm{OGr}(5, U_{10})$, it identifies one (and only one) point of the Grassmannian $\mathrm{OGr}(1, U_{10})$, thus providing an inverse to the morphism $\mathcal{W} \to V$. As a result, the orbit closure is singular in codimension 5.

In the relative case, consider a variety X endowed with two vector bundles G, H of respective rank two and ten, with $G^* \otimes S_+(H)$ globally generated. Then, if

$$s \in \mathrm{H}^0(X, G^* \otimes S_+(H)),$$

we can define $D_Y(s) \subset X$. If s is general, it is a subvariety of codimension 8 resolved by $\mathscr{Z}(\tilde{s})$, where $\tilde{s} \in \mathrm{H}^0(\mathrm{OGr}(4, H), T_{\frac{1}{2}}(1))$ (as explained in Section 3.2). By making the choice $X = \mathrm{Gr}(2, V_8)$, $G = \mathcal{U}_X$ and $H = V_{10} \otimes \mathcal{O}_X$, it is straightforward to see that the desingularization $\mathscr{Z}(\tilde{s})$ of $D_Y(s)$ is exactly the same as $\mathscr{Z}_{S_{12}}(\tilde{s})$ defined after Corollary 3.1.6 Moreover, as $D_Y(s)$ is a fourfold, it is smooth and isomorphic to $\mathscr{Z}(\tilde{s})$. □

Contrary to the Beauville-Donagi case, $D_Y(s)$ does not live in a family of dimension 20 inside $\mathrm{Gr}(2, V_8)$. Indeed, it is possible to use the resolution of the variety Y (see [KW13]) in order to compute the cohomology $\mathrm{H}^0(D_Y(s), T_{\mathrm{Gr}}|_{D_Y(s)})$, which is an obstruction to the completeness of the family. It turns out that

$$\mathrm{H}^0(D_Y(s), T_{\mathrm{Gr}(2,V_8)}|_{D_Y(s)}) \cong \mathbb{C}^2,$$

and therefore the family can only be of codimension 2, i.e. of dimension 19.

So far, using orbital degeneracy loci we have been able to construct many special varieties, whose geometry would be worth being investigated more closely. More precisely, the situation is the following:

— in [BFMT17a], we have constructed six families of Fano degeneracy loci of dimension 3 by using the orbit closure of partially decomposable forms (two examples) and nilpotent orbit closures (four examples); as the classification of Fano threefolds is known, we have been able to identify all the threefolds we have found. Moreover, we have constructed eighteen families of almost Fano nilpotent degeneracy loci of dimension 3, among which two remain mysterious (see [BFMT17a, Table 7]);

— in [BFMT17a], we have found five families of Fano degeneracy fourfolds (for which we have computed the Hodge numbers in Section 3.2.1);

— in [BFMT17a] we have constructed a dozen families of Calabi-Yau degeneracy threefolds, five from the orbit closure of partially decomposable forms and for which we have computed the Hodge numbers, and the rest from nilpotent orbit closures;

— the largest number of degeneracy loci we have constructed to date are fourfolds of Calabi-Yau type. This is because we were essentially searching for new examples of hyper-Kähler fourfolds among loci with trivial canonical bundle. In [BFMT17a] we have found several dozens Calabi-Yau degeneracy fourfolds from the orbit closure of partially decomposable forms (~ 40 examples) and nilpotent orbit closures (~ 20 examples); in [BFMT17b] we have found two tens extra Calabi-Yau degeneracy fourfolds by using orbit closures inside parabolic representations. All these examples deserve a more detailed analysis;

— finally, in this section, we have showed that orbital degeneracy loci can be successfully used to construct some families of hyper-Kähler fourfolds. All the examples are (deformations of) Hilbert schemes of two points on a $K3$ surface, but they suggest that other kinds of interesting hyper-Kähler varieties can be found among orbital degeneracy loci. We believe that similar constructions as the ones used here will permit to construct such hyper-Kähler degeneracy loci.

The work done in [BFMT17a] and [BFMT17b] already shows that ODL provide an efficient way of producing (special) varieties. However, we would like to underline the fact that much more can and needs to be done; for instance, in

[BFMT17b] we have just given a sample of parabolic degeneracy loci, and we intend to enlarge this list in the future, with the hope of finding some remarkable ODL.

In the next two sections we will study quiver degeneracy loci, and we will construct eight additional examples among smooth and singular, Fano and Calabi-Yau varieties of dimension 3 and 4.

3.3. Crepant resolutions of orbit closures in Quiver representations of type A_n and D_4

We study resolutions of singularities of orbit closures in quiver representations. We chose to analyze the case of quiver orbits because of the nice properties they have. In particular, when the quivers are of finite type, a complete description of such orbits is known ([Gab72]). We dealt with the quivers of type A_n and D_4. We consider certain resolutions of singularities which have already been constructed by Reineke, and we determine under which conditions they are crepant. Our motivation to search for crepant desingularizations is that we want to apply Proposition 3.2.12 to construct special varieties; indeed, the desingularizations we will consider are actually Kempf collapsings.

3.3.1. Quiver Representations

Let $Q = (S, A)$ be a quiver; S is the set of vertices of the quiver, and A the set of arrows. Each arrow $a \in A$ starts from the vertex $a(0) \in S$ and ends on the vertex $a(1) \in S$. A quiver is said to be *source-sink* if for each vertex $s \in S$, either all the arrows that are connected to s start from it, or they end on it.

If T is the type of a certain semisimple Lie algebra (e.g. $T = A_n, B_n, \ldots$) then we will say that the quiver Q is of type T if the underlying graph of Q (which is obtained by replacing arrows with edges) is the Dynkin diagram of type T. We will always assume that the quiver has no loops, meaning that its graph has no loops, so that the possible Dynkin types are A_n, D_n, E_6, E_7, E_8. Actually, for the definition of quivers of type B_n, C_n, G_2, F_4, one needs a more general definition of quiver (i.e. that of a *valued quiver*, as is defined for example in [DR76]).

If Q is of type A_n, we will say that it is a *one-way* quiver if all the arrows point in the same direction.

Definition 3.3.1. *A representation ϕ of a quiver $Q = (S, A)$ is defined by the choice of a vector space V_s of dimension d_s for all $s \in S$ and of a morphism $\phi_a : V_{a(0)} \to V_{a(1)}$ for every $a \in A$. The data $d = \{d_s\}_{s \in S}$ is usually referred to as the dimension vector of the representation.*

Definition 3.3.2. *Consider two representations $\phi = ((V_s)_{s\in S}, (\phi_a)_{a\in A})$ and $\psi = ((W_s)_{s\in S}, (\psi_a)_{a\in A})$ of a quiver Q. A morphism $\alpha : \phi \to \psi$ between them is the data* *of morphisms $\alpha_s : V_s \to W_s$ for every $s \in S$ such that for every $a \in A$ the following diagram commutes:*

$$\begin{tikzcd} V_{a(0)} \arrow[r, "\phi_a"] \arrow[d, "\alpha_{a(0)}"] & V_{a(1)} \arrow[d, "\alpha_{a(1)}"] \\ W_{a(0)} \arrow[r, "\psi_a"] & W_{a(1)} \end{tikzcd}$$

The parameter space $\mathcal{R}_d$ for representations of a quiver Q with a given dimension vector d (and fixed spaces $(V_s)_{s\in S}$) is

$$\mathcal{R}_d := \bigoplus_{a\in A} \mathrm{Hom}(V_{a(0)}, V_{a(1)}) \ ,$$

on which there is a natural action of

$$G := \prod_{s\in S} \mathrm{GL}(V_s) \ .$$

Remark 3.3.3. One can look at $\mathcal{R}_d$ as a representation of the group G. It is straightforward to see that orbits in $\mathcal{R}_d$ under the action of G are in one-to-one correspondence with isomorphism classes of representations with dimension vector d.

We address now the problem of classifying the orbits in such a parameter space $\mathcal{R}_d$, which is equivalent to classifying isomorphism classes of representations of Q. For some special type of quivers, Gabriel's theorem ([Gab72]) gives this classification:

Definition 3.3.4. *A quiver Q is said to be of finite type if there exists only a finite number of isomorphism classes of indecomposable representations of Q.*

Theorem 3.3.5 ([Gab72])**.** *A quiver Q is of finite type if and only if Q is of type A_n, D_n, E_6, E_7, E_8 or a finite disjoint unions of those. Moreover in this case, there is a bijection between isomorphism classes of indecomposable representations and positive roots of the Dynkin diagram underlying Q.*

Corollary 3.3.6. *If Q is of finite type, there are only finitely many orbits in $\mathcal{R}_d$ under the action of G for a fixed d.*

Example 3.3.7. Consider a quiver of type A_2 (with the arrow toward the second vertex, for example). The parameter space for representations $\mathcal{R}_d = \mathcal{R}_{d_1,d_2}$ is the space of matrices $\mathrm{Hom}(V_1, V_2)$, where $\dim(V_i) = d_i$, $i = 1, 2$. There are three indecomposable representations; we denote by α_i the i-th simple root of A_2, and the indecomposable representations associated to the positive roots as ϕ_{α_1}, ϕ_{α_2}, $\phi_{\alpha_1+\alpha_2}$. Their dimension vectors are respectively $(1,0)$, $(0,1)$, $(1,1)$. Notice that these vectors are given by the coefficients of the positive root in the basis of simple roots; this is actually a general fact which comes from a more precise statement of Gabriel's theorem.

Orbits in $\mathcal{R}_{d_1,d_2}$ correspond to the possible direct sums of the indecomposable representations with dimension vector (d_1, d_2). For $0 \leq r \leq min(d_1, d_2)$, they are of the form $r\phi_{\alpha_1+\alpha_2} \oplus (d_1 - r)\phi_{\alpha_1} \oplus (d_2 - r)\phi_{\alpha_2}$. The corresponding orbit $\mathcal{O}_r$ is the orbit in $\mathrm{Hom}(V_1, V_2)$ under the action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ of matrices of rank r.

Reineke's resolutions

In the following we study the orbit closures in $\mathcal{R}_d$ for a fixed quiver Q with a fixed dimension vector d under the action of $G = \prod_{s\in S} \mathrm{GL}(V_s)$. More precisely, we give a description of a desingularization of the orbit closures, which was first found by Reineke in [Rei03]. In Section 3.3.2, we will see some concrete examples of those desingularizations. The power of Reineke's construction is that it gives a desingularization of each orbit closure; by working out the examples, we will see that in some cases these desingularizations are easy to understand and very natural.

We will follow Reineke's paper [Rei03]. The desingularizations are indexed by monomials in S.

Definition 3.3.8. *A monomial in S is a couple $(\vec{s}, \vec{a})$, where $\vec{s} = (s_1, \ldots, s_\tau) \in S^\tau$, $\vec{a} = (a_1, \ldots, a_\tau) \in \mathbb{N}^\tau$ for a certain τ.*

To each monomial we will associate a flag variety $F_{(\vec{s},\vec{a})}$ and a vector bundle $W_{(\vec{s},\vec{a})}$ on it. Consider the vector space $V = \oplus_{s\in S} V_s$. For $s \in S$, a subspace (or more generally, a quotient of a subspace) of V is said to be pure of type s if it is contained in V_s (if it is generated by a subspace of V_s). Then

$$F_{(\vec{s},\vec{a})} = \left\{ \begin{array}{c} \text{Flags } 0 = F^\tau \subset \cdots \subset F^1 \subset F^0 = V \text{ s.t.} \\ F^{i-1}/F^i \text{ is pure of type } s_i \text{ and dimension } a_i \end{array} \right\}.$$

In fact, it is possible to write this variety as a product of usual flag varieties: it suffices to separate the contributions of the different V_s's. For $s \in S$, let us define $(a_1^s, \ldots, a_\mu^s) = (a_{i_1}, \ldots, a_{i_\mu})$, where $(i_1, \ldots, i_\mu)$ are all the occurrences of s in $\vec{s}$ (i.e. $s_{i_1} = \cdots = s_{i_\mu} = s$). Then :

$$F_{(\vec{s},\vec{a})} \cong \prod_{s\in S} \{\text{Flags in } V_s \text{ whose i-th successive quotient has dimension } a_i^s\} =$$

$$= \prod_{s\in S} F(a_1^s, a_1^s + a_2^s, \ldots, V_s).$$

The vector bundle $W_{(\vec{s},\vec{a})}$ is given by the elements in $\mathcal{R}_d$ which are compatible with the flag variety $F_{(\vec{s},\vec{a})}$; more precisely, over a point

$$F^\bullet = \{0 = F^\tau \subset \cdots \subset F^1 \subset F^0 = V\} \in F_{(\vec{s},\vec{a})},$$

the fiber of this bundle is defined as:

$$(W_{(\vec{s},\vec{a})})_{F^\bullet} = \{w \in \mathcal{R}_d \subset End(V) \text{ s.t. } w(F^k) \subset F^k \text{ for } 0 \leq k \leq \tau\}.$$

We denote by $\pi_{(\vec{s},\vec{a})} : W_{(\vec{s},\vec{a})} \to \mathcal{R}_d$ the natural projection.

Remark 3.3.9. By the description of $F_{(\vec{s},\vec{a})}$ as a product of flag varieties, it is clear that it is a homogeneous variety under the action of G, and the vector bundle $W_{(\vec{s},\vec{a})}$ is G-homogeneous. Being the projection morphism $\pi_{(\vec{s},\vec{a})}$ G-equivariant, its image is G-stable. Moreover, it is irreducible and closed (since $\pi_{(\vec{s},\vec{a})}$ is projective). If Q is of finite type, there is a finite number of orbits in $\mathcal{R}_d$, and therefore $\pi_{(\vec{s},\vec{a})}(W_{(\vec{s},\vec{a})})$ is the closure of one of them. Actually, for every orbit closure in $\mathcal{R}_d$, a resolution of singularities of the type $W_{(\vec{s},\vec{a})}$ for a certain monomial $(\vec{s},\vec{a})$ always exists (and is not unique!), as explained by the following theorem.

Theorem 3.3.10 ([Rei03])**.** *For each representation $M \in \mathcal{R}_d$, there exists a monomial $(\vec{s},\vec{a})(M)$ such that $\pi_{(\vec{s},\vec{a})(M)}$ is a desingularization of the orbit closure $\overline{\mathcal{O}}_M$ of M, and is an isomorphism when restricted to the preimage of $\mathcal{O}_M$.*

The theorem is proven by showing that each representation M admits a filtration (of indecomposable representations) of a certain type $(\vec{s},\vec{a})(M)$, and then constructing from this data a desingularization of $\overline{\mathcal{O}}_M$.

Example 3.3.11. Before considering non-trivial cases, let us convince ourselves that despite the heavy notation, we are dealing with familiar objects. We start from the situation described in Example 3.3.7. We want to construct a desingularization of $Y_r = \overline{\mathcal{O}}_r \subset \mathrm{Hom}(V_1, V_2)$ by using Reineke's method. A monomial that defines a resolution of singularities of Y_r is given by $(s_1, d_1 - r)$ (where s_1 is the first vertex). Then:

$$F := F_{(s_1,d_1-r)} \cong \mathrm{Gr}(d_1 - r, V_1)$$

and, if $P \in F$,

$$(W_{(s_1,d_1-r)})_P = \{M \in \mathrm{Hom}(V_1, V_2) \text{ s.t. } M|_P = 0\}.$$

Therefore, $W_{(s_1,d_1-r)} = \mathcal{Q}^* \otimes V_2$, where $\mathcal{Q}$ is the quotient bundle over F. This is a well known resolution of a determinantal variety. Other choices of the monomial are possible, and give other desingularizations of the same orbit closure.

3.3.2. Crepant resolutions

The aim of this section is to find resolutions of orbit closures in quiver representations, and conditions under which they are crepant. We will start with quivers of type A_3, and then we will deal with quivers of type A_n. Finally, in order to go beyond the A_n case, we will give some results on quivers of type D_4.

The quiver A_3

Let us consider a quiver $Q = (S, A)$ of type A_3, where $S = \{s_1, s_2, s_3\}$, and $A = \{a_1, a_2\}$. We also fix vector spaces V_1, V_2, V_3 of dimension vector $d = d_1, d_2, d_3$. According to the direction of the arrows, three different quivers occur. We study them separately.

Quiver with $a_1(1) = a_2(1) = s_2$

The quiver is represented by the following picture:

Figure 3.1. – Quiver of type A_3 with $a_1(1) = a_2(1) = s_2$

$$Q \qquad \overset{s_1}{\bullet} \xrightarrow[a_1]{} \overset{s_2}{\bullet} \xleftarrow[a_2]{} \overset{s_3}{\bullet}$$

This configuration has been studied in [Sut13]. $\mathcal{R}_d$ is the representation $\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_3, V_2)$. Fix three integers r_1, r_2, p_1. Under the action of $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$, all the orbits in $\mathcal{R}_d$ are of the form

$$\mathcal{O}_{r_1,r_2,p_1} = \left\{ \begin{array}{ll} \phi \in \mathcal{R}_d \text{ s.t.} & \dim(\mathrm{Im}\,\phi_{a_1}) = r_1 \ , \ \dim(\mathrm{Im}\,\phi_{a_2}) = r_2 \ , \\ & \dim(\mathrm{Im}\,\phi_{a_1} + \mathrm{Im}\,\phi_{a_2}) = p_1 \end{array} \right\}$$

for all geometrically possible r_1, r_2, p_1 (e.g. $p_1 \geq \max\{r_1, r_2\}$). This means that what defines an orbit is the rank of the morphisms, and the relative position of the images, i.e. the dimension of the sum (we will see that in the other quivers too, a similar description of the orbits will hold).

Remark 3.3.12. If needed, for a given element ϕ in the orbit, we will denote by U_i, U_{ij}, U_{123} respectively the image of ϕ_{a_i}, the intersection $\mathrm{Im}\,\phi_{a_i} \cap \mathrm{Im}\,\phi_{a_j}$ and the intersection $\mathrm{Im}\,\phi_{a_1} \cap \mathrm{Im}\,\phi_{a_2} \cap \mathrm{Im}\,\phi_{a_3}$ (the last will make sense for quivers of type D_4); their respective dimensions will be denoted by u_i, u_{ij}, u_{123}. Moreover, we will denote by S_{ij}, S_{123} respectively the sum $\mathrm{Im}\,\phi_{a_i} + \mathrm{Im}\,\phi_{a_j}$ and the sum $\mathrm{Im}\,\phi_{a_1} + \mathrm{Im}\,\phi_{a_2} + \mathrm{Im}\,\phi_{a_3}$ (again, the last will make sense for quivers of type D_4), of dimensions s_{ij}, s_{123}.

When we consider the closure of such an orbit we obtain:

Lemma 3.3.13. *The orbit closure $\overline{\mathcal{O}}_{r_1,r_2,p_1}$ is given by*

$$\overline{\mathcal{O}}_{r_1,r_2,p_1} = \left\{ \begin{array}{ll} \phi \in \mathcal{R}_d \ s.t. & \dim(\mathrm{Im}\,\phi_{a_1}) \leq r_1 \ , \ \dim(\mathrm{Im}\,\phi_{a_2}) \leq r_2 \ , \\ & \dim(\mathrm{Im}\,\phi_{a_1} + \mathrm{Im}\,\phi_{a_2}) \leq p_1 \end{array} \right\}.$$

Proof. The inequalities come from the fact that the dimension of the kernels of ϕ_{a_1}, ϕ_{a_2}, $\phi_{a_1} \oplus \phi_{a_2}$ can only be greater than those of the elements in the orbit.

When $u_1 + u_2 \geq p_1$, the upper bound for s_{12} is p_1 because one can suppose that, moving inside $\mathcal{O}_{r_1,r_2,p_1}$, the images of the two morphisms collapse inside their intersection in a complementary way; if $u_1 + u_2 < p_1$, the upper bound is reached when the intersection of the images of the two morphisms is zero.

Moreover, whenever the points ϕ with given u_1, u_2 and s_{12} belong to the orbit closure, then also the points ψ with $\dim(\operatorname{Im}\psi_{a_1}) = u_1$, $\dim(\operatorname{Im}\psi_{a_1}) = u_2$ and $\dim(\operatorname{Im}\psi_{a_1} + \operatorname{Im}\psi_{a_2}) \leq s_{12}$ belong to it, because suitable subspaces of the images of the morphisms ϕ_{a_1} and ϕ_{a_2} can collapse onto each other, so that the dimension of the intersection raises. $\square$

Remark 3.3.14. Recall that each orbit corresponds to an equivalence class of quiver representations. Moreover, equivalence classes of indecomposable representations correspond to positive roots of A_3 (Theorem 3.3.5), and are indexed by their dimension vectors (see Example 3.3.7). Therefore, each representation is of the form

$$a(0,1,0) \oplus b(1,1,0) \oplus c(0,1,1) \oplus d(1,1,1) \oplus e(1,0,0) \oplus f(0,0,1)$$

for some integers a, b, c, d, e, f. It corresponds to the orbit $\overline{\mathcal{O}}_{r_1,r_2,p_1}$, with $d_1 = b + d + e$, $d_2 = a + b + c + d$, $d_3 = c + d + f$, $r_1 = b + d$, $r_2 = c + d$, $p_1 = b + c + d$ (a similar interpretation of the orbits holds for the other A_3 cases, even though we will not give further details for them).

In [Sut13], Sutar used the geometric technique (see [Wey03]) and Reineke's resolutions to classify all Gorenstein orbits:

Theorem 3.3.15 ([Sut13])**.** *The orbit closure $\overline{\mathcal{O}}_{r_1,r_2,p_1}$ in $\mathcal{R}_d$ is Gorenstein if and only if one of the following conditions hold:*

(i) $d_1 = d_3 = p_1$, $d_2 = r_1 + r_2$;

(ii) $d_3 = r_1 = p_1$, $d_2 = d_1 + r_2$;

(iii) $r_1 = p_1$, $r_2 = d_3$, $d_2 = d_1 + d_3$;

(iv) $d_1 = r_2 = p_1$, $d_2 = d_3 + r_1$;

(v) $r_2 = p_1$, $r_1 = d_1$, $d_2 = d_1 + d_3$.

Remark 3.3.16. Conditions (iv) and (v) are equivalent to conditions (ii) and (iii) when the two vertices of Q are exchanged.

Remark 3.3.17. Notice that condition (ii) implies that $\operatorname{Im}\phi_{a_2} \subset \operatorname{Im}\phi_{a_1}$, and condition (iii) implies that $\operatorname{Im}\phi_{a_2} \subset \operatorname{Im}\phi_{a_1}$ and $\operatorname{Ker}\phi_{a_2} = \{0\}$.

We now use Reineke's construction to find desingularizations for all orbits in $\mathcal{R}_d$; therefore they will be total spaces of a vector bundle $\mathcal{W} \subset \mathcal{R}_d \times F$ over a homogeneous variety F. In practice, we will distinguish between three types of orbits, each of which admits a different kind of resolution. As we are interested in crepant resolutions, we try to find those resolutions $\pi : Z \to \overline{\mathcal{O}}$ which are not dominated by some other desingularization $\pi' : Z' \to \overline{\mathcal{O}}$ (i.e. such that there exists no morphism $f : Z' \to Z$ with the property that $\pi' = \pi \circ f$). This condition guided our choice of desingularization, even though there are other possibilities one could consider.

Let us define the variety

$$F_i = \mathrm{Gr}(d_1 - r_1, V_1) \times \mathrm{Gr}(p_1, V_2) \times \mathrm{Gr}(d_3 - r_2, V_3).$$

Let us denote $\mathcal{U}_i$, $\mathcal{Q}_i$ the tautological and quotient bundle over the Grassmannian of subspaces in V_i; then

$$W_i = (\mathcal{Q}_1^* \oplus \mathcal{Q}_3^*) \otimes \mathcal{U}_2 \subset (V_1^* \oplus V_3^*) \otimes V_2.$$

The total space of $\mathcal{W}_i$ is a desingularization of $\overline{\mathcal{O}}_{r_1,r_2,p_1}$ when $p_1 \neq r_1, r_2$. Indeed, the variety F_i is parametrizing the planes $\mathrm{Ker}\,\phi_1 \subset V_1$, $\mathrm{Ker}\,\phi_2 \subset V_3$ and $\mathrm{Im}\,\phi_1 + \mathrm{Im}\,\phi_2 \subset V_2$ for all the elements $\phi \in \mathcal{O}_{r_1,r_2,p_1}$.

Remark 3.3.18. This total space is actually one of Reineke's resolutions. In the notation of the previous section, define:

$$(\vec{s}, \vec{a}) = ((s_1, s_3, s_2, s_1, s_3), (d_1 - r_1, d_3 - r_2, p_1, d_1, d_3)).$$

Then, $(F_{(\vec{s},\vec{a})}, W_{(\vec{s},\vec{a})})$ desingularizes the orbit of elements $\phi \in \mathcal{R}_d$ such that $\mathrm{Ker}(\phi_{a_1})$ contains a space of dimension $d_1 - r_1$, $\mathrm{Ker}(\phi_{a_2})$ contains a space of dimension $d_3 - r_2$, $\mathrm{Im}(\phi_{a_1})$ and $\mathrm{Im}(\phi_{a_2})$ are both contained in the same space of dimension p_1. This means that $(F_{(\vec{s},\vec{a})}, W_{(\vec{s},\vec{a})})$ desingularizes $\overline{\mathcal{O}}_{r_1,r_2,p_1}$, and one can easily see that it is equal to (F_i, W_i).

The next two desingularizations admit the same interpretation, respectively for

$$(\vec{s}, \vec{a}) = ((s_3, s_2, s_1, s_3), (d_3 - r_2, r_1, d_1, d_3))$$

and for

$$(\vec{s}, \vec{a}) = ((s_2, s_1, s_3), (r_1, d_1, d_3)).$$

In the following of this work, we will not explicit the vectors $(\vec{s}, \vec{a})$ for the resolutions we will use, but it should be kept in mind that they can be interpreted as belonging to Reineke's construction.

In the case $p_1 = r_1$ we consider

$$F_{ii} = \mathrm{Gr}(r_1, V_2) \times \mathrm{Gr}(d_3 - r_2, V_3)$$

with the bundle

$$W_{ii} = (V_1^* \oplus \mathcal{Q}_3^*) \otimes \mathcal{U}_2,$$

whose total space is a desingularization of the orbit closure $\overline{\mathcal{O}}_{r_1,r_2,r_1}$ when $r_2 \neq d_3$. In this case, as $p_1 = r_1$, we do not need to fix the kernel of ϕ_1 because we are already fixing its image in V_2.

Finally, if $p_1 = r_1$ and $r_2 = d_3$, the resolution of $\overline{\mathcal{O}}_{r_1,r_2,p_1}$ is given by the total space of W_{iii} over F_{iii}, where:

$$F_{iii} = \mathrm{Gr}(r_1, V_2),$$

$$W_{iii} = (V_1^* \oplus V_2^*) \otimes \mathcal{U}_2.$$

Remark 3.3.19. The resolutions Sutar used in her work are not in general the same as those we chose to use. Indeed, she used Reineke's resolutions $(F_{(\vec{s},\vec{a})}, W_{(\vec{s},\vec{a})})$ such that

$$\xi^* = (F_{(\vec{s},\vec{a})} \times \mathcal{R}_d)/W_{(\vec{s},\vec{a})} = (\mathcal{U}_1^* \oplus \mathcal{U}_3^*) \otimes Q_2.$$

For these resolutions the expression of ξ in terms of irreducible bundles is very easy. This is particularly useful to apply the geometric technique, for which the cohomology of $\wedge^i \xi$ has to be computed.

For what concerns the crepant condition for those resolutions, we have the following result:

Proposition 3.3.20. *Let us take the orbit closure $\overline{\mathcal{O}}_{r_1,r_2,p_1} \subset \mathcal{R}_d$, and consider its resolution of singularities of the form described above. The orbit closure is Gorenstein if and only if the resolution is crepant.*

Proof. Let us suppose the resolution is given by W_i over F_i; we prove that it is crepant when the orbit satisfies condition (i) of Theorem 3.3.15. The proof for W_{ii} over F_{ii} (condition (ii)) and W_{iii} over F_{iii} (condition (iii)) is similar.

The crepancy condition of a Kempf collapsing gives in our case $\det(W_i) = K_{F_i}$. We have

$$\det(W_i) = \mathcal{O}_1(-p_1) \otimes \mathcal{O}_2(-r_1 - r_2) \otimes \mathcal{O}_3(-p_1)$$

and

$$K_{F_i} = \mathcal{O}_1(-d_1) \otimes \mathcal{O}_2(-d_2) \otimes \mathcal{O}_3(-d_3).$$

By equating the different terms, we get condition (i). □

Remark 3.3.21. The resolution given by W_{iii} over F_{iii} is actually the resolution of determinantal varieties for $\mathrm{Hom}(V_1 \oplus V_3, V_2)$ of rank r_1. Therefore, the corresponding orbit is a determinantal orbit. Indeed, the Gorenstein condition is the same as the one that holds for determinantal varieties ($d_1 + d_3 = d_2$).

Quiver with $a_1(0) = a_2(0) = s_2$

This quiver is represented by the following picture:

Figure 3.2. – Quiver of type A_3 with $a_1(0) = a_2(0) = s_2$

$$Q \qquad s_1 \xleftarrow{a_1} s_2 \xrightarrow{a_2} s_3$$

The representation we are dealing with is $\mathcal{R}_d = \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_2, V_3)$. Fix three integers k_1, k_2, q_1. Then the orbits are

$$\mathcal{O}_{k_1,k_2,q_1} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \text{ s.t. } \dim(\mathrm{Ker}\,\phi_{a_1}) \geq k_1 \ , \ \dim(\mathrm{Ker}\,\phi_{a_2}) = k_2 \ , \\ \dim(\mathrm{Ker}\,\phi_{a_1} \cap \mathrm{Ker}\,\phi_{a_2}) = q_1 \end{array} \right\}$$

for $d_2 - d_1 \leq k_1 \leq d_2$, $d_2 - d_3 \leq k_2 \leq d_2$, and $q_1 \leq \min\{k_1, k_2\}$. Again, what matters is the relative position of the two subspaces (kernels) in V_2.

Lemma 3.3.22. *The orbit closure $\overline{\mathcal{O}}_{k_1,k_2,q_1}$ is given by*

$$\overline{\mathcal{O}}_{k_1,k_2,q_1} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \ s.t. \ \dim(\mathrm{Ker}\,\phi_{a_1}) \geq k_1 \ , \ \dim(\mathrm{Ker}\,\phi_{a_2}) \geq k_2 \ , \\ \dim(\mathrm{Ker}\,\phi_{a_1} \cap \mathrm{Ker}\,\phi_{a_2}) \geq q_1 \end{array} \right\}.$$

Proof. Consider the dual situation, i.e. dual morphisms $\phi^*_{a_1}$ and $\phi^*_{a_2}$, and reason as in the proof of Lemma 3.3.13. $\square$

Depending on the choice of k_1, k_2, q_1, there are two different kinds of resolutions. On one hand, if $q_1 \neq k_1, k_2$, then we have resolution (i) given by the total space of

$$W_i := \mathcal{Q}_2^* \otimes (\mathcal{U}_1 \oplus \mathcal{U}_3) \quad \text{over} \quad F_i := \mathrm{Gr}(d_2 - k_1, V_1) \times \mathrm{Gr}(q_1, V_2) \times \mathrm{Gr}(d_2 - k_2, V_3).$$

On the other hand, if $q_1 = k_1$ (and similarly for $q_1 = k_2$), we have resolution (ii) given by

$$W_{ii} := \mathcal{Q}_2^* \otimes (V_1 \oplus \mathcal{U}_3) \quad \text{over} \quad F_{ii} := \mathrm{Gr}(q_1, V_2) \times \mathrm{Gr}(d_2 - k_2, V_3).$$

The following proposition describes when resolutions of type (i) and (ii) are crepant.

Proposition 3.3.23. *Resolutions of type (i) are crepant when $d_1 = d_3 = d_2 - q_1$, $k_1 + k_2 = d_2$.*

Resolutions of type (ii) are crepant when $q_1 = k_1$, $d_1 = k_1$, $d_3 = d_2 - k_1$.

Proof. The proof is similar to the one of Proposition 3.3.20. $\square$

Remark 3.3.24. The quiver studied in this section can be thought of as being the *dual* of the one described in Figure 3.1. Indeed, the representations (orbits, desingularizations) of one can be obtained from the other by considering the dual situation, i.e. by passing from morphisms ϕ_{a_1} and ϕ_{a_2} to their duals $\phi_{a_1}^*$ and $\phi_{a_2}^*$. The same can be said for the quivers that will be described by Figure 3.6 and Figure 3.7.

Quiver with $a_1(0) = s_1$, $a_2(0) = s_2$

This quiver is represented by the following picture:

Figure 3.3. – Quiver of type A_3 with $a_1(0) = s_1$, $a_2(0) = s_2$

$$Q \qquad \overset{s_1}{\bullet} \xrightarrow[a_1]{} \overset{s_2}{\bullet} \xrightarrow[a_2]{} \overset{s_3}{\bullet}$$

The representation we are dealing with is $\mathcal{R}_d = \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_3)$. Fix three integers r_1, k_2, u_1. Then, the orbits are

$$\mathcal{O}_{r_1,k_2,u_1} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \text{ s.t. } \dim(\mathrm{Im}\,\phi_{a_1}) = r_1 \ , \ \dim(\mathrm{Ker}\,\phi_{a_2}) = k_2 \ , \\ \dim(\mathrm{Im}\,\phi_{a_1} + \mathrm{Ker}\,\phi_{a_2}) = u_1 \end{array} \right\}$$

for $r_1 \leq \min\{d_2, d_1\}$, $d_2 - d_3 \leq k_2 \leq d_2$, and $\min\{r_1, k_2\} \leq u_1 \leq \min\{r_1 + k_2, d_2\}$. Again, what matters is the relative position of the two subspaces (image and kernel) in V_2.

Lemma 3.3.25.

$$\overline{\mathcal{O}}_{r_1,k_2,u_1} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \ s.t. \ \dim(\mathrm{Im}\,\phi_{a_1}) \leq r_1 \ , \ \dim(\mathrm{Ker}\,\phi_{a_2}) \geq k_2 \ , \\ \dim(\mathrm{Im}\,\phi_{a_1} + \mathrm{Ker}\,\phi_{a_2}) \leq u_1 + \dim(\mathrm{Ker}\,\phi_{a_2}) - k_2 \end{array} \right\}.$$

Proof. The only non trivial condition is the last one. The upper bound of the inequality is attained, for fixed ϕ_{a_1}, by collapsing the morphism ϕ_{a_2} so that its kernel becomes bigger while $\mathrm{Im}\,\phi_{a_1} \cap \mathrm{Ker}\,\phi_{a_2}$ remains unchanged. On the other hand, when $\mathrm{Im}\,\phi_{a_1} \not\subset \mathrm{Ker}\,\phi_{a_2}$, the dimension of $\mathrm{Im}\,\phi_{a_1}$ can be reduced without changing $\mathrm{Im}\,\phi_{a_1} + \mathrm{Ker}\,\phi_{a_2}$. Then, if the dimensions of $\mathrm{Ker}\,\phi_{a_2}$ and $\mathrm{Im}\,\phi_{a_1}$ are fixed, one can collapse these spaces onto each other; this "movement" just reduces the dimension of $\mathrm{Im}\,\phi_{a_1} + \mathrm{Ker}\,\phi_{a_2}$. Hence all the possibilities which satisfy the inequality correspond to points in the orbit closure. □

Depending on the choice of r_1, k_2, u_1, there are two different kinds of resolutions. On one hand, if $u_1 \neq k_2$, then we have resolution (i) given by the total space of

$$W_i := (\mathcal{Q}_1^* \otimes \mathcal{U}_{2,2}) \oplus ((V_2/\mathcal{U}_{2,1})^* \otimes V_3) \quad \text{over} \quad F_i := \mathrm{Gr}(d_1 - r_1, V_1) \times F(k_2, u_1, V_2).$$

where $F(k_2, u_1, V_2)$ is the flag variety with tautological bundles $\mathcal{U}_{2,1} \subset \mathcal{U}_{2,2}$ of rank k_2, u_1. On the other hand, if $u_1 = k_2$, we have resolution (ii) given by

$$W_{ii} := (\mathcal{Q}_1^* \otimes \mathcal{U}_2) \oplus (\mathcal{Q}_2^* \otimes V_3) \quad \text{over} \quad F_{ii} := \mathrm{Gr}(d_1 - r_1, V_1) \times \mathrm{Gr}(k_2, V_2).$$

Proposition 3.3.26. *Resolutions of type (i) are crepant when $k_2 = d_2 - r_1 = d_1 - u_1$, $d_1 = d_3$.*

Resolutions of type (ii) are crepant when $u_1 = k_2$, $d_1 - r_1 = d_2 - k_2$, $d_3 = 2(d_2 - k_2)$.

Proof. The proof is similar to the one of Proposition 3.3.20. □

One-way and source-sink quivers of type A_n

In this section we find crepant resolutions for certain orbit closures in $\mathcal{R}_d$ for a quiver of type A_n, in a similar way we have proceeded for the A_3 quivers. It is a natural generalization of the results of the previous section.

Let us suppose $Q = (S, A)$, is a quiver of type A_n with vertices $S = \{s_1, \ldots, s_n\}$ and arrows $A = \{a_1, \ldots, a_{n-1}\}$. The vector spaces $V_1, \ldots, V_n$ of dimensions $d = (d_1, \ldots, d_n)$ are those appearing in the definition of $\mathcal{R}_d$. For later use, let us denote by $F(\alpha_1, \alpha_2, V_i)$ the flag variety in V_i with tautological bundles $\mathcal{U}_{i,1} \subset \mathcal{U}_{i,2}$ of ranks α_1, α_2 (for the Grassmannian, we will write $\mathcal{U}_{i,1}$ or $\mathcal{U}_{i,2}$ for the tautological bundle, according to the symmetries of the formula it will appear in).

One-way quiver of type A_n

This quiver is represented by the following picture (all the arrows point in the same direction):

Figure 3.4. – One-way quiver of type A_n

$$Q \qquad \begin{tikzcd} \overset{s_1}{\bullet} \arrow[r, "a_1"'] & \overset{s_2}{\bullet} \arrow[r, dashed, no head] & \overset{s_{n-1}}{\bullet} \arrow[r, "a_{n-1}"'] & \overset{s_n}{\bullet} \end{tikzcd}$$

Fix integer vectors $k = (k_1, \ldots, k_{n-1})$, $t = (t_2, \ldots, t_{n-1})$, and consider the orbits given by

$$\mathcal{O}_{k,t} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \text{ s.t. } \dim(\operatorname{Ker} \phi_{a_i}) = k_i \text{ for } 1 \leq i \leq n-1 \,, \\ \dim(\operatorname{Im} \phi_{a_i} \cap \operatorname{Ker} \phi_{a_{i+1}}) = t_{i+1} \text{ for } 1 \leq i \leq n-2 \end{array} \right\}.$$

Remark 3.3.27. Not all the orbits in $\mathcal{R}_d$ are of the form $\mathcal{O}_{k,t}$ for some k, t. In order to consider other orbits, one should also fix the dimension of other characteristic subspaces, for example of $\operatorname{Ker} \phi_{i+1} \cap \operatorname{Im}(\phi_i \circ \phi_{i-1})$, just to name one. By using notations similar to Example 3.3.7 and Remark 3.3.14, the orbits $\mathcal{O}_{k,t}$ correspond to representations of the form

$$\bigoplus_i \alpha_i(0, \dots, 0, 1, 0, \dots, 0) \oplus \bigoplus_j \beta_j(0, \dots, 0, 1, 1, 0, 0, \dots, 0) \oplus$$

$$\oplus \bigoplus_l \gamma_l(0, \dots, 0, 0, 1, 1, 1, 0, \dots, 0).$$

We study those orbits because the desingularization of their closure is the naive generalization of the ones in the A_3 case. A similar argument holds for the other A_n cases.

Suppose $t_i \neq k_i$ for $2 \leq i \leq n-1$. A resolution of singularities of $\overline{\mathcal{O}}_{k,t}$ is given by the total space of the vector bundle

$$W = ((V_{n-1}/\mathcal{U}_{n-1,1})^* \otimes V_n) \oplus \bigoplus_{i=1}^{n-1} ((V_i/\mathcal{U}_{i,1})^* \otimes \mathcal{U}_{i+1,2})$$

over the variety

$$F = \operatorname{Gr}(k_1, V_1) \times \prod_{i=2}^{n-1} \times F(k_i, d_{i-1} - k_{i-1} + k_i - t_i, V_i).$$

Proposition 3.3.28. *Consider an orbit closure $\overline{\mathcal{O}}_{k,t}$ with $t_i \neq k_i$ for $2 \leq i \leq n-1$. Then, the resolution of singularities defined by W, F as above is crepant if and only if there exists r such that*

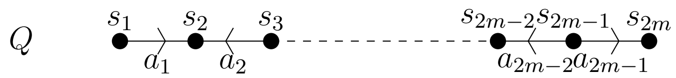
$$d_1 = d_n = d_i - t_i \text{ for } 2 \leq i \leq n-1 \quad \text{and} \quad r = d_i - k_i \text{ for } 1 \leq i \leq n-1.$$

Proof. As in the proof of Proposition 3.3.20, the statement comes from the computation of the relative canonical bundle of the desingularization, and by imposing the crepant equation appearing in Remark 1.4.13. ☐

Source-sink quiver of type A_{2m}

This quiver is represented by the following picture (all nodes are either a source or a sink for the arrows):

Figure 3.5. – Source-sink quiver of type A_{2m}



Fix integer vectors $r = (r_1, \dots, r_{2m-1})$, $p = (p_1, \dots, p_{m-1})$, $q = (q_1, \dots, q_{m-1})$ and consider the orbits given by

$$\mathcal{O}_{r,p,q} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \text{ s.t. } \dim(\operatorname{Im}\phi_{a_i}) = r_i \text{ for } 1 \leq i \leq 2m-1\ , \\ \dim(\operatorname{Im}\phi_{a_{2i-1}} + \operatorname{Im}\phi_{a_{2i}}) = p_i \text{ for } 1 \leq i \leq m-1\ , \\ \dim(\ker\phi_{a_{2i+1}} \cap \ker\phi_{a_{2i}}) = q_i \text{ for } 1 \leq i \leq m-1 \end{array} \right\}.$$

Suppose $p_i \neq r_{2i-1}, r_{2i}$ and $q_i \neq d_{2i+1} - r_{2i+1}, d_{2i+1} - r_{2i}$ for $1 \leq i \leq m-1$. A resolution of singularities of $\overline{\mathcal{O}}_{r,p,q}$ is given by the total space of the vector bundle

$$W = ((V_{2m-1}/\mathcal{U}_{2m-1,2})^* \otimes V_{2m}) \oplus$$

$$\oplus \bigoplus_{i=1}^{m-1} (((V_{2i-1}/\mathcal{U}_{2i-1,2})^* \otimes \mathcal{U}_{2i,2}) \oplus ((V_{2i+1}/\mathcal{U}_{2i+1,1})^* \otimes \mathcal{U}_{2i,1}))$$

over the variety

$$F = \operatorname{Gr}(d_1 - r_1, V_1) \times \prod_{i=1}^{m-1} (F(r_{2i}, p_i, V_{2i}) \times F(q_i, d_{2i+1} - r_{2i+1}, V_{2i+1})).$$

Proposition 3.3.29. *Consider an orbit closure $\overline{\mathcal{O}}_{r,p,q}$ with $p_i \neq r_{2i-1}, r_{2i}$ and $q_i \neq d_{2i+1} - r_{2i+1}, d_{2i+1} - r_{2i}$ for $1 \leq i \leq m-1$. Then, the resolution of singularities defined by W, F as above is crepant if and only if*

$$d_1 = d_{2m} = p_i \text{ for } 1 \leq i \leq m-1 \quad , \quad q_i = d_{2i+1} - d_1 \text{ for } 1 \leq i \leq m-1 \quad \text{and}$$

$$r_i = d_{i+1} - r_{i+1} \text{ for } 1 \leq i \leq 2m-2.$$

Proof. The proof is similar to the one of Proposition 3.3.28. □

Source-sink quiver of type A_{2m+1}, type I

The quiver is represented by the following picture (all nodes are either a source or a sink for the arrows):

Figure 3.6. – Source-sink quiver of type A_{2m+1}, type I

$$Q \qquad s_1 \xrightarrow{a_1} s_2 \xleftarrow{a_2} s_3 \cdots\cdots s_{2m-1} \xrightarrow{a_{2m-1}} s_{2m} \xleftarrow{a_{2m}} s_{2m+1}$$

Fix integer vectors $r = (r_1, \dots, r_{2m})$, $p = (p_1, \dots, p_m)$, $q = (q_1, \dots, q_{m-1})$ and consider the orbits given by

$$\mathcal{O}_{r,p,q} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \text{ s.t. } \dim(\operatorname{Im}\phi_{a_i}) = r_i \text{ for } 1 \leq i \leq 2m \,, \\ \dim(\operatorname{Im}\phi_{a_{2i-1}} + \operatorname{Im}\phi_{a_{2i}}) = p_i \text{ for } 1 \leq i \leq m \,, \\ \dim(\ker\phi_{a_{2i+1}} \cap \ker\phi_{a_{2i}}) = q_i \text{ for } 1 \leq i \leq m-1 \end{array} \right\}.$$

Suppose $p_i \neq r_{2i-1}, r_{2i}$ and $q_j \neq d_{2j+1} - r_{2j+1}, d_{2j+1} - r_{2j}$ for $1 \leq i \leq m$ and $1 \leq j \leq m-1$. A resolution of singularities of $\overline{\mathcal{O}}_{r,p,q}$ is given by the total space of the vector bundle

$$W = \bigoplus_{i=1}^{m-1} (((V_{2i-1}/\mathcal{U}_{2i-1,2})^* \otimes \mathcal{U}_{2i,2}) \oplus ((V_{2i+1}/\mathcal{U}_{2i+1,1})^* \otimes \mathcal{U}_{2i,1}))\oplus$$

$$\oplus((V_{2m-1}/\mathcal{U}_{2m-1,2})^* \otimes \mathcal{U}_{2m,2}) \oplus (V_{2m+1}^* \otimes \mathcal{U}_{2m,1})$$

over the variety

$$F = \operatorname{Gr}(d_1 - r_1, V_1) \times F(r_{2m}, p_m, V_{2m}) \times$$

$$\times \prod_{i=1}^{m-1} (F(r_{2i}, p_i, V_{2i}) \times F(q_i, d_{2i+1} - r_{2i+1}, V_{2i+1})).$$

Proposition 3.3.30. *Consider an orbit closure $\overline{\mathcal{O}}_{r,p,q}$ with $p_i \neq r_{2i-1}, r_{2i}$ and $q_j \neq d_{2j+1} - r_{2j+1}, d_{2j+1} - r_{2j}$ for $1 \leq i \leq m$ and $1 \leq j \leq m-1$. Then, the resolution of singularities defined by W, F as above is crepant if and only if*

$$d_1 = d_{2m+1} = p_i \text{ for } 1 \leq i \leq m \quad , \quad q_i = d_{2i+1} - d_1 \text{ for } 1 \leq i \leq m-1 \quad \text{and}$$

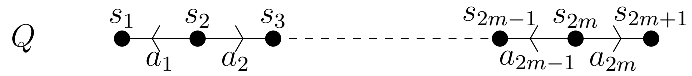
$$r_i = d_{i+1} - r_{i+1} \text{ for } 1 \leq i \leq 2m-1.$$

Proof. The proof is similar to the one of Proposition 3.3.28. □

Source-sink quiver of type A_{2m+1}, type II

The quiver is represented by the following picture (all nodes are either a source or a sink for the arrows):

Figure 3.7. – Source-sink quiver of type A_{2m+1}, type II



Fix integer vectors $r = (r_1, \ldots, r_{2m})$, $p = (p_1, \ldots, p_{m-1})$, $q = (q_1, \ldots, q_m)$ and consider the orbits given by

$$\mathcal{O}_{r,p,q} = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \text{ s.t. } \dim(\operatorname{Im}\phi_{a_i}) = r_i \text{ for } 1 \le i \le 2m \ , \\ \dim(\operatorname{Im}\phi_{a_{2i}} + \operatorname{Im}\phi_{a_{2i+1}}) = p_i \text{ for } 1 \le i \le m-1 \ , \\ \dim(\ker\phi_{a_{2i}} \cap \ker\phi_{a_{2i-1}}) = q_i \text{ for } 1 \le i \le m \end{array} \right\}.$$

Suppose $p_i \neq r_{2i+1}, r_{2i}$ and $q_i \neq d_{2i} - r_{2i-1}, d_{2i} - r_{2i}$ for all possible i. A resolution of singularities of $\overline{\mathcal{O}}_{r,p,q}$ is given by the total space of the vector bundle

$$W = \bigoplus_{i=1}^{m-1} (((V_{2i}/\mathcal{U}_{2i,1})^* \otimes \mathcal{U}_{2i-1,1}) \oplus ((V_{2i}/\mathcal{U}_{2i,2})^* \otimes \mathcal{U}_{2i+1,2}))\oplus$$

$$\oplus((V_{2m}/\mathcal{U}_{2m,1})^* \otimes \mathcal{U}_{2m-1,1}) \oplus ((V_{2m}/\mathcal{U}_{2m,2})^* \otimes \mathcal{U}_{2m+1,2})$$

over the variety

$$F = \operatorname{Gr}(r_1, V_1) \times F(q_m, d_{2m} - r_{2m}, V_{2m}) \times$$

$$\times \prod_{i=1}^{m-1} (F(q_i, d_{2i} - r_{2i}, V_{2i}) \times F(r_{2i+1}, p_i, V_{2i+1})).$$

Proposition 3.3.31. *Consider an orbit closure $\overline{\mathcal{O}}_{r,p,q}$ with $p_i \neq r_{2i+1}, r_{2i}$ and $q_i \neq d_{2i} - r_{2i-1}, d_{2i} - r_{2i}$ for all possible i. Then, the resolution of singularities defined by W, F as above is crepant if and only if*

$$d_1 = d_{2m+1} = p_i \textit{ for } 1 \le i \le m-1 \quad , \quad q_i = d_{2i} - d_1 \textit{ for } 1 \le i \le m \quad \textit{and}$$

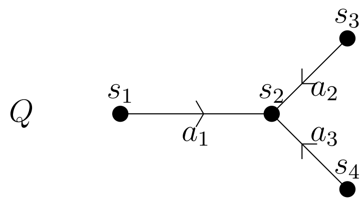
$$r_i = d_{i+1} - r_{i+1} \textit{ for } 1 \le i \le 2m-1.$$

A quiver of type D_4

The study of quivers of type D_4 presents some interesting difficulties, especially involving the construction of desingularizations for the orbit closures. We will study the crepancy condition for the resolutions of singularities we will consider among Reineke's resolutions.

Let us begin with a quiver $Q(S, A)$ of type D_4 with $S = (s_1, s_2, s_3, s_4)$, $A = (a_1, a_2, a_3)$. We will study the quiver with all the arrows pointing toward the central vertex s_2 (this is the analogue of the quiver of type A_3 studied by Sutar). The quiver is represented by the following picture:

Figure 3.8. – Quiver of type D_4



The vector spaces appearing in the definition of $\mathcal{R}_d$ are $V_1, \ldots, V_4$ of dimensions $d_1, \ldots, d_4$. Fix integers r_i for $i = 1, 2, 3$, r_{ij} for $1 \leq i < j \leq 3$, and r_{123}. Define $x := \sum_i r_i - \sum_{i<j} r_{ij} + r_{123}$. All the orbits are of the following form, for geometrically possible r:

$$\mathcal{O}_r = \left\{ \begin{array}{c} \phi \in \mathcal{R}_d \text{ s.t. } \dim(\operatorname{Im}\phi_{a_i}) = r_i \text{ for } i = 1, 2, 3\,, \\ \dim(\operatorname{Im}\phi_{a_i} \cap \operatorname{Im}\phi_{a_j}) = r_{ij} \text{ for } 1 \leq i < j \leq 3\,, \\ \dim(\operatorname{Im}\phi_{a_1} \cap \operatorname{Im}\phi_{a_2} \cap \operatorname{Im}\phi_{a_3}) = r_{123} \end{array} \right\}.$$

In words, we are fixing the relative position of the images of the morphisms.

Remark 3.3.32. Finding desingularizations of the orbit closures $\overline{\mathcal{O}}_r$ by hand may be tricky (if one does not rely on Reineke's construction). For example, one would be tempted to write a Kempf collapsing of a vector bundle over the following flag variety F_0:

$$\left\{ \begin{array}{c} (\{P_i\}_i, \{P_{ij}\}_{i,j}, P_{123}) \in \prod_i \mathrm{Gr}(r_i, V_2) \times \prod_{i<j} \mathrm{Gr}(r_{ij}, V_2) \times \mathrm{Gr}(r_{123}, V_2) \\ \text{s.t. } P_{123} \subset P_{ij} \text{ for all } i, j \quad \text{and} \quad P_{ij} \subset P_i \text{ for all } i, j \end{array} \right\}.$$

Here, the desingularization is given by the morphisms ϕ such that $(\{U_i\}_i, \{U_{ij}\}_{i,j}, U_{123}) \in F_0$. However, we do not know a priori if this variety is smooth. The problem lies in the fact that the chain of inclusions between the P_i's, P_{ij}'s, P_{123} is not linear (e.g. $P_2 \supset P_{12} \subset P_1 \supset P_{13} \subset P_3 \supset P_{23} \subset P_2$). If it was so, the variety could be seen as a composition of Grassmannian bundles over a certain flag variety, and would therefore be smooth.

Remark 3.3.33. Another problem that appears is to construct resolutions which, in some particular cases, are crepant. For example, in the case of matrices (quiver of type A_2, see Example 3.3.7), consider the determinantal variety Y_r (of matrices of rank at most r). A resolution given by

$$\mathcal{Q}_1^* \otimes \mathcal{U}_2 \quad \text{over} \quad \mathrm{Gr}(d_1 - r, V_1) \times \mathrm{Gr}(r, V_2)$$

is never crepant. Similarly, in our case, some resolutions are never crepant. For instance, we could take the Kempf collapsing with vector bundle over the flag variety which fixes, for each element ϕ, the spaces $U_{123} \subset U_{12} \subset U_{12} + U_{13} \subset U_{12} + U_{13} + U_{23} \subset V_2$ (and the corresponding subspaces in V_1, V_3, V_4), but these collapsings are never crepant. In the following, we will describe three types of resolutions which are sometimes crepant, and we will see exactly when they are.

Suppose that, for the elements in the orbit $\phi \in \mathcal{O}_r$, there is no equality between the spaces U_i, U_{ij}, U_{123}. Then, we consider the Kempf collapsing (i) given by the vector bundle

$$W_i = ((\bigoplus_{i=1,3,4} \mathcal{U}_{i,2}/\mathcal{U}_{i,1})^* \otimes \mathcal{U}_{2,1}) \oplus ((\bigoplus_{i=1,3,4}^{3} V_i/\mathcal{U}_{i,2})^* \otimes (V_2/\mathcal{U}_{2,1}))$$

over

$$F_i = F(d_1 - r_1, d_1 - r_1 + r_{12} + r_{13} - r_{123}, V_1) \times \mathrm{Gr}(r_{12} + r_{13} + r_{23} - 2r_{123}, V_2) \times$$

$$\times F(d_3 - r_2, d_3 - r_2 + r_{12} + r_{23} - r_{123}, V_3) \times F(d_4 - r_3, d_4 - r_3 + r_{13} + r_{23} - r_{123}, V_4).$$

The motivation for this choice is that the base variety F_i parametrizes, for the elements inside $\overline{\mathcal{O}}_r$, the subspaces $\operatorname{Ker} \phi_{a_1} \subset \phi_{a_1}^{-1}(U_{12} + U_{13}) \subset V_1$ (and similarly for the other spaces V_3, V_4) and $U_{12} + U_{13} + U_{23} \subset V_2$.

Remark 3.3.34. Even though it may seem we are losing some information (e.g. not fixing the dimension of U_{123}), we are not. For instance, by taking a general point ϕ of this resolution, $U_{12} + U_{13}$ and $U_{12} + U_{23}$ are well defined subspaces of dimension $r_{12} + r_{13} - r_{123}$ and $r_{12} + r_{23} - r_{123}$ inside $U_{12} + U_{13} + U_{23}$. Therefore, they intersect in a subspace of dimension at least

$$r_{12} + r_{13} + r_{23} - r_{123} - (r_{12} + r_{13} - r_{123}) - (r_{12} + r_{23} - r_{123}) = r_{12},$$

which is exactly the bound we want for the dimension of U_{12}. In the same way, the dimension of the intersection of $U_{12} + U_{13}$, $U_{12} + U_{13}$ and $U_{12} + U_{13}$ is at least

$$2(r_{12} + r_{13} + r_{23} - r_{123}) - (r_{12} + r_{13} - r_{123}) +$$

$$-(r_{12} + r_{23} - r_{123}) - (r_{13} + r_{23} - r_{123}) = r_{123},$$

which is the required bound for the dimension of U_{123}. A similar argument will hold for the other desingularizations below.

Remark 3.3.34 tells us that the image (inside $\mathcal{R}_d$) of this collapsing is contained in $\overline{\mathcal{O}}_r$. To see if it is a desingularization, let us take an element $\phi \in \mathcal{O}_r$. Then, its preimage inside W_i lies over the (unique) point of the flag variety F_i whose explicit expression is:

$$(\operatorname{Ker} \phi_{a_1} \subset \phi_{a_1}^{-1}(U_{12} + U_{13}) \subset V_1) \times (U_{12} + U_{13} + U_{23} \subset V_2) \times$$

$$\times(\operatorname{Ker}\phi_{a_2} \subset \phi_{a_2}^{-1}(U_{12}+U_{23}) \subset V_3) \times (\operatorname{Ker}\phi_{a_3} \subset \phi_{a_3}^{-1}(U_{13}+U_{23}) \subset V_4).$$

Therefore the morphism $W_i \to \overline{\mathcal{O}}_r$ is generically one-to-one, and as a consequence it makes W_i a desingularization of $\overline{\mathcal{O}}_r$.

The second resolution (ii) we consider is obtained by just fixing r_i for $i = 1, 2, 3$ and the dimension of $U_1 + U_2 + U_3$; therefore the dimensions of U_{ij} and U_{123} are the minimal possible for the generic element $\phi \in \mathcal{O}_r$. Then, the orbit closure $\overline{\mathcal{O}}_r$ is resolved by the total space of

$$W_{ii} = (V_1^* \otimes \mathcal{U}_{2,1}) \oplus (((V_3/\mathcal{U}_{3,1}) \oplus (V_4/\mathcal{U}_{4,1}))^* \otimes \mathcal{U}_{2,2})$$

over

$$F_{ii} = F(r_1, x, V_2) \times \operatorname{Gr}(d_3 - r_2, V_3) \times \operatorname{Gr}(d_4 - r_3, V_4).$$

This Kempf collapsing is of the same type of the one that could be used for the quivers of type A_2, and therefore it should be straightforward to see that it is a desingularization. Notice that this resolution is not symmetric: indeed, a particular role is played by V_1, because in F_{ii} we are parametrizing the image of ϕ_{a_1} (while for ϕ_{a_2}, ϕ_{a_3} we are parametrizing the kernels).

Finally, in the third resolution (iii) we fix again r_i for $i = 1, 2, 3$, and the dimension of $U_1 + U_2 + U_3$; as before, the dimensions of U_{ij} and U_{123} are the minimal possible for the generic element $\phi \in \mathcal{O}_r$. The resolution of the orbit closure is given by

$$W_{iii} = ((V_1/\mathcal{U}_1) \oplus (V_3/\mathcal{U}_3) \oplus (V_4/\mathcal{U}_4))^* \otimes \mathcal{U}_2$$

over

$$F_{iii} = \operatorname{Gr}(d_1 - r_1, V_1) \times \operatorname{Gr}(x, V_2) \times \operatorname{Gr}(d_3 - r_2, V_3) \times \operatorname{Gr}(d_4 - r_3, V_4).$$

As before, it is straightforward to see that it is a desingularization.

Proposition 3.3.35. *Let $\overline{\mathcal{O}}_r$ be an orbit closure in $\mathcal{R}_d$ which admits one of the three resolutions (i), (ii), (iii).*

The resolution of type (i) is never crepant.

The resolution of type (ii) and the resolution of type (iii) are crepant when $d_2 = \sum_i r_i$, $d_1 = d_3 = d_4 = x$.

Proof. The proof is similar to the one of Proposition 3.3.20. We just remark that the condition for (i) to be crepant is $d_1 = d_2/3 = d_3 = d_4 = r_i = r_{jk} = r_{123}$ for all i, $j < k$. But this is not possible, the definition of F_i requires that $r_1 < d_1$, $r_2 < d_3$, $r_3 < d_4$. □

3.4. Quiver degeneracy loci

In this section, we use the results we found on crepancy of resolutions of quiver orbit closures to construct some examples of orbital degeneracy loci. As already pointed out, the fact that the resolution is crepant allows us to compute the canonical bundle of these loci. We exhibit a sample of constructions of varieties (especially fourfolds) with trivial canonical bundle.

All the computations in cohomology, in particular the computation of the Euler characteristic of the trivial bundle, have been done with `Macaulay2`([GS]). With this software in our cases it was possible to explicitly construct the resolution of the orbital degeneracy loci and perform the computations we need in the cohomology ring of the variety.

3.4.1. Quiver degeneracy loci of type A_3

We begin by considering the case of quivers of type A_3 described in Figure 3.1 (refer to it for the notations). If we want to consider ODL, we need to fix a smooth projective variety X, and three vector bundles E_1, E_2, E_3 of dimensions d_1, d_2, d_3 on it, such that $\mathrm{Hom}(E_1, E_2) \oplus \mathrm{Hom}(E_3, E_2)$ is globally generated. Then, suppose that s is a general section of this bundle, and fix an orbit closure $Y = \overline{\mathcal{O}}_{r_1,r_2,p_1}$ inside $\mathcal{R}_d = \mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \oplus \mathrm{Hom}(\mathbb{C}^{d_3}, \mathbb{C}^{d_2})$. Recall that $D_Y(s)$ is the locus of points $x \in X$ which are sent by the section s inside $Y \subset (\mathrm{Hom}(E_1, E_2) \oplus \mathrm{Hom}(E_3, E_2))_x \cong \mathcal{R}_d$.

Theorem 3.4.1. *Let $D_Y(s)$ be defined as above, where $Y = \overline{\mathcal{O}}_{r_1,r_2,p_1} \subset \mathcal{R}_d$. Then $K_{D_Y(s)}$, $\mathrm{codim}_X(D_Y(s))$ and a lower bound for $\mathrm{codim}_{D_Y(s)} \mathrm{Sing}(D_Y(s))$ are given by the formulas in Table 3.2.*

Table 3.2. – ODL from a quiver of type A_3 with $a_1(1) = a_2(1) = s_2$. We use the following variables: $\eta_1 = d_1 - r_1$, $\eta_2 = d_2 - p_1$, $\eta_3 = d_3 - r_2$.

Case (Thm. 3.3.15)	$K_{D_Y(s)/X}$	$\mathrm{codim}_X\, D_Y(s)$	$\mathrm{codim}_{D_Y(s)}\, \mathrm{Sing}$
(i)	$\det E_1^{-r_2} \otimes$ $\det E_2^{p_1} \otimes \det E_3^{-r_1}$	$\eta_1^2 + \eta_2^2 + \eta_3^2 +$ $+ \eta_2(\eta_1 + \eta_3)$	$\geq \min \left\{ \begin{array}{c} 2\eta_1 + 1 \\ 2\eta_3 + 1 \\ 2\eta_2 + 1 \end{array} \right\}$
(ii)	$\det E_1^{-d_2+p_1} \otimes$ $(\det E_2^* \otimes \det E_3)^{-d_2+r_2}$	$\eta_3^2 + \eta_2^2 + \eta_3\eta_2$	$\geq \min \left\{ \begin{array}{c} 2\eta_3 + 1 \\ 2\eta_2 + 1 \end{array} \right\}$
(iii)	$(\det E_1 \otimes \det E_2^* \otimes$ $\det E_3)^{-d_2+p_1}$	η_2^2	$\geq 2\eta_2 + 1$

Proof. By [BFMT17a][Prop. 2.3], we have that $\operatorname{codim}_X(D_Y(s)) = \operatorname{codim}_{\mathcal{R}_d} Y$ and $\operatorname{codim}_{D_Y(s)} \operatorname{Sing}(D_Y(s)) = \operatorname{codim}_Y \operatorname{Sing}(Y)$. The first quantity can be computed directly from the resolution of singularities we have. For the codimension of the singularities, we explain the case of the resolution W_i over F_i (the others are similar).

The singularities are contained in the locus in the orbit closure Y where the map $\pi : W_i \to Y \subset \mathcal{R}_d$ is not an isomorphism. As Y is normal ([BZ01]), the morphism π restricted to the fiber over such a point is a contraction. Take a point $y \in Y$. If $\dim(\operatorname{Ker} y_1) = d_1 - r_1$, $dim(\operatorname{Ker} y_2) = d_3 - r_2$, $\dim(\operatorname{Im} y_1 + \operatorname{Im} y_2) = p_1$, then $\pi^{-1}y$ is a single point. Therefore, if y is in the singular locus, then either $\dim(\operatorname{Ker} y_1) > d_1 - r_1$ or $\dim(\operatorname{Ker} y_2) > d_3 - r_2$ or $\dim(\operatorname{Im} y_1 + \operatorname{Im} y_2) < p_1$. As a consequence, the singular locus is contained in the union of three orbit closures, i.e.

$$\operatorname{Sing}(Y) \subset \overline{\mathcal{O}}_{r_1-1,r_2,p_1} \cup \overline{\mathcal{O}}_{r_1,r_2-1,p_1} \cup \overline{\mathcal{O}}_{r_1,r_2,p_1-1}, \tag{3.2}$$

whose dimension are easy to compute from their resolution, giving the bound for $\operatorname{codim}_{D_Y(s)} \operatorname{Sing}$ of Table 3.2.

Finally, to compute the canonical bundle, recall that, from [BFMT17a], a crepant resolution of $D_Y(s)$ is given by $\pi|_{\mathscr{Z}(\tilde{s})} : \mathscr{Z}(\tilde{s}) \to D_Y(s)$. This is the zero locus of the section $\tilde{s}$ (which is constructed from s) of the bundle

$$\mathcal{Q}_{W_i} := ((E_1^* \oplus E_3^*) \otimes E_2)/((\mathcal{Q}_1^* \oplus \mathcal{Q}_3^*) \otimes \mathcal{U}_2)$$

over the Grassmannian bundle

$$F_i(E_1, E_2, E_3) := \operatorname{Gr}(d_1 - r_1, E_1) \times \operatorname{Gr}(p_1, E_2) \times \operatorname{Gr}(d_3 - r_2, E_3)$$

over X. This, together with the adjunction formula, gives

$$K_{\mathscr{Z}(\tilde{s})} = (\pi^*(K_X) \otimes K_{F_i(E_1,E_2,E_3)/X} \otimes \det(\mathcal{Q}_{W_i}))|_{\mathscr{Z}(\tilde{s})}.$$

An easy computation shows that

$$K_{\mathscr{Z}(\tilde{s})} = \pi|^*_{\mathscr{Z}(\tilde{s})}((K_X \otimes \det E_1^{-r_2} \otimes \det E_2^{p_1} \otimes \det E_3^{-r_1})|_{D_Y(s)}),$$

which implies that

$$K_{D_Y(s)} = K_X \otimes \det E_1^{-r_2} \otimes \det E_2^{p_1} \otimes \det E_3^{-r_1}.$$

□

Remark 3.4.2. In order to compute exactly the codimension of the singular locus it would be necessary, for instance, to know that the desingularization morphism does not contract any divisor. In this case, the morphism restricts to an isomorphism over the smooth locus of Y, and equality holds in (3.2) (thus giving equality in the last column of Table 3.2).

Suppose that the resolution is given by $W = \mathcal{U} \otimes W'$ over $F = \mathrm{Gr}(a, v) \times F'$, for W' a vector bundle of rank w over the variety F'. Then there is a locus E contracted by the morphism $\pi : W \to \mathcal{R}_d$, whose general fiber is $\mathbf{P}^{w-a}$, and whose image is resolved by $\mathcal{U} \otimes W'$ over $\mathrm{Gr}(a-1, v) \times F'$. By a simple computation, one gets that

$$\operatorname{codim}(E) = w - a + 1.$$

Therefore, if the contracted locus $\tilde{E}$ of π is the union of such E's, and for each of them $w > a$, no divisor is contained in $\tilde{E}$. This is the case for example of (F_i, W_i) when $r_1, r_2 < p_1 < r_1 + r_2$. Similarly, one can work out the case of orbit closures admitting other desingularizations.

The following are some explicit examples of such loci.

Example 3.4.3. Take $X = \mathrm{Gr}(4, 8)$, $E_1 = 2(\mathcal{O}_X(-1) \oplus \mathcal{O}_X)$, $E_2 = \mathcal{Q} \oplus \mathcal{O}_X$, $E_3 = 3\mathcal{O}_X$, and the orbit closure $Y = \overline{\mathcal{O}}_{3,1,3}$. Then $D_Y(s)$ is a smooth fourfold with trivial canonical bundle and $\chi(\mathcal{O}_{D_Y(s)}) = 2$.

Example 3.4.4. Take $X = \mathrm{IGr}(2, 8)$, $E_1 = \mathcal{U} \oplus \mathcal{O}_X(-1)$, $E_2 = \mathcal{U}^* \oplus 2\mathcal{O}_X$, $E_3 = 2\mathcal{O}_X$, and the orbit closure $Y = \overline{\mathcal{O}}_{2,1,2}$. Then $D_Y(s)$ is a fourfold with trivial canonical bundle, singular in codimension 3, and whose desingularisation satisfies $\chi(\mathcal{O}_{\mathscr{Z}(\tilde{s})}) = 2$.

Example 3.4.5. Take $X = \mathrm{OGr}(2, 9)$, $E_1 = 3\mathcal{O}_X$, $E_2 = \mathcal{U}^* \oplus 2\mathcal{O}_X$, $E_3 = \mathcal{U}$, and the orbit closure $Y = \overline{\mathcal{O}}_{2,1,2}$. Then $D_Y(s)$ is a fourfold with trivial canonical bundle singular in codimension 3 and whose desingularisation satisfies $\chi(\mathcal{O}_{\mathscr{Z}(\tilde{s})}) = 2$.

We next consider the quiver described in Figure 3.2 (refer to it for the notations). In the relative setting, we fix a smooth projective variety X, three vector bundles E_1, E_2, E_3 of ranks d_1, d_2, d_3 such that $\mathrm{Hom}(E_2, E_1) \oplus \mathrm{Hom}(E_2, E_3)$ is globally generated. Then, suppose that s is a general section of this bundle, and fix an orbit closure $Y = \overline{\mathcal{O}}_{k_1,k_2,q_1}$ inside $\mathcal{R}_d = \mathrm{Hom}(\mathbb{C}^{d_2}, \mathbb{C}^{d_1}) \oplus \mathrm{Hom}(\mathbb{C}^{d_2}, \mathbb{C}^{d_3})$. Recall that $D_Y(s)$ is the locus of points $x \in X$ which are sent by the section s inside $Y \subset (\mathrm{Hom}(E_2, E_1) \oplus \mathrm{Hom}(E_2, E_3))_x \cong \mathcal{R}_d$.

Theorem 3.4.6. *Let $D_Y(s)$ be defined as above, where $Y = \overline{\mathcal{O}}_{k_1,k_2,q_1} \subset \mathcal{R}_d$. Then $K_{D_Y(s)}$, $\operatorname{codim}_X(D_Y(s))$ and a lower bound for $\operatorname{codim}_{D_Y(s)} \operatorname{Sing}(D_Y(s))$ are given in Table 3.3.*

Proof. The proof is similar to the one of Theorem 3.4.1. □

Finally, let Q be the quiver appearing in Figure 3.3 (refer to it for the notations). In the relative setting, we fix a smooth projective variety X, three vector bundles E_1, E_2, E_3 of dimension d_1, d_2, d_3 such that $\mathrm{Hom}(E_1, E_2) \oplus \mathrm{Hom}(E_2, E_3)$ is globally generated. Then, suppose that s is a general section of this bundle, and fix an orbit closure $Y = \overline{\mathcal{O}}_{r_1,k_2,u_1}$ inside $\mathcal{R}_d = \mathrm{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \oplus \mathrm{Hom}(\mathbb{C}^{d_2}, \mathbb{C}^{d_3})$. Recall that $D_Y(s)$ is the locus of points $x \in X$ which are sent by the section s inside $Y \subset (\mathrm{Hom}(E_1, E_2) \oplus \mathrm{Hom}(E_2, E_3))_x \cong \mathcal{R}_d$.

Table 3.3. – ODL from a quiver of type A_3 with $a_1(0) = a_2(0) = s_2$. We use the following variables: $\eta_1 = d_2 - k_1$, $\eta_2 = d_2 - k_2$.

Case	$K_{D_Y(s)/X}$	$\operatorname{codim}_X D_Y(s)$	$\operatorname{codim}_{D_Y(s)} \operatorname{Sing}$
(i)	$\det E_1^{d_2-k_2} \otimes \det E_2^{-d_2+q_1} \otimes \det E_3^{d_2-k_1}$	$\eta_1^2 + \eta_2^2 + q_1^2 + -q_1(\eta_1 + \eta_2)$	$\geq \min \left\{ \begin{array}{c} 2(\eta_1 - q_1) + 1 \\ 2(\eta_2 - q_1) + 1 \\ 2q_1 + 1 \end{array} \right\}$
(ii)	$\det E_1^{q_1} \otimes \det E_3^{d_1} \otimes \det E_2^{-d_1-q_1}$	$(d_3 - \eta_2)^2 - q_1^2 + (d_3 - \eta_2)q_1$	$\geq \min \left\{ \begin{array}{c} 2(d_3 - \eta_2) + 1 \\ 2q_1 + 1 \end{array} \right\}$

Theorem 3.4.7. *Let $D_Y(s)$ be defined as above, where $Y = \overline{\mathcal{O}}_{r_1,k_2,u_1} \subset \mathcal{R}_d$. Then $K_{D_Y(s)}$, $\operatorname{codim}_X(D_Y(s))$ and a lower bound for $\operatorname{codim}_{D_Y(s)} \operatorname{Sing}(D_Y(s))$ are given in Table 3.4.*

Table 3.4. – ODL from a quiver of type $A_3$with $a_1(0) = s_1$, $a_2(0) = s_2$. We use the following variables: $\eta_1 = k_2 - d_1 + r_1$, $\eta_2 = d_2 - u_1$.

Case	$K_{D_Y(s)/X}$	$\operatorname{codim}_X D_Y(s)$	$\operatorname{codim}_{D_Y(s)} \operatorname{Sing}$
(i)	$\det E_1^{-k_2} \otimes \det E_3^{k_2}$	$k_2^2 + (d_1 - r_1)^2$	$\geq \min \left\{ \begin{array}{c} 2(d_1 - r_1) + 1 \\ 2(d_2 - d_1) + 1 \\ k_2 + 1 \end{array} \right\}$
(ii)	$\det E_1^{-d_2+r_1} \otimes \det E_3^{k_2}$	$2(d_1 - r_1)^2 + \eta_1\eta_2 + +(d_1 - r_1)(\eta_1 + \eta_2)$	$\geq 2(d_1 - r_1) + 1$

3.4.2. Quiver degeneracy loci of type D_4

For notations, we refer to Figure 3.8. In order to construct the degeneracy loci, we fix a smooth projective variety X, four vector bundles E_1, E_2, E_3, E_4 of dimension d_1, d_2, d_3, d_4 such that $\operatorname{Hom}(E_1, E_2) \oplus \operatorname{Hom}(E_3, E_2) \oplus \operatorname{Hom}(E_4, E_2)$ is globally generated. Then, suppose that s is a general section of this bundle, and fix an orbit closure $Y = \overline{\mathcal{O}}_r$ inside $\mathcal{R}_d = \operatorname{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \oplus \operatorname{Hom}(\mathbb{C}^{d_3}, \mathbb{C}^{d_2}) \oplus \operatorname{Hom}(\mathbb{C}^{d_4}, \mathbb{C}^{d_2})$. Recall that $D_Y(s)$ is the locus of points $x \in X$ which are sent by the section s inside $Y \subset (\operatorname{Hom}(E_1, E_2) \oplus \operatorname{Hom}(E_3, E_2) \oplus \operatorname{Hom}(E_4, E_2))_x \cong \mathcal{R}_d$.

Theorem 3.4.8. *Let $D_Y(s)$ be defined as above, where $Y = \overline{\mathcal{O}}_r \subset \mathcal{R}_d$ is resolved by the resolution of type (ii) or (iii). Then*

$$K_{D_Y(s)/X} = \det E_1^{r_1-d_2} \otimes \det E_2^{2x} \otimes \det E_3^{r_2-d_2} \otimes \det E_4^{r_3-d_2},$$

$$\operatorname{codim}_X(D_Y(s)) = \sum_i r_i^2 + x^2,$$

$$\operatorname{codim}_{D_Y(s)} \operatorname{Sing}(D_Y(s)) \geq \min \left\{ \begin{array}{c} d_1 + x - 2r_1 + 1 \\ d_2 + r_1 + r_2 + r_3 - 2x + 1 \\ d_3 + x - 2r_2 + 1 \\ d_4 + x - 2r_3 + 1 \end{array} \right\}.$$

Proof. The proof is similar to the one of Theorem 3.4.1. □

The easiest way to construct varieties with trivial canonical bundle in this case is to assume that E_1, E_3, E_4 are trivial and E_2 is globally generated. We give two explicit examples of ODL.

Example 3.4.9. Take X to be the intersection of a hypersurface in $|\mathcal{O}(1)|$ and a hypersurface in $|\mathcal{O}(2)|$ inside $\mathrm{Gr}(3,7)$; moreover take $E_1 = 2\mathcal{O}_X$, $E_2 = \mathcal{U}^*$, $E_3 = 2\mathcal{O}_X$, and $E_4 = 2\mathcal{O}_X$. The orbit closure chosen will be $Y = \overline{\mathcal{O}}_{r_1,r_2,r_3,x} = \overline{\mathcal{O}}_{1,1,1,2}$. Then $D_Y(s)$ is a singular (over a finite number of points) threefold whose resolution of singularities is of type Calabi-Yau.

This singular variety is a hypersurface inside two singular (over a curve) almost Fano fourfolds F_1, F_2, which are the corresponding degeneracy loci when the base variety X is a hypersurface in respectively $|\mathcal{O}(1)|$ and $|\mathcal{O}(2)|$ inside $\mathrm{Gr}(3,7)$. We computed some invariants of their resolutions $\tilde{F}_1$ and $\tilde{F}_2$:

Table 3.5. – Some invariants of $\tilde{F}_1$ and $\tilde{F}_2$

i	$(-K_{\tilde{F}_i})^4$	$\chi(\Omega^1_{\tilde{F}_i})$	$\chi(\Omega^2_{\tilde{F}_i})$	$\chi(-K_{\tilde{F}_i})$
1	224	−4	8	51
2	28	−16	94	12

Example 3.4.10. Take X the intersection of two hypersurfaces in $|\mathcal{O}(1)|$ inside $\mathrm{Gr}(3,7)$; moreover take $E_1 = 2\mathcal{O}_X$, $E_2 = \mathcal{U}^*$, $E_3 = 2\mathcal{O}_X$, and $E_4 = 2\mathcal{O}_X$. The orbit closure chosen will be $Y = \overline{\mathcal{O}}_{r_1,r_2,r_3,x} = \overline{\mathcal{O}}_{1,1,1,2}$. Then $D_Y(s)$ is a singular (over a finite number of points) almost Fano threefold F whose resolution $\tilde{F}$ has the following invariants: $(-K_{\tilde{F}})^3 = 14$, $\chi(\Omega^1_{\tilde{F}}) = -2$, $\chi(-K_{\tilde{F}}) = 10$.

Example 3.4.11. Take X a hypersurface in $|\mathcal{O}(3)|$ inside $\mathrm{Gr}(3,7)$; moreover take $E_1 = 2\mathcal{O}_X$, $E_2 = \mathcal{U}^*$, $E_3 = 2\mathcal{O}_X$, and $E_4 = 2\mathcal{O}_X$. The orbit closure chosen will be $Y = \overline{\mathcal{O}}_{r_1,r_2,r_3,x} = \overline{\mathcal{O}}_{1,1,1,2}$. Then $D_Y(s)$ is a singular (over a curve) fourfold whose desingularisation has trivial canonical bundle and $\chi(\mathcal{O}_{\mathscr{Z}(\tilde{s})}) = 2$.

4. Bisymplectic Grassmannians

The last chapter of this thesis is dedicated to the analysis of a certain class of Fano varieties inside ordinary Grassmannians, i.e. bisymplectic Grassmannians. They parametrize subspaces of a given vector space isotropic with respect to two symplectic forms. They belong to the larger class of *multisymplectic* Grassmannians, to which the symplectic Grassmannian and the Fano threefold V_{22} belong (see Example 1.5.1) as well, but have some additional properties which make them worthy to be investigated more in detail.

More precisely, they are always Fano, we are able to describe the general element in their moduli space, and we will show that they admit the action of a torus with a finite number of fixed points. Therefore, the techniques of equivariant cohomology can be employed in this situation. We will see however how using equivariant techniques for bisymplectic Grassmannian presents some additional difficulties with respect to the case of homogeneous varieties (we will work out the case of symplectic Grassmannians).

Finally, we will compute explicitly the (equivariant) cohomology in the special case of $\mathrm{I}_2\mathrm{Gr}(2,6)$. This is the smallest non trivial bisymplectic Grassmannian, but its analysis will already underline all the specific features of these varieties. Although in the general case, due to the intrinsic complexity of the combinatorics of the problem, we were not able to determine the cohomology via a general formula, we hope that the study of $\mathrm{I}_2\mathrm{Gr}(2,6)$ will guide us in the future to find one.

One can also be interested in other questions concerning bisymplectic Grassmannians, such as the study of their derived categories or quantum cohomology. As the analysis of bisymplectic Grassmannians is far from being complete, we added some possible research directions on them in Section 4.4.

4.1. First properties of $\mathrm{I}_2\mathrm{Gr}(k,2n)$

Let us consider the Grassmannian $\mathrm{Gr}(k,2n)$ of k-dimensional subspaces inside a vector space of dimension $2n$. From now on, if not otherwise stated, we will assume that $2 \leq k \leq n$. By fixing a skew-symmetric form ω over $\mathbb{C}^{2n}$, one can consider the subvariety $\mathrm{IGr}(k,2n)$ inside $\mathrm{Gr}(k,2n)$ of isotropic subspaces with respect to ω. If ω is non-degenerate, $\mathrm{IGr}(k,2n)$ is smooth, and it is a rational homogeneous variety for the natural action of $\mathrm{Sp}(2n) \subset \mathrm{GL}(2n)$. Denoting by $\mathcal{U}$ the tautological bundle over the Grassmannian, the variety $\mathrm{IGr}(k,2n)$ can be seen as the zero locus of a general section of $\wedge^2\mathcal{U}^*$ over $\mathrm{Gr}(k,2n)$; indeed notice that, by the Borel-Weyl Theorem, $\mathrm{H}^0(\mathrm{Gr}(k,2n),\wedge^2\mathcal{U}^*) \cong \wedge^2(\mathbb{C}^{2n})^*$. We will refer to $\mathrm{IGr}(k,2n)$ as the isotropic (or symplectic) Grassmannian.

Let us now fix two skew-symmetric forms ω_1, ω_2 over $\mathbb{C}^{2n}$.

Definition 4.1.1. *The bisymplectic Grassmannian is the subvariety* $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ *inside* $\mathrm{Gr}(k, 2n)$ *of subspaces isotropic with respect to* ω_1 *and* ω_2. *Equivalently, the points in* $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ *are isotropic with respect to the pencil* $\langle \omega_1, \omega_2 \rangle$.

Remark 4.1.2. As we will see later, there is not only one isomorphism class of bisymplectic Grassmannians. Indeed, the definition depends on the choice of a pencil $\langle \omega_1, \omega_2 \rangle$. However, we will still refer to *the* bisymplectic Grassmannian in the following.

Of course, $\mathrm{I}_2\mathrm{Gr}(k, 2n) \subset \mathrm{IGr}(k, 2n)_i$, where $\mathrm{IGr}(k, 2n)_i$ is the symplectic Grassmannian with respect to ω_i, $i = 1, 2$. The fact that $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ is not empty is a consequence of the fact that $\mathrm{I}_2\mathrm{Gr}(n, 2n) \neq \emptyset$ (see Example 4.1.3). Moreover, $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ can be seen as the zero locus of a section of $(\wedge^2 \mathcal{U}^*)^{\oplus 2}$ over $\mathrm{Gr}(k, 2n)$; by Bertini's theorem, if this section is general, the bisymplectic Grassmannian is smooth. Moreover in this case, its dimension is $2k(n - k) + k$ and, by the adjunction formula, its canonical bundle is

$$K_{\mathrm{I}_2\mathrm{Gr}(k,2n)} = \mathcal{O}(-2n + 2k - 2);$$

therefore, $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ is a Fano variety. In the next sections, we will study under which conditions the bisymplectic Grassmannians are smooth (i.e. for what kind of pencils). Before doing so, let us deal with the case $k = n$.

Example 4.1.3 ($k = n$). In [Kuz15], Kuznetsov proves that the variety $\mathrm{I}_2\mathrm{Gr}(n, 2n)$ is smooth exactly when the pencil $\langle \omega_1, \omega_2 \rangle$ intersects the Pfaffian divisor $D \subset \mathbf{P}(\wedge^2(\mathbb{C}^{2n})^*)$ (of degree n) in n distinct points; in this case, the two forms are simultaneously block diagonalizable (with blocks of size 2×2), and there exists an isomorphism

$$\mathrm{I}_2\mathrm{Gr}(n, 2n) \cong (\mathbf{P}^1)^n. \tag{4.1}$$

Therefore, the automorphism group of $\mathrm{I}_2\mathrm{Gr}(n, 2n)$ is $(\mathrm{PGL}(2))^n \times \mathfrak{S}_n$ (where $\mathfrak{S}_n$ is the group of permutations of n elements). Surprisingly enough, from the isomorphism one realises that $\mathrm{I}_2\mathrm{Gr}(n, 2n)$ has no small deformations.

From now on, the zero locus of a section s will be denoted by $\mathscr{Z}(s)$. Moreover, let us denote by $V = \mathbb{C}^{2n}$.

4.1.1. Smoothness

We have already said that if the pencil of skew-symmetric forms $\langle \omega_1, \omega_2 \rangle$ is general, the corresponding bisymplectic Grassmannian is smooth. In this section we make more precise what the word *general* stands for in this situation. Essentially, we use the same proof of the analogous result for $k = n$ in [Kuz15].

We also include a weaker result (Proposition 4.1.4) which has the advantage that its proof can be applied to other interesting cases (e.g. ortho-symplectic Grassmannians). Recall that $\mathrm{I}_2\mathrm{Gr}(k, V)$ can be seen as the zero locus of a section of $(\wedge^2 \mathcal{U}^*)^{\oplus 2}$ over $\mathrm{Gr}(k, V)$.

Proposition 4.1.4. *If (ω_1, ω_2) is general in $(\wedge^2 V^*)^{\oplus 2}$, then ω_1 and ω_2 are simoultaneously block diagonalizable.*

Proof. The set of invertible forms is a dense open subset of $\wedge^2(V)^*$, so we can suppose ω_1 and ω_2 to be invertible. If ω_1 and ω_2 are simoultaneously block diagonalizable, then there exist n 2-planes $K_1, \dots, K_n$ which are orthogonal to each other with respect to ω_1 and ω_2. These planes are also unique modulo permutations. We can therefore suppose that $K_i = \langle x_i, x_{-i} \rangle$, and

$$\omega_1 = \sum_{i=1}^{n} l_i x_i \wedge x_{-i},$$

$$\omega_2 = \sum_{i=1}^{n} \lambda_i x_i \wedge x_{-i}.$$

Once more, the l_i's and the λ_i's are unique.

To resume, invertible simoultaneously block diagonalizable two-forms are parametrized by an open subset of $\mathrm{Gr}(2, V)^{2n} \times \mathbb{C}^{2n}$. This variety has dimension $4n^2 - 2n$, which is equal to the dimension of $(\wedge^2(V)^*)^{\oplus 2}$, therefore proving that two general forms (ω_1, ω_2) are simoultaneosly block diagonalizable. □

We need the following lemma on the smoothness of the zero locus of a section of $\wedge^2 \mathcal{U}^*$ over $\mathrm{Gr}(k, V)$. We will denote by $[P]$ a point in $\mathrm{Gr}(k, V)$ represented by the k-dimensional vector subspace $P \subset V$.

Lemma 4.1.5. *Let $\omega_0 \in \wedge^2 V^*$ be a section of $\wedge^2 \mathcal{U}^*$ over $\mathrm{Gr}(k, V)$. Suppose $\mathscr{Z}(\omega_0)$ has the expected dimension. Then $\mathscr{Z}(\omega_0)$ is singular at a point $[P] \in \mathrm{Gr}(k, V)$ if and only if $\dim(P \cap \mathrm{Ker}(\omega_0)) \geq 2$.*

Proof. The expected dimension of $\mathscr{Z}(\omega_0)$ is $k(2n-k) - \binom{k}{2}$. To prove the lemma, we show that the tangent space has dimension greater than the expected dimension exactly at the points described by the statement of the lemma.

Let $[P] \in \mathscr{Z}(\omega_0)$. We have:

$$T_{\mathscr{Z}(\omega_0),[P]} \hookrightarrow T_{\mathrm{Gr}(k,V),[P]} \cong \mathrm{Hom}(P, V/P).$$

Moreover, let $K := P \cap \mathrm{Ker}(\omega_0)$, and $j := \dim(K)$. The two-form ω_0 can be seen as a linear map $\omega_0 : V \to V^*$; by using this morphism, and a point $\phi \in T_{\mathrm{Gr}(k,V),[P]}$, we get a chain of morphisms:

$$K \hookrightarrow P \xrightarrow{\phi} V/P \overset{pr}{\twoheadrightarrow} V/P^{\perp} \xrightarrow{\omega_0} (P/K)^* \overset{i}{\hookrightarrow} P^*.$$

If $K = \emptyset$, then $\phi \in T_{\mathscr{Z}(\omega_0),[P]}$ if and only if $\eta_\phi := (i \circ \omega_0 \circ pr \circ \phi) \in S^2 P^*$, i.e. the skew-symmetric part $(\eta_\phi)_a$ of η_ϕ is zero.

If $K \neq \emptyset$, the situation is just slightly more complicated. Indeed, an explicit computation gives that $\phi \in T_{\mathscr{Z}(\omega_0),[P]}$ if and only if $(\eta_\phi)|_K = 0$, $\eta_\phi(P) \subset (P/K)^*$

(this condition is trivial), and the induced morphism $P/K \to (P/K)^*$ is symmetric. This gives

$$\dim(T_{\mathscr{X}(\omega_0),[P]}) = \dim(\mathscr{X}(\omega_0)) + \frac{j(j-1)}{2},$$

which is strictly greater than $\dim(\mathscr{X}\omega_0)$ if and only if $j \geq 2$. $\square$

Let $D \subset \mathbf{P}(\wedge^2 V^*)$ be the Pfaffian divisor of degree n.

Proposition 4.1.6. *Let $\Omega = \langle \omega_1, \omega_2 \rangle \subset \mathbf{P}(\wedge^2(V^*))$ be a pencil of skew-symmetric forms such that $\mathscr{X}(\Omega) \subset \mathrm{Gr}(k, V)$ has the expected dimension. If $\mathscr{X}(\Omega)$ is smooth then $\Omega \cap D = p_1, \ldots, p_n$, where the p_i's are n distinct points such that:*

1. $\dim(\mathrm{Ker}(p_i)) = 2$ *for* $1 \leq i \leq n$*;*
2. $V = \mathrm{Ker}(p_1) \oplus \cdots \oplus \mathrm{Ker}(p_n)$.

Having proved Lemma 4.1.5, the proof of this result is exactly the same as the one used in [Kuz15]. We report it for the sake of completeness.

Proof. Let us denote by $K_i := \mathrm{Ker}(p_i)$, $p_i \in \Omega \cap D$. Assume $\dim(K_i) > 2$, and take $p \neq p_i$, $p \in \Omega$. Then there exists a 2-dimensional subspace $K \subset K_i$ isotropic with respect to p. If $K^\perp$ is the orthogonal of K with respect to p, let us denote by $V' := K^\perp / K$. It is a vector space of dimension $2n - 4$. For dimensional reasons, there exists a $(k-2)$-dimensional subspace $U' \subset V'$ isotropic with respect to p and p_i. Then

$$U := U' + K \subset V$$

is a k-dimensional space isotropic with respect to p and p_i. Therefore U belongs to $\mathscr{X}(\Omega)$. But $\dim(U \cap K_i) \geq 2$; moreover $\mathscr{X}(\Omega) = \mathscr{X}(p) \cap \mathscr{X}(p_i)$, and $\mathscr{X}(p_i)$ is singular at U by the lemma. Therefore $\mathscr{X}(\Omega)$ is singular at U. This proves that $\dim(\mathrm{Ker}(p_i)) = 2$ for $1 \leq i \leq n$.

The same argument holds if $K_i = K_j$ for $j \neq i$ (choose $K = K_j$); therefore all the K_i's are distinct. To prove that the intersection $\Omega \cap D$ is transversal, notice that the tangent space T_{D,p_i} is given by all the skew-symmetric forms that vanish on K_i; for what we have just stated, Ω cannot be tangent to D, and therefore the intersection is transversal.

Finally, to prove the direct sum decomposition, fix a non-degenerate form $p \in \Omega$. Then, if $i \neq j$, $K_i \perp K_j$ with respect to p, because p is a linear combination of p_i and p_j, and $p_i(K_i, K_j) = 0 = p_j(K_j, K_i)$. $\square$

Remark 4.1.7. The proof actually shows that if $\mathscr{X}(\Omega)$ is smooth then all the forms in Ω are simoultaneously block diagonalizable. This can be seen as a proof of the fact that two general skew-symmetric forms are simultaneously block diagonalizable. Moreover, as any non-degenerate form is conjugate to the standard one, one can suppose that Ω is generated by ω_1 and ω_2 with:

$$\omega_1 = \sum_{i=1}^{n} x_i \wedge x_{-i}$$

$$\omega_2 = \sum_{i=1}^{n} \lambda_i x_i \wedge x_{-i}$$

where $\langle x_i, x_{-i}\rangle = (K_i)^*$ for $1 \leq i \leq n$, and the λ_i's are all distinct.

4.1.2. Small deformations

We now deal with the problem of understanding what are the small deformations of smooth bisymplectic varieties. We already saw that $\mathrm{I}_2\mathrm{Gr}(n,2n)$ has no deformation (this comes from the isomorphism (4.1)), but this is not the case for $\mathrm{I}_2\mathrm{Gr}(k,2n)$, $k<n$ in general. The study will be conducted by computing (the dimension of)

$$\mathrm{H}^1(\mathrm{I}_2\mathrm{Gr}(k,2n), T_{\mathrm{I}_2\mathrm{Gr}(k,2n)}).$$

We will use the fact that bisymplectic Grassmannians $X = \mathrm{I}_2\mathrm{Gr}(k,V) = \mathscr{Z}(\Omega)$ are zero loci of a section of $F = (\wedge^2\mathcal{U}^*)^{\oplus 2}$ inside $G = \mathrm{Gr}(k,V)$, and we will compute the cohomology by using the Koszul complex

$$0 \to \det(F^*) \to \cdots \to \wedge^2 F^* \to F^* \to \mathcal{O}_G \to \mathcal{O}_X \to 0$$

and the normal exact sequence

$$0 \to T_X \to T_G|_X \to F|_X \to 0.$$

We will denote a (completely reducible) homogeneous vector bundle F over the Grassmannian G by a sequence of integers $a = (a_k,\dots,a_1;a_{2n},\dots,a_{k+1})$, which means that $F = \mathbb{S}_{a_k,\dots,a_1}\mathcal{U}^* \otimes \mathbb{S}_{a_{2n},\dots,a_{k+1}}\mathcal{Q}^*$ (where $\mathbb{S}.$ denotes the Schur functor). For instance $T_G = (1,0,\dots,0,-1)$, and $F = (1,1,0,\dots,0)^{\oplus 2}$. The cohomology of an irreducible homogeneous vector bundle E is given by Bott's theorem: if $\delta = (2n,2n-1,\dots,2,1)$, the vector bundle associated to a has cohomology if and only if $a+\delta$ has all distinct integers; moreover, if $\sigma \in \mathfrak{S}_n$ is the permutation of length l such that $\sigma(a+\delta)$ is a decreasing sequence of integers, then the only non-trivial cohomology of the vector bundle E is

$$\mathrm{H}^l(G,E) \cong V_{\sigma(a+\delta)-\delta},$$

where V_λ denotes the representation of $GL(V)$ with weight λ.

If $\lambda = (\lambda_k \geq \cdots \geq \lambda_1)$ is a sequence of integers, we can represent it as a Young diagram of the following form:

$$
\begin{array}{cccccccc}
\boxed{1} & \boxed{2} & \boxed{3} & \dots & \dots & \dots & \boxed{\lambda_k} \\
\boxed{1} & \boxed{2} & \boxed{3} & \dots & \dots & \boxed{\lambda_{k-1}} & \\
\vdots & \vdots & \vdots & \vdots & & & \\
\boxed{1} & \boxed{2} & \dots & \boxed{\lambda_1} & & &
\end{array}
$$

In order to decompose the tensor product of two irreducible representations in irreducible factors, one can use the Littlewood-Richardson rule. It says that, if λ, μ are two weights, then

$$V_\lambda \otimes V_\mu = \bigoplus_\nu c^\nu_{\lambda,\mu} V_\nu,$$

where $c^\nu_{\lambda,\mu}$ is the number of ways the Young diagram of μ can be expanded to the Young diagram of ν by a strict λ-expansion. For the definition of a λ-expansion, refer to [FH91]; we will just recall that it is a particular way to add the boxes of λ to the Young diagram of μ obtaining a new Young diagram.

Remark 4.1.8. A consequence of the definition of a λ-expansion is the following: when multiplying two Young diagrams μ and λ, suppose that a box was in the i-th row of λ; then in each Young diagram of the product, if it appears in the j-th row, then $j \geq i$. We will use this remark later on.

Proposition 4.1.9. *The non-trivial cohomology groups of $F|_X$ are:*

$$\mathrm{H}^0(G, F|_X) \cong ((\wedge^2 V^*)^{\oplus 2}/\mathbb{C}^4)/\mathbb{C}^{2n-4} \textit{ for } k = n,$$

$$\mathrm{H}^0(G, F|_X) \cong (\wedge^2 V^*)^{\oplus 2}/\mathbb{C}^4 \textit{ for } k < n.$$

Moreover, the quotient factor $\mathbb{C}^4$ is the 4-dimensional space

$$\langle(\omega_1,\omega_1),(\omega_1,\omega_2),(\omega_2,\omega_1),(\omega_2,\omega_2)\rangle \subset (\wedge^2 V^*)^{\oplus 2}, \tag{4.2}$$

where $\Omega = \langle \omega_1, \omega_2\rangle$.

Proof. By using the Koszul complex, in order to compute the cohomology of $F|_X$, we need to compute the cohomology groups

$$\mathrm{H}^p(G, \wedge^q F^* \otimes F).$$

Let us suppose that an irreducible factor of the bundle $\wedge^q F^* \otimes F$ is associated to the weight $a = (a_k, \dots, a_1; a_{2n}, \dots, a_{k+1})$; as $\wedge^q F^* \otimes F$ is a Schur functor applied to $\mathcal{U}$ and is independent of $\mathcal{Q}$, we have that $a_{2n} = \cdots = a_{k+1} = 0$. It is easy to verify that if $p = 0$ the only cohomology groups appear for $q = 0, 1$; more precisely:

$$\mathrm{H}^0(G, F) \cong (\wedge^2 V^*)^{\oplus 2} \text{ and } \mathrm{H}^0(G, F^* \otimes F) \cong \mathbb{C}^4,$$

where the explicit comutation gives that $\mathrm{H}^0(G, F^* \otimes F)$ is the subspace described in (4.2).

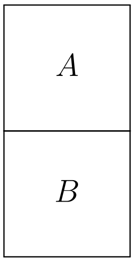
By Bott's theorem and the fact that $a_{2n} = \cdots = a_{k+1}$, we know that if $p \neq 0$, the cohomology of a is concentrated in degree $p = j(2n-k)$ for a certain integer $j > 0$; indeed, to order $a + \delta$, if the permutation σ moves an integer $i \leq k$, then σ has to move i past the integers $k+1, \ldots, 2n$. In order to have cohomology in degree $j(2n-k)$, a needs to satisfy:

$$-(2n-k+j) \geq a_j \geq \cdots \geq a_1 \tag{4.3}$$

and

$$a_k \geq \cdots \geq a_{j-1} \geq -j. \tag{4.4}$$

This means that the Young diagram of the dual weight $a^* = (-a_1, \ldots, -a_k)$ should be of the form:

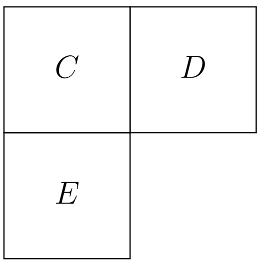


where A is a Young diagram containing the $j \times (n-k+j)$ rectangle (of height j and length $(n-k+j)$), and B is a Young diagram contained in a $(k-j) \times j$ rectangle.

Now, a^* is the weight of a factor of $\wedge^q(\wedge^2(\mathcal{U}^*)^{\oplus 2}) \otimes \mathcal{U}$. By classical formulas for composing Schur functors,

$$\wedge^q(F^*) = \bigoplus_{r+s=q} \wedge^r \wedge^2 \mathcal{U} \otimes \wedge^s \wedge^2 \mathcal{U},$$

and the weights of each irreducible factor of $\wedge^r \wedge^2 \mathcal{U}$ are of the form $b = (b_k, \ldots, b_1)$, with $\sum_i b_i = 2r$ and Young diagram:



with C a $(h+1) \times h$ rectangle for a certain $1 \leq h \leq k-1$, and D and E two Young diagrams dual to one another.

The fact that A contains a $j \times (n-k+j)$ rectangle means that (the Young diagram of) a factor of $\wedge^r(\wedge^2\mathcal{U}) \otimes \wedge^s(\wedge^2\mathcal{U})$ must contain on the top left corner a $j \times (n-k+j)$ rectangle. Similarly, as B is contained in a $(k-j) \times j$ rectangle, (the Young diagram of) the same factor of $\wedge^r(\wedge^2\mathcal{U}) \otimes \wedge^s(\wedge^2\mathcal{U})$ must be contained in a $(k-j) \times j$ rectangle to which one can add two unit boxes (we need to tensor $\wedge^q F^*$ by F).

Such a factor is a subspace of the tensor product of two irreducible terms $b_1 \subset \wedge^r(\wedge^2\mathcal{U})$ and $b_2 \subset \wedge^s(\wedge^2\mathcal{U})$. Let b_1 correspond to C_1, D_1, E_1 and b_2 correspond to C_2, D_2, E_2 as above. Moreover, let x_1 (respectively x_2) be the number of unit boxes in the first j rows of b_1 (resp. b_2). Then there are $x_1 + j$ (resp. $x_2 + j$) unit boxes in the first j columns of b_1 (resp. b_2). By the Littlewood-Richardson rule (and Remark 4.1.8), conditions (4.3) and (4.4) become:

$$x_1 + x_2 \geq j(2n - k + j),$$

$$x_1 + x_2 + 2j - 2j^2 \leq j(k-j) + 2.$$

By subtracting the two equations, one obtains:

$$k \geq n + 2 - 2/j.$$

This equation is satisfied for $j \geq 1$ only when $j = 1$, and $k = n$. These conditions imply that $p = n$, $q = n+1$, and

$$\mathrm{H}^n(\mathrm{Gr}(n, 2n), \wedge^{n+1} F^* \otimes F) \cong \mathbb{C}^{2n-4}.$$

From these computations, the statement follows at once. $\square$

Proposition 4.1.10. *The non-trivial cohomology groups of $T_G|_X$ are:*

$$\mathrm{H}^0(G, T_G|_X) \cong \mathfrak{sl}(V)/\mathbb{C}^{n-1} \textit{ for } k = n,$$

$$\mathrm{H}^0(G, T_G|_X) \cong \mathfrak{sl}(V) \textit{ for } k < n.$$

Proof. The proof of this proposition is similar to the previous one. By the Koszul complex, we need to compute the cohomology groups

$$\mathrm{H}^p(G, \wedge^q F^* \otimes T_G),$$

where T_G is the tangent bundle of G and is represented by the weight $(1, 0, \ldots, 0, -1)$. Let us deal with the case $p = 0$: the only cohomology different from zero is when $q = 0$ and

$$\mathrm{H}^0(G, T_G) = \mathfrak{sl}(V).$$

Let us suppose $p \neq 0$; then, by Bott's theorem, $p = jn - \epsilon$ for $\epsilon \in \{0, 1\}$ and $1 \leq j \leq k$. We study the two cases $\epsilon = 0$ and $\epsilon = 1$ separately.

If $\epsilon = 0$, the same kind of argument we used in the previous proof gives that there should exist two integers x_1 and x_2 such that

$$x_1 + x_2 \geq j(2n - k + j + 1)$$

and

$$x_1 + x_2 + 2j - 2j^2 \leq j(k - j) + 1.$$

This means

$$2k \geq 2n + 3 - 1/j,$$

which is never possible for $j \geq 1$, $k \leq n$.

If $\epsilon = 1$, there should exist two integers x_1 and x_2 such that

$$x_1 + x_2 \geq j(2n - k + j) - 1$$

and

$$x_1 + x_2 + 2j - 2j^2 \leq j(k - j) + 1.$$

This means

$$k \geq n + 1 - 1/j,$$

which is possible only for $j = 1$, $k = n$. In this case we have

$$\mathrm{H}^{n-1}(\mathrm{Gr}(n, 2n), \wedge^n F^* \otimes T_G) \cong \mathbb{C}^{n-1}.$$

From these computations, the statement follows at once. □

As we are interested in the case $k < n$, we need to study the morphism

$$\mathfrak{sl}(V) \to (\wedge^2 V^*)^{\oplus 2}/\mathbb{C}^4.$$

This is just the induced morphism of the differential of the natural action of $SL(V)$ over $(\wedge^2 V^*)^{\oplus 2}$. The first step is to understand the differential of this action:

Lemma 4.1.11. *The differential of the natural action of $SL(V)$ over $(\wedge^2 V^*)^{\oplus 2}$ is part of the following exact sequence:*

$$0 \to \mathfrak{sl}(2)^n \to \mathfrak{sl}(V) \to (\wedge^2 V^*)^{\oplus 2} \to \mathbb{C}^{n+1} \to 0$$

Proof. The action of an element $M \in SL(V)$ on $(\wedge^2 V^*)^{\oplus 2}$ is given by:

$$M \mapsto (M^t \omega_1 M, M^t \omega_2 M),$$

where $\Omega = \langle \omega_1, \omega_2 \rangle$. Let us suppose that:

$$\omega_1 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \text{ and } \omega_2 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},$$

where $J = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, and the λ_i's are pairwise distinct. Then the differential of the action above is given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \Big(\begin{pmatrix} -B + B^t & A + D^t \\ -A - D^t & C - C^t \end{pmatrix}, \begin{pmatrix} -BJ + JB^t & AJ + JD^t \\ -AJ - JD^t & CJ - JC^t \end{pmatrix} \Big).$$

It is straightforward to verify that the kernel of this morphism is given by

$$\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ s.t. } C = JCJ^{-1}, B = JBJ^{-1}, -D^t = A = JAJ^{-1} \} =$$

$$= \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ s.t. } -D^t = A \text{ and } A, B, C \text{ are diagonal matrices} \} \cong \mathfrak{sl}(2)^n.$$

□

The following theorem puts everything together and gives the small deformations of bisymplectic Grassmannians for $k < n$.

Theorem 4.1.12. *The following isomorphisms hold:*

$$\mathrm{H}^0(X, T_X) \cong \mathfrak{sl}(2)^n,$$

$$\mathrm{H}^1(X, T_X) \cong \mathbb{C}^{n-3}.$$

Proof. Let $\mathbb{C}^4 \subset (\wedge^2 V^*)^{\oplus 2}$ be as in (4.2). We study its preimage W inside $\mathfrak{sl}(V)$. We want to prove that $W \cong \mathfrak{sl}(2)^n$. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in W.$$

Then M must satisfy $C = JCJ^{-1}$, $B = JBJ^{-1}$, i.e. B, C are diagonal matrices. Moreover

$$A + D^t = \operatorname{diag}(a + b\lambda_1, \dots, a + b\lambda_n) \text{ and } AJ + JD^t = \operatorname{diag}(c + d\lambda_1, \dots, c + d\lambda_n)$$

for certain real numbers a, b, c, d. Therefore D is determined by A, and

$$AJ - JA = \operatorname{diag}(-b\lambda_1^2 + (d - a)\lambda_1 + c, \dots, -b\lambda_n^2 + (d - a)x_\lambda + c). \tag{4.5}$$

This implies that A must be diagonal, and D too. Equation (4.5) also implies that $-b\lambda_i^2 + (d - a)\lambda_i + c = 0$ for $i = 1, \dots, n$. By regarding a, b, c, d as variables, one obtains a system of n equations in four variables. As the λ_i's are general for a general Ω, one has that if $n \geq 3$, then the only solution is $(a, b, c, d) = (a, 0, 0, a)$. But then, as $M \in \mathfrak{sl}(V)$, its trace is zero, which means that $a = 0$. This implies that $W \cong \mathfrak{sl}(2)^n$, which proves the theorem. □

Remark 4.1.13. The fact that $\mathrm{H}^0(X, T_X) \cong \mathfrak{sl}(2)^n$ should not be surprising; indeed, by Proposition 4.1.6 we know that the forms in Ω can be simultaneously block diagonalized, the blocks being the 2-dimensional subspaces K_i appearing in the proof of the same proposition. A consequence of this is the fact that, for $1 \leq i \leq n$, the group $\mathrm{PGL}(K_i) \subset PGL(\mathbf{P}(\wedge^2 V^*))$ fixes the pencil Ω. Therefore, it is contained in the automorphism group of $\mathscr{X}(\Omega) = X$. The fact that these are the only automorphisms of X modulo a finite group is a consequence of the previous theorem. To state it more intrinsically, we can write:

$$T_{\mathrm{Aut}(X)} \cong \mathrm{H}^0(X, T_X) \cong \mathfrak{sl}(K_1) \oplus \cdots \oplus \mathfrak{sl}(K_n).$$

Moreover, this observation implies that a n-dimensional torus acts on X, which we will use in Section 4.3.

Remark 4.1.14. When $n = 3$ (and $k = 2$), the variety X has no small deformations. This is related to the fact that $(\wedge^2 V^*) \otimes \mathbb{C}^2$ is a prehomogeneous space for the action of $\mathrm{SL}(V) \times \mathrm{SL}(2) \times \mathbb{C}^*$; more precisely, it is the parabolic space associated to (E_7, α_3) (see Section 1.4). This implies that there are just a finite number of orbits, and therefore that all pencils Ω in a dense subset of $\mathbf{P}(\wedge^2 V^*)$ are conjugated under the action of $\mathrm{PGL}(V)$. As a consequence, there are only finitely many isomorphism classes of varieties of the form $\mathscr{X}(\Omega)$.

Moreover, by Proposition 4.1.6, if $\mathscr{X}(\Omega)$ is smooth, Ω intersects the Pfaffian divisor D in three points p_1, p_2, p_3; by changing coordinates if necessary, we can suppose that $p_1 = [x_1 \wedge x_{-1} + x_2 \wedge x_{-2}]$ and $p_2 = [x_2 \wedge x_{-2} + x_3 \wedge x_{-3}]$, where $(x_{\pm 1}, x_{\pm 2}, x_{\pm 3})$ is a basis of V^*. Thus, there is only one smooth isomorphism class of $\mathrm{I}_2\mathrm{Gr}(2, 6)$.

Remark 4.1.15. What the theorem tells us is that the moduli space $\mathcal{M}_{bisym(k,n)}$ of bisymplectic Grassmannians should have dimension $n - 3$. This is the same as the dimension of the moduli space $\mathcal{M}_n$ of n points inside $\mathbf{P}^1$. It is not difficult to define a rational morphism

$$\mathcal{M}_{bisym(k,n)} \dashrightarrow \mathcal{M}_n.$$

Indeed, to any smooth variety $X = \mathscr{X}(\Omega)$, we can associate $\Omega \cap D$, which are n distinct points inside $\Omega \cong \mathbf{P}^1$ (modulo $\mathrm{PGL}(2)$!). More explicitly, if Ω is generated by the two forms in Remark 4.1.7, we can use the following identification:

$$\Omega \cong \mathbf{P}^1 \quad , \quad (a\omega_1 + b\omega_2) \mapsto [a, b].$$

Then to $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ we associate the point in $\mathcal{M}_n$ corresponding to the n points $[-\lambda_1, 1], \ldots, [-\lambda_n, 1]$ inside $\mathbf{P}^1$. The morphism between the moduli spaces corresponds therefore to the fact that the bisymplectic Grassmannians are parametrized by the λ_i's; an interesting and natural question is wether this morphism is birational or not.

4.2. Cohomology of $\mathrm{IGr}(k, 2n)$

In this section we make a digression from the central topic of the chapter, and we deal with the well studied symplectic Grassmannians. We do this essentially for two reasons. On one hand, it can be useful to start from a classical framework, where (almost) everything is well understood, and only later pass to the rather new context of bisymplectic Grassmannians, which presents some additional difficulties. On the other hand, there is a strong relationship between symplectic Grassmannians and bisymplectic Grassmannians, which is even stronger when we see them as varieties on which a torus acts. Of course, the fact that $\mathrm{IGr}(k, V)$ is a homogeneous variety (for $\mathrm{Sp}(V)$) is one of the principal reasons why things are easier to deal with in this situation.

4.2.1. Torus action, weights, Schubert varieties

Let $T \cong (\mathbb{C}^*)^n$ be a maximal torus inside $Sp(V)$; as $\mathrm{Aut}(\mathrm{IGr}(k, V)) \cong Sp(V)/\mathbb{Z}_2$, then T acts on $\mathrm{IGr}(k, V)$. For simplicity, we will assume from now on that T is the diagonal torus $\mathrm{diag}(t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1}) \subset \mathrm{Sp}(V)$. The first important fact is the following:

Proposition 4.2.1. *There are only $2^k\binom{n}{k}$ fixed points for the action of T on $\mathrm{IGr}(k, V)$. They are parametrized by subsets $I \subset \{\pm 1, \ldots, \pm n\}$ such that $I \cap (-I) = \emptyset$.*

Proof. Let V be generated by the vectors $v_n, \ldots, v_1, v_{-1}, \ldots, v_{-n}$, and the skew-symmetric form given by $w = \sum_{i=1}^n x_i \wedge x_{-i}$. A point $p \in \mathrm{IGr}(k, V) \subset \mathrm{Gr}(k, V)$ can be represented by a $2n \times k$ matrix (a_{ij}), $i = n, \ldots, 1, -1, \ldots, -n$, and $j = 1, \ldots, k$. It is well known that an open covering of the Grassmannian is given by the set of matrices where a certain square $k \times k$ minor is invertible. If the minor is identified by $I \subset \{\pm 1, \ldots, \pm n\}$, by inverting the I-th minor, on the corresponding open subset (which we denote by U_I) coordinates are given by the remaining entries of the matrix (x_{ij}), $i \notin I$, $j = 1, \ldots, k$.

The open subset U_I is fixed by T. Let $I = \{h_1 > \cdots > h_k\}$; then one has:

$$(t_n, \ldots, t_1)((x_{ij})) = (\frac{t_i}{t_{h_j}} x_{ij}),$$

where by convention, if $i < 0$, then $t_i = t_{-i}^{-1}$. Therefore, it is clear that the only fixed point for each U_I is $p_I := (0_{ij})$. Such a point p_I belongs to the isotropic Grassmannian if and only if $I \cap (-I) = \emptyset$. ☐

Definition 4.2.2. *We will say that a subset $I \subset \{\pm 1, \ldots, \pm n\}$ is admissible if $I \cap (-I) = \emptyset$.*

Therefore, by the Bialynicki-Birula decomposition (refer to Section 1.2), by fixing a general one dimensional torus $\tau \subset T$, we can associate to each fixed point p_I (where $I \subset \{\pm 1, \ldots, \pm n\}$ is admissible) a Schubert variety σ'_I, which is the closure of a Schubert cell isomorphic to an affine space. The Schubert cell is defined as the set of points which accumulate towards p_I under the action of τ. The condition that τ needs to satisfy in order to give the decomposition is that it acts with a finite number of fixed points. For instance, let

$$\tau = \mathrm{diag}(t^n, \ldots, t, t^{-1}, \ldots, t^{-n}) \subset T. \tag{4.6}$$

Lemma 4.2.3. *The one dimensional torus τ acts with a finite number of fixed points over* $\mathrm{IGr}(k, V)$*.*

Proof. We use the same notation as in the proof of the previous lemma. Then we have

$$t((x_{ij})) = (\frac{t^i}{t^{h_j}} x_{ij}).$$

As $i \notin I = \{h_1 > \cdots > h_k\}$, then $(i - h_j) \neq 0$ for all the coefficients x_{ij}. Therefore, as before, the only fixed points are the points where $(x_{ij}) = (0_{ij})$, i.e. the points p_I for I admissible. $\square$

Remark 4.2.4. From the previous proof, it is clear that a 1-dimensional torus $\tau' \subset T$ acts on $\mathrm{IGr}(k, V)$ with a finite number of fixed points if and only if, whenever $i \neq j$, the eigenvalue of τ' for the eigenvector v_i is different from that of τ' for the eigenvector v_j.

Remark 4.2.5. The symplectic Grassmannian $\mathrm{IGr}(k, V)$ is a homogeneous variety under the action of $\mathrm{Sp}(V)$, and as such it has a natural Bruhat decomposition in orbits under the action of a Borel subgroup of $\mathrm{Sp}(V)$. It turns out that the Bruhat decomposition and the Bialynicki-Birula one are the same (Section 1.2).

This has a very important consequence for us (see Remark 1.2.4). It implies that if a fixed point p_J belongs to a Schubert variety σ'_I, then actually $\sigma'_J = \overline{B.p_J} \subset \overline{B.p_I} = \sigma'_I$. This fact is crucial when trying to compute the equivariant cohomology of $\mathrm{IGr}(k, V)$, as we will see. However, this property will not hold in the bisymplectic case, and it is one of the main reasons why computing the equivariant cohomology for $\mathrm{I_2Gr}(k, V)$ becomes more difficult.

As p_I is fixed, the torus T acts on the vector space $T_I := T_{\mathrm{IGr}(k,V),p_I}$. Let $\epsilon_i \in \Xi(T)$ be the character of T given by $\mathrm{diag}(t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1}) \mapsto t_i$. If $i < 0$, we denote by ϵ_i the character $-\epsilon_{-i}$.

Lemma 4.2.6. *The weights of the action of T on T_I are*

$$-\epsilon_i - \epsilon_j \text{ for } i \leq j \in I \text{ and}$$

$$\epsilon_i - \epsilon_j \text{ for } i \notin I \cup (-I), \;\; j \in I.$$

Proof. The tangent space to the Grassmannian is equal to $\mathcal{U}^* \otimes \mathcal{Q}$. Therefore, the weights of $T_{\mathrm{Gr}(k,V),p_I}$ are

$$-\epsilon_i - \epsilon_j \text{ for } i,j \in I\ ,$$

$$\epsilon_i - \epsilon_j \text{ for } i \notin I \cup (-I),\ \ j \in I.$$

The normal bundle of the isotropic Grassmannian inside $\mathrm{Gr}(k,V)$ is the bundle $\wedge^2\mathcal{U}^*$, whose weights are

$$-\epsilon_i - \epsilon_j \text{ for } i < j \in I.$$

Therefore, by using the exact normal sequence, one obtains the statement. □

The weights of the action of τ are easily deduced from Lemma 4.2.6; indeed, under the identification $\Xi(\tau) \cong \mathbb{Z}$, it is sufficient to notice that $\epsilon_i \mapsto i$ under the morphism $j^* : \Xi(T) \to \Xi(\tau)$ induced by the natural inclusion $j : \tau \to T$. The tangent space $T_{\sigma'_I,p_I}$ is the τ-invariant subspace of T_I whose weights with respect to τ are negative.

Definition 4.2.7. *From now on, we will say that $\xi \in \Xi(T)$ is τ-positive (and we will denote it by $\xi > 0$) if $j^*(\xi) > 0$.*

Therefore, given a certain subset I, it is not difficult to compute the codimension of σ'_I as:

$$\operatorname{codim}(\sigma'_I) = \#\{(i,j) \text{ s.t. } i \notin I \cup (-I)\ , j \in I \text{ and } i > j\} +$$

$$+\#\{(i,j) \in I \times I \text{ s.t. } |j| \leq i\}.$$

In order to compute the equivariant cohomology of $\mathrm{IGr}(k,V)$, which we will do in the next section, we still need two results. The first concerns T-invariant curves, and the second is about the inclusions of fixed points in Schubert varieties $p_J \in \sigma'_I$. Recall that T-invariant curves are rational curves whose intersection with the fixed locus has cardinality 2; these two fixed points will be denoted by p_0 and p_∞.

Lemma 4.2.8. *There is only a finite number of T-invariant curves inside* $\mathrm{IGr}(k,V)$. *They are of two types:*

type α: curves with $p_0 = p_I$ and $p_\infty = p_J$, where $\#(I \cap J) = k-1$;

type β: curves with $p_0 = p_I$ and $p_\infty = p_J$, where $\#(I \cap J) = k-2$, $I - J = \{a_1, a_2\}$, $J - I = \{-a_2, -a_1\}$.

Proof. Without any loss of generality, we can suppose that one of the fixed points is p_I, with $I = \{1,\dots,k\}$. We have already seen that the action of the torus T is given in local coordinates in a neighbourhood of p_I by:

$$(t_n,\dots,t_1)((x_{ij})) = (\frac{t_i}{t_j}x_{ij}).$$

The image $T((x_{ij}))$ is a curve if and only if it depends on one parameter. This can happen only in the following two situations:

type α: all the coordinates (x_{ij}) are zero except for one, for instance $x_{\bar{i},\bar{j}}$. Then the other fixed point in the curve is p_J, with $J = (I - \{\bar{j}\}) \cup \{\bar{i}\}$;

type β: the coordinates $x_{-ij} = -x_{-ji}$ are the only ones different from zero, with $i, j \in I$. This is the second type of curves, with $a_1 = i$, $a_2 = j$.

Therefore, one sees that for each fixed point, there is a finite number of T-invariant curves passing through it. $\square$

Definition 4.2.9. *Let $I = \{a_k \geq \cdots \geq a_1\}$ and $J = \{b_k \geq \cdots \geq b_1\}$. If $a_i \geq b_i$ for $1 \leq i \leq k$, then we will say that I is greater or equal than J, and we will denote this by $I \geq J$.*

Lemma 4.2.10. *For two admissible subsets I and J, the fact that $I \geq J$ is equivalent to $p_J \in \sigma'_I$.*

Proof. First, let us suppose that $p_J \in \sigma'_I$. If $\mathring{\sigma}'_I$ denotes the Schubert cell corresponding to I, let us consider

$$U_{\sigma'_I} := U_I \cap \mathring{\sigma}'_I = \{p \in U_I \cap \mathrm{IGr}(k, V) \text{ s.t. } \lim_{t\to\infty} t(p) = p_I \text{ for } t \in \tau\}.$$

Actually, we have that

$$U_{\sigma'_I} = \mathring{\sigma}'_I.$$

Indeed, the action of τ on the Plücker coordinate q_I corresponding to I is given by

$$t(q_I) = t^{\sum_{i\in I} i} q_I.$$

Therefore τ fixes the complementary U_I^c of $U_I = \{q_I \neq 0\}$, thus implying that $U_I^c \cap \mathring{\sigma}'_I = \emptyset$.

As already stated, the torus τ acts as:

$$t((x_{ij})) = (\frac{t^i}{t^{a_j}} x_{ij}).$$

Suppose that there exists $(\bar{i}, \bar{j})$ such that $\bar{i} > a_{\bar{j}}$ and $\bar{i}$ is maximal with the property that $x_{\bar{i}\bar{j}} \neq 0$. Then $\lim_{t\to\infty} t((x_{ij})) \neq p_I$ for $t \in \tau$ because $\lim_{t\to\infty} t(x_{\bar{i}\bar{j}}) \neq 0$. Therefore $U_{\sigma'_I}$ is contained in the set

$$\{(x_{ij}) \in U_I \cap \mathrm{IGr}(k, V) \text{ s.t. if } \bar{i} > a_{\bar{j}} \text{ then } x_{\bar{i}\bar{j}} = 0\}.$$

As a consequence, it is easy to verify that if the closure of $U_{\sigma'_I}$ (which is equal to σ'_I) contains a fixed point p_J, then $I \geq J$.

Now, fix I, J such that $I \geq J$. Let us suppose that we have proved that there exists a chain of T-invariant curves $p_J \to p_{J_1} \to \cdots \to p_I$ such that the curve

$p_{J_h} \to p_{J_{h+1}}$ has $p_{J_{h+1}}$ as accumulation point for the action of τ. This implies that $p_{J_h} \in \sigma'_{J_{h+1}}$ for any h. Moreover by Remark 4.2.5, if $p_{J_h} \in \sigma'_{J_{h+1}}$, then the entire orbit of p_{J_h} is contained in $\sigma'_{J_{h+1}}$. This implies that $\sigma'_{J_h} \subset \sigma'_{J_{h+1}}$; by induction we get that $p_J \in \sigma'_I$.

As a consequence, we just need to prove the existence of a chain of curves from p_J to p_I. This will be done in several steps.

1. First, we can suppose that $I \cap J = \emptyset$, i.e. we can replace I with $I' = I - J$ and J with $J' = J - I$.
2. If $a_i \notin -J$, then we can use a curve of type α and replace J with $J' = (J - \{b_i\}) \cup \{a_i\}$. Therefore we can suppose that $I = -J$.
3. If $a_i = -b_i$, we can use a curve of type α to replace J with $J' = (J - \{-a_i\}) \cup \{a_i\}$, i.e. we can suppose that for any i, $a_i \neq -b_i$.
4. If $a_i > 0$, and $b_j = -a_i$ for $j < i$, we can use a curve of type α to replace J with $J' = (J - \{-a_i\}) \cup \{a_i\}$; here the condition $j < i$ is essential in order to ensure that $I \geq J'$.
5. From the previous steps, we can suppose that $I = -J$, and if $b_j = -a_i$ and $a_i > 0$, then $j > i$. Moreover, the a_i's and the b_i's are ordered as a decreasing sequence. This implies that $a_1 > 0$, $a_1 = -b_k$, and $b_1 > 0$, $b_1 = -a_k$. Therefore, there exists a curve of type β which goes from p_J to $p_{J'}$, with $J' = (J - \{b_1, b_k\}) \cup \{a_1, a_k\}$. One can apply this step until $I = J'$, and conclude the proof.

$\square$

Poincaré duality

It is possible to understand Poincaré duality for Schubert varieties. In fact, the following holds:

Proposition 4.2.11. *The basis given by the classes of Schubert varieties σ'_I, for I admissible, is Poincaré self-dual; the dual of σ'_I is σ'_{-I}.*

Proof. Recall that the Bialynicki-Birula decomposition Theorem actually provides two cell decompositions, one for τ and the other for τ^{-1} (see Remark 1.2.2). Let us denote by $\sigma'^{+}_{\bullet} = \sigma'_{\bullet}$ the first one and by $\sigma'^{-}_{\bullet}$ the second one.

Let us fix $\sigma'^{+}_{I} = \sigma'_I$, with I admissible. Consider the subgroup $\mathrm{SL}_2^n \subset \mathrm{Sp}_{2n}$ of diagonal 2×2 blocks which fixes the symplectic Grassmannian. One can find a rational curve γ inside SL_2^n such that, under its conjugation action, the torus τ is sent to τ^{-1}. For instance, one can take the rotation

$$\gamma(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}^n .$$

This curve, when acting on $\mathrm{IGr}(k, V)$, sends p_I to p_{-I} and gives a rational equivalence between the class of $\sigma_I'^+$ and the class of $\sigma_{-I}'^-$ for every I admissible.

Now, consider $\sigma_I'^+$ and $\sigma_J'^+$ of complementary dimension inside $\mathrm{IGr}(k, V)$. If $\sigma_I'^+ \cap \sigma_J'^+ = \emptyset$, then $\sigma_I'^+ \sigma_J'^+ = 0$ in cohomology. Therefore, let us assume that $\sigma_I'^+ \cap \sigma_J'^+ \neq \emptyset$. By Remark 4.2.5, we can assume that $\sigma_J'^+ \subset \sigma_I'^+$. It is easy to verify that $\operatorname{codim}(\sigma_I'^+) = \dim(\sigma_{-I}'^+)$, and therefore we have that $\operatorname{codim}(\sigma_J'^+) = \operatorname{codim}(\sigma_{-I}'^+)$.

As a consequence, if $J \neq -I$, then $\sigma_{-I}'^+ \cap \sigma_J'^+ = \emptyset$, and this implies that $\sigma_{-I}'^- \cap \sigma_J'^+ = \emptyset$ as well. As a result, in cohomology we have

$$\sigma_I'^+ \sigma_J'^+ = \sigma_{-I}'^- \sigma_J'^+ = 0.$$

On the other hand it is straightforward to verify that

$$\sigma_I'^+ \sigma_{-I}'^+ = \sigma_{-I}'^- \sigma_{-I}'^+ = 1$$

because $\sigma_{-I}'^- \cap \sigma_{-I}'^+ = p_I$. □

4.2.2. Equivariant cohomology: an easy situation

In this section we study the equivariant cohomology for the torus T of $\mathrm{IGr}(k, V)$. We begin by recalling some basic facts about equivariant cohomology. A reference for this subject is [Bri98]; the general results we will cite can be found in [GKM98] or [Bri97].

Let X be a smooth variety on which a torus $T \cong (\mathbb{C}^*)^n$ acts with finitely many fixed points $X^T = \{p_1, \ldots, p_r\}$. Denote by $\Xi(T) \cong \mathbb{Z}^n$ the character group of T. Moreover, let $\tau \in T$ be a general 1-dimensional torus such that its fixed locus is equal to X^T; then the Bialynicki-Birula decomposition for τ provides varieties σ_{p_i} for all $1 \leq i \leq r$ which are a basis for the ordinary cohomology $\mathrm{H}^*(X, \mathbb{Z})$.

The equivariant cohomology ring $\mathrm{H}^*_T(X)$ is an algebra over the polynomial ring $\mathrm{H}^*_T(\mathrm{pt}) \cong \mathbb{C}[\Xi(T)] = (\Xi(T)) \otimes_{\mathbb{Z}} \mathbb{C}$ via the push-forward map of the natural inclusion of a point pt inside X. An additive basis for this algebra is given by the (equivariant) classes $[\sigma_{p_i}]$ for $1 \leq p_i \leq r$.

Denote by $\mathrm{H}^*(X) := \mathrm{H}^*(X, \mathbb{C})$. The pullback map $i^* : \mathrm{H}^*_T(X) \to \mathrm{H}^*_T(X^T)$ of the natural inclusion $i : X^T \to X$ is injective; therefore

$$\mathrm{H}^*_T(X) = \Xi(T) \otimes_{\mathbb{Z}} \mathrm{H}^*(X) \cong \Xi(T) \otimes_{\mathbb{Z}} \bigoplus_{p_i} \mathbb{C}[\sigma_{p_i}]$$

can be seen as a subring of

$$\mathrm{H}^*_T(X^T) \cong \Xi(T) \otimes_{\mathbb{Z}} \mathrm{H}^*(X^T) \cong \Xi(T) \otimes_{\mathbb{Z}} \bigoplus_{p_i} \mathbb{C} p_i \cong \mathbb{C}[\Xi(T)]^{\oplus r}.$$

Via this inclusion, we will denote by $f_{\sigma_i} \in \mathbb{C}[\Xi(T)]^{\oplus r}$ the class of $[\sigma_i] \in \mathrm{H}^*_T(X)$, and by $f_{\sigma_i}(p_j) = (i \circ i_j)^*[\sigma_i]$, where $i_j : p_j \to X^T$ is the natural inclusion. Clearly, if $\epsilon_1, \ldots, \epsilon_n$ is a $\mathbb{Z}$-basis of $\Xi(T)$, then $f_{\sigma_i}(p_j) \in \mathrm{H}_T^*(p_j)$ is a polynomial in $\epsilon_1, \ldots, \epsilon_n$. Therefore, in order to understand the equivariant cohomology of X, we need to find the polynomials $f_{\sigma_i}(p_j)$. The following results hold:

Theorem 4.2.12. *The polynomials $f_{\sigma_i}(p_j)$ satisfy the following properties:*

1. *$f_{\sigma_i}(p_j)$ is a homogeneous polynomial of degree* $\operatorname{codim}(\sigma_i)$*;*
2. *$f_{\sigma_i}(p_j) = 0$ if $p_j \notin \sigma_i$;*
3. *$f_{\sigma_i}(p_j)$ is the product of the T-characters of the normal bundle $N_{\sigma_i/X,p_j}$ whenever σ_i is smooth at p_j;*
4. *If there exists a T-equivariant curve between p_j and p_k whose character is χ, then χ divides $f_{\sigma_i}(p_j) - f_{\sigma_i}(p_k)$ for $1 \leq i \leq r$.*

Theorem 4.2.13. *If there is only a finite number of T-invariant curves inside X, then the equivariant cohomology $\mathrm{H}_T^*(X)$ is the subset of $\mathbb{C}[\Xi(T)]^{\oplus r}$ consisting of elements $f = (f_1, \ldots, f_r)$ satisfying the last condition in Theorem 4.2.12, i.e.:*

$$\text{if there exists a } T\text{-equivariant curve between } p_j \text{ and } p_k \\ \text{whose character is } \chi, \text{ then } \chi \text{ divides } f_j - f_k. \tag{4.7}$$

Moreover, from the equivariant cohomology, it is possible to recover the ordinary cohomology $\mathrm{H}^*(X)$:

Theorem 4.2.14. *The classical cohomology $\mathrm{H}^*(X)$ can be recovered from the equivariant cohomology $\mathrm{H}_T^*(X)$ as*

$$\mathrm{H}^*(X) \cong \mathrm{H}_T^*(X)/(\epsilon_1, \cdots, \epsilon_n).$$

Therefore, the finiteness of the number of T-invariant curves inside $\mathrm{IGr}(k, V)$ (Lemma 4.2.8) and Theorem 4.2.13 give:

Theorem 4.2.15. *The relations in (4.7) are enough to determine the equivariant cohomology of* $\mathrm{IGr}(k, V)$*.*

One should be careful: being able to determine the equivariant cohomology of $\mathrm{IGr}(k, V)$ does not imply that we are able to identify the equivariant classes $[\sigma_I']$ in general. However, as $\mathrm{IGr}(k, V)$ is homogeneous, the following proposition ensures that we can in this case:

Proposition 4.2.16. *Let $X = G/P$ be a homogeneous rational variety under the action of a simple group G. Then the maximal torus T inside a Borel subgroup $B \subset G$ acts with a finite number of fixed points on X. Moreover, if there is only a finite number of T equivariant curves, then the equivariant classes of Schubert varieties inside $\mathrm{H}_T^*(X)$ are determined by the relations $1, 2, 3$ in Theorem 4.2.12.*

Proof. The first part of the theorem is classical from general theory on homogeneous varieties. For the second part, consider a 1-dimensional torus $\tau \subset T$ which acts positively on the set of positive roots of B. Let us denote by σ_i, $i = 1, \ldots, r$ the Schubert varieties with respect to the action of τ. By Remark 4.2.5, these varieties are equal to the orbits $B.p_i$ for $i = 1, \ldots, r$, where the p_i's are the T-fixed points inside X. Moreover, if $p_j \in \sigma_i$, then $\sigma_j \subset \sigma_i$ and therefore $\operatorname{codim}(\sigma_j) > \operatorname{codim}(\sigma_i)$.

The polynomials $f_{\sigma_i}(p_j)$ of the equivariant class of a Schubert variety σ_i satisfy the relations in Theorem 4.2.12. Moreover, by the finiteness of the number of T-invariant curves, we have that if two T-invariant curves with characters χ_1, χ_2 meet p_i, then χ_1 and χ_2 must be prime to each other.

Let us consider an element

$$g = (g_1 \ldots, g_r) \in \mathrm{H}^*_T(X) \subset \mathbb{C}[\Xi(T)]^{\oplus r}$$

satisfying the relations $1, 2, 3$ in Theorem 4.2.12. Then $f_{\sigma_i} - g$ is zero over all points p_j such that $\operatorname{codim}(\sigma_j) \leq \operatorname{codim}(\sigma_i)$. We want to prove that $f_{\sigma_i} - g = 0$. Let us suppose that $f_{\sigma_i} - g \neq 0$. Then we can find a point $p_h \in \sigma_i$ such that $(f_{\sigma_i} - g)(p_h) \neq 0$ and $\operatorname{codim}(\sigma_h)$ is minimal. Condition (4.7) and the finiteness of the number of T-invariant curves implies that $(f_{\sigma_i} - g)(p_h)$ must be divisible by $f_{\sigma_h}(p_h)$; but

$$\deg((f_{\sigma_i} - g)(p_h)) = \operatorname{codim}(\sigma_i) < \operatorname{codim}(\sigma_h) = \deg(f_{\sigma_h}(p_h)),$$

which gives a contradiction. □

In the following, we present an effective way to compute explicitly the equivariant classes of the Schubert varieties of the symplectic Grassmannian. The method uses an inductive argument, and was already used in [Man16]. We will always suppose $k < n$. Moreover, if no ambiguity arises, we will denote σ_I and p_I by I. The first step is the following lemma:

Lemma 4.2.17. *The Schubert variety σ'_H, $H = \{n, \ldots, n-k+2, n-k\}$, corresponds to the unique generator of $\rho(\mathrm{IGr}(k, V))$. Moreover, in the equivariant cohomology, it is represented by the degree 1 polynomials*

$$f_H(I) = \sum_{i \in I} -\epsilon_i + \sum_{i=1}^{k} (\epsilon_{n-i+1}).$$

Proof. It is well known that $\rho(\mathrm{IGr}(k, V)) \cong \mathbb{Z}$. Moreover, by counting the number of positive weights of T_H with respect to τ, one obtains $\operatorname{codim}(\sigma'_H) = 1$. The first assertion follows. For the second one, notice that by Proposition 4.2.16 f_H is uniquely determined by requiring that $f_H(\{n, \ldots, n-k+2, n-k+1\}) = 0$, and $f_H(H) = \epsilon_{n-k+1} - \epsilon_{n-k}$, which are satisfied by the formula in the statement, together with condition (4.7). □

The inductive method proceeds as follows. Let us fix a Schubert variety σ'_I. Then

$$\text{if } p_J \notin \sigma'_I, \text{ then } f_I(J) = 0.$$

Moreover, $f_I(I)$ is just the product of the (positive) τ-weights of T_I (because σ'_I is smooth at p_I). Notice that these two assertions are general, and will hold for the bisymplectic Grassmannian as well.

Finally, the polynomial $f_I(\cdot)(f_H(\cdot) - f_H(H))$ has support over the points $p_J \in \sigma'_I$, $J \neq I$. By applying Lemma 4.2.10, we obtain:

$$f_I(\cdot)(f_H(\cdot) - f_H(I)) = \sum_{J \in I_{-1}} a_{I,J} f_J(\cdot), \tag{4.8}$$

where $I_{-1} = \{J \text{ s.t. } I \geq J, \text{ and } \operatorname{codim}(\sigma'_J) = \operatorname{codim}(\sigma'_I) + 1\}$. The condition on the codimension is a consequence of the fact that $\deg(f_J) = \operatorname{codim}(\sigma'_J)$. If we are able to determine the coefficients $a_{I,J}$ for $J \in I_{-1}$, then by induction on the dimension of σ'_I we can recover all the equivariant cohomology of $\mathrm{IGr}(k, V)$. As a matter of fact, the following results hold:

Lemma 4.2.18. *Let $I \geq J$ and*

$$\operatorname{codim}(\sigma'_I) = \operatorname{codim}(\sigma'_J) - 1. \tag{4.9}$$

Then three cases are possible:

1. *$I - J = \{1\}$ and $J - I = \{-1\}$;*
2. *$I - J = \{i\}$ and $J - I = \{i - 1\}$;*
3. *$I - J = \{i, 1 - i\}$ and $J - I = \{-i, i - 1\}$.*

Proof. A consequence of the proof of Lemma 4.2.10 is that, as $p_J \in \sigma'_I$, there exists a chain of T-invariant curves joining p_J to p_I. The condition on the codimension then implies that the chain is actually composed of one single curve. This curve is either of type α either of type β. For later use, let

$$W_I = \{\tau\text{-positive weights in the normal bundle at } p_I\},$$

and similarly for W_J.

1. The curve is of type α, $I - J = \{i\}$ and $J - I = \{-i\}$, $i > 0$. We want to prove that the codimension condition implies $i = 1$. Let us compute $\operatorname{codim}(\sigma'_I) - \operatorname{codim}(\sigma'_J)$. We have:

   $$W_I - W_J \subset \{\epsilon_x - \epsilon_i, -\epsilon_j - \epsilon_i \text{ for } x \notin (I \cup -I), i \neq j \in I\},$$

   $$W_J - W_I \subset \{\epsilon_x + \epsilon_i, -\epsilon_j + \epsilon_i \text{ for } x \notin (I \cup -I), i \neq j \in I, \text{ and } 2\epsilon_i\}.$$

   As $i > 0$, if $x - i > 0$ (respectively $-j - i > 0$), then $x + i > 0$ ($-j + i > 0$). Moreover, $2\epsilon_i \in W_J$ and $2\epsilon_i \notin W_I$. Suppose that $i > 1$. Then there exists

$x \notin (I \cup -I)$ such that $i > x > -i$, or there exists $i \neq j \in I$ such that $i > j > -i$. In the first case $\epsilon_x - \epsilon_i < 0$ and $\epsilon_x + \epsilon_i > 0$; in the second case $-\epsilon_j - \epsilon_i < 0$ and $-\epsilon_j - \epsilon_i > 0$. In both cases Equation (4.9) cannot be satisfied. Therefore $i = 1$.

2. The curve is of type α, $I - J = \{i\}$ and $J - I = \{j\}$, $i > j \neq -i$. We want to prove that the codimension condition implies $j = i - 1$. We have:

$$W_I - W_J \subset \{\epsilon_x - \epsilon_i, -\epsilon_h - \epsilon_i, -\epsilon_h + \epsilon_j \text{ for } x \notin (I \cup -I), x \neq \pm j, h \in I\},$$

$$W_J - W_I \subset$$

$$\subset \{\epsilon_x - \epsilon_j, -\epsilon_h - \epsilon_j, -\epsilon_h + \epsilon_i \text{ for } x \notin (I \cup -I), x \neq \pm i, h \in I, \text{ and } \epsilon_i - \epsilon_j\}.$$

As $i > j$, if $x - i > 0$ (respectively $-h - i > 0$, $-h + j > 0$), then $x + i > 0$ ($-h + i > 0$, $-h + i > 0$). Moreover, $\epsilon_i - \epsilon_j \in W_J$ and $\epsilon_i - \epsilon_j \notin W_I$. Suppose that $i - 1 > j$. Then there exists $x \notin (I \cup I)$ such that $i > x > j$, or there exists $h \in I$ such that $i > h > j$, or there exists $h \in I$ such that $i > -h > j$. In the first case $\epsilon_x - \epsilon_i < 0$ and $\epsilon_x - \epsilon_j > 0$; in the second case $-\epsilon_h + \epsilon_j < 0$ and $-\epsilon_h + \epsilon_i > 0$; in the third case $-\epsilon_h - \epsilon_i < 0$ and $-\epsilon_h - \epsilon_j > 0$. In all the cases Equation (4.9) cannot be satisfied. Therefore $j = i - 1$.

3. The curve is of type β, $I - J = \{i, j\}$ and $J - I = \{-i, -j\}$, $|i| > |j|$, $i > 0$. We want to prove that the codimension condition implies $j = -i + 1$. First of all, if $j > 0$, then we can connect p_J to p_I with a chain composed of two curves of type α (sending first $\{i\} \to \{-i\}$ and then $\{j\} \to \{-j\}$); this implies that $\operatorname{codim}(\sigma'_J) - \operatorname{codim}(\sigma'_I) \geq 2$. So we can suppose that $j < 0$. We have:

$$W_I - W_J \subset \left\{ \begin{array}{c} \epsilon_x - \epsilon_i, \epsilon_x - \epsilon_j, -\epsilon_h - \epsilon_i, -\epsilon_h - \epsilon_j \\ \text{for } x \notin (I \cup -I), h \in I, h \neq \pm i, \pm j \text{ and } -2\epsilon_j \end{array} \right\},$$

$$W_J - W_I \subset \left\{ \begin{array}{c} \epsilon_x + \epsilon_i, \epsilon_x + \epsilon_j, -\epsilon_h + \epsilon_i, -\epsilon_h + \epsilon_j \\ \text{for } x \notin (I \cup -I), h \in I, h \neq \pm i, \pm j \text{ and } 2\epsilon_i, \epsilon_i + \epsilon_j \end{array} \right\}.$$

Moreover, $2\epsilon_i, \epsilon_i + \epsilon_j \in W_J$ and $2\epsilon_i, \epsilon_i + \epsilon_j \notin W_I$, while $-2\epsilon_j \in W_I$ and $-2\epsilon_j \notin W_J$. It is possible to verify by hand that Equation (4.9) is true if and only if for any x and any h as above the following conditions are satisfied:

$$x \notin \{(-j, i) \cup (-i, j)\} \text{ and } h \notin \{(-j, i) \cup (-i, j)\}.$$

Therefore $j = -i + 1$.

□

Proposition 4.2.19. *Suppose I and J are as in Lemma 4.2.18. Then:*

1. $a_{I,J} = 1$;
2. $a_{I,J} = 1$;
3. $a_{I,J} = 2$.

Proof. As, for all $J \in I_{-1}$, the codimension of σ'_J is the same, there exist no $J, J' \in I_{-1}$ such that $p_J \in \sigma'_{J'}$. Therefore $f_{J'}(J) = 0$. As a consequence, by applying Equation (4.8) to p_J, we have:

$$f_I(J)(f_H(J) - f_H(I)) = a_{I,J} f_J(J).$$

This determines $f_I(J)$ modulo the constant $a_{I,J}$ (as by induction we already know $f_J(J)$). To determine this constant, we use the fact that there is a T-invariant curve between p_J and p_I, and so, if χ is the character of this curve,

$$\chi \text{ divides } f_I(I) - f_I(J).$$

1. In this case $\chi = f_H(J) - f_H(H) = 2\epsilon_1$. Let f be the polynomial of maximal degree in which the variable ϵ_1 does not appear, such that $f_I(I) = fp$ and $f_I(J) = fq$ for suitable polynomials p, q. Then it is straightforward to see that:

$$(f_I(I) - f_I(J))/f = \prod_{x \notin (I \cup -I), x > 1} (\epsilon_x - \epsilon_1) \prod_{h \in I, h > 1} (-\epsilon_h - \epsilon_1) +$$

$$-a_{I,J} \prod_{x \notin (I \cup -I), x > 1} (\epsilon_x + \epsilon_1) \prod_{h \in I, h < -1} (-\epsilon_h + \epsilon_1).$$

   As this polynomial is equal to zero when we let $2\epsilon_1 = 0$, we obtain $a_{I,J} = 1$.
2. In this case $j = i - 1$, $\chi = f_H(J) - f_H(H) = \epsilon_i - \epsilon_j$. Let f be the polynomial of maximal degree in which the variables ϵ_i, ϵ_j do not appear, such that $f_I(I) = fp$ and $f_I(J) = fq$ for suitable polynomials p, q. Then it is straightforward to see that:

$$(f_I(I) - f_I(J))/f = \prod_{x \notin (I \cup -I), x > i} (\epsilon_x - \epsilon_i) \prod_{h \in I, h < -i} (-\epsilon_h - \epsilon_i) +$$

$$-a_{I,J} \prod_{x \notin (I \cup -I), x > i} (\epsilon_x - \epsilon_j) \prod_{h \in I, h < -i} (-\epsilon_h - \epsilon_j).$$

   As this polynomial is equal to zero when we let $\epsilon_i = \epsilon_j$, we obtain $a_{I,J} = 1$.
3. In this case $j = -i + 1$, $\chi = f_H(J) - f_H(H) = 2(\epsilon_i + \epsilon_j)$. Let f be the polynomial of maximal degree in which the variables ϵ_i, ϵ_j do not appear, such that $f_I(I) = fp$ and $f_I(J) = fq$ for suitable polynomials p, q. Then it is straightforward to see that:

$$(f_I(I) - f_I(J))/f = \prod_{x \notin (I \cup -I), x > i} (\epsilon_x - \epsilon_i) \prod_{x \notin (I \cup -I), x > -i} (\epsilon_x - \epsilon_j)$$

$$\prod_{h \in I, h < -i} (-\epsilon_h - \epsilon_i) \prod_{h \in I, h \leq -j} (-\epsilon_h - \epsilon_j) +$$

$$-\frac{a_{I,J}}{2} \prod_{x \notin (I \cup -I), x > -i} (\epsilon_x + \epsilon_i) \prod_{x \notin (I \cup -I), x > i} (\epsilon_x + \epsilon_j)$$
$$\prod_{h \in I, -j \neq h \leq i} (-\epsilon_h + \epsilon_i) \prod_{h \in I, h < -j} (-\epsilon_h + \epsilon_j).$$

Notice that in this case we obtain a factor $\frac{1}{2}$ because χ is twice the character $\epsilon_i + \epsilon_j$ of the normal bundle $N_{\sigma_I / X, p_I}$. As the polynomial is equal to zero when we let $\epsilon_i = -\epsilon_j$, we obtain $a_{I,J} = 2$.

□

Putting everything together, we obtain:

Theorem 4.2.20. *Equation* (4.8) *and Proposition 4.2.19 determine inductively the equivariant classes of Schubert varieties inside* $\mathrm{IGr}(k, V)$.

Further, from the equivariant cohomology, one can recover the classical cohomology of $\mathrm{IGr}(k, V)$.

4.3. Cohomology of $\mathrm{I}_2\mathrm{Gr}(k, 2n)$

In this section we study the cohomology structure of the bisymplectic Grassmannians. The situation is more involved than that of $\mathrm{IGr}(k, V)$. As an application of the existence of an action of a torus with a finite set of fixed points, we compute the Betti numbers of $\mathrm{I}_2\mathrm{Gr}(k, V)$. Then we compute explicitly the cohomology of $\mathrm{I}_2\mathrm{Gr}(2, 6)$ as an application of the study of its equivariant cohomology.

4.3.1. Torus action, weights, Schubert varieties

From the results in Section 4.1.1, and more precisely from Remark 4.1.7, one can suppose that the pencil Ω of skew-symmetric forms defining $\mathrm{I}_2\mathrm{Gr}(k, V)$ is generated by ω_1 and ω_2, with:

$$\omega_1 = \sum_{i=1}^{n} x_i \wedge x_{-i}$$

$$\omega_2 = \sum_{i=1}^{n} \lambda_i x_i \wedge x_{-i}$$

where $\langle x_i, x_{-i} \rangle = (K_i)^*$ for $1 \leq i \leq n$, and the λ_i's are all distinct. The torus T introduced in the previous section fixes Ω, and therefore acts on $\mathrm{I}_2\mathrm{Gr}(k, V)$.

Proposition 4.3.1. *There are* $2^k \binom{n}{k}$ *fixed points for the action of* T *on* $\mathrm{IGr}(k, V)$. *They are parametrized by the subsets* $I \subset \{\pm 1, \ldots, \pm n\}$ *such that* $I \cap (-I) = \emptyset$.

Proof. The bisymplectic Grassmannian is contained in $\mathrm{IGr}(k, V)$, and the torus acting on the two varieties is the same. Then, the lemma follows from Proposition 4.2.1 and from the fact that all the p_I's also belong to $\mathrm{I_2Gr}(k, V)$. $\square$

Therefore, by the Bialynicki-Birula decomposition, by fixing a general 1-dimensional torus $\tau \subset T$, we can associate to each fixed point p_I a variety σ_I (see Section 1.2). We will refer to σ_I (respectively its cell) as a Schubert variety (resp. cell) of the bisymplectic Grassmannian, in analogy with the notation in the homogeneous case.

Lemma 4.3.2. *The* 1*-dimensional torus* τ *introduced in* (4.6) *acts with a finite number of fixed points over* $\mathrm{IGr}(k, V)$.

Proof. This is a consequence of Lemma 4.2.3. $\square$

As p_I is fixed, the torus T acts on the tangent space $T_I := T_{\mathrm{IGr}(k,V),p_I}$. We have:

Lemma 4.3.3. *The weights of the action of* T *on* T_I *are*

$$-2\epsilon_i \text{ for } i \in I \text{ and}$$

$$\epsilon_i - \epsilon_j \text{ for } i \notin I \cup (-I), \;\; j \in I.$$

Proof. This is similar to the proof of the analogous result for $\mathrm{IGr}(k, V)$. Only notice that in this case the normal bundle of the bisymplectic Grassmannian inside $\mathrm{Gr}(k, V)$ is the bundle $(\wedge^2 \mathcal{U}^*)^{\oplus 2}$, whose weights are

$$-\epsilon_i - \epsilon_j \text{ for } i \neq j \in I.$$

Therefore, by using the exact normal sequence, one obtains the statement. $\square$

The weights of the action of τ are easily deduced from Lemma 4.3.3; indeed, under the identification $\Xi(\tau) \cong \mathbb{Z}$, it is sufficient to notice that $\epsilon_i \mapsto i$ under the morphism $\Xi(T) \to \Xi(\tau)$ induced by the inclusion. The tangent space T_{σ_I,p_I} is the τ-invariant subspace of T_I whose weights with respect to τ are negative. Therefore, given a certain subset I, it is possible to compute the codimension of σ_I as:

$$\mathrm{codim}(\sigma_I) = \#\{(i,j) \text{ s.t. } i \notin I \cup (-I)\,, j \in I, \text{ and } j > i\} + \#\{j \in I \text{ s.t. } j < 0\}.$$

4.3.2. Betti numbers

From what we have said in the previous section, it is a priori possible to determine the Betti numbers of the variety $\mathrm{I_2Gr}(k, V)$ by hand. However, in this section, we give a recursive formula to do so, which just requires knowing the Betti numbers of, for instance, $\mathrm{I_2Gr}(2, 4) \cong \mathbf{P}^1 \times \mathbf{P}^1$ and of the projective space in order to recover those of all bisymplectic Grassmannians. Afterwards, we will give a geometric interpretation for the terms in the recursive formula.

The decomposition of $\mathrm{I}_2\mathrm{Gr}(k,V)$ in Schubert cells isomorphic to an affine space implies that the odd Bettinumbers are all equal to zero. Let $\{b^i_{k,n}\}_i$ be the even Betti numbers of $\mathrm{I}_2\mathrm{Gr}(k,2n)$ (where i is the codimension). We will denote by $S_{k,n}$ the sequence of integers:

$$S_{k,n} = (b^0_{k,n}, \dots, b^{\dim(\mathrm{I}_2\mathrm{Gr}(k,2n))}_{k,n}, 0, \dots, 0, \dots).$$

Of course, the decomposition of $\mathrm{I}_2\mathrm{Gr}(k,2n)$ in Schubert cells isomorphic to the affine space and whose closures are the σ_I's implies that $b^i_{k,n}$ is equal to the number of subsets I such that $\mathrm{codim}(\sigma_I) = i$. We will denote by $[h]$ the shift on the right by h. For instance, $S_{k,n}[1] = (0, b^0_{k,n}, \dots, b^{\dim(\mathrm{I}_2\mathrm{Gr}(k,2n))}_{k,n}, 0, \dots, 0, \dots)$.

Theorem 4.3.4. *The following recursive formula holds for the Betti numbers of* $\mathrm{I}_2\mathrm{Gr}(k,2(n+1))$*:*

$$S_{k,n+1} = S_{k,n}[k] + S_{k-1,n} + S_{k-1,n}[1+2(n+1-k)]. \tag{4.10}$$

Proof. Let us consider the set $\mathcal{I}_{k,n+1}$ of subsets $I \subset \{\pm 1, \dots, \pm n+1\}$ such that $p_I \in \mathrm{I}_2\mathrm{Gr}(k,2(n+1))$. Then we have a partition of $\mathcal{I}_{k,n+1}$ into three parts which are the images of three morphisms:

$$\eta_1 : \mathcal{I}_{k,n} \to \mathcal{I}_{k,n+1} \ , \ I \mapsto I,$$

$$\eta_2 : \mathcal{I}_{k-1,n} \to \mathcal{I}_{k,n+1} \ , \ I \mapsto I \cup \{n+1\},$$

$$\eta_3 : \mathcal{I}_{k-1,n} \to \mathcal{I}_{k,n+1} \ , \ I \mapsto I \cup \{-n-1\}.$$

Moreover:

- η_1. if $\mathrm{codim}_{\mathrm{I}_2\mathrm{Gr}(k,2n)}(\sigma_I) = c$, then $\mathrm{codim}_{\mathrm{I}_2\mathrm{Gr}(k,2(n+1))}(\sigma_I) = c+k$. This is because the normal bundle of σ_I at p_I in $\mathrm{I}_2\mathrm{Gr}(k,2(n+1))$ contains the k additional τ-positive weights $\epsilon_{n+1} - \epsilon_i$ for $i \in I$;
- η_2. if $\mathrm{codim}_{\mathrm{I}_2\mathrm{Gr}(k-1,2n)}(\sigma_I) = c$, then $\mathrm{codim}_{\mathrm{I}_2\mathrm{Gr}(k,2(n+1))}(\sigma_{I\cup\{n+1\}}) = c$. This is because the normal bundle of $\sigma_{I\cup\{n+1\}}$ at $p_{I\cup\{n+1\}}$ in $\mathrm{I}_2\mathrm{Gr}(k,2(n+1))$ contains no additional τ-positive weights;
- η_3. if $\mathrm{codim}_{\mathrm{I}_2\mathrm{Gr}(k-1,2n)}(\sigma_I) = c$, then $\mathrm{codim}_{\mathrm{I}_2\mathrm{Gr}(k,2(n+1))}(\sigma_{I\cup\{-n-1\}}) = c+1+2(n+1-k)$. This is because the normal bundle of $\sigma_{I\cup\{-n-1\}}$ at $p_{I\cup\{-n-1\}}$ in $\mathrm{I}_2\mathrm{Gr}(k,2(n+1))$ contains the $1+2(n+1-k)$ additional τ-positive weights $2\epsilon_{n+1}$, $\epsilon_i - \epsilon_{n+1}$ for $i \notin (I \cup -I)$.

Therefore, by using the partition, one obtains the three factors in the RHS of Equation (4.10). □

Example 4.3.5. We give here a list of examples of Betti numbers of bisymplectic Grassmannians for small k, n:

$$
\begin{aligned}
S_{2,3} &= (1,1,2,4,2,1,1,0,\dots);\\
S_{3,4} &= (1,1,2,6,6,6,6,2,1,1,0,\dots);\\
S_{4,5} &= (1,1,2,7,8,12,18,12,8,7,2,1,1,0,\dots);\\
S_{2,4} &= (1,1,2,2,3,6,3,2,2,1,1,0,\dots).
\end{aligned}
$$

Let

$$S_{k,n}(t) = \sum_i b^i_{k,n} t^i$$

be the Hilbert polynomial of $\mathrm{I}_2\mathrm{Gr}(k,2n)$. The following equalities are given by the isomorphisms $\mathrm{I}_2\mathrm{Gr}(1,2n) \cong \mathbf{P}^{2n-1}$ and $\mathrm{I}_2\mathrm{Gr}(n,2n) \cong (\mathbf{P}^1)^n$:

$$S_{n,n}(t) = (1+t)^n\ ,$$

$$S_{1,n}(t) = \frac{1-t^{2n}}{1-t}\ ,$$

$$S_{0,n}(t) = 1.$$

Equation (4.10) gives the following identity:

$$S_{k,n+1}(t) = t^k S_{k,n}(t) + (1+t^{1+2(n+1-k)}) S_{k-1,n}(t).$$

The following propositions allow to compute the Hilbert polynomial in two additional cases.

Proposition 4.3.6. *The Hilbert polynomial of* $\mathrm{I}_2\mathrm{Gr}(2,2n)$ *is equal to*

$$S_{2,n}(t) = \frac{(1-t^{2n-1})(1-t^{2n-2})}{(1-t)(1-t^2)} + (n-1)t^{2n-3}.$$

Proof. We prove the formula by induction on n. When $n=2$, $S_{2,2}(t) = 1+t+t^2+t = (1+t)^2$. Suppose the formula is true for n, we prove it for $n+1$. We have

$$
\begin{aligned}
S_{2,n+1}(t) = t^2\frac{(1-t^{2n-1})(1-t^{2n-2})}{(1-t)(1-t^2)} + (n-1)t^{n-1} + \frac{(1+t^{2n-1})(1-t^{2n})}{1-t} = \\
= \frac{(1-t^{2n+1})(1-t^{2n})}{(1-t)(1-t^2)} + nt^{2n-1}.
\end{aligned}
$$

□

Remark 4.3.7. This case is particularly easy because $\mathrm{I}_2\mathrm{Gr}(2,2n)$ is a codimension 2 complete intersection inside $\mathrm{Gr}(2,2n)$; all its Betti numbers except the middle term can be derived from those of $\mathrm{Gr}(2,2n)$ by applying Lefschetz hyperplane theorem.

Proposition 4.3.8. *The Hilbert polynomial of* $\mathrm{I}_2\mathrm{Gr}(k, 2k+2)$ *is equal to*

$$S_{k,k+1}(t) = \sum_{i+j=k} t^i(1+t)^i(1+t^3)^j$$

Proof. Let us define

$$S(x,y,t) = \sum_{k,m\geq 0} x^k y^m S_{k,k+m}(t).$$

Then we have

$$S(x,y,t) = \sum_{k,m\geq 0} x^k y^m S_{k,k+m}(t) = \sum_{k\geq 0} x^k(1+t)^k + \sum_{k,m\geq 0} x^k y^{m+1} S_{k,k+m+1}(t) =$$

$$= \sum_{k\geq 0} x^k(1+t)^k + \sum_{k,m\geq 0} x^k y^{m+1} t^k S_{k,m}(t) + (1+t^{2m+3}) \sum_{k,m\geq 0} x^k y^{m+1} S_{k-1,k+m}(t) =$$

$$= \sum_{k\geq 0} x^k(1+t)^k + yS(xt,y,t) + xS(x,y,t) - \sum_{k\geq 0} x^{k+1}(1+t)^k + xtS(x,yt^2,t) - \sum_{k\geq 0} x^{k+1}t(1+t)^k,$$

from which we obtain the relation:

$$S(x,y,t) = 1 + yS(xt,y,t) + xS(x,y,t) + xtS(x,yt^2,t).$$

Let us rewrite $S(x,y,t) = \sum_{i\geq 0} S^{[i]}(x,t)y^i$. The previous relation gives:

$$S^{[1]}(x,t) = \frac{S^{[0]}(xt,t)}{1-x-xt^3}.$$

By developing this expression, and using the fact that $S^{[0]}(x,t) = \sum_{i\geq 0}(1+t)^i x^i$, one obtains the statement of the proposition. $\square$

Remark 4.3.9. The series $S(x,y,t)$ and the relation it satisfies can be used a priori to find inductively the Hilbert polynomials of bisymplectic Grassmannians for $1 \leq k \leq n$.

Remark 4.3.10. By using the recursive formula, it is possible to prove that for any $1 < k < n$ we have: $b^0_{k,n} = 1$, $b^1_{k,n} = 1$, $b^2_{k,n} = 2$. In particular,

$$\rho(\mathrm{I}_2\mathrm{Gr}(k,V)) \cong \mathbb{Z}.$$

The following lemma will be useful in the sequel.

Lemma 4.3.11. *The Schubert variety* σ_H, $H = \{n, \dots, n-k+2, n-k\}$, *corresponds to the unique generator of* $\rho(\mathrm{I}_2\mathrm{Gr}(k,V))$. *Moreover, in the equivariant cohomology, it is represented by the degree* 1 *polynomials*

$$f_H(I) = \sum_{i\in I} -\epsilon_i + \sum_{i=1}^{k} \epsilon_{n-i+1}.$$

Remark 4.3.12. The Schubert variety σ_H is a hyperplane section of $\mathcal{O}(1)$ inside $\mathrm{I}_2\mathrm{Gr}(k,V)$. Indeed, it is the restriction to $\mathrm{I}_2\mathrm{Gr}(k,V)$ of the hyperplane section $\sigma'_H \subset \mathrm{IGr}(k,V)$.

Proof. The codimension of σ_H is 1, and $\rho(\mathrm{I}_2\mathrm{Gr}(k,V)) \cong \mathbb{Z}$, therefore the first assertion follows. Moreover the same argument used in the proof of Proposition 4.2.16 shows that f_H in the equivariant cohomology is uniquely determined by the fact that $f_H(\{n,\dots,n-k+2,n-k+1\}) = 0$ and $f_H(H) = -\epsilon_{n-k} + \epsilon_{n-k+1}$. These conditions, together with condition (4.7), are satisfied by the formula in the statement. □

Remark 4.3.13. For what concerns Poincaré duality, in cohomology the basis given by the classes of Schubert varieties σ_I, for I admissible, is not self-dual, as it was the case for the symplectic Grassmannian. One can still prove that σ_I is rationally equivalent to σ^-_{-I} as in the proof of Proposition 4.2.11; this implies that $\sigma_I\sigma_{-I} = 1$. However, if $J \neq -I$ and $\mathrm{codim}(\sigma_I) = \dim(\sigma_J)$, one cannot conclude that $\sigma_I\sigma_J = 0$; indeed, refer to Remark 4.3.22.

4.3.3. Equivariant cohomology

In order to compute the equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(k,V)$, we need to understand what are the T-invariant curves in $\mathrm{I}_2\mathrm{Gr}(k,V)$.

Lemma 4.3.14. *There is only a finite number of T-invariant curves inside $\mathrm{I}_2\mathrm{Gr}(k,V)$. They are all the curves of type α.*

Proof. The set of T-invariant curves inside $\mathrm{I}_2\mathrm{Gr}(k,V)$ is contained in the set of T-invariant curves inside $\mathrm{IGr}(k,V)$. Among these, all the curves of type α belong to the bisymplectic Grassmannian too, while the curves of type β do not. □

As a consequence of Lemma 4.3.14 and of Theorem 4.2.13, we obtain:

Theorem 4.3.15. *The relations in Theorem 4.2.12 are enough to determine the equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(k,V)$.*

It would be nice to have a way to compute effectively this equivariant cohomology, as in the case of the symplectic Grassmannian. However, the bisymplectic case presents some difficulties which are harder to deal with, as it is illustrated by the following explicit example.

The example of $\mathrm{I}_2\mathrm{Gr}(2,6)$

Let us determine the Schubert classes in the equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(2,6)$. We will do so by using the inductive method introduced to study the equivariant cohomology of $\mathrm{IGr}(k,2n)$. The problem for the bisymplectic Grassmannian is that it may very well happen that $p_J \in \sigma_I$ even though $\mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I)$

(contrary to $\mathrm{IGr}(k, 2n)$ where $p_J \in \sigma'_I$ implies necessarily $\sigma'_J \subset \sigma'_I$!). Therefore, a priori, we do not know over which set to carry the summation in (4.8).

In order to understand this problem, recall that if $p_J \in \sigma'_I$, then $I \geq J$. As a consequence, if $p_J \in \sigma_I$, then also $I \geq J$. Moreover $\geq$ is a partial order relation on the admissible subsets of $\{\pm 1, \dots, \pm n\}$ of cardinality k. We define the relation $\geq_\in$ on admissible subsets: $I \geq_\in J$ if and only if there exist admissible subsets $J = J_1, J_2, \dots, J_u = I$ such that $p_{J_i} \in \sigma_{J_{i+1}}$ for $i = 1, \dots, u-1$. This relation is by construction reflexive and transitive. Moreover, it is skew-symmetric because if $I \neq J$, $I \geq_\in J$ and $J \geq_\in I$, then this would imply that $J \geq I \geq J$ and $I \neq J$, which is a contradiction by the definition of $\geq$. As a result, $\geq_\in$ is a partial order relation on admissible subsets of $\{\pm 1, \dots, \pm n\}$.

Lemma 4.3.16. *There exist polynomials $a_{I,J} \in \mathbb{C}[\epsilon_1, \dots, \epsilon_n]$ of degree $\mathrm{codim}(\sigma_J) - \mathrm{codim}(\sigma_I) - 1$ such that*

$$f_I(\,\cdot\,)(f_H(\,\cdot\,) - f_H(I)) = \sum_{J \in I_{\geq_\in -1}} a_{I,J} f_J(\,\cdot\,), \tag{4.11}$$

where $I_{\geq_\in -1} = \{J \text{ s.t. } I \geq_\in J, \text{ and } \mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I) + 1\}$.

Proof. We already know that

$$f_I(\,\cdot\,)(f_H(\,\cdot\,) - f_H(I)) = \sum_J a_{I,J} f_J(\,\cdot\,)$$

for some polynomials $a_{I,J}$ because the classes of Schubert varieties generate the equivariant cohomology over $\Xi(T)$. We want to prove that $a_{I,J} = 0$ if $J \notin I_{\geq_\in -1}$. Indeed, let $L \notin I_{\geq_\in -1}$ be a subset which is maximal for the partial order relation $\geq_\in$ such that $a_{I,L} \neq 0$. Then, by evaluating the previous equation at p_L, we obtain

$$0 = a_{I,L} f_L(L).$$

But $f_L(L)$ is the product of the weights of the normal bundle of σ_L at p_L, and we have $f_L(L) \neq 0$. This gives a contradiction. The assertion on the degree of $a_{I,J}$ is a consequence of the fact that $f_L(\bullet)$ is a homogeneous polynomial of degree $\mathrm{codim}(\sigma_L)$ for any admissible L. $\square$

A consequence of the lemma is that in general we are looking for coefficients $a_{I,J}$'s which are not constants, but actual polynomials; determining even one of them may need the use of a lot of relations, and not only, as for the symplectic Grassmannian, essentially two.

A second problem which arises is the fact that it becomes harder to determine the inclusions of the fixed points inside the Schubert varieties. In order to determine these inclusions, we need to work with the explicit equations.

Let us show it concretely. Figure 4.1 represents the graph of T-invariant curves in $\mathrm{I_2Gr}(2, 6)$: a vertex labelled by the ordered subset I corresponds to the fixed point p_I, while an arrow corresponds to a T-invariant curve between two points. The codimension increases towards the bottom of the graph. Notice that there are two couples of points which are joined by a T-invariant curve, and whose dimensions of the associated Schubert varieties are the same, which could not happen in the case of $\mathrm{IGr}(k, V)$.

Now, let us determine the relevant inclusions of fixed points for the computation of the cohomology:

Lemma 4.3.17. *Figure 4.2 represents the inclusions inside* $\mathrm{I}_2\mathrm{Gr}(2, 6)$*: an arrow goes from* p_I *to* p_J *if and only if* $p_J \in \sigma_I$ *and* $\mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I) + 1$.

Figure 4.1. – T-invariant curves in $\mathrm{I}_2\mathrm{Gr}(2, 6)$

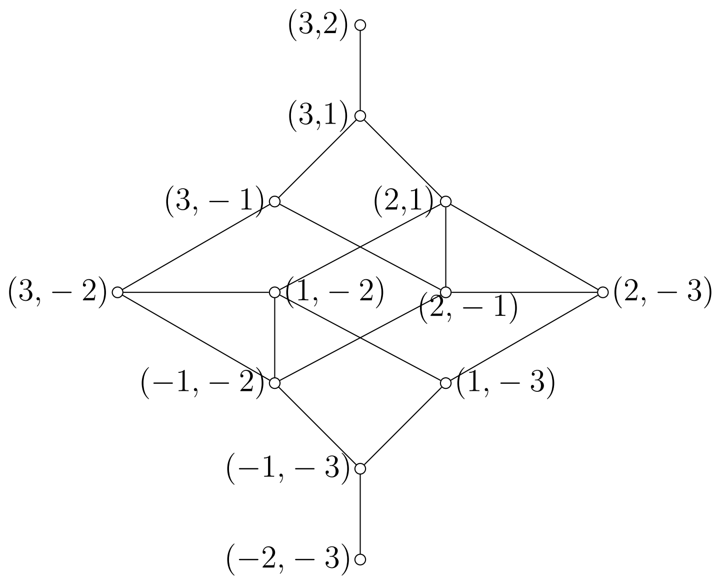
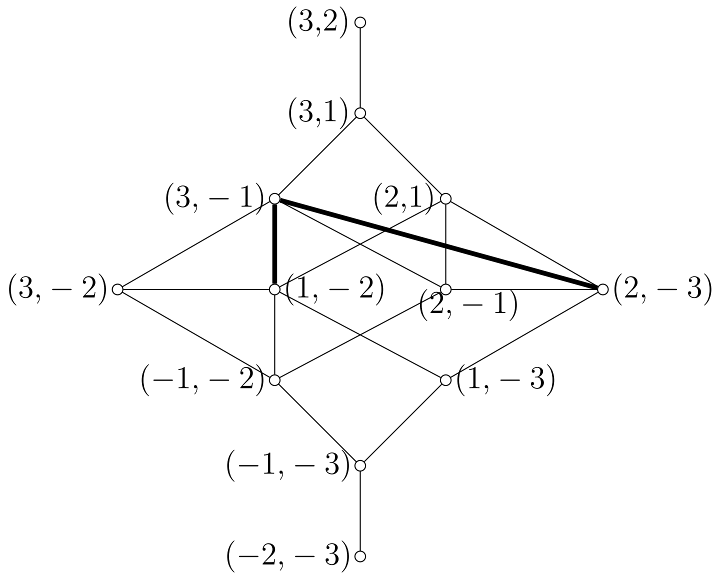


Figure 4.2. – Inclusions of fixed points in $\mathrm{I}_2\mathrm{Gr}(2,6)$



Proof. If there exists a T-invariant curve between two points p_I and p_J and $I \geq J$, then $p_J \in \sigma_I$. Moreover, if I is not greater ($\geq$) than J, then $p_J \notin \sigma_I$. We study case by case when $I \geq J$, there is no T-invariant curve between p_I and p_J, and $\mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I) + 1$. Before doing so, let us fix some notation. We denote by q_I the Plücker coordinates on the Grassmannian $\mathrm{Gr}(2,V)$. Then $\mathrm{Gr}(2,V) \subset \mathbf{P}(\wedge^2 V)$ is defined by the quadratic equations

$$q_{(a,b)}q_{(c,d)} - q_{(a,c)}q_{(b,d)} + q_{(b,c)}q_{(a,d)} = 0 \text{ for } a,b,c,d \in \{\pm 1, \dots, \pm n\}. \tag{4.12}$$

Moreover the two equations defining the bisymplectic Grassmannian (and coming from ω_1 and ω_2) are:

$$\sum_{i=1}^{n} q_{(i,-i)} = 0 \text{ and } \sum_{i=1}^{n} \lambda_i q_{(i,-i)} = 0. \tag{4.13}$$

Finally, the Schubert variety σ_I is defined by the relations

$$q_J = 0 \text{ for } I \ngeq J,$$

while in a neighbourhood of p_I we can suppose that $q_I \neq 0$. This implies, for example, that $\sigma_{3,-2}$ and $\sigma_{2,-1}$ are contained in the locus where $q_{3,-3} = q_{2,-2} = q_{1,-1} = 0$ (use (4.13)).

$p_{(1,-2)} \in \sigma_{(3,-1)}$ The relations in (4.13) give that

$$\sigma_{(3,-1)} \subset \{q_{(1,-1)} = \alpha q_{(2,-2)} \text{ and } q_{(3,-3)} = \beta q_{(2,-2)}\}$$

for $\alpha, \beta \neq 0$ depending on the λ_i's and that can be computed explicitely. The points

$$p(t) = \langle v_3 + t^{-1}v_2 + \alpha t v_1, v_{-1} + t^2 v_{-2} + \beta t v_{-3} \rangle$$

belong to $\sigma_{(3,-1)}$, and we have $\lim_{t\to\infty} p(t) = p_{(1,-2)}$.

$p_{(2,-3)} \in \sigma_{(3,-1)}$ The points $p(t) = \langle v_3 + tv_2 + \alpha t^2 v_1, v_{-1} + tv_{-2} + \beta t^2 v_{-3} \rangle$ belong to $\sigma_{(3,-1)}$, and we have $\lim_{t\to\infty} p(t) = p_{(2,-3)}$.

$p_{(1,-3)} \notin \sigma_{(3,-2)}$ Use the relations in (4.12) with $a = 1, b = -3, c = 3, d = -2$ to see that $\sigma_{(3,-2)} \subset \{q_{(1,-3)} = 0\}$.

$p_{(1,-3)} \notin \sigma_{(2,-1)}$ Use the relations in (4.12) with $a = 1, b = -3, c = 2, d = -1$ to see that $\sigma_{(2,-1)} \subset \{q_{(1,-3)} = 0\}$.

$p_{(2,-3)} \notin \sigma_{(3,-2)}$ Use the relations in (4.12) with $a = 2, b = -3, c = 3, d = -2$ to see that $\sigma_{(3,-2)} \subset \{q_{(2,-3)} = 0\}$.

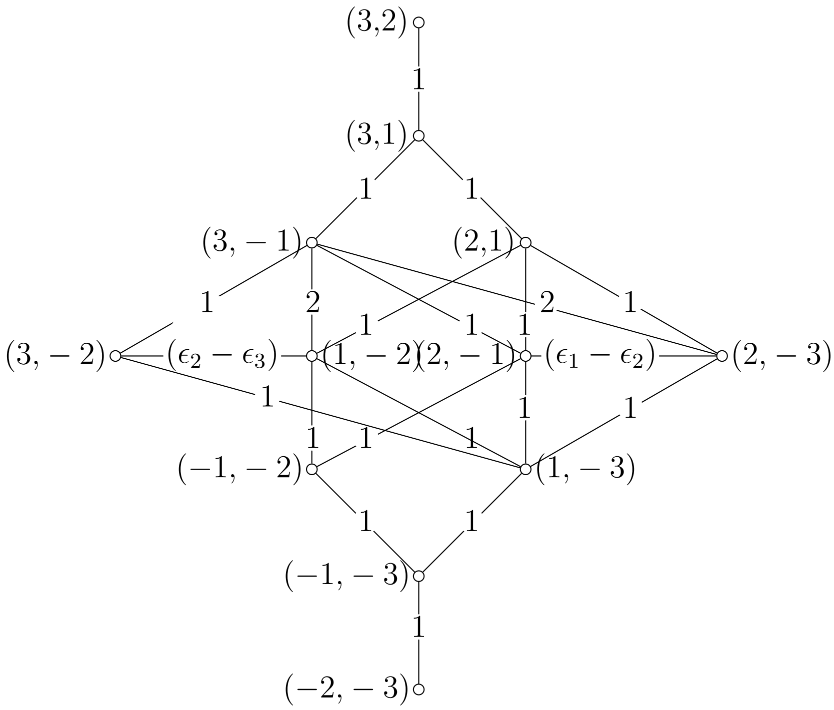
$p_{(1,-2)} \notin \sigma_{(2,-1)}$ Use the relations in (4.12) with $a = 1, b = -2, c = 2, d = -1$ to see that $\sigma_{(2,-1)} \subset \{q_{(1,-2)} = 0\}$.

$\square$

Notice that the two graphs in Figure 4.1 and Figure 4.2 are different.

Proposition 4.3.18. *The coefficients $a_{I,J}$ that appear in Equation (4.11) for $\mathrm{I}_2\mathrm{Gr}(2,6)$ are uniquely determined by the relations in Theorem 4.2.12. They are reported in Figure 4.3.*

Figure 4.3. – Coefficients $a_{I,J}$ in $\mathrm{I}_2\mathrm{Gr}(2,6)$



Proof. First suppose that there is a T-invariant curve between p_I and p_J, with $\mathrm{codim}(\sigma_J) = \mathrm{codim}(\sigma_I) + 1$, and J is a maximal element among those in $I_{\geq \epsilon - 1}$. Then, by evaluating Equation (4.11) at p_J, one determines $f_I(J)$ as a function of $a_{I,J}$; moreover, as $f_I(I) - f_I(J)$ must be a multiple of the character of the curve between p_I and p_J, it is possible to determine $a_{I,J}$, which turns out to be equal to one (essentially, it is the same computation as in the symplectic case for curves of type α). We determine the remaining coefficients one by one.

$a_{(3,-2)},$ · Let $I = (3,-2)$. The only coefficients different from zero are $a_{I,(1,-2)}$, $a_{I,(-1,-2)}$, $a_{I,(1,-3,)}$. As $p_{(1,-3)} \notin \sigma_I$, then $f_I(1,-3) = 0$, and the existence of a T-invariant curve between $p_{(1,-2)}$ and $p_{(1,-3)}$ implies that $\epsilon_2 - \epsilon_3$ divides $f_I(1,-2)$. Therefore, by applying (4.11) to $p_{(1,-2)}$, we get $a_{I,(1,-2)} = \alpha(\epsilon_2 - \epsilon_3)$ for a certain constant α. Then the existence of a T-invariant curve between p_I and $p_{(1,-2)}$ gives a relation that implies $\alpha = 1$.

By applying (4.11) to $p_{(1,-3)}$, we get $a_{I,(1,-3)} = 1$; by applying it to $p_{(-1,-2)}$ and using the T-invariant curve between $p_{(-1,-2)}$ and p_I, we get $a_{I,(-1,-2)} = 0$.

$a_{(2,-1)},$ · Let $I = (2,-1)$. The only coefficients different from zero are $a_{I,(2,-3)}$, $a_{I,(-1,-2)} = 1$, $a_{I,(1,-3,)}$. As $p_{(1,-3)} \notin \sigma_I$, then $f_I(1,-3) = 0$, and the existence of a T-invariant curve between $p_{(2,-3)}$ and $p_{(1,-3)}$ implies that $\epsilon_1 - \epsilon_2$ divides

$f_I(2,-3)$. Therefore, by applying (4.11) to $p_{(2,-3)}$, we get $a_{I,(2,-3)} = \alpha(\epsilon_1 - \epsilon_2)$ for a certain constant α. Then the existence of a T-invariant curve between p_I and $p_{(2,-3)}$ gives the relation determining $\alpha = 1$.
By applying (4.11) to $p_{(1,-3)}$, we get $a_{I,(1,-3)} = 1$.

$a_{(2,1)}$, · Let $I = (2,1)$. The only coefficients different from zero are $a_{I,(2,-1)} = 1$, $a_{I,(1,-2)} = 1$, $a_{I,(2,-3)}$. By applying (4.11) to $p_{(-2,-3)}$ and using the T-invariant curve between $p_{(2,-3)}$ and p_I, we get $a_{I,(2,-3)} = 1$.

$a_{(3,-1)}$, · Let $I = (2,1)$. The only coefficients different from zero are $a_{I,(3,-2)} = 1$, $a_{I,(2,-1)} = 1$, $a_{I,(1,-2)}$, $a_{I,(2,-3)}$. By applying (4.11) to $p_{(1,-2)}$, we get $a_{I,(1,-2)} = 2$. By applying (4.11) to $p_{(2,-3)}$, we get $a_{I,(2,-3)} = 2$.

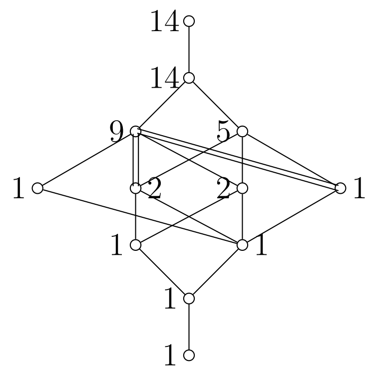
□

Thus, we have proved:

Theorem 4.3.19. *Equation* (4.11) *and Proposition 4.3.18 determine inductively the equivariant classes of all the Schubert varieties inside* $\mathrm{I}_2\mathrm{Gr}(2,6)$.

Remark 4.3.20. The constant coefficients $a_{I,J}$ determine the multiplication of a Schubert variety with the hyperplane section in the ordinary cohomology, i.e. a Pieri type formula for $\mathrm{I}_2\mathrm{Gr}(2,6)$. In particular, our computations are coherent with the fact that the degree of $\mathrm{I}_2\mathrm{Gr}(2,6)$ is 14, as we know because it is the degree of $\mathrm{Gr}(2,6)$.

Figure 4.4. – Degrees of Schubert varieties



From the equivariant cohomology, one can recover the classical cohomology of $\mathrm{I}_2\mathrm{Gr}(2,6)$ (Theorem 4.2.14). We will use the following notations:

$$\sigma_1 := \sigma_{3,1}\ ,\ \sigma_2 := \sigma_{2,1}\ ,\ \sigma_3 := \sigma_{3,-2}\ ,\ \sigma_3' := \sigma_{2,-3},$$

with

$$\deg(\sigma_1) = 14\ ,\ \deg(\sigma_2) = 5\ ,\ \deg(\sigma_3) = 1\ ,\ \deg(\sigma_3') = 1.$$

Theorem 4.3.21. *A presentation of the cohomology of the bisymplectic Grassmannian* $\mathrm{I_2Gr}(2,6)$ *is given by:*

$$\mathrm{H}^*(\mathrm{I_2Gr}(2,6),\mathbb{Z}) \cong \mathbb{Z}[\sigma_1,\sigma_2,\sigma_3,\sigma_3']/I,$$

where I *is the ideal generated by the following elements:*

$$\begin{array}{ccccc}
2\sigma_1^4-2\sigma_1^2\sigma_2-3\sigma_1\sigma_3' & , & \sigma_2\sigma_3' & , \\
\sigma_1\sigma_3-\sigma_1\sigma_3' & , & \sigma_3\sigma_3'-\sigma_1^3\sigma_3' & , \\
\sigma_2^2-\sigma_1^4+2\sigma_1^2\sigma_2+2\sigma_1\sigma_3' & , & \sigma_3^2 & , \\
\sigma_1^5-14\sigma_1^2\sigma_3' & , & \sigma_3'^2 & , \\
\sigma_2\sigma_3 & , & \sigma_1^4\sigma_3' & .
\end{array}$$

Proof. First, we prove that $\sigma_1,\sigma_2,\sigma_3,\sigma_3'$ generate the cohomology by showing that they generate all the Schubert classes σ_I. This is a consequence of the following formulas, which can be derived directly from Figure 4.4:

$$\begin{gathered}
\sigma_{(3,-1)}=\sigma_1^2-\sigma_2,\\
\sigma_{(2,-1)}=3\sigma_1\sigma_2-\sigma_1^3+\sigma_3,\\
\sigma_{(1,-2)}=\sigma_1^3-2\sigma_1\sigma_2-\sigma_3-\sigma_3',\\
\sigma_{(-1,-2)}=\sigma_1^4-2\sigma_1^2\sigma_2-3\sigma_1\sigma_3',\\
\sigma_{(1,-3)}=\sigma_1\sigma_3',\\
\sigma_{(-1,-3)}=\sigma_1^2\sigma_3',\\
\sigma_{(-2,-3)}=\sigma_1^3\sigma_3'.
\end{gathered}$$

The relations generating I involving the product of σ_1 with other classes can be derived from Figure 4.4 too. For the remaining relations, they can be derived from the following identities, which hold in the equivariant cohomology, and can be verified by computing explicitly the classes σ_I:

$$\begin{gathered}
\sigma_2^2=\sigma_2(\epsilon_3-\epsilon_1)(\epsilon_3-\epsilon_2)+\sigma_{(1,-2)}(\epsilon_3-\epsilon_1)+\sigma_{(2,-1)}(\epsilon_3-\epsilon_2)+\\
+\sigma_3'(\epsilon_3-\epsilon_2)+\sigma_1\sigma_{(1,-2)},\\
\sigma_2\sigma_3=(\epsilon_2+\epsilon_3)(\sigma_{(1,-2)}(\epsilon_2-\epsilon_3)+\sigma_{(1,-3)}),\\
\sigma_2\sigma_3'=2\epsilon_3(\sigma_3'(\epsilon_3-\epsilon_2)+\sigma_{(1,-3)}),\\
\sigma_3\sigma_3'=\sigma_{(-2,-3)},\\
\sigma_3^2=2\epsilon_2(\sigma_3(\epsilon_1+\epsilon_2)(\epsilon_2-\epsilon_1)+\sigma_{(1,-2)}(\epsilon_1+\epsilon_3)(\epsilon_3-\epsilon_2)+\\
-\sigma_{(-1,-2)}(\epsilon_3-\epsilon_2)-\sigma_{(1,-3)}(\epsilon_1+\epsilon_3)+\sigma_{(-1,-3)}),\\
\sigma_3'^2=2\epsilon_3(\sigma_3'(\epsilon_3-\epsilon_1)(\epsilon_3+\epsilon_1)+\sigma_{(1,-3)}(\epsilon_1+\epsilon_2)-\sigma_{(-1,-3)}).
\end{gathered}$$

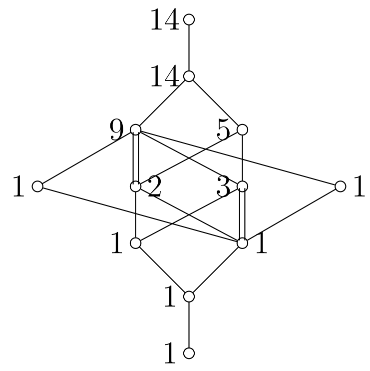
□

Remark 4.3.22. The basis given by the Schubert classes inside $\mathrm{I_2Gr}(2,6)$ is not self-dual. For instance, the non zero products of codimension 3 Schubert classes are as follows:

$$\sigma_{(3,-2)}\sigma_{(2,-3)} = 1,$$
$$\sigma_{(1,-2)}\sigma_{(2,-1)} = 1,$$
$$\sigma_{(3,-2)}\sigma_{(2,-1)} = -1.$$

A self-dual basis in codimension 3 would be given by $\sigma_{(3,-2)}, \sigma_{(2,-3)}, \sigma_{(1,-2)}, \sigma_x = \sigma_{(2,-1)} + \sigma_{(2,-3)}$. In this basis, the degree diagram is the one shown in Figure 4.5. Notice that the diagram is symmetric with respect to a central reflection; this is a consequence of the fact that the additive basis chosen in this case is self-dual.

Figure 4.5. – Degree of classes in a self-dual basis; the codimension 3 classes are, from left to right: $\sigma_{(3,-2)}, \sigma_{(2,-3)}, \sigma_x, \sigma_{(1,-2)}$



Remark 4.3.23. The group of permutations $\mathfrak{S}_n$ acts on the cohomology of the bisymplectic Grassmannians, even though it does not act on the varieties themselves; the action is a consequence of a monodromy phenomenon.

Let X be a bisymplectic Grassmannian $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ defined by the forms

$$\omega_1 = \sum_{i=1}^{n} x_i \wedge x_{-i} \text{ and } \omega_2 = \sum_{i=1}^{n} \lambda_i x_i \wedge x_{-i}.$$

Let η be an element of the group of permutations $\mathfrak{S}_n$. There exists a curve γ inside the space of pencils of bisymplectic forms that goes from $\Omega = \langle \omega_1, \omega_2 \rangle$ to $\eta.\Omega = \langle \omega_1, \eta.\omega_2 \rangle$, where

$$\eta.\omega_2 = \sum_{i=1}^{n} \lambda_{\eta(i)} x_i \wedge x_{-i}.$$

Following the curve, one obtains a continuous deformation γ such that $\gamma(0) = X = \gamma(1)$, and which sends a Schubert variety σ_I to $\eta.\sigma_I$, where the action on σ_I is induced by the one of $\mathfrak{S}_n$ on the pencils. As the cohomology is locally constant, the action on Schubert varieties induces an action in cohomology. In the following we show concretely what it means in the case when $k = 2, n = 3$.

As the irreducible representations of $\mathfrak{S}_3$ given by Schubert classes with codimension different from 3 are only 1-dimensional, we will focus on codimension 3 Schubert varieties. They admit the following explicit description:

$$\alpha_2 := \sigma_{(3,-2)} = v_{-2} \wedge \mathbf{P}(\langle v_{\pm 3}, v_{\pm 1}\rangle),$$

$$\beta_1 := \sigma_{(1,-2)} = \{x \in \mathbf{P}(\langle v_{-2}, v_{-3}\rangle) \wedge \mathbf{P}(\langle v_{\pm 1}, v_{-2}, v_{-3}\rangle) \text{ s.t. } x \neq 0\},$$
$$\beta_2 := \sigma_{(2,-1)} = \{x \in \mathbf{P}(\langle v_{-1}, v_{-3}\rangle) \wedge \mathbf{P}(\langle v_{\pm 2}, v_{-1}, v_{-3}\rangle) \text{ s.t. } x \neq 0\},$$
$$\alpha_3 := \sigma_{(2,-3)} = v_{-3} \wedge \mathbf{P}(\langle v_{\pm 2}, v_{\pm 1}\rangle).$$

Moreover, inside the cohomology of $\mathrm{I}_2\mathrm{Gr}(2,6)$ there are two more remarkable classes:

$$\alpha_1 := v_{-1} \wedge \mathbf{P}(\langle v_{\pm 3}, v_{\pm 2}\rangle),$$
$$\beta_3 := \{x \in \mathbf{P}(\langle v_{-1}, v_{-2}\rangle) \wedge \mathbf{P}(\langle v_{\pm 3}, v_{-1}, v_{-2}\rangle) \text{ s.t. } x \neq 0\}.$$

Actually, there are also classes $\alpha_{-1}, \alpha_{-2}, \alpha_{-3}, \beta_{-1}, \beta_{-2}, \beta_{-3}$, but one can prove that in cohomology $\alpha_i = \alpha_{-i}$ and $\beta_i = \beta_{-i}$ for $i = 1, 2, 3$. The action of $\mathfrak{S}_3$ on the α_i's and the β_i's is the expected one. By using the products of the codimension 3 Schubert varieties and the symmetries given by $\mathfrak{S}_3$, one can prove that

$$\alpha_1 - \alpha_2 = \beta_1 - \beta_2,$$

$$\alpha_2 - \alpha_3 = \beta_2 - \beta_3.$$

To summarize, the action of $\mathfrak{S}_3$ on $\mathrm{H}^i(\mathrm{I}_2\mathrm{Gr}(2,6),\mathbb{Z})$ is trivial if $i \neq 6$, and $\mathrm{H}^6(\mathrm{I}_2\mathrm{Gr}(2,6),\mathbb{Z})$ decomposes in the sum of two trivial representations generated by the classes of $\sigma_H^3 = \alpha_2 + 3\beta_1 + 2\beta_2 + 3\alpha_3$ and $\sigma_{(2,1)}\sigma_H = \beta_1 + \beta_2 + \alpha_3$, and one natural representation given by the action on $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

4.4. Conclusions

Even though we have shown that bisymplectic Grassmannians present interesting properties, there is still much more work to be done. We have a local description of the moduli space of these varieties, and we know which conditions to impose on the forms ω_1 and ω_2 in order to have smooth bisymplectic Grassmannians, but at the level of cohomology the work is still incomplete. We added this brief section in order to summarize what we have done, and to suggest what should or could be done in the future.

Our only general result concerning the equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(k,2n)$ is Theorem 4.3.15, which is far from being satisfying. At this level one does not even know if the equivariant classes of Schubert varieties can be determined only by using the relations in Theorem 4.2.12. What one would like to have is an analogous of Theorem 4.2.16 or, even better, of Theorem 4.2.20 to give explicit formulas to compute inductively the equivariant classes of Schubert varieties.

With the example of $\mathrm{I}_2\mathrm{Gr}(2,6)$ we have shown that, in this case, it is possible to determine the equivariant classes of Schubert varieties only by imposing the relations in Theorem 4.2.12. However, we have also brought to the surface what are the features of bisymplectic Grassmannians that make it difficult to determine the equivariant cohomology. We have identified the most important feature in the fact that a Schubert variety σ_1 may contain a fixed point p_2 even if the corresponding Schubert variety σ_2 has dimension greater or equal than σ_1 (as it happens for $\sigma_{(3,-2)}$ and $\sigma_{(1,-2)}$, or $\sigma_{(2,-1)}$ and $\sigma_{(2,-3)}$ inside $\mathrm{I}_2\mathrm{Gr}(2,6)$). Another property that should be understood better is the graph of the inclusions of fixed points inside $\mathrm{I}_2\mathrm{Gr}(k,2n)$. Also, an interesting role may be played by the $\mathfrak{S}_n$-symmetries (see Remark 4.3.23).

A possible direction to follow in the future is the study of the quantum cohomology of bisymplectic Grassmannians. For classical Grassmannians quantum cohomology is known (see [Buc03] or [Tam05]). In the quasi-homogeneous case some results have been obtained recently: for instance, for odd symplectic Grassmannians of lines one can refer to [Pec13], and more generally for non-homogeneous horospherical spaces to [GPPS18]. An effective way to find the quantum cohomology of $\mathrm{I}_2\mathrm{Gr}(k,2n)$ would be to exploit the equivariant situation. Indeed, it is possible to define an *equivariant quantum cohomology*, which is a deformation of equivariant cohomology as the quantum cohomology is a deformation of classical cohomology, and use it to recover the quantum cohomology. Of course, the combinatorics involved in the problem may be very complicated, but a priori this seems feasible.

The cohomology is not the only object of study worthy to be considered. An interesting field to explore would concern the derived category of bisymplectic Grassmannians. In particular, to study it one could use homological projective duality (see [Kuz07]), which has already proven to be a successful tool for $\mathrm{I}_2\mathrm{Gr}(2,6)$. In what follows, we sketch the situation for the Grassmannian $\mathrm{Gr}(2,6)$.

For a "sufficiently nice" projective variety $X \subset \mathbf{P}(V)$ with projective dual $Y \subset \mathbf{P}(V^*)$, derived Lefschetz type theorems can be proven to hold; they relate the derived category of linear sections of X to that of linear sections of $Y \subset \mathbf{P}(V^*)$. More precisely, if $L \subset V$ is a linear subspace, and $L^\perp \subset \mathbf{P}(V^*)$ its annihilator, one can reconstruct the derived category of $Y_{L^\perp} := Y \cap \mathbf{P}(L^\perp)$ from that of $X_L := X \cap \mathbf{P}(L)$, and vice versa.

For instance, it is known that the projective dual variety of $X = \mathrm{Gr}(2,V_6) \subset \mathbf{P}(\wedge^2 V_6)$ is the Pfaffian cubic hypersurface $Y \subset \mathbf{P}(\wedge^2 V_6^*)$ (see [Kuz16]). Then, if $L \cong \mathbb{C}^9$, one obtains that the bounded derived category $\mathcal{D}^b(Y_{L^\perp})$ of coherent sheaves on the Pfaffian cubic fourfold $Y_{L^\perp}$ admits a semiorthogonal decomposition given by

$$\mathcal{D}^b(Y_{L^\perp}) = \langle \mathcal{O}_{Y_{L^\perp}}(-3), \mathcal{O}_{Y_{L^\perp}}(-2), \mathcal{O}_{Y_{L^\perp}}(-1), \mathcal{D}^b(X_L) \rangle,$$

where X_L is a $K3$ surface of degree 14 (see [Kuz16, Theorem 10.4]). If $L \cong \mathbb{C}^{13}$, we get that the derived category of $\mathrm{I}_2\mathrm{Gr}(2,6)$ is related to that of three points on the Pfaffian hypersurface cut out by a $\mathbf{P}^1$; in [Kuz16, Corollary 10.2] a full exceptional collection for $\mathcal{D}^b(\mathrm{I}_2\mathrm{Gr}(2,6))$ is given explicitly.

Bisymplectic Grassmannians may be a good example to test Dubrovin's conjecture (see [Dub98]). The conjecture states that the big quantum cohomology of a smooth projective variety is semisimple if and only if its bounded derived category admits a full exceptional collection. Dubrovin's conjecture, which relates two a priori very different types of information, has been proved for isotropic Grassmannians of lines (see [MKM$^+$17]) and for other kinds of quasi homogeneous varieties (see [GPPS18]).

Finally, let us once more remark that there is at least another class of varieties which share some interesting features with bisymplectic Grassmannians. We refer to ortho-symplectic Grassmannians, i.e. the varieties of linear subspaces of a fixed vector space which are isotropic with respect to a skew-symmetric form and a symmetric form. For such varieties, it is possible to prove the analogous result of Proposition 4.1.4, which implies that ortho-symplectic Grassmannians admit the action of a torus with a finite number of fixed points. Therefore, one could lead a study similar to the one done for bisymplectic Grassmannians for these varieties too. We do not exclude that other *kinds* of Grassmannians may share similar properties as those possessed by bisymplectic and ortho-symplectic ones.

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Appendix

A. Euler characteristic

We refer to the notation in Chapter 2, more precisely in Section 2.1, 2.2 and 2.3. We recall that Y is the zero locus of a section of the bundle $\mathcal{F}$ inside a classical Grassmannian. The computation of the Euler characteristic of the trivial bundle $\mathcal{O}_Y$ can be done in two different ways. The first one is applicable in general only for the symplectic and odd orthogonal Grassmannians, while the second one can be applied to any generalized Grassmannian (and actually is lighter in terms of computational time).

We explain the first method in Section A.1. For the second method, we use the fact that the cohomology ring of the generalized Grassmannians is well known. In Section A.2.1 we recall Borel's presentation of the cohomology ring of orthogonal Grassmannians, and in Section A.2.2 we explain how to integrate products of Chern classes of homogeneous vector bundles.

A.1. Computation in the symplectic and odd orthogonal case

For symplectic and odd orthogonal Grassmannians one knows that one can choose multiplicative generators of the cohomology to be the Chern classes of the quotient bundle $\mathcal{Q}$. This is the tautological quotient bundle of the classical Grassmannian in which the symplectic and odd orthogonal Grassmannians embed naturally (as a reference, one can see [Tam05, Theorem 8, Theorem 12]). For instance, $\mathrm{IGr}(k, 2n)$ embeds in $\mathrm{Gr}(k, 2n)$ as the zero locus of a section of $\Lambda^2\mathcal{U}^*$. Then, suppose to be able to express the Chern classes of the bundle $\mathcal{F}$ which defines Y in $\mathrm{IGr}(k, 2n)$ in terms of the Chern classes of $\mathcal{Q}$, which live in the cohomology of $\mathrm{Gr}(k, 2n)$. Then, the computation can be made in this last space. Indeed, $[Y] = \mathrm{c}_{\mathrm{top}}(\mathcal{F})$ in the cohomology ring of IGr, $[\mathrm{IGr}] = \mathrm{c}_{\mathrm{top}}(\Lambda^2\mathcal{U}^*)$ in the cohomology ring of Gr, and

$$\int_Y (\,\cdot\,) = \int_{\mathrm{IGr}} (\,\cdot\,)[Y] = \int_{\mathrm{Gr}} (\,\cdot\,)[Y][\mathrm{IGr}].$$

Therefore, one has:

$$\chi(\mathcal{O}_Y) = \int_Y \mathrm{td}(T_Y) = \int_{\mathrm{IGr}} \frac{\mathrm{td}(T_{\mathrm{IGr}})}{\mathrm{td}(\mathcal{F})} \mathrm{c}_{\mathrm{top}}(\mathcal{F}) =$$

$$= \int_{\mathrm{Gr}} \frac{\mathrm{td}(T_{\mathrm{Gr}})}{\mathrm{td}(\mathcal{F})} \frac{\mathrm{c}_{\mathrm{top}}(\Lambda^2(\mathcal{U}^*))}{\mathrm{td}(\Lambda^2(\mathcal{U}^*))} \mathrm{c}_{\mathrm{top}}(\mathcal{F}).$$

In the second equality we have used the fact that td is multiplicative with respect to short exact sequences, and we have applied this to the normal sequence for Y:

$$0 \to T_Y \to T_{\mathrm{IGr}}|_Y \to \mathcal{F}|_Y \to 0$$

Then, as in the case of the classification of fourfolds in the classical Grassmannian, one can use the MACAULAY2 package SCHUBERT2 to do the computation. In the symplectic Grassmannian concretely there is essentially one bundle for which one needs to find the relation with the Chern classes of $\mathcal{Q}$, namely $(\mathcal{U}^{\perp}/\mathcal{U})(1)$. This is given by the exact sequence:

$$0 \to \mathcal{U}^{\perp}/\mathcal{U}(1) \to \mathcal{Q}(1) \to \mathcal{U}^{*}(1) \to 0$$

For the orthogonal Grassmannian one has to consider also the bundles which correspond to half integer sequences, and in particular the spin bundles. By relating the exterior and symmetric powers of these bundles with the "classical" bundles in the different cases, we can do the computation. For example, for $\mathrm{OGr}(n, 2n+1)$, the spin bundle is just the "square root" of $\mathcal{O}(1)$, so its first Chern class is half the one of the ample line bundle $\mathcal{O}(1)$.

A.2. Cohomology of the even orthogonal Grassmannian

The method explained earlier cannot be used in general for the even orthogonal Grassmannian (nor for the exceptional Grassmannians), because its cohomology is a little bit more complicated. In this case in fact, the Chern classes of the tautological quotient bundle $\mathcal{Q}$ do not generate multiplicatively the cohomology ring. One way to proceed is to use a "good" presentation of the cohomology, for which it is relatively easy to understand what the Chern classes of the homogeneous bundles are, and so to compute the integral in the equation mentioned above. The picture we are going to present applies actually, with appropriate modifications, to the other cases too, and to exceptional Grassmannians as well.

A.2.1. Borel's presentation of the cohomology ring

We present the multiplicative structure of the cohomology ring of the even orthogonal Grassmannian; it is usually referred to as *Borel's presentation* of the cohomology. Borel's presentation holds for any homogeneous space (as a reference see [Dem73]); when we compute the invariants for the classifications in exceptional Grassmannians, we use it as well. However, we just explicit the construction for the even orthogonal Grassmannian, as the exceptional cases are similarly treated.

The idea is to inject the cohomology ring into one which is better understood, namely that of a complete flag variety. This has the property that every irreducible homogeneous bundle has rank one. Therefore it will always be possible to split completely a not necessarily irreducible bundle and to compute its Chern class as the product of the Chern classes of the line bundles of the splitting.

To be more precise, one considers the projection map

$$\pi : \mathrm{SO}(2n)/B \to \mathrm{SO}(2n)/P_k = \mathrm{OGr}(k, 2n),$$

where B is a Borel subgroup contained in P_k. The homogeneous space $\mathrm{SO}(2n)/B$ is the complete flag $\mathrm{OF}(1, \dots, n, \mathbb{C}^{2n})$ of isotropic planes in $\mathbb{C}^{2n}$. The fiber of π over a point $[R] \in \mathrm{OGr}(k, 2n)$ is isomorphic to $\mathrm{F}(1, .., k, R) \times \mathrm{OF}(1, \dots, n - k, R^{\perp}/R)$, where the first factor is the usual complete flag in R. As π is a fibration, the pull-back morphism π^* is an injection. Therefore, after pulling back, one can work in the cohomology of the flag variety G/B, where $G = \mathrm{SO}(2n)$.

What one gains, as already anticipated, is the fact that this cohomology is well known and every homogeneous vector bundle can be split as the sum of line bundles. Indeed, let us denote by $X(T)$ the characters of the maximal torus T in B. Then one has a morphism (defined in [Dem73, Section 3]):

$$c : S_{\mathbb{Q}}[X(T)] \to \mathrm{H}^*_{\mathbb{Q}}(G/B)$$

from the symmetric algebra on the characters with rational coefficients to the rational cohomology of G/B. This morphism is surjective, and so identifies a quotient $S_{\mathbb{Q}}[X(T)]/I$ with $\mathrm{H}^*_{\mathbb{Q}}(G/B)$. The ideal I can be computed explicitly as the ideal generated by invariant polynomials (without constant terms) under the natural action of the Weyl group W of G on $S_{\mathbb{Q}}[X(T)]$. On the other hand, we will need to be able to write explicitly the morphism c. One has:

$$c(f) = \sum_{w \in W | l(w) = \deg(f)} \Delta_w(f) X^w$$

for f homogeneous in $S_{\mathbb{Q}}[X(T)]$, where X^w is the Schubert cohomology class corresponding to the Weyl element w (in the usual *Schubert presentation* of the cohomology ring of homogeneous spaces). Moreover, given a reduced decomposition of $w = s_{i_1} \dots s_{i_{l(w)}}$ in terms of simple reflections, $\Delta_w = \Delta_{s_{i_1}} \circ \dots \circ \Delta_{s_{i_{l(w)}}}$, where

$$\Delta_{s_i}(f) = \frac{f - s_i(f)}{\alpha_i},$$

α_i being the i-th simple root. The value of $\Delta_w(f)$ doesn't depend on the chosen reduced decomposition (again, refer to [Dem73, Theorem 1]).

A.2.2. Chern class of homogeneous bundles

Now, having this in mind, the last step to do the computation of the Euler characteristic is to express the Chern classes of a homogeneous vector bundle on G/P_k in $\mathrm{H}^*(G/B)$. As already pointed out, a homogeneous completely reducible bundle splits in $\mathrm{H}^*(G/B)$ as the sum of line bundles. These line bundles correspond to representations of B, as explained in Section 1.1, i.e. to elements of $X(T)$. Fix $\mathcal{F}$ on G/P_k coming from a representation V of P_k. Then one has the weight space decomposition $V = \oplus_{\mu \in X(T)} V_\mu^{m_\mu}$. As a consequence,

$$\pi^*(\mathcal{F}) \sim \oplus_{\mu \in X(T)} \mathcal{L}_\mu^{m_\mu}$$

where $\mathcal{L}_\mu$ is the line bundle corresponding to $\mu \in X(T)$. Here, the symbol $\sim$ stands for *"are the same as T-homogeneous bundles"*, which implies that they have the same Chern classes. The last ingredient is the fact that the Chern class of $\mathcal{L}_\mu$ is represented inside $S_{\mathbb{Q}}[X(T)]$ by $1+\mu$. As a consequence

$$\mathrm{Chern}(\pi^*(\mathcal{F})) = \prod_{\mu \in X(T)} (1+\mu)^{m_\mu}$$

Knowing this, we can compute the Chern class of the bundle, products of cohomology classes, integrations, etc. In particular, the integration on G/P_k of a class f of the right degree is given by computing $\Delta_{w_0}(f)$, where w_0 is the longest element in $W/W(P_k)$.

Example A.1. Here we report the code to use in order to compute the Euler characteristic for the case $(ow6)$ in Table B.4, as an example:

—Definition of $S_{\mathbb{Q}}[X(T)]$

$S = QQ[a, b, c, d, e];$

—Chern class of the tangent bundle and todd class (first terms)

$ctan = (1+a+b+d)*(1+a+b+e)*(1+a+b-d)*(1+a+b-e)*(1+a+c+d)*(1+a+c+e)*(1+a+c-d)*(1+a+c-e)*(1+c+b+d)*(1+c+b+e)*(1+c+b-d)*(1+c+b-e)*(1+a+b)*(1+a+c)*(1+b+c);$

$ctan1 = part(1, ctan);$

$ctan2 = part(2, ctan);$

$ctan3 = part(3, ctan);$

$ctan4 = part(4, ctan);$

$tdtan1 = ctan1//2;$

$tdtan2 = (ctan1 * ctan1 + ctan2)//12;$

$tdtan3 = (ctan1 * ctan2)//24;$

$tdtan4 = (-ctan1 * ctan1 * ctan1 * ctan1 + 4 * ctan1 * ctan1 * ctan2 + 3 * ctan2 * ctan2 + ctan1 * ctan3 - ctan4)//720;$

—Chern class of the vector bundle $\mathcal{F}$ and todd class (first terms)

$cF = (1 + ((a + b + c + d + e)//2)) * (1 + ((a + b + c - d - e)//2)) * (1 + ((a + b + c + d + e)//2)) * (1 + ((a + b + c - d - e)//2)) * (1 + ((a + b + c + d + e)//2)) * (1 + ((a + b + c - d - e)//2)) * (1 + ((a + b + c + d + e)//2)) * (1 + ((a + b + c - d - e)//2)) * (1 + a + b) * (1 + b + c) * (1 + a + c);$

$cF1 = part(1, cF);$

$cF2 = part(2, cF);$

$cF3 = part(3, cF);$

$cF4 = part(4, cF);$

$cF11 = part(11, cF);$

$tdF1 = cF1//2;$

$tdF2 = (cF1 * cF1 + cF2)//12;$

$tdF3 = (cF1 * cF2)//24;$

$tdF4 = (-cF1 * cF1 * cF1 * cF1 + 4 * cF1 * cF1 * cF2 + 3 * cF2 * cF2 + cF1 * cF3 - cF4)//720;$

—Definition of (the first terms of) $cr = \frac{todd(T_{OG})}{todd(\mathcal{F})}$

$cr1 = tdtan1 - tdF1;$

$cr2 = tdtan2 - tdF2 - tdF1 * cr1;$

$cr3 = tdtan3 - tdF3 - tdF1 * cr2 - tdF2 * cr1;$

$cr4 = tdtan4 - tdF4 - tdF1 * cr3 - tdF2 * cr2 - tdF3 * cr1;$

—Definition of the class int to be integrated

$int = cr4 * cF11;$

— *Computation of* $\Delta_{w_0}(int)$

$intfifteen = (int - sub(int, d => c, c => d))//(c - d);$

$intfourteen = (intfifteen - sub(intfifteen, c => b, b => c))//(b - c);$

$intthirteen = (intfourteen - sub(intfourteen, e => d, d => e))//(d - e);$

$inttwelve = (intthirteen - sub(intthirteen, b => c, c => b))//(b - c);$

$inteleven = (inttwelve - sub(inttwelve, d => -e, e => -d))//(d + e);$

$intten = (inteleven - sub(inteleven, d => c, c => d))//(c - d);$

$intnine = (intten - sub(intten, d => e, e => d))//(d - e);$

$inteight = (intnine - sub(intnine, b => c, c => b))//(b - c);$

$intseven = (inteight - sub(inteight, d => c, c => d))//(c - d);$

$intsix = (intseven - sub(intseven, a => b, b => a))//(a - b);$

$intfive = (intsix - sub(intsix, c => b, b => c))//(b - c);$

$intfour = (intfive - sub(intfive, c => d, d => c))//(c - d);$

$intthree = (intfour - sub(intfour, e => d, d => e))//(d - e);$

$inttwo = (intthree - sub(intthree, e => -d, d => -e))//(d + e);$

$intone = (inttwo - sub(inttwo, c => d, d => c))//(c - d)$

Remark A.2 (The case ($ob5$), Table B.4)**.** It is the only case in which the Euler characteristic is not equal to 2, but to 4. In order to understand better why this happens, we studied in more detail the cohomology of $\mathcal{O}_Y$ by using the Koszul complex associated to the bundle $\mathcal{F}$. The method is standard (see Section 2.1.2), the only difficulty in this case is to express the bundle $\Lambda^k \mathcal{F}^*$ as a sum of irreducible homogeneous bundles, but this can be done using the program LiE ([vCL]). What one finds is that the variety $(ob5)$ is not connected, and actually it consists of two connected components, which are therefore Calabi-Yau varieties. Two questions which can be asked is whether these two components are isomorphic, and if there is a more geometric explanation for the existence of these two components, as it is the case for the variety of zeroes of $S^2\mathcal{Q}$ in $\mathrm{Gr}(m, 2m)$.

B. Tables classification theorems

In this appendix we report all tables of the varieties we have found, as indicated in the classification theorems in Chapter 2. The labelling of the varieties follows essentially the subdivision of the classification's proofs in lemmas and propositions. In the following we explicit some rules we adopted.

For the ordinary (respectively symplectic, orthogonal) Grassmannians $\mathrm{Gr}(k, n)$ ($\mathrm{IGr}(k, m)$, $\mathrm{OGr}(k, m)$) with $k \leq n$, if $k = 2$ then the varieties are labelled by the letter (b) (resp. (sb), (ob)), if $k = 3$ they are labelled by (c) (resp. (sc), (oc)), and if $k \geq 4$, labels start with the letter (d) (resp. (sd), (od)).

Some special cases are to be taken into account. For the odd orthogonal Grassmannians, varieties inside $\mathrm{OGr}(k-2, 2k+1)$ are labelled by (ox) and varieties inside $\mathrm{OGr}(k, 2k+1)$ by (oy). For the even orthogonal Grassmannians, varieties inside $\mathrm{OGr}(k-2, 2k)$ are labelled by (ow) and varieties inside $\mathrm{OGr}(k, 2k)$ by (oz). Finally, varieties inside $\mathrm{OGr}(k-1, 2k)$ are labelled by the letter (oe).

In the tables concerning $K3$ surfaces, we have marked with $M.$ the varieties which have already been studied by Mukai. In particular: cases $M.(b10)$, $M.(oy5)$, $M.(b13)$, $M.(c6)$ are in [Muk88]; case $M.(c9)$ is in [Muk06]; cases $M.(ox5)$, $M.(d3)$ are in [Muk92]. There are many other cases which have not been examined yet, and they are worth being considered in more detail, as we intend to do in the next future.

For what concerns the notations for tables in the exceptional cases, we refer to Section 2.4. We recall that H denotes the positive generator of the Picard group of the Grassmannian, and irreducible bundles are represented by the highest weight of the irreducible representation they are associated to. We reported just the invariants interesting (in our opinion) for each dimension. On the other hand we were not able to compute some invariants for varieties of big codimension, due to computational weight $(c.w.)$.

case	bundle $\mathcal{F}$	$\mathrm{Gr}(k,n)$	$\chi(\mathcal{O}_Y)$
(a)	complete intersection of hypersurfaces	$\mathbb{P}^n$	
(b1)	$\mathcal{O}(1) \oplus \mathcal{O}(4)$	$\mathrm{Gr}(2,5)$	2
(b2)	$\mathcal{O}(2) \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,5)$	2
(b2.1)	$\mathcal{U}^*(2)$	$\mathrm{Gr}(2,5)$	2
(b4)	$S^2\mathcal{U}^* \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,6)$	2
(b5)	$\mathcal{U}^*(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,6)$	2
(b6)	$\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)^{\oplus 2}$	$\mathrm{Gr}(2,6)$	2
(b6.1)	$\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,6)$	2
(b6.2)	$\mathcal{U}^*(1)^{\oplus 2}$	$\mathrm{Gr}(2,6)$	2
(b12)	$S^3\mathcal{U}^*$	$\mathrm{Gr}(2,6)$	3
(b3)	$\mathcal{Q}(1) \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,7)$	2
(b7)	$\mathcal{O}(1)^{\oplus 5} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,7)$	2
(b8)	$S^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,7)$	2
(b8.1)	$\mathcal{U}^*(1) \oplus \mathcal{O}(1)^{\oplus 4}$	$\mathrm{Gr}(2,7)$	2
(b9)	$S^2\mathcal{U}^* \oplus \mathcal{U}^*(1) \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,7)$	2
(b10)	$\Lambda^4\mathcal{Q} \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,7)$	2
(b10.1)	$\Lambda^5\mathcal{Q} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,8)$	2
(b10.2)	$\Lambda^5\mathcal{Q} \oplus \mathcal{U}^*(1)$	$\mathrm{Gr}(2,8)$	2
(b11)	$\mathcal{O}(1)^{\oplus 8}$	$\mathrm{Gr}(2,8)$	2
(b13)	$S^2(\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(2,8)$	2
(b14)	$S^2(\mathcal{U}^*) \oplus \mathcal{O}(1)^{\oplus 5}$	$\mathrm{Gr}(2,8)$	2
(b10.3)	$\Lambda^6\mathcal{Q} \oplus S^2\mathcal{U}^*$	$\mathrm{Gr}(2,9)$	2
(b10.4)	$\Lambda^6\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{Gr}(2,9)$	2
(c1)	$\mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,6)$	2
(c1.1)	$\mathcal{Q}(1) \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(3,6)$	2
(c2)	$\Lambda^2\mathcal{Q} \oplus \mathcal{O}(1) \oplus \mathcal{O}(3)$	$\mathrm{Gr}(3,6)$	2
(c2.1)	$\Lambda^2\mathcal{Q} \oplus \mathcal{O}(2)^{\oplus 2}$	$\mathrm{Gr}(3,6)$	2
(c4)	$S^2\mathcal{U}^* \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,7)$	2
(c5)	$\Lambda^3\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 4}$	$\mathrm{Gr}(3,7)$	2
(c5.1)	$\Lambda^3\mathcal{Q} \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,7)$	2
(c6)	$(\Lambda^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,7)$	2
(c6.1)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 5}$	$\mathrm{Gr}(3,7)$	2
(c7)	$\Lambda^3\mathcal{Q} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,8)$	2
(c7.1)	$\Lambda^4\mathcal{Q} \oplus S^2\mathcal{U}^*$	$\mathrm{Gr}(3,8)$	2
(c7.2)	$\Lambda^4\mathcal{Q} \oplus (\Lambda^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{Gr}(3,8)$	2
(c7.3)	$S^2\mathcal{U}^* \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(3,8)$	2
(c7.4)	$(\Lambda^2\mathcal{U}^*)^{\oplus 3} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(3,8)$	2

Table B.1. – Fourfolds in ordinary Grassmannians, see Theorem 2.1.1

case	bundle $\mathcal{F}$	$\mathrm{Gr}(k,n)$	$\chi(\mathcal{O}_Y)$
(d6)	$S^2\mathcal{U}^* \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{Gr}(4,8)$	4
(d9)	$\Lambda^2(\mathcal{U}^*) \oplus \Lambda^3(\mathcal{U}^*) \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(4,8)$	2
(d9.1)	$\Lambda^3\mathcal{Q} \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(4,8)$	2
(d1)	$\Lambda^3\mathcal{U}^* \oplus (\Lambda^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{Gr}(4,9)$	2
(d8)	$\Lambda^2\mathcal{U}^* \oplus \Lambda^3\mathcal{Q}$	$\mathrm{Gr}(4,9)$	2
(d5)	$(S^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{Gr}(4,10)$	0
(d7)	$\Lambda^3\mathcal{Q}$	$\mathrm{Gr}(4,10)$	3
(d2)	$(\Lambda^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(5,10)$	2
(d2.1)	$\Lambda^2\mathcal{Q} \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(2)$	$\mathrm{Gr}(5,10)$	2
(d4)	$\Lambda^2\mathcal{U}^* \oplus S^2\mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{Gr}(5,11)$	2
(d3)	$(\Lambda^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(6,12)$	2
(d3.1)	$\Lambda^2\mathcal{Q} \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(6,12)$	2

Fourfolds in ordinary Grassmannians, continues from Table B.1

case	bundle $\mathcal{F}$	$\mathrm{IGr}(k,2n)$	$\chi(\mathcal{O}_Y)$
(sb0)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus \mathcal{O}(3)$	$\mathrm{IGr}(2,6)$	2
(sb0.1)	$(\mathcal{U}^\perp/\mathcal{U})(2) \oplus \mathcal{O}(1)$	$\mathrm{IGr}(2,6)$	2
(sb1)	$\Lambda^2(\mathcal{U}^\perp/\mathcal{U})(1) \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{IGr}(2,8)$	2
(sb2)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{IGr}(2,8)$	2
(sb2.1)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus \mathcal{U}^* \oplus \mathcal{O}(2)$	$\mathrm{IGr}(2,8)$	2
(sb3)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus S^2\mathcal{U}^*$	$\mathrm{IGr}(2,8)$	2
(sc0.1)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus (\Lambda^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{IGr}(3,8)$	2
(sc0.2)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus S^2\mathcal{U}^*$	$\mathrm{IGr}(3,8)$	2
(sc0.3)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus \mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{IGr}(3,8)$	2
(sc0.4)	$(\mathcal{U}^\perp/\mathcal{U})(1)^{\oplus 2} \oplus \mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{IGr}(3,8)$	2

Table B.2. – Fourfolds in symplectic Grassmannians, see Theorem 2.2.1

case	bundle $\mathcal{F}$	$\mathrm{OGr}(k,n)$	$\chi(\mathcal{O}_Y)$
(ob0)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,9)$	2
(ob0.1)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{U}^*(1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,9)$	2
(ox1)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(3)$	$\mathrm{OGr}(2,7)$	2
(ox2)	$\mathcal{T}_{+\frac{1}{2}}(2) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,7)$	2
(ox3)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 3} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{OGr}(3,9)$	2
(ox4)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{OGr}(3,9)$	2
(ox5)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus (\Lambda^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{OGr}(3,9)$	2
(ox6)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2\mathcal{U}^*$	$\mathrm{OGr}(3,9)$	2
(oy1)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{+\frac{1}{2}}(3)$	$\mathrm{OGr}(3,7)$	2
(oy1.1)	$\mathcal{T}_{+\frac{1}{2}}(2)^{\oplus 2}$	$\mathrm{OGr}(3,7)$	2
(oy2)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \Lambda^3\mathcal{U}^*$	$\mathrm{OGr}(4,9)$	2
(oy3)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus (\mathcal{T}_{+\frac{1}{2}}(1) \otimes \mathcal{U}^*)$	$\mathrm{OGr}(4,9)$	2
(oy4)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 5} \oplus \mathcal{T}_{+\frac{1}{2}}(2)$	$\mathrm{OGr}(4,9)$	2
(oy5)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 4} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{OGr}(4,9)$	2
(oy6)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \Lambda^2(\mathcal{U}^*)$	$\mathrm{OGr}(6,13)$	2

Table B.3. – Fourfolds in odd orthogonal Grassmannians, see Theorem 2.2.1

case	bundle $\mathcal{F}$	$\mathrm{OGr}(k,n)$	$\chi(\mathcal{O}_Y)$
(ob1)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(3)$	$\mathrm{OGr}(2,10)$	2
(ob1.2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(3)$	$\mathrm{OGr}(2,10)$	2
(ob2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 5}$	$\mathrm{OGr}(2,10)$	2
(ob3)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2(\mathcal{U}^*) \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{OGr}(2,10)$	2
(ob4)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 5}$	$\mathrm{OGr}(2,12)$	2
(ob4.1)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2(\mathcal{U}^*) \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{OGr}(2,12)$	2
(ob5)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(3)$	$\mathrm{OGr}(2,14)$	4
(ow1)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(3)$	$\mathrm{OGr}(2,8)$	2
(ow1.2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(3)$	$\mathrm{OGr}(2,8)$	2
(ow2)	$(\mathcal{U}^{\perp}/\mathcal{U})(1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$	2
(ow3)	$(\mathcal{T}_{+\frac{1}{2}}(1) \otimes \mathcal{U}^*) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$	2
(ow4)	$(1,1;1;1) \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{OGr}(2,8)$	2
(ow5)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,8)$	2
(ow9)	$\mathcal{T}_{+\frac{1}{2}}(2) \oplus \mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$	2
(ow10)	$\mathcal{T}_{-\frac{1}{2}}(2) \oplus \mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$	2
(ow11)	$\mathcal{U}^*(1) \oplus \mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$	2
(ow6)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 4} \oplus \Lambda^2(\mathcal{U}^*)$	$\mathrm{OGr}(3,10)$	2
(ow7)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{T}_{-\frac{1}{2}}(1)^{\oplus 2} \oplus \Lambda^2(\mathcal{U}^*)$	$\mathrm{OGr}(3,10)$	2
(ow8)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 3} \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \Lambda^2(\mathcal{U}^*)$	$\mathrm{OGr}(3,10)$	2
(ow12)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 5} \oplus \mathcal{O}(1)$	$\mathrm{OGr}(3,10)$	2
(ow13)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 3} \oplus \mathcal{T}_{-\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(1)$	$\mathrm{OGr}(3,10)$	2
(ow14)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 4} \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(3,10)$	2
(oz3)	$\mathcal{O}(\frac{1}{2}) \oplus \mathcal{O}(\frac{5}{2})$	$\mathrm{OGr}(4,8)$	2
(oz7)	$\mathcal{O}(\frac{3}{2}) \oplus \mathcal{O}(\frac{3}{2})$	$\mathrm{OGr}(4,8)$	2
(oz1)	$\mathcal{U}^*(-\frac{1}{2}) \oplus \mathcal{O}(\frac{5}{2})$	$\mathrm{OGr}(5,10)$	2
(oz4)	$\mathcal{O}(\frac{1}{2})^{\oplus 4} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{OGr}(5,10)$	2
(oz5)	$\mathcal{O}(\frac{1}{2})^{\oplus 5} \oplus \mathcal{O}(\frac{3}{2})$	$\mathrm{OGr}(5,10)$	2
(oz6)	$\mathcal{O}(\frac{1}{2}) \oplus \mathcal{U}^*(\frac{1}{2})$	$\mathrm{OGr}(5,10)$	2
(oz2)	$\mathcal{U}^*(-\frac{1}{2}) \oplus \mathcal{O}(\frac{1}{2})^{\oplus 4} \oplus \mathcal{O}(1)$	$\mathrm{OGr}(6,12)$	2

Table B.4. – Fourfolds in even orthogonal Grassmannians, see Theorem 2.2.1

case	bundle $\mathcal{F}$	$\mathrm{OGr}(n-1,2n)$	$\chi(\mathcal{O}_Y)$
(oe1)	$(\mathcal{U}^* \otimes \mathcal{L}_+) \oplus \mathcal{L}_- \oplus \mathcal{L}_-^{\otimes 2}$	$\mathrm{OGr}(3,8)$	2
(oe2)	$(\mathcal{L}_+^{\otimes 2})^{\oplus 2} \oplus \mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_-^{\oplus 2}$	$\mathrm{OGr}(3,8)$	2
(oe3)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_+^{\otimes 2}$	$\mathrm{OGr}(3,8)$	2
(oe4)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{L}_-(1) \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$	2
(oe5)	$\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{L}_- \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$	2
(oe6)	$\mathcal{O}(1) \oplus \mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_+^{\otimes 2} \oplus \mathcal{L}_- \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$	2
(oe7)	$\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+^{\otimes 2}$	$\mathrm{OGr}(3,8)$	2
(oe8)	$\mathcal{O}(2) \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+^{\oplus 2}$	$\mathrm{OGr}(3,8)$	2
(oe9)	$\mathcal{L}_+^{\otimes 2}(1) \oplus \mathcal{L}_-^{\oplus 3} \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$	2
(oe10)	$\mathcal{L}_-^{\oplus 4} \oplus \mathcal{L}_+^{\otimes 4}$	$\mathrm{OGr}(3,8)$	2
(oe11)	$\mathcal{O}(1) \oplus \mathcal{L}_+(1) \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$	2
(oe12)	$\mathcal{L}_+^{\otimes 2} \oplus \mathcal{L}_-^{\oplus 3} \oplus \mathcal{L}_+(1)$	$\mathrm{OGr}(3,8)$	2
(oe13)	$\mathcal{L}_-(1) \oplus \mathcal{L}_+^{\otimes 2} \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$	2
(oe14)	$\mathcal{O}(1) \oplus \mathcal{L}_-^{\oplus 3} \oplus \mathcal{L}_+^{\otimes 3}$	$\mathrm{OGr}(3,8)$	2
(oe15)	$\mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_+^{\otimes 3} \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$	2
(oe16)	$\mathcal{L}_-^{\oplus 5} \oplus \mathcal{L}_+^{\oplus 5}$	$\mathrm{OGr}(4,10)$	2
(oe17)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+^{\oplus 2}$	$\mathrm{OGr}(4,10)$	2

Table B.5. – Fourfolds in $\mathrm{OGr}(n-1,2n)$, see Theorem 2.2.21

case	bundle $\mathcal{F}$	$\mathrm{Gr}(k,n)$
(a)	complete intersection of hypersurfaces	$\mathbb{P}^n$
(b1)	$\mathcal{O}(4)$	$\mathrm{Gr}(2,4)$
(b2)	$\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,5)$
(b3)	$\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,5)$
(b4)	$\Lambda^2\mathcal{Q}(1)$	$\mathrm{Gr}(2,5)$
(b5)	$\mathcal{U}^*(1) \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,5)$
(b6)	$\mathcal{Q}(1) \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,6)$
(b9)	$\Lambda^3\mathcal{Q} \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,6)$
(b12)	$S^2\mathcal{U}^* \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,6)$
(b13)	$\mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,6)$
(b14)	$S^2\mathcal{U}^* \oplus \mathcal{U}^*(1)$	$\mathrm{Gr}(2,6)$
(b15)	$\mathcal{U}^*(1) \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{Gr}(2,6)$
(b8)	$\Lambda^4\mathcal{Q} \oplus \mathcal{U}^*(1)$	$\mathrm{Gr}(2,7)$
(b10)	$\Lambda^4\mathcal{Q} \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,7)$
(b16)	$(S^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,7)$
(b17)	$S^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 4}$	$\mathrm{Gr}(2,7)$
(b18)	$\mathcal{O}(1)^{\oplus 7}$	$\mathrm{Gr}(2,7)$
(b7)	$\Lambda^5\mathcal{Q} \oplus S^2\mathcal{U}^*$	$\mathrm{Gr}(2,8)$
(b11)	$\Lambda^5\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{Gr}(2,8)$
(c2)	$\mathcal{Q}(1) \oplus \Lambda^2\mathcal{Q}$	$\mathrm{Gr}(3,6)$
(c3)	$\mathcal{Q}(1) \oplus \Lambda^2\mathcal{U}^*$	$\mathrm{Gr}(3,6)$
(c4)	$\Lambda^2\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,6)$
(c8)	$\mathcal{O}(1)^{\oplus 6}$	$\mathrm{Gr}(3,6)$
(c6)	$\Lambda^3\mathcal{Q}^{\oplus 2} \oplus \mathcal{O}(1)$	$\mathrm{Gr}(3,7)$
(c7)	$\Lambda^3\mathcal{Q} \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(3,7)$
(c9)	$\mathcal{O}(1)^{\oplus 3} \oplus S^2\mathcal{U}^*$	$\mathrm{Gr}(3,7)$
(c10)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{Gr}(3,7)$
(c1)	$\Lambda^3\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(3,8)$
(c11)	$(S^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{Gr}(3,8)$
(c12)	$S^2\mathcal{U}^* \oplus (\Lambda^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{Gr}(3,8)$
(c13)	$(\Lambda^2\mathcal{U}^*)^{\oplus 4}$	$\mathrm{Gr}(3,8)$
(d3)	$(\Lambda^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(4,8)$
(d3.1)	$\Lambda^2\mathcal{U}^* \oplus \Lambda^2(\mathcal{Q}) \oplus \mathcal{O}(2)$	$\mathrm{Gr}(4,8)$
(d4)	$S^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{Gr}(4,8)$
(d1)	$\Lambda^2\mathcal{U}^* \oplus S^2\mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{Gr}(4,9)$
(d2)	$(\Lambda^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(5,10)$
(d2.1)	$\Lambda^2\mathcal{U}^* \oplus \Lambda^2\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(5,10)$

Table B.6. – Threefolds in ordinary Grassmannians, see Theorem 2.3.1

case	bundle $\mathcal{F}$	$\mathrm{IGr}(k,2n)$
(sb1)	$(\mathcal{U}^{\perp}/\mathcal{U})(2) \oplus \mathcal{U}^*$	$\mathrm{IGr}(2,6)$
(sb2)	$(\mathcal{U}^{\perp}/\mathcal{U})(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{IGr}(2,6)$
(sb3)	$(\mathcal{U}^{\perp}/\mathcal{U})(1) \oplus \mathcal{U}^*(1)$	$\mathrm{IGr}(2,6)$
(sb4)	$\Lambda^2(\mathcal{U}^{\perp}/\mathcal{U})(1) \oplus \mathcal{O}(1) \oplus \mathcal{U}^*$	$\mathrm{IGr}(2,8)$
(sb5)	$(\mathcal{U}^{\perp}/\mathcal{U})(1) \oplus \mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{IGr}(2,8)$
(sc1)	$(\mathcal{U}^{\perp}/\mathcal{U})(1) \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{IGr}(3,8)$

Table B.7. – Threefolds in symplectic Grassmannians, see Theorem 2.3.2

case	bundle $\mathcal{F}$	$\mathrm{OGr}(k,n)$
(ob1)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2\mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,9)$
(ob2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 4}$	$\mathrm{OGr}(2,9)$
(ob3)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2\mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,11)$
(ob4)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 4}$	$\mathrm{OGr}(2,11)$
(ox1)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{+\frac{1}{2}}(2)$	$\mathrm{OGr}(2,7)$
(ox2)	$\mathcal{T}_{+\frac{1}{2}}(1) \otimes \mathcal{U}^*$	$\mathrm{OGr}(2,7)$
(ox3)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,7)$
(ox4)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{U}^*(1)$	$\mathrm{OGr}(2,7)$
(ox5)	$S^2\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,7)$
(ox6)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 4} \oplus \mathcal{O}(1)$	$\mathrm{OGr}(3,9)$
(ox7)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 3} \oplus \Lambda^2\mathcal{U}^*$	$\mathrm{OGr}(3,9)$
(oy2)	$\mathcal{O}(\frac{1}{2}) \oplus \mathcal{O}(\frac{3}{2}) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(3,7)$
(oy3)	$\mathcal{O}(\frac{1}{2})^{\oplus 6} \oplus \mathcal{O}(1)$	$\mathrm{OGr}(4,9)$
(oy1)	$\mathcal{O}(\frac{1}{2}) \oplus \Lambda^2\mathcal{U}^*$	$\mathrm{OGr}(5,11)$

Table B.8. – Threefolds in odd orthogonal Grassmannians, see Theorem 2.3.2

case	bundle $\mathcal{F}$	$\mathrm{OGr}(k,n)$
(ob5)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,10)$
(ob6)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{U}^*(1)$	$\mathrm{OGr}(2,10)$
(ob7)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,10)$
(ob8)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{U}^*(1)$	$\mathrm{OGr}(2,10)$
(ob9)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,14)$
(ob10)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{U}^*(1)$	$\mathrm{OGr}(2,14)$
(ow1)	$\mathcal{T}_{+\frac{1}{2}}(2) \oplus \mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2}$	$\mathrm{OGr}(2,8)$
(ow2)	$\mathcal{T}_{+\frac{1}{2}}(2) \oplus \mathcal{T}_{-\frac{1}{2}}(1)^{\oplus 2}$	$\mathrm{OGr}(2,8)$
(ow3)	$\mathcal{T}_{+\frac{1}{2}}(2) \oplus \mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1)$	$\mathrm{OGr}(2,8)$
(ow4)	$(\mathcal{T}_{+\frac{1}{2}}(1) \otimes \mathcal{U}^*) \oplus \mathcal{T}_{+\frac{1}{2}}(1)$	$\mathrm{OGr}(2,8)$
(ow5)	$(\mathcal{T}_{+\frac{1}{2}}(1) \otimes \mathcal{U}^*) \oplus \mathcal{T}_{-\frac{1}{2}}(1)$	$\mathrm{OGr}(2,8)$
(ow6)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,8)$
(ow7)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,8)$
(ow8)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{U}^*(1)$	$\mathrm{OGr}(2,8)$
(ow9)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{U}^*(1)$	$\mathrm{OGr}(2,8)$
(ow10)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2\mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$
(ow11)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus (1,1;1;1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$
(ow12)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus (1,1;1;-1) \oplus \mathcal{O}(1)$	$\mathrm{OGr}(2,8)$
(ow13)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus (\mathcal{U}^{\perp}/\mathcal{U})(1)$	$\mathrm{OGr}(2,8)$
(ow14)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 4}$	$\mathrm{OGr}(2,8)$
(ow15)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 6}$	$\mathrm{OGr}(3,10)$
(ow16)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 5} \oplus \mathcal{T}_{-\frac{1}{2}}(1)$	$\mathrm{OGr}(3,10)$
(ow17)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 4} \oplus \mathcal{T}_{-\frac{1}{2}}(1)^{\oplus 2}$	$\mathrm{OGr}(3,10)$
(ow18)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 3} \oplus \mathcal{T}_{-\frac{1}{2}}(1)^{\oplus 3}$	$\mathrm{OGr}(3,10)$
(oz1)	$\mathcal{O}(\frac{1}{2})^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{OGr}(4,8)$
(oz2)	$\mathcal{O}(\frac{1}{2}) \oplus \mathcal{O}(2) \oplus \Lambda^4\mathcal{U}^*(-\frac{1}{2})$	$\mathrm{OGr}(5,10)$
(oz3)	$\mathcal{O}(\frac{3}{2}) \oplus \mathcal{O}(1) \oplus \Lambda^4\mathcal{U}^*(-\frac{1}{2})$	$\mathrm{OGr}(5,10)$
(oz4)	$\mathcal{O}(\frac{1}{2})^{\oplus 6} \oplus \mathcal{O}(1)$	$\mathrm{OGr}(5,10)$
(oz5)	$\mathcal{O}(\frac{1}{2})^{\oplus 6} \oplus \Lambda^5\mathcal{U}^*(-\frac{1}{2})$	$\mathrm{OGr}(6,12)$

Table B.9. – Threefolds in even orthogonal Grassmannians, see Theorem 2.3.2

case	bundle $\mathcal{F}$	$\mathrm{OGr}(n-1,2n)$
(oe1)	$\mathcal{L}_-(2) \oplus \mathcal{L}_+$	$\mathrm{OGr}(2,6)$
(oe2)	$\mathcal{L}_-^{\otimes 3} \oplus \mathcal{L}_+^{\otimes 3}$	$\mathrm{OGr}(2,6)$
(oe3)	$\mathcal{L}_-(1) \oplus \mathcal{L}_+(1)$	$\mathrm{OGr}(2,6)$
(oe4)	$\mathcal{L}_-^{\otimes 2}(1) \oplus \mathcal{L}_+^{\otimes 2}$	$\mathrm{OGr}(2,6)$
(oe5)	$\mathcal{L}_-^{\oplus 3} \oplus (\mathcal{U}^* \otimes \mathcal{L}_+)$	$\mathrm{OGr}(3,8)$
(oe6)	$\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+^{\oplus 2}$	$\mathrm{OGr}(3,8)$
(oe7)	$\mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_+^{\otimes 2} \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+^{\oplus 2}$	$\mathrm{OGr}(3,8)$
(oe8)	$\mathcal{L}_-(1) \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+^{\oplus 3}$	$\mathrm{OGr}(3,8)$
(oe9)	$\mathcal{O}(1) \oplus \mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_- \oplus \mathcal{L}_+^{\oplus 3}$	$\mathrm{OGr}(3,8)$
(oe10)	$(\mathcal{L}_-^{\otimes 2})^{\oplus 2} \oplus \mathcal{L}_+^{\oplus 4}$	$\mathrm{OGr}(3,8)$
(oe11)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1) \oplus \mathcal{L}_- \oplus \mathcal{L}_+$	$\mathrm{OGr}(3,8)$
(oe12)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_+^{\oplus 2}$	$\mathrm{OGr}(3,8)$
(oe13)	$\mathcal{L}_-^{\oplus 4} \oplus \mathcal{L}_+ \oplus \mathcal{L}_+^{\otimes 3}$	$\mathrm{OGr}(3,8)$

Table B.10. – Threefolds in $\mathrm{OGr}(n-1,2n)$, see Theorem 2.3.3

case	bundle $\mathcal{F}$	$\mathrm{Gr}(k,n)$	$\chi(\mathcal{O}_Y)$	$\deg(\mathcal{F})$
(a)	complete intersection of hypersurfaces	$\mathbb{P}^n$		
(b7)	$\mathcal{O}(1) \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,4)$	2	6
(b8)	$\mathcal{O}(2)^{\oplus 2}$	$\mathrm{Gr}(2,4)$	2	8
(b1)	$\mathcal{Q}(1) \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,5)$	2	14
(b2)	$\Lambda^2\mathcal{Q} \oplus \mathcal{O}(3)$	$\mathrm{Gr}(2,5)$	2	6
(b9)	$S^2\mathcal{U}^* \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,5)$	2	16
M.(b10)	$\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(2,5)$	2	10
(b3)	$\Lambda^3\mathcal{Q} \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$	$\mathrm{Gr}(2,6)$	2	12
(b4)	$\Lambda^3\mathcal{Q} \oplus \mathcal{U}^*(1)$	$\mathrm{Gr}(2,6)$	2	14
(b12)	$S^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{Gr}(2,6)$	2	20
M.(b13)	$\mathcal{O}(1)^{\oplus 6}$	$\mathrm{Gr}(2,6)$	2	14
(b11)	$(S^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{Gr}(2,6)$	0	32
(b5)	$S^2\mathcal{U}^* \oplus \Lambda^4\mathcal{Q}$	$\mathrm{Gr}(2,7)$	2	24
(b6)	$\mathcal{O}(1)^{\oplus 3} \oplus \Lambda^4\mathcal{Q}$	$\mathrm{Gr}(2,7)$	2	18
(c2)	$\Lambda^2\mathcal{U}^* \oplus \Lambda^2\mathcal{Q} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,6)$	2	12
(c4)	$S^2\mathcal{U}^* \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,6)$	4	32
(c5)	$(\Lambda^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{Gr}(3,6)$	2	12
M.(c6)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 4}$	$\mathrm{Gr}(3,6)$	2	16
(c3)	$S^2\mathcal{U}^* \oplus \Lambda^3\mathcal{Q}$	$\mathrm{Gr}(3,7)$	2	48
M.(c9)	$\Lambda^3\mathcal{Q} \oplus (\Lambda^2\mathcal{U}^*)^{\oplus 2}$	$\mathrm{Gr}(3,7)$	2	24
(c7)	$S^2\mathcal{U}^* \oplus \Lambda^2\mathcal{U}^* \oplus \mathcal{O}(1)$	$\mathrm{Gr}(3,7)$	2	48
(c8)	$(\Lambda^2\mathcal{U}^*)^{\oplus 3} \oplus \mathcal{O}(1)$	$\mathrm{Gr}(3,7)$	2	22
(c1)	$\Lambda^3\mathcal{Q} \oplus \Lambda^2\mathcal{U}^*$	$\mathrm{Gr}(3,8)$	2	36
(d1)	$S^2\mathcal{U}^* \oplus \Lambda^3\mathcal{U}^*$	$\mathrm{Gr}(4,8)$	4	96
(d1.1)	$S^2\mathcal{U}^* \oplus \Lambda^3\mathcal{Q}$	$\mathrm{Gr}(4,8)$	4	96
(d2)	$(\Lambda^2\mathcal{U}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(4,8)$	2	24
(d2.1)	$\Lambda^2\mathcal{U}^* \oplus \Lambda^2\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{Gr}(4,8)$	2	24
M.(d3)	$(\Lambda^2\mathcal{U}^*)^{\oplus 3}$	$\mathrm{Gr}(4,9)$	2	38

Table B.11. – Surfaces in ordinary Grassmannians, see Theorem 2.3.4

case	bundle $\mathcal{F}$	$\mathrm{IGr}(k,2n)$	$\chi(\mathcal{O}_Y)$	$\deg(\mathcal{F})$
(sb1)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus \mathcal{U}^* \oplus \mathcal{O}(2)$	$\mathrm{IGr}(2,6)$	2	12
(sb2)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus S^2\mathcal{U}^*$	$\mathrm{IGr}(2,6)$	2	24
(sb3)	$(\mathcal{U}^\perp/\mathcal{U})(1)^{\oplus 2} \oplus \mathcal{O}(1)$	$\mathrm{IGr}(2,6)$	2	24
(sb4)	$(\mathcal{U}^\perp/\mathcal{U})(1) \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{IGr}(2,6)$	2	18

Table B.12. – Surfaces in symplectic Grassmannians, see Theorem 2.3.5

case	bundle $\mathcal{F}$	$\mathrm{OGr}(k,n)$	$\chi(\mathcal{O}_Y)$	$\deg(\mathcal{F})$
(ob1)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,9)$	2	12
(ob2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,13)$	4	24
(ox1)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,7)$	2	12
(ox2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2\mathcal{U}^*$	$\mathrm{OGr}(2,7)$	2	24
(ox3)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2\mathcal{T}_{+\frac{1}{2}}(1)$	$\mathrm{OGr}(2,7)$	2	24
(ox4)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{OGr}(2,7)$	2	18
M.(ox5)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 5}$	$\mathrm{OGr}(3,9)$	2	34
(oy2)	$\mathcal{O}(\frac{3}{2}) \oplus \mathcal{O}(\frac{1}{2})^{\oplus 3}$	$\mathrm{OGr}(3,7)$	2	6
(oy3)	$\mathcal{U}^*(\frac{1}{2}) \oplus \mathcal{O}(\frac{1}{2})$	$\mathrm{OGr}(3,7)$	2	12
(oy4)	$\mathcal{O}(\frac{1}{2})^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$	$\mathrm{OGr}(3,7)$	2	8
(oy1)	$\mathcal{O}(\frac{1}{2})^{\oplus 2} \oplus \Lambda^2\mathcal{U}^*$	$\mathrm{OGr}(4,9)$	2	24
M.(oy5)	$\mathcal{O}(\frac{1}{2})^{\oplus 8}$	$\mathrm{OGr}(4,9)$	2	12

Table B.13. – Surfaces in odd orthogonal Grassmannians, see Theorem 2.3.5

case	bundle $\mathcal{F}$	$\mathrm{OGr}(k,n)$	$\chi(\mathcal{O}_Y)$	$\deg(\mathcal{F})$
(ob3)	$\mathcal{T}_{+\frac{1}{2}}(2)$	Quadric in $\mathbb{P}^7$	2	12
(ob4)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{OGr}(2,10)$	2	18
(ob4.2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{OGr}(2,10)$	2	20
(ob5)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus S^2\mathcal{U}^*$	$\mathrm{OGr}(2,10)$	2	24
(ob5.2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus S^2\mathcal{U}^*$	$\mathrm{OGr}(2,10)$	2	24
(ob6)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{OGr}(2,14)$	4	36
(ob7)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus S^2\mathcal{U}^*$	$\mathrm{OGr}(2,14)$	4	48
(ow1)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{OGr}(2,8)$	2	20
(ow2)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus (1,1;1;1)$	$\mathrm{OGr}(2,8)$	0	32
(ow3)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus S^2\mathcal{U}^*$	$\mathrm{OGr}(2,8)$	2	24
(ow4)	$(\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 3} \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,8)$	2	16
(ow1.2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(1)^{\oplus 3}$	$\mathrm{OGr}(2,8)$	2	18
(ow2.2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus (1,1;1;1)$	$\mathrm{OGr}(2,8)$	2	24
(ow2.3)	$\mathcal{T}_{-\frac{1}{2}}(1)^{\oplus 2} \oplus (1,1;1;1)$	$\mathrm{OGr}(2,8)$	2	24
(ow3.2)	$\mathcal{T}_{+\frac{1}{2}}(1) \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus S^2\mathcal{U}^*$	$\mathrm{OGr}(2,8)$	2	24
(ow4.2)	$\mathcal{T}_{+\frac{1}{2}}(1)^{\oplus 2} \oplus \mathcal{T}_{-\frac{1}{2}}(1) \oplus \mathcal{O}(2)$	$\mathrm{OGr}(2,8)$	2	12
(oz3)	$\mathcal{U}^*(\frac{1}{2})$	$\mathrm{OGr}(4,8)$	2	12
(oz4)	$\mathcal{O}(\frac{1}{2})^{\oplus 2} \oplus \mathcal{O}(\frac{3}{2}) \oplus \Lambda^4\mathcal{U}^*(-\frac{1}{2})$	$\mathrm{OGr}(5,10)$	2	6
(oz5)	$\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(\frac{1}{2}) \oplus \Lambda^4\mathcal{U}^*(-\frac{1}{2})$	$\mathrm{OGr}(5,10)$	2	8
(oz7)	$\mathcal{O}(1) \oplus \Lambda^5\mathcal{U}^*(-\frac{1}{2})^{\oplus 2}$	$\mathrm{OGr}(6,12)$	2	12

Table B.14. – Surfaces in even orthogonal Grassmannians, see Theorem 2.3.5

case	bundle $\mathcal{F}$	$\mathrm{OGr}(n-1,2n)$	$\chi(\mathcal{O}_Y)$	$\deg(\mathcal{O}(1))$	$\deg(\mathcal{L}_+)$	$\deg(\mathcal{L}_-)$
(oe1)	$\mathcal{L}_-^{\otimes 2} \oplus (\mathcal{U}^* \otimes \mathcal{L}_+)$	$\mathrm{OGr}(2,6)$	2	26	4	6
(oe2)	$\mathcal{L}_+ \oplus \mathcal{L}_-^{\otimes 3} \oplus \mathcal{L}_+^{\otimes 2}$	$\mathrm{OGr}(2,6)$	2	18	0	6
(oe3)	$\mathcal{L}_-(1) \oplus \mathcal{L}_- \oplus \mathcal{L}_+^{\otimes 2}$	$\mathrm{OGr}(2,6)$	2	18	4	2
(oe4)	$\mathcal{O}(1) \oplus \mathcal{L}_-(1) \oplus \mathcal{L}_+$	$\mathrm{OGr}(2,6)$	2	16	2	4
(oe5)	$\mathcal{L}_- \oplus \mathcal{L}_- \oplus \mathcal{L}_+^{\otimes 2}(1)$	$\mathrm{OGr}(2,6)$	2	10	4	0
(oe6)	$\mathcal{O}(2) \oplus \mathcal{L}_- \oplus \mathcal{L}_+$	$\mathrm{OGr}(2,6)$	2	12	2	2
(oe7)	$\mathcal{O}(1) \oplus \mathcal{L}_-^{\otimes 2} \oplus \mathcal{L}_+^{\otimes 2}$	$\mathrm{OGr}(2,6)$	2	24	4	4
(oe8)	$\Lambda^2\mathcal{U}^* \oplus \mathcal{L}_-^{\oplus 2} \oplus \mathcal{L}_+^{\oplus 2}$	$\mathrm{OGr}(3,8)$	2	36	8	8
(oe9)	$\mathcal{O}(1) \oplus \mathcal{L}_-^{\oplus 3} \oplus \mathcal{L}_+^{\oplus 3}$	$\mathrm{OGr}(3,8)$	2	28	6	6
(oe10)	$\mathcal{L}_+^{\otimes 2} \oplus \mathcal{L}_-^{\oplus 4} \oplus \mathcal{L}_+^{\oplus 2}$	$\mathrm{OGr}(3,8)$	2	28	8	4

Table B.15. – Surfaces in $\mathrm{OGr}(n-1,2n)$, see Theorem 2.3.6

Table B.16. – Classification for exceptional Grassmannians with $G = G_2$

$G(i)$	$\dim(Y)$	F_λ	$\chi(Y)$	$\deg(H)$
$G_2(1)$	2	$H^{\otimes 2} \oplus (0,1)$	2	12
$G_2(1)$	2	$(0,2)$	2	24
$G_2(1)$	2	$H^{\oplus 3}$	2	18
$G_2(1)$	3	$H \otimes (0,1)$	0	–
$G_2(1)$	3	$H^{\otimes 2} \oplus H$	0	–
$G_2(1)$	4	$H^{\otimes 3}$	2	–
$G_2(2)$	2	$H^{\otimes 2} \oplus H^{\otimes 2} \oplus H$	2	8
$G_2(2)$	2	$H^{\otimes 3} \oplus H^{\oplus 2}$	2	6
$G_2(2)$	2	$H^{\otimes 2} \oplus (1,0)$	2	12
$G_2(2)$	3	$H^{\otimes 3} \oplus H^{\otimes 2}$	0	–
$G_2(2)$	3	$H^{\otimes 4} \oplus H$	0	–
$G_2(2)$	3	$(1,0) \otimes H$	0	–
$G_2(2)$	4	$H^{\otimes 5}$	2	–

Table B.17. – Classification for exceptional Grassmannians with $G = F_4$

$G(i)$	$\dim(Y)$	F_λ	$\chi(Y)$	$\deg(H)$
$F_4(1)$	2	$H^{\otimes 2} \oplus (0,0,0,1)^{\oplus 2}$	2	12
$F_4(1)$	4	$H^{\oplus 5} \oplus (0,0,0,1)$	2	–
$F_4(1)$	6	$H^{\otimes 2} \oplus H^{\otimes 2} \oplus H \oplus (0,0,0,1)$	2	–
$F_4(1)$	6	$H^{\otimes 3} \oplus H \oplus (0,0,0,1)$	2	–
$F_4(2)$	6	$(1,0,0,0) \oplus (0,0,0,1)^{\oplus 4}$	2	–
$F_4(3)$	6	$(0,0,0,1)^{\oplus 7}$	2	–
$F_4(4)$	4	$H^{\oplus 11}$	2	–
$F_4(4)$	4	$H^{\oplus 4} \oplus (1,0,0,0)$	2	–
$F_4(4)$	6	$H^{\otimes 3} \oplus H^{\oplus 8}$	2	–
$F_4(4)$	6	$H^{\otimes 2} \oplus H^{\otimes 2} \oplus H^{\oplus 7}$	2	–
$F_4(4)$	6	$H^{\otimes 2} \oplus H^{\otimes 2} \oplus (1,0,0,0)$	2	–
$F_4(4)$	6	$H^{\otimes 3} \oplus H \oplus (1,0,0,0)$	2	–

Table B.18. – Classification for exceptional Grassmannians with $G = E_6, E_7, E_8$

$G(i)$	$\dim(Y)$	F_λ	$\chi(Y)$	$\deg(H)$
$E_6(1)$	4	$H^{\oplus 12}$	2	–
$E_6(1)$	6	$H^{\oplus 9} \oplus H^{\otimes 3}$	2	–
$E_6(1)$	6	$H^{\otimes 2} \oplus H^{\otimes 2} \oplus H^{\oplus 8}$	2	–
$E_6(2)$	2	$H^{\otimes 2} \oplus (1,0,0,0,0,0)^{\oplus 3}$	2	12
$E_6(2)$	2	$H^{\otimes 2} \oplus (1,0,0,0,0,0)^{\oplus 2} \oplus (0,0,0,0,0,1)$	2	12
$E_6(2)$	4	$H^{\oplus 5} \oplus (1,0,0,0,0,0)^{\oplus 2}$	2	–
$E_6(2)$	4	$H^{\oplus 5} \oplus (1,0,0,0,0,0) \oplus (0,0,0,0,0,1)$	2	–
$E_6(2)$	6	$H^{\otimes 2} \oplus H^{\otimes 2} \oplus H \oplus (1,0,0,0,0,0)^{\oplus 2}$	2	–
$E_6(2)$	6	$H^{\otimes 3} \oplus H^{\oplus 2} \oplus (1,0,0,0,0,0)^{\oplus 2}$	2	–
$E_6(2)$	6	$H^{\otimes 2} \oplus H^{\otimes 2} \oplus H \oplus (1,0,0,0,0,0) \oplus (0,0,0,0,0,1)$	2	–
$E_6(2)$	6	$H^{\otimes 3} \oplus H^{\oplus 2} \oplus (1,0,0,0,0,0) \oplus (0,0,0,0,0,1)$	2	–
$E_6(5)$	3	$(0,1,0,0,0,0)^{\oplus 4} \oplus (0,0,0,0,0,1)$	$c.w.$	–
$E_6(5)$	4	$(1,0,0,0,0,0)^{\oplus 4} \oplus H$	$c.w.$	–
$E_6(5)$	4	$(1,0,0,0,0,0)^{\oplus 3} \oplus (0,0,0,0,0,1)^{\oplus 3}$	$c.w.$	–
$E_6(5)$	6	$(1,0,0,0,0,0)^{\oplus 3} \oplus (0,0,0,0,0,1) \oplus H^{\otimes 2}$	$c.w.$	–
$E_6(5)$	6	$(1,0,0,0,0,0)^{\oplus 2} \oplus (0,0,0,0,0,1)^{\oplus 4} \oplus H$	$c.w.$	–
$E_6(5)$	6	$(1,0,0,0,0,0) \oplus (0,0,0,0,0,1)^{\oplus 7}$	$c.w.$	–
$E_7(1)$	4	$(0,0,0,0,0,0,1)^{\oplus 2} \oplus H^{\oplus 5}$	$c.w.$	–
$E_7(1)$	6	$(0,0,0,0,0,0,1)^{\oplus 2} \oplus H^{\otimes 2} \oplus H^{\otimes 2} \oplus H$	$c.w.$	–
$E_7(1)$	6	$(0,0,0,0,0,0,1)^{\oplus 2} \oplus H^{\otimes 3} \oplus H^{\oplus 2}$	$c.w.$	–

C. Table of c_λ

We prove here a result that is needed in Section 2.4.2.

Lemma C.1. *Let F_λ be a homogeneous irreducible vector bundle over $G(i)$, where $\lambda = \sum_j \lambda_j \omega_j$. Then the coefficient c_λ of Lemma 2.4.1 is given by the formulas in Table C.19.*

Proof. From an explicit description of the simple roots as vectors in $\mathbb{C}^{\operatorname{rank} G} \cong \mathfrak{h}^*$, the problem is reduced to an easy computation in linear algebra. Indeed, as $\langle \omega_i \rangle \perp E_i = \langle \alpha_1, ..., \widehat{\alpha}_i, ..., \alpha_n \rangle$, we have:

$$c_\lambda = \frac{K(\omega_i, \lambda)}{K(\omega_i, \omega_i)}.$$

Moreover, by definition of ω_j and H_{α_k},

$$\omega_j(H_{\alpha_k}) = 2\frac{K(\omega_j, \alpha_k)}{K(\alpha_k, \alpha_k)} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}.$$

Therefore, if $\omega_i = \sum_k \mu_k \alpha_k$, we have

$$c_\lambda = \frac{K(\omega_i, \lambda)}{K(\omega_i, \omega_i)} = \frac{\sum_{k,j} K(\mu_k \alpha_k, \lambda_j \omega_j)}{K(\omega_i, \omega_i)} = \sum_k \mu_k \lambda_k \frac{K(\alpha_k, \alpha_k)}{2K(\omega_i, \omega_i)}.$$

It is quite easy to find the μ_i's; after doing so, this formula allows to compute c_λ. □

Table C.19. – Formulas for c_λ with $\lambda = (\lambda_1, \ldots, \lambda_n)$, refer to Lemma C.1

$G(i)$	c_λ
$G_2(1)$	$\lambda_1 + \frac{1}{2}(\lambda_2)$
$G_2(2)$	$\lambda_2 + \frac{1}{2}(3\lambda_1)$
$F_4(1)$	$\lambda_1 + \frac{1}{2}(3\lambda_2 + 2\lambda_3 + 1\lambda_4)$
$F_4(2)$	$\lambda_2 + \frac{1}{6}(3\lambda_1 + 4\lambda_3 + 2\lambda_4)$
$F_4(3)$	$\lambda_3 + \frac{1}{6}(4\lambda_1 + 8\lambda_2 + 3\lambda_4)$
$F_4(4)$	$\lambda_4 + \frac{1}{2}(2\lambda_1 + 4\lambda_2 + 3\lambda_3)$
$E_6(1)$	$\lambda_1 + \frac{1}{4}(3\lambda_2 + 5\lambda_3 + 6\lambda_4 + 4\lambda_5 + 2\lambda_6)$
$E_6(2)$	$\lambda_2 + \frac{1}{2}(\lambda_1 + 2\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6)$
$E_6(3)$	$\lambda_3 + \frac{1}{10}(5\lambda_1 + 6\lambda_2 + 12\lambda_4 + 8\lambda_5 + 4\lambda_6)$
$E_6(4)$	$\lambda_4 + \frac{1}{6}(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 4\lambda_5 + 2\lambda_6)$
$E_7(1)$	$\lambda_1 + \frac{1}{4}(4\lambda_2 + 6\lambda_3 + 8\lambda_4 + 6\lambda_5 + 4\lambda_6 + 2\lambda_7)$
$E_7(2)$	$\lambda_2 + \frac{1}{7}(4\lambda_1 + 10\lambda_3 + 16\lambda_4 + 12\lambda_5 + 8\lambda_6 + 4\lambda_7)$
$E_7(3)$	$\lambda_3 + \frac{1}{12}(6\lambda_1 + 8\lambda_2 + 16\lambda_4 + 12\lambda_5 + 8\lambda_6 + 4\lambda_7)$
$E_7(4)$	$\lambda_4 + \frac{1}{12}(4\lambda_1 + 6\lambda_2 + 8\lambda_3 + 9\lambda_5 + 6\lambda_6 + 3\lambda_7)$
$E_7(5)$	$\lambda_5 + \frac{1}{15}(6\lambda_1 + 9\lambda_2 + 12\lambda_3 + 24\lambda_4 + 10\lambda_6 + 5\lambda_7)$
$E_7(6)$	$\lambda_6 + \frac{1}{4}(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 6\lambda_4 + 5\lambda_5 + 2\lambda_7)$
$E_7(7)$	$\lambda_7 + \frac{1}{3}(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 6\lambda_4 + 5\lambda_5 + 4\lambda_6)$
$E_8(1)$	$\lambda_1 + \frac{1}{4}(5\lambda_2 + 7\lambda_3 + 10\lambda_4 + 8\lambda_5 + 6\lambda_6 + 4\lambda_7 + 2\lambda_8)$
$E_8(2)$	$\lambda_2 + \frac{1}{8}(5\lambda_1 + 10\lambda_3 + 15\lambda_4 + 12\lambda_5 + 9\lambda_6 + 6\lambda_7 + 3\lambda_8)$
$E_8(3)$	$\lambda_3 + \frac{1}{14}(7\lambda_1 + 10\lambda_2 + 20\lambda_4 + 16\lambda_5 + 12\lambda_6 + 8\lambda_7 + 4\lambda_8)$
$E_8(4)$	$\lambda_4 + \frac{1}{30}(10\lambda_1 + 15\lambda_2 + 20\lambda_3 + 24\lambda_5 + 18\lambda_6 + 12\lambda_7 + 6\lambda_8)$
$E_8(5)$	$\lambda_5 + \frac{1}{20}(8\lambda_1 + 12\lambda_2 + 16\lambda_3 + 24\lambda_4 + 15\lambda_6 + 10\lambda_7 + 5\lambda_8)$
$E_8(6)$	$\lambda_6 + \frac{1}{12}(6\lambda_1 + 9\lambda_2 + 12\lambda_3 + 18\lambda_4 + 15\lambda_5 + 8\lambda_7 + 4\lambda_8)$
$E_8(7)$	$\lambda_7 + \frac{1}{6}(4\lambda_1 + 6\lambda_2 + 8\lambda_3 + 12\lambda_4 + 10\lambda_5 + 8\lambda_6 + 3\lambda_8)$
$E_8(8)$	$\lambda_8 + \frac{1}{2}(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 6\lambda_4 + 5\lambda_5 + 4\lambda_6 + 3\lambda_7)$