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Propriétés qualitatives de quelques systèmes de la mécanique des fluides incompressibles

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À la mémoire de mon père.

À

ma chère mère,

ma chère épouse,

mes petites Chame et Taliya.

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Abstract

The purpose of this thesis is to study the qualitative properties of the solutions of some systems of incompressible fluids mechanics. The thesis is divided into three chapters.

The first chapter is dedicated to the issue of local existence and uniqueness of a solution to the 2D Euler incompressible system. We prove a theorem of local existence and uniqueness in a large space of initial vorticities. This extends the results (the local existence part more precisely) by Bernicot and Keraani [**2**] on the subject. Some laws of composition in these spaces (with Lebesgue measure preserving homeomorphisms) are given and used to prove the principal theorem of this chapter.

The second chapter is concerned with the profile decomposition for the 3D fractional Navier-Stokes system. We prove some structure theorem which highlights the role of the invariances group of this system and we use it to establish some qualitative properties of the global solutions of fractional Navier-Stokes.

In the last chapter we study the asymptotic behavior of the solutions of fractional 3D Navier-Stokes. We prove that the critical Sobolev norm of the solution vanishes at infinity if it is global and blows up if it develops singularities at the finite time. A suitable profile decomposition is the main tool for our analysis throughout this chapter.

Keywords. 3D Fractional Navier-Stokes system, 2D incompressible Euler system, profile decomposition, local and global theory, paradiffrential calculus..

Résumé

L'objet de cette thèse est l'étude des propriétés qualitatives des solutions de quelques équations de la mécanique de fluides incompressibles. Elle est divisée en trois chapitres.

Le premier chapitre est consacré à la question d'existence locale est d'unité pour le système d'Euler 2D incompressible. On montre une théorème d'existence locale et d'unité dans un espace large de tourbillons initiaux. Ceci généralise la partie existence locale du travail de Bernicot Keraani [**2**] sur le sujet. Une lois de compositions dans ces espaces (avec les homéomorphisme préservant la mesure de Lebesgue) est donnée et utilisée pour la preuve du théorème principal de ce chapitre.

Le deuxième chapitre est consacré à la décomposition en profils pour le système de Navier-Stokes fractionnaire en 3D dans la boule maximale d'existence globale. On montre une théorème de structures qui mettent en évidence le rôle du groupe des invariances de ce système et on l'utilise pour établir des propriétés qualitatives des solutions globales.

Enfin, dans le dernier chapitre, on utilise une décomposition en profil plus générale pour établir des résultats sur le comportement asymptotiques des solutions du Navier-Stokes fractionnaire en 3D. On montre que la norme de Sobolev critique des solutions globales converge vers 0 et que celle des solutions singulières explose en s'approchant du temps d'explosion fini.

Mots-Clefs. Système d'Euler incompressible, estimations a priori, Navier-Stokes fractionnaire, décomposition en profil, calcul paradifférentiel.

Contents

Introduction

Le modèle de base décrivant la dynamique d'un fluide visqueux incompressible mouvant dans l'espace est donné par le fameux système de Navier-Stokes. Il s'agit d'un système dans lequel l'équation de continuité (dite aussi équation d'incompressibilité) est couplée au bilan de la quantité de mouvement. Plus précisément en dimension $d \geq 2$, si $u = (u^1, u^2, .., u^d)$ désigne la vitesse du fluide et p sa pression[1] alors le système de Navier-Stokes s'écrit

$$\text{(NS)} \qquad \begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases}$$

Ici le paramètre $\nu > 0$ est une constante appelée coefficient de viscosité et $\operatorname{div} u = \sum_{j=1}^{d} \partial_{x_j} u^j$.

Si l'on néglige l'effet des forces de frottement visqueux[2] on retrouve le modèle d'Euler concevant le fluide comme parfait:

$$\text{(E)} \qquad \begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases}$$

L'étude mathématique du système de Navier-Stokes a été initiée par J. Leray dans son article célèbre [**15**]. Il démontre, en utilisant une méthode de compacité, que pour toute donnée initiale u^0 de divergence nulle et appartenant à l'espace $L^2(\mathbb{R}^d)$, il existe une solution globale au système (NS_ν) appartenant à l'espace d'énergie $L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$. Par ailleurs l'unicité de telles solutions faibles n'est connue qu'en dimension deux d'espace.

[1]En mécanique des fluides incompressible, la pression est le multiplicateur de Lagrange assurant la contrainte d'incompressibilité.

[2]Ceci correspond à un nombre de Mach très faible.

Dans les années 60', H. Fujita et T. Kato **[9]** ont construit des solutions locales en temps (qui sont uniques) lorsque les données initiales appartiennent à l'espace de Sobolev homogène $\dot{H}^{\frac{d}{2}-1}$. Dans toute sa généralité, l'existence globale de ces solutions reste l'un des problèmes majeurs des EDPs. Nous mentionnons que des résultats similaires ont été validés dans d'autres espaces fonctionnels $L^d, \dot{B}_{p,\infty}^{-1+\frac{d}{p}}, BMO^{-1}$, on pourra consulter **[15]** pour une discussion plus complète. Remarquons que ces espaces sont invariants par le *scaling* de l'équation: pour $\lambda > 0$, si $u(t,x)$ est solution de (NS) alors $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ l'est aussi.

Concernant le système d'Euler incompressible (E), il est bien connu suite au travail de T. Kato **[16]** qu'il est localement bien posé dans H^s pour $s > d/2+1$. L'unique solution maximale $\in \mathcal{C}([0,T^*[, H^s)$ satisfait l'alternative suivante: soit $T^* = \infty$ (existence globale) soit $T^* < \infty$ (explosion en temps fini) et

$$\|u(t,\cdot)\|_{H^s} \to \infty \qquad \text{quand} \quad t \to T^*\,.$$

L'existence locale découle d'une méthode d'énergie classique aboutissant à l'estimation *a priori* suivante:

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|_{H^s} &\leq C\|\nabla u(t)\|_{L^\infty}\|u(t,\cdot)\|_{H^s}\\ &\leq C\|u(t,\cdot)\|_{H^s}^2.\end{aligned}$$

La dernière inégalité est l'injection de Sobolev $H^{s-1} \hookrightarrow L^\infty$ si $s > d/2+1$. Remarquons que la première inégalité et le lemme de Gronwall donnent

$$\|u(t,\cdot)\|_{H^s} \leq \|u^0\|_{H^s} e^{C\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}.$$

Ceci permet d'obtenir un meilleur critère d'explosion:

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\nabla u(\tau)\|_{L^\infty} d\tau = \infty.$$

Ce critère fut raffiné d'une manière remarquable dans l'article célèbre de Beale, Kato et Majda **[3]**. Ils ont montré que l'explosion en temps fini est équivalente à une accumulation de son tourbillon.

DÉFINITION 0.1 (Tourbillon). *Si u est un champ de vecteurs, son tourbillon, noté ω, est la partie antisymétrique de la matrice ∇u. En dimension 3, $\omega = rot(u)$ et en dimension 2, $\omega = \partial_1 u^2 - \partial_2 u^1$ est un scalaire.*

En dimension $d \geq 3$, le tourbillon vérifie l'équation

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0.$$

En dimension 2 le tourbillon vérifie l'équation de transport

$$\partial_t \omega + (u.\nabla)\omega = 0\,.$$

Rappelons aussi que modulo des hypothèses de décroissance à l'infini la vitesse u peut être retrouvée à partir de la vorticité grâce à la loi de Biot-Savart:

$$u(x) = c_d \int_{\mathbb{R}^d} K(x-y) \cdot \omega(y) dy,$$

avec

$$K(x) = \begin{cases} \frac{x^\perp}{|x|^2} & \text{si} \quad d = 2, \\ \frac{x}{|x|^3} \wedge & \text{si} \quad d = 3. \end{cases}$$

Dans [3] il est prouvé que si $u \in \mathcal{C}([0, T^*[, H^s)$ est la solution maximale du système d'Euler incompressible (E) et ω son tourbillon alors

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\omega(\tau)\|_{L^\infty} d\tau = \infty\,.$$

En dimension 2D le tourbillon ω est transporté par le flot et prend la forme

$$\omega(t,x) = \omega^0(\mathtt{X}^{-1}(t,x))\,,$$

où $\mathtt{X} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ est le flot décrivant les trajectoires des particules par le biais de l'EDO suivante

$$\mathtt{X}(t,x) = x + \int_0^t u(\tau, \mathtt{X}(\tau, x)) d\tau\,.$$

L'incompressibilité du fluide implique que $x \mapsto \mathtt{X}(t,x)$ est un difféomorphisme qui préserve la mesure de Lebesgue[3], et en conséquence toutes les normes L^p du tourbillon sont préservées, d'où l'existence globale pour des données H^s, avec $s > 2$.

[3]Ceci découle de la formule: $\dfrac{\partial J}{\partial t} = (\mathrm{div}_x\, u)|_{\mathtt{X}(t,x)} J(x,t)$.

En dimension $d \geq 3$, l'équation du tourbillon contient le terme d'étirement $\omega \cdot \nabla u$ qui affecte profondément la dynamique du fluide et pourrait être une source d'amplification de la vorticité, et l'on ne sait pas encore s'il y a des singularités qui se développent en temps fini ou non.

Les espaces de Besov $B^{\frac{d}{p}+1}_{p,q}$ sont tous[4] critiques, car leurs parties homogènes ont la même homogénéité que $\dot{H}^{\frac{d}{2}+1}$. Toutefois, le cas $q = 1$ est très particulier car l'espace $B^{\frac{d}{p}+1}_{p,1}$ s'injecte continûment dans l'espace des fonctions Lipschitziennes Lip$= \{u \in L^\infty, \nabla u \in L^\infty\}$. Parmi ces espaces de Besov critiques la théorie d'existence locale n'est validée que dans ce dernier cas *i.e.* pour des données initiales appartenant à $B^{\frac{d}{p}+1}_{p,1}$ [**6**]. L'existence globale de ces solutions en dimension 2 est due à Vishik [**24**]. La preuve utilise de manière incontournable la structure de l'équation de transport régissant l'évolution du tourbillon selon l'estimation logarithmique suivante (voir [**24**]): pour tout $f \in B^0_{\infty,1}$ et pour tout difféomorphisme $\mathtt{X}$ préservant la mesure de Lebesgue, on a

$$\|f \circ \mathtt{X}\|_{B^0_{\infty,1}} \leq C\|f\|_{B^0_{\infty,1}} \Big(1 + \ln_+(\|\nabla \mathtt{X}\|_{L^\infty}\|\nabla \mathtt{X}^{-1}\|_{L^\infty})\Big).$$

Ceci permet d'avoir

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\|\omega^0\|_{B^0_{\infty,1}}\big(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\big).$$

Un contrôle de la norme L^∞ de la vitesse et un argument de Gronwall permettent de contrôler la norme Lipschitz de la vitesse, et donc de prouver l'existence globale.

1. Existence locale pour Euler 2D avec données peu régulières

L'objet de la première partie de cette thèse est l'étude de la question d'existence/unicité pour le système d'Euler 2D avec des données peu régulières (tourbillons non bornés).

[4]Remarquons que ces espaces forment une chaîne croissante en p et en q.

En effet, la forme particulière de l'équation du tourbillon en 2D a permis à Yudovich de développer une théorie d'existence (globale notamment) et d'unicité pour une classe de solutions faibles pour Euler 2D.
Rappelons qu'en 2D le système d'Euler peut s'écrire sous:

$$\text{(1)} \qquad \partial_t \omega + (u \cdot \nabla)\omega = 0, \qquad u = \frac{x^\perp}{2\pi|x|^2} * \omega.$$

On dit que (u, w) est une solution faible de l'équation (1) dans $\mathbb{R}^2$, si:

(1) $u = K * \omega, \quad \text{avec} \quad K(x) = \dfrac{x^\perp}{2\pi|x|^2}.$ (Lois de Bio Savart)

(2) Pour tout $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R})$, on a

$$\int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + u.\nabla\varphi)\omega(t,x)\, dx\, dt + \int_{\mathbb{R}^2} \varphi(0,x)\omega_0(x)\, dx = 0.$$

Dans [**7, 8, 19**], les auteurs prouvent l'existence d'une unique solution faible. Contrairement à l'argument du point fixe, la méthode de compacité ne garantit pas l'unicité des solutions construite, et pour cela les deux questions sont traitées séparément.
Yudovich [**25**] ainsi que Serfati [**20**] ont prouvé l'existence d'une solution unique dans un domaine borné, sous l'hypothèse $(u_0, \omega_0) \in L^2 \times L^\infty$ dans [**25**], et pour (u_0, ω_0) borné dans [**20**]. DiPerna et Majda [**10**] considèrent $\omega_0 \in L^1 \cap L^p$, alors que Giga et al. [**13**] considèrent $\omega_0 \in L^p$, avec $2 < p < \infty$. Dans [**5**], Chae montre l'existence de solutions pour Euler pour ω_0 dans $L \ln^+ L$ de Support compact. Récemment, Taniuchi dans [**21**] prouve l'existence globale de solutions pour $(u_0, \omega_0) \in L^\infty \times \mathrm{BMO}$. Dans [**23, 26**], les auteurs montrent l'existence d'une unique solution faible pour classes de tourbillons non bornées.
Dans la première partie de cette thèse on montre l'existence locale et l'unicité pour les équations d'Euler 2D avec des tourbillons initiaux dans un espace large qui contient strictement L^∞. Ces espaces sont des versions généralisées des espaces BMO-logarithmique qui a été introduite dans [**2**].

DÉFINITION 0.2. *Soit* $\Psi :]0, +\infty \longrightarrow]0, +\infty[$ *est une fonction régulier, qui vérifie les hypothèses:*

(H1) Ψ *est croissante.*

(H2) *Il existe* $C > 0$ *telle que*

$$\Psi(r+s) \leq C(\Psi(r)+\Psi(s)), \qquad \forall\,(r,s) \in]0,+\infty[\times]0,+\infty[.$$

(H3) *Il existe une fonction lisse g telle que*

$$\lambda \mapsto \sum_{j\geq g(\lambda)} e^{-\frac{j}{\lambda}}\Psi(1+\ln(1+\lambda))$$

est bien défini une fonction lisse sur $]1,+\infty[$.

Pour une fonction localement intégrable à valeur complexe sur $\mathbb{R}^2$ *on définit*

$$\|f\|_{\widetilde{\mathrm{B}}\mathrm{MO}} := \|f\|_{\mathrm{BMO}} + \sup_{B_1,B_2} \frac{|\mathrm{Avg}_{B_2}(f) - \mathrm{Avg}_{B_1}(f)|}{\Psi\big(1+\frac{1+|\ln r_2|}{1+|\ln r_1|}\big)},$$

Où le supremum port sur toutes les paires de boules $B_2 = B(x_2, r_2)$ *et* $B_1 = B(x_1, r_1)$ *dans* $\mathbb{R}^2$ *avec* $0 < r_1 \leq 1$ *et* $2B_2 \subset B_1$.

Le premier théorème est un théorème d'existence locale.

THÉORÈME 0.3 (**[1]**). *Soit* $\omega_0 \in L^p \cap \widetilde{\mathrm{B}}\mathrm{MO}$, $p \in]1,2[$. *Il existe* $T > 0$, *et une unique solution faible* (v,ω) *à l'équation d 'Euler 2D incompressible sur* $[0,T]$:

$$\omega \in L^\infty([0,T], L^p \cap \widetilde{\mathrm{B}}\mathrm{MO}).$$

1.1. Espaces fonctionnels. Rappelons d'abord que l'ensemble des champs de vecteur log- lipschitzien sur $\mathbb{R}^2$, notée LL, est l'ensemble des champs de vecteurs v tels que

$$\|v\|_{LL} := \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|\big(1+|\ln|x-y||\big)} < \infty.$$

DÉFINITION 0.4 (**[2]**). *Soit* ψ *un homéomorphisme, posons*

$$\|\psi\|_* := \sup_{x\neq y} \Phi\big(|\psi(x)-\psi(y)|, |x-y|\big),$$

oú Φ *est défini sur* $]0,+\infty[\times]0,+\infty[$ *par*

$$\Phi(r,s) = \begin{cases} \max\{\frac{1+|\ln(s)|}{1+|\ln r|}; \frac{1+|\ln r|}{1+|\ln(s)|}\}, & \text{if} \quad (1-s)(1-r) \geq 0, \\ (1+|\ln s|)(1+|\ln r|), & \text{if} \quad (1-s)(1-r) \leq 0. \end{cases}$$

PROPOSITION 0.5 ([**2**]). *Soit u un champ de vecteurs lisses et de divergence nulle et soit ψ son flôt:*

$$\partial_t \psi(t,x) = u(t,\psi(t,x)), \qquad \psi(0,x) = x.$$

Alors, pour tout $t \geq 0$

$$\|\psi(t,\cdot)\|_* \leq \exp(\int_0^t \|u(\tau)\|_{LL} d\tau).$$

L'ingrédient principal dans la demonstration du Théorème 1.9 est la loi de composition suivante:

THÉORÈME 0.6 ([**1**]). *Il existe une fonction lisse G telle que*

$$\|f o \psi\|_{\tilde{\mathrm{B}}\mathrm{MO} \cap L^p} \leq G(\|\psi\|_*)\|f\|_{\tilde{\mathrm{B}}\mathrm{MO} \cap L^p},$$

pour toute homéomorphisme ψ qui préserve la mesure de Lebesgue.

La preuve suit les arguments développés dans [**2**].

2. Décomposition en profils pour le système de Navier-Stokes fractionnaire

La décomposition en profils est une technique introduite par P. Gérard [**13**] pour étudier le défaut de compacité des injections de Sobolev critiques $\dot{H}^s \hookrightarrow L^p$, avec $p = \frac{2d}{d-2s}$. Ensuite, elle fut utilisée par plusieurs auteurs pour étudier des équations d'évolution non linéaires critiques (voir [**5, 17, 13, 11, 22**] par exemple). Dans la seconde partie de cette thèse on applique cet outil pour étudier le système de Navier-Stokes fractionnaire.

2.1. Idée générale sur la décomposition en profils. Soit $\mathcal{H}$ un espace de Hilbert et $\mathcal{L}(\mathcal{H})$ l'espace de Banach des opérateurs bornés dans $\mathcal{H}$. Soit $G \subset \mathcal{L}(\mathcal{H})$ un groupe d'isométries qui vérifie l'hypothèse suivante:

HYPOTHÈSE (**A**). *Toute suite de G qui ne converge pas faiblement vers* 0 *admet une sous-suite qui converge fortement.*

EXAMPLE. $\mathcal{H} = H^1(\mathbb{R}^d)$ *et* $G = \{\tau_y,\ y \in \mathbb{R}^d\}$ *groupe des translations.*

Deux suites (g_n) et (g'_n) à valeurs dans G sont dites *orthogonales* si $g_n^{-1}g'_n \rightharpoonup 0$.
Pour toute suite bornée $\mathbf{v} = (v_n)$ de $\mathcal{H}$, on note $\mathcal{P}(\mathbf{v})$ l'ensemble des $v \in \mathcal{H}$ tel qu'il existe une sous-suite (v_{n_k}) et une suite $(g_k) \subset G$, telles que $g_k(v_{n_k}) \rightharpoonup v$. Avec ceci, on définit

$$\eta(\mathbf{v}) = \sup\{\|v\|, \quad v \in \mathcal{P}(\mathbf{v})\}.$$

Le théorème de concentration-compacité suivant est une modification (proposée par P. Gérard) d'un théorème annoncé dans **[21]**. Il s'agit d'un argument général qui représente l'étape zéro des décompositions en profils.

THEOREM 0.7. *Pour toute suite* $\mathbf{u} = (u_n)$ *bornée de* $\mathcal{H}$*, il existe une sous-suite* $\mathbf{u}' = (u'_n)$ *de* $\mathbf{u}$*, une famille* $\mho^j$ *de* $\mathcal{H}$ *et, pour tout* j*, une suite* $\mathbf{g^j} = (g_n^j)$ *de* G*, telles que*

1. *Si* $j \neq k$*, les suites* $\mathbf{g^j}$ *et* $\mathbf{g^k}$ *sont orthogonales.*
2. *Pour tout* $\ell \geq 1$*, la suite* $\mathbf{r}^\ell = (r_n^\ell)$ *définie par*

$$u'_n = \sum_{j=1}^{\ell} g_n^j \mho^j + r_n^\ell$$

satisfait

$$\eta(\mathbf{r}^\ell) \underset{\ell\to\infty}{\longrightarrow} 0.$$

3. *Pour tout* $\ell \geq 1$*, on a la relation de la presque-orthogonalité*

$$\|u'_n\|^2 = \sum_{j=1}^{\ell} \|\mho^j\|^2 + \|r_n^\ell\|^2 + o(1), \qquad n \to \infty.$$

REMARQUES.

1. Ce théorème est vrai pour tout groupe G d'isométries. Evidemment, plus G est grand plus la structure des profils est riche. Par exemple, si $G = \{I\}$ alors le théorème en haut est réduit à la précompacité faible de bornés de $\mathcal{H}$.
2. Le groupe G est imposé par le contexte de l'utilisation d'un tel théorème. Dans la plupart des cas il s'agit de la situation suivante: $\mathcal{X}$ est un espace de Banach et $\Psi : \mathcal{H} \longrightarrow \mathcal{X}$ une application linéaire continue, tels que

$\|\Psi(g(u))\| = \|\Psi(u)\|$, pour tout $(u,g) \in \mathcal{H} \times G$. Dans ce cas-là, le groupe G crée un défaut de compacité pour l'application linéaire Ψ. En effet, si $p \in \mathcal{H} \setminus Ker(\Psi)$, alors pour toute suite (g_n) de G telle que[5] $g_n \rightharpoonup 0$, la suite $(g_n(p))$ converge faiblement vers 0. Cependant, la suite $\Psi(g_n(p))$ n'est pas relativement compacte dans $\mathcal{X}$ car sa norme est constante.

La partie difficile de cette technique est la preuve de l'inverse: prouver que tel ou tel groupe est le seul responsable de ce défaut de compacité. Là-dessus on n'a pas le choix, il faut trouver le «bon» groupe G. D'une manière générale, et dans le contexte du théorème 0.7, on a à prouver

$$\eta(\mathbf{r}^\ell) \to 0 \Longrightarrow \limsup_{n\to\infty} \Psi(\mathbf{r}_\mathbf{n}^\ell) \to 0, \qquad \ell \to \infty.$$

Une fois prouvé, ceci décrirait complètement le défaut de compacité de Ψ: toute suite bornée dans $\mathcal{H}$ peut s'écrire, à une sous-suite près, comme une somme presque orthogonale de famille de suites[6] $g_n^j(\mho^j)$ et un reste dont l'image par Ψ est compacte dans $\mathcal{X}$. En d'autres termes, l'application Ψ est compacte modulo le groupe G: si $(u_n)_{n\geq 0}$ est une suite bornée de $\mathcal{H}$, telle que $g_n(u_n) \rightharpoonup 0$ faiblement, pour toute suite $(g_n) \subset G$, alors $\Psi(u_n) \to 0$ fortement dans $\mathcal{X}$.

2.2. Le système de Navier-Stokes fractionnaire. Le système de Navier-Stokes à diffusion fractionnaire (NS_α) décrit l'évolution d'un fluide incompressible, visqueux et à densité constante (égale à 1) et à dissipation fractionnaire:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu\,(-\Delta)^\alpha u + \nabla p = 0, \\ \nabla . u = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{2}$$

où le paramètre $\nu > 0$ représente la viscosité cinématique du fluide et $(-\Delta)^\alpha$ désigne l'opérateur de Laplace d'ordre α ($\frac{1}{2} < \alpha < 1$), dont sa transformée

[5]On suppose qu'une telle suite existe (si G est fini alors une telle suite ne peut pas exister).

[6]C'est-à-dire des suites très particulières obtenues par l'action d'une suite d'isométries sur un vecteur fixe.

de Fourier en espace est définie par:

$$\mathcal{F}(-\Delta)^{\alpha}\psi(\xi) = |\xi|^{2\alpha}\hat{\psi}(\xi).$$

Grâce à la condition d'incompressibilité du fluide on a

$$(u \cdot \nabla)u = div(u \otimes u), \qquad \text{où} \quad div(u \otimes u)^j = \sum_{i=1}^{d} \partial_i(u^j u^i) = div(u^j u),$$

avec le produit tensoriel de deux vecteurs $a = (a^i)_{i=1}^d$ et $b = (b^i)_{i=1}^d$ est une matrice notée par $a \otimes b$, de coefficients $(a \otimes b)_{ij} = a^i b^j$. On suppose en particulier que la constante de viscosité du fluide ν égale à 1. De plus, la pression p et le champ de vitesse u vérifient l'équation de Poisson:

$$-\Delta p = div(u \cdot \nabla u),$$

et grâce à la condition d'incompressibilité du fluide a pression p peut être retrouvée de la manière suivante:

$$p = (-\Delta)^{-1} \sum_{j,k} \partial_j \partial_k (u^j u^k).$$

On définit alors le projecteur de Leray $\mathbb{P}$ par:

$$\mathbb{P} = Id - \nabla \Delta^{-1} div,$$

dont sa transformée de Fourier $\mathcal{F}(\mathbb{P})$ est une matrice de coefficients:

$$\mathcal{F}(\mathbb{P}_{i,j}) = \delta_{i,j} - \frac{\xi_i \xi j}{|\xi|^2}.$$

$\mathbb{P}$ est un multiplicateur de Fourier d'ordre 0, continu de $\dot{H}^s$ dans $\dot{H}^s$ pour tout $s \in \mathbb{R}$ et continu de L^p dans L^p pour tout $p \in]1, +\infty[$.
En conclusion, en tenant compte des ces dernières définitions, on trouve alors une forme équivalente à (2), donnée par:

$$(3) \qquad \begin{cases} \partial_t u + (-\Delta)^{\alpha} u = Q(u, u), \\ u_{|t=0} = u_0, \end{cases}$$

où Q est un opérateur bilinéaire défini par:

$$Q(v, w) = -\frac{1}{2}\mathbb{P}\big(div(v \otimes w) + div(w \otimes v)\big).$$

On résout la version intégrale du système (3) qui est la suivante :

$$u(t) = e^{-t(-\Delta)^\alpha} u_0 - \int_0^s e^{-(t-s)(-\Delta)^\alpha} Q(u,u)(s)ds. \tag{4}$$

Par conséquent, on constate que u apparaît comme un point fixe (dans un espace approprié) de l'application

$$u \mapsto e^{-t(-\Delta)^\alpha} u_0 + B(u,u),$$

où

$$B(u,u) = -\int_0^s e^{-(t-s)(-\Delta)^\alpha} Q(u,u)(s)ds. \tag{5}$$

L'existence locale et l'unicité de ces solutions sont assurées par un théorème de point fixe de Picard dans un espace approprié.
Si on suppose que u a la régularité nécessaire alors on aura la loi de conservation suivante dite égalité d'énergie:

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{\alpha}{2}} u(s,.)\|_{L^2([0,t],L^2)}^2 = \frac{1}{2}\|u_0\|_{L^2}^2.$$

D'autre part, les équations de Navier-Stokes fractionnaire jouissent d'une invariance d'échelle. En effet, si u est une solution de NS_α sur $[0,T] \times \mathbb{R}^3$ pour une donnée initiale u_0, alors pour tout $\lambda > 0$, le paire (u_λ, p_λ) défini:

$$u_\lambda(t,x) = \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x), \qquad p_\lambda(t,x) = \lambda^{4\alpha-2} p(\lambda^{2\alpha} t, \lambda x),$$

une solution de NS_α sur $[0, \lambda^{-2\alpha} T] \times \mathbb{R}^3$ pour une donnée initiale

$$u_{0,\lambda} = \lambda^{2\alpha-1} u_0(\lambda x).$$

2.3. Préliminaires et espaces fonctionnels. Pour commencer nous allons rappeler quelques espaces fonctionnels qui serviront tout au long de ce travail. La construction de ces espaces est basée sur les opérateurs de troncature en fréquence dits aussi de Littlewood-Paley, et qui sont reliés à la partition dyadique de l'unité (on pourra consulter **[7]** pour plus de détails): il existe deux fonctions $\chi \in \mathcal{D}(\mathbb{R}^d)$ et $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\}$ qui sont radiales et positives, telles que

(i) $\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1,$

(ii) $\operatorname{supp} \varphi(2^{-j}\cdot) \cap \operatorname{supp} \varphi(2^{-k}\cdot) = \varnothing$, si $|j-k| \geq 2$,

(iii) $j \geq 1 \Rightarrow \operatorname{supp} \chi \cap \operatorname{supp} \varphi(2^{-j}) = \varnothing$.

NOTATIONS. Les opérateurs de Littlewood-Paley non homogènes sont définis par

$$\Delta_{-1}u = \chi(\mathrm{D})u;\ \forall j \in \mathbb{N},\ \Delta_j u = \varphi(2^{-j}\mathrm{D})u \quad \text{et} \qquad S_j u = \sum_{k=-1}^{j-1} \Delta_k u\,.$$

La version homogène est définie par

$$\forall j \in \mathbb{Z}, \quad \dot{\Delta}_j u = \varphi(2^{-j}\mathrm{D})v \quad \text{et} \quad \dot{S}_j = \sum_{k \leq j-1} \dot{\Delta}_k u\,.$$

Le calcul paradifférentiel introduit par J.-M. Bony **[3]** distingue dans un produit de deux distributions trois termes: deux termes de paraproduit et un terme de reste

$$uv = T_u v + T_v u + R(u,v),$$

où

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \quad \text{et} \quad R(u,v) = \sum_j \Delta_j u \widetilde{\Delta}_j v,$$

$$\text{avec} \quad \widetilde{\Delta}_j = \sum_{i=-1}^{1} \Delta_{j+i}.$$

$T_u v$ est appelé le paraproduit de v par u et $R(u,v)$ le reste.
Rappelons maintenant les espaces fonctionnels suivants.

DÉFINITION 0.8 (Espaces de Besov). • *Etant donnés* $(p_1, p_2) \in [1,+\infty]^2$ *et* $s \in \mathbb{R}$, *l'espace de Besov non homogéne* $B^s_{p,r}$ *est l'espace des distributions tempérées* u, *telles que*

$$\|u\|_{B^s_{p,r}} := \Big(2^{js}\|\Delta_j u\|_{L^{p_1}}\Big)_{\ell^{p_2}} < +\infty.$$

• *Soient* $T > 0$ *et* $r \geq 1$, *on note par* $L^r_T B^s_{p,r}$ *l'espace des distributions* u, *définies sur* $[0,T] \times \mathbb{R}^d$, *telles que*

$$\|u\|_{L^r_T B^s_{p,r}} := \Big\|\Big(2^{js}\|\Delta_j u\|_{L^{p_1}}\Big)_{\ell^{p_2}}\Big\|_{L^r_T} < \infty.$$

- *Si, avec les mêmes notations, on intègre en temps avant de prendre la norme ℓ^{p_2} on obtient l'espace $\widetilde{L^r_T}B^s_{p,r}$ muni de la norme*

$$\|u\|_{\widetilde{L^r_T}B^s_{p,r}} := \Big(2^{js}\|\Delta_j u\|_{L^r_T L^{p_1}}\Big)_{\ell^{p_2}}.$$

Pour un espace de Banach E, nous avons noté par $\|f\|_{L^r_T E} := \big(\int_0^T \|f(t)\|^r_E dt\big)^{\frac{1}{r}}$, si r est fini et en faisant la modification classique si r est infini.

Pour des valeurs particulières de (p, r, s) on retrouve des espaces usuels:
– Espaces de Sobolev: $H^s = B^s_{2,2}$.
– Espaces de Hölder: $C^s = B^s_{\infty,\infty}$, pour $s \in \mathbb{R} \setminus \mathbb{N}$.

Les relations entre ces espaces sont établies dans le lemme suivant qui est une conséquence directe de l'inégalité de Minkowski.

LEMME 0.9. *Soient $s \in \mathbb{R}, \varepsilon > 0$, $r \geq 1$ et $(p_1, p_2) \in [1, \infty]^2$. Alors on a les injections continues suivantes*

$$L^r_T B^s_{p,r} \hookrightarrow \widetilde{L^r_T}B^s_{p,r} \hookrightarrow L^r_T B^{s-\varepsilon}_{p,r}, \text{ si } \quad r \leq p_2,$$
$$L^r_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L^r_T}B^s_{p,r} \hookrightarrow L^r_T B^s_{p,r}, \text{ si } \quad r \geq p_2.$$

Les inégalités de Bernstein sont des estimations sur les norme L^p des fonctions localisées en fréquence.

LEMME 0.10. *Il existe une constante C tel que pour tout $q, k \in \mathbb{N}$, $1 \leq a \leq b$ et pour $f \in L^a(\mathbb{R}^d)$,*

$$\begin{aligned}\sup_{|\alpha|=k} \|\partial^\alpha S_q f\|_{L^b} &\leq C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))}\|S_q f\|_{L^a},\\ C^{-k}2^{qk}\|\Delta_q f\|_{L^a} &\leq \sup_{|\alpha|=k}\|\partial^\alpha \Delta_q f\|_{L^a} \leq C^k 2^{qk}\|\Delta_q f\|_{L^a}.\end{aligned}$$

2.4. Théorie locale Navier-Stokes fractionnaire. Dans cette partie on revisite la théorie d'existence locale pour Navier-Stokes fractionnaire. Les preuve suit ceux valables pour Navier-Stokes ordinaires. On n'a pas trouvé de références claires sur la question alors on a refait les preuves par souci de complétude.

2.4.1. *Cas de données $\dot{H}^{s_\alpha}$.* On rappelle qu'on travaille sur la forme intégrale (3) pour une donnée initiale $u_0 \in \dot{H}^{s_\alpha}$, avec $s_\alpha = \frac{5}{2} - 2\alpha$.

THÉORÈME 0.11. *Soient $s_\alpha = \frac{5}{2} - 2\alpha$ et $\mathbf{q}_\alpha = \frac{4\alpha}{2\alpha-1}$, avec $\frac{1}{2} < \alpha < 1$. Pour toute $u_0 \in \dot{H}^{s_\alpha}(\mathbb{R}^3)$ une fonction vectorielle solénoïdale, il existe un temps positif T tel que le système* (3) *admet une unique solution u vérifiant*

$$u \in \tilde{L}^{\mathbf{q}_\alpha}([0,T], \dot{H}^{2-\alpha}).$$

De plus, u appartient à l'espace d'énergie E_T^α défini par:

$$E_T^\alpha = \mathcal{C}\big([0,T], \dot{H}^{s_\alpha}\big) \cap L^2\big([0,T], \dot{H}^{s_\alpha+\alpha}\big).$$

La démonstration du Théorème 0.11 utilise le théorème de point fixe de Picard dans un espace approprié. Cet espacé est relié aux estimées des solutions de l'équation de la chaleur fractionnaire. Le lemme suivant décrit l'effet régularisant de l'équation de la chaleur avec un Laplacian fractionnaire.

LEMME 0.12. *Soit u un champ de vecteur régulier à divergence nulle vérifiant*

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u + \nabla p = f \\ u_{|t=0} = u^0 \end{cases}$$

sur l'intervalle $[0,T]$. Alors, pour tout $p \geq r \geq 1$ et $s \in \mathbb{R}$, on a

$$\|u\|_{C([0,T];\dot{B}^s_{q,\ell}) \cap \tilde{L}^p_T \dot{B}^{s+\frac{2\alpha}{p}}_{q,\ell}} \lesssim \|u^0\|_{\dot{B}^s_{q,\ell}} + \|f\|_{\tilde{L}^r_T \dot{B}^{s-2\alpha+\frac{2\alpha}{r}}_{q,\ell}}.$$

Le point important est le fait que la forme bilinéaire B définie par (5), est une application continue sur $\tilde{L}^{\mathbf{q}_\alpha}([0,T], \dot{H}^{2-\alpha})$.

2.4.2. *Cas de données L^{p_α}.* Un deuxième résultat fondamental présente l'existence de solutions des équations de Navier-Stokes fractionnaire (2) pour une donnée initiale $u_0 \in L^{p_\alpha}$, avec $p_\alpha = \frac{3}{2\alpha-1}$.

THÉORÈME 0.13. *Soit $p_\alpha = \frac{3}{2\alpha-1}$, avec $\frac{1}{2} < \alpha < 1$. Pour tout $u_0 \in L^{p_\alpha}(\mathbb{R}^3)$ un champ de vecteur à divergence nulle, il existe un temps positif T tel que le modèle* NS_α *(2) admet une unique solution u vérifiant*

$$u \in \mathcal{C}([0,T], L^{p_\alpha}(\mathbb{R}^3)) \quad et \quad t^{\frac{2\alpha-1}{2\alpha}}\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \in L^\infty(]0,T]).$$

De plus, si $\|u_0\|_{L^{p_\alpha}(\mathbb{R}^3)}$ est suffisamment petite alors la solution u est globale en temps.

La démonstration du Théorème 0.13 repose à nouveau sur une application du théorème de point fixe de Picard, dans des espaces fonctionnel de type $Y_{q,T}$ défini par:

$$Y_{q,T} := \{f \in L^\infty_{loc}(]0,T], L^q(\mathbb{R}^3)/ \quad \|f\|_{Y_{q,T}} < \infty\},$$

où

$$\|f\|_{Y_{q,T}} = \sup_{0<t\leq T} t^{\frac{3}{2\alpha}(\frac{1}{p_\alpha}-\frac{1}{q})}\|f\|_{L^q}.$$

tout en se basant sur la proposition suivante.

PROPOSITION 0.14. *Soient $p, q, r \in [p_\alpha, \infty]$ vérifiant $\frac{1}{q} \leq \frac{2}{r} < \frac{1}{p_\alpha} + \frac{1}{q}$. Alors il existe une constante $C > 0$ telle que*

$$\|B(u,u)\|_{Y_{q,T}} \leq C\|u\|^2_{Y_{r,T}}.$$

De plus, on a

$$\lim_{t\longrightarrow 0^+} \|e^{-t(-\Delta)^\alpha} f\|_{Y_{q,T}} = 0.$$

Le noyau de la chaleur fractionnaire n'a pas la propriété de décroissance du noyau de la chaleur ($\alpha = 1$) néanmoins on a:

LEMME 0.15. *Pour tout $\beta > 1$, l'application*

$$\psi : x \longmapsto \int_{\mathbb{R}^3} e^{ix\xi} e^{-|\xi|^\beta} d\xi$$

et $\nabla\psi$ appartiennent à $L^p(\mathbb{R}^3)$, pour tout $p \in [1, +\infty]$.

2.5. Décomposition en profils des solutions de NS_α**.** L'objectif principal de cette partie est la décomposition en profils des solutions des équations de Navier-Stokes fractionnaire (2).
Afin d'aboutir à cette décomposition en profils, on commence dans un premier temps par établir la décomposition en profils des solutions de l'équation de la chaleur fractionnaire linéaire associée à NS_α:

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0, & t > 0, x \in \mathbb{R}^d, \\ u_{|t=0} = u_0. \end{cases} \tag{6}$$

Tout d'abord, on note par $H_\alpha(u_0)$ la solution du système (6), associée à la donnée initiale u_0. Ensuite, on a besoin d'introduire quelques définitions.

DÉFINITION 0.16. (1) *On appelle* **échelle***, toute suite* $\mathbf{h} = (h_n)_{n\geq o}$ *de réels positifs et* **coeur***, toute suite* $\mathbf{x} = (x_n)_{n\geq 0}$ *dans* $\mathbb{R}\times\mathbb{R}^3$.

(2) *étant donnés deux échelles* h, h' *et deux coeurs* x, x'*, on dit que les couples* $(\mathbf{h}, \mathbf{x})$ *et* $(\mathbf{h}^{'}, \mathbf{x}^{'})$ *sont orthogonaux si:*

$$\frac{h_n}{h_n^{'}} + \frac{h_n^{'}}{h_n} + |\frac{x_n - x_n^{'}}{h_n}| \underset{n\to\infty}{\longrightarrow} +\infty.$$

La décomposition en profils pour l'équation (6) est obtenue à partir d'une décomposition en profils établie par P. Gérard [**13**], pour une suite bornée dans les espaces homogènes de Sobolev. On présente en particulier cette décomposition en profils établie par P. Gérard, en dimension trois de l'espace.

THÉORÈME 0.17 ([**13**]). *Soient* $s \in]0, \frac{3}{2}[$ *et* $(\varphi_n)_{n\geq 0}$ *une suite bornée dans* $\dot{H}^s(\mathbb{R}^3)$*. Alors, à une extraction de sous-suite près,* (φ_n) *se décompose de la façon suivante :*

$$\varphi_n(x) = \sum_{j=1}^{\ell} \frac{1}{(h_n^j)^{\frac{3}{2}-s}} \Phi^j(\frac{x - x_n^j}{h_n^j}) + w_n^\ell(x),$$

où

(1) $\Phi^j \in \dot{H}^s(\mathbb{R}^3), \quad \forall j \in \mathbb{N}^*,$

(2) *la suite* w_n^ℓ *est uniformément bornée en* ℓ *dans* $\dot{H}^s(\mathbb{R}^3)$*, vérifiant*

$$\limsup_{n\to\infty} \|w_n^\ell\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \underset{\ell\to\infty}{\longrightarrow} 0$$

(3) *la suite* $(\mathbf{h}_n^j, \mathbf{x}_n^j) \in (\mathbb{R}_*^+, \mathbb{R}^3)^{\mathbb{N}}$ *vérifie la propriété d'orthogonalité suivante; pour tout* $j, k \in \mathbb{N}$ *tel que* $j \neq k$,

$$\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} + \frac{|x_n^j - x_n^k|}{h_n^j} \longrightarrow +\infty, \qquad n \to \infty.$$

De plus, pour tout entier $\ell \geq 1$,

$$\|\varphi_n\|_{\dot{H}^{s_\alpha}}^2 = \sum_{j=1}^{\ell} \|\Phi^j\|_{\dot{H}^{s_\alpha}}^2 + \|w_n^\ell\|_{\dot{H}^{s_\alpha}}^2 + o(1), \quad n \to \infty.$$

Soit (u_n) une suite définie par $u_n = H_\alpha(\varphi_n)$ vérifiant

$$u_n(t,x) = \mathrm{e}^{-t(-\Delta)^\alpha}\varphi_n(x) = \widehat{\mathrm{e}^{t(-\Delta)^\alpha}} \star \varphi_n(x).$$

Dans la suite on utilise la notation suivante: pour tout $T > 0$

$$\|\psi\|^2_{E^\alpha_T} = \|\psi\|^2_{L^\infty([0,T[,\dot{H}^{s_\alpha})} + \|\psi\|^2_{L^2([0,T[,\dot{H}^{s_\alpha+\alpha})}.$$

En utilisant Théorème 0.17, on établit la décomposition en profils pour l'équation de la chaleur fractionnaire (6).

THÉORÈME 0.18 ([**2**]). *Soient $(\phi)_n$ une suite de champs de vecteurs à divergence nulle, bornée dans $\dot{H}^{s_\alpha}(\mathbb{R}^3)$ et (Φ^j, Γ^j) les profils qui lui ont associées via Théorème 0.17. On pose : $v_n = H_\alpha(\phi_n)$ et $V^j = H_\alpha(\Phi^j)$. Alors, à une extraction de sous-suite près, on a pour tout $\ell \in \mathbb{N}^*$, $t \in \mathbb{R}^+$, $x \in \mathbb{R}^3$,*

$$v_n(t,x) = \sum_{j=1}^{\ell} \Big(\frac{1}{h_n^j}\Big)^{\frac{3}{p_\alpha}} V^j\Big(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x - x_n^j}{h_n^j}\Big) + w_n^\ell(t,x),$$

où (w_n^ℓ) est uniformément bornée dans E^α_∞ en ℓ, et vérifie

$$\limsup_{n\to\infty} \|w_n^\ell\|_{L^\infty(\mathbb{R}^+, L^{p_\alpha}(\mathbb{R}^3))} \underset{\ell\to\infty}{\longrightarrow} 0.$$

De plus, pour tout $\ell \in \mathbb{N}$, on a

$$\|u_n\|^2_{E^\alpha_\infty} = \sum_{j=1}^{\ell} \|V^j\|^2_{E^\alpha_\infty} + \|w_n^\ell\|^2_{E^\alpha_\infty} + o(1), \qquad n \to \infty.$$

Finalement, on établit la décomposition en profils des solutions des équations de Navier-Stokes fractionnaire globales. Pour cela on se place dans la boule maximale pour l'existence globale.
On définit:

$$C_{L^{p_\alpha}} := \sup\{\rho > 0 : \quad \mathbb{B}_\rho \cap \dot{H}^{s_\alpha} \subset \mathcal{E}\},$$

où

$$\mathbb{B}_\rho := \{\phi \in L^{p_\alpha} : \|\phi\|_{L^{p_\alpha}} < \rho\}$$

et

$$\mathcal{E} = \{u_0 \in \dot{H}^{s_\alpha} : \mathrm{NS}_\alpha(u_0) \in E^\alpha_\infty\}.$$

La décomposition en profil pour les solutions du système de Navier-Stokes fractionnaire sont valables pour $\alpha \in [\frac{5}{6}, 1[$. Plus précisément, on a le théorème suivant.

THÉORÈME 0.19 ([**2**]). *Soient $\alpha \in [\frac{5}{6}, 1[$ et $\rho < C_{L^{p_\alpha}}$. Soient $(\phi)_n$ une suite de champs de vecteurs à divergence nulle, bornée dans $\dot{H}^{s_\alpha} \cap \mathbb{B}_\rho$ et (Φ^j, Γ^j) les profils qui lui ont associées via Théorème 0.17. On note $u_n = \mathrm{NS}_\alpha(\varphi_n)$ et $U^j = \mathrm{NS}_\alpha(\Phi^j)$. Alors, à une extraction de sous-suite près, la suite (u_n) se décompose de la façon suivante, pour tout $n \in \mathbb{N}$, $\ell \in \mathbb{N}^*$, $t \in \mathbb{R}^+$ et $x \in \mathbb{R}^3$,*

$$u_n(t,x) = \sum_{j=1}^{\ell} (\frac{1}{h_n^j})^{2\alpha-1} U^j(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x - x_n^j}{h_n^j}) + w_n^\ell(t,x) + r_n^\ell(t,x),$$

où w_n^ℓ est comme dans Théorème 0.18 et le reste r_n^ℓ vérifie

$$\limsup_{n\to\infty} \|r_n^\ell\|_{E_\infty^\alpha} \underset{\ell\to\infty}{\longrightarrow} 0.$$

La preuve de ce théorème utilise une analyse perturbative et la difficulté dépend de la valeur de α. Grosso modo, si $\alpha \in [\frac{7}{8}, 1[$ alors la preuve peut est similaire à celui pour Navier-Stokes ordinaire ($\alpha = 1$). Pour $\alpha \in [\frac{5}{6}, \frac{7}{8}[$ l'analyse est plus compliquée. Les cas $\alpha \in]\frac{1}{2}, 1[$ restent ouverts.
Comme applications on peut citer les deux corollaires suivantes.

COROLLAIRE 0.20. *Il existe une fonction croissante $\mathbb{R}^+ \times [0, C_{L^{p_\alpha}}[\to [0, \infty[$, telle que pour toute solution u (22) avec $u_0 \in \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha}$, on a*

$$\|u\|_{E_\infty \alpha} \leq A(\|u_0\|_{\dot{H}^{s_\alpha}}, \|u_0\|_{L^{p_\alpha}}).$$

Une description précise de A reste un problème ouvert. On sait seulement que $A(t) \sim t$ pour t petit.
La seconde application est du Théorème 0.19.

COROLLAIRE 0.21. *L'application $F : \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha} \longrightarrow C(\mathbb{R}, \dot{H}^{s_\alpha})$ qui associe $\varphi \in \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha}$ à l'unique solution u de (22) avec donnée initiale φ est Lipschitzienne dans $\mathbb{B}_\rho \cap \dot{H}^{s_\alpha}$ pour tout $\rho < C_{L^{p_\alpha}}$.*

3. Comportement asymptotique des solutions globale de fractional Navier-Stokes

Dans cette partie on étudie les propriétés des solutions de Navier-Stokes fractionnaire. La restriction sur la dissipation est plus restrictive.

THEOREM 0.22. *Soit* $\alpha \in [\frac{5}{6}, 1[$. *Soit* $u \in \mathcal{C}([0, \infty[; \dot{H}^{s_\alpha})$ *une solution globale de* (2). *Alors,* $u \in E^{\alpha}_{\infty}$ *et*

$$\lim_{t \to \infty} \|u(t)\|_{\dot{H}^{s_\alpha}} = 0.$$

Le cas $\alpha \in]\frac{1}{2}, \frac{5}{6}[$ est un problème ouvert. En fait, même si on suppose que $u_0 \in H^{s_\alpha}$ et bénificier de la conservation d'énergie on ne peut pas conclure car les quantités contrôlées par l'énergie sont à un niveau de régularité inférieur à $\dot{H}^{s_\alpha}$.
Une conséquence directe du Théorème 0.22 est le corollaire suivant.

COROLLAIRE 0.23. *Soit* $\alpha \in [\frac{5}{6}, 1[$. *L'ensemble des données initiales* u_0 *engendrant des solutions globales (via Théorème* (0.11)*) est un ouvert de* $\dot{H}^{s_\alpha}$*).*

La théorie d'existence locale ne dit rien sur le comportement ponctuelle de la norme $\dot{H}^{s_\alpha}$ de la solution lorsque le temps s'approche du temps d'explosion. La question pour Navier-Stokes a été résolue dans **[8]**, **[?]** et GKP.

HYPOTHÈSE (H). *La solution nulle est l'unique solution* $u = \mathrm{NS}_\alpha(u_0)$ *de* (2) *pour laquelle il existe une suite croissante et bornée* $t_n \in [0, T^*[$ *telle que*

$$\sup_n \|u\|_{L^\infty([0,t_n],L^{p_\alpha})} < \infty$$

et

$$u(t_n, .) \longrightarrow 0 \qquad \text{dans} \quad L^2_{loc}(\mathbb{R}^3).$$

THÉORÈME 0.24. *[***[3]***] Soit* $\alpha \in [\frac{5}{6}, 1[$ *et supposons que l'hypothèse* (H) *est vérifiée. Soit* $u_0 \in \dot{H}^{s_\alpha}$ *et u est la solution de* (22) *qui lui est associée. Si le temps d'existence maximale* T^* *est finie alors*

$$\sup_{t \in [0, T^*[} \|u(t)\|_{\dot{H}^{s_\alpha}} = +\infty.$$

L'ingrédient principal dans la preuve du Théorème 0.24 est le théorème suivant qui est une version raffiné du Théorème 0.19 (la preuve est néanmoins suit le même calcul que celui développé dans la preuve de ce théorème).

THÉORÈME 0.25 ([**3**]). *Soit* $\alpha \in [\frac{5}{6}, 1[$. *Soient* $(\phi)_n$ *une suite de champs de vecteurs à divergence nulle, bornée dans* $\dot{H}^{s_\alpha}$ *et* (Φ^j, Γ^j) *les profils qui lui sont associées via Théorème 0.17. On note* $u_n = \mathrm{NS}_\alpha(\varphi_n)$ *et* $U^j = \mathrm{NS}_\alpha(\Phi^j)$.

Pour tout $a_n > 0$ *les assertions suivantes sont équivalentes.*

(i) *Pour tout* $j \geq 1$, *on a*

$$\limsup_{n\to\infty} \|U^j\|_{E^\alpha_{\tilde{a}^j_n}} < \infty,$$

où

$$\tilde{a}^j_n \stackrel{\text{def}}{=} (h^j_n)^{2\alpha} a_n.$$

(ii)

$$\limsup_{n\to\infty} \|u_n\|_{E^\alpha_{a_n}} < \infty.$$

De plus, si (i) *ou* (ii) *est vérifiée, alors*

$$u_n = \sum_{j=1}^{\ell} \Gamma^n_j U^j + w^\ell_n + r^\ell_n,$$

et

$$\limsup_{n\to\infty} \|r^\ell_n\|_{E^\alpha_{a_n}} \underset{\ell\to\infty}{\longrightarrow} 0.$$

Ici,

$$\Gamma^n_j U^j(t,x) := (\frac{1}{h^j_n})^{2\alpha-1} U^j(\frac{t}{(h^j_n)^{2\alpha}}, \frac{x - x^j_n}{h^j_n}).$$

CHAPTER 1

Local theory for the Euler 2D with large class of initial data

1. Introduction

We consider the Euler system related to an incompressible inviscid fluid with constant density, namely

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0, \qquad x \in \mathbb{R}^d, t > 0, \\ \nabla . u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{7}$$

Here, the vector field $u = (u_2, u_1, ..., u_d)$ is a function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ denoting the velocity of the fluid and the scalar function P stands for the pressure. The second equation of the system $\nabla . u = 0$ is the condition of incompressibility. Mathematically, it guarantees the preservation of Lebesgue measure by the particle-trajectory mapping (the classical flow associated to the velocity vector fields). It is worthy of noting that the pressure can be recovered from the velocity via an explicit Calderón-Zygmund type operator (see [**7**] for instance).

The question of local well-posedness of (7) with smooth data was resolved by many authors in different spaces (see for instance [**7, 8**]). In this context, the vorticity $\omega = \operatorname{curl} u$ plays a fundamental role. In fact, the well-known BKM criterion [**3**] ensures that the development of finite time singularities for these solutions is related to the blow-up of the L^∞ norm of the vorticity near the maximal time existence. A direct consequence of this result is the global well-posedness of the two-dimensional Euler solutions with smooth initial data, since the vorticity satisfies the transport equation

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \tag{8}$$

and then all its L^p norms are conserved.

Another class of solutions requiring lower regularity on the velocity can be considered: the weak solutions (see for instance [**19**, Chap 4]). They solve a weak form of the equation in the distribution sense, placing the equations in large spaces and using duality. The divergence form of Euler equations allows to put all the derivative on the test functions and so to obtain

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi + (u \cdot \nabla)\varphi).u \, dxdt + \int_{\mathbb{R}^d} \varphi(0,x) u_0(x) \, dx = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$ with $\nabla.\varphi = 0$. In the two dimensional space and when the regularity is sufficient to give a sense to Biot-Savart law, then one can consider an alternative weak formulation: the vorticity-stream weak formulation. It consists in resolving the weak form of (8) supplemented with the Biot-Savart law:

$$u = K * \omega, \quad \text{with} \quad K(x) = \frac{x^\perp}{2\pi|x|^2}. \tag{9}$$

In this case, (v, ω) is a weak solution to the vorticity-stream formulation of the 2D Euler equation with initial data ω_0 if (9) is satisfied and

$$\int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + u \nabla \varphi) \omega(t,x) dxdt + \int_{\mathbb{R}^2} \varphi(0,x) \omega_0(x) dx = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R})$.
The questions of existence/uniqueness of weak solutions have been extensively studied and a detailed account can be found in the books [**7, 8, 19**]. We emphasize that, unlike the fixed-point argument, the compactness method does not guarantee the uniqueness of the solutions and then the two issues (existence/uniqueness) are usually dealt with separately. These questions have been originally addressed by Yudovich in [**25**] where the existence and uniqueness of weak solution to 2D Euler systems (in bounded domain) are proved under the assumptions: $u_0 \in L^2$ and $\omega_0 \in L^\infty$. Serfati [**20**] proved the uniqueness and existence of a solution with initial velocity and vorticity which are only bounded (without any integrability condition). There is an extensive literature on the existence of weak solution to Euler system, possibly without uniqueness, with unbounded vorticity. DiPerna-Majda [**10**] proved the existence of weak solution for $\omega_0 \in L^1 \cap L^p$

with $2 < p < \infty$. The L^1 assumption in DiPerna-Majda's paper has been removed by Giga-Miyakawa-Osada [**13**]. Chae [**5**] proved an existence result for ω_0 in $L\ln^+ L$ with compact support. More recently, Taniuchi [**21**] has proved the global existence (possibly without uniqueness nor regularity persistence) for $(u_0, \omega_0) \in L^\infty \times \mathrm{BMO}$. The papers [**23**] and [**26**] are concerned with the questions of existence and uniqueness of weak solutions for larger classes of vorticity. Both have intersections with the present paper and we will come back to them at the end of this section (Remark 1.11). A framework for measure-valued solutions can be found in [**9**] and [**11**] (see also [**12**] for more detailed references).
Roughly speaking, the proof of uniqueness of weak solutions requires a uniform, in time, bound of the log-Lipschitzian norm of the velocity. This "almost" Lipschitzian regularity of the velocity is enough to assure the existence and uniqueness of the associated flow (and then of the solution). Initial conditions of the type $\omega_0 \in L^\infty(\mathbb{R}^2)$ (or $\omega_0 \in \mathrm{BMO}, B^0_{\infty,\infty}, ...$) guarantee the log-Lipschitzian regularity of u_0. However, the persistence of such regularity when time varies requires an *a priori* bound of these quantities for the approximate-solution sequences. This is trivially done (via the conservation law) in the L^∞ case but not at all clear for the other cases. The main issue in this context is the action of Lebesgue measure preserving homeomorphisms on these spaces. In fact, it is easy to prove that all these spaces are invariant under the action of such class of homeomorphisms, but the optimal form of the constants (depending on the homeomorphisms and important for the application) are not easy to find. It is worth of mentioning, in this context, that the proof by Vishik [**24**] of the global existence for (7) in the borderline Besov spaces is based on a refined result on the action of Lebesgue measure preserving homeomorphisms on $B^0_{\infty,1}$.
In [**2**] Berncot and Keraani established the global existence and uniqueness in a $BM0$ type space (some Banach space which is strictly imbricated between L^∞ and BMO). In [**1**] we prove the local existence and uniqueness in a more general spaces. These space are constructed in the same spirit of these in [**2**].

2. Functional spaces

Before going any further, let us introduce this functional space (details about BMO spaces can be found in the book of Grafakos **[14]**).

DEFINITION 1.1. *Let* $\Psi\ :]0,+\infty\ \longrightarrow]0,+\infty[$ *a regular function satisfying the following assumptions*[1]*:*

(H1) Ψ *is increasing.*
(H2) *There exist* $C>0$ *such that*

$$\Psi(r+s)\leq C(\Psi(r)+\Psi(s)),\qquad \forall\,(r,s)\in]0,+\infty[\times]0,+\infty[.$$

(H3) *There exists a smooth function* g *such that*

$$\lambda\mapsto\sum_{j\geq g(\lambda)}e^{-\frac{j}{\lambda}}\Psi(1+\ln(1+\lambda))$$

is well defined smooth function on $]1,+\infty[$.

For a complex-valued locally integrable function on $\mathbb{R}^2$*, set*

$$\|f\|_{\tilde{\mathrm{B}}\mathrm{MO}}:=\|f\|_{\mathrm{BMO}}+\sup_{B_1,B_2}\frac{|\mathrm{Avg}_{B_2}(f)-\mathrm{Avg}_{B_1}(f)|}{\Psi\big(1+\frac{1+|\ln r_2|}{1+|\ln r_1|}\big)},$$

where the supremum is taken aver all pairs of balls $B_2=B(x_2,r_2)$ *and* $B_1=B(x_1,r_1)$ *in* $\mathbb{R}^2$ *with* $0<r_1\leq 1$ *and* $2B_2\subset B_1$*. Here and subsequently, we denote*

$$\mathrm{Avg}_D(g):=\frac{1}{|D|}\int_D g(x)dx,$$

for every $g\in L^1_{loc}$ *and every non negligible set* $D\subset\mathbb{R}^2$*. Also, for a ball* B *and* $\lambda>0$*,* λB *denotes the ball that is concentric with* B *and whose radius is* λ *times the radius of* B*.*

[1] It is easy to see that $\Psi(r)=r^m\ln(1+r)^n$ satisfies theses assumptions for every $(m,n)\in\mathbb{N}^*$.

We recall that It is worth of noting that if B_2 and B_1 are two balls such that $2B_2 \subset B_1$ then[2]

$$|\mathrm{Avg}_{B_2}(f) - \mathrm{Avg}_{B_1}(f)| \lesssim \ln(1 + \frac{r_1}{r_2})\|f\|_{\mathrm{BMO}}. \tag{10}$$

In the definition of $\widetilde{\mathrm{B}}\mathrm{MO}$ we replace the term $\ln(1 + \frac{r_1}{r_2})$ by $\varepsilon(r_1, r_2)$, which is smaller. This puts more constraints on the functions belonging to this space[3] and allows us to derive some crucial property on the composition of them with Lebesgue measure preserving homeomorphisms, which is the heart of our analysis.

Let us now recall that the set of log-Lipschitzian vector fields on $\mathbb{R}^2$, denoted by LL, is the set of bounded vector fields v such that

$$\|v\|_{LL} := \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x - y|\big(1 + \big|\ln|x - y|\big|\big)} < \infty.$$

The importance of this notion lies in the fact that if the vorticity belong to the Yudovich type space (say $L^1 \cap L^\infty$) then the velocity is no longer Lipschitzian, but log-Lipschitzian. In this case we still have existence and uniqueness of flow but a loss of regularity may occur. Actually, this loss of regularity is unavoidable and its degree is related to the norm $L^1_t(LL)$ of the velocity. The reader is referred to section 3.3 in **[1]** for more details about this issue.

To capture this behavior, and overcome the difficulty generated by it, we introduce the following definition.

DEFINITION 1.2. *For every homeomorphism ψ, we set*

$$\|\psi\|_* := \sup_{x \neq y} \Phi\big(|\psi(x) - \psi(y)|, |x - y|\big),$$

[2] Throughout this chapter the notation $A \lesssim B$ means that there exists a positive universal constant C such that $A \leq CB$.

[3] Here, we identify all functions whose difference is a constant. In section 2, we will prove that $\widetilde{\mathrm{B}}\mathrm{MO}$ is complete and strictly imbricated between BMO and L^∞. The "L" in $\widetilde{\mathrm{B}}\mathrm{MO}$ stands for "logarithmic".

where Φ *is defined on* $]0,+\infty[\times]0,+\infty[$ *by*

$$\Phi(r,s)=\begin{cases}\max\{\frac{1+|\ln(s)|}{1+|\ln r|};\frac{1+|\ln r|}{1+|\ln(s)|}\}, & \text{if} \quad (1-s)(1-r)\geq 0,\\ (1+|\ln s|)(1+|\ln r|), & \text{if} \quad (1-s)(1-r)\leq 0.\end{cases}$$

Since Φ is symmetric then $\|\psi\|_*=\|\psi^{-1}\|_*\geq 1$. It is clear also that every homeomorphism ψ satisfying

$$\frac{1}{C}|x-y|^\alpha\leq|\psi(x)-\psi(y)|\leq C|x-y|^\beta,$$

for some $\alpha,\beta,C>0$ has its $\|\psi\|_*$ finite (see Proposition 63 for a reciprocal property).
The definition above is motivated by this proposition (and by Theorem 1.7 below as well). The following two propositions are taken from **[2]**

PROPOSITION 1.3. *Let* u *be a smooth divergence-free vector fields and* ψ *be its flow:*

$$\partial_t\psi(t,x)=u(t,\psi(t,x)),\qquad \psi(0,x)=x.$$

Then, for every $t\geq 0$

$$\|\psi(t,\cdot)\|_*\leq\exp(\int_0^t\|u(\tau)\|_{LL}d\tau).$$

As an application we obtain the following useful lemma.

PROPOSITION 1.4. *For every* $r>0$ *and a homeomorphism* ψ *one has*

$$4\psi(B(x_0,r))\subset B(\psi(x_0),g_\psi(r)),$$

where[4]*,*

$$g_\psi(r):=\begin{cases}4e^{\|\psi\|_*}r^{\|\psi\|_*}, & \text{if} \quad r\geq 1,\\ 4\max\{er^{\frac{1}{\|\psi\|_*}};e^{\|\psi\|_*}r\}, & \text{if} \quad 0<r<1.\end{cases}$$

In particular,

$$|\ln\Big(\frac{1+|\ln g_\psi(r)|}{1+|\ln r|}\Big)|\lesssim 1+\ln\big(1+\|\psi\|_*\big). \tag{11}$$

REMARK 1.5. The estimate (11) remains valid when we multiply $g_\psi(r)$ by any positive constant.

[4]This notation means that for every ball $B\subset\psi(B(x_0,r))$ we have $4B\subset B(\psi(x_0),g_\psi(r))$.

3. The $\widetilde{\mathrm{B}}\mathrm{MO}$ space

Let us now detail some properties of the space $\widetilde{\mathrm{B}}\mathrm{MO}$ introduced in the first section of this chapter .

PROPOSITION 1.6. *The following properties hold true.*
(i) *The space* $\widetilde{\mathrm{B}}\mathrm{MO}$ *is a Banach space included in* BMO *and strictly containing* $L^\infty(\mathbb{R}^2)$.
(ii) *For every* $g \in C_0^\infty(\mathbb{R}^2)$ *and* $f \in \widetilde{\mathrm{B}}\mathrm{MO}$ *one has*

$$\|g * f\|_{\widetilde{\mathrm{B}}\mathrm{MO}} \leq \|g\|_{L^1} \|f\|_{\widetilde{\mathrm{B}}\mathrm{MO}}. \tag{12}$$

PROOF. (i) Completeness of the space. Let $(f_n)_n$ be a Cauchy sequence in $\widetilde{\mathrm{B}}\mathrm{MO}$. Since BMO is complete then this sequences converges in BMO and then in L^1_{loc}. Using the definition and the the convergence in L^1_{loc}, we get that the convergence holds in $\widetilde{\mathrm{B}}\mathrm{MO}$
(ii) Stability by convolution. (12) follows from the fact that for all $r > 0$

$$x \mapsto \mathrm{Avg}_{B(x,r)}(g * f) = (g * \mathrm{Avg}_{B(\cdot,r)}(f))(x).$$

$\square$

The advantage of using the space $\widetilde{\mathrm{B}}\mathrm{MO}$ lies in the following logarithmic estimate which is the main ingredient for proving Theorem 1.9.

THEOREM 1.7. *There exists a a smooth function G such that such that*

$$\|f \mathrm{o} \psi\|_{\widetilde{\mathrm{B}}\mathrm{MO} \cap L^p} \leq G(\|\psi\|_*) \|f\|_{\widetilde{\mathrm{B}}\mathrm{MO} \cap L^p},$$

for any Lebesgue measure preserving homeomorphism ψ.

PROOF OF THEOREM 1.7. Of course we are concerned with ψ such that $\|\psi\|_*$ is finite (if not the inequality is empty). Without loss of generality one can assume that $\|f\|_{\widetilde{\mathrm{B}}\mathrm{MO} \cap L^p} = 1$. Since ψ preserves Lebesgue measure then the L^p-part of the norm is conserved. For the two other parts of the norm, we will proceed in two steps. In the first step we consider the BMO term of the norm and in the second one we deal with the other term.

Step 1. Let $B = B(x_0, r)$ be a given ball of $\mathbb{R}^2$. By using the L^p-norm we need only to deal with balls whose radius is smaller than a universal constant δ_0 (we want r to be small with respect to the constants appearing in Whitney covering lemma below). Since ψ is a Lebesgue measure preserving homeomorphism then $\psi(B)$ is an open connected[5] set with $|\psi(B)| = |B|$. By Whitney covering lemma, there exists a collection of balls $(O_j)_j$ such that:
- The collection of double ball is a bounded covering:

$$\psi(B) \subset \bigcup 2O_j.$$

- The collection is disjoint and, for all j,

$$O_j \subset \psi(B).$$

- The Whitney property is verified:

$$r_{O_j} \simeq d(O_j, \psi(B)^c).$$

• *Case 1*: $r \leq \frac{1}{4} e^{-\|\psi\|_*}$. In this case

$$g_\psi(r) \leq 1.$$

We set $\tilde{B} := B(\psi(x_0), g_\psi(r))$. Since ψ preserves Lebesgue measure we get

$$\begin{aligned} \mathrm{Avg}_B |f \mathrm{o} \psi - \mathrm{Avg}_B(f \mathrm{o} \psi)| &= \mathrm{Avg}_{\psi(B)} |f - \mathrm{Avg}_{\psi(B)}(f)| \\ &\leq 2\mathrm{Avg}_{\psi(B)} |f - \mathrm{Avg}_{\tilde{B}}(f)|. \end{aligned}$$

Using the notations above

$$\begin{aligned} \mathrm{Avg}_{\psi(B)} |f - \mathrm{Avg}_{\tilde{B}}(f)| &\lesssim \frac{1}{|B|} \sum_j |O_j| \mathrm{Avg}_{2O_j} \big| f - \mathrm{Avg}_{\tilde{B}}(f) \big| \\ &\lesssim I_1 + I_2, \end{aligned}$$

[5] We have also that $\psi(B)^C = \psi(B^C)$ and $\psi(\partial B) = \partial(\psi(B))$.

with

$$\begin{aligned} I_1 &= \frac{1}{|B|}\sum_j |O_j|\big|\mathrm{Avg}_{2O_j}(f) - \mathrm{Avg}_{2O_j}(f)\big| \\ I_2 &= \frac{1}{|B|}\sum_j |O_j|\big|\mathrm{Avg}_{2O_j}(f) - \mathrm{Avg}_{\tilde{B}}(f)\big|. \end{aligned}$$

On one hand, since $\sum |O_j| \leq |B|$ then

$$\begin{aligned} I_1 &\leq \frac{1}{|B|}\sum_j |O_j|\|f\|_{\mathrm{BMO}} \\ &\leq \|f\|_{\mathrm{BMO}}. \end{aligned}$$

On the other hand, sinc $4O_j \subset \tilde{B}$ (remember Lemma 1.4) and $r_{\tilde{B}} \leq 1$, it ensues that

$$\begin{aligned} I_2 &\lesssim \frac{1}{|B|}\sum_j |O_j|\Psi\Big(1+\ln\big(\frac{1-\ln 2r_j}{1-\ln g_\psi(r)}\big)\Big) \\ &\lesssim \frac{1}{|B|}\sum_j |O_j|\Psi\Big(1+\ln\big(\frac{1-\ln r_j}{1-\ln g_\psi(r)}\big)\Big). \end{aligned}$$

Thanks to (11) and (H1) we get

$$\begin{aligned} \Psi\Big(1+\ln\big(\frac{1-\ln r_j}{1-\ln g_\psi(r)}\big)\Big) &\lesssim \Psi\Big(1+\ln\big(\frac{1-\ln r_j}{1-\ln r}\big)\Big) + \Psi\Big(1+\ln\big(\frac{1-\ln r}{1-\ln g_\psi(r)}\big)\Big) \\ &\lesssim 1+\Psi\Big(1+\ln\big(\frac{1-\ln r_j}{1-\ln r}\big)\Big) + \Psi\big(1+\ln(1+\|\psi\|_*\big). \end{aligned} \tag{13}$$

Thus it remains to prove that

$$II := \frac{1}{|B|}\sum_j |O_j|\Psi(1+\ln\big(\frac{1-\ln r_j}{1-\ln r}\big)) \lesssim G(\|\psi\|_*). \tag{14}$$

for some smooth function. For every $k \in \mathbb{N}$ we set

$$u_k := \sum_{e^{-(k+1)}r < r_j \leq e^{-k}r} |O_j|,$$

so that

$$II \leq \frac{1}{|B|}\sum_{k\geq 0} u_k \Psi\big(1+\ln\big(\frac{k+2-\ln r}{1-\ln r}\big)\big). \tag{15}$$

We need the following lemma.

LEMMA 1.8. *There exists a universal constant* $C > 0$ *such that*

$$u_k \leq Ce^{-\frac{k}{\|\psi\|_*}} r^{1+\frac{1}{\|\psi\|_*}},$$

for every $k \in \mathbb{N}$.

Coming back to (15). Let N a large integer to be chosen later. We split the sum in the right hand side of (15) into two parts

$$II \lesssim \sum_{k \leq N} (...) + \sum_{k > N} (.....) := II_1 + II_2.$$

Since $\sum u_k \leq |B|$ then

$$II_1 \leq \Psi(1 + \ln\big(\frac{N + 2 - \ln r}{1 - \ln r}\big)). \tag{16}$$

On the other hand

$$\begin{aligned} II_2 &\leq \sum_{k>N} e^{-\frac{k+(\|\psi\|_*-1)\ln(r)}{\|\psi\|_*}} r^{\frac{1}{\|\psi\|_*}-1} \Psi(1 + \ln\big(\frac{k+2-\ln r}{1-\ln r}\big)) \\ &\leq \sum_{k>N+(\|\psi\|_*-1)\ln(r)} e^{-\frac{k}{\|\psi\|_*}} \Psi(1 + \ln\big(\frac{k + (\|\psi\|_* + 1)|\ln r|}{1 + |\ln r|}\big)) \\ &\lesssim \sum_{k>N+(\|\psi\|_*-1)\ln(r)} e^{-\frac{k}{\|\psi\|_*}} \Psi(1 + \ln\big(k + \|\psi\|_*\big)). \end{aligned}$$

Taking $N = g(\|\psi\|_*) + (\|\psi\|_* - 1)|\ln(r)| + 1$ yields

$$II_2 \lesssim \sum_{k > g(\|\psi\|_*)} e^{-\frac{k}{\|\psi\|_*}} \Psi(1 + \ln\big(k + \|\psi\|_*\big)). \tag{17}$$

This gives also

$$II_1 \lesssim \Psi(1 + \ln\Big(g(\|\psi\|_*) + \|\psi\|_*\Big). \tag{18}$$

Putting (17) and (18) together

$$II \lesssim \Psi(1 + \ln\Big(g(\|\psi\|_*) + \|\psi\|_*\Big) + \sum_{k > g(\|\psi\|_*)} e^{-\frac{k}{\|\psi\|_*}} \Psi(1 + \ln\big(k + \|\psi\|_*\big)).$$

The RHS is a regular function on $\|\psi\|_*$ thanks to the assumption $H3$. This ends the proof of (14).

• *Case 2:* $\delta_0 \geq r \geq \frac{1}{4}e^{-\|\psi\|_*}$. In this case

$$|\ln r| \lesssim \|\psi\|_*.$$

Since ψ preserves Lebesgue measure, we get

$$\begin{aligned} I &:= \operatorname{Avg}_B|f\text{o}\psi - \operatorname{Avg}_B(f\text{o}\psi)| \\ &\leq 2\operatorname{Avg}_{\psi(B)}|f|. \end{aligned}$$

Let $\tilde{O}_j$ denote the ball which is concentric to O_j and whose radius is equal to 1 (we use the same Whitney covering as above). Without loss of generality we can assume $\delta_0 \leq \frac{1}{4}$. This guarantees $4O_j \subset \tilde{O}_j$ and yields by definition

$$\begin{aligned} I &\lesssim \frac{1}{|B|}\sum_j |O_j|\operatorname{Avg}_{2O_j}|f - \operatorname{Avg}_{\tilde{O}_j}(f)| + \frac{1}{|B|}\sum_j |O_j||\operatorname{Avg}_{\tilde{O}_j}(f)| \\ &\lesssim \frac{1}{|B|}\sum_j |O_j|\Psi\Big(1+\ln\big(1-\ln 2r_j\big)\Big)\|f\|_{\tilde{\mathrm{B}}\mathrm{MO}} + \frac{1}{|B|}\sum_j |O_j|\|f\|_{L^p} \\ &\lesssim 1 + \frac{1}{|B|}\sum_j |O_j|\Psi\big(1+\ln\big(1-\ln r_j\big)\big). \end{aligned}$$

As before, and using the fact that $|\ln r| \lesssim \|\psi\|_*$, one writes

$$\begin{aligned} I &\lesssim \frac{1}{|B|}\sum_{k\geq 0} u_k\Psi\big(1+\ln\big(k+2-\ln r\big)\big) \\ &\lesssim \frac{1}{|B|}\sum_{k\geq 0} u_k\Psi\big(1+\ln\big(k+2+\|\psi\|_*\big)\big) \end{aligned}$$

A similar analysis as before leads to the desired result.
The outcome of this first step of the proof is

$$\|f\text{o}\psi\|_{\mathrm{BMO}\cap L^p} \lesssim G(\|\psi\|_*)\|f\|_{\tilde{\mathrm{B}}\mathrm{MO}\cap L^p}.$$

Step 2. This step of the proof deals with the second term in the $\tilde{\mathrm{B}}$MO-norm. It is shorter than the first step because it makes use of the arguments developed above. Take $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ in $\mathbb{R}^2$ with $r_1 \leq 1$ and $2B_2 \subset B_1$. There are three cases to consider.
• *Case 1:* $r_1 \lesssim e^{-\|\psi\|_*}$ (so that $g_\psi(r_2) \leq g_\psi(r_1) \leq \frac{1}{2}$).

We set $\tilde{B}_i := B(\psi(x_i), g_\psi(r_i)), i = 1, 2$ and

$$J := \frac{|\mathrm{Avg}_{B_2}(f o\psi) - \mathrm{Avg}_{B_1}(f o\psi)|}{\Psi(1 + \ln\big(\frac{1-\ln r_2}{1-\ln r_1}\big))}.$$

Since the denominator is bigger than 1 one get

$$J \leq J_1 + J_2 + J_3,$$

with

$$\begin{aligned} J_1 &= |\mathrm{Avg}_{\psi(B_2)}(f) - \mathrm{Avg}_{\tilde{B}_2}(f)| + |\mathrm{Avg}_{\psi(B_1)}(f) - \mathrm{Avg}_{\tilde{B}_1}(f)| \\ J_2 &= \frac{|\mathrm{Avg}_{\tilde{B}_2}(f) - \mathrm{Avg}_{2\tilde{B}_1}(f)|}{\Psi(1 + \ln\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big))} \\ J_3 &= |\mathrm{Avg}_{\tilde{B}_1}(f) - \mathrm{Avg}_{2\tilde{B}_1}(f)|. \end{aligned}$$

Since $2\tilde{B}_2 \subset 2\tilde{B}_1$ and $r_{2\tilde{B}_1} \leq 1$ then

$$J_2 \leq \frac{\Psi(1 + \ln\big(\frac{1-\ln g_\psi(r_2))}{1-\ln(2g_\psi(r_1))}\big)}{\Psi(1 + \ln\big(\frac{1-\ln r_2}{1-\ln r_1}\big))}\|f\|_{\tilde{\mathrm{B}}\mathrm{MO}}.$$

Using similar argument than (13) (and remembering Remark 1.5) we infer

$$\ln\Big(\frac{1 - \ln g_\psi(r_2)}{1 - \ln(2g_\psi(r_1))}\Big) \lesssim 1 + \ln(1 + \|\psi\|_*) + \ln\Big(\frac{1 - \ln r_2}{1 - \ln r_1}\Big).$$

Thus, (H2) implies

$$J_2 \lesssim \Psi\big(1 + \ln(1 + \|\psi\|_*)\big).$$

The estimation (10) yields

$$J_3 \lesssim \|f\|_{\mathrm{BMO}}.$$

The term J_1 can be handled exactly as in the analysis of case 1 of step 1.

• *Case 2:* $e^{-\|\psi\|_*} \lesssim r_2$. In this case we write

$$J \leq \mathrm{Avg}_{\psi(B_2)}|f| + \mathrm{Avg}_{\psi(B_1)}|f|.$$

Both terms can be handled as in the analysis of case 2 of the proof of BMO-part in step 1.

• *Case 3:* $r_2 \lesssim e^{-\|\psi\|_*}$ and $r_1 \gtrsim e^{-\|\psi\|_*}$. Again since the denominator is bigger than 1 we get

$$J \leq \mathrm{Avg}_{\psi(B_2)}|f - \mathrm{Avg}_{\tilde{B}_2}(f)| + \frac{|\mathrm{Avg}_{\tilde{B}_2}(f)|}{\Psi(1 + \ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right))} + \mathrm{Avg}_{\psi(B_1)}|f| = J_1 + J_2 + J_3.$$

The terms J_1 and J_3 can be controlled as before. The second term is controlled as follows (we make appear the average on $B(\psi(x_2), 1)$ and use Lemma 1.4 with $\|f\|_{L^p} \leq 1$)

$$\begin{aligned} J_2 &\leq \frac{\Psi(1 + \ln(1 - \ln r_2))}{\Psi(1 + \ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right))} \\ &\lesssim \Psi(1 + \ln(1 + |\ln r_1|)) \\ &\lesssim \Psi(1 + \ln(1 + \|\psi\|_*)). \end{aligned}$$

□

4. Local existence and uniqueness in $L^p \cap \tilde{\mathrm{B}}\mathrm{MO}$

The main result in this chapter is the the following local existence-uniqueness theorem.

THEOREM 1.9. *Under the assumptions in the Definition above. Take $\omega_0 \in L^p \cap \tilde{\mathrm{B}}\mathrm{MO}$ with $p \in]1, 2[$. Then there exists $T > 0$ and a unique weak solution (v, ω) to the vorticity-stream formulation of the 2D Euler equation on $[0, T]$:*

$$\omega \in L^\infty([0, T], L^p \cap \tilde{\mathrm{B}}\mathrm{MO}). \tag{19}$$

Some remarks are in order.

REMARK 1.10. The proof gives more, namely $\omega \in \mathcal{C}(\mathbb{R}_+, L^q)$ for all $p \leq q < \infty$. Combined with the Biot-Savart law[6] this yields $u \in \mathcal{C}(\mathbb{R}_+, W^{1,r}) \cap \mathcal{C}(\mathbb{R}_+, L^\infty)$ for all $\frac{2p}{2-p} \leq r < \infty$.

[6] If $\omega_0 \in L^p$ with $p \in]1, 2[$ then a classical Hardy-Littlewood-Sobolev inequality gives $u \in L^q$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

REMARK 1.11. The essential point of Theorem 1.9 is that it provides an initial space which is strictly larger than $L^p \cap L^\infty$ (it contains unbounded elements) which is a space of existence, uniqueness and persistence of regularity at once. We emphasize that the bound (19) is crucial since it implies that u is, uniformly in time, log-Lipschitzian which is the main ingredient for the uniqueness. Once this bound established the uniqueness follows from the work by Vishik [**23**]. In this paper Vishik also gave a result of existence (possibly without regularity persistence) in some large space characterized by growth of the partial sum of the L^∞-norm of its dyadic blocs. We should also mention the result of uniqueness by Yudovich [**26**] which establish uniqueness (for bounded domain) for some space which contains unbounded functions. Note also that the example of unbounded function, given in [**26**], belongs actually to the space $\widetilde{\mathrm{BMO}}$ (see Proposition 1.6 below). Our approach is different from those in [**23**] and [**26**] and uses a classical harmonic analysis “à la stein" without making appeal to the Fourier analysis (para-differential calculus).

REMARK 1.12. The main ingredient of the proof of (19) is a logarithmic estimate in the space $L^p \cap \widetilde{\mathrm{BMO}}$ (see Theorem 1.7 below). It would be desirable to prove this result for BMO instead of $\widetilde{\mathrm{BMO}}$. Unfortunately, as it is proved in [**2**], the corresponding estimate with BMO is optimal (with the bi-Lipschitzian norm instead of the log-Lipschitzian norm of the homeomorphism) and so the argument presented here seem to be not extendable to BMO.

PROOF OF THEOREM 1.9. The proof falls naturally into three parts.

A priori estimate. The following estimate follow directly from Proposition 1.3 and Theorem 1.7.

PROPOSITION 1.13. *Let u be a smooth solution of* (7) *and ω its vorticity. Then, there exists a constant $T > 0$ depending only on the norm $L^p \cap \widetilde{\mathrm{BMO}}$ of ω_0 such that*

$$\|\omega(t)\|_{\widetilde{\mathrm{BMO}}} \leq 2\|\omega(0)\|_{\widetilde{\mathrm{BMO}}},$$

for every $[0, T]$.

PROOF. One has $\omega(t,x) = \omega_0(\psi_t^{-1}(x))$ where ψ_t is the flow associated to the velocity u. Since u is smooth then $\psi_t^{\pm 1}$ is Lipschitzian for every $t \geq 0$. This implies in particular that $\|\psi_t^{\pm 1}\|_*$ is finite for every $t \geq 0$. Theorem 1.7 and Proposition 1.3 yield together

$$\begin{aligned}
\|\omega(t)\|_{\widetilde{\mathrm{BMO}}} &\leq C\|\omega_0\|_{\widetilde{\mathrm{BMO}}\cap L^p} G(\|\psi_t^{-1}\|_*) \\
&\leq C\|\omega_0\|_{\widetilde{\mathrm{BMO}}\cap L^p} G(\exp(\int_0^t \|u(\tau)\|_{LL} d\tau)).
\end{aligned}$$

On has has

$$\begin{aligned}
\|u(t)\|_{LL} &\leq \|\omega(t)\|_{L^2} + \|\omega(t)\|_{B^0_{\infty,\infty}} \\
&\leq C(\|\omega_0\|_{L^2} + \|\omega(t)\|_{BM0}).
\end{aligned}$$

The first estimate is classical (see **[1]** for instance) and the second one is just the conservation of the L^2-norm of the vorticity and the continuity of the embedding BMO $\hookrightarrow B^0_{\infty,\infty}$.

Consequently, we deduce that

$$\|u(t)\|_{LL} \leq C_0(1 + G(\exp(\int_0^t \|u(\tau)\|_{LL} d\tau)),$$

We set

$$\zeta(t) = \int_0^t \|u(\tau)\|_{LL} d\tau, \qquad H(r) = \int_0^r \frac{1}{G(\exp(s))} ds$$

This gives

$$H(\zeta(t)) \leq C_0 t,$$

On chooses T such that

$$C_0 T < \int_0^\infty \frac{1}{G(\exp(s))} ds.$$

This guarantees that

$$\zeta(t) \leq H^{-1}(C_0 t),$$

for every $t \in [0,T]$. This yields in particular

$$\|\omega(t)\|_{\widetilde{\mathrm{BMO}}} \leq C\|\omega_0\|_{\widetilde{\mathrm{BMO}}\cap L^p} G(\exp(H^{-1}(C_0 t))).,\qquad \forall\, t \in [0,T],$$

as claimed.

□

Existence. Let $\omega_0 \in L^p \cap \tilde{\mathrm{B}}\mathrm{MO}$ and $u_0 = k * \omega_0$, with $K(x) = \frac{x^\perp}{2\pi|x|^2}$. We take $\rho \in \mathcal{C}_0^\infty$, with $\rho \geq 0$ and $\int \rho(x)dx = 1$ and set

$$\omega_0^n = \rho_n * \omega_0, \qquad u_0^n = \rho_n * u_0,$$

where $\rho_n(x) = n^2\rho(nx)$. Obviously, ω_0^n is a C^∞ bounded function for every $n \in \mathbb{N}^*$. Furthermore, thanks to (12),

$$\|\omega_0^n\|_{L^p} \leq \|\omega_0\|_{L^p} \qquad \text{and} \qquad \|\omega_0^n\|_{\tilde{\mathrm{B}}\mathrm{MO}} \leq \|\omega_0\|_{\tilde{\mathrm{B}}\mathrm{MO}}.$$

The classical interpolation result between Lebesgue and BMO spaces (see **[14]** for more details) implies that

$$\|\omega_0^n\|_{L^q} \leq \|\omega_0^n\|_{L^p\cap\mathrm{BMO}} \leq \|\omega_0\|_{L^p\cap\mathrm{BMO}}, \qquad \forall\, q \in [p, +\infty[.$$

Since, $\omega_0^n \in L^p \cap L^\infty$ then there exists a unique weak solution u^n with

$$\omega_n \in L^\infty(\mathbb{R}_+, L^p \cap L^\infty).$$

according to the classical result of Yudovich **[25]**. According to Proposition 1.13 one has

$$\|u^n(t)\|_{LL} + \|\omega^n(t)\|_{L^p\cap\tilde{\mathrm{B}}\mathrm{MO}} \leq C, \qquad \forall\, t \in [0, T]. \tag{20}$$

With this uniform estimate in hand, we can perform the same analysis as in the case $\omega_0 \in L^p \cap L^\infty$ (see paragraph 8.2.2 in **[8]** for more explanation). For the convenience of the reader we briefly outline the main arguments of the proof.
If one denotes by $\psi_n(t, x)$ the associated flow to u^n then

$$\|\psi_n^{\pm 1}(t)\|_* \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+. \tag{21}$$

This yields the existence of explicit time continuous functions $\beta(t) > 0$ and $C(t)$ such that

$$|\psi_n^{\pm 1}(t, x_2) - \psi_n^{\pm 1}(t, x_1)| \leq C(t)|x_2 - x_1|^{\beta(t)}, \qquad \forall\, (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Moreover,

$$|\psi_n^{\pm 1}(t_2, x) - \psi_n^{\pm 1}(t_1, x)| \leq |t_2 - t_1|\|u^n\|_{L^\infty} \leq C_0|t_2 - t_1|, \qquad \forall\, (t_1, t_2) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Here, we have used the Biot-Savart law to get

$$\|u^n(t)\|_{L^\infty} \lesssim \|\omega^n(t)\|_{L^p\cap L^3} \leq \|\omega_0\|_{L^p\cap L^3}.$$

The family $\{\psi_n,\, n \in \mathbb{N}\}$ is bounded and equicontinuous on every compact $[0,T] \times \bar{B}(0,R) \subset \mathbb{R}_+ \times \mathbb{R}^2$. The Arzela-Ascoli theorem implies the existence of a limiting particle trajectories $\psi(t,x)$. Performing the same analysis for $\{\psi_n^{-1},\, n \in \mathbb{N}\}$ we figure out that $\psi(t,x)$ is a Lebesgue measure preserving homeomorphism . Also, passing to the limit[7] in (21) leads to

$$\|\psi_t\|_* = \|\psi_t^{-1}\|_* \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+.$$

One defines,

$$\omega(t,x) = \omega_0(\psi_t^{-1}(x)), \qquad u(t,x) = (k *_x \omega(t,.))(x).$$

We easily check that for every $q \in [p, +\infty[$ one has

$$\begin{array}{rcll} \omega^n(.,t) & \longrightarrow & \omega(.,t) & \text{in } L^q. \\ u^n(.,t) & \longrightarrow_x & u(.,t) & \text{uniformly.} \end{array}$$

The last claim follows from the fact that

$$\|u^n(t) - u(t)\|_{L^\infty} \lesssim \|\omega^n(t) - \omega(t)\|_{L^p\cap L^3}.$$

All this allows us to pass to the limit in the integral equation on ω^n and then to prove that (u,ω) is a weak solution to the vorticity-stream formulation of the 2D Euler system. Furthermore, the convergence of $\{\omega^n(t)\}$ in L^1_{loc} and (20) imply together that

$$\|\omega(t)\|_{L^p\cap \tilde{\mathrm{B}}\mathrm{MO}} \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+.$$

as claimed.

[7] We take the pointwise limit in the definition formula and then take the supremum.

The continuity of ψ and the preservation of Lebesgue measure imply that $t \mapsto \omega(t)$ is continuous[8] with values in L^q for all $q \in [p, +\infty[$. This implies in particular that $u \in \mathcal{C}([0, +\infty[, L^r(\mathbb{R}^d))$ for every $r \in [\frac{2p}{2-p}, +\infty]$.

4.1. Uniqueness. Since the vorticity remains bounded in BMO space then the uniqueness of the solutions follows from Theorem 7.1 in **[23]**. Another way to prove that is to add the information $u \in \mathcal{C}([0, +\infty, L^\infty(\mathbb{R}^d))$ (which is satisfied for the solution constructed above) to the theorem and in this case the uniqueness follows from Theorem 7.17 in **[1]**.

$\square$

[8]By approximation we are reduced to the following situation: $g_n(x) \to g(x)$ pointwise and

$$\|g_n\|_{L^q} = \|g\|_{L^q}.$$

This is enough to deduce that $g_n \to g$ in L^q (see Theorem 1.9 in **[18]** for instance).

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CHAPTER 2

Profile decomposition for the fractional Navier-Stokes equations

1. Introduction

In this chapter we consider the Navier-Stokes system with fractional dissipation (NS_α):

$$(22)\quad \begin{cases} \partial_t u(t,x) + u.\nabla u + \nu(-\Delta)^\alpha u(t,x) + \nabla p = 0, & x \in \mathbb{R}^d, t > 0, \\ \nabla.u = 0, \\ u_{|t=0} = u_0. \end{cases}$$

The operator $(-\Delta)^\alpha$ is the α-order Laplace operator which can be defined by the Fourier transform in the space variable:

$$\mathcal{F}(-\Delta)^\alpha \psi(\xi) = |\xi|^{2\alpha}\hat{\psi}(\xi).$$

The vectors field $u = (u_1, u_2, u_3)$ is the velocity field, p the scalar pressure and the coefficient ν is the kinematic viscosity. The parameter α is taken $\frac{1}{2} < \alpha < 1$. This encloses the case $\alpha = 1$ and, in all cases, the problem is supercritical (the case $\alpha = \frac{5}{4}$ is critical with respect tho the conservation of the energy). These generalizations have been introduced by Lions in 1960s. Equation 22 is invariant under the scaling:

$$(23)\qquad u_\lambda(t,x) = \lambda^{2\alpha-1}u(\lambda^{2\alpha}t, \lambda x), \qquad p_\lambda(t,x) = \lambda^{4\alpha-2}p(\lambda^{2\alpha}t, \lambda x).$$

This induce us to investigate the solution of (22) in the critical spaces whose norm is invariant under scaling (23). Thus, a natural candidate is homogeneous Sobolev space $\dot{H}^{s_\alpha}$ with $s_\alpha = \frac{5}{2} - 2\alpha$ or the Lebesgue space $L^{\frac{3}{2\alpha-1}}$. These spaces (and many others) have the same scaling of equation (22) and it appears to be a critical space for the locally well-posed question.

The 3-dimensional incompressible Navier-Stokes ($\alpha = 1$) has a long history as a challenging open problem in nonlinear partial differential equations (see **[15]** for instance). The Navier-Stokes system with fractional dissipation has been addressed by many authors. For $\alpha \geq \frac{5}{4}$, the system is globally well-posed, a logarithmic improvement was proved by T. Tao **[27]**.
Recently, the fractional Navier-Stokes system regains some interests. In **[21, 23]** Wu obtained lower bounds for the integral involving $(-\Delta)^\alpha$ by combining pointwise inequalities for $(-\Delta)^\alpha$ with Bernstein inequalities for fractional derivatives. As an application of these lower bounds, he established the existence and uniqueness of solutions to the generalized Navier-Stokes equations in Besov spaces for $\alpha > 0$.
Li and Zhai **[20]** have studied the well-posedness and regularity of the generalized Navier-Stokes equations with initial data in a new critical space

$$Q^{\beta,-1}_{\alpha,\infty} = \nabla \cdot (Q^\beta_\alpha(\mathbb{R}^n))^n; \, \beta \in (\frac{1}{2}, 1)$$

which is larger than some known critical homogeneous Besov spaces. Also, Zhai **[26]** studied the well-posedness for the fractional Navier-Stokes equations in critical spaces $G^{-(2\beta-1)_n}(\mathbb{R}^n)$ and $BMO^{-(2\beta-1)}(\mathbb{R}^n)$ which are close to the largest critical space

$$\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n); \beta \in (\frac{1}{2}, 1).$$

The profile decomposition is a refined tool of compactness-concentration type which was introduced by P. Gérard **[13]** to study the defect of compactness for the critical Sobolev embedding. Afterwards, many authors used it to study some critical evolution equations (see **[2]**, **[17]**, **[4]** for instance). The profile decomposition for the Navier-Stokes has been performed by Isabelle Gallagher in **[10]**. In this chapter some qualitative properties are proved and the interest of this aspect was recently renewed by the work of C. Kenig and Koch **[22]** and Gallagher *et al* **[12]**.
It is worth noting that the pressure filed can be explicitly determined by Poisson equation:

$$-\Delta p = div(u.\nabla u)$$

then, due to the incompressibility of the fluid, we can write

$$p = (-\Delta)^{-1} \sum_{i,j} \partial_i \partial_j (u^i u^j).$$

The Leray projector $\mathbb{P} = Id - \nabla \Delta^{-1} div$ is the orthogonal projector onto divergence free vector fields. In Fourier variable, the $\widehat{\mathbb{P}}$ can be represented as a matrix with coefficients given by

$$\widehat{\mathbb{P}}_{i,j} = \delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2}.$$

It follows that the system (22) can be written

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = Q(u,u), & x \in \mathbb{R}^3, t > 0, \\ u_{|t=0} = u_0. \end{cases}$$

where the bilinear operator Q is defined by

$$Q(v,w) := -\frac{1}{2} \mathbb{P}(div(v \otimes w) + div(w \otimes v)).$$

The solutions we consider, which called *mild* solutions, are the solutions of the following integral equation

$$u(t) = e^{-t(-\Delta)^\alpha} u_0 - \int_0^s e^{-(t-s)(-\Delta)^\alpha} Q(u,u)(s) ds.$$

These solutions are fixed points (in the appropriate spaces) of the map

$$u \longrightarrow e^{-t(-\Delta)^\alpha} u_0(x) + B(u,u)$$

with

$$B(u,u) = -\int_0^s e^{-(t-s)(-\Delta)^\alpha} Q(u,u) ds.$$

2. Preliminaries and functional spaces

Let us introduce the well-known Littlewood-Paley decomposition and the corresponding cut-off operators, there exist a radial positive function $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that

$$\begin{aligned} \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) &= 1 \qquad \forall\, \xi \in \mathbb{R}^d \setminus \{0\}, \\ \mathrm{Supp}\varphi(2^{-q}\cdot) \cap \mathrm{Supp}\varphi(2^{-j}\cdot) &= \emptyset, \qquad \forall\, |q-j| \geq 2. \end{aligned}$$

For every $q \in \mathbb{Z}$ and $v \in \mathcal{S}'(\mathbb{R}^d)$ we set

$$\Delta_q v = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad S_q = \sum_{j=-\infty}^{q-1} \Delta_j.$$

Bony's decomposition **[3]** consist of splitting the product uv into three parts[1]:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad R(u,v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$ we define the homogeneous Besov space $\dot{B}^s_{p,r}$ as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ *modulo* polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} = \left\| \left(2^{qs} \|\Delta_q u\|_{L^p} \right)_{q\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

In the case $p = r = 2$, the space $\dot{B}^s_{2,2}$ turns out to be the classical homogeneous Sobolev space $\dot{H}^s$.
We also define the following spaces[2]

DEFINITION 2.1. *Let p, r and a be in $[1,\infty]$ and s in $\mathbb{R}$. The space $\widetilde{L}^p_T(\dot{B}^s_{a,r})$ is the space of distributions u such that*

$$\|u\|_{\widetilde{L}^p_T(\dot{B}^s_{a,r})} = \left\| \left(2^{js} \|\Delta_j u\|_{L^p_T} \right)_{j\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

We will make frequent use of Bernstein's inequalities (see **[1]** for instance).

[1] It should be said that this decomposition is true in the class of distributions for which

$$\sum_{q\in\mathbb{Z}} \Delta_q = I.$$

For example, polynomial functions do not belong to this class.

[2] These spaces were introduced by Chemin and Lerner **[7]**.

LEMMA 2.2. *There exists a constant C such that for any $q, k \in \mathbb{N}$, $1 \leq a \leq b$ and for $f \in L^a(\mathbb{R}^d)$,*

$$\begin{aligned}\sup_{|\alpha|=k} \|\partial^\alpha S_q f\|_{L^b} &\leq C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))} \|S_q f\|_{L^a},\\ C^{-k} 2^{qk} \|\Delta_q f\|_{L^a} &\leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q f\|_{L^a} \leq C^k 2^{qk} \|\Delta_q f\|_{L^a}.\end{aligned}$$

The last result that we recall is the following one giving the parabolic regularity in the particular case which is relevant for us (see **[1]** for instance).

LEMMA 2.3. *Let u be a smooth divergence free vector field solving*

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u + \nabla p = f \\ u_{|t=0} = u^0, \end{cases}$$

on some time interval $[0, T]$. Then, for every $p \geq r \geq 1$ and $s \in \mathbb{R}$, we have

$$\|u\|_{C([0,T];\dot{B}^s_{q,\ell}) \cap \tilde{L}^p_T \dot{B}^{s+\frac{2\alpha}{p}}_{q,\ell}} \lesssim \|u^0\|_{\dot{B}^s_{q,\ell}} + \|f\|_{\tilde{L}^r_T \dot{B}^{s-2\alpha+\frac{2\alpha}{r}}_{q,\ell}}.$$

3. The local well-posedness of the Cauchy problem

This section is dedicated to the local existence and uniqueness theory for the fractional Navier-Stokes problem where we recall the theory of local well-posedness of the Cauchy problem (22). To make the chapter self-contained we give a the proof of all these results.

3.1. Local theory in $\dot{H}^{s_\alpha}$. Here, one takes $\frac{1}{2} < \alpha < 1$ and denotes

$$s_\alpha = \frac{5}{2} - 2\alpha, \qquad p_\alpha = \frac{3}{2\alpha - 1}.$$

Recall that $\dot{H}^{s_\alpha}$ denotes the inhomogeneous Sobolev space with the norm

$$\|u\|^2_{\dot{H}^{s_\alpha}} := \int |\xi|^{2s_\alpha} |\widehat{u}(\xi)|^2 d\xi.$$

The main theorem of this subsection is presented down below,

THEOREM 2.4. *Let $\frac{1}{2} < \alpha < 1$ and set $s_\alpha = \frac{5}{2} - 2\alpha$ and $\mathbf{q}_\alpha = \frac{4\alpha}{2\alpha-1}$. For every $u_0 \in \dot{H}^{s_\alpha}(\mathbb{R}^3)$ a Solenoidal vectorial function, there exists a positive time T such* *that the system* (22) *has a unique solution u in $\tilde{L}^{\mathbf{q}_\alpha}([0,T],\dot{H}^{2-\alpha})$. Moreover, the unique solution u belongs to the energy space :*

$$E_T^\alpha = \mathcal{C}\big([0,T],\dot{H}^{s_\alpha}\big)\cap L^2\big([0,T],\dot{H}^{s_\alpha+\alpha}\big).$$

PROOF. Let us first denote

$$\mathcal{B}(u,v) = -\int_0^s e^{-(t-s)(-\Delta)^\alpha}Q(u,v)ds.$$

To prove the result above it is enough to show that the established mapping

$$u\mapsto e^{-t(-\Delta)^\alpha}u_0+\mathcal{B}(u,u), \tag{24}$$

has a unique fixed point in the Banach space $\tilde{L}^{\mathbf{q}_\alpha}([0,T],\dot{H}^{2-\alpha})$ for some $T>0$. In order to do that, we will define the following bilinear estimate.

PROPOSITION 2.5. *For $\alpha\in]\frac{1}{2},1[$ one denotes $\mathbf{q}_\alpha=\frac{4\alpha}{2\alpha-1}$. A constant $C>0$ exists such that for every $T>0$ one has*

$$\|\mathcal{B}(u,u)\|_{\tilde{L}^{\mathbf{q}_\alpha}([0,T],\dot{H}^{2-\alpha})}\le C\|u\|^2_{\tilde{L}^{\mathbf{q}_\alpha}([0,T],\dot{H}^{2-\alpha})},$$

for every $u\in\tilde{L}^{\mathbf{q}_\alpha}([0,T],\dot{H}^{2-\alpha})$.

PROOF OF PROPOSITION 2.5. Using Lemma 2.3, with $q=\ell=2$, $p=\mathbf{q}_\alpha$ and $r=\frac{\mathbf{q}_\alpha}{2}$, we get

$$\|B(u,u)\|_{\tilde{L}^{\mathbf{q}_\alpha}([0,T],\dot{H}^{2-\alpha})}\le C\|Q(u,u)\|_{\tilde{L}^{\frac{\mathbf{q}_\alpha}{2}}([0,T],\dot{H}^{\frac{3}{2}-2\alpha})}.$$

The product laws in homogeneous Sobolev spaces[3] yields easily

$$\begin{aligned}\|\nabla.(u\otimes u)\|_{\tilde{L}^{\frac{\mathbf{q}_\alpha}{2}}([0,T],\dot{H}^{\frac{3}{2}-2\alpha})} &\le C\|u\otimes u\|_{\tilde{L}^{\frac{\mathbf{q}_\alpha}{2}}([0,T],\dot{H}^{\frac{5}{2}-2\alpha})}\\ &\le C\|u\|^2_{\tilde{L}^{\mathbf{q}_\alpha}([0,T],\dot{H}^{2-\alpha})},\end{aligned}$$

which is the desired result. □

Now we turn back to the proof of the Theorem 2.4, we will use the following classical fixed point theorem.

[3]Recall the product laws

$$\dot{H}^{s_1}(\mathbb{R}^d)\cdot\dot{H}^{s_2}(\mathbb{R}^d)\hookrightarrow\dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$$

with $s_1,s_2\in]-\frac{d}{2},\frac{d}{2}[$ and $s_1+s_2>0$.

LEMMA 2.6. *Let X be a Banach space, $\mathcal{B}$ a continuous bilinear map from $X \times X$ to X, and h a positive real number such that*

$$\kappa < \frac{1}{4\|\mathcal{B}\|},$$

then for any x_0 in the ball $B_X(0,\kappa)$, the equation

$$x = x_0 + \mathcal{B}(x,x)$$

has a unique solution in the ball $B_X(0,2\kappa)$.

According to Proposition 2.5, the bilinear operator $\mathcal{B}$ maps continuously $X_T = \tilde{L}^{\mathfrak{q}_\alpha}([0,T],\dot{H}^{2-\alpha})$ on itself. Furthermore, taking $r = \frac{4\alpha}{2\alpha-1}$ in the last inequality of Lemma 2.3 we get

$$\|e^{-t(-\Delta)^\alpha}u_0\|_{\tilde{L}^{\mathfrak{q}_\alpha}([0,T],\dot{H}^{2-\alpha})} \leq C\|u_0\|_{\dot{H}^{s_\alpha}}, \tag{25}$$

for any positive $T > 0$.
Then we continue the proof considering two cases:
• *Small initial data.*
If $u_0 \in \dot{H}^{s_\alpha}$ satisfies

$$\|u_0\|_{\dot{H}^{s_\alpha}} \leq \frac{1}{4C\|B\|},$$

then the Fixed Point theorem lead us to the existence of one unique global solution in $\tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,+\infty[,\dot{H}^{2-\alpha})$.
• *Large initial data.*
Let u_0 be an arbitrary initial data of $\dot{H}^{s_\alpha}$, we prove that for a suitable $T > 0$

$$\|e^{-t(-\Delta)^\alpha}u_0\|_{X_T} \leq \frac{1}{4C\|B\|},$$

then using the Fixed Point theorem to get the result.
More precisely, We split u_0 into two parts: a small part in $\dot{H}^{s_\alpha}$ and a large part with a compactly supported Fourier transform. For that, we fix some positive real number $L = L(u_0)$ such that

$$\Big(\int_{|\xi|\geq L} |\xi|^{2s_\alpha}|\hat{u}(\xi)|^2\Big)^{\frac{1}{2}} \leq \frac{1}{8c} \quad \text{with} \quad c > \|\mathcal{B}\|.$$

Using (25) and defining $u_0^b = F^{-1}(\mathbb{1}_{B(0,L)})$, we get

$$\|e^{-t(-\Delta)^\alpha} u_0\|_{X_T} \leq \frac{1}{8c} + \|e^{-t(-\Delta)^\alpha} u_0^b\|_{X_T}.$$

Now, one can estimate

$$\begin{aligned} \|e^{t(-\Delta)^\alpha} u_0^b\|_{X_T} &\leq L^{\frac{2\alpha-1}{2}} \|e^{t(-\Delta)^\alpha} u_0^b\|_{L([0,T],\dot{H}^{s_\alpha})} \\ &\leq L^{\frac{2\alpha-1}{2}} T^{\frac{2\alpha-1}{4\alpha}} \|u_0\|_{\dot{H}^{s_\alpha}}. \end{aligned}$$

Thus, if

$$T \leq \Big(\frac{1}{8cL^{\frac{2\alpha-1}{2}}\|u_0\|_{\dot{H}^{s_\alpha}}}\Big)^{\frac{2\alpha-1}{4\alpha}}$$

then we get the existence and uniqueness of the solution in the ball with center 0 and radius $\frac{1}{2c}$ in the space X_T.
Finally, since $u \in \tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,T],\dot{H}^{2-\alpha}))$, it follows that $Q(u,u) \in L^{\frac{2\alpha}{2\alpha-1}}([0,T],\dot{H}^{\frac{3}{2}-2\alpha})$. Taking $p=2$, and $r=\frac{2\alpha}{2\alpha-1}$ in Lemma 2.3 we obtain

$$u \in \mathbf{C}([0,T],\dot{H}^{s_\alpha}) \cap L^2([0,T],\dot{H}^{s_\alpha+\alpha}).$$

□

We call $\mathrm{NS}_\alpha(u_0)$ the unique solution of the system (22) associated with the initial data $u_0 \in \dot{H}^{s_\alpha}$. In relation of the Theorem 2.4 we define the function spaces

$$E_T^\alpha = \mathbf{C}([0,T],\dot{H}^{s_\alpha}(\mathbb{R}^3)) \cap L^2([0,T],\dot{H}^{s_\alpha+\alpha}(\mathbb{R}^3))$$

and

$$E_\infty^\alpha = \mathbf{C}(\mathbb{R}^+,\dot{H}^{s_\alpha}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+,\dot{H}^{s_\alpha+\alpha}(\mathbb{R}^3)),$$

We also define the sets of initial data yielding solutions of the system (2.4) in E_T^α and E_∞^α respectively,

$$D_T = \{u_0 \in \dot{H}^{s_\alpha} | \mathrm{NS}_\alpha(u_0) \in E_T^\alpha\}$$

and

$$D_\infty = \{u_0 \in \dot{H}^{s_\alpha} | \mathrm{NS}_\alpha(u_0) \in E_\infty^\alpha\}.$$

3.2. Local theory in L^{p_α}. In this paragraph we are interested in solving the system of NS_α when the initial data lies in the space L^{p_α} with

$$p_\alpha = \frac{3}{2\alpha - 1}.$$

The space L^{p_α} has the same scaling as (22) and it is critical for the local well-posedness question.
To deal with the Cauchy problem in L^p spaces we need the following lemma on the integrability of the kernel of the fractional heat equation.

LEMMA 2.7. *For every $\beta > 1$, the function defined on $\mathbb{R}^3$ by*

$$\psi(x) = \int_{\mathbb{R}^3} e^{ix\xi} e^{-|\xi|^\beta} d\xi$$

and its gradient belong to $L^p(\mathbb{R}^3)$ for every $p \in [1, +\infty]$.

PROOF OF LEMMA 2.7. First let us note that, since $\xi \mapsto e^{-|\xi|^\beta}$ and $\xi \mapsto \xi e^{-|\xi|^\beta}$ are L^1-functions, then theirs Fourier transforms are bounded. This implies in particular that ψ and $\nabla\psi$ belong to $L^\infty(\mathbb{R}^3)$.
Second, using integration by parts we have, for $x \neq 0$,

$$\begin{aligned}
|\psi(x)| &= |\int_{\mathbb{R}^3} e^{ix\xi} e^{-|\xi|^\beta} d\xi| \\
&= |\frac{1}{|x|^4} \int_{\mathbb{R}^3} (\triangle_\xi)^2 e^{ix\xi} e^{-|\xi|^\beta} d\xi| \\
&= |\frac{1}{|x|^4} \int_{\mathbb{R}^3} e^{ix\xi} (\triangle_\xi)^2 e^{-|\xi|^\beta} d\xi| \\
&\leq \frac{1}{|x|^4} \int_{\mathbb{R}^3} |(\partial_r^2 + \frac{2}{r}\partial_r)^2 e^{-r^\beta}| d\xi \\
&\leq \frac{C_\beta}{|x|^4} \int_0^\infty (r^{\beta-2} + r^{2\beta-2} + r^{3\beta-2} + r^{4\beta-2}) e^{-r^\beta} dr.
\end{aligned}$$

The convergence of this integral is guaranteed at infinity. At 0 the convergence is due to the assumption that $\beta > 1$. In particular, one obtains

$$|\psi(x)| \lesssim \frac{1}{|x|^4}.$$

combined with the fact that $\psi \in L^\infty$, this bound yields the desired result.
A similar argument proves that, for $i = 1, 2, 3$,

$$\psi : x \longmapsto \int_{\mathbb{R}^3} e^{ix\xi} \xi_i e^{-|\xi|^\beta} d\xi$$

belongs to L^1, the same result happens then for he gradient $\nabla\psi$. $\square$

As a direct consequence of the previous lemma we get

LEMMA 2.8. *Let q, r be real numbers such that $1 \leq r \leq q \leq \infty$. Then, there is a constant C, such for any $t > 0$ and any $f \in L^r(\mathbb{R}^3)$, we have*

$$\|e^{-t(-\Delta)^\alpha} f\|_{L^q(\mathbb{R}^3)} \leq t^{\frac{3}{2\alpha}(\frac{1}{q} - \frac{1}{r})} \|f\|_{L^r(\mathbb{R}^3)}$$

and

$$\|\nabla e^{-t(-\Delta)^\alpha} f\|_{L^q(\mathbb{R}^3)} \leq C t^{-\frac{1}{2\alpha} + \frac{3}{2\alpha}(\frac{1}{q} - \frac{1}{r})} \|f\|_{L^r(\mathbb{R}^3)}.$$

PROOF OF LEMMA 2.8. One can write

$$e^{-t(-\Delta)^\alpha} f = \frac{1}{t^{\frac{3}{2\alpha}}} \phi\left(\frac{\cdot}{t^{\frac{1}{2\alpha}}}\right) \star f$$

with $\hat{\psi} = e^{-|\xi|^{2\alpha}}$.
Using Lemma 2.7 and Young inequality we get

$$\begin{aligned} \|e^{-t(-\Delta)^\alpha} f\|_{L^q(\mathbb{R}^3)} &\leq t^{\frac{3}{2\alpha}(\frac{1}{q} - \frac{1}{r})} \|\psi\|_{L^p} \|f\|_{L^r(\mathbb{R}^3)} \\ &\leq C_p t^{\frac{3}{2\alpha}(\frac{1}{q} - \frac{1}{r})} \|f\|_{L^r(\mathbb{R}^3)}, \end{aligned}$$

where

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{r} + 1.$$

This gives the desired estimates.
The second part of the proof is similar. Using Lemma 2.7 and the fact

$$\nabla\left(e^{-t(-\Delta)^\alpha} f\right) = \frac{1}{t^{\frac{4}{2\alpha}}} \nabla\phi\left(\frac{\cdot}{t^{\frac{1}{2\alpha}}}\right) \star f.$$

we can argue as before to get the claimed estimate. $\square$

The main result is the following theorem.

THEOREM 2.9. *Let u_0 be a divergence free vector field in the space $L^{p_\alpha}(\mathbb{R}^3)$. Then there exists a positive time T such that the system* (22) *has a unique solution u satisfying*

$$u \in \mathcal{C}([0,T], L^{p_\alpha}(\mathbb{R}^3), \quad \textit{and} \quad t^{\frac{2\alpha-1}{2\alpha}} \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \in L^\infty(]0,T]).$$

Moreover, if $\|u_0\|_{L^{p_\alpha}(\mathbb{R}^3)}$ is small then the solution u is global in time.

PROOF OF THEOREM 2.9. The proof of the theorem is based on the fixed point theorem using Lemma 2.8 and integral form of the system (NS_α)

$$u(t,x) = e^{-t(-\Delta)^\alpha} u_0(x) + B(u,u)(t,x),$$

with

$$B(u,u)(t,x) = \int_0^t e^{-(t-s)(-\Delta)^\alpha} \mathcal{P}\nabla.(u \otimes u)(s,x) ds.$$

For every $q > p_\alpha$ and $T > 0$ one denotes

$$Y_{q,T} := \{f \in L^\infty_{loc}(]0,T], L^q(\mathbb{R}^3)| \quad \|f\|_{Y_{q,T}} < \infty\},$$

where

$$\|f\|_{Y_{q,T}} = \sup_{0<t\leq T} t^{\frac{3}{2\alpha}(\frac{1}{p_\alpha}-\frac{1}{q})} \|f\|_{L^q}.$$

The fixed point argument is based on the following :

PROPOSITION 2.10. *Let $p,q,r \in [p_\alpha, \infty]$ such that $\frac{1}{q} \leq \frac{2}{r} < \frac{1}{p_\alpha} + \frac{1}{q}$. Then there is $C > 0$ such that*

$$\|B(u,u)\|_{Y_{q,T}} \leq C\|u\|^2_{Y_{r,T}}.$$

Furthermore, we have

$$\lim_{t \longrightarrow 0^+} \|e^{-t(-\Delta)^\alpha} f\|_{Y_{q,T}} = 0. \tag{26}$$

PROOF OF PRPOSITION 2.10. According to Lemma 2.8 we have

$$\|B(u(t),u(t))\|_{L^q(\mathbb{R}^3)} \leq \int_0^t (t-s)^{-\frac{1}{2\alpha}-\frac{3}{2\alpha}(\frac{2}{r}-\frac{1}{q})} \|P(u(s)\otimes u(s))\|_{L^{\frac{r}{2}}} ds.$$

Since $r \in]2, +\infty[$ then $\mathcal{P}$ is continuous on $L^{\frac{r}{2}}$. This yields for every $t \in [0, T]$

$$\begin{aligned} \|B(u(t),u(t))\|_{L^q(\mathbb{R}^3)} &\leq \int_0^t (t-s)^{-\frac{1}{2\alpha}-\frac{3}{2\alpha}(\frac{2}{r}-\frac{1}{q})}\|u(s)\|_{L^r}^2 ds \\ &\leq \|u\|_{Y_{q,T}}^2 \int_0^t (t-s)^{-\frac{1}{2\alpha}-\frac{3}{2\alpha}(\frac{2}{r}-\frac{1}{q})} s^{\frac{3}{\alpha}(-\frac{1}{p_\alpha}+\frac{1}{r})} ds \\ &\leq \|u\|_{Y_{q,T}}^2 t^{\frac{3}{2\alpha}(\frac{1}{p_\alpha}-\frac{1}{q})}. \end{aligned}$$

A direct computation yields

$$\int_0^t (t-s)^{-\frac{1}{2\alpha}-\frac{3}{2\alpha}(\frac{2}{r}-\frac{1}{q})} s^{\frac{3}{\alpha}(-\frac{1}{p_\alpha}+\frac{1}{r})} ds \leq C t^{\frac{3}{2\alpha}(\frac{1}{p_\alpha}-\frac{1}{q})},$$

which gives

$$\|B(u(t),u(t))\|_{L^q(\mathbb{R}^3)} \leq C\|u\|_{Y_{q,T}}^2 t^{\frac{3}{2\alpha}(\frac{1}{p_\alpha}-\frac{1}{q})}.$$

as claimed.

Now, let us move to the proof of the second estimate. Since $f \in L^{p_\alpha}$ it is well known that for any $\varepsilon > 0$ there is a function ϕ in S such that $\|f-\phi\|_{L^q} \leq \varepsilon$. Lemma 2.8 enables us to write

$$\|e^{-t(-\Delta)^\alpha} f\|_{Y_{q,T}} \leq C\varepsilon + T^{\frac{3}{2\alpha}(\frac{1}{P_\alpha}-\frac{1}{q})}\|\phi\|_{L^q(\mathbb{R}^3)}.$$

□

Coming back the the proof of Theorem 2.9. One may write the integral form of The system (NS_α)

$$u(t) = e^{-t(-\Delta)^\alpha} u_0 + B(u(t), u(t)),$$

with

$$B(u(t),u(t)) = \int_0^t e^{-(t-s)(-\Delta)^\alpha} \mathcal{P}(u(s) \otimes u(s)) ds.$$

Now, Proposition 2.10 , in the particular case $r = q$, yields that B is continuous in the space $Y_{q,T}$ for $q \in]p_\alpha, \infty[$. The existence and the uniqueness of the global solution are given by the point fixed theorem and the assumption that $\|e^{-t(-\Delta)^\alpha} u_0\|_{Y_{q,\infty}} \leq \varepsilon_0$. However, since

$$\|e^{-t(-\Delta)^\alpha} u_0\|_{Y_{q,\infty}}\| \leq \|u_0\|_{L^{p_\alpha}}$$

then this smallness condition is fulfilled if $\|u_0\|_{L^{p_\alpha}}$ is small.
Thanks to (26) the smallness of $\|e^{-t(-\Delta)^\alpha}u_0\|_{Y_{q,T}}$ is guaranteed for T small enough. This yields, by the same fixed point theorem, a local unique solution on $[0,T]$.
To complete the proof we have to show that both $e^{-t(-\Delta)^\alpha}u_0$ and $B(u,u)$ belong to $C(]0,T],L^{p_\alpha})\cap Y_{\infty,T}$. For that we go back to the Proposition 2.10 and take $q=p_\alpha$ and $r=2p_\alpha$, however, if we take $q=\infty$ and $r>2p_\alpha$, as desired. $\square$

The following proposition is important in the sequel.

PROPOSITION 2.11. *Assume that $\alpha\in[\frac{5}{6},1[$. Let $u_0\in\dot{H}^{s_\alpha}$ which is small in L^{p_α} then the associated solution given by Theorem 2.4 is global.*

PROOF. First, Theorem 2.9 implies the existence of global solution $\tilde{u}$ in L^{p_α} with

$$\|\tilde{u}(t)\|_{L^{p_\alpha}}\leq 2\|u_0\|_{L^{p_\alpha}}\leq 2\epsilon.$$

Second, Theorem 2.4 yields a maximal solution $u\in\dot{H}^{s_\alpha}$ on some interval $[0,T^*[$. Since $L^{p_\alpha}\subset\dot{H}^{s_\alpha}$ then the uniqueness yields

$$\tilde{u}=u,\qquad \text{on}\quad [0,T^*[.$$

We are going to prove that $T^*=+\infty$ and that

$$\|u\|_{E_\infty}\leq 2\|u_0\|_{\dot{H}^{s_\alpha}}.$$

An energy yields for every $t\in[0,T^*[$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^{s_\alpha}}+\|u(t)\|^2_{\dot{H}^{s_\alpha+\alpha}} &= \langle u(t).\nabla u(t),u(t)\rangle_{\dot{H}^{s_\alpha}}\\ &= \langle u(t).\nabla u(t),|D|^{2s_\alpha}u(t)\rangle_{L^2}.\end{aligned}$$

By Hölder inequality we get

$$|\langle u.\nabla u,u(t)\rangle_{\dot{H}^{s_\alpha}}|\leq\|u(t)\|_{L^{p_\alpha}}\|\nabla u(t)\|_{L^{p_1}}\||D|^{2s_\alpha}u(t)\|_{L^{p_2}}.$$

with

$$p_1=\frac{3}{\alpha},\qquad p_2=\frac{3}{4-3\alpha}$$

The Sobolev embedding yields

$$\|\nabla u(t)\|_{L^{p_1}} + \||D|^{2s_\alpha} u(t)\|_{L^{p_2}} \leq \|u(t)\|^2_{\dot{H}^{s_\alpha+\alpha}}.$$

Thus, for every $t \in [0, T^*[$,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^{s_\alpha}} + \|u(t)\|^2_{\dot{H}^{s_\alpha+\alpha}} \leq 2C\epsilon\|u(t)\|^2_{\dot{H}^{s_\alpha+\alpha}}.$$

For ϵ small enough, we get

$$\frac{d}{dt}\|u(t)\|^2_{\dot{H}^{s_\alpha}} + \|u(t)\|^2_{\dot{H}^{s_\alpha+\alpha}} \leq 0.$$

Finally we obtain, for every $t \in [0, T^*[$,

$$\|u(t)\|^2_{\dot{H}^{s_\alpha}} + \int_0^t \|u(\tau)\|^2_{\dot{H}^{s_\alpha+\alpha}} d\tau \leq \|u_0\|^2_{\dot{H}^{s_\alpha}}.$$

This yields a bound to the energy of the solution u and prevent it to develop singularities at finite time. Thus, $T^* = \infty$ and

$$\|u\|_{E_\infty} \leq 2\|u_0\|_{\dot{H}^{s_\alpha}}$$

as claimed.

□

4. Profile decomposition for the fractional Navier-Stokes system

Our purpose in this section is to give profile decomposition for the fractional Navier-Stokes system. We begin by the associated linear fractional heat equation.

4.1. Profile decomposition for the fractional heat equation.

$$\begin{cases} \partial_t v(t,x) + (-\Delta)^\alpha v(t,x) = 0, & x \in \mathbb{R}^d, t > 0, \\ v_{|t=0} = v_0. \end{cases} \tag{27}$$

In the following, we shall denote $H_\alpha(u_0)$ the solution of the system (27), associated with the data u_0. The profile decomposition for the equation (27) is obtained from the one established by P. Gérard **[13]**, for the bounded sequence in the homogeneous Sobolev spaces. Before stating our main result, we introduce some definitions in addition to the result of P. Gérard.

DEFINITION 2.12. (1) *We call* **scale**, *every sequence* $\mathbf{h} = (h_n)_{n\geq o}$ *of positive numbers and* **core**, *every sequence* $\mathbf{x} = (x_n)_{n\geq 0}$ *in* $\mathbb{R}\times\mathbb{R}^3$.

(2) *We say that two pairs* $(\mathbf{h},\mathbf{x})$ *and* $(\mathbf{h}^{'},\mathbf{x}^{'})$ *are orthogonal if*

$$\frac{h_n}{h_n^{'}} + \frac{h_n^{'}}{h_n} + +|\frac{x_n - x_n^{'}}{h_n}| \underset{n\to\infty}{\longrightarrow} +\infty.$$

The following theorem is the profile decomposition in homogeneous Sobolev spaces P. Gérard (stated in 3D space dimensions).

THEOREM 2.13 (**[13]**). *Let* $s \in]0,\frac{3}{2}[$ *and* $(\varphi_n)_{n\geq 0}$ *be a bounded sequence in* $\dot{H}^s(\mathbb{R}^3)$. *Then there exist a subsequence* (v_n) *(still denoted by* (ϕ_n)*), a sequence* $(\mathbf{h}^j)_{j\geq 1}$ *of scales, a sequence* $(\mathbf{x}^j)_{j\geq 1}$ *of cores and a sequence* $(\Phi^j)_{j\geq 1}$ *of* $\dot{H}^s(\mathbb{R}^3)$ *, such that*

(1) *the pairs* $(\mathbf{h}^j,\mathbf{x}^j)$ *are pairwise orthogonal;*

(2) *for every* $\ell \geq 1$ *and every* $x \in \mathbb{R}^3$*, one has*

$$\varphi_n(x) = \sum_{j=1}^{\ell} \frac{1}{(h_n^j)^{\frac{3}{2}-s}} \Phi^j(\frac{x - x_n^j}{h_n^j}) + w_n^\ell(x),$$

with

(28) $$\limsup_{n\to\infty} \|w_n^\ell\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \longrightarrow 0, \text{ as } \quad \ell\to\infty.$$

Moreover, for all $\ell \in \mathbb{N}^*$ *:*

(29) $$\|\varphi_n\|^2_{\dot{H}^{s_\alpha}} = \sum_{j=1}^{\ell} \|\Phi^j\|^2_{\dot{H}^{s_\alpha}} + \|w_n^\ell\|^2_{\dot{H}^{s_\alpha}} + o(1), \text{ as } \quad n\to\infty.$$

The next theorem gives the profile decomposition of the fractional heat equation.

THEOREM 2.14. *let* $(\phi_n)_{n\in\mathbb{N}}$ *is a sequence in a divergence free vector field, bounded in* $\dot{H}^{s_\alpha}(\mathbb{R}^3)$ *and* Φ^j *denote the profile of the decomposition of* (ϕ_n). *If* $u_n = \mathrm{e}^{-t(-\Delta)^\alpha}\phi_n$ *and* $V^j = \mathrm{e}^{-t(-\Delta)^\alpha}\Phi^j$ *then for all* $\ell \in \mathbb{N}^*$, $(t,x) \in \mathbb{R}^+\times\mathbb{R}^3$ *we have*

$$u_n(t,x) = \sum_{j=1}^{\ell} (\frac{1}{h_n^j})^{\frac{3}{p_\alpha}} V^j(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x - x_n^j}{h_n^j}) + \omega_n^\ell(t,x),$$

with

$$\limsup_n \|w_n^\ell\|_{L^\infty(\mathbb{R}^+,L^{p_\alpha}(\mathbb{R}^3))}\longrightarrow 0, \ as \quad \ell\to\infty.$$

Moreover, for all $\ell \in \mathbb{N}$,

$$\|u_n\|^2_{E^\alpha_\infty} = \sum_{j=1}^{\ell} \|V^j\|^2_{E^\alpha_\infty} + \|w_n^\ell\|^2_{E^\alpha_\infty} + o(1), \qquad n\to\infty.$$

PROOF OF THEOREM 2.14. The proof is based on the following lemmas. The first one is the classical energy estimate for the fractional heat equation.

LEMMA 2.15. *Let $u_0 \in \dot{H}^{s_\alpha}(\mathbb{R}^3)$. Then*

$$\|H_\alpha(u_0)\|_{E^\alpha_\infty} = \frac{1}{2}\|u_0\|_{H^{s_\alpha}}.$$

PROOF OF LEMMA 2.15. Denote $u = H_\alpha(u_0)$. By multiplying the fractional heat equation by u it result

$$\frac{1}{2}\frac{d}{ds}\|u\|^2_{\dot{H}^{s_\alpha}} + \langle(-\Delta)^\alpha u(s,.), u(s,.)\rangle_{\dot{H}^{s_\alpha}} = 0.$$

Thus,

$$\frac{1}{2}\frac{d}{ds}\|u\|^2_{\dot{H}^{s_\alpha}} + \|u(s,.)\|^2_{\dot{H}^{s_\alpha+\alpha}} = 0.$$

By integration we get, for all $t\in\mathbb{R}^+$,

$$\frac{1}{2}\|u(t)\|^2_{\dot{H}^{s_\alpha}} + \|u\|^2_{L^2([0,t],\dot{H}^{s_\alpha+\alpha})} = \frac{1}{2}\|u_0\|^2_{\dot{H}^{s_\alpha}},$$

as claimed. □

The proof of the next lemma is a direct application of Young's inequality and the last lemma.

LEMMA 2.16. *There exists a constant $C>0$ such that*

$$\|H_\alpha(u_0)\|_{L^\infty((\mathbb{R}^+),L^{p_\alpha}(\mathbb{R}^3))} \leq C\|u_0\|_{L^{p_\alpha}(\mathbb{R}^3)},$$

for all $u_0 \in L^{p_\alpha}(\mathbb{R}^3)$.

Recall the decomposition of profiles obtained for the initial data sequence (ϕ_n), in H^s for all $\ell \in \mathbb{N}^*$, and $x \in \mathbb{R}^3$, we have

$$\phi_n(x) = \sum_{j=1}^{\ell} \phi_n^j(x) + w_n^\ell(x),$$

with $\phi_n^j(x) = (\frac{1}{h_n^j})^{\frac{3}{p_\alpha}} \Phi^j(\frac{x-x_n^j}{h_n^j})$.
Using the invariances of the Heat equation and the notations in Theorem **??** we get, for all $(\ell, n) \in \mathbb{N}^2$ and $x \in \mathbb{R}^3$,

$$u_n(t,x) = \sum_{j=1}^{\ell} (\frac{1}{h_n^j})^{\frac{3}{p_\alpha}} V^j(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x - x_n^j}{h_n^j}) + \omega_n^\ell(t,x),$$

with

$$\omega_n^\ell(t,x) = \mathrm{e}^{-t(-\Delta)^\alpha} w_n^\ell.$$

The sequence (h_n^j, x_n^j) satisfies the orthogonality property since it came from profile decomposition of (ϕ_n). w_n^ℓ is a bounded sequence in $\dot{H}^{s_\alpha}$ and uniformly in ℓ so w_n^ℓ is a bounded sequence in E_∞^α uniformly in l from

$$\|H_\alpha(u_0)\|_{E_\infty^\alpha} = \|u_0\|_{\dot{H}^{s_\alpha}}.$$

On the other hand lemma 2.16 gives

$$\|w_n^\ell\|_{L^\infty(\mathbb{R}^+, L^{p_\alpha})} \leq \|w_n^\ell\|_{L^{p_\alpha}},$$

consider the limit we obtained

$$\lim_{\ell\to\infty} (\limsup_{n\to\infty} \|\omega_n^\ell\|_{L^\infty(\mathbb{R}^+, L^{p_\alpha})}) = 0.$$

It remains to prove that

$$\|u_n\|_{E_\infty^\alpha}^2 = \sum_{j=1}^{\ell} \|V^j\|_{E_\infty^\alpha}^2 + \|w_n^\ell\|_{E_\infty^\alpha}^2 + o(1), \text{ as } \quad n \to \infty.$$

This a direct application of Lemma (2.15) and Theorem 2.13. $\square$

4.2. Profile decomposition for the fractional Navier-Stokes in the maximal ball. This section is the main contribution in this chapter and it concerns the profile decomposition for the solutions of fractional Navier-Stokes system. We start by the initial data lying in some maximal ball for the global wellposedness.

DEFINITION 2.17. *We define $C_{L^{p_\alpha}}$ as the supremum of initial data size for which the global existence for* (22) *holds:*

$$C_{L^{p_\alpha}} := \sup\{\rho > 0 : \quad \mathbb{B}_\rho \cap \dot{H}^{s_\alpha} \subset \mathcal{E}\},$$

where

$$\mathbb{B}_\rho := \{\phi \in L^{p_\alpha} : \|\phi\|_{L^{p_\alpha}} < \rho\}$$

and

$$\mathcal{E} = \{u_0 \in \dot{H}^{s_\alpha} : \mathrm{NS}_\alpha(u_0) \in E^\alpha_\infty\}.$$

It is an open problem to prove that $C_{L^{p_\alpha}} = \infty$, i.e. global wellposedness of the IVP (22) for any data in $\dot{H}^{s_\alpha}(\mathbb{R}^3)$.

THEOREM 2.18. *Assume that $\alpha \in [\frac{5}{6}, 1[$ and $\rho < C_{L^{p_\alpha}}$. Let (φ_n) be a family of divergence free vector fields, bounded in $\dot{H}^{s_\alpha} \cap \mathbb{B}_\rho$ and (u_n) the family of solutions of the system* (22) *associated with the initial data φ_n. Then, with the notations of Theorem 2.13 and up to the extraction of subsequence, the solutions u_n can be decomposed :*

$$u_n = \sum_{j=1}^{\ell} \Gamma^j_n(U^j) + w^\ell_n + r^\ell_n, \qquad \forall (n,\ell) \in \mathbb{N}^2.$$

where $U^j \overset{def}{=} \mathrm{NS}_\alpha(\Phi^j)$ and

$$\lim_{\ell\to\infty} \big(\limsup_{n\to\infty} \|r^\ell_n\|_{E^\alpha_\infty} \big) = 0. \tag{30}$$

REMARK 2.19. The validation of this theorem for $\alpha \in]\frac{1}{2}, \frac{5}{6}[$ remains an open problem. However, if Proposition 2.11 is true for $\alpha \in [\frac{5}{8}, \frac{5}{6}[$ then Theorem 2.18 can be extended to that interval.

Finally, as a consequence of Theorem 2.18, we obtain the following *a priori* estimates.

COROLLARY 2.20. *Assume that* $\alpha \in [\frac{5}{6}, 1[$. *There exists a function* $\mathbb{R}^+ \times [0, C_{L^{p_\alpha}}[\to [0, \infty[$, *such that for every solution* u (22) *with* $u_0 \in \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha}$, *we have*

$$\|u\|_{E_\alpha^\infty} \leq A(\|u_0\|_{\dot{H}^{s_\alpha}}, \|u_0\|_{L^{p_\alpha}}). \tag{31}$$

The second consequence of Theorem2.18 is the following

COROLLARY 2.21. *Assume that* $\alpha \in [\frac{5}{6}, 1[$. *The application* $F : \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha} \longrightarrow C(\mathbb{R}, \dot{H}^{s_\alpha})$ *which mapping every* $\varphi \in \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha}$ *to the associated solution* u *of* (22) *with initial data* φ *is Lipschitz on* $\mathbb{B}_\rho \cap \dot{H}^{s_\alpha}$ *for every* $\rho < C_{L^{p_\alpha}}$.

5. Proof of Theorem 2.18

Let us start with this orthogonally result.

PROPOSITION 2.22. *Let* $T \in \mathbb{R}^+ \cup \{\infty\}$ *be given, and let* F^1 *and* F^2 *be two divergence vector fields belonging to* E_T^α *for* $i = 1, 2$. *Let us consider* $\Gamma_n^1 = (h_n^1, x_n^1)$ *and* $\Gamma_n^2 = (h_n^2, x_n^2)$ *two orthogonal sequences of* $(\mathbb{R}_*^+ \times \mathbb{R}^3)^{\mathbb{N}}$, *we also suppose* $h_n^1 \leq h_n^2$. *If we denote* $F_n^i = \Gamma_n^i F^i$ *for* $i = 1, 2$.
Then, we have the following orthogonality results:

$$\sup_{t \in [0, (h_n^1)^{2\alpha} T]} \Big(\langle F_n^1(t,.), F_n^2(t,.) \rangle_{\dot{H}^{s_\alpha}(\mathbb{R}^3)} + \|F_n^1(t,.) F_n^2(t,.)\|_{L^{\frac{p_\alpha}{2}}} \Big) \longrightarrow 0, \qquad \text{as } n \to \infty.$$

as well as

$$\lim_{n \to \infty} \langle F_n^1(t,.), F_n^2(t,.) \rangle_{L^2([0,(h_n^1)^{2\alpha} T], \dot{H}^{s_\alpha + \alpha}(\mathbb{R}^3)} = 0.$$

PROOF. By definition

$$\forall j \in \{1, 2\}, \quad F_n^j(t, x) = (\frac{1}{h_n^j})^{2\alpha - 1} F^j(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x - x_n^j}{h_n^j}),$$

By assumption, we have that $\phi^j \in E_T^\alpha$, and so $\phi_n^j \in E^\alpha_{(h_n^j)^{2\alpha} T}$. Without loss of generality, we can assume that f^1, f^2 are continuous and compactly supported. Therefore, we have

$$\langle F_n^1(t,.), F_n^2(t,.) \rangle_{\dot{H}^{s_\alpha}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (h_n^1 h_n^2)^{-\frac{3}{2}} (\Lambda^{s_\alpha} F^1)(\frac{t}{(h_n^1)^{2\alpha}}, \frac{x - x_n^1}{h_n^1})(\Lambda^{s_\alpha} F^2)(\frac{t}{(h_n^2)^{2\alpha}}, \frac{x - x_n^2}{h_n^2}) dx$$

where $\Lambda = \sqrt{-\triangle}$.
If $\lim_{n\to\infty}(\frac{h_n^1}{h_n^2} + \frac{h_n^2}{h_n^1}) = +\infty$ then one can assume, for example, $\lim_{n\to\infty}\frac{h_n^1}{h_n^2} = 0$, then the change of variables

$$x = x_n^1 + h_n^1 y, \quad t = (h_n^1)^{2\alpha} s$$

yields

$$\langle F_n^1(t,.), F_n^2(t,.)\rangle_{\dot{H}^{s_\alpha}} = (\frac{h_n^1}{h_n^2})^{\frac{3}{2}} \int (\Lambda^{s_\alpha} F^1)(s,y)(\Lambda^{s_\alpha} F^2)((\frac{h_n^1}{h_n^2})^{2\alpha} s, \frac{h_n^1}{h_n^2} y + y_n) dx,$$

where

$$y_n = \frac{x_n^1 - x_n^2}{h_n^2}.$$

Thus,

$$|\langle F_n^1(t,.), F_n^2(t,.)\rangle_{\dot{H}^{s_\alpha}}| \leq (\frac{h_n^1}{h_n^2})^{\frac{3}{2}} \|\Lambda^{s_\alpha} F^1\|_{L_s^\infty L_y^1} \|\Lambda^{s_\alpha} F^2\|_{L_s^\infty L_y^\infty}.$$

Now, if $\lim_{n\to\infty}(\frac{h_n^1}{h_n^2} + \frac{h_n^2}{h_n^1}) \neq \infty$, then h_n^1 and h_n^2 are proportional (to simplify one can assume that $h_n^1 = h_n^2$). In this case the orthogonality is reduced to

$$|y_n| \to +\infty.$$

So the similar computation gives

$$\langle F_n^1(t,.), F_n^2(t,.)\rangle_{\dot{H}^{s_\alpha}} = \int (\Lambda^{s_\alpha} F^1)(s,y)(\Lambda^{s_\alpha} F^2)(s, y + y_n) dy \longrightarrow 0.$$

The last limit is due to the fact that F^2 is compactly supported and the use of dominated convergence.
The other orthogonality relation can be proved similarly.

□

Let us first remark that U^j are global for all $j \in \mathbb{N}^*$. Actually, Proposition 2.22 applied to the decomposition given by Theorem 2.14 yields

$$\|\Phi_j\|_{L^{p_\alpha}} \leq \sup_n \|\varphi_n\|_{L^{p_\alpha}} \leq \rho < C_{L^{p_\alpha}}, \qquad \forall j \in \mathbb{N}^*,$$

which, by definition of $C_{L^{p_\alpha}}$, implies that $U^j := \mathrm{NS}_\alpha(\Phi^j)$ is global.

Now, we set, for every $\ell \in \mathbb{N}^*$ and every $n \in \mathbb{N}$,

$$r_n^\ell := v_n - \sum_{j=1}^{\ell} U_n^j - \omega_n^\ell,$$

where $\omega_n^\ell \stackrel{\text{def}}{=} H_\alpha(w_n^\ell)$ as in Théorème 2.14 and

$$U_n^j = \Gamma_n^j U^j.$$

We have to prove that

$$\lim_{\ell\to\infty}\big(\lim_{n\to\infty}\sup \|r_n^\ell\|_{E_\infty^\alpha}\big) = 0.$$

The function r_n^ℓ satisfies the following system:

$$\begin{cases} \partial_t r_n^\ell + (-\Delta)^\alpha r_n^\ell + \mathbb{P}(r_n^\ell \cdot \bigtriangledown r_n^\ell) + Q(r_n^\ell, f_n^\ell) = g_n^\ell, & x \in \mathbb{R}^3, t > 0,\\ r^\ell_{n|t=0} = 0, \end{cases}$$

where

$$f_n^\ell := \sum_{j=1}^{\ell} U_n^j + w_n^\ell,$$

and

$$g_n^\ell := \sum_{\substack{1\le k,j\le \ell\\ j\neq k}} Q(U_n^j, U_n^k) - \sum_{j\le l} Q(U_n^j, w_n^\ell) - P(w_n^\ell \cdot \bigtriangledown w_n^\ell).$$

Using Lemma 2.3 we get for every interval $J = [a, b]$

$$\begin{aligned} (32)\quad \|r_n^\ell\|_{E(J)} \le C\big(&\|r_n^\ell(a)\|_{\dot H^{s_\alpha}} + \\ &+ \|r_n^\ell\|_{E(J)}^2 + \|r_n^\ell\|_{E(J)}\|f_n^\ell\|_{L^2(J,\dot H^\beta)} + \|g_n^\ell\|_{L^2(J,\dot H^\beta)}\big), \end{aligned}$$

where C denotes a universal constant (depends only on α). Here and in the sequel, we have used the notation

$$\beta = \frac{5}{2} - 3\alpha.$$

We prepare several propositions. In the sequel one denotes

$$\mathcal{U}_n^\ell = \sum_{j=1}^{\ell} U_n^j.$$

LEMMA 2.23. *The families $\{\mathcal{U}_n^\ell\}$ and $\{w_n^\ell\}$ are uniformly bounded in the energy space E_∞^α.*

PROOF OF LEMMA 2.23 . We know from Proposition 2.14 that the family $\{w_n^\ell\}$ is uniformly bounded in E_∞^α. It remains to prove that, $\sum_{j=1}^{\ell} U_n^j$ is uniformly bounded in E_∞^α. By Proposition 2.22 we can write that

$$\sum_{j=1}^{\infty} \|\Phi^j\|_{\dot{H}^{s_\alpha}}^2 \leq \limsup_{n\to\infty} \|\phi_n\|_{\dot{H}^{s_\alpha}}^2 \leq C.$$

Thus, the small data theory implies that there exists $j_0 \geq 1$ such that

$$\|U^j\|_{E_\infty^\alpha} \leq 2\|\Phi^j\|_{\dot{H}^{s_\alpha}}^2, \qquad \forall\, j \geq j_0.$$

On the other hand, the pairwise orthogonality implies that

$$\begin{aligned}
\limsup_{n\to\infty} \|\sum_{j=1}^{\ell} U_n^j\|_{E_\infty^\alpha}^2 &= \sum_{j=1}^{\ell} \|U^j\|_{E_\infty^\alpha}^2 \\
&\leq \sum_{j=1}^{j_0} \|U^j\|_{E_\infty^\alpha}^2 + 2\sum_{j=j_0}^{\ell} \|\Phi^j\|_{\dot{H}^{s_\alpha}}^2 \\
&\leq \sum_{j=1}^{j_0} \|U^j\|_{E_\infty^\alpha}^2 + 2C.
\end{aligned}$$

□

The core of the proof of Theorem 2.18 is the following proposition.

PROPOSITION 2.24. *Under the assumptions of Theorem* (2.18) *one has*

$$\limsup_{n} \|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^\beta)} \longrightarrow 0, \qquad \text{as} \quad \ell \longrightarrow \infty. \tag{33}$$

where $\beta = \frac{5}{2} - 3\alpha$.

Let us postpone the proof of this proposition and finish the proof of Theorem 2.18. For a fixed n and ℓ one divides the interval $[0,T]$ into $p = p(n,\ell)$ small intervals

$$[0,+\infty[= \underbrace{[0,a^1]}_{I^1} \cup \underbrace{[a^1,a^2]}_{I^2} \cup ... \cup \underbrace{[a^{p-1},+\infty[}_{I^p},$$

so that

$$C\|f_n^\ell\|_{L^2(I_n^i,\dot H^\beta)}\leq\frac{1}{2}.$$

We set

$$X_k=\|r_n^\ell\|_{E(I_k)},\qquad k=1,...,p.$$

The estimate (32) and Lemma 2.23 yield

$$X_{k+1}\leq C\big(X_k+X_{k+1}^2+\|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}\big).$$

Now, the function $H:t\mapsto\|r_n^\ell\|_{E([0,t])}$ is obviously continuous on I^1 and satisfies

$$H(t)\leq CH^2(t)+C\|g_n^\ell\|_{L^2([0,t],\dot H^\beta)},\qquad H(0)=0.$$

For n and ℓ large enough we get

$$H(t)\leq 2C\|g_n^\ell\|_{L^2([0,t],\dot H^\beta)},\qquad\forall\, t\in I^1.$$

Thus, we infer

$$X_1\leq 2C\|g_n^\ell\|_{L^2([0,t],\dot H^\beta)}.$$

On $[a^1,a^2]$ the function $H:t\mapsto\|r_n^\ell\|_{E([a^1,t])}$ is continuous and satisfies

$$\begin{aligned}H(t)&\leq CX_1+CH^2(t)+C\|g_n^\ell\|_{L^2([0,t],\dot H^\beta)}\\&\leq 2C^2\|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}+CH^2(t)+C\|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}.\end{aligned}$$

Thus, for n and ℓ large enough we get

$$X_2\leq 4C^2\|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}.$$

Repeating this argument we obtain

$$X_k\leq C^{k+1}\|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)},\qquad k=1,...,p.$$

Summing up all these terms one gets, for n and ℓ large enough,

$$\|r_n^\ell\|_{E_\infty^\alpha}\leq\sum_{k=0}^{p}X_k\leq\frac{C^{p+2}}{C-1}\|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}.$$

However, one has

$$p\simeq\|f_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}^2.$$

Thus,

$$\|r_n^\ell\|_{E_\infty^\alpha}\leq C\exp(C\|f_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}^2)\|g_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}.$$

Finally, combining Lemma 2.23 and (33) yield together the claimed result.

PROOF OF PROPOSITION 2.24. We set

$$\mathcal{U}_n^\ell = \sum_{j=1}^{\ell} U_n^j.$$

One has trivially

$$g_n^\ell = \sum_{j\neq k} Q(U_n^j, U_n^k) + Q(\mathcal{U}_n^\ell, w_n^\ell) + \mathbb{P}(w_n^\ell.\nabla w_n^\ell).$$

We have divided the proof of (33) into a sequence of lemmas.

LEMMA 2.25. *Let* $p \in [p_\alpha, \infty[$ *with* $p_\alpha = \frac{3}{2\alpha-1}$ *and take* $q = \frac{2\alpha}{2\alpha-1-\frac{3}{p}}$. *Then, we have*

$$\limsup_n \|w_n^\ell\|_{L^q(\mathbb{R}^+,L^p)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty.$$

PROOF OF LEMMA 2.25. Since $\{w_n^\ell\}$ is bounded in the energy space then by interpolation and Sobolev embeddings, then it's uniformly bounded in

$$L^q(\mathbb{R}^+, L^p(\mathbb{R}^3))$$

for

$$q > \frac{2\alpha}{2\alpha-1}, \quad \frac{1}{q} = -\frac{1}{3} + \frac{2}{3}(1-\frac{1}{q})\alpha.$$

By interpolation between this fact and fact

$$\limsup_n \|w_n^\ell\|_{L^\infty(\mathbb{R}^+,L^{p_\alpha})} \longrightarrow 0, \text{ as } \quad \ell \to \infty,$$

we get the result.

□

LEMMA 2.26. *Under the definitions above, one has*

$$\limsup_n \|w_n^\ell \nabla G_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^\beta)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty,$$

for both case $G_n^\ell = w_n^\ell$ *or* $G_n^\ell = \mathcal{U}_n^\ell$.

PROOF OF LEMMA 2.26. By interpolation on has

$$\|w_n^\ell \nabla G_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} \leq \|w_n^\ell \nabla G_n^\ell\|^\theta_{L^{\frac{2\alpha}{2\alpha-1}}(\mathbb{R}^+,\dot H^{\beta+\alpha-1})} \|w_n^\ell \nabla G_n^\ell\|^{1-\theta}_{L^{\frac{4\alpha}{8\alpha-5}}(\mathbb{R}^+,L^2)},$$

with

$$\theta = \frac{5-6\alpha}{3-4\alpha}.$$

By a similar argument as above, one can estimate

$$\begin{aligned}\|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{2\alpha}{2\alpha-1}}(\mathbb{R}^+,\dot H^{\beta+\alpha-1})} &\leq \|w_n^\ell\|_{L^{\frac{4\alpha}{2\alpha-1}}(\mathbb{R}^+,\dot H^{2-\alpha})} \|G_n^\ell\|_{L^{\frac{4\alpha}{2\alpha-1}}(\mathbb{R}^+,\dot H^{2-\alpha})}\\ &\leq C.\end{aligned}$$

It remains to prove that

$$\limsup_n \|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-5}}(\mathbb{R}^+,L^2)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty.$$

To do so, we use a Hôlder inequality to get

$$\|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-5}}(\mathbb{R}^+,L^2)} \leq \|w_n^\ell\|_{L^{\frac{4\alpha}{6\alpha-5}}(\mathbb{R}^+,L^{\frac{6}{3-2\alpha}})} \|\nabla G_n^\ell\|_{L^2(\mathbb{R}^+,L^{\frac{3}{\alpha}})}.$$

By Sobolev inequlaity we infer

$$\begin{aligned}|\nabla G_n^\ell\|_{L^2(\mathbb{R}^+,L^{\frac{3}{\alpha}})} &\leq C\|\nabla G_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^{\frac{3}{2}-\alpha})}\\ &\leq C\|G_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^{\frac{5}{2}-\alpha})}\\ &\leq C\|G_n^\ell\|_{E_\alpha^\infty}\\ &\leq C.\end{aligned}$$

The last inequality comes from (2.23).
Combing the two last inequalities together we get

$$\|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-5}}(\mathbb{R}^+,L^2)} \leq C\|w_n^\ell\|_{L^{\frac{4\alpha}{6\alpha-5}}(\mathbb{R}^+,L^{\frac{6}{3-2\alpha}})}.$$

The needed result is a direct consequence of Lemma 2.25.

□

REMARK 2.27. In order to use the smallness in the Lebesgue norm of w_n^ℓ, we need to use the interpolation between Soblev space to keep $\|\nabla.(w_n^\ell \otimes w_n^\ell\|_{L^2}$ or $\|(w_n^\ell \otimes w_n^\ell\|_{L^2}$ (the last one is obtained by an interpolation between energy espace and remove the ∇ operator of the term $\nabla.(w_n^\ell \otimes w_n^\ell)$ by taking an interpolation between $\dot{H}^{-1}$). If one use the second option then we are left with $\|w_n^\ell\|_{L^4}$ to control which is possible only if $4 \geq p_\alpha$ (the minimal Lebesgue regularity of the remainder w_n^ℓ), that is $\alpha \geq \frac{7}{8}$. Actually, for $\alpha \geq \frac{7}{8}$ the calculous used in **[10]** can be adapted without difficulties to give the result.

The next lemma deals with the third term in f_n^ℓ.

LEMMA 2.28. *Under the assumptions of Theorem 2.18 we have*

$$\limsup_n \|\mathcal{U}_n^\ell.\nabla w_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^\beta)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty,$$

where $\beta = \dfrac{5}{2} - 3\alpha$.

PROOF OF LEMMA 2.28. Recall that

$$\sum_{j=1}^{\infty} \|U^j\|^2_{\dot{H}^{s_\alpha}} < \infty.$$

Let $\epsilon > 0$ small then there exists ℓ_0 such that

$$\sum_{j=\ell_0}^{\infty} \|U^j\|^2_{\dot{H}^{s_\alpha}} < \epsilon.$$

By the orthogonality we get then

$$\begin{aligned} \limsup_n \|\sum_{j\geq \ell_0} U_n^j\|^2_{L^\infty(\mathbb{R}^+,\dot{H}^{s_\alpha})} &= \sum_{j\geq\ell_0} \|U^j\|^2_{L^\infty(\mathbb{R}^+,\dot{H}^{s_\alpha})} \\ &\leq 2\epsilon. \end{aligned}$$

First, by the product laws in Sobolev spaces we get

$$\|\sum_{j\geq\ell_0} U_n^j.\nabla w_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^\beta)} \leq \|\sum_{j\geq\ell_0} U_n^j\|_{L^\infty(\mathbb{R}^+,\dot{H}^{s_\alpha})}\|w_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^{s_\alpha+\alpha})}.$$

Since $\|w_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^{s_\alpha+\alpha})}$ is bounded, then

$$\|\sum_{j\geq\ell_0} U_n^j.\nabla w_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^\beta)} \leq C\sqrt{\epsilon}. \tag{34}$$

Next, we need to show that

$$\limsup_n \|U_n^j.\nabla w_n^\ell\|_{L^2(\mathbb{R}^+,\dot{H}^\beta)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty, \quad \forall\ 1 \leq j \leq \ell_0.$$

Let us prove it that for $j=1$ (the other terms are similar).
By a change of variables one has

$$\|U_n^1.\nabla w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} = \|U^1.\nabla \tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} \tag{35}$$

with

$$\tilde w_n^\ell(t,x) = (h_n^1)^\alpha w_n^\ell\big((h_n^1)^2 t, h_n^1 x + x_n^1\big).$$

Notice[4] that for any $\epsilon>0$ there is $V\in C_c^\infty([0,\infty[,\mathcal{S}(\mathbb{R}^3))$ such that

$$\|U^1-U\|_{L^2(\mathbb{R}^+,\dot H^\beta)}<\epsilon.$$

It follows that

$$\|U^1.\nabla\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} \le \|(U^1-U).\nabla\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} + \|U.\nabla\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)}.$$

Now, the first term in the last inequality can be controlled as above

$$\|(U^1-\tilde U).\nabla\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} \le \|(U^1-\tilde U)\|_{L^\infty(\mathbb{R}^+,\dot H^{s_\alpha})}\|\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^{s_\alpha+\alpha})}.$$

But

$$\|\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^{s_\alpha+\alpha})} = \|w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^{s_\alpha+\alpha})}.$$

Consequently,

$$\|(U^1-\tilde U).\nabla\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} \le \epsilon C.$$

Next, for the second one, by interpolation one has

$$\|U\nabla\tilde w_n^\ell\|_{L^2(\mathbb{R}^+,\dot H^\beta)} \le \|U\tilde w_n^\ell\|^{\frac{\alpha-\frac12}{3-2\alpha}}_{L^{\frac{4\alpha}{8\alpha-7}}(\mathbb{R}^+,L^2)} \|U\nabla\tilde w_n^\ell\|^{\frac{\frac72-3\alpha}{3-2\alpha}}_{L^{\frac{4\alpha}{4\alpha-1}}(\mathbb{R}^+,\dot H^{\beta+\alpha-\frac12})}$$

As in the proof of (**??**) one can control

$$\|U\nabla\tilde w_n^\ell\|_{L^{\frac{4\alpha}{4\alpha-1}}(\mathbb{R}^+,\dot H^{\beta+\alpha-\frac12})}\le C,$$

uniformly in n and ℓ.

---

[4]Actually, using the Fourier side, one can approximate U^1 by $U\in C_c^\infty([0,\infty[,\mathcal{S}(\mathbb{R}^3))$ such that $\hat U\in C_c^\infty([0,\infty[,\mathcal{D}(\mathbb{R}^3\setminus\{0\}))$. The projection of such function on the free divergence vectors fields is still an approximation of U^1. Indeed on has

$$\|U^1-U\|^2_{L^2(\mathbb{R}^+,\dot H^\beta)} = \|U^1-\mathcal{P}U\|^2_{L^2(\mathbb{R}^+,\dot H^\beta)} + \|(I-\mathcal{P})U\|^2_{L^2(\mathbb{R}^+,\dot H^\beta)}.$$

Now, since $\hat U$ is supported away from 0 then $\mathcal{P}U\in C_c^\infty([0,\infty[,\mathcal{D}(\mathbb{R}^3\setminus\{0\}))$.

Finally,

$$\begin{aligned}\|U.\tilde{w}_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-7}}(\mathbb{R}^+,L^2)} &\leq C\|U\|_{L^{\frac{4\alpha}{8\alpha-7}}(\mathbb{R}^+,L^{\frac{6}{5-4\alpha}})}\|\tilde{w}_n^\ell\|_{L^\infty(\mathbb{R}^+,L^{p_\alpha})} \\ &\leq C(V)\|w_n^\ell\|_{L^\infty(\mathbb{R}^+,L^{p_\alpha})}.\end{aligned}$$

This ends the proof of Lemma 2.28.

□

This ends the proof of Proposition 2.24. □

5.1. Proof of Corollary 2.20. $u_0 \in \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha}$, Assume that the *a priori* estimate (31) fails, then there exist some sequence (u_n) of solution to (22), such that $(u_n(0,.)$ is bounded in $u_0 \in \mathbb{B}_\rho \cap \dot{H}^{s_\alpha}$, with $\rho < C_{L^{p_\alpha}}$

$$\|\!|u_n|\!\|_{E^\alpha_\infty} \underset{n\to\infty}{\longrightarrow} +\infty. \tag{36}$$

Theorem 2.18 applied to $(u_n(0,.))$ shows that there exist a subsequence $(u_n^{'})$ of (u_n), such that

$$u_n^{'} = \sum_{j=1}^{\ell} (\frac{1}{h_n^j})^{\frac{3}{p_\alpha}} V^j(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x-x_n^j}{h_n^j}) + w_n^\ell(t,x) + r_n^\ell(t,x) \tag{37}$$

with

$$\limsup_{n\to\infty} \|\!|w_n^\ell + r_n^\ell|\!\|_{E^\alpha_\infty} \leq C \tag{38}$$

for every $l \geq 1$. Hence,

$$\limsup_{n\to\infty} \|\!|u_n^{'}|\!\|_{E^\alpha_\infty} \leq \limsup_{n\to\infty} \|\!|w_n^\ell + r_n^\ell|\!\|_{E^\alpha_\infty} + \sum_{j=1}^{\ell} \|\!|U^j|\!\|_{E^\alpha_\infty} < +\infty. \tag{39}$$

This implies that $(u_n^{'})$ is bounded in $\|\!|\,|\!\|_{\mathbb{R}}$ norm, which contradicts (36) and proves the existence of some function A satisfying (31). This completes the proof of Corollary 2.20. □

5.2. Proof of Corollary 2.21. We have to prove that the real tangent map of F at every point of $E_0 := \mathbb{B}_{C_{L^{p_\alpha}}} \cap \dot{H}^{s_\alpha}$ is bounded on $E_\rho := \mathbb{B}_\rho \cap \dot{H}^{s_\alpha}$. For this we start by computing the real tangent map on an arbitrary element ϕ of E_0, i.e $\frac{d}{d\epsilon} F(\phi + \epsilon\tilde{\phi})_{|\epsilon=0}$.

PROPOSITION 2.29. *For every $\phi \in E_0$ and $\tilde{\phi} \in \dot{H}^1(\mathbb{R}^3)$ we have*

$$\frac{d}{d\epsilon} F(\phi + \epsilon\tilde{\phi})_{|\epsilon=0} = v \tag{40}$$

where v is the unique solution in $\mathcal{Y}(\mathbb{R})$ of the following IVP

$$\begin{cases} \partial_t v + (-\Delta)^\alpha v = -\mathbb{P}(u.\nabla v + v.\nabla u) \\ v(0,x) = \tilde{\phi}(x) \end{cases} \tag{41}$$

Proof. The proof is divided into 2 steps. In the first one we prove that the initial values problem (41) has a unique solution $v \in \mathcal{Y}(\mathbb{R})$ satisfying

$$\|v\|_{E^\alpha_\infty} \leq \exp\left(\|u\|^{\frac{4\alpha}{2\alpha-1}}_{L^{\frac{4\alpha}{2\alpha-1}}([0,+\infty[,\dot{H}^{2-\alpha})} + \|u\|^2_{L^2([0,+\infty[,\dot{H}^{\frac{5}{2}-\alpha})} \right) \|\tilde{\phi}\|_{\dot{H}^{s_\alpha}}. \tag{42}$$

In the second one we shall prove that

$$\left\|\frac{F(\phi + \epsilon\tilde{\phi}) - F(\phi)}{\epsilon} - v\right\|_{E^\alpha_\infty} \underset{\epsilon\to 0}{\longrightarrow} 0. \tag{43}$$

Step1. The equation (41) is equivalent to the following integral equation

$$\begin{aligned} v(t) &= e^{t(-\Delta)^\alpha}\tilde{\phi} - \int_0^t e^{(t-s)(-\Delta)^\alpha} \mathbb{P}(u.\nabla v + v.\nabla u) ds \\ &:= Av. \end{aligned} \tag{44}$$

The solution can be obtained by a standard method of contraction. By using Lemma 2.3, with $p = r = 2$ and $s = s_\alpha$, we get

$$\|Av\|_{E^\alpha_{T_1}} \leq C\|\tilde{\phi}\|_{\dot{H}^{s_\alpha}} + C_1 \|u.\nabla v + v.\nabla u\|_{L^2([0,T_1],\dot{H}^{\frac{5}{2}-3\alpha})}, \tag{45}$$

for every $0 \leq T_1 < +\infty$.
A product law yields

$$\|u.\nabla v + v.\nabla u\|_{L^2([0,T_1],\dot{H}^{\frac{5}{2}-3\alpha})} \leq (\|u\|_{L^{\frac{4\alpha}{2\alpha-1}}([0,T_1],\dot{H}^{2-\alpha})} + \|u\|_{L^2([0,T_1],\dot{H}^{\frac{5}{2}-\alpha})}) \|v\|_{E^\alpha_{T_1}}$$

We choose T_1 too small such that

$$C_1(\|u\|_{L^{\frac{4\alpha}{2\alpha-1}}([0,T_1],\dot H^{2-\alpha})} + \|u\|_{L^2([0,T_1],\dot H^{\frac{5}{2}-\alpha})}) < 1/2. \tag{46}$$

(45) and (46) assert that the operator A defined by (44) is a contraction in $\mathcal{Y}([0,T_1])$. Hence, a unique solution v of (41) exists on $[0,T_1]$. Moreover,

$$\|v\|_{E^\alpha_{T_1}} \leq 2C\|\tilde\phi\|_{\dot H^{s_\alpha}}.$$

Applying the same process on small intervals I_i partition of $\mathbb{R}$ we obtain a unique solution v on E^α_∞ of the initial values problem (41) satisfying

$$\|v\|_{E^\alpha_\infty} \leq C2^N\|\tilde\phi\|_{\dot H^1}, \tag{47}$$

where N denotes the number of the intervals I_i partition of $\mathbb{R}$ such that (46) holds for everyI_i. However, it is clear that that N can be chosen so that

$$N \leq C_2\left(\|u\|^{\frac{4\alpha}{2\alpha-1}}_{L^{\frac{4\alpha}{2\alpha-1}}([0,+\infty[,\dot H^{2-\alpha})} + \|u\|^2_{L^2([0,+\infty[,\dot H^{\frac{5}{2}-\alpha})}\right). \tag{48}$$

Combining (47) and (48), the claim (42) follows.

Step 2. Let u_ϵ denotes the solution of (22) with initial data $\phi_\epsilon = \phi + \epsilon\tilde\phi$. Set

$$r_\epsilon = \frac{u_\epsilon - u}{\epsilon} - v.$$

It is easy to see that r_ϵ satisfies the difference equation

$$\begin{cases} \partial_t r_\epsilon + (-\Delta)^\alpha r_\epsilon = -\mathbb{P}(u.\nabla r_\epsilon + r_\epsilon.\nabla u) + F_\epsilon \\ r_\epsilon(0,x) = 0 \end{cases} \tag{49}$$

with

$$F_\epsilon = \epsilon(v.\nabla v + v.\nabla r_\epsilon + r_\epsilon\nabla v + r_\epsilon.\nabla r_\epsilon).$$

By using Lemma (2.3) and the product laws, we obtain

$$\begin{aligned}\|r_\epsilon\|_{E^\alpha_{T_1}} \leq C_1(\|u\|_{L^{\frac{4\alpha}{2\alpha-1}}([0,T_1],\dot H^{2-\alpha})} + \|u\|_{L^2([0,T_1],\dot H^{\frac{5}{2}-\alpha})})\|r_\epsilon\|_{E^\alpha_{T_1}} + \\ + |\epsilon|(\|r_\epsilon\|_{E^\alpha_{T_1}}\|v\|_{E^\alpha_{T_1}} + \|r_\epsilon\|^2_{E^\alpha_{T_1}}),\end{aligned}$$

for every $0 \leq T_1 < \infty$.

Combining the last estimate with (41) we get

$$\|r_\epsilon\|_{E^\alpha_{T_1}} \le C_1(\|u\|_{L^{\frac{4\alpha}{2\alpha-1}}([0,T_1],\dot H^{2-\alpha})} + \|u\|_{L^2([0,T_1],\dot H^{\frac{5}{2}-\alpha})})\|r_\epsilon\|_{E^\alpha_{T_1}} + + |\epsilon|(\|r_\epsilon\|_{E^\alpha_{T_1}}\Phi(t) + \|r_\epsilon\|^2_{E^\alpha_{T_1}}). \tag{50}$$

with

$$\Phi(t) = \exp\left(\|u\|^{\frac{4\alpha}{2\alpha-1}}_{L^{\frac{4\alpha}{2\alpha-1}}([0,+\infty[,\dot H^{2-\alpha})} + \|u\|^2_{L^2([0,+\infty[,\dot H^{\frac{5}{2}-\alpha})}\right).$$

If we choose T_1 such that (46) holds and by a standard bootstrap argument we conclude from (50)

$$\|r_\epsilon\|_{E^\alpha_{T_1}} \le \tilde C|\epsilon|.$$

By applying the same inequalities successively on small interval I_i partitions of $\mathbb{R}$, such that (46) holds for every I_i, we obtain finally

$$\|r_\epsilon\|_{E^\alpha_\infty} \le \tilde C_1|\epsilon|. \tag{51}$$

This yields (45) and, then, concludes the proof of Proposition 2.29. □

Let us now achieve the proof of Corollary 2.21. By Proposition 2.29 and (45) we get finally

(52)

$$\|\frac{d}{d\epsilon}F(\phi+\epsilon\tilde\phi)_{|\epsilon=0}\|_{E^\alpha_\infty} \le \exp\left(\|u\|^{\frac{4\alpha}{2\alpha-1}}_{L^{\frac{4\alpha}{2\alpha-1}}([0,+\infty[,\dot H^{2-\alpha})} + \|u\|^2_{L^2([0,+\infty[,\dot H^{\frac{5}{2}-\alpha})}\right)\|\tilde\phi\|_{\dot H^{s_\alpha}}.$$

By Hölder on can estimate

$$\|u\|^{\frac{4\alpha}{2\alpha-1}}_{L^{\frac{4\alpha}{2\alpha-1}}([0,+\infty[,\dot H^{2-\alpha})} + \|u\|^2_{L^2([0,+\infty[,\dot H^{\frac{5}{2}-\alpha})} \le \|u\|_{E^\alpha_\infty} + \|u\|^{\frac{2\alpha}{2\alpha-1}}_{E^\alpha_\infty}.$$

However, Corollary (2.20) asserts that if ϕ varies in $E_\lambda = \mathbb{B}_\rho \cap \dot H^{s_\alpha}$, with $\rho < C_{L^{p_\alpha}}$ the term $\|u\|_{E^\alpha_\infty}$ remains bounded. This concludes the proof of Corollary 2.21.

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CHAPTER 3

Asymptotic behavior for the fractional Navier-Stokes system

1. Introduction

In this chapter we deal with the issue of symptomatic behavior behavior of solutions of Navier-Stokes system with fractional dissipation (NS_α):

$$\begin{cases} \partial_t u(t,x) + u.\nabla u + \nu(-\Delta)^\alpha u(t,x) + \nabla p = 0, & x \in \mathbb{R}^d, t > 0, \\ \nabla.u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{53}$$

The operator $(-\Delta)^\alpha$ is the α-order Laplace operator which can be defined by the Fourier transform in the space variable:

$$\mathcal{F}(-\Delta)^\alpha \psi(\xi) = |\xi|^{2\alpha}\hat{\psi}(\xi).$$

The vectors field $u = (u_1, u_2, u_3)$ is the velocity field, p the scalar pressure and the coefficient ν is the kinematic viscosity. The parameter α is taken $\frac{1}{2} < \alpha < 1$. This encloses the case $\alpha = 1$ and, in all cases, the problem is supercritical (the case $\alpha = \frac{5}{4}$ is critical with respect the conservation of the energy). These generalizations have been introduced by Lions in 1960s.
The equation is invariant under the scaling:

$$u_\lambda(t,x) = \lambda^{2\alpha-1}u(\lambda^{2\alpha}t, \lambda x), \qquad p_\lambda(t,x) = \lambda^{4\alpha-2}p(\lambda^{2\alpha}t, \lambda x). \tag{54}$$

This induces us to investigate problem (53) in the critical spaces whose norm is invariant under scaling (54). Thus, a natural candidate is homogeneous Sobolev space $\dot{H}^{s_\alpha}$ with $s_\alpha = \frac{5}{2} - 2\alpha$ or the Lebesgue space $L^{\frac{3}{2\alpha-1}}$. These spaces (and many others) has the same scaling than (53) and appears to be critical space for the locally well-posed question.

The 3-dimensional incompressible Navier-Stokes ($\alpha = 1$) has a long history as a challenging open problem in nonlinear partial differential equations (see **[15]** for instance). The Navier-Stokes system with fractional dissipation has been addressed by many authors. For $\alpha \geq \frac{5}{4}$, the system is globally well-posed, a logarithmic improvement was proved by T. Tao **[27]**.
Recently, the fractional Navier-Stokes system regains some interests. In **[21, 23]** Wu obtained lower bounds for the integral involving $(-\Delta)^\alpha$ by combining pointwise inequalities for $(-\Delta)^\alpha$ with Bernstein inequalities for fractional derivatives. As an application of these lower bounds, he established the existence and uniqueness of solutions to the generalized Navier-Stokes equations in Besov spaces for $\alpha > 0$. Li and Zhai **[20]** have studied the well-posedness and regularity of the generalized Navier-Stokes equations with initial data in a new critical space $Q^{\beta,-1}_{\alpha,\infty} = \nabla \cdot (Q^\beta_\alpha(\mathbb{R}^n))^n$, $\beta \in (\frac{1}{2}, 1)$ which is larger than some known critical homogeneous Besov spaces. Also Zhai **[26]** studied the well-posedness for the fractional Navier-Stokes equations in critical spaces$G^{-(2\beta-1)_n}(\mathbb{R}^n)$ and $BMO^{-(2\beta-1)}(\mathbb{R}^n)$ which are close to the largest critical space $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$. for $\beta \in (\frac{1}{2}, 1)$
The profile decomposition is a refined tool of compactness-concentration type which was introduced by P. Gérard **[13]** to study the defect of compactness for the critical Sobolev embedding. Afterwards, many authors used it to study some critical evolution equations (see **[5]**, **[14]**, **[4]** for instance). The profile decomposition for the Navier-Stokes has been performed by Isabelle Gallagher in **[10]**. In this chapter some qualitative properties are proved and the interest of this aspect was recently renewed by the works of C. Kenig and Koch **[22]** and Gallagher *et al* **[12]**.
It is worth noting that the pressure filed can be explicitly determined by Poisson equation:

$$-\Delta p = div(u.\nabla u)$$

then, due to the incompressibility of the fluid we can write

$$p = (-\Delta)^{-1} \sum_{i,j} \partial_i \partial_j (u^i u^j).$$

The Leray projector $\mathbb{P} = Id - \nabla\Delta^{-1}div$ is the orthogonal projector onto divergence free vector fields. In Fourier variable the $\widehat{\mathbb{P}}$ can be represented as a matrix with coefficients given by

$$\widehat{\mathbb{P}}_{i,j} = \delta_{i,j} - \frac{\xi_i\xi_j}{|\xi|^2}.$$

It follows that the system (53) can be written

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = Q(u,u), & x \in \mathbb{R}^3, t > 0, \\ u_{|t=0} = u_0. \end{cases}$$

where the biliniear operator Q defined by

$$Q(v,w) := -\frac{1}{2}\mathbb{P}(div(v \otimes w) + div(w \otimes v)).$$

The solutions we consider, which are called *mild* solutions, are the solutions of the following integral equation

$$u(t) = e^{-t(-\Delta)^\alpha}u_0 - \int_0^s e^{-(t-s)(-\Delta)^\alpha}Q(u,u)(s)ds.$$

These solutions are fixed points (in the appropriate spaces) of the map

$$u \longrightarrow e^{-t(-\Delta)^\alpha}u_0(x) + B(u,u)$$

with

$$B(u,u) = -\int_0^s e^{-(t-s)(-\Delta)^\alpha}Q(u,u)ds.$$

We recall the following local wellposedness result.

THEOREM 3.1. *Let $\frac{1}{2} < \alpha < 1$ and set $s_\alpha = \frac{5}{2} - 2\alpha$ and $\mathbf{q}_\alpha = \frac{4\alpha}{2\alpha-1}$. For every $u_0 \in \dot{H}^{s_\alpha}(\mathbb{R}^3)$ a Solenoidal vectorial function, there exists a positive time T such that the system* (53) *has a unique solution u in $\tilde{L}^{\mathbf{q}_\alpha}([0,T], \dot{H}^{2-\alpha})$. Moreover, the unique solution u belongs to the energy space :*

$$E_T^\alpha = \mathcal{C}\big([0,T], \dot{H}^{s_\alpha}\big) \cap L^2\big([0,T], \dot{H}^{s_\alpha+\alpha}\big).$$

ASSUMPTION (A). *The null function is the only solution $u = \mathrm{NS}_\alpha(u_0)$ of* (22) *for which there exists a bounded increasing sequence $t_n \in [0, T^*[$ such that*

$$\sup_n \|u\|_{L^\infty([0,t_n],L^{p_\alpha})} < \infty$$

and

$$u(t_n,.) \longrightarrow 0 \qquad \text{dans} \quad L^2_{loc}(\mathbb{R}^3).$$

THEOREM 3.2. *Take $\alpha \in [\frac{5}{6},1[$ and assume that (A) is true. Let $u \in \mathcal{C}([0,\infty[;\dot{H}^{s_\alpha})$ a global solution of* (53). *Then*

$$\lim_{t\to\infty} \|u(t)\|_{\dot{H}^{s_\alpha}} = 0, \tag{55}$$

and $u \in E^\alpha_\infty$.

REMARK 3.3. The case $\alpha \in]\frac{1}{2},\frac{5}{6}[$ is still open. Actually, even if we assume that $u_0 \in H^{s_\alpha}$ one canot conclude because the a priori bound given by the energy is not sufficient to prove the asymptotic behavior.

A direct consequence of Theorem 3.2 is the following:

COROLLARY 3.4. *Assume $\alpha \in [\frac{5}{6},1[$.The set of initial data u_0 such that the solution u given by Theorem* (3.1) *is global is an open set in $\dot{H}^{s_\alpha}$).*

THEOREM 3.5. *Let $\alpha \in [\frac{5}{6},1[$. Let $u_0 \in \dot{H}^{s_\alpha}$ and u is the unique solution of* (22). *If the maximal time of existence T^* is finite then*

$$\sup_{t\in[0,T^*[} \|u(t)\|_{\dot{H}^{s_\alpha}} = +\infty.$$

REMARK 3.6. This theorem was proved for the Navier-Stokes in **[8]** and in **[22]** with a different approach. Here we follow the ideas of **[22]**.

2. Proof of Theorem 3.2

Let us first assume $u_0 \in H^{s_\alpha}$. In this case the energy is well-defined and conserved

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u(\tau)\|^2_{L^2} d\tau = \|u_0\|^2_{L^2}.$$

for every $t \in [0,+\infty[$.
This implies that

$$u \in L^\infty([0,+\infty[,L^2) \cap L^2([0,+\infty[,\dot{H}^\alpha).$$

Since $s_\alpha \leq \alpha$ (remember that $\alpha \in [\frac{5}{6},1[$) then by interpolation

$$u \in L^{\frac{4\alpha}{5-4\alpha}}([0,+\infty[,\dot{H}^{s_\alpha}).$$

This implies in particular that there exists t_* such that

$$\|u(t_*)\|_{\dot{H}^{s_\alpha}} \leq \epsilon$$

with ϵ small enough.
The small data theory implies that

$$\|u(t)\|_{\dot{H}^{s_\alpha}} \leq 2\epsilon,$$

for $t \in [t_*, +\infty[$.
For the general case we split the initial data into two parts:

$$u_0 = w_0 + v_0$$

with

$$\|w_0\|_{\dot{H}^{s_\alpha}} \leq \varepsilon \quad \text{and} \quad v_0 \in L^2.$$

This is can be done trivially by multiplying u_0 by a suitable function with a localized Fourier transform. More precisely, we take w_0 so that

$$\hat{w}_0 = 1_{B(0,\delta)}\hat{u}_0$$

with $\delta = \delta(\epsilon)$ is small enough.
We know, by the small data theory, that there exists a global solution w to the fractional Navier-Stokes equations (53) with data w_0. Furthermore, one has

$$\|w\|_{\tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,\infty[,\dot{H}^{2-\alpha})} \leq 2\varepsilon$$

Now let us define $v = u - w$ which satisfies the following system:

$$(56) \quad \begin{cases} \partial_t v(t,x) + (-\Delta)^\alpha v(t,x) = Q(u,v) + Q(v,w), & x \in \mathbb{R}^3, t > 0, \\ v_{|t=0} = v_0. \end{cases}$$

We know that

$$v \in \mathcal{C}([0,\infty[; \dot{H}^{s_\alpha}) \cap \tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,\infty[, \dot{H}^{2-\alpha}).$$

$$v \in \tilde{L}^{\infty}([0,T*]; \dot{H}^{s_\alpha}) \cap \tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,T*], \dot{H}^{2-\alpha}).$$

An energy estimate in L^2 (using the fact $\nabla.v = 0$) yields

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \|(-\Delta)^{\frac{\alpha}{2}} v(t)\|_{L^2}^2 = 2\langle Q(v(t), w(t)).v(t)\rangle.$$

By using Plancherel identity and then the Cauchy Schwartz inequality we get

$$\begin{aligned}|\langle Q(v(t),w(t)).v(t)\rangle| &\leq \|vw\|_{\dot H^{1-\alpha}}\|v\|_{\dot H^{\alpha}} \\ &\leq \|vw\|^2_{\dot H^{1-\alpha}} + \frac{1}{2}\|v\|^2_{\dot H^{\alpha}}.\end{aligned}$$

On the other hand, the product law yields

$$\begin{aligned}\|vw\|_{\dot H^{1-\alpha}} &\leq C\|v\|_{\dot H^{\alpha}}\|vw\|_{\dot H^{\frac{5}{2}-2\alpha}} \\ &\leq 2C\epsilon\|v\|_{\dot H^{\alpha}}.\end{aligned}$$

Putting all these estimate together and taking ϵ small enough we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2} + \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}v(t)\|^2_{L^2} \leq 0.$$

This implies that

$$v \in L^\infty([0,+\infty[, L^2) \cap L^2([0,+\infty[, \dot H^{\alpha}).$$

Since $s_\alpha \leq \alpha$ (remember that $\alpha \in [\frac{5}{6}, 1[$) then by interpolation

$$u \in L^{\frac{4\alpha}{5-4\alpha}}([0,+\infty[, \dot H^{s_\alpha}).$$

This implies in particular that there exists t_* such that

$$\|v(t_*)\|_{\dot H^{s_\alpha}} \leq \epsilon$$

with ϵ small enough. In particular, one has

$$\|u(t_*)\|_{\dot H^{s_\alpha}} \leq \|w(t_*)\|_{\dot H^{s_\alpha}} + \|v(t_*)\|_{\dot H^{s_\alpha}} \leq 3\epsilon.$$

The small data theory implies that

$$\|u(t)\|_{\dot H^{s_\alpha}} \leq 2\epsilon,$$

for $t \in [t_*, +\infty[$.

3. Proof of Theorem 3.5

We first consider the case $T^* < \infty$. Therefore, we suppose that

$$u \in \mathcal{C}([0,T^*];\dot{H}^{s_\alpha}).$$

Then, it suffices to show that

$$u \in \tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,T^*],\dot{H}^{2-\alpha}) \qquad \text{for} \qquad \alpha \in]\frac{1}{2},1[.$$

According to Lemma 2.3 one has

$$\|u\|_{\tilde{L}_t^{\frac{4\alpha}{2\alpha-1}}(\dot{H}^{2-\alpha})} \lesssim \|u_0\|_{\dot{H}^{s_\alpha}} + \|Q(u,u)\|_{\tilde{L}_t^{\frac{2\alpha}{2\alpha-1}}(,\dot{H}^{\frac{3}{2}-2\alpha})}. \tag{57}$$

Next, by density of smooth function in $\mathcal{C}([0,T^*];\dot{H}^{s_\alpha})$, there existe some decompostion

$$u = u_1 + u_2$$

such that

$$\|u_1\|_{\tilde{L}_t^{\frac{4\alpha}{2\alpha-1}}(\dot{H}^{2-\alpha})} \le C$$

and u_2 is as smooth as we want. By virtue of product laws in Sobolev Spaces, we write

$$\begin{aligned}
\|Q(u,u)\|_{\tilde{L}_t^{\frac{2\alpha}{2\alpha-1}}(,\dot{H}^{2-\alpha})} &\lesssim \|Q(u,u_1)\|_{\tilde{L}_t^{\frac{2\alpha}{2\alpha-1}}(,\dot{H}^{\frac{3}{2}-2\alpha})} + \|Q(u,u_2)\|_{\tilde{L}_t^{\frac{2\alpha}{2\alpha-1}}(,\dot{H}^{\frac{3}{2}-2\alpha})} \\
&\lesssim \|u\|_{\tilde{L}_t^{\frac{4\alpha}{2\alpha-1}}(\dot{H}^{2-\alpha})}\|u_1\|_{\tilde{L}_t^{\frac{4\alpha}{2\alpha-1}}(\dot{H}^{2-\alpha})} + \|u\|_{\tilde{L}_t^{\frac{4\alpha}{2\alpha-1}}(,\dot{H}^{2-\alpha})}\|u_2\|_{\tilde{L}_t^{\frac{4\alpha}{2\alpha-1}}(,\dot{H}^{2-\alpha})}
\end{aligned}$$

$$\|u\|_{\tilde{L}_t^{\frac{4\alpha}{2\alpha-1}}(\dot{H}^{2-\alpha})} \lesssim C. \tag{58}$$

Passing to the limite $t \to T^*$ we get

$$\|u\|_{\tilde{L}_{T^*}^{\frac{4\alpha}{2\alpha-1}}(\dot{H}^{2-\alpha})} \lesssim C.$$

As desired.
Now, the case $T^* = \infty$. Under the hypothesis that $\lim_{t\to\infty}\|u(t)\| = 0$ there existe T such that

$$\|u(T)\|_{\dot{H}^{2-\alpha}} \le C_0,$$

for some small enough constant C_0. Acocoding to Theorem 2.4, we have that

$$u \in \tilde{L}^{\frac{4\alpha}{2\alpha-1}}([T,\infty[,\dot{H}^{2-\alpha}).$$

On the other hand, the previous case imply that

$$u \in \tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,T],\dot{H}^{2-\alpha}).$$

we then conclude that

$$u \in \tilde{L}^{\frac{4\alpha}{2\alpha-1}}([0,\infty],\dot{H}^{2-\alpha}).$$

This completes the proof of the theorem.

3.1. Profile decomposition for the fractional Navier-Stokes. The following theorem is a profile decomposition solutions of (53) with bounded family of initial data. We start by recalling the following The following theorem is the profile decomposition in homogeneous Sobolev spaces P. Gérard (stated in 3D space dimensions).
We start by recalling the following theorem which is the profile decomposition in homogeneous Sobolev spaces P. Gérard (stated in 3D space dimensions) then we present theorem 3.8 as a profile decomposition solutions of (53) with bounded family of initial data.

THEOREM 3.7 (**[13]**). *Let $s \in]0,\frac{3}{2}[$ and $(\varphi_n)_{n\geq 0}$ be a bounded sequence in $\dot{H}^s(\mathbb{R}^3)$. Then there exist a subsequence (still denoted by (ϕ_n)), which satisfies the following properties: There exist*
(i) *a sequence $(\Phi^j)_{j\geq 1}$ of $\dot{H}^s(\mathbb{R}^3)$,*
(ii) *a family of sequences $\{h_n^j, x_n^j\} \subset \mathbb{R}_+^* \times \mathbb{R}^3$ with*

$$\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} + \frac{|x_n^j - x_n^k|}{h_n^j} \to \infty, \qquad \forall\, j \neq k,$$

such that for every $\ell \geq 1$ and every $x \in \mathbb{R}^3$, one has

$$\varphi_n(x) = \sum_{j=1}^{\ell} \frac{1}{(h_n^j)^{\frac{3}{2}-s}} \Phi^j\Big(\frac{x - x_n^j}{h_n^j}\Big) + w_n^\ell(x),$$

with

$$\limsup_{n\to\infty} \|w_n^\ell\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \longrightarrow 0, \text{ as } \quad \ell \to \infty.$$

Furthermore, for all $\ell \in \mathbb{N}^*$ *:*

$$\|\varphi_n\|^2_{\dot{H}^{s_\alpha}} = \sum_{j=1}^{\ell} \|\Phi^j\|^2_{\dot{H}^{s_\alpha}} + \|w_n^\ell\|^2_{\dot{H}^{s_\alpha}} + o(1), \text{ as } \quad n \to \infty.$$

THEOREM 3.8. *Assume that* $\alpha \in [\frac{5}{6}, 1[$. *Let* (φ_n) *be a family of divergence free vector fields, bounded in* $\dot{H}^{s_\alpha}$ *and* u_n *the the solution of the system* (22) *associated with the initial data* φ_n. *Let* $\{\Phi^j, \Gamma^j\}$ *be the family of linear profiles associated to* (φ_n) *via Theorem 3.7 and denote* $U^j := \mathrm{NS}_\alpha(\Phi^j)$.

For every $a_n > 0$ *the following statements are equivalent:*

(i) *For every* $j \geq 1$, *we have*

$$\limsup_{n\to\infty} \|U^j\|_{E^\alpha_{\tilde{a}_n^j}} < \infty, \tag{59}$$

where

$$\tilde{a}_n^j \stackrel{\text{def}}{=} (h_n^j)^{2\alpha} a_n.$$

(ii)

$$\limsup_{n\to\infty} \|u_n\|_{E^\alpha_{a_n}} < \infty. \tag{60}$$

Moreover, if (i) *or* (ii) *holds , then*

$$u_n = \sum_{j=1}^{\ell} \Gamma_j^n U^j + w_n^\ell + r_n^\ell, \tag{61}$$

and

$$\lim_{\ell\to\infty} \big(\lim_{n\to\infty} \sup \|r_n^\ell\|_{E^\alpha_{a_n}} \big) = 0. \tag{62}$$

REMARK 3.9. There exists a family $(T^j)_{j\in\mathbb{N}}$ of elements of $\mathbb{R}^+ \cup \{\infty\}$ and a finite subset $J \subset \mathbb{N}$, such that

$$\forall j \in \mathbb{N}, \quad U^j \in E^\alpha_{T^j} \quad \text{and} \quad \forall j \in \mathbb{N} \setminus J, T^j = +\infty, \tag{63}$$

The condition (59) in this case is equivalent to

$$\limsup\, (h_n^j)^{2\alpha} a_n \geq T^j, \quad \forall j \in J.$$

REMARK 3.10. The implication (i)$\Longrightarrow$(ii) shows that the length of the interval of existence of u^n is bounded from below by the smallest of the length of the interval of existence of each profile. This is a direct effect of the pairwise orthogonality of the family $\{h_n^j, x_n^j\}$; the sum of the linear profiles is decoupled when n goes to infinity and there is no interaction of the profiles inducing a smaller interval of existence than that associated to every profile. This works on the contrast sense : the implication (ii)$\Longrightarrow$(i) proves that there is no interaction between the profiles which generates a solution for a larger interval of existence than one of the profile separately.

PROOF. The proof is divided in two parts.

Part 1. In this part we shall prove the statement (i)$\Longrightarrow$(ii) of Theorem 3.8.
Let $a_n > 0$ and assume

$$\limsup_{n\to\infty} \|U^j\|_{E^\alpha_{\tilde{a}_n^j}} < \infty,$$

where

$$\tilde{a}_n^j \stackrel{\text{def}}{=} (h_n^j)^{2\alpha} a_n.$$

and $U^j := \mathrm{NS}_\alpha(\Phi^j)$ where $\{\Phi^j, h_n^j, x_n^j\}$ is the family of linear profiles given by Theorem 2.13. Our purpose is to prove that

$$\limsup_{n\to\infty} \|u_n\|_{E^\alpha_{a_n}} < \infty.$$

We set

$$r_n^l = u_n - \sum_{j=1}^{\ell} U_n^j - w_n^\ell. \tag{64}$$

$$\begin{cases} \partial_t r_n^\ell + (-\Delta)^\alpha r_n^\ell + \mathbb{P}(r_n^\ell \cdot \nabla r_n^\ell) + Q(r_n^\ell, f_n^\ell) = g_n^\ell, & x \in \mathbb{R}^3, t > 0, \\ r^\ell_{n|t=0} = 0, \end{cases}$$

where

$$f_n^\ell := \sum_{j=1}^{\ell} U_n^j + w_n^\ell,$$

and

$$g_n^\ell := \sum_{\substack{1\le k,j\le \ell\\ j\neq k}} Q(U_n^j, U_n^k) - \sum_{j\le l} Q(U_n^j, w_n^\ell) - P(w_n^\ell\cdot\nabla w_n^\ell).$$

We shall use (59) to prove that

$$\limsup_{n\to\infty} \|r_n^\ell\|_{E^\alpha_{a_n}} \underset{\ell\to\infty}{\longrightarrow} 0. \tag{65}$$

Once proved, (62) yields

$$\limsup_{n\to\infty} \|u_n\|_{E^\alpha_{a_n}} \le \sum_{j=1}^{\ell_0} \limsup_{n\to\infty} \|U_n^j\|_{E^\alpha_{a_n}} + 1$$

for some ℓ_0 fixed. By assumption, the right-hand side term is bounded and (60) is then satisfied. Our purpose, then, is to prove (65).

Using Lemma 2.3 we get for every interval $J=[a,b]\subset[0,a_n]$

$$\begin{aligned}\|r_n^\ell\|_{E(J)} \le C\big(&\|r_n^\ell(a)\|_{\dot H^{s_\alpha}}+ \\ &+\|r_n^\ell\|^2_{E(J)}+\|r_n^\ell\|_{E(J)}\|f_n^\ell\|_{L^2(J,\dot H^\beta)}+\|g_n^\ell\|_{L^2(J,\dot H^\beta)}\big),\end{aligned} \tag{66}$$

where C denotes a universal constant (depends only on α). Here and in the sequel, we have used the notation

$$\beta = \frac{5}{2} - 3\alpha.$$

We prepare several propositions. In the sequel one denotes

$$\mathcal{U}_n^\ell = \sum_{j=1}^{\ell} U_n^j.$$

LEMMA 3.11. *The families $\{\mathcal{U}_n^\ell\}$ and $\{w_n^\ell\}$ are uniformly bounded in the energy space $E^\alpha_{a_n}$.*

PROOF OF LEMMA 3.11 . We know from Proposition 2.14 that the family $\{w_n^\ell\}$ is uniformly bounded in $E^\alpha_{a_n}$. It remains to prove that, $\sum_{j=1}^\ell U_n^j$ is uniformly bounded in $E^\alpha_{a_n}$. By Proposition 2.22 we can write that

$$\sum_{j=1}^\infty \|\Phi^j\|^2_{\dot H^{s_\alpha}} \le \limsup_{n\to\infty} \|\phi_n\|^2_{\dot H^{s_\alpha}} \le C.$$

Thus, the small data theory implies that there exist $j_0 \geq 1$ such that

$$\|U^j\|_{E^\alpha_\infty} \leq 2\|\Phi^j\|^2_{\dot{H}^{s_\alpha}}, \qquad \forall\, j \geq j_0.$$

On the other hand, the pairwise orthogonality implies that

$$\begin{aligned}
\limsup_{n\to\infty} \|\sum_{j=1}^{\ell} U^j_n\|^2_{E^\alpha_{a_n}} &= \sum_{j=1}^{\ell} \|U^j\|^2_{E^\alpha_{\tilde{a}_n}} \\
&\leq \sum_{j=1}^{j_0} \|U^j\|^2_{E^\alpha_{\tilde{a}_n}} + 2\sum_{j=j_0}^{\ell} \|\Phi^j\|^2_{\dot{H}^{s_\alpha}} \\
&\leq \sum_{j=1}^{j_0} \|U^j\|^2_{E^\alpha_{\tilde{a}_n}} + 2C.
\end{aligned}$$

□

The core of the proof of Theorem 2.18 is the following proposition.

PROPOSITION 3.12. *Under the assumptions of Theorem* (2.18) *one has*

$$\limsup_{n\to\infty} \|g^\ell_n\|_{L^2([0,a_n],\dot{H}^\beta(\mathbb{R}^3))} \longrightarrow 0, \qquad \textit{as} \quad \ell \longrightarrow \infty. \tag{67}$$

where $\beta = \frac{5}{2} - 3\alpha$.

Let us postpone the proof of this proposition and finish the proof of Theorem 2.18. For a fixed n and ℓ, one divides the interval $[0,T]$ into $p = p(n,\ell)$ small intervals

$$[0,a_n] = \underbrace{[0,a^1]}_{I^1} \cup \underbrace{[a^1,a^2]}_{I^2} \cup \ldots \cup \underbrace{[a^{p-1},+a_n]}_{I^p},$$

so that

$$C\|f^\ell_n\|_{L^2(I^i_n,\dot{H}^\beta)} \leq \frac{1}{2}.$$

We set

$$X_k = \|r^\ell_n\|_{E(I_k)}, \qquad k = 1,\ldots,p.$$

The estimate (66) and Lemma 3.11 yield

$$X_{k+1} \leq C\big(X_k + X^2_{k+1} + \|g^\ell_n\|_{L^2([0,a_n],\dot{H}^\beta)}\big).$$

Now, the function $H : t \mapsto \|r_n^\ell\|_{E([0,t])}$ is obviously continuous on I^1 and satisfies

$$H(t) \le CH^2(t) + C\|g_n^\ell\|_{L^2([0,t],\dot{H}^\beta)}, \qquad H(0) = 0.$$

For n and ℓ large enough we get

$$H(t) \le 2C\|g_n^\ell\|_{L^2([0,t],\dot{H}^\beta)}, \qquad \forall\, t \in I^1.$$

Thus, we infer

$$X_1 \le 2C\|g_n^\ell\|_{L^2([0,t],\dot{H}^\beta)}.$$

On $[a^1, a^2]$ the function $H : t \mapsto \|r_n^\ell\|_{E([a^1,t])}$ is continuous and satisfies

$$\begin{aligned} H(t) &\le CX_1 + CH^2(t) + C\|g_n^\ell\|_{L^2([0,t],\dot{H}^\beta)} \\ &\le 2C^2\|g_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)} + CH^2(t) + C\|g_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)}. \end{aligned}$$

Thus, for n and ℓ large enough we get

$$X_2 \le 4C^2\|g_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)}.$$

Repeating this argument we obtain

$$X_k \le C^{k+1}\|g_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)}, \qquad k = 1, ..., p.$$

Summing up all these terms one gets, for n and ℓ large enough,

$$\|r_n^\ell\|_{E^\alpha_{a_n}} \le \sum_{k=0}^{p} X_k \le \frac{C^{p+2}}{C-1}\|g_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)}.$$

However, one has

$$p \simeq \|f_n^\ell\|^2_{L^2([0,a_n],\dot{H}^\beta)}.$$

Thus,

$$\|r_n^\ell\|_{E^\alpha_{a_n}} \le C\exp(C\|f_n^\ell\|^2_{L^2([0,a_n],\dot{H}^\beta)})\|g_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)}.$$

Finally, combining Lemma 3.11 and (67) yield together the claimed result. □

PROOF OF PROPOSITION 3.12. We set

$$\mathcal{U}_n^\ell = \sum_{j=1}^{\ell} U_n^j.$$

One has trivially

$$g_n^\ell = \sum_{j \neq k} Q(U_n^j, U_n^k) + Q(\mathcal{U}_n^\ell, w_n^\ell) + \mathbb{P}(w_n^\ell . \nabla w_n^\ell).$$

We have divided the proof of (67) into a sequence of lemmas.

LEMMA 3.13. *Let $p \in [p_\alpha, \infty[$ with $p_\alpha = \frac{3}{2\alpha-1}$ and take $q = \frac{2\alpha}{2\alpha-1-\frac{3}{p}}$. Then, we have*

$$\limsup_n \|w_n^\ell\|_{L^q(\mathbb{R}^+, L^p)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty.$$

PROOF OF LEMMA 3.13. Since $\{w_n^\ell\}$ is bounded in the energy space then by interpolation and Sobolev embeddings, it's uniformly bounded in

$$L^q(\mathbb{R}^+, L^p(\mathbb{R}^3))$$

for

$$q > \frac{2\alpha}{2\alpha - 1}, \quad \frac{1}{q} = -\frac{1}{3} + \frac{2}{3}(1 - \frac{1}{q})\alpha.$$

By interpolation between this fact and the fact

$$\limsup_n \|w_n^\ell\|_{L^\infty(\mathbb{R}^+, L^{p_\alpha})} \longrightarrow 0, \text{ as } \quad \ell \to \infty,$$

we get the result.

□

LEMMA 3.14. *Under the definitions above, one has*

$$\limsup_n \|w_n^\ell \nabla G_n^\ell\|_{L^2([0,a_n], \dot{H}^\beta)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty,$$

for both case $G_n^\ell = w_n^\ell$ or $G_n^\ell = \mathcal{U}_n^\ell$.

PROOF OF LEMMA 3.14. By interpolation,

$$\|w_n^\ell \nabla G_n^\ell\|_{L^2([0,a_n], \dot{H}^\beta)} \leq \|w_n^\ell \nabla G_n^\ell\|^\theta_{L^{\frac{2\alpha}{2\alpha-1}}([0,a_n], \dot{H}^{\beta+\alpha-1})} \|w_n^\ell \nabla G_n^\ell\|^{1-\theta}_{L^{\frac{4\alpha}{8\alpha-5}}([0,a_n], L^2)},$$

with

$$\theta = \frac{5 - 6\alpha}{3 - 4\alpha}.$$

By a similar argument as above,

$$\begin{aligned}\|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{2\alpha}{2\alpha-1}}([0,a_n],\dot{H}^{\beta+\alpha-1})} &\leq \|w_n^\ell\|_{L^{\frac{4\alpha}{2\alpha-1}}([0,a_n],\dot{H}^{2-\alpha})}\|G_n^\ell\|_{L^{\frac{4\alpha}{2\alpha-1}}([0,a_n],\dot{H}^{2-\alpha})}\\ &\leq C.\end{aligned}$$

It remains to prove that

$$\limsup_n \|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-5}}([0,a_n],L^2)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty.$$

To do so, we use a Hôlder inequality to get

$$\|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-5}}([0,a_n],L^2)} \leq \|w_n^\ell\|_{L^{\frac{4\alpha}{6\alpha-5}}([0,a_n],L^{\frac{6}{3-2\alpha}})}\|\nabla G_n^\ell\|_{L^2([0,a_n],L^{\frac{3}{\alpha}})}.$$

By Sobolev inequlaity we infer

$$\begin{aligned}\|\nabla G_n^\ell\|_{L^2([0,a_n],L^{\frac{3}{\alpha}})} &\leq C\|\nabla G_n^\ell\|_{L^2([0,a_n],\dot{H}^{\frac{3}{2}-\alpha})}\\ &\leq C\|G_n^\ell\|_{L^2([0,a_n],\dot{H}^{\frac{5}{2}-\alpha})}\\ &\leq C\|G_n^\ell\|_{E_\alpha^\infty}\\ &\leq C.\end{aligned}$$

The last inequality comes from (3.11).
Combing the two last inequalities together we get

$$\|w_n^\ell \nabla G_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-5}}([0,a_n],L^2)} \leq C\|w_n^\ell\|_{L^{\frac{4\alpha}{6\alpha-5}}([0,a_n],L^{\frac{6}{3-2\alpha}})}.$$

The needed result is a direct consequence of Lemma 3.13.

□

The next lemma deals with third term in f_n^ℓ.

LEMMA 3.15. *Under the assumptions of Theorem 2.18 we have*

$$\limsup_n \|\mathcal{U}_n^\ell.\nabla w_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty,$$

where $\beta = \dfrac{5}{2} - 3\alpha$.

PROOF OF LEMMA 3.15. Recall that

$$\sum_{j=1}^{\infty} \|U^j\|^2_{\dot{H}^{s_\alpha}} < \infty.$$

Let $\epsilon > 0$ small, there exists ℓ_0 such that

$$\sum_{j=\ell_0}^{\infty} \|U^j\|^2_{\dot{H}^{s_\alpha}} < \epsilon.$$

By orthogonality,

$$\begin{aligned}\limsup_n \|\sum_{j\geq\ell_0} U_n^j\|^2_{L^\infty([0,a_n],\dot{H}^{s_\alpha})} &= \sum_{j\geq\ell_0} \|U^j\|^2_{L^\infty([0,a_n],\dot{H}^{s_\alpha})} \\ &\leq 2\epsilon.\end{aligned}$$

Using the product laws in Sobolev spaces,

$$\|\sum_{j\geq\ell_0} U_n^j.\nabla w_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)} \leq \|\sum_{j\geq\ell_0} U_n^j\|_{L^\infty([0,a_n],\dot{H}^{s_\alpha})}\|w_n^\ell\|_{L^2([0,a_n],\dot{H}^{s_\alpha+\alpha})}.$$

Also, Since $\|w_n^\ell\|_{L^2([0,a_n],\dot{H}^{s_\alpha+\alpha})}$ is bounded then

$$\|\sum_{j\geq\ell_0} U_n^j.\nabla w_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)} \leq C\sqrt{\epsilon}. \tag{68}$$

Next, we need to show that

$$\limsup_n \|U_n^j.\nabla w_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)} \longrightarrow 0, \qquad as \quad \ell \longrightarrow \infty, \quad \forall\ 1 \leq j \leq \ell_0.$$

Let us prove that for $j = 1$ (the other terms are similar).
Applying the change of variables technique,

$$\|U_n^1.\nabla w_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)} = \|U^1.\nabla \tilde{w}_n^\ell\|_{L^2([0,\tilde{a}_n],\dot{H}^\beta)}$$

with

$$\tilde{w}_n^\ell(t,x) = (h_n^1)^\alpha w_n^\ell\big((h_n^1)^2 t, h_n^1 x + x_n^1\big).$$

By assumptions, and up extarcting if it is necessary,

$$\tilde{a}_n \leq \rho < T^1,$$

for all $n \in \mathbb{N}$. Thus,

$$\|U_n^1.\nabla w_n^\ell\|_{L^2([0,a_n],\dot{H}^\beta)} = \|U^1.\nabla \tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^\beta)}$$

Notice that for any $\epsilon > 0$ there is $V \in C_c^\infty([0,\rho[,\mathcal{S}(\mathbb{R}^3))$ such that

$$\|U^1 - U\|_{L^2([0,\rho],\dot{H}^\beta)} < \epsilon.$$

It follows that

$$\|U^1.\nabla \tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^\beta)} \leq \|(U^1 - U).\nabla \tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^\beta)} + \|U.\nabla \tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^\beta)}.$$

Now, the first term in the last inequality can be controlled as above

$$\|(U^1 - \tilde{U}).\nabla \tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^\beta)} \leq \|(U^1 - \tilde{U})\|_{L^\infty([0,\rho],\dot{H}^{s_\alpha})}\|\tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^{s_\alpha+\alpha})}.$$

But

$$\|\tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^{s_\alpha+\alpha})} \leq \|w_n^\ell\|_{L^2([0,\infty[,\dot{H}^{s_\alpha+\alpha})}.$$

Consequently,

$$\|(U^1 - \tilde{U}).\nabla \tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^\beta)} \leq \epsilon C.$$

Next, for the second one, by interpolation one has

$$\|U\nabla \tilde{w}_n^\ell\|_{L^2([0,\rho],\dot{H}^\beta)} \leq \|U\tilde{w}_n^\ell\|^{\frac{\alpha-\frac{1}{2}}{3-2\alpha}}_{L^{\frac{4\alpha}{8\alpha-7}}([0,\rho],L^2)}\|U\nabla \tilde{w}_n^\ell\|^{\frac{\frac{7}{2}-3\alpha}{3-2\alpha}}_{L^{\frac{4\alpha}{4\alpha-1}}([0,\rho],\dot{H}^{\beta+\alpha-\frac{1}{2}})}.$$

As in the proof above one can control

$$\|U\nabla \tilde{w}_n^\ell\|_{L^{\frac{4\alpha}{4\alpha-1}}([0,\rho],\dot{H}^{\beta+\alpha-\frac{1}{2}})} \leq C,$$

uniformly in n and ℓ.
Finally,

$$\begin{aligned}\|U.\tilde{w}_n^\ell\|_{L^{\frac{4\alpha}{8\alpha-7}}([0,\rho],L^2)} &\leq C\|U\|_{L^{\frac{4\alpha}{8\alpha-7}}([0,\rho],L^{\frac{6}{5-4\alpha}})}\|\tilde{w}_n^\ell\|_{L^\infty([0,\rho],L^{p_\alpha})}\\ &\leq C(V)\|w_n^\ell\|_{L^\infty(\mathbb{R}^+,L^{p_\alpha})}.\end{aligned}$$

This ends the proof of Lemma 3.15 and consequently it proves the Proposition 3.12 □

Part 2. In this part we prove the statement (ii)$\Longrightarrow$(i) of Theorem 3.8.
Let $\{a_n\}$ be a family of open intervals containing 0 such that (60) holds. We shall proceed by contradiction to prove that (59) holds too.
First, in view of Remark 3.9 and up to a permutation, we may assume that there exists ℓ_0 such that the family $\{U^j\}_{j>\ell_0}$ is global and the family

$\{U^j\}_{1\leq j\leq \ell_0}$ is blow-up (if all the profiles are global then then the problem is trivial). Thus, we have only to treat a finite family of profiles $\{U^j\}_{1\leq j\leq l_0}$.

Let T^j denote the maximal time of existence of U^j. The failure of (59) means that, for every $M > 0$, there exists some intervals $b^n \leq a_n$ so that

$$M \leq \sup_{1\leq j\leq \ell_0} (\limsup_{n\to\infty} \|U^j\|_{E^\alpha_{\tilde{b}^j_n}}) < \infty,$$

where

$$\tilde{b}^j_n \stackrel{\text{def}}{=} (h^j_n)^{2\alpha} b_n.$$

Since the family $\{U^j\}_{j>l_0}$ is global, the sequence (b_n) satisfies of the statement (i) of Theorem 3.8. Thus, the first part of the proof yields

$$\limsup_{n\to\infty} \|u_n\|_{E^\alpha_{b_n}} = \limsup_{\ell\to\infty} \left(\limsup_{n\to\infty} \| \sum_{j=1}^{\ell} \Gamma^j_n U^j \|_{E^\alpha_{b_n}} \right) < \infty. \tag{69}$$

The pairwise orthogonality of the family $\{\Gamma^j\}$ implies

$$\limsup_{\ell\to\infty} \left(\limsup_{n\to\infty} \| \sum_{j=1}^{\ell} \Gamma^j_n U^j \|^2_{E^\alpha_{\tilde{b}^j_n}} \right) = \sum_{j=1}^{\infty} \limsup_{n\to\infty} \|U^j\|^2_{E^\alpha_{\tilde{b}^j_n}}. \tag{70}$$

This yields

$$\limsup_{n\to\infty} \|u_n\|^2_{E^\alpha_{b_n}} \leq \limsup_{n\to\infty} \|u_n\|^2_{E^\alpha_{a_n}} \leq C_1. \tag{71}$$

where C_1 is fixed as the bound (60). Combining (69), (70) and (71), we get

$$\sum_{j=1}^{\infty} \limsup_{n\to\infty} \|U^j\|^2_{E^\alpha_{\tilde{b}^j_n}} \leq C_1^4,$$

which implies, in particular, that

$$M \leq \sup_{1\leq j\leq \ell_0} (\limsup_{n\to\infty} \|U^j\|_{E^\alpha_{\tilde{a}^j_n}}) \leq C_1$$

This fact is impossible when M is larger than C_1.
Thus, (59) holds. This concludes the proof of Theorem 3.8.

□

4. Proof of Theorem 3.5

On define

$$A_c = \sup\{A > 0; \|u\|_{L^\infty([0,T^*(u)],\dot{H}^{s_\alpha}(\mathbb{R}^3))} \le A \Longrightarrow T^*(u) = +\infty\}.$$

The existence of the critical element is guaranteed by the following theorem.

PROPOSITION 3.16. *Suppose $A_c < +\infty$. Then there exists some $u_{0,c} \in \dot{H}^{s_\alpha}(\mathbb{R}^3)$, $u_c = \mathrm{NS}_\alpha(u_{0,c})$ the unique solution of NS associated with the initial data $u_{0,c}$ such that*

$$\|u_c\|_{L^\infty([0,T^*(u)],\dot{H}^{s_\alpha}(\mathbb{R}^3))} = A_c, \quad T^*(u_c) < +\infty.$$

PROOF OF PROPOSITION 3.16. Assume that $A_c < +\infty$ then there exists a decreasing sequeses A_n and family of divergence free functions $\varphi_n \in \dot{H}^{s_\alpha}(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} A_n = A_c$$

and $u_n = \mathrm{NS}_\alpha(\varphi_n)$ blows up at finite time $T^*(u_n)$

$$\|u_n\|_{L^\infty([0,T^*(u_n)],\dot{H}^{s_\alpha}(\mathbb{R}^3))} \leqslant A_n, \infty,$$

for every $n \in \mathbb{N}$.
We may assume that

$$\|\varphi_n\|_{\dot{H}^{s_\alpha}(\mathbb{R}^3)} \le A_n \le 2A_c.$$

Also, by scaling one can assume that all u_n are defined on $[0,1]$and

$$\lim_{n\to\infty} \|\varphi_n\|_{E_1^\alpha} = +\infty. \tag{72}$$

Now, we apply the profile decomposition (Theorem 2.13) to the sequences $\{\varphi_n\}$

$$\varphi_n(x) = \sum_{j=1}^{\ell} (\frac{1}{h_n^j})^{2\alpha-1}\Phi^j(\frac{x-x_n^j}{h_n^j}) + \psi_n^\ell(x).$$

Next, apply the profile decomposition (Theorem 3.8) to the sequences $\mathtt{u}_n$ to get for every $n \in \mathbb{N}$, every $t \le \tau_n$, for every $\ell \in \mathbb{N}^*$ and every $x \in \mathbb{R}^3$

$$u_n(t,x) = \sum_{j=1}^{\ell} (\frac{1}{h_n^j})^{2\alpha-1}U^j(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x-x_n^j}{h_n^j}) + w_n^\ell(t,x) + r_n^\ell(t,x),$$

where $w_n^\ell = H_\alpha(\psi_n^\ell)$ and $U^j = \mathrm{NS}_\alpha(\Phi^j)$.
Also, from the proof of Theorem 3.8 (see Remark 3.9) we know that there exists a finite (or empty) subset $J \subset \mathbb{N}$ such that

$$T^*(U^j) = +\infty \qquad j \in \mathbb{N} \setminus J.$$

According to Theorem 3.2 we have

$$\|U^j\|_{E^\alpha_\infty} < +\infty \qquad j \in \mathbb{N} \setminus J.$$

Applying the implication $(ii) \Longrightarrow (i)$ of Theorem 3.8 to the sequences $\{u_n\}$ on $[0,1]$ (remember (72)) we get that there exists at least one integer j such that $T^*(U^j) < +\infty$.
Therefore, J is not empty and hence we may re-order the profiles in the decomposition such that for some j_1, define $T_j^* = T^*(U^j)$

$$\|U^j\|_{E^\alpha_{T_j^*}} = +\infty \qquad \text{for} \qquad 1 \leq j \leq j_1$$

and

$$\|U^j\|_{E^\alpha_{T_j^*}} < +\infty \qquad \text{for} \qquad j > j_1.$$

By definition of A_c one has, for every $1 \leq j \leq j_1$,

$$\|U^j\|_{L^\infty([0,T^*(Uj)],\dot{H}^{s_\alpha}(\mathbb{R}^3))} = A_j \geq A_c. \tag{73}$$

We need the following orthogonality result.

LEMMA 3.17 ([**22**]). *Fix some $k \in \mathbb{N}^*$ and choose $t_n \leq min_{1\leq j\leq j_1}(h_n^j)^{2\alpha}(T^*(U^j) - \frac{1}{k})$ then up to subsequences (depending on k)*

$$\|u_n(t_n)\|^2_{\dot{H}^{s_\alpha}} = \sum_1^{j_1} \|U_n^j(t_n)\|^2_{\dot{H}^{s_\alpha}} + \|\omega_n^{j_1}(t_n)\|^2_{\dot{H}^{s_\alpha}} + o(1), \quad n \to \infty.$$

We choose a suitable t_n that guarantees (as n goes to infinity and after rearrging the term if necessary)

$$\|u_n(t_n)\|_{\dot{H}^{s_\alpha}} \longrightarrow A_c, \qquad \|U_n^1(t_n)\|_{\dot{H}^{s_\alpha}} \longrightarrow A_1,$$

and, for $2 \leq j \leq j_1$ (if this set in not empty)

$$\|U_n^j(t_n)\|_{\dot{H}^{s_\alpha}} \longrightarrow \tilde{A}_j \in]0, A_j].$$

In particular, Lemma 3.17 yields

$$A_c^2 \geq A_1^2 + \sum_2^{j_1} \tilde{A}_j^2.$$

Thanks to (73) we infer

$$j_1 = 1 \qquad \text{and} \quad A_1 = A_c.$$

Thus, U^1 is the candidate for the critical element. This concludes the proof of the proposition. □

Let us come back to the proof of Theorem 3.5.
Take a minimal a critical element u given by Proposition 3.16 (blowups at finite time T^* in particular) and take $t_n \uparrow T^*$. One denotes

$$\phi_n(x) = u(t_n, x).$$

By assumptions, we have

$$\|\phi_n\|_{\dot{H}^{s_\alpha})} \leq A_c, \qquad \forall n \in \mathbb{N}.$$

The family of associated solutions u_n are of course

$$u_n(t,x) = u(t + t_n, x),$$

with lifespan $T^*(u_n) = T^* - t_n$.
Applying Theorem 3.8 to the family $\{\phi_n\}$ on $[0, T^* - t_n[$ we get

$$(74) \qquad u(t + t_n, x) = \sum_{j=1}^{\ell} (\frac{1}{h_n^j})^{\frac{3}{p\alpha}} U^j(\frac{t}{(h_n^j)^{2\alpha}}, \frac{x - x_n^j}{h_n^j}) + w_n^\ell(t,x) + r_n^\ell(t,x).$$

The implication $(ii) \implies (i)$ of Theorem 3.8 yields j_1 (let us take it equal to 1) such that

$$\lim \|U^1\|_{E^\alpha_{(h_n^1)^{2\alpha}(T^* - t_n)}} = \infty.$$

This means that

$$\lim\inf (h_n^1)^{2\alpha}(T^* - t_n) \geq T^*(U^1).$$

Furthermore, a similar argument based on the orthogonality used in the proof of Proposition 3.16 above yields that there exists only one singular profile U^1 and that (after changing notations):

$$u(t_n, x) = (\frac{1}{h_n})^{\frac{3}{p_\alpha}} V(\frac{x - x_n}{h_n}) + \varepsilon_n(x).$$

with

$$h_n \geq \frac{C}{(T^* - t_n)^{\frac{1}{2\alpha}}} \quad \text{and} \qquad \lim_{n \to \infty} \|\varepsilon_n\|_{L^{p_\alpha}} = 0.$$

This implies that u and t_n satisfies the assumptions (A) and so $u \equiv 0$ which is impossible.

■

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