U S PC
Université Sorbonne
Paris Cité

université PARIS DIDEROT PARIS 7

Thèse de Doctorat

de l'Université Sorbonne Paris Cité

Préparée à l'Université Paris Diderot

École Doctorale de Sciences Mathématiques de Paris Centre - ED 386

---

# ABOUT $E_\infty$-STRUCTURES IN $\mathcal{L}$-ALGEBRAS

# SUR LES $E_\infty$-STRUCTURES DANS LES $\mathcal{L}$-ALGÈBRES

---

PAR JESÚS SÁNCHEZ

**Thèse de Doctorat de Mathématiques**

Dirigée par Alain PROUTÉ

Présentée et soutenue publiquement à Paris le 6 décembre 2016

Présidente du jury: Mme Muriel Livernet, Université Paris-Diderot
Rapporteur: Mr Clemens Berger, Université de Nice
Rapporteur: Mr Martin Markl, Czech Academy
Examinateur: Mr Dimitri Ara, Université de Marseille
Examinateur: Mr Benoit Fresse, Université Lille 1
Examinateur: Mr Loïc Merel, Université Paris-Diderot
Directeur de thèse: Mr Alain Prouté, Université Paris-Diderot

Institut de Mathématiques de Jussieu-
Paris Rive Gauche
Université Paris Diderot-Paris 7
Campus des Grands Moulins
Bâtiment Sophie Germain, case 7012
75205 Paris Cedex 13

# Acknowledgement

I WOULD LIKE to thank Mr Berger and Mr Markl for reviewing this manuscript and give a favorable approval to this thesis. I thank Mme Livernet, Mr Ara, Mr Fresse and Mr Merel for kindly accepting to be part of the jury.

To my advisor Alain Prouté for all the dedication, patience and pertinent remarks to my work. He teach me a lot besides the love for mathematics. Humble and working man, his humanity is beyond conventional standards.

My sincere gratitude to the *Universidad de Costa Rica* for supporting this research and my academic career. To the *Escuela de Matématicas* of Costa Rica for giving me the opportunity to complete my doctoral studies. I thank professor William Ugalde and professor Mario Villalobos, for their unconditional support. I would like to thank the professors Eugenio Chinchilla, Jorge Guier, Ronald Bustamante, Pedro Mendez and Orietta Proti, for their friendship, making possible the beginning and achievement of my studies.

I'm thankful to all the people working in *OAICE* for their dedication and advices in all the aspects of my time in France. I want to thank in particular Vivian Madrigal, who along these years worked with my case.

I would like to thank Alexander and Ermelinde for their gentle welcome and help during my first year in Paris.

To my close friend Gloria Zemborain, which support and encouragement helped me in the most difficult times.

Through my experience as doctoral student in *Paris 7* all these people did of my research time something very special and unforgettable. I would like to thank all my awesome friends in *IMJ-PRG* for been so *pura vida* with me, so thanks Victoria, Marco, Aurelien, Annalaura, Charlotte, Assia, Baptiste, Nicolas, Kevin, Alexandra, Julie, David, Charles, Élie, Omar, Leonard, Pooneh, Martin, Antoine, Jeremy, Reda, Claire, Sakina, Jérôme, Daniela, Jorge, Juan Pablo, Véronique, Arnaud and René.

To the physicians Camila(and Harvey), Homero, Anaïs(and Thibault), Rebeca, Apsara and Rossana. They teach me how close traditions are between Costa Rica and Venezuela. I so thankful for all the Venezuelan new year dinners.

To the people of *ACRF* for all the beautiful experiences in Paris, Silvia, Carlos, Harold, Renán, Erica, Tania, Maria Laura, Maria José Julio, Angie, Guillaume, Montserrat, Carolina, Jessica and Elvin.

To all my long distance friends, Gabriel, Alejandra, Javier, Carmen, Griscelda, Maikol, Marianela, Priscila E., Priscila S., Carolina and Daylin.

To all the wonderful people I met in Paris, Paolo, Mélodie, Sandra, Ioanna, Arianna, Luca, Manuela, Gloria K., Carlotta, Elena, Thibaut, Emmanuela, Giulio, Ariana, Luca, Carolina, Camille and Sabrina.

Finally, I would like to thank my family, Geovanni, Jesús, Elizabeth, Nayara, Elsie, William, and Steven. To Katherine for the beauty, the rivers, the souls, the dreams, the movies, the walks and talks.

* * *

# Summary of the Thesis

In this thesis we recall the notion of $\mathcal{L}$-algebra. $\mathcal{L}$-algebras are intended as algebraic models for homotopy types. $\mathcal{L}$-algebras were introduced by Alain Prouté in several talks since the eighties. The principal objective of this thesis is the description of an $E_\infty$-coalgebra structure on the main element of an $\mathcal{L}$-algebra. This can be seen as a generalization of the $E_\infty$-coalgebra structure on the chain complex associated to a simplicial set given by Smith in [Smi94]. We construct an $E_\infty$-operad, denoted $\mathcal{K}$, used to construct the $E_\infty$-coalgebra on the main element of a $\mathcal{L}$-algebra. This $E_\infty$-coalgebra structure shows that the canonical $\mathcal{L}$-algebra associated to a simplicial set contains at least as much homotopy information as the $E_\infty$-coalgebras usually associated to simplicial sets.

**Keywords**
Differential graded modules, $\mathcal{L}$-algebras, symmetric operads, $E_\infty$-coalgebras.

# Résumé de la Thèse

Dans cette thèse nous rappelons la notion de $\mathcal{L}$-algèbre, qui a pour objet d'être un modèle algébrique des types d'homotopie. L'objectif principal de cette thèse est la description d'une structure de $E_\infty$-coalgèbre sur l'élément principal d'une $\mathcal{L}$-algèbre. Ceci peut être vu comme une généralisation de la structure de $E_\infty$-coalgèbre sur le complexe des chaînes d'un ensemble simplicial, telle que décrite par Smith dans [Smi94]. Nous construisons une $E_\infty$-opérade, notée $\mathcal{K}$, utilisée pour construire la $E_\infty$-coalgèbre sur l'élément principal d'une $\mathcal{L}$-algèbre. Cette structure de $E_\infty$-coalgèbre montre que la $\mathcal{L}$-algèbre canoniquement associée à un ensemble simplicial contient au moins autant d'information homotopique que la $E_\infty$-coalgèbre couramment associée à un ensemble simplicial.

**Mots-clefs**
Modules différentiels gradués, $\mathcal{L}$-algèbres, opérades symétriques, $E_\infty$-coalgèbres.

*A mi mamá*

# Contents

# Introduction: In English

In this introductory chapter we explain the context where the $\mathcal{L}$-algebras are placed. We start with a review of some of the techniques used to study homotopy type of spaces, beginning with minimal models in rational homotopy theory and continuing with the use of $A_\infty$-algebras in the case of fields with positive characteristic and the description of the notion of operad. In the final part we discuss $\mathcal{L}$-algebras and the results proven in this thesis.

**Sullivan Minimal Models** A concept of minimal model in rational homotopy theory was introduced by Dennis Sullivan at the end of 1960's (see [Jam99], §27). Simply connected spaces can be *rationalized*, which means that we can replace a space $X$ with a rational version of it, $X_{\mathbb{Q}}$, such that $H_*(X;\mathbb{Q}) = H_*(X_{\mathbb{Q}})$. A simply connected space $Y$ is said to be *rational* when its reduced homology (or $\pi_*(Y)$, or the reduced homology of its loop space $\Omega Y$) is a $\mathbb{Q}$-vector space. Given a continuous map $\phi : X \to Z$, we can state the existence of a (up to homotopy) unique induced morphism between the rationalizations of $X$ and $Z$. With this, the rational homotopy type of a simply connected space is defined as the weak homotopy type of its rationalization.

This simplification of a space implies some loss of information, for instance, the homotopy groups of the sphere $S^2$ are non-zero in infinitely many degrees, but the rational homotopy groups vanish in all degrees above 3. Nevertheless, the advantage of the approximation by a rational model, is the facility for computations while ordinary homotopy theory is too complicated. This is due to the discovery of an explicit algebraic formulation for rational homotopy by Quillen and Sullivan ([Sul77], [Qui69]). They established an equivalence of categories between the homotopy category of rational spaces and their categories of minimal models. Sullivan found a functor $A_{PL}$ that associates a commutative cochain algebra $A_{PL}(X)$ to $X$. The algebras $A_{Pl}(X)$ and $C^*(X)$ are linked by a zig-zag of quasi-isomorphisms, so that in particular they have the same cohomology $H^*(X) = H(A_{PL}(X))$.

The transition from topological spaces to commutative cochain algebras established by the functor $A_{PL}$ allows us to focus in the study of commutative cochain algebras. In this category shows up a special kind of commutative cochain algebras called Sullivan algebras. These algebras live in each isomorphism class, and under special conditions on the space $X$, have a minimal representative uniquely determined up to isomorphism, called Sullivan minimal model.

If simply connected topological spaces $X$ and $Y$ have the same rational homotopy type, then the cochain algebras $A_{PL}(X)$ and $A_{PL}(Y)$ are weakly equivalent, and by the unicity of the minimal models, they have the same minimal model. So, if we restrict ourselves to simply connected spaces with rational homology of finite type, there is a bijection between the rational homotopy types and the isomorphism classes of minimal Sullivan algebras on $\mathbb{Q}$.

**Commutative Cochain Problem** The cochain ring $C^*(X;\mathsf{k})$ product, that is, the cup product of cochains, usually is not commutative. In the graded context, commutativity means that $x \cup y = (-1)^{|x||y|} y \cup x$, where $|x|$ and $|y|$ are the degrees of $x$ and $y$. Essentially, the commutative cochain problem is functorially finding a commutative differential graded algebra $A^*(X)$ on the ring $k$, in such a way that there exits a zig-zag of quasi-isomorphisms between $A^*(X)$ and $C^*(X;k)$ (see [GM81], §9). This problem was solved for the rational case by Sullivan.

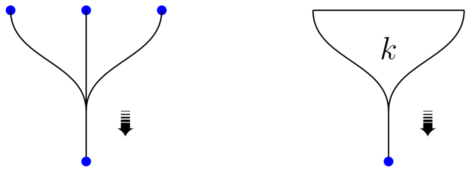
Steenrod proposed a type of cohomology operations linked to the cup product (see [Ste47]). The Steenrod squares $Sq^i$ are defined on the cohomology ring with coefficient in $\mathbb{Z}/2\mathbb{Z}$. They take the class $x$ of a cocycle of degree $n$ in to a class $Sq^i(x)$ of degree $n+i$. When $n = i$, $Sq^i(x)$ is just the cup product $x \cup x$. The construction of the Steenrod squares depends strongly on the non commutativity of the cochain ring $C^*(X;\mathbb{Z}/2\mathbb{Z})$ and a consequence of their existence is that there is no solution for the commutative cochain problem on the $\mathbb{Z}/2\mathbb{Z}$, and consequently on $\mathbb{Z}$ (see for instance [Cen89]). The same problem arises for $\mathbb{Z}/p\mathbb{Z}$, for $p$ an odd prime.

$A_\infty$**-algebras** Introduced by Stasheff ([Smi86], [Sta63]), the $A_\infty$-algebras are graded chain complexes $(A,d)$ together with operations $\mu_n : A^{\otimes n} \to A$, $n \geqslant 2$, of degree n-2, satisfying some conditions. $A_\infty$-algebras can be seen as a generalization of differential graded algebras. In fact, for a DGA-algebra the operations satisfy $\mu_n = 0$ for $n \geqslant 3$ and the category $DGA$ is a full subcategory of the category of $A_\infty$-algebras.

In [Kad80] Kadeishvili describes the construction of the $A_\infty$-algebra structure on the algebra of homology of chain complexes and, after some generalizations in the $A_\infty$-algebras category, he gives a description of a fiber space using $A_\infty$-algebras. In 1986, this approach was used by Prouté in [Pro11] with the idea of making an explicit computation of the homology of a fiber space with fiber $K(\mathbb{Z}/p\mathbb{Z}, n)$. The idea behind his technique, is to express the chain complex of the total space in the fiber bundle, by something having a description by operations in the category of $A_\infty$-algebras, because in some especial cases there are already established methods to compute minimal models in the $A_\infty$-algebras category and, naturally $A_\infty$-structures arise when the fiber bundle has as fiber a space of the type $K(\pi, n)$. Associativity is not the whole story, we also need to relax the commutativity, which led May in [May72] to the notion of operad. We will be specially interested in $E_\infty$-operads.

**Symmetric operads** An operad can be thought as a framework to model algebraic structures. In this part we explore the intuition behind the concept of operad. Operads can be defined in any symmetric monoidal category $\mathcal{C}$, in particular we are interested in the category DGA-$\mathsf{k}$-Mod. Then, all the constructions will be made having in mind this category. An operad $\mathcal{P}$ is composed of a collection $\{P(i)\}_{i \geqslant 0}$ of objects of $\mathcal{C}$, which is subject to several conditions that we will discuss along this introduction. The elements of each object $P(k)$ can be seen as abstract operations with $k$ inputs and one output, referred as elements of arity $k$. In the following picture are represented two element of $\mathcal{P}$, the first is an element of arity 3

($\in P(3)$), and the other is an element of arity $k$ ($\in P(k)$).

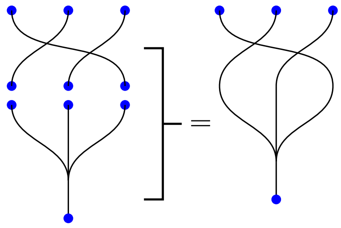


(1)

Each operad $\mathcal{P}$ will have a distinguished element in arity 1, which is supposed to represent the identity application, called the unit of $\mathcal{P}$. It is defined to as a morphism $\eta : 1 \to P(1)$ and is represented by a stick with one input and one output.

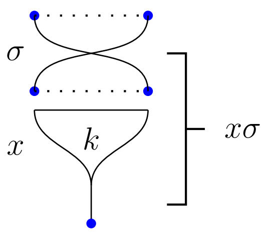
(2)

Each $P(k)$ is equipped with an action by the symmetric group $\Sigma_k$. Graphically, this action is represented by the shuffle of inputs. For instance, consider $\sigma \in \Sigma_3$ given by $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$. The action of $\sigma$ on an element of $P(3)$, is represented as follows.



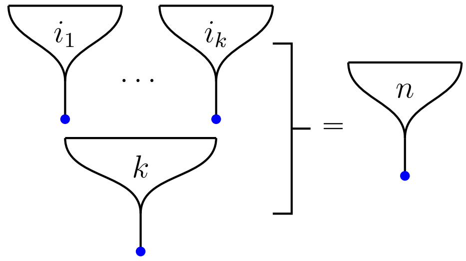
(3)

That is, if the element of $P(3)$ is an operation $f(x_1, x_2, x_3)$ then $f\sigma(x_1, x_2, x_3) = f(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) = f(x_2, x_3, x_1)$. In the last picture the bracket is used to represent the act of applying the action by $\sigma$ on an element of $P(3)$. In general, we represent action of $\sigma \in \Sigma_k$ on and element of $P(k)$ by the following picture.



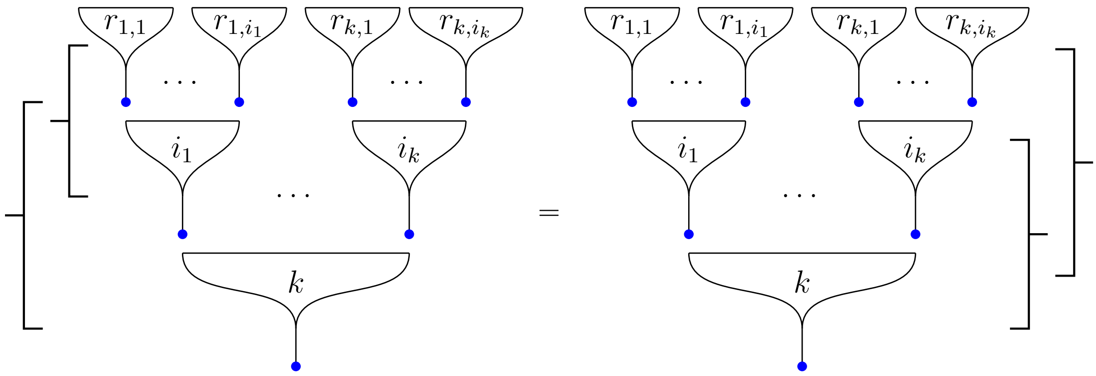

(4)

Another important component of an operad $\mathcal{P}$ are the compositions. Since what we are modeling are operations, we need to code how the composition of operations behaves. Let $f \in P(k)$, then we could compose this operation with $k$ (one for each input) operations of $\mathcal{P}$, resulting in an operation of arity equal to the sum of arities of the operations in each input of $f$. The compositions are given by morphisms of the form $\gamma : P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \to P(n)$, with $n = i_1 + \cdots + i_k$. We also represent the act of applying the composition by a bracket.



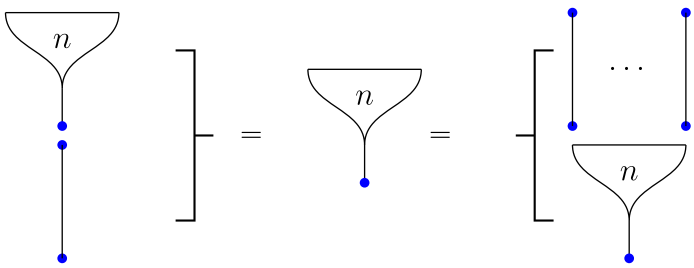
(5)

The data given by the action of symmetric groups, the unit and the compositions have to satisfy some conditions.The first condition is about the associativity of the composition, in the sens that our abstract compositions in $\mathcal{P}$ does not depend of the order in which is made. The following picture represents this situation.



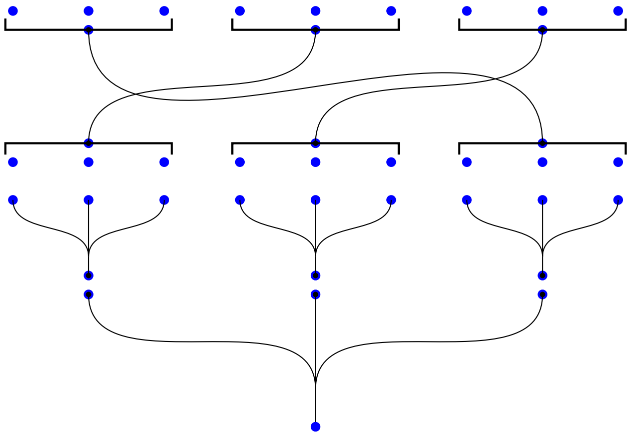
(6)

The left side says the compositions are made first in the two upper levels and then, the resulting operations are compose with the base. The right side indicates the compositions starts with the two lower levels of operations and the resulting operation is compose with the $i_1 + \cdots + i_k$ operations on top. For the unit in $P(1)$, we demands that it does not affect the result of compositions. In other words, if we have an operation with $k$ inputs, the composition with $k$ times the unit, gives as result the same operation. And if we compose the unit in its only input with any operation, the unit doesn't change this operation.



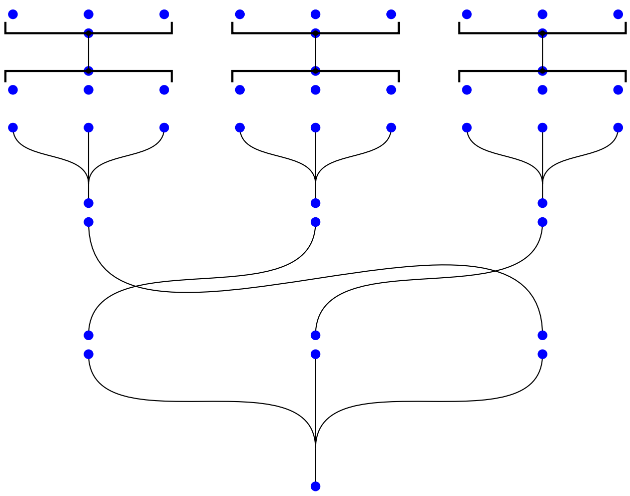
(7)

Finally we require right actions of symmetric groups to satisfy some equivariance condition with respect to compositions. The first condition applies when in a composition we have over the inputs of the resulting operation, a permutation acting in such a way that it respects the blocks of inputs of each part of the composition. For instance, consider the permutation $\sigma = \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$, and an element of $P(9)$ obtained by the composition of an element of $P(3)$ with three elements of $P(3)$, and affected by the associated action of $\sigma$ in $\Sigma_9$.



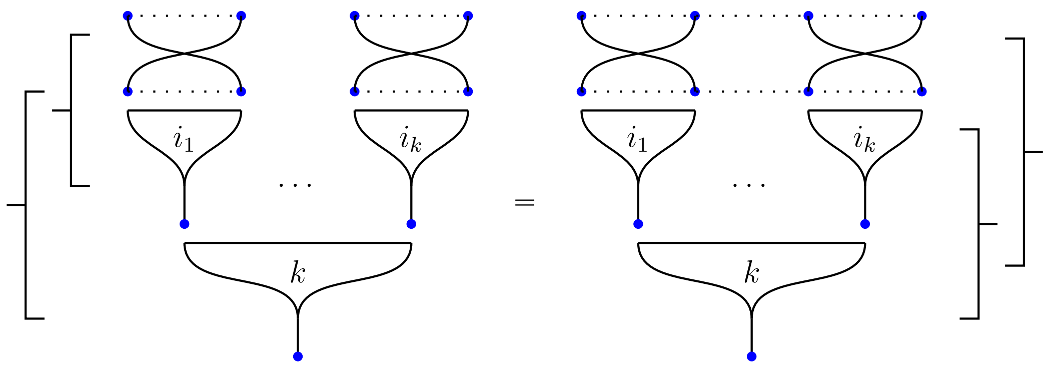
(8)

Now, if we try to arrange the mix in the upper part of the picture made by the permutation, in such a way that we put face to face the blocks by moving the corresponding operations in the inputs of the base operation. Then, the shuffle made by the permutation will be now placed over the inputs of the base operation, that is, the action on the resulting composite is now converted in to an action on the operation at the base.



(9)

This process of moving the action to the bottom is wanted so as not to affect the resulting operation, that is, both expression are the same in an operad. The second equivariance condition apply when the action of the symmetric group on a composite affects individually the inputs of each operation in the composition.




(10)

In the left side first the actions are applied on each component and then the composition is performed. In the right side, the composition is first made and then on the resulting operation is applied a single permutation which is obtained by putting together all the others permutations. Both processes are supposed to give the same result in an operad.

**$\mathcal{L}$-algebras and $E_\infty$-structures** There are several attempts for generalizing Sullivan's ideas to arbitrary coefficients. In particular, A. Prouté proposes another approach using higher homotopy techniques formalized as $\mathcal{L}$-algebras. His ideas have been part of several talks since the eighties, but never published. The $\mathcal{L}$-algebras can be seen as an adaptation of Segal's ideas in a rather simple way.

Besides, for the analysis of infinite loop spaces in [Seg74], Segal introduce the notion of $\Gamma$-space. His point of view is essentially based on the idea that the relatively big family of higher homotopies needed for $E_\infty$-techniques, can be coded in a different way. This higher homotopy techniques introduce $E_\infty$-spaces in order to state a recognition principle for infinite loop spaces (see [BV68]). The higher homotopies of an $E_\infty$-space can be replaced by a small family of homotopy equivalences, much easier to describe, from which the higher homotopies can be recovered just by choosing homotopy inverses.

$\mathcal{L}$-algebras are similar to $\Gamma$-spaces, but instead of applying to spaces they apply to singular chain complexes. However, this is technically somewhat different, essentially because in the theory of $\Gamma$-spaces, the cartesian product of topological spaces is a product (in the categorical sens), unlike the tensor product of modules which we must use in this dual situation, is not a sum. Fortunately, this gap is compensated by the good properties of the Eilenberg-Mac Lane transformation, which satisfies several commutation properties exactly, not only up to homotopy.

This thesis is dedicated in a first part to the description of several properties of the category of $\mathcal{L}$-algebras. In the second part of this work we focus on the description of the $E_\infty$-coalgebra on the main element of an $\mathcal{L}$-algebra. Intuitively this structure is reflected in the fact that all the coproducts on the main element constructed from the morphisms of the $\mathcal{L}$-algebra must be homotopic. This is the case when we consider the canonical $\mathcal{L}$-algebra of a simplicial set. In the monograph [Smi94], Smith constructs a natural $E_\infty$-coalgebra structure on the chain complexes. From his construction, we give an alternative way to describe this $E_\infty$-coalgebra (see section 4.2), by finding a more direct way to construct an $E_\infty$-operad that acts on the chain complexes. Our $E_\infty$-operad $\mathcal{R}$ used for this propose could be useful to describe explicitly the $E_\infty$-algebras. For the $E_\infty$-coalgebra structure on the main element of an $\mathcal{L}$-algebra, we design an $E_\infty$-operad $\mathcal{K}$ using a special technique that we call polynomial operads. Then, the main result of this thesis can be stated as follows.

❐ **Main Result**

---

*There exists a functor $\mathcal{F} : \mathcal{L}\text{-}Alg \to \mathcal{K}\text{-}CoAlg$, which associates a $(E_\infty)$ $\mathcal{K}$-coalgebra $\mathcal{F}(A)$ to each $\mathcal{L}$-algebra $A$, in such a way that the underlying DGA-$\mathsf{k}$-module of $\mathcal{F}(A)$ is $A[1]$, and for all $n \geqslant 1$, the operad morphism $\mathcal{K} \to Coend(A[1])$ given by $\mathcal{F}$, sends every $\mathsf{k}[\Sigma_n]$-generator $x \in K(n)_0$ to a morphism of DGA-$\mathsf{k}$-modules $\overline{x}$ such that $\mu \circ \overline{x}$ is homotopic to $s_0$, with $\mu$ given by the structural quasi-isomorphism of $A$ and $s_0$ by the image of $A$ of the only morphism in $\mathcal{L}$ of the form $([n], \alpha) : [n] \to [1]$.*

**Organization of the thesis** Oriented towards the description of the $E_\infty$-coalgebra acting on the main element of an $\mathcal{L}$-algebra, this thesis is organized in the following parts :

❏ **Chapter 1**: In this chapter we review the principal concepts used along this work. They include for instance, the properties of augmented differential graded modules and symmetric coequalizers. In the last part, we recall the properties of the Eilenberg-Mac Lane transformation and a version of the acyclic models theorem used in its characterization.

❏ **Chapter 2**: This chapter is devoted to the study of operads. Its role is to be used to justify the construction that will be made in the next chapters. We where principally interested in the construction of the free operad on a $\mathbb{S}$-module and the existence of small colimits in the category of operads. Even if we work with symmetric operads, that is, with actions by the symmetric groups, we include some results about non-symmetric operads in order to perform a construction of an operad presented in chapter 5.

❏ **Chapter 3**: It is about $\mathcal{L}$-algebras. We introduce this concept in details and discuss its construction in the general setting of monoidal categories, to stay after that in the category of differential graded modules. The rest of this chapter is dedicated to the study of the principal properties of $\mathcal{L}$-algebras.

❏ **Chapter 4**: In this chapter we study the $E_\infty$-coalgebra structure given in [Smi94] on the chain complexes associated to simplicial sets, also we construct a different operad to the one presented in [Smi94] and proof that it gives an $E_\infty$-structure to chain complexes. In fact, our operad is in somehow a free version of the operad in [Smi94].

❏ **Chapter 5**: We present a technique to construct operads that we call polynomial operads. Next, this technique is used to construct a $E_\infty$-operad $\mathcal{K}$. Then we proof the existence of an $E_\infty$-coalgebra structure on the main element of an $\mathcal{L}$-algebra using $\mathcal{K}$. Finally, we establish the functoriality of this construction.

# Introduction: En Français

Dans cette introduction nous expliquons le contexte dans lequel les $\mathcal{L}$-algèbres se placent. Nous commençons par une revue de quelques techniques utilisés dans l'étude du type d'homotopie des espaces comme les modèles minimaux en homotopie rationnelle et puis nous continuons avec l'utilisation des $A_\infty$-algèbres dans le cas des corps de caractéristique positive et une description du concept d'opérade. Dans la partie finale on parle de $\mathcal{L}$-algèbres et des résultats prouvés dans cette thèse.

**Modèles Minimaux de Sullivan** À la fin des années 60, un concept de modèle minimal dans la théorie de l'homotopie rationnelle fut introduit par Dennis Sullivan (voir [Jam99], §27). Les espaces simplement connexes peuvent être rationalisés, c'est-à-dire qu'on peut remplacer un espace $X$ par une version rationnelle de cet espace, $X_\mathbb{Q}$, telle que $H_*(X;\mathbb{Q}) = H_*(X_\mathbb{Q})$. Un espace simplement connexe $Y$ est dit rational quand son homologie réduit (ou $\pi_*(Y)$, ou bien l'homologie réduit de son espace de lacets $\Omega Y$) est un $\mathbb{Q}$-espace vectoriel. Pour une application continue $\phi : X \to Z$, nous pouvons établir l'existence (à homotopie près) d'un unique morphisme induit entre les rationalisations de $X$ et $Z$. Ensuite, le type d'homotopie d'un espace simplement connexe est définit comme le type d'homotopie faible de sa rationalisation.

Cette simplification d'un espace implique une perte d'information, par exemple, les groupes d'homotopie de la sphère $S^2$ ne sont pas nuls dans une infinité de degrés, mais, les groupes d'homotopie rationnelle sont nuls dans tous les dégrées au dessus de 3. Néanmoins, l'avantage de l'approximation par un modèle rationnel c'est la facilité de calcul tandis que la théorie ordinaire de l'homotopie est plus complexe. C'est dû à la découverte d'une formulation explicite pour l'homotopie rationnelle par Quillen et Sullivan ([Sul77], [Qui69]). Ils ont établit une équivalence de catégories entre la catégorie homotopique des espaces rationels et leur catégorie des modèles minimaux. Sullivan a trouvé un functeur $A_{PL}$ lequel associe une algèbre de cochaînes commutatives $A_{PL}(X)$ à $X$. Les algèbres $A_{PL}(X)$ et $C^*(X)$ sont liées par un zigzag de quasi-isomorphismes, et en particulier ont le même type de cohomologie $H^*(X) = H(A_{PL}(X))$.

Le passage d'espaces topologiques à des algèbres commutatives de cochaînes est établi par un foncteur $A_{PL}$ qui nous permet de nous concentrer sur l'étude des algèbres de cochaînes commutatives. Dans cette catégorie on trouve un type spécial d'algèbres commutatives de cochaînes qui sont appellés algèbres de Sullivan. Ces algèbres appartiennent à chaque classe d'isomorphisme, et sous certaines conditions sur l'espace $X$, elles sont un représentant unique déterminé à homotopie près, elles s'appellent modèles minimaux de Sullivan.

Si des espaces topologiques simplement connexes $X$ et $Y$ ont le même type d'homotopie rationnelle, alors leur algèbres de cochaînes $A_{PL}(X)$ et $A_{PL}(Y)$ sont quasi-isomorphes, et par l'unicité des modèles minimaux, ils ont le même modèle minimal. Alors, si on se concentre sur les espaces simplement connexes de type d'homologie rationnelle finie, il existe une bijection entre le type de l'homotopie rationnelle et les classes d'isomorphismes des algèbres minimales de Sullivan sur $\mathbb{Q}$.

**Le problèmes de Cochaînes Commutatives** Le produit de l'anneau de cochaînes $C^*(X;\mathsf{k})$, c'est-à-dire, le cup produit de cochaînes, normalement n'est pas commutatif. Dans le cas gradué, la commutativité signifie que $x \cup y = (-1)^{|x||y|} y \cup x$, où $|x|$ et $|y|$ sont les dégrés de $x$ et $y$. Essentiellement, le problème des cochaînes commutatives consiste à trouver d'une manière fonctorielle une algèbre commutative différentielle $A^*(X)$ sur l'anneau $\mathsf{k}$, de tel manière qu'il existe un zig-zag de quasi-isomorphismes entre $A^*(X)$ et $C^*(X:\mathsf{k})$ (voir [GM81], §9). Ce problème est résolu par Sullivan dans le cas rational.

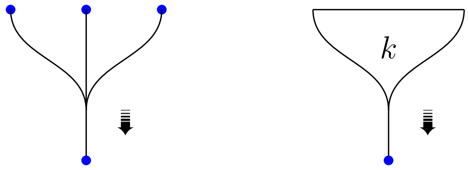
Steenrod propose un type d'opérations cohomologiques liées au cup produit (voir [Ste47]). Les carrés de Steenrod $Sq^i$ sont définis sur l'anneau de cohomologie à coefficients dans $\mathbb{Z}/2\mathbb{Z}$. Ils prennent la classe $x$ d'un cocycle de degré $n$ dans une classe $Sq^i(x)$ de degré $n+i$. Quand $n = i$, $Sq^i(x)$ est simplement le cup produit $x \cup x$. La construction des carrés de Steenrod dépend fortement de la non conmutativité de l'anneau de cochaînes $C^*(X;\mathbb{Z}/2\mathbb{Z})$, et une conséquence de leur existence est la non existence d'une solution au problème des cochaînes commutatives sur l'anneau $\mathbb{Z}/2\mathbb{Z}$, et en conséquence sur $\mathcal{Z}$ (voir par exemple [Cen89]). Le même problème est présent dans $\mathbb{Z}/p\mathbb{Z}$, quand $p$ est un nombre premier impair.

$A_\infty$**-algèbres** Introduites par Stasheff ([Smi86], [Sta63]), les $A_\infty$-algèbres sont des complexes de chaînes $(A, d)$ avec des opérations $\mu_n : A^{\otimes n} \to A$, $n \geqslant 2$, de dégrée n-2, qui satisfont certains conditions. Les $A_\infty$-algèbres peuvent être vues comme une généralisation des algèbres différentielles graduées. En fait, pour une DGA-algèbre les opérations vont satisfaire $\mu_n = 0$ pour $n \geqslant 3$ et la catégorie des DGA-algèbres est une sous-catégorie de la catégorie des $A_\infty$-algèbres.

Dans [Kad80] Kadeishvili décrit la construction de la structure de $A_\infty$-algèbre sur l'algèbre d'homologie du complexe des chaînes et, après quelques généralisations sur la catégorie des $A_\infty$-algèbres, il donne une description de l'espace fibré en utilisant des $A_\infty$-algèbres. En 1986, cet approche fut utilisé par Prouté dans [Pro11] avec l'idée de faire des calculs explicites de l'homologie des espaces fibrés, où la fibre était $K(\mathbb{Z}/p\mathbb{Z}, n)$. L'idée derrière cette technique, c'est d'exprimer le complexe de chaînes de l'espace total du fibré par quelque chose avec une description en utilisant des opérations dans la catégorie des $A_\infty$-algèbres, parce que dans quelques cas spéciaux, il existait des méthodes établis pour faire la calcul des modèles minimaux dans la catégorie des $A_\infty$-algèbres, et les $A_\infty$-structures se présentent d'une manière naturelle quand le fibré a comme fibre un espace du type $K(\pi, n)$. L'associativité n'est pas toute l'histoire, nous avons besoin aussi de relaxer homotopiquement la commutativité, ce qui a conduit May dans [May72] à la notion d'opérade. Nous somme particulièrement intéressés par les $E_\infty$-opérades.

**Opérades symétriques** Une opérade peut être vue comme un cadre pour modéliser des structures algébriques. Dans cette partie nous expliquons l'intuition derrière le concept d'opérade. Les opérades peuvent être définies sur n'importe quelle catégorie monoïdale symétrique $\mathcal{C}$, nous sommes intéressés par la catégorie DGA-$\mathsf{k}$-Mod. Toutes les constructions seront faites en pensant à cette catégorie. Une opérade $\mathcal{P}$ est composée d'une collection $\{P(i)\}_{i \geqslant 0}$ d'objets de $\mathcal{C}$, laquelle est soumise à plusieurs conditions que nous allons discuter au long de cette introduction.

Les éléments de chaque objet $P(k)$ peuvent être vus comme des opérations abstraites avec $k$ entrées et une seule sortie, ils sont dit d'aritée $k$. Dans l'image suivante sont représentés deux éléments de $\mathcal{P}$, le premier est un élément d'arité 3 ($\in P(3)$), et l'autre est un élément d'arité $k$ ($\in P(k)$).



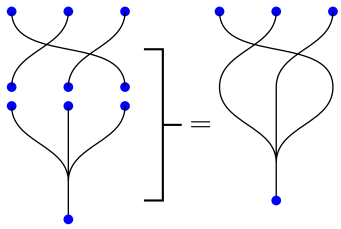

(11)

Chaque opérade $\mathcal{P}$ a un élément distingué en arité 1, lequel est censé représenter l'opération identité, appelé l'unité de $\mathcal{P}$. Il est défini comme un morphisme $\eta : 1 \to P(1)$ et il est représenté par un bâton avec une entrée et une sortie-

(12)

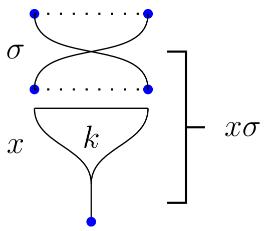
Chaque $P(k)$ est équipé d'une action du groupe symétrique $\Sigma_k$. Graphiquement, cette action est représentée par un mélange des entrées. Par exemple, on considère $\sigma \in \Sigma_3$ donné par

$\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$. L'action de $\sigma$ sur un élément de $P(3)$ est représentée de la manière suivante.




(13)

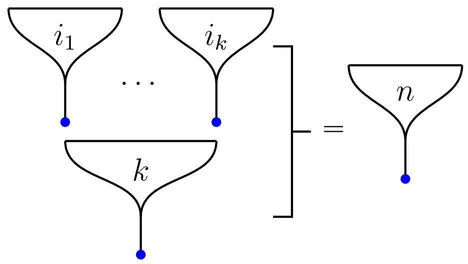
C'est-à-dire, si l'élément de $P(3)$ est une opération $f(x_1, x_2, x_3)$, alors $f\sigma(x_1, x_2, x_3) = f(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) = f(x_2, x_3, x_1)$. Dans la dernière image les crochets sont utilisés pour représenter l'acte d'appliquer l'action par $\sigma$ sur un élément de $P(3)$. En général, nous représentons une action de $\sigma \in \Sigma_k$ sur un élément de $P(k)$ par l'image suivante.




(14)

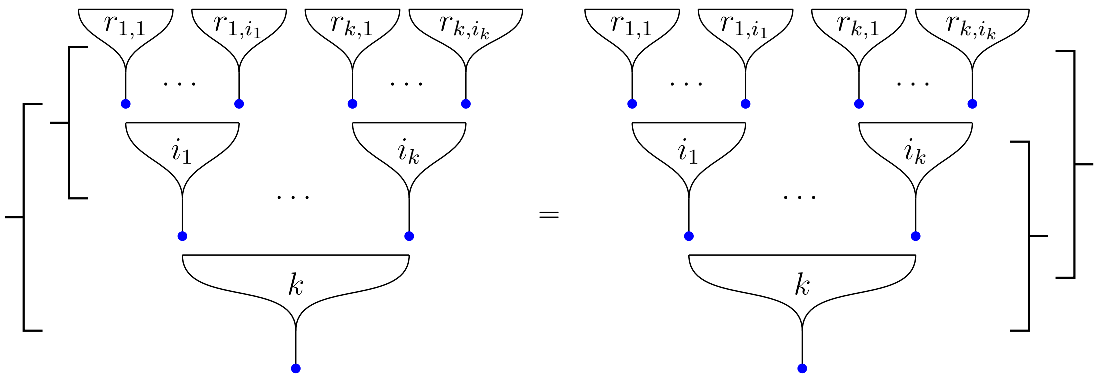
Une autre partie importante de l'opérade $\mathcal{P}$ est la notion de composition. Comme nous sommes en train de modéliser des opérations, nous devons coder comment les composés des opérations vont se comporter. Soit $f \in P(k)$, alors nous pouvons faire le composé de cette opération avec $k$ (une pour chaque entrée) opérations de $\mathcal{P}$, alors on va obtenir une opération d'arité égale à la somme des arités des opérations dans chaque entrée. Les composés sont donné par des morphismes de la forme

$\gamma : P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \to P(n)$, with $n = i_1 + \cdots + i_k$. Nous allons aussi représenter l'acte d'appliquer les composées par un crochet.



(15)

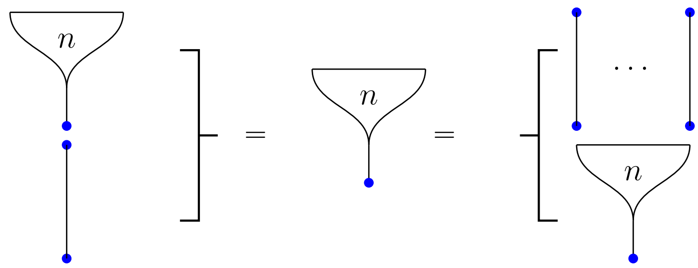
Les donnés produites par l'action des groupe symétriques, l'unité et les composés doivent satisfaire quelques conditions. La première de ces conditions est l'associativité de la composition, dans le sens que nos opérations abstraites dans $\mathcal{P}$ ne vont pas dépendre de l'ordre dans lequel elles sont faites. L'image suivante représente cette situation.



(16)

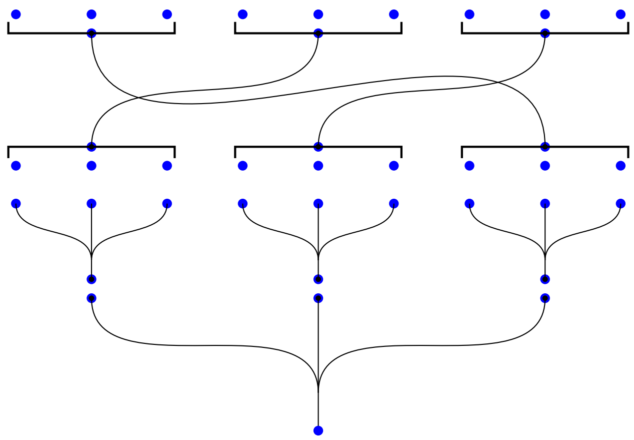
La partie de gauche dit que les composés sont fait d'abord dans les deux niveaux supérieurs, et après, les opération qui en résultent, sont composées avec la base. La partie de droite dit que la composition est faite d'abord dans la base, et l'opération qui en résulte est ensuite composée avec les $i_1 + \cdots + i_k$ opérations supérieures.

Pour l'unité $P(1)$, nous demandons qu'elle n'affecte pas le résultat des compositions. C'est-à-dire, si nous avons une opération avec $k$ entrées, les composés avec $k$ fois l'unité vont donner la même opération. Et si on fait le composé de l'unité dans sa seule entrée avec n'importe quelle autre opération, on obtient cette même opération.



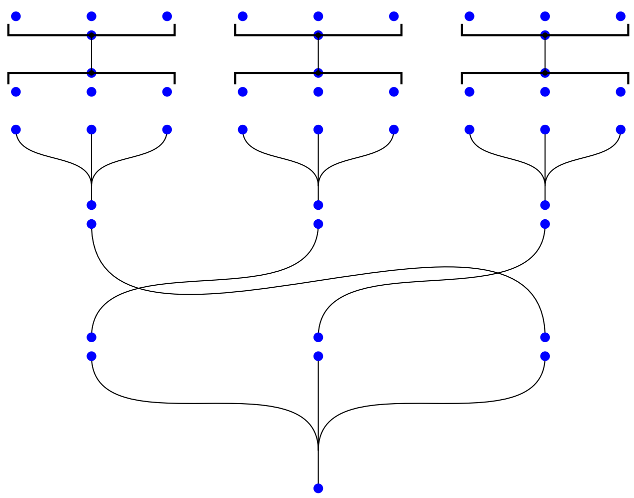
(17)

Finalement, nous demandons que les actions à droite des groupes symétriques satisfassent quelques conditions d'équivariance par rapport aux compositions. La première condition s'applique quand dans un composé nous avons sur les entrées de l'opération résultante, une permutation qui agit de telle manière qu'elle respecte les blocs des entrées de chaque partie du composé. Par exemple, on considère la permutation $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$, et un élément de $P(9)$ obtenue par la composition d'un élément de $P(3)$ avec trois éléments de $P(3)$, et tous affectés par l'action de $\Sigma_9$ associé à $\sigma$.



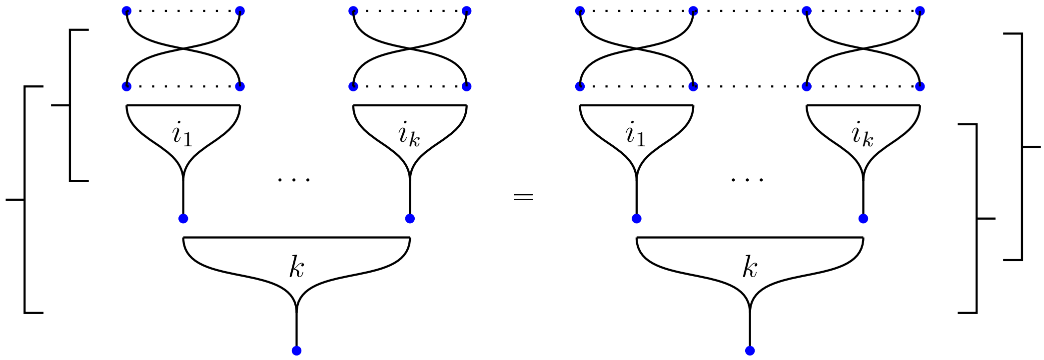
(18)

Maintenant, si nous essayons de ranger le mélange fait par la permutation dans la partie supérieure de l'image, de telle manière qu'on place face à face les blocs en faisant bouger les opérations correspondantes dans les entrées de l'opération de la base. Alors, le mélange fait par la permutation sera converti en une action sur l'opération de la base.



(19)

On veut que ce processus de faire bouger l'action au-dessous n'affecte pas l'opération résultante, c'est-à-dire, que les deux opérations sont les mêmes dans une opérade. La deuxième condition d'équivariance s'applique quand l'action du groupe symétrique sur le composé affecte d'une manière individuelle les entrées de chaque opération du composé.



(20)

Dans la partie de gauche d'abord les actions sont appliquées sur chaque composante et puis la composition est faite. Dans la partie de droite, la composition est faite d'aborde, puis sur les opérations qui en résultent est appliquée une seule permutation obtenue par le rassemblement des toutes les autres permutations. Les deux processus sont alors censés de donner le même résultat dans une opérade.

**$\mathcal{L}$-algèbres et $E_\infty$-structures** Par ailleurs, pour l'analyse des espaces de lacets infinis dans [Seg74], Segal introduit la notion de $\Gamma$-espace. Son point de vue est essentiellement basé sur l'idée qu'une famille relativement grande d'homotopies d'ordre supérieur nécessaires pour la $E_\infty$-technique, peut être codée d'une manière différente. Ces techniques d'homotopie d'ordre supérieur introduisent les $E_\infty$-espaces pour établir un principe d'identification pour les espaces de lacets infinis(see [BV68]). Les homotopies d'ordre supérieur d'un $E_\infty$-espace peuvent être remplacées par une petite famille d'équivalences d'homotopie, plus facile à décrire, depuis laquelle les homotopies d'ordre supérieur peuvent être récupérées juste en choisissant des inverses homotopiques.

Les $\mathcal{L}$-algèbres sont similaire aux $\Gamma$-espaces, mais au lieu de s'appliquer à des espaces, elles s'appliquent à des complexes de chaînes singulières. Cependant, c'est une technique différente, essentiellement parce que dans la théorie des $\Gamma$-espaces, le produit cartésien des espaces topologiques est un vrai produit(dans le sens catégorique), contrairement au produit tensoriel des modules, qu'on doit utiliser dans cette situation duale, et qui n'est pas une somme. Heureusement, ce problème est compensé par les bonnes propriétés de la transformation d'Eilenberg-Mac Lane, laquelle satisfait plusieurs propriétés de commutativité d'une manière exacte, et pas seulement à homotopie près.

Cette thèse est dédié dans sa première partie à la description de plusieurs propriétés de la catégorie des $\mathcal{L}$-algèbres. Dans sa deuxième partie on se focalise dans la description de la $E_\infty$-coalgèbre sur l'élément principal d'une $\mathcal{L}$-algèbre. D'une manière intuitive, cette structure est reflété dans le fait que tous les coproduits sur l'élément principal qui sont construit à partir des morphismes dans la structure d'une $\mathcal{L}$-algèbre, doivent être homotopes. C'est le cas quand nous considérons la $\mathcal{L}$-algèbre canonique d'un ensemble simplicial. Dans la monographie [Smi94], Smith construit une structure naturelle de $E_\infty$-coalgèbre sur les complexes de chaînes. En s'inspirant de cette construction, on donne une manière alternative de décrire cette structure de $E_\infty$-coalgèbre (voir section 4.2), en trouvant une manière plus directe de construire une $E_\infty$-opérade qui agit sur les complexes de chaînes. Notre $E_\infty$-opérade

$\mathcal{R}$ utilisée pour ce propos peut être utile pour décrire d'une manière explicite les $E_\infty$-coalgèbres. Pour la structure de $E_\infty$-coalgèbre sur l'élément principal d'une $\mathcal{L}$-algèbre, nous construisons une $E_\infty$-opérade $\mathcal{K}$ en utilisant une technique spéciale que nous appelons opérade polynomiale. Le résultat principal de cette thèse peut être énoncé comme suit.

❒ **Main Result**

---

*Il existe un foncteur $\mathcal{F} : \mathcal{L}\text{-}Alg \to \mathcal{K}\text{-}CoAlg$, lequel associe une $(E_\infty)$ $\mathcal{K}$-coalgèbre $\mathcal{F}(A)$ à chaque $\mathcal{L}$-algèbre $A$, de tel manière que le module sous-jacent à $\mathcal{F}(A)$ soit $A[1]$, et tel que pour tout $n \geqslant 1$, le morphisme d'opérades $\mathcal{K} \to Coend(A[1])$ donné par $\mathcal{F}$, envoie chaque $\mathsf{k}[\Sigma_n]$-générateur $x \in K(n)_0$ vers un morphisme de DGA-$\mathsf{k}$-modules $\overline{x}$ tel que $\mu \circ \overline{x}$ soit homotopique à $s_1$, où $\mu$ est donné par le quasi-isomorphisme structurel de $A$, et $s_1$ est donné par l'image par $A$ du seul morphisme dans $\mathcal{L}$ de la forme $([n], \alpha) : [n] \to [1]$.*

**Organisation de la thèse** Orientée vers la description de la $E_\infty$-coalgèbra agissant sur l'élément principale d'une $\mathcal{L}$-algèbre, cette thèse s'organise comme suit :

❏ **Chapitre 1**: Dans ce chapitre nous faisons une revue des principaux concepts qui seront utilisés au long de ce travail. Ils comprennent, par exemple, les propriétés des modules différentiels gradués ainsi comme les notions catégoriques de monade et coégaliseur réflexif. Nous rappelons les propriétés de la transformation d'Eilenberg-Mac Lane et une version du théorème des modèles acycliques qui permet de la caractériser.

❏ **Chapitre 2**: Ce chapitre est dédié à l'étude des opérades. Son rôle est de servir à la justification des constructions que nous ferons dans les chapitres suivants. Nous nous sommes intéressés à la construction de l'opérade libre sur un $\mathbb{S}$-module et à l'existence des colimites sur la catégorie des opérades. Nous incluons quelques résultats sur les opérades non symétriques envisageant une construction présenté dans le chapitre 5.

❏ **Chapitre 3**: Nous introduisons les $\mathcal{L}$-algèbres. Nous allons discuter leur construction dans le cadre général des catégories monoïdales, pour après rester dans la catégorie des modules différentiels gradués.

❏ **Chapitre 4**: Dans ce chapitre nous avons étudié la structure de $E_\infty$-coalgèbre donné dans [Smi94] sur les complexes de chaînes. Nous y construisons une opérade différente de celle présentée dans [Smi94] et prouvons qu'elle une $E_\infty$-structure au complexe de chaînes. En fait, notre opérade constitue en quelque sorte une version libre de l'opérade dans [Smi94].

❏ **Chapitre 5**: Nous présentons une technique pour construire des opérades qu'on appelle des opérades polynomiales. Ensuite, cette technique est utilisée pour construire une $E_\infty$-opérade $\mathcal{K}$, avec laquelle on prouve l'existence d'une $E_\infty$-coalgèbre sur l'élément principal des $\mathcal{L}$-algèbres. Enfin, on prouve la fonctorialité de cette construction.

# Chapter 1

# Preliminaries

The main purpose of this chapter is to fix notations and make a review of the principal results used in this thesis. Some proofs will be omitted, since they can be found in the existing literature. In such a case, we will include the required references. Most of the proofs that we judge to include are those containing relevant details that improve the reading of this thesis or those with proofs not presented in the references.

The first three sections are dedicated to the rappels of the theory of differential graded modules, algebras and coalgebras. Then, the following three sections deal with the categorical notions required for the study of operads and $\mathcal{L}$-algebras. The final sections include generalities about chain complexes, a enhanced version of the acyclic models theorem and the principal properties of the Eilenberg-Mac Lane transformation.

## 1.1 Graded modules and Koszul Convention

Along this thesis $\mathsf{k}$ will denote a field, having in mind the finite field $\mathbb{Z}/p\mathbb{Z}$ where $p$ is a prime number, or the field of rational numbers $\mathbb{Q}$. The modules on $\mathsf{k}$ will be simply called modules, but if we need to be more specific they will referred as $\mathsf{k}$-modules. Some propositions can be stated in more general setting than fields, in such cases, we will used the symbol $\Lambda$ for the ring. Tensor product $\otimes_{\mathsf{k}}$ is written $\otimes$. We identify the tensor products $\mathsf{k} \otimes M$ and $M \otimes \mathsf{k}$ with $M$ by $1 \otimes m = m \otimes 1 = m$, for every $m \in M$ and where 1 is the unit of $\mathsf{k}$.

**Definition 1.1.1.** A $\mathsf{k}$-module $M$ is said to be a graded $\mathsf{k}$-module if there is a family $\{M_i\}_{i\in\mathbb{Z}}$ of $\mathsf{k}$-submodules of $M$, such that :

$$M = \bigoplus_{i\in\mathbb{Z}} M_i \tag{1.1}$$

An element $x \in M_i$ is called homogeneous of degree $i$, in symbols $|x| = i$. A graded $\mathsf{k}$-module is said to be positively graded if $M_i = 0$ for every $i < 0$, negatively graded if $M_i = 0$ for every $i > 0$, bounded below if there exists $j$ integer such that $M_i = 0$ for every $i < j$ and bounded above if there exists $j$ integer such that $M_i = 0$ for every $i > j$.

**Definition 1.1.2.** Let $M$ and $N$ be two graded $\mathsf{k}$-modules. A morphism of graded $\mathsf{k}$-modules, $f : M \to N$ is a collection of homogeneous linear maps $\{f_i : M_i \to N_{i+p}\}_{i\in\mathbb{Z}}$.

In such a case, the morphism $f$ is said to be of degree $p$. The category of graded $\mathsf{k}$-modules is denoted $\mathsf{k}$-Mod.

**Definition 1.1.3.** Let $M$ and $N$ be two graded $\mathsf{k}$-modules. The tensor product $M \otimes N$ is the graded $\mathsf{k}$-module defined by :

$$(M \otimes N)_i = \bigoplus_{p+q=i} M_i \otimes N_i \tag{1.2}$$

➪ *Remark* 1.1.4 (Koszul convention). Working in graded contexts leads to state sign conventions. They apply when the positions of graded symbols in an expression are permuted. For instance, when two symbols of degree $p$ and $q$ are permuted, the resulting expression will be multiplied by $(-1)^{pq}$. In fact, the Koszul sign is the signature of the permutation of odd degree letters.

**Definition 1.1.5.** Let $f : M \to R$ and $g : N \to S$ be two morphisms of graded $\mathsf{k}$-modules. We define the morphism $f \otimes g : M \otimes N \to R \otimes S$ by the formula,

$$(f \otimes g)(m \otimes n) = (-1)^{|g||m|} f(m) \otimes g(n) \tag{1.3}$$

**Proposition 1.1.6.** *Let $f : M \to R$, $h : R \to T$, $g : N \to S$ and $k : S \to U$ be morphisms of graded $\mathsf{k}$-modules, then :*

$$(h \circ f) \otimes (k \circ g) = (-1)^{|f||k|} (h \otimes k) \circ (f \otimes g) \tag{1.4}$$

□

**Definition 1.1.7.** Let $M$ and $N$ be two graded $\mathsf{k}$-modules. $Hom(M, N)$ is defined to be the graded $\mathsf{k}$-module given by all the morphisms of graded $\mathsf{k}$-modules of every degree from $M$ to $N$, that is, the elements of grade $i$ in $Hom(M, N)$ are,

$$Hom(M, N)_i = \prod_{n \in \mathbb{Z}} Hom(M_n, N_{n+i}) \tag{1.5}$$

**Definition 1.1.8.** The morphisms of graded $\mathsf{k}$-modules $f : M \to R$, $g : N \to S$ induce the morphisms of graded $\mathsf{k}$-modules $f^*$ and $g_*$ between $Hom(M, N)$ and $Hom(R, S)$, defined by :

$$f^*(h) = (-1)^{|h||f|} h \circ f \tag{1.6}$$

$$g_*(h) = g \circ h \tag{1.7}$$

**Proposition 1.1.9.** *We have the relations between morphism of graded $\mathsf{k}$-modules,*

$$(g \circ f)^* = (-1)^{|f||g|} f^* \circ g^* \tag{1.8}$$

*and*

$$(g \circ f)_* = g_* \circ f_* \tag{1.9}$$

□

➪ *Remark* 1.1.10. We can identify $\mathsf{k}$ with a graded $\mathsf{k}$-module by setting $\mathsf{k}_0 = \mathsf{k}$ and $\mathsf{k}_i = 0$ for $i \neq 0$. In this case we say that $\mathsf{k}$ is a graded module concentrated in degree zero.

## 1.2 Differential Graded Modules

**Definition 1.2.1.** Let $M$ be a graded $\mathsf{k}$-module. A differential is a homogeneous morphism $\partial : M \to M$ of degree $-1$, such that $\partial^2 = 0$. If $M$ has a differential it is called a differential graded $\mathsf{k}$-module or simply DG-$\mathsf{k}$-module.

**Definition 1.2.2.** Let $M$, $N$ be DG-$\mathsf{k}$-modules. A morphism $f$ from $M$ to $N$ is a homogeneous homomorphism such that $\partial f = (-1)^{|f|} f\partial$. The category of differential graded modules is denoted DG-$\mathsf{k}$-Mod.

**Definition 1.2.3.** Let $M$ be a DG-$\mathsf{k}$-module.

1. An augmentation $\epsilon$ of $M$ is a degree 0 morphism of DG-$\mathsf{k}$-modules $\epsilon : M \to \mathsf{k}$.

2. A coaugmentation $\eta$ of $M$ is a degree 0 morphism of DG-$\mathsf{k}$-modules $\eta : \mathsf{k} \to M$.

**Definition 1.2.4.** A DG-$\mathsf{k}$-module $M$ is said to be a DGA-$\mathsf{k}$-module † if it is provided with an augmentation $\epsilon$ and a coaugmentation $\eta$ such that,

$$\epsilon \circ \eta = 1_{\mathsf{k}} \tag{1.10}$$

A morphism of DGA-$\mathsf{k}$-modules $f : M \to N$ is a morphism of DG-$\mathsf{k}$-modules such that $\epsilon f = \epsilon$ and $f\eta = \eta$. The category of DGA-$\mathsf{k}$-modules is denoted DGA-$\mathsf{k}$-Mod.

**Definition 1.2.5.** Let $M$ be a positively or negatively graded DGA-$\mathsf{k}$-module.

1. $M$ is said to be connected if $\epsilon : M_0 \to \mathsf{k}$ is an isomorphism.

2. $M$ is said to be simply connected if also satisfies $M_1 = M_{-1} = 0$.

**Proposition 1.2.6.** *Let $M$ and $N$ be DGA-$\mathsf{k}$-modules.*

1. *The tensor product $M \otimes N$ is a DGA-$\mathsf{k}$-module if we define the differential by $\partial_{M\otimes N} = \partial_M \otimes 1 + 1 \otimes \partial_N$, the augmentation by $\epsilon_{M\otimes N} = \epsilon_M \otimes \epsilon_N$ and the coaugmentation by $\eta_{M\otimes N} = \eta_M \otimes \eta_N$.*

2. *$Hom(M, N)$ is a DG-$\mathsf{k}$-module if the differential is defined by $\partial_{Hom(M,N)} = (\partial_N)_* - (\partial_M)^*$. We use no notion of augmentation and coaugmentation for $Hom(M, N)$.*

□

➪ *Remark* 1.2.7. The explicit formulas for the expressions in 1.2.6 are the following.

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^{|x|} x \otimes \partial y \tag{1.11}$$
$$\epsilon(x \otimes y) = \epsilon(x)\epsilon(y),\ \eta(1) = \eta(1) \otimes \eta(1) \tag{1.12}$$
$$\partial(f) = \partial f - (-1)^{|f|} f\partial \tag{1.13}$$

† DGA for differential graded with augmentation.

**Proposition 1.2.8.** *The following canonical homomorphism are morphism of DG-*$\mathsf{k}$*-modules.*

$$\circ : Hom(N, P) \otimes Hom(M, N) \to Hom(M, P) \tag{1.14}$$
$$\otimes : Hom(M, P) \otimes Hom(N, Q) \to Hom(M \otimes N, P \otimes Q) \tag{1.15}$$

*And we have the following relations between differentials.*

$$\partial(g \circ f) = (\partial g) \circ f + (-1)^{|f|} g \circ (\partial f) \tag{1.16}$$
$$\partial(f \otimes g) = (\partial f) \otimes g + (-1)^{|f|} f \otimes (\partial g) \tag{1.17}$$

□

**Definition 1.2.9.** Let $f : M \to N$ be a morphism of DG-$\mathsf{k}$-modules of degree $l$. The mapping cone of $f$ is the DG-$\mathsf{k}$-module $C(f)$ defined by,

$$C(f)_n = M_{n-l-1} \oplus N_n \tag{1.18}$$
$$\partial(x, y) = (-(-1)^l \partial x, f(x) + \partial y) \tag{1.19}$$

➪ *Remark* 1.2.10. The differential of the mapping cone can be expressed using matrices,

$$\partial(x, y) = \begin{pmatrix} -(-1)^l\partial & 0 \\ f & \partial \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (-(-1)^l \partial x, f(x) + \partial y) \tag{1.20}$$

➪ *Remark* 1.2.11. The canonical inclusion $i : N \to C(f)$ and the canonical projection $j : C(f) \to M$, fit together in the exact sequence,

$$0 \longrightarrow N \xrightarrow{i} C(f) \xrightarrow{j} M \longrightarrow 0 \tag{1.21}$$

**Proposition 1.2.12.** *For every morphism $f : M \to N$ of DGA-*$\mathsf{k}$*-modules we have the exact sequence,*

$$\begin{tikzcd} & H_*(C(f)) \arrow[dr, "j_*"] & \\ H_*(N) \arrow[ur, "i_*"] & & H_*(M) \arrow[ll, "f_*"] \end{tikzcd} \tag{1.22}$$

□

**Definition 1.2.13.** Let $f : M \to N$ and $g : M \to N$ morphism of DG-$\mathsf{k}$-modules of degree $k$. An homotopy from $f$ to $g$ is a homomorphism $h \in Hom(M, N)$ of degree $k + 1$ such that $\partial(h) = g - h$.

➪ *Remark* 1.2.14. We have the following consequences of definition 1.2.13 and proposition 1.2.6.

1. The homotopy $h$ satisfies $g - f = \partial h - (-1)^{|h|}\partial = \partial h + (-1)^k h\partial$.
2. If $\partial(f) = 0$ then $f$ is a morphism of DG-$\mathsf{k}$-modules.
3. Homotopy is an equivalence relation.

**Definition 1.2.15.** Let $M$ be a DGA-$\mathsf{k}$-module.

1. $M$ is said to be inessential if there is a $h \in Hom(M, M)$ homotopy from 0 to $1_M$ of degree 1. In other words, $h$ satisfies,

$$\partial h + h\partial = 1 \tag{1.23}$$

in this case, $h$ is called a contracting chain homotopy of $M$.

2. M is said to be null homotopic if $H_*(M) = 0$.

**Definition 1.2.16.** Let $P$, $M$ and $N$ in DGA-$\mathsf{k}$-Mod. $P$ is called projective if, for every epimorphism $g : M \to N$, every morphism $f : P \to N$ can be lifted to a morphism $l : P \to M$ such that $gl = f$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "l"'] \arrow[d, "f"] \\ M \arrow[r, "g"'] & N \end{tikzcd} \tag{1.24}$$

➪ *Remark* 1.2.17. Every free module is projective and, on a field, every projective module is free (see [Wei95], §2).

**Proposition 1.2.18.** *Let $M$ be a DGA-$\mathsf{k}$-module.*

1. *If $M$ has a contracting chain homotopy then $M$ is null homotopic.*

2. *If $M$ projective, null homotopic and bounded below, then it has a contracting chain homotopy.*

*Proof.* First affirmation follows from formula 1.23. The contracting chain homotopy for the second affirmation is constructed by induction on the degree, starting in degree $j$ where $M_i = 0$ for $i < j$. □

**Definition 1.2.19.** Let $f : M \to N$ be a DGA-$\mathsf{k}$-morphism. Then $f$ is said to be an homotopy equivalence is $fg$ is homotopic to $1_N$ and $gf$ is homotopic to $1_M$. In this case, $g$ is called a homotopy inverse of $f$.

**Proposition 1.2.20.** *Let $f : M \to N$ be a DGA-$\mathsf{k}$-morphism inducing an isomorphism in homology. If $M$ and $N$ are projective and bounded below then $f$ is an homotopy equivalence.*

*Proof.* The hypothesis implies that $C(f)$ is projective, null homotopic and bounded below, then by proposition 1.2.18 there exists a contracting chain homotopy $H : C(f) \to C(f)$ such that $\partial h + h\partial = 1$. Using matrices we have the following.

$$1 = \partial h + h\partial$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix} h + h \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix} = \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix} \begin{pmatrix} \lambda_1 & g \\ \mu & \lambda_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & g \\ \mu & \lambda_2 \end{pmatrix} \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix}$$

Then we get the equations,

$$1 = -\partial\lambda_1 - \lambda_1\partial + gf \tag{1.25}$$
$$0 = -\partial g + g\partial \tag{1.26}$$
$$1 = f\lambda_1 + \partial\mu - \mu\partial + \lambda_2 f \tag{1.27}$$
$$0 = fg + \partial\lambda_2 + \lambda_2\partial \tag{1.28}$$

The equation 1.26 says that $g : N \to M$ is a morphism of DG-$\mathsf{k}$-modules, the equation 1.25 that $gf$ and 1 are homotopic, and the equation 1.28 that $fg$ and 1 are homotopic. Then $g$ is an homotopy inverse of $f$. □

**Definition 1.2.21.** Let $M$ be a DGA-$\mathsf{k}$-module.

1. $M$ is said to be acyclic if $\epsilon : M \to \mathsf{k}$ induces an isomorphism in homology.

2. $M$ is said to be contractible is $\epsilon : M \to \mathsf{k}$ is a homotopy equivalence.

**Proposition 1.2.22.** *Let $M$, $N$ DGA-$\mathsf{k}$-modules.*

*1. If $M$ is acyclic, projective and bounded below, then $M$ is contractible.*

*2. If $M$ and $N$ are contractible, then $M \otimes N$ is also contractible.*

□

➪ *Remark* 1.2.23. If $h$ is a contraction for $M$ and $k$ a contraction for $N$, then $h \otimes 1 + \eta\epsilon \otimes k$ is a contraction for $M \otimes N$.

**Definition 1.2.24.** Let $M$, $N$ be DGA-$\mathsf{k}$-modules. We denote by $T$ the isomorphism from $M \otimes N$ to $N \otimes M$ given by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$.

## 1.3 DGA-algebras and DGA-Coalgebras

**Definition 1.3.1.** Let $A$ and $C$ be DGA-$\mathsf{k}$-modules.

1. $A$ is said to be a DGA-$\mathsf{k}$-algebra if it is equipped with a morphism of DGA-$\mathsf{k}$-modules $\mu : A \otimes A \to A$, called the product, satisfying the following properties.

   (a) $\mu$ is associative: $\mu(1 \otimes \mu) = \mu(\mu \otimes 1)$.

   (b) The coaugmentation $\eta : \mathsf{k} \to A$ is a bilateral unit for $\mu$, that is $\mu(\eta \otimes 1) = \mu(1 \otimes \eta)$.

   If $\mu$ also satisfy $\mu T = \mu$, $A$ is called commutative.

2. $C$ is said to be a DGA-$\mathsf{k}$-coalgebra if it is equipped with a morphism of DGA-$\mathsf{k}$-modules $\Delta : C \to C \otimes C$, called coproduct, which satisfy the following properties.

   (a) $\Delta$ is associative: $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$.

   (b) The augmentation $\epsilon : C \to \mathsf{k}$ is a bilateral unit for $\Delta$, in other words, $(1 \otimes \epsilon)\Delta = (\epsilon \otimes 1)\Delta$.

   $C$ is said commutative if $\Delta$ also satisfies $T\Delta = \Delta$.

**Definition 1.3.2.** Let $A$, $A'$ be DGA-$\mathsf{k}$-algebras and $C$, $C'$ be DGA-$\mathsf{k}$-coalgebras.

1. A morphism $f : A \to A'$ of DGA-$\mathsf{k}$-algebras is a morphism of DGA-$\mathsf{k}$-modules which commutes with the products, that is $f\mu = \mu(f \otimes f)$. The category of DGA-$\mathsf{k}$-algebras is denoted DGA-$\mathsf{k}$-Alg.

2. A morphism $g : C \to C'$ of DGA-$\mathsf{k}$-coalgebras is a morphism of DGA-$\mathsf{k}$-modules which commutes with the coproducts, that is $\Delta f = (f \otimes f)\Delta$. The category of DGA-$\mathsf{k}$-coalgebras is denoted DGA-$\mathsf{k}$-Coalg.

**Proposition 1.3.3.** *Let $A$, $A'$ be DGA-$\mathsf{k}$-algebras and $C$, $C'$ DGA-$\mathsf{k}$-coalgebras.*

1. *The tensor product $A \otimes A'$ is a DGA-$\mathsf{k}$-algebra if we define the product by $\mu = (\mu_A \otimes \mu_{A'})(1 \otimes T \otimes 1)$.*

$$(A \otimes A') \otimes (A \otimes A') \xrightarrow{1 \otimes T \otimes 1} A \otimes A \otimes A' \otimes A' \xrightarrow{\mu_A \otimes \mu_{A'}} A \otimes A' \tag{1.29}$$

2. *The tensor product $C \otimes C'$ is a DGA-$\mathsf{k}$-coalgebra if we define the coproduct by $\Delta = (1 \otimes T \otimes 1)(\Delta \otimes \Delta)$.*

$$C \otimes C' \xrightarrow{\Delta_C \otimes \Delta_{C'}} C \otimes C \otimes C' \otimes C' \xrightarrow{1 \otimes T \otimes 1} (C \otimes C') \otimes (C \otimes C). \tag{1.30}$$

□

**Definition 1.3.4.** Let $A$ be a DGA-$\mathsf{k}$-algebra and $C$ be a DGA-$\mathsf{k}$-coalgebra. Let $M$ and $N$ be DGA-$\mathsf{k}$-modules.

1. $M$ is said to be a left DGA-$A$-module, if it is equipped with a DGA-$\mathsf{k}$-morphism $\mu : A \otimes M \to M$ which satisfies the following conditions.
   (a) Associativity: $\mu(1 \otimes \mu) = \mu(\mu_A \otimes 1)$.
   (b) Unit: $\mu(\eta_A \otimes 1) = 1$.
2. $N$ is said to be a left DGA-$C$-comodule, is it is equipped with a DGA-$\mathsf{k}$-morphism $\Delta : N \to C \otimes N$, which satisfy the following conditions.
   (a) Associativity: $(\Delta_C \otimes 1)\Delta = (1 \otimes \Delta)\Delta$.
   (b) Unit: $(\epsilon_C \otimes 1)\Delta = 1$.

Analogously, we define a right DGA-$A$-module and a right DGA-$C$-comodule.

## 1.4 Monads

In this sections we discuss some of the categorical concepts used in the chapter on operads and in the chapter on $\mathcal{L}$-algebras.

**Definition 1.4.1.** A category is said to be a monoidal category if it is equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $1 \in \mathcal{C}$, a natural transformation $\alpha$ from $\otimes(\mathrm{Id} \times \otimes)$ to $\otimes(\otimes \times \mathrm{Id})$, and natural transformations $\lambda : \otimes(1 \times -) \to \mathrm{Id}$ and $\rho : \otimes(- \times 1) \to \mathrm{Id}$, satisfying the following properties.

1. For all objects $X, Y, Z$ in $\mathcal{C}$ the morphisms

$$X \otimes (Y \otimes Z) \xrightarrow{\alpha_{X,Y,Z}} (X \otimes Y) \otimes Z \tag{1.31}$$

are isomorphisms.

2. For all objects $X$ in $\mathcal{C}$ the morphisms,

$$1 \otimes X \xrightarrow{\lambda_X} X \xleftarrow{\rho_X} X \otimes 1 \tag{1.32}$$

are isomorphisms.

3. For all objects $X, Y$ in $\mathcal{C}$ the following diagram is commutative.

$$\begin{tikzcd} X \otimes (1 \otimes Y) \arrow[rr, "\alpha_{X,1,Y}"] \arrow[dr, bend right, "\mathrm{Id}\otimes\lambda_Y"'] & & (X \otimes 1) \otimes Y \arrow[dl, bend left, "\rho_X\otimes\mathrm{Id}"] \\ & X \otimes Y & \end{tikzcd} \tag{1.33}$$

4. For all objects $W, X, Y, Z$ in $\mathcal{C}$ the following diagram is commutative.

$$\begin{tikzcd} & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W\otimes X,Y,Z}"] & \\ W \otimes (X \otimes (Y \otimes Z)) \arrow[ur, "\alpha_{W,X,Y\otimes Y}"] \arrow[d, "\mathrm{Id}\otimes\alpha_{X,Y,Z}"'] & & ((W \otimes X) \otimes Y) \otimes Z \\ W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[u, "\alpha_{X,Y,Z}\otimes\mathrm{Id}"'] \end{tikzcd} \tag{1.34}$$

**Definition 1.4.2.** Let $\mathcal{C}$ be a monoidal category. We call $\mathcal{C}$ symmetric monoidal category if it is equipped with a natural transformation $s : \otimes \to \otimes \circ t$ † which satisfies the following properties.

1. For every objects $X, Y$ in $\mathcal{C}$ the morphism

$$s_{X,Y} : X \otimes Y \to Y \otimes X \tag{1.35}$$

is an isomorphism.

2. The following diagram is commutative for every object $X, Y$ in $\mathcal{C}$.

$$\begin{tikzcd} X \otimes Y \arrow[rr, "="] \arrow[dr, "s_{X,Y}"'] & & X \otimes Y \\ & Y \otimes X \arrow[ur, "s_{Y,X}"'] & \end{tikzcd} \tag{1.36}$$

† Here $t$ is the canonical twisting morphism $(X, Y) \to (Y, X)$.

3. For all objects $X, Y, X$ in $\mathcal{C}$ the following diagram is commutative.

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha_{X,Y,X}"] \arrow[d, "\mathrm{Id} \otimes S_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "S_{X \otimes Y, Z}"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha_{Z,X,Y}"] \\
X \otimes (Z \otimes Y) \arrow[r, "\alpha_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[r, "S_{X,Z} \otimes \mathrm{Id}"'] & (Z \otimes X) \otimes Y
\end{tikzcd} \tag{1.37}$$

**Definition 1.4.3.** Let $\mathcal{C}$ be a monoidal category. A monoid is an object $M$ of $\mathcal{C}$ together with morphisms $\mu : M \otimes M \to M$ and $\eta : 1 \to M$ making the following diagrams commutative.

$$\begin{tikzcd}
 & (M \otimes M)M \arrow[r, "\mu \otimes 1"] & M \otimes M \arrow[dd, "\mu"] \\
M \otimes (M \otimes M) \arrow[ur, "\alpha_{M,M,M}"] \arrow[d, "1 \otimes \mu"'] & & \\
M \otimes M \arrow[rr, "\mu"'] & & M
\end{tikzcd} \tag{1.38}$$

$$\begin{tikzcd}
1 \otimes M \arrow[r, "\eta \otimes \mathrm{Id}"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "\mu"] & M \otimes 1 \arrow[l, "\mathrm{Id} \otimes \eta"'] \arrow[dl, "\rho_M"] \\
 & M &
\end{tikzcd} \tag{1.39}$$

**Definition 1.4.4.** A monoidal category $\mathcal{C}$ is said to be an strict monoidal category if the natural transformations $\alpha$, $\lambda$, $\rho$ in definition 1.4.1 are identities. In other words, is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and an object $1 \in \mathcal{C}$ such that :

1. For all objects $x, y, z \in \mathcal{C}$ we have $x \otimes (y \otimes z) = (x \otimes y) \otimes z$.

2. For every object $x \in \mathcal{C}$, $1 \otimes x = x = x \otimes 1$.

➪ *Remark* 1.4.5. Let $End(\mathcal{C})$ be the category of endofunctors of $\mathcal{C}$, that is, the objects are functors from $\mathcal{C}$ to $\mathcal{C}$, and the morphism are the natural transformations between them. This category with the composition of functors and the identity functor of $\mathcal{C}$, forms an strict monoidal category.

**Definition 1.4.6.** Let $\mathcal{C}$ be a strict monoidal category. A monoid is an object $M$ of $\mathcal{C}$ together with morphisms $\mu : M \otimes M \to M$ and $\eta : 1 \to M$, which satisfy $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$ and $\mu(1 \otimes \eta) = \mu(\eta \otimes 1) = 1$.

**Definition 1.4.7.** Let $\mathcal{C}$ be a category. A monad in $\mathcal{C}$ is a triplet $(T, \mu, \eta)$ where $T : \mathcal{C} \to \mathcal{C}$ is a functor, and $\mu : T \circ T \to T$, $\eta : \mathrm{Id} \to T$ are natural transformations which make commutative the following diagrams.

$$
\begin{tikzcd}
T \circ T \circ T \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T \circ T \arrow[d, "\mu"] \\
T \circ T \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "\mathrm{Id}"'] & T \circ T \arrow[d, "\mu"] & T \arrow[l, "\eta_T"'] \arrow[dl, "\mathrm{Id}"] \\
& T &
\end{tikzcd}
\tag{1.40}
$$

➪ *Remark* 1.4.8. Recall that the natural transformation $\mu_T : T \circ T \circ T \to T \circ T$ associates to each object $X \in \mathcal{C}$ the arrow that $\mu$ associates to $T(X)$. $T\mu : T \circ T \circ T \to T \circ T$, associates to each object $X \in C$, the arrow image by $T$ of the arrow that $\mu$ associates to $X$. $\eta T : T \to T \circ T$, associates to each object $X \in C$, the arrow that $\eta$ associates to $T(X)$, and that $T\eta : T \to T \circ T$, associates to each object $X \in \mathcal{C}$, the image by $T$ of the arrow that $\eta$ associates to $X$.

➪ *Remark* 1.4.9. In the category $End(\mathcal{C})$ a monad is a monoid in $End(\mathcal{C})$.

**Definition 1.4.10.** Let $\mathcal{C}$ be a category. A morphism of monads in $\mathcal{C}$, $F : (T, \mu, \eta) \to (T', \mu', \eta')$ is a natural transformation $F : T \to T'$ which makes the following diagram commutative.

$$
\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, "\mathrm{Id}_1"'] & T \arrow[d, "T"] & T \circ T \arrow[l, "\mu"'] \arrow[d, "F^2"] \\
1 \arrow[r, "\eta'"'] & T' & T' \circ T' \arrow[l, "\mu'"]
\end{tikzcd}
\tag{1.41}
$$

The category of monads in $\mathcal{C}$ is denoted by $Monad_{\mathcal{C}}$.

➪ *Remark* 1.4.11. Recall that the natural transformation $F^2 : T \circ T \to T' \circ T'$, associates to each object $X \in \mathcal{C}$ the diagonal of the following commutative diagram.

$$
\begin{tikzcd}
T(T(X)) \arrow[r, "F_{T(X)}"] \arrow[d, "T(F_X)"'] \arrow[dr, dashed, "F^2_X"] & T'(T(X)) \arrow[d, "T(F_{X'})"] \\
T(T'(X)) \arrow[r, "F_{T'(X)}"'] & T'(T'(X))
\end{tikzcd}
\tag{1.42}
$$

➪ *Remark* 1.4.12. For every category $\mathcal{C}$, the monad given by the triplet $(\mathrm{Id}, \mathrm{Id}_{\mathrm{Id}}, \mathrm{Id}_{\mathrm{Id}})$ is called the identity monad and is an initial object of $Monad_{\mathcal{C}}$.

**Definition 1.4.13.** Let $(T, \mu, \eta)$ be a monad in a category $\mathcal{C}$. An algebra on $T$, or $T$-algebra, is a pair $(X, h)$ where $X$ is an object of $\mathcal{C}$ and $h$ is a morphism from $T(X)$ to $X$, which make the commutative the following diagrams.

$$
\begin{tikzcd}
(T \circ T)(X) \arrow[r, "\mu_X"] \arrow[d, "T(h)"'] & T(X) \arrow[d, "h"] \\
T(X) \arrow[r, "h"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{Id}_X"'] & T(X) \arrow[d, "h"] \\
& X
\end{tikzcd}
\tag{1.43}
$$

**Definition 1.4.14.** Let $(T, \mu, \eta)$ be a monad on a category $\mathcal{C}$. A morphism of $T$-algebras from $(X, h)$ to $(X', h')$ is a morphism in $\mathcal{C}$, $f : X \to X'$ such that the following diagram commute.

$$\begin{tikzcd} T(X) \arrow[r, "h"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\ T(X') \arrow[r, "h'"'] & X' \end{tikzcd} \tag{1.44}$$

The category of $T$-algebras is denoted by $T$-Alg.

**Definition 1.4.15.** Let $(T, \mu, \eta)$ be a monad on the category $\mathcal{C}$. For every object $X \in \mathcal{C}$, the pair $(T(X), \mu_X)$ is a $T$-algebra called the free $T$-algebra on $X$.

➪ *Remark* 1.4.16. The free $T$-algebra on $X$ satisfies the following universal property: for every $T$-algebra on $X$, $(X, h)$ there exists an unique morphism of $T$-algebras from $(X, h)$ to $(T(X), \mu_X)$. Indeed, the morphism is $h$.

## 1.5 Adjunctions

**Definition 1.5.1.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. Then, $F$ is said to be left adjoint of $G$, denoted by $F \vdash G$, if there exists $\theta$ bijection,

$$\mathcal{D}(F(X), Y) \xrightarrow{\theta} \mathcal{C}(X, G(Y)) \tag{1.45}$$

natural in $X$ and $Y$.

**Definition 1.5.2.** Let $F \vdash G : \mathcal{C} \to \mathcal{D}$ be a pair of adjoint functors.

1. The natural transformation $\eta : 1_{\mathcal{C}} \to GF$ given by $\eta_X = \theta(1_{F(X)})$, is called the unit of the adjunction.

2. The natural transformation $\epsilon : FG \to 1_{\mathcal{D}}$ given by $\epsilon_Y = \theta^{-1}(1_{G(Y)})$, is called the counit of the adjunction.

**Proposition 1.5.3.** *Let $F \vdash G : \mathcal{C} \to \mathcal{D}$ be a pair of adjoint functors, with unit $\eta$ and counit $\epsilon$. Then the following diagrams are commutative.*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} GFG \arrow[d, "G\epsilon"'] & G \arrow[l, "\eta_G"'] \arrow[dl, "1_G"] \\ G & \end{tikzcd} \tag{1.46}$$

*The equations given by the commutativity of these diagrams,*

$$\epsilon_F \cdot F\eta = 1_F \tag{1.47}$$
$$G\epsilon \cdot \eta_G = 1_G \tag{1.48}$$

*are called triangular equations.*

*Proof.* Let $X$ be an object of $\mathcal{C}$. For the commutativity of first diagram we have to show that $1_{F(X)} = \epsilon_{F(X)}F(\eta_X)$. By definition $\eta_X = \theta(1_{F(X)})$, then by using the fact that $\theta$ is a bijection, we only have to check that $\theta(\epsilon_{F(X)}F(\eta_X)) = \eta_X$.

$$\begin{aligned}\theta(\epsilon_{F(X)}F(\eta_X)) &= \theta(\epsilon_{F(X)})\theta(F(\eta_X)) \\ &= \theta(\epsilon_{F(X)})\eta_X && \text{(by naturality of } \theta) \\ &= \theta(\theta^{-1}(1_{GF(X)}))\eta_X \\ &= \eta_X\end{aligned}$$

The commutativity of the other diagram is similar. □

**Definition 1.5.4.** Let $G : \mathcal{D} \to \mathcal{C}$ be a covariant functor and $X$ an object of $\mathcal{C}$. An universal arrow from $X$ to $G$ is an morphism of the form $\psi : X \to GF(X)$ such that, for every morphism $f : X \to G(Y)$ there is an unique morphism of $\mathcal{D}$, denoted $\theta^{-1}(f)$, from $F(X)$ to $Y$, which satisfies $G(\theta^{-1}(f))\eta = f$.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & GF(X) \arrow[d, "G(\theta^{-1}(f))"] \\ & G(Y) \end{tikzcd} \tag{1.49}$$

**Theorem 1.5.5.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor such that for every object $X \in \mathcal{C}$ there exists an universal arrow $\eta_X : X \to G(F(X))$. Then the application $F$ between the objects of $\mathcal{C}$ and $\mathcal{D}$ extends uniquely to a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F \vdash G$.*

□

➪ *Remark* 1.5.6. For a proof of theorem 1.5.5 see [Mac98], §4 Theorem 2. Nevertheless, the extension of $F$ on arrows is made in the following way: let $f : X \to Y$ morphism in $\mathcal{C}$ and consider the following diagram.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & G(F(X)) \arrow[d, dashed, "G(F(f))"] \\ Y \arrow[r, "\eta_Y"] & G(F(Y)) \end{tikzcd} \tag{1.50}$$

The existence and unicity of $F(f)$ making the diagram commutative, is guaranteed by the universal property of $\eta_X$.

**Theorem 1.5.7.** *Every left adjoint preserves colimits and every right adjoint preserves limits.*

□

➪ *Remark* 1.5.8. For a proof see §5 of [Mac98], or §9 of [Awo06].

## 1.6 Reflexive Coequalizers

In a category $\mathcal{C}$ coequalizer of two morphisms $f, g : X \to Y$ is the colimit on the diagram formed by them. We will denote this coequalizer $Ceq(f, g)$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] & Ceq(f,g) \end{tikzcd} \tag{1.51}$$

Equivalently the coequalizer of $f$ and $g$ is an initial object in the category of morphisms $l$ left equalizing $f$ and $g$, that is $l \circ f = l \circ g$. We are interested in a special kind of coequalizer, which are called reflexive coequalizers. In the next chapter, we will see that reflexive coequalizer play an important role in the proof of existence of small colimits in the category of operads.

**Definition 1.6.1.** Let $\mathcal{D}_0$ denote the category generated by the diagram,

$$\begin{tikzcd} x_0 \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & x_1 \arrow[l, bend right, "s"'] \end{tikzcd} \tag{1.52}$$

where the arrows satisfy $is = 1 = js$. For every category $\mathcal{C}$, we call reflexive pair in $\mathcal{C}$, any diagram in $\mathcal{C}$ on the category $\mathcal{D}_0$. In other words, a reflexive pair is a pair of parallel arrows having a common section.

**Proposition 1.6.2.** *Let $f, g : X \to Y$ be two morphisms in a category $\mathcal{C}$. If there is a morphism $s : Y \to X$ in $\mathcal{C}$ such that $f \circ s = g \circ s = 1_Y$ then the coequalizer of $f$ and $g$ (if exists) is isomorphic to the colimit on the diagram formed by $f$, $g$ and $s$.*

*Proof.* It is well know that $q$ in diagram 1.51 is an epimorphism. Let $(B, \alpha : X \to B, \beta : Y \to B)$ be the colimit on the diagram formed by $f$, $g$ and $s$. Then $\alpha$ and $\beta$ satisfy $\beta f = \alpha = \beta g$ and $\alpha s = \beta$. We also have that $\alpha$ is an epimorphism. Indeed, if $r, s : B \to Z$ are two arrows such that $r\alpha = s\alpha$ then $(Z, r\alpha f : X \to Z, r\alpha : Y \to Z)$ is a cocone on $f$, $g$ and $s$ (because $fs = 1$), which implies by the universal property of colimits that $r\alpha$ is uniquely factorized by $r$ through $\alpha$. The same applies for $s\alpha$, but $r\alpha = s\alpha$ then $r = s$ and $\alpha$ is an epimorphism. To show that $B$ and $Ceq(f, g)$ are isomorphic, first note that $\alpha$ left equalize $f$ and $g$. Indeed $\alpha f = \beta = \alpha g$. Then it exists an unique arrow $h : Ceq(f, g) \to B$ such that $hq = \alpha$. Now $(Ceq(f, g), qf : X \to Ceq(f, g), q : Y \to Ceq(f, g))$ is a cocone on $f$, $g$ and $s$, because $qfs = q1 = q$. Then it exists an unique arrow $\overline{h} : B \to Ceq(f, g)$ such that $\overline{h}\alpha = q$ and $\overline{h}\beta = fq$. But $q$ and $\alpha$ are epimorphisms, so we have that $h\overline{h}\alpha = hq = \alpha$ implies $h\overline{h} = 1$ and that $\overline{h}hq = \overline{h}\alpha = q$ implies $\overline{h}h = 1$. Then $B$ and $Ceq(f, g)$ are isomorphic. $\square$

➪ *Remark* 1.6.3. Proposition 1.6.2 says that the morphism $s$ does not change the coequalizer.

**Definition 1.6.4.** Let $f, g : X \to Y$ be two morphism in a category $\mathcal{C}$. If there is a morphism $s : Y \to X$ in $\mathcal{C}$ such that $f \circ s = g \circ s = 1_Y$ then $Ceq(f, g)$ is called the reflexive coequalizer of $f$ and $g$.

**Definition 1.6.5.** Let $F : C \to D$ be a covariant functor. $F$ is said to be final if satisfies the following conditions for every object $X \in \mathcal{D}$.

1. There is a morphism from $X$ to an object of the form $F(Y)$.
2. For every pair of such morphisms from $X$, $\alpha : X \to F(Y)$ and $\alpha' : X \to F(Y')$, there exists a finite sequence $g_1, \ldots, g_k$ of morphisms of $\mathcal{C}$ making the following diagram commutative.

$$\begin{tikzcd} & & & X \arrow[llld, "\alpha"'] \arrow[lld] \arrow[rd] \arrow[rrd, "\alpha'"] & & \\ F(Y) \arrow[r, "F(g_1)"'] & F(Y_1) & \cdots \arrow[l, "F(g_2)"'] \arrow[r, "F(g_{k-1})"'] & & F(Y_{k-1}) & F(Y') \arrow[l, "F(g_k)"'] \end{tikzcd} \tag{1.53}$$

➪ *Remark* 1.6.6. Another way to define a functor $F : \mathcal{C} \to \mathcal{D}$ as final is saying that for every $X \in \mathcal{D}$ the comma category $X/F$ is not empty and connected (See [Mac98], §9).

**Proposition 1.6.7.** *Consider the category $\mathcal{D}_0$ from definition 1.6.1. Then, for every $n \geqslant 1$, the diagonal functor from $\mathcal{D}_0$ to the product category $\mathcal{D}_0^n$, is final.*

*Proof.* Let $D : \mathcal{D}_0 \to \mathcal{D}_0^n$ be the diagonal functor. Let $X$ be an object of $\mathcal{D}_0^n$, then it has the form $X = (x_{i_1}, \dots, x_{i_n})$, with $i_j \in 0, 1$. There is a morphism $f$ from $X$ to $D(x_1)$ given by $f = (f_{i_1}, \dots, f_{i_n})$, where $f_{i_j} = 1_{x_1}$ if $i_j = 1$ and $f_{i_j} = f$ if $i_j = 0$. Note that we can still have a morphisms from $X$ to $D(x_1)$, by taking arbitrarily $f$ of $g$ in the entries $f_{i_j}$ when $i_j = 0$. But the only morphism from $X$ to $D(x_0)$ is given by $s = (s_{i_1}, \dots, s_{i_n})$, where $s_{i_j} = 1_{x_0}$ if $i_j = 0$ and $s_{i_j} = s$ if $i_j = 1$. Now we check the second condition in definition 1.6.5. Let $\alpha, \alpha'$ two morphisms from $X$ to $D(x_1)$. Let $\beta = (b_{i_1}, b_{i_k})$ be the morphism from $X$ to $D(x_0)$ defined by $b_{i_j} = s$ if $i_j = 1$ and $b_{i_j} = 1_{x_0}$ if $i_j = 0$. Then we have the following commutative diagram.

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[d, "\beta"] \arrow[dr, "\alpha'"] & \\ D(x_1) \arrow[r, "D(s)"'] & D(x_0) & D(x_1) \arrow[l, "D(s)"] \end{tikzcd} \tag{1.54}$$

This suffices to show that $D$ is a final functor. □

Final functors are useful for computing colimits, as the following proposition shows. For a proof, we refer to [Mac98], §9.

**Proposition 1.6.8.** *Let $F : \mathcal{D} \to \mathcal{C}$ be a diagram on the category $\mathcal{C}$ and $I : \mathcal{D}' \to \mathcal{D}$ be a final functor such that the colimit of $F \circ I$ exists. Then the colimit of $F$ exists and is canonically isomorphic to the colimit of $F \circ I$.*

□

## 1.7 Simplicial Sets

$\mathcal{L}$-algebras are defined in the ambient category of differential graded modules or chain complexes. They are intended to be models for homotopy types. In this thesis we use simplicial sets instead of topological spaces. Simplicial sets can be seen as a combinatorial version of topological spaces. We will restrict our attention to the category of simplicial sets and employ the word *space* to refer to them.

**Definition 1.7.1.** Let $\Delta$ be category where the objects are the totally ordered sets $[n] = \{0, \dots, n\}$, with $n$ non negative integer, and the morphisms are the nondecreasing applications between them. $\Delta$ is called the simplicial category.

**Definition 1.7.2.** In the category $\Delta$ we identify two important families of morphism.

1. For every $n \geqslant 1$, $0 \leqslant i \leqslant n$, $\delta_i : [n-1] \to [n]$, is the application defined by $\delta_i(j) = j$ if $i < j$ and $\delta_i(j) = j + 1$ is $j \leqslant i$. They are called face applications.

2. For every $n \geqslant 0$, $0 \leqslant i \leqslant n$, $\sigma_i : [n+1] \to [n]$, is the application defined by $\sigma_i(j) = j$ if $j \leqslant i$ and $\sigma_i(j) = j-1$ if $j > i$. They are called degeneracy applications.

➪ *Remark* 1.7.3. The face and degeneracy applications can be used to describe the morphisms in $\Delta$, in the sens that every non identity morphism have an unique factorization as composite of face and degeneracies (see [May93], §1).

**Definition 1.7.4.** Let $\mathcal{C}$ be any category. A simplicial object $X$ in $\mathcal{C}$ is a contravariant functor $X : \Delta \to \mathcal{C}$, and the morphisms of simplicial objects are the natural transformations of functors. The category of simplicial objects of $\mathcal{C}$ is written $s\mathcal{C}$. In particular, if $\mathcal{C}$ is the category of sets **Set**, the category of simplicial sets is denoted **sSet**.

**Definition 1.7.5.** The ensemble simplicial $\Delta_n$ is the simplicial set defined by the contravariant functor given by the diagram,

$$\begin{tikzcd} {[p]} \arrow[r, mapsto] & \Delta([p],[n]) = \Delta_n(p) \arrow[d, "g^*"] \\ {[q]} \arrow[u, "g"] \arrow[r, mapsto] & \Delta([q],[n]) = \Delta_n(q) \end{tikzcd} \tag{1.55}$$

The element $e_n \in \Delta_n$ given by the identity application of $[n]$ is called the universal $n$-simplex.

➪ *Remark* 1.7.6. $\Delta_n$ is just the representable functor from $\Delta^{op}$ to **Set** determined by $[n] \in \Delta$.

➪ *Remark* 1.7.7. Every face application $\delta_i : [n] \to [n+1]$, $1 \leqslant i \leqslant n+1$, induce a simplicial morphism $\delta_{i*} : \Delta_n \to \Delta_{n+1}$, given by $f \mapsto \delta_i f$. They are called face morphisms.

The following proposition is a particular case of the Yoneda lemma (see [Pro10], $S$2.3.2)

**Proposition 1.7.8** (Yoneda lemma)**.** *For every simplicial set $X$ and $n$ non negative integer, the application $f \mapsto f(e_n)$ is a bijection from **sSet**$(\Delta_n, X)$ to $X_n$, natural in $n$.*

□

➪ *Remark* 1.7.9. This bijection implies that $X$ is canonically isomorphic to the simplicial set given by the functor defined on objects as $[n] \mapsto \mathbf{sSet}(\Delta_n, X)$ and on morphisms by $g \mapsto (h \mapsto hg_*)$. Then the face applications of $X$, $\partial_i : X_{p+1} \to X_p$ correspond to the face applications from $\mathbf{sSet}(\Delta_{p+1}, X)$ to $\mathbf{sSet}(\Delta_p, X)$, which are given by $g \mapsto g \circ \delta_{i*}$.

➪ *Remark* 1.7.10. To each simplicial set $X$ is associated the simplicial $\mathsf{k}$-module $S_*(X)$ which $n$ component is the free $\mathsf{k}$-module on $X_n$, and the face and degeneracy applications are the linear extensions of the corresponding application of $X$. With it, is formed a chain complex or DGA-$\mathsf{k}$-module by taking as differential the alternated sum of faces $\partial = \sum_k (-1)^k \partial_k$. This chain complex is denoted $C_*(X)$.

## 1.8 Acyclic Models Theorem

Prouté in [Pro83] introduced a unicity criterion complementing the classical acyclic models theorem. This gives a simple characterization of the Eilenberg-Mac Lane transformation, which is discussed in section 1.9. We include here the version of the acyclic models theorem containing the mentioned unicity criterion, that was presented by Prouté in 2012 in an algebraic topology course at the university Paris-Diderot.

**Definition 1.8.1.** Let $\mathcal{C}$ be a category, let $\mathcal{M}$ a small subcategory of $\mathcal{C}$ and a covariant functor $F : \mathcal{C} \to$ DGA-$\mathsf{k}$-Mod. Let $e = \{e_M\}_{M\in\mathcal{M}}$ be a family indexed by the objects of $\mathcal{M}$ such that for every object $M \in \mathcal{M}$, $e_M$ is an homogeneous element of $F(M)$. We say that $F$ is free on $(\mathcal{M}, e)$ if the following conditions are satisfied.

1. For every object $X \in \mathcal{C}$ the associated DGA-$\mathsf{k}$-module $F(X)$ admits as a base the family,
$$\{F(f)(e_M)\}_{\substack{M\in\mathcal{M}\\ f\in\mathcal{C}(M,X)}} \tag{1.56}$$
2. For every object $M \in \mathcal{M}$, the submodule of $F(M)$ generated by the family
$$\{F(f)(e_N)\}_{\substack{N\in\mathcal{M}\\ f\in\mathcal{M}(N,M)}} \tag{1.57}$$
is a DGA-$\mathsf{k}$-submodule of $F(M)$.

➪ *Remark* 1.8.2. An example of free functor on a subcategory is the normalized chain complex functor $C_* :$ **sSet** $\to$ DGA-$\mathsf{k}$-Mod. The subcategory $\mathcal{M}$ of **sSet** can be any category whose objects are the $n$-simplexes $\Delta_n$, $n \geqslant 0$, and the set of morphisms contains at least the face operations $\delta_{i*} : \Delta_n \to \Delta_{n+1}$ (see remark 1.7.7). The element $e_{\Delta_n}$ is taken to be the universal $n$-simplex $e_n$ of $\Delta_n$.

For every simplicial set $X$ and $n \geqslant 0$, $C_n(X)$ is by definition a free module on non degenerated element of $X_n$, then by remark 1.7.9 it is generated by the set $\{f_*(e_n)\}_{f\in\mathbf{sSet}(\Delta_n,X)}$, with $f_*(e_n)$ non degenerated $^\dagger$. Moreover, for every non degenerated element $C_p(\Delta_n)$ of the form $f_*(e_p)$, with $f : \Delta_p \to \Delta_n$ in $\mathcal{M}$, we have $\partial(f_*(e_p)) = f_*(\partial(e_p))$. But, as we saw in remark 1.7.9, the expression $f_*\partial$ is a linear combination of applications of the form $(f \circ \delta_i)_*$, where $\delta_i : \Delta_{i-1} \to \Delta_i$ are the face operations. Then, the composites $f \circ \delta_i$ are in $\mathcal{M}$ and we can conclude that the submodule generated by the non degenerated elements of the form $f(e_p)$, with $f \in \mathcal{M}$, is stable by $\partial$. Finally, it is stable by the augmentation $\epsilon$, because $\epsilon(e_n) = 1$.

**Theorem 1.8.3.** *[Acyclic Models Theorem] Let $\mathcal{C}$ be a category. Consider $F$ and $G$ covariant functors from $\mathcal{C}$ to DGA-$\mathsf{k}$-Mod. Let $\mathcal{M}$ be a small subcategory of $\mathcal{C}$. Suppose that,*

1. *$F$ is free on $(\mathcal{M}, e)$.*
2. *There is a functor $A : \mathcal{M} \to$ DGA-$\mathsf{k}$-Mod, such that $A(M)$ is an acyclic DGA-$\mathsf{k}$-submodule of $G(M)$, natural in $M$, that is, for every morphism $f : M \to N$ in $\mathcal{M}$, $G(f) : G(M) \to G(N)$ sends $A(M)$ to $A(N)$.*

$^\dagger$We use $f_*$ to denote $C_n(f)$

*Under these hypothesis we have the following.*

1. *There exists a natural transformation $\varphi : F \to G$ such that $\varphi(e_M) \in A(M)$ for every object $M \in \mathcal{M}$.*

2. *Between two such natural transformations, there exists an natural homotopy $h$ such that $h(e_M) \in A(M)$ for every object $M \in \mathcal{M}$.*

3. *If for every object $M \in \mathcal{M}$ we have that $A_{|e_M|+1}(M) = 0$, then the natural transformation $\varphi : F \to G$ given by the first point, is unique.*

*Proof.* For every integer $i$ let $\mathcal{M}_i$ be the set of objects $M \in \mathcal{M}$ such that $|e_M| = i$. Set $F_{-1}(X) = G_{-1}(X) = A_{-1}(X) = \mathsf{k}$, and rename $\partial$ the augmentation $\epsilon$. Then we take $\varphi_X : F_{-1}(X) \to G_{-1}(X)$ as the identity of $\mathsf{k}$. Now, suppose that we have construct $\varphi$ for all $0 \leqslant j < i$, that is, for all $j < i$ we have the following.

1. $\varphi_X : F_j(X) \to G_j(X)$ is defined, linear and natural on $X$.

2. $\varphi_X \partial = \partial \varphi_X : F_j(X) \to G_{j-1}(X)$.

3. $\varphi_N(e_N) \in A_j(N)$ for every object $N \in \mathcal{M}_j$.

Then for every object $M \in \mathcal{M}_i$, $\varphi_M(\partial(e_M)) \in A_{i-1}(M)$. Indeed, $e_M = F(1_M)(e_M)$, which means it belong to the submodule of $F(M)$ generated by $\{F(f)(e_{N_i})\}_{\substack{N_i \in \mathcal{M} \\ \mathcal{M}(N_i, M)}}$. This submodule is stable by $\partial$, then $\partial(e_M)$ can be written as finite sum,

$$\partial(e_M) = \sum_k \lambda_k F(f_k)(e_{N_k}) \tag{1.58}$$

with $N_k \in \mathcal{M}_{i-1}$ and $f_k \in \mathcal{M}(N_k, M)$, and not only $f_k \in \mathcal{C}(N_k, M)$. Then we have,

$$\varphi_M(\partial(e_M)) = \sum_k \lambda_k \varphi_M(F(f_k)(e_{N_k})) = \sum_k \lambda_k G(f_k)(\varphi_{N_k}(e_{N_k})) \tag{1.59}$$

by using the naturality of $\varphi$ with $f_k : N_k \to M$. But $\varphi_{N_k}(e_{N_k}) \in A_{i-1}(N_k)$ and because $f_k$ is a morphism of $\mathcal{M}$ together with the naturality condition for the inclusions of the form $A(M) \subset G(M)$ $M \in \mathcal{M}$, $G(f_k)(\varphi_{N_k}(e_{N_k})) \in A_{i-1}(M)$.

By linearity we only need to define $\varphi_X(F(f)(e_M))$ for every object $M \in \mathcal{M}_i$ and $f \in \mathcal{C}(M, X)$. The morphism $\varphi$ must be natural, so we need to have the following commutative diagram.

$$\begin{tikzcd} F_i(M) \arrow[r, "\varphi_M"] \arrow[d, "F(f)"'] & G_i(M) \arrow[d, "G(f)"] \\ F_i(X) \arrow[r, "\varphi_X"'] & G_i(X) \end{tikzcd} \tag{1.60}$$

Then $\varphi_X(F(f)(e_M))$ must be $G(f)(\varphi_M(e_M))$ and we only need to define $\varphi_M(e_M)$. By the fact that $\varphi_M(\partial(e_M)) \in A_{i-1}(M)$, $\partial(\varphi_M(\partial(e_M))) = \varphi_M(\partial(\partial(e_M))) = 0$ and by the exactness of $A(M)$, there exists in $a \in A_i(M)$ such that $\partial(a) = \varphi_M(\partial(e_M))$. And we take $\varphi_M(e_M) = a$. This defines $\varphi$ in degree $i$.

The naturality of $\varphi : F_i \to G_i$ follows from the next computation. Take $g : X \to Y$ any morphism in $\mathcal{C}$.

$$\begin{aligned} G(g)(\varphi_X(F(f)(e_M))) &= G(g)(G(f)(\varphi_M(e_M))) && \text{(by definition of } \varphi_X) \\ &= G(gf)(\varphi_M(e_M)) \\ &= \varphi_Y(F(gf)(e_M)) && \text{(by definition of } \varphi_Y) \\ &= \varphi_Y(F(g)(F(f)(e_M))) \end{aligned}$$

Now we have to check that $\varphi_X \partial = \partial \varphi_X : F_i(X) \to G_{i-1}(X)$. Let $f : M \to X$ any morphism.

$$\begin{aligned} \partial\varphi_X(F(f)(e_M)) &= \partial G(f)(\varphi_M(e_M)) && \text{(by naturality of } \varphi) \\ &= G(f)\partial(\varphi_M(e_M)) && (G(f) \text{ DGA morphism}) \\ &= G(f)\varphi_M(\partial(e_M)) && \text{(by construction of } \varphi_M(e_M)) \\ &= \varphi_X(F(f)(\partial(e_M))) && \text{(by naturality of } \varphi) \\ &= \varphi_X(\partial(F(f)(e_M))) && (F(f) \text{ morphism of DGA-}\mathsf{k}\text{-modules}) \end{aligned}$$

We proceed with the second point the theorem. Let $\varphi$ and $\psi$ two natural transformations from $F$ to $G$ such that $\varphi_M(e_M)$ and $\phi_M(e_M)$ are in $A(M)$ for every object $M \in \mathcal{M}$. The natural homotopy $h$ from $\varphi$ to $\psi$ is constructed degree by degree. In dimension $-1$ we take $h = 0$. Suppose we have construct $h$ for every $j < i$. Then we have the following hypothesis.

1. $h_X : F_j(X) \to G_{j+1}(X)$ is defined, linear and natural in $X$.
2. $\partial h_X + h_X\partial = \varphi_X - \psi_X : F_j(X) \to G_j(X)$.
3. $h_N(e_N) \in A_{j+1}(N)$ for every object $N \in \mathcal{M}_j$.

For every object $M \in \mathcal{M}_i$, let $\alpha \in A_i(M)$ with $\alpha = h_M(\partial(e_M)) - \varphi_M(e_M) + \psi_M(e_M)$. We only need to define $h_M(e_M)$. But, by hypothesis we have $|\partial(e_M)| < i$ and

$$\partial h_M(\partial(e_M)) + h_M(\partial\partial(e_M)) = \varphi_M(\partial(e_M)) - \psi_M(\partial(e_M)) \tag{1.61}$$

that is, $\partial(\alpha) = 0$. Then, there exists an element $h_M(e_M)$ in $A_{i+1}(M)$ such that $\partial(h_M(e_M)) = \alpha$. Like the first point, it can be checked without difficulties that $h$ is natural and $\partial h + h\partial = \varphi - \psi$.

Finally, for the third point, let $\varphi$ and $\psi$ be natural transformations from $F$ to $G$, equal in degree $-1$. Suppose that they are equal in degree $j < i$ and let $M$ be an object of $\mathcal{M}$ such that $|e_M| = i$. By hypothesis we have that $\varphi_M(\partial(e_M)) = \psi_M(\partial(e_M))$ because $|\partial(e_M)| < i$. But $A_{i+1}(M) = 0$, the acyclicity of $A(M)$ implies that $\partial : A_i(M) \to A_{i-1}(M)$ is injective. But $\partial(\varphi_M(e_M)) = \varphi_M(\partial(e_M)) = \psi_M(\partial(e_M)) = \partial(\psi_M(e_M))$ and then, $\varphi_M(e_M) = \psi_M(e_M)$ for every object $M \in \mathcal{M}_i$ and then $\varphi = \psi$ in degree $i$. $\square$

## 1.9 The Eilenberg-Mac Lane Transformation

Introduced in [SE53],§5, in order to study the properties of spaces $K(\pi, n)$, the Eilenberg-Mac Lane transformation is one of the fundamental elements of the theory of $\mathcal{L}$-algebras. In this section we describe its principal properties. The unicity condition in theorem 1.8.3 is used to describe the Eilenberg-Mac Lane transformation in the context of simplicial sets, as states the following proposition.

Denote **Ordf** the category where the objects are the finite ordered sets and the arrows the increasing applications. An $n$-simplex is an increasing application $[n] \to X$. If this application is injective, we say that the $n$-simplex is non degenerated. The set of non degenerated simplexes of $X$ is a formal finite polyhedron with vertexes in $X$. In this way, every finite ordered set can be seen as a simplicial set $\overline{X}$ in the obvious way. Every $n$-simplex $\sigma : [n] \to X$ induce a morphism $\sigma : \overline{[n]} \to \overline{X}$. Observe that for every ordered sets $X$, $Y$, $\overline{X} \times \overline{Y} \cong \overline{X \times Y}$, where the order of $X \times Y$ is given by $(x, y) \leqslant (x', y')$ when $x \leqslant x'$ and $y \leqslant y'$.

For any simplex $x$ of $\overline{X}$, $\partial_i(x)$ is non degenerated if $x$ is non degenerated. Thus, the submodule $A_*(\overline{X})$ of $C_*(\overline{X})$ generated by the non degenerated simplexes of $\overline{X}$, is a DGA-$\mathsf{k}$-submodule of $C_*(\overline{X})$.

**Lemma 1.9.1.** *If the ordered set $X$ has a smallest element, $A_*(\overline{X})$ is acyclic.*

*Proof.* Denote by $a$ the smallest element of $X$. $A_*(\overline{X})$ contains the DG-$\mathsf{k}$-submodule $\Lambda[a]$ generated by the 0-simplex $\{a\}$. which is isomorphic to $\Lambda$ concentrated in degree 0. We have to show that $B = A_*(\overline{X})/\Lambda[a]$ is null homotopic. We can decompose $B$ as the direct sum $B = B' \oplus B''$, where $B'$ has as base the simplexes with vertex $a$ (and necessarily another vertex), and $B''$ has as base the simplexes where $a$ is not a vertex. Clearly $B''$ is stable by $\partial$, and the component $B' \to B''$ of $\partial$ is bijective, because it sends a simplex with vertex $a$ and having at least another vertex to the simplex obtained by removing $a$. So we have that $A_*(\overline{X})/\Lambda[a]$ is null homotopic because it is the cone of an isomorphism.

□

**Proposition 1.9.2.** *There exists a unique natural transformation from the functor $\otimes \circ (C_* \times C_*)$ to the functor $C_* \circ \times$.*

$$C_*(X) \otimes C_*(Y) \xrightarrow{\nabla} C_*(X \times Y) \tag{1.62}$$

*such that for every $p, q$ in $\mathbb{N}$, $\nabla(e_p \otimes e_q) \in A_*(\overline{[p]} \times \overline{[q]})$. It is called the Eilenberg-Mac Lane transformation.*

➪ *Remark* 1.9.3. In other words, the Eilenberg-Mac Lane transformation is the only natural transformation that sends the tensorial products of universal simplexes $e_p \otimes e_q$ to linear combinations of non degenerated simplexes.

*Proof.* The existence and unicity of the Eilenberg-Mac Lane transformation follows from the points 1 and 3 of the acyclic models theorem (34). Using the notation of the theorem 1.8.3, the category $\mathcal{C}$ is **sSet** × **sSet**, the subcategory $\mathcal{M}$ has as objects the pairs $(\overline{[p]}, \overline{[q]})$ and as arrows the couples $(f, g)$, where $f : \overline{[p]} \to \overline{[p']}$ and $g : \overline{[q]} \to \overline{[q']}$ are induced by the injective applications $[p] \to [p']$ and $[q] \to [q']$. The element $e_{(\overline{[p]}, \overline{[q]})}$ is $e_p \otimes e_q$ (it belongs to $C_*(\overline{[p]}) \otimes C_*(\overline{[q]})$). Clearly the functor

$(X, Y) \to C_*(X) \otimes C_*(Y)$ is free over $(\mathcal{M}, e)$, after noting that if $f : [p] \to [n]$ is an injective increasing application (that is, an arrow of $\mathcal{M}$), the faces of $f_*(e_p)$ are of the form $g_*(e_{p-1})$ for some arrow $g : [p-1] \to [n]$ of $\mathcal{M}$. In fact, $g$ has the form $f \circ \delta_i$.

Now we take $A(\overline{[p]}, \overline{[q]})$ as the DG-k-submodule $A_*(\overline{[p]} \times \overline{[q]})$ of $C_*(\overline{[p]} \times \overline{[q]})$, which is clearly natural in $(\overline{[p]}, \overline{[q]})$ (recall it is defined over $\mathcal{M}$ and not over **sSet** $\times$ **sSet**). We have $A_i(\overline{[p]}, \overline{[q]}) = 0$ for $i > p + q$ because in $[p] \times [q]$ there is no non degenerated simplex of strictly bigger dimension than $p + q$. Finally, $A_*(\overline{[p]}, \overline{[q]})$ is acyclic by the fact that $[0] \times [0]$ is the smallest element of $[p] \times [q]$. □

**Lemma 1.9.4.** *If $X$ and $Y$ are ordered sets, then when $x$ and $y$ are non degenerated simplexes of $\overline{X}$ and $\overline{Y}$ respectively, $\nabla(x \otimes y)$ belongs to $A_*(\overline{X} \times \overline{Y})$.*

*Proof.* By naturality of $\nabla$, we have commutative diagram,

$$\begin{tikzcd} C_*(\overline{[p]}) \otimes C_*(\overline{[q]}) \arrow[r, "\nabla"] \arrow[d, "x_* \otimes y_*"'] & C_*(\overline{[p]} \times \overline{[q]}) \arrow[d, "(x \times y)_*"] \\ C_*(\overline{X}) \otimes C_*(\overline{Y}) \arrow[r, "\nabla"'] & C_*(\overline{X} \times \overline{Y}) \end{tikzcd} \tag{1.63}$$

It suffices to verify that $(x \times y)_*(z) \in A_*(\overline{X} \times \overline{Y})$ for every non degenerated simplex $z$ of $\overline{[p]} \times \overline{[q]}$. But this is an immediate consequence of the fact that $x$ and $y$ and induced by the injective increasing applications $[p] \to X$ and $[q] \to Y$.

□

**Lemma 1.9.5.** *The Eilenberg-Mac Lane transformation satisfies :*

$$\nabla \circ (\nabla \otimes 1) = \nabla \circ (1 \otimes \nabla) \qquad \textit{(associativity)} \tag{1.64}$$

$$t_* \circ \nabla = \nabla \circ T \qquad \textit{(commutativity)} \tag{1.65}$$

$$p_{1*} \circ \nabla = 1 \otimes \epsilon \tag{1.66}$$

$$p_{2*} \circ \nabla = \epsilon \otimes 1 \tag{1.67}$$

*Proof.* The natural transformation $\nabla \otimes 1$ sends $e_p \otimes e_q \otimes e_r$ to a finite sum of tensors with the form $x \otimes y$, where $x$ and $y$ are non degenerated simplices. The same happens with $\nabla \circ (\nabla \otimes 1)$ by lemma 1.9.4. The same argument is applied to $\nabla \circ (1 \otimes \nabla)$, and by the fact that $\overline{[p]} \times \overline{[q]} \times \overline{[r]}$ is acyclic ($\overline{[p]} \times \overline{[q]} \times \overline{[r]}$ has as smaller element $[0, 0, 0]$) and don't has non degenerated simplices of dimension strictly bigger than $p + q + r$, the point 3 of theorem acyclic models shows that $\nabla \circ (\nabla \otimes 1) = \nabla(1 \otimes \nabla)$. As $T$, $t_*$, $\epsilon \otimes 1$, $1 \otimes \epsilon$, $p_{1*}$ and $p_{2*}$ preserve the non degenerated simplices or they send them to 0, the other properties are immediate by the same method. □

Usually $\Delta$ is described by its explicit formula (see for example [May93], [Hes07]). But the unicity criterion simplifies the verification of its principal properties.

There exists an an homotopy inverse for $\nabla$. In the case when simplicial sets have an extra group structure, the Eilenberg-Mac Lane transformation helps to carry this structure to the associated chain complex, such operation on chains is called Pontrjagin product (see [Pon39]).

**Definition 1.9.6.** Let $H$ be a simplicial group $H$. The Pontrjagin product $\mathcal{P} : C_*(H) \otimes C_*(H) \to C_*(H)$ is defined to be the following composite.

$$C_*(H) \otimes C_*(H) \xrightarrow{\nabla} C_*(H \times H) \xrightarrow{m_*} C_*(H) \tag{1.68}$$

Where $m_*$ is the induced morphism by the product $m : H \times H \to H$.

**Proposition 1.9.7.** *Let $H$, $K$ simplicial groups. The Pontrjagin product $\mathcal{P} : C_*(H) \otimes C_*(H) \to C_*(H)$ satisfies the following properties.*

1. *$\mathcal{P}$ is associative.*
2. *If $H$ is commutative, then $\mathcal{P}$ is commutative.*
3. *The chain complex $C_*(H)$ is a DGA-$\mathsf{k}$-algebra with $\mathcal{P}$.*
4. *The Eilenberg-Mac Lane transformation $\nabla : C_*(H) \otimes C_*(K) \to C_*(H \times K)$ is a morphism of DGA-$\mathsf{k}$-algebras.*

□

➪ *Remark* 1.9.8. Consider the commutative diagram,

$$\begin{tikzcd}
C_*(H) \otimes C_*(K) \arrow[r, "\nabla"] \arrow[d, "i_* \otimes j_*"'] & C_*(H \times K) \arrow[d, "(i \times j)_*"] \arrow[ddr, bend left, "Id"] & \\
C_*(H \times K) \otimes C_*(H \times K) \arrow[r, "\nabla"] \arrow[drr, bend right, "\mathcal{P}"'] & C_*(H \times K \times H \times K) \arrow[dr, "(m_{H \times K})_*"] & \\
& & C_*(H \times K)
\end{tikzcd} \tag{1.69}$$

where $i : H \to H \times K$ and $j : K \to H \times K$ are defined by $i(h) = (h, 1)$ and $j(k) = (1, k)$. And where $H$ and $K$ are simplicial groups. This diagrams says that we can recover the Eilenberg-Mac Lane transformation from the Pontrjagin product, because $\nabla = \mathcal{P}(i_* \otimes j_*)$.

Back to the third point of acyclic models theorem, we can't assure anymore the unicity of a natural transformation,

$$\Phi : C_*(X \times Y) \to C_*(X) \otimes C_*(Y) \tag{1.70}$$

In this case the criterion of theorem 1.8.3 fails (see [Pro83], remark I). But in fact, it is well know that this kind of transformations are not unique. Indeed, if such a transformation $\Phi$ is unique in the case where $\mathsf{k} = \mathbb{Z}/2\mathbb{Z}$, the unicity will imply that the cup product $\cup_1$ is zero, which is not true, because the existence of the Steenrod squares (see [Cen89]).

Nevertheless, one important choice for an homotopical inverse of the Eilenberg-Mac Lane transformation, is the Alexander-Whitney transformation,

$$\Psi : C_*(X \times Y) \to C_*(X) \otimes C_*(Y) \tag{1.71}$$

$\Psi$ has the particularity of being associative, but not commutative. With $\Psi$ we can equipe $C_*(X)$ with a DGA-$\mathsf{k}$-coalgebra structure.

**Definition 1.9.9.** Let $X$ be a simplicial set. The Alexander-Whitney diagonal $\Delta$ is defined to be the composite of the Alexander-Whitney transformation with the induced chain morphism by the diagonal $\delta : X \to X \times X$.

$$\Delta : C_*(X) \xrightarrow{\delta_*} C_*(X \times X) \xrightarrow{\Psi} C_*(X) \otimes C_*(X) \tag{1.72}$$

**Proposition 1.9.10.** *Let $X$, $Y$ be a simplicial sets. Then the Alexander-Whitey diagonal satisfies the following properties.*

1. *$\Delta$ is explicitly described by the formula on the canonical base elements of $C_*(X)$.*

$$\Delta(x) = \sum_{i=0}^{|x|} \overline{\partial}^i(x) \otimes \partial_0^{|x|-i}(x) \tag{1.73}$$

   *Where $\overline{\partial}$ is the last face operator given by $\overline{\partial}(x) = \partial_{|x|}(x)$.*

2. *$\Delta$ is associative.*
3. *With $\Delta$, the chain complex $C_*(X)$ is a DGA-$\mathsf{k}$-coalgebra.*
4. *The Eilenberg-Mac Lane transformation $\nabla : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ is a morphism of DGA-$\mathsf{k}$-coalgebras.*

□

In [Pro84] Prouté explores a characterization of $\Delta$ by a property of its image, in the sense that the choice for $\Psi$ is limited by a subcomplex of $C_*(X) \otimes C_*(X)$. He also uses this characterization to prove the fourth point of proposition 1.9.10 in a very simple way.

# Chapter 2

# Operads

## 2.1 Operads

In this thesis we deal mostly with symmetric operads, which as we saw in the introduction, are equipped with an action by the symmetric groups. In this section we introduce the classical definition of operad together with the variation called partial definition, and we discuss two fundamental examples, the endomorphism operad an the coendomorphism operad. In the following diagrams the signs given by the Koszul convention are omitted in order to simplify the writing.

**Definition 2.1.1** (Operad)**.** An operad $\mathcal{P}$ is a collection of DGA-$\mathsf{k}$-modules $\{\mathcal{P}(n)\}_{n\geqslant 0}$ together with,

1. A morphism $\eta : \mathsf{k} \to \mathcal{P}(1)$, called the unit of $\mathcal{P}$.

2. For every $n$, a right action by the symmetric group $\Sigma_n$ over $\mathcal{P}(n)$, that is, a morphism of DGA-$\mathsf{k}$-modules making of $P(n)$ a right DGA-$\mathsf{k}[\Sigma_n]$-module.

$$P(n) \otimes \mathsf{k}[\Sigma_n] \longrightarrow P(n) \tag{2.1}$$

3. For each tuple $(k, i_1, \dots, i_k)$, a morphisms of DG-$\mathsf{k}$-modules,

$$\gamma_{(k,i_1,\dots,i_k)} : \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \to \mathcal{P}(n) \tag{2.2}$$

   where $n = i_1 + \cdots + i_k$ and $n, k, i_j \geqslant 0$. Usually this morphism will be simply written $\gamma$.

These applications are required to satisfy the following conditions.

1. The morphisms $\gamma$ are associative, in the sense that the following diagram com-

mutes.

$$\begin{tikzcd}
P(k)\otimes\left[\bigotimes_{p=1}^{k}P(i_p)\right]\otimes\left[\bigotimes_{p=1}^{k}\bigotimes_{q=1}^{i_p}P(r_{p,q})\right] \arrow[r, "\gamma\otimes 1"] \arrow[dd, "\text{shuffle}"'] & P(n)\otimes\left[\bigotimes_{p=1}^{k}\bigotimes_{q=1}^{i_s}P(r_{p,q})\right] \arrow[d, "\gamma"] \\
 & P(r) \\
P(k)\otimes\bigotimes_{p=1}^{k}\left[P(i_p)\otimes\bigotimes_{q=1}^{i_p}P(r_{p,q})\right] \arrow[r, "1\otimes\gamma^{\otimes k}"'] & P(k)\otimes\bigotimes_{p=1}^{k}P(r_p) \arrow[u, "\gamma"']
\end{tikzcd} \tag{2.3}$$

Where $n=\sum_{p=1}^{k} i_p$, $r=\sum_{p=1}^{k}\sum_{q=1}^{i_p} r_{p,q}=\sum_{p=1}^{k} r_p$ and the vertical left arrow is just a permutation of factors.

2. The unit $\eta:\mathsf{k}\to P(1)$ make the following diagrams commute.

$$\begin{tikzcd}
\mathcal{P}(n)\otimes\mathsf{k}^{\otimes n} \arrow[r, "\cong"] \arrow[d, "1\otimes\eta^{\otimes n}"'] & \mathcal{P}(n) \\
\mathcal{P}(n)\otimes\mathcal{P}(1)^{\otimes n} \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{k}\otimes\mathcal{P}(n) \arrow[r, "\cong"] \arrow[d, "\eta\otimes 1"'] & \mathcal{P}(n) \\
\mathcal{P}(1)\otimes\mathcal{P}(n) \arrow[ur, "\gamma"'] &
\end{tikzcd} \tag{2.4}$$

3. The actions of the symmetric groups satisfy the following two condition about equivariance.

$$\begin{tikzcd}
\mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k) \arrow[r, "\gamma"] \arrow[d, "\sigma\otimes\sigma^{-1}"'] & P(n) \arrow[d, "\sigma(i_1,\dots,i_n)"] \\
\mathcal{P}(k)\otimes\mathcal{P}(i_{\sigma(1)})\otimes\cdots\otimes\mathcal{P}(i_{\sigma(k)}) \arrow[r, "\gamma"] & P(n)
\end{tikzcd} \tag{2.5}$$

Where $n=i_1+\cdots+i_k$ and the arrow $\sigma\otimes\sigma^{-1}$ consists of the right action by $\sigma$ over $P(k)$ and the left action by $\sigma^{-1}$ over the tensor product $P(i_1)\otimes\cdots\otimes P(i_k)$.

$$\begin{tikzcd}
\mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k) \arrow[r, "1\otimes\tau_1\otimes\cdots\otimes\tau_n"] \arrow[d, "\gamma"'] & \mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k) \arrow[d, "\gamma"] \\
\mathcal{P}(n) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_n"] & \mathcal{P}(n)
\end{tikzcd} \tag{2.6}$$

where $n=i_1+\cdots+i_k$ and the action $1\otimes\tau_1\otimes\cdots\otimes\tau_n$ is the identity of $P(k)$ over the first factor and the right action by $\tau_j$ over the factor $P(i_j)$.

There is another approach to the definition of operads. The variation resides in the way the description of the composition operation is made. For an operad $\mathcal{P}$, instead of describe what happens when an operation of $P(k)$ is composed in each of its $k$ entries with operations of arities $i_1,\dots,i_k$, the partial definition of operads only describe the composition with another operation of arity $m$ in one of its entries, which will give as result an operation of $P(k+m-1)$.

**Definition 2.1.2** (Partial Definition of Operad)**.** An operad $\mathcal{P}$ is a collection of DGA-$\mathsf{k}$-modules $\{P(n)\}_{n\geqslant 0}$ together with an unit $\eta : \mathsf{k} \to P(1)$, a right action of $\Sigma_n$ over $P(n)$ for $n \geqslant 1$ and a collection of DGA-$\mathsf{k}$-morphisms,

$$\circ_i : P(k) \otimes P(m) \to P(k+m-1) \tag{2.7}$$

for $1 \leqslant i \leqslant k$, which satisfy the following conditions.

1. $\alpha \circ_i (\beta\sigma) = (\alpha \circ_i \beta)\sigma'$, where $\alpha \in P(k)$, $\beta \in P(m)$, $\sigma \in \Sigma_m$ and $\sigma' \in \Sigma_{k+m-1}$ given by the direct sum of $k$ terms $1\oplus\cdots\oplus\sigma\oplus\cdots\oplus 1$, with $\sigma$ in the $i$ position.

2. $(\alpha\sigma) \circ_i \beta = (\alpha \circ_{\sigma(i)} \beta)\sigma''$, where $\alpha \in P(k)$, $\beta \in P(m)$, $\sigma \in \Sigma_n$ and $\sigma'' \in \Sigma_{k+m-1}$, acting like $\sigma$ over $k$ blocks of lenght $1, \ldots, 1, m, 1, \ldots, 1$, with $m$ the block in the position $i$.

3. $(\alpha \circ_i \beta) \circ_{i-1+j} \gamma = \alpha \circ_i (\beta \circ_j \gamma)$, where $\alpha \in P(l)$, $\beta \in P(m)$, $\gamma \in P(n)$ and with $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant m$.

4. $(\alpha \circ_i \beta) \circ_{k+m-1} \gamma = (-1)^{|\beta||\gamma|}(\alpha \circ_k \gamma) \circ_i \beta$, where $\alpha \in P(l)$, $\beta \in P(m)$, $\gamma \in P(n)$ and with $1 \leqslant i < k \leqslant l$.

**Proposition 2.1.3.** *The definition of operads given in 2.1.2 and 2.1.1 are equivalents.*

*Proof.* The partial composition $\circ_i$ is obtained from the composition $\gamma$ of definition 2.1.1 by the formula,

$$\alpha \circ_i \beta = \gamma(\alpha \otimes 1 \otimes \ldots \otimes \beta \otimes \ldots \otimes 1) \tag{2.8}$$

with $\beta$ in the $i$ position between the 1's.

Conversely, the composition $\gamma$ is obtained from the partial compositions by the formula,

$$\gamma(\alpha \otimes \beta_1 \otimes \cdots \otimes \beta_k) = \alpha \circ_k \beta_k \circ_{k-1} \cdots \circ_1 \beta_1 \tag{2.9}$$

It is straightforward to show that these operations satisfy the conditions of the respective operad definition. $\square$

Two fundamental examples of operads are the endomorphism operad and the co-endomorphism operad. Their behavior was used to model the definition of operads.

**Definition 2.1.4** (Endomorphism Operad)**.** For every $M \in$ DGA-$\mathsf{k}$-Mod, the operad $End(M)$ of endomorphisms of $M$ is defined by:

1. For every $n \geqslant 0$, $End(M)(n) = Hom(M^{\otimes n}, M)$, that is the DGA-$\mathsf{k}$-module of homogeneous applications from $M^{\otimes n}$ to $M$.

2. The unit $\eta : \mathsf{k} \to End(M)(1)$ is defined by $\eta(1) = 1_M$, the identity of $M$.

3. The right action of $\Sigma_n$ over $End(M)$ is induced by the left action of $\Sigma_n$ over $M^{\otimes n}$.

4. The composition applications,

$$\gamma : End(M)(k) \otimes End(M)(i_1) \otimes \cdots \otimes End(M)(i_k) \to End(M)(n) \tag{2.10}$$

where $n = i_1 + \cdots + i_k$, are given by,

$$\gamma(f_k \otimes f_{i_1} \otimes \cdots \otimes f_{i_k}) = f_k \circ (f_{i_1} \otimes \cdots \otimes f_{i_k}) \tag{2.11}$$

**Definition 2.1.5** (Coendomorphism Operad)**.** For every $N \in$ DGA-k-Mod, the operad $Coend(N)$ of coendomorphisms of $N$ is defined by:

1. For every $n \geqslant 0$, $Coend(N)(n) = Hom(N, N^{\otimes n})$, that is the DGA-k-module of homogeneous applications from $N$ to $N^{\otimes n}$.
2. The unit $\eta : \mathsf{k} \to Coend(N)(1)$ is defined by $\eta(1) = 1_N$, the identity morphism of $N$.
3. The right action of $\Sigma_n$ over $Coend(N)$ is induced by the right action of $\Sigma_n$ over $N^{\otimes n}$.
4. The composition applications,

$$\gamma : Coend(M)(k) \otimes Coend(N)(i_1) \otimes \cdots \otimes Coend(N)(i_k) \to Coend(N)(n) \tag{2.12}$$

where $n = i_1 + \cdots + i_k$, are given by,

$$\gamma(f_k \otimes f_{i_1} \otimes \cdots \otimes f_{i_k}) = (-1)^{|f_k|(|f_{i_1}|+\cdots+|f_{i_k}|)}(f_{i_1} \otimes \cdots \otimes f_{i_k}) \circ f_k \tag{2.13}$$

➪ *Remark* 2.1.6. The reader can easily check that $End(M)$ and $Coend(N)$ satisfy the conditions in definition 2.1.1.

**Definition 2.1.7.** Let $P$ and $Q$ be two operads. A morphism $\phi$ from $P$ to $Q$, is a collection of morphism of DGA-k-modules,

$$\phi_n : P(n) \to Q(n) \tag{2.14}$$

which satisfy the following conditions.

1. The morphism $f_1 : P(1) \to Q(1)$ preserves the units of the operads, that is $f\eta = \eta$.

$$\begin{tikzcd} P(1) \arrow[rr, "f_1"] & & Q(1) \\ & \mathsf{k} \arrow[ul, "\eta"] \arrow[ur, "\eta"'] & \end{tikzcd} \tag{2.15}$$

2. The morphisms $f_n : P(n) \to Q(n)$ are $\Sigma_n$-equivariants.

3. $f$ preserve the compositions of the operads, that is, the following diagram is commutative.

$$\begin{tikzcd} P(k)\otimes P(i_1)\otimes\cdots\otimes P(i_k) \arrow[r, "\gamma_P"] \arrow[d, "f_k\otimes f_{i_1}\otimes\cdots\otimes f_{i_k}"'] & P(n) \arrow[d, "f_n"] \\ Q(k)\otimes Q(i_1)\otimes\cdots\otimes Q(i_k) \arrow[r, "\gamma_Q"] & Q(n) \end{tikzcd} \tag{2.16}$$

The category of operads over DGA-$\mathsf{k}$-Mod is denoted $\mathcal{OP}$.

We finish this section with two more examples of operads.

*Example* 2.1.8. The operad $\mathcal{N}$ is given by $\mathcal{N}(n) = \mathsf{k}$ for every $n$ non negative. Here $\mathsf{k}$ is seen as a DGA-$\mathsf{k}$-module concentrated in degree zero. The unit $\eta$ is the identity of $\mathsf{k}$, the action is the trivial action of $\Sigma_n$ over $\mathsf{k}$ and the composites, if we denote $a_i$ the generator of degree zero of $N(i)$, are simply given by the rule,

$$\gamma : a_k\otimes a_{i_1}\otimes\cdots\otimes a_{i_k}\to a_n \tag{2.17}$$

where $n = i_1+\cdots+i_k$.

*Example* 2.1.9. Making the action of the last example free will produce an operad that we denote $\mathcal{M}$. The components of $\mathcal{M}$ are $M(n) = \mathsf{k}[\Sigma_n]$ for every non negative $n$. The $M(n)$ are graded differential modules concentrated in degree zero. The action by the symmetric group group is clear, and the composite operations are defined over the generators in degree zero as before, but respecting the symmetric group action, in the sense that:

$$\gamma(a_k\otimes a_{i_1}\sigma_{i_1}\otimes\cdots\otimes a_{i_k}\sigma_{i_1}) = a_n(\sigma_{i_1}\oplus\cdots\oplus\sigma_{i_k}) \tag{2.18}$$

and

$$\gamma(a_k\sigma\otimes a_{i_{\sigma^{-1}(1)}}\otimes\cdots\otimes a_{i_{\sigma^{-1}(k)}}) = a_n(T_{i_1,\ldots,i_k}\sigma) \tag{2.19}$$

where $n = i_1+\cdots+i_k$.

## 2.2 Algebras and Coalgebras over an Operad

The most important feature in the theory of operads are its representations. That is, when the abstract operations of the operads are interpreted as concrete application over an object in the ground category, which is DGA-$\mathsf{k}$-Mod in our case. This passage from the abstract to the concrete is made through morphisms of type $P(n)\to Hom(A^{\otimes n}, A)$. In this sens, an element of arity $n$ of $\mathcal{P}$ is realized as an $n$-ary operation over $A$. This association must be coherent with respect the composition operation in the operad, and with respect the symmetric groups actions. For instance, if we have that $c = \gamma(a\otimes b_1\otimes b_2)$ in $\mathcal{P}$, then the associated operations over $A$, $\mu_a$, $\mu_{b_1}$, $\mu_{b_2}$ and $\mu_c$ must be related in the sense that $\mu_c = \mu_a\circ(\mu_{b_1}\otimes\mu_{b_2})$.

**Definition 2.2.1.** Let $\mathcal{P}$ be an operad.

1. An algebra over the operad $\mathcal{P}$, or $\mathcal{P}$-algebra, is a DGA-$\mathsf{k}$-Mod $A$, together with a morphism of operads from $\mathcal{P}$ to $End(A)$.

2. An coalgebra over the operad $\mathcal{P}$, or $\mathcal{P}$-coalgebra, is a DGA-$\mathsf{k}$-Mod $C$, together with a morphism of operad from $\mathcal{P}$ to $Coend(C)$.

It is well know that in the symmetric monoidal category DGA-$\mathsf{k}$-Mod, the functor $-\otimes Y$ is left adjunct of the functor $Hom(Y, -)$ for every $Y$ DGA-$\mathsf{k}$-module. Denote $\theta$ the natural bijection given by this adjunction,

$$\theta : Hom(X \otimes Y, Z) \to Hom(X, Hom(Y, Z)) \tag{2.20}$$

Then for a morphism of operads $f : \mathcal{P} \to End(A)$, each component $f_n : P(n) \to Hom(A^{\otimes n}, A)$ determines a morphism of DGA-$\mathsf{k}$-modules $\varphi_n : P(n) \otimes A \to A^{\otimes n}$, given by $\varphi_n = \theta^{-1}(f_n)$. It is not hard to verify that this has as consequence the following equivalent definitions for algebras and coalgebras over an operad.

**Proposition 2.2.2.** *1. Equivalently, a $\mathcal{P}$-algebra $A$ is a collection $\{\varphi_n\}_{n\geqslant 1}$ of morphisms of DGA-$\mathsf{k}$-modules $\varphi_n : P(n) \otimes C^{\otimes n} \to C$, which satisfy the following conditions.*

*(a) Associativity. The following diagram is commutative,*

$$\begin{tikzcd}
P(k)\otimes P(i_1)\otimes\cdots\otimes P(i_k)\otimes C^{\otimes n} \arrow[rr, "\gamma\otimes 1"] \arrow[d, "shuffle"'] & & P(n)\otimes C^{\otimes n} \arrow[dd, "\varphi_n"] \\
P(k)\otimes P(i_1)\otimes C^{\otimes i_1}\otimes\cdots\otimes P(i_k)\otimes C^{\otimes i_k} \arrow[d, "1\otimes\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}"'] & & \\
P(k)\otimes C^{\otimes k} \arrow[rr, "\varphi_k"'] & & C
\end{tikzcd} \tag{2.21}$$

*where $n = i_1 + \cdots + i_k$.*

*(b) Unit and equivariance. The following diagrams are commutative for every $\sigma \in \Sigma_n$.*

$$\begin{tikzcd}
P(1)\otimes C \arrow[r, "\varphi_1"] & C \\
\mathsf{k}\otimes C \arrow[u, "\eta\otimes 1"] \arrow[ur, "\cong"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
P(n)\otimes C^{\otimes n} \arrow[r, "\varphi_n"] \arrow[d, "\sigma\otimes\sigma^{-1}"'] & C \arrow[d, "="] \\
P(n)\otimes C^{\otimes n} \arrow[r, "\varphi_n"] & C
\end{tikzcd} \tag{2.22}$$

*2. Equivalently, a $\mathcal{P}$-coalgebra $C$ is a collection $\{\varphi_n\}_{n\geqslant 1}$ of morphisms of DGA-$\mathsf{k}$-modules $\varphi_n : P(n) \otimes C \to C^{\otimes n}$, which satisfy the following conditions.*

*(a) Associativity. The following diagram is commutative, where $n = i_1 + \cdots +$*

$i_k$

$$\begin{tikzcd} P(k)\otimes P(i_1)\otimes\cdots\otimes P(i_k)\otimes C \arrow[r, "\gamma\otimes 1"] \arrow[d, "shuffle"'] & P(n)\otimes C \arrow[ddd, "\varphi_n"] \\ P(k)\otimes C\otimes P(i_1)\otimes\cdots\otimes P(i_k) \arrow[d, "\varphi_k\otimes 1"'] & \\ C^{\otimes k}\otimes P(i_1)\otimes\cdots\otimes P(i_k) \arrow[d, "shuffle"'] & \\ P(i_1)\otimes C\otimes\cdots\otimes P(i_k)\otimes C \arrow[r, "\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}"] & C^{\otimes n} \end{tikzcd} \tag{2.23}$$

(b) *Unit and equivariance. The following diagrams are commutative, where* $\sigma \in \Sigma_n$.

$$\begin{tikzcd} P(1)\otimes C \arrow[r, "\varphi_1"] & C \\ \mathsf{k}\otimes C \arrow[u, "\eta\otimes 1"] \arrow[ur, "\cong"'] & \end{tikzcd} \qquad \begin{tikzcd} P(n)\otimes C \arrow[r, "\varphi_n"] \arrow[d, "\sigma\otimes 1"'] & C^{\otimes n} \arrow[d, "\sigma"] \\ P(n)\otimes C \arrow[r, "\varphi_n"] & C^{\otimes n} \end{tikzcd} \tag{2.24}$$

□

**Proposition 2.2.3.** *Let $\mathcal{N}$ and $\mathcal{M}$, be the operads of examples 2.1.8 and 2.1.9, respectively. Then we have the following isomorphism of categories.*

1. *The category of $\mathcal{N}$-algebras (coalgebras) is isomorphic to the category of commutative DGA-$\mathsf{k}$-algebras (coalgebras).*

2. *The category of $\mathcal{M}$-algebras (coalgebras) is isomorphic to the category of DGA-$\mathsf{k}$-algebras (coalgebras).*

*Proof.* Let $A$ be an $\mathcal{N}$-algebra. Then we have a collection of DGA-$\mathsf{k}$-morphism,

$$f_n : \mathsf{k} \to Hom(A^{\otimes n}, A) \tag{2.25}$$

which determines the a DGA-$\mathsf{k}$-morphisms $f_n(1) : A^{\otimes n} \to A$. By the operad structure of $\mathcal{N}$ we have that $f_2(1) \circ (f_1(1) \otimes f_2(1)) = f_3(1) = f_2(1) \circ (f_2(1) \otimes f_1(1))$, which says that $f_2(1)$ is an associative product for $A$. Using that the action of the symmetric group is trivial in $\mathcal{N}$, we have $f_2(1) = f_2(1\tau) = f_2(1)\tau$, which implies that the product $f_2(1)$ is commutative. The rest of the proof is similar and left to the reader. □

## 2.3 Free Operads

When the composition in an operad $\mathcal{P}$ is forgotten we obtain a collection $\{P(n)\}_{n\geqslant 0}$ of DGA-$\mathsf{k}$-modules with right actions by the symmetric groups. In this section we explore the inverse process: from a collection of this kind, generate an operadic structure over it. The operads obtained in this way are called free operads and satisfy an universal property, that every morphism between the depart collection to any other collection with an operadic structure, can be uniquely extended to a morphism of operads from the free operad. We will call the sequences of DGA-$\mathsf{k}$-modules with symmetric group actions, $\mathbb{S}$-modules.

The construction of free operads is widely described in the principal references about operads(see for instance [LV12], [MSS07], [BM06] and [Rez96]). There are many different ways to define free operads, our presentation tries to keep operads as close as we can from the classical definition (see 2.1.1).The reason relies in the way we constructed the operad $\mathcal{K}$ which describes the $E_\infty$-coalgebra associated to an $\mathcal{L}$-algebra.

In the following we will see that operads can be regarded as monoids over the category of $\mathbb{S}$-modules in order to construct free operads. This way of defining operads is an instance of a more general point of view, where the operads are defined as monads over the category of endofunctors of the category of DGA-$\mathsf{k}$-modules and where the $\mathbb{S}$-modules are identified with Schur functors(see [LV12]). Here we keep the $\mathbb{S}$-modules in his natural state and the composite of functors will appear as a special operation of $\mathbb{S}$-modules.

At the end of this section we also include the case when the operadic structure does not have the actions by symmetric groups, the idea is to facilitate the description of the construction in the next chapter of an $E_\infty$-operad used to describe the $E_\infty$-coalgebra over the chain complexes.

**Definition 2.3.1.** Let $\mathbb{S}$ be the groupoid where the objects are the ordered sets $[n] = \{1, \ldots, n\}$, with $n$ positive integer and $[0] = \varnothing$. The morphisms of $\mathbb{S}$ are given by $(n, m) = \varnothing$, if $n \neq m$, and $\mathbb{S}(n, n) = \Sigma_n$, the $n$-symmetric group.

**Definition 2.3.2.** An $\mathbb{S}$-module $M$ is a functor from the category $\mathbb{S}$ to the category DGA-$\mathsf{k}$-Mod. The morphisms $\mathbb{S}(n, n)$ are interpreted as a right action by $\Sigma_n$ over $M(n)$. The category of $\mathbb{S}$-modules and natural transformations is denoted $\mathbb{S}$-Mod.

➪ *Remark* 2.3.3. The category $\mathbb{S}$-Mod has all colimits and limits because it is a category of diagrams over DGA-$\mathsf{k}$-Mod.

**Definition 2.3.4.** We denote $U$ the forgetful functor from the category of operads to the category of $\mathbb{S}$-modules.

Before starting with the construction of free operads over a symmetric we are going to sketch how this kind of object would looks like. Let $M$ be an $\mathbb{S}$-module, the free operad $F(M)$ associated to $M$, as $\mathbb{S}$-module will contain $M(n)$ as a $\mathbb{S}$-submodule and all the possible tensors of type $M(k) \otimes M(i_1) \otimes \cdots \otimes M(i_k)$, where $i_1 + \cdots + i_k = n$, because they represent the formal compositions.

In order to satisfy the equivariance axiom 2.5 we need for every $\sigma \in \Sigma_k$ the relation,

$$M(k)\sigma \otimes M(i_1) \otimes \cdots \otimes M(i_k) = M(k) \otimes M(i_{\sigma(1)}) \otimes \cdots \otimes M(i_{\sigma(k)}) \tag{2.26}$$

which can is obtained by taking the tensor product over $\mathsf{k}[\Sigma_k]$,

$$M(k) \otimes_{\Sigma_k} M(i_1) \otimes \cdots \otimes M(i_k) \tag{2.27}$$

Now, consider the second equivariance axiom 2.6. In the left part of 2.27, we could have actions in each factor by the respective symmetric group,

$$M(i_1)\tau_1 \otimes \cdots \otimes M(i_k)\tau_k \tag{2.28}$$

with $\tau_j \in \Sigma_{i_j}$ and $i_1 + \cdots + i_k = n$. The permutations $\tau_j$ all together can be seen as the permutation of $\Sigma_n$ given by $\tau_1 \oplus \cdots \oplus \tau_k$ acting at right of $M(i_1) \otimes \cdots \otimes M(i_k)$. This kind of permutations form the subgroup $\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}$ of $\Sigma_n$. Then we can write,

$$M(i_1)\tau_1 \otimes \cdots \otimes M(i_k)\tau_k = (M(i_1) \otimes \cdots \otimes M(i_k))\,(\tau_1 \oplus \cdots \oplus \tau_k) \tag{2.29}$$

The process of put at right the permutations $\tau_j$ is then expressed by the following tensor product.

$$(M(i_1) \otimes \cdots \otimes M(i_k)) \otimes_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} \mathsf{k}[\Sigma_n] \tag{2.30}$$

Here we put $\mathsf{k}[\Sigma_n]$ instead of $\mathsf{k}[\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}]$, in order to considerate all the other permutations of $\Sigma_n$ that acts over the $i_1 + \cdots + i_k$ inputs but cannot be expressed by a sum of type $\tau_1 \oplus \cdots \oplus \tau_k$. Then we can simplify the expression to obtain,

$$(M(i_1) \otimes \cdots \otimes M(i_k)) \otimes \mathsf{k}[(\Sigma_{i_1} \times \cdots \times \Sigma_{i_k})\backslash\Sigma_n] \tag{2.31}$$

The quotient $(\Sigma_{i_1} \times \cdots \times \Sigma_{i_k})\backslash\Sigma_n$ can be represented by the set of $(i_1, \ldots, i_k)$-shuffles of $\Sigma_n$, which is written $Sh(i_1, \ldots, i_k)$. Recall that a $(i_1, \ldots, i_k)$-shuffle, where $i_1+\cdots+i_k = n$, is an element of $\Sigma_n$ sending $(1, \ldots, n)$ to $(\mu_1^1, \ldots, \mu_{i_1}^1, \ldots, \mu_1^k, \ldots, \mu_{i_k}^k)$ such that $\mu_1^j < \ldots < \mu_{i_j}^j$ for all $1 \leqslant j \leqslant k$. Then 2.31 is written,

$$M(i_1) \otimes \cdots \otimes M(i_k) \otimes \mathsf{k}[Sh(i_1, \ldots, i_k)] \tag{2.32}$$

Which together with the part $M(k)$ gives the following expression.

$$M(k) \otimes_{\Sigma_k} M(i_1) \otimes \cdots \otimes M(i_k) \otimes \mathsf{k}[Sh(i_1, \ldots, i_k)] \tag{2.33}$$

Our free operad will need this for arbitrary $n$ and all possible sums $i_1+\cdots+i_k = n$, that is, we need to consider the direct sum,

$$\bigoplus_{n\geqslant 0}\bigoplus_{k\geqslant 0} M(k) \otimes_{\Sigma_k} \left( \bigoplus_{i_1+\cdots+i_k=n} M(i_1) \otimes \cdots \otimes M(i_k) \otimes \mathsf{k}[Sh(i_1, \ldots, i_k)] \right) \tag{2.34}$$

This expression represents the first stage of all possible compositions between the elements of $M$ when they are interpreted as applications. The next stage of compositions is to consider when each of $M(i_j)$ in 2.34 comes from another arbitrary composite, and so on. In order to manage all the possible levels of composites we need to introduce some operations over the $\mathbb{S}$-modules.

**Definition 2.3.5.** Let $M$, $N$ be $\mathbb{S}$-modules. We define the tensor product of $M$ and $N$ as the $\mathbb{S}$-module $M \otimes N$ given by the formula,

$$(M \otimes N)(n) = \bigoplus_{i+j=n} M(i) \otimes M(j) \otimes \mathsf{k}[Sh(i,j)] \tag{2.35}$$

**Proposition 2.3.6.** *The tensor product of* $\mathbb{S}$*-modules is associative and for every* $\mathbb{S}$*-module* $M$ *satisfies* $M = M \otimes \mathsf{k} = \mathsf{k} \otimes M$*. Where* $\mathsf{k}$ *is seen as a* $\mathbb{S}$*-module concentrated in arity* $0$*.*

*Proof.* Let $M$, $N$ and $P$ be $\mathbb{S}$-modules.

$$\begin{aligned}
((M \otimes N) \otimes P)(n) &= \bigoplus_{i+j=n} (M \otimes N)(i) \otimes P(j) \otimes \mathsf{k}[Sh(i,j)] \\
&= \bigoplus_{i+j=n} \bigoplus_{r+s=i} M(r) \otimes N(s) \otimes \mathsf{k}[Sh(r,s)] \otimes P(j) \otimes \mathsf{k}[Sh(i,j)] \\
&= \bigoplus_{i+j=n} \bigoplus_{r+s=i} (M(r) \otimes N(s) \otimes_{\Sigma_r \times \Sigma_s} \mathsf{k}[\Sigma_i]) \otimes_{\Sigma_i \times \Sigma_j} P(j) \otimes \mathsf{k}[\Sigma_n] \\
&= \bigoplus_{r+s+j=n} M(r) \otimes N(s) \otimes P(j) \otimes_{\Sigma_r \times \Sigma_s \times \Sigma_j} \mathsf{k}[\Sigma_n] \\
&= \bigoplus_{r+i=n} \bigoplus_{s+j=i} M(r) \otimes \big(N(s) \otimes P(j) \otimes_{\Sigma_s \times \Sigma_j} \mathsf{k}[\Sigma_i]\big) \otimes_{\Sigma_r \times \Sigma_i} \mathsf{k}[\Sigma_n] \\
&= (M \otimes (N \otimes P))(n)
\end{aligned}$$

The rest of the proof is left to the reader. □

➪ *Remark* 2.3.7. Note that in formula 2.34 we have,

$$\bigoplus_{i_1+\cdots+i_k=n} M(i_1) \otimes \cdots \otimes M(i_k) \otimes \mathsf{k}[Sh(i_1, \ldots, i_k)] = M^{\otimes k}(n) \tag{2.36}$$

where $M^{\otimes n}$ is $n$ times the tensor product of $\mathbb{S}$-modules.

**Definition 2.3.8.** Let $M$, $N$ be $\mathbb{S}$-modules. We define the composition of $M$ with $N$ as the $\mathbb{S}$-module,

$$M \circ N = \bigoplus_{k \geqslant 0} M(k) \otimes_{\Sigma_k} N^{\otimes k} \tag{2.37}$$

➪ *Remark* 2.3.9. The formula 2.34 can be written,

$$\begin{aligned}
&\bigoplus_{k \geqslant 0} M(k) \otimes_{\Sigma_k} \left( \bigoplus_{n \geqslant 0} \bigoplus_{i_1+\cdots+i_k=n} M(i_1) \otimes \cdots \otimes M(i_k) \otimes \mathsf{k}[Sh(i_1, \ldots, i_k)] \right) \\
&= \bigoplus_{k \geqslant 0} M(k) \otimes_{\Sigma_k} (M^{\otimes k}) = M \circ M
\end{aligned} \tag{2.38}$$

The composition of $M \circ M$ represents the first stage of formal compositions and $M^{\circ k}$ can be used to represents $k$ stages of formal compositions.

**Proposition 2.3.10.** *Let* $f : M \to N$ *and* $f' : M' \to N'$ *be morphisms of* $\mathbb{S}$*-modules, then the morphism given by* $(f \circ f')(x \otimes y_1 \otimes \cdots \otimes y_k) = f(x) \otimes f'(y_1) \otimes \cdots \otimes f'(y_k)$ *is a morphism of* $\mathbb{S}$*-modules from* $M \circ M'$ *to* $N \circ N'$*.*

□

The $\mathbb{S}$-modules can be identify with endofunctors of the category DGA-$\mathsf{k}$-Mod in such a way that composite of $\mathbb{S}$-modules coincide with the composition of functors (see [LV12], §5). This kind of functors are called Schur functors. The next proposition is a consequence of this identification.

**Proposition 2.3.11.** *The category of $\mathbb{S}$-modules with the composite $\circ$ and $I = (0, \mathsf{k}, 0, \dots)$ is a monoidal category.*

□

In fact, operads are instances of monoids over the category $\mathbb{S}$-modules.

**Proposition 2.3.12.** *Every operad determines a monoid in $\mathbb{S}$-Mod and conversely.*

*Proof.* Observe that an application of $\mathbb{S}$-modules $\eta : I \to M$ is not zero only in arity 1, then it determine an application $\eta : \mathsf{k} \to M(1)$ and conversely. A morphism $\mu : M \circ M \to M$ of $\mathbb{S}$-modules in arity $n$ is given by an equivariant morphism of DGA-$\mathsf{k}$-modules $\mu_n$,

$$\mu_n : \bigoplus_{k \geqslant 0} \bigoplus_{i_1 + \cdots + i_k = n} M(k) \otimes_{\Sigma_k} (M(i_1) \otimes \cdots \otimes M(i_k) \otimes \mathsf{k}[Sh(i_1, \dots, i_k)]) \to M(n) \tag{2.39}$$

which determined by the collection of equivariant morphisms,

$$\gamma : M(k) \otimes_{\Sigma_k} (M(i_1) \otimes \cdots \otimes M(i_k) \otimes \mathsf{k}[Sh(i_1, \dots, i_k)]) \to M(n) \tag{2.40}$$

and each morphism $\gamma$ is characterized as a morphism,

$$\gamma : M(k) \otimes M(i_1) \otimes \cdots \otimes M(i_k) \to M(n) \tag{2.41}$$

satisfying the equivariance conditions 2.5 and 2.6. □

Before the construction of free operads one more operation with $\mathbb{S}$-modules is needed.

**Definition 2.3.13.** Let $M$, $N$ be $\mathbb{S}$-modules. We define the direct sum of $M$ and $N$ by the formula,

$$(M \oplus N)(n) = M(n) \oplus N(n) \tag{2.42}$$

**Proposition 2.3.14.** *The forgetful functor $U : \mathcal{OP} \to \mathbb{S}$-Mod has a left adjoint $F : \mathbb{S}$-Mod$\to \mathcal{OP}$. We call $F$ the free operad functor.*

*Proof.* Let $M$ be a $\mathbb{S}$-module, then by proposition 2.3.12 we only need exhibit the free operad as a monoid $(F(M), \mu, \eta)$ in $\mathbb{S}$-Mod, We are going to describe the construction of $F(M)$, $\mu$, $\eta$ and the unit and counit of the adjunction, the verifications that they satisfy the required properties are not hard, for details see §5.4 in [LV12].

First, we construct inductively for each $n$ a $\mathbb{S}$-module $F(M)_n$ as follows.

$$F(M)_0 = I \tag{2.43}$$
$$F(M)_1 = I \oplus M \tag{2.44}$$
$$F(M)_2 = I \oplus (M \circ (I \oplus M)) = I \oplus (M \circ F(M)_1) \tag{2.45}$$
$$F(M)_{n+1} = I \oplus (M \circ F(M)_n) \tag{2.46}$$

Let $i_0$ be the inclusion of $I$ in $F(M)_1$. Using the identity $M = M \circ I$ and the morphism $1_M \circ i_0 : M \circ I \to M \circ F(M)_1$ we obtain the morphism $i_1 = 1_I \oplus (1_M \circ i_0) : F(M)_1 \to F(M)_2$. Repeating this process we got the morphisms,

$$i_n : F(M)_n \to F(M)_{n+1} \tag{2.47}$$

defined by induction with the formula $i_{n+1} = 1_I \oplus (1_M \circ i_n)$. The $\mathbb{S}$-modules $F(M)_n$ code all the possible $n$ stage compositions of elements of $M$. In order to put together all this information in a single $\mathbb{S}$-module, we take the colimit over the diagram given by the morphisms $i_j$.

$$F(M) = \operatorname*{colim}_n F(M)_n \tag{2.48}$$

The differential over $F(M)$ is determined by the differential of $M$ and extended in the obvious way.

Now, we have to define $\mu$ and the unit $\eta$ for $F(M)$. The unit is given by the inclusion $\eta : I \to F(M)$. The morphism $\mu : F(M) \circ F(M) \to F(M)$ is determined by a collection of maps $\mu_{n,m} : F(M)_n \circ F(M)_m \to F(M)_{n+m}$, defined by induction over $n$ by taking $\mu_{0,m} = 1_{F(M)_m}$ and for $n > 0$, $\mu_{n,m}$ is given by the composition,

$$\begin{aligned} F(M)_n \circ F(M)_m &= (I \oplus M \circ F(M)_{n-1}) \circ F(M)_m \\ &\xrightarrow{\cong} F(M)_m \oplus (M \circ F(M)_{n-1}) \circ F(M)_m \\ &\xrightarrow{\cong} F(M)_m \oplus M \circ (F(M)_{n-1} \circ F(M)_m) \\ &\xrightarrow{1 \oplus 1 \circ \mu_{n-1,m}} F(M)_m \oplus M \circ F(M)_{n+m-1} \\ &\xrightarrow{i + i'} F(M)_{n+m} \end{aligned}$$

where $i$ is the inclusion of $F(M)_m$ in $F(M)_{n+m}$, and $i'$ the inclusion of $F(M)_{n+m-1}$ as the second factor of $F(M)_{n+m}$.

Let $\mathcal{P}$ be an operad, the counit $\epsilon : FU \to 1$ of the adjunction is determined by morphisms $\epsilon_n : FU(\mathcal{P})_n \to \mathcal{P}$ defined by induction as follows. $\epsilon_0 : I \to \mathcal{P}$ is determined by the unit $\eta$ of $\mathcal{P}$, $\epsilon_1 = \eta + 1 : I \oplus U\mathcal{P} \to \mathcal{P}$ and $\epsilon_{n+1} = \eta + \gamma(1 \circ \epsilon_n) : FU(P)_n = I \oplus (U\mathcal{P} \circ UF(P)_n) \to \mathcal{P}$. Finally, for $M \in \mathbb{S}$-Mod, the unit of the adjunction $\eta : 1 \to UF$, is determined by the inclusions in the second factor $M \circ F(M)_{n-1} \to F(M)_n$. □

➪ *Remark* 2.3.15. Summarizing, the adjunction given by proposition 2.3.14, defines for every operad $\mathcal{Q}$ and $\mathbb{S}$-module $M$, the natural bijection,

$$\theta : \mathcal{OP}(F(M), \mathcal{Q}) \to \mathbb{S}\text{-Mod}(M, U(\mathcal{Q})) \tag{2.49}$$

The unit and counit of the adjunction are denoted $\eta$ and $\epsilon$ respectively. For the unit $\eta$ we have the morphisms,

$$\eta : 1_{\mathbb{S}\text{-Mod}} \to UF \tag{2.50}$$
$$\eta_M : M \to UF(M) \tag{2.51}$$

And for counit $\epsilon$ we have,

$$\epsilon : FU \to 1_{\mathcal{OP}} \tag{2.52}$$
$$\epsilon_{\mathcal{P}} : FU(\mathcal{P}) \to \mathcal{P} \tag{2.53}$$

**Definition 2.3.16.** A non symmetric operad is defined as an operad but without considering the actions of symmetric groups. The category of non symmetric operads is denoted $n\mathcal{OP}$

**Definition 2.3.17.** An $\mathbb{N}$-module is a functor from the groupoid $\mathfrak{N}$ with objects the sets $[0] = \varnothing$ and $[n] = \{1, .., n\}$ for $n > 0$, and morphisms the identity applications, to the category DGA-$\mathsf{k}$-Mod. The category of $\mathbb{N}$-modules is denoted $\mathbb{N}$-Mod.

**Definition 2.3.18.** Let $G$ be the forgetful functor from $\mathbb{S}$-Mod to $\mathbb{N}$-Mod.

**Proposition 2.3.19.** *The forgetful functor $G : \mathbb{S}$-Mod $\to \mathbb{N}$-Mod has a left adjoint $H : \mathbb{N}$-Mod $\to \mathbb{S}$-Mod, called the free $\mathbb{S}$-module functor.*

*Proof.* For $M \in \mathbb{N}$-Mod, $H(M)$ is defined as the $\mathbb{S}$-module with $n$ component given by $M(n) \otimes \mathsf{k}[\Sigma_n]$, the verification that it satisfies the proposition is straightforward. □

**Definition 2.3.20.** Let $\mathfrak{U}$ be the forgetful functor from the category $n\mathcal{OP}$ to $\mathbb{N}$-Mod.

**Proposition 2.3.21.** *The forgetful functor $nU : n\mathcal{OP} \to \mathbb{N}$-Mod have a left adjunct $nF : \mathbb{N}$-Mod $\to n\mathcal{OP}$, called the free non symmetric operad functor.*

*Proof.* The non symmetric case is similar to the symmetric case, only without the considerations about the $\Sigma_n$ actions. For an $\mathbb{N}$-module $N$, in order to construct a free ns operad over $N$ we need a direct sum, tensor product and composite of $\mathbb{N}$-modules. They are defined as follows,

$$(N \oplus E)(n) = N(n) \oplus E(n) \tag{2.54}$$
$$(N \otimes E)(n) = \bigoplus_{i+j=n} N(i) \otimes E(j) \tag{2.55}$$
$$(N \circ E) = \bigoplus_{k \geqslant 0} N(k) \otimes E^{\otimes k} \tag{2.56}$$

Note that,

$$N \circ E = \bigoplus_{k \geqslant 0} \bigoplus_{n \geqslant 0} \bigoplus_{i_1 + \cdots + i_k = n} N(k) \otimes E(i_1) \otimes \cdots \otimes E(i_k) \tag{2.57}$$

As the symmetric case, the ns operads are monoids over the monoidal category of $\mathbb{N}$-modules, where the monoidal structure is given by the composite. The steps for the construction of $nF$ are the same as the symmetric case. □

**Definition 2.3.22.** Let $\mathcal{G}$ be the forgetful functor from the category of operads $\mathcal{OP}$ to the category of non symmetric operads $n\mathcal{OP}$. $\mathcal{G}$ associate each to each operad the non symmetric operads obtained by dropping the symmetric groups actions.

**Proposition 2.3.23.** *The forgetful functor $\mathcal{G} : \mathcal{OP} \to n\mathcal{OP}$ has a left adjoint $\mathcal{H} : n\mathcal{OP} \to \mathcal{OP}$.*

*Proof.* For an non symmetric operad $\mathcal{P}$ its associated free symmetric operad is given by $\mathcal{P} \otimes \mathsf{k}[\Sigma]$. The verifications are straightforward and left to the reader. □

**Proposition 2.3.24.** *The relations between these forgetful functors and its associated free functors are reunite in the following commutative diagrams.*

$$\begin{tikzcd} \mathcal{OP} \arrow[r, "\mathcal{G}"] \arrow[d, "U"'] & n\mathcal{OP} \arrow[d, "nU"] \\ \mathbb{S}\text{-}Mod \arrow[r, "G"'] & \mathbb{N}\text{-}Mod \end{tikzcd} \qquad \begin{tikzcd} \mathcal{OP} & n\mathcal{OP} \arrow[l, "\mathcal{H}"'] \\ \mathbb{S}\text{-}Mod \arrow[u, "F"] & \mathbb{N}\text{-}Mod \arrow[l, "H"] \arrow[u, "nF"'] \end{tikzcd} \tag{2.58}$$

*Proof.* Immediate, by using the fact that composition of adjoint functors is again an adjunction, and by unicity of the adjunction their images are isormorphic. □

➪ *Remark* 2.3.25. The commutative diagram in proposition 2.3.24 suggests that we can use the construction of free non symmetric operads to describe the free operads over an symmetric sequence in which the actions of symmetric groups are free, in the sense that for a $\mathbb{S}$-module $M$, the free operad $F(M)$ is canonically isomorphic to the operad $(\mathcal{H} \circ nF \circ G)(M)$. Which is the case when the are working with $\mathbb{S}$-modules with components $\Sigma_n$-free bar resolutions. We will use this point of view to describe the $E_\infty$-coalgebra structure on the chain complex associated to a simplicial set.

## 2.4 Colimits of Operads

In this section we show that the category of operads have all small colimits. The existence of this kind of structure will be used in chapter 5 to construct the $E_\infty$-operad $\mathcal{K}$.

The tensor product of DGA-$\mathsf{k}$-modules preserves small colimits in each component, then it satisfies the following lemma ([Fre16]).

**Lemma 2.4.1.** *Let $F : \mathcal{C}^n \to \mathcal{C}$ be a covariant functor. If $F$ preserves reflexive coequalizers in each component, then $F$ preserves reflexive coequalizers in $C^n$. That is, if for every every $1 \leqslant i \leqslant n$ and every $X_i \overset{\leftarrow}{\rightrightarrows} Y_i$ reflexive diagram in $\mathcal{C}$, the morphism given by the universal property of coequalizers from the coequalizer of the diagram in $\mathcal{C}$,*

$$F(A_1, \ldots, A_{i-1}, X_i, A_{i+1}, \ldots, A_n) \overset{\leftarrow}{\rightrightarrows} F(A_1, \ldots, A_{i-1}, Y_i, A_{i+1}, \ldots, A_n) \tag{2.59}$$

*to $F(A_1, \ldots, A_{i-1}, Ceq(X_i \overset{\leftarrow}{\rightrightarrows} Y_i), A_{i+1}, \ldots, A_n)$, is an isomorphism, then for every collection of reflexive diagrams $\{X_i \overset{\leftarrow}{\rightrightarrows} Y_i\}_{1 \leqslant i \leqslant n}$ the morphism from the coequalizer of the diagram in $\mathcal{C}$,*

$$F(X_1, \ldots, X_n) \overset{\leftarrow}{\rightrightarrows} F(Y_1, \ldots, Y_n) \tag{2.60}$$

*to $F(Ceq(X_1 \overset{\leftarrow}{\rightrightarrows} Y_1), \ldots, Ceq(X_n \overset{\leftarrow}{\rightrightarrows} Y_n))$ is an isomorphism.*

*Proof.* The collection of reflexive diagrams $\{X_i \overset{\leftarrow}{\rightrightarrows} Y_i\}_{1\leqslant i\leqslant n}$ defines a collection of functors $\{T_i : \mathcal{D}_0 \to \mathcal{C}\}_{1\leqslant i\leqslant n}$. We use the notation $\operatorname*{colim}_{\alpha\in\mathcal{D}_0} T_i(\alpha)$ for $Ceq(X_i \overset{\leftarrow}{\rightrightarrows} Y_i)$. Then the hypothesis can be written as

$$\operatorname*{colim}_{\alpha\in\mathcal{D}_0} F(A_1,\dots,A_{i-1},T_i(\alpha),A_{i+1},\dots,A_n) \underset{\longrightarrow}{\cong} F(A_1,\dots,A_{i-1},\operatorname*{colim}_{\alpha\in\mathcal{D}_0} T_i(\alpha),A_{i+1},\dots,A_n) \tag{2.61}$$

By proposition 1.6.7 the diagonal $D : \mathcal{D}_0 \to \mathcal{D}_0^n$ is a final functor. Consider the functor $T_1 \times \cdots \times T_n : \mathcal{D}_0^n \to \mathcal{C}^n$. Then the proposition 1.6.8 says that there is an isomorphism from the colimit of $F(T_1\times\cdots\times T_n)D$ to the colimit of $F(T_1\times\cdots\times T_n)$, and we have

$$\begin{aligned}
&Ceq(F(X_1,\dots,X_n) \overset{\leftarrow}{\rightrightarrows} F(Y_1,\dots,Y_n))\\
&= \operatorname*{colim}_{\alpha\in\mathcal{D}_0} F(T_1\times\cdots\times T_n)D(\alpha) && (2.62)\\
&\underset{\longrightarrow}{\cong} \operatorname*{colim}_{(\alpha_1,\dots,\alpha_n)\in\mathcal{D}_0^n} F(T_1(\alpha_1),\dots,T_n(\alpha_n)) \qquad \text{(by 1.6.8)} && (2.63)\\
&\cong \operatorname*{colim}_{\alpha_1\in\mathcal{D}_0}\dots\operatorname*{colim}_{\alpha_n\in\mathcal{D}_0} F(T_1(\alpha_1),\dots,T_n(\alpha_n)) && (2.64)\\
&\underset{\longrightarrow}{\cong} F(\operatorname*{colim}_{\alpha_1\in\mathcal{D}_0} T_1(\alpha_1),\dots,\operatorname*{colim}_{\alpha_n\in\mathcal{D}_0} T_n(\alpha_n)) \qquad \text{(by hypothesis)} && (2.65)\\
&= F(Ceq(X_1 \overset{\leftarrow}{\rightrightarrows} Y_1),\dots,Ceq(X_n \overset{\leftarrow}{\rightrightarrows} Y_n))
\end{aligned}$$

□

In the following proposition we construct an operad using the classic definition of operads in 2.1.1.

**Proposition 2.4.2.** *In the category $\mathcal{OP}$ the forgetful $U : \mathcal{OP} \to \mathbb{S}$-Mod functor creates reflexive coequalizers.*

*Proof.* Let $\mathcal{P} \overset{\leftarrow}{\rightrightarrows} \mathcal{Q}$ a reflexive pair in $\mathcal{OP}$. We are going to construct the reflexive coequalizer $\mathcal{O}$ of this diagram in $\mathcal{OP}$. For that, first we define components of the operad $\mathcal{O}$ by $\mathcal{O}(n) = Coeq(P(n) \overset{\leftarrow}{\rightrightarrows} Q(n))$. This coequalizer exists because $\mathbb{S}$-Mod, as well as DGA-k-Mod, has all small colimits. To define the composition $\gamma$ of $\mathcal{O}$ consider the following morphism of DGA-k-modules.

$$\begin{array}{c}
Coeq\Big[P(k)\otimes P(i_1)\otimes\cdots P(i_k) \overset{\leftarrow}{\rightrightarrows} Q(k)\otimes Q(i_1)\otimes\cdots\otimes Q(i_k)\Big]\\
\Big\downarrow \psi\\
Coeq\Big[P(k) \overset{\leftarrow}{\rightrightarrows} Q(k)\Big]\otimes Coeq\Big[P(i_k) \overset{\leftarrow}{\rightrightarrows} Q(i_k)\Big]\otimes\cdots\otimes Coeq\Big[P(i_k) \overset{\leftarrow}{\rightrightarrows} Q(i_k)\Big]
\end{array} \tag{2.66}$$

By lemma 2.4.1, this morphism is an isomorphism, then we can take its inverse

$\psi^{-1}$ and define $\gamma$ to be the following composition.

$$
\begin{array}{c}
O(k) \otimes O(i_1) \otimes \cdots \otimes O(i_k) = \\
Coeq\left[P(k) \overset{\leftarrow}{\rightrightarrows} Q(k)\right] \otimes Coeq\left[P(i_k) \overset{\leftarrow}{\rightrightarrows} Q(i_k)\right] \otimes \cdots \otimes Coeq\left[P(i_k) \overset{\leftarrow}{\rightrightarrows} Q(i_k)\right] \\
\Big\downarrow \psi^{-1} \\
Coeq\left[P(k) \otimes P(i_1) \otimes \cdots P(i_k) \overset{\leftarrow}{\rightrightarrows} Q(k) \otimes Q(i_1) \otimes \cdots \otimes Q(i_k)\right] \\
\Big\downarrow \overline{\gamma} \\
Coeq\left[P(i_1 + \cdots i_k) \overset{\leftarrow}{\rightrightarrows} Q(i_1 + \cdots i_k)\right] = O(i_1 + \cdots + i_k)
\end{array}
\tag{2.67}
$$

Where $\overline{\gamma}$ is the induced morphism by the compositions of operads $\mathcal{P}$ and $\mathcal{Q}$. For define the unit of $\mathcal{O}$, consider the following commutative diagram obtained from $\mathcal{P} \overset{\leftarrow}{\rightrightarrows} \mathcal{Q}$ and the coequalizer properties.

$$
\begin{tikzcd}
 & P(1) \arrow[d, shift right] \arrow[d, shift left] \arrow[dr, "i_{P(1)}"] & \\
\mathsf{k} \arrow[ur, "\eta_{\mathcal{P}}"] \arrow[dr, "\eta_{\mathcal{Q}}"'] & & O(1) \\
 & Q(1) \arrow[u, bend right] \arrow[ur, "i_{Q(1)}"'] &
\end{tikzcd}
\tag{2.68}
$$

Thus the unit for $\mathcal{O}$ is defined by the composite $i_{P(1)}\eta_{\mathcal{P}} : \mathsf{k} \to O(1)$. Note that the choice $i_{Q(1)}\eta_{\mathcal{Q}}$ gives the same result, as a consequence of reflexive arrow existence. It is not hard to check that $\mathcal{O}$ with this structure satisfies the axioms of operads and the universal property for coequalizer. $\square$

**Proposition 2.4.3.** *Let $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ two small collections of objects in a category $\mathcal{C}$ such that the colimits $\underset{i\in I}{colim}\, P_i$ and $\underset{i\in I}{colim}\, Q_i$ exist. Denote for $i \in I$ the cocone edges $p_i : P_i \to \underset{i\in I}{colim}\, P_i$ and $q_i : Q_i \to \underset{i\in I}{colim}\, Q_i$.*

1. *Every collection of morphisms $f_i : P_i \to \underset{i\in I}{colim}\, Q_i$, $i \in I$, determines a morphism $\overline{f} : \underset{i\in I}{colim}\, P_i \to \underset{i\in I}{colim}\, Q_i$, such that $f_i = \overline{f} \circ p_i$ for every $i \in I$.*

2. *Every collection of morphism $f_i : P_i \to Q_i$, $i \in I$, determines a morphism $f : \underset{i\in I}{colim}\, P_i \to \underset{i\in I}{colim}\, Q_i$, such that $q_i \circ f = f_i \circ p_i$ for every $i \in I$.*

*Proof.* The collection $f_i : P_i \to \operatorname{colim}_{i\in I} Q_i$, $i \in I$ exhibit $\operatorname{colim}_{i\in I} Q_i$ as a cocone over the diagram $\{P_i\}_{i\in I}$, then by the universal property of coproducts, there exists an unique morphism in $\mathcal{C}$, $\overline{f} : \operatorname{colim}_{i\in I} P_i \to \operatorname{colim}_{i\in I} Q_i$ such that $f_i = \overline{f} \circ p_i$ for every $i \in I$.

For the second statement, compose every $f_i : P_i \to Q_i$ with the respective cocone edge $q_i : Q_i \to \operatorname{colim}_{i\in I} Q_i$, then we have a collection $g_i : P_i \to \operatorname{colim}_{i\in I} Q_i$, with $g_i = q_i \circ f_i$ for $i \in I$. And applying the first part we get that this determines $f = \overline{g} : \operatorname{colim}_{i\in I} P_i \to \operatorname{colim}_{i\in I} Q_i$. $\square$

**Proposition 2.4.4.** *The category of operads has all small colimits.*

*Proof.* Let $\{P_i\}_{i\in I}$ be a small collection of operads. With it is possible construct a reflexive pair in $\mathcal{OP}$, and by proposition 2.4.2 its reflexive coequalizer exists in $\mathcal{OP}$. The last part of the proof consists in checking that this reflexive coequalizer is the colimit of $\{P_i\}_{i\in I}$.

From $\{P_i\}_{i\in I}$ we obtain in $\mathbb{S}$-Mod the collection $\{U(P_i)\}_{i\in I}$, denote its colimit $\operatorname*{colim}_{i\in I} U(P_i)$, and $\alpha_i : U(P_i) \to \operatorname*{colim}_{i\in I} U(P_i)$ the cocone edges.

The morphisms $\alpha_i : U(P_i) \to \operatorname*{colim}_{i\in I} U(P_i)$ induce the morphisms $UF(\alpha_i) : UFU(P_i) \to UF(\operatorname*{colim}_{i\in I} U(P_i))$, which determines the following morphism.

$$\operatorname*{colim}_{i\in I} UFU(P_i) \xrightarrow{\underline{d_0}} UF(\operatorname*{colim}_{i\in I} U(P_i)) \tag{2.69}$$

Consider the unit $\epsilon$ and counit $\eta$ of the adjunction $F \vdash U$, and the following composite in $\mathbb{S}$-Mod.

$$UFU(P_i) \xrightarrow{U(\epsilon_{P_i})} U(P_i) \xrightarrow{\alpha_i} \operatorname*{colim}_{i\in I} U(P_i) \xrightarrow{\eta} UF(\operatorname*{colim}_{i\in I} U(P_i)) \tag{2.70}$$

These compositions determines the morphism in $\mathbb{S}$-Mod,

$$\operatorname*{colim}_{i\in I} UFU(P_i) \xrightarrow{\underline{d_1}} UF(\operatorname*{colim}_{i\in I} U(P_i)) \tag{2.71}$$

By the universal property of free operads $\underline{d_0}$ and $\underline{d_1}$ will determine the morphisms $d_0$ and $d_1$ in $\mathcal{OP}$ in the following commutative diagram.

$$\begin{tikzcd}
UF\left(\operatorname*{colim}_{i\in I} UFU(P_i)\right) \arrow[dr, shift left, "U(d_1)"] \arrow[dr, shift right, "U(d_0)"'] & & F\left(\operatorname*{colim}_{i\in I} UFU(P_i)\right) \arrow[dr, shift left, "d_1"] \arrow[dr, shift right, "d_0"'] & \\
\operatorname*{colim}_{i\in I} UFU(P_i) \arrow[u, "\eta"] \arrow[r, shift left, "\underline{d_1}"] \arrow[r, shift right, "\underline{d_0}"'] & UF(\operatorname*{colim}_{i\in I} U(P_i)) & & F(\operatorname*{colim}_{i\in I} U(P_i))
\end{tikzcd} \tag{2.72}$$

Now we give the contraction $s$ for $d_0$ and $d_1$. With the counit $\eta$ consider the morphisms $\eta_{U(P_i)} : U(P_i) \to UFU(P_i)$. By the colimit properties 2.4.3 they determine a morphism of $\mathbb{S}$-modules $\beta : \operatorname*{colim}_{i\in I} U(P_i) \to \operatorname*{colim}_{i\in I} UFU(P_i)$, and we take $s = F(\beta)$. Thus we have the following diagram in $\mathcal{OP}$.

$$\begin{tikzcd}
F\left(\operatorname*{colim}_{i\in I} UFU(P_i)\right) \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & F(\operatorname*{colim}_{i\in I} U(P_i)) \arrow[l, bend right=40, "s"']
\end{tikzcd} \tag{2.73}$$

Before take the reflexive coproduct of this diagram we have to check that $d_1 s = d_0 s = 1$. To show this we only have to check that over their components defined over $U(P_i)$, $d_1 s$ and $d_0 s$ are both equal to the identity.

The morphism $s$ is determined by $\eta_{U(P_i)} : U(P_i) \to UFU(P_i)$, $d_0$ by $UF(\alpha_i) : UFU(P_i) \to UF(\operatorname*{colim}_{i\in I} U(P_i))$. The naturality of $\eta$ makes the following diagram commutative.

$$
\begin{tikzcd}
U(P_i) \arrow[r, "\eta_{U(P_i)}"] \arrow[d, "\alpha_i"'] & UFU(P_i) \arrow[d, "UF(\alpha_i)"] \\
\operatorname*{colim}_{i\in I} U(P_i) \arrow[r, "\eta"] & UF(\operatorname*{colim}_{i\in I} U(P_i))
\end{tikzcd}
\tag{2.74}
$$

Then we have that $UF(\alpha_i)\eta_{U(P_i)} = \eta\alpha_i : U(P_i) \to UF(\operatorname*{colim}_{i\in I} U(P_i))$, which induces the identity over $UF(\operatorname*{colim}_{i\in I} U(P_i))$. For $d_1 s$, $d_1$ is determined by the composition 2.70, then we have $d_1 s$ is determined by the composition $\eta\alpha_i U(\epsilon_{P_i})\eta_{U(P_i)}$, which by the triangular equations of the unit and counit (proposition 1.5.3), is equal to $\eta\alpha_i$, which as before induce the identity over $UF(\operatorname*{colim}_{i\in I} U(P_i))$. Then by proposition 2.4.2 there exist the coequalizer of the diagram 2.73,

$$
\begin{tikzcd}
F\left(\operatorname*{colim}_{i\in I} UFU(P_i)\right) \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & F(\operatorname*{colim}_{i\in I} U(P_i)) \arrow[l, bend right, "s"'] \arrow[r, "q"] & Q
\end{tikzcd}
\tag{2.75}
$$

The operad $Q$ will be the colimit of the collection $\{P_i\}_{i\in I}$. We only have to check the existence of operad morphisms from each $P_i$ to $Q$ and the universal property for colimits. In order to do that, first we are going to see the information that been an coequalizer of $d_0$ and $d_1$ gives.

Let $R$ be an operad and an operad morphism $f : F(\operatorname*{colim}_{i\in I} U(P_i)) \to R$ such that $fd_0 = fd_1$. This morphism is determined by its components $h_i : U(P_i) \to U(R)$ given by the compositions,

$$
\begin{tikzcd}
U(P_i) \arrow[r, "\alpha_i"] \arrow[rr, bend right, "h_i"'] & \operatorname*{colim}_{i\in I} U(P_i) \arrow[r, "\theta(f)"] & U(R)
\end{tikzcd}
\tag{2.76}
$$

The morphisms $fd_0$ and $fd_1$ are determined by morphisms from $UFU(P_i)$ to $U(R)$ in $\mathbb{S}$-Mod, so we will describe in terms of their components the relation $fd_1 = fd_0$. In the case of $fd_0$ recall that $d_0$ is determined by the morphisms $UF(\alpha_i) : UFU(P_i) \to UF(\operatorname*{colim}_{i\in I} U(P_i))$ and consider the following commutative diagram.

$$
\begin{tikzcd}
UFU(P_i) \arrow[dr, "UF(\alpha_i)"] \arrow[dddrr, dashed, bend left, "U(\theta^{-1}(h_i))"] & & \\
& UF(\operatorname*{colim}_{i\in I} U(P_i)) \arrow[ddr, "U(f)"] & \\
& \operatorname*{colim}_{i\in I} U(P_i) \arrow[u, "\eta"] \arrow[dr, "\theta(f)"] & \\
U(P_i) \arrow[uuu, "\eta_{U(P_i)}"] \arrow[ur, "\alpha_i"] \arrow[rr, "h_i"'] & & U(R)
\end{tikzcd}
\tag{2.77}
$$

The quadrilateral is commutative by the naturality of the counit. Then the composite at the diagonal $U(f)UF(\alpha_i)$ is determined by the bottom side, that is $h_i$, and the bijection of the adjunction $F \vdash U$ says that $U(f)UF(\alpha_i)$ is equal to $U(\theta^{-1}(h_i))$, which means that $fd_0$ is determined by $U(\theta^{-1}(h_i)) : UFU(P_i) \to U(R)$.

For $fd_1$, recall that $d_1$ is determined by the composition 2.70, then $fd_1$ is determined by the composition,

$$UFU(P_i) \xrightarrow{U(\epsilon_{P_i})} U(P_i) \xrightarrow{\alpha_i} \operatorname*{colim}_{i\in I} U(P_i) \xrightarrow{\eta} UF(\operatorname*{colim}_{i\in I} U(P_i)) \xrightarrow{U(f)} U(R) \tag{2.78}$$

By 2.77 we have that $U(f)\eta\alpha_i = \theta(f)\alpha_i = h_i$, then $fd_1$ is determined by the composition $U(\epsilon_{P_i})h_i : UFU(P_i) \to U(R)$. Together with the result for $fd_0$, says that $fd_1 = fd_0$ if and only if the following diagram is commutative.

$$\begin{tikzcd} UFU(P_i) \arrow[d, "U(\epsilon_{P_i})"'] \arrow[dr, "U(\theta^{-1}(h_i))"] & \\ U(P_i) \arrow[r, "h_i"'] & U(R) \end{tikzcd} \tag{2.79}$$

This diagram is commutative if and only if $h_i$ is a morphism of operads, in other words, if there is a morphism of operads $f_i : P_i \to R$ such that $U(f_i) = h_i$.

Suppose that 2.79 is commutative. We need to proof that $h_i$ preserves the operadic structure on $P_i$, that is we have to check the conditions of definition 2.1.7. To avoid confusion we denote $\lambda$ the unit of an operad in this part.

1. The unit.

$$\begin{aligned} h_iU(\lambda_{P_i}) &= h_iU(\epsilon_{P_i}\lambda_{FU(P_i)}) && (2.80)\\ &= h_iU(\epsilon_{P_i})U(\lambda_{FU(P_i)}) && (2.81)\\ &= U(\theta^{-1}(h_i))U(\lambda_{FU(P_i)}) && (2.82)\\ &= U(\theta^{-1}(h_i)\lambda_{FU(P_i)}) = U(\lambda_R) && (2.83)\\ & && (2.84) \end{aligned}$$

2. Equivariance follows by the fact that all are morphisms of $\mathbb{S}$-modules.

3. The composition.

$$\begin{aligned}
h_iU(\gamma_{P_i}) &= h_iU(\gamma_{P_i}1_{P_i}) && (2.85)\\
&= h_iU(\gamma_{P_i})1_{U(P_i)} && (2.86)\\
&= h_iU(\gamma_{P_i})U(\epsilon_{P_i})\eta_{U(P_i)} && (2.87)\\
&= h_iU(\gamma_{P_i}\epsilon_{P_i})\eta_{U(P_i)} && (2.88)\\
&= h_iU(\epsilon_{P_i}\gamma_{FU(P_i)})\eta_{U(P_i)} && (2.89)\\
&= h_iU(\epsilon_{P_i})U(\gamma_{FU(P_i)})\eta_{U(P_i)} && (2.90)\\
&= U(\theta^{-1}(h_i))U(\gamma_{FU(P_i)})\eta_{U(P_i)} && (2.91)\\
&= U(\theta^{-1}(h_i)\gamma_{FU(P_i)})\eta_{U(P_i)} && (2.92)\\
&= U(\gamma_R\theta^{-1}(h_i))\eta_{U(P_i)} && (2.93)\\
&= U(\gamma_R)U(\theta^{-1}(h_i))\eta_{U(P_i)} && (2.94)\\
&= U(\gamma_R)h_i && (2.95)
\end{aligned}$$

Conversely, suppose there is a collection of morphisms of operads $\{f_i : P_i \to R\}_{i\in I}$, such that $U(f_i) = h_i$ for every $i \in I$. Then the following triangle is commutative by the naturality of $\theta^{-1}$.

$$\begin{tikzcd}
FU(P_i) \arrow[d, "\epsilon_{P_i}"'] \arrow[dr, "\theta^{-1}(U(f_i))"] & \\
P_i \arrow[r, "f_i"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{OP}(FU(P_i), R) & \mathbb{S}\text{-Mod}(U(P_i), U(R)) \arrow[l, "\theta^{-1}"'] \\
\mathcal{OP}(FU(P_i), P_i) \arrow[u, "f_{i*}"] & \mathbb{S}\text{-Mod}(U(P_i), U(P_i)) \arrow[l, "\theta^{-1}"'] \arrow[u, "U(f_i)_*"']
\end{tikzcd}
\tag{2.96}$$

Then the diagram 2.79 is commutative. Now we pass to verify that $Q$ is the colimit of the collection of operads $\{P_i\}_{i\in I}$. We saw that this collection induces morphisms $h_i : U(P_i) \to U(Q)$ of $\mathbb{S}$-modules that satisfy 2.79, then they define morphisms of operads $f_i : P_i \to Q$, such that $U(f_i) = h_i$. These morphisms are the cocone edges.

Any collection of operad morphisms $f_i : P_i \to R$, defines a morphisms of operads $f$ from $F(\operatorname{colim}_{i\in I} U(P_i))$ to $R$ such that $fd_0 = fd_1$. Then there is an unique morphism of operads $g : Q \to R$ such that $gq = f$. The morphism $g$ commutes with the cocone edges, and this exhibit $Q$ as the colimit of $\{P_i\}_{i\in I}$. $\square$

# Chapter 3

# $\mathcal{L}$-Algebras

The central notion of this thesis is the algebraic structure called $\mathcal{L}$-algebra. Introduced by Alain Prouté in several talks since the eighties and never published (Max Planck Institut-Bonn 1986, Louvain-la Neuve 1987, Freie Universität-Berlin 1988, Seminar Keller-Maltsiniotis-Paris 2010), $\mathcal{L}$-algebras have been thought to be highly related to the homotopy type of spaces by using an internal structure that models the diagonals which determines invariants like Steenrod operations. $\mathcal{L}$-algebras are similar to Segal's $\Gamma$-structures (see [Seg74]), but in an algebraic context instead of a topological context. It happens that the Eilenberg-Mac Lane transformation plays a central role in $\mathcal{L}$-algebras, where it is the prototype (motivation) of the product of $\mathcal{L}$-algebras. The present chapter introduces the concept of $\mathcal{L}$-algebras and contains establishes its principal properties. It is interesting to notice the existence of a preprint of Tom Leinster (see [Lei00]), which present a similar definition.

## 3.1 The Category $\mathcal{L}$

We saw in the chapter of preliminaries that simplicial sets are described as contravariant functors from the simplicial category $\Delta$ to the category of sets. In this way the simplicial relations are coded by the category $\Delta$, which allows an easy extension of the concept of simplicial set to other categories and gives the definition of simplicial object. With $\mathcal{L}$-algebras our principal interest is to model the relations describing the behavior of diagonals in chain complexes. This can be done by using an approach similar to the technique used to define simplicial objects, that is, defining $\mathcal{L}$-algebras as contravariant functors from a suitable category. This category will be denoted $\mathcal{L}$.

**Definition 3.1.1.** We define $\mathcal{L}$ to be the category where the objects are the totally ordered sets $[n] = \{1, \ldots, n\}$ for $n > 0$ and $[0] = \varnothing$, the empty set†. The arrows of $\mathcal{L}$ are all the partial maps between these sets. The composition is simply the composition of partial maps of sets.

➪ *Remark* 3.1.2. We can describe any arrow $\alpha : [n] \to [m]$ of $\mathcal{L}$ by a pair $(D, f)$, where $D$ is a subset of $[n]$ and $f$ is an everywhere defined map from $D$ to $[m]$. The set $D$ is called the domain of $f$ and is denoted by $Dom(f)$. Then, the composition in $\mathcal{L}$ of two arrows $(Dom(f), f) : [n] \to [m]$ and $(Dom(g), g) : [m] \to [p]$, will be the pair $(Dom(g \circ f), g \circ f) : [n] \to [p]$, where $Dom(g \circ f) = f^{-1}(Dom(g))$.

†We make the abuse of using the same notation for the objects in the category $\Delta$

➪ *Remark* 3.1.3. Note that $\mathcal{L}$ contains as a subcategory a copy of the simplicial category $\Delta$, by taking the embedding $\{0, \ldots, n\} \mapsto \{1, \ldots, n+1\}$, but clearly it's not a full subcategory. Also, the set of morphisms $\mathcal{L}([n], [n])$ include the set $\Sigma_n$ of permutations of $n$ elements.

➪ *Remark* 3.1.4. The objects of the category $\Gamma$ (see [Seg74]) are the finite sets, and a morphism from $x$ to $y$ is an application $f : x \to \mathcal{P}(y)$ † such that $z_1 \neq z_2$ implies $f(z_1) \cap f(z_2) = \emptyset$. Then we have the isomorphisms of categories $\Gamma^{\mathrm{op}} \cong \mathcal{L}$ and $\mathcal{L}^{\mathrm{op}} \cong \Gamma$.

**Proposition 3.1.5.** *The category $\mathcal{L}$ equipped with the sum functor,*

$$+ : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \tag{3.1}$$

*defined for objects by,*

$$[n] + [m] := [n+m] \tag{3.2}$$

*and for arrows $\alpha : [n] \to [p]$ and $\beta : [m] \to [q]$, as the sum $\alpha + \beta : [n+m] \to [p+q]$ given by,*

$$(\alpha + \beta)(n) = \begin{cases} \alpha(x) & \text{if } x \leqslant n \\ p + \beta(x - n) & \text{if } x > n \end{cases} \tag{3.3}$$

*is a cocartesian category. The object $[0]$ is the zero object of $\mathcal{L}$, that is, an object which is at the same time terminal and initial. In both cases the universal map has empty domain.*

*Proof.* Let $i_1 : [n] \to [n+m] \leftarrow [m] : i_2$ the cocone in $\mathcal{L}$ where the application $i_1, i_2$ are the inclusion $i_1(x) = x$ for $x \in [n]$ and $i_2(y) = (y+n)$ for $y \in [m]$. To show that $\mathcal{L}$ is a cocartesian category it suffice to show that this cocone is initial. Let $\alpha : [n] \to [r] \leftarrow [m] : \beta$ any cocone from $[n]$ and $[m]$. Let $\gamma : [n+m] \to [r]$ defined by $\gamma(z) = \alpha(z)$ if $1 \leqslant z \leqslant n$ and $\gamma(z) = \beta(z-n)$ if $n < z \leqslant n+m$. Then it is clear that $\gamma : [n+m] \to [r]$ is the only application in $\mathcal{L}$ such that $\gamma \circ i_i = \alpha$ and $\gamma \circ i_2 = \beta$. □

**Definition 3.1.6.** When a cocartesian category has a zero object is called pointed cocartesian category. Furthermore, if the zero and the sum are explicitly given, the category is called strict pointed cocartesian category.

➪ *Remark* 3.1.7. $\mathcal{L}$ is a strict pointed cocartesian category.

**Proposition 3.1.8.** *The sum defined in $\mathcal{L}$ is strictly associative, i.e.*

$$([n] + [m]) + [p] = [n] + ([m] + [p]) \tag{3.4}$$

*for all objects in $\mathcal{L}$, and $(f+g)+h = f+(g+h)$ for all morphisms.*

*Proof.* The proof is immediate. □

† $\mathcal{P}(y)$ is the set of subsets of $y$.

The sum and composition of $\mathcal{L}$ can be used to generate all the morphism in $\mathcal{L}$ from a small set of morphism in $\mathcal{L}$. The required morphisms of $\mathcal{L}$ are introduced in the following definition.

**Definition 3.1.9.** In $\mathcal{L}$ we identify the following arrows.

1. The face operator $d_i : [n] \to [n+1]$ is defined for $1 \leqslant i \leqslant n+1$ by:

$$d_i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \geqslant i \end{cases} \tag{3.5}$$

   When $n = 0$, the only face operator $d_1 : [0] \to [1]$ is the universal morphism from $[0]$.

2. The degeneracy operator $s_i : [n] \to [n-1]$ is defined for $1 \leqslant i \leqslant n$ by:

$$s_i(x) = \begin{cases} x & \text{if } x \leqslant i \\ x-1 & \text{if } x > i \end{cases} \tag{3.6}$$

   In the case $n = 1$, the only degeneracy operator $s_1 : [1] \to [0]$ is the universal morphism to $[0]$.

3. In $\mathcal{L}$, any injective map $i : [n] \to [m]$ of the form $([n], i)$ has a unique minimal retraction, denoted by $\bar{i} : [m] \to [n]$, in other words, $\bar{i}$ is the only morphism with domain given by the image of $i$ and which satisfies the relation $\bar{i} \circ i = 1_{[n]}$. In particular, the minimal retraction associated to the face operator $d_i$ will be denoted $\zeta_i$. For $d_1 : [0] \to [1]$, its minimal retraction $\zeta_1 : [1] \to [0]$ coincide with $s_1 : [1] \to [0]$.

➪ *Remark* 3.1.10. The operator $d_i : [n] \to [n+1]$ is the only increasing injection ignoring $i \in [n+1]$.

$$\begin{array}{ccccccccccc} [n] & = & \{1, & 2, & \cdots & i-1, & i, & & \cdots & n-1, & n\} \\ {\scriptstyle d_i}\big\downarrow & & \big\downarrow & \big\downarrow & & \big\downarrow & \searrow & & & \searrow & \searrow \\ [n+1] & = & \{1, & 2, & \cdots & i-1, & i, & i+1, & \cdots & n, & n+1\} \end{array} \tag{3.7}$$

The operator $s_i$ is the only decreasing surjection crashing $i$ and $i+1$ in the same element of $[n-1]$.

$$\begin{array}{cccccccccc} [n] & = & \{1, & 2, & \cdots & i, & i+1, & i+2, & \cdots & n-2, & n\} \\ {\scriptstyle s_i}\big\downarrow & & \big\downarrow & \big\downarrow & & \big\downarrow & \swarrow & \swarrow & & \swarrow & \swarrow \\ [n-1] & = & \{1, & 2, & \cdots & i, & i+1, & & \cdots & n-2, & n-1\} \end{array} \tag{3.8}$$

The operator $\zeta_i$ is like $d_i$ but with inversed arrows, that is, the only decreasing injection without $i$ in its domain.

$$\begin{array}{ccccccccccc} [n] & = & \{1, & 2, & \cdots & ,i-1, & i, & & \cdots & n-1, & n\} \\ {\scriptstyle \zeta_i}\big\uparrow & & \big\uparrow & \big\uparrow & & \big\uparrow & \nwarrow & & & \nwarrow & \nwarrow \\ [n+1] & = & \{1, & 2, & \cdots & ,i-1, & i, & i+1, & \cdots & ,n, & n+1\} \end{array} \tag{3.9}$$

**Proposition 3.1.11.** *Let* $1$ *be the identity of* $[1]$, $d_1$ *the only face operator from* $[0]$ *to* $[1]$ *and* $\tau : [2] \to [2]$ *the only non trivial permutation of* $[2]$. *Then we have the following decompositions.*

1. *Every face operator* $d_i : [n] \to [n+1]$ *can be expressed as the sum,*

$$d_i = 1 + \overset{(i-1)}{\cdots} + 1 + d_1 + 1 + \overset{(n-i+1)}{\cdots} + 1 \tag{3.10}$$

2. *Any transposition* $\sigma : [n] \to [n]$, *that is, a permutation that exchanges two consecutive elements* $i$, $i+1$ *and leaves the rest fixed, can be expressed by the sum,*

$$\sigma = 1 + \overset{(i-1)}{\cdots} + 1 + \tau + 1 + \overset{(n-i-1)}{\cdots} + 1 \tag{3.11}$$

*Proof.* The face operator $d_1 : [n] \to [n+1]$ is equal to the sum $d_1 + 1_{[n]}$, as the following pictures shows.

$$\begin{array}{ccccccc}
[n] & = & \{1,\ 2,\ \cdots\ ,n\} & = & \varnothing\ +\ \{1,\ 2,\ \cdots\ ,n\} & = & [n] \\
{\scriptstyle d_1}\downarrow & & \searrow\ \ \searrow\qquad\searrow & & \downarrow\qquad\downarrow\ \ \downarrow\quad\ \downarrow & & \downarrow{\scriptstyle d_1+1_{[n]}} \\
[n+1] & = & \{1,\ 2,\ 3,\ \cdots\ ,n+1\} & = & \{1\}\ +\ \{1,\ 2,\ \cdots\ ,n\} & = & [n+1]
\end{array} \tag{3.12}$$

Also we have that $1_{[n]} = 1 + \overset{(n)}{\cdots} + 1$, then $d_1 : [n] \to [n+1]$ is equal to $d_1 + 1 + \overset{(n)}{\cdots} + 1$. So we can express the face operator $d_i : [n] \to [n+1]$ as the sum $1_{[i-1]} + d_1$, with $d_1 : [n-i+1] \to [n+i+2]$, and obtain that,

$$d_i = 1_{[i-1]} + d_1 + 1_{[n-i+1]} = 1 + \overset{(i-1)}{\cdots} + 1 + d_1 + 1 + \overset{(n-i+1)}{\cdots} + 1 \tag{3.13}$$

The transposition $\sigma : [n] \to [n]$ can be written like $1_{[i-1]} + \tau + 1_{[n-i-1]}$ as the following picture shows.

$$\begin{array}{ccccccccc}
[n] & = & \{1,\ 2,\ \cdots\ i-1,\ i,\ i+1,\ i+2,\ \cdots\ n\} & = & [i-1] & + & [2] & + & [n-i-1] \\
{\scriptstyle \sigma}\downarrow & & \downarrow\ \ \downarrow\qquad\ \downarrow\quad\ \times\quad\ \ \downarrow\qquad\quad\downarrow & & \downarrow{\scriptstyle 1_{[i-1]}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle 1_{[n-i-1]}} \\
[n-1] & = & \{1,\ 2,\ \cdots\ i-1,\ i,\ i+1,\ i+2,\ \cdots,\ n\} & = & [i-1] & + & [2] & + & [n-i-1]
\end{array} \tag{3.14}$$

Then we have the decomposition,

$$\sigma = 1 + \overset{(i-1)}{\cdots} + 1 + \tau + 1 + \overset{(n-i-1)}{\cdots} + 1 \tag{3.15}$$

□

**Proposition 3.1.12.** *All the arrows of $\mathcal{L}$ can be generated using the sum $+ : \mathcal{L}\times\mathcal{L} \to \mathcal{L}$ and compositions of the following five arrows.*

$$\begin{tikzcd}
{[0]} \arrow[r, bend left, "d_1"] & {[1]} \arrow[l, bend left, "\zeta_1"] \arrow[loop above, "1"] & {[2]} \arrow[l, "s_1"] \arrow[loop above, "\tau"]
\end{tikzcd} \tag{3.16}$$

*Proof.* The result follows from the considerations below.

1. Any $i : [p] \to [n]$ increasing injective defined everywhere application, can be expressed as a composition of face operators, then by proposition 3.1.11, $i$ is expressed as compositions of sums of 1 and $d_1$. Then its minimal retraction will be expressed as compositions of sums of 1 and $\zeta_1$.

2. Any permutation of $[n]$ is a composition of transpositions, then by proposition 3.1.11, it is a composition of applications of the form $1+\cdots+1+\tau+1+\cdots+1$.

3. Any defined everywhere application from $[n]$ to $[1]$ sending every number to 1, can be expressed as a composition of applications of the form $s_1 + \cdots + s_1$ or $s_1 + \cdots + s_1 + 1$.

4. Any morphism $\alpha : [n] \to [m]$ of $\mathcal{L}$ can be expressed as a composite $\sigma \circ i \circ \beta \circ j$, where the minimal retraction of $j$ is increasing injective defined everywhere, $\beta$ is defined everywhere increasing, $i$ is increasing injective defined everywhere and $\sigma$ is a permutation of $[m]$.

□

Actually, we can characterize the category $\mathcal{L}$ as the free strictly associative pointed cocartesian category on one object, as the following proposition shows.

**Proposition 3.1.13.** *Let $\mathcal{C}$ be a strictly associative pointed cocartesian category, and $X$ an object of $\mathcal{C}$. Then there an unique functor $F : \mathcal{L} \to \mathcal{C}$ preserving zero and coproducts and such that $F([1]) = X$.*

*Proof.* Indeed, $F([n])$ must be the $n$-fold sum $X + \overset{(n)}{\cdots} + X$ and $F([0])$ must be the zero object of $\mathcal{C}$. The five arrows above have mandatory images by F, this means that $F(1) = 1_X$, $d_1$ and $\zeta_1$ are send to the unique arrows $0 \to X$ and $X \to 0$, where 0 is the zero object of $\mathcal{C}$, the image of $s_1$ is the codiagonal of $[1]$, that is the morphisms $[1] + [1] \to [1]$ obtained by the universal property of coproduct,

$$\begin{tikzcd}
 & {[1]+[1]} \arrow[dd, dashed, "s_1"] & \\
{[1]} \arrow[ur, "i_1"] \arrow[dr, "1"'] & & {[1]} \arrow[ul, "i_2"'] \arrow[dl, "1"] \\
 & {[1]} &
\end{tikzcd} \tag{3.17}$$

so its image by $F$ must be $X + X \to X$, the codiagonal of $X$, which is well defined because $\mathcal{C}$ is cocartesian, that is, the sum is well defined. And $\sigma$ which is the canonical twisting arrow of the sum $[1] + [1]$,

$$\begin{tikzcd}
& {[1]+[1]} \arrow[dd, dashed, "s_1"] & \\
{[1]} \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & {[1]} \arrow[ul, "i_2"'] \arrow[dl, "i_1"] \\
& {[1]+[1]} &
\end{tikzcd} \tag{3.18}$$

should be send to the canonical twisting arrow of $X + X$. □

➪ *Remark* 3.1.14. In the same sense of this definition, the opposite category $\mathcal{L}^{\text{op}}$ of $\mathcal{L}$ is characterized as the free strictly associative pointed cartesian category on the object $[1]$.

## 3.2 $\mathcal{L}$-Algebras

In this section we present the definition of $\mathcal{L}$-algebras. An $\mathcal{L}$-algebra is a contravariant functor from $\mathcal{L}$ to a category with a notion of homology and a natural transformation $\mu$, which will be called the product of the $\mathcal{L}$-algebra. The homotopy coherence is concentrated in the fact that $\mu$ induces isomorphisms in homology. Then we will deal with categories equipped with quasi-isomorphisms, that is, a distinguished class of arrows, called quasi-isomorphisms, which forms a subcategory of the given category. The only categories of this kind that we will use are the already mentioned DGA-$\mathsf{k}$-Mod and DGA-$\mathsf{k}$-Alg, where being an quasi-isomorphisms means inducing an isomorphism in homology.

**Definition 3.2.1** ($\mathcal{L}$-algebra)**.** Let $(\mathcal{C}, \otimes, k, T)$ be a strict symmetric monoidal category with quasi-isomorphisms. An $\mathcal{L}$-algebra $A$ with values in the category $\mathcal{C}$ consists of a functor,

$$A : \mathcal{L}^{\text{op}} \to \mathcal{C} \tag{3.19}$$

together with a natural transformation $\mu : \otimes \circ (\mathcal{A} \times \mathcal{A}) \to \mathcal{A} \circ +$.

$$\begin{tikzcd}
\mathcal{L} \times \mathcal{L} \arrow[r, bend left=50, "\otimes \circ (\mathcal{A} \times \mathcal{A})"{name=U}] \arrow[r, bend right=50, "\mathcal{A} \circ +"'{name=D}] & \mathcal{C}
\arrow[from=U, to=D, phantom, "\Downarrow \mu"]
\end{tikzcd} \tag{3.20}$$

The morphism in $\mathcal{C}$ that $\mu$ associates to each pair $([n], [m])$ of $\mathcal{L} \times \mathcal{L}$, goes from $A[n] \otimes A[m]$ to $A[n+m]$ and is written $\mu_{[n],[m]}$. The image of any arrow $\alpha$ of $\mathcal{L}$ by the functor $\mathcal{A}$, $\mathcal{A}(\alpha)$, is simply written again as $\alpha$, but this image goes in the opposite direction of the original arrow in $\mathcal{L}$. Then, for every pair of arrows $\alpha : [p] \to [n]$ and $\beta : [q] \to [m]$ in $\mathcal{L}$ we have the following commutative diagram.

$$\begin{tikzcd} A[n] \otimes A[m] \arrow[r, "\mu_{[m],[n]}"] \arrow[d, "\alpha\otimes\beta"'] & A[n+m] \arrow[d, "\alpha+\beta"] \\ A[p] \otimes A[q] \arrow[r, "\mu_{[p],[q]}"] & A[p+q] \end{tikzcd} \tag{3.21}$$

The functor $\mathcal{A}$ and the natural transformation $\mu$ are required to satisfy the following conditions.

1. **Associativity:** $\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$. Equivalently, for every $[n], [m]$ and $[p]$ in $\mathcal{L}$ the following diagram commutes.

$$\begin{tikzcd} A[n] \otimes A[m] \otimes A[p] \arrow[r, "\mu_{[n],[m]}\otimes 1"] \arrow[d, "1\otimes\mu_{[m],[p]}"'] & A[n+m] \otimes A[p] \arrow[d, "\mu_{[n+m],[p]}"] \\ A[n] \otimes A[m+p] \arrow[r, "\mu_{[n],[m+p]}"] & A[n+m+p] \end{tikzcd} \tag{3.22}$$

2. **Commutativity:** Let $[n]$, $[m]$ in $\mathcal{L}$, and $\tau : [m+n] \to [n+m]$ be the twisting morphism of $[m] + [n]$, then the following diagram commutes.

$$\begin{tikzcd} A[n] \otimes A[m] \arrow[r, "\mu"] \arrow[d, "T"'] & A[n+m] \arrow[d, "\tau"] \\ A[m] \otimes A[n] \arrow[r, "\mu"] & A[m+n] \end{tikzcd} \tag{3.23}$$

3. **Unit:** The image of $[0]$ by $\mathcal{A}$ is $k$ and $\mu_{[0],[n]} = \mu_{[n],[0]} = 1$. In terms of commutative diagrams we have,

$$\begin{tikzcd} A[0] \otimes A[n] \arrow[rr, "\mu_{[0],[n]}"] & & A[n] \\ & A[n] \arrow[ul, "\cong"] \arrow[ur, "1"'] & \end{tikzcd} \qquad \begin{tikzcd} A[n] \otimes A[0] \arrow[rr, "\mu_{[n],[0]}"] & & A[n] \\ & A[n] \arrow[ul, "\cong"] \arrow[ur, "1"'] & \end{tikzcd} \tag{3.24}$$

4. **Coherence:** For every pair $[n]$, $[m]$ of objects of $\mathcal{L}$, the morphism

$$\mu_{[n],[m]} : A[n] \otimes A[m] \to A[n+m] \tag{3.25}$$

is a quasi-isomorphism, that is, $\mu_{[n],[m]}$ induces an isomorphism in homology.

➪ *Remark* 3.2.2. The natural transformation of an $\mathcal{L}$-algebra is called the product of $\mathcal{A}$ or the structural quasi-isomorphism of $\mathcal{A}$. Also, in order to simplify the expressions we drop the indexes of $\mu_{[n],[m]}$ and simply write $\mu$ when necessary.

➪ *Remark* 3.2.3. In an $\mathcal{L}$-algebra $\mathcal{A}$ the morphisms induced by the structural quasi-isomorphism and the images by $\mathcal{A}$ of morphism in $\mathcal{L}$ like faces, degeneracies and permutations, maybe can be visualized with following diagram of morphism in $\mathcal{C}$.

$$\begin{tikzcd}
\mathsf{k}=A[0] \arrow[r, shift left, "s_1=\eta"] & A[1] \arrow[l, shift left, "d_1=\epsilon"] \arrow[r, bend left=40, "s_1"] \arrow[r, bend left=15, "\xi_1{,}\xi_2"] \arrow[loop below, "1"] & A[2] \arrow[l, bend left=15, "d_1{,}d_2"] \arrow[loop below, "1{,}\tau"] \arrow[r, bend left=40, "s_1{,}s_2"] \arrow[r, bend left=15, "\xi_1{,}\xi_2{,}\xi_3"] & A[3] \arrow[l, bend left=15, "d_1{,}d_2{,}d_3"] \arrow[loop below, "\sigma"] \arrow[r, bend left=40] \arrow[r, bend left=15] & \cdots \arrow[l, bend left=15] \\
 & A[1] & A[2] & A[3] & \\
 & & A[1]\otimes A[1] \arrow[u, "\mu_{1,1}"] & A[1]\otimes A[2] \arrow[u, "\mu_{1,2}"] & \\
 & & & A[2]\otimes A[1] \arrow[uu, bend right=60, "\mu_{2,1}"'] &
\end{tikzcd} \tag{3.26}$$

Where $\tau$ is the non trivial permutation of $\Sigma_2$ and $\sigma$ is any permutation of $\Sigma_3$.

➪ *Remark* 3.2.4. There is an "degenerated" case of $\mathcal{L}$-algebra. It happens when $\mu$ is taken to be the identity. This implies that $A[n] = A[1]^{\otimes n}$ for every $n \leqslant 1$ and that the application $s_0 : A[1] \to A[1] \otimes A[1]$ is a commutative coproduct . Indeed, in $\mathcal{L}$ we have the following commutative diagram.

$$\begin{tikzcd}
{[1]} & {[2]} \arrow[l, "s_1"'] \arrow[d, "\tau"] \\
 & {[2]} \arrow[ul, "s_1"]
\end{tikzcd} \tag{3.27}$$

Which after applying $\mathcal{A}$ and put together with the commutativity of $\mu$, gives,

$$\begin{tikzcd}
A[1] \arrow[r, "s_1"'] \arrow[dr, "s_1"'] & A[2] \arrow[d, "\tau"] & A[1]\otimes A[1] \arrow[l, "\mu=1"] \arrow[d, "T"] \\
 & A[2] & A[1]\otimes A[1] \arrow[l, "\mu=1"]
\end{tikzcd} \tag{3.28}$$

making $T = \tau$ and $T \circ s_1 = s_1 : A[1] \to A[1] \otimes A[1]$. The fact that the coproduct is commutative implies that all the higher homotopies of the diagonals can be taken as zero, so this kind of $\mathcal{L}$-algebras are not very interesting for us. The $\mathcal{L}$-algebras are supposed to model the behavior of systems of diagonals like the one found in the chain complex associated to a simplicial set. In that case the diagonals obtained from homotopy inverses of the Eilenberg-Mac Lane transformation are not commutative, because of the existence of Steenrod operations.

Now we pass to the notion of morphism of $\mathcal{L}$-algebras in order to complete the introduction of $\mathcal{L}$-algebras as a category.

**Definition 3.2.5.** Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-algebras with products $\mu_{\mathrm{A}}$ and $\mu_{\mathrm{B}}$ respectively. A morphism of $\mathcal{L}$-algebras $f : \mathcal{A} \to \mathcal{B}$ is a natural transformation from $\mathcal{A}$ to $\mathcal{B}$ which satisfies the following conditions.

1. It preserves the product of $\mathcal{L}$-algebras in the sens that $f\mu_{\mathrm{A}} = \mu_{\mathrm{B}}(f \otimes f)$, that is, the following diagram is commutative.

$$\begin{tikzcd} A[n] \otimes A[m] \arrow[r, "\mu_{\mathcal{A}}"] \arrow[d, "f_{[n]}\otimes f_{[m]}"'] & A[n+m] \arrow[d, "f_{[n+m]}"] \\ B[n] \otimes B[m] \arrow[r, "\mu_{\mathcal{B}}"] & B[n+m] \end{tikzcd} \tag{3.29}$$

2. The morphism $f_{[0]} : A[0] \to B[0]$ is the identity of $k$.

$$A[0] = \mathsf{k} \xrightarrow{f_{[0]}} B[0] = \mathsf{k} \tag{3.30}$$

**Proposition 3.2.6** (The category of $\mathcal{L}$-algebras)**.** *Let $\mathcal{C}$ be a category as 3.2.1. Then the $\mathcal{L}$-algebras with values in $\mathcal{C}$ together with the morphisms of $\mathcal{L}$-algebras form a category. This category will be denoted $\mathcal{L}(\mathcal{C})$.*

*Proof.* Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three $\mathcal{L}$-algebras, $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, be two morphisms of $\mathcal{L}$-algebras. It suffice to check that the composition of natural transformation $g \circ f$ is a morphism of $\mathcal{L}$-algebras. The first condition of 3.2.5 is consequence of the following commutative diagrams for $f$ and $g$.

$$\begin{tikzcd} A[n] \otimes A[m] \arrow[r, "\mu_{\mathcal{A}}"] \arrow[d, "f_{[n]}\otimes f_{[m]}"'] & A[n+m] \arrow[d, "f_{[n+m]}"] \\ B[n] \otimes B[m] \arrow[r, "\mu_{\mathcal{B}}"] \arrow[d, "g_{[n]}\otimes g_{[m]}"'] & B[n+m] \arrow[d, "g_{[n+m]}"] \\ C[n] \otimes C[m] \arrow[r, "\mu_{\mathcal{C}}"] & C[n+m] \end{tikzcd} \tag{3.31}$$

Making the left and right compositions we obtain the commutative diagram we want.

$$\begin{tikzcd} A[n] \otimes A[m] \arrow[r, "\mu_{\mathcal{A}}"] \arrow[d, "(g\circ f)_{[n]}\otimes (g\circ f)_{[m]}"'] & A[n+m] \arrow[d, "(g\circ f)_{[n+m]}"] \\ C[n] \otimes C[m] \arrow[r, "\mu_{\mathcal{C}}"] & C[n+m] \end{tikzcd} \tag{3.32}$$

The second condition also is a direct verification, because we only have to check that $(g \circ f)_{[0]}$ is the identity of $\mathsf{k}$.

$$\begin{tikzcd} A[0] = \mathsf{k} \arrow[r, "f_{[0]}"] \arrow[rr, bend right, "(g\circ f)_{[0]}"'] & B[0] = \mathsf{k} \arrow[r, "g_{[0]}"] & C[0] = \mathsf{k} \end{tikzcd} \tag{3.33}$$

□

Even if the definition of $\mathcal{L}$ is established in total generality on strict symmetric monoidal categories with quasi-isomorphism, we will restrict our attention to the category of differential graded modules or the category of differential graded algebras.

**Definition 3.2.7.** Let $A$ an $\mathcal{L}$-algebra with values in the category $\mathcal{C}$.

1. If $\mathcal{C}$ is the category of DGA-$\mathsf{k}$-modules, $A$ is called $\mathcal{L}$-algebra.

2. If $\mathcal{C}$ is the category of DGA-$\mathsf{k}$-algebras, $A$ is called multiplicative $\mathcal{L}$-algebra.

➪ *Remark* 3.2.8. $\mathcal{L}$-algebras are designed to represent the 0-reduced simplicial sets and multiplicative $\mathcal{L}$-algebras will represent the 0-reduced simplicial groups.

## 3.3 Monoidal Structure of $\mathcal{L}(\mathcal{C})$

In this section $\mathcal{C}$ represents the category DGA-$\mathsf{k}$-Mod or DGA-$\mathsf{k}$-Alg.

**Proposition 3.3.1.** *Let $T : \mathcal{L}^{op} \to \mathcal{C}$ be the functor defined by $T[n] = \mathsf{k}$ for every $n \geqslant 0$ and $T(\alpha) = 1_{\mathsf{k}}$ for every morphism in $\mathcal{L}$. Together with the natural transformation $\mu : \otimes \circ (T \times T) \to T \circ +$ defined by $\mu_{[n],[m]} = 1_{\mathsf{k}}$ for all $[n],[m] \in \mathcal{L}$, the functor $T$ is an $\mathcal{L}$-algebra.*

*Proof.* The proof is evident. □

**Definition 3.3.2.** The $\mathcal{L}$-algebra in proposition 3.3.1, is called the trivial $\mathcal{L}$-algebra with values in $\mathcal{C}$ and it is denoted $\mathsf{k}$.

**Proposition 3.3.3.** *Let $\mathcal{C}$ be the category DGA-$\mathsf{k}$-modules. Then the trivial $\mathcal{L}$-algebra $\mathsf{k}$ is a zero object in $\mathcal{L}(\mathcal{C})$.*

*Proof.* We have that $\mathsf{k}$ is a zero object of DGA-$\mathsf{k}$-Mod, then, for any $\mathcal{L}$-algebra $A$, this defines unique DGA-$\mathsf{k}$-morphisms $i_{[n]} : \mathsf{k} \to A[n]$ and $p_{[n]} : A[n] \to \mathsf{k}$ $(n \geqslant 0)$, which coincide with the coaugmentation and augmentation of $A[n]$, respectively. The associated natural transformations $i : \mathsf{k} \to A$ and $p : A \to \mathsf{k}$ are morphisms of $\mathcal{L}$-algebras by the commutativity of the following diagram,

$$\begin{tikzcd} & \mathsf{k} \arrow[dl, "\eta_n\otimes\eta_m"'] \arrow[dr, "\eta_{n+m}"] & \\ A[n]\otimes A[m] \arrow[rr, "\mu"'] & & A[n+m] \\ & \mathsf{k} \arrow[ul, "\epsilon_n\otimes\epsilon_m"] \arrow[ur, "\epsilon_{n+m}"'] & \end{tikzcd} \tag{3.34}$$

because $\mu$ is a morphism of DGA-$\mathsf{k}$-modules. □

**Proposition 3.3.4.** *Let $A$ and $B$ be two $\mathcal{L}$-algebras. Let $P$ be the functor $P : \mathcal{L}^{op} \to \mathcal{C}$ defined by,*

1. *$P[n] = A[n] \otimes B[n]$ for all $[n] \in \mathcal{L}$.*

2. *$P(\alpha) = A(\alpha) \otimes B(\alpha)$ for all $\alpha$ morphism in $\mathcal{L}$.*

*Let $\mu_{\mathcal{P}} : \otimes\circ(\mathcal{P}\times\mathcal{P}) \to \mathcal{P}\circ +$ be the natural transformation given by the following composition.*

$$\begin{tikzcd}
P[n]\otimes P[m] \arrow[r, "="] \arrow[dd, "\mu_{\mathcal{P}}"'] & A[n]\otimes B[n]\otimes A[m]\otimes B[m] \arrow[d, "1\otimes T\otimes 1"] \\
 & A[n]\otimes A[m]\otimes B[n]\otimes B[m] \arrow[d, "\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}}"] \\
P[n+m] \arrow[r, "="] & A[n+m]\otimes B[n+m]
\end{tikzcd} \tag{3.35}$$

*Then $\mathcal{P}$ is an $\mathcal{L}$-algebra.*

*Proof.* Clearly $\mu_{\mathcal{P}}$ satisfy the unit axiom. The commutativity follows from the commutative diagram,

$$\begin{tikzcd}
A[n]\otimes B[n]\otimes A[m]\otimes B[m] \arrow[r, "T^{\sigma}"] \arrow[d, "1\otimes T\otimes 1"'] & A[m]\otimes B[m]\otimes A[n]\otimes B[n] \arrow[d, "1\otimes T\otimes 1"] \\
A[n]\otimes A[m]\otimes B[n]\otimes B[m] \arrow[r, shift left, "T\otimes T"] \arrow[d, "\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}}"'] & A[m]\otimes A[n]\otimes B[m]\otimes B[n] \arrow[l, shift left, "T\otimes T"] \arrow[d, "\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}}"] \\
A[n+m]\otimes B[n+m] \arrow[r, "\tau\otimes\tau"] & A[n+m]\otimes B[n+m]
\end{tikzcd} \tag{3.36}$$

where $\sigma = \left(\begin{smallmatrix}1&2&3&4\\3&4&1&2\end{smallmatrix}\right)$. The upper square is commutative by direct evaluation, and bottom square by the commutativity of $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{B}}$. The associativity of $\mu_{\mathcal{P}}$ can be verified directly,

$$\begin{aligned}
\mu_{\mathcal{P}}(1\otimes\mu_{\mathcal{P}}) &= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes T\otimes 1)((\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes T\otimes 1)\otimes 1\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes T\otimes 1)(\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}}\otimes 1\otimes 1)(1\otimes T\otimes 1\otimes 1\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(\mu_{\mathcal{A}}\otimes T(\mu_{\mathcal{B}}\otimes 1)\otimes 1)(1\otimes T\otimes 1\otimes 1\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(\mu_{\mathcal{A}}\otimes(1\otimes\mu_{\mathcal{B}})(T\otimes 1)(1\otimes T)\otimes 1)(1\otimes T\otimes 1\otimes 1\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(\mu_{\mathcal{A}}\otimes 1\otimes\mu_{\mathcal{B}}\otimes 1)(1\otimes 1\otimes(T\otimes 1)(1\otimes T)\otimes 1)(1\otimes T\otimes 1\otimes 1\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes\mu_{\mathcal{A}}\otimes 1\otimes\mu_{\mathcal{B}})(1\otimes(1\otimes T)(T\otimes 1)\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes T\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes\mu_{\mathcal{A}}(1\otimes T)(T\otimes 1)\otimes\mu_{\mathcal{B}})(1\otimes 1\otimes 1\otimes T\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes T(1\otimes\mu_{\mathcal{A}})\otimes\mu_{\mathcal{B}})(1\otimes 1\otimes 1\otimes T\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes T\otimes 1)(1\otimes 1\otimes\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes 1\otimes 1\otimes T\otimes 1)\\
&= (\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes T\otimes 1)(1\otimes 1\otimes(\mu_{\mathcal{A}}\otimes\mu_{\mathcal{B}})(1\otimes T\otimes 1))\\
&= \mu_{\mathcal{P}}(1\otimes\mu_{\mathcal{P}})
\end{aligned}$$

Finally, that $\mu_{\mathcal{P}}$ satisfies the coherence condition of 3.2.1 follows from the fact that the tensor product of two quasi-isomorphisms is again a quasi-isomorphism, when $\mathsf{k}$ is a field. □

**Definition 3.3.5.** The $\mathcal{L}$-algebra $\mathcal{P}$ in 3.3.4 is called the tensor product of $\mathcal{A}$ and $\mathcal{B}$ and will be denoted $\mathcal{A}\otimes\mathcal{B}$.

**Proposition 3.3.6.** *The functor tensor product of two $\mathcal{L}$-algebras induces a functor $\otimes : \mathcal{L}(C) \times \mathcal{L}(C) \to \mathcal{L}(C)$.*

*Proof.* We need to check that for every pair of morphisms of $\mathcal{L}$-algebras, $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{C} \to \mathcal{D}$, there is a morphism of $\mathcal{L}$-algebras $f \otimes g : \mathcal{A} \otimes \mathcal{C} \to \mathcal{B} \otimes \mathcal{D}$.

Define $f \otimes g$ as $(f \otimes g)_n = f_n \otimes g_n : A[n] \otimes C[n] \to B[n] \otimes D[n]$. Let $\alpha : [m] \to [n]$ morphism of $\mathcal{L}$, and consider the following diagram.

$$\begin{tikzcd} A[n] \otimes C[n] \arrow[r, "(f\otimes g)_n"] \arrow[d, "\alpha\otimes\alpha"'] & B[n] \otimes D[n] \arrow[d, "\beta\otimes\beta"] \\ A[m] \otimes C[m] \arrow[r, "(f\otimes g)_n"] & B[m] \otimes D[m] \end{tikzcd} \tag{3.37}$$

This diagram is commutative because,

$$\begin{aligned} (f \otimes g)_m \circ (\alpha \otimes \alpha) &= (f_m \otimes g_m) \circ (\alpha \otimes \alpha) \\ &= (f_m \circ \alpha) \otimes (g_m \otimes \alpha) \\ &= (\beta \circ f_n) \otimes (\beta \circ g_n) \text{ ($f$ and $g$ are morphism of $\mathcal{L}$-algebras)} \\ &= (\beta \otimes \beta) \circ (f_n \otimes g_n) \\ &= (\beta \otimes \beta) \circ (f \otimes g)_n \end{aligned} \tag{3.38}$$

Now we have to check that $f \otimes g$ preserves the quasi-isomorphism $\mu$. For that, consider the following diagram.

$$\begin{tikzcd} (\mathcal{A} \otimes \mathcal{C})(n) \otimes (\mathcal{A} \otimes \mathcal{C})(m) \arrow[r, "(f\otimes g)_n \otimes (f\otimes g)_m"] \arrow[d, "\mu_{\mathcal{A}\otimes\mathcal{C}}"] & (\mathcal{B} \otimes \mathcal{D})(n) \otimes (\mathcal{B} \otimes \mathcal{D})(m) \arrow[d, "\mu_{\mathcal{B}\otimes\mathcal{D}}"] \\ (\mathcal{A} \otimes \mathcal{C})(n+m) \arrow[r, "(f\otimes g)_{n+m}"] & (\mathcal{B} \otimes \mathcal{D})(n+m) \end{tikzcd} \tag{3.39}$$

The commutativity follows because,

$$\begin{aligned} (f \otimes g)_{n+m} \circ \mu_{\mathcal{A}\otimes\mathcal{C}} &= (f_{n+m} \otimes g_{n+m})(\mu_{\mathcal{A}} \otimes \mu_{\mathcal{C}})(1 \otimes T \otimes 1) \\ &= (f_{n+m}\mu_{\mathcal{A}} \otimes g_{n+m}\mu_{\mathcal{C}})(1 \otimes T \otimes 1) \\ &= (\mu_{\mathcal{B}}(f_n \otimes f_m) \otimes \mu_{\mathcal{D}}(g_n \otimes g_m))(1 \otimes T \otimes 1) \\ &= (\mu_{\mathcal{B}} \otimes \mu_{\mathcal{D}})(f_n \otimes f_m \otimes g_n \otimes g_m)(1 \otimes T \otimes 1) \\ &= (\mu_{\mathcal{B}} \otimes \mu_{\mathcal{D}})(1 \otimes T \otimes 1)(f_n \otimes g_n \otimes f_m \otimes g_m) \\ &= \mu_{\mathcal{B}\otimes\mathcal{D}} \circ ((f \otimes g)_n \otimes (f \otimes g)_m) \end{aligned} \tag{3.40}$$

□

**Proposition 3.3.7.** *Then category $\mathcal{L}(\mathcal{C})$ is a strict symmetric monoidal category with unit. The product is given by the tensor product of $\mathcal{L}$ defined in 3.3.5 and the unit is the trivial $\mathcal{L}$-algebra* $\mathsf{k}$.

*Proof.* It is a straightforward succession of verifications. □

## 3.4 Homology of $\mathcal{L}$-Algebras

**Definition 3.4.1.** Let $\mathcal{A}$ be an $\mathcal{L}$-algebra. The module $A[1]$ is called the main element of $\mathcal{A}$. The associated forgetful functor from $\mathcal{L}(\mathcal{C})$ to $\mathcal{C}$ is denoted by $U$. In fact we have a collection indexed by $n \geqslant 0$ of forgetful functors $U_n : \mathcal{L}(\mathcal{C}) \to \mathcal{C}$, with $U_n(\mathcal{A}) = A[n]$.

**Definition 3.4.2.** Let $A$ be an $\mathcal{L}$-algebra. The homology of $A$ is defined to be the homology of its main element.

➪ *Remark* 3.4.3. The homology of $\mathcal{L}$-algebras is equal to the composition of functors $H_* \circ U$, where $H_*$ is the homology functor in $\mathcal{C}$.

**Definition 3.4.4.** Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of $\mathcal{L}$-algebras with values in $\mathcal{C}$. The morphism $f$ is called quasi-isomorphism if the induced morphism $U(f)$ in $\mathcal{C}$ by the forgetful functor, is a quasi-isomorphism.

**Proposition 3.4.5.** *Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of $\mathcal{L}$-algebras with values in $\mathcal{C}$. If $U_k(f)$ is a quasi-isomorphism in $\mathcal{C}$ for some $k$, then $U_{kn}(f)$ is a quasi-isomorphism for every $n \geqslant 0$. In particular if $f$ is a quasi-isomorphism then $U_n(f)$ is a quasi-isomorphism for every $n \geqslant 1$.*

*Proof.* We proceed by induction. The hypothesis says that $f_k : A[k] \to B[k]$ is a quasi-isomorphism. Now, the following diagram is commutative because $f$ is a morphism of $\mathcal{L}$-algebras

$$\begin{tikzcd} A[k] \otimes A[k(n-1)] \arrow[r, "f_k \otimes f_{k(n-1)}"] \arrow[d, "\mu_{\mathcal{A}}"'] & B[k] \otimes B[k(n-1)] \arrow[d, "\mu_{\mathcal{B}}"] \\ A[kn] \arrow[r, "f_{kn}"] & B[kn] \end{tikzcd} \tag{3.41}$$

The tensor product $f_k \otimes f_{k(n-1)}$ is a quasi-isomorphism since $\mathsf{k}$ is a field. Then $f_{kn}$ is a quasi-isomorphism.

□

**Definition 3.4.6.** The equivalence relation on $\mathcal{L}$-algebras spanned by quasi-isomorphisms will be called again quasi-isomorphism.

## 3.5 Canonical $\mathcal{L}$-Algebras

The concept of $\mathcal{L}$ is inspired by the fact that any natural diagonal of chain complexes $C_*(X) \to C_*(X) \otimes C_*(X)$, is determined by a zig-zag of natural morphisms $C_*(X) \to C_*(X \times X) \leftarrow C_*(X) \otimes C_*(X)$, where the first arrows is the morphism induced by the simplicial diagonal $X \to X \times X$ and the second arrow is the Eilenberg-Mac Lane transformation. In this section we will proceed to describe the $\mathcal{L}$-algebra structure on the chain complexes that have as product the Eilenberg-Mac Lane transformation. Moreover there is a completely canonical way to associate to each simplicial set (not necessarily 0-reduced) an $\mathcal{L}$-algebra whose main element is its chain complex.

**Proposition 3.5.1.** *Let $\boldsymbol{sSet}_*$ be the category of pointed simplicial set. Then for every simplicial set $X$, there is an unique functor $S_X : \mathcal{L}^{op} \to \boldsymbol{sSet}_*$ preserving zeros, mapping sums to products, and $[1]$ to $X$.*

*Proof.* **sSet**$_*$ is a strict pointed cartesian category, then by proposition 3.1.13 the result follows. ☐

**Proposition 3.5.2.** *Let $X$ be a pointed simplicial set. Then composition,*

$$C_* \circ S_X : \mathcal{L}^{op} \to \mathcal{C} \tag{3.42}$$

*together with the Eilenberg-Mac Lane transformation is an $\mathcal{L}$-algebra.*

*Proof.* It follows easily from the definitions and properties of Eilenberg-Mac Lane transformation. ☐

**Definition 3.5.3.** The $\mathcal{L}$-algebra associated to the pointed simplicial set $X$ will be called the canonical $\mathcal{L}$-algebra of $X$, and denoted by $\mathcal{A}_X$.

➪ *Remark* 3.5.4. Let's see how $\mathcal{A}_X$ looks like. For $n \leqslant 1$ we have,

$$\mathcal{A}_X[n] = C_*(X^n) \tag{3.43}$$

The product for $\mathcal{A}_X$, $\mu_{\mathcal{A}_X} : \otimes \circ (\mathcal{A}_X \times \mathcal{A}_X) \to \mathcal{A}_X \circ +$ is the Eilenberg-Mac Lane transformation,

$$\nabla_{n,m} : C_*(X^n) \otimes C_*(X^m) \to C_*(X^{n+m}) \tag{3.44}$$

Let $*$ be the base point of $X$, then the images by $\mathcal{A}_X$ for a morphism $\alpha : [m] \to [n]$ in $\mathcal{L}$ is given by the following formula.

$$\begin{array}{ccc} \mathcal{A}_X[n] = C_*(X^n) & \xrightarrow{\mathcal{A}_X(\alpha)} & \mathcal{A}_X[m] = C_*(X^m) \\ (x_1, \ldots, x_n) & \longmapsto & (x_{\alpha(1)}, \ldots, x_{\alpha(m)}) \end{array} \tag{3.45}$$

Where $x_{\alpha(j)} = *$ for each $j$ not in $Dom(\alpha)$.

In the case of a simplicial group (who will be pointed by its unit), we have an extra structure.

**Proposition 3.5.5.** *Let $H$ be a simplicial group, then $\mathcal{A}_H$ is a multiplicative $\mathcal{L}$-algebra.*

*Proof.* For every $n \geqslant 0$, $H^n$ and $\mathcal{A}_H[n] = C_*(H^n)$ is a differential graded algebra (with the Pontrjagin product). Since the Eilenberg-Mac Lane is a morphism of algebras, the functor $\mathcal{A}$ maps simplicial groups to multiplicative $\mathcal{L}$-algebras. ☐

**Proposition 3.5.6.** *Let $X = *$ the simplicial point. Then $\mathcal{A}_*$ is the trivial $\mathcal{L}$-algebra.*

*Proof.* For every $n$, $C_*(*) = \mathsf{k}$, and the application $\mathcal{A}_*(f)$ are always the identity of $\mathsf{k}$. The Eilenberg-Mac Lane transformation is then the identity of $\mathsf{k}$. ☐

# Chapter 4

# $E_\infty$-structures on $C_*(X)$

In [Smi94] Smith describes an $E_\infty$-coalgebra structure on the chain complex of a simplicial set when the coefficients ring is $\mathbb{Z}$. In order to do this, he uses an $E_\infty$-operad, denoted $\mathfrak{S}$, with components $\mathrm{R}\Sigma_n$, the $\Sigma_n$-free bar resolution of $\mathbb{Z}$. The morphisms $f_n : \mathrm{R}\Sigma_n \otimes C_*(X) \to C_*(X)^{\otimes n}$ determined by the operad $\mathfrak{S}$ contains the family of higher diagonals on $C_*(X)$ starting at an homotopic version of the iterated Alexander-Whitney diagonal (given by $x \mapsto f_n([\,]_n \otimes x)$). The construction made by Smith can be seen as a version of the Barratt-Eccles operad (see [BE74]). Moreover, Berger and Fresse (see [BF04]) construct a explicit coaction over the normalized chain complex associated to a simplicial set by the Barrat-Eccles operad that extend the structure given by the Alexander-Whitney diagonal.

In this chapter we review the construction of the $E_\infty$-operad $\mathfrak{S}$ given by Smith in [Smi00][†] and his proof that $C_*(X)$ is an $E_\infty$-coalgebra using this operad. Next, we give an alternative proof of the $E_\infty$-structure on the chain complex of an simplicial set by using an operad $\mathcal{R}$ constructed by us that simplify the task. The method used to construct $\mathcal{R}$ gives an simply way to produce $E_\infty$-operads.

The operad $\mathcal{R}$ presents similarities with the bar-cobar resolution of Ginzburg-Kapranov (see [GK94]). Berger and Moerdij (see [BM07]) show that this resolution can identified with the $W$ construction of Boardman and Vogt (see [BV73]), given as a result that applied to the Barratt-Eccles operad, the $W$ construction gives a cofibrant resolution of it. Then, the construction of $\mathcal{R}$ can be seen as a middle point between the Barratt-Eccles operad and its $W$ construction.

The ground category in this chapter is DGA-$\mathbb{Z}$-Mod. To simplify the notation it will be written DGA-Mod. All the operads are operads on DGA-Mod.

## 4.1 The Operad $\mathfrak{S}$

In this section we make a review the technique presented by Smith in order to exhibit the $E_\infty$-coalgebra on chain complexes associated to simplicial sets, originally published in his monograph [Smi94]. In fact, we present the improved version of [Smi00]. His results are based on the construction of an particular $E_\infty$-operad denoted $\mathfrak{S}$.

**Definition 4.1.1.** An operad $\mathcal{P}$ is called $E_\infty$-operad if for every $k > 0$ the component $P(k)$ is a $\Sigma_k$-free resolution of $\mathbb{Z}$.

[†] An updated version of [Smi94]

➪ *Remark* 4.1.2. We already saw an example of $E_\infty$-operad, the operad $\mathcal{M}$ in example 2.1.9.

The following lemma included in [Smi00] is important for the construction of $\mathfrak{S}$. It is based on the Cartan theory of constructions (see [Car55]).

**Lemma 4.1.3.** *Let $M_1$, $M_2$ be DGA-modules which satisfy the following conditions.*

1. *$M_1 = A_1 \otimes N_1$, with $N_1$ $\mathbb{Z}$-free and $A_1$ DGA-algebra.*

2. *$M_2$ is left DGA-$A_2$-module such that,*

   (a) *There is a sub DG-module $N_2 \subset M_2$ with $\partial_{M_2}|_{N_2}$ injective.*

   (b) *There is a contracting chain homotopy $\varphi : M_2 \to M_2$ with $\varphi(M_2) \subset N_2$.*

*Then, every DGA-morphism $f_0 : M_1 \to M_2$ en dimension 0 such that $f_0(N_1) \subset N_2$ can be extended to an unique DGA-morphism $f : M_1 \to M_2$ satisfying,*

1. *$f(N_1) \subset N_2$.*

2. *$f(a \otimes x) = g(a)f(n)$, where $g : A_1 \to A_2$ morphism of DG-modules such that $a \otimes x \mapsto g(a)f(n)$ DGA-morphism.*

□

**Definition 4.1.4.** Let $\mathfrak{S}$ be the $E_\infty$-operad given by :

1. The $n$ component $\mathrm{R}\Sigma_n$ of its underlying $\mathbb{S}$-module is the $\Sigma_n$-free bar resolution of $\mathbb{Z}$.

2. The compositions (in the sens of definition 2.1.2) $\mathrm{R}\Sigma_n \circ_i \mathrm{R}\Sigma_m \to \mathrm{R}\Sigma_{n+m+1}$ are the only DG-morphism satisfying the condition,

$$(1 \otimes A(\Sigma_n, 1)) \otimes_i (1 \otimes A(S_m, 1)) \subseteq 1 \otimes A(\Sigma_{n+m-1}) \tag{4.1}$$

   where $A(\Sigma_k, 1)$ in degree $j$ is generated as $\mathbb{Z}[\Sigma_j]$-module by the elements of the form $1[a_1|\cdots|a_i]^\dagger$.

➪ *Remark* 4.1.5. The unicity of the composition $\circ_i$ and the fact that they satisfies the operad conditions follows easily from the lemma 4.1.3. The contracting chain homotopy $\varphi$ for $\mathrm{R}\Sigma_n$ is given by $\varphi(1[a_1|\cdots|a_i]) = 0$ and $\varphi(a[a_1|\cdots|a_i]) = 1[a|a_1|\cdots|a_i]$.

➪ *Remark* 4.1.6. The $\mathfrak{S}$-coalgebra on a chain complex associated to a pointed simply connected 2-reduced simplicial set $C_*(X)$ is made by defining morphisms $f_n : \mathrm{R}\Sigma_n \otimes C_*(X) \to C_*(X)^{\otimes n}$ by using acyclic models.

†Standard notation of bar resolution

## 4.2 The Operad $\mathcal{R}$

In this section we present an alternative to the operad $\mathfrak{S}$ given by Smith [Smi94]. Roughly speaking, we take the $\mathbb{S}$-module where the components are the $\Sigma_n$-free bar resolutions of $\mathbb{Z}$, then we take the free operad on this $\mathbb{S}$-module and finally we quotient this operad by an suitable operad ideal $\mathcal{I}$, which makes that our operad will have only one generator of degree 0 in each component. The resulting $E_\infty$-operad is denoted $\mathcal{R}$ and in the following section we proof that $C_*(X)$ is an $\mathcal{R}$-coalgebra.

To construct the operad $\mathcal{R}$ we will need the notion of ideal of an operad (see [GK94] §2.1).

**Definition 4.2.1.** Let $\mathcal{P}$ be an operad on the category of DGA-$\mathbb{Z}$-modules, with composition $\gamma$. Let $\mathcal{I}$ be a sub $\mathbb{S}$-module of $U(\mathcal{P})$ which satisfies $\gamma(x \otimes y_1 \otimes \cdots \otimes y_k) \in \mathcal{I}$ whenever some of the elements $x, y_1, \ldots, y_k$ belongs to $\mathcal{I}$. $\mathcal{I}$ is called an operadic ideal of $\mathcal{P}$.

**Definition 4.2.2.** Let $\mathcal{P}$ be an operad and $\mathcal{I}$ an operadic ideal of $\mathcal{P}$. We define the quotient operad $\mathcal{P}/\mathcal{I}$ as the operad with components given by $(\mathcal{P}/\mathcal{I})(k) = P(k)/I(k)$ for every $k \geqslant 0$, and composition induced by the composition of $\mathcal{P}$.

➪ *Remark* 4.2.3. Clearly the operad structure $\mathcal{P}/\mathcal{I}$ is well defined by the properties of the ideal, which allows the pass to the quotient of the composition in $\mathcal{P}$.

**Definition 4.2.4.** Let $S$ be the be the $\mathbb{S}$-module on the category of DGA-modules, with components $S(n) = \mathrm{R}\Sigma$, the $\Sigma_n$-free bar resolution of $\mathbb{Z}$.

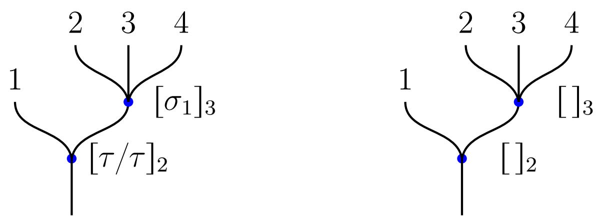
**Definition 4.2.5.** Let $S$ be the $\mathbb{S}$-module defined in 4.2.4.

1. Let $\mathcal{J}$ be the operadic ideal of $F(S)$ (see section 2.3) generating by the elements of degree zero of $F(S)$ of the form $x - y$, where $x$ and $y$ are not null.

2. The operad $\mathcal{R}$ is defined to be the quotient operad $F(S)/\mathcal{J}$.

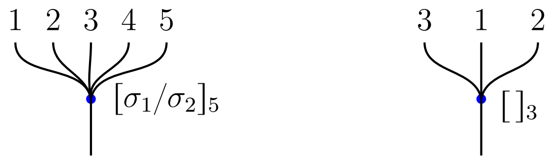
Before continuing we make a description using trees of the operad $F(S)$, the ideal $\mathcal{J}$ and the operad $\mathcal{R}$. By proposition 2.3.24 the operad $F(S)$ is canonically isomorphic to an operad of the form $\mathbb{Z}[\Sigma_n] \otimes \mathfrak{P}(n)$ where $\mathfrak{P}$ is a free non symmetric operad. Then the operad $F$ can be described by labeled rooted planar trees where the vertices with $i$ inputs are labeled by $\Sigma_i$-generating elements of $\mathrm{R}\Sigma_i$, that is element of the form $[\sigma_1/\cdots/\sigma_k]_i$, where the permutations $\sigma_j$ belong to $\Sigma_i$. Also the leaves of a tree in $F(n)$ are labeled from left to right by $1, \ldots, n$. The degree of a tree in $F(n)$ is equal to the sum of the degrees of the elements that label its vertices. This description is illustrate by the following pictures.

1. Trees in $F(S)(4)$ of degree 3 and 0 respectively:




(4.2)

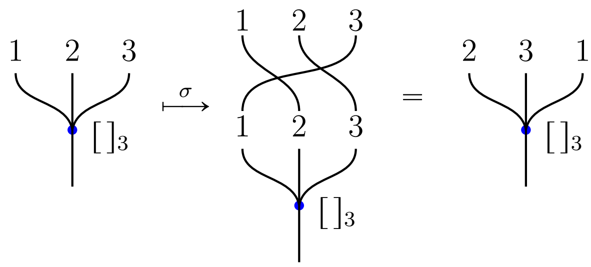
2. Trees with only one vertex are called corollas. Here we have an example of $F(S)(5)$ with degree 2 and one of $F(S)(3)$ of degree 0.




(4.3)

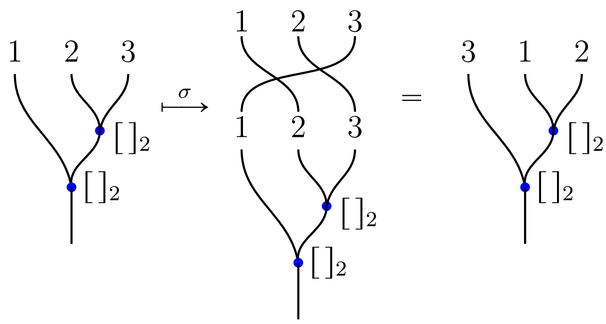
For $n \geqslant 2$, the action by $\sigma \in \Sigma_n$ on this kind of elements in $F(S)(n)$, changes the labeling of leaves to $\sigma(1)^{-1}, \ldots, \sigma^{-1}(n)$. In the following pictures we make two examples with the permutation $\tau(1,2) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$

1. Corolla in $F(S)(3)$.




(4.4)

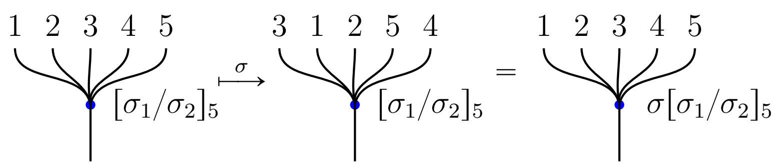
2. Element in $F(S)(3)$ with two vertices.




(4.5)

In order to include all the elements of the resolutions $\mathrm{R}\Sigma_n$ as labels we make the following identifications.

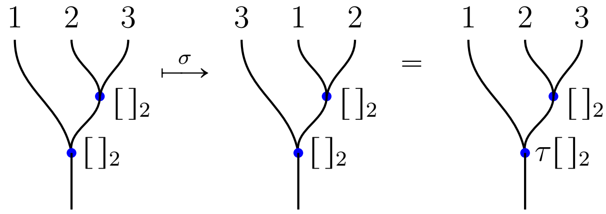
1. Every corolla in $F(S)(n)$ with only vertex labeled by an element of the form $[\sigma_1/\ldots/\sigma_k]$ (with $\sigma_j \in \Sigma_n$) under the action of $\sigma \in \Sigma_n$ is identify with the corolla whose only vertex is labeled by $\sigma[\sigma_1, \ldots, \sigma_k]$ and with leaves labeled from left to right by $1, \ldots, n$.




(4.6)

In this case $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{smallmatrix}\right)$.

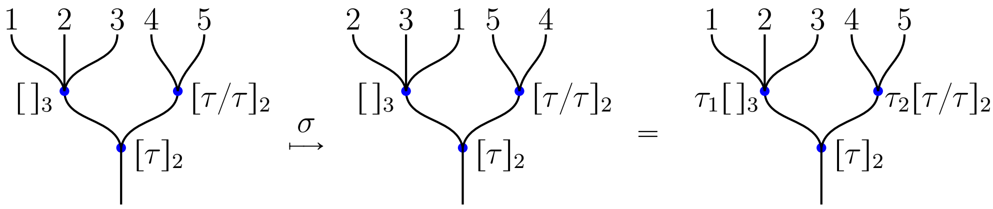
2. Let $t$ be a tree of $F(S)(n)$ whose base vertex (the first vertex from bottom to top) have $k \leqslant n$ inputs and its labeled by $[\sigma_1/\ldots/\sigma_k]$(with $\sigma_j \in \Sigma_k$). For $1 \leqslant j \leqslant k$, let $i_j$ be the number of leaves of the subtree of $t$ over the $i$ input. If $t$ is under the action of $\sigma \in \Sigma_n$ of the form $\theta(i_1,\ldots,i_k)$ (with $\theta \in \Sigma_k$), then we made the identification $t\sigma = t'$, where $t'$ is exactly as $t$ but its base vertex is labeled by $\theta(\sigma_1,\ldots,\sigma_k)$.




(4.7)

Where $\sigma = \tau(1,2) = \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$.

3. Let $t$ a tree of $F(n)$ like the last case. Suppose for each $1 \leqslant i \leqslant k$ the subtree of $t$ over the input $i$ have its base vertex labeled by an element of the form $[x_i]$ (where $x_i$ represents some sequence of permutations of the corresponding type). If $t$ is under the action of $\sigma \in \Sigma_n$ of the form $\tau_1 \oplus \cdots \oplus \tau_k$ (with $\tau_j \in \Sigma_{i_j}$) then we made the identification $t\sigma = t'$, where $t'$ is a tree exactly like $t$ but for each $1 \leqslant i \leqslant k$ the label in the base vertex of the subtree over the $i$ input is changed by $t[x_i]$.



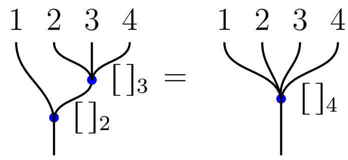

(4.8)

Where $\sigma = \tau_1 \oplus \tau_2 = \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&2&3&4&5\\3&1&2&5&4\end{smallmatrix}\right)$.

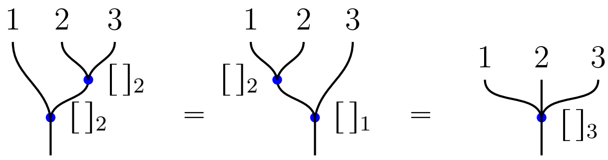
This identification on the labeling is extended for the rest of the trees by induction on the subtrees. With it the description of the form of trees in $F(S)$ is complete. The next step is the differential on $F(S)$. But this follows easily form the natural inclusion $S \hookrightarrow F(S)$, which sends every element of the form $\sigma[\sigma_1/\ldots/\sigma_k] \in \mathrm{R}\Sigma_n$ to the corolla of $F(S)_n$, which vertex is labeled by this element. Then the differential on corollas behaves like the differential on $S$, and then we extend the differential to all F(S) in the obvious way.

The composition for $F(S)$ is given by the grafting of trees, in the sense that $t \in F(S)(n)$ and $t' \in F(S)(m)$, then the tree $t' \circ_i t$, with $1 \leqslant i \leqslant n$, is obtained by glue together the $i$ input of $t$ and the root of $t'$. The labeling for the resulting tree, if both are in $T$, is just from left to right, $1,\ldots,n+m-1$. If only $t$ is affected by an action $\sigma \in \Sigma_n$, then do the composite with the non affected version of $t$, and multiply the resulting tree by the action $\sigma(1,\ldots,m,\ldots,1)$, that is only one block of length $m$ in the position $i$. In the case where both are affected by actions, $t$ by $\sigma \in \Sigma_n$ and $t'$ by $\sigma' \in \Sigma_m$. Then do the composition with the non affected versions and multiply the result by $\sigma(1,\ldots,m,\ldots,1)$ (as before) and then by $1 \oplus \cdots \oplus \sigma' \oplus \cdots \oplus 1$ ($n$ summands and $\sigma'$ in the position $\sigma^{-1}(i)$).

The operadic ideal $\mathcal{J}$ is used to identify all the subtrees of degree 0 in $F(S)(n)$ coming from $F(n)$ to the $n$ corolla of $F(S)(n)$. The following pictures shows some examples.




(4.9)




(4.10)

Then the operad $\mathcal{R}$ only have in arity $n$ one $\Sigma_n$-generating element of degree 0, which is the $n$ corolla labeled by $[\,]_n$.

**Proposition 4.2.6.** *The operad $\mathcal{R}$ is an $E_\infty$-operad.*

*Proof.* It suffices to exhibit in each arity an contracting chain homotopy. In arity $n$, the contracting chain homotopy $\Phi_n : R(n) \to R(n)$ is obtained by extending on $R(n)$ the contracting chain homotopy from the component of the operad $S$, $\mathrm{R}\Sigma_n$ in the obvious way. $\square$

## 4.3 The Chain Complexe $C_*(X)$ as $E_\infty$-Coalgebra

Consider the diagram given by the universal property of the coaugmentation of the adjunction $F \vdash U$.

$$\begin{tikzcd} S \arrow[r, "\epsilon"] \arrow[dr, "i"'] & F(S) \arrow[d, "p"] \\ & \mathfrak{S} \end{tikzcd} \tag{4.11}$$

Where the morphism $i$ is the identity of $\mathbb{S}$-modules. The morphism of operads $p : F(S) \to \mathfrak{S}$ is given by the universal property of $\epsilon$. It is easy to see that $p$ respect the ideal $\mathcal{J}$ because $p$ is essentially the contraction of vertices of trees. Then $p$ pass to the quotient and we obtain a morphism of operads $\overline{p} : \mathcal{R} \to \mathfrak{S}$, which implies that every $\mathfrak{S}$-coalgebra is an $\mathcal{R}$-coalgebra.

We can also use the lemma 4.1.3 to show that chain complexes are $\mathcal{R}$-coalgebras. We only have to observe that $R(n)$ can be expressed as the tensor product of $\mathbb{Z}[\Sigma_n]$ with the trees with vertices labeled by elements of the form $1[\sigma_1|\ldots|\sigma_j]$, and that the grafting of two trees of this type is again a tree of this form.

Recently, in [DV15] Vallette and Dehling describe an operad similar to $\mathcal{R}$. Moreover, they show that this operad can be used to state explicitly (by the use relations) the definition of $E_\infty$-algebras. Which is the case for $A_\infty$-algebras.

# Chapter 5

# $E_\infty$-Structures Associated to $\mathcal{L}$-Algebras

Using a homotopy inverse of the structural quasi-isomorphism $\mu$ of an $\mathcal{L}$-algebra $A$ we can define a coproduct on its main element $A[1]$. Indeed, we only have to take the composition of an homotopy inverse of $\mu : A[1] \otimes A[1] \to A[2]$ with the morphism $s_1 : A[1] \to A[2]$. Observe that this coproduct in general is not associative. But, the structure of $\mathcal{L}$-algebra makes this coproduct associative and commutative up to homotopy. Moreover, the homotopies also satisfy to be associative and commutative up to homotopy, and this property is maintained on the next levels of homotopies, generating a system of higher homotopies. The classical case where this happens is in the context of chain complexes associated to a simplicial set. We saw in the last chapter that the information of higher homotopies can be organized into an $E_\infty$-coalgebra. Such a structure was exhibited in two different ways, using the operad $\mathfrak{S}$ (see [Smi94]) and alternatively using the operad $\mathcal{R}$ designed by us. This chapter is dedicated to the generalization of these descriptions in the context of $\mathcal{L}$-algebras with values in the category DGA-$\mathsf{k}$-Mod, in other words, we will prove that the main element of an $\mathcal{L}$-algebra $A$ is equipped with an $E_\infty$-coalgebra structure describing the system of higher homotopies associated to the coproducts induced by the structural quasi-isomorphism of $A$.

The main difference with the case of chain complexes associated to a simplicial set, where the process begins with the Alexander-Whitney diagonal, which is an associative coproduct, is that in general we don't have the associativity. Then, in order to model the higher homotopies we have to consider an $E_\infty$-operad that will have several generators in degree 0, and not only one like the operads $\mathfrak{S}$ and $\mathcal{R}$. In the section 5.4, we will construct an $E_\infty$-operad that we denote $\mathcal{K}$. The construction is made by infinitely many steps, in the sense that we construct a sequence of operads $\{\mathcal{K}_n\}_{n\geqslant 2}$, in such a way that $\mathcal{K}_i$ is a suboperad of $\mathcal{K}_{i+1}$. The operads $\mathcal{K}_i$ are not $E_\infty$-operads, but they will be almost $E_\infty$-operads, in the sense that until arity $i$ they will satisfy the $E_\infty$-conditions. Finally, the $E_\infty$-operad $\mathcal{K}$ is obtained by taking the inductive limit of this sequence of operads.

One of the characteristics of this construction is the use of a technique that we call polynomial operads. It will create a new operad from an $\mathbb{S}$-module containing an $\mathbb{S}$-submodule with an operadic structure, in such a way that this operadic structure is preserved in the resulting operad. This done by using amalgamated sums in the category of operads. The section 5.3 is completely dedicated to the description of this technique.

In the final part of this chapter we exhibit the main element $A[1]$ of an $\mathcal{L}$-algebra as a $\mathcal{E}_\infty$-coalgebra. Again, this will be possible due to the sequence of operads that define $\mathcal{K}$, in the sense that it will be sufficient to exhibit $A[1]$ as a $\mathcal{K}_i$-coalgebra for each $i$, because the universal property of colimits will induce the $\mathcal{K}$-coalgebra structure on $A[1]$. Moreover, our construction that $A[1]$ is a $E_\infty$-coalgebra is functorial. This proves that an $\mathcal{L}$-algebra quasi-isomorphic to $\mathcal{A}(X)$ contains at least as many homotopy information as a $E_\infty$-coalgebra structure on $C_*(X)$, such as the one described by J. Smith.

In [Man06], Mandell describes an $E_\infty$-algebra structure on the normalized cochain complex associated to a simplicial space, which under some finiteness hypothesis gives an invariant for the weak homotopy type of the space. Our results suggest that $\mathcal{L}$-algebras are also pertinent in order to describe the weak homotopy type of spaces.

## 5.1 $E_\infty$-Operads

In the last chapter we defined $E_\infty$-operads for the case of differential graded modules with coefficients in $\mathbb{Z}$. For coefficients in a field $\mathsf{k}$, we take the obvious adaptation.

**Definition 5.1.1** ($E_\infty$-Operad)**.** An operad $\mathcal{P}$ on the category DGA-$\mathsf{k}$-Mod is called $E_\infty$-operad if each component $P(n)$ is a $\mathsf{k}[\Sigma_n]$-free resolution of $\mathsf{k}$.

**Definition 5.1.2** ($E_\infty$-algebra and $E_\infty$-coalgebra)**.** We call $E_\infty$-algebra any $\mathcal{P}$-algebra with $\mathcal{P}$ an $E_\infty$-operad. And in the same way, an $E_\infty$-coalgebra is an $\mathcal{P}$-coalgebra where the operad $\mathcal{P}$ is an $E_\infty$-operad.

We introduce a notion of morphism between $E_\infty$-coalgebras which is well suited for our purpose.

**Definition 5.1.3.** Let $\mathcal{P}$ be an $E_\infty$-operad on the category DGA-$\mathsf{k}$-Mod, and let $A, B$ $\mathcal{P}$-coalgebras. A morphism $f : A \to B$ of $\mathcal{P}$-coalgebras is a morphism of DGA-$\mathsf{k}$-Mod which preserves the $\mathcal{P}$-coalgebra structure up to homotopy, that is, the following diagram

$$\begin{tikzcd}
\mathcal{P}(n) \otimes A \arrow[r, "\varphi_n^A"] \arrow[d, "1\otimes f"'] & A^{\otimes n} \arrow[d, "f^{\otimes n}"] \\
\mathcal{P}(n) \otimes B \arrow[r, "\varphi_n^B"'] & B^{\otimes n}
\end{tikzcd} \tag{5.1}$$

is commutative up to homotopy for every $n > 0$, where $\varphi_n^A$ and $\varphi_n^B$ are the associated morphisms of the $\mathcal{P}$-coalgebra structure of $A$ and $B$, respectively. The category of coalgebras on the operad $\mathcal{P}$ is denoted $\mathcal{P}$-CoAlg.

## 5.2 The Lifting Theorem

In this section we include a basic tool that will be needed along this chapter. The symbol $\Lambda$ is used to represent any ring.

**Lemma 5.2.1.** *Let $f : L \to N$ be a homogeneous morphism of DG-$\Lambda$-modules of degree $k \in \mathbb{Z}$. Let $L'$ be a DG-$\Lambda$-submodule of $L$ and $P$ graded submodule of $L$ such that $P$ is projective and bounded below, and in each degree we have the decomposition $L_i = L'_i \oplus P_i$. Write $f'$ for the restriction of $f$ to $L'$. Suppose that the homology of $N$ is zero and that $f'$ is null-homotopic by a homotopy $h : L' \to N$. Then there exists a homotopy $H$ on $L$ extending $h$, which makes $f$ null-homotopic.*

*Proof.* Let $p \in \mathbb{Z}$ the inferior bound of $P$, then for $i \leqslant p$, $P_i = 0$ and $L_i = L'_i$. Then we take $H_i = h_i$ for $i \leqslant p$. Now suppose we have defined $H_i$ on $L_i$ for all $i < n$, such that $H_i = h_i$ on $L'_i$ and satisfying $f_i = \partial H_i + (-1)^k H_{i-1}\partial$.

$$\begin{tikzcd}
 & & N_{n+k+1} \arrow[d, "\partial"] \\
L'_n \arrow[r, hook] \arrow[d, "\partial"] \arrow[urr, "h_n"] & L_n = L'_n \oplus P_n \arrow[r, "f_n"] \arrow[d, "\partial"] \arrow[ur, dashed, "H_n"] & N_{n+k} \arrow[d, "\partial"] \\
L'_{n-1} \arrow[r, hook] \arrow[d, "\partial"] \arrow[urr, "h_{n-1}"] & L_{n-1} = L'_{n-1} \oplus P_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\partial"] \arrow[ur, "H_{n-1}"] & N_{n+k-1} \\
L'_{n-2} \arrow[r, hook] \arrow[urr, "h_{n-2}"] & L_{n-2} = L'_{n-2} \oplus P_{n-2} \arrow[ur, "H_{n-2}"] &
\end{tikzcd} \tag{5.2}$$

For every homogeneous element $x \in L'_n$ we take $H_n(x) = h_n(x)$. Note that the image $f_n - (-1)^k H_{n-1}\partial : L_n \to N_{n+k}$ belongs to the image $\partial : N_{n+k+1} \to N_{n+k}$, indeed

$$\begin{aligned}
\partial(f_n - (-1)^n H_{n-1}\partial) &= \partial f_n - (-1)^n \partial H_{n-1}\partial \\
&= \partial f_n - (-1)^n (f_{n-1} - (-1)^n H_{n-2}\partial)\partial \text{ (by hypotesis)} \\
&= \partial f_n - (-1)^n f_{n-1}\partial + H_{n-2}\partial\partial \\
&= \partial f_n - (-1)^n f_{n-1}\partial = 0
\end{aligned} \tag{5.3}$$

Thus for $x \in P_n$, we can choose the desired element $H_n(x) \in N_{n+k+1}$ using the fact that $P$ is projective and $H_*(N) = 0$. ☐

**Theorem 5.2.2** (Relative Lifting Theorem). *Let $f : M \to N$ and $\varphi : L \to N$ morphisms of DG-$\Lambda$-modules of degree $l$ and $k$, respectively. Suppose that $f$ is a quasi-isomorphism, and let $L'$ be a DG-$\Lambda$-submodule of $L$ and $P$ graded submodule of $L$ such that $P$ is projective and bounded below, and in each degree we have the decomposition $L_i = L'_i \oplus P_i$. Write $\varphi'$ for the restriction of $\varphi$ to $L'$. Suppose there is a morphism $\alpha' : L' \to M$ of DG-$\Lambda$-modules of degree $k - l$ that lifts $\varphi'$ up to homotopy along $f$. Then there exists an extension $\alpha : L \to M$ of $\alpha'$, that lifts $\varphi$ up* *to homotopy along $f$. Moreover, the homotopy can be choose to be an extension of the homotopy associated to $\alpha'$.*

$$\begin{tikzcd}
M \arrow[rr, "f"] & & N \\
& L \arrow[ul, dashed, "\alpha"] \arrow[ur, "\varphi"'] & \\
& L' \arrow[u, hook] \arrow[uul, bend left, "\alpha'"] \arrow[uur, bend right, "\varphi'"'] &
\end{tikzcd} \tag{5.4}$$

*Proof.* Let $C(f)$ be the mapping cone of $f$. Let $u : N \to C(f)$ the inclusion $x \mapsto \begin{pmatrix} 0 \\ x \end{pmatrix}$ and $h' : L' \to N$ the homotopy from $\varphi'$ to $f \circ \alpha'$. Then we can easily check that $\begin{pmatrix} \alpha' \\ h \end{pmatrix} : L' \to C(f)$ is a homotopy to 0 of $u \circ \varphi' = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}$. The lemma 5.2.1 says there exists a homotopy to zero $\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} : L \to C(f)$ of $u \circ \varphi$ extending $\begin{pmatrix} \alpha' \\ h \end{pmatrix}$. So we have,

$$\begin{aligned}
\begin{pmatrix} 0 \\ \varphi \end{pmatrix} &= \partial_{C(f)} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} + (-1)^k \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \partial_L \\
&= \begin{pmatrix} -(-1)^l \partial_M & 0 \\ f & \partial_N \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} + (-1)^k \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \partial_L \\
&= \begin{pmatrix} -(-1)^l \partial_M H_1 + (-1)^k H_1 \partial_L \\ f H_1 + \partial_N H_2 + (-1)^k H_2 \partial_L \end{pmatrix}
\end{aligned} \tag{5.5}$$

This gives the following equations.

$$\begin{aligned}
\partial_M H_1 &= (-1)^{l+k} H_1 \partial_L \\
\varphi - f H_1 &= \partial_N H_2 + (-1)^k H_2 \partial_L
\end{aligned} \tag{5.6}$$

The first says that $H_1$ is a morphism of DG-$\Lambda$-modules and the second that $H_2$ is a homotopy from $fH_1$ to $\varphi$. Finally, we take $\alpha = H_1$ as the lift of $\varphi$ along $f$. □

## 5.3 Polynomial Operads

The polynomial operads construction is a technique used to create an operad from an $\mathbb{S}$-module with an $\mathbb{S}$-submodule having an operadic structure, in such a way that this operadic structure is preserved. Recall that we denote by $U$ the forgetful functor from operads to $\mathbb{S}$-modules.

**Definition 5.3.1.** $\mathfrak{C}$ is the category such that,

1. The objects are pairs of the form $(\mathcal{E}, M)$, where $M$ is a $\mathbb{S}$-module and $\mathcal{E}$ is an operad such that $U(\mathcal{E})$ is a $\mathbb{S}$-submodule of $M$. The canonical inclusion is denoted by $i_{\mathcal{E}} : U(\mathcal{E}) \to M$.

2. A morphism from $(\mathcal{E}, M)$ to $(\mathcal{F}, N)$, is a pair $(f, \overline{f})$ with $f : \mathcal{E} \to \mathcal{F}$ morphism of operads, and $\overline{f} : M \to N$ morphism of $\mathbb{S}$-modules, such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "\overline{f}"] & N \\ U(\mathcal{E}) \arrow[u, "i_{\mathcal{E}}"] \arrow[r, "U(f)"'] & U(\mathcal{F}) \arrow[u, "i_{\mathcal{F}}"'] \end{tikzcd} \tag{5.7}$$

➪ *Remark* 5.3.2. Essentially, a morphism from $(\mathcal{E}, M)$ to $(\mathcal{F}, N)$ in $\mathfrak{C}$ is morphism of $\mathbb{S}$-modules from $M$ to $N$ that sends $U(\mathcal{E})$ to $U(\mathcal{F})$ and respects the operadic structure of $\mathcal{E}$.

**Definition 5.3.3.** We define $\mathfrak{U} : \mathcal{OP} \to \mathfrak{C}$ to be the functor forgetful which sends every operad $\mathcal{E}$ to the pair $(\mathcal{E}, U(\mathcal{E}))$. That is, every operad is sent to the pair formed by itself and its underlying $\mathbb{S}$-module.

**Theorem 5.3.4.** *The functor $\mathfrak{U} : \mathcal{OP} \to \mathfrak{C}$ has a left adjoint. We denote this adjoint by $\mathfrak{J}$, and the image of $(\mathcal{E}, M)$ under $\mathfrak{J}$ by $\mathcal{E}[M]$, that we call the polynomial operad on $M$ with coefficients in $\mathcal{E}$.*

*Proof.* We can associate to every $(\mathcal{E}, M) \in \mathfrak{C}$ the following diagram in $\mathcal{OP}$,

$$\begin{tikzcd} FU(\mathcal{E}) \arrow[r, "\epsilon_{\mathcal{E}}"] \arrow[d, "F(i_{\mathcal{E}})"'] & \mathcal{E} \\ F(M) & \end{tikzcd} \tag{5.8}$$

where $\epsilon : FU \to 1_{\mathcal{OP}}$ is the counit of the adjunction $F \vdash U : \mathbb{S} \to \mathcal{OP}$. This association is functorial. Indeed, for every morphism in $\mathfrak{C}$, $(f, \overline{f}) : (\mathcal{E}, M) \to (\mathcal{D}, N)$, we have the following commutative diagram.

$$\begin{tikzcd} FU(\mathcal{E}) \arrow[r, "\epsilon_{\mathcal{E}}"] \arrow[d, "F(i_{\mathcal{E}})"'] \arrow[dr, "FU(f)"] & \mathcal{E} \arrow[dr, "f"] & \\ F(M) \arrow[dr, "F(\overline{f})"'] & FU(\mathcal{D}) \arrow[r, "\epsilon_{\mathcal{D}}"'] \arrow[d, "F(i_{\mathcal{D}})"] & \mathcal{D} \\ & F(N) & \end{tikzcd} \tag{5.9}$$

The commutativity of this diagram follows from the naturality of the counit $\epsilon$ and the diagram from the definition of $(f, \overline{f})$ as a morphism in $\mathfrak{C}$. Thus we have a functor $Cm$ from $\mathfrak{C}$ to the category of diagrams in $\mathcal{OP}$ of the form $\bullet \longleftarrow \bullet \longrightarrow \bullet$ Then, we define the functor $\mathfrak{J} : \mathfrak{C} \to \mathcal{OP}$ to be the composition of $Cm$ with the functor of colimits on $\mathcal{OP}$ (see proposition 2.4.4).

In order to prove that we have the adjunction $\mathfrak{J} \vdash \mathfrak{U} : \mathfrak{C} \to \mathcal{OP}$, we use the proposition 1.5.5. That is, we will construct for every object $(\mathcal{E}, M) \in \mathfrak{C}$ an universal arrow $\Psi$ from $(\mathcal{E}, M)$ to $\mathfrak{U}\mathfrak{J}(\mathcal{E}, M) = (\mathcal{E}[M], U(\mathcal{E}[M]))$. We proceed first by defining the arrow, then check that it is a morphism in $\mathfrak{C}$ and finally that it satisfies the universal property. To improve clarity, despite proof's length, before each verification we will include the used relevant commutative diagrams.

Let $(\mathcal{E}, M)$ be an object in $\mathfrak{C}$ and consider the following diagram given by the colimit $\mathfrak{J}(\mathcal{E}, M)$.

$$\begin{tikzcd} FU(\mathcal{E}) \arrow[r, "\epsilon_\mathcal{E}"] \arrow[d, "F(i_\mathcal{E})"'] & \mathcal{E} \arrow[d, "\alpha"] \\ F(M) \arrow[r, "\beta"'] & \mathcal{E}[M] \end{tikzcd} \tag{5.10}$$

Now consider the couple of arrows $(\alpha, \theta(\beta))$, where $\theta$ is the isomorphism,

$$\mathcal{OP}(F(M), P) \xrightarrow{\theta} \mathbb{S}(M, U(P)) \tag{5.11}$$

given by the adjunction $F \vdash U$. This couple will be our universal arrow $\Psi$. But before we have to check that $\Psi$ is an arrow in $\mathfrak{C}$, that is, the following diagram commute.

$$\begin{tikzcd} M \arrow[r, "\theta(\beta)"] & U(\mathcal{E}[M]) \\ U(\mathcal{E}) \arrow[u, "i_\mathcal{E}"] \arrow[r, "U(\alpha)"'] & U(\mathcal{E}[M]) \arrow[u, "1"'] \end{tikzcd} \tag{5.12}$$

We will need the following commutative diagrams.

1. $\theta$ naturality:

$$\begin{tikzcd} \mathcal{E}[M] & \mathcal{OP}(F(M), \mathcal{E}[M]) \arrow[r, "\theta"] & \mathbb{S}(M, U(\mathcal{E}[M])) & \beta \arrow[r, mapsto] & \theta(\beta) = U(\beta)\eta_M \\ F(M) \arrow[u, "\beta"] & \mathcal{OP}(F(M), F(M)) \arrow[u, "\beta_*"] \arrow[r, "\theta"] & \mathbb{S}(M, UF(M)) \arrow[u, "U(\beta)_*"'] & 1_{F(M)} \arrow[u, mapsto] \arrow[r, mapsto] & \theta(1_{F(M)}) = \eta_M \arrow[u, mapsto] \end{tikzcd} \tag{5.13}$$

2. $\eta$ naturality:

$$\begin{tikzcd} M \arrow[r, "\eta_M"] & UF(M) \\ U(\mathcal{E}) \arrow[u, "i_\mathcal{E}"] \arrow[r, "\eta_{U(\mathcal{E})}"'] & UFU(\mathcal{E}) \arrow[u, "UF(i_\mathcal{E})"'] \end{tikzcd} \tag{5.14}$$

3. Triangular equation for $\eta$ an $\epsilon$:

$$\begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[dr, "1"'] & UFU \arrow[d, "U\epsilon"] \\ & U \end{tikzcd} \quad \Longrightarrow_{\text{for all } \mathcal{P} \in \mathcal{OP}} \quad \begin{tikzcd} U(\mathcal{P}) \arrow[r, "\eta_{U(\mathcal{P})}"] \arrow[dr, "1"'] & UFU(\mathcal{P}) \arrow[d, "U(\epsilon_\mathcal{P})"] \\ & U(\mathcal{P}) \end{tikzcd} \tag{5.15}$$

Now, we check that the diagram 5.12 is commutative.

$$
\begin{aligned}
\theta(\beta)i_{\mathcal{E}} &= U(\beta)\eta_M i_{\mathcal{E}} && \text{(by diagram 5.13)}\\
&= U(\beta)UF(i_{\mathcal{E}})\eta_{U(\mathcal{E})} && \text{(by diagram 5.14)}\\
&= U(\alpha)U(\epsilon_{\mathcal{E}})\eta_{U(\mathcal{E})} && \text{(by diagram 5.10)}\\
&= U(\alpha) && \text{(by diagram 5.15)}
\end{aligned}
$$

To verify that $(\alpha, \theta(\beta))$ satisfies the universal property, we have to show that given any $\mathcal{Q}$ operad and any morphism $(g, \overline{g}) : (\mathcal{E}, M) \to (\mathcal{Q}, U(\mathcal{Q}))$, there is an unique morphism $(h, \overline{h})$ in $\mathfrak{C}$ making the following diagram commutative.

$$
\begin{tikzcd}
(\mathcal{E}, M) \arrow[r, "{(\alpha,\theta(\beta))}"] \arrow[d, "{(g,\overline{g})}"'] & (\mathcal{E}[M], U(\mathcal{E}[M])) \arrow[dl, "{(h,\overline{h})}"] \\
(\mathcal{Q}, U(\mathcal{Q}))
\end{tikzcd}
\tag{5.16}
$$

For that consider following diagram associated to the pair $(g, \overline{g})$.

$$
\begin{tikzcd}
FU(\mathcal{E}) \arrow[r, "\epsilon_{\mathcal{E}}"] \arrow[d, "F(i_{\mathcal{E}})"'] & \mathcal{E} \arrow[d, "\alpha"] \arrow[ddr, bend left, "g"] & \\
F(M) \arrow[r, "\beta"'] \arrow[drr, bend right, "\theta^{-1}(\overline{g})"'] & \mathcal{E}[M] \arrow[dr, dashed, "h"] & \\
& & \mathcal{Q}
\end{tikzcd}
\tag{5.17}
$$

Where $h$ is the morphism we want to construct and the arrow $\theta^{-1}(\overline{g})$ is given by the bijection $\theta$ of the adjunction $F \vdash U$.

$$
\begin{tikzcd}
\mathcal{OP}(F(M), \mathcal{Q}) \arrow[r, "\theta"] & \mathbb{S}(M, U(\mathcal{Q})) \\
\theta^{-1}(\overline{g}) & \overline{g} \arrow[l, mapsto]
\end{tikzcd}
\tag{5.18}
$$

To construct $h$ we use the universal property of the colimit $\mathcal{E}[\mathcal{M}]$, that is, if $g\epsilon_{\mathcal{E}} = \theta^{-1}(\overline{g})F(i_{\mathcal{E}})$ in diagram 5.17, there exists an unique operad morphism $h$ from $\mathcal{E}[M]$ to $Q$, such that $h\alpha = g$ and $h\beta = \theta^{-1}(\overline{g})$. Both arrows, $g\epsilon_{\mathcal{E}}$ and $\theta^{-1}(\overline{g})F(i_{\mathcal{E}})$, go from $FU(\mathcal{E})$ to $Q$. Then we will use the universal property of the unit $\eta$ of $F \vdash U$ to show they are the same arrow. Which says in particular that there exists only one morphism of operad $\rho : FU(\mathcal{E}) \to \mathcal{Q}$ such that the following diagram is commutative.

$$
\begin{tikzcd}
U(\mathcal{E}) \arrow[r, "\eta_{U(\mathcal{E})}"] \arrow[dr, "U(g)"'] & UFU(\mathcal{E}) \arrow[d, dashed, "U(\rho)"] \\
& U(\mathcal{Q})
\end{tikzcd}
\tag{5.19}
$$

Then we only have to check that $U(g\epsilon_{\mathcal{E}})\eta_{U(\mathcal{E})}$ and $U(\theta^{-1}(\overline{g})F(i_{\mathcal{E}}))\eta_{U(\mathcal{E})}$ are equal to $U(g)$. But before that we make a list with some of the necessary commutative diagrams.

1. Naturality of $\theta^{-1}$:

$$
\begin{tikzcd}
\mathcal{OP}(F(M),\mathcal{Q}) & \mathbb{S}(M,U(\mathcal{Q})) \arrow[l, "\theta^{-1}"'] & \epsilon_{\mathcal{Q}}F(\overline{g}) = \theta^{-1}(\overline{g}) & \overline{g} \arrow[l, mapsto] \\
\mathcal{OP}(FU(\mathcal{Q}),\mathcal{Q}) \arrow[u, "F(\overline{g})^*"] & \mathbb{S}(U(\mathcal{Q}),U(\mathcal{Q})) \arrow[l, "\theta^{-1}"] \arrow[u, "\overline{g}^*"'] & \epsilon_{\mathcal{Q}} = \theta^{-1}(1_{U(\mathcal{Q})}) \arrow[u, mapsto] & 1_{U(\mathcal{Q})} \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}
\tag{5.20}
$$

2. Definition of $(g,\overline{g})$ as morphism in $\mathfrak{C}$:

$$
\begin{tikzcd}
M \arrow[r, "\overline{g}"] & U(\mathcal{Q}) & F(M) \arrow[r, "F(\overline{g})"] & FU(\mathcal{Q}) \\
U(\mathcal{E}) \arrow[u, "i_{\mathcal{E}}"] \arrow[r, "U(g)"'] & U(\mathcal{Q}) \arrow[u, "1"'] \arrow[r, phantom, "\Longrightarrow"] & FU(\mathcal{E}) \arrow[u, "F(i_{\mathcal{E}})"] \arrow[ur, "FU(g)"'] &
\end{tikzcd}
\tag{5.21}
$$

3. $\epsilon$ naturality:

$$
\begin{tikzcd}
FU(\mathcal{Q}) \arrow[r, "\epsilon_{\mathcal{Q}}"] & Q \\
FU(\mathcal{E}) \arrow[u, "FU(g)"] \arrow[r, "\epsilon_{\mathcal{E}}"] & \mathcal{E} \arrow[u, "g"']
\end{tikzcd}
\tag{5.22}
$$

4. $\eta$ naturality:

$$
\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[d, "\overline{g}"'] & UF(M) \arrow[d, "UF(\overline{g})"] \\
U(\mathcal{Q}) \arrow[r, "\eta_{U(\mathcal{Q})}"'] & UFU(\mathcal{Q})
\end{tikzcd}
\tag{5.23}
$$

Now the verifications.

$$
\begin{aligned}
U(g\epsilon_{\mathcal{E}})\eta_{U(\mathcal{E})} &= U(g)U(\epsilon_{\mathcal{E}})\eta_{U(\mathcal{E})} \\
&= U(g) && \text{(by diagram 5.15)}
\end{aligned}
$$

And now, the other one.

$$
\begin{aligned}
U(\theta^{-1}(\overline{g})F(i_{\mathcal{E}}))\eta_{U(\mathcal{E})} &= U(\epsilon_{\mathcal{Q}}F(\overline{g})F(i_{\mathcal{E}}))\eta_{U(\mathcal{E})} && \text{(by diagram 5.20)} \\
&= U(\epsilon_{\mathcal{Q}}FU(g))\eta_{U(\mathcal{E})} && \text{(by diagram 5.21)} \\
&= U(g\epsilon_{\mathcal{E}})\eta_{U(\mathcal{E})} && \text{(by diagram 5.22)} \\
&= U(g)U(\epsilon_{\mathcal{E}})\eta_{U(\mathcal{E})} \\
&= U(g) && \text{(by diagram 5.15)}
\end{aligned}
$$

Then by the colimits universal property, there exist an unique operad morphism $h : \mathcal{E}[M] \to \mathcal{Q}$ such that $h\alpha = g$ and $h\beta = \theta^{-1}(\overline{g})$. Defining $\overline{h}$ as $U(h)$, the pair $(h, \overline{h})$ is clearly a morphism in $\mathfrak{C}$. Now, even if we already have $h\alpha = g$, to be sure $(h, \overline{h})$ makes the diagram 5.16 commutative we need to check that $\overline{h}\theta(\beta) = \overline{g}$.

$$
\begin{aligned}
\overline{h}\theta(\beta) &= U(h)\theta(\beta) \\
&= U(h)U(\beta)\eta_M && \text{(by diagram 5.13)} \\
&= U(\theta^{-1}(\overline{g}))\eta_M && \text{(by property of } h\text{)} \\
&= U(\epsilon_{\mathcal{Q}}F(\overline{g}))\eta_M && \text{(by diagram 5.20)} \\
&= U(\epsilon_{\mathcal{Q}})UF(\overline{g})\eta_M \\
&= U(\epsilon_{\mathcal{Q}})\eta_{U(\mathcal{Q})}\overline{g} && \text{(by diagram 5.23)} \\
&= \overline{g} && \text{(by diagram 5.15)}
\end{aligned}
$$

The unicity for $(h, \overline{h})$ follows from the unicity of $h$ and the fact that every morphism $(f, \overline{f})$ of $\mathfrak{C}$ from $(\mathcal{E}[M], U(\mathcal{E}[M]))$ to $(Q, U(Q))$ satisfies $\overline{f} = U(f)$.

□

➪ *Remark* 5.3.5. The universal arrow

$$\Psi : (\mathcal{E}, M) \to \mathfrak{U}(\mathfrak{I}(\mathcal{E}, M)) = (\mathcal{E}[M], U(\mathcal{E}[M])) \tag{5.24}$$

associated to every pair $(\mathcal{E}, M) \in \mathcal{C}$ in the proof of theorem 5.3.4, extends to the unit of the adjunction $\mathfrak{I} \dashv \mathfrak{U} : \mathcal{C} \to \mathcal{OP}$. We keep the notation $\Psi$ for this unit.

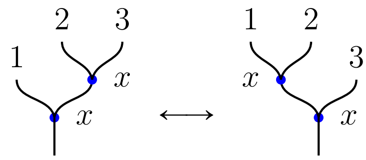
**Proposition 5.3.6.** *Let $(\mathcal{E}, M) \in \mathfrak{C}$ and $\mathcal{A} \in \mathcal{OP}$. For every morphism $(f, \overline{f}) : (\mathcal{E}, M) \to \mathfrak{U}(\mathcal{A}) = (\mathcal{A}, U(\mathcal{A}))$, there exists an unique morphism of operads $\varphi : \mathcal{E}[M] \to \mathcal{A}$, such that $U(\varphi)\Psi = \overline{f}$. So we have the following commutative diagram.*

$$
\begin{tikzcd}
(\mathcal{E}, M) \arrow[r, "\Psi"] \arrow[dr, "(f,\overline{f})"'] & (\mathcal{E}[M], U(\mathcal{E}[M])) \arrow[d, "(\varphi, U(\varphi))"] \\
& (\mathcal{A}, U(\mathcal{A}))
\end{tikzcd} \tag{5.25}
$$

*Proof.* This is just the universal property for the unit $\Psi : 1_{\mathfrak{C}} \to \mathfrak{U}\mathfrak{I}$. □

## 5.4 The Operad $\mathcal{K}$

In this section is constructed a collection of operads $\{\mathcal{K}_n\}_{n \geqslant 2}$ in such a way its inductive limit is an $E_\infty$-operad. This operad will be denoted by $\mathcal{K}$. In order to do that, we begin with an $\mathbb{S}$-module concentrated in arity 2, then $\mathcal{K}_2$ is taken to be the free operad on it. $\mathcal{K}_2$ as $\mathbb{S}$-module will have a $\mathsf{k}[\Sigma_2]$-free resolution of $\mathsf{k}$ in its second component, which is formed by the abstract binary operations coded by $\mathcal{K}_2$. But the rest of components of $\mathcal{K}_2$ are not necessarily free resolutions of $\mathsf{k}$. Indeed, $\mathcal{K}_2$ fails to have the homotopy linking the following trees of degree 0 in $\mathcal{K}_2(3)$,




(5.26)

where $x$ is any generating element of degree 0 in $\mathcal{K}_2(2)$. To overcome this difficulty, $\mathcal{K}_2$ is extended to an operad $\mathcal{K}_3$ having the missing homotopies in the component $\mathcal{K}_2(3)$. This processes requires the use of the polynomial operads technique discussed in the previous section. Step by step the homotopies are completed to finally give, as an inductive limit, an operad with all the homotopies we want, in other words, the homotopies to have an $E_\infty$-operad, the operad $\mathcal{K}$.

**Proposition 5.4.1** (Acyclic Extension)**.** *Let $M$ be a $\mathsf{k}[\Sigma_n]$-free finitely generated DGA-$\mathsf{k}$-module. Then there exists a $\mathsf{k}[\Sigma_n]$-free finitely generated acyclic DGA-$\mathsf{k}$-module $N$ such that*

1. $M_0 = N_0$.
2. *$M$ is a DGA-$\mathsf{k}$-submodule of $N$.*

*Proof.* For the modules on $\mathsf{k}[\Sigma_n]$, we consider the adjunction $L \vdash U : \mathbf{Set} \to \mathrm{Mod}_{\mathsf{k}[\Sigma_n]}$, where $U$ is the forgetful functor. For every module $N$, the counit gives the surjection $\epsilon_N : LU(N) \to N$, which will be denoted $p : PN \to N$. Given a DGA-$\mathsf{k}$-module $M$ we denote $ZM$ its submodule of cycles. On $ZM$ the differential is 0, then we extend the meaning of $P$ to graded modules, we keep the same notation for the extended morphism $p : PZM \to ZM$. Consider the composition $d = i \circ p : PZM \to M$, where $i$ is the canonical inclusion of $ZM$ in $M$. $ZM$ can seen as a submodule of $PZM$, then $p : PZM \to ZM$ is a retraction for this inclusion and if $m \in ZM$, then $d(m) = m$.

Observe that in the mapping cone of $d : i \circ p : PZM \to M$, $C(d)$, all the cycles of $M$ are now boundaries and also on $C(d)$ will appear new cycles. Indeed, let $m \in M$ cycle, recall that the differential of $C(d)$ is given by $\begin{pmatrix} 0 & 0 \\ d & \partial_M \end{pmatrix}$. Then in $C(d)$ we have $\partial \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ m \end{pmatrix}$, which means that $m$ is a boundary. If it happens that $m$ is already a boundary in $M$, that is there exists $n$ such that $\partial_M(n) = m$, then $\begin{pmatrix} m \\ -n \end{pmatrix}$ is a cycle in $C(d)$. From this also notice that if all the cycles of $M$ have degree at least $k$, then all the cycles in $C(d)$ will have at least degree $k+1$.

Let $M$ $\mathsf{k}[\Sigma_n]$-free finitely generated DGA-$\mathsf{k}$-module, and denote $W$ the kernel of the augmentation $\epsilon : M \to \mathsf{k}$ and consider the $\mathsf{k}[\Sigma_n]$-linear morphisms for $n \geqslant 1$,

$$PZC(d_n) \xrightarrow{d_{n+1}} C(d_n) \tag{5.27}$$

where $d_1$ is $d : PZW \to W$, and $d_{n+1}$ is $d : PZC(d_n) \to C(d_n)$. Then we have that $W$ is included in $C(d_1)$ and $C(d_n)$ is included in $C(d_{n+1})$. With this we can define a DGA-k-module $N$ that satisfies the conditions of the theorem by taking the colimit of the following diagram,

$$M \longleftarrow W \longrightarrow C(d_1) \longrightarrow C(d_2) \longrightarrow \cdots \tag{5.28}$$

where all the arrows are the respective canonical inclusions. Observe that we can reduce the size of this acyclic extension by considering in the first step only the cycles of degree 0 of $W$, and for the construction of $d_{n+1}$, considering only the cycles of degree $n$ of the last mapping cone. □

**Definition 5.4.2.** Let $M$ be a $\mathsf{k}[\Sigma_n]$-free finitely generated DGA-k-module. The acyclic extension of $M$ is the associated DGA-k-module given by proposition 5.4.1. It will be denoted by $X(M)$.

**Definition 5.4.3.** Let $M$ be a DGA-k-module $\mathsf{k}[\Sigma_2]$-free resolution of k. For $n \geqslant 2$, $\mathcal{K}_n$ is the operad defined by induction as follows.

1. $\mathcal{K}_2 = F(M)$, where $F$ is the free operad functor and $M$ is seen as a $\mathbb{S}$-module concentrated in arity 2.
2. $\mathcal{K}_{n+1} = \mathfrak{J}(\mathcal{K}_n, M_n) = \mathcal{K}_n[M_n]$, where $\mathfrak{J}$ is the functor defined in proposition 5.3.4 and $M_n$ is the $\mathbb{S}$-module given by:

$$M_n(i) = \begin{cases} \mathcal{K}_n(i) & i \neq n+1 \\ X(\mathcal{K}_n(n+1)) & i = n+1 \end{cases} \tag{5.29}$$

➪ *Remark* 5.4.4. Between the operads of the collection $\{\mathcal{K}_n\}_{n\geqslant 2}$, we have canonical inclusions of operads $\mathcal{K}_n \hookrightarrow \mathcal{K}_{n+1}$, given by the arrow $\alpha_n$ of the following commutative diagram from the construction of $\mathcal{K}_{n+1}$.

$$\begin{tikzcd} FU(\mathcal{K}_n) \arrow[r, "\epsilon_{\mathcal{K}_n}"] \arrow[d, "F(i_{\mathcal{K}_n})"'] & \mathcal{K}_n \arrow[d, "\alpha_n"] \\ F(M_n) \arrow[r, "\beta_n"'] & \mathcal{K}_n[M_n] = \mathcal{K}_{n+1} \end{tikzcd} \tag{5.30}$$

➪ *Remark* 5.4.5. By construction $K_2 = F(M)$ only contains operations with arity 2 or more, because $M$ is concentrated in arity 2. Then, the operations of arity $\geqslant 3$ are obtained by composition of operation of arity 2. $K_2$ is not acyclic for arities $\geqslant 3$. The next step is make acyclic only $K_2(3)$, for that we construct the inclusion $K_2 \hookrightarrow M_2$, which is strict only in arity 3. The new operations are not decomposable in terms of operations of arity 2 of $K_2$, and $K_3 = K_2[M_2]$ will be formed by the compositions all operations in $K_2$ and the new arity 3 operations. Then $K_2$ coincide with $K_3$ in arity 2, and in arity 3 we have the inclusion $K_2(3) \hookrightarrow K_3(3)$. A similar reasoning apply for the general case, in other words the extension $K_n \to K_{n+1}$ is the identity for arities $\leqslant n+1$.

**Definition 5.4.6.** The operad $\mathcal{K}$ is defined to be the inductive limit of the collection of operads,

$$\mathcal{K}_2 \overset{\alpha_2}{\hookrightarrow} \mathcal{K}_3 \overset{\alpha_3}{\hookrightarrow} \cdots \overset{\alpha_{n-1}}{\hookrightarrow} \mathcal{K}_n \overset{\alpha_n}{\hookrightarrow} \cdots \tag{5.31}$$

**Proposition 5.4.7.** *The operad $\mathcal{K}$ is an $E_\infty$-operad.*

*Proof.* We have $\mathcal{K}(2) = \mathcal{K}_2(2) = M_2$, a $\mathsf{k}[\Sigma_2]$-free resolution of $\mathsf{k}$, and by construction $\mathcal{K}(n+1) = \mathcal{K}_{n+1}(n+1) = M_n = X(\mathcal{K}_n(n+1))$, which is acyclic and $\mathsf{k}[\Sigma_{n+1}]$-free. □

## 5.5 $E_\infty$-Structures in $\mathcal{L}$-Algebras

The following theorem is the principal objective of this thesis. This theorem exhibits the main element of an $\mathcal{L}$-algebra as an $E_\infty$-coalgebra.

The outline of the theorem is, first we prove that the main element $A[1]$ of an $\mathcal{L}$-algebra $\mathcal{A}$ have a $\mathcal{K}_n$-coalgebra structure for all $n > 1$. Then, using the fact that our operad $\mathcal{K}$ is the inductive limit of these operads, we prove that $A[1]$ will have a structure of $\mathcal{K}$-coalgebra, in other words, $A[1]$ is an $E_\infty$-coalgebra.

**Theorem 5.5.1** (Main Theorem)**.** *Let $\mathcal{K}$ be the $E_\infty$-operad defined in 5.4.6. Then, there exists a functor $\mathcal{F}$*

$$\mathcal{L}\text{-}Alg \xrightarrow{\mathcal{F}} \mathcal{K}\text{-}CoAlg \tag{5.32}$$

*which associates a $\mathcal{K}$-coalgebra $\mathcal{F}(\mathcal{A})$ to each $\mathcal{L}$-algebra $\mathcal{A}$ and satisfies the following conditions.*

1. *The underlying DGA-$\mathsf{k}$-module of $\mathcal{F}(\mathcal{A})$ is $A[1]$.*

2. *For every $n \geqslant 1$, the morphism $\varphi_n : K(n) \otimes A[1] \to A[1]^{\otimes n}$, given by the $\mathcal{K}$-coalgebra structure defined on $A[1]$ by $\mathcal{F}$, makes the following diagram commutative up to homotopy,*

$$\begin{tikzcd} A[1]^{\otimes n} \arrow[rr, "\mu"] & & A[n] \\ & K(n)\otimes A[1] \arrow[ul, "\varphi_n"] \arrow[ur, "s_1(\epsilon\otimes 1)"'] & \end{tikzcd} \tag{5.33}$$

*where $\mu$ is given by the structural quasi-isomorphism of $\mathcal{A}$ and $s_1$ is the image by $\mathcal{A}$ of the only morphism in $\mathcal{L}$ of the form $([n], \alpha) : [n] \to [1]$.*

➪ *Remark* 5.5.2. Clearly, the composition of $\mathcal{F}$ with the canonical $\mathcal{L}$-algebra functor (see definition 3.5.3), associates an $E_\infty$-coalgebra to each simplicial set.

*Proof of theorem 5.5.1.* We use the fact that the operad $E_\infty$-operad $\mathcal{K}$ is the inductive limit of the sequence of operads,

$$\mathcal{K}_2 \subset \cdots \subset \mathcal{K}_n \subset \cdots \subset \mathcal{K} \tag{5.34}$$

in order to proceed by induction. We first show for all $n \geqslant 2$ that $A[1]$ has an structure of $\mathcal{K}_n$-coalgebra which satisfies the second condition of the theorem. That is, there exists an operad morphism $\overline{F}_n : \mathcal{K}_n \to Coend(A[1])$, such that the associated morphisms $\varphi_i : K_n(i) \otimes A[1] \to A[1]^{\otimes i}$, makes the following diagram commutative up to homotopy,

$$\begin{tikzcd} A[1]^{\otimes i} \arrow[rr, "\mu"] & & A[i] \\ & K_n(i) \otimes A[1] \arrow[ul, "\varphi_i"] \arrow[ur, "s_1(\epsilon\otimes 1)"'] & \end{tikzcd} \tag{5.35}$$

where $\mu$ is given by the structural quasi-isomorphism of $\mathcal{A}$ and $s_1$ is the image by $\mathcal{A}$ of the only morphism in $\mathcal{L}$ of the form $([i], \alpha) : [i] \to [1]$.

**Case $\mathcal{K}_2$:** Recall that $\mathcal{K}_2$ is the free operad on the $\mathbb{S}$-module $M_2$ concentrated in arity 2. To show that $A[1]$ is a $\mathcal{K}_2$-coalgebra, we define an $\Sigma_2$-equivariant morphism from $M_2(2)$ to $Coend(A[1])(2)$ using the relative lifting theorem 5.2.2 in order to satisfy the condition on $\mathcal{K}_2$ and then, the $\mathcal{K}_2$-coalgebra structure is obtained as a consequence of the universal property of free operads.

Defining a $\Sigma_2$ morphism from $M_2(2)$ to $Coend(A[1])(2)$ is equivalent to define a morphism of DGA-$\mathsf{k}[\Sigma_2]$-modules,

$$\overline{\varphi}_2 : K_2(2) \otimes A[1] \to A[1] \otimes A[1] \tag{5.36}$$

Recall that $M_2(2) = K_2(2)$, the action of $\Sigma_2$ on $A[1] \otimes A[1]$ is the permutation of factors and the action of $\Sigma_2$ on $K_2(2) \otimes A[1]$ maps $x \otimes a$ to $x\sigma \otimes a$. Now consider the following diagram,

$$\begin{tikzcd} A[1] \otimes A[1] \arrow[r, "\mu"] & A[2] \\ & K_2(2) \otimes A[1] \arrow[ul, dotted, "\varphi_2"] \arrow[u, "s_0\circ(\epsilon\otimes 1)"'] \end{tikzcd} \tag{5.37}$$

where $\epsilon$ is the augmentation of $K_2(2)$, $s_0 : A[1] \to A[2]$ is the image by $\mathcal{A}$ of the only arrow in $\mathcal{L}$ of the form $([2], \alpha) : [2] \to [1]$ and $\mu$ is the structural quasi-isomorphism of $\mathcal{A}$.

The DGA-$\mathsf{k}[\Sigma_2]$-morphism $\varphi_2$ that makes the diagram commutative up to homotopy is obtained with the theorem 5.2.2 by taking $L' = 0$. This complete the existence of a $\Sigma_2$-equivariant morphism from $M_2(2)$ to $Coend(A[1])(2)$ and therefore, we have a morphism $F_2$ of $\mathbb{S}$-modules from $M_2$ to $Coend(A[1])$, which behaves on $M_2(2)$ as $\varphi_2$ and as 0 on $M_2(i)$, $i \neq 2$.

Now, consider the following diagram,

$$\begin{tikzcd} M_2 \arrow[r, hook] \arrow[dr, "F_2"'] & \mathcal{K}_2 = F(M_2) \arrow[d, dashed, "\overline{F}_2"] \\ & Coend(A[1]) \end{tikzcd} \tag{5.38}$$

where the upper arrow is given by the inclusion of $\mathbb{S}$-modules. The universal property of the free operad $\mathcal{K}_2$ says there is an unique morphism of operads $\overline{F}_2$ making the diagram commutative. This morphism $\overline{F_2}$ gives the $\mathcal{K}_2$-coalgebra structure on $A[1]$ that we wanted.

**Case $\mathcal{K}_n$:** Suppose we have a sequence of operad morphisms $\overline{F}_2, \ldots, \overline{F}_{n-1}$, such that, for $i < n$, $\overline{F}_i : \mathcal{K}_i \to Coend(A[1])$ and $\overline{F}_i$ satisfies the second condition of the theorem. As we have seen in the construction of $K_i$'s, the operad $\mathcal{K}_{n-1}$ as $\mathbb{S}$-module, can be embedded as a direct factor in a $\mathbb{S}$-module $M_{n-1}$ with component $M_{n-1}(n)$ acyclic and $\mathsf{k}[\Sigma_n]$-free. Then we have $M_{n-1} = \mathcal{K}_{n-1} \oplus P$, where the component $P(j)$ is $\mathsf{k}[\Sigma_j]$-free for $j > 0$ .

Observe that this defines an object $(\mathcal{K}_{n-1}, M_{n-1})$ and morphism in $\mathfrak{C}$,

$$F_{n-1} : (K_{n-1}, M_{n-1}) \to (Coend(A[1]), U(Coend(A[1]))) \tag{5.39}$$

which behaves as $\overline{F}_{n-1}$ on $\mathcal{K}_{n-1}$ and as 0 on $P$. In order to satisfy the second condition of the theorem we focus our attention in the $\Sigma_j$-equivariant morphism given by $\overline{F}_{n-1}$ on the component $\mathcal{K}_{n-1}(j)$, $j > 0$. We will extend this morphism on the components $M_{n-1}(j)$ or equivalently, define a DGA-$\mathsf{k}[\Sigma_j]$-morphism $\overline{\phi}_j$ from $M_{n-1}(j) \otimes A[1]$ to $A[1]^{\otimes j}$. In order to do that, consider the diagram,

$$\begin{tikzcd}
A[1]^{\otimes j} \arrow[rr, "\mu"] & & A[j] \\
& M_{n-1}(j) \otimes A[1] \arrow[ul, dotted, "\overline{\phi}_j"'] \arrow[ur, "s_0\circ(\epsilon\otimes 1)"'] & \\
& K_{n-1}(j) \otimes A[1] \arrow[u, hook] \arrow[uul, bend left, "\phi_j"] \arrow[uur, bend right, "s_0\circ(\epsilon\otimes 1)"'] &
\end{tikzcd} \tag{5.40}$$

where $\phi_j$ is the $\mathsf{k}[\Sigma_n]$-morphism induced by $\overline{F}_{n-1}$. By hypothesis, the morphism $\phi_j$ makes commutative up to homotopy the outer triangle of the diagram. Then the existence of $\overline{\phi}_j$ follows after applying the relative lifting theorem 5.2.2 with $L = M_{n-1}(j) \otimes A[1]$ and $L' = K_{n-1}(j) \otimes A[1]$.

Observe that the $\Sigma_j$-equivariant morphism from $M_{n-1}(j)$ to $Coend(A[1])(j)$ induced by $\overline{\phi}_j$ behaves like $\overline{F}_{n-1}$ over $K_{n-1}(j)$. Denote by $F_{n-1}$ the morphism of $\mathbb{S}$-modules given by this data. Then $(\overline{F}_{n-1}, F_{n-1}) : (\mathcal{K}_{n-1}, M_{n-1}) \to (Coend(A[1]), U(Coend(A[1])))$ is a morphism of $\mathfrak{C}$, and consider the following diagram.

$$\begin{tikzcd}
(\mathcal{K}_{n-1}, M_{n-1}) \arrow[r, "\Psi"] \arrow[dr, "F_{n-1}"'] & (\mathcal{K}_{n-1}[M_{n-1}], U(\mathcal{K}_{n-1}[M_{n-1}])) \arrow[d, dashed, "(\overline{F}_n, U(\overline{F}_n)"] \\
& (Coend(A[1]), U(Coend(A[1])))
\end{tikzcd} \tag{5.41}$$

The operad morphism $\overline{F}_n$ making the diagram commutative follows by proposition 5.3.6. Then, $\overline{F}_n$ gives the $\mathcal{K}_n$-coalgebra structure on $A[1]$ needed to complete the inductive step.

$A[1]$ **is a $\mathcal{K}$-coalgebra:** We now proceed with the final part of the proof and exhibit $A[1]$ as a $\mathcal{K}$-coalgebra. Consider the following cocone of operads onthe diagram given by the operads $\{\mathcal{K}_i\}_{i\geqslant 2}$.

$$\begin{tikzcd}
\mathcal{K} \arrow[rrr, dashed, "\overline{F}"] & & & Coend(A[1]) \\
\mathcal{K}_2 \arrow[u] \arrow[urrr, "\overline{F}_2"] \arrow[r, hook] & \cdots \arrow[r, hook] & \mathcal{K}_n \arrow[ull] \arrow[ur, "\overline{F}_n"] \arrow[r, hook] & \cdots
\end{tikzcd} \tag{5.42}$$

The inductive part of the proof exhibited the operad $Coend(A[1])$, together with the morphisms $\overline{F}_n$'s, as a cocone onthe $\mathcal{K}_i$'s. By definition $\mathcal{K}$ is also a cocone on the $\mathcal{K}_i$'s. Then the universal property of colimits says that there exists an unique morphism of operads $\overline{F}$ from $\mathcal{K}$ to $Coend(A[1])$ commutative on these two cocones. The morphism $\overline{F} : \mathcal{K} \to Coend(A[1])$ exhibit $A[1]$ as an $\mathcal{K}$-coalgebra with the conditions stated by the theorem.

**Functoriality:** Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of $\mathcal{L}$ and consider the following diagram.

$$\begin{tikzcd}
A[n]^{\otimes n} \arrow[rrr, "f_1^{\otimes n}"] \arrow[dr, "\mu^{\mathcal{A}}"] & & & B[n]^{\otimes n} \arrow[dl, "\mu^{\mathcal{B}}"'] \\
& A[n] \arrow[r, "f_n"] & B[n] & \\
K[n]\otimes A[1] \arrow[uu, bend left, "\varphi_n^{\mathcal{A}}"] \arrow[ur, "s(\epsilon\otimes 1)"] \arrow[rrr, "1\otimes f_1"'] & & & K[n]\otimes B[1] \arrow[uu, bend right, "\varphi_n^{\mathcal{B}}"'] \arrow[ul, "s(\epsilon\otimes 1)"']
\end{tikzcd} \tag{5.43}$$

The two triangles are commutative up to homotopy by the second condition of the theorem and the inner diagrams are commutative because $f$ is a morphism of $\mathcal{L}$-algebras. The commutative up to homotopy of the outer diagram follows from this and the fact that $\mu$ is a quasi-isomorphism. This shows that our construction is functorial and completes the proof. $\square$

* * *

# Bibliography


[Awo06] S. Awodey. *Category theory.* Oxford Logic Guides. Ebsco Publishing, 2006.

[BE74] M. G. Barratt and P. J. Eccles. $\Gamma^+$-structures-I: a free group functor for stable homotopy theory. *Topology*, 13(1):25 – 45, 1974.

[BF04] C. Berger and B. Fresse. Combinatorial operad actions on cochains. *Mathematical Proceedings of the Cambridge Philosophical Society*, 137:135–174, July 2004, arXiv:math/0109158.

[BM06] C. Berger and I. Moerdijk. The Boardman-Vogt resolution of operads in monoidal model categories. *Topology*, 45(5):807 – 849, 2006.

[BM07] C. Berger and I. Moerdijk. Resolution of coloured operads and rectification of homotopy algebras. *Contemporary mathematics*, 431:31–58, December 2007, arXiv:math/0512576.

[BV68] J. M. Boardman and R. M. Vogt. Homotopy-everything $H$-spaces. *Bull. Amer. Math. Soc.*, 74(6):1117–1122, November 1968.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1973.

[Car55] H. Cartan. *Séminaire Henri Cartan: 1954-55. Algèbre d'Eilenberg-Mac Lane et homotopie. 7e année.* Secrétariat de Mathématique, 1955.

[Cen89] B. Cenkl. Cohomology operations from higher products in the de Rham complex. *Pacific Journal of Mathematics*, 140(1):21–33, 1989.

[DV15] M. Dehling and B. Vallette. Symmetric homotopy theory for operads. *ArXiv e-prints*, March 2015, arXiv:1503.02701.

[Fre16] B. Fresse. Homotopy of operads & Grothendiek-Teichmüller groups. unpublished, 2016, `http://math.univ-lille1.fr/~fresse/OperadHomotopyBook/`.

[GK94] V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 10 1994.

[GM81] P. Griffiths and J. Morgan. *Rational homotopy theory and differential forms.* Progress in Mathematics. Birkhäuser Boston, 1981.

[Hes07] K. Hess. The cobar construction: a modern perspective. *Institut de Géométrie, Algèbre et Topologie, École Polytechnique Fédérale de Lausanne*, 2007.

[Jam99] I. M. James. *History of Topology.* Elsevier Science, 1999.

[Kad80] T. Kadeishvili. On the homology theory of fibre spaces. *Russian Mathematical Surveys*, 35(3):231, 1980.

[Lei00] T. Leinster. Homotopy Algebras for Operads. *ArXiv Mathematics e-prints*, February 2000, arXiv:math/0002180.

[LV12] J. L. Loday and B. Vallette. *Algebraic operads.* Grundlehren der mathematischen Wissenschaften. Springer, 2012.

[Mac98] S. Mac Lane. *Categories for the working mathematician.* Springer, 2 edition, 1998.

[Man06] M. Mandell. Cochains and homotopy type. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 103(1):213–246, 2006.

[May72] P. May. *The geometry of iterated loop space.* Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg, 1972.

[May93] P. May. *Simplicial objects in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, 1993.

[MSS07] M. Markl, S. Shnider, and J. D. Stasheff. *Operads in algebra, topology and physics*. Mathematical Surveys and Monographs. American Mathematical Society, 2007.

[Pon39] L. Pontrjagin. Homologies in compact lie groups. *Rec. Math. [Mat. Sbornik] N.S.*, 6(48):389–422, 1939.

[Pro83] A. Prouté. Sur la transformation d'Eilenberg-Maclane. *C. R. Acad. Sc. Paris*, 297:193–194, 1983.

[Pro84] A. Prouté. Sur la diagonal d'Alexander-Whitney. *C. R. Acad. Sc. Paris*, 299:391–392, 1984.

[Pro10] A. Prouté. Introduction à la logique catégorique, 2010, `http://www.logique.jussieu.fr/~alp/cours_2010.pdf`.

[Pro11] A. Prouté. $A_\infty$-structures, modèles minimaux de Baues-Lemaire et Kadeishvili, 2011, `http://www.tac.mta.ca/tac/reprints/articles/21/tr21abs.html`,.

[Qui69] D. Quillen. Rational homotopy theory. *Annals of Mathematics*, 90(2):205–295, 1969.

[Rez96] C. Rezk. Spaces of algebra structures and cohomology of operads. Ph.D thesis, 1996.

[SE53] S. Mac Lane S. Eilenberg. On the groups $H(\Pi, n)$, I. *Annals of Mathematics*, 58(1):55–106, 1953.

[Seg74] G. Segal. Catégories and cohomology théories. *Topology*, 13(3):293 – 312, 1974.

[Smi86] V. A. Smirnov. Homotopy theory of coalgebras. *Mathematics of the USSR-Izvestiya*, 27(3):575, 1986.

[Smi94] J. R. Smith. *Iterating the cobar construction*. Number 524 in American Mathematical Society: Memoirs of the American Mathematical Society. American Mathematical Society, 1994.

[Smi00] J. R. Smith. Operads and algebraic homotopy. *ArXiv Mathematics e-prints*, April 2000, arXiv:math/0004003.

[Sta63] J. D. Stasheff. Homotopy associativity of H-spaces. II. *Transactions of the American Mathematical Society*, 108(2):293–312, 1963.

[Ste47] N. E. Steenrod. Products of cocycles and extensions of mappings. *Annals of Mathematics*, 48(2):290–320, 1947.

[Sul77] D. Sullivan. Infinitesimal computations in topology. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 47(1):269–331, 1977.

[Wei95] C. A. Weibel. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

# Index