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OPÉRATEUR DE RANKIN-COHEN ET MATRICE DE FUSION

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Résumé

Ce travail est consacré a l'étude des déformations équivariantes des orbites co-adjointes du groupe de Lie SL(2,$\mathbb{R}$). Nous établissons un lien entre des méthodes de quantification basées sur les crochets de Rankin-Cohen et les matrices de fusion pour les modules de Verma. Par ailleurs nous formalisons et étudions la notion associée d'algèbre de Rankin-Cohen qui contrôle l'associativité de ces déformations.

Mots clés

Crochets de Rankin-Cohen, Quantification, Star-produits, Dualité de Shapovalov, Matrices de Fusion, Théorie d'algèbres et groupes de Lie.

Abstract

This work is devoted to the study of equivariant star-product on coadjoint orbits of the Lie group SL(2,$\mathbb{R}$). We establish a correspondence between two quantization methods. The first is based on the Rankin-Cohen brackets and the second is based in the canonical element associate to the Shapovalov form and fusion matrices for Verma modules. Furthermore we formalize and study the associated notion of non-commutative algebra that controls the associativity of these deformations.

Keywords

Rankin-Cohen brackets, Quantization, Star-products, Shapovalov form, Fusion matrix, Lie algebras and Lie groups Theory.

...A mi hermosa Familia

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Introduction

Par quantification on entend le formalisme mathématique sous-jacent à la mécanique quantique.

La formulation hamiltonienne associe à un système mécanique une variété symplectique M vue comme l'espace de phase de ce dernier. Les observables du système sont les fonctions régulières $f \in C^\infty(M)$ sur la variété M. Si f est une observable, son évolution dans le temps est déterminée par l'équation de Hamilton :

$$\frac{d}{dt} f(t) = -\{H, f\},$$

où H est une fonction régulière de M appelée hamiltonien du système et $\{\ ,\ \}$ est le crochet de Poisson associé à la structure symplectique de M.

La mécanique quantique, en sa formulation due à Heisenberg, associe à un système quantique un espace de Hilbert $\mathscr{H}$ vu comme l'espace de phase de ce dernier. Les observables du système sont des opérateurs auto-adjoints dans $\mathscr{H}$. La dynamique du système est définie en termes du hamiltonien $\widetilde{H}$ du système. Si $\tilde{f}$ est une observable dans $\mathscr{H}$, l'évolution de $\tilde{f}$ dans le temps est donnée par l'équation de Hamilton :

$$\frac{d}{dt} \tilde{f}(t) = \frac{\mathbf{i}}{\hbar} [\widetilde{H}, \tilde{f}],$$

où $\hbar$ dénote la constante de Planck et $[\ ,\ ]$ le crochet d'opérateurs.

De point de vue mathématique [Dir47, Wey50, vN55], la quantification est une correspondance $\mathcal{Q} : f \to \tilde{f}$ entre les observables de chaque système telle que

$$\mathcal{Q}(1) = \mathrm{id} \qquad \text{et} \qquad [\mathcal{Q}(f), \mathcal{Q}(g)] = \mathbf{i}\,\hbar\,\mathcal{Q}(\{f, g\}).$$

Sous certaines conditions d'irréductibilité il est bien connu qu'une telle application $\mathcal{Q}$ n'existe pas. Néanmoins, il existe plusieurs méthodes pour faire face à ce problème. D'abord, il est clair que les systèmes mécaniques classiques sont décrits par des algèbres commutatives (typiquement algèbres de fonctions régulières sur M). Par contre, les systèmes mécaniques quantiques sont décrits par des algèbres non commutatives (algèbres d'opérateurs dans un espace de Hilbert $\mathscr{H}$). Dans notre travail, nous étudierons suivant [BFF⁺78, Dri86, Fed94, UU96, Gut11] principalement quatre méthodes qui nous permettent d'établir un passage entre une algèbre commutative et une série d'algèbres non commutatives, à savoir,

a) Drinfeld twist.

b) Calcul symbolique équivariant.

c) Quantification par déformation.

d) Quantification géométrique.

L'étude des quantifications qui respectent des symétries (quantifications équivariantes) fait appel à la théorie des représentations des groupes de Lie. Le but principal de ce travail est d'étudier et de comparer différentes constructions de quantifications équivariantes sur les orbites co-adjointes du groupe de Lie $G = \mathrm{SL}(2, \mathbb{R})$. D'après les résultats de Gutt et Fedosov entre autres [Fed94, BBG98], il est bien connu qu'il n'existe qu'une seule classe d'équivalence de déformations (équivariantes ou pas) sur les orbites co-adjointes de ce groupe. Toutefois, il existe plusieurs méthodes principalement différentes de construction de telles déformations équivariantes. Nous nous intéressons et comparons deux d'entre elles. La première quantification de l'orbite co-adjointe $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1)$ est basée sur la théorie classique et quantique des équations dynamiques de Yang-Baxter [GN84, EV99, ES01] et plus précisément elle est déterminée par l'élément canonique F_λ associé à la dualité de Shapovalov $\langle\ ,\ \rangle_{\mathrm{Sh}}$ entre les modules de Verma correspondants. Cette déformation utilise les idées de Drinfeld sur les déformations formelles sur les groupes de Lie-Poisson, exprimées en termes de l'équation de Yang-Baxter. En 2005 Alekseev et Lachowska [AL05] ont donné une construction générale d'une telle déformation pour une large classe d'orbites co-adjointes.

La deuxième déformation formelle est déterminée par les crochets de Rankin-Cohen (RC). Les crochets de RC ont été introduits par D. Zagier suivant les travaux de Rankin et Cohen [Ran57, Coh75, Zag94] comme la solution à la question suivante : quels sont les opérateurs différentiels qui préservent l'ensemble des formes modulaires ? En théorie des représentations les crochets de RC peuvent être également vus comment des opérateurs de brisure de symétrie d'un produit tensoriel entre deux représentations irréductibles [DvP07, KP15a, KP15b].

En 1996 P. Cohen, Y. Manin, et D. Zagier [CMZ97] montrent que la série formelle $\sum_n \mathrm{RC}_n \hbar^n$ définit un produit associatif sur l'espace des formes modulaires de poids arbitraire.

Ces dernières années, les opérateurs de Rankin-Cohen ont été largement étudiés. Ceci est principalement dû au fait que ces opérateurs bi-différentiels ont des applications importantes dans plusieurs domaines de recherche. On peut citer la théorie des formes modulaires, formes quasimodulaires, équations différentielles de Ramanujan et Chazy [DR14, CL11, MR09, Zag94], quantification équivariante [BTY07, CMZ97, CM04, DvP07, OS00, OR03, Pev12, Pev08, UU96] et structures d'anneau dans les espaces de représentations [CM04, Zag94, DvP07].

Ces deux constructions sont différentes. Cependant, nous établissons un lien explicite entre ces deux méthodes (Théorème 4.85). Pour cela nous utilisons les matrices de fusion ([EV99, ES01]) et les résultats de A. Unterberger et J. Unterberger [UU96] sur l'interprétation des crochets de RC sur l'espace de Bergman en termes d'un calcul symbolique covariant sur l'espace de Lobachevsky imaginaire.

Par ailleurs, nous mettons en évidence comment les propriétés combinatoires des algèbres de Rankin-Cohen contrôlent l'associativité de tels produits formels (Proposition 3.17).

La structure de cette thèse est la suivante.

Le premier chapitre de cette thèse contient une introduction des objets et notions nécessaires pour la suite. Nous faisons un bref rappel sur les algèbres de Hopf, groupes et algèbres de Lie ainsi que la théorie de représentations de ces derniers.

Dans le deuxième chapitre nous rappelons trois différentes méthodes de quantification. La quantification géométrique comme généralisation du calcul de Weyl. La quantification par déformation comme généralisation du star-produit de Moyal. La théorie de Drinfeld des déformations formelles basée sur les solutions à l'équation de Yang-Baxter.

Comme une généralisation de l'algèbre des formes modulaires de poids arbitraire, nous introduisons et développons dans le troisième chapitre la notion d'algèbre de Rankin-Cohen. Cette notion diffère de celle que Zagier a introduite dans [Zag94]. Nous dirons qu'une algèbre commutative et associative A munie d'une dérivation X est une algèbre de Rankin-Cohen si elle a une décomposition $\mathbb{Z}$-graduée $A = \bigoplus_n A_n$ en des représentations irréductibles du groupe de Lie $\mathrm{SL}(2,\mathbb{R})$ telle que les crochets de Rankin-Cohen définis par :

$$\mathrm{RC}_k(a,b) = \sum_{i+j=k} (-1)^i \binom{\alpha_a + k - 1}{j} \binom{\alpha_b + k - 1}{i} X^i(a)\, X^j(b), \qquad (a,b) \in A_{\alpha_a} \times A_{\alpha_b} \quad (1)$$

forment une déformation formelle et $\mathrm{SL}(2,\mathbb{R})$-invariante

$$\sum_k \mathrm{RC}_k \hbar^k, \tag{2}$$

dans l'anneau $A[[\hbar]]$ des séries formelles à coefficients dans A et en l'indéterminée $\hbar$. Nous introduisons aussi la notion de "twist" de Rankin-Cohen. Le twist t est un objet combinatoire qui nous indique le degré de "déformation" de (2), c'est-à-dire, t est un "twist" de A si le produit défini par $\sum_k t_k \mathrm{RC}_k \hbar^k$, est encore une déformation formelle et $\mathrm{SL}(2,\mathbb{R})$-invariante dans $A[[\hbar]]$. On sait (voir [CMZ97]) que l'application $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ définie par

$$t_k^\kappa(i,j) = \left(-\frac{1}{4}\right)^i \sum_{r \geq 0} \binom{i}{2r} \frac{\binom{-\frac{1}{2}}{r}\binom{\kappa - \frac{3}{2}}{r}\binom{\frac{1}{2} - \kappa}{r}}{\binom{-i-\frac{1}{2}}{r}\binom{-j-\frac{1}{2}}{r}\binom{k+i+j-\frac{3}{2}}{r}}. \tag{3}$$

est un twist pour l'algèbre des formes modulaires de poids arbitraire. Sous certains conditions sur l'algèbre A, nous retrouvons des conditions nécessaires et suffisantes pour qu'une fonction t soit un twist de A (Théorème 3.21). De cette façon nous arrivons au résultat principal du chapitre (Théorème 3.41). Considérons l'espace $\mathscr{H}_n^2(\Pi)$ de Bergman de poids n, *i.e.* l'espace de Hilbert des fonctions holomorphes sur le demi-plan de Poincaré $\Pi \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ qui sont de carré intégrables par rapport à la mesure

$$d\mu_n(z) = y^{n-2} dx dy \qquad (z = x + \mathbf{i}\, y).$$

La formule

$$(\rho_n(\gamma^{-1})f)(z) = (cz+d)^{-n} f\left(\frac{az+b}{cz+d}\right), \qquad (z \in \Pi, f \in \mathscr{H}_n^2(\Pi), \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G)$$

définit une représentation irréductible de G dans $\mathscr{H}_n^2(\Pi)$. Si (e^-, h, e^+) est la base standard de l'algèbre de Lie $\mathfrak{g}$ de G, l'action infinitésimal $(d\rho_n(e^-), d\rho_n(h), d\rho_n(e^+))$ de $\mathfrak{g}$ dans $\mathscr{H}_n^2(\Pi)$ munit l'espace $\mathscr{H}^+ = \bigoplus_{n\in\mathbb{N}} \mathscr{H}_{2n}^2(\Pi)$ d'une dérivation invariante

$$X = d\rho_n(e^+) = -\frac{\partial}{\partial z}.$$

On définit dans $\mathscr{H}^+$ les crochets de RC comme dans l'équation (1). On a

Théorème. *Pour tout $\kappa \in \mathbb{C}$ la fonction $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ définit en (3) est un twist de $\mathscr{H}^+$. En particulière, si $\kappa = \frac{1}{2}$ ou $\kappa = \frac{3}{2}$ nous retrouvons la déformation* $\mathrm{SL}(2,\mathbb{R})$*-covariante*

$$f \star_{\mathrm{RC}} g = \sum_{k\in\mathbb{N}} \mathrm{RC}_k(f,g)\hbar^k \qquad (f, g \in \mathscr{H}^+).$$

Autrement dit l'espace $\mathscr{H}^+$ est une algèbre de Rankin-Cohen.

L'objectif du quatrième chapitre est d'établir un lien entre les déformation $\star_{\mathrm{RC}}$ et celle donnée par Alekseev et Lachowska que l'on note $\star_{\mathrm{S}}$. Considérons $\lambda \in \mathbb{C}$ tel que la dualité de Shapovalov $\langle\ ,\ \rangle_\lambda : U(\mathfrak{n}_-) \times U(\mathfrak{n}_+) \to \mathbb{C}$ soit non singulière. L'objet fondamental pour la construction de $\star_{\mathrm{S}}$ est l'élément canonique F_λ associée à $\langle\ ,\ \rangle_\lambda$. Si (e^-, h, e^+) est une $\mathfrak{sl}_2$-triple de $\mathfrak{g}$, alors F_λ est donné explicitement par

$$F_\lambda = \sum_{k\in\mathbb{N}} \frac{(e^-)^k \otimes (e^+)^k}{\langle (e^-)^k, (e^+)^k\rangle_\lambda} = \sum_{k\in\mathbb{N}} \frac{(-1)^k (e^-)^k \otimes (e^+)^k}{k!\,\lambda(\lambda-1)\cdots(\lambda-(k-1))},$$

qui appartient à $U(\mathfrak{n}_-)\,\widehat{\otimes}\,U(\mathfrak{n}_+)$ une complétion du produit tensoriel $U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+)$. Dans la proposition 4.27 nous exposons résultats précis à propos de F_λ. Nous montrons de façon détaillée que l'élément F_λ satisfait les conditions d'associativité et unicité de Drinfeld, (Proposition 4.47), c'est-à-dire

Théorème. *Si la forme Shapovalov $\langle\ ,\ \rangle_\lambda$ est non singulière, nous avons que*

$$(\mathrm{id} \otimes \pi \otimes \mathrm{id})\,(\Delta \otimes \mathrm{id})F_\lambda(F_\lambda \otimes 1) = (\mathrm{id} \otimes \pi \otimes \mathrm{id})\,(\mathrm{id} \otimes \Delta)F_\lambda(1 \otimes F_\lambda)$$
$$(\varepsilon \otimes \mathrm{id})F_\lambda = 1 = (\mathrm{id} \otimes \varepsilon)F_\lambda$$

dans $U(\mathfrak{n}_-)\,\widehat{\otimes}(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h})\,\widehat{\otimes}\,U(\mathfrak{n}_+)$. Ici, ε dénote la counité dans $U(\mathfrak{g})$ et $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}$ est la projection naturelle de $U(\mathfrak{g})$ vers l'idéal à gauche engendré par $\mathfrak{h}$.

La fonction $\lambda \mapsto F_\lambda$ est méromorphe en $\mathbb{C}$ et holomorphe à l'infini (Proposition 4.27). Par conséquence, si $\hbar$ est dans un voisinage de zéro, alors $F_{\hbar^{-1}}$ admet un développement asymptotique de la forme

$$F_{\hbar^{-1}} = a_0 + a_1\hbar + a_2\hbar^2 + \cdots + a_n\hbar^n + \cdots \qquad (a_i \in U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+))^{\mathfrak{g}}$$

Autrement dit, $F_{\hbar^{-1}}$ est bien un élément dans $U(\mathfrak{n}_-)\otimes U(\mathfrak{n}_+)[[\hbar]]$ l'anneau des séries formelles à coefficients dans $U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+)$ en l'indéterminée $\hbar$. Alors

Théorème ([AL05]). *L'élément $(\pi \otimes \pi)(F_{\hbar^{-1}})$ est inversible dans*

$$(U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h} \,\widehat{\otimes}\, U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h}))^{\mathfrak{h}}[[\hbar]]$$

et satisfait les conditions d'associativité et unicité de la proposition 4.47. De plus, l'opérateur bi-différentiel $\star_{\mathrm{S}}$ associé à cet élément est un star-produit G-invariant sur l'orbite co-adjointe $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$.

Dans la Proposition 4.49, nous décrivons explicitement les éléments a_n du développement asymptotique de $F_{\hbar^{-1}}$ et précisons le résultats d'Alekseev et Lachowska.

Par la suite nous donnons une "réalisation" du star-produit $\star_{\mathrm{S}}$. Pour cela, nous étudions le calcul symbolique d'Unterberger [UU96] sur l'espace de Lobachevsky imaginaire $\Pi_L \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$. Considérons l'espace de Hilbert $L^2(\Pi_L, d\mu_L)$ des fonctions de carré intégrable sur Π_L par rapport à la mesure invariante

$$d\mu_L(s,t) = (s-t)^{-2} ds dt.$$

L'action quasi-régulière

$$(\gamma^{-1}\cdot f)(s,t) = f\left(\frac{as+b}{cs+d}, \frac{at+b}{ct+d}\right), \qquad \left((s,t) \in \Pi_L, f \in L^2(\Pi_L), \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G\right)$$

détermine une représentation de G sur $L^2(\Pi_L, d\mu_L)$. La décomposition de $L^2(\Pi_L)$ [Str73, Far79] en espaces invariants sous l'action quasi-régulière de G est donnée par la décomposition spectral de l'opérateur de Laplace-Beltrami

$$\Box = (s-t)^2 \frac{\partial^2}{\partial s \partial t}.$$

Le spectre de $\Box$ consiste en une partie continue ainsi qu'une partie discrète. La partie discrète est donnée par $E_{-n(n+1)}(\Box)$ les espaces propres de $\Box$ avec valeur propre $-n(n+1)$, $n \in \mathbb{N}$.

Si ζ appartient à Π le demi-plan de Poincaré, alors la partie "anti-holomorphe" E_n^+ est un G-module irréductible. Il est donné par

$$E_n^+ = \overline{\mathrm{Span}_{\mathbb{C}}(\gamma \cdot X^n_{\zeta,(1,0)} | \, \gamma \in G)},$$

où $X_{\zeta,(i,k)} : \Pi_L \to \mathbb{C}$ est défini par

$$X^{\pm}_{\zeta,(i,k)}(s,t) = \frac{1}{(s-\bar{\zeta})^k(t-\bar{\zeta})^{i-k}} \pm \frac{1}{(t-\bar{\zeta})^k(s-\bar{\zeta})^{i-k}} \qquad (i \in \mathbb{N}, 0 \leq k \leq i, \ (s,t) \in \Pi_L).$$

Nous utilisons les fonctions $X_{\zeta,(i,k)} : \Pi_L \to \mathbb{C}$ comme un outil pour exprimer la base totale $\{v_{n,k} | k \in \mathbb{N}\}$ de E_n^+ (Proposition 4.66) ainsi que le produit dans $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} E_n^+$ des fonctions $v_{n,i}v_{m,j}$ (Lemme 4.71 et Proposition 4.72.)

Le résultat principal de cette thèse (Théorème 4.85) exprime les crochets de Rankin-Cohen en termes des matrices de fusion (Proposition 4.82). Alors la condition d'associativité de F_λ codifie l'associativité du produit $\sum_k \mathrm{RC}_k \hbar^k$. Nous avons,

Théorème. *Soient M_λ^- et M_μ^- deux modules de Verma de plus bas poids λ et de plus bas poids μ. Si $-\mu \notin \mathbb{N}$ et $-\mu - \lambda \notin \mathbb{N}$, alors*

$$J^+_{M_\lambda^-, M_\mu^-}(0)(u \otimes v_\mu^-) = F_{-\mu}(u \otimes v_\mu^-) = \sum_{n \in \mathbb{N}} RC_n(v_\lambda^- \otimes v_\mu^-)\hbar^n,$$

où u est donné par $u = \sum_{n \in \mathbb{N}} u_n(\lambda, \mu)\hbar^n = \sum_{n \in \mathbb{N}} (-1)^n \binom{\mu+n-1}{n} (e^+)^n v_\lambda^- \hbar^n \in \widehat{M}_\lambda^-$.

Notations and conventions

If A, B and C are expressions or symbols, for convenience we use the following convention

$$A^{\pm} = B \pm C^{\mp}$$

to describe the following relations:

$$a) \quad A^{+} = B + C^{-} \qquad\qquad b) \quad A^{-} = B - C^{+}$$

Unless otherwise indicated, along this dissertation we use the following notations:

- $\mathbb{N}$ denote the set of non-negatives integers $\{0, 1, 2, \ldots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.
- If X is a set and $0 \in X$, we denote $X^{\times} = X \smallsetminus \{0\}$. For instance, $\mathbb{N}^{\times}$ is the set of positive integers $\{1, 2, \ldots\}$.
- $\mathbf{i} = \sqrt{-1}$.
- $\Bbbk$ denote a commutative field and $\mathbb{K}$ denote a commutative ring.
- G denote a group (a topological group in the most cases). If X is a set and G acts on X (*i.e.* X is a G-module) we denote, $g \cdot x$ the action of $g \in G$ on $x \in X$.
- $\Lambda^{(i,j)}(V; W)$ denote the vector space of skew-symmetric i-multilinear mappings on V with values in $W^j = W \times W \times \cdots \times W$ (j times).
- $\binom{z}{k}$ denotes $\frac{z(z-1)\cdots(z-k+1)}{k!}$ if $z \in \mathbb{C}$ and $k \in \mathbb{N}$.

Also if there is no confusion, we say vector space, algebra, morphism, ... instead $\Bbbk$-vector space, $\Bbbk$-algebra, $\Bbbk$-morphism, ... and we denote $\mathrm{Hom}(V, W)$, $V \otimes W$, ... instead $\mathrm{Hom}_{\Bbbk}(V, W)$, $V \otimes_{\Bbbk} W$...

Chapter 1

Preliminaries

Dans ce chapitre nous faisons un bref rappel des objets et notions nécessaires pour la suite.

1.1 *Algèbres de Hopf. Une algèbre de Hopf est une bigèbre (Définition 1.8) qui possède un antipode (endomorphisme qui généralise la notion de passage à l'inverse dans un groupe). Depuis de l'introduction des groupes quantiques par Drinfeld et Jimbo dans les années 80, les algèbres de Hopf ont été intensivement étudiées par mathématiciens et physiciens avec divers intérêts et horizons. Leur structure, qui peut être considérée comme une généralisation de la structure du groupe, est sous-jacente à des différentes domaines mathématiques. Les algèbres de Hopf quasi-triangulaires apparaissent comme un instrument adapté pour construire des solutions à l'équation quantique de Yang-Baxter et servent à la renormalisation en théorie quantique de champs ; en topologie, elles sont liées à la construction des invariants de nœuds et 3-variétés, etc. La littérature sur des différents aspects de la théorie des groupes quantiques est abondante.*

1.2 *Groupes de Lie. Un groupe de Lie est un groupe doté d'une structure de variété différentielle de dimension finie tel que les opérations du groupe soient différentiables. C'est un outil indispensable à de nombreux domaines de recherche en mathématiques, ainsi qu'en physique théorique. Nous nous intéressons principalement au groupe de Lie simple* $\mathrm{SL}(2,\mathbb{R})$*, qui nous sert de modèle.*

1.3 *Algèbres de Lie. Une algèbre de Lie est un espace vectoriel muni d'une application bilinéaire antisymétrique que satisfait l'identité de Jacobi (Eq. 1.5). Les catégories des groupes et algèbres de Lie sont étroitement liées. Tout groupe de Lie détermine une unique algèbre de Lie. La réciproque est vraie pour un certain type des groupes de Lie. Les représentations linéaires d'une algèbre de Lie nous aide à connaître la structure* *de l'algèbre. Finalement, nous introduisons la notion de bigèbre de Lie, fondamentale pour trouver des déformations de l'algèbre enveloppente d'une algèbre de Lie.*

In this chapter we introduce and recall some notions and results that we use throughout this dissertation.

1.1 Hopf algebras

In this section we discuss the notion of a Hopf algebra. Our main reference is [DNR01].

1.1.1 Algebras and coalgebras

1.1 Definition. A $\mathbb{K}$-algebra is a pair (A, m), where A is a $\mathbb{K}$-module and $m : A \otimes A \to A$ is a $\mathbb{K}$-morphism, called the multiplication map. We say that the algebra (A, m) is a unital associative algebra if there exists a $\mathbb{K}$-morphism $u : \mathbb{K} \to A$, called the unity map, such that the following diagrams commute:

a) Associativity:

$$\begin{tikzcd} & A \otimes A \otimes A \arrow[dl, "m\otimes id_A"'] \arrow[dr, "id_A\otimes m"] & \\ A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

b) Unity:

$$\begin{tikzcd} & A \otimes A \arrow[dd, "m"] & \\ \mathbb{K} \otimes A \arrow[ur, "u\otimes id_A"] \arrow[dr, leftrightarrow] & & A \otimes \mathbb{K} \arrow[ul, "id_A\otimes u"'] \arrow[dl, leftrightarrow] \\ & A & \end{tikzcd}$$

We say that the algebra (A, m) is commutative if $m \circ \tau = m$. Here τ denotes the flip map:

$$\tau(a \otimes b) = b \otimes a. \tag{1.1}$$

By inverting the morphism in the above diagrams we get the dual notion:

1.2 Definition. A $\mathbb{K}$-coalgebra is a pair (C, Δ), where C is a $\mathbb{K}$-module and $\Delta : C \to C \otimes C$ is a $\mathbb{K}$-morphism, called the comultiplication map. We say that the coalgebra (C, Δ) is a counital coassociative coalgebra if there exists a $\mathbb{K}$-morphism $\varepsilon : C \to \mathbb{K}$, the counit map, such that the following diagrams commute:

a) Coassociativity:

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[dr, "\Delta\otimes id_C"'] & & C \otimes C \arrow[dl, "id_C\otimes\Delta"] \\ & C \otimes C \otimes C & \end{tikzcd}$$

b) Counity:

$$\begin{tikzcd} & C \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, "\Delta"'] & \\ C \otimes \mathbb{K} & & \mathbb{K} \otimes C \\ & C \otimes C \arrow[ul, "id_C\otimes\varepsilon"] \arrow[ur, "\varepsilon\otimes id_C"'] & \end{tikzcd}$$

The coalgebra (C, Δ) is cocommutative if $\tau \circ \Delta = \Delta$.

1.3 Notation (Sweedler's Notation). Let (C, Δ, ε) be a coalgebra. For any $c \in C$, the element $\Delta(c) = \sum_i c_{1i} \otimes c_{2i}$ in $C \otimes C$ is denoted by:

$$\Delta(c) = c_{(1)} \otimes c_{(2)}. \tag{1.2}$$

In particular, the coassocitivity is given by

$$(c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}.$$

Then according the Sweedler's notation we write $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ instead of $(c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)}$ or $c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}$

1.4 Definition. *i*) Let (A, m_A, u_A) and (B, m_B, u_B) be two unital associative $\mathbb{K}$-algebras. A morphism of $\mathbb{K}$-modules $f : A \to B$ is a morphism of algebras if the following diagrams commute:

a)

$$\begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m_A"'] & B \otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

b)

$$\begin{tikzcd} & \mathbb{K} \arrow[dl, "u_A"'] \arrow[dr, "u_B"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

ii) Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be two counital coassociative $\mathbb{K}$-coalgebras. A morphism of $\mathbb{K}$-modules $g : C \to D$ is a morphism of coalgebras if the following diagrams commute:

a)

$$\begin{tikzcd} C \otimes C \arrow[r, "g\otimes g"] & D \otimes D \\ C \arrow[u, "\Delta_C"] \arrow[r, "g"'] & D \arrow[u, "\Delta_D"'] \end{tikzcd}$$

b)

$$\begin{tikzcd} & C \arrow[dl, "g"'] \arrow[dr, "\varepsilon_C"] & \\ D \arrow[rr, "\varepsilon_D"'] & & \mathbb{K} \end{tikzcd}$$

1.5 Definition. *i*) Let (A, m, u) a unital associative $\mathbb{K}$-algebra. A left A-module is a pair (M, ρ), where M is a $\mathbb{K}$-module and $\rho : A \otimes M \to M$ is a morphism of $\mathbb{K}$-modules such that the following diagrams commute:

a)

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{id}_A \otimes \rho"] \arrow[d, "m \otimes \mathrm{id}_M"'] & A \otimes M \arrow[d, "\rho"] \\ A \otimes M \arrow[r, "\rho"'] & M \end{tikzcd}$$

b)

$$\begin{tikzcd} A \otimes M \arrow[r, "\rho"] & M \arrow[dl, leftrightarrow] \\ \mathbb{K} \otimes M \arrow[u, "u \otimes \mathrm{id}_M"] & \end{tikzcd}$$

The map ρ is called the action map of the A-module M.

ii) Let (C, Δ, ε) be a counital coassociative $\mathbb{K}$-coalgebra. A right C-comodule is a pair (M, ρ), where M is a $\mathbb{K}$-module and $\rho : M \to M \otimes C$ is a morphism of $\mathbb{K}$-modules such that the following diagrams commute:

a)

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "\rho \otimes \mathrm{id}"] \\ M \otimes C \arrow[r, "\mathrm{id} \otimes \Delta"'] & M \otimes C \otimes C \end{tikzcd}$$

b)

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, leftrightarrow] & M \otimes C \arrow[d, "\mathrm{id} \otimes \varepsilon"] \\ & M \otimes \mathbb{K} \end{tikzcd}$$

The map ρ is called the coaction map of the C-comodule M.

1.1.2 Representations of Hopf algebras

1.6 Proposition. *Let (A, m, u) be a unital associative algebra and (A, Δ, ε) be a counital coassociative coalgebra. Then the following assertions are equivalent:*

i) The linear maps m and u are morphisms of coalgebras.

ii) The linear maps Δ and ε are morphisms of algebras.

Proof. See [DNR01, Proposition 4.1.1]. □

1.7 Remark. In Sweedler's notations, the condition (ii) is given by

$$(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)},$$
$$\varepsilon(xy) = \varepsilon(x)\,\varepsilon(y),$$
$$\Delta(1) = 1 \otimes 1,$$

for any $x, y \in A$.

1.8 Definition. A $\mathbb{K}$-bialgebra is a quintuple $(A, m, u, \Delta, \varepsilon)$ where (A, m, u) is an algebra, (A, Δ, ε) is a coalgebra such that m and u are morphisms of coalgebras (and then by Proposition 1.6 it follows that Δ and ε are the morphisms of algebras).

1.9 Definition. Let $(A, m, u, \Delta, \varepsilon)$, $(A', m', u', \Delta', \varepsilon')$ be two bialgebras. A linear map $f : A \to A'$ is a morphism of bialgebras if $f : (A, m, u) \to (A', m', u')$ is a morphism of algebras and $f : (A, \Delta, \varepsilon) \to (A', \Delta', \varepsilon')$ is a morphism of coalgebras.

Let (C, Δ, ε) be a $\mathbb{K}$-coalgebra and (A, m, u) be a $\mathbb{K}$-algebra. The ring $\mathrm{Hom}_{\mathbb{K}}(C, A)$ has a $\mathbb{K}$-algebra structure with the convolution product given by

$$f * g = m(f \otimes g)\Delta \qquad (f, g \in \mathrm{Hom}(C, A)).$$

Indeed, let $f, g, h \in \mathrm{Hom}(C, A)$ and $c \in C$ the following equations

$$\begin{aligned}
&((f * g) * h)(c)\\
&= (f * g)(c_{(1)})h(c_{(2)})\\
&= f((c_{(1)})_{(1)})g((c_{(1)})_{(2)})h(c_{(2)})\\
&= f(c_{(1)})g((c_{(2)})_{(1)})h((c_{(2)})_{(2)})\\
&= f(c_{(1)})(g * h)(c_{(2)})\\
&= (f * (g * h))(c),
\end{aligned}$$

show that the convolution product is associative.

1.10 Remark. The element $u\,\varepsilon \in \mathrm{Hom}_{\mathbb{K}}(C, A)$ is the identity element of the convolution product.

1.11 Definition (Hopf Algebra)**.** Let A a $\mathbb{K}$-bialgebra. We say that A is a Hopf algebra over $\mathbb{K}$ if there exists a morphism of $\mathbb{K}$-modules $S : A \to A$, called the antipode, which is the inverse element of the convolution product. In this case the following diagram commutes:

Antipode:

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "S\otimes id"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "m"'] & & \mathbb{K} \arrow[rr, "u"] & & A \\
 & A \otimes A \arrow[rr, "id\otimes S"'] & & A \otimes A \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

In Sweedler's notation the antipode condition is expressed by:

$$S(a_{(1)})a_{(2)} = \varepsilon(a)1_A = a_{(1)}S(a_{(2)}) \qquad (a \in A).$$

1.12 Lemma. *Let (A, S), (A', S') be two Hopf algebras. If $f : A \to A'$ is a morphism of bialgebras, then f preserves the antipode condition, i.e. for any $a \in A$ we have that*

$$f(S(a)) = S'(f(a)).$$

Proof. See [DNR01, Proposition 4.2.5]. $\square$

1.13 Remark. We give some examples of Hopf algebras. Here $\Bbbk$ denote a commutative field.

1. Let G a group with identity element e. The group algebra over $\Bbbk$ is the vector space $\Bbbk G$ of all linear combinations of finitely many elements of G with coefficients in $\Bbbk$, hence of all elements of the form

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_n g_n,$$

where $\alpha_i \in \Bbbk$ and $g_i \in G$ for all $i = 1, ..., n$. This element can be denoted in general by

$$\sum_{g \in G} \alpha_g u_g$$

where it is assumed that $\alpha_g = 0$ for all but finitely many elements of g. $\Bbbk G$ is a Hopf algebra with the following operations:

$$m(u_g \otimes u_h) = u_{gh} \qquad u(1) = u_e$$

$$\Delta(u_g) = u_g \otimes u_g \qquad \varepsilon(u_g) = 1$$

$$S(u_g) = u_{g^{-1}}$$

for any $g, h \in G$.

2. Let G a finite group with identity elemet e. We consider the unital associative commutative algebra $\mathcal{F}(G, \Bbbk)$ of functions from G to $\Bbbk$. The map

$$\mathcal{F}(G, \Bbbk) \otimes \mathcal{F}(G, \Bbbk) \to \mathcal{F}(G \times G, \Bbbk)$$
$$f \otimes h \mapsto f \otimes h = [(x, y) \mapsto f(x)h(y)],$$

is an isomorphism of algebras. Then $\mathcal{F}(G, \Bbbk)$ is a Hopf algebra, with

$$\Delta(f)(x, y) = f(xy) \qquad \varepsilon(f) = f(e)$$

$$S(f)(x) = f(x^{-1})$$

for any $x, y \in G$, $f \in \mathcal{F}(G, \Bbbk)$.

3. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. The universal enveloping algebra[1] $U(\mathfrak{g})$ is a cocommutative Hopf algebra

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \varepsilon(x) = 0$$

$$S(x) = -x$$

for any $x \in \mathfrak{g}$.

Finally we introduce the notion of action of a Hopf algebra over a unital commutative algebra.

1.14 Definition. Let A be a $\mathbb{K}$-algebra and H be a $\mathbb{K}$-bialgebra. We say that H acts on A (or that A is a left H-module algebra) if the following conditions hold:

i) A is a left H-module, with action

$$H \otimes A \to A, \quad h \otimes a \mapsto h \cdot a.$$

ii) The multiplication map $m_A : A \otimes A \to A$, $m_A(a \otimes b) = ab$, and the unity map $u_A : \mathbb{K} \to A$ of A are the morphisms of H-modules. Here, $A \otimes A$ is the left H-module with action $h \cdot (a \otimes b) = (h_{(1)} \cdot a) \otimes (h_{(2)} \cdot b)$.

[1](Definition 1.16)

1.15 Remark. In the Sweedler's notation, the condition (ii) is expressed by

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b),$$
$$h \cdot (1_A) = \varepsilon(h) 1_A,$$

for any $h \in H$ and $a, b \in A$.

1.2 Lie groups

In this section we briefly introduce some notions and results from Lie theory. We use the books [Hel78, Kna86, HT92] as references.

In this section $\mathbb{k}$ denote $\mathbb{R}$ the real number field or $\mathbb{C}$ the complex number field.

Let G be a Lie group that is a group that is also a finite-dimensional smooth manifold with the property that the group operations are compatible with the smooth structure, *i.e.* the following map

$$G \times G \to G \quad \text{given by } (x, y) \mapsto xy^{-1}$$

is a smooth map of the product manifold $G \times G$ into G. We say that a smooth map $\psi : G \to H$ between two Lie groups is a Lie group morphism if ψ is also a group homomorphism.

1.2.1 Matrix groups

The group $\mathrm{GL}_n(\mathbb{k})$ of $n \times n$ invertible matrices together with the operation of ordinary matrix multiplication is a Lie group called the general linear group of degree n. A closed subgroup of the general linear group $\mathrm{GL}_n(\mathbb{k})$ is call a matrix group and this is one of the most important source of examples of Lie groups.

We say that G is a linear connected reductive group if it is a closed connected subgroup of $\mathrm{GL}_n(\mathbb{k})$ which is stable under conjugate transpose. Then the inverse conjugate transpose Θ is an automorphism of G called the Cartan involution ($\Theta^2 = 1$). We consider the subgroup of G

$$K = \{g \in G | \Theta(g) = g\}.$$

It is well-known that K is a maximal compact subgroup of G (see [Kna86, Section 1]). A linear connected reductive group G is semisimple if in addition the center of G

$$\mathcal{Z}_G = \{g \in G | hg = gh \quad \forall h \in G\}$$

is finite. We say that a matrix group G is a linear simple Lie group if it is connected non-abelian and G does not have nontrivial connected normal subgroups.

Throughout this dissertation we extensively use the particular exemple of such a group, namely

$$\mathrm{SL}(2, \Bbbk) = \{g \in \mathrm{GL}_2(\Bbbk) | \det g = 1\}.$$

If $\Bbbk = \mathbb{C}$ we have that $K = \{g \in G | \Theta(g) = g\}$ is isomorphic to

$$SU(2, \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\},$$

the special unitary group. Now, if $\Bbbk = \mathbb{R}$ the maximal compact subgroup is isomorphic to the group of rotations

$$SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}.$$

1.2.2 Representations of Lie groups

A representation of G (or G-module) is a pair (ρ, V) where V is a topological vector space over $\mathbb{C}$ and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphisms such that for every $v \in V$ the map $g \mapsto \rho(g)(v) = g \cdot v$ is continuous. Here $\mathrm{GL}(V)$ denotes the group of invertible continuous operators on V with continuous inverse.

If V is a Hilbert space, a very important class of representations are the unitary ones, that is representations (ρ, V) such that for every $g \in G$ the map $\rho(g) : v \mapsto g \cdot v$ is an unitary operator on V: $\rho(g) \circ \rho(g)^* = \rho(g)^* \circ \rho(g) = \mathrm{id}$, where $\rho(g)^*$ is the corresponding adjoint operator. Unitary one-dimensional representations $G \to \mathbb{C}$ are called unitary characters.

A continuous linear map $T : V_1 \to V_2$ between two G-modules (ρ_1, V_1), (ρ_2, V_2) is an intertwining operator (or G-equivariant operator) if for any $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{T} & V_2 \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ V_1 & \xrightarrow[T]{} & V_2 \end{array}$$

We say that two representations (ρ_1, V_1), (ρ_2, V_2) of G are G-equivalents if there is a non-zero intertwining operator $T : V_1 \to V_2$ which is a topological isomorphism.

A submodule of (ρ, V) is a closed subspace U of V such that for every $g \in G$ $\rho(g)(U) \subset U$. In particular V and $\{0\}$ are the submodules of V. A proper submodule of V is a submodule distinct from V and $\{0\}$. One says that a representation (ρ, V) is simple (or irreducible) if it has no proper submodules.

We say that a representation (ρ, V) has a direct sum decomposition if there exists a sequence of G-submodules V_i of V such that

$$V = \bigoplus_i V_i. \tag{1.3}$$

If the number of summands is infinite, the sum is understood in a topological sense: the algebraic sum should be dense in V. We say that (ρ, V) is completely reducible or semisimple, if V has a direct sum decomposition and the submodules V_i in decomposition (1.3) are irreducible.

Let (ρ, V) be a representation of G. We consider the topological dual space to V, $V^* = \{f : V \to \mathbb{C} \,|\, f \text{ is a continuous linear map}\}$, and the action $\rho^* : G \to \mathrm{GL}(V^*)$ is given by

$$\rho^*(g)(v^*) : v \mapsto v^*(\rho(g^{-1}(v))).$$

The pair (ρ^*, V^*) is a representation of G called the contragredient representation to (ρ, V).

Let (ρ_1, V_1) and (ρ_2, V_2) be two finite dimensional representations of G. The tensor product of V_1 and V_2 is also a representation of G: the action $\rho_1 \otimes \rho_2$ of G on $V_1 \otimes V_2$ is given by

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2), \tag{1.4}$$

for any $g \in G$ and any $(v_1, v_2) \in V_1 \times V_2$. Moreover, if V_1 and V_2 are Hilbert spaces, $V_1 \otimes V_2$ has an structure of pre-Hilbert space:

$$\langle v_1 \otimes v_2, \tilde{v}_1 \otimes \tilde{v}_2 \rangle = \langle v_1, \tilde{v}_1 \rangle_1 \langle v_2, \tilde{v}_2 \rangle_2. \qquad (v_1, v_2), (\tilde{v}_1, \tilde{v}_2) \in V_1 \times V_2.$$

The topological completion $V_1 \check{\otimes} V_2$ of $V_1 \otimes V_2$ is a representation of G with action given by the formula (1.4). In both cases: V_1, V_2 are finite dimensional or V_1, V_2 are Hilbert spaces, we say that $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$ or $(\rho_1 \otimes \rho_2, V_1 \check{\otimes} V_2)$ is the tensor product representation of (ρ_1, V_1) and (ρ_2, V_2).

1.3 Lie algebras

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra over $\Bbbk$, that is a $\Bbbk$-vector space $\mathfrak{g}$ together with an antisymmetric bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfied the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{for any } x, y, z \in \mathfrak{g}\,. \tag{1.5}$$

The bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the Lie bracket of $\mathfrak{g}$. A linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is a morphism of Lie algebras if ψ preserves the Lie brackets, *i.e*

$$\psi([x, y]) = [\psi(x), \psi(y)] \qquad \text{for any } x, y \in \mathfrak{g}\,.$$

Any associative algebra has also a Lie algebra structure with the so-called commutator Lie bracket

$$[a, b] := ab - ba \qquad (a, b \in A). \tag{1.6}$$

In particular, the algebra $\mathfrak{gl}(V) = \{f : V \to V | f \text{ is a linear map}\}$ of endomorphisms of a vector space V is a Lie algebra with the commutator Lie bracket

$$[X, Y] = X \circ Y - Y \circ X \qquad (X, Y \in \mathfrak{g}).$$

We say that a pair (ρ, V) is a representation of $\mathfrak{g}$ (or a $\mathfrak{g}$-module) if V is a vector space over $\Bbbk$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a morphism of Lie algebras.

1.16 Definition. The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative $\Bbbk$-algebra $T(\mathfrak{g})/I$ where $T(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n}$ is the tensor algebra of $\mathfrak{g}$ and I is the left ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$, with $X, Y \in \mathfrak{g}$.

A linear map $T : V_1 \to V_2$ between two $\mathfrak{g}$-modules (ρ_1, V_1), (ρ_2, V_2) is a morphism of $\mathfrak{g}$-modules if for any $X \in \mathfrak{g}$ the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\rho_1(X)"'] & V_2 \arrow[d, "\rho_2(X)"] \\ V_1 \arrow[r, "T"'] & V_2 \end{tikzcd}$$

We say that two representations (ρ_1, V_1), (ρ_2, V_2) of $\mathfrak{g}$ are $\mathfrak{g}$-equivalents if there is a morphism of $\mathfrak{g}$-modules $T : V_1 \to V_2$ which is a vector space isomorphism.

1.17 Remark. By [Dix96, Proposition 2.1.9 page 69] the canonical map $\mathfrak{g} \to U(\mathfrak{g})$ is injective, in other words, we can see the Lie algebra $\mathfrak{g}$ included its universal enveloping algebra $U(\mathfrak{g})$. Consequently and according to [Dix96, Corollary 2.2.2 page 70], representations of $\mathfrak{g}$ can be extend to those of $U(\mathfrak{g})$ and are actually in 1-1 correspondence with representations of $U(\mathfrak{g})$.

A submodule of (ρ, V) is a subspace U of V such that $\rho(X)(U) \subset U$ for every $X \in \mathfrak{g}$. In particular V and $\{0\}$ are the submodules of V. A proper submodule of V is a submodule distinct from V and $\{0\}$. One says that a representation (ρ, V) is simple (or irreducible) if it has no proper submodules.

We say that (ρ, V) is completely reducible or semisimple, if V has a direct sum decomposition of simple submodules, *i.e.* there exists a sequence V_i of simple $\mathfrak{g}$-submodules of V such that

$$V = \bigoplus_i V_i.$$

We say that (ρ, V) is decomposable if we can write V as an algebraic direct sum of two nontrivial $\mathfrak{g}$-invariant subspaces. Otherwise it is indecomposable.

The contragredient representation (ρ^*, V^*) to (ρ, V) is the representation $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ given by

$$\rho^*(X)(v^*) : v \mapsto v^*(\rho(-X)(v)).$$

Here V^* denote the algebraic dual space of V.

Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of $\mathfrak{g}$. The tensor product representation of (ρ_1, V_1) and (ρ_2, V_2) is the representation $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$ with action $\rho_1 \otimes \rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$

given by

$$(\rho_1 \otimes \rho_2)(X)(v_1 \otimes v_2) = \rho_1(X)(v_1) \otimes v_2 + v_1 \otimes \rho_2(X)(v_2), \tag{1.7}$$

for any $X \in \mathfrak{g}$ and any $(v_1, v_2) \in V_1 \times V_2$.

Representations of a Lie group G give rise to representations of its Lie algebra $\mathfrak{g}$. Let (ρ, V) a representation of G, we say that $v \in V$ is a smooth vector for G if the map $g \mapsto \rho(g)(v)$ is a smooth function from G to V. The set V^∞ of smooth vectors for G is a dense subspace of V. The derived representation of ρ is the representation $(d\rho, V^\infty)$ of $\mathfrak{g}$ given by

$$\begin{aligned} d\rho(X)(v) &= \left.\frac{d}{dt}\right|_{t=0} \rho(\exp(tX))v, \\ &= \lim_{t\to 0} \frac{\rho(\exp(tX))(v) - v}{t}, \end{aligned} \tag{1.8}$$

for $X \in \mathfrak{g}$ and $v \in V^\infty$. Here $\exp : \mathfrak{g} \to G$ denote the exponential map which allows us to recapture the local group structure on G from the Lie algebra $\mathfrak{g}$. In particular, if G is a matrix group the exponential map coincides with the matrix exponential of G and is given by the ordinary series expansion:

$$\exp(X) = \mathrm{e}^X = \sum_{n\in\mathbb{N}} \frac{X^n}{n!} \tag{1.9}$$

for any $X \in \mathfrak{g} \subseteq M_n(\Bbbk)$.

1.18 Remark. The exponential map is a diffeomorphism from some neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G$ with $\exp(0) = e$. Besides, if (V, ρ) is representation of G the formula (1.8) can be expressed by the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}(V) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \mathrm{e}} \\ \mathfrak{g} & \xrightarrow[d\rho]{} & \mathfrak{gl}(V) \end{array}$$

We empathized that a representation of $\mathfrak{g}$ does not give necessarily rise to a representation of G.

1.3.1 Polarized Lie algebras and Verma Modules

In this subsection we assume that $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, that is $\mathfrak{g}$ has an algebraic sum decomposition of vector spaces

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i \tag{1.10}$$

such that for any $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. We also assume that for any $i \in \mathbb{Z}^\times$ the component $\mathfrak{g}_i$ is finite dimensional. We denote $\mathfrak{n}_+ = \bigoplus_{i>0} \mathfrak{g}_i$, $\mathfrak{n}_- = \bigoplus_{i<0} \mathfrak{g}_i$ and $\mathfrak{p}_+ = \bigoplus_{i\geq 0} \mathfrak{g}_i$, $\mathfrak{p}_- = \bigoplus_{i\leq 0} \mathfrak{g}_i$.

1.19 Definition. A $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ is a polarized Lie algebra if on $\mathfrak{g}_0 = \mathfrak{h}$ there is a nonsingular character *i.e.* a character $\chi : \mathfrak{g}_0 \to \mathbb{C}$ such that the bilinear map $\widetilde{\chi} : \mathfrak{n}_+ \times \mathfrak{n}_- \to \mathbb{C}$ given by $(u, v) \mapsto \chi([u, v]_0)$ is non-degenerate. Here $x \mapsto x_0$ denotes the projection onto the zero graded component $\mathfrak{g}_0$.

1.20 Remark. Let $\mathfrak{g}$ be a polarized Lie algebra. We consider the vector space $\mathbb{C}_\chi^+ = \mathbb{C}\, v_\chi$ of dimension 1 span by the vector v_χ. The space $\mathbb{C}_\chi^+$ is actually a representation of $\mathfrak{p}_+$ given by

$$x \cdot v_\chi = \begin{cases} \chi(x) v_\chi & \text{if } x \in \mathfrak{h}, \\ 0 & \text{if } x \in \mathfrak{n}_+ \,. \end{cases}$$

Mutatis mutandis, we consider $\mathbb{C}_\chi^- = \mathbb{C}\, v_\chi$ the representation of $\mathfrak{p}_-$ of dimension 1.

Some examples of polarized Lie algebras:

1. One of the most import examples for us is the case when $\mathfrak{g}$ is a finite dimensional semi-simple complex Lie algebra. To build a $\mathbb{Z}$-gradation on $\mathfrak{g}$, we take $\mathfrak{h} = \mathrm{Span}(h_1, \ldots, h_n)$ a Cartan subalgebra of $\mathfrak{g}$, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a set of simple roots and $Q = \sum_i \mathbb{Z}\, \alpha_i$ its $\mathbb{Z}$-lattice. Then we consider the principal gradation of $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$,

$$\mathfrak{g}_i = \bigoplus_{\alpha\in Q, \mathrm{ht}(\alpha)=i} \mathfrak{g}[\alpha],$$

   with $\mathfrak{g}[\alpha] := \{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}$ the weight space α, and $\mathrm{ht}(\sum_i c_i\alpha_i) = \sum_i c_i$ the height of $\alpha = \sum_i c_i\alpha_i \in Q$. Thus all regular characters of $\mathfrak{h}$ (i. e. $\chi : \mathfrak{h} \to \mathbb{C}$, $\chi(h_i) \neq 0$), determinate a non-degenerate bilinear map $\widetilde{\chi} : \mathfrak{n}_+ \times \mathfrak{n}_- \to \mathbb{C}$. This construction applies *verbatim* to any Kac-Moody algebra.

2. Another example is when $\mathfrak{g} = \mathfrak{H}_n$ is the *Heisenberg Lie algebra*, that is, the Lie algebra generated by c, p_i, q_i; $i = 1, \ldots, n$, with relations:

$$[c, c] = [c, p_i] = [c, q_i] = 0, \qquad [p_i, q_j] = \delta_{ij} c.$$

One can consider the gradation such that for all $i \in \{1, \cdots, n\}$,

$$\deg(p_i) = -\deg(q_i) = 1, \qquad \deg(c) = 0.$$

Then, as in the previous example, all regular characters of $\mathfrak{g}_0$ are the non-singular characters.

3. Our last example is when $\mathfrak{g}$ is the *Virasoro algebra* $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, $\mathfrak{g}_n = \mathbb{C}\, L_n$ if $n \in \mathbb{Z} \smallsetminus \{0\}$ and $\mathfrak{g}_0 = \mathbb{C}\, L_0 \oplus \mathbb{C}\, c$. The Lie bracket of $\mathfrak{g}$ is given by:

$$[c, c] = [c, L_n] = 0, \quad [L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n(n^2-1)}{12}c.$$

So if $\chi : \mathfrak{g}_0 \to \mathbb{C}$ is a character such that $\chi(L_0 + \frac{(n^2-1)}{24}c) = z \in \mathbb{C} \smallsetminus \{0\}$, then for all $n \neq 0$

$$\widetilde{\chi}(L_n, L_{-n}) = 2nz$$

define a non-degenerate bilinear map.

Shapovalov form. Let $\mathfrak{g}$ be a polarized Lie algebra, we recall that $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ has a Hopf algebra structure with comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ given in the part 3 of the Remark 1.13.

For each $\lambda \in \mathbb{C}$, we consider $M^+_{\chi_\lambda}$ the generalized $\mathfrak{g}$-Verma module of highest weight $\chi_\lambda = \lambda\, \chi$, and $M^-_{\chi_\lambda}$ the generalized $\mathfrak{g}$-Verma Module of lowest weight χ_λ given by:

$$M^{\pm}_{\chi_\lambda} := \mathrm{Ind}^{U(\mathfrak{g})}_{U(\mathfrak{p}_\pm)} \mathbb{C}_{\chi_\lambda} \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\pm)} \mathbb{C}_{\chi_\lambda}.$$

If there is no possible confusion, we denote $M^{\pm}_{\lambda}$ and v_λ instead $M^{\pm}_{\chi_\lambda}$ and v_{χ_λ}.

By Poincaré-Birkhoff-Witt theorem we know that the map $x \mapsto x \otimes v_\lambda := xv_\lambda$ defines an isomorphism of $U(\mathfrak{n}_-)$-modules (resp. $U(\mathfrak{n}_+)$-modules) between the vector spaces $U(\mathfrak{n}_-)$ and M^+_λ (resp. $U(\mathfrak{n}_+)$ and M^-_λ). Moreover, and also by P-B-W theorem, the algebra $U(\mathfrak{g})$ has a vector space decomposition

$$U(\mathfrak{g}) \simeq U(\mathfrak{g}_0) \oplus N \tag{1.11}$$

with N the bilateral ideal $\mathfrak{n}_-\, U(\mathfrak{g}) + U(\mathfrak{g})\, \mathfrak{n}_+$ of $U(\mathfrak{g})$. We denote $\mathrm{HC} : U(\mathfrak{g}) \to U(\mathfrak{g}_0)$ the projection (Harish-Chandra projection) given by the decomposition (1.11).

1.21 Definition. For any $\lambda \in \mathbb{C}$, the map $\langle\ ,\ \rangle_{\mathrm{Sh}} : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ given by

$$(yv_\lambda, xv_{-\lambda}) \mapsto \chi_\lambda(\mathrm{HC}(S(x)y)),$$

defines a bilinear form called the Shapovalov form on M_λ. By abuse of notations, we also call Shapovalov form to the bilinear map $\langle\ ,\ \rangle_\lambda : U(\mathfrak{n}_-) \times U(\mathfrak{n}_+) \to \mathbb{C}$ induced by the equation

$$\langle y, x\rangle_\lambda = \langle yv_\lambda, xv_{-\lambda}\rangle_{\mathrm{Sh}}$$

where $x \in \mathfrak{n}_+$ and $y \in \mathfrak{n}_-$.

1.22 Proposition. *i) The Shapovalov form is $U(\mathfrak{g})$-invariant that is for any $(u, v) \in M_\lambda^+ \times M_{-\lambda}^-$ and $a \in U(\mathfrak{g})$ we have*

$$\langle au, v\rangle_{\mathrm{Sh}} = \langle u, S(a)v\rangle_{\mathrm{Sh}}.$$

ii) The Verma modules M_λ^+ and $M_{-\lambda}^-$ are irreducible if and only if the Shapovalov form is non-degenerate.

Proof. See Section 3.1 in [AL05]. □

1.3.2 Lie algebra Cohomology

Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. For any $x \in \mathfrak{g}$ and $y_1 \otimes \cdots \otimes y_k \in \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (k times) the Lie algebra $\mathfrak{g}$ acts on the tensor product $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (k times) in the following way:

$$\begin{aligned} x \cdot (y_1 \otimes \cdots \otimes y_k) &= \mathrm{ad}_x^k(y_1 \otimes \cdots \otimes y_k) \\ &= x \cdot y_1 \otimes y_2 \otimes \cdots \otimes y_k + \cdots + y_1 \otimes y_2 \otimes \cdots \otimes x \cdot y_k \\ &= [x, y_1] \otimes y_2 \otimes \cdots \otimes y_k + \cdots + y_1 \otimes y_2 \otimes \cdots \otimes [x, y_k]. \end{aligned}$$

1.23 Definition. Let $k \in \mathbb{N}$ and (V, ρ) be a representation of $\mathfrak{g}$. The vector space of skew-symmetric k-multilinear mappings on $\mathfrak{g}$ with values in V,

$$\Lambda^k(\mathfrak{g}; V) = \{u : \mathfrak{g}^k \to V | u \text{ is a skew-symmetric } k\text{-multilinear map}\},$$

is called the space of k-cochains on $\mathfrak{g}$ with values in V. Here a 0-cochain is an element in V.

1.24 Definition. The coboundary of a k-cochain u on $\mathfrak{g}$ with values in V is the $(k+1)$-cochain $\eth_k u$, with values in V defined by

$$\begin{aligned}\eth_k u(x_0,\dots x_k) = &\sum_{i=0}^{k}(-1)^i \rho(x_i)(u((x_0,\dots,\hat{x}_i,\dots x_k))) \\ &+\sum_{i<j}(-1)^{i+j}u([x_i,x_j],x_0,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots x_k),\end{aligned} \tag{1.12}$$

for $x_0, x_1, \dots x_k \in \mathfrak{g}$, where $\hat{x}_i$ indicates that the element x_i is omitted.

We obtain a sequence

$$\cdots \xrightarrow{\eth_{k-1}} \Lambda^k(\mathfrak{g};V) \xrightarrow{\eth_k} \Lambda^{k+1}(\mathfrak{g};V) \xrightarrow{\eth_{k+1}} \cdots$$

which satisfies the condition $\eth_k \circ \eth_{k+1} = 0$ for all $k \in \mathbb{N}$. Then for any $k \in \mathbb{N}$, $\operatorname{Im}\eth_{k-1} \subseteq \operatorname{Ker}\eth_k$. A k-cochain u is called a k-cocycle on $\mathfrak{g}$ with values in V if $u \in \operatorname{Ker}\eth_k \subseteq \Lambda^k(\mathfrak{g};V)$, and u is called a k-coboundary of $\mathfrak{g}$ with values in V if $u \in \operatorname{Im}\eth_{k-1} \subseteq \Lambda^k(\mathfrak{g};V)$. We denote $Z^k(\mathfrak{g};V) = \operatorname{Ker}\eth_k$ the space of k-cocycles and $B^k(\mathfrak{g};V) = \operatorname{Im}\eth_{k-1}$ the space of k-coboundaries.

The kth cohomology of $\mathfrak{g}$ with values in a representation (ρ, V) of $\mathfrak{g}$ is the space $H^k(\mathfrak{g};V)$ of k-cocycles modulo k-coboundaries:

$$H^k(\mathfrak{g};V) = Z^k(\mathfrak{g};V)/B^k(\mathfrak{g};V). \tag{1.13}$$

1.3.3 Lie bialgebras

In this section we review the notion of Lie bialgebra.

1.25 Definition. A Lie coalgebra of finite dimension over $\Bbbk$ is a vector space $\mathfrak{g}$ of finite dimension over $\Bbbk$ together with a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

i) $\tau \circ \delta = -\delta$, where τ is the flip (1.1).

ii) The transpose map ${}^t\delta : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ given by

$$\begin{aligned}{}^t\delta : \mathfrak{g}^* \otimes \mathfrak{g}^* &\to \mathfrak{g}^* \\ f \otimes g &\mapsto (f \otimes g)(\delta),\end{aligned}$$

is a Lie bracket in $\mathfrak{g}^*$.

Now we can introduce the notion of Lie bialgebra.

1.26 Definition. A Lie bialgebra of finite dimension over $\Bbbk$ is a triple $(\mathfrak{g}, [\ ,\ ], \delta)$ where $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra and $(\mathfrak{g}, \delta)$ is a Lie coalgebra such that δ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. In this case, we say that δ is the Lie cobracket of $\mathfrak{g}$.

Now, we consider $\mathfrak{g}$ a Lie algebra with Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and $U(\mathfrak{g})$ its universal enveloping algebra. Let $r = r_{(1)} \otimes r_{(2)}$ be an element in $\mathfrak{g} \otimes \mathfrak{g}$, written in the summation convention. We denote

$$\begin{aligned} r_{12} &= r_{(1)} \otimes r_{(2)} \otimes 1 = r \otimes 1 \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}, \qquad (1.14)\\ r_{23} &= 1 \otimes r_{(1)} \otimes r_{(2)} = 1 \otimes r \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g},\\ r_{31} &= r_{(2)} \otimes 1 \otimes r_{(1)} \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g},\\ &\dots \end{aligned}$$

and

$$[|r, r|] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}].$$

1.27 Definition. We say that $r \in \mathfrak{g} \otimes \mathfrak{g}$ is an r-matrix of $\mathfrak{g}$ if it is a solution of the classical Yang-Baxter equation (CYBE):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad \text{in } U(\mathfrak{g}). \qquad \text{(CYBE)}$$

1.28 Proposition. *Let $(\mathfrak{g}, [\ ,\ ])$ be a finite-dimensional Lie algebra, r be an element in $\mathfrak{g} \otimes \mathfrak{g}$ and $\delta_r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ be the map defined by $\delta_r(x) = x \cdot r$. A necessary and sufficient condition for $(\mathfrak{g}, [\ ,\ ], \delta_r)$ to define a Lie bialgebra is that $r_{12} + r_{21}$ and $[|r, r|]$ are ad-invariant.*

Proof. See [KS97, Proposition page 17]. □

In particular, if r is an anti-symmetric r-matrix ($r_{12} = -r_{21}$), we have that $(\mathfrak{g}, [\ ,\ ], \delta_r) = (\mathfrak{g}, r)$ is a Lie bialgebra. In this case we say that $(\mathfrak{g}, r)$ is a triangular Lie bialgebra.

Chapter 2

Quantization

La quantification est le processus de formation d'un système mécanique quantique à partir d'un système mécanique classique.

La formulation hamiltonienne associe à un système mécanique une variété symplectique M vue comme l'espace de phase de ce dernier. Les observables du système sont les fonctions régulières $f \in C^\infty(M)$ sur la variété M. Si f est une observable, son évolution dans le temps est déterminée par l'équation de Hamilton :

$$\frac{d}{dt} f(t) = -\{H, f\},$$

où H est une fonction régulière de M appelée hamiltonien du système et $\{\ ,\ \}$ est le crochet de Poisson associé à la structure symplectique de M.

La mécanique quantique, en sa formulation due à Heisenberg, associe à un système quantique un espace de Hilbert $\mathscr{H}$ vu comme l'espace de phase de ce dernier. Les observables du système sont des opérateurs auto-adjoints dans $\mathscr{H}$. La dynamique du système est définie en termes du hamiltonien $\widetilde{H}$ du système. Si $\tilde{f}$ est une observable dans $\mathscr{H}$, l'évolution de $\tilde{f}$ dans le temps est donnée par l'équation de Hamilton :

$$\frac{d}{dt} \tilde{f}(t) = \frac{\mathbf{i}}{\hbar}[\widetilde{H}, \tilde{f}],$$

où $\hbar$ dénote la constante de Planck et $[\ ,\ ]$ le crochet d'opérateurs.

De point de vue mathématique [Dir47, Wey50, vN55], la quantification est une correspondance $\mathcal{Q} : f \to \tilde{f}$ entre les observables de chaque système telle que

$$\mathcal{Q}(1) = \mathrm{id} \qquad \textit{et} \qquad [\mathcal{Q}(f), \mathcal{Q}(g)] = \mathbf{i}\,\hbar\, \mathcal{Q}(\{f, g\}).$$

La formulation originale d'un système mécanique classique prend $M = \mathbb{R}^n \times \mathbb{R}^n$ *comme l'espace de phase. La formulation originale de quantification consiste à assigner aux fonctions régulières* f *en* $C^\infty(M)$, *des opérateurs auto-adjoints* $\mathcal{Q}_f$ *dans l'espace de Hilbert* $L^2(\mathbb{R}^n)$ *tels que*

(C1) *La correspondance* $f \mapsto \mathcal{Q}_f$ *est linéaire.*

(C2) $\mathcal{Q}_1 = \mathrm{id}$, *où 1 est l'élément unité dans* $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

(C3) *Pour toute fonction* $\phi : \mathbb{R} \to \mathbb{R}$ *telle que* $\mathcal{Q}_{\phi\circ f}$ *et* $\phi(\mathcal{Q}_f)$ *sont bien définies, on a*

$$\mathcal{Q}_{\phi\circ f} = \phi(\mathcal{Q}_f).$$

(C4) *Si* $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n)$ *sont des coordonnées dans* $\mathbb{R}^n$, *les opérateurs* $\mathcal{Q}_{p_i}$ *et* $\mathcal{Q}_{q_i}$ *correspondant à la coordonnée* p_i, q_i $(i = 1, \ldots n)$ *sont données par*

$$\mathcal{Q}_{q_i}\,\psi = q_i\psi, \qquad \mathcal{Q}_{p_i}\,\psi = -\,\mathbf{i}\,\hbar\frac{\partial\psi}{\partial q_i} \qquad \text{for any } \psi \in L^2(\mathbb{R}^n, dq).$$

Ici $\hbar$ *est un paramètre qui dépend de la quantification* $\mathcal{Q}$, *il correspond à la constante de Planck.*

(C5) *Pour tout* $f, g \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, *on a*

$$[\mathcal{Q}_f, \mathcal{Q}_g] = \mathbf{i}\,\hbar\,\mathcal{Q}_{\{f,g\}}$$

où

$$\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

est le crochet de Poisson.

Dans cette thèses nous discutons trois différents quantifications :

2.1 *Quantification géométrique. Dans cette section nous rappelons un type de quantification due à A. Unterberger et J. Unterberger [UU96] qui utilisent les idées classiques de quantification de à H. Weyl. Nous utilisons cette quantification dans le Chapitre 4, section 4.2.*

2.2 *Quantification par déformation. Motivée par le développement asymptotique du produit de Moyal, nous donnons un bref rappel de la quantification par déformation introduit par Bayen, Flato, Fronsdal, Lichnerowicz et Sternheimer [BFF+78]. L'idée est de remplacer le produit commutatif de l'algèbre $C^\infty(M)$, par une série formelle d'opérateurs bi-différentiels déformant, d'une facon associative le produit usual par le crochet de Poisson de M.*

2.3 *Groupes quantiques. Dans cette section nous sommes intéressés par la quantification définie par Drinfeld [Dri85] sur les groupes de Lie-Poisson, exprimée en termes de l'équation de Yang-Baxter. En 2005 Alekseev et Lachowska [AL05] ont donné une construction générale d'une telle déformation pour une large classe d'orbites co-adjointes.*

In very vague terms quantization is the process of forming a quantum mechanical system starting from a classical mechanical one. On the other side, if we start with a quantum theory and go back to its classical counterpart, we talk about the process of dequantization. The original mathematical formulation of quantization (due to Heisenberg, Dirac, Weyl, von Neumann [Dir47, Wey50, vN55]) consists in assigning to the observables of a classical mechanical system, which are real-valued functions on the space $\mathbb{R}^n \times \mathbb{R}^n$ (the phase space), self-adjoint operators $\mathcal{Q}_f$ on the Hilbert space $L^2(\mathbb{R}^n)$ in such way that

(C1) The correspondence $f \mapsto \mathcal{Q}_f$ is linear.

(C2) $\mathcal{Q}_1 = \mathrm{id}$, where 1 is the unit element in the real-valued functions on $\mathbb{R}^n \times \mathbb{R}^n$.

(C3) For any function $\phi : \mathbb{R} \to \mathbb{R}$ for which $\mathcal{Q}_{\phi\circ f}$ and $\phi(\mathcal{Q}_f)$ are well-defined, we have

$$\mathcal{Q}_{\phi\circ f} = \phi(\mathcal{Q}_f).$$

(C4) If $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n)$ are in $\mathbb{R}^n$, the operators $\mathcal{Q}_{p_i}$ and $\mathcal{Q}_{q_i}$ corresponding to the coordinate functions p_i, q_i $(i = 1, \ldots n)$ are given by

$$\mathcal{Q}_{q_i}\,\psi = q_i\psi, \qquad \mathcal{Q}_{p_i}\,\psi = -\,\mathbf{i}\,\hbar\frac{\partial\psi}{\partial q_i} \qquad \text{for any } \psi \in L^2(\mathbb{R}^n, dq),$$

where $\hbar$ is a parameter that depends on the quantization map $\mathcal{Q}$, corresponding to the Planck constant.

(C5) For any f, g two observable functions we have that

$$[\mathcal{Q}_f, \mathcal{Q}_g] = \mathbf{i}\,\hbar\,\mathcal{Q}_{\{f,g\}}$$

where

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

is the Poisson bracket of f and g.

2.1 Remark. Unfortunately (or not), it is well-known that any three of the axioms $(C1)$, $(C3)$, $(C4)$ and $(C5)$ are inconsistent (see [AE05, Section 1.2]). So there are two traditional approaches to handle this situation. The first approach is to keep the four axioms $(C1)$, $(C2)$, $(C4)$ and $(C5)$, but restrict the space of observables. In this way, the domain $\mathrm{Dom}(\mathcal{Q})$ of definition of the mapping $\mathcal{Q} : f \mapsto \mathcal{Q}_f$ is called the space of quantizable observables. Ideally, it should include at least the space $C^\infty(\mathbb{R}^{2n})$ of real-valued smooth functions on $\mathbb{R}^{2n}$ or some other convenient function space.

The second approach is to keep $(C1)$, $(C2)$ and $(C4)$, but change the condition $(C5)$ to hold only asymtotically as the Planck constant $\hbar$ tends to zero. This approach is the celebrated Weyl calculus of pseudodifferential operators. Where the quantization map $\mathcal{Q} : L^2(\mathbb{R}^{2n}) \to HS(L^2(\mathbb{R}^n))$ is given by

$$(\mathcal{Q}(f)u)(x) = \hbar^n \iint_{\mathbb{R}^n \times \mathbb{R}^n} f\left(\frac{x+\varepsilon}{2}, \eta\right) \exp\left(\frac{2\,\mathbf{i}\,\pi}{\hbar} \langle x - \varepsilon, \eta \rangle\right) u(\varepsilon) d\,\varepsilon\, d\eta \qquad (2.1)$$

for $f \in L^2(\mathbb{R}^{2n}), u \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

In this dissertation we discuss and use four different approaches to quantization:

a) Geometric quantization (orbit method, ...).

b) Deformation quantization (formal deformation, star products ...).

c) Quantum groups (Drinfeld twist,...).

d) Symbolic Calculus

We take as reference [AE05, CKTB05]

2.1 Geometric quantization

Geometric quantization deals with the problem of construction of an explicit Hilbert space and an algebra of operators on it. We start with a symplectic manifold (the phase space) (M, ω), *i.e.* a finite-dimensional smooth manifold M together with a closed 2-form $\omega \in \Omega^2(M)$ on M which is non-degenerate as a bilinear form on each tangent space. We denote $A = C^\infty(M)$ the algebra of real-valued smooth functions on M. For any function $f \in A$, the Hamiltonian vector field X_f is given by $\omega(-, X_f) = df$ and the Poisson bracket between two elements f and g of A is defined by

$$\{f, g\} = -\omega(X_f, X_g). \tag{2.2}$$

According to Definition 2.2, the space A together with the Poisson bracket $\{\ ,\ \}$ given in (2.2) is a Poisson algebra.

The goal of geometric quantization is to assign to (M, ω) a pair $(H_M, \mathcal{Q}_M)$ where $\mathcal{Q}_M : f \mapsto \mathcal{Q}_M(f)$ is a map from $\mathrm{Dom}(\mathcal{Q}_M)$ a convenient subalgebra of A of self-adjoint linear operators on a convenient Hilbert space H_M in such a way that

(Q1) The correspondence $f \mapsto \mathcal{Q}_M(f)$ is linear.

(Q2) $\mathcal{Q}_M(1) = \mathrm{id}$, where 1 is the unit element in $\mathrm{Dom}(\mathcal{Q}_M) \subseteq A$.

(Q3) The correspondence is functorial, *i.e.* if (N, υ) is an other symplectic manifold with quantization map $\mathcal{Q}_N : f \mapsto \mathcal{Q}_N(f)$ and Hilbert space H_N, then for any $\phi : (M, \omega) \to (N, \upsilon)$ diffeomorphism (which send ω into υ) the composition with ϕ should map $\mathrm{Dom}(\mathcal{Q}_N)$ into $\mathrm{Dom}(\mathcal{Q}_M)$ and there should be a unitary operator $u_\phi : H_M \to H_N$ such that

$$\mathcal{Q}_M(f \circ \phi) = u_\phi^* \, \mathcal{Q}_N(f) u_\phi \qquad \text{for all } f \in \mathrm{Dom}(\mathcal{Q}_N).$$

(Q4) If (M, ω) is $\mathbb{R}^{2n}$ together with its standard symplectic form, then the operators $\mathcal{Q}_{p_i}$ and $\mathcal{Q}_{q_i}$ corresponding to the coordinate functions p_i, q_i of $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{R}^n \times \mathbb{R}^n$ are given by

$$\mathcal{Q}_{q_i}\, \psi = q_i \psi, \qquad \mathcal{Q}_{p_i}\, \psi = -\,\mathbf{i}\,\hbar \frac{\partial \psi}{\partial q_i} \qquad \text{for } \psi \in L^2(\mathbb{R}^n, dq).$$

(Q5) For all $f, g \in \mathrm{Dom}(\mathcal{Q}_M) \subseteq A$ we have that

$$[\mathcal{Q}_M(f), \mathcal{Q}_M(g)] = \mathbf{i}\,\hbar\,\mathcal{Q}_M(\{f, g\}). \tag{2.3}$$

Here $\hbar$ is a parameter that depends of the geometric quantization $(H_M, \mathcal{Q}_M)$.

2.1.1 Unterberger's quantization

Now we briefly recall the classical approach to quantification procedure which comes from fundamental construction due to H. Weyl.

Consider the map $\mathcal{Q}_\lambda : L^2(M) \to HS(L^2(\mathbb{R}))$ defined by

$$(\mathcal{Q}_\lambda(f)u)(s) = c_\lambda \int_{\mathbb{R}} f\,(s,t)\,|s-t|^{-1-\mathbf{i}\,\lambda}\,(\theta u)\,(t)dt, \tag{2.4}$$

where θ is the convolution operator by the distribution that coincides on $\mathbb{R}^\times$ with the function $s \mapsto c_\lambda |s|^{-1+\mathbf{i}\,\lambda}$ and

$$c_\lambda = \frac{1}{2}(2\pi)^{\mathbf{i}\,\lambda}\left[\Gamma(\mathbf{i}\,\lambda)\cosh\left(\frac{\pi\lambda}{2}\right)\right]^{-1}.$$

Notice that θ is the so-called Knapp-Stein intertwining operator for the principal series representations of $\mathrm{SL}(2,\mathbb{R})$. Therefore, the map $\mathcal{Q}_\lambda$ is equivariant with respect to this series of representations:

$$\pi_\lambda(g)\,\mathcal{Q}_\lambda(f)\pi_\lambda(g^{-1}) = \mathcal{Q}_\lambda(f(g^{-1}\circ)) \qquad (g \in G).$$

The composition of operators on $L^2(\mathbb{R}^n)$ defines an associative and non-commutative product on $L^2(M)$ by

$$\mathcal{Q}_\lambda(f_1 \# f_2) = \mathcal{Q}_\lambda(f_1) \circ \mathcal{Q}_\lambda(f_2) \qquad (f_1, f_2 \in L^2(M)). \tag{2.5}$$

This algebraic structure plays a crucial role in the rest of our work.

2.2 Deformation quantization

Motivated by the asymptotic expansion for the Moyal product, we give a brief reminder on the deformation quantization in the sense given by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [BFF+78]. The idea of the deformation quantization is to avoid difficulties of geometric quantization by changing the formula (2.3) in the axiom $(Q3)$ to the formula

$(Q'3)$ For all $f, g \in \mathrm{Dom}(\mathcal{Q}_M) \subseteq A$ we have that

$$[\mathcal{Q}_M(f), \mathcal{Q}_M(g)] = \mathbf{i}\,\hbar\,\mathcal{Q}_M(\{f,g\}) + O(\hbar^2). \tag{2.6}$$

2.2.1 Formal deformation and Star products

Let A be an unital commutative algebra over the field $\Bbbk$.

2.2 Definition. We say that A is a Poisson algebra if there exists a $\Bbbk$-bilinear operator $\{\,,\,\} : A \times A \to A$ which forms a Lie bracket on A and for any $a \in A$ the map $\{a, -\} : A \to A$ is a $\Bbbk$-derivation, *i.e.* for any $x, y \in A$,

$$\{a, xy\} = \{a, x\}y + x\{a, y\}.$$

In this case, we say that $\{\,,\,\}$ is a Poisson bracket of A.

Let $(A, \{\,,\,\}_A)$ and $(B, \{\,,\,\}_B)$ be two Poisson algebras over $\Bbbk$. A $\Bbbk$-linear map $\psi : A \to B$ which satisfies, for all $a, b \in A$,

i) $\psi(ab) = \psi(a)\psi(b)$,

ii) $\psi\{a, b\}_A = \{\psi(a), \psi(b)\}_B$,

is called a morphism of Poisson algebras. A morphism $\psi : A \to B$ of Poisson algebras is called an isomorphism of Poisson algebras if ψ is also bijective.

2.3 Definition. Let A be a commutative algebra, $A_\hbar = A[[\hbar]]$ be the formal power series in a variable $\hbar$ with coefficients in A and $\{\Pi_n : A \times A \to A\}_{n\in\mathbb{N}}$ be a family of $\Bbbk$-bilinear operators. We say that the family $\Pi = (\Pi_n)_{n\in\mathbb{N}}$ is a formal deformation if

i) Π_0 is the original multiplication on A.

ii) $\{a, b\} = \frac{1}{2}(\Pi_1(a, b) - \Pi_1(b, a))$ is a Poisson bracket of A.

iii) The $\hbar$-linear[1] extension of the $\Bbbk$-linear map

$$\Pi : A \otimes A \to A[[\hbar]] \quad (a \otimes b) \mapsto \sum_{n\in\mathbb{N}} \Pi_n(a, b)\hbar^n. \tag{2.7}$$

is an associative product on the space $A_\hbar = A[[\hbar]]$.

[1] A $\Bbbk$-linear map $Q : A_\hbar \to B_\hbar$ is a $\hbar$-linear if for any $a \in A$ and any $n \in \mathbb{N}$, $Q(a\hbar^n) = Q(a)\hbar^n$.

iv) The identity element 1 of A is the identity element of $(A_\hbar, \Pi)$.

2.4 Remark. (i) For short we write $a \star b$ instead of $\Pi(a,b)$.

(ii) If a group G acts on A and all $\Pi_n : A \times A \to A$ are G-equivariant (*i.e.* $\Pi_n(g \cdot a, g \cdot b) = g \cdot \Pi_n(a,b)$ for any $g \in G$, $a, b \in A$ and $n \in \mathbb{N}$), we say that $\Pi = (\Pi_n)$ is a G-invariant formal deformation.

(iii) The product $\Pi : A_\hbar \times A_\hbar \to A_\hbar$ defined in (2.7) is associative if and only if for any $a, b, c \in A$ and any $p \in \mathbb{N}$ the following condition holds

$$\sum_{n+m=p} \Pi_n(a, \Pi_m(b,c)) = \sum_{n+m=p} \Pi_n(\Pi_m(a,b), c). \tag{2.8}$$

(iv) If A has a formal deformation then by condition (ii) in Definition 2.3 A has a structure of a Poisson algebra. According to Kontsevich [Kon03], we know that the reciprocal is also true: every smooth Poisson algebra admits a formal deformation $\Pi = (\Pi_n)$ such that Π_0 is the original multiplication on A and $\frac{1}{2}(\Pi_1(a,b) - \Pi_1(b,a))$ is the original Poisson bracket of A.

2.5 Definition. Let A, B be two $\mathbb{k}$-algebras equipped with two formal deformations $\star_A$ and $\star_B$ respectively. We say that $\star_A$ and $\star_B$ are equivalents if there exists $Q : (A_\hbar, \star_A) \to (B_\hbar, \star_B)$ an $\hbar$-linear algebra isomorphism between the unital associative algebras $(A_\hbar, \star_A)$ and $(B_\hbar, \star_B)$.

2.6 Proposition. *Let A, B be two commutative $\mathbb{k}$-algebras equipped respectively with two formal deformations $\Pi = \star$ and $\widetilde{\Pi} = \tilde{\star}$. The formal deformations $\star$ and $\tilde{\star}$ are equivalents if and only if there exists a sequence $\{Q_n : A \to B\}_{n \in \mathbb{N}}$ of $\mathbb{k}$-linear maps such that $Q_0 : A \to B$ is a $\mathbb{k}$-linear isomorphism and for any $a, b \in A$, $n \in \mathbb{N}$ the following condition holds*

$$\sum_{i+j=n} Q_i(\Pi_j(a,b)) = \sum_{i+j+k=n} \widetilde{\Pi}_k(Q_i(a), Q_j(b)). \tag{2.9}$$

Proof. It is clear that $Q : A_\hbar \to B_\hbar$ is an $\hbar$-linear isomorphism if and only if there exists a sequence $\{Q_n : A \to B\}_{n \in \mathbb{N}}$ of $\mathbb{k}$-linear maps such that $Q = \sum_{n \in \mathbb{N}} Q_n \hbar^n$ and $Q_0 : A \to B$ is bijective. Now, we suppose that $Q = \sum_{n \in \mathbb{N}} Q_n \hbar^n$ is an $\hbar$-linear map. For any $a, b \in A$ we have that

$$
\begin{aligned}
Q(a\star b) &= Q\left(\sum_{l\in\mathbb{N}}(\mathit{\Pi}_l(a,b)\hbar^l\right) \\
&= \sum_{n\in\mathbb{N}}\left(\sum_{i+j=n} Q_i(\mathit{\Pi}_j(a,b))\right)\hbar^n
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
Q(a)\tilde{\star}Q(b) &= \sum_{l\in\mathbb{N}}\widetilde{\mathit{\Pi}}_l(Q(a),Q(b))\hbar^l \\
&= \sum_{n\in\mathbb{N}}\left(\sum_{i+j+k=n}\widetilde{\mathit{\Pi}}_k(Q_i(a),Q_j(b))\right)\hbar^n.
\end{aligned}
\tag{2.11}
$$

From equations (2.10) and (2.11) we have that $Q(a\star b) = Q(a)\tilde{\star}Q(b)$ if and only if the condition (2.9) holds. $\square$

2.7 Remark. If $\star_A = \star$ and $\star_B = \tilde{\star}$ are equivalents ($Q : (A_\hbar, \star_A) \to (B_\hbar, \star_B)$), the equation (2.9) for $n = 0$ and $n = 1$ implies that $Q_0 : A \to B$ is an isomorphism between the Poisson algebra A with bracket $\{a_1, a_2\}_A = \frac{1}{2}(\mathit{\Pi}_1(a_1, a_2) - \mathit{\Pi}_1(a_2, a_1))$ and the Poisson algebra B with bracket $\{b_1, b_2\}_B = \frac{1}{2}(\widetilde{\mathit{\Pi}}_1(b_1, b_2) - \widetilde{\mathit{\Pi}}_1(b_2, b_1))$.

Let $Q = \sum_n Q_n\hbar^n : (A_\hbar, \star_A) \to (B_\hbar, \star_B)$ be an equivalence between two $\Bbbk$-algebras A, B equipped respectively with two equivalent formal deformations $\star_A$ and $\star_B$. If a group G acts on A and B, we say that $\star_A$ and $\star_B$ are G-equivalent if, in addition, for each $n \in \mathbb{N}$ the map $Q_n : A \to B$ is G-equivariant.

A particular example of such a construction is given by the space of smooth functions on a symplectic manifold. More precisely, let (M, ω) be a symplectic manifold, that is a finite-dimensional smooth manifold M together a closed 2-form $\omega \in \Omega^2(M)$ on M which is non-degenerate as a bilinear form on each tangent space. Now, we consider the algebra of real-valued smooth functions $A = C^\infty(M)$. The bilinear map

$$\{f, g\} = \langle (df \otimes dg), \omega\rangle$$

defines a Poisson bracket on M.

2.8 Definition. Let (M, ω) be a symplectic manifold. We say that a formal deformation $\mathit{\Pi} = (\mathit{\Pi}_n)_{n\in\mathbb{N}}$ of $A = C^\infty(M)$ is a star-product on M if for each $n \in \mathbb{N}$ the bilinear map $\mathit{\Pi}_n : A \times A \to A$ is a bidifferential operator on M.

If Π is a star-product on (M, ω), we denote

$$a \star b = \Pi(a, b) \quad \text{for } a, b \in A.$$

The star commutator $[a, b]_\star$ is defined by $a \star b - b \star a$, which makes $A_\hbar = A[[\hbar]]$ into a Lie algebra.

2.9 Definition. Two star products $\star$ and $\tilde{\star}$ on (M, ω) are equivalent if there is a sequence $\{Q_n : A \to A\}_{n \in \mathbb{N}^\times}$ of differential operators on A such that the series $Q = \mathrm{id} + Q_1\hbar^1 + Q_2\hbar^2 \cdots$ satisfies the relation:

$$Q(a \star b) = Q(a) \mathbin{\tilde{\star}} Q(b),$$

for all $a, b \in A$.

2.10 Remark. Then two star products $\star$ and $\tilde{\star}$ on (M, ω) are equivalent if there is a formal deformation equivalence $Q = (Q_n) : (A_\hbar, \star) \to (A_\hbar, \tilde{\star})$ between $\star$ and $\tilde{\star}$ such that for any $n \in \mathbb{N}^\times$ the operator $Q_n : A \to A$ is a differentiable map on M.

2.2.2 Hochschild Cohomology

Let (M, ω) be a symplectic manifold and $A = C^\infty(M)$. For any $k \geq 1$, we say that C is a differential k-cochain (ou k-cochain) on A if C is a k-multilinear map from A^k to A such that C is a differentiable operator on each argument. We denote $C^k(A, A)$ the space of differential k-cochains on A. The kth Hochschild coboundary operator $\partial_k : C^k(A, A) \to C^{k+1}(A, A)$ is the linear operator from k-cochains to $(k+1)$-cochains defined by

$$\begin{aligned}(\partial_k C)(a_0, \cdots, a_k) =& a_0 C(a_1, \cdots, a_k) + \\ & \sum_{r=1}^{k} (-1)^r C(a_0, \cdots, a_{r-2}, a_{r-1}a_r, a_{r+1}, \cdots a_k) \\ & + (-1)^{k-1} C(a_0, \cdots, a_{k-1}) a_k\end{aligned}$$

where $C \in C^k(A, A)$ and $(a_0, \cdots, a_k) \in A^{k+1}$.

A k-cochain C is called a k-cocycle if $\partial_k C = 0$, and a k-coboundary if there is a $(k-1)$-cochain B such that $\partial_{k-1} B = C$. We denote $Z^k(A, A)$ the space of k-cocycles on A and $B^k(A, A)$ the space of k-coboundaries on A.

2.11 Remark. One has that for any $k \in \mathbb{N}$, $\partial_{k+1} \circ \partial_k = 0$ so $B^k(A, A) \subseteq Z^k(A, A)$.

2.12 Definition. The kth differential Hochschild cohomology space of A is the space $HH^k(A, A)$ of differential k-cocycles modulo differential k-coboundaries, *i.e.*

$$HH^k(A, A) = Z^k(A, A)/B^k(A, A).$$

Let C be a k-cochain and B be a p-cochain. The tensor product $C \otimes B \in C^{k+p}(A, A)$ is the $(k+p)$-cochain defined by

$$(C \otimes B)(a_1, \cdots, a_{k+p}) = C(a_1, \cdots, a_k)B(a_{k+1}, \cdots, a_{k+p}).$$

for any $(a_1, \cdots, a_{k+p}) \in A^{k+p}$.

2.3 Quantum groups

In this section, our aim is to introduce the notion of deformation on a particular class of Lie algebras. We follow the books [Kas95, CP95] for our presentation.

Let $\Bbbk$ be a field and $\mathbb{K}$ be an unital associative and commutative ring.

2.3.1 Classical and Quantum Yang-Baxter equation

Let V be a $\Bbbk$-vector space. We say that an automorphism $c \in \mathrm{Aut}(V \otimes V)$ is an R-matrix of V if it is a solution to the quantum Yang-Baxter equation (QYBE):

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c). \tag{QYBE}$$

2.3.1 Example. Let V be a vector space, the flip $\tau_{V,V} : V \otimes V \to V \otimes V$ given by:

$$\begin{aligned} \tau_{V,V} : V \otimes V &\to V \otimes V \\ v_1 \otimes v_2 &\mapsto v_2 \otimes v_1, \end{aligned}$$

is a R-matrix. Indeed, it satisfies the Coxeter relation

$$(12)(23)(12) = (23)(12)(23),$$

in the symmetric group $\mathcal{S}_3$.

2.13 Definition. Let $(A, m, u, \Delta, \varepsilon)$ be a bialgebra. We say that A is quasi-cocommutative if there exists an invertible element $R \in A \otimes A$, called a universal R-matrix, such that for any $x \in A$:

$$\Delta^{op}(x) = R\Delta(x)R^{-1},$$

where $\Delta^{op} := \tau_{A,A} \circ \Delta$.

2.14 Notation. If $R = \sum_i R_i' \otimes R_i'' \in A \otimes A$, then we denote

$$\begin{aligned} R_{12} &= \sum_i R_i' \otimes R_i'' \otimes 1 = R \otimes 1 \in A \otimes A \otimes A, \\ R_{23} &= \sum_i 1 \otimes R_i' \otimes R_i'' = 1 \otimes R \in A \otimes A \otimes A, \\ R_{31} &= \sum_i R_i'' \otimes 1 \otimes R_i' \in A \otimes A \otimes A, \\ &\dots \end{aligned}$$

2.15 Definition. Let $(A, m, u, \Delta, \varepsilon, R)$ be a quasi-cocommutative $\Bbbk$-bialgebra. We say that $(A, m, u, \Delta, \varepsilon, R)$ is braided if the universal R-matrix satisfies the following relations:

$$\begin{aligned} (\Delta \otimes \mathrm{id}_A)(R) &= R_{13}R_{23}, \\ (\mathrm{id}_A \otimes \Delta)(R) &= R_{13}R_{12}. \end{aligned}$$

2.16 Proposition. *Let (A, R) be a braided quasi-cocommutative bialgebra and V be an A-module. Then the automorphism*

$$\begin{aligned} c^R_{V,V} : V \otimes V &\to V \otimes V \\ v_1 \otimes v_2 &\mapsto \tau_{V,V}(R(v_1 \otimes v_2)), \end{aligned}$$

satisfies the quantum Yang-Baxter equation, i.e. $c^R_{V,V}$ is a R-matrix of V.

Proof. Voir [Kas95, Proposition VIII.3.1 pg 178]. □

2.3.2 Deformations of Hopf algebras

We consider $\mathbb{K} = \Bbbk[[\hbar]]$ the ring of formal power series in $\hbar$ with coefficients in the field $\Bbbk$. The ring $\mathbb{K}$ has a Banach structure with the norm:

$$\|f\| = \begin{cases} 2^{-\omega(f)} & \text{if } f \neq 0 \\ 0 & \text{otherwise ;} \end{cases} \tag{2.12}$$

where $f = \sum_{n\in\mathbb{N}} a_n \hbar^n \in \mathbb{K}$ and $\omega(f) = n$ if for any $k < n$ $a_k = 0$ and $a_n \neq 0$.

Let M be a $\mathbb{K}$-module. For any $n \in \mathbb{N}$, we consider the submodule $M_n = M/\hbar^n M$ of M and $p_n : M_n \to M_{n-1}$ the canonical $\mathbb{K}$-linear projection:

$$p_n : M_n = M/\hbar^n M \to M_{n-1} = M/\hbar^{n-1} M.$$

We say that $(M_n, p_n)_{n\in\mathbb{N}}$ is a projective system of $\mathbb{K}$-modules associated to M.

The projective limit

$$\widehat{M} = \varprojlim M_n = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} M_n, \quad p_n(x_n) = x_{n-1} \text{ for all } n \in \mathbb{N} \right\}$$

possesses a natural topology called the $\hbar$-adic topology. It is obtained as follows. Put the discrete topology on each submodule M_n. The projective topology on $\widehat{M}$ is the restriction of the direct product topology on $\prod_{n\in\mathbb{N}} M_n$.

2.17 Remark. *i)* The $\Bbbk[[\hbar]]$-morphisms of M are continuous maps.

ii) The family $\{\hbar^n \widehat{M}\}_{n\in\mathbb{N}}$ of left ideals of $\widehat{M}$ is a family of open neighbourhoods containing 0. Moreover,

$$\widehat{M}/\hbar^n \widehat{M} \simeq M/\hbar^n M.$$

iii) The projections $q_n : M \to M_n$ induce a unique $\mathbb{K}$-linear map $q : M \to \widehat{M}$ such that $\pi_n \circ q = q_n$ for all $n \in \mathbb{N}$. Here $\pi_n : \widehat{M} \to M_n$ denotes the natural projection. The kernel of q is given by

$$\mathrm{Ker}(q) = \bigcap_{n\geq 1} \hbar^n M.$$

2.18 Definition. A $\Bbbk[[\hbar]]$-module M is separated if the map q is injective and it is complete if q is surjective.

2.19 Remark. According to [Kas95, section XVI.2 page 388], if q is bijective the topological space $\widehat{M}$ is complete in the sense of metric spaces. Indeed, if q is surjective so M is a dense subspace of $\widehat{M}$. Now, if $\mathrm{Ker}(q) = \bigcap_{n\geq 1} \hbar^n M = \{0\}$ we defined the following norm on M. If $x \in M$, and $x \neq 0$, there exists i with $x \in \hbar^i M$ but $x \in \hbar^{i+1} M$; define $\|x\| = 2^{-i}$, if $x = 0$, define $\|x\| = 0$. It turns out that $\widehat{M}$ is a completion of M, *i.e.* M is dense in $\widehat{M}$ and every Cauchy sequence in M converges in $\widehat{M}$. In this case, we say that $\widehat{M}$ is a $\hbar$-completion of M.

2.3.2 Example. Let V be a $\Bbbk$-vector space. We consider the space $V[[\hbar]]$ of formal power series in the indeterminate $\hbar$ with coefficients in V is the space with elements of form

$$\sum_{n\in\mathbb{N}} v_n\,\hbar^n \in V[[\hbar]]$$

with $v_n \in V$ and $n \in \mathbb{N}$.

We recall that a $\Bbbk[[\hbar]]$-module M is torsion free if $\hbar m \neq 0$ when m is any non zero element of M. We say that M is topologically free if it is separated, complete and torsion-free $\Bbbk[[\hbar]]$-module.

2.20 Proposition. *The space $V[[\hbar]]$ is a topologically free $\Bbbk[[\hbar]]$-module.*

Proof. Voir [Kas95, Proposition XVI.2.4 pg 390]. □

2.21 Definition. Let M and N be two $\Bbbk[[\hbar]]$-modules. The topological tensor product $M\widehat{\otimes}N$ between M and N is defined by

$$M\widehat{\otimes}N = (M\otimes_{\mathbb{K}} N)\widehat{\ } = \varprojlim (M\otimes_{\mathbb{K}} N)/\hbar^n(M\otimes_{\mathbb{K}} N). \tag{2.13}$$

The topological tensor product $\widehat{\otimes}$ satisfies the following $\Bbbk[[\hbar]]$-linear isomorphisms

$$(M\widehat{\otimes}N)\widehat{\otimes}P \simeq M\widehat{\otimes}(N\widehat{\otimes}P),$$
$$M\widehat{\otimes}N \simeq N\widehat{\otimes}M,$$
$$\mathbb{K}\,\widehat{\otimes}M \simeq \widehat{M} \simeq M\widehat{\otimes}\,\mathbb{K}\,.$$

If V is a $\Bbbk$-vector space finite-dimensional, then $V[[\hbar]] \simeq V\otimes_{\mathbb{K}}\mathbb{K}$ but if it is infinite-dimensional, then the set $V[[\hbar]]$ is strictly bigger than $V\otimes_{\mathbb{K}}\mathbb{K}$. However, we have that

$$V[[\hbar]] \simeq V\,\widehat{\otimes}_{\mathbb{K}}\,\mathbb{K}\,.$$

If M and N are two topologically free $\Bbbk[[\hbar]]$-modules the topological tensor product $M\,\widehat{\otimes}\,N$ is a $\hbar$-adic completion of $M\otimes_{\mathbb{K}} N$. More precisely,

2.22 Proposition. *If M and N are two topologically free $\Bbbk[[\hbar]]$-modules, then so is $M\widehat{\otimes}N$. In particular, if V and W are two vector spaces we have*

$$V[[\hbar]]\widehat{\otimes}W[[\hbar]] \simeq (V\otimes W)[[\hbar]].$$

Proof. See [Kas95, Proposition XVI.3.2 pg 391] □

Let $f : M \to N$ and $f' : M' \to N$ be two morphisms of $\Bbbk[[\hbar]]$-modules, the $\Bbbk[[\hbar]]$-linear map

$$f\widehat{\otimes}f' : M\widehat{\otimes}M' \to N\widehat{\otimes}N' \tag{2.14}$$

is the natural extension of the map $f \otimes f' : M \otimes M' \to N \otimes N'$ such that the following diagram commutes:

$$\begin{tikzcd} M \otimes M' \arrow[r, "f\otimes f'"] \arrow[d] & N \otimes N' \arrow[d] \\ M\widehat{\otimes}M' \arrow[r, "f\widehat{\otimes}f'"'] & N\widehat{\otimes}N' \end{tikzcd}$$

We want to extend the notion of Hopf algebra when the algebra has a topological structure. Essentially we replace $\widehat{\otimes}$ by $\otimes$ in the associativity, coassociativity, unity, county and antipode definitions.

2.23 Definition. Let A_h be a topological module over $\mathbb{K} = \Bbbk[[\hbar]]$. We say that A_h is a topological Hopf algebra over $\mathbb{K}$ if $\widehat{A}_h = A_h$ and there exist $\mathbb{K}$-morphisms (continuous)

$$m_h : A_h\widehat{\otimes}A_h \to A_h, \qquad u_h : \mathbb{K} \to A_h,$$

$$\Delta_h : A_h\widehat{\otimes}A_h \to A_h, \qquad \varepsilon_h : A_h \to \mathbb{K},$$

$$S_h : A_h \to A_h$$

such that, replacing $\widehat{\otimes}$ by $\otimes$, the sextuple $(A_h, m_h, u_h, \Delta_h, \varepsilon_h, S_h)$ satisfies the associativity, coassociativity, unity, county and antipode diagrams from the Hopf algebra definitions on $\mathbb{K}$ (Definitions 1.1,1.2, 1.11).

2.24 Remark. A topological Hopf algebra A_h is not a Hopf algebra, because $A_h\widehat{\otimes}A_h$ is not equivalent to $A_h \otimes A_h$. We have that

$$A_h\widehat{\otimes}A_h \simeq A_h \otimes A_h$$

if A_h is a topologically free $\mathbb{K}$-module.

Let $(A_h, m_h, u_h, \Delta_h, \varepsilon_h, S_h)$ and $(A'_h, m'_h, u'_h, \Delta'_h, \varepsilon'_h, S'_h)$ be two topological Hopf algebras over $\mathbb{K} = \Bbbk[[\hbar]]$. A $\mathbb{K}$-morphism $f : A_h \to A'_h$ is a morphism of topological Hopf algebras if

$$f \circ m_h = m'_h \circ (f\widehat{\otimes}f), \qquad f \circ u_h = u'_h,$$

$$(f\widehat{\otimes}f) \circ \Delta_h = \Delta'_h \circ f, \qquad \varepsilon'_h \circ f = \varepsilon_h,$$

$$f \circ S_h = S'_h \circ f.$$

Finally, we introduce the notion of deformation of Hopf algebras.

2.25 Definition. Let $(A, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra over $\Bbbk$. A deformation of A is a topological Hopf algebra over $\mathbb{K} = \Bbbk[[\hbar]]$, $(A_h, m_h, u_h, \Delta_h, \varepsilon_h, S_h)$, such that

i) There exists a isomorphism $f : A_h \to A[[\hbar]]$ between the $\Bbbk[[\hbar]]$-modules A_h and $A[[\hbar]]$.

ii) The following diagrams commute:

a)

$$\begin{tikzcd} A_h\widehat{\otimes}A_h/hA_h\widehat{\otimes}A_h \arrow[r, "\widehat{m_h}"] \arrow[d, "\hat{f}\otimes\hat{f}"'] & A_h/hA_h \arrow[d, "\hat{f}"] \\ A\otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

b)

$$\begin{tikzcd} A_h\widehat{\otimes}A_h/hA_h\widehat{\otimes}A_h \arrow[d, "\hat{f}\otimes\hat{f}"'] & A_h/hA_h \arrow[l, "\widehat{\Delta_h}"'] \arrow[d, "\tilde{f}"] \\ A\otimes A & A \arrow[l, "\Delta"] \end{tikzcd}$$

where $\hat{f}$, $\widehat{m_h}$ and $\widehat{\Delta_h}$ are the natural maps obtained from the maps f, m_h and Δ_h. In this case, the condition (a) is written by $m_h \equiv m \pmod{\hbar}$ and the condition (b) is written by $\Delta_h \equiv \Delta \pmod{\hbar}$.

Finally, we introduce the notion of deformation on a Lie bialgebra (Definition 1.26).

2.26 Definition. Let $\Bbbk$ a field, $\mathbb{K} = \Bbbk[[\hbar]]$ be the formal power series in $\hbar$ with coefficients in $\Bbbk$ and $(\mathfrak{g}, \delta)$ be a $\Bbbk$-bialgebra. A deformation of the Lie bialgebra $(\mathfrak{g}, \delta)$ is a deformation $(U_h(\mathfrak{g}), m_h, u_h, \Delta_h, \varepsilon_h, S_h)$ of $\Bbbk$-Hopf algebra $(U(\mathfrak{g}), m, u, \Delta, \varepsilon, S)$ such that for all $x \in \mathfrak{g}$:

$$\delta(x) \equiv \frac{\Delta_h(a) - \Delta_h^{op}(a)}{h} \pmod{\hbar},$$

where a is an element in $U_h(\mathfrak{g})$ equivalent to x modulo $\hbar$.

2.27 Proposition. *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and let $r \in \mathfrak{g}\otimes\mathfrak{g}$ be a skew-symmetric r-matrix of $\mathfrak{g}$, i.e. r is a skew-symmetric solution of the classical Yang-Baxter equation* (CYBE). *Then, there exists a deformation $U_\hbar(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ whose classical limit is $\mathfrak{g}$ with the Lie bialgebra structure defined by r. Moreover, $U_\hbar(\mathfrak{g})$ is a triangular Hopf algebra and is isomorphic to $U(\mathfrak{g})[[\hbar]]$ as an algebra over $\mathbb{R}[[\hbar]]$.*

Proof. See [CP95, Theorem 6.2.9 page 183]. □

As a corollary we have

2.28 Theorem. *Let G be a connected and simply-connected Lie group, $\mathfrak{g}$ its Lie algebra and $U(\mathfrak{g})[[\hbar]]$ the space of formal power series in $\hbar$ with coefficients in the universal enveloping algebra of $\mathfrak{g}$. A sufficient condition for the existence of an invariant star product on G is that there is an element invertible $\mathcal{F} \in U(\mathfrak{g})[[\hbar]]\widehat{\otimes}U(\mathfrak{g})[[\hbar]]$ such that*

i) $\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\,\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\,\mathcal{F}$,

ii) $(\varepsilon \otimes \mathrm{id})\,\mathcal{F} = 1 = (\mathrm{id} \otimes \varepsilon)\,\mathcal{F}$.

Proof. See [CP95, Lemma 6.2.10 page 184]. □

2.3.3 Invariant star products on homogeneous spaces

In order to extend the Theorem 2.28 to homogeneous spaces, we recall the following definitions. We say that M is a G-homogeneous space if M is a smooth manifold and G is a Lie group which acts transitively and continuously on M. In this case, there exists H a closed Lie subgroup such that $M \simeq G/H$ as smooth manifolds. Let $A = C^\infty(M) \simeq C^\infty(G/H)$ be the algebra of real-valued smooth functions of $M \simeq G/H$. The left action of G on A is given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x) \qquad\qquad (f \in A, g \in G, x \in M).$$

A k-differential operator $\phi : A^k \to A$ on M is G-invariant if for all $g \in G$, $(f_1, \dots, f_k) \in A^k$

$$g \cdot \phi(f_1, \dots, f_k) = \phi(g \cdot f_1, \dots, g \cdot f_k).$$

Let us denote $\mathcal{D}^k_G(M)$ the space of G-invariant k-differential operators on $M \simeq G/H$.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ be the Lie algebra of H and $U(\mathfrak{g})\,\mathfrak{h}$ be the left ideal generated by $\mathfrak{h} \subset U(\mathfrak{g})$. We consider $((U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h})^{\otimes k})^{\mathfrak{h}}$ the subalgebra of $(U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h})^{\otimes k}$ of invariants with respect to $\mathfrak{h}$, that is the elements $(x_1 \otimes \dots \otimes x_k) \in (U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h})^{\otimes k}$ with $x_i \in U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h}$, which satisfied the condition

$$h \cdot (x_1 \otimes \dots \otimes x_k) = (h \cdot x_1 \otimes \ldots \otimes x_k) + \dots + (x_1 \otimes \ldots \otimes h \cdot x_k) = 0$$

for all $h \in \mathfrak{h}$. We have

2.29 Proposition. *There is a natural bijection between*

$$\mathcal{D}^k_G(M) \simeq ((U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h})^{\otimes k})^{\mathfrak{h}}.$$

Proof. See [AL05]. □

Finally, we obtain a result analogous to the Theorem 2.28.

2.30 Proposition. *Let $M \simeq G/H$ be a G-homogeneous space. Then M admits an invariant star product if there exists an invertible element $\mathcal{F} \in (U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h} \otimes U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ such that*

i) $\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\,\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\,\mathcal{F}$*,*

ii) $(\varepsilon \otimes \mathrm{id})\,\mathcal{F} = 1 = (\mathrm{id} \otimes \varepsilon)\,\mathcal{F}$*.*

Proof. See [AL05]. □

Chapter 3

Rankin-Cohen algebras

Dans ce chapitre nous étudions les crochets de Rankin-Cohen (RC). Les crochets RC ont été introduits par D. Zagier suivant les travaux de Rankin et Cohen [Ran57, Coh75, Zag94] comme solution à la question suivante : quels sont les opérateurs bi-différentiels qui préservent l'ensemble des formes modulaires ? En théorie des représentations les crochets RC peuvent être également vus comment des opérateurs de brisure de symétrie d'un produit tensoriel entre deux représentations irréductibles [DvP07, KP15a, KP15b].

3.1 *Algèbres de Rankin-Cohen. Dans cette section nous commençons l'étude de l'algèbre des formes modulaires M_*. En 1994 Zagier [Zag94] introduit la notion d'algèbre de Rankin-Cohen comme une généralisation de l'algèbre M_*. Dans la section 3.1.2 nous introduisons une version d'algèbre de RC différente de celle donnée par Zagier.*

3.1 Definition. Soit $A = \bigoplus_{n\in\mathbb{N}} A_n$ une algèbre commutative $\mathbb{Z}$-graduée telle quelle pour tout $n \in \mathbb{N}$ (ρ_n, A_n) soit une représentation irréductible de $G = \mathrm{SL}(2, \mathbb{R})$ et soit $X : A \to A$ une A_0-dérivation de A ($X(ab) = aX(b) + X(a)b$ et $X(a_0 a) = a_0 X(a)$ pour tout $a, b \in A$, $a_0 \in A_0$) nous définissons l'opérateur bilinéaire

$$\mathrm{RC}_k(a, b) = \sum_{i+j=k} (-1)^i \binom{\alpha_a + k - 1}{j} \binom{\alpha_b + k - 1}{i} X^i(a)\, X^j(b),$$

où $a \in A_{\alpha_a}, b \in A_{\alpha_b}$ sont deux éléments homogènes et k est un entier naturel. Nous disons que (A, X) est une algèbre de Rankin-Cohen si le produit

$$\mathrm{RC}(a, b) = \sum_{k\in\mathbb{N}} \mathrm{RC}_k(a, b)\hbar^k$$

est associatif dans l'anneau $A[[\hbar]]$ des séries formelles sur A en une indéterminée $\hbar$ et pour tout $k \in \mathbb{N}$, il satisfait l'équation

$$\mathrm{RC}_k(\rho_{\alpha_a}(\gamma)(a), \rho_{\alpha_b}(\gamma)(b)) = \rho_{\alpha_a+\alpha_b+\nu_k}(\gamma)(\mathrm{RC}_k(a, b)) \qquad \forall (a, b) \in A_{\alpha_a} \times A_{\alpha_b}, \gamma \in G.$$

Autrement dit, (A, X) est une algèbre de Rankin-Cohen si le produit RC est une déformation formelle et G-invariante de A.

Dans la section 3.1.3 nous introduisons la notion de "twist".

3.2 Definition. Soit $A = \bigoplus_{n\in\mathbb{N}} A_n$ une algèbre commutative $\mathbb{Z}$-graduée telle quelle pour tout $n \in \mathbb{N}$, A_n soit une représentation irréductible de $G = \mathrm{SL}(2, \mathbb{R})$, soit $X : A \to A$ une A_0-dérivation de A et soit $t : \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ une application que satisfait les conditions suivantes :

$$\text{pour tout } \alpha_1, \alpha_2 \in \mathbb{Z},\ t_0(\alpha_1, \alpha_2) = 1, \tag{C1}$$

$$\text{pour tout } \alpha_1, \alpha_2 \in \mathbb{Z}^\times,\ t_1(\alpha_1, \alpha_2) = z \neq 0, \tag{C2}$$

$$\text{pour tout } k \in \mathbb{N}, \alpha_1, \alpha_2 \in \mathbb{Z},\ t_k(\alpha_1, \alpha_2) = t_k(\alpha_2, \alpha_1). \tag{C3}$$

Nous considérons l'application bilinéaire

$$\mathrm{tRC}_k(a, b) = t_k(\alpha_a, \alpha_b)\mathrm{RC}_k(a, b),$$

où $a \in A_{\alpha_a}, b \in A_{\alpha_b}$ sont deux éléments homogènes. Nous disons que t est un RC "twist" de (A, X) si le produit

$$\mathrm{tRC}(a, b) = \sum_{k\in\mathbb{N}} t(\alpha_a, \alpha_b)\mathrm{RC}_k(a, b)\hbar^k$$

est associatif dans $A[[\hbar]]$ et satisfait l'équation

$$\mathrm{tRC}_k(\rho_{\alpha_a}(\gamma)(a), \rho_{\alpha_b}(\gamma)(b)) = \rho_{\alpha_a+\alpha_b+\nu_k}(\gamma)(\mathrm{tRC}_k(a, b)) \qquad \forall (a, b) \in A_{\alpha_a} \times A_{\alpha_b}, \gamma \in G.$$

Autrement dit, t est un RC "twist" de (A, X) si le produit tRC est une déformation formelle et G-invariante de A.

Dans [CMZ97] les auteurs montrent que la fonction $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ donnée par

$$t_k^\kappa(i, j) = \left(-\frac{1}{4}\right)^i \sum_{r\geq 0} \binom{i}{2r} \frac{\binom{-\frac{1}{2}}{r}\binom{\kappa-\frac{3}{2}}{r}\binom{\frac{1}{2}-\kappa}{r}}{\binom{-i-\frac{1}{2}}{r}\binom{-j-\frac{1}{2}}{r}\binom{k+i+j-\frac{3}{2}}{r}}. \tag{3.2}$$

détermine (un twist) une déformation formelle et $\mathrm{SL}(2, \mathbb{R})$-invariante $\sum_k t_k \mathrm{RC}_k \hbar^k$, de l'algèbre des formes modulaires de poids arbitraire. Dans le Théorème 3.21 nous décrivons des conditions nécessaires et suffisantes pour qu'une telle fonction t soit un twist.

Soit $Q_{(\alpha_1,\alpha_2,\alpha_3)}(i,j,k,r,s)$ l'entier défini par :

$$(-1)^j\binom{i+j}{i}\binom{\alpha_1+\alpha_2+\nu_{r+s}+i+j+k-1}{k}\binom{\alpha_3+i+j+k-1}{i+j}$$
$$\binom{\alpha_1+r+s-1}{s}\binom{\alpha_2+r+s-1}{r}.$$

Considérons $t:\mathbb{N}^3\to\mathbb{C}$ une application et $P_{(\alpha_1,\alpha_2,\alpha_3)}(x,y,z)$ le nombre complexe défini par :

$$\sum_{I(x,y,z)} t_{r+s}(\alpha_1,\alpha_2)t_{i+j+k}(\alpha_1+\alpha_2+\nu_{r+s},\alpha_3)Q_{(\alpha_1,\alpha_2,\alpha_3)}(i,j,k,r,s), \tag{3.3}$$

où $I(x,y,z)$ est l'ensemble de 5-uplets (i,j,k,r,s) tels que $i+r=x$, $j+s=y$, $k=z$. Nous dirons qu'une application $t:\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$ satisfait la condition (Ct) si pour tout $x,y,z\in\mathbb{N}$ et $\alpha_1,\alpha_2,\alpha_3\in\operatorname{supp}(A)\subset\mathbb{Z}$, on a

$$P_{(\alpha_1,\alpha_2,\alpha_3)}(x,y,z)=(-1)^yP_{(\alpha_3,\alpha_2,\alpha_1)}(z,y,x). \tag{Ct}$$

Théorème. *Soit $A=\bigoplus_{n\in\mathbb{N}}A_n$ une algèbre commutative $\mathbb{Z}$-graduée telle que pour tout $n\in\mathbb{N}$, A_n soit une représentation irréductible de $G=\mathrm{SL}(2,\mathbb{R})$, soit $X:A\to A$ une A_0-dérivation de A et soit $t:\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$ une application qui satisfait les conditions suivantes :*

$$\text{pour tout } \alpha_1,\alpha_2\in\mathbb{Z},\ t_0(\alpha_1,\alpha_2)=1, \tag{C1}$$

$$\text{pour tout } \alpha_1,\alpha_2\in\mathbb{Z}^{\times},\ t_1(\alpha_1,\alpha_2)=z\neq 0, \tag{C2}$$

$$\text{pour tout } k\in\mathbb{N},\alpha_1,\alpha_2\in\mathbb{Z},\ t_k(\alpha_1,\alpha_2)=t_k(\alpha_2,\alpha_1). \tag{C3}$$

Soit $\{RC_k:A\times A\to A|k\in\mathbb{N}\}$ une famille de crochets de Rankin-Cohen d'ordre k et degré ν_k. Une condition suffisante pour que l'application $t:\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$ soit un RC twist de (A,X) est donnée par

$$P_{(\alpha_1,\alpha_2,\alpha_3)}(x,y,z)=(-1)^yP_{(\alpha_3,\alpha_2,\alpha_1)}(z,y,x), \tag{Ct}$$

pour tout $x,y,z\in\mathbb{N}$ et $\alpha_1,\alpha_2,\alpha_3\in\operatorname{supp}(A)\subset\mathbb{Z}$.

3.2 L'algèbre de Bergman. Dans cette section nous donnons un exemple d'une algèbre de Rankin-Cohen. Il s'agit de l'algèbre de Bergman pondérée.

Soient $n \in 2\,\mathbb{N}$, $\Pi = \{z = x + \mathbf{i}\,y \in \mathbb{C}\,|y > 0\}$ *le demi-plan de Poincaré et* $\mathcal{O}(\Pi) = \{f : \Pi \to \mathbb{C}\,|f \text{ is holomorphic}\}$ *l'espace de fonctions holomorphes dans* Π. *Nous considérons*

$$\mathscr{H}^2_n(\Pi) := \mathrm{L}^2(\Pi, d\mu_n) \cap \mathcal{O}(\Pi)$$

l'espace de Hilbert des fonctions holomorphes de carré sommable dans Π *par rapport à la mesure*

$$d\mu_n(z) = y^{n-2}dxdy \qquad\qquad (z = x + \mathbf{i}\,y \in \Pi)$$

Le produit scalaire dans $\mathscr{H}^2_n(\Pi)$ *est donné par*

$$(f, g)_n = \int_\Pi f(z)\overline{g(z)}d\mu_n(z).$$

Nous dirons que $\mathscr{H}^2_n(\Pi)$ *est l'espace de Bergman de poids* $n \in 2\,\mathbb{N}$. *Les représentations de la série discrète holomorphe de* G, $(\mathscr{H}^2_n(\Pi), \rho_n)$ *sont définies par*

$$(\rho_n(\gamma^{-1})f)(z) = (cz + d)^{-n} f\left(\frac{az + b}{cz + d}\right), \tag{3.5}$$

pour tout $z \in \Pi, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$.

Soit $\mathscr{H}^+ = \bigoplus_{n\in\mathbb{N}} \mathscr{H}^2_{2n}(\Pi)$ *la somme directe de* $\mathscr{H}^2_{2n}(\Pi)$ $(\mathscr{H}^2_0(\Pi) = \mathbb{C})$. *On munit l'espace de Hilbert* $\mathscr{H}^+$ *de l'action* $\rho = \oplus_{n\in\mathbb{N}}\rho_{2n}$ (ρ_0 *est l'action triviale dans* $\mathbb{C}$). *Plus précisément, si* $f = (f_{2n})_{n\in\mathbb{N}}$ *est un élément dans* $\mathscr{H}^+$ *avec* $f_n \in \mathscr{H}^2_{2n}(\Pi)$ *alors pour tout* $\gamma \in \mathrm{SL}(2, \mathbb{R})$ *la représentation* ρ *est donnée par* $\rho(\gamma)(f) = (\rho_{2n}(f_{2n}))_{n\in\mathbb{N}}$. *Par conséquence,* $\rho = (\rho, \mathscr{H}^+)$ *est une représentation unitaire de* $\mathrm{SL}(2, \mathbb{R})$. *En plus* $\mathscr{H}^+$ *est une algèbre commutative avec le produit usuel. Nous l'appelons l'algèbre de Bergman à l'espace* $\mathscr{H}^+$.

Ensuite, nous montrons que pour tout $\kappa \in \mathbb{C}$ *la fonction* $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ *est aussi un twist pour l'algèbre de Bergman :*

Théorème. *Pour tout* $\kappa \in \mathbb{C}$ *la fonction* $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ *définie par Eq.* (3.2) *est un twist de* $\mathscr{H}^+$. *En particulier, si* $\kappa = \frac{1}{2}$ *ou* $\kappa = \frac{3}{2}$ *nous retrouvons la déformation* $\mathrm{SL}(2, \mathbb{R})$-*équivariante*

$$f \star_{\mathrm{RC}} g = \sum_{k\in\mathbb{N}} \mathrm{RC}_k(f, g)\hbar^k \qquad\qquad (f, g \in \mathscr{H}^+).$$

Autrement dit l'espace $\mathscr{H}^+$ *est une algèbre de Rankin-Cohen.*

The Rankin-Cohen bracket is a bidifferential operator that has been extensively studied in recent years. It has many applications in various areas as theory of modular and quasimodular forms, Ramanujan and Chazy differentials equations [DR14, CL11, MR09, Zag94], covariant quantization [BTY07, CMZ97, DvP07, OS00, Pev12, Pev08, UU96].

Throughout this chapter, we assume that A is a complex associative and commutative algebra with unity 1.

3.1 Rankin-Cohen algebras

In this chapter we study the notion of Rankin-Cohen algebra as a "generalization" of algebra of modular forms. The definition of Rankin-Cohen algebra is different from the one given by Zagier [Zag94], however as in Zagier's definition the number of conditions is infinite (see Definition 3.10).

We begin this chapter with the remainder on modular and quasimodular forms.

3.1.1 Modular and quasimodular forms

Consider the modular subgroup $\mathrm{SL}(2,\mathbb{Z}) \subseteq \mathrm{SL}(2,\mathbb{R})$ whose matrices entries are integers numbers and Π the Poincaré upper half-plane

$$\Pi = \{z = x + \mathbf{i}\, y \in \mathbb{C} \,|y > 0\}.$$

A modular form of weight $k \in 2\,\mathbb{N}$, $k \geq 4$, is a holomorphic function on Π satisfying

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)$$

for any $z \in \Pi$ and any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and having Fourier expansion

$$f(z) = \sum_{n\geq 0} \hat{f}(n) \exp(2\pi\, \mathbf{i}\, nz).$$

We denote by M_k the finite dimensional space of modular forms of weight k. The algebra of modular forms is defined as the graded algebra

$$M_* = \bigoplus_{k\geq 2} M_{2k}. \tag{3.6}$$

We recall also the definition of Eisentsein series. For $k \geq 2$, even, the Eisentsein series of weight k is given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \in \mathbb{N}^\times} \sigma_{k-1}(n) \exp(2\pi \mathbf{i}\, nz), \tag{3.7}$$

where the rational numbers B_k are defined by their exponential generating series

$$\sum_{n \in \mathbb{N}} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1},$$

and σ_{k-1} is the divisor function defined by

$$\sigma_{k-1}(n) = \sum_{\substack{d|n \\ d>0}} d^{k-1} \qquad (n \in \mathbb{N}^\times).$$

It is well-known (see [Zag94, DR14]) that the algebra of modular forms M_* is an polynomial algebra in the two algebraically independent Eisentsein series E_4 and E_6,

$$M_* = \mathbb{C}[E_4, E_6]. \tag{3.8}$$

Let X be the normalised complex derivation

$$X = \frac{1}{2\pi \mathbf{i}} \frac{\partial}{\partial z}.$$

The algebra M_* is not stable under X. However, we can extend the notion of modular form in such a way it becomes stable under complex derivation. More precisely, we consider the algebra of quasimodular forms $M_*^{\leq \infty}$, that is, the polynomial algebra generated by the three algebraically independent Eisentsein series E_2, E_4 and E_6:

$$M_*^{\leq \infty} = \mathbb{C}[E_2, E_4, E_6] = M_*[E_2]. \tag{3.9}$$

The action of X on $M_*^{\leq \infty}$ is given by the Ramanujan differential equations:

$$\begin{aligned} X(E_2) &= \frac{1}{12}\left(E_2^2 - E_4\right), \\ X(E_4) &= \frac{1}{3}\left(E_4 E_2 - E_6\right), \\ X(E_6) &= \frac{1}{2}\left(E_6 E_2 - E_4^2\right). \end{aligned}$$

Another way to avoid this problem consists in the change of differential operators. Let $f \in M_{2n}$, $g \in M_{2m}$ be two modular forms and k be a positive integer. The Rankin-Cohen[1] bracket $\mathrm{RC}_k(f, g)$ of order k between f and g is given by

$$\mathrm{RC}_k(f, g) = \sum_{r=0}^{k} (-1)^r \binom{2n + k - 1}{k - r} \binom{2m + k - 1}{r} f^{(r)}\, g^{(k-r)}, \tag{3.10}$$

[1] We also say the classical Rankin-Cohen bracket of order k.

where $f^{(r)}$ means $X^r(f)$.

Then, one may easily show that $\mathrm{RC}_k(f,g) \in M_{2(n+m+k)}$ is again a modular form.

3.1.2 Rankin-Cohen brackets

In this section, we will introduce a generalization of the Rankin-Cohen bracket given in Eq. (3.10).

3.3 Definition. We say that (A, X) is a quasi-RC algebra if

i) A is $\mathbb{Z}$-graded, that is it has a sum direct decomposition of complex vector spaces

$$A = \bigoplus_{n\in\mathbb{Z}} A_n, \tag{3.11}$$

such that for any $(i,j) \in \mathbb{Z}\times\mathbb{Z}$, we have $A_iA_j \subseteq A_{i+j}$. We denote by $\mathrm{supp}(A) = \{n \in \mathbb{Z}\,|A_n \neq 0\}$ the support of A.

ii) each homogeneous component A_n of decomposition Eq. (3.11) is an irreducible representation of $G = \mathrm{SL}(2,\mathbb{R})$ denoted ρ_n. Hence, $\rho = \bigoplus_{n\in\mathbb{N}} \rho_n : G \to \mathrm{GL}(A)$ is a representation of G.

iii) X is a derivation, that is a linear map $X : A \to A$ such that for any $a, b \in A$,

$$X(ab) = aX(b) + X(a)b.$$

iv) $X : A \to A$ is an A_0-map, that is for any $a_0 \in A_0$ and $a \in A$,

$$X(a_0a) = a_0X(a).$$

3.4 Remark. It is clear that A_0 is a subalgebra of A. Besides, the derivation $X : A \to A$ is an A_0-map if and only if $A_0 \subseteq \mathrm{Ker}\, X$.

3.5 Definition. Let $k \in \mathbb{N}$ and (A, X) be a quasi-RC algebra. The Rankin-Cohen bracket of order k relative to (A, X) is the bilinear map $\mathrm{RC}_k : A \times A \to A$ defined by

$$\mathrm{RC}_k(a,b) = \sum_{i+j=k} (-1)^i \binom{\alpha_a + k - 1}{j}\binom{\alpha_b + k - 1}{i} X^i(a)\, X^j(b), \tag{3.12}$$

where $a, b \in A$ are two homogeneous elements of degree[2] $\deg(a) = \alpha_a$ and $\deg(b) = \alpha_b$ respectively.

We list some proprieties of the Rankin-Cohen bracket relative to (A, X) (or RC bracket if there is no confusion).

3.6 Proposition. *Let (A, X) be a quasi-RC algebra and $\mathrm{RC}_k : A \times A \to A$ be the RC bracket defined in (3.12). We have that*

i) RC_0 is the original multiplication on A.

ii) For all $a, b \in A$, $\mathrm{RC}_k(a, b) = (-1)^k \,\mathrm{RC}_k(b, a)$. In particular,

$$\{a, b\} := \frac{1}{2}(\mathrm{RC}_1(a, b) - \mathrm{RC}_1(b, a)) = \mathrm{RC}_1(a, b).$$

iii) $\mathrm{RC}_k : A \times A \to A$ is an A_0-bilinear map.

iv) For any $x \in A$ and $k \in \mathbb{N}^\times$, we have $A_0 \subseteq \mathrm{Ker}\,\mathrm{RC}_k(x, -)$.

Proof. (i) and (ii) are evident. To show (iii) we take x, a homogeneous in A and $a_0 \in A_0$, then

$$\begin{aligned}
\mathrm{RC}_k(x, a_0 a) &= \sum_{i+j=k} (-1)^i \binom{\alpha_x + k - 1}{j} \binom{\alpha_{a_0 a} + k - 1}{i} X^i(x)\, X^j(a_0 a) \\
&= a_0 \sum_{i+j=k} (-1)^i \binom{\alpha_x + k - 1}{j} \binom{\alpha_a + k - 1}{i} X^i(x)\, X^j(a) \\
&= a_0 \mathrm{RC}_k(x, a).
\end{aligned}$$

Now, if we take $a = 1$ and $k \geq 1$ in the previous equation, we obtain

$$\begin{aligned}
\mathrm{RC}_k(x, a_0) &= \sum_{i+j=k} (-1)^i \binom{\alpha_x + k - 1}{j} \binom{k - 1}{i} X^i(x)\, X^j(a_0) \\
&= (-1)^k \binom{k - 1}{k} X^k(x) a_0 \\
&= 0.
\end{aligned}$$

so we have (iv). $\square$

[2] In general, we denote $\deg(x) = \alpha_x \in \mathbb{Z}$ the degree of x an non-zero homogeneous element of A, *i.e.* $x \in A_{\alpha_x}$.

3.7 Definition. Let (A, X) be a quasi-RC algebra and $\{\mathrm{RC}_k : A \times A \to A | k \in \mathbb{N}\}$ be the family of bilinear maps given by the Eq. (3.12).

i) We say that the RC bracket $\mathrm{RC}_k : A \times A \to A$ of order k is homogeneous of degree $\nu_k \in \mathbb{N}$ if for any a, b homogeneous in A, one has that $\mathrm{RC}_k(a, b)$ is also homogeneous and

$$\deg(\mathrm{RC}_k(a, b)) = \deg(a) + \deg(b) + \nu_k.$$

Here $A \otimes A = \oplus_n \tilde{A}_n$ is provided with the $\mathbb{Z}$-graduation $\tilde{A}_n = \oplus_{i+j=n} A_i \otimes A_j$.

ii) We say that the homogeneous RC bracket $\mathrm{RC}_k : A \times A \to A$ of order k and degree $\nu_k \in \mathbb{N}$ is G-equivariant if for any $\gamma \in G$ and any a, b homogeneous in A, we have

$$\mathrm{RC}_k(\rho_{\alpha_a}(\gamma)(a), \rho_{\alpha_b}(\gamma)(b)) = \rho_{\alpha_a + \alpha_b + \nu_k}(\gamma)(\mathrm{RC}_k(a, b)).$$

iii) We say that the quasi-RC algebra (A, X) is G-homogeneous if for all $k \in \mathbb{N}$ the RC bracket $\mathrm{RC}_k : A \times A \to A$ of order k relative to (A, X) is G-equivariant and homogeneous of degree ν_k.

3.8 Proposition. *Let (A, X) be a quasi-RC algebra. If the RC bracket $\mathrm{RC}_1 : A \times A \to A$ of order 1 relative to (A, X) is G-homogeneous, then*

i) For all $x \in A$, $\mathrm{RC}_1(x, -)$ is a derivation.

ii) RC_1 satisfies the Jacobi identity.

In other words, RC_1 is a Poisson bracket on A (Definition 2.2).

Proof. Let x, a_1, a_2, a_3 be elements homogeneous in A. For short we denote $\{a_1, a_2\} = \mathrm{RC}_1(a_1, a_2)$, $X(a_1) = a_1'$ and $\alpha_{a_i} = \alpha_i$, $i = 1, 2, 3$.
We have

$$\begin{aligned} \{x, a_1 a_2\} &= \alpha_x \, x a_1' a_2 + \alpha_x \, x a_1 a_2' - \alpha_1 \, x' a_1 a_2 - \alpha_2 \, x' a_1 a_2 \\ &= \{x, a_1\} a_2 + a_1 \{x, a_2\}, \end{aligned} \tag{3.13}$$

which implies $\{x,-\}$ is a derivation. To show (ii), we take

$$\begin{aligned}
\{\{a_1,a_2\},a_3\} &= \alpha_{\{a_1,a_2\}}\alpha_1\ a_1a_2'a_3' - \alpha_{\{a_1,a_2\}}\alpha_2\ a_1'a_2a_3' \\
&\quad -\alpha_1\alpha_3\ a_1'a_2'a_3 - \alpha_1\alpha_3\ a_1a_2''a_3 \\
&\quad +\alpha_2\alpha_3\ a_1''a_2a_3 + \alpha_2\alpha_3\ a_1'a_2'a_3, \\
\{\{a_2,a_3\},a_1\} &= \alpha_{\{a_2,a_3\}}\alpha_2\ a_1'a_2a_3' - \alpha_{\{a_2,a_3\}}\alpha_3\ a_1'a_2'a_3 \\
&\quad -\alpha_1\alpha_2\ a_1a_2'a_3' - \alpha_1\alpha_2\ a_1a_2a_3'' \\
&\quad +\alpha_1\alpha_3\ a_1a_2''a_3 + \alpha_1\alpha_3\ a_1a_2'a_3',
\end{aligned}$$

and

$$\begin{aligned}
\{\{a_3,a_1\},a_2\} &= \alpha_{\{a_3,a_1\}}\alpha_3\ a_1'a_2'a_3 - \alpha_{\{a_3,a_1\}}\alpha_1\ a_1a_2'a_3' \\
&\quad -\alpha_2\alpha_3\ a_1'a_2a_3' - \alpha_2\alpha_3\ a_1''a_2a_3 \\
&\quad +\alpha_1\alpha_2\ a_1a_2a_3'' + \alpha_1\alpha_2\ a_1'a_2a_3'
\end{aligned}$$

The coefficient of term $a_1a_2'a_3'$ in the expression

$$\{\{a_1,a_2\},a_3\} + \{\{a_2,a_3\},a_1\} + \{\{a_3,a_1\},a_2\} \tag{3.14}$$

is given by $\alpha_1(\alpha_{\{a_1,a_2\}} - \alpha_{\{a_3,a_1\}} + \alpha_3 - \alpha_2) = 0$. Then the coefficient of term $a_1a_2'a_3'$ is zero. Analogously, we can show that the coefficients of others terms in (3.14) are also zero. Thus the G-homogeneous RC_1 bracket of order 1 satisfies the Jacobi identity. □

3.9 Remark. By proposition 3.8, we have that if (A,X) is a quasi-RC algebra and the RC bracket $\mathrm{RC}_1 : A\otimes A \to A$ of order 1 relative to (A,X) is homogeneous (not necessarily G-homogeneous), then (A,RC_1) is a Poisson algebra. Thus, according to Kontsevich [Kon03], we know that (A,RC_1) admits a formal deformation $\Pi = (\Pi_n)$ such that Π_0 is the original multiplication on A and $\Pi_1 = \mathrm{RC}_1$. We can firstly ask if all Π_n, $n\in\mathbb{N}$, are G-equivariant ?. Secondly, we ask if the map Π_n $(n\in\mathbb{N})$ and the RC bracket of order n are related in any way ?

Aspirated in the Zagier's [Zag94] Rankin-Cohen algebra definition we introduce the next notion.

3.10 Definition. Let (A,X) be a G-homogeneous quasi-RC algebra equipped with $\{\mathrm{RC}_k\,|k\in\mathbb{N}\}$ the family of G-homogeneous RC brackets relatives to (A,X). We say that (A,X) is a

Rankin-Cohen algebra if

$$\mathrm{RC} = \sum_{k\in\mathbb{N}} \mathrm{RC}_k\, \hbar^k$$

is an associative product on $A[[\hbar]]$ the ring of formal power series in a variable $\hbar$ with coefficients in A. In other words, RC defined a G-invariant formal deformation of A.

3.11 Remark. Let (A, X) be a quasi-RC algebra. If X is a $\mathbb{Z}$-graded map of degree ν ($X : A_* \to A_{*+\nu}$,) then the RC bracket $\mathrm{RC}_k : A \otimes A \to A$ of order k relative to (A, X) is a $\mathbb{Z}$-graded map of degree $\nu_k = k\nu$. If $\nu = 2$ and $A_0 = \mathbb{C}$, then $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a particular case of Zagier's notion [Zag94] of a standard Rankin-Cohen algebra where each A_n is a G-module irreducible.

3.1.3 Rankin-Cohen twist

From now on until the end of chapter we suppose that (A, X) is a quasi-RC algebra and RC_k is the RC bracket of order k relative to (A, X).

Let's "deform" the Rankin-Cohen bracket definition. For that, we consider the following function

$$\begin{aligned} t : \mathbb{N}\times\mathbb{Z}\times\mathbb{Z} &\to \mathbb{C} \\ (k, \alpha_1, \alpha_2) &\mapsto t_k(\alpha_1, \alpha_2), \end{aligned}$$

which satisfies the following conditions

$$\text{For all } \ \alpha_1, \alpha_2 \in \mathbb{Z},\ t_0(\alpha_1, \alpha_2) = 1, \tag{C1}$$

$$\text{For all } \ \alpha_1, \alpha_2 \in \mathbb{Z}^\times,\ t_1(\alpha_1, \alpha_2) = z \neq 0, \tag{C2}$$

$$\text{For all } k \in \mathbb{N}, \alpha_1, \alpha_2 \in \mathbb{Z},\ t_k(\alpha_1, \alpha_2) = t_k(\alpha_2, \alpha_1). \tag{C3}$$

3.12 Definition. Let (A, X) be a quasi-RC algebra and $t : \mathbb{N}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be a function which satisfies (C1)-(C3). The t-RC bracket of order k relative to (A, X, t) is the bilinear map $\mathrm{tRC}_k : A \times A \to A$ given by

$$\mathrm{tRC}_k(a, b) = t_k(\alpha_a, \alpha_b)\mathrm{RC}_k(a, b), \tag{3.16}$$

where a and b are homogeneous elements of degree α_a and α_b.

3.13 Proposition. *The t-RC bracket* $\mathrm{tRC}_k : A \times A \to A$ *of order* k *relative to* (A, X, t) *defined in Eq.* (3.16) *satisfies the conditions* $(i) - (iv)$ *of the Proposition 3.6. Moreover, if the RC bracket of order* k *relative to* (A, X) *is* G*-homogeneous, then* tRC_k *is also* G*-homogeneous.*

Proof. Straightforward. □

3.14 Proposition. *Let* (A, X) *be a quasi-RC algebra. If the RC bracket of order 1 relative to* (A, X) *is homogeneous, then the t-RC bracket* $\mathrm{tRC}_1 : A \times A \to A$ *of order* 1 *relative to* (A, X, t) *is a Poisson bracket on* A.

Proof. Straightforward. □

3.15 Definition. Let (A, X) be a G-homogeneous quasi-RC algebra and $t : \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a function which satisfies (C1)-(C3). We say (A, X, t) is a Rankin-Cohen twist algebra (or t-RC algebra) if the family $\{\mathrm{tRC}_k : A \times A \to A | k \in \mathbb{N}\}$ gives rise to a G-invariant formal deformation of A. In this case, we say that $t : \mathbb{N} \times \mathrm{supp}(A) \times \mathrm{supp}(A) \to \mathbb{C}$ is a RC twist for (A, X).

3.16 Remark. We have that (A, X) is a RC algebra if and only if $t \equiv 1$ is a RC twist for (A, X).

We want to interpret the fact that t defines a RC twist in terms of combinatorial conditions.

Let $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$, $(x, y, z) \in \mathbb{N}^3$, $(i, j, k, r, s) \in \mathbb{N}^5$, $\{\nu_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Z}$ and $t : \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a map which satisfies (C1)-(C3). We denote by $Q_{(\alpha_1,\alpha_2,\alpha_3)}(i, j, k, r, s)$ the following integer

$$(-1)^j \binom{i+j}{i} \binom{\alpha_1 + \alpha_2 + \nu_{r+s} + i + j + k - 1}{k} \binom{\alpha_3 + i + j + k - 1}{i + j}$$
$$\binom{\alpha_1 + r + s - 1}{s} \binom{\alpha_2 + r + s - 1}{r},$$

and by $P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z)$ the complex number

$$\sum_{I(x,y,z)} t_{r+s}(\alpha_1, \alpha_2) t_{i+j+k}(\alpha_1 + \alpha_2 + \nu_{r+s}, \alpha_3) Q_{(\alpha_1,\alpha_2,\alpha_3)}(i, j, k, r, s), \tag{3.17}$$

where $I(x, y, z)$ is the set of 5-tuples (i, j, k, r, s) such that $i + r = x$, $j + s = y$, $k = z$.

3.17 Proposition. *Let (A, X) be a G-homogeneous quasi-RC algebra equipped with $\{RC_k : A \times A \to A | k \in \mathbb{N}\}$ the family of Rankin-Cohen brackets of order k and degree ν_k. Then a sufficient condition for a map $t : \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ which satisfies (C1)-(C3) to be a RC twist for (A, X) is that for all $x, y, z \in \mathbb{N}$ and $\alpha_1, \alpha_2, \alpha_3 \in \operatorname{supp}(A) \subset \mathbb{Z}$,*

$$P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z) = (-1)^y P_{(\alpha_3,\alpha_2,\alpha_1)}(z, y, x). \tag{Ct}$$

Proof. For short we denote $tRC_k(a, b) = t_k(\alpha_a, \alpha_b)RC_k(a, b) = a \star_k b$ and $X^i(a) = a^{(i)}$. We have that the product $\sum_k tRC_k\hbar^k$ is associative if and only if for all $a, b, c \in A$ the following condition holds

$$\sum_{n+m=p} (a \star_m b) \star_n c = \sum_{n+m=p} a \star_n (b \star_m c), \qquad (\forall p \in \mathbb{N}) \tag{3.18}$$

In particular if a, b, c are homogeneous in A we have that $(a \star_m b) \star_n c$ is

$$\sum_{I(n,m)} (-1)^{i+r} t_m(\alpha_a, \alpha_b) t_n(\alpha_a + \alpha_b + \nu_m, \alpha_c) Q_{(\alpha_a,\alpha_b,\alpha_c)}(i, j, k, r, s) a^{(i+r)} b^{(j+s)} h^{(k)}$$

where $I(n, m)$ is the set of $(i, j, k, r, s) \in \mathbb{N}^5$ such that $i + j + k = n$ and $r + s = m$. For any $p \in \mathbb{N}$, one has $\sum_{n+m=p}(a \star_m b) \star_n c$ is

$$\begin{aligned}\sum_{i+j+k+r+s=p} & (-1)^{i+r} P_{(\alpha_a,\alpha_b,\alpha_c)}(i + r, j + s, k) a^{(i+r)} b^{(j+s)} c^{(k)} \\ &= \sum_{x+y+z=p} (-1)^x P_{(\alpha_a,\alpha_b,\alpha_c)}(x, y, z) a^{(x)} b^{(y)} c^{(z)}\end{aligned}$$

Analogously, for any $p \in \mathbb{N}$, one has $\sum_{n+m=p} a \star_n (b \star_m c) = \sum_{n+m=p}(-1)^{n+m}(c \star_m b) \star_n a$ is

$$\begin{aligned}\sum_{i+j+k+r+s=p} & (-1)^{i+j+k+r+s}(-1)^k P_{(\alpha_c,\alpha_b,\alpha_a)}(i + r, j + s, k) c^{(i+r)} b^{(j+s)} a^{(k)} \\ &= \sum_{x+y+z=p} (-1)^{x+y} P_{(\alpha_c,\alpha_b,\alpha_a)}(z, y, x) a^{(x)} b^{(y)} c^{(z)}.\end{aligned}$$

Now, it is clear that if t satisfies the condition (Ct), then $\{\mathrm{tRC}_k : A \times A \to A\}_{k\in\mathbb{N}}$ given by (3.16) is a G-invariant formal deformation of A. □

3.18 Remark. One may wonder if the reciprocal statement in Proposition 3.17 holds.

3.19 Definition. Let A be a complex associative and commutative $\mathbb{Z}$-graded algebra with unity 1,

$$A = \bigoplus_{n\in\mathbb{Z}} A_n,$$

and let X be a derivation on A. We say that (A, X) satisfies the condition (CA) if for any $p \in \mathbb{N}$ and any $(\alpha_1, \alpha_2, \alpha_3) \in \operatorname{supp}(A)^3$ there exist a_1, a_2, a_3 homogeneous in A of degree α_1, α_2 and α_3 respectively such that the vector space generated by the set $\{a_1^{(x)} a_2^{(y)} a_3^{(z)} | x+y+z = p\}$ is of the maximal dimension, that is,

$$\dim_{\mathbb{C}} \operatorname{Span}\{a_1^{(x)} a_2^{(y)} a_3^{(z)} | x+y+z = p\} = \frac{(p+1)(p+2)}{2}. \tag{CA}$$

3.20 Remark. It is clear that if (A, X) satisfies the condition (CA), then A is infinite dimensional. Moreover, if X is a homogeneous graded derivation $(X : A_* \to A_{*+l})$ then each homogeneous component A_n is infinite dimensional.

3.21 Theorem. *Let (A, X) be a G-homogeneous quasi-RC algebra with homogeneous components of infinite dimension and $\{RC_k : A \times A \to A | k \in \mathbb{N}\}$ be the family of G-homogeneous Rankin-Cohen brackets of order k and degree ν_k relative to (A, X). If (A, X) satisfies the condition (CA), then a sufficient and necessary condition for a map $t : \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ which satisfies (C1)-(C3) to be a* RC *twist for A is that for all $x, y, z \in \mathbb{N}$ and $\alpha_1, \alpha_2, \alpha_3 \in \operatorname{supp}(A) \subseteq \mathbb{Z}$,*

$$P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z) = (-1)^y P_{(\alpha_3,\alpha_2,\alpha_1)}(z, y, x).$$

Proof. It only remains to show the necessary condition. For short we denote $tRC_k(a, b) = t_k(\alpha_a, \alpha_b)RC_k(a, b) = a \star_k b$ and $X^i(a) = a^{(i)}$. Let $p \in \mathbb{N}$ and $(\alpha_1, \alpha_2, \alpha_3) \in \operatorname{supp}(A)^3$, hence (A, X) satisfies the condition (CA) there is $a_1, a_2\ a_3$ homogeneous in A of degree α_1, α_2 and α_3 respectively such that the vector space generated by the set $\{a_1^{(x)} a_2^{(y)} a_3^{(z)} | x+y+z = p\}$ is of the maximal dimension.

By Proposition 3.17 if the product $\sum_k tRC_k \hbar^k$ is associative, then for any $n \in \mathbb{N}$ and any a, b, c homogeneous in A we have

$$\sum_{x+y+z=n} (-1)^x P_{(\alpha_a,\alpha_b,\alpha_c)}(x, y, z) a^{(x)} b^{(y)} c^{(z)} = \sum_{x+y+z=n} (-1)^{x+y} P_{(\alpha_c,\alpha_b,\alpha_a)}(z, y, x) a^{(x)} b^{(y)} c^{(z)}.$$

In particular, for $p \in \mathbb{N}$ and (a_1, a_2, a_3) we have

$$\sum_{x+y+z=p} (-1)^x P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z) a_1^{(x)} a_2^{(y)} a_3^{(z)} = \sum_{x+y+z=p} (-1)^{x+y} P_{(\alpha_3,\alpha_2,\alpha_1)}(z, y, x) a_1^{(x)} a_2^{(y)} a_3^{(z)},$$

since the set $\{a_1^{(x)} a_2^{(y)} a_3^{(z)} | x+y+z = p\}$ is linearly independent, we have that

$$(-1)^x P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z) = (-1)^{x+y} P_{(\alpha_3,\alpha_2,\alpha_1)}(z, y, x).$$

□

The following result is due to Cohen-Manin-Zagier [CMZ97].

3.22 Proposition. *Let $\kappa \in \mathbb{C}$ and $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ the function defined by*

$$t_k^\kappa(i,j) = \left(-\frac{1}{4}\right)^i \sum_{r\geq 0} \binom{i}{2r} \frac{\binom{-\frac{1}{2}}{r}\binom{\kappa-\frac{3}{2}}{r}\binom{\frac{1}{2}-\kappa}{r}}{\binom{-i-\frac{1}{2}}{r}\binom{-j-\frac{1}{2}}{r}\binom{k+i+j-\frac{3}{2}}{r}}. \tag{3.19}$$

Then the multiplication $f \star_\kappa g$ on M_* defined by

$$f \star_\kappa g = \sum_{k\in\mathbb{N}} t_k^\kappa(i,j)\mathrm{RC}_k(f,g) \qquad (f \in M_{2i}, g \in M_{2j})$$

is associative.

Proof. See [CMZ97, Theorem 1, page 10] □

3.23 Corollary. *For any $\kappa \in \mathbb{C}$, the functions t^κ satisfy the* (Ct) *condition on M_*.*

Proof. We use the notations of Proposition 3.17. According to Proposition 3.22 t^κ gives rise to an associative product on M_*, then for any $(a,b,c) \in M_{\alpha_1}\times M_{\alpha_2}\times M_{\alpha_3}$ with $\alpha_1, \alpha_2, \alpha_3 \in 2\,\mathbb{N}$ we have that $f_p(a,b,c) = 0$, where

$$\begin{aligned} f_p(a,b,c) = &\sum_{x+y+z=p} (-1)^x P_{(\alpha_1,\alpha_2,\alpha_3)}(x,y,z) a^{(x)} b^{(y)} c^{(z)} \\ &- \sum_{x+y+z=p} (-1)^{x+y} P_{(\alpha_3,\alpha_b,\alpha_1)}(z,y,x) a^{(x)} b^{(y)} c^{(z)}. \end{aligned}$$

We want to show that the coefficients of $X^x(a)X^y(b)X^z(c)$ in the expansion of $f_p(a,b,c)$ equal zero. We proceed by contradiction. We assume that there is $(a_0,b_0,c_0) \in M_{\alpha_1}\times M_{\alpha_2}\times M_{\alpha_3}$ and (x_0,y_0,z_0), $(x_0+y_0+z_0 = p)$, such that

$$P_{(\alpha_1,\alpha_2,\alpha_3)}(x_0,y_0,z_0) \neq P_{(\alpha_3,\alpha_b,\alpha_1)}(z_0,y_0,x_0).$$

The functions a_0, b_0, c_0 are invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$, then in a small open set contained in the fundamental domain there is $(a_1,b_1,c_1) \in M_{\alpha_1}\times M_{\alpha_2}\times M_{\alpha_3}$ such that

$$\begin{aligned} &X^{(k)}(a_1) \neq 0 \quad \text{if } 0 \leq k \leq x_0 \qquad X^{(k)}(a_1) = 0 \quad \text{if } k > x_0 \\ &X^{(k)}(b_1) \neq 0 \quad \text{if } 0 \leq k \leq y_0 \qquad X^{(k)}(b_1) = 0 \quad \text{if } k > y_0 \\ &X^{(k)}(c_1) \neq 0 \quad \text{if } 0 \leq k \leq z_0 \qquad X^{(k)}(c_1) = 0 \quad \text{if } k > z_0. \end{aligned}$$

But

$$0 = f_p(a_1,b_1,c_1) = \left(P_{(\alpha_1,\alpha_2,\alpha_3)}(x_0,y_0,z_0) - P_{(\alpha_3,\alpha_b,\alpha_1)}(z_0,y_0,x_0)\right) a_1^{(x_0)} b_1^{(y_0)} c_1^{(z_0)}.$$

This contradiction finishes the proof. □

3.2 The Bergman algebra

In this section we give an explicit example of Rankin-Cohen algebra.

3.2.1 Reproducing kernel and weighted Bergman spaces

Let D be a domain (connected and open subset) in $\mathbb{C}^n$ and $\mathcal{H}(D)$ be the Hilbert space of holomorphic functions on D, equipped with the standard scalar product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}d\mu(z) \qquad (f, g \in \mathcal{H}(D))$$

where μ denote the Lebesgue measure on $\mathbb{C}^n$.

3.24 Definition. Let H be a closed linear subspace of $\mathcal{H}(D)$. A complex-valued function $K : D \times D \to \mathbb{C}$ is a reproducing kernel for H if for any $w \in D$ the function $K_w : D \to \mathbb{C}$ given by $K_w(z) = K(z, w)$ is a holomorphic on D and

$$f(w) = \langle f, K_w\rangle$$

for all $f \in H$ and $w \in D$.

3.25 Proposition. *The Hilbert space H has a reproducing kernel K if and only if, for all $w \in D$, the linear map $H \to \mathbb{C}$ given by $f \mapsto f(w)$ is continuous on H. If it exists, the reproducing kernel is unique.*

Proof. See [FK94, Proposition IX.2.1 pag 164]. □

Here we list some properties of the reproducing kernel.

3.26 Proposition. *Let $K : D \times D \to \mathbb{C}$ be a reproducing kernel for H. Then*

i) $K(z, w) = \overline{K(w, z)}$ *and* $K(z, z) \geq 0$ *for all* $z, w \in D$*,* $K(z, z) = 0$ *if and only if* $f(z) = 0$ *for all* $f \in H$*.*

ii) *For all* $z_1, \ldots, z_l \in D$*,* $\alpha_1, \ldots, \alpha_l \in \mathbb{C}$*, we have*

$$\sum_{i,j=1}^{l} K(z_i, z_j)\, \alpha_j\, \overline{\alpha_i} \geq 0.$$

iii) *For any orthonormal basis* $\{\psi_k\}$ *of* H*, we have*

$$K(z,w) = \sum_k \psi_k(z)\overline{\psi_k(w)},$$

with the series convergent for every $z, w \in D$*. Here* $\overline{\psi_k(w)}$ *means the complex conjugate* $\psi_k(w)$*.*

iv) $K(z,w)$ *is holomorphic in* z *and antiholomorphic in* w*, in other words, the functions holomorphic in* z *and* $\overline{w}$*.*

v) $K(z,w)$ *is uniquely determined by knowing* $K(z,z)$ *for all* $z \in D$*.*

Proof. See [FK94, Proposition IX.2.2 pag 164]. □

If a group G acts on D, we say that the reproducing kernel $K : D \times D \to \mathbb{C}$ for H is G-invariant if for all $g \in G$, $(z,w) \in D \times D$ we have

$$K(g \cdot z, g \cdot w) = K(z,w).$$

3.27 Definition. Let p be a measurable positive function on D. The weighted Bergman space of D with weight p, or p-weighted Bergman space of D, denote by $\mathscr{H}^2_p(D)$, is the space of holomorphic functions on D such that

$$\|f\|^2 = \int_D |f(z)|^2 p(z) d\mu(z) < \infty.$$

That is

$$\mathscr{H}^2_p(D) = L^2(D, p\mu) \cap \mathcal{H}(D).$$

3.28 Remark. If p is bounded from below on any compact subset of D, then $\mathscr{H}^2_p(D)$ is continuously embedded in the space $L^1_{loc}(D,\mu)$ of locally integrable functions on D. Thus by [FK94, Proposition IX.2.3 pag 166] we have that $\mathscr{H}^2_p(D)$ has a reproducing kernel and $\mathscr{H}^2_p(D)$ is a closed space of $L^2(D, p\mu)$. The reproducing kernel K for $\mathscr{H}^2_p(D)$ is called the p-weighted Bergman kernel of D.

3.2.2 The Bergman representation of weight $2n$

Let $\Pi = \{z = x + \mathbf{i}\, y \in \mathbb{C} \,|\, y > 0\}$ be the complex upper half-plane, $\mathcal{O}(\Pi) = \{f : \Pi \to \mathbb{C} \,|\, f \text{ is holomorphic}\}$ be the space of holomorphic functions on Π and $\mathcal{O}(\overline{\Pi}) = \{f : \Pi \to$

$\mathbb{C}\,|f$ is anti-holomorphic$\}$ be the space of anti-holomorphic functions on Π.

The group $G = \mathrm{SL}(2,\mathbb{R})$ acts on Π by fractional linear transformations

$$\gamma\cdot z = \frac{az+b}{cz+d} \qquad (z\in\Pi, \gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in G). \tag{3.20}$$

We denote $J_\gamma(z)$ the Jacobian of the transformation $z\mapsto \gamma\cdot z$,

$$J_\gamma(z) = \frac{\partial}{\partial z}\left(\frac{az+b}{cz+d}\right) = \frac{\det(\gamma)}{(cz+d)^2} = \frac{\det(\gamma)}{(\jmath_\gamma(z))^2}.$$

Here $\jmath_\gamma$ denotes the map $z\mapsto cz+d$ if $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in G$.

We have that the Jacobian (and also the map $\jmath$) satisfies the following 1-cocycle condition

$$J_{\gamma\alpha}(z) = J_\gamma(\alpha\cdot z)J_\alpha(z) \quad \text{ for all } \alpha,\gamma\in G, z\in\mathbb{C}\,.$$

The fractional linear action of G on Π is transitive, so if $K = \mathrm{Stab}_G(\mathbf{i}) = \{g\in G | g\cdot\mathbf{i} = \mathbf{i}\}$ denotes the isotropy group of $\mathbf{i}\in\Pi$ which is a maximal compact subgroup of G, thus we have a diffeomorphism

$$G/\mathrm{Stab}_G(\mathbf{i}) = G/K \simeq \Pi.$$

On the complex upper half-plane we have a G-invariant measure given by the formula

$$d\mu(z) = y^{-2}dxdy.$$

Then for each integer $n\geq 2$ we let

$$d\mu_n(z) = y^n d\mu(z).$$

Now, according Remark 3.28 we can consider the weighted Bergman space of Π with weight $p(y) = y^n$, that is the Hilbert space

$$\mathscr{H}^2_p(\Pi) := \mathscr{H}^2_n(\Pi) = \mathrm{L}^2(\Pi, d\mu_n)\cap\mathcal{O}(\Pi)$$

of holomorphic square-integrable functions on Π with respect to measure $d\mu_n$. The inner product on $\mathscr{H}^2_n(\Pi)$ is given by

$$(f,g)_n = \int_\Pi f(z)\overline{g(z)}d\mu_n(z).$$

Now, if $f\in\mathscr{H}^2_n(\Pi), z\in\Pi$ the formula

$$(\rho_n(\gamma^{-1})f)(z) = (cz+d)^{-n} f\left(\frac{az+b}{cz+d}\right) = \jmath_\gamma^{-n}(z)f(\gamma\cdot z), \tag{3.21}$$

give us a representation of G on the Hilbert space $\mathscr{H}^2_n(\Pi)$. Moreover

3.29 Proposition. *For all $n \geq 2$, the representation $(\rho_n, \mathscr{H}_n^2(\Pi))$ given by (3.21) is unitary and irreducible.*

Proof. See [Lan85, Theorem 2, Theorem 3 pg 183-184] □

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of $G = \mathrm{SL}(2, \mathbb{R})$. We fix a basis of $\mathfrak{g}$

$$e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutators relations[3]

$$[h, e^\pm] = \pm 2 e^\pm \quad \text{and} \quad [e^+, e^-] = h. \tag{3.22}$$

Then if X is in $\mathfrak{g}$ and $f \in \mathscr{H}_n^2(\Pi)$, the infinitesimal action of representation ρ_n is given by

$$d\rho_n(X) f = \left.\frac{d}{dt}\right|_{t=0} \rho_n(\exp(tX)) f, \tag{3.23}$$

so we have

$$\begin{aligned} (d\rho_n(e^+) f)(z) &= -f'(z) \\ (d\rho_n(h) f)(z) &= -(nf(z) + 2z f'(z)) \\ (d\rho_n(e^-) f)(z) &= nz f(z) + z^2 f'(z) \end{aligned}$$

Now we consider $D = \{w = u + \mathbf{i}\, v \in \mathbb{C} \,|\, |w| < 1\}$ the open unit disk centred at the origin and the Cayley matrix

$$\mathscr{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathbf{i} \\ -\mathbf{i} & 1 \end{pmatrix} = (\overline{\mathscr{C}})^{-1} \in \mathrm{SL}(2, \mathbb{C}).$$

The Cayley transform $(z \mapsto \mathscr{C} \cdot z)$ is a biholomorphic equivalence between Π and D. If $\gamma \in \mathrm{SL}(2, \mathbb{R})$, then we denote $\tilde{\gamma} = \mathscr{C}\, \gamma\, \mathscr{C}^{-1}$ the element in the group

$$\mathrm{SU}(1,1) = \mathscr{C}\, \mathrm{SL}(2, \mathbb{R})\, \mathscr{C}^{-1} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

Thus the fractional linear action (3.20) induces a transitive action of $\mathrm{SU}(1,1)$ on D. Then the formula

$$d\tilde{\mu}(w) = d\mu(\mathscr{C}^{-1} \cdot w) = \left(\frac{2}{1 - |w|^2} \right)^2 du dv$$

[3] In general a sl_2-triplet is a triple (e^+, h, e^-) of elements of a Lie algebra that satisfy the commutation relations (3.22).

provides on D an $\mathrm{SU}(1,1)$-invariant measure.

Let $n \geq 2$ be an integer, we set

$$d\tilde{\mu}_n(w) = \left(\frac{1-|w|^2}{2}\right)^{n-2} dudv = \left(\frac{1-|w|^2}{2}\right)^{n} d\tilde{\mu}(w).$$

Similarly, we consider the Hilbert space $\mathscr{H}_n^2(D) = \mathrm{L}^2(D, d\tilde{\mu}_n) \cap \mathcal{O}(D)$ of holomorphic square-integrable functions on D with respect to measure $d\tilde{\mu}_n$. The inner product on $\mathscr{H}_n^2(D)$ is given by

$$(\tilde{f}, \tilde{g})_n = \int_D \tilde{f}(w)\overline{\tilde{g}(w)} d\tilde{\mu}_n(w).$$

3.30 Lemma. *Let w be an element in D. We have $d\mu_n(\mathscr{C}^{-1} \cdot w) = |J^n_{\mathscr{C}^{-1}}(w)| d\tilde{\mu}_n(w)$*

Proof.

$$\begin{aligned} d\mu_n(\mathscr{C}^{-1} \cdot w) &= \mathrm{Im}^n(\mathscr{C}^{-1} \cdot w) d\mu(\mathscr{C}^{-1} \cdot w) \\ &= \left(\frac{1-|w|^2}{|\,\mathbf{i}\, w+1|^2}\right)^n d\tilde{\mu}(w) \\ &= |J^n_{\mathscr{C}^{-1}}(w)| \left(\frac{1-|w|^2}{2}\right)^n d\tilde{\mu}(w) \end{aligned}$$

□

3.31 Proposition. *The map $T_n : \mathscr{H}_n^2(D) \to \mathscr{H}_n^2(\Pi)$ given by*

$$T_n(\tilde{f})(z) = \jmath_{\mathscr{C}}^{-n}(z)\tilde{f}(\mathscr{C} \cdot z) \qquad (\tilde{f} \in \mathscr{H}_n^2(D), z \in \Pi)$$

is a unitary transformation with inverse

$$T_n^{-1}(f)(w) = \jmath_{\mathscr{C}^{-1}}^{-n}(w) f(\mathscr{C}^{-1} \cdot w) \qquad (f \in \mathscr{H}_n^2(\Pi), w \in D).$$

Proof. Let $\tilde{f}, \tilde{g}$ be two functions in $\mathscr{H}_n^2(D)$ and let $w = \mathscr{C} \cdot z$. Then

$$\begin{aligned} (T_n(\tilde{f}), T_n(\tilde{g}))_n &= \int_\Pi \jmath_{\mathscr{C}}^{-n}(z)\tilde{f}(\mathscr{C} \cdot z)\overline{\jmath_{\mathscr{C}}^{-n}(z)}\overline{\tilde{g}(\mathscr{C} \cdot z)} d\mu_n(z) \\ &= \int_D |J_{\mathscr{C}}^{-n}(\mathscr{C}^{-1} \cdot w)| \tilde{f}(w)\overline{\tilde{g}(w)} d\mu_n(\mathscr{C}^{-1} \cdot w) \\ &= (\tilde{f}, \tilde{g})_n \end{aligned}$$

□

3.32 Remark. SU(1,1) acts on $\mathscr{H}^2_n(D)$ by $\tilde{\rho}_n$ given by

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\rho_n} & \mathrm{GL}(\mathscr{H}^2_n(\Pi)) \\ {\scriptstyle \mathscr{C}(-)\mathscr{C}^{-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle T_n^{-1}(-)T_n} \\ \mathrm{SU}(1,1) & \xrightarrow[\tilde{\rho_n}]{} & \mathrm{GL}(\mathscr{H}^2_n(D)) \end{array}$$

i.e. $\tilde{\rho}_n(\tilde{\gamma}) := T_n^{-1}\rho_n(\gamma)T_n$ for any $\gamma \in \mathrm{SL}(2,\mathbb{R})$. Then by Proposition 3.29 for any integer $n \geq 2$ the representation $(\tilde{\rho}_n, \mathscr{H}^2_n(D))$ of SU(1,1) is unitary and irreducible. We say that $\mathscr{H}^2_n(\Pi)$ (respectively, $\mathscr{H}^2_n(D)$) is the n-weighted Bergman space on Π (respectively, on D).

If $X \in \mathfrak{sl}_2(\mathbb{R})$, the element $\tilde{X} = \mathscr{C}\,X\,\mathscr{C}^{-1}$ belongs to $\mathfrak{su}(1,1)$ the Lie algebra of SU(1,1). The infinitesimal action of $\tilde{\rho}_n$ is given by

$$d\tilde{\rho}_n(\mathscr{C}\,X\,\mathscr{C}^{-1}) = T_n^{-1} d\rho_n(X) T_n.$$

Thus in the basis $\{\tilde{e}^+, \tilde{h}, \tilde{e}^-\}$ of $\mathfrak{su}(1,1)$, we have

$$\begin{aligned}
(d\tilde{\rho}_n(\tilde{e}^+)\tilde{f})(w) &= \frac{w-\mathbf{i}}{2}\left[n\tilde{f}(w) + (w-\mathbf{i})\tilde{f}'(w)\right] \\
(d\tilde{\rho}_n(\tilde{h})\tilde{f})(w) &= -\,\mathbf{i}\left[nw\tilde{f}(w) + (w^2+1)\tilde{f}'(w)\right] \\
(d\tilde{\rho}_n(\tilde{e}^-)\tilde{f})(w) &= \frac{w+\mathbf{i}}{2}\left[n\tilde{f}(w) + (w+\mathbf{i})\tilde{f}'(w)\right]
\end{aligned}$$

for any $\tilde{f}$ in $\mathscr{H}^2_n(D)$ and w in D.

Now on $\mathscr{H}^2_n(\Pi)$ we introduce the operators $\mathscr{L}_\pm, \mathscr{L}_0$ defined by

$$\begin{aligned}
(\mathscr{L}_+\, f)(z) &= \frac{z+\mathbf{i}}{2}\left[nf(z) + (z+\mathbf{i})f'(z)\right] \\
(\mathscr{L}_0\, f)(z) &= \mathbf{i}\left[nzf(z) + (z^2+1)f'(z)\right] \\
(\mathscr{L}_-\, f)(z) &= \frac{z-\mathbf{i}}{2}\left[nf(z) + (z-\mathbf{i})f'(z)\right]
\end{aligned}$$

for $f \in \mathscr{H}^2_n(\Pi)$ and $z \in \Pi$.

Similarly, on $\mathscr{H}^2_n(D)$ we consider the operators $\widetilde{\mathscr{L}}_\pm = T_n^{-1}\,\mathscr{L}_\pm\,T_n$ and $\widetilde{\mathscr{L}}_0 = T_n^{-1}\,\mathscr{L}_0\,T_n$, *i.e.*

$$\begin{aligned}
(\widetilde{\mathscr{L}}_+\tilde{f})(w) &= -\tilde{f}'(w) \\
(\widetilde{\mathscr{L}}_0\tilde{f})(w) &= -(n\tilde{f}(w) + 2w\tilde{f}'(w)) \\
(\widetilde{\mathscr{L}}_-\tilde{f})(w) &= nw\tilde{f}(w) + w^2\tilde{f}'(w)
\end{aligned}$$

for $\tilde{f} \in \mathscr{H}^2_n(D)$ and w in D.

3.33 Proposition. *We have that the triple* $(\mathscr{L}_+, \mathscr{L}_0, \mathscr{L}_-)$ *and* $(\widetilde{\mathscr{L}}_+, \widetilde{\mathscr{L}}_0, \widetilde{\mathscr{L}}_-)$ *are the* $\mathfrak{sl}_2(\mathbb{R})$*-triples. Moreover, we have that*

$$\begin{aligned} d\tilde{\rho}_n(\tilde{e}^+) &= \frac{1}{2}\left(\widetilde{\mathscr{L}}_+ + \widetilde{\mathscr{L}}_- + \mathbf{i}\,\widetilde{\mathscr{L}}_0\right) \\ d\tilde{\rho}_n(\tilde{h}) &= \mathbf{i}\left(\widetilde{\mathscr{L}}_+ - \widetilde{\mathscr{L}}_-\right) \\ d\tilde{\rho}_n(\tilde{e}^-) &= \frac{1}{2}\left(\widetilde{\mathscr{L}}_+ + \widetilde{\mathscr{L}}_- - \mathbf{i}\,\widetilde{\mathscr{L}}_0\right) \end{aligned}$$

Proof. The proof follows from the equations:

$$\begin{aligned} \tilde{e}^+ &= \frac{1}{2}\left(e^+ + e^- + \mathbf{i}\,h\right) \\ \tilde{h} &= \mathbf{i}\left(e^+ - e^-\right) \\ \tilde{e}^- &= \frac{1}{2}\left(e^+ + e^- - \mathbf{i}\,h\right). \end{aligned}$$

□

For all $k \in \mathbb{N}$, we consider the holomorphic functions $\psi_k \in \mathscr{H}_n^2(D)$ defined by $\psi_k(w) = (\widetilde{\mathscr{L}}_-^k(1))(w) = n(n+1)(n+2)\cdots(n+k-1)w^k = \mathrm{ph}_+(n,k)w^k$, where $\mathrm{ph}_+(n,k) = k!\binom{n+k-1}{k}$ is the (rising) Pochhammer symbol.

In addition the vectors $\psi_k \in \mathscr{H}_n^2(D)$ and $w_k(n) = T_n(\psi_k) \in \mathscr{H}_n^2(\Pi)$ satisfy the relations

$$\begin{aligned} \widetilde{\mathscr{L}}_+\psi_k &= -k(n+k-1)\psi_{k-1}, & \mathscr{L}_+\, w_k(n) &= -k(n+k-1)w_{k-1}(n) \\ \widetilde{\mathscr{L}}_0\psi_k &= -(n+2k)\psi_k, & \mathscr{L}_0\, w_k(n) &= -(n+2k)w_k(n) \\ \widetilde{\mathscr{L}}_-\psi_k &= \psi_{k+1}, & \mathscr{L}_-\, w_k(n) &= w_{k+1}(n). \end{aligned}$$

3.34 Proposition. *For all* $k \in \mathbb{N}$ *the holomorphic functions* ψ_k *are the eigenvectors on* $\tilde{K} = SU(1)$*, with character* $n + 2k$*. Consequently, we have that* $\mathscr{H}_n^2(D) = \widehat{\bigoplus}_{k\in\mathbb{N}}\mathbb{C}\,\psi_k$ *is an orthogonal decomposition, with lowest weight vector* $\psi_0 = 1$*, of weight* n*.*

Proof. See [Lan85, Theorem 3]. □

3.35 Corollary. *The holomorphic functions* $w_k(n) = T_n(\psi_k) \in \mathscr{H}_n^2(\Pi)$ *are the eigenvectors on* $K = SO(2)$*, with character* $n + 2k$ *and* $\mathscr{H}_n^2(\Pi) = \widehat{\bigoplus}_{k\in\mathbb{N}}\mathbb{C}\,w_k(n)$ *is an orthogonal decomposition, with lowest weight vector* $w_0(n) = \jmath_{\mathscr{C}}^{-n}$*, of weight* n*.*

3.36 Remark. We can extend the representation $d\rho_n : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{gl}(\mathscr{H}_n^2(\Pi))$ to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$: If $x \in \{e^+, h, e^-\}$, for all $a, b \in \mathbb{R}$ we defined $d\rho_n((a + \mathbf{i}\, b)x) = a(d\rho(x)) + \mathbf{i}\, b(d\rho(x))$. In this way $d\rho_n : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathscr{H}_n^2(\Pi))$ is a complex representation of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ unitary and irreducible. Moreover, as $\{\mathscr{L}_+, \mathscr{L}_0, \mathscr{L}_-\}$ is an $\mathfrak{sl}_2(\mathbb{C})$-triple then $\mathscr{H}_n^2(\Pi)$ is a generalized $\mathfrak{sl}_2(\mathbb{C})$-Verma module of highest weight $-n$.

3.2.3 The Bergman algebra

Let $\mathscr{H}^+ = \bigoplus_{n\in\mathbb{N}} \mathscr{H}_{2n}^2(\Pi)$ be the direct sum of $\mathscr{H}_{2n}^2(\Pi)$ $(\mathscr{H}_0^2(\Pi) = \mathbb{C})$. We endow the Hilbert space $\mathscr{H}^+$ with the action $\rho = \oplus_{n\in\mathbb{N}} \rho_{2n}$ (ρ_0 is the trivial action on $\mathbb{C}$). Precisely, if $f = (f_{2n})_{n\in\mathbb{N}}$ is an element in $\mathscr{H}^+$ with $f_n \in \mathscr{H}_{2n}^2(\Pi)$ then for all γ in $\mathrm{SL}(2,\mathbb{R})$ the representation ρ is defined by $\rho(\gamma)(f) = (\rho_{2n}(f_{2n}))_{n\in\mathbb{N}}$. Then $\rho = (\rho, \mathscr{H}^+)$ is an unitary representation of $\mathrm{SL}(2,\mathbb{R})$. Moreover the $\mathscr{H}^+$ is a commutative algebra with the point-wise multiplication. We call to $\mathscr{H}^+$ the Bergman algebra.

3.37 Lemma. *For all $f \in \mathscr{H}_n^2(\Pi)$, the holomorphic function $\partial f = \frac{\partial}{\partial z} f$ belongs to the weighted Bergman space $\mathscr{H}_{n+2}^2(\Pi)$. In other words, $\partial : \mathscr{H}_*^2(\Pi) \to \mathscr{H}_{*+2}^2(\Pi)$ is a homogeneous map of degree 2.*

Proof. It follows from Lemme 3.1 and Lemme 3.2 in [UU96]. □

3.38 Conjecture. *The Bergman algebra $\mathscr{H}^+$ satisfies the condition (CA).*

Let $k \in \mathbb{N}$ and $f = (f_0, f_2, \ldots, f_{2i}, \ldots)$ and $g = (g_0, g_2, \ldots, g_{2j}, \ldots)$ be two elements in $\mathscr{H}^+$. We consider the following bilinear map

$$\mathcal{RC}_k(f, g) = (p_0, \ldots, p_{2k-2}, p_{2k}, \ldots, p_{2k+2n}, \ldots)$$

with $p_0 = \ldots = p_{2k-2} = 0$ and $p_{2k+2n} = \sum_{i+j=n} \mathrm{RC}_k(f_{2i}, g_{2j})$, for all $n \in \mathbb{N}$. Here RC_k denotes the "classical" Rankin-Cohen bracket given by

$$\mathrm{RC}_k(f_{2i}, g_{2j}) = \sum_{r=0}^{k} (-1)^r \binom{2i+k-1}{r} \binom{2j+k-1}{k-r} \partial^{k-r} f_{2i}\, \partial^r g_{2j}. \tag{3.24}$$

We have that

3.39 Proposition. *The Rankin-Cohen bracket $\mathrm{RC}_k : \mathscr{H}_i^2(\Pi) \times \mathscr{H}_j^2(\Pi) \to \mathscr{H}_{i+j+2k}^2(\Pi)$ is a $\mathrm{SL}(2,\mathbb{R})$-equivariant map, i.e. for all $(f, g) \in \mathscr{H}_i^2(\Pi) \times \mathscr{H}_j^2(\Pi)$ and $\gamma \in \mathrm{SL}(2,\mathbb{R})$ we have*

$$\mathrm{RC}_k(\rho_i(\gamma)f, \rho_j(\gamma)g) = \rho_{i+j+2k}(\gamma)\mathrm{RC}_k(f, g).$$

Proof. See [UU96]. □

Then we have that following corollary of Proposition 3.39.

3.40 Proposition. *For $k \in \mathbb{N}$, the map $\mathcal{RC}_k : \mathscr{H}^+ \times \mathscr{H}^+ \to \mathscr{H}^+$ is G-equivariant, i.e. for any $\gamma \in G$, and any $(f, g) \in \mathscr{H}^+ \times \mathscr{H}^+$ we have*

$$\mathcal{RC}_k(\rho(\gamma)(f), \rho(\gamma)(g)) = \rho(\gamma)(\mathcal{RC}_k(f, g)).$$

Proof. It follows by the Proposition 3.39. □

3.41 Theorem. *For any $\kappa \in \mathbb{C}$, let $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ be given by*

$$t_k^\kappa(i, j) = \left(-\frac{1}{4}\right)^i \sum_{r \geq 0} \binom{i}{2r} \frac{\binom{-\frac{1}{2}}{r}\binom{\kappa-\frac{3}{2}}{r}\binom{\frac{1}{2}-\kappa}{r}}{\binom{-i-\frac{1}{2}}{r}\binom{-j-\frac{1}{2}}{r}\binom{k+i+j-\frac{3}{2}}{r}}. \tag{3.25}$$

Then t^κ is a RC twist for $\mathscr{H}^+$.

Proof. According to proof of Proposition 3.17 the functions t^κ give rise to an associative product if and only if for any $p \in \mathbb{N}$ and $(a_1, a_2, a_3) \in \mathscr{H}^2_{\alpha_1}(\Pi) \times \mathscr{H}^2_{\alpha_2}(\Pi) \times \mathscr{H}^2_{\alpha_3}(\Pi)$, $\alpha_1, \alpha_2, \alpha_3 \in 2\,\mathbb{N}$, we have that $f_p(a_1, a_2, a_3) = 0$, where

$$\begin{aligned} f_p(a_1, a_2, a_3) = & \sum_{x+y+z=p} (-1)^x P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z) a_1^{(x)} a_2^{(y)} a_3^{(z)} \\ & - \sum_{x+y+z=p} (-1)^{x+y} P_{(\alpha_3,\alpha_2,\alpha_1)}(z, y, x) a_1^{(x)} a_2^{(y)} a_3^{(z)}. \end{aligned}$$

But by Proposition 3.22 we already know that that is true. Thus for any $f_1, f_2, f_3 \in M_{\alpha_1} \times M_{\alpha_2} \times M_{\alpha_3}$ and any $p \in \mathbb{N}$

$$\sum_{x+y+z=p} (-1)^x P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z) f_1^{(x)} f_2^{(y)} f_3^{(z)} = \sum_{x+y+z=p} (-1)^{x+y} P_{(\alpha_3,\alpha_2,\alpha_1)}(z, y, x) f_1^{(x)} f_2^{(y)} f_3^{(z)}.$$

Now, by Corollary 3.23 we have that

$$(-1)^x P_{(\alpha_1,\alpha_2,\alpha_3)}(x, y, z) = (-1)^{x+y} P_{(\alpha_3,\alpha_2,\alpha_1)}(z, y, x).$$

Thus Proposition 3.17 yields the result. □

In particular if $\kappa = \frac{1}{2}$ or $\kappa = \frac{3}{2}$, we have the following corollary of Theorem 3.41

3.42 Theorem. *The product $\mathcal{RC} : \mathscr{H}^+_\hbar \otimes \mathscr{H}^+_\hbar \to \mathscr{H}^+_\hbar$ given by $\mathcal{RC} = \sum_n \mathcal{RC}_n \hbar^n$ is an associative product $SL(2, \mathbb{R})$-equivariant on the ring $\mathscr{H}^+_\hbar = \mathscr{H}^+[[\hbar]]$ of formal power series in the variable $\hbar$ over $\mathscr{H}^+$.*

Chapter 4

Two $\mathrm{SL}(2,\mathbb{R})$-invariants deformations on the imaginary Lobachevsky space

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L'objectif de ce chapitre est d'établir un lien explicite entre la déformation formelle de Rankin-Cohen et celle donnée par Alekseev et Lachowska dans [AL05] que l'on note $\star_{\mathrm{S}}$.

Nous notons $\Bbbk$ le corps $\mathbb{C}$ des nombres complexes ou le corps $\mathbb{R}$ des nombres réels et λ un élément dans $\mathbb{C}$.

4.1 *Star-produit invariant d'Alekseev-Lochowska. L'objet fondamental de la construction de $\star_{\mathrm{S}}$ est l'élément canonique F_λ associé à la dualité de Shapovalov $\langle\ ,\ \rangle_\lambda : U(\mathfrak{n}_-) \times U(\mathfrak{n}_+) \to \mathbb{C}$. Nous développons une construction détaillée de F_λ pour le cas $\mathfrak{sl}_2(\mathbb{R})$. Nous fixons une base de $\mathfrak{g}_\Bbbk = \mathfrak{sl}_2(\Bbbk)$*

$$e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

avec les relations de commutation suivantes :

$$[h, e^\pm] = \pm 2 e^\pm \quad \textit{et} \quad [e^+, e^-] = h.$$

Dans la base $\{(e^\pm)^k\}_{k\in\mathbb{N}}$ de l'algèbre enveloppante $U(\mathfrak{n}_\pm)$ la dualité de Shapovalov $\langle\ ,\ \rangle_\lambda$ s'exprime par

$$\langle (e^-)^i, (e^+)^j \rangle_\lambda = \delta_{ij}(-1)^i i!(\lambda)_i, \tag{4.1}$$

où $(\lambda)_i = \lambda(\lambda-1)\cdots(\lambda-(i-1))$ est le symbole de Pochhammer. Dans ce cas, l'élément de Shapovalov est donné par

$$F_\lambda = \sum_{k\in\mathbb{N}} \frac{(e^-)^k \otimes (e^+)^k}{\langle (e^-)^k, (e^+)^k \rangle_\lambda} = \sum_{k\in\mathbb{N}} \frac{(-1)^k (e^-)^k \otimes (e^+)^k}{k!\,\lambda(\lambda-1)\cdots(\lambda-(k-1))},$$

où $\lambda \in \mathbb{C} \setminus \mathbb{N}$ *et* (e^-, h, e^+) *est un* $\mathfrak{sl}_2$*-triplet de* $\mathfrak{sl}_2(\mathbb{R})$*. L'élément canonique* F_λ *satisfait les propriétés suivantes.*

Théorème. *Soit* $F : \mathbb{C} \to U(\mathfrak{n}_-) \widehat{\otimes} U(\mathfrak{n}_+)$ *l'application donnée par*

$$\lambda \mapsto F_\lambda.$$

a) L'application F *est méromorphe dans* $\mathbb{C}$ *et holomorphe à l'infini. Les seuls pôles de* F *sont d'ordre 1 et ils sont localisées en* $n \in \mathbb{N}$*.*

b) Le résidu de F *en* $\lambda = n$ *est donné par*

$$\mathrm{Res}_{\lambda=n} F_\lambda = \frac{(-1)^{n+1}}{n!} \sum_{k>n} \frac{(e^-)^k \otimes (e^+)^k}{k!(k-(n+1))!}.$$

c) Pour tout $\lambda \in \mathbb{C}^\times$*, tel que* $1/\lambda \notin \mathbb{N}$*, on a*

$$F_{1/\lambda} = 1 \otimes 1 - \lambda(e^- \otimes e^+) - \sum_{k>1} \frac{\lambda^k [(e^-)^k \otimes (e^+)^k]}{k!(\lambda - 1)(2\lambda - 1)\cdots((k-1)\lambda - 1)}. \tag{4.2}$$

d) Pour tout $\lambda \in \mathbb{C} \setminus \mathbb{N}$*, on a*

$$\frac{d}{d\lambda} F_\lambda = \sum_{k>0} (-1)^{k+1} \frac{(e^-)^k \otimes (e^+)^k}{k!(\lambda)_k} \sum_{l=0}^{k-1} \frac{1}{\lambda - l}.$$

En suivant Etingof et Schiffman [ES01], nous introduisons dans la section 4.1.5 la notion de matrice de fusion $J_{V,W}$*.*

Soient V*,* W *deux représentations de* $\mathfrak{g}_\Bbbk$ *formellement* h*-diagonalisables (Définition 4.32) tels que le produit tensoriel* $V \otimes W$ *soit aussi formellement* h*-diagonalisable. Soient* $(v, w) \in \widehat{V}[\lambda_v] \times \widehat{W}[\mu_w]$ *une paire d'éléments* h*-homogènes de* V *et* W *(i.e.* $h \cdot v = \lambda_v v$ *et* $h \cdot w = \mu_w w$*). Si* $\lambda - \mu_w$ *et* $\lambda - \lambda_v - \mu_w$ *sont dans* $\mathbb{C} \setminus \mathbb{N}$*, nous notons* $\phi_\lambda^{w,v}$ *la composition de deux morphismes suivants :*

$$M_\lambda^+ \xrightarrow{\phi_\lambda^w} M_{\lambda-\mu_w}^+ \widehat{\otimes} W \xrightarrow{\phi_{\lambda-\mu_w}^v \otimes \mathrm{id}} M_{\lambda-\lambda_v-\mu_w}^+ \widehat{\otimes} V \widehat{\otimes} W.$$

On note $J_{V,W}^+(\lambda)(v \otimes w)$ *l'unique élément dans* $V \widehat{\otimes} W[\lambda_v + \mu_w]$ *tel que*

$$\phi_\lambda^{w,v} = \phi_\lambda^{J_{V,W}^+(\lambda)(v \otimes w)}.$$

De même, si $\lambda_v - \lambda$ *et* $\lambda_v + \mu_w - \lambda$ *sont dans* $\mathbb{C} \setminus \mathbb{N}$*, on introduit la composition* $\phi_\lambda^{v,w}$ *par :*

$$M_\lambda^- \xrightarrow{\phi_\lambda^v} V \widehat{\otimes} M_{\lambda-\lambda_v}^- \xrightarrow{\mathrm{id} \otimes \phi_{\lambda-\lambda_v}^w} V \widehat{\otimes} W \widehat{\otimes} M_{\lambda-\lambda_v-\mu_w}^-.$$

On note $J_{V,W}^-(\lambda)(v \otimes w)$ *l'unique élément dans* $V \widehat{\otimes} W[\lambda_v + \mu_w]$ *telles que*

$$\phi_\lambda^{v,w} = \phi_\lambda^{J_{V,W}^-(\lambda)(v\otimes w)}.$$

Théorème (Définition)**.** *Soient* V*,* W *deux représentations de* $\mathfrak{g}_\Bbbk$ *formellement h-diagonalisables tels que le produit tensoriel* $V \otimes W$ *soit aussi formellement h-diagonalisable et soit* Λ_V*,* Λ_W *l'ensemble des poids des représentations* V *et* W*.*

i) Si $\lambda \notin \Lambda_W + \mathbb{N}$ *et* $\lambda \notin \Lambda_V + \Lambda_W + \mathbb{N}$*, alors il existe un endomorphisme de* $V \widehat{\otimes} W$ *donné par*

$$\begin{aligned} J_{V,W}^+(\lambda) : V \widehat{\otimes} W &\to V \widehat{\otimes} W \\ (v, w) &\mapsto J_{V,W}^+(\lambda)(v \otimes w). \end{aligned}$$

Plus précisément,

$$\begin{aligned} J_{V,W}^+(\lambda)(v \otimes w) &= F_{\lambda-\mu_w}(v \otimes w) \\ &= \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k!(\lambda-\mu_w)_k}(e^-)^k v \otimes (e^+)^k w. \end{aligned}$$

ii) De même, si $\lambda \notin \Lambda_V - \mathbb{N}$ *et* $\lambda \notin \Lambda_V + \Lambda_W - \mathbb{N}$*, alors il existe un endomorphisme de* $V \widehat{\otimes} W$ *donné par*

$$\begin{aligned} J_{V,W}^-(\lambda) : V \widehat{\otimes} W &\to V \widehat{\otimes} W \\ (v, w) &\mapsto J_{V,W}^-(\lambda)(v \otimes w). \end{aligned}$$

Plus précisément,

$$\begin{aligned} J_{V,W}^-(\lambda)(v \otimes w) &= F_{\lambda_v-\lambda}(v \otimes w) \\ &= \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k!(\lambda_v-\lambda)_k}(e^-)^k v \otimes (e^+)^k w. \end{aligned}$$

iii) Les applications $J_{V,W}^+(\lambda)$ *et* $J_{V,W}^-(\lambda)$ *sont appelées matrices de fusion de poids* λ*.*

On utilise les matrices de fusion pour montrer que F_λ satisfait la condition d'associativité de Drinfeld (Proposition 4.47). Ensuite, dans la Proposition 4.49, nous décrivons explicitement les éléments a_n du développement asymptotique de $F_{\hbar^{-1}}$ et précisons le résultats d'Alekseev et Lachowska.

4.2 *Calcul symbolique d'Unterberger. Dans cette section, nous considérons le calcul symbolique $\#_\lambda$ d'Unterberger [UU96] dans l'algèbre $E = \bigoplus_{n\in\mathbb{N}} E_n^+$ des espaces propres E_n^+ "anti-holomorphes" de valeur propre $-n(n-1)$ de l'opérateur de Laplace-Beltrami de l'hyperboloïde à une nappe $\mathrm{SL}(2,2,\mathbb{R})/\mathrm{SO}(1,1) \simeq \Pi_L := \{(s,t)|s,t \in \mathbb{R}\cup\{\infty\}, s \neq t\}$. Le produit $\#_\lambda$ a une représentation intégrale (voir [UU94]). En effet, nous considérons le rapport anharmonique de quatre points dans $\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$:*

$$[s,x,y,t] = \frac{(s-y)(x-t)}{(x-y)(s-t)}.$$

Alors, le produit $\#_\lambda$ est donné par

$$(f\#_\lambda g)(s,t) = c_\lambda c_{-\lambda} \int_{\Pi_L} f(s,x)g(y,t)|[s,y,x,t]|^{-1-\mathbf{i}\,\lambda} d\mu_L(x,y).$$

où $c_\lambda = \frac{1}{2}(2\pi)^{\mathbf{i}\,\lambda}\left[\Gamma(\mathbf{i}\,\lambda)\cosh(\frac{\pi\,\lambda}{2})\right]^{-1}$ et $d\mu_L$ est la mesure G-invariante dans Π_L donné par

$$d\mu_L(s,t) = (s-t)^{-2} ds dt.$$

Le calcul symbolique $\#_\lambda$ est lié aux crochets de Rankin-Cohen (Proposition 4.77).

Théorème. *Soit $\lambda \in \mathbb{R}^\times$ et $(f,g) \in E_n^+ \times E_m^+$ si le produit $f\#_\lambda g$ s'exprime par*

$$f\#_\lambda g = \sum_{k\geq 2} h_k,$$

avec $h_k \in E_{n+m+k}^+$. Alors il existe une suite $\{\Phi_k(n,m,\lambda)|k \geq 2\} \subseteq \mathbb{C}$ telle que pour tout $k \geq 2$

$$Q_{n+m+k}(h_k) = \Phi_k(n,m,\lambda) RC_k(Q_n(f), Q_m(g)).$$

Ici, $Q_n : E_n \to \mathscr{H}_{2n}^2(\Pi)$ est l'isomorphisme d'espaces de Hilbert définie en l'équation (4.69).

Nous conjecturons que les coefficients de Unterberger $\Phi_k(n,m,\lambda)$ sont étroitement liés au twist de Zagier t^κ (Conjecture 4.79).

4.3 Crochets de Rankin-Cohen et matrices de fusion. Dans cette section nous établissons le théorème principal de cette thèse (Théorème 4.85) qui exprime les crochets de Rankin-Cohen en termes des matrices de fusion (Proposition 4.82). Alors la condition d'associativité de F_λ codifie l'associativité du produit $\sum_k \mathrm{RC}_k \hbar^k$. Nous avons,

Théorème. *Soient M_λ^- et M_μ^- deux modules de Verma de plus bas poids λ et de plus bas poids μ, respectivement. Si $-\mu \notin \mathbb{N}$ et $-\mu - \lambda \notin \mathbb{N}$, alors*

$$J^+_{M_\lambda^-, M_\mu^-}(0)(u \otimes v_\mu^-) = F_{-\mu}(u \otimes v_\mu^-) = \sum_{n \in \mathbb{N}} RC_n(v_\lambda^- \otimes v_\mu^-)\hbar^n,$$

où u est donné par $u = \sum_{n\in\mathbb{N}} u_n(\lambda,\mu)\hbar^n = \sum_{n\in\mathbb{N}}(-1)^n \binom{\mu+n-1}{n}(e^+)^n v_\lambda^- \hbar^n \in \widehat{M_\lambda^-}$.

The purpose of this chapter is to compare three different approaches to the covariant quantization of coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$. The key ingredient of all constructions is the representation theory of this Lie group. Thus, we start by reviewing some fundamental facts about it.

Throughout this chapter $\Bbbk$ is $\mathbb{C}$ or $\mathbb{R}$, $G = \mathrm{SL}(2,\mathbb{R})$ is the simple real Lie group of 2×2 matrices with determinant 1, $\mathfrak{g}_\mathbb{R} = \mathfrak{sl}_2(\mathbb{R})$ be its Lie algebra, $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$ be the complexification of $\mathfrak{g}_\mathbb{R}$ and finally let $\mathfrak{g} = \mathfrak{g}_\Bbbk = \mathfrak{sl}_2(\Bbbk)$. We refer to [HT92] and [UU96] for more details.

4.1 Alekseev-Lochowska's invariant $\star$-product

In this first section we deal with an invariant $\star$-product on a suitable coadjoint orbit. This invariant star product introduced by Alekseev-Lochowska in the article [AL05] follows the original construction by Etingof, Varchenko and Schiffmann [EV99, ES01].

4.1.1 Representation theory on $\mathrm{SL}(2,\mathbb{R})$

We fix a basis of $\mathfrak{g}_\Bbbk = \mathfrak{sl}_2(\Bbbk)$

$$e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.3}$$

with commutators relations

$$[h, e^\pm] = \pm 2e^\pm \quad \text{and} \quad [e^+, e^-] = h. \tag{4.4}$$

We denote by H the following subgroup of G

$$H = \exp(\mathbb{R}\, h) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right) | a \in \mathbb{R}^{\times} \right\},$$

and by K the maximal compact subgroup of G

$$K = \left\{ \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right) | \theta \in \mathbb{R} \right\}.$$

The group G admits an Iwasawa decomposition

$$G = KAN,$$

where $A = \left\{ \left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix} \right), y \in \mathbb{R}^{+} \right\} \subset H$ and $N = \{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right), x \in \mathbb{R} \}$.

The element $\mathcal{C}$ of $U(\mathfrak{g}_{\Bbbk})$ defined by

$$\mathcal{C} = h^2 + 2(e^+e^- + e^-e^+) = h^2 + 2h + 4e^-e^+, \tag{4.5}$$

is called the Casimir element of $\mathfrak{g}_{\Bbbk}$, it belongs and actually generates the center $\mathcal{Z}(\mathfrak{g}_{\Bbbk})$ of $U(\mathfrak{g}_{\Bbbk})$.

We list some combinatorial proprieties of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$.

4.1 Lemma. *Let $i, j, k \in \mathbb{N}$, and $a_k^{\pm}$ be the element in $U(\mathfrak{sl}_2(\Bbbk))$ defined[1] by*

$$a_k^+ = e^-e^+ + (k+1)(h-k), \qquad a_k^- = e^+e^- - (k+1)(h+k)$$

We have that

a) $a_i^+ a_j^+ = a_j^+ a_i^+$ *and* $a_i^- a_j^- = a_j^- a_i^-$,

b) $[h, (e^{\pm})^k] = \pm 2k(e^{\pm})^k$,

c) $(e^+)^k(e^-) = a_{k-1}^+(e^+)^{k-1}$,

d) $(e^+)(e^-)^k = (e^-)^{k-1}a_{k-1}^+$,

e) In general,

$$(e^+)^i(e^-)^j = \begin{cases} (e^-)^{j-i}\left(\prod_{k=1}^{i} a_{j-k}^+\right), & \text{if } j \geq i \quad (4.6) \\ \left(\prod_{k=1}^{j} a_{i-k}^+\right)(e^+)^{i-j}, & \text{if } j \leq i. \quad (4.7) \end{cases}$$

[1] Here $k \in \mathbb{C}$ means the element $k \cdot 1_{U(\mathfrak{g})} \in U(\mathfrak{g})$.

and

$$(e^-)^i(e^+)^j = \begin{cases} (e^+)^{j-i}\left(\prod_{k=1}^{i} a^-_{j-k}\right), & \text{if } j \geq i \qquad (4.8)\\ \left(\prod_{k=1}^{j} a^-_{i-k}\right)(e^-)^{i-j}, & \text{if } j \leq i. \qquad (4.9)\end{cases}$$

Proof. (a) is evident. The items (b), (c) and (d) are easily proved by induction on k. Now for e), we assume that $j > i$. We fix $j \in \mathbb{N}$, and we proceed by induction on i. If $i = 1$, we have

$$(e^+)(e^-)^j = (e^-)^{j-1}a^+_{j-1},$$

which is (d). Now we assume that for i the formula (4.6) is true. Then

$$\begin{aligned}(e^+)^{i+1}(e^-)^j &= e^+((e^+)^i(e^-)^j)\\ &= e^+\left[(e^-)^{j-i}\left(\prod_{k=1}^{i} a^+_{j-k}\right)\right]\\ &= (e^+(e^-)^{j-i})\left(\prod_{k=1}^{i} a^+_{j-k}\right)\\ &= (e^-)^{j-(i+1)}\left(\prod_{k=1}^{i+1} a^+_{j-k}\right)\end{aligned}$$

The proof of second condition in (e) goes along the same lines. $\square$

4.2 Remark. We say that a basis (x_{-1}, x_0, x_1) of $\mathfrak{sl}_2(\mathbb{k})$ is an sl(2)-triplet if it satisfies the commutation relations:

$$[x_0, x_{\pm 1}] = \pm 2x_{\pm 1} \quad \text{and} \quad [x_1, x_{-1}] = x_0. \tag{4.10}$$

The Lemma 4.1 is also true if we replace (e^-, h, e^+) by any other sl(2)-triplet.

4.1.1.1 Formal representations

Let V be a representation of $\mathfrak{g}_{\mathbb{k}} = \mathfrak{sl}_2(\mathbb{k})$ such that it admits a direct sum decomposition of complex vector spaces

$$V = \bigoplus_{k\in\mathbb{Z}} V_k. \tag{4.11}$$

The algebraic product of the family $\{V_k | k \in \mathbb{Z}\}$

$$\prod_{k\in\mathbb{Z}} V_k,$$

is the space elements v which have a unique expression of the form

$$v = \sum_{k \in \mathbb{Z}} v_k$$

with $v_k \in V_k$ (may have infinite length). It is also a representation of $\mathfrak{g}_{\Bbbk}$, indeed if $x \in \mathfrak{g}_{\Bbbk}$ and $v = \sum_{k \in \mathbb{Z}} v_k$ the action of $\mathfrak{g}_{\Bbbk}$ on $\prod_{k \in \mathbb{Z}} V_k$, is given by

$$x \cdot v = \sum_{k \in \mathbb{Z}} x \cdot v_k.$$

This representation is, in fact, a completion of V in the sense of Section 2.3.2. Indeed for any $n \in \mathbb{N}$, we take $M_{2n} = \bigoplus_{k=-n}^{n} V_k$, $M_{2n+1} = \bigoplus_{k=-n-1}^{n-1} V_k$ $(M_n \subseteq M_{n-1})$ and $p_n : M_n \to M_{n-1}$ the natural projection, then

$$\widehat{V} = \varprojlim M_n \simeq \prod_{k \in \mathbb{Z}} V_k.$$

4.3 Definition. We call $\widehat{V}$ the formal representation of V relative to the decomposition (4.11).

4.4 Remark. We emphasize two particular cases. Firstly, if the decomposition (4.11) of V is trivial, *i.e.* $V = \bigoplus_{k \in \mathbb{Z}} V_k$ with $V_0 = V$ and $V_k = \{0\}$ if $k \neq 0$, then for all $n \in \mathbb{N}$, $M_n = V$ and $\widehat{V} = \varprojlim M_n$ is isomorphic to $V[[\hbar]]$ the complete space of formal series with coefficients in V and an indeterminate $\hbar$.

Secondly, if V is a module $\mathbb{Z}$-graded, *i.e.* the decomposition (4.11) is compatible with decomposition of $\mathfrak{g}_{\Bbbk}$ (see Definition 4.17). Then we can see $\widehat{V} = \prod_{k \in \mathbb{Z}} V_k$ as a proper submodule of $V[[\hbar]]$.

Let V and W be two $\mathfrak{g}_{\Bbbk}$-modules. If $V = \bigoplus_{k \in \mathbb{Z}} V_k$ has an algebraic sum decomposition, the completed tensor product (or formal tensor product) is given by

$$V \widehat{\otimes} W = \prod_{k \in \mathbb{Z}} (V_k \otimes W).$$

In particular, if W has also an algebraic sum decomposition, we have that

$$V \widehat{\otimes} W = \prod_{i,j \in \mathbb{Z}} V_i \otimes W_j.$$

In both cases the action of $U(\mathfrak{g}_{\Bbbk})$ on the formal tensor product $V \widehat{\otimes} W$ is given by

$$x \cdot v \otimes w = \Delta(x)(v \otimes w), \qquad (x \in U(\mathfrak{g}_{\Bbbk}), v \in V, w \in W).$$

where Δ is the comultiplication of $U(\mathfrak{g}_{\Bbbk})$.

4.1.1.2 The h-diagonalizable $\mathfrak{g}_{\mathbb{R}}$-modules

Let (ρ, V) be a complex representation of the Lie algebra $\mathfrak{g}_{\Bbbk}$. For each $\lambda \in \mathbb{C}$, we consider the generalized h-eigenspace of V:

$$V[\lambda] = \{v \in V | (\exists n \in \mathbb{N})((h - \lambda \,\mathrm{id})^n v = 0)\}.$$

4.5 Definition. Let (ρ, V) be a complex representation of the Lie algebra $\mathfrak{g}_{\Bbbk}$.

a) The representation (ρ, V) is called h-admissible if

   i) V has an algebraic sum decomposition

$$V = \bigoplus_{\lambda \in \mathbb{C}} V[\lambda],$$

   ii) For all $\lambda \in \mathbb{C}$ the generalized h-eigenspace $V[\lambda]$ in the decomposition (i) is finite dimensional.

b) The representation (ρ, V) is called h-diagonalizable if

   i) V has an algebraic sum decomposition

$$V = \bigoplus_{\lambda \in \mathbb{C}} V[\lambda], \tag{4.12}$$

   ii) For all $\lambda \in \mathbb{C}$ the eigenspace $V[\lambda]$ in the decomposition (i) is a genuine eigenspace, *i.e.*

$$V[\lambda] = \{v \in V | h \cdot v = \lambda \, v\}.$$

c) We say that (ρ, V) is h-semisimple if it is h-admissible and h-diagonalizable.

d) The representation (ρ, V) is called quasisimple if the Casimir element $\mathcal{C}$ acts as a multiple of the identity map on V.

4.6 Remark. *i)* Let V be a representation of $\mathfrak{g}_{\Bbbk}$ and Λ be the spectrum of the action h on V:

$$\Lambda = \{\lambda \in \mathbb{C} \,| (\exists v \in V \setminus \{0\})(h \cdot v = \lambda \, v)\}. \tag{4.13}$$

We say that $\lambda \in \Lambda$ is a weight if $V[\lambda] \neq 0$. The spectrum Λ is actually a lattice in $\mathbb{C}$. Thus, if V is h-diagonalizable and

$$\Lambda = \{\lambda_k \,| k \in \mathbb{N}\},$$

then the decomposition of Eq. (4.12) is given by

$$V = \bigoplus_{k \in \mathbb{N}} V[\lambda_k].$$

ii) Let $V = \bigoplus_{k \in \mathbb{N}} V[\lambda_k]$ be a h-admissible $\mathfrak{g}_{\mathbb{k}}$-module. It is clear that the dual space

$$V^* = \mathrm{Hom}(V, \mathbb{C}) = \prod_{k \in \mathbb{N}} (V[\lambda_k])^*$$

may not be h-admissible. To avoid this inconvenient we define the admissible dual V^*_{adm} of V by

$$V^*_{adm} = \bigoplus_{k \in \mathbb{N}} (V[\lambda_k])^* = \bigoplus_{k \in \mathbb{N}} B_{\lambda_k},$$

where B_{λ_k} is a finite-dimensional subspace of $V^*[-\lambda_k]$. Here the action of $U(\mathfrak{g}_{\mathbb{k}})$ on V^* and V^*_{adm} is given respectively by

$$(x \cdot v^*)(w) = v^*(S(x) \cdot w), \qquad (x \in U(\mathfrak{g}_{\mathbb{k}}), v^* \in V^*, w \in V)$$

where S denotes the antipode of $U(\mathfrak{g}_{\mathbb{k}})$.

Next we give some examples of a collection of isomorphism classes of indecomposable quasisimple h-semisimple $\mathfrak{g}_{\mathbb{R}}$-modules.

4.7 Definition. Let $\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

i) (*Lowest weight modules*) For any $\lambda \in \mathbb{C}$, we consider the complex vector space V^-_λ with basis[2] $\{v^-_j | j \in \mathbb{N}\}$, and relations

$$\begin{aligned}
h \cdot v^-_j &= (\lambda + 2j) v^-_j, && j \in \mathbb{N},\\
e^+ \cdot v^-_j &= v^-_{j+1}, && j \in \mathbb{N},\\
e^- \cdot v^-_j &= -j(\lambda + j - 1) v^-_{j-1}, && j \in \mathbb{N}^\times,\\
e^- \cdot v^-_0 &= 0, &&\\
\mathcal{C} \cdot v &= \lambda(\lambda - 2) v, && v \in V^-_\lambda.
\end{aligned}$$

The element v^-_0 is called the lowest λ-weight vector of the lowest weight module V^-_λ.

[2] Here, if there is no possible confusion, we denote $v^\pm_j$ instead of $v^\pm_{\lambda,j}$.

ii) (*Highest weight modules*) For any $\lambda \in \mathbb{C}$, we consider the vector space V_λ^+ with basis $\{v_j^+ | j \in \mathbb{N}\}$, and relations

$$\begin{aligned} h \cdot v_j^+ &= (\lambda - 2j)v_j^+, && j \in \mathbb{N}, \\ e^- \cdot v_j^+ &= v_{j+1}^+, && j \in \mathbb{N}, \\ e^+ \cdot v_j^+ &= j(\lambda - j + 1)v_{j-1}^+, && j \in \mathbb{N}^\times, \\ e^+ \cdot v_0^+ &= 0, && \\ \mathcal{C} \cdot v &= \lambda(\lambda + 2)v, && v \in V_\lambda^+ \end{aligned}$$

The element v_0^+ is called the highest λ-weight vector of the highest weight module V_λ^+.

iii) (*Lowest and highest weight modules*) For any $n \in \mathbb{N}$, we consider the vector space V_n with basis $\{v_j | 0 \leq j \leq n\}$, and relations

$$\begin{aligned} h \cdot v_j &= (-n + 2j)v_j, && 0 \leq j \leq n, \\ e^+ \cdot v_j &= v_{j+1}, && 0 \leq j < n, \\ e^- \cdot v_0 &= 0 = e^+ \cdot v_n, && \\ e^- \cdot v_j &= j(n - j + 1)v_{j-1}, && 0 < j \leq n, \\ \mathcal{C} \cdot v &= n(n + 2)v, && v \in V_n. \end{aligned}$$

The element v_0 is a lowest n-weight vector and $v_n = (e^+)^n \cdot v_0$ is a highest n-weight.

iv) Let μ, λ be two complex numbers, we consider the vector space $W(\mu, \lambda)$ with basis $\{v_j | j \in \mathbb{Z}\}$, and relations

$$\begin{aligned} h \cdot v_j &= (\lambda + 2j)v_j, && j \in \mathbb{Z}, \\ e^+ \cdot v_j &= v_{j+1}, && j \in \mathbb{Z}, \\ e^- \cdot v_j &= \frac{1}{4}(\mu - (\lambda + 2(j-1))^2 - 2(\lambda + 2(j-1)))v_{j-1}, && j \in \mathbb{Z}, \\ \mathcal{C} \cdot v &= \mu v, && v \in W(\mu, \lambda). \end{aligned}$$

v) Let μ, λ be two complex numbers, we consider the vector space $\overline{W}(\mu, \lambda)$ with basis $\{\bar{v}_j | j \in$

$\mathbb{Z}\}$, and relations

$$\begin{aligned}
h \cdot \bar{v}_j &= (\lambda + 2j)\bar{v}_j, && j \in \mathbb{Z},\\
e^- \cdot \bar{v}_j &= \bar{v}_{j-1}, && j \in \mathbb{Z},\\
e^+ \cdot \bar{v}_j &= \frac{1}{4}(\mu - (\lambda + 2(j+1))^2 + 2(\lambda + 2(j+1)))\bar{v}_{j+1}, && j \in \mathbb{Z},\\
\mathcal{C} \cdot v &= \mu v, && v \in \overline{W}(\mu, \lambda).
\end{aligned}$$

vi) Let μ^+, μ^- be two complex numbers, we consider the vector space $U(\mu^+, \mu^-)$ with basis $\{v_j | j \in \mathbb{Z}\}$, and relations

$$\begin{aligned}
h \cdot v_j &= (\mu^+ - \mu^- + 2j)v_j, && j \in \mathbb{Z},\\
e^+ \cdot v_j &= (\mu^+ + j)v_{j+1}, && j \in \mathbb{Z},\\
e^- \cdot v_j &= (\mu^- - j)v_{j-1}, && j \in \mathbb{Z},\\
\mathcal{C} \cdot v &= (\mu^+ + \mu^-)(\mu^+ + \mu^- - 2)v, && v \in U(\mu^+, \mu^-).
\end{aligned}$$

4.8 Remark. According to [Kir08, Lemma 4.4] any complex representation of $\mathfrak{sl}_2(\mathbb{R})$ has a unique structure of representation of $\mathfrak{sl}_2(\mathbb{C})$ and for any V and W representations of $\mathfrak{sl}_2(\mathbb{R})$, we have

$$\mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{R})}(V, W) = \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(V, W).$$

That is, the category of complex representations of $\mathfrak{sl}_2(\mathbb{R})$ and the category of complex representations of $\mathfrak{sl}_2(\mathbb{C})$ are equivalents.

4.9 Definition. We say V is a standard module of $\mathfrak{sl}_2(\Bbbk)$ if V is one of the $\mathfrak{sl}_2(\Bbbk)$-modules of Definition 4.7. According to structure Theorem [HT92, Theorem 1.3.1 page 64] any indecomposable quasisimple, h-admissible, and h-semisimple module is isomorphic to some indecomposable standard module.

4.10 Proposition. *i) If $\lambda \notin -\mathbb{N}$, then the lowest weight module V_λ^- is irreducible.*

ii) If $\mu \notin \mathbb{N}$, then the highest weight module V_μ^+ is irreducible.

iii) For any $n \in \mathbb{N}$ the $\mathfrak{g}_\mathbb{R}$-module V_n is irreducible. Moreover if V is an irreducible $\mathfrak{g}_\mathbb{R}$-module of dimension $n+1$ then V is $\mathfrak{g}$-isomorphic to V_n.

Proof. See [HT92, Proposition 1.1.12 and 1.2.6]. □

4.11 Proposition. *Let $\mu^+, \mu^- \in \mathbb{C}$ and $U(\mu^+, \mu^-)$ be the $\mathfrak{sl}_2(\mathbb{R})$-module defined in (4.7)-(iv). We have that $U(\mu^+, \mu^-)$ is irreducible if neither of the μ^+ or μ^- is an integer.*

Proof. See [HT92, Proposition 1.2.9 pg 63]. □

4.12 Proposition. *Let $V_\lambda^- = \mathrm{Span}\{v_j^- | j \in \mathbb{N}\}$ and $V_\mu^- = \mathrm{Span}\{\tilde{v}_j^- | j \in \mathbb{N}\}$ be two $\mathfrak{g}_\mathbb{R}$-modules of lowest weight λ and μ, respectively. If $\lambda + \mu \notin -\mathbb{N}$, we have*

i) The tensor product $V_\lambda^- \otimes V_\mu^-$ has a direct sum decomposition

$$V_\lambda^- \otimes V_\mu^- = \bigoplus_{n\in\mathbb{N}} V_\lambda^- \otimes V_\mu^-[\lambda + \mu + 2n] \simeq \bigoplus_{n\in\mathbb{N}} V_{\lambda+\mu+2n}. \tag{4.14}$$

ii) The h-eigenspace $V_\lambda^- \otimes V_\mu^-[\lambda + \mu + 2n]$ of $V_\lambda^- \otimes V_\mu^-$ with eigenvalue $\lambda + \mu + 2n$ is finite dimensional with basis $\{v_i^- \otimes \tilde{v}_j^- | i, j \in \mathbb{N}, i + j = n\}$.

iii) The element $r_n^-(\lambda, \mu) = r_n^-$ given by

$$\begin{aligned} r_n^- &= \sum_{i+j=n} (-1)^i \binom{\lambda + n - 1}{j} \binom{\mu + n - 1}{i} v_i^- \otimes \tilde{v}_j^- \\ &= \sum_{i+j=n} (-1)^i \binom{\lambda + n - 1}{j} \binom{\mu + n - 1}{i} (e^+)^i \cdot v_0^- \otimes (e^+)^j \cdot \tilde{v}_0^- \end{aligned} \tag{4.15}$$

is a lowest $\lambda + \mu + 2n$-weight vector, i.e. $r_n^- \in V_\lambda^- \otimes V_\mu^-[\lambda + \mu + 2n]$ and $e^- \cdot r_n^- = 0$.

Proof. In [HT92, Proposition 2.1.1], the e^--null vector of h-eigenvalue $\lambda + \mu + 2n$ is given by

$$u_n = \sum_{i+j=n} (-1)^i \binom{n}{j} \frac{1}{(\lambda + i - 1)!(\mu + j - 1)!} (e^+)^i \cdot v_0^- \otimes (e^+)^j \cdot \tilde{v}_0^-.$$

It is easy to show that $\frac{(\lambda+n-1)!(\mu+n-1)!}{n!} u_n = r_n^-$. □

4.13 Remark. For consistency with the Rankin-Cohen elements of the Chapter 3, we keep the notation $r_n^-(\lambda, \mu) = r_n^-$.

There is an analogous version for the case of highest weight modules.

4.14 Proposition. *Let $V_\lambda^+ = \mathrm{Span}\{v_j^+ | j \in \mathbb{N}\}$ and $V_\mu^+ = \mathrm{Span}\{\tilde{v}_j^+ | j \in \mathbb{N}\}$ be two $\mathfrak{g}_\mathbb{R}$-modules of highest weight λ and μ, respectively. If $\lambda + \mu \notin \mathbb{N}$, we have*

i) The tensor product $V_\lambda^+ \otimes V_\mu^+$ has a direct sum decomposition

$$V_\lambda^+ \otimes V_\mu^+ = \bigoplus_{n\in\mathbb{N}} V_\lambda^+ \otimes V_\mu^+[\lambda + \mu - 2n] \simeq \bigoplus_{n\in\mathbb{N}} V_{\lambda+\mu-2n}. \tag{4.16}$$

ii) *The h-eigenspace $V_\lambda^+ \otimes V_\mu^+[\lambda+\mu-2n]$ of $V_\lambda^+ \otimes V_\mu^+$ with eigenvalue $\lambda+\mu-2n$ is finite dimensional with basis $\{v_i^+ \otimes \tilde{v}_j^+ | i,j \in \mathbb{N}, i+j=n\}$.*

iii) *The element $r_n^+(\lambda,\mu) = r_n^+$ given by*

$$\begin{aligned} r_n^+ &= \sum_{i+j=n} (-1)^i \binom{-\lambda+n-1}{j} \binom{-\mu+n-1}{i} v_i^- \otimes \tilde{v}_j^- \\ &= \sum_{i+j=n} (-1)^i \binom{-\lambda+n-1}{j} \binom{-\mu+n-1}{i} (e^+)^i \cdot v_0^- \otimes (e^+)^i \cdot \tilde{v}_0^- \end{aligned} \tag{4.17}$$

is a highest $\lambda+\mu-2n$-weight vector, i.e. $r_n^+ \in V_\lambda^+ \otimes V_\mu^+[\lambda+\mu-2n]$ and $e^+ \cdot r_n^+ = 0$.

Now the tensor product $V_\lambda^+ \otimes V_\mu^-$ is not an h-admissible module. However, it has a $\mathbb{Z}$-graduation of h-eigenspaces. To see that, we consider the basis

$$\tilde{e}^\pm = \frac{1}{2}\left(e^+ + e^- \pm \mathbf{i}\, h\right)$$
$$\tilde{h} = \mathbf{i}\left(e^+ - e^-\right)$$

of $\mathfrak{su}(1,1) \simeq \mathfrak{sl}_2(\mathbb{R})$ the Lie algebra of the Lie group

$$\mathrm{SU}(1,1) = \mathscr{C}\,\mathrm{SL}(2,\mathbb{R})\,\mathscr{C}^{-1} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| a,b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\},$$

where $\mathscr{C}$ denotes the Cayley matrix

$$\mathscr{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathbf{i} \\ -\mathbf{i} & 1 \end{pmatrix} = (\overline{\mathscr{C}})^{-1} \in \mathrm{SL}(2,\mathbb{C}).$$

Here, we recall[3] that if $X \in \mathfrak{sl}_2(\mathbb{R})$, the element $\tilde{X} = \mathscr{C}\, X\, \mathscr{C}^{-1}$ belongs to $\mathfrak{su}(1,1)$.

Let $\tilde{V}_\lambda^+ = \mathrm{Span}\{\tilde{v}_{\lambda,j}^+ | j \in \mathbb{N}\}$ and $\tilde{V}_\mu^- = \mathrm{Span}\{\tilde{v}_{\mu,j}^- | j \in \mathbb{N}\}$ be two $\mathfrak{su}(1,1)$-modules of highest weight λ and lowest weight μ respect relatively to the basis $\{\tilde{e}^-, \tilde{h}, \tilde{e}^+\}$. We recall that $\tilde{v}_{\lambda,j}^+$ and $\tilde{v}_{\mu,j}^-$ are given by

$$\tilde{v}_{\lambda,j}^+ = (\tilde{e}^-)^j v_\lambda^+ \qquad \tilde{v}_{\mu,j}^- = (\tilde{e}^+)^j v_\mu^-. \tag{4.18}$$

By Remark 4.2 the action of $\mathfrak{su}(1,1) \simeq \mathfrak{g}_{\mathbb{R}}$ on $\tilde{V}_\lambda^+$ and $\tilde{V}_\mu^-$ is given by (i) and (ii) of Definition 4.7.

[3] See Remark 3.32

4.15 Proposition. *Consider $\tilde{V}_\lambda^+$ an $\mathfrak{su}(1,1)$-module of highest weight λ and $\tilde{V}_\mu^-$ an $\mathfrak{su}(1,1)$-module of lowest weight μ. Then for each $\alpha \in \mathbb{C}$ there is a unique (up to a scalar multiple) formal h eigenvector in $(\tilde{V}_\lambda^+)^\vee$ (resp. $(\tilde{V}_\mu^-)^\vee$) which has eigenvalue α. Moreover, if $\{\tilde{v}_{\lambda,j}^+ | j \in \mathbb{N}\}$ is a basis of $\tilde{h}$ eigenvectors (as in Equation (4.18)) which spans $\tilde{V}_\lambda^+$, then the formal vectors*

$$u_\lambda^{++} = \sum_{k\in\mathbb{N}} \frac{(\mathbf{i})^k}{k!} \tilde{v}_{\lambda,k}^+ \qquad u_\lambda^{+-} = \sum_{k\in\mathbb{N}} \frac{(-\mathbf{i})^k}{k!} \tilde{v}_{\lambda,k}^+ \tag{4.19}$$

$$u_\lambda^{-+} = \sum_{k\in\mathbb{N}} \frac{(-\mathbf{i})^k}{k!} \tilde{v}_{\lambda,k}^- \qquad u_\lambda^{--} = \sum_{k\in\mathbb{N}} \frac{(\mathbf{i})^k}{k!} \tilde{v}_{\lambda,k}^- \tag{4.20}$$

satisfy the relations:

$$e^+ \cdot u_\lambda^{\pm +} = 0 = e^- \cdot u_\lambda^{\pm -}, \quad h \cdot u_\lambda^{-\pm} = \pm \lambda\, u_\lambda^{-\pm}, \quad h \cdot u_\lambda^{+\pm} = \mp \lambda\, u_\lambda^{+\pm}.$$

Proof. See [HT92, Proposition 3.1.3 page 78]. □

4.16 Proposition. *Let V_λ^+ be a $\mathfrak{g}_\mathbb{R}$-module of highest weight λ and $V_\mu^- = \mathrm{Span}\{v_{\mu,j}^- | j \in \mathbb{N}\}$ be a $\mathfrak{g}_\mathbb{R}$-module of lowest weight μ.*

i) The h-eigenspace $V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2n]$ of eigenvalue $\lambda + \mu + 2n$, $n \in \mathbb{Z}$, is spanned by the set $\{v_{\lambda,i}^+ \otimes v_{\mu,j}^- | j - i = n\}$ and it is infinite dimensional. Moreover, $V_\lambda^+ \otimes V_\mu^-$ has a h-decomposition

$$V_\lambda^+ \otimes V_\mu^- = \bigoplus_{n\in\mathbb{Z}} V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2n], \tag{4.21}$$

such that

$$\begin{aligned} e^- \cdot V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2n] &\subseteq V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2(n-1)] \\ h \cdot V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2n] &\subseteq V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2n] \\ e^+ \cdot V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2n] &\subseteq V_\lambda^+ \otimes V_\mu^-[\lambda + \mu + 2(n+1)] \end{aligned}$$

ii) Let $\{\tilde{v}_{\lambda,j}^+ | j \in \mathbb{N}\}$ and $\{\tilde{v}_{\mu,j}^- | j \in \mathbb{N}\}$ be the $\tilde{h}$-basis in the spaces $\tilde{V}_\lambda^+$ and $\tilde{V}_\mu^-$ given in Equation (4.18). Then the formal vector $u^+ \in V_\lambda^+ \check{\otimes} V_\mu^-$ given by

$$\begin{aligned} u^+ &= u_\lambda^{++} \otimes u_\mu^{-+} \\ &= \sum_{i,j\in\mathbb{N}} \frac{(\mathbf{i})^i(-\mathbf{i})^j}{i!j!} \tilde{v}_{\lambda,i}^+ \otimes \tilde{v}_{\mu,j}^- \\ &= \sum_{n\in\mathbb{Z}} (-\mathbf{i})^n R_n(\lambda, \mu), \end{aligned} \tag{4.22}$$

is the unique h eigenvector with eigenvalue $\mu - \lambda$ such that

$$e^+ \cdot u^+ = 0.$$

Here we have written

$$R_n(\lambda,\mu) = \sum_{j-i=n} \frac{\tilde{v}^+_{\lambda,i} \otimes \tilde{v}^-_{\mu,j}}{i!j!} \tag{4.23}$$

iii) Let $\{\tilde{v}^+_{\lambda,j} | j \in \mathbb{N}\}$ and $\{\tilde{v}^-_{\mu,j} | j \in \mathbb{N}\}$ be the $\tilde{h}$-basis in the spaces $\tilde{V}^+_\lambda$ and $\tilde{V}^-_\mu$ given in Equation (4.18). Then the formal vector $u^- \in V^+_\lambda \check{\otimes} V^-_\mu$ given by

$$\begin{aligned} u^- &= u^{+-}_\lambda \otimes u^{--}_\mu \\ &= \sum_{i,j\in\mathbb{N}} \frac{(-\mathbf{i})^i(\mathbf{i})^j}{i!j!} \tilde{v}^+_{\lambda,i} \otimes \tilde{v}^-_{\mu,j} \\ &= \sum_{n\in\mathbb{Z}} (\mathbf{i})^n R_n(\lambda,\mu), \end{aligned} \tag{4.24}$$

is the unique h eigenvector with eigenvalue $\lambda - \mu$ such that

$$e^- \cdot u^- = 0.$$

iv) For $n \in \mathbb{Z}$ and $R_n(\lambda,\mu) = R_n$ as in Eq. (4.23), we have

$$\tilde{e}^- \cdot R_n = -(\mu + n - 1)R_{n-1},$$

$$\tilde{h} \cdot R_n = (\lambda + \mu + 2n)R_n,$$

$$\tilde{e}^+ \cdot R_n = (\lambda + n + 1)R_{n+1}.$$

Therefore $\{R_n(\lambda,\mu) | n \in \mathbb{Z}\}$ spans an $sl(2)$ module of the type $U(\lambda+1, 1-\mu)$ (See Definition (4.7)-(iv)).

Proof. The prove of all items $(i) - (iv)$ can be found in [HT92, Sections 2.3, 3.1 and 3.2 pages 73-81]. □

4.1.2 $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{k})$-modules

We fix in $\mathfrak{g}_\mathbb{k} = \mathfrak{sl}_2(\mathbb{k})$ the following $\mathbb{Z}$-graduation (see Eq. (1.10)):

$$\mathfrak{g}_\mathbb{k} = (\mathfrak{g}_\mathbb{k})_{-1} \oplus (\mathfrak{g}_\mathbb{k})_0 \oplus (\mathfrak{g}_\mathbb{k})_1 = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{4.25}$$

where $(\mathfrak{g}_{\Bbbk})_{\pm 1} = \mathfrak{n}_{\pm} = \Bbbk\, e^{\pm}$ and $(\mathfrak{g}_{\Bbbk})_0 = \mathfrak{h} = \Bbbk\, h$.

If $\prod_{k\in\mathbb{Z}} V_k$ is a representation of the Lie algebra $\mathfrak{g}_{\Bbbk}$, we say that it is compatible with the graduation of $\mathfrak{g}_{\Bbbk}$ if for any $(i,j) \in \{-1,0,1\} \times \mathbb{N}$ we have that $(\mathfrak{g}_{\Bbbk})_i \cdot V_j \subset V_{i+j}$.

4.17 Definition. A complex representation V of the Lie algebra $\mathfrak{g}_{\Bbbk}$ which has an algebraic sum decomposition of complex vector spaces

$$V = \bigoplus_{k\in\mathbb{Z}} V_k \tag{4.26}$$

is a $\mathbb{Z}$-graded module if the decomposition (4.26) of V is compatible with the graduation of $\mathfrak{g}_{\Bbbk}$.

Let $V = \prod_{k\in\mathbb{Z}} V_k$ and $W = \prod_{k\in\mathbb{Z}} W_k$ be two representations of $\mathfrak{g}_{\Bbbk}$ compatible with the graduation of $\mathfrak{g}_{\Bbbk}$. A $\mathfrak{g}_{\Bbbk}$-morphism $f : V \to W$ is a graded morphism if for all $k \in \mathbb{Z}$ we have $f(V_k) \subseteq W_k$.

According to Definition 4.7, any standard module of $\mathfrak{sl}_2(\Bbbk)$ is h-semisimple and the weight decomposition of h eigenvectors is a $\mathbb{Z}$-graduation. By Propositions 4.12 and 4.14, we know that $V_\lambda^+ \otimes V_\mu^+$, $V_\lambda^+ \otimes V_\mu^+$ are also h-semisimple and their weight decompositions of h eigenvectors are $\mathbb{Z}$-graduations. However, the tensor product $V_\lambda^+ \otimes V_\mu^-$ is not h-semisimple but it is h-diagonalizable as show the following proposition.

4.18 Proposition. *Let V_λ^+ be a $\mathfrak{g}_{\Bbbk}$-module of highest weight λ and V_μ^- be a $\mathfrak{g}_{\Bbbk}$-module of lowest weight μ. Then the tensor product $V_\lambda^+ \otimes V_\mu^-$ is diagonalizable such that its h-decomposition is a $\mathbb{Z}$-graduation.*

Proof. By Proposition 4.16, we know that

$$V_\lambda^+ \otimes V_\mu^- = \bigoplus_{n\in\mathbb{Z}} V_\lambda^+ \otimes V_\mu^-[\lambda+\mu+2n],$$

and $\{V_\lambda^+ \otimes V_\mu^-[\lambda+\mu+2n] | n \in \mathbb{Z}\}$ is indeed a $\mathbb{Z}$-graduation of V. $\square$

4.1.3 Verma modules and Shapovalov form on $\mathfrak{sl}_2(\Bbbk)$

In preparation for the main result we review some properties of Verma modules.

Let $\mathfrak{p}_+$ be the Lie subalgebra of $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$ generated by h and e^+. If z is a non zero complex number, the linear map $\chi : \Bbbk h \to \mathbb{C}$ determined by $h \mapsto z$ is a nonsingular character of $\mathfrak{h} = \Bbbk h$. Then $\mathfrak{g}$ is a polarized Lie algebra (Definition 1.19) with direct sum decomposition (4.25)

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ .$$

For any $\tilde{\lambda} \in \mathbb{C}$, $\chi_{\tilde{\lambda}} : \mathfrak{h} \to \mathbb{C}$ defined a character of $\mathfrak{h}$ by $\chi_{\tilde{\lambda}}(h) = \tilde{\lambda} z := \lambda$. Then the space $\mathbb{C}_\lambda^+ = \mathbb{C}\, v_\lambda^+$ (Remark 1.20) is a representation of $\mathfrak{p}_+$ given by

$$h \cdot v_\lambda^+ = \lambda\, v_\lambda^+, \qquad \text{and} \qquad e^+ \cdot v_\lambda^+ = 0.$$

Correspondingly, we define the Lie subalgebra $\mathfrak{p}_- \subset \mathfrak{g}$ generated by h and e^-, and $\mathbb{C}_\lambda^- := \mathbb{C}\, v_\lambda^-$ the representation of dimension 1 of $\mathfrak{p}_-$. The Verma module M_λ^+ of highest (resp. M_λ^- of lowest) weight with highest (resp. lowest) weight λ is defined by

$$M_\lambda^\pm = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\pm)} \mathbb{C}_\lambda^\pm .$$

According to Poincaré-Birkhoff-Witt Theorem $M_\lambda^\pm$ is isomorphic as vector space to $U(\mathfrak{n}_\mp)$. The isomorphism $U(\mathfrak{n}_\mp) \to M_\lambda^\pm$ is explicitly given by

$$u \to u \otimes v_\lambda^\pm = u v_\lambda^\pm,$$

see for instance [Kir08, Proposition 8.14 page 168].

The space $U(\mathfrak{n}_\pm)$ has a vector space decomposition

$$U(\mathfrak{n}_\pm) = \Bbbk \oplus \Bbbk\, e^\pm \oplus \cdots \oplus \Bbbk (e^\pm)^n \oplus \cdots \tag{4.27}$$

Then[4] $\{(e^\pm)^n\}_{n\in\mathbb{N}}$ is a basis of the universal enveloping space $U(\mathfrak{n}_\pm)$. We consider the vectors

$$v_{\lambda,k}^\pm = \frac{1}{k!}(e^\mp)^k \otimes v_\lambda^\pm = \frac{1}{k!}(e^\mp)^k v_\lambda^\pm$$

which span the $\Bbbk$-vector space $M_\lambda^\pm$.

4.19 Proposition. *The action of $\mathfrak{g}$ on $M_\lambda^\pm$ in the $\Bbbk$-basis $(v_{\lambda,k}^\pm)_{k\in\mathbb{N}}$ is given by:*

$$e^- \cdot v_{\lambda,k}^+ = (k+1) v_{\lambda,k+1}^+,$$

$$h \cdot v_{\lambda,k}^+ = (\lambda - 2k) v_{\lambda,k}^+,$$

$$e^+ \cdot v_{\lambda,k}^+ = (\lambda - (k-1)) v_{\lambda,k-1}^+.$$

[4] Here $(e^\pm)^0 = 1$ is the unit element in $U(\mathfrak{n}_\pm)$

and

$$e^{-}\cdot v^{-}_{\lambda,k} = -(\lambda+(k-1))v^{-}_{\lambda,k-1},$$
$$h\cdot v^{-}_{\lambda,k} = (\lambda+2k)v^{-}_{\lambda,k},$$
$$e^{+}\cdot v^{-}_{\lambda,k} = (k+1)v^{-}_{\lambda,k+1}.$$

Proof. Straightforward. □

4.20 Remark. In our particular setting, we have that $V^{\pm}_{\lambda}\simeq M^{\pm}_{\lambda}$ where the $\mathfrak{sl}_2(\Bbbk)$-isomorphism is given by

$$(e^{\mp})^{k}v^{\pm}_{\lambda}\to \frac{1}{k!}(e^{\mp})^{k}v^{\pm}_{\lambda}.$$

We keep the notation $V^{\pm}_{\lambda}$ if the basis of h-eigenvectors is given by $(e^{\mp})^{k}v^{\pm}_{\lambda}$ and $M^{\pm}_{\lambda}$ if the basis of h-eigenvectors is given by $\frac{1}{k!}(e^{\mp})^{k}v^{\pm}_{\lambda}$.

We have that for each $k\in\mathbb{N}$, $v^{\pm}_{\lambda,k}\in M^{\pm}_{\lambda}[\lambda\mp 2k]$. Namely, $M^{\pm}_{\lambda}[\lambda\mp 2k]=\Bbbk\, v^{\pm}_{\lambda,k}$ because the dimension of $M^{\pm}_{\lambda}[\lambda\mp 2k]$ is 1 (see [Kir08, Proposition 8.14 page 168]). Then $M^{\pm}_{\lambda}$ has an h-decomposition

$$M^{\pm}_{\lambda}=\bigoplus_{k\in\mathbb{N}} M^{\pm}_{\lambda}[\lambda\mp 2k] \tag{4.28}$$

which also is in this case a $\mathbb{Z}$-graduation. Now by (4.28) it is clear that for any $\lambda\in\mathbb{C}$ the Verma modules $M^{\pm}_{\lambda}$ are h-semisimples $\mathbb{Z}$-graded representations of $\mathfrak{g}_{\Bbbk}$.

Now we are ready to introduce the Shapovalov form (Definition 1.21) for the case $\mathfrak{sl}_2(\mathbb{C})$. We recall that the Shapovalov form $\langle\ ,\ \rangle_{\mathrm{Sh}}: M^{+}_{\lambda}\times M^{-}_{-\lambda}\to\mathbb{C}$ is given by

$$(yv_{\lambda}, xv_{-\lambda})\mapsto \chi_{\lambda}(\mathrm{HC}(S(x)y)),$$

where $x\in U(\mathfrak{n}_{+})$, $y\in U(\mathfrak{n}_{-})$, $\mathrm{HC}: U(\mathfrak{g})\to U(\mathfrak{g}_0)$ is the Harish-Chandra projection and $S: U(\mathfrak{g})\to U(\mathfrak{g})$ is the antipode of $U(\mathfrak{g})$.

4.21 Proposition. *Let M^{+}_{λ} and $M^{-}_{-\lambda}$ be the Verma modules of $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$ of highest weight λ and lowest weight $-\lambda$ respectively. The Shapovalov bilinear form $\langle\ ,\ \rangle_{\mathrm{Sh}}: M^{+}_{\lambda}\times M^{-}_{-\lambda}\to\mathbb{C}$ in Definition 1.21 is given by*

$$\left\langle v^{+}_{\lambda,i}, v^{-}_{-\lambda,j}\right\rangle_{\mathrm{Sh}}=\delta_{ij}(-1)^{i}\binom{\lambda}{i}.$$

Where δ_{ij} stands for the Kronecker symbol.

Proof. Let $N = \mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\,\mathfrak{n}_+$ be the bilateral ideal of $U(\mathfrak{g})$. By the Lemma 4.1 if $i \neq j$ we have that $(e^+)^i(e^-)^j = 0$ modulo N and

$$(e^+)^k(e^-)^k = a_0 \cdot a_1 \cdots a_{k-1}.$$

if $i = j = k$. Here the a_k elements are defined by

$$a_k = e^-e^+ + (k+1)(h-k) \qquad\qquad k \in \mathbb{N}\,.$$

Then modulo N,

$$\begin{aligned}(e^+)^k(e^-)^k &= h(h-1)(2(h-2))\cdots(k(h-(k-1)))\\ &= k!h(h-1)\cdots(h-(k-1)).\end{aligned}$$

Now we have

$$\begin{aligned}\left\langle v^+_{\lambda,i}, v^-_{-\lambda,j}\right\rangle_{\mathrm{Sh}} &= \left\langle \frac{1}{i!}(e^+)^i v^+_\lambda, \frac{1}{j!}(e^+)^j v^-_{-\lambda}\right\rangle_{\mathrm{Sh}}\\ &= \frac{1}{i!j!}\chi_{\tilde\lambda}(\mathrm{HC}(S((e^+)^i)(e^-)^j))\\ &= \delta_{ij}\frac{(-1)^i}{i!}\chi_{\tilde\lambda}(h(h-1)\cdots(h-(i-1))))\\ &= \delta_{ij}\frac{(-1)^i}{i!}\,\lambda(\lambda-1)\cdots(\lambda-(i-1)))\\ &= \delta_{ij}(-1)^i\binom{\lambda}{i}.\end{aligned}$$

□

4.22 Definition. For any $\lambda \in \mathbb{C}$, we define a pairing $\langle\ ,\ \rangle_\lambda : U(\mathfrak{n}_-)\otimes U(\mathfrak{n}_+) \to \mathbb{C}$ given by

$$y \otimes x \mapsto \langle y, x\rangle_\lambda = \left\langle y v^+_\lambda, x v^-_{-\lambda}\right\rangle_{\mathrm{Sh}} = \chi_{\tilde\lambda}(\mathrm{HC}(S(x)y)) \tag{4.29}$$

By abuse of notations, we also call to the pairing $\langle\ ,\ \rangle_\lambda$ the Shapovalov form.

4.23 Corollary. *In the basis $\{(e^\pm)^k\}_{k\in\mathbb{N}}$ of $U(\mathfrak{n}_\pm)$ the pairing $\langle\ ,\ \rangle_\lambda$ is expressed by*

$$\langle (e^-)^i, (e^+)^j\rangle_\lambda = \delta_{ij}(-1)^i i!(\lambda)_i, \tag{4.30}$$

where $(\lambda)_i = \lambda(\lambda-1)\cdots(\lambda-(i-1))$ is the Pochhammer symbol (falling factorial).

Proof. We have that

$$\begin{aligned}\langle (e^-)^i, (e^+)^j\rangle_\lambda &= \chi_{\tilde{\lambda}}(\mathrm{HC}(S((e^+)^i)(e^-)^j)) \\ &= \delta_{ij}(-1)^i i!\,\lambda(\lambda-1)\cdots(\lambda-(i-1)) \\ &= \delta_{ij}(-1)^i i!(\lambda)_i.\end{aligned}$$

as we wanted to show. □

4.24 Proposition. *For all $\lambda \in \mathbb{C} \setminus \mathbb{N}$, the pairing $\langle\ ,\ \rangle_\lambda : U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+) \to \mathbb{C}$ defined in (4.29) is nonsingular.*

Proof. Follows from equation (4.30). □

4.1.4 The canonical element F_λ

Throughout this section we fix λ in $\mathbb{C} \setminus \mathbb{N}$ such that the pairing

$$\langle\ ,\ \rangle_\lambda : U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+) \to \mathbb{C}$$

is nonsingular.

The universal enveloping algebra $U(\mathfrak{n}_-)$ is isomorphic to the polynomial algebra of $\Bbbk$ in the indeterminate e^-, $\Bbbk[e^-]$. If V is a complex representation of $\mathfrak{g}_\Bbbk$ the completed tensor product $U(\mathfrak{n}_-) \mathbin{\widehat{\otimes}} V$ from Definition 2.21 is isomorphic to the formal representation:

$$\begin{aligned}U(\mathfrak{n}_-) \mathbin{\widehat{\otimes}} V &= \varprojlim (U(\mathfrak{n}_-) \otimes_{\Bbbk[e^-]} V)/(e^-)^n (U(\mathfrak{n}_-) \otimes_{\Bbbk[e^-]} V) \\ &\simeq \prod_{n\in\mathbb{N}} (e^-)^n \otimes V.\end{aligned}$$

That is, an element x in $U(\mathfrak{n}_-) \mathbin{\widehat{\otimes}} V$ is the form

$$x = \sum_{n\in\mathbb{N}} (e^-)^n \otimes v_n,$$

with $v_n \in V$.

Similarly, we can consider the completed tensor product

$$\begin{aligned}U(\mathfrak{n}_+) \mathbin{\widehat{\otimes}} V &= \varprojlim (U(\mathfrak{n}_-) \otimes_{\Bbbk[e^+]} V)/(e^+)^n (U(\mathfrak{n}_-) \otimes_{\Bbbk[e^+]} V) \\ &\simeq \prod_{n\in\mathbb{N}} (e^+)^n \otimes V.\end{aligned}$$

Let

$$1, \frac{(e^+)}{\langle (e^-), (e^+) \rangle_\lambda}, \frac{(e^+)^2}{\langle (e^-)^2, (e^+)^2 \rangle_\lambda}, \cdots, \frac{(e^+)^n}{\langle (e^-)^n, (e^+)^n \rangle_\lambda}, \cdots$$

be the dual basis of $\{(e^-)^n\}_{n\in\mathbb{N}} \subset U(\mathfrak{n}_-)$ relatively to the Shapovalov form $\langle\ ,\ \rangle_\lambda$.

4.25 Definition. For any $\lambda \in \mathbb{C} \setminus \mathbb{N}$, we consider the element F_λ in $U(\mathfrak{n}_-) \widehat{\otimes} U(\mathfrak{n}_+)$ defined by

$$\begin{aligned} F_\lambda &= \sum_{k\in\mathbb{N}} \frac{(e^-)^k \otimes (e^+)^k}{\langle (e^-)^k, (e^+)^k \rangle_\lambda} \\ &= \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k!(\lambda)_k} (e^-)^k \otimes (e^+)^k. \end{aligned} \tag{4.31}$$

The follow proposition (Prop. 4.26) show us that F_λ does not depend on the choice of a basis. Then we call F_λ the canonical element corresponding to the Shapovalov form $\langle\ ,\ \rangle_\lambda$.

4.26 Proposition. *Let $\lambda \in \mathbb{C} \setminus \mathbb{N}$. The map*

$$\begin{aligned} \psi_\lambda : U(\mathfrak{n}_-) \widehat{\otimes} U(\mathfrak{n}_+) &\to \mathrm{End}(\widehat{(U(\mathfrak{n}_+))}) \\ a \otimes b &\mapsto [c \mapsto \langle a, c \rangle_\lambda b] \end{aligned}$$

is an isomorphism of vector spaces such that $F_\lambda \mapsto \mathrm{id}$.

Proof. For each $(i, j, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, let $a_{i,j}, b_k \in \Bbbk$. The expression of ψ_λ in the basis $\{(e^-)^k\}_{k\in\mathbb{N}}$ and $\{(e^+)^k\}_{k\in\mathbb{N}}$ of $U(\mathfrak{n}_-)$ and $U(\mathfrak{n}_+)$, respectively, is given by

$$\begin{aligned} &\psi_\lambda \left(\sum_{i,j\in\mathbb{N}} a_{i,j} (e^-)^i \otimes (e^+)^j \right) \left(\sum_{k\in\mathbb{N}} b_k (e^+)^k \right) \\ &= \sum_{i,j,k\in\mathbb{N}} a_{i,j} b_k \langle (e^-)^i, (e^+)^k \rangle_\lambda (e^+)^j \\ &= \sum_{j\in\mathbb{N}} \left(\sum_{i\in\mathbb{N}} (-1)^i i! (\lambda)_i b_i a_{i,j} \right) (e^+)^j. \end{aligned} \tag{4.32}$$

From the equation (4.32) we can directly conclude that ψ_λ is an isomorphism and $\psi_\lambda(F_\lambda) = \mathrm{id}$. □

In the next proposition we list some proprieties of the canonical element F_λ.

4.27 Proposition. *Let $F : \mathbb{C} \to U(\mathfrak{n}_-) \widehat{\otimes} U(\mathfrak{n}_+)$ be the function defined by*

$$\lambda \mapsto F_\lambda.$$

a) The function F is meromorphic on $\mathbb{C}$ and holomorphic at $\lambda = \infty$. The only poles of F are of order one and are located at integers points $n \in \mathbb{N}$.

b) The residue of F at $\lambda = n$ is given by

$$\operatorname{Res}_{\lambda=n} F_\lambda = \frac{(-1)^{n+1}}{n!} \sum_{k>n} \frac{(e^-)^k \otimes (e^+)^k}{k!(k-(n+1))!}.$$

c) For any $\lambda \in \mathbb{C}^\times$, such that $1/\lambda \notin \mathbb{N}$ one has

$$F_{1/\lambda} = 1 \otimes 1 - \lambda(e^- \otimes e^+) - \sum_{k>1} \frac{\lambda^k[(e^-)^k \otimes (e^+)^k]}{k!(\lambda-1)(2\lambda-1)\cdots((k-1)\lambda-1)}. \tag{4.33}$$

d) For any $\lambda \in \mathbb{C} \smallsetminus \mathbb{N}$, we have that

$$\frac{d}{d\lambda} F_\lambda = \sum_{k>0} (-1)^{k+1} \frac{(e^-)^k \otimes (e^+)^k}{k!(\lambda)_k} \sum_{l=0}^{k-1} \frac{1}{\lambda - l}.$$

Proof. (a) For any $\lambda \in \mathbb{C}$, it is clear that F_λ is a rational function with singularities in the set $\mathbb{N}$. Let $n \in \mathbb{N}$ and $W \subset \mathbb{C}$ be an open neighbourhood of n such that $W \cap \mathbb{N} = \{n\}$. We consider the function $f : W \to U(\mathfrak{n}_-) \mathbin{\widehat{\otimes}} U(\mathfrak{n}_+)$ given by

$$\begin{aligned} f(\lambda) &= (\lambda - n) F_\lambda \\ &= \sum_{k=0}^{n} (-1)^k \frac{(\lambda-n)}{k!(\lambda)_k} (e^-)^k \otimes (e^+)^k + \frac{(-1)^{n+1}}{(n+1)!(\lambda)_n} (e^-)^{n+1} \otimes (e^+)^{n+1} \\ &\quad + \frac{1}{(\lambda)_n} \sum_{k>n+1} \frac{(-1)^k}{k!} \frac{(e^-)^k \otimes (e^+)^k}{(\lambda-(n+1))\cdots(\lambda-(k-1))}. \end{aligned} \tag{4.34}$$

Since the function f is holomorphic in W and $f(n)$ is nonzero, then F at $\lambda = n$ has a simple pole.

(b) We take the limit of f as λ tends to n in the formula (4.34),

$$\begin{aligned} \lim_{\lambda \to n} f(\lambda) &= (-1)^{n+1} \frac{(e^-)^{n+1} \otimes (e^+)^{n+1}}{(n+1)!n!} + \frac{1}{n!} \sum_{k>n+1} \frac{(-1)^k}{k!} \frac{(e^-)^k \otimes (e^+)^k}{(-1)(-2)\cdots(n-(k-1))} \\ &= \frac{(-1)^{n+1}}{n!} \sum_{k>n} \frac{(e^-)^k \otimes (e^+)^k}{k!(k-(n+1))!}. \end{aligned} \tag{4.35}$$

In particular if $n = 0$ we have

$$\operatorname{Res}_{\lambda=0} F_\lambda = \lim_{\lambda \to 0} \lambda F_\lambda = -\sum_{k>0} \frac{(e^-)^k \otimes (e^+)^k}{k!(k-1)!}.$$

(c) Replacing $\lambda \to 1/\lambda$ in the equation (4.31) we obtain

$$F_{1/\lambda} = 1 \otimes 1 - \lambda(e^- \otimes e^+) - \sum_{k>1} \frac{\lambda^k[(e^-)^k \otimes (e^+)^k]}{k!(\lambda-1)(2\lambda-1)\cdots((k-1)\lambda-1)}.$$

Thus in particular we have that

$$\lim_{\lambda\to 0} F_{1/\lambda} = 1 \otimes 1.$$

(d) Straightforward. □

Proposition 4.27 is a precise formulation of Proposition 3.2 in [AL05] for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

Similarly, if $\lambda \in \mathbb{C} \smallsetminus \mathbb{N}$, and V is a representation of $\mathfrak{g}_{\Bbbk}$, we can consider the completed tensor product

$$M_\lambda^{\pm} \mathbin{\widehat{\otimes}} V = \prod_{n\in\mathbb{N}} v_{\lambda,n}^{\pm} \otimes V.$$

Thus we can consider the following element.

4.28 Definition. For any $\lambda \in \mathbb{C} \smallsetminus \mathbb{N}$, we define $\mathcal{F}_\lambda = F_\lambda(v_\lambda^+ \otimes v_{-\lambda}^-) \in M_\lambda^+ \mathbin{\widehat{\otimes}} M_{-\lambda}^-$ the canonical element corresponding to the Shapovalov form $\langle\ ,\ \rangle_{\mathrm{Sh}}$. It is given by

$$\begin{aligned}\mathcal{F}_\lambda &= F_\lambda(v_\lambda^+ \otimes v_{-\lambda}^-) \\ &= \sum_{k\in\mathbb{N}} \frac{v_{\lambda,k}^+ \otimes v_{-\lambda,k}^-}{\left\langle v_{\lambda,k}^+, v_{-\lambda,k}^- \right\rangle_{\mathrm{Sh}}} \\ &= \sum_{k\in\mathbb{N}} (-1)^k \binom{\lambda}{k}^{-1} v_{\lambda,k}^+ \otimes v_{-\lambda,k}^-.\end{aligned} \tag{4.36}$$

We have a corollary of Proposition 4.26.

4.29 Corollary. *Let $\lambda \in \mathbb{C} \smallsetminus \mathbb{N}$. The map*

$$\begin{aligned} M_\lambda^+ \mathbin{\widehat{\otimes}} M_{-\lambda}^- &\to \mathrm{End}((M_{-\lambda}^-)^\wedge) \\ a \otimes b &\mapsto [c \mapsto \langle a, c\rangle_{\mathrm{Sh}}\, b]\end{aligned}$$

is an isomorphism of vector space such that $\mathcal{F}_\lambda \mapsto \mathrm{id}$.

Proof. It follows from the Proposition 4.26. □

4.30 Proposition. *Let $\lambda \in \mathbb{C} \smallsetminus \mathbb{N}$. Then, the canonical element $\mathcal{F}_\lambda$ is $U(\mathfrak{sl}_2(\mathbb{C}))$-invariant, i.e. for any $x \in \mathfrak{sl}_2(\mathbb{C})$*

$$x \cdot \mathcal{F}_\lambda = 0.$$

Proof. Straightforward. □

4.31 Remark. We define

a) The "residue" of $\mathcal{F}$ at $\lambda = n$ by

$$\begin{aligned}\operatorname{Res}_{\lambda=n} \mathcal{F}_\lambda &= \operatorname{Res}_{\lambda=n} F_\lambda(v_\lambda^+ \otimes v_\lambda^-)\\ &= (-1)^{n+1}(n+1) \sum_{k>n} \binom{k}{n+1} v_{\lambda,k}^+ \otimes v_{-\lambda,k}^-.\end{aligned}$$

In particular if $n = 0$ we have

$$\operatorname{Res}_{\lambda=0} \mathcal{F}_\lambda = -\sum_{k>0} k(v_{\lambda,k}^+ \otimes v_{-\lambda,k}^-).$$

b) For any $\lambda \in \mathbb{C}^\times$, we have that

$$\begin{aligned}\mathcal{F}_{1/\lambda} &= F_{1/\lambda}(v_\lambda^+ \otimes v_\lambda^-) \\ &= v_\lambda^+ \otimes v_\lambda^- - \lambda(v_{\lambda,1}^+ \otimes v_{-\lambda,1}^-) \\ &\quad - \sum_{k>1} \frac{\lambda^k k!}{(\lambda-1)(2\lambda-1)\cdots((k-1)\lambda-1)}[v_{\lambda,k}^+ \otimes v_{-\lambda,k}^-].\end{aligned} \tag{4.37}$$

c) For any $\lambda \in \mathbb{C} \smallsetminus \mathbb{N}$, we define

$$\begin{aligned}\frac{d}{d\lambda}\mathcal{F}_\lambda &= \frac{d}{d\lambda}F_\lambda(v_\lambda^+ \otimes v_\lambda^-)\\ &= \sum_{k>0}(-1)^{k+1}k!\frac{v_{\lambda,k}^+ \otimes v_{-\lambda,k}^-}{(\lambda)_k}\sum_{l=0}^{k-1}\frac{1}{\lambda-l}.\end{aligned}$$

4.1.5 Fusion Matrix

In this section we introduce, according to Etingof and Schiffman [ES01], the notion of a fusion matrix which is a key object in the construction of a certain kind of invariant star products done in [AL05, EV99].

If V is a $\mathfrak{g}$-module, we denote $V^{\mathfrak{g}}$ the subspace of invariants with respect to $\mathfrak{g}$, *i.e.* the subspace of all elements v such that $x \cdot v = 0$ for any $x \in \mathfrak{g}$. If $v \in V^{\mathfrak{g}}$, by abuse of notations we also denote v the $\mathfrak{g}$-homomorphism $v : 1 \mapsto v$ between the trivial $\mathfrak{g}$-module $\mathbb{C}$ and the space of invariants $V^{\mathfrak{g}}$.

Throughout this section we assume that $\widehat{V}$ the formal representation of V is either $V[[\hbar]]$ if the decomposition of V (maybe a trivial one) is not compatible with decomposition of $\mathfrak{g}_{\Bbbk}$ or $\widehat{V} = \prod_{k\in\mathbb{Z}} V_k$ if V is a $\mathbb{Z}$-graded module (see Remark 4.4).

4.32 Definition. We say that V is formally h-diagonalizable if the formal representation $\widehat{V}$ admits an algebraic product decomposition

$$\widehat{V} = \prod_{k\in\mathbb{N}} \widehat{V}[\lambda_k],$$

where $\Lambda = \{\lambda_k | k \in \mathbb{N}\}$ is the spectrum of the action h on V and $V[\lambda_k]$ is a genuine eigenspace of h with eigenvalue λ_k.

4.33 Proposition. *If the representation V is h-diagonalizable, then V is formally h-diagonalizable.*

Proof. It follows immediately from the definition. $\square$

4.34 Proposition. *Let $\lambda \in \mathbb{C} \smallsetminus \mathbb{N}$, τ be the flip map ($\tau(a \otimes b) = b \otimes a$) and V be a representation of $\mathfrak{sl}_2(\Bbbk)$.*

i) The map

$$\begin{aligned} (M^+_\lambda \,\widehat{\otimes}\, V)^{\mathfrak{g}} &\to \mathrm{Hom}_{\mathfrak{g}}(M^-_{-\lambda}, \widehat{V}) \\ w &\mapsto \tilde{w} \end{aligned}$$

is a vector space isomorphism. Here $\tilde{w}$ denotes the composition map

$$(\mathrm{id} \otimes \langle\ ,\ \rangle_{\mathrm{Sh}}) \circ (\tau(w) \otimes \mathrm{id}).$$

ii) The map

$$\begin{aligned} (V \,\widehat{\otimes}\, M^-_{-\lambda})^{\mathfrak{g}} &\to \mathrm{Hom}_{\mathfrak{g}}(M^+_\lambda, \widehat{V}) \\ w &\mapsto \tilde{w} \end{aligned}$$

is a vector space isomorphism. Here $\tilde{w}$ denotes the composition map

$$(\langle\ ,\ \rangle_{\mathrm{Sh}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau(w)).$$

Proof. Let $w \in (M_\lambda^+ \widehat{\otimes} V)^{\mathfrak{g}}$. The flip map is a $\mathfrak{g}$-homomorphism, thus $\tau(w) \in (V \widehat{\otimes} M_\lambda^+)^{\mathfrak{g}}$. By Proposition 1.22 the Shapovalov form $\langle\ ,\ \rangle_{\mathrm{Sh}}$ is invariant, so $\tilde{w} = (\mathrm{id} \otimes \langle\ ,\ \rangle_{\mathrm{Sh}}) \circ (\tau(w) \otimes \mathrm{id})$ is actually an element in $\mathrm{Hom}_{\mathfrak{g}}(M_{-\lambda}^-, \widehat{V})$. Now, the map $w \mapsto \tilde{w}$ is clearly injective. For the surjectivity, if $\phi \in \mathrm{Hom}_{\mathfrak{g}}(M_{-\lambda}^-, \widehat{V})$ it is easy to show that $v_\lambda^+ \otimes \phi(v_{-\lambda}^-)$ is $\mathfrak{g}$-invariant and

$$\phi = v_\lambda^+ \otimes \phi(v_{-\lambda}^-) \mapsto (v_\lambda^+ \otimes \phi(v_{-\lambda}^-))^{\tilde{}}.$$

□

The following theorem is a specific case $\mathfrak{sl}_2(\mathbb{k})$, of Theorem 8 in [EV99] for $\mathfrak{sl}_2(\mathbb{k})$.

4.35 Theorem. *Let V be a formally h-diagonalizable representation of $\mathfrak{sl}_2(\mathbb{k})$.*

i) Let M_λ^+ and M_μ^+ be two Verma modules of highest weight λ and μ, respectively. If M_μ^+ is irreducible, then $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, M_\mu^+ \widehat{\otimes} V)$ and $\widehat{V}[\lambda - \mu]$ are isomorphic as vector spaces. The isomorphism is given by

$$\begin{aligned} \widehat{V}[\lambda-\mu] &\to \mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, M_\mu^+ \widehat{\otimes} V) \\ u &\mapsto \phi_\lambda^u, \end{aligned}$$

where ϕ_λ^u is defined by

$$\phi_\lambda^u(v_\lambda^+) = F_\mu(v_\mu^+ \otimes u).$$

ii) Let M_λ^- and M_μ^- be two Verma modules of lowest weight λ and μ, respectively. If M_μ^- is irreducible, then $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^-, V \widehat{\otimes} M_\mu^-)$ and $\widehat{V}[\lambda - \mu]$ are isomorphic as vector spaces. The isomorphism is given by

$$\begin{aligned} \widehat{V}[\lambda-\mu] &\to \mathrm{Hom}_{\mathfrak{g}}(M_\lambda^-, V \widehat{\otimes} M_\mu^-) \\ u &\mapsto \phi_\lambda^u, \end{aligned}$$

where ϕ_λ^u is defined by

$$\phi_\lambda^u(v_\lambda^-) = F_{-\mu}(u \otimes v_\mu^-).$$

Proof. According to the Frobenius reciprocity Theorem we have that

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, M_\mu^+ \widehat{\otimes} V) \simeq \mathrm{Hom}_{\mathfrak{p}_+}(\mathbb{C}_\lambda^+, M_\mu^+ \widehat{\otimes} V).$$

Now, the space $\mathrm{Hom}_{\mathfrak{p}_+}(\mathbb{C}_\lambda^+, M_\mu^+ \widehat{\otimes} V)$ can be described as the space of all $x \in M_\mu^+ \widehat{\otimes} V$ such that the $\mathfrak{p}_+$-submodule of $M_\mu^+ \widehat{\otimes} V$ generated by x is isomorphic to $\mathbb{C}_\lambda^+$. After tensoring with $\mathbb{C}^+_{-\lambda}$ this submodule gives a trivial module. Thus the space $\mathrm{Hom}_{\mathfrak{p}_+}(\mathbb{C}^+_\lambda, M^+_\mu \widehat{\otimes} V)$ is isomorphic to the space $(M^+_\mu \widehat{\otimes} V \otimes \mathbb{C}^+_{-\lambda})^{\mathfrak{p}_+}$. By Proposition 4.34 we have that

$$(M^+_\mu \widehat{\otimes} V \otimes \mathbb{C}^+_{-\lambda})^{\mathfrak{p}_+} \simeq \mathrm{Hom}_{\mathfrak{p}_+}(M^-_{-\mu}, \widehat{V} \otimes \mathbb{C}^+_{-\lambda}).$$

Again by Frobenius reciprocity Theorem, we have

$$\mathrm{Hom}_{\mathfrak{p}_+}(M^-_{-\mu}, \widehat{V} \otimes \mathbb{C}^+_{-\lambda}) \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_{-\mu}, \widehat{V} \otimes \mathbb{C}_{-\lambda}) \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_{-\mu} \otimes \mathbb{C}_\lambda, \widehat{V}) \simeq \widehat{V}[\lambda - \mu].$$

□

4.36 Corollary. *Let V be a formally h-diagonalizable representation of $\mathfrak{sl}_2(\Bbbk)$ and $u \in \widehat{V}[\lambda - \mu]$ be a element h-homogeneous of $\widehat{V}$.*

i) If M^+_μ is irreducible, then the map ϕ^u_λ of the Theorem 4.35-(i) is given by $\phi^u_\lambda(v^+_{\lambda,k}) = \sum_{i\in\mathbb{N}} v^+_{\mu,i} \otimes u_{i,k}(\mu)$ with

$$u_{i,k}(\mu) = \sum_{r=0}^{\min(i,k)} \binom{i}{r} \frac{(-1)^{i-r}}{(k-r)!(\mu)_{i-r}} (e^-)^{k-r}(e^+)^{i-r}u.$$

ii) If M^-_μ is irreducible, then the map ϕ^u_λ of the Theorem 4.35-(ii) is given by $\phi^u_\lambda(v^-_{\lambda,k}) = \sum_{i\in\mathbb{N}} u_{i,k}(\mu) \otimes v^-_{\mu,i}$ with

$$u_{i,k}(\mu) = \sum_{r=0}^{\min(i,k)} \binom{i}{r} \frac{(-1)^{i-r}}{(k-r)!(-\mu)_{i-r}} (e^+)^{k-r}(e^-)^{i-r}u.$$

Proof. Straightforward. □

4.37 Remark. *a*) By Proposition 4.33 if V is h-diagonalizable then V is formally h-diagonalizable. So Theorem 4.35 applies to any h-diagonalizable representation of $\mathfrak{g}_\Bbbk$. For instance, it is true if V is a standard module of Definition 4.7.

b) Let V be an h-diagonalizable $\mathbb{Z}$-graded representation of $\mathfrak{sl}_2(\Bbbk)$. We assume that V is a bounded from above, *i.e.* the graded component of V corresponding to k is equal zero if $k \gg 0$. In this case Theorem 8 in [EV99] can be applied. Then, for any $u \in V[\lambda - \mu]$, the element $\phi^u_\lambda(v^+_{\lambda,k}) = \sum_{i\in\mathbb{N}} v^+_{\mu,i} \otimes u_{i,k}(\mu)$ belongs, actually, to $M^+_\mu \otimes V$. Similarly, we have the same phenomenon (*mutatis mutandis*) if V is bounded from below, *i.e.* the graded component of V corresponding to k equal zero if $k \ll 0$.

Let V, W be two formally h-diagonalizable representations of $\mathfrak{g}_{\Bbbk}$ such that the tensor product $V \otimes W$ is also formally h-diagonalizable. Let $(v,w) \in \widehat{V}[\lambda_v] \times \widehat{W}[\mu_w]$ be a couple of elements h-homogeneous of V and W (*i.e.* $h \cdot v = \lambda_v\, v$ and $h \cdot w = \mu_w w$). If $\lambda - \mu_w$ and $\lambda - \lambda_v - \mu_w$ are elements in $\mathbb{C} \setminus \mathbb{N}$, we denote $\phi_\lambda^{w,v}$ the composition map

$$M_\lambda^+ \xrightarrow{\phi_\lambda^w} M_{\lambda-\mu_w}^+ \widehat{\otimes}\, W \xrightarrow{\phi^v_{\lambda-\mu_w} \otimes \mathrm{id}} M^+_{\lambda-\lambda_v-\mu_w} \widehat{\otimes}\, V \widehat{\otimes}\, W.$$

We denote $J^+_{V,W}(\lambda)(v \otimes w)$ the unique element in $V \widehat{\otimes}\, W[\lambda_v + \mu_w]$ such that

$$\phi_\lambda^{w,v} = \phi_\lambda^{J^+_{V,W}(\lambda)(v\otimes w)}.$$

Similarly, if $\lambda_v - \lambda$ and $\lambda_v + \mu_w - \lambda$ are elements in $\mathbb{C} \setminus \mathbb{N}$, we denote $\phi_\lambda^{v,w}$ the composition map

$$M_\lambda^- \xrightarrow{\phi_\lambda^v} V \widehat{\otimes}\, M^-_{\lambda-\lambda_v} \xrightarrow{\mathrm{id} \otimes \phi^w_{\lambda-\lambda_v}} V \widehat{\otimes}\, W \widehat{\otimes}\, M^-_{\lambda-\lambda_v-\mu_w}.$$

We denote $J^-_{V,W}(\lambda)(v \otimes w)$ the unique element in $V \widehat{\otimes}\, W[\lambda_v + \mu_w]$ such that

$$\phi_\lambda^{v,w} = \phi_\lambda^{J^-_{V,W}(\lambda)(v\otimes w)}.$$

4.38 Proposition-Definition. *Let V, W be two formally h-diagonalizable representations of $\mathfrak{g}_{\Bbbk}$ such that the tensor product $V \otimes W$ is also formally h-diagonalizable and let Λ_V, Λ_W be the set of weights of representation V and W respectively.*

i) If $\lambda \notin \Lambda_W + \mathbb{N}$ and $\lambda \notin \Lambda_V + \Lambda_W + \mathbb{N}$, then there is an endomorphism of $V \widehat{\otimes}\, W$ given by

$$\begin{aligned} J^+_{V,W}(\lambda) : V \widehat{\otimes}\, W &\to V \widehat{\otimes}\, W \\ (v,w) &\mapsto J^+_{V,W}(\lambda)(v \otimes w). \end{aligned}$$

More precisely,

$$\begin{aligned} J^+_{V,W}(\lambda)(v \otimes w) &= F_{\lambda-\mu_w}(v \otimes w) \\ &= \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k!(\lambda-\mu_w)_k} (e^-)^k v \otimes (e^+)^k w. \end{aligned}$$

ii) Similarly, if $\lambda \notin \Lambda_V - \mathbb{N}$ and $\lambda \notin \Lambda_V + \Lambda_W - \mathbb{N}$, then there is an endomorphism of $V \widehat{\otimes}\, W$ given by

$$\begin{aligned} J^-_{V,W}(\lambda) : V \widehat{\otimes}\, W &\to V \widehat{\otimes}\, W \\ (v,w) &\mapsto J^-_{V,W}(\lambda)(v \otimes w). \end{aligned}$$

More precisely,

$$\begin{aligned} J^-_{V,W}(\lambda)(v\otimes w) &= F_{\lambda_v-\lambda}(v\otimes w) \\ &= \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k!(\lambda_v-\lambda)_k}(e^-)^k v\otimes (e^+)^k w. \end{aligned}$$

iii) Both maps $J^+_{V,W}(\lambda)$ and $J^-_{V,W}(\lambda)$ are called the λ-fusion matrix between V and W.

Proof. Straightforward. □

4.39 Remark. The condition $\lambda\notin\Lambda_W+\mathbb{N}$ in proposition 4.38 guarantees the existence of $F_{\lambda-\lambda_w}(v\otimes w)$ in $V\mathbin{\widehat{\otimes}}W$. On the other hand, the condition $\lambda\notin\Lambda_V+\Lambda_W+\mathbb{N}$ is a condition of unicity, that is if $\lambda\notin\Lambda_W+\mathbb{N}$ and $\lambda\notin\Lambda_V+\Lambda_W+\mathbb{N}$, the Verma module $M^+_{\lambda-\lambda_v-\lambda_w}$ is irreducible thus by Theorem 4.35 there exist an unique $\phi\in\mathrm{Hom}_{\mathfrak{g}}(M^+_\lambda, M^+_{\lambda-\lambda_v-\mu_w}\mathbin{\widehat{\otimes}}V\mathbin{\widehat{\otimes}}W)$ of the form

$$\phi(v^+_\lambda)=v^+_{\lambda-\lambda_v-\mu_w}\otimes F_{\lambda_v-\lambda}(v\otimes w)+\cdots$$

4.40 Proposition. *The fusion matrix $J^\pm_{V,W}(\lambda)$ is of the form*

$$J^\pm_{V,W}(\lambda)=\mathrm{id}+N^\pm,$$

where $N^\pm$ is an endomorphism of $V\mathbin{\widehat{\otimes}}W$. In particular, we have that it is invertible and

$$(J^\pm_{V,W}(\lambda))^{-1}=\mathrm{id}-N^\pm+(N^\pm)^2-\cdots.$$

Proof. It follows immediately from the definition. □

4.41 Proposition. *Let V, W be two representation of $\mathfrak{g}_{\Bbbk}$ such that the completed tensor product $V\mathbin{\widehat{\otimes}}W$ is formally h-diagonalizable*

$$V\mathbin{\widehat{\otimes}}W=\prod_{k\in\mathbb{Z}}V\mathbin{\widehat{\otimes}}W[\lambda_k].$$

If the product decomposition of $V\mathbin{\widehat{\otimes}}W$ is compatible with the graduation of $\mathfrak{g}_{\Bbbk}$, i.e.

$$h\cdot V\mathbin{\widehat{\otimes}}W[\lambda_k]\subseteq V\mathbin{\widehat{\otimes}}W[\lambda_k],\quad e^\pm\cdot V\mathbin{\widehat{\otimes}}W[\lambda_k]\subseteq V\mathbin{\widehat{\otimes}}W[\lambda_{k\pm1}],\qquad (\forall k\in\mathbb{Z})$$

then the fusion matrix $J^\pm_{V,W}(\lambda)$ (if it exists) is a $\mathbb{Z}$-graded morphism.

Proof. It follows immediately from the definition. □

4.42 Remark. Under assumptions of Theorem 8 in [EV99] (See Remark 4.37), according to Lemma 11 in [EV99] the endomorphism $N : V \widehat{\otimes} W \to V \widehat{\otimes} W$ of Proposition 4.40 is locally nilpotent, that is for each $v \in V[\lambda_v]$ there exist $n_v \in \mathbb{N}$ such that

$$J^+_{V,W}(\lambda)(v \otimes w) = \sum_{k=0}^{n_v} \frac{(-1)^k}{k!(\lambda - \mu_w)_k} (e^-)^k v \otimes (e^+)^k w$$

or

$$J^-_{V,W}(\lambda)(v \otimes w) = \sum_{k=0}^{n_v} \frac{(-1)^k}{k!(\lambda_v - \lambda)_k} (e^-)^k v \otimes (e^+)^k w.$$

Now, we consider the Verma module M^+_ν of highest weight ν and M^-_μ of lowest weight μ. By Proposition 4.18 the tensor product $M^+_\nu \otimes M^-_\mu$ is diagonalizable. Then under some assumptions on λ the λ-fusion matrices do exist between these two modules. More precisely,

4.43 Proposition. *Let λ, ν, μ complex numbers. We have*

i) If $\lambda - \mu \notin \mathbb{N}$ and $\lambda - \mu - \nu \notin \mathbb{Z}$, then the λ-fusion matrix $J^+_{M^+_\nu, M^-_\mu}(\lambda)$ exist and

$$J^+_{M^+_\nu, M^-_\mu}(\lambda)(v^+_{\nu,i} \otimes v^-_{\mu,j}) = F_{\lambda-\mu-2j}(v^+_{\nu,i} \otimes v^-_{\mu,j}), \qquad (i,j \in \mathbb{N}).$$

ii) If $\nu - \lambda \notin \mathbb{N}$ and $\lambda - \mu - \nu \notin \mathbb{Z}$, then the λ-fusion matrix $J^-_{M^+_\nu, M^-_\mu}(\lambda)$ exist and

$$J^-_{M^+_\nu, M^-_\mu}(\lambda)(v^+_{\nu,i} \otimes v^-_{\mu,j}) = F_{\nu-\lambda-2i}(v^+_{\nu,i} \otimes v^-_{\mu,j}), \qquad (i,j \in \mathbb{N}).$$

Proof. It follows from Proposition 4.38. □

4.44 Remark. In particular, if we take $\lambda = 0$, $\nu = -\mu$ the condition $\nu \in \mathbb{C} \setminus \mathbb{N}$ guarantees the existence of F_ν the canonical element corresponding to the Shapovalov form. We have

$$\mathcal{F}_\nu = J_{M^+_\nu, M^-_{-\nu}}(0)(v^+_\nu \otimes v^-_{-\nu}).$$

However, it fails on the unicity condition. For instance, we consider $\zeta \in \mathbb{C}^\times$ and the morphisms $\phi, \tilde{\phi} \in \mathrm{Hom}_{\mathfrak{g}}(M^+_0, M^+_0 \widehat{\otimes} M^+_\nu \widehat{\otimes} M^-_{-\nu})$ given by

$$\phi(v^+_0) = \sum_{k \in \mathbb{N}} v^+_{0,k} \otimes u_k \qquad \text{and} \qquad \tilde{\phi}(v^+_0) = \sum_{k \in \mathbb{N}} v^+_{0,k} \otimes \tilde{u}_k,$$

where $u_0 = \tilde{u}_0 = \mathcal{F}_\nu$, $u_1 = e^+ \mathcal{F}_\nu$, $\tilde{u}_1 = \zeta(e^+ \cdot \mathcal{F}_\nu)$ and $u_{k+1} = \tilde{u}_{k+1} = \frac{e^{k+1}}{k(k+1)} \cdot \mathcal{F}_\nu$ if $k \geq 1$.

4.1.6 Alekseev-Lochowska's invariant $\star$-product

In all this section $\mathfrak{g}_{\Bbbk}$ denotes the Lie algebra $\mathfrak{sl}_2(\Bbbk)$. Consider $W_0 = \mathbb{C}\, w_0$ the trivial representation of $\mathfrak{h} = \Bbbk\, h$ and let

$$W = \mathrm{Ind}_{U(\mathfrak{h})}^{U(\mathfrak{g}_{\mathbb{C}})} W_0 = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{h})} W_0 \tag{4.38}$$

be the induced $U(\mathfrak{g}_{\mathbb{C}})$-module generated by the vector $w = 1 \otimes w_0$.

4.45 Lemma. *The induced module W defined in* (4.38) *is isomorphic as a vector space to $U(\mathfrak{g}_{\mathbb{C}})/U(\mathfrak{g}_{\mathbb{C}})\, \mathfrak{h} \simeq U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+)$.*

Proof. We choice the order $e^- < e^+ < h$ in the standard basis of the Lie algebra $\mathfrak{g}_{\Bbbk}$. By Poincaré-Birkhoff-Witt Theorem we have that the elements

$$(e^-)^i (e^+)^j h^k \qquad\qquad (i, j, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$$

form a basis of the universal enveloping algebra $U(\mathfrak{g}_{\Bbbk})$. Since $(e^-)^i (e^+)^j h^k \otimes w_0 = 0$ if $k \neq 0$, then there is an isomorphism between W and $U(\mathfrak{g}_{\Bbbk})/U(\mathfrak{g}_{\Bbbk})\, \mathfrak{h}$. Moreover, the map

$$\begin{aligned} U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+) &\to W \\ (e^-)^i \otimes (e^+)^j &\mapsto (e^-)^i (e^+)^j \otimes w_0 \end{aligned}$$

is an isomorphism of vector spaces. □

4.46 Remark. The action of $\mathfrak{g}_{\Bbbk}$ on $U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+)$ induced by the isomorphism $U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+) \to W$ is given by

$$\begin{aligned} e^- \cdot (e^-)^i \otimes (e^+)^j &= (e^-)^{i+1} \otimes (e^+)^j \\ h \cdot (e^-)^i \otimes (e^+)^j &= 2(j - i)(e^-)^i \otimes (e^+)^j \\ e^+ \cdot (e^-)^i \otimes (e^+)^j &= (e^-)^i \otimes (e^+)^{j+1} - i(2j - (i-1))(e^-)^{i-1} \otimes (e^+)^j \end{aligned}$$

Let W_l be the vector space generated by the vectors $\{(e^-)^i \otimes (e^+)^j | j - i = l \in \mathbb{Z}\}$, we have

$$e^- \cdot W_l \subseteq W_{l-1}$$

$$h \cdot W_l \subseteq W_l$$

$$e^+ \cdot W_l \subseteq W_{l+1}$$

Then, $W = \bigoplus_{l \in \mathbb{Z}} W_l$ is a $\mathbb{Z}$-graded representation of $\mathfrak{sl}_2(\Bbbk)$.

According to the above remark, we have that $W = \mathrm{Ind}_{U(\mathfrak{h})}^{U(\mathfrak{g}_{\mathbb{C}})} W_0$ is isomorphic to the space $M_0^+ \otimes M_0^-$.

We use the following Proposition (see [AL05, Proposition 4.5]).

4.47 Proposition. *Let $\lambda \in \mathbb{C} \setminus \mathbb{N}$ and $p : U(\mathfrak{g}) \to U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}$ be the natural projection to the left ideal generated by $\mathfrak{h}$. We have that*

$$(\mathrm{id} \otimes p \otimes \mathrm{id})\ [(\Delta \otimes \mathrm{id})F_\lambda(F_\lambda \otimes 1)] = (\mathrm{id} \otimes p \otimes \mathrm{id})\ [(\mathrm{id} \otimes \Delta)F_\lambda(1 \otimes F_\lambda)] \tag{4.39}$$

in $U(\mathfrak{n}_-) \widehat{\otimes} (U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}) \widehat{\otimes} U(\mathfrak{n}_+)$.

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{N}$ such that the Shapovalov form $\langle\ ,\ \rangle_\lambda$ is nonsingular and let W be the induced module defined in (4.38). Then the $\mathfrak{sl}_2(\mathbb{k})$-morphisms

$$M_0^+ \xrightarrow{\phi_0^{v_{-\lambda}^-}} M_\lambda^+ \widehat{\otimes} M_{-\lambda}^- \xrightarrow{\phi_\lambda^w \otimes \mathrm{id}} M_\lambda^+ \widehat{\otimes} W \widehat{\otimes} M_{-\lambda}^- \tag{4.40}$$

and

$$M_0^- \xrightarrow{\phi_0^{v_\lambda^+}} M_\lambda^+ \widehat{\otimes} M_{-\lambda}^- \xrightarrow{\mathrm{id} \otimes \phi_\lambda^w} M_\lambda^+ \widehat{\otimes} W \widehat{\otimes} M_{-\lambda}^- \tag{4.41}$$

coincide. Indeed, firstly by definition of fusion matrix, we have that the composition map in (4.40) is $\phi_0^{J_{W \widehat{\otimes} M_{-\lambda}^-}(0)(w \otimes v_{-\lambda}^-)}$. Secondly, in the basis $\{v_k^\pm = \frac{(e^\mp)^k}{k!} v_0 | k \in \mathbb{N}\}$ of $M_0^\pm$ we have that

$$\phi_0^{v_{-\lambda}^-}(v_k^+) = \delta_{0,k}\, \mathcal{F}_\lambda = \phi_0^{v_\lambda^+}(v_k^-).$$

Then the maps $\phi_0^{v_{-\lambda}^-}$ and $\phi_0^{v_\lambda^+}$ given in Eq. (4.40) and (4.41) factor through into the quotient map $\mathbb{C} \to M_\lambda^+ \widehat{\otimes} M_{-\lambda}^-$ defined by

$$1 \to \mathcal{F}_\lambda\,.$$

Thus we can see $\phi_0^{v_\lambda^+}$ as a $\mathfrak{g}_{\mathbb{k}}$-morphism between M_0^+ and $M_\lambda^+ \widehat{\otimes} M_{-\lambda}^-$. Now by Proposition 4.18 the tensor product $W \otimes M_{-\lambda}^- \simeq M_0^+ \otimes M_0^- \otimes M_{-\lambda}^-$ is diagonalizable and the element $w \otimes v_{-\lambda}^- = v_0^+ \otimes v_0^- \otimes v_{-\lambda}^-$ has weight $-\lambda$. Then by Theorem 4.35 the composition map

$$\psi = (\mathrm{id} \otimes \phi_\lambda^w) \circ (\phi_0^{v_\lambda^+}) : M_0^+ \to M_\lambda^+ \widehat{\otimes} W \widehat{\otimes} M_{-\lambda}^-$$

is determined by

$$\begin{aligned}\psi(v_0^+) &= F_\lambda(v_\lambda^+ \otimes (w \otimes v_{-\lambda}^-)) \\ &= v_\lambda^+ \otimes (w \otimes v_{-\lambda}^-) + (e^-) \cdot v_\lambda^+ \otimes (e^+) \cdot (w \otimes v_{-\lambda}^-) + \cdots\end{aligned}$$

Consequently, ψ fixes the h-homogeneous element $w \otimes v^-_{-\lambda}$ so $\psi = \phi_0^{J_{W \widehat{\otimes} M^-_{-\lambda}}(0)(w \otimes v^-_{-\lambda})}$.

Now if we write $\mathcal{F}_\lambda$ in terms of F_λ, we have that the composition in Eq. (4.40) is in fact the product

$$(\mathrm{id} \otimes p \otimes \mathrm{id})\ [(\Delta \otimes \mathrm{id}) F_\lambda (F_\lambda \otimes 1)]$$

and the composition in Eq. (4.41) is

$$(\mathrm{id} \otimes p \otimes \mathrm{id})\ [(\mathrm{id} \otimes \Delta) F_\lambda (1 \otimes F_\lambda)]\,,$$

in $U\,\mathfrak{n}_- \tilde{\otimes} U\,\mathfrak{g}\,/U\,\mathfrak{g} \cdot \mathfrak{h}\,\tilde{\otimes} U\,\mathfrak{n}_+$, what concludes the proof. □

We recall the construction of the canonical element F_λ relative to the Shapovalov form. We take $\lambda \in \mathbb{C} \setminus \mathbb{N}$, χ a nonsingular character of $\mathfrak{h}$ and set $\chi_\lambda = \lambda\,\chi$. The canonical element

$$F_{\chi_\lambda} = F_\lambda$$

is an invariant in the completed space $U\,\mathfrak{n}_- \,\widehat{\otimes}\, U\,\mathfrak{n}_+$. In particular, we can see F_λ as an element in $(U\,\mathfrak{n}_- \,\widehat{\otimes}\, U\,\mathfrak{n}_+)^{\mathfrak{g}}[[\lambda, \lambda^{-1}]]$ the space of Laurent series with coefficients in $(U\,\mathfrak{n}_- \,\widehat{\otimes}\, U\,\mathfrak{n}_+)^{\mathfrak{g}}$ and in the indeterminate λ. Now by Proposition 4.27, F_λ is holomorphic in $\lambda = 1/\hbar$ at infinity, this is the same as to say that the function $F_{\hbar^{-1}}$ is holomorphic at $\hbar = 0$. Then, if $\hbar$ is in an open neighbourhood of zero the canonical element $F_{\hbar^{-1}}$ is actually in

$$(U\,\mathfrak{n}_- \,\widehat{\otimes}\, U\,\mathfrak{n}_+)^{\mathfrak{g}}[[\hbar]].$$

Then we have the following theorem which is, for the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$, the key result in [AL05].

4.48 Theorem. *Let $\pi : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h} \otimes U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h}$ be the natural projection. The element $D = \pi(F_{\hbar^{-1}})$ is invertible in*

$$(U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h} \,\widehat{\otimes}\, U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{h}))^{\mathfrak{h}}[[\hbar]]$$

and satisfies the associativity conditions of the Proposition 2.30. In other words, D is a G-invariant star product on the space $G/H \simeq \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1)$.

Proof. See [AL05, Theorem 4.9]. □

By [AL05, Example 4.16], we have that the invariant star-product D of Theorem 4.48 is given by

$$D = 1 \otimes 1 - \frac{e^- \otimes e^+}{z}\hbar + \sum_{n\geq 1}(-1)^{n+1}\frac{\hbar^{n+1}}{(n+1)!}\left[\frac{(e^-)^{n+1}\otimes(e^+)^{n+1}}{z(z-\hbar)\dots(z-n\hbar)}\right]. \tag{4.42}$$

More precisely,

4.49 Proposition. *Let $z \in \mathbb{C}^\times$ and $\{a_n\}_{n\geq 2}$ be the sequence in $U(\mathfrak{n}_-)\otimes U(\mathfrak{n}_+)$ given by*

$$z^n a_n = \sum_{i=0}^{n-2}\frac{(e^-)^{i+2}\otimes(e^+)^{i+2}}{(i+2)!}\sum_{k=1}^{i+1}(-1)^k\frac{k^{n-1}}{k!(i+1-k)!}. \tag{4.43}$$

We have that

$$D = 1\otimes 1 - \frac{e^-\otimes e^+}{z}\hbar - \sum_{n\geq 2} a_n\hbar^n \tag{4.44}$$

is an invariant $\star$-product on the space $G/H \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$.

Proof. We take $Y = e^-\otimes e^+$, $x = \hbar/z$ and

$$f(x) = \sum_{n\geq 1}(-1)^{n+1}\frac{Y^{n+1}}{(n+1)!}\left[\frac{x^{n+1}}{(1-x)\dots(1-nx)}\right].$$

The rational polynomial $(-1)^n/[(1-x)\dots(1-nx)]$ has a partial-fraction decomposition

$$\frac{(-1)^n}{(1-x)\dots(1-nx)} = \sum_{k=1}^{n}\frac{A_{n,k}}{1-kx},$$

where $A_{n,k}$ are following integers:

$$A_{n,k} = (-1)^k\frac{k^n}{k!(n-k)!} = (-1)^k\binom{n}{k}\frac{k^n}{n!}$$

for $n\geq 1$ and $1\leq k\leq n$. Now for all x and k, we have that $\frac{1}{1-kx}$ belongs to $U(\mathfrak{g})\widehat{\otimes}U(\mathfrak{g})$ the completed space of $U(\mathfrak{g})\otimes U(\mathfrak{g})$. Then

$$\frac{1}{1-kx} = \sum_{l\in\mathbb{N}} k^l x^l.$$

Consequently,

$$\begin{aligned}
f(x) &= -\sum_{n\geq 1}\sum_{l\in\mathbb{N}}\sum_{k=1}^{n}\frac{Y^{n+1}}{(n+1)!}A_{n,k}k^l x^{n+l+1}\\
&= -\sum_{r\in\mathbb{N}}\left(\sum_{n+l=r}\sum_{k=1}^{n+1}\frac{Y^{n+2}}{(n+2)!}A_{n+1,k}k^l\right)x^{r+2}\\
&= -\sum_{n\in\mathbb{N}}\left(\sum_{i=0}^{n-2}\frac{(e^-)^{i+2}\otimes(e^+)^{i+2}}{(i+2)!}\sum_{k=1}^{i+1}(-1)^k\frac{k^{n-1}}{k!(i+1-k)!}\right)x^{n+2}\\
&= -\sum_{n\in\mathbb{N}} z^{n+2}a_{n+2}x^{n+2}.
\end{aligned}$$

Finally, it is clear that

$$D = 1 \otimes 1 - (e^- \otimes e^+)x + f(x).$$

□

Let M_λ^- be the Verma module of lowest weight λ and M_μ^- be the Verma module of lowest weight μ. We want to express the invariant star-product of Proposition 4.49 in the total basis $\{v_{\lambda,i}^- | i \in \mathbb{N}\}$ of M_λ^- and $\{v_{\mu,j}^- | j \in \mathbb{N}\}$ of M_μ^-.

4.50 Corollary. *Under the assumptions of Proposition 4.49, we have*

$$z^n a_n(v_{\lambda,i}^- \otimes v_{\mu,j}^-) = \sum_{l=0}^{\min(i-2,n-2)} (-1)^l c_l(n)(\lambda+i-l-2)\cdots(\lambda+i-1)\binom{l+j+2}{j} v_{\lambda,i-l-2}^- \otimes v_{\mu,l+j+2}^-,$$

where $c_l(n) = \sum_{k=1}^{l+1}(-1)^k \frac{k^{n-1}}{k!(l+1-k)!}$.

Proof. Straightforward. □

The fusion matrix $J(\lambda)$ is the key tool for an explicit construction of solutions to the quantum dynamical Yang-Baxter equations (QDYB) (also known as the Gervais-Neveu-Felder equations) which is a generalization of quantum Yang-Baxter equations (QYBE). We refer the reader to [ABRR98, EV99, ES01] for more details.

4.2 Unterberger's Symbolic Calculus

Completely different approach to the quantization problem based on equivariant operator calculus was developed by A. Unterberger and J. Unterberger. We follow their article [UU96] in order to introduce such symbolic calculus on the imaginary Lobachevsky space.

4.2.1 The imaginary Lobachevsky space

Let $\Pi_L = \{(s,t) | s,t \in \mathbb{R} \cup \{\infty\}, s \neq t\}$ be the imaginary Lobachevsky space. The Lie group $G = SL(2,\mathbb{R})$ acts on Π_L by:

$$\gamma \cdot (s,t) = \left(\frac{as+b}{cs+d}, \frac{at+b}{ct+d}\right)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $(s,t) \in \Pi_L$. The formula

$$d\mu_L(s,t) = (s-t)^{-2} ds dt \tag{4.45}$$

defines a G-invariant measure on Π_L.

The Lie group G acts on the Hilbert space $L^2(\Pi_L) = L^2(\Pi_L, d\mu_L)$ of square-integrable functions on Π_L (with respect to the measure μ_L) by

$$(\varphi(\gamma^{-1})f)(s,t) = (\gamma^{-1}\cdot f)(s,t) = f(\gamma\cdot(s,t)), \qquad (\gamma \in G, (s,t) \in \Pi_L, f \in L^2(\Pi_L)).$$

4.51 Proposition. *Let (e^+, h, e^-) be the standard basis in $\mathfrak{sl}_2(\mathbb{R})$ defined in the formula (4.3). The infinitesimal action of the representation φ*

$$d\varphi(X)f = \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(tX))f, \tag{4.46}$$

in the basis (e^+, h, e^-) is given by

$$(d\varphi(e^+)f)(s,t) = -\left(\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right) \tag{4.47}$$

$$(d\varphi(h)f)(s,t) = -2\left(s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t}\right) \tag{4.48}$$

$$(d\varphi(e^-)f)(s,t) = \left(s^2\frac{\partial f}{\partial s} + t^2\frac{\partial f}{\partial t}\right) \tag{4.49}$$

Proof. Direct computation. □

4.52 Proposition. *The Laplace-Beltrami operator on Π_L is given by*

$$\Box = (s-t)^2\frac{\partial^2}{\partial s \partial t}.$$

Proof. We recall that the Casimir operator in $\mathfrak{sl}_2(\mathbb{R})$ is given by

$$\mathcal{C} = h^2 + 2(e^+e^- + e^-e^+) = h^2 + 2h + 4e^-e^+. \tag{4.50}$$

Then after a simple computation we obtain that

$$\begin{aligned}
\mathcal{C} &= h^2 + 2(e^+e^- + e^-e^+) \\
&= -4\left(s^2 - 2st + t^2\right)\frac{\partial^2}{\partial s\partial t} \\
&= -4\,(s-t)^2\,\frac{\partial^2}{\partial s\partial t} \\
&= -4\Box.
\end{aligned}$$

□

The decomposition of $L^2(\Pi_L)$ under the action of G is given by the spectral decomposition of the operator $\Box$ (see for example [Str73] or [Far79]). The spectrum of $\Box$ consists of a continuous part together with a discrete part. The discrete part is given by:

$$L^2_{\text{discr}}(\Pi_L) = \bigoplus_{n\geq 1} E_{-n(n-1)}(\Box),$$

where $E_{-n(n-1)}(\Box)$ is the eigenspace of $\Box$ with eigenvalue $-n(n-1)$;

$$E_{-n(n-1)}(\Box) = \{f \in L^2(\Pi_L) |\ \Box(f) = -n(n-1)f\}.$$

Let us fix an element ζ in $\Pi = \{z = x + \mathbf{i}\, y \in \mathbb{C}\,|y > 0\}$ the upper half-plane. For any $n \in \mathbb{N}^\times$, we consider the square-integrable smooth function g^n_ζ given by

$$g^n_\zeta(s,t) = \left(\frac{s-t}{(s-\bar\zeta)(t-\bar\zeta)}\right)^n = \left(\frac{1}{(t-\bar\zeta)} - \frac{1}{(s-\bar\zeta)}\right)^n. \tag{4.51}$$

It belongs to the eigenspace $E_{-n(n-1)}(\Box)$. Now, we can separate $E_{-n(n-1)}(\Box)$ into two irreducible G-modules the "anti-holomorphic" part and the "holomorphic" part (see Proposition 4.56).

4.53 Proposition. *For all $n \in \mathbb{N}^\times$, we consider the spaces*[5]

$$E^+_n = \overline{\text{Span}_{\mathbb{C}}(g^n_z | z \in \Pi)} \qquad \textit{and} \qquad E^-_n = \overline{\text{Span}_{\mathbb{C}}(g^n_z | \bar z \in \Pi)}.$$

Then the spaces E^+_n and E^-_n are two irreducible G-modules and

$$E_{-n(n-1)}(\Box) = E^+_n \oplus E^-_n.$$

Proof. See for instance [UU96]. □

4.54 Remark. Here the action of G on each g^n_z $(n \in \mathbb{N}^\times, z \in \Pi)$ is given by

$$\gamma \cdot g^n_z = (\jmath_\gamma(\bar z))^{-2n} g^n_{\gamma\cdot z},$$

where $\gamma \cdot z$ is the transitive action of G on Π given by

$$\gamma \cdot z = \frac{az+b}{cz+d}, \qquad z \in \Pi, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

and $\jmath_\gamma(z) = cz + d$.

[5] Here $\overline{X}$ means the topological closure of X in $L^2(\Pi_L)$.

4.55 Proposition. *Let $n \in \mathbb{N}^\times$ and ζ an element fix in Π. Then*

$$E_n^+ = \overline{\mathrm{Span}_{\mathbb{C}}(\gamma \cdot g_\zeta^n | \, \gamma \in G)}, \qquad E_n^- = \overline{\mathrm{Span}_{\mathbb{C}}(\gamma \cdot g_{\bar\zeta}^n | \, \gamma \in G)}.$$

Proof. The action of G on Π is transitive, then for all $g_{z'}^n \in E_n^+$, $z' \in \Pi$, there exist $\gamma \in G$ and $\alpha \in \mathbb{C}$ such that $\alpha(\gamma \cdot g_z^n) = g_{z'}^n$. $\square$

Now let us fix a pair $(s,t) \in \Pi_L$ and $\zeta \in \Pi$. We consider the function

$$\begin{aligned} g^n(s,t) :& \Pi \to \mathbb{C} \\ & \zeta \mapsto g_\zeta^n(s,t). \end{aligned}$$

The smooth function $g^n(s,t)$ is called the function associated to g^n in Π. So, according to Proposition 4.55 all functions in E_n^+ (resp. E_n^-) can be considered as associated functions in Π (resp. $\overline{\Pi} = \{z = x + \mathbf{i}\, y \in \mathbb{C} \,|y < 0\}$).

4.56 Proposition. *We have that $f \in E_n^+$ if and only if $f \in E_{-n(n-1)}(\square)$ and for all $(s,t) \in \Pi_L$ the associated function $f(s,t) : \Pi \to \mathbb{C}$ is antiholomorphic in Π. Similarly, $f \in E_n^-$ if and only if $f \in E_{-n(n-1)}(\square)$ and for all $(s,t) \in \Pi_L$ the associated function $f(s,t) : \Pi \to \mathbb{C}$ is holomorphic in Π.*

Proof. If $f \in E_n^+$, there is a sequence $\{\gamma_k \in G\}_{k\in\mathbb{N}}$ and $\{\alpha_k \in \mathbb{C}\}_{k\in\mathbb{N}}$ such that

$$f = \sum_{k\in\mathbb{N}} \alpha_k(\gamma_k \cdot g_z^n),$$

then the function associated to f in $(s,t) \in \Pi_L$ is $f(s,t) = \sum_{k\in\mathbb{N}} \alpha_k(\gamma_k \cdot g^n(s,t))$ which is a antiholomorphic function in Π. Conversely, if $f \in E_{-n(n-1)}(\square)$ we have two options either $f \in E_n^+$ or $f \in E_n^-$, thus the associated function $f(s,t)$ is antiholomorphic or holomorphic. The case $f \in E_n^-$ is treated in a similar way. $\square$

Let us consider the differential operators on Π_L:

$$\begin{aligned} (D_\zeta^+ f)(s,t) &= -\left(\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right), \\ (D_\zeta^0 f)(s,t) &= -2\left((s-\bar\zeta)\frac{\partial f}{\partial s} + (t-\bar\zeta)\frac{\partial f}{\partial t}\right), \\ (D_\zeta^- f)(s,t) &= \left((s-\bar\zeta)^2\frac{\partial f}{\partial s} + (t-\bar\zeta)^2\frac{\partial f}{\partial t}\right). \end{aligned}$$

4.57 Proposition. *For any $\zeta \in \Pi$, the operators $D_\zeta^+, D_\zeta^0, D_\zeta^-$ are differentials operators on E_n^+ which satisfy the $\mathfrak{sl}_2$ relations:*

$$[D_\zeta^0, D_\zeta^\pm] = \pm 2 D_\zeta^\pm \qquad [D_\zeta^+, D_\zeta^-] = D_\zeta^0.$$

That is, $(D_\zeta^+, D_\zeta^0, D_\zeta^-)$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple (see Definition (4.10)).

Proof. Straightforward. □

One has that every g_ζ^n is an anti-holomorphic function such that

$$D_\zeta^0 g_\zeta^n = 2n g_\zeta^n \quad \text{and} \quad D_\zeta^- g_\zeta^n = 0.$$

In other words, we have that g_ζ^n is a highest weight vector of weight $2n$ in E_n^+. We set,

$$v_{n,k} = (D_\zeta^+)^k g_\zeta^n.$$

4.58 Proposition. *The $\mathfrak{sl}_2(\mathbb{R})$-triple $(D_\zeta^+, D_\zeta^0, D_\zeta^-)$ acts on the $v_{n,k}$'s vectors in the following way:*

$$\begin{aligned}
D_\zeta^+ v_{n,k} &= v_{n,k+1}, \\
D_\zeta^0 v_{n,k} &= (2n+2k) v_{n,k}, \\
D_\zeta^- v_{n,k} &= -k(2n + (k-1)) v_{n,k-1}.
\end{aligned}$$

Proof. Straightforward. □

Then, by Proposition 4.58 we have that $(v_{n,k})_{k\in\mathbb{N}}$ is a total basis of E_n^+:

$$E_n^+ = \overline{\mathrm{Span}_{\mathbb{C}}(v_{n,k} | k \in \mathbb{N})}.$$

For all $n \geq 1$, we consider $L_n = p_{n,1} + D_\zeta^+$ the differential operator on Π_L where $p_{n,1} : \Pi_L \to \mathbb{C}$ denote the smooth function

$$p_{n,1}(s,t) = n \left(\frac{1}{s - \bar{\zeta}} + \frac{1}{t - \bar{\zeta}} \right).$$

Then by induction we define the functions $p_{n,k} : \Pi_L \to \mathbb{C}$

$$p_{n,0} = 1 \quad \text{and} \quad p_{n,k+1} = L_n^{k+1}(1) = L_n(p_{n,k}) = p_{n,1} p_{n,k} + D_\zeta^+ p_{n,k}.$$

4.59 Proposition. *For all $n \in \mathbb{N}^\times$, $k \in \mathbb{N}$, we have that*

$$v_{n,k} = v_{n,0} p_{n,k}. \tag{4.52}$$

Proof. Let us fix $n \in \mathbb{N}^\times$. We will prove by induction on k. When $k = 0$ the formula (4.52) is obvious. Assume that for k the formula (4.52) works. Then

$$\begin{aligned} v_{n,k+1} &= D_\zeta^+ v_{n,k} \\ &= D_\zeta^+(v_{n,0} p_{n,k}) = v_{n,1} p_{n,k} + v_{n,0} D_\zeta^+(p_{n,k}) \\ &= v_{n,0}(p_{n,1} p_{n,k} + D_\zeta^+(p_{n,k})) \\ &= v_{n,0} p_{n,k+1}. \end{aligned}$$

□

Now for any $\zeta \in \Pi$ and $i \in \mathbb{N}$, we consider the functions $X^\pm_{\zeta,(i,k)} : \Pi_L \to \mathbb{C}$ defined by

$$X^\pm_{\zeta,(i,k)}(s,t) = \frac{1}{(s-\bar{\zeta})^k (t-\bar{\zeta})^{i-k}} \pm \frac{1}{(t-\bar{\zeta})^k (s-\bar{\zeta})^{i-k}} \qquad (0 \le k \le i,\ (s,t) \in \Pi_L). \tag{4.53}$$

If there is no danger of confusion, we denote $X^\pm_{i,k}$ instead $X^\pm_{\zeta,(i,k)}$ and $X_{i,k}$ instead $X^+_{i,k}$.

Now, we list some proprieties of $X^\pm_{\zeta,(i,k)}$.

4.60 Proposition. *Let $\zeta \in \Pi$, $a \in \mathbb{R}^\times$, $i, i_1, i_2 \in \mathbb{N}$, $(0 \le k \le i)$, $(0 \le k_1 \le i_1)$, $(0 \le k_2 \le i_2)$ we have that*

i) $X^+_{0,0} \equiv 2$ *and* $X^-_{0,0} \equiv 0$.

ii) *For every* $(s,t) \in \Pi_L$,

$$X^-_{\zeta,(1,0)}(s,t) = g_\zeta(s,t) = \frac{1}{(t-\bar{\zeta})} - \frac{1}{(s-\bar{\zeta})}.$$

iii) $X^\pm_{i,k} = \pm X^\pm_{i,i-k}$, *in other words for every* $(s,t) \in \Pi_L$

$$X^\pm_{i,k}(s,t) = \pm X^\pm_{i,k}(t,s).$$

iv) $X^\pm_{a\zeta,(i,k)}(as,at) = \frac{1}{a^i} X^\pm_{\zeta,(i,k)}(s,t)$, *for every* $(s,t) \in \Pi_L$.

v) $X^\pm_{i_1,k_1} X^\pm_{i_2,k_2} = X^+_{i_1+i_2,k_1+k_2} \pm X^+_{i_1+i_2,i_1-k_1+k_2}$.

vi) $X^+_{i_1,k_1}X^-_{i_2,k_2} = X^-_{i_1+i_2,k_1+k_2} + X^-_{i_1+i_2,i_1-k_1+k_2}$.

vii) $D^+_\zeta X^\pm_{i,k} = kX^\pm_{i+1,k+1} + (i-k)X^\pm_{i+1,k}$.

viii) $L_n(X^\pm_{i,k}) = (n+i-k)X^\pm_{i+1,k} + (n+k)X^\pm_{i+1,k+1}$.

Proof. Straightforward. □

4.61 Proposition. *For all $n \in \mathbb{N}^\times$ and $i \in \mathbb{N}$, we have*

$$p_{n,i} = \frac{i!}{2}\sum_{k=0}^{i} a_{(n,i,k)}X_{i,k}, \tag{4.54}$$

where

$$a_{(n,i,k)} = \binom{n-1+k}{k}\binom{n-1+i-k}{i-k}. \tag{4.55}$$

Proof. Let us fix $n \in \mathbb{N}^\times$. The formula (4.54) is true when $i = 0$ because $X_{0,0} = 2$. Now first we will show by induction on $i \geq 1$ that for all $(n,i) \in \mathbb{N}^\times \times \mathbb{N}^\times$,

$$p_{n,i} = (i-1)!\sum_{k=0}^{i} a_{(n,i,k)}(i-k)X_{i,k}. \tag{4.56}$$

When $i = 1$ the formula (4.56) is true because $a_{(n,1,0)} = n$. Assume that for i the formula (4.56) is valid. Then

$$\begin{aligned}
p_{n,i+1} &= L_n(p_{n,i}) \\
&= (i-1)!\sum_{k=0}^{i} a_{(n,i,k)}(i-k)[(n+i-k)X_{i+1,k} + (n+k)X_{i+1,k+1}] \\
&= (i-1)!\left[\sum_{k=0}^{i} a_{(n,i,k)}(i-k)(n+i-k)X_{i+1,k}\right. \\
&\qquad\qquad \left. + \sum_{k=0}^{i} a_{(n,i,i-k)}k(n+i-k)X_{i+1,i-k+1}\right] \\
&= i!\sum_{k=0}^{i} a_{(n,i,k)}(n+i-k)X_{i+1,k} \\
&= i!\sum_{k=0}^{i+1} a_{(n,i+1,k)}(i+1-k)X_{i+1,k}.
\end{aligned}$$

Then, we know that $a_{(n,i,k)} = a_{(n,i,i-k)}$ and $X_{i,k} = X_{i,i-k}$, so replacing k by $(i-k)$ the formula (4.56) becomes

$$p_{n,i} = (i-1)!\sum_{k=0}^{i} a_{(n,i,k)}kX_{i,k}. \tag{4.57}$$

Thus, adding (4.56) + (4.57), we obtain the desired result:

$$2p_{n,i} = i! \sum_{k=0}^{i} a_{(n,i,k)} X_{i,k}.$$

□

4.62 Corollary. *For all $n \in \mathbb{N}^\times$ and $i \in \mathbb{N}$, we have*

$$v_{n,i} = v_{n,0} p_{n,i} = \frac{i!}{2} v_{n,0} \sum_{k=0}^{i} a_{(n,i,k)} X_{i,k}.$$

In particular, for $n = 1$ we have

$$v_{1,i} = v_{1,0} p_{1,i} = \frac{i!}{2} g_\zeta \sum_{k=0}^{i} X_{i,k}.$$

Proof. It follows of expression $v_{n,i} = v_{n,0} p_{n,i}$ and the Proposition 4.61. □

4.63 Proposition. *For all $n, m \in \mathbb{N}^\times$ and $k \in \mathbb{N}$, we have*

$$\frac{v_{n+m,k}}{k!} = \sum_{i+j=k} \frac{v_{n,i}}{i!} \frac{v_{m,j}}{j!}. \tag{4.58}$$

Proof. Let us fix $n, m \in \mathbb{N}^\times$. We proceed by induction on k. When $k = 0$ the result is obvious. Assume that the formula (4.58) is true for a given k. Then,

$$\begin{aligned}
\frac{v_{n+m,k+1}}{(k+1)!} &= \frac{1}{(k+1)} D_\zeta^+ \left(\frac{v_{n+m,k}}{k!} \right) \\
&= \frac{1}{(k+1)} D_\zeta^+ \left(\sum_{i+j=k} \frac{v_{n,i}}{i!} \frac{v_{m,j}}{j!} \right) \\
&= \frac{1}{(k+1)} \left(\sum_{i+j=k} \frac{1}{i!j!} v_{n,i+1} v_{m,j} + \sum_{i+j=k} \frac{1}{i!j!} v_{n,i} v_{n,j+1} \right) \\
&= \frac{1}{(k+1)} \left(\sum_{i+j=k+1} \frac{1}{(i-1)!j!} v_{n,i} v_{m,j} + \sum_{i+j=k+1} \frac{1}{i!(j-1)!} v_{n,i} v_{n,j} \right) \\
&= \sum_{i+j=k+1} \frac{v_{n,i}}{i!} \frac{v_{m,j}}{j!}.
\end{aligned}$$

□

4.64 Lemma. *For any $n \in \mathbb{N}$, we have that*

$$a) \quad (X_{2,0}^+)^n = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} X_{2n,2k}^+ \qquad \textit{and} \qquad b) \quad (X_{2,1}^+)^n = 2^{n-1} X_{2n,n}^+. \tag{4.59}$$

Proof. a) For $n=0$ and $n=1$ the Eq. (4.59) is true. Now we assume that equation is true for n and we follow the proof by induction. Then

$$\begin{aligned}
(X^+_{2,0})^{n+1} &= X^+_{2,0}\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}X^+_{2n,2k} \\
&= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}[X^+_{2n+2,2k}+X^+_{2n+2,2k+2}] \\
&= \frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}X^+_{2n+2,2k}+\sum_{k=1}^{n+1}\binom{n}{k-1}X^+_{2n+2,2k}\right] \\
&= \frac{1}{2}\sum_{k=0}^{n+1}\binom{n+1}{k}X^+_{2n+2,2k}.
\end{aligned}$$

b) It follows immediately from the definition of $X^+_{2,1}$. □

4.65 Proposition. *For any $n\in\mathbb{N}$, we set*

$$b_{(n,2r)} = \sum_{j=0}^{\min(r,n-r)} 2^{2j-1}\binom{n}{2j}\binom{n-2j}{r-j} \qquad (0\le r\le n) \tag{4.60}$$

$$b_{(n,2r+1)} = -\sum_{j=0}^{\min(r,n-r-1)} 2^{2j}\binom{n}{2j+1}\binom{n-2j-1}{r-j} \qquad (0\le r\le n-1). \tag{4.61}$$

We have that

$$a)\ v_{2n,0}=\sum_{r=0}^{2n}b_{(n,r)}X^+_{2n,r} \quad and \quad b)\ v_{2n+1,0}=\sum_{r=1}^{2n}[b_{(n,r)}-b_{(n,r-1)}]X^-_{2n+1,r}.$$

Proof. *a*) Let $n\in\mathbb{N}$. By Binomial Theorem and Lemma 4.64, we have that

$$\begin{aligned}
v_{2n,0} &= (X_{2,0}-X_{2,1})^n \\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i}X^{n-i}_{2,0}X^i_{2,1} \\
&= \sum_{i=0}^{n}\sum_{k=0}^{n-i}(-1)^i 2^{i-1}\binom{n}{i}\binom{n-i}{k}X_{2n,2k+i} \\
&= \sum_{r=0}^{2n}b_{(n,r)}X_{2n,r},
\end{aligned}$$

where

$$b_{(n,r)} = \sum_{(i,k)\in I(n,r)}\alpha_n(i,k),$$

with $\alpha_n(i,k)=(-1)^i2^{i-1}\binom{n}{i}\binom{n-i}{k}$ and

$$I(n,r)=\{(i,k)|0\le i\le n, 0\le k\le n-i, 2k+i=r\}.$$

Now, it is clear that

$$b_{(n,2r)} = \sum_{j=0}^{\min(r,n-r)} \alpha_n(2j, r-j) = \sum_{j=0}^{\min(r,n-r)} 2^{2j-1}\binom{n}{2j}\binom{n-2j}{r-j}$$

and

$$b_{(n,2r+1)} = \sum_{j=0}^{\min(r,n-r-1)} \alpha_n(2j+1, r-j) = -\sum_{j=0}^{\min(r,n-r-1)} 2^{2j}\binom{n}{2j+1}\binom{n-2j-1}{r-j}$$

b) It follows from (a) and

$$v_{2n+1,0} = v_{1,0}v_{2n,0} = X^-_{1,0}v_{2n.0}.$$

$\square$

4.66 Proposition. *Let $i \in \mathbb{N}$, $a_{(n,i,k)}$ be the integer given in Eq. (4.55) and $b_{(n,r)}$ be the integer given in Eq. (4.60). We have*

a) For every $n \geq 1$

$$v_{2n,i} = i! \sum_{l=0}^{2n+i} c_l(n,i)X^+_{2n+i,l},$$

with

$$c_l(n,i) = \sum_{k=0}^{\min(i,2n,l,2n+i-l)} b_{(n,l-k)}a_{(2n,i,k)}$$

b) For every $n \geq 0$

$$v_{2n+1,i} = i! \sum_{l=0}^{2n+i} \tilde{c}_l(n,i)X^-_{2n+i+1,l},$$

with

$$\tilde{c}_l(n,i) = \sum_{k=0}^{\min(i,2n,l-1,2n+i-l)} [b_{(n,l-k)} - b_{(n,l-k-1)}]a_{(2n+1,i,k)}.$$

Proof. Straightforward. $\square$

We consider the spaces

$$\mathcal{A}^+ = \overline{\mathrm{Span}_{\mathbb{C}}\{X^+_{n,i} | n \geq 2, 0 \leq i \leq n\}} \oplus \mathbb{C}\,.$$

and

$$\mathcal{A}^- = \overline{\mathrm{Span}_{\mathbb{C}}\{X^-_{n,i} | n \geq 1, 0 \leq i \leq n\}}.$$

4.67 Proposition. *We have that*

$$\mathcal{A}^+ = \bigoplus_{n\in\mathbb{N}} E^+_{2n} \qquad \textit{and} \qquad \mathcal{A}^- = \bigoplus_{n\in\mathbb{N}} E^+_{2n+1}.$$

4.68 Proposition. *i) If $f \in \mathcal{A}^+$, then for any $(s,t) \in \Pi_L$ we have $f(s,t) = f(t,s)$.*

ii) If $f \in \mathcal{A}^-$, then for any $(s,t) \in \Pi_L$ we have $f(s,t) = -f(t,s)$.

iii) If $f, g \in \mathcal{A}^\pm$, then $fg \in \mathcal{A}^+$.

iv) If $f \in \mathcal{A}^+$ and $g \in \mathcal{A}^-$ then $fg \in \mathcal{A}^-$.

According to Propositions 4.67 and 4.68, we have that

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^- = \bigoplus_{n \in \mathbb{N}} E_n^+, \tag{4.62}$$

is an algebra with the point-wise product. Here $E_0^+ = \mathbb{C}$.

4.69 Lemma. *For any $n, m \in \mathbb{N}^\times$ and $i, j \in \mathbb{N}$, we have that*

$$p_{n,i} p_{m,j} = \frac{i!j!}{2} \sum_{r=0}^{i+j} \left(\sum_{k+l=r} a_{(n,i,k)} a_{(m,j,l)} \right) X_{i+j,r}^+, \tag{4.63}$$

where $a_{(n,i,k)}$ is defined in Eq. (4.55).

Proof. By Proposition 4.61, we know that

$$p_{n,i} = \frac{i!}{2} \sum_{k=0}^{i} a_{(n,i,k)} X_{i,k} \quad \text{and} \quad p_{m,j} = \frac{j!}{2} \sum_{l=0}^{j} a_{(m,j,l)} X_{j,l}.$$

Then

$$\begin{aligned} p_{n,i} p_{m,j} &= \frac{i!j!}{4} \left(\sum_{k=0}^{i} a_{(n,i,k)} X_{i,k} \right) \left(p_{m,j} = \frac{j!}{2} \sum_{l=0}^{j} a_{(m,j,l)} X_{j,l} \right) \\ &= \frac{i!j!}{4} \left(\sum_{k=0}^{i} \sum_{l=0}^{j} a_{(n,i,k)} a_{(m,j,l)} [X_{i+j,k+l} + X_{i+j,i-k+l}] \right) \\ &= \frac{i!j!}{2} \left(\sum_{k=0}^{i} \sum_{l=0}^{j} a_{(n,i,k)} a_{(m,j,l)} X_{i+j,k+l} \right) \\ &= \frac{i!j!}{2} \sum_{r=0}^{i+j} \left(\sum_{k+l=r} a_{(n,i,k)} a_{(m,j,l)} \right) X_{i+j,r}. \end{aligned}$$

□

4.70 Corollary. *For any $i, j \in \mathbb{N}$, if $i \leq j$ we have that*

$$p_{1,i} p_{1,j} = \frac{i!j!}{2} \left\{ 2 \sum_{r=0}^{i-1} (r+1) X_{i+j,r}^+ + (i+1) \sum_{r=i}^{j} X_{i+j,r}^+ \right\} \tag{4.64}$$

Proof. It follows from Lemma 4.69 and the fact that $a_{(1,i,k)} = 1$ for any i and $0 \le k \le i$. $\square$

4.71 Lemma. *Let $i, j \in \mathbb{N}$. If $i \le j$ we have*

$$v_{1,i}v_{1,j} = i!j!\left(X^{+}_{i+j+2,0} - X^{+}_{i+j+2,i+1}\right).$$

Proof. By Lemma 4.70, we have that

$$\begin{aligned}
v_{1,i}v_{1,j} &= v_{2,0}p_{1,i}p_{1,j} \\
&= \frac{i!j!}{2}(X_{2,0} - X_{2,1})\left\{2\sum_{r=0}^{i-1}(r+1)X_{i+j,r} + (i+1)\sum_{r=i}^{j} X_{i+j,r}\right\} \\
&= \frac{i!j!}{2}\{2[X_{i+j+2,0} - (i+1)X_{i+j+2,i} + iX_{i+j+2,i+1}] \\
&\qquad\qquad +2(i+1)[X_{i+j+2,i} - X_{i+j+2,i+1}]\} \\
&= i!j!\left(X_{i+j+2,0} - X_{i+j+2,i+1}\right).
\end{aligned}$$

$\square$

4.72 Proposition. *For any $n, m \in \mathbb{N}^{\times}$, $i, j \in \mathbb{N}$, there is a sequence*

$$\{\alpha_k \,|0 \le k \le n+m+i+j-1\} \subseteq \mathbb{Q}$$

such that

$$v_{n,i}v_{m,j} = \sum_{k=0}^{n+m+i+j-1} \alpha_k\, v_{n+m+i+j-k,k}. \tag{4.65}$$

Proof. It follows from Proposition 4.66. $\square$

4.73 Remark. Under assumptions of Proposition 4.72, we have

i) If $n+m$ is even and $i+j-k$ is odd, thus $\alpha_k = 0$.

ii) If $n+m$ is odd and $i+j-k$ is even, thus $\alpha_k = 0$.

4.2.2 Unterberger's symbolic calculus

In this section, we introduce a family of invariant symbolic calculi on the imaginary Lobachevsky space according to [UU96].

Let $\pi_{\mathbf{i}\,\lambda}$ be the unitary principal series representations of the Lie group $G = \mathrm{SL}(2, \mathbb{R})$, defined for $\lambda \in \mathbb{R}$ by

$$(\pi_{\mathbf{i}\,\lambda}(\gamma^{-1})u)(s) = |\,\mathrm{J}_\gamma(s)|^{-1-\mathbf{i}\,\lambda}u(\gamma \cdot s) \qquad (\gamma \in G, u \in L^2(\mathbb{R}), s \in \mathbb{R})$$

where $\mathrm{J}_\gamma(s) = cs + d$ and $\gamma \cdot s = \frac{as+b}{cs+d}$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

The Lobachevsky space Π_L is isomorphic to G/H, where H is the subgroup $\mathrm{SO}(1, 1)$. The Lie group G acts on the Hilbert space $L^2(\Pi_L) = L^2(\Pi_L, d\mu_L)$ of square-integrable functions on Π_L (with respect to the measure μ_L) by

$$(\varphi(\gamma^{-1})f)(s, t) = (\gamma^{-1} \cdot f)(s, t) = f(\gamma \cdot (s, t)), \qquad (\gamma \in G, (s, t) \in \Pi_L, f \in L^2(\Pi_L)).$$

Now, for any $f \in L^2(\Pi_L)$ and any $\lambda \in \mathbb{R}^\times$ we define the operator $\mathbf{Op}_\lambda(f)$ on the imaginary Lobachevsky space $G/H \simeq \Pi_L$ given by:

$$(\mathbf{Op}_\lambda(f)u)(s) = c_{-\lambda} \int_{\Pi_L} f(s, t)|s - t|^{-1-\mathbf{i}\,\lambda}u(\tau)|\tau - t|^{-1+\mathbf{i}\,\lambda}d\tau dt, \qquad (u \in L^2(\mathbb{R}), s \in \mathbb{R}) \tag{4.66}$$

where c_λ is the normalizing constant

$$c_\lambda = \frac{1}{2}(2\pi)^{\mathbf{i}\,\lambda}\left[\Gamma(\mathbf{i}\,\lambda)\cosh(\frac{\pi\,\lambda}{2})\right]^{-1}.$$

We have

4.74 Proposition. *For all $\lambda \in \mathbb{R}^\times$, the operator $\mathbf{Op}_\lambda : L^2(\Pi_L) \to HS(L^2(\mathbb{R}))$ is an isometry from the Hilbert space of square integrable functions $L^2(\Pi_L)$, called symbols, into the space of Hilbert-Schmidt operators on the configuration space $L^2(\mathbb{R})$. Moreover, $\mathbf{Op}_\lambda$ respects symmetries, that is, for any $\gamma \in G$, $f \in L^2(\Pi_L)$ we have*

$$\pi_{\mathbf{i}\,\lambda}(\gamma)\,\mathbf{Op}_\lambda(f)\pi_{\mathbf{i}\,\lambda}(\gamma^{-1}) = \mathbf{Op}_\lambda(f \circ \gamma^{-1}).$$

Proof. See [UU96]. □

The usual composition of Hilbert-Schmidt operators gives rise to a natural associative non-commutative product $\#_\lambda$ on the Hilbert space $L^2(\Pi_L)$ such that:

$$\mathbf{Op}_\lambda(f \#_\lambda g) = \mathbf{Op}_\lambda(f) \circ \mathbf{Op}_\lambda(g). \tag{4.67}$$

The product $\#_\lambda$ has an integral representation (see [UU94]). To see that we consider the cross- ratio of four points in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$[s,x,y,t] = \frac{(s-y)(x-t)}{(x-y)(s-t)}.$$

Then, the product $\#_\lambda$ is given by

$$(f\#_\lambda g)(s,t) = c_\lambda c_{-\lambda} \int_{\Pi_L} f(s,x)g(y,t)|[s,y,x,t]|^{-1-\mathbf{i}\lambda} d\mu_L(x,y). \tag{4.68}$$

where $d\mu_L$ is the G-invariant measure on Π_L given in Eq. (4.45).

According to [UU96] the commutative and associative algebra

$$\mathcal{A} = \bigoplus_{n\in\mathbb{N}} E_n^+,$$

defined in Eq. (4.62), is also an algebra with associative product $\#_\lambda$. Indeed,

4.75 Proposition. *Let $(n,m) \in \mathbb{N}^\times \times \mathbb{N}^\times$ and $(f,g) \in E_n^+ \times E_m^+$. The product $f\#_\lambda g$ can be written as a series $\sum_{k\in\mathbb{N}} h_k$, $h_k \in E_{n+m+k+2}^+$ (convergent as a series of pairwise orthogonal elements of $L^2(\Pi_L)$).*

Proof. See [UU96, Theorem 3.6]. □

The elements h_k in the Proposition 4.75 are given by the Rankin-Cohen brackets

$$\mathrm{RC}_k(f,g) = \sum_{r=0}^{k} (-1)^r \binom{2n+k-1}{r}\binom{2m+k-1}{k-r} \partial^{k-r} f\, \partial^r g \qquad ((f,g) \in \mathscr{H}^2_{2n}(\Pi) \times \mathscr{H}^2_{2m}(\Pi))$$

on the Bergman algebra $\mathscr{H}^+ = \bigoplus_{n\in\mathbb{N}} \mathscr{H}^2_{2n}(\Pi)$ (see Section 3.2.2). To show that the authors prove in [UU96] that each E_n^+ is isometric to the Hilbert space $\mathscr{H}^2_{2n}(\Pi)$. More precisely,

4.76 Proposition. *Given $n \in \mathbb{N}^\times$, set*

$$\alpha_n = 2^{-2(n-1)} \binom{2(n-1)}{n-1} \pi^2.$$

and define the operator Q_n by

$$(Q_n f)(z) = \alpha_n^{-1} \int_{\Pi_L} f(s,t) g_z^n(s,t) d\mu_L(s,t). \tag{4.69}$$

for every $f \in L^2(\Pi_L)$, $z \in \Pi$ and g_z^n as in Eq. (4.51). Then the operator $((2n-1)\,\alpha_n\,/4\pi)^{1/2} Q_n$ is an isometry from E_n^+ onto $\mathscr{H}^2_{2n}(\Pi)$. It acts as an intertwining operator between the quasi-regular representation of G in E_n^+ (given by $\gamma \cdot f = f \circ \gamma^{-1}$) and the representation ρ_{2n} (Eq. (3.21)) of G in $\mathscr{H}^2_{2n}(\Pi)$. Its inverse is given by the formula

$$f(s,t) = \frac{2n-1}{4\pi} \int_{\Pi} (Q_n f)(z) g_z^n(s,t) (\mathrm{Im}(z))^{2n} d\mu(z). \tag{4.70}$$

Proof. See [UU96, Proposition 2.2]. □

According to Proposition 3.39, if $\tilde{f} \in \mathscr{H}^2_{2n}(\Pi)$ and $\tilde{g} \in \mathscr{H}^2_{2m}(\Pi)$, then the Rankin-Cohen bracket $RC_k(\tilde{f}, \tilde{g})$ actually belongs to the Bergman space $\mathscr{H}^2_{2(n+m+k)}(\Pi)$ and

$$\mathrm{RC}_k(\rho_{2n}(\gamma)\tilde{f}, \rho_{2m}(\gamma)\tilde{g}) = \rho_{2(n+m+k)}(\gamma)(\mathrm{RC}_k(\tilde{f}, \tilde{g})),$$

for any $\gamma \in \mathrm{SL}(2, \mathbb{R})$.

4.77 Proposition. *Let $\lambda \in \mathbb{R}^\times$ and $(f, g) \in E^+_n \times E^+_m$ such that the product $f \#_\lambda g$ is expressed by*

$$f \#_\lambda g = \sum_{k \geq 2} h_k,$$

where the h_k's elements are in E^+_{n+m+k}. Then, there is a sequence $\{\Phi_k(n, m, \lambda) | k \geq 2\} \subseteq \mathbb{C}$ such that for all $k \geq 2$

$$Q_{n+m+k}(h_k) = \Phi_k(n, m, \lambda) RC_k(Q_n(f), Q_m(g)). \tag{4.71}$$

Proof. See [UU96, Proposition 3.6]. □

4.78 Remark. According to [UU96, Theorem 4.2], the explicit expression for coefficients $\Phi_k(n, m, \lambda)$ is

$$\Phi_k(n, m, \lambda) = 2^{2(k-2)} \frac{(2n-1)!(2m-1)!(n+m+k-3)!}{(n-1)!(2m+k-3)!(2n+2m+2k-6)!} \tag{4.72}$$
$$\sum_{l=0}^{k-2} (-1)^{k-l} \binom{k-2}{l} \frac{(n+l-1)!(2n+2m+k-4+l)!\Gamma(n+l-\mathbf{i}\,\lambda)\Gamma(m-\mathbf{i}\,\lambda)}{(2n+l-1)!(n+m-1+l)!\Gamma(n+m+l-\mathbf{i}\,\lambda)\Gamma(-\mathbf{i}\,\lambda)}.$$

Here we replace n by $m-1$, n by $m-1$ and k by $k-2$ in the expression given in [UU96, Theorem 4.2].

By some computations in GP/PARI the coefficients $\Phi_k(n, m, \lambda)$ are related to the coefficients $t^\kappa_k(i, j)$ defined in Eq. (3.25), in the follow way:

4.79 Conjecture. *For any $n, m \geq 1$, $k \geq 2$ and $\lambda \in \mathbb{R}^\times$, we have that*

$$\Phi_k(n, m, \lambda) = 4^k \frac{\Gamma(n+1-\mathbf{i}\,\lambda)\Gamma(m+1-\mathbf{i}\,\lambda)}{\Gamma(n+m+k-\mathbf{i}\,\lambda)\Gamma(-\mathbf{i}\,\lambda)} t_k^{-\mathbf{i}\,\lambda+1}(n+1, m+1). \tag{4.73}$$

We recall from Subsection 4.2.1 that $\{v_{n,k} | k \in \mathbb{N}\}$ is a total basis of E^+_n. It is given by

$$v_{n,k} = (D^+_\zeta)^k g^n_\zeta.$$

4.80 Proposition. *Let $n, m \geq 1$ and $\lambda \in \mathbb{R}^\times$, we have*

$$v_{n,0} \#_\lambda v_{m,0} = \Phi_2(n, m, \lambda) v_{n+m,0}.$$

Proof. See [UU96, Proposition 4.1]. □

4.3 Rankin-Cohen brackets and Fusion matrices

It is well-known [BBG98] that there is only one equivalence class of deformations (covariant or not) on co-adjoint orbits of group $G = \mathrm{SL}(2, \mathbb{R})$. The orbits $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1)$ and $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ are quite different. However, we have find one relation between Rankin-Cohen deformation and fusion matrices. We use as reference [EV99, ES01, Pev12].

Let M_λ^- and M_μ^- be two Verma modules of lowest weight λ and μ. We recall the definition (see Proposition 4.12) of $r_n^-(\lambda, \mu)$ the unique (up to constant multiples) e^--null vector of h eigenvalue $\lambda + \mu + 2n$

$$r_n^-(\lambda, \mu) = \sum_{i+j=n} (-1)^i \binom{\lambda + n - 1}{j} \binom{\mu + n - 1}{i} (e^+)^i \cdot v_\lambda^- \otimes (e^+)^j \cdot v_\mu^-.$$

We consider the the bilinear operator $RC_n^{(\lambda,\mu)}$ defined by

$$RC_n^{(\lambda,\mu)}(v_\lambda^- \otimes v_\mu^-) = r_n^-(\lambda, \mu).$$

If there is no confusion, we denote RC_n instead $RC_n^{(\lambda,\mu)}$. We can considerer a morphism between $M_{\lambda+\mu+2n}^-$ and $M_\lambda^- \otimes M_\mu^-$. More precisely,

4.81 Proposition. *In the basis $\{v_{\lambda+\mu+2n,k}^- | k \in \mathbb{N}\}$ of $M_{\lambda+\mu+2n}^-$, the map $R : M_{\lambda+\mu+2n}^- \to M_\lambda^- \otimes M_\mu^-$ given by*

$$R(v_{\lambda+\mu+2n,k}^-) = \frac{1}{k!} \Delta(e^+)^k \; r_n^-(\lambda, \mu) = \frac{1}{k!} \Delta(e^+)^k \; RC_n(v_\lambda^- \otimes v_\mu^-),$$

is a $\mathfrak{g}_\Bbbk$-morphism.

Proof. Clear for the definition. □

According to Theorem 4.35, we remark that

4.82 Proposition. *Let λ, μ be two complex numbers and $n \in \mathbb{N}$. If $-\mu \notin \mathbb{N}$, then we have that*

$$F_{-\mu}(u_n \otimes v_\mu^-) = R(v_{\lambda+\mu+2n}^-) = r_n^-(\lambda, \mu) = RC_n(v_\lambda^- \otimes v_\mu^-), \tag{4.74}$$

where $u_n := u_n(\lambda, \mu) = (-1)^n \binom{\mu+n-1}{n} (e^+)^n v_\lambda^-$.

Proof. It follows by Theorem 4.35 and the unicity of morphism. □

Now, we want express the Rankin-Cohen bracket RC_n as a fusion matrix. To see that, we find the fusion matrix between two Verma modules.

4.83 Proposition. *Let λ, ν, μ be three complex numbers. We have*

i) If $\lambda - \mu \notin \mathbb{N}$ and $\lambda - \mu - \nu \notin \mathbb{N}$, then the λ-fusion matrix $J^+_{M_\nu^-, M_\mu^-}(\lambda)$ exist and

$$\begin{aligned} J^+_{M_\nu^-, M_\mu^-}(\lambda)(v_{\nu,i}^- \otimes v_{\mu,j}^-) &= F_{\lambda-\mu-2j}(v_{\nu,i}^- \otimes v_{\mu,j}^-), \qquad (i, j \in \mathbb{N}) \\ &= \sum_{k=0}^{i} \frac{(-1)^k}{k!(\lambda-\mu-2j)_k} (e^-)^k v_{\nu,i}^- \otimes (e^+)^k v_{\mu,j}^-. \end{aligned}$$

ii) If $\lambda - \mu \notin \mathbb{Z}$ and $\lambda - \mu - \nu \notin \mathbb{Z}$, then the λ-fusion matrix $J^-_{M_\nu^-, M_\mu^-}(\lambda)$ exist and

$$\begin{aligned} J^-_{M_\nu^-, M_\mu^-}(\lambda)(v_{\nu,i}^- \otimes v_{\mu,j}^-) &= F_{\nu-\lambda+2i}(v_{\nu,i}^- \otimes v_{\mu,j}^-), \qquad (i, j \in \mathbb{N}) \\ &= \sum_{k=0}^{i} \frac{(-1)^k}{k!(\nu-\lambda+2i)_k} (e^-)^k v_{\nu,i}^- \otimes (e^+)^k v_{\mu,j}^-. \end{aligned}$$

Proof. It follows from Proposition 4.38. □

In particular,

4.84 Proposition. *Let λ, μ be two complex numbers. If $-\mu \notin \mathbb{N}$ and $-\mu - \lambda \notin \mathbb{N}$, then the λ-fusion matrix $J^+_{M_\lambda^-, M_\mu^-}(0)$ exist and*

$$J^+_{M_\lambda^-, M_\mu^-}(0)(u_n \otimes v_{\mu,j}^-) = RC_n^{(\lambda,\mu)}(v_\lambda^- \otimes v_{\mu,j}^-),$$

where $u_n = u_n(\lambda, \mu) = (-1)^n \binom{\mu+n-1}{n} (e^+)^n v_\lambda^-$. In particular, if $j = 0$ we have

$$J^+_{M_\lambda^-, M_\mu^-}(0)(u_n \otimes v_\mu^-) = RC_n(v_\lambda^- \otimes v_\mu^-) = r_n^-(\lambda, \mu).$$

Proof. It follows from Proposition 4.83. □

The Verma module M_λ^- is a $\mathbb{Z}$-graded module, thus according to Subsection 4.1.1.1, we can consider the completion

$$\widehat{M}_\lambda^- = \prod_{n\in\mathbb{N}} \Bbbk\, v_{\lambda,n}\hbar^n,$$

where $\{v_{\lambda,n}^- | n \in \mathbb{N}\}$ is the total basis of M_λ^-. The completed tensor product $M_\lambda^- \mathbin{\widehat{\otimes}} M_\mu^-$ is given by

$$M_\lambda^- \mathbin{\widehat{\otimes}} M_\mu^- = \widehat{M}_\lambda^- \otimes_{\Bbbk[[\hbar]]} M_\mu^- = \prod_{n\in\mathbb{N}} (v_{\lambda,n} \otimes w_n)\hbar^n,$$

where w_n is an element in M_μ^-.

4.85 Theorem. *Let M_λ^- and M_μ^- be two Verma modules of lowest weight λ and lowest weight μ. If $-\mu \notin \mathbb{N}$ and $-\mu - \lambda \notin \mathbb{N}$, then*

$$J^+_{M_\lambda^-, M_\mu^-}(0)(u \otimes v_\mu^-) = F_{-\mu}(u \otimes v_\mu^-) = \sum_{n\in\mathbb{N}} RC_n(v_\lambda^- \otimes v_\mu^-)\hbar^n,$$

where u is given by

$$u = \sum_{n\in\mathbb{N}} u_n(\lambda,\mu)\hbar^n = \sum_{n\in\mathbb{N}} (-1)^n \binom{\mu+n-1}{n} (e^+)^n v_\lambda^- \hbar^n \in \widehat{M}_\lambda^-.$$

Proof. It follows from Proposition 4.84. □

Conclusions

Dans le troisième chapitre, nous introduisons et développons la notion d'algèbre de Rankin-Cohen (Définition 3.10) comme une généralisation de l'algèbre des formes modulaires de poids arbitraire. Cette notion diffère de celle que Zagier a introduite dans [Zag94].

En accord avec [CMZ97], on sait que l'application $t^{\kappa}:\mathbb{N}^3\to\mathbb{C}$ donnée par

$$t_k^{\kappa}(i,j)=\left(-\frac{1}{4}\right)^i\sum_{r\geq 0}\binom{i}{2r}\frac{\binom{-\frac{1}{2}}{r}\binom{\kappa-\frac{3}{2}}{r}\binom{\frac{1}{2}-\kappa}{r}}{\binom{-i-\frac{1}{2}}{r}\binom{-j-\frac{1}{2}}{r}\binom{k+i+j-\frac{3}{2}}{r}}, \tag{4.75}$$

détermine une déformation formelle et $\mathrm{SL}(2,\mathbb{R})$-invariante $\sum_k t_k \mathrm{RC}_k \hbar^k$, dans l'algèbre des formes modulaires de poids arbitraire. Étant donné une algèbre de RC (Définition 3.10), nous dirons que l'application $t:\mathbb{N}^3\to\mathbb{C}$ est une “twist” de Rankin-Cohen de A, si le produit défini par $\sum_k t_k \mathrm{RC}_k \hbar^k$, est encore une déformation formelle et $\mathrm{SL}(2,\mathbb{R})$-invariante dans $A[[\hbar]]$.

Dans le théorème principal du chapitre nous montrons que l'algèbre de Bergman $\mathscr{H}^+$ est une algèbre de RC. En plus, nous montrons que pour tout $\kappa\in\mathbb{C}$ l'application t^{κ} (Eq. 4.75) est un twist de $\mathscr{H}^+$. Pour la démonstration, nous utilisons une condition combinatoire d'associativité intrinsèquement associée au twist et à l'algèbre A. Nous décrivons cette condition. Soit $Q_{(\alpha_1,\alpha_2,\alpha_3)}(i,j,k,r,s)$ l'entier

$$(-1)^j\binom{i+j}{i}\binom{\alpha_1+\alpha_2+\nu_{r+s}+i+j+k-1}{k}\binom{\alpha_3+i+j+k-1}{i+j}$$
$$\binom{\alpha_1+r+s-1}{s}\binom{\alpha_2+r+s-1}{r}.$$

Considérons $t:\mathbb{N}^3\to\mathbb{C}$ une application et $P_{(\alpha_1,\alpha_2,\alpha_3)}(x,y,z)$ le nombre complexe

$$\sum_{I(x,y,z)} t_{r+s}(\alpha_1,\alpha_2)t_{i+j+k}(\alpha_1+\alpha_2+\nu_{r+s},\alpha_3)Q_{(\alpha_1,\alpha_2,\alpha_3)}(i,j,k,r,s), \tag{4.76}$$

où $I(x,y,z)$ est l'ensemble de 5-tuples (i,j,k,r,s) tels que $i+r=x$, $j+s=y$, $k=z$. Nous dirons qu'une application $t:\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$ satisfasse la condition (Ct) si pour tout $x,y,z\in\mathbb{N}$ et $\alpha_1,\alpha_2,\alpha_3\in\mathrm{supp}(A)\subset\mathbb{Z}$, on a

$$P_{(\alpha_1,\alpha_2,\alpha_3)}(x,y,z)=(-1)^y P_{(\alpha_3,\alpha_2,\alpha_1)}(z,y,x). \tag{Ct}$$

Le théorème suivant est le résultat principal du troisième chapitre.

Théorème. *Pour tout* $\kappa \in \mathbb{C}$ *la fonction* $t^\kappa : \mathbb{N}^3 \to \mathbb{C}$ *défini en* (3) *est un twist de* $\mathscr{H}^+$*. En particulier, si* $\kappa = \frac{1}{2}$ *ou* $\kappa = \frac{3}{2}$ *nous retrouvons la déformation* $\mathrm{SL}(2, \mathbb{R})$*-équivariante*

$$f \star_{\mathrm{RC}} g = \sum_{k \in \mathbb{N}} \mathrm{RC}_k(f, g)\hbar^k \qquad\qquad (f, g \in \mathscr{H}^+).$$

Autrement dit l'espace $\mathscr{H}^+$ *est une algèbre de Rankin-Cohen.*

Dans le quatrième chapitre nous établissons un lien entre les déformation formelle de Rankin-Cohen et celle donnée par Alekseev et Lachowska que l'on note $\star_{\mathrm{S}}$.

L'objet fondamental pour la construction de $\star_{\mathrm{S}}$ est l'élément canonique F_λ associée à la dualité de Shapovalov $\langle\ ,\ \rangle_\lambda : U(\mathfrak{n}_-) \times U(\mathfrak{n}_+) \to \mathbb{C}$ (Définition 4.25). Ensuite, on utilise les matrices de fusion (Définition 4.38) pour montrer que F_λ satisfasse la condition d'associativité de Drinfeld (Proposition 4.47). Dans la Proposition 4.49, nous décrivons explicitement les éléments a_n du développement asymptotique de $F_{\hbar^{-1}}$ et précisons le résultats d'Alekseev et Lachowska.

Le calcul symbolique $\#_\lambda$ (Eq. 4.67) d'Unterberger est lié aux crochets de Rankin-Cohen 4.77. Nous conjecturons que les coefficients de Unterberger $\Phi_k(n, m, \lambda)$ sont étroitement liés au twist de Zagier t^κ (Conjecture 4.79).

Finalement, nous montrons le théorème principal de cette thèse (Théorème 4.85) qui exprime les crochets de Rankin-Cohen en termes des matrices de fusion (Proposition 4.82). Alors la condition d'associativité de F_λ codifie l'associativité du produit $\sum_k \mathrm{RC}_k \hbar^k$.

Théorème. *Soient* M_λ^- *et* M_μ^- *deux modules de Verma de plus bas poids* λ *et de plus bas poids* μ*. Si* $-\mu \notin \mathbb{N}$ *et* $-\mu - \lambda \notin \mathbb{N}$*, alors*

$$J^+_{M_\lambda^-, M_\mu^-}(0)(u \otimes v_\mu^-) = F_{-\mu}(u \otimes v_\mu^-) = \sum_{n \in \mathbb{N}} RC_n(v_\lambda^- \otimes v_\mu^-)\hbar^n,$$

où u *est donné par* $u = \sum_{n \in \mathbb{N}} u_n(\lambda, \mu)\hbar^n = \sum_{n \in \mathbb{N}} (-1)^n \binom{\mu+n-1}{n} (e^+)^n v_\lambda^- \hbar^n \in \widehat{M}_\lambda^-$.

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OPERATEURS DE RANKIN-COHEN ET MATRICES DE FUSION

Ce travail est consacré a l'étude des déformations équivariantes des orbites co-adjointes du groupe de Lie SL(2,$\mathbb{R}$). Nous établissons un lien entre des méthodes de quantification basées sur les crochets de Rankin-Cohen et les matrices de fusion pour les modules de Verma. Par ailleurs nous formalisons et étudions la notion associée d'algèbre de Rankin-Cohen qui contrôle l'associativité de ces déformations.

Crochets de Rankin-Cohen, Quantification, Star-produits, Dualité de Shapovalov, Matrices de Fusion, Théorie d'algèbres et groupes de Lie.

Titre en anglai

This work is devoted to the study of equivariant star-product on coadjoint orbits of the Lie group SL(2,$\mathbb{R}$). We establish a correspondence between two quantization methods. The first is based on the Rankin-Cohen brackets and the second is based in the canonical element associate to the Shapovalov form and fusion matrices for Verma modules. Furthermore we formalize and study the associated notion of non-commutative algebra that controls the associativity of these deformations.

Rankin-Cohen brackets, Quantization, Star-products, Shapovalov form, Fusion matrix, Lie algebres and Lie group Theory.

Discipline : MATHEMATIQUES

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