Département de formation doctorale en automatique
UFR Sciences et Technologies

École doctorale IAEM Lorraine

Observateurs des systèmes singuliers incertains : Application au contrôle et au diagnostic

THÈSE

présentée et soutenue publiquement le 16 Juillet 2015

pour l'obtention du

Doctorat de l'Université de Lorraine

(Spécialité Automatique, Traitement du Signal et Génie Informatique)

par

Gloria Lilia OSORIO GORDILLO

Composition du jury

Rapporteurs :	Michel KINNAERT	Professeur, Université Libre de Bruxelles
	Efraín ALCORTA GARCÍA	Professeur, Université Autonome de Nuevo León
Examinateurs :	Holger VOOS	Professeur, Université de Luxembourg
	Mohamed DAROUACH	Professeur, Université de Lorraine, CRAN, Directeur de thèse
	Latifa BOUTAT-BADDAS	Maître de Conférences, Université de Lorraine, CRAN, Co-directrice de thèse
	Carlos Manuel ASTORGA ZARAGOZA	Professeur, Centre National de Recherche et Développement Technologique, CENIDET, Co-directeur de thèse

Centre de Recherche en Automatique de Nancy
CRAN - CNRS UMR 7039

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Reviewers :	Holger VOOS	Professor, Université de Luxembourg
	Mohamed DAROUACH	Professor, Université de Lorraine, CRAN, Thesis director
	Latifa BOUTAT-BADDAS	Associate Professor, Université de Lorraine,CRAN, Co-director of thesis
	Carlos Manuel ASTORGA ZARAGOZA	Professor, Centro Nacional de Investigación y Desarrollo Tecnológico, CENIDET, Co-director of thesis

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To my family...

Acknowledgments

First, I would like to express my deepest and most sincere gratitude to my thesis supervisors doctors Mohamed Darouach, Latifa Boutat-Baddas and Carlos-Manuel Astorga-Zaragoza, who supported me and pushed me in the course of my thesis with patience, motivation and enthusiasm. Their guidance helped me all the time of research and writing of this thesis.

I would also like to thank to the members of the thesis review committee, in France to doctors Michel Kinnaert, Efraín Alcorta García and Holger Voos. In Mexico to doctors Carlos Daniel García Beltrán, Gerardo Vicente Guerrero, Manuel Adam Medina and Juan Reyes Reyes. To all of them thank you for the valuable comments that have enriched this work.

I would like to thank to members of CRAN that made my stay in Longwy a very important experience for my training as a researcher. Especially to Marouane Alma, Michel Zasadzinski, Harouna Soley Ali, Ali Zemouche, Mohamed Boutayeb, Cèdric Delattre, Joëlle Pinelli and Nathalie Clèment with whom I shared knowledge and experiences of professional and personal type that were invaluable.

I also like to thank to PhD students : Yassine Boukal, Hao Nguyen Dang, Florian Seve, Ghazi Bel Haj Frej, Bessem Bhiri and Asma Barbata who supported me in diverse situations, really thank you. In a particular way I would like to express my gratitude to my college and friend Nan Gao, with who I shared long time of studies and also moments of life.

I would also like to express my sincere thanks to members of CENIDET : professors, PhD students, master students and administrative staff, who during the course of my education were guiding me and supporting me. Sincerely thank you.

I would also like to thank to MEXPROTEC guys who supported me in the difficult task of being away from home, but also accompanied me on the great adventure of living in another country. Thank you for your friendship.

To my friends from Oaxaca, Mexico, that despite the distance our friendship has prevailed, their words of encouragement and unconditional support have been an important factor for the culmination of this work.

Certainly, I would like to especially thank to Rodolfo Amalio Vargas Méndez for his unconditional support, patience and love. For sharing laughs, travels and life itself. Really thank you.

Of course, the deepest thanks and heartfelt goes to my parents and brothers, who have always supported me in every decision that I have made, that motivated me to keep going, that no matter the distance they always had me in their thoughts, that to taught me that life is not easy but you have to enjoy it. You always fill my life with happiness, this success is also yours.

I would also like to extend my thanks to my grandparents, who always gave me a reason to return home.

Publications

Published journals

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH, L. BOUTAT-BADDAS and C.-M. ASTORGA-ZARAGOZA, (2015). Dynamical observer-based fault detection and isolation for linear singular systems. *Systems Science & Control Engineering : An Open Access Journal*, 3 :189-197.

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH and C.-M. ASTORGA-ZARAGOZA, (2015). H_∞ dynamical observers design for linear descriptor systems. *European Journal of Control*.

Published conferences

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH and C.-M. ASTORGA-ZARAGOZA, (2014). H_∞ Dynamical observer design for linear descriptor systems. American Control Conference (ACC14), Portland, Oregon, USA.

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH, L. BOUTAT-BADDAS and C.-M. ASTORGA-ZARAGOZA, (2014). H_∞ Dynamical observer-based control for descriptor systems. 13th European Control Conference (ECC14), Strasbourg, France.

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH, C.-M. ASTORGA-ZARAGOZA and L. BOUTAT-BADDAS, (2014). Dynamical observer design for discrete-time descriptor linear systems. XVI Congreso Latinoamericano de Control Automático (CLCA 2014), Cancún, Mexico.

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH, C.-M. ASTORGA-ZARAGOZA and L. BOUTAT-BADDAS, (2015). Fault diagnosis for discrete-time descriptor linear systems. 9th IFAC Symposium on Fault Detection, Supervision and Safety for Technical Processes (SAFEPROCESS'15), Paris, France.

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH, C.-M. ASTORGA-ZARAGOZA and L. BOUTAT-BADDAS, (2015). Observateurs dynamiques généralisés pour les systèmes singuliers linéaires à temps discret. 6èmes Journées Doctorales/Journées Nationales MACS, Bourges, France.

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH, L. BOUTAT-BADDAS and C.-M. ASTORGA-ZARAGOZA, (2015). Generalized dynamic observers for LPV singular systems. 1st IFAC Workshop on Linear Parameter Varying Systems (LPVS 2015), Grenoble, France.

- **G.-L. OSORIO-GORDILLO**, M. DAROUACH, L. BOUTAT-BADDAS and C.-M. ASTORGA-ZARAGOZA, (2015). H_∞ generalized dynamic observer for LPV singular systems. Congreso Nacional de Control Automático (AMCA 2015), Cuernavaca, Morelos, Mexico.

Abstract

Keywords : Observers, Descriptor systems, Uncertainties, Fault diagnosis, Control.

In this thesis the observer design for uncertain linear descriptor systems and their applications to control and fault diagnosis is studied.

Descriptor systems can be considered as a generalization of dynamical systems. This class of systems include algebraic and differential equations.

The observer used in this work has a new structure more general than those presented in the literature. The observer structure proposed has additional degrees of freedom, which provides it robustness in face to variations not considered in the model.

The new observer structure used in this thesis, named as generalized dynamic observer (GDO), is designed for different classes of descriptor systems. The asymptotic stability of the observer is proved by Lyapunov analysis through a set of linear matrix inequalities (LMIs).

In all cases, the LMI obtained from the Lyapunov analysis is treated by the elimination lemma. The use of the elimination lemma is essential in the development of the stability analysis of the observers, since it allows to obtain the GDO structure. Proportional observers (PO) and proportional-integral observers (PIO) can be considered as particular cases of our observer.

The thesis is organized as follows :

In the general introduction, the problem formulation is presented, the objectives of the thesis are pointed out, the scope of the investigation and the main contributions are also presented.

Chapter 1 introduces descriptor systems as the class of systems considered in this work and presents a review of the state of the art focused on the observers design for these systems. Also we introduce the GDO as an observer with structure more general than that of the PO and the PIO.

Chapter 2 develops the GDO for descriptor systems with or without disturbances. Extension of these approaches for discrete-time descriptor systems with or without disturbances are also presented.

In Chapter 3, the robust approach of the GDO is treated for parametric uncertain descriptor systems, where the uncertainty is bounded, and for linear parameter varying (LPV) descriptor systems, where the parameters vary inside a polytope.

Chapter 4 presents the GDO application to observer-based control with the objective to stabilize descriptor systems that normally are unstable. An extension of this approach to disturbed descriptor systems is also developed.

Chapter 5 presents the GDO application to fault diagnosis, which is divided in two parts. The first one is to detect and isolate faults by using a GDO that provides residuals that are able to represent only the presence of one fault, so that we can isolate multiple faults. And the second part is to estimate the faults by using a GDO with a modified structure. These approaches are developed for descriptor systems and for uncertain descriptor systems.

The last part is dedicated to general conclusions and some perspectives.

Résumé

Mots-clés : Observateurs, Systèmes singuliers, Incertitudes, Diagnostic de défauts, Commande.

Dans cette thèse, la conception d'observateurs pour les systèmes singuliers linéaires incertains et leurs applications au contrôle et au diagnostic. En effet, nous avons développé des méthodes de reconstruction d'état et d'estimation de défauts est étudié.

Les systèmes algèbro-différentiels ou systèmes singuliers peuvent être considérés comme une généralisation des systèmes dynamiques. Ils constituent un puissant outil de modélisation dans la mesure où ils peuvent décrire des processus régis à la fois par des équations différentielles (dynamiques) et des équations algébriques (statiques).

La nouvelle structure d'observateurs utilisée dans cette thèse est nommée l'Observateur Dynamique Généralisé (ODG), elle est plus générale que celle d'Observateurs Proportionnels (OP) et d'Observateurs Proportionnels Intégrals (OPI). Cette structure présente une estimation d'état alternative qui peut être considérée comme plus générale que les OP et les OPI, ceux-ci pouvant être considérés comme des cas particuliers de cette structure. L'approche proposée repose sur la paramétrisation des solutions des équations de Sylvester pour éliminer le biais entre l'erreur d'observation et la paire (entrée état).

La thèse est organisée comme suit :

Dans l'introduction générale, nous présentons la problématique et les objectifs de la thèse ainsi que les principales contributions.

Dans le premier chapitre, nous présentons la classe des systèmes singuliers considérée. Nous faisons des rappels sur l'analyse de stabilité et l'utilisation des outils numériques LMI avec lesquels nous vérifions l'existence de conditions de stabilité. Ensuite, nous présentons les méthodes de reconstruction d'état des systèmes singuliers linéaires à savoir l'ODG, l'OP et l'OPI.

Dans le deuxième chapitre, nous présentons en détail la procédure de synthèse d'ODG pour les systèmes singuliers continus avec et sans perturbations. Ensuite, nous faisons une extension aux systèmes singuliers en temps discret avec et sans perturbations.

Dans le chapitre 3, nous donnons les conditions d'existence et de stabilité robuste de l'ODG pour les systèmes singuliers à paramètres incertains, où l'incertitude est bornée.

Dans le chapitre 4, nous présentons une méthode de synthèse de commande stabilisante par retour d'état basée observateur pour une classe de systèmes singuliers linéaires avec et sans perturbations.

Le chapitre 5, est consacré au diagnostic. L'étude que nous avons menée est traitée en deux étapes : La première étape est consacrée à la détection et l'isolation des défauts en utilisant un ODG. Cet observateur génère des résidus qui sont en mesure de représenter seulement la présence d'un défaut, de sorte que nous pouvons localiser des défauts multiples. Enfin, la deuxième étape est consacrée à l'estimation des défauts en utilisant un ODG avec une structure modifiée. Ces approches sont développées pour les systèmes singuliers et pour les systèmes singuliers incertains avec ou sans perturbations.

Nous terminerons ce mémoire de thèse par une conclusion générale et quelques perspectives.

Table of contents

List of figures

List of tables

Notation and acronyms

Sets and norms

$\mathbb{R}$, $\mathbb{C}$	Set of all real numbers (resp. complex).
$\mathbb{R}^+$, $\mathbb{R}^-$	Set of all real positive (negative) of real numbers.
$\mathbb{C}^+$, $\mathbb{C}^-$	The open right (left) half complex plane.
$\mathbb{R}^n$, $\mathbb{C}^n$	Set of n-dimensional real vectors (resp. complex).
$\mathbb{R}^{n\times m}$	Set of $n \times m$ dimensional real matrices.
$\mathcal{L}_2[0,\infty)$	Signals having finite energy over the infinite time interval $[0,\infty)$.
$\|a\|$	The absolute value of scalar a.
$\\|x\\|_\infty$	The norm-∞ of signal x.
$\\|x\\|_2$	The norm-2 of signal x.
$\\|G\\|_\infty$	The norm-∞ of system $G(s)$.

Notation related to vectors and matrices

$A > 0$, $A \geq 0$	Real symmetric (semi) positive-definite matrix A.
$A < 0$, $A \leq 0$	Real symmetric (semi) negative-definite matrix A.
I, 0	Identity matrix (resp. zero matrix) of appropriate dimension.
I_n	Identity matrix of dimension $n \times n$.
A^{-1}	Inverse of matrix $A \in \mathbb{R}^{n\times n}$, $\mathsf{det}(A) \neq 0$.
A^+	Any generalized inverse of matrix A, verifying $AA^+A = A$.
A^T	Transpose of matrix A.
$A^\perp$	Any maximal row rank matrix such that $A^\perp A = 0$ and $A^\perp A^{\perp T} > 0$.
$\mathsf{eig}(A)$	Set of all eigenvalues of matrix A.
$\mathsf{det}(A)$	Determinant of matrix $A \in \mathbb{R}^{n\times n}$.
$\mathsf{rank}(A)$	Rank of matrix $A \in \mathbb{R}^{n\times m}$.
$\mathcal{I}m(A)$	Image space of matrix A : $\{y$ such that $y = Ax\}$.
$\mathscr{R}(A)$	Denote the row space of a matrix A, $\mathcal{I}m(A) = \{Ax, x \in \mathbb{R}^n\}$, and $\mathscr{R}(A) = \mathcal{I}m(A^T)$.
$\mathsf{Re}(A)$	Real part of A.
$\mathsf{diag}(A_1,\ldots,A_p)$	Diagonal matrix with elements $A_i \in \mathbb{R}^{n\times n},\ i = 1,\ldots,p$ on its diagonal.
$\mathsf{dim}(A)$	Dimension of A.
$(*)$	The transpose elements in the symmetric positions.

Acronyms

BRL	Bounded Real Lemma.
FE	Fault Estimation.
FDI	Fault Detection and Isolation.
GDO	Generalized Dynamic Observer.
IAE	Integral Absolute Error.
LMI	Linear Matrix Inequality.
LPV	Linear Parameter Varying.
PO	Proportional Observer.
PIO	Proportional-Integral Observer.
SISO	Single-Input Single-Output.

General introduction

1 Problem formulation

The problematic to be addressed is the observer design for uncertain descriptor systems. Descriptor systems were introduced by Luenberger in 1977, to represent the systems for which, the results obtained in the domain of control and observation of standard systems can not be applied. This class of systems can represent the physical phenomena that the model by ordinary differential equations can not describe. This special kind of model is found in different areas as chemical, mineral industries, robotics and electrical fields (Dai, 1989).

In many works on control of dynamic systems, the state vector is assumed to be accessible for measurement. However, in practical cases, this assumption is not always verified. Either by technical or economic reasons it is difficult, or even impossible to measure all the system state variables, is there where the necessity of estimation arises.

The necessity to know completely the state variables of a system is often a requirement in modeling development, identification, fault diagnosis or control of systems. All these developments need to have as much as possible knowledge of the system state variables. Which places the observers design as one of the principal tools to provide the estimation of system state variables.

The first work on the problem of reconstruction of state variables was devoted to linear systems (Luenberger, 1971). Since then, many theoretical results have been proposed and are widely used in control and fault diagnosis.

In matter of design of observers for descriptor systems there are several works. In Müller and Hou (1993), Darouach (2009a), Zhou and Lu (2009), Darouach (2012) and references therein, a proportional observer (PO) for descriptor systems is proposed, on the other hand in Koenig (2005), Söffker et al. (1995) the authors present the proportional-integral observer (PIO) approach. In Kaczorek (1979), Beale and Shafai (1989) and Söffker et al. (1995) the authors show the performance of the PIO, compared to the PO in presence of disturbances and uncertainties, being the PIO estimation better than the PO. Then, it is apparent that a modified structure with additional degrees of freedom in the observer might provide best estimation and robustness in face to variations.

In the estimation by a PO there always exists a static estimation error in presence of disturbances, in order to deal with this disadvantage of the PO, PIO were introduced with an integral gain of the output error (difference between the estimated output and the output) in their structure, this change in the structure achieves steady-state accuracy in their estimations.

A new structure of observer was developed by Goodwin and Middleton (1989) and Marquez (2003), known as generalized dynamic observer (GDO). This structure presents an alternative state estimation which can be considered as more general than PO and PIO.

2 Objectives of the thesis

The goal of the present research is to develop a generalized dynamic observer (GDO) for uncertain descriptor systems to provide state estimation. Using the proposed observer an observer-based control is develop to stabilize uncertain descriptor systems that are naturally unstable. Another goal of this research is to develop a scheme of fault diagnosis for uncertain descriptor systems using the observer proposed. The objectives of the research are summarized bellow :

- Propose a new method for the synthesis of observers for uncertain descriptor systems, establishing sufficient conditions for its convergence and stability.
- Develop an observer-based control to stabilize uncertain descriptor systems that are naturally unstable.
- Design a fault diagnosis scheme using the generalized dynamic observer.

3 Scope and delimitation

- Just norm bounded uncertainties are considered for descriptor systems, such that

$$\begin{aligned} E\dot{x}(t) &= (A + \Delta A(t))x(t) + Bu(t) \\ y(t) &= (C + \Delta C(t))x(t) \end{aligned}$$

  is the general kind of uncertain descriptor systems treated in this work.

- The GDO for uncertain descriptor systems has the following general representation :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \\ \dot{v}(t) &= S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \end{aligned}$$

- Observer-based feedback control is considered to stabilize descriptor systems that normally are unstable. By using the observer-based control approach when the state variables are not available to be measured.
- Uncertain descriptor systems with actuator faults are considered for the fault diagnosis scheme, as is shown below :

$$\begin{aligned} E\dot{x}(t) &= (A + \Delta A(t))x(t) + Gf(t) \\ y(t) &= Cx(t) \end{aligned}$$

- The main contribution of this thesis is the synthesis of new observer schemes for uncertain descriptor systems on order to be used for control and fault diagnosis purposes. In that sense, just simulations are used to show the approaches developed in this work using numerical examples.

4 Main contributions

The main contribution of this thesis is the GDO structure proposed, since it is more general than the PO and PIO generalizing those reported in the literature. The structure of the dynamical observer contains an additional variable which represents the integral of the state of the observer and the output. It is a generalization of the integral of the output error (difference between the estimated output and the output). Its role is to increase steady state accuracy and to improve robustness in estimation performances against disturbances and modeling errors. It also provides additional degrees of freedom in the observer design which can be used to increase the stability margin.

5 Outline of the thesis

This thesis consists of five principal chapters. The following paragraphs gives more details about the content of each part :

Chapter 1 is dedicated to the description of descriptor systems, some practical examples and definitions of descriptor systems are also presented. A general bibliography review of observers design for descriptor systems is treated, and then we present the GDO which is considered in the rest of the thesis. Additionally, some useful tools for stability analysis in dynamic systems are introduced.

Chapter 2 is devoted to the H_∞ GDO design for disturbed descriptor systems in continuous-time without or with disturbances, and discrete-time descriptor systems without or with disturbances are treated. For all approaches, sufficient conditions to guarantee the asymptotic convergence of the estimate state to the real state are given in terms of linear matrix inequalities (LMIs). Particular cases of the GDO as the PO and the PIO are also developed. Simulation examples are presented to illustrate our approach compared to the PO and the PIO.

Chapter 3 is dedicated to the robust H_∞ GDO design for disturbed descriptor systems. In this chapter descriptor systems with parametric uncertainties are considered. Uncertain descriptor systems without or with disturbances, and linear parameter varying (LPV) descriptor systems without or with disturbances are presented. The PO and PIO as particular cases of GDO are also designed. A numerical example is presented to illustrate the effectiveness of the proposed method.

Chapter 4 is dedicated to the application of the H_∞ GDO to control in the sense of stability. The classes of systems considered are : descriptor systems without or with disturbances, and uncertain descriptor systems without or with disturbances. Particular cases of the GDO as the PO and the PIO are also developed. A numerical example is shown to test our approach compared to the PO and the PIO.

Chapter 5 is dedicated to the GDO application to fault diagnosis for descriptor systems and for uncertain descriptor systems. This problem is boarded through two observers, the fist one devoted to fault detection and isolation (FDI), and the second one is devoted to simultaneous estimation of state variables and faults.

Finally, the **conclusions and perspectives** is dedicated to general conclusions and to expose some problems that could be addressed in future works.

Introduction et résumé détaillé de la thèse

6 Formulation du problème

Les systèmes algébro-différentiels ou systèmes singuliers ont été introduits par Luenberger (1977). Ils peuvent être considérés comme une généralisation des systèmes dynamiques. Ils constituent un puissant outil de modélisation dans la mesure où ils peuvent décrire des processus régis à la fois par des équations différentielles (dynamiques) et par des équations algébriques (statiques). Ce formalisme représente des phénomènes physiques dont le modèle ne peut pas être décrit par des équations différentielles ordinaires. On les rencontre dans des domaines aussi variés que les industries chimiques et minérales, la robotique et le domaine électrique (Dai, 1989).

Dans de nombreux travaux sur le contrôle des systèmes dynamiques, le vecteur d'état est supposé accessible à la mesure. Cependant, dans de nombreux cas pratiques, cette hypothèse n'est pas toujours vérifiée. Soient pour des raisons techniques ou économiques, il est difficile, voire impossible de mesurer toutes les variables d'état du système.

La nécessité de connaître complètement les variables d'état d'un système est souvent une exigence dans le domaine de la modélisation, l'identification, le diagnostic de défauts ou de contrôle des systèmes. Tous ces domaines ont besoin de connaitre autant que possible les variables d'état du système. C'est pourquoi, au cours de ces dernières décennies, une partie importante des activités de recherche en automatique s'est focalisée sur le problème de l'observation de l'état des systèmes dynamiques.

Les premiers travaux sur le problème de la reconstruction du vecteur d'état ont été consacrés aux systèmes linéaires standards (Luenberger, 1971). Depuis, de nombreux résultats théoriques ont été proposés et sont largement utilisés dans le contrôle et le diagnostic des déffauts.

En ce qui concerne la conception d'observateurs pour les systèmes singuliers, il existe plusieurs travaux : Dans Müller and Hou (1993), Darouach (2009a), Zhou and Lu (2009) et Darouach (2012) un Observateur Proportionnel (OP) pour les systèmes singuliers est proposé. D'autre part dans Koenig (2005) et Söffker et al. (1995), les auteurs présentent l'Observateur Proportionnel Intégral (OPI). Dans Kaczorek (1979), Beale and Shafai (1989) et Söffker et al. (1995), les auteurs montrent la performance de l'OPI, par rapport à l'OP en présence de perturbations et d'incertitudes.

Une nouvelle structure d'observateurs pour les systèmes linéaires standards, connue sous le nom d'Observateur Dynamique Généralisée (ODG) a été développée par Goodwin and Middleton (1989) et Marquez (2003). Cette structure présente un état d'estimation alternatif, et en est plus générale que celle de l'OP et de l'OPI.

7 Conception d'observateurs dynamiques généralisées pour les systèmes singuliers incertains

En géneral, en traitant avec les systèmes réels, il est nécessaire de manipuler le vecteur d'état $x(t)$. Les exemples incluent les applications de contrôle, la détection de défauts et la surveillance des systèmes. Malheureusement, le plus souvent , il est difficile et trop coûteux, voire impossible de mesurer $x(t)$. Dans ce cas, l'observateur peut être utilisé pour obtenir une estimation $\hat{x}(t)$ du vecteur d'état $x(t)$.

Comme les systèmes réels sont exposés à des perturbations, au bruit du capteur et aux erreurs de modélisation, la conception d'observateurs devient alors problématique.

La Figure 1 montre le schéma général utilisé pour mettre en œuvre un observateur pour les systèmes singuliers incertains. Les entrées et sorties du système sont les entrées de l'observateur. L'objectif principal de l'observateur est que la condition suivante $\lim_{t\to\infty} e(t) = 0$ soit satisfaite, où $e(t)$ est l'erreur d'estimation, c'est-á-dire (c-à-d). la différence entre l'état et sa valeur estimé.

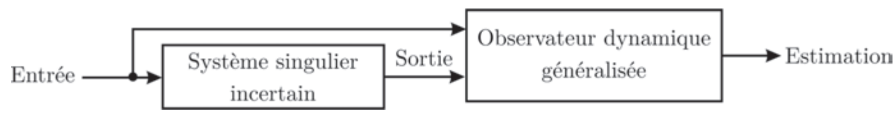

FIGURE 1 – Schéma d'un observateur.

La classe des systèmes singuliers incertains considérée est décrite par :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Dw(t) \tag{1a}$$
$$y(t) = (C_1 + \Delta A(t)) + D_1 w(t) \tag{1b}$$

où $x(k) \in \mathbb{R}^n$ est le semi-vecteur d'état, $w(t) \in \mathbb{R}^{n_w}$ est le vecteur de perturbations à énergie finie et $y(k) \in \mathbb{R}^p$ la sortie mesurée du système. La matrice $E \in \mathbb{R}^{n_1 \times n}$ est singulière. Les matrices $A \in \mathbb{R}^{n_1 \times n}$, $C_1 \in \mathbb{R}^{n_y \times n_1}$, $D \in \mathbb{R}^{n_1 \times n_w}$ et $D_1 \in \mathbb{R}^{n_y \times n_w}$ sont constantes et connues. $\Delta A(t)$ et $\Delta C(t)$ sont des matrices inconnues représentant les paramètres incertains variants dans le temps, telles que :

$$\Delta A(t) = \mathcal{M}_1 \Gamma(t) \mathcal{G} \tag{2a}$$
$$\Delta C(t) = \mathcal{M}_2 \Gamma(t) \mathcal{G} \tag{2b}$$

où $\mathcal{M}_1$, $\mathcal{M}_2$ et $\mathcal{G}$ sont des matrices constantes et connues et $\Gamma(t)$ est une matrice inconnue variant dans le temps qui satisfait :

$$\Gamma(t)^T \Gamma(t) \leq I, \ \forall t \in [0, \infty) \tag{3}$$

La structure de l'ODG considérée est :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) \tag{4a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + My(t) \tag{4b}$$
$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{4c}$$

où $\zeta(t) \in \mathbb{R}^{q_0}$ représente le vecteur d'état de l'observateur, $v(t) \in \mathbb{R}^{q_1}$ est un vecteur auxiliaire et $\hat{x}(t) \in \mathbb{R}^n$ est l'estimation de $x(t)$. N, F, H, L, M, S, P et Q sont des matrices inconnues de dimensions appropriées.

Dans la Section 3.2.3, nous présentons le théorème qui considère le système (1) sans perturbation, c-à-d. $w(t) = 0$. L'objectif est de trouver les matrices de l'observateur de telle sorte que l'observateur (4) soit stable et $\hat{x}(t)$ converge asymptotiquement vers $x(t)$.

D'autre part, le cas où $w(t) \neq 0$ est traité, dans la Section 3.2.4, l'objectif est de trouver les matrices de l'observateur de telle sorte que l'énergie de l'erreur d'estimation soit minimale pour toutes les perturbations d'énergie bornée. En d'autres termes $\|G_{we}\|_\infty < \gamma$, où G_{we} est la fonction de transfert de la perturbation $w(t)$ vers $e(t)$, et γ est un scalaire positif. Ceci est équivalent au critère de performance suivant :

$$J = \int_0^\infty \left[e(t)^T e(t) - \gamma^2 w(t)^T w(t)\right] dt. \tag{5}$$

8 Conception de la commande des systèmes singuliers incertains

Le problème de commande ou de stabilisation est l'un des problèmes le plus important dans la théorie du contrôle. Cela est dû au fait que la stabilité est toujours la première condition qui doit être satisfaite pour chaque problème de conception.

La commande basée sur des observateurs consiste à concevoir un dispositif qui garantit que la boucle fermée entre le système et l'observateur soit stable.

La Figure 2 montre le schéma utilisé pour stabiliser un système singulier incertain qui est normalement instable. Les variables détats estimés sont multipliés par une matrice de gain et ils constituent la loi de contrôle qui permet de stabiliser le système.

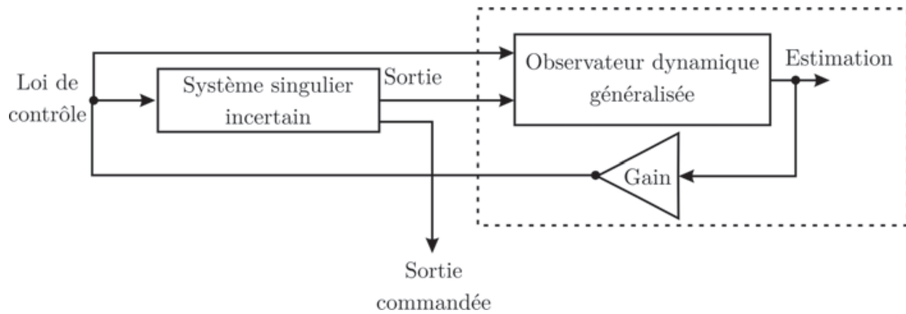

FIGURE 2 – Contrôle base sur l'observateur.

La classe des systèmes singuliers incertains perturbés considérée est :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Bu(t) + Dw(t) \tag{6a}$$
$$y(t) = (C_1 + \Delta C(t))x(t) + D_1 w(t) \tag{6b}$$
$$z(t) = C_2 x(t) + D_2 w(t) \tag{6c}$$

où $x(t) \in \mathbb{R}^n$ est le semi-vecteur d'état, $u(t) \in \mathbb{R}^m$ est la commande, $w(t) \in \mathbb{R}^{n_w}$ est le vecteur de perturbations à énergie finie, $y(t) \in \mathbb{R}^{n_y}$ est la sortie mesurée du système et $z(t) \in \mathbb{R}^s$ est la sortie commandée. Les matrices $E \in \mathbb{R}^{n_1 \times n}$, $A \in \mathbb{R}^{n_1 \times n}$, $B \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{n \times n_w}$, $C_1 \in \mathbb{R}^{n_y \times n_1}$, $D_1 \in \mathbb{R}^{n_y \times n_w}$, $C_2 \in \mathbb{R}^{s \times n}$ et $D_2 \in \mathbb{R}^{s \times n_w}$ sont constantes et connues. Soient, $\mathsf{rank}(E) = \varrho < n$ et $E^{\perp} \in \mathbb{R}^{\varrho_1 \times n}$ est une matrice de rang plein ligne tel que $E^{\perp}E = 0$, alors, dans ce cas $\varrho_1 = n - \varrho$.

$\Delta A(t)$ et $\Delta C(t)$ sont des matrices inconnues représentant les paramètres incertains variants dans le temps, telles que :

$$\Delta A(t) = \mathcal{M}_1 \Gamma(t) \mathcal{G} \tag{7a}$$
$$\Delta C(t) = \mathcal{M}_2 \Gamma(t) \mathcal{G} \tag{7b}$$

où $\mathcal{M}_1$, $\mathcal{M}_2$ et $\mathcal{G}$ sont des matrices réelles connues et $\Gamma(t)$ est une matrice inconnue variant dans le temps qui satisfait :

$$\Gamma(t)^T \Gamma(t) \leq I, \ \forall t \in [0, \infty). \tag{8}$$

La commande basée ODG considérée est :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{9a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + M \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{9b}$$
$$\hat{x}(t) = P\zeta(t) + Q \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{9c}$$
$$u(t) = -\kappa \hat{x}(t) \tag{9d}$$

où $\zeta(t) \in \mathbb{R}^{q_0}$ représente le vecteur d'état de la commande basée observateur, $v(t) \in \mathbb{R}^{q_1}$ est un vecteur auxiliaire et $\hat{x}(t) \in \mathbb{R}^n$ est l'estimation de $x(t)$. N, F, J, H, L, M, S, P, Q et κ sont des matrices inconnues de dimensions appropriées.

La Section 4.3.3 présente le théorème qui montre comment obtenir les matrices de l'observateur lorsque $w(t) = 0$, de telle sorte que la boucle fermée entre le système (6) et l'observateur (1.34a) soit stable.

D'autre part, la Section 4.3.4 traite le cas où $w(t) \neq 0$. L'objectif de la commande H_∞ basée ODG est de déterminer les matrices de l'observateur tel que pour un scalaire $\gamma > 0$, la sortie commandée $z(t)$ converge asymptotiquement vers zéro en boucle fermée pour $w(t) = 0$, et pour $w(t) \neq 0$ le critère de performance H_∞ suivant :

$$J = \int_0^\infty \left[z(t)^T z(t) - \gamma^2 w(t)^T w(t) \right] dt. \tag{10}$$

soit négatif, $\forall w(t) \in \mathcal{L}_2[0,\ \infty)$, ce qui est équivalent à $\|G_{wz}\|_\infty < \gamma$, où G_{wz} est la fonction de transfert de la perturbation $w(t)$ vers la sortie commandée $z(t)$ et γ est un scalaire positif.

9 Conception d'un dispositif de diagnostic des défauts des systèmes singuliers incertains

Pendant des siècles, la seule façon de détecter les défauts et leurs emplacements est l'intervention humaine : ce qui sous entend l'observation des changements de forme, de la couleur, des sons inhabituels, des vibrations.... Plus tard, les appareils de mesure ont été introduits, ce qui a fourni des informations plus précises sur des variations physiques importantes. Cependant, ces dispositifs (capteurs) sont également sensibles aux défauts, ce qui soulève le dilemme des fausses alarmes.

En général, les défauts sont des déviations du comportement normal dans le système ou dans son instrumentation. Il y a beaucoup de raisons de l'apparition de défauts. Pour ne citer que quelques exemples :

- Mauvaise conception, montage incorrect
- Mauvaise opération, manque d'entretien
- Vieillissement, l'usure pendant le fonctionnement, etc.

Pour diagnostiquer un système, il faut mettre en œuvre les tâches suivantes :

- Détection du défaut : c-à-d. voir ce qui ne va pas dans le système surveillé
- Isolation du défaut : c-à-d. la détermination de la position exacte du défaut
- Estimation du défaut : c-à-d. la détermination de la grandeur de l'erreur

Le dispositif de diagnostic des défauts consiste en la détermination du type de défaut avec autant de détails que possible, à savoir la grandeur du défaut, le lieu et l'heure de la détection.

La Figure 3 schématise le dispositif de diagnostic des défauts pour les systèmes singuliers incertains, où deux observateurs sont mis en œuvre. Le premier observateur est dédié à la détection des défauts, la génération des résiduus et l'isolement des défauts. Le deuxième observateur adaptatif donne l'estimation des défauts. L'avantage de l'approche proposée est d'obtenir des résidus indépendants, c-à-d. que chaque résidu correspond à une seule faute, de telle façon que nous puissions isoler les défauts multiples.

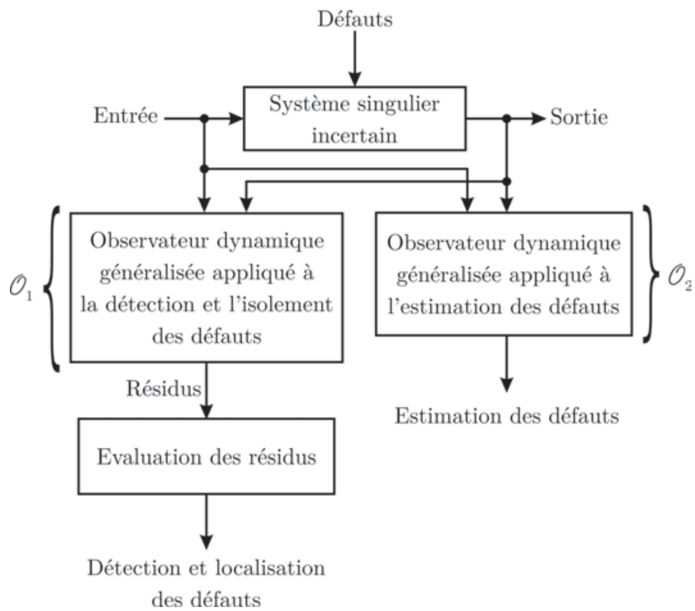

FIGURE 3 – Schéma de diagnostic appliqué.

La classe des systèmes singuliers incertains considéréé est décrite par le système suivant :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Gf(t) \tag{11a}$$
$$y(t) = C_1 x(t) \tag{11b}$$

où $x(t) \in \mathbb{R}^n$ est le semi-vecteur d'état, $f(t) \in \mathbb{R}^{n_f}$ est le vecteur des défauts et $y(t) \in \mathbb{R}^{n_y}$ est la sortie mesurée du système. $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n_f}$ et $C_1 \in \mathbb{R}^{n_y\times n}$ sont des matrices constantes et connues. $\Delta A(t)$ est une matrice inconnue qui représente les paramètres incertains variants dans le temps, telle que :

$$\Delta A(t) = \mathcal{M}\Gamma(t)\mathcal{G} \tag{12}$$

où $\mathcal{M}$ and $\mathcal{G}$ sont matrices réelles connues et $\Gamma(t)$ est une matrice inconnue variant dans le temps qui satisfait :

$$\Gamma(t)^T\Gamma(t) \leq I, \ \forall t \in [0, \infty) \tag{13}$$

L'ODG considéré pour la détection et l'isolement des défauts est :

$$\mathcal{O}_1 := \begin{cases} \dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) & (14a) \\ \dot{v}(t) = S\zeta(t) + Lv(t) + My(t) & (14b) \\ \hat{x}(t) = P\zeta(t) + Qy(t) & (14c) \\ r(t) = W(C_1\hat{x}(t) - y(t)) & (14d) \end{cases}$$

où $\zeta(t) \in \mathbb{R}^{q_0}$ représente le vecteur d'état de l'observateur, $v(t) \in \mathbb{R}^{q_1}$ est un vecteur auxiliaire, $\hat{x}(t) \in \mathbb{R}^n$ est l'estimation de $x(t)$ et $r(t) \in \mathbb{R}^{n_f}$ est le vecteur des résidus. N, F, H, S, L, M, P, Q et W sont des matrices inconnues de dimensions appropriées.

L'objectif de la détection et l'isolement des défauts est de rendre la transfert function des résidus vers $G_{rf}(s)$ diagonale, c-à-d :

$$G_{rf}(s) = \mathsf{diag}(g_1(s), \ldots, g_{n_f}(s)) \tag{15}$$

de telle sorte que la stabilité de l'observateur (14a) soit garantie. Dans la Section 5.3, le théorème qui donne une solution à cet objectif est présenté.

Afin de compléter le dispositif de diagnostic des défauts, nous consisérons l'ODG suivant pour estimer simultanément le vecteur d'état et des défauts :

$$\mathcal{O}_2 := \begin{cases} \dot{\zeta}(t) = N(\zeta(t) + TG\hat{f}(t)) + Hv(t) + Fy(t) + TG\hat{f}(t) & (16a) \\ \dot{v}(t) = S(\zeta(t) + TG\hat{f}(t)) + Lv(t) + My(t) & (16b) \\ \hat{x}(t) = P(\zeta(t) + TG\hat{f}(t)) + Qy(t) & (16c) \\ \dot{\hat{f}}(t) = \Phi(C_1\hat{x}(t) - y(t)) & (16d) \end{cases}$$

où $\zeta(t) \in \mathbb{R}^{q_0}$ représente le vecteur d'état de l'observateur, $v(t) \in \mathbb{R}^{q_1}$ est un vecteur auxiliaire, $\hat{x}(t) \in \mathbb{R}^n$ est l'estimation de $x(t)$ et $\hat{f}(t)$ est l'estimation de $f(t)$. N, F, H, L, M, S, P, Q, T et Φ sont des matrices inconnues de dimensions appropriées.

Le théorème de la Section 5.3.3 donne les conditions d'inégalités matricielles linéaires (LMI) pour obtenir les matrices de l'observateur de telle sorte que l'observateur (16a) soit stable et que $\hat{x}(t)$ et $\hat{f}(t)$ respectivement, convergent asymptotiquement vers $x(t)$ et $f(t)$.

10 Organisation de la thèse

Ce mémoire de thèse se compose de cinq chapitres principaux. Les paragraphes suivants donnent plus de détails sur le contenu de chaque partie :

Le **Chapitre 1** est consacré à la description des systèmes singuliers. Des exemples pratiques, des définitions et des propriétés des systèmes singuliers sont également présentés. Un examen bibliographique général de la conception des observateurs pour les systèmes singuliers est donné. Nous avons présenté aussi l'ODG considéré dans ce travail de thèse. En outre, nous avons fait un rapel des outils mathématiques pour l'analyse de la stabilité des systèmes dynamiques.

Le **Chapitre 2** est consacré à la conception de l'ODG H_∞ pour les systèmes singuliers. Des conditions suffisantes pour garantir la convergence asymptotique sont données sous forme de LMIs dans le cas des systèmes singuliers en temps continu et discret avec ou sans perturbations. Des cas particuliers de l'ODG tels que l'OP et l'OPI sont également développés. Des exemples de simulations sont proposés pour illustrer notre approche et pour comparer nos résultats avec l'OP et l'OPI.

Le **Chapitre 3** est dédié à la conception d'ODG H_∞ robuste vis-à-vis des incertutudes pour les systèmes singuliers. Dans ce chapitre, nous traitons les systèmes singuliers avec des incertitudes paramétriques variantes dans le temps. L'approche LPV (Linear Parameter-Varying) a été proposée comme une extension de l'approche H_∞ dans le contexte des systèmes dépendant de paramètres variants dans le temps. Des cas particuliers de l'ODG tels que l'OP et l'OPI sont également développés. Un exemple pratique est présenté pour illustrer l'efficacité de la méthode proposée et pour comparer nos résultats avec l'OP et l'OPI.

Le **Chapitre 4** est dédié à la commande H_∞ basée ODG pour les systèmes singuliers incertains avec ou sans perturbations. Des cas particuliers de l'ODG tels que l'OP et l'OPI sont également développés. Un exemple numérique est présenté pour illustrer l'efficacité de la méthode proposée et pour comparer nos résultats avec l'OP et l'OPI.

Le **Chapitre 5** est consacré au diagnostic des pannes pour les systèmes singuliers incertains avec ou sans perturbations. Ce problème est résolu en utilisant deux observateurs, le premier sert à la détection et de l'isolement des défauts, et le second est consacré à l'estimation simultanée de l'état et de défauts.

Finalement, la section **conclusions et perspectives** est dédiée à des conclusions générales et expose certains problèmes qui pourraient être abordés dans les travaux à venir.

Chapter 1

Overview of Descriptor systems

Contents

1.1 Introduction

In this chapter the state of the art of observers design for descriptor systems is presented. Some mathematical preliminaries that will be used later are also given. Section 1.2 is dedicated to present descriptor systems and some practical examples where this system structure can be used. Section 1.3 presents a general classification of uncertainties on the system. In Section 1.4 a general bibliography review of observers for descriptor systems is presented. Based on the analysis of the observers structure reported in the literature a new observer structure is introduced in Section 1.5. In Section 1.6 some basic proprieties of descriptor systems are presented. Finally, in Section 1.7 some basic tools for the stability analysis in dynamic systems are presented.

1.2 Presentation of descriptor systems

Descriptor systems, also known as singular systems or differential-algebraic systems, is a class of systems that can be considered as a generalization of dynamical systems. The descriptor system representation is a powerful modeling tool since it can describe processes governed by both, differential equations and algebraic equations. So it represents the physical phenomena that the model by ordinary differential equations can not describe. These systems were introduced by Luenberger (1977) from a control theory point of view and since, great efforts have been made to investigate descriptor systems theory and its applications.

A nonlinear descriptor system can be represented by the following set of equations :

$$\begin{aligned} E\dot{x}(t) =& g(x(t), u(t), d(t)) \\ y(t) =& h(x(t)) \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the input, $d(t) \in \mathbb{R}^{n_d}$ are the perturbations and $y(t) \in \mathbb{R}^{n_y}$ is the output. $g(\cdot)$ and $h(\cdot)$ are nonlinear continuous functions and infinitely differentiable. E is a singular matrix with constant parameters. The system (1.1) can be linearized in such a way that we get the following descriptor linear system in continuous-time case :

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + Bu(t) + Gd(t) \\ y(t) =& C_1x(t) + D_1u(t) \end{aligned} \tag{1.2}$$

where matrices A, B, G, C_1 and D_1 are known matrices of appropriate dimensions.

The discrete-time representation of system (1.2) can be expressed as :

$$\begin{aligned} Ex(t+1) =& Ax(t) + Bu(t) + Gd(t) \\ y(t) =& Cx(t) \end{aligned} \tag{1.3}$$

In the continuous-time case the semi-state vector $x(t)$, input $u(t)$, perturbation $d(t)$ and output $y(t)$ are real-valued vector functions. In the discrete-time case $x(t)$, $u(t)$, $d(t)$ and $y(t)$ are real-valued vector sequences.

1.2.1 Practical examples

Descriptor systems have a great theoretical and practical importance since they describe a large class of systems encountered in chemical, mineral, mechanical and electrical systems.

In the following sections some descriptor models of practical systems are shown.

1.2.1.1 Hydraulic system

Model of the change of the height of water in three tanks, with an input flow in the first tank and with the third tank leaking.

The pressures at the bottoms of the tanks 1, 2 and 3 are represented as p_1, p_2 and p_3, respectively. The pipe from tank 1 branches off to tanks 2 and 3. The pressure at the pipe branch is given as p_B.

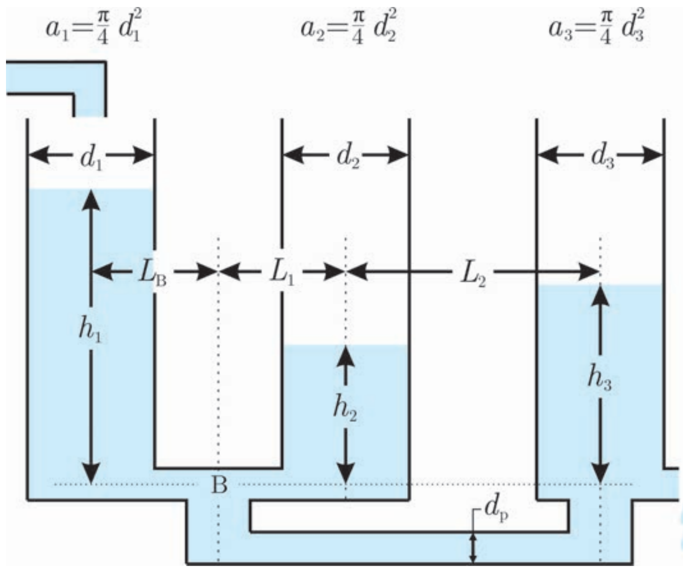

FIGURE 1.1 – Hydraulic system.

By using the Hagen–Poiseuille equation the flow rates between the tanks and the pipe branch can be written as :

$$F_{1B} = (p_1(t) - p_B(t)) \frac{\pi d_p^4}{128 \eta L_B} \tag{1.4a}$$

$$F_{B2} = (p_B(t) - p_2(t)) \frac{\pi d_p^4}{128 \eta L_1} \tag{1.4b}$$

$$F_{B3} = (p_B(t) - p_3(t)) \frac{\pi d_p^4}{128 \eta L_2} \tag{1.4c}$$

where η is the dynamic viscosity, L_i, $\forall i \in [1, 2, B]$ are the lengths of pipes, and d_p is the diameter of the pipe.

All the fluid leaving tank 1 should enter in tank 2 and tank 3. This is presented as a constraint.

$$F_{1B} = F_{B2} + F_{B3} \tag{1.5}$$

The pressure in each tank is given by :

$$p_1(t) = \rho g h_1(t) \tag{1.6a}$$

$$p_2(t) = \rho g h_2(t) \tag{1.6b}$$

$$p_3(t) = \rho g h_3(t) \tag{1.6c}$$

where h_i, $\forall i \in [1, 2, 3]$ is the depth in each tank, ρ is the density of the liquid, and g is the gravity acceleration.

The rate at which the fluid leaves or enters the tank is directly proportional to the rate of change of the height of the fluid in the tank. This relation comes through the analysis of mass conservation of incompressible fluids :

$$a_1 \dot{h}_1(t) = -F_{1B} + F_{in} \tag{1.7a}$$

$$a_2 \dot{h}_2(t) = F_{B2} \tag{1.7b}$$

$$a_3 \dot{h}_3(t) = F_{B3} - \tfrac{a_3}{10} h_3(t) \tag{1.7c}$$

where a_1, a_2 and a_3 are the cross sectional areas of tanks 1, 2 and 3 respectively. F_{in} is the input flow in tank 1, and the term $\frac{a_3}{10}h_3(t)$ represents the leaking in the tank 3.

Using equations (1.4) - (1.7) the state-space representation of the three interconnected tanks can be constructed as

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \\ \dot{h}_3(t) \\ \dot{p}_B(t) \\ \dot{p}_1(t) \\ \dot{p}_2(t) \\ \dot{p}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & k_1 & -k_1 & 0 & 0 \\ 0 & 0 & 0 & k_2 & 0 & -k_2 & 0 \\ 0 & 0 & -\frac{a_3}{10} & k_3 & 0 & 0 & -k_3 \\ -\rho g & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\rho g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\rho g & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -(k_1+k_2+k_3) & k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ p_B(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_{in} \tag{1.8}$$

where $k_1 = \dfrac{\pi d_p^4}{128\eta L_B}$, $k_2 = \dfrac{\pi d_p^4}{128\eta L_1}$ and $k_3 = \dfrac{\pi d_p^4}{128\eta L_2}$.

1.2.1.2 Mechanical system

Model of the mass-spring-damper system which includes a rigid bar that can prevent the motion of the second mass. The exciting force is applied to mass 1.

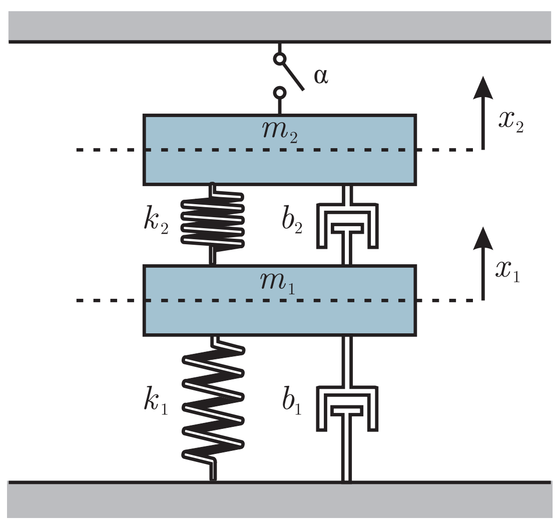

FIGURE 1.2 – Mechanical system.

The positions of masses m_1 and m_2 are represented by x_1 and x_2, respectively. The spring coefficients are given as k_1 and k_2. The damper coefficients are given as b_1 and b_2.

Using Newton's second law the forces acting on each mass can be written as :

$$m_1\ddot{x}_1(t) = -b_1\dot{x}_1(t) - k_1x_1(t) + b_2(\dot{x}_2(t) - \dot{x}_1(t)) + k_2(x_2(t) - x_1(t)) + u(t) \tag{1.9}$$
$$m_2\ddot{x}_2(t) = -b_2(\dot{x}_2(t) - \dot{x}_1(t)) - k_1(x_2(t) - x_1(t)) + \alpha\mu(t) \tag{1.10}$$

and the constraint equation

$$0 = \alpha(x_1(t) + x_2(t)) + (1-\alpha)\mu(t) \tag{1.11}$$

where α represents the state of the switch $1 = \textit{closed}$ and $0 = \textit{open}$, and $\mu(t)$ is the force absorbed.

Using equations (1.9) - (1.11) the state-space representation of the mechanical system can be constructed as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -(k_1+k_2) & k_2 & -(b_1+b_2) & b_2 & 0 \\ k_1 & -k_1 & b_2 & -b_2 & \alpha \\ \alpha & \alpha & 0 & 0 & (1-\alpha) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ \mu(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{1.12}$$

1.2.1.3 Electric system

Model of an electric system controlled by the voltage v.

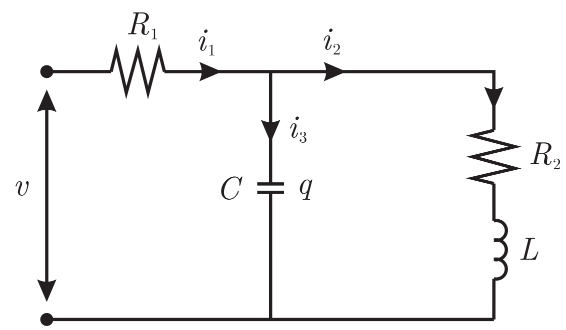

FIGURE 1.3 – Electric system.

The currents i_1 and i_2 are measured through the resistors R_1 and R_2. The electric charge in the capacitor C is denoted as q, and L represents the inductance.

The following relations are obtained by using current and voltage Kirchhoff's laws

$$\dot{q}(t) = i_3(t) \tag{1.13}$$

$$L\dot{i}_2(t) = \frac{1}{C}q(t) - i_2(t)R_2 \tag{1.14}$$

and the constraint equation

$$0 = R_1 i_2(t) + R_1 i_3(t) + \frac{1}{C}q(t) - v(t). \tag{1.15}$$

Using equations (1.13) - (1.15) the state-space representation of the electrical circuit can be expressed as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ \dot{i}_2(t) \\ \dot{i}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{C} & -R_2 & 0 \\ \frac{1}{C} & R_1 & R_1 \end{bmatrix} \begin{bmatrix} q(t) \\ i_2(t) \\ i_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} v(t) \tag{1.16}$$

1.3 Uncertain systems

The basic assumption of availability of precise model of the plant is almost never realized. There are almost always inconsistencies of the model and the real plant called model uncertainties.

The term uncertainty refers to the difference or errors between models and real processes. Because of the mathematical simplification in modeling processes and the inevitable measurement error involved in any engineering practice, it is important for analysis design of a practical control systems to incorporate uncertainties into the model of the studied systems.

There are three main reasons for model uncertainties (Gopal and Vancil, 1963) :

1. Model uncertainties occur because of an incomplete knowledge of the system.
2. The use of a model with no greater precision as that required by the performance to be achieved.
3. The model is based upon consideration of some physical phenomena thought to represent the salient features of the plant.

In any cases, the model at hand yields merely an approximation of the original system performance.

For control purposes the uncertainties are classified in structured and unstructured parameter uncertainties (Aberkane et al., 2008).

1.3.1 Structured uncertainties

This type of uncertainties arise from the linearization of a nonlinear system around a fixed operating point. They could be sub-classified in two classes :

1. Norm Bounded Uncertainty.
   The admissible parameter uncertainties are modeled as :
   $$\Delta A(t) = M_1 F(t) N_1$$
   where M_1 and N_1 are known real constant matrices, and $F(t)$ is an unknown real-valued time-varying matrix, satisfying the condition :
   $$F(t)^T F(t) \leq I, \ \ \forall t \geq 0$$
2. Linear Combination Uncertainty.
   This form of structured uncertainty is modeled as :
   $$\Delta A(t) = \sum_{i=1}^{p_\Delta} N_i \alpha_i(t)$$
   where N_i are known matrices and $\alpha_i(t)$ represent the bounded uncertain parameters, i.e.
   $$|\alpha_i(t)| \leq \overline{\alpha_i}, \ \ \forall i \in [1, \ldots, p_\Delta]$$
   where $\overline{\alpha_i}$ are given positive scalars. The matrix N_i can be written as :
   $$N_i = k_i(t) l_i(t)^T$$
   where $k_i(t)$ and $l_i(t)$ are matrices of appropriate dimensions.

1.3.2 Unstructured uncertainties

The unstructured uncertainties are associated with unmodeled dynamics, truncation of high frequency modes, or nonlinearity in the system. These parameter uncertainties are modeled as :

$$|\Delta A_i| \leq M_i, \ \ M_i = [m_{ij}]_k; \ m_{ij} \geq 0$$

where $|\cdot|$ represents the modulus of entries of the corresponding matrix and M_i is known constant matrix with all positive elements.

1.4 Bibliography review. Observers for descriptor systems

A descriptor system is a set of equations that are the result of modeling a system. These equations represent a general class of phenomena that is evolving in time, where some of the variables are related in a dynamical way, whereas others are purely static (Luenberger, 1977).

The so-called state observer is reconstructing the state variables of the system asymptotically. Based on this, an observer should satisfy the following two necessary conditions (Dai, 1989) :

1. The inputs of the observer should be the control input and the measured output of system.
2. Its output should satisfy the asymptotically condition $\lim_{t\to\infty} e(t) = 0$, where $e(t)$ is the estimation error, i.e. $\hat{x}(t) - x(t)$, the difference between the state and its estimate.

The general scheme of an observer is shown in Figure 1.4.

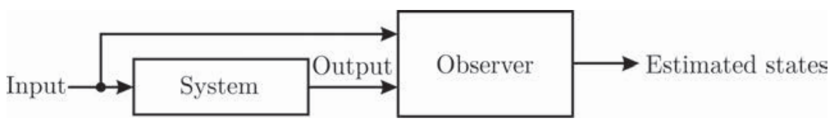

FIGURE 1.4 – Observers general scheme.

Some applications of observers for descriptor systems can be found in Boulkrone et al. (2009), where a nonlinear observer for descriptor systems is designed to estimate the state variables and unknown inputs in a wastewater treatment plant. Discrete-time descriptor systems are used in Araujo et al. (2012) to design observers for state estimation in an experimental hydraulic tank system. The observers design for descriptor systems have been widely studied in Müller and Hou (1993), Darouach (2009b), Zhou and Lu (2009), and Darouach (2012).

As for observers design for a standard and descriptor systems with unknown inputs has been trated in Hou and Müller (1992), Darouach et al. (1994), Darouach and Boutayeb (1995), Hou and Müller (1999), and Koenig (2005). The design of unknown input observer is a crucial problem since, in many practical cases, all input signals cannot be known. Moreover, this class of observers is widely used in the area of fault diagnosis, even if all the inputs are known (see Darouach et al. (1994), Darouach (2009a), and references therein). All these results use the proportional observers (POs). In the estimation by a POs there always exist a static error estimation in presence of disturbances. In order to deal with the disadvantage of POs, proportional-integral observers (PIOs) were introduced with an integral gain in their structure, which achieve steady-state accuracy in their estimations. The first results on the PIOs were presented by Wojciechowski (1978) for single-input single-output (SISO) systems. Its extension to multivariable systems was presented in Kaczorek (1979), Beale and Shafai (1989), and Söffker et al. (1995) where the authors show the performances of the PIO compared to PO in presence of disturbances and uncertainties. Some recently results on the PIO for systems with unknown inputs are presented in Gao et al. (2008) and Koenig (2005).

On the other hand, there are various papers that deal with estimation or control of uncertain systems. In Otsuka and Saito (2012) a linear system with disturbances is considered, uncertain parameters are taken in a polytopic form and a disturbance decoupling is address by Lyapunov functions and a set of LMIs. A general class of uncertainties named as integral quadratic constraints (IQC) are treated in Zasadzinski et al. (2006) a robust reduced order filter is designed for uncertain systems using a Luenberger observer for this end, the elimination theorem is used to analyze the stability of the observer. In Nagy-Kiss et al. (2012) a linear multi-model system is obtained from a nonlinear system via linearization, the estimation is achieved by a PIO with simultaneous state and unknown input estimation.

As for uncertain descriptor systems, in Corradini et al. (2012) the design of a robust fault detection H_∞ filter for descriptor systems is boarded, it show a descriptor system that consider the uncertainty as an unknown input, and using a Luenberger observer decouple faults. In Chadli and Darouach (2011) an admissibility condition is boarded to uncertain switched descriptor systems, and the authors consider a norm-bounded uncertainties in a polytopic form. In

Yang et al. (2007) a class of uncertain descriptor systems with state and input delays based on Takagi-Sugeno model is address using the cost control approach in order to stabilize the system but also to guarantee an adequate level of performance. In Zhu et al. (2004) the robust output feedback stabilization problem for descriptor time delay systems with norm-bounded parametric uncertainties is considered, the stability is ensured by the elimination theorem trough a set of LMIs. Xu et al. (2001) consider the problem of robust stabilization for uncertain discrete-time descriptor systems subject to norm-bounded uncertainties in the state matrix, it is controlled by state feedback controller such that the resulting closed-loop system is regular, causal as well as stable for all admissible uncertainties.

As shown, numerous works have focused on the design and implementation of algorithms for estimation and control for descriptor systems. Different techniques for state estimation have been developed and these depend largely on the structure of the model, the information available in the process and the relations that can be established between them. Considering this, some of the most representative observers for singular systems are presented below in order to show the main differences between them and the structure of the observer subsequently proposed.

1.4.1 Descriptor state observer

Descriptor state observers is typically derived from the system representation. An additional term is included in order to ensure that the estimated state converges to the real state. Specifically, the additional term consists in the difference between the system output and the estimated output and then multiplied by a matrix, this is added to the state of the observer to produce the so-called descriptor state observer. Some results of descriptor state observers in descriptor systems are found in Hou and Müller (1995); Gao and Ho (2006) and Duan et al. (2010).

In Dai (1989) a singular observer for singular systems is presented. The following system is considered :

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + Bu(t) \\ y(t) =& C_1 x(t) \end{aligned} \tag{1.17}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^{n_y}$ is the output vector. $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C_1 \in \mathbb{R}^{n_y \times n}$ are constant matrices. It is assumed that system (1.17) is regular and $\varrho = \mathsf{rank}(E) < n$.

Considering that system (1.17) is detectable, the following observer is proposed :

$$E\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G(y(t) - \hat{y}(t)) \tag{1.18a}$$

$$\hat{y}(t) = C_1 \hat{x}(t) \tag{1.18b}$$

such that $\lim_{t \to \infty} (\hat{x}(t) - x(t)) = 0$.

Let $e(t) = \hat{x}(t) - x(t)$ be the estimation error between real and estimated states. Its derivative is described by :

$$E\dot{e}(t) = (A - GC_1)e(t). \tag{1.19}$$

Thus, if $\lim_{t \to \infty} e(t) = 0$ the estimated state $\hat{x}(t)$ converges asymptotically toward $x(t)$.

If $\mathsf{rank}(E) < n$, observer (1.18) is called *descriptor state observer*. Otherwise, when $\mathsf{rank}(E) = n$, $E = I_n$ is assumed without loss of generality, so the observer (1.18) is called a *Luenberger observer*.

1.4.2 Proportional observers

Commonly, the estimation of the different variables of interest is preformed through a Luenberger observer, also known as proportional observer (PO). Some results on POs for descriptor systems are shown in Hou and Müller (1995); Darouach and Boutayeb (1995); Wang and Zou (2001) and Darouach (2012).

In Darouach (2012) a functional PO design for linear descriptor systems is presented. The approach is based on a new definition of partial impulse observability. The following system is considered :

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + Bu(t) \\ y(t) =& C_1 x(t) \\ z(t) =& Lx(t) \end{aligned} \tag{1.20}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^{n_y}$ is the output vector and $z(t) \in \mathbb{R}^p$ is the vector to be estimated. $E \in \mathbb{R}^{n_1 \times n}$, $A \in \mathbb{R}^{n_1 \times n}$, $B \in \mathbb{R}^{n_1 \times m}$, $C_1 \in \mathbb{R}^{n_y \times n}$ and $L \in \mathbb{R}^{p \times n}$ are known constant matrices.

Definition 1.1. *The system* (1.20) *with $u(t) = 0$, or the triplet (C_1, E, A) is said to be partially impulse observable with respect to L if $y(t)$ is impulse free for $t \geq 0$, only if $Lx(t)$ is impulse free for $t \geq 0$.*

The reduced-order observer for the singular system (1.20) is given by :

$$\dot{\zeta}(t) = N\zeta(t) + F \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \tag{1.21a}$$

$$\hat{z}(t) = P\zeta(t) + Q \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{1.21b}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ is the state of the observer, $\hat{z}(t) \in \mathbb{R}^p$ is the estimate of $z(t)$.

Now, consider the following conditions
(i) $NTE - TA + F \begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} = 0$
(ii) $\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ E^{\perp}A \\ C_1 \end{bmatrix} = L$
(iii) $H = TB$
so, there exists a matrix T of appropriate dimension such that $\varepsilon(t) = \zeta(t) - TEx(t)$, and its derivative is given by :

$$\dot{\varepsilon}(t) = N\varepsilon(t) + \left(NTE - TA + F \begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} \right) x(t) + (H - TB)u(t) \tag{1.22}$$

and from equation (1.21b) we get :

$$z(t) = P\varepsilon(t) + \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ E^{\perp}A \\ C_1 \end{bmatrix} x(t) \tag{1.23}$$

If conditions (i) - (iii) hold and N is a stability matrix then $\lim_{t \to \infty} (\hat{z}(t) - z(t)) = 0$ for any initial conditions. The observer matrices are obtained by solving Sylvester equations (i) - (iii).

In this way the author proposes an algorithm to obtain a reduced-order functional observer for descriptor systems, likewise, some particular cases are discussed in the paper. Sufficient conditions for the existence and stability of these observers are given.

1.4.3 Proportional-Integral observers

Compared with POs, proportional-integral observers (PIOs) provide more robust estimation against model uncertainties and better disturbance attenuation. As in the conventional systems, in view of the advantages of integral actions, some researches have introduced the integral term in observer design for descriptor systems (Koenig, 2005; Wu and Duan, 2008; Share et al., 2010).

Koenig and Mammar (2002) present a method to design a proportional-integral observer for unknown input descriptor systems. The following system is considered :

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + Bu(t) + Nd(t) \\ y(t) =& C_1 x(t) \end{aligned} \tag{1.24}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d(t) \in \mathbb{R}^{n_d}$, and $y(t) \in \mathbb{R}^{n_y}$ are the semi-state vector, the input vector, the unknown input and the output vector, respectively. $E \in \mathbb{R}^{n_1 \times n}$, $A \in \mathbb{R}^{n_1 \times n}$, $B \in \mathbb{R}^{n_1 \times m}$, $N \in \mathbb{R}^{n_1 \times n_d}$ and $C_1 \in \mathbb{R}^{n_y \times n}$ are known constant matrices.

Considering that $\mathsf{rank}(N) = n_d$ and $\mathsf{rank}(C_1) = n_y$, and that there exists a nonsingular matrix P such that

$$P\begin{bmatrix} E & N \end{bmatrix} = \begin{bmatrix} E_0 & N_0 \\ 0 & 0 \end{bmatrix}, \; PA = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \text{ and } PB = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}$$

where $E_0 \in \mathbb{R}^{\varrho \times n}$ and $\mathsf{rank}(E_0) = \varrho$. So, thus the following restricted system equivalent (r.s.e) is obtained :

$$\begin{aligned} \begin{bmatrix} E_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{d}(t) \end{bmatrix} &= \begin{bmatrix} A_0 & N_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(t) \\ y_0(t) &= \begin{bmatrix} C_0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \end{aligned} \tag{1.25}$$

where $y_0 = \begin{bmatrix} -B_1 u(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^{m+n_1-\varrho}$ and $C_0 = \begin{bmatrix} A_1 \\ C_1 \end{bmatrix} \in \mathbb{R}^{(m+n_1-\varrho)\times n}$.

The following PIO for the descriptor system (1.25) is considered :

$$\dot{\zeta}(t) = F\zeta(t) + L_1 y_0(t) + L_2 y_0(t) + Ju(t) + T_1 N\hat{d}(t) \tag{1.26a}$$

$$\dot{\hat{d}}(t) = L_3(y_0(t) - \hat{y}_0(t)) \tag{1.26b}$$

$$\hat{x}(t) = M_1\zeta(t) + T_2 y_0(t) \tag{1.26c}$$

$$\hat{y}_0(t) = C_0\hat{x}(t) \tag{1.26d}$$

Equation (1.26b) describes the integral loop added to the proportional equation (1.26a). Now, defining $e(t) = x(t) - \hat{x}(t)$ and $e_d(t) = d(t) - \hat{d}(t)$ the estimation error system expressed in terms of $e(t)$ and $e_d(t)$ is :

$$\begin{bmatrix} \dot{e}(t) \\ \dot{e_d}(t) \end{bmatrix} = \begin{bmatrix} F & T_1 N \\ -L_3 C_0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ e_d(t) \end{bmatrix} + \begin{bmatrix} T_1 A_0 - FT_1 E_0 - L_1 C_0 - L_2 C_0 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} T_1 B_0 - J \\ 0 \end{bmatrix} u(t) \tag{1.27}$$

where $F = T_1 A_0 - L_2 C_0$. Letting $L_1 = FT_2$, $J = TB_0$, and autonomous system is obtained. Estimation errors converge asymptotically to zero if matrix $\begin{bmatrix} T_1 A_0 - L_2 C_0 & T_1 N \\ -L_3 C_0 & 0 \end{bmatrix}$ is stable. This matrix becomes $\begin{bmatrix} T_1 A_0 & T_1 N \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L_2 \\ L_3 \end{bmatrix} \begin{bmatrix} C_0 & 0 \end{bmatrix}$, and it can be stabilized by the gain $\begin{bmatrix} L_2 \\ L_3 \end{bmatrix}$, if and only if the pair $\left(\begin{bmatrix} T_1 A_0 & T_1 N \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} C_0 & 0 \end{bmatrix} \right)$ is detectable.

In this research the authors propose a PIO for descriptor systems with an unknown input, the particular case of reduced-order PIO is also boarded.

1.5 Presentation of the generalized dynamic observer

Consider the following descriptor system

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + Bu(t) \\ y(t) =& C_1 x(t) \end{aligned} \tag{1.28}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^{n_y}$ represents the measured output vector. Now, consider the following generalized dynamic observer (GDO) for system (1.28)

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{1.29a}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + M \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{1.29b}$$

$$\hat{x}(t) = P\zeta(t) + Q \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{1.29c}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$.

GDO was developed by Goodwin and Middleton (1989) and Marquez (2003) where they see systems as mapping from input-to-state and look for small estimation errors in presence of persistent excitations. This approach allows to consider a general class of perturbations acting on the nominal system. In this context, an observer can be seen as a filter, so the error dynamics $e(t)$ has some desirable frequency domain characteristics.

GDO presents a more general alternative of state estimation than PO and PIO. Which can be only considered as particular cases of this structure. The idea of including additional dynamics in the observer structure was presented by Goodwin and Middleton (1989).

The observer (1.29) is in a general form and generalizes the existing ones. In fact :

– For $H = 0$, $S = 0$, $M = 0$ and $L = 0$ the observer reduces to the PO for descriptor systems (see for example Darouach (2012) and references therein).

$$\dot{\zeta}(t) = N\zeta(t) + F \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{1.30a}$$

$$\hat{x}(t) = P\zeta(t) + Q \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix}. \tag{1.30b}$$

– For $H = 0$, $S = 0$, $M = 0$, $L = 0$, $F = \begin{bmatrix} 0 & F_a \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & Q_a \end{bmatrix}$, then the following observer is obtained :

$$\dot{\zeta}(t) = N\zeta(t) + F_a y(t) + Ju(t) \tag{1.31a}$$

$$\hat{x}(t) = P\zeta(t) + Q_a y(t) \tag{1.31b}$$

which is the form used for the unknown input PO for descriptor systems (Darouach et al., 1996).

– For $L = 0$, $S = -C_1$ and $M = -C_1 Q + \begin{bmatrix} 0 & I \end{bmatrix}$, then the following observer is obtained

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{1.32a}$$

$$\dot{v}(t) = y(t) - C_1\hat{x}(t) \tag{1.32b}$$

$$\hat{x}(t) = P\zeta(t) + Q \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{1.32c}$$

which is the form used for the unknown input PIO for descriptor systems.

1.5.1 Importance of the integral term

This section shows the error estimation behavior in steady-state for the PO, PIO and GDO.

Consider the following disturbed descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + \eth(t) \\ y(t) &= C_1 x(t) \end{aligned} \tag{1.33}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output and $\eth \in \mathbb{R}^n$ is the perturbation vector.

Now, consider the PO of equation (1.31)

$$\dot{\zeta}(t) = N\zeta(t) + Fy(t) + Ju(t) \tag{1.34a}$$
$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{1.34b}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer and $\hat{x} \in \mathbb{R}^n$ is the estimate of $x(t)$.

Defining the transformed error $\varepsilon(t) = \zeta(t) - TEx(t)$, its derivative is given by :

$$\dot{\varepsilon}(t) = N\varepsilon(t) - T\eth(t) + (NTE + FC_1 - TA)x(t) + (J - TB)u(t) \tag{1.35}$$

by using definition of $\varepsilon(t)$, equation (1.34b) can be written as :

$$\hat{x}(t) = P\varepsilon(t) + \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ C_1 \end{bmatrix} x(t) \tag{1.36}$$

in order to obtain equation (1.35) independent of the input $u(t)$ and the state $x(t)$ we consider that

(a) $NTE + FC_1 - TA = 0$
(b) $J = TB$

and to obtain the estimation error as $\hat{x}(t) - x(t)$ we also consider

(c) $\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ C_1 \end{bmatrix} = I$

then we get :

$$\dot{\varepsilon}(t) = N\varepsilon(t) - T\eth(t) \tag{1.37}$$

and equation (1.36) becomes :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon(t) \tag{1.38}$$

In order to show the advantage of the integral observer over the proportional observer, let us show the steady-state error by applying Laplace transformation to (1.38) :

$$\bar{\mathbf{E}}(s) = P\bar{\mathscr{E}}(s) \tag{1.39}$$

where $\bar{\mathbf{E}}(s)$ and $\bar{\mathscr{E}}(s)$ are the Laplace transform of functions $e(t)$ and $\varepsilon(t)$, respectively.

The Laplace transform of (1.37) is given by :

$$\bar{\mathscr{E}}(s) = (sI - N)^{-1}(\boldsymbol{\varepsilon}(0) - T\bar{\boldsymbol{\eth}}(s)) \tag{1.40}$$

where $\boldsymbol{\varepsilon}(0)$ is the initial value of the transformed error in the time domain and $\bar{\boldsymbol{\eth}}(s)$ is the Laplace transform of function $\eth(t)$.

Now, by inserting (1.40) into (1.39) we get :

$$\bar{\mathbf{E}}(s) = P(sI - N)^{-1}(\boldsymbol{\varepsilon}(0) - T\bar{\boldsymbol{\eth}}(s)) \tag{1.41}$$

Considering $\eth(t)$ as a constant step with amplitude $\mathbf{d_C}$, we get its Laplace transform as $\bar{\boldsymbol{\eth}}(s) = \dfrac{\mathbf{d_C}}{s}$, also consider that N is stable and of full rank, i.e. N is nonsingular.

To find the steady-state value of the estimation error, we apply the final value theorem to equation (1.41) since N is stable, and we obtain :

$$\begin{aligned}\lim_{s\to 0} s\bar{\mathbf{E}}(s) &= \lim_{s\to 0}\left[P(sI - N)^{-1}\left(s\boldsymbol{\varepsilon}(0) - Ts\frac{\mathbf{d_C}}{s}\right)\right]\\ &= (PN^{-1}T)\mathbf{d_C}\end{aligned} \tag{1.42}$$

The objective of the estimation is to get the estimation error zero in steady-state. This can be achieved if the perturbation is zero. Hence, PO cannot eliminate the bias in the state estimates in presence of constant perturbations.

As in the PO there always exists a static estimation error in presence of constant perturbations, it is important to consider an additional integral term provided by the PIO to deal with the effect of the perturbation in the estimation error. The following analysis shows the estimation error for the PIO in steady-state.

Consider the following PIO

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) + Ju(t) \tag{1.43a}$$

$$\dot{v}(t) = y(t) - C_1\hat{x}(t) \tag{1.43b}$$

$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{1.43c}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x} \in \mathbb{R}^n$ is the estimate of $x(t)$.

By considering Sylvester equations (a) - (c) the estimation error obtained is :

$$e(t) = P\varepsilon(t) \tag{1.44}$$

the dynamics of the transformed error $\dot{\varepsilon}(t)$ and equation (1.43b) become :

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) - T\eth(t) \tag{1.45}$$

$$\dot{v}(t) = -C_1P\varepsilon(t) \tag{1.46}$$

Now, by applying the Laplace transformation we get :

$$\bar{\mathbf{E}}(s) = P\bar{\mathscr{E}}(s) \tag{1.47}$$

with

$$\bar{\mathscr{E}}(s) = (sI - N)^{-1}(\boldsymbol{\varepsilon}(0) + H\bar{\mathbf{V}}(s) - T\bar{\boldsymbol{\eth}}(s)) \tag{1.48}$$

where $\bar{\mathbf{V}}(s)$ is the transform of function $v(t)$.

From equation (1.46) we get :

$$\bar{\mathbf{V}}(s) = \frac{1}{s}(\mathbf{v}(0) - C_1P\bar{\mathscr{E}}(s)) \tag{1.49}$$

where $\mathbf{v}(0)$ is the initial value of $v(t)$ in the time domain.

By replacing (1.48) into (1.47) we get :

$$\bar{\mathbf{E}}(s) = P(sI - N)^{-1}(\boldsymbol{\varepsilon}(0) + H\bar{\mathbf{V}}(s) - T\bar{\boldsymbol{\eth}}(s)) \tag{1.50}$$

Since N is stable, applying the final value theorem to (1.50) and considering $\bar{\boldsymbol{\eth}}(s) = \dfrac{\mathbf{d_c}}{s}$, we obtain :

$$\lim_{s\to 0} s\bar{\mathbf{E}}(s) = \lim_{s\to 0}\left(P(sI-N)^{-1}\left(s\boldsymbol{\varepsilon}(0) + Hs\bar{\mathbf{V}}(s) - Ts\frac{\mathbf{d_c}}{s}\right)\right) \tag{1.51a}$$

$$= -PN^{-1}\left(H\lim_{s\to 0} s\bar{\mathbf{V}}(s) - T\mathbf{d_c}\right) \tag{1.51b}$$

the objective of estimation error equal to zero can be satisfied if $H\lim\limits_{s\to 0} s\bar{\mathbf{V}}(s) = T\mathbf{d_c}$.

As can be seen, the integral term $\bar{\mathbf{V}}(s)$ is in charge to cancel the effect of the disturbance in the estimation error. So that, the estimation error becomes zero in steady-state if condition $H\lim\limits_{s\to 0} s\bar{\mathbf{V}}(s) = T\mathbf{d_c}$ is satisfied.

Now, we show the behavior of the estimation error in steady-state of the GDO.

Consider the following GDO :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) + Ju(t) \tag{1.52a}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + My(t) \tag{1.52b}$$

$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{1.52c}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$.

By considering Sylvester equations (a) - (c) and $MC_1 + STE = 0$, the following estimation error is obtained :

$$e(t) = P\varepsilon(t) \tag{1.53}$$

and the dynamics of the transformed error $\dot{\varepsilon}(t)$ and equation (1.52b) become :

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) - T\eth(t) \tag{1.54}$$

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) \tag{1.55}$$

Now, by applying the Laplace transformation we get :

$$\bar{\mathbf{E}}(s) = P\bar{\mathscr{E}}(s) \tag{1.56}$$

where

$$\bar{\mathscr{E}}(s) = (sI-N)^{-1}(\boldsymbol{\varepsilon}(0) + H\bar{\mathbf{V}}(s) - T\bar{\boldsymbol{\eth}}(s)) \tag{1.57}$$

and from (1.55) we get :

$$\bar{\mathbf{V}}(s) = (sI-L)^{-1}(\mathbf{v}(0) + S\bar{\mathscr{E}}(s)) \tag{1.58}$$

By replacing (1.57) into (1.56) we get :

$$\bar{\mathbf{E}}(s) = P(sI-N)^{-1}(\boldsymbol{\varepsilon}(0) + H\bar{\mathbf{V}}(s) - T\bar{\boldsymbol{\eth}}(s)) \tag{1.59}$$

Applying the final value theorem since N is stable to (1.59) and considering $\bar{\boldsymbol{\eth}}(s) = \dfrac{\mathbf{d_C}}{s}$, we obtain :

$$\lim_{s\to 0} s\bar{\mathbf{E}}(s) = \lim_{s\to 0}\left(P(sI-N)^{-1}(s\boldsymbol{\varepsilon}(0) + Hs\bar{\mathbf{V}}(s) - Ts\frac{\mathbf{d_c}}{s})\right) \tag{1.60a}$$

$$= -PN^{-1}\left(H\lim_{s\to 0} s\bar{\mathbf{V}}(s) - T\mathbf{d_c}\right) \tag{1.60b}$$

As in the PIO the objective of estimation error equal to zero can be satisfied if $H\lim\limits_{s\to 0} s\bar{\mathbf{V}}(s) = T\mathbf{d_c}$.

We can conclude that the integral term in the PIO and GDO allows cancel the effect of the disturbance in the estimation error in steady-state. The main difference between the PIO and the GDO is that, the PIO has to manipulate just one matrix, while the GDO has two available matrices to achieve the objective of error estimation equal to zero.

In other words, the GDO has more degrees of freedom than the PIO. This freedom can be used to improve the estimation performance through the observer design by adding other performances as the pole placement in some domain.

1.6 Basic properties of descriptor systems

Consider the following descriptor system

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + Bu(t) \\ y(t) =& C_1 x(t) \end{aligned} \tag{1.61}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^{n_y}$ represents the measured output vector. Matrices E, A, B and C_1 are real and of appropriate dimensions.

1.6.1 Regularity of descriptor systems

Definition 1.2. *(Yip and Sincovec, 1981) Consider E and A as square matrices. The regularity property in descriptor systems guarantees the existence and uniqueness of solutions. The system* (1.61) *is called regular if there exists a constant scalar $\gamma \in \mathbb{C}$ such that*

$$\mathsf{det}(\gamma E - A) \neq 0$$

or equivalently, the polynomial $\mathsf{det}(sE - A)$ is not identically zero. In this case, we also say that the pair $(E,\ A)$, or the matrix pencil $sE - A$, is regular.

Remark 1.1. *In Ishihara and Terra (2001) and Darouach (2012) the authors show that the regularity of the matrix pair (E, A) is a property not needed for observer design, instead of the regularity property described above for square systems, this property is replaced by $\mathsf{normal-rank}\begin{bmatrix}\gamma E - A & BU(s) & Ex(0)\end{bmatrix} = \mathsf{normal-rank}(\lambda E - A)$ in the rectangular descriptor systems. Where the $\mathsf{normal-rank}$ of the matrix pencil $sE - A$ is defined as the rank of $(\lambda E - A)$ for almost all $\lambda \in \mathbb{C}$ and $U(s)$ is the Laplace transformation of $u(t)$ (see Ishihara and Terra (2001) and Ionescu et al. (1999) and references therein).*

1.6.2 Stability of descriptor systems

Definition 1.3. *(Duan, 2010) Stability of a dynamical system describes the response behavior of the system at infinity with respect to initial condition disturbances, and is well regarded as one of the most important properties of dynamical systems.*

Consider the descriptor linear system (1.61) *regular, it is stable if and only if*

$$\mathsf{eig}(E,\ A) \subset \mathbb{C}^- = \{s | s \in \mathbb{C},\ \mathsf{Re}(s) < 0\}$$

where $\mathsf{eig}(E,\ A)$ is defined as the roots of $\mathsf{det}(sE - A) = 0$, which must lie in the stable region, i.e. the open left-half plane for the continuous-time systems or the interior of unit disk for discrete-time systems.

1.6.3 Impulse-free behavior

Definition 1.4. *(Duan, 2010) If the state response of a descriptor linear systems, starting from an arbitrary initial value, does not contain impulse terms, then the system is called impulse-free.*

The following statements are equivalent :
- *The pair $(E,\ A)$ is impulse-free.*
- $\mathsf{deg}(\mathsf{det}(sE-A)) = \mathsf{rank}(E)$.
- $\mathsf{rank}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \mathsf{rank}(E)$.

1.6.4 Admissibility

Definition 1.5. *The pair $(E,\ A)$ is said to be admissible if it is regular, impulse-free and stable.*

Lemma 1.1. *System* (1.61) *or the pair $(E,\ A)$ is admissible if and only if there exists a nonsingular matrix Θ such that $E^T\Theta = \Theta^T E \geq 0$ and $A^T\Theta + \Theta^T A < 0$.*

1.6.5 Observability

Definition 1.6. *(Duan, 2010) Observability is defined as the ability to reconstruct the state from the system inputs and the measured outputs.*

System (1.61) *is observable if the following condition*

$$\mathsf{rank}\begin{bmatrix} sE-A \\ C_1 \end{bmatrix} = n,\ \forall s \in \mathbb{C},\ s \textit{ finite}$$

and

$$\mathsf{rank}\begin{bmatrix} E \\ C_1 \end{bmatrix} = n$$

are satisfied.

1.6.6 Impulse observability

Definition 1.7. *(Boukas, 2008) Impulse observability guarantees the ability to uniquely determine the impulse behavior in $x(t)$ from information of the impulse behavior in the output $y(t)$. System* (1.61) *is called impulse observable if*

$$\mathsf{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C_1 \end{bmatrix} = n + \mathsf{rank}(E).$$

1.6.7 Detectability

Definition 1.8. *(Dai, 1989) The system* (1.61) *is detectable if and only if all its outputs transmission zeros are stable, that is*

$$\mathsf{rank}\begin{bmatrix} sE-A \\ C_1 \end{bmatrix} = n,\ \forall s \in \mathbb{C}^+,\ s \textit{ finite}.$$

1.6.8 Stabilizability

Definition 1.9. *(Duan, 2010) System* (1.61) *is stabilizable if there exists a state feedback controller $u(t)$ such that the resulted closed-loop system is stable.*

The regular system (1.61) *is stabilizable if and only if*

$$\mathsf{rank}\begin{bmatrix} sE-A & B \end{bmatrix} = n,\ \forall s \in \mathbb{C}^+,\ s \textit{ finite}.$$

1.6.9 Controllability

Definition 1.10. *(Duan, 2010) System* (1.61) *is controllable if their poles can be arbitrarily assigned by state feedback.*

System (1.61) *is controllable if conditions*

$$\mathsf{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n, \ \forall s \in \mathbb{C}, \ s \text{ finite} \qquad \text{and} \qquad \mathsf{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \mathsf{rank}(E)$$

are satisfied.

1.7 Tools for the stability analysis of dynamic systems

In this section, some useful results are presented, they will be used in the sequel of the thesis.

1.7.1 Stability analysis using Lyapunov

Various types of stability may be discussed for the solutions of differential equations describing dynamical systems. The most important type is the one concerning the stability of solutions near a point of equilibrium. This may be discussed by the theory of Lyapunov.

Lyapunov, in his original work in 1892, he proposed two methods to study the stability of systems. The first method develops the solution in a series which is then proved that converges in the limits. The second method, makes use of a Lyapunov function $V(x(t))$ which has an analogy to the potential function of classical dynamics.

Definition 1.11. *(Lyapunov function) A generalized energy or Lyapunov function $V(x(t))$ of an autonomous system with the state equation $\dot{x} = f(x(t))$ is a scalar-valued function with the following properties :*

1. *scalar function*
$$V(x(t)) : \mathbb{R}^n \to \mathbb{R}$$
2. *positive definiteness*
$$V(x(t)) > 0$$
3. *dissipativity*
$$\frac{d}{dt}V(x(t)) = \frac{\partial V}{\partial x}\frac{d(x(t))}{dt} < 0$$

Theorem 1.1. *(Lyapunov) A system $\dot{x} = f(x(t))$ is asymptotically stable (in the strong sense) if there exists a Lyapunov function with the properties above.*

1.7.2 H_∞ norm and $\mathcal{L}_2$ gain

Consider the following linear system

$$G := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = C_1 x(t) + D_1 u(t) \end{cases} \tag{1.62}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $y(t) \in \mathbb{R}^{n_y}$ is the output vector and $u(t) \in \mathbb{R}^m$ is the input vector.

The H_∞ norm of the transfer function $G(s)$ is the distance in the complex plane from the origin to the furthest point of the Nyquist plot of $G(s)$, in addition this also appears in the maximum magnitude diagram Bode of $G(jw)$.

Definition 1.12. *(H_∞ norm)*
The H_∞ norm of the transfer function $G(s)$ is the peak value of $G(jw)$ as a function of frequency

$$\|G(s)\|_\infty = \sup_w \overline{\sigma}(G(jw)) \tag{1.63}$$

where $\overline{\sigma}$ denotes the largest singular value of the transfer matrix $G(s)$.

Definition 1.13. *($\mathcal{L}_2$ gain) Let $\gamma > 0$ be a fixed scalar. Suppose there exists a positive definite symmetric matrix P such that the quadratic function $V(x(t)) = x(t)^T Px(t)$ satisfies, for some $\epsilon > 0$, the inequality*

$$\frac{\partial V}{\partial x}(Ax(t) + Bu(t)) \leq -\epsilon[x(t)^T x(t)] + \gamma^2[u(t)^T u(t)] - [y(t)^T y(t)] \tag{1.64}$$

for all $x(t) \in \mathbb{R}^n$ and all $u(t) \in \mathbb{R}^m$. Observe that this property can only hold if the system is asymptotically stable. For $u(t) = 0$ this reduces in fact to

$$\frac{\partial V}{\partial x}Ax(t) \leq -\epsilon[x(t)^T x(t)]$$

which implies asymptotic stability.

Suppose that the input u(t) of system (1.62) *is a function in $\mathcal{L}_2[0, \infty)$. Integrating the inequality* (1.64) *on the interval $[0, T]$, for any initial state $x(0)$ we obtain :*

$$V(x(T)) \leq V(x(0)) + \gamma^2 \int_0^T [u(t)^T u(t)]dt - \int_0^T [y(t)^T y(t)]dt$$

from which it is deduced that the response $x(t)$ of the system is defined for all $t \in [0, \infty)$ and bounded. Now, suppose $x(0) = 0$ and note that the previous inequality becomes :

$$V(x(T)) \leq \gamma^2 \int_0^T [u(t)^T u(t)]dt - \int_0^T [y(t)^T y(t)]dt$$

for any $T > 0$. Since $V(x(T)) \geq 0$, we deduce that

$$[\|y(t)\|_2]^2 \leq \gamma^2 \left[\|u(t)\|_2\right]^2,$$

or

$$\|y(t)\|_2 \leq \gamma\|u(t)\|_2.$$

In other words, for any $u(t) \in \mathcal{L}_2[0, \infty)$, the response of the system from the initial state $x(0) = 0$ is defined for all $t \geq 0$, produces an output $y(t)$ which is a function in $\mathcal{L}_2[0, \infty)$ and the ratio between the output and the input is bounded by γ. For this reason, the system is said to have a finite $\mathcal{L}_2$ gain, bounded above by the number γ.

1.7.2.1 Bounded Real Lemma

H_∞ theory is one of the most sophisticated fields to design robust control systems, commonly based on the bounded real lemma (BRL) (Xie, 2008).

Lemma 1.2. *Given the following linear system :*

$$\begin{aligned} \dot{x}(t) =& Ax(t) + Bw(t) \\ z(t) =& C_1x(t) + D_1w(t) \end{aligned} \tag{1.65}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, $w(t) \in \mathcal{L}_2[0, \infty)$ is the disturbance signal and $z(t) \in \mathbb{R}^{n_z}$ is the objective function. Matrices A, B, C_1 and D_1 are constant matrices of appropriate dimensions. For a prescribed scalar $\gamma > 0$, the following performance index is defined :

$$J = \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t)]dt \tag{1.66}$$

Then, from Boyd et al. (1994), it follows that $J < 0$, for all nonzero $w(t) \in \mathcal{L}_2[0, \infty)$, if and only if there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying the inequality :

$$\begin{bmatrix} A^T P + PA & XB & C_1^T \\ B^T P & -\gamma^2 I & D_1^T \\ C_1 & D_1 & -I \end{bmatrix} < 0, \tag{1.67}$$

where the symmetric positive matrix P is usually called as Lyapunov matrix.

1.7.2.2 Discrete-time Bounded Real Lemma

The following discrete-time BRL will be used subsequently.

Lemma 1.3. *(Skelton et al., 1998)Let a stable linear discrete-time system described by*

$$\begin{aligned} x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= C_1x(t) + D_1w(t) \end{aligned} \tag{1.68}$$

with transfer function $G(z) = C_1(zI - A)^{-1} + D_1$ *and let* γ *be a given positive scalar. Then* $\|G_{wy}\|_\infty < \gamma$ *if and only is there exists a matrix* $X = X^T > 0$ *such that*

$$\begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \tag{1.69}$$

1.7.3 Matrix properties

In this section, some lemmas about linear matrix inequalities (LMIs) transformation are presented.

Lemma 1.4. *(Schur complement)(Boyd et al., 1994) Let* A, B *and* D *be matrices of appropriate dimensions. Then the following statements are equivalent :*
(i) $\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} < 0.$
(ii) $D < 0$ *and* $A - BD^{-1}B^T < 0.$
(iii) $A < 0$ *and* $D - B^TA^{-1}B < 0.$

Remark 1.2. *If* D *is nonsingular, then* $\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \leq 0$ *is equivalent to* $D < 0$ *and* $A - BD^{-1}B^T \leq 0.$

Lemma 1.5. *(Elimination lemma)(Skelton et al., 1998) Let matrices* B, C, $D = D^T$ *be given, then the following statements are equivalent :*
(i) There exists a matrix Λ *satisfying*

$$B\Lambda C + (B\Lambda C)^T + D < 0.$$

(ii) The following two conditions hold

$$\begin{aligned} B^{\perp}DB^{\perp T} < 0 \text{ or } BB^T > 0 \\ C^{T\perp}DC^{T\perp T} < 0 \text{ or } C^TC > 0. \end{aligned}$$

Suppose that the statement (ii) holds. Let r_b *and* r_c *be the ranks of* B *and* C*, respectively, and* $(B_l,\ B_r)$ *and* $(C_l,\ C_r)$ *be any full rank factors of* B *and* C*, i.e.* $B = B_lB_r$, $C = C_lC_r$*. Then the matrix* Λ *in statement (i) is given by*

$$\Lambda = B_r^+KC_l^+ + Z - B_r^+B_rZC_lC_l^+$$

where Z *is an arbitrary matrix and*

$$\begin{aligned} K =& -R^{-1}B_l^T\vartheta C_r^T(C_r\vartheta C_r^T)^{-1} + S^{1/2}L(C_r\vartheta C_r^T)^{-1/2} \\ S =& R^{-1} - R^{-1}B_l^T[\vartheta - \vartheta C_r^T(C_r\vartheta C_r^T)^{-1}C_r\vartheta]B_lR^{-1} \end{aligned}$$

where L *is an arbitrary matrix such that* $\|L\|_2 < 1$ *and* R *is an arbitrary positive definite matrix such that*

$$\vartheta = (B_rR^{-1}B_l^T - D)^{-1} > 0.$$

Lemma 1.6. *(Boukas and Liu, 2002) Given constant matrices* H *and* D*, a symmetric constant matrix* S *and an unknown constant matrix* $\Delta(t)$ *of appropriate dimensions satisfying the inequality* $\Delta(t)^T\Delta(t) \leq I$*. The following two inequalities are equivalent :*

(i) $S + D\Delta(t)H + H^T\Delta(t)^TD^T < 0,$
(ii) $S + \epsilon DD^T + \epsilon^{-1}H^TH < 0$ *for some* $\epsilon > 0.$

1.8 Conclusion

This chapter was focused on the presentation of descriptor systems and on some of their basic properties. Also some models of real physical processes with this particular kind of representation were developed. On the other hand a bibliography review of observers for descriptor systems is presented making evident the differences between the observers structure previously reported in the literature and the observer structure proposed in this thesis. Additionally, some concepts concerning the stability analysis of dynamic systems were introduced.

Chapter 2

H_∞ generalized dynamic observers design

Contents

2.1 Introduction

In this chapter the H_∞ GDO design for descriptor systems with or without disturbances is boarded, for both continuous-time and for discrete-time cases. In the same way, special cases of the GDO as the PIO and the PO are developed in order to compare their results in simulation. In Section 2.2 the class of system considered is presented, also a more general consideration of impulse observability is introduced. Section 2.3 shows the way to extend the approaches presented to simultaneous semi-state and unknown input estimation. In Section 2.4 the problematic is posed through the observer error dynamics, also the observer parameterization for both cases, without disturbances or with disturbances is boarded. In Section 2.5 the GDO for continuous-time descriptor systems free of disturbances is developed. In Section 2.6 the extension to H_∞ GDO is carried out. Section 2.7 deals with the design of the GDO for discrete-time descriptor systems without disturbances, and Section 2.8 extends those results to the discrete-time descriptor systems with disturbances case.

2.2 Class of disturbed descriptor systems considered

Consider the following descriptor system described by :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1 x(t) + D_1 w(t) \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the input, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector of bounded energy and $y(t) \in \mathbb{R}^{n_y}$ represents the measured output vector. Matrix $E \in \mathbb{R}^{n_1 \times n}$ and when $n_1 = n$ matrix E is singular. Matrices $A \in \mathbb{R}^{n_1 \times n}$, $B \in \mathbb{R}^{n_1 \times m}$, $D \in \mathbb{R}^{n_1 \times n_w}$, $C_1 \in \mathbb{R}^{n_y \times n}$ and $D_1 \in \mathbb{R}^{n_y \times n_w}$. Let $\mathsf{rank}(E) = \varrho < n$ and $E^\perp \in \mathbb{R}^{\varrho_1 \times n_1}$ be a full row rank matrix such that $E^\perp E = 0$, in this case $\varrho_1 = n_1 - \varrho$.

In the sequel we assume that

Assumption 2.1.

$$\mathsf{rank} \begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = n.$$

This assumption is equivalent to the impulse observability from Definition 1.7 (see Darouach (2009b)).

Assumption 2.2.

$$\mathsf{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = n, \ \forall s \in \mathbb{C}^+, \ s \textit{ finite}.$$

The relation between Assumption 2.1 and the impulse observability is given by the following lemma.

Lemma 2.1. *The following conditions are equivalent :*

1. *System* (2.1) *is impulse observable.*
2. $\mathsf{rank} \begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = n.$
3. $\mathsf{rank} \begin{bmatrix} E & A \\ \hline 0 & E \\ 0 & C_1 \end{bmatrix} = n + \mathsf{rank}(E).$

Proof. (Zerrougui, 2011) Consider that matrix $\begin{bmatrix} E^\perp \\ EE^+ \end{bmatrix}$ is of full column rank, where E^+ is any generalized inverse of matrix E, such that $EE^+E = E$, then

$$\begin{aligned} \mathsf{rank} \begin{bmatrix} E & A \\ \hline 0 & C_1 \\ 0 & E \end{bmatrix} &= \mathsf{rank} \begin{bmatrix} E^\perp & 0 \\ EE^+ & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E & A \\ \hline 0 & C_1 \\ 0 & E \end{bmatrix} = \mathsf{rank} \begin{bmatrix} 0 & E^\perp A \\ E & EE^+A \\ 0 & C_1 \\ 0 & E \end{bmatrix} \\ &= \mathsf{rank} \begin{bmatrix} 0 & E^\perp A \\ E & EE^+A \\ 0 & C_1 \\ 0 & E \end{bmatrix} \begin{bmatrix} I & -E^+A \\ 0 & I \end{bmatrix} = \mathsf{rank} \begin{bmatrix} 0 & E^\perp A \\ E & 0 \\ 0 & C_1 \\ 0 & E \end{bmatrix} \\ &= \mathsf{rank} \begin{bmatrix} E \\ E^\perp A \\ C_1 \end{bmatrix} + \mathsf{rank}(E) \end{aligned}$$

□

2.3 Extension to simultaneous semi-state and unknown input estimation

In this section we show how to extend the approaches that will be developed below to the simultaneous estimation of the semi-state and the unknown input.

Consider the following descriptor system with unknown input :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) + Gd(t) \\ y(t) &= C_1x(t) + D_1w(t) \end{aligned} \tag{2.2}$$

where $G \in \mathbb{R}^{n_1 \times n_d}$ and $d(t) \in \mathbb{R}^{n_d}$ is the unknown input vector, which can be differentiable, i.e. $\dot{d}(t)$ could exists.

By augmenting the semi-state $x(t)$ with the unknown input $d(t)$, the system (2.2) can be written as :

$$\begin{aligned} \mathbf{E}\dot{\chi}(t) &= \mathbf{A}\chi(t) + Bu(t) + Dw(t) \\ y(t) &= \mathbf{C}\chi(t) + D_1w(t) \end{aligned} \tag{2.3}$$

where $\chi(t) = \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$, $\mathbf{E} = \begin{bmatrix} E & 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} A & G \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix}$.

The following lemma deals with the impulse observability of system (2.3).

Lemma 2.2. *The system* (2.3) *is impulse observable if and only if*

$$\mathsf{rank}\begin{bmatrix} E & A & G \\ 0 & C_1 & 0 \\ 0 & E & 0 \end{bmatrix} - \mathsf{rank}(E) = n + n_d \tag{2.4}$$

Proof. From Assumption 2.1 the system (2.3) is impulse observable if and only if $\mathsf{rank}\left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{E}^{\perp}\mathbf{A} \\ \mathbf{C} \end{array}\right] = n + n_d$ or equivalently

$$\mathsf{rank}\left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{E}^{\perp}\mathbf{A} \\ \mathbf{C} \end{array}\right] = \mathsf{rank}\begin{bmatrix} E & 0 \\ E^{\perp}A & E^{\perp}G \\ C_1 & 0 \end{bmatrix} = n + n_d. \tag{2.5}$$

On the other hand

$$\begin{aligned} \mathsf{rank}\begin{bmatrix} E & A & G \\ 0 & C_1 & 0 \\ 0 & E & 0 \end{bmatrix} &= \mathsf{rank}\begin{bmatrix} E^{\perp} & 0 \\ EE^{+} & 0 \\ 0 & I \end{bmatrix}\left[\begin{array}{ccc} E & A & G \\ \hline 0 & C_1 & 0 \\ 0 & E & 0 \end{array}\right] = \mathsf{rank}\begin{bmatrix} 0 & E^{\perp}A & E^{\perp}G \\ E & EE^{+}A & EE^{+}G \\ 0 & C_1 & 0 \\ 0 & E & 0 \end{bmatrix} \\ &= \mathsf{rank}\begin{bmatrix} 0 & E^{\perp}A & E^{\perp}G \\ E & EE^{+}A & EE^{+}G \\ 0 & C_1 & 0 \\ 0 & E & 0 \end{bmatrix}\begin{bmatrix} I & -E^{+}A & -E^{+}G \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \mathsf{rank}\begin{bmatrix} 0 & E^{\perp}A & E^{\perp}G \\ E & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & E & 0 \end{bmatrix} \\ &= \mathsf{rank}\begin{bmatrix} E^{\perp}A & E^{\perp}G \\ C_1 & 0 \\ E & 0 \end{bmatrix} + \mathsf{rank}(E) \end{aligned} \tag{2.6}$$

From equations (2.5) and (2.6) we obtain equation (2.4) which proves the lemma. □

As long as Lemma 2.2 is satisfied, we can see that the transformation (2.3) has the same structure of system (2.1), so, the observers design procedure can be extended to system (2.3).

2.4 Problem formulation

Consider the following GDO for system (2.1)

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{2.7a}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{2.7b}$$

$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{2.7c}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. Matrices N, F, J, H, L, M, S, P and Q are unknown matrices of appropriate dimensions which must be determined such that $\hat{x}(t)$ converges asymptotically to $x(t)$.

Now, we can give the following lemma.

Lemma 2.3. *There exists an observer of the form* (2.7) *for the system* (2.1) *if the following two statements hold.*

1. *There exists a matrix T of appropriate dimension such that the following conditions are satisfied :*

   (a) $NTE + F\begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} - TA = 0$

   (b) $J = TB$

   (c) $M\begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} + STE = 0$

   (d) $\begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} TE \\ \hline E^{\perp}A \\ C_1 \end{bmatrix} = I_n$

2. *The matrix $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz when $w(t) = 0$.*

Proof. Let $T \in \mathbb{R}^{q_0 \times n_1}$ be a parameter matrix and define the error $\varepsilon(t) = \zeta(t) - TEx(t)$, then its derivative is given by :

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) + \left(NTE + F\begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} - TA\right)x(t) + (J - TB)u(t) + \left(F\begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix} - TD\right)w(t) \tag{2.8}$$

by using the definition of $\varepsilon(t)$, equations (2.7b) and (2.7c) can be written as :

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + \left(M\begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} + STE\right)x(t) + M\begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}w(t) \tag{2.9}$$

$$\hat{x}(t) = P\varepsilon(t) + \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} TE \\ \hline E^{\perp}A \\ C_1 \end{bmatrix}x(t) + Q\begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix} \tag{2.10}$$

Now, if conditions $(a) - (d)$ of Lemma 2.3 are satisfied the following observer error dynamics is obtained from (2.8) and (2.9)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix}\begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} F\begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix} - TD \\ M\begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix} \end{bmatrix} w(t) \tag{2.11}$$

and from equation (2.10) we get :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon(t) + Q \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix} w(t) \tag{2.12}$$

in this case if $w(t) = 0$ and $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz, then $\lim_{t\to\infty} e(t) = 0$. □

The problem of the GDO design is reduced to determine the matrices N, F, J, H, L, M, S, P, Q and T such that the conditions of Lemma 2.3 are satisfied.

Remark 2.1. *The GDO* (2.7) *is of dimension $q_0 + q_1$. As can be seen from equation* (2.7c) *only the state $\zeta(t) \in \mathbb{R}^{q_0}$ is used in the determination of $\hat{x}(t)$. A more general form could be used for the estimate state $\hat{x}(t)$, i.e.*

$$\hat{x}(t) = P\zeta(t) + Q \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + \Phi v(t)$$

where Φ is a parameter matrix of appropriate dimension.

By using the results of Lemma 2.3, we obtain :

$$e(t) = P\varepsilon(t) + \Phi v(t)$$

for $w(t) = 0$. We can see from the equation above that, for any parameter matrix Φ we have $e(t) \to 0$ when $\lim_{t\to\infty} \varepsilon(t) = 0$. Then, for simplicity, we can take $\Phi = 0$.

2.4.1 Determination of the observer parameters

Before giving the solution to the constraints $(a) - (d)$ of Lemma 2.3 the following definitions should be considered.

Definition 2.1. *(Bernstein, 2009) Generalized inverse. Let $A \in \mathbb{R}^{n\times m}$. If $\mathsf{rank}(A) = m$, then A^+ is a left inverse of A, it satisfies $A^+A = I$. If $\mathsf{rank}(A) = n$, then A^+ is a right inverse of A, it satisfies $AA^+ = I$. Both, left and right inverses satisfies $AA^+A = A$.*

Lemma 2.4. *Consider the following equation of a non homogeneous system :*

$$\mathcal{A}\mathcal{X} = \mathcal{B} \tag{2.13}$$

where $\mathcal{A} \in \mathbb{R}^{n\times m}$ is a constant matrix, $\mathcal{B} \in \mathbb{R}^{n\times p}$ is a constant matrix and $\mathcal{X} \in \mathbb{R}^{m\times p}$ is the vector to determine.

The equation (2.13) *admits a solution, if and only if*

$$\mathsf{rank}(\mathcal{A}) = \mathsf{rank}\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$$

in this case, the general solution to equation (2.13) *is given by :*

$$\mathcal{X} = \mathcal{A}^+\mathcal{B} - (I - \mathcal{A}^+\mathcal{A})Z \tag{2.14}$$

where Z is an arbitrary matrix of appropriate dimension.

Equivalently for a system with the form $\mathcal{X}\mathcal{A} = \mathcal{B}$, the necessary and sufficient condition for the existence of a solution is :

$$\mathsf{rank}(\mathcal{A}) = \mathsf{rank}\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix}$$

in this case, the general solution is given by :

$$\mathcal{X} = \mathcal{B}\mathcal{A}^+ - Y(I - \mathcal{A}\mathcal{A}^+) \tag{2.15}$$

where Y is an arbitrary matrix of appropriate dimension.

Now, the parameterization of the all solutions to the algebraic constraints $(a) - (d)$ of Lemma 2.3 are given.

Lemma 2.5. *Let $R \in \mathbb{R}^{q_0 \times n}$ be a full row rank matrix such that the matrix $\Sigma = \begin{bmatrix} R \\ \hline E^\perp A \\ C_1 \end{bmatrix}$ is of full column rank, then under Assumption 2.1, the general solution to constraints $(a) - (d)$ of Lemma 2.3 is given by :*

$$T = T_1 - Z_1 T_2 \tag{2.16}$$
$$N = N_1 - Z_1 N_2 - Y_1 N_3 \tag{2.17}$$
$$F = F_1 - Z_1 F_2 - Y_1 F_3 \tag{2.18}$$
$$S = -Y_2 N_3 \tag{2.19}$$
$$M = -Y_2 F_3 \tag{2.20}$$
$$P = P_1 - Y_3 N_3 \tag{2.21}$$
$$Q = Q_1 - Y_3 F_3 \tag{2.22}$$

where Z_1, Y_1, Y_2 and Y_3 are arbitrary matrices of appropriate dimensions.

Proof. Let $R \in \mathbb{R}^{q_0 \times n}$ be a full row rank matrix such that the matrix $\Sigma = \begin{bmatrix} R \\ \hline E^\perp A \\ C_1 \end{bmatrix}$ is of full column rank and let $\Omega = \begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix}$. Conditions (c) and (d) of Lemma 2.3 can be written as :

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} \begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \tag{2.23}$$

the necessary and sufficient condition for equation (2.23) to have a solution is :

$$\mathsf{rank} \begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \end{bmatrix} = \mathsf{rank} \begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \\ 0 \\ I_n \end{bmatrix} = n.$$

Now, since $\mathsf{rank} \begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \end{bmatrix} = n$, there always exist matrices $T \in \mathbb{R}^{q_0 \times n_1}$ and $K \in \mathbb{R}^{q_0 \times (\varrho_1 + n_y)}$ such that :

$$TE + K \begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} = R \tag{2.24}$$

which can be written as :

$$\begin{bmatrix} T & K \end{bmatrix} \Omega = R \tag{2.25}$$

by using Lemma 2.4, $\mathsf{rank} \begin{bmatrix} \Omega \\ R \end{bmatrix} = \mathsf{rank}(\Omega)$ or equivalently $R\Omega^+\Omega = R$.Then the general solution of equation (2.25) is given by :

$$\begin{bmatrix} T & K \end{bmatrix} = R\Omega^+ - Z_1(I_{n_1+\varrho_1+n_y} - \Omega\Omega^+) \tag{2.26}$$

where Ω^+ is the generalized inverse of Ω (see Definition 2.1).

Equation (2.26) is equivalent to :

$$T = T_1 - Z_1 T_2 \tag{2.27}$$
$$K = K_1 - Z_1 K_2 \tag{2.28}$$

where $T_1 = R\Omega^+ \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$, $T_2 = (I_{n_1+\varrho_1+n_y} - \Omega\Omega^+) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$, $K_1 = R\Omega^+ \begin{bmatrix} 0 \\ I_{\varrho_1+n_y} \end{bmatrix}$, $K_2 = (I_{n_1+\varrho_1+n_y} - \Omega\Omega^+) \begin{bmatrix} 0 \\ I_{\varrho_1+n_y} \end{bmatrix}$ and Z_1 is an arbitrary matrix of appropriate dimension.

Now, define the following matrices :
$N_1 = T_1 A\Sigma^+ \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $N_2 = T_2 A\Sigma^+ \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $N_3 = (I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $\widetilde{K}_1 = T_1 A\Sigma^+ \begin{bmatrix} 0 \\ I_{\varrho_1+n_y} \end{bmatrix}$, $\widetilde{K}_2 = T_2 A\Sigma^+ \begin{bmatrix} 0 \\ I_{\varrho_1+n_y} \end{bmatrix}$, $\widetilde{K}_3 = (I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \begin{bmatrix} 0 \\ I_{\varrho_1+n_y} \end{bmatrix}$, $F_1 = T_1 A\Sigma^+ \begin{bmatrix} K \\ I_{\varrho_1+n_y} \end{bmatrix}$, $F_2 = T_2 A\Sigma^+ \begin{bmatrix} K \\ I_{\varrho_1+n_y} \end{bmatrix}$ and $F_3 = (I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \begin{bmatrix} K \\ I_{\varrho_1+n_y} \end{bmatrix}$.
By inserting the equivalence of TE from equation (2.24) into condition (a) of Lemma 2.3 it leads to :

$$N\left(R - K \begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix}\right) + F \begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} = TA \tag{2.29a}$$
$$NR + \widetilde{K} \begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} = TA \tag{2.29b}$$

where $\widetilde{K} = F - NK$, and equation (2.29b) can be written as :

$$\begin{bmatrix} N & \widetilde{K} \end{bmatrix} \Sigma = TA \tag{2.30}$$

The general solution of equation (2.30) is given by :

$$\begin{bmatrix} N & \widetilde{K} \end{bmatrix} = TA\Sigma^+ - Y_1(I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \tag{2.31}$$

by replacing matrix T from equation (2.27) into equation (2.31) it gives :

$$N = N_1 - Z_1 N_2 - Y_1 N_3 \tag{2.32}$$
$$\widetilde{K} = \widetilde{K}_1 - Z_1 \widetilde{K}_2 - Y_1 \widetilde{K}_3 \tag{2.33}$$

where Y_1 is an arbitrary matrix of appropriate dimension.

As matrices N, T, K and $\widetilde{K}$ are known, we can deduce the form of matrix F as :

$$F = \widetilde{K} + NK \tag{2.34a}$$
$$= \widetilde{K}_1 + N_1 K - Z_1(\widetilde{K}_2 + N_2 K) - Y_1(\widetilde{K}_3 + N_3 K) \tag{2.34b}$$
$$= F_1 - Z_1 F_2 - Y_1 F_3 \tag{2.34c}$$

On the other hand, from equation (2.24) we obtain :

$$\begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \end{bmatrix} = \begin{bmatrix} I_{q_0} & -K \\ 0 & I_{\varrho_1+n_y} \end{bmatrix} \Sigma \tag{2.35}$$

inserting equation (2.35) into equation (2.23) we get :

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} \begin{bmatrix} I_{q_0} & -K \\ 0 & I_{\varrho_1+n_y} \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ I_n \end{bmatrix}. \tag{2.36}$$

Since matrix Σ is of full column rank and $\begin{bmatrix} I_{q_0} & -K \\ 0 & I_{\varrho_1+n_y} \end{bmatrix}^{-1} = \begin{bmatrix} I_{q_0} & K \\ 0 & I_{\varrho_1+n_y} \end{bmatrix}$, the general solution to equation (2.36) is given by :

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} = \left(\begin{bmatrix} 0 \\ I_n \end{bmatrix} \Sigma^+ - \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix} (I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \right) \begin{bmatrix} I_{q_0} & K \\ 0 & I_{\varrho_1+n_y} \end{bmatrix} \tag{2.37}$$

where Y_2 and Y_3 are arbitrary matrices of appropriate dimensions.

Then matrices S, M, P and Q can be determined as :

$$S = -Y_2 N_3 \tag{2.38}$$
$$M = -Y_2 F_3 \tag{2.39}$$
$$P = P_1 - Y_3 N_3 \tag{2.40}$$
$$Q = Q_1 - Y_3 F_3 \tag{2.41}$$

where $P_1 = \Sigma^+ \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$ and $Q_1 = \Sigma^+ \begin{bmatrix} K \\ I_{\varrho_1+n_y} \end{bmatrix}$. $\square$

2.4.1.1 Parameterization for the robust case

When a system is subject to disturbances as is shown in equation (2.1), a bilinearity in the matrix F is involved in the stability analysis of the observer. By developing matrix F we obtain :

$$F = T_1 A \Sigma^+ \begin{bmatrix} K_1 - Z_1 K_2 \\ I_{\varrho_1+n_y} \end{bmatrix} - Z_1 T_2 A \Sigma^+ \begin{bmatrix} K_1 - Z_1 K_2 \\ I_{\varrho_1+n_y} \end{bmatrix} - Y_1 (I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \begin{bmatrix} K_1 - Z_1 K_2 \\ I_{\varrho_1+n_y} \end{bmatrix}$$

where the unknown matrices are Z_1 and Y_1. In order to avoid this bilinearity an adaptation in the parameterization is carried.

Let $\bar{K}_2 = K_2 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$ and $Z_1 = Z(I_{n_1+\varrho_1+n_y} - \bar{K}_2 \bar{K}_2^+)$, where Z is an arbitrary matrix of appropriate dimensions, so that $F \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$, from the observer error system (2.11) becomes :

$$F \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} = \mathcal{F}_{d_1} - Z \mathcal{F}_{d_2} - Y_1 \mathcal{F}_{d_3} \tag{2.42}$$

where $\mathcal{F}_{d_1} = T_1 A \Sigma^+ \begin{bmatrix} K_1 \\ I_{\varrho_1+n_y} \end{bmatrix} \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$, $\mathcal{F}_{d_2} = (I_{n_1+\varrho_1+n_y} - \bar{K}_2 \bar{K}_2^+) T_2 A \Sigma^+ \begin{bmatrix} K_1 \\ I_{\varrho_1+n_y} \end{bmatrix} \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$ and
$\mathcal{F}_{d_3} = (I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \begin{bmatrix} K_1 \\ I_{\varrho_1+n_y} \end{bmatrix} \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$.

In the same way, we obtain the following expressions for T, K, N, $M \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$ and $Q \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$

$$T = T_1 - Z \mathcal{T}_2 \tag{2.43}$$
$$K = K_1 - Z \mathcal{K}_2 \tag{2.44}$$
$$N = N_1 - Z \mathcal{N}_2 - Y_1 N_3 \tag{2.45}$$
$$M \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} = -Y_2 \mathcal{F}_{d_3} \tag{2.46}$$
$$Q \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} = \mathcal{Q}_{d_1} - Y_3 \mathcal{F}_{d_3} \tag{2.47}$$

where $\mathcal{T}_2 = (I_{n_1+\varrho_1+n_y} - \bar{K}_2\bar{K}_2^+)T_2$, $\mathcal{K}_2 = (I_{n_1+\varrho_1+n_y} - \bar{K}_2\bar{K}_2^+)K_2$, $\mathcal{N}_2 = (I_{n_1+\varrho_1+n_y} - \bar{K}_2\bar{K}_2^+)N_2$ and $\mathcal{Q}_{d_1} = \Sigma^+ \begin{bmatrix} K_1 \\ I_{\varrho_1+n_y} \end{bmatrix} \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}$. Where we have used the fact that $\bar{K}_2\bar{K}_2^+\bar{K}_2 = \bar{K}_2$ (see Definition 2.1).

In order to study the observer stability, the observer error dynamics (2.11) - (2.12) can be written as :

$$\begin{aligned} \dot{\varphi}(t) &= (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) + (\mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2)w(t) \\ e(t) &= \mathbb{P}\varphi(t) + \mathbb{Q}w(t) \end{aligned} \tag{2.48}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{B}_1 = \begin{bmatrix} \mathcal{F}_{d_1} - T_1D_1 + Z(\mathcal{T}_2D_1 - \mathcal{F}_{d_2}) \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} P_1 - Y_3N_3 & 0 \end{bmatrix}$, $\mathbb{Q} = \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

In this case the robust parameterization of Section 2.4.1.1 is used, for $w(t) \neq 0$.

Matrices Z and $\mathbb{Y}$ can be obtained by pole placement such that matrix $(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)$ is stable when $w(t) = 0$. In the sequel we will present an LMI based approach for the determination of these parameter matrices.

2.5 Generalized dynamic observer design for descriptor systems, $w(t)$=0

In this section we consider $w(t) = 0$, then system (2.1) becomes :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_1x(t) \end{aligned} \tag{2.49}$$

with the GDO :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{2.50a}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{2.50b}$$

$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{2.50c}$$

and the error dynamics (2.48) becomes :

$$\begin{aligned} \dot{\varphi}(t) &= (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) \\ e(t) &= \mathbb{P}\varphi(t) \end{aligned} \tag{2.51}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z_1N_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$. $Y_3 = 0$ is taken for simplicity. Since the system is free of disturbances, we have used the parameterization of Lemma 2.5.

From the above results we can give the following lemma.

Lemma 2.6. *The following statements are equivalent :*

(1) $\mathsf{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \ \forall s \in \mathbb{C}, \ \mathsf{Re}(s) \geq 0.$

(2) The pair $\left[\begin{bmatrix} N_2 \\ N_3 \end{bmatrix}, \ N_1\right]$ *is detectable.*

(3) $\operatorname{rank}\begin{bmatrix} sR - T_1A \\ E^\perp A \\ C \\ T_2A \end{bmatrix} = \operatorname{rank}(\Sigma),\ \forall s \in \mathbb{C},\ \operatorname{Re}(s) \geq 0.$

Proof. The proof start by showing that condition *(1)* is equivalent to condition *(3)*. In fact, we have

$$\operatorname{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I & 0 \\ -E^\perp & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} sE - A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} sE - A \\ E^\perp A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} sE - A \\ sE^\perp A \\ sC \\ E^\perp A \\ C \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} s\Omega - \begin{bmatrix} A \\ 0 \end{bmatrix} \\ E^\perp A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \begin{bmatrix} R \\ \hline E^\perp A \\ C \end{bmatrix}\Omega^+ & 0 \\ (\Omega\Omega^+ - I) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} s\Omega - \begin{bmatrix} A \\ 0 \end{bmatrix} \\ \hline E^\perp A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} s\Sigma - \Sigma\Omega^+\begin{bmatrix} A \\ 0 \end{bmatrix} \\ (I - \Omega\Omega^+)\begin{bmatrix} A \\ 0 \end{bmatrix} \\ E^\perp A \\ C \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} sR - T_1A \\ s\begin{bmatrix} E^\perp A \\ C \end{bmatrix} - \begin{bmatrix} E^\perp A \\ C \end{bmatrix}\Omega^+\begin{bmatrix} A \\ 0 \end{bmatrix} \\ E^\perp A \\ C \\ T_2A \end{bmatrix} = \operatorname{rank}\begin{bmatrix} sR - T_1A \\ \begin{bmatrix} E^\perp A \\ C \end{bmatrix}\Omega^+\begin{bmatrix} A \\ 0 \end{bmatrix} \\ E^\perp A \\ C \\ T_2A \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} sR - T_1A \\ E^\perp A \\ C \\ T_2A \end{bmatrix}$$

we have used the fact that $\Omega^+\Omega = I$, the last inequality results from the fact that

$$\mathscr{R}\left(\begin{bmatrix} E^\perp A \\ C \end{bmatrix}\Omega^+\begin{bmatrix} A \\ 0 \end{bmatrix}\right) \subset \mathscr{R}(T_2A)$$

Now, we can show that condition *(3)* is equivalent to condition *(2)*. From the definition of matrix Σ we have the following equality

$$\operatorname{rank}\begin{bmatrix} sR - T_1A \\ E^\perp A \\ C \\ T_2A \end{bmatrix} = \operatorname{rank}\begin{bmatrix} s\begin{bmatrix} I & 0 \end{bmatrix}\Sigma - T_1A\Sigma^+\Sigma \\ \begin{bmatrix} I & 0 \end{bmatrix}\Sigma \\ T_2A\Sigma^+\Sigma \end{bmatrix} = \operatorname{rank}\begin{bmatrix} s\begin{bmatrix} I & 0 & 0 \end{bmatrix} - T_1A\Sigma^+ \\ \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ T_2A\Sigma^+ \end{bmatrix}\Sigma$$

By using the results of lemma 2.5, condition *(3)* is equivalent to $\operatorname{rank}\begin{bmatrix} s\begin{bmatrix} I & 0 \end{bmatrix} - T_1A\Sigma^+ \\ \begin{bmatrix} 0 & I \end{bmatrix} \\ T_2A\Sigma^+ \end{bmatrix}\Sigma = \operatorname{rank}(\Sigma)$ if and only if matrix $\begin{bmatrix} s\begin{bmatrix} I & 0 \end{bmatrix} - T_1A\Sigma^+ \\ \begin{bmatrix} 0 & I \end{bmatrix} \\ T_2A\Sigma^+ \end{bmatrix}$ is of full column rank or equivalently the matrix $\begin{bmatrix} sI - T_1A\Sigma^+\begin{bmatrix} I \\ 0 \end{bmatrix} & \star \\ 0 & I \\ T_2A\Sigma^+\begin{bmatrix} I \\ 0 \end{bmatrix} & \star \\ (I - \Sigma\Sigma^+)\begin{bmatrix} I \\ 0 \end{bmatrix} & \star \end{bmatrix}$ if of full column rank, where $\star$ represent matrices without any importance. This condition is equivalent to the matrix $\begin{bmatrix} sI - N_1 \\ \hline N_2 \\ N_3 \end{bmatrix}$ of full column rank, which proves the lemma. □

From equation (2.51) we must determine matrices Z_1 and $\mathbb{Y}$ such that $(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)$ is Hurwitz. The following lemma gives the condition to guarantee this stability.

Lemma 2.7. *There exists a parameter matrix $\mathbb{Y}$ such that $(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)$ is Hurwitz if and only if the pair $(\mathbb{A}_2,\ \mathbb{A}_1)$ is detectable or equivalently* $\mathsf{rank}\begin{bmatrix} sI - \mathbb{A}_1 \\ \mathbb{A}_2 \end{bmatrix} = q_0 + q_1,\ \forall s \in \mathbb{C},\ \mathsf{Re}(s) \geq 0.$

Now, we have

$$\begin{aligned}
\mathsf{rank}\begin{bmatrix} sI - \mathbb{A}_1 \\ \mathbb{A}_2 \end{bmatrix} &= \mathsf{rank}\begin{bmatrix} sI - N_1 + Z_1 N_2 & 0 \\ 0 & sI_{q_1} \\ N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} \\
&= \mathsf{rank}\begin{bmatrix} sI - N_1 + Z_1 N_2 \\ N_3 \end{bmatrix} + q_1 \\
&= q_0 + q_1,\ \forall s \in \mathbb{C},\ \mathsf{Re}(s) \geq 0.
\end{aligned}$$

This is exactly the condition of the detectability of the pair $(N_3,\ N_1 - Z_1 N_2)$, which proves the lemma.

The following theorem gives the LMIs conditions to the determination of all GDO matrices for the continuous-time descriptor system (2.49).

Theorem 2.1. *Under Assumptions 2.1 and 2.2 there exist two parameter matrices Z_1 and $\mathbb{Y}$ such that observer error system* (2.51) *is asymptotically stable if there exists a symmetric positive definite matrix $X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix}$, with $X_1 = X_1^T$ such that the following LMI is satisfied.*

$$N_3^{T\perp}(N_1^T X_1 + X_1 N_1 - N_2^T W_1^T - W_1 N_2) N_3^{T\perp T} < 0. \tag{2.52}$$

In this case matrix $W_1 = X_1 Z_1$ and matrix $\mathbb{Y}$ is parameterized as follows :

$$\mathbb{Y} = X^{-1}(\mathcal{K}\mathcal{C}_l^+ + \mathcal{Z}(I - \mathcal{C}_l\mathcal{C}_l^+)) \tag{2.53}$$

where

$$\mathcal{K} = \mathcal{R}^{-1}\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \tag{2.54a}$$
$$\vartheta = (\mathcal{R}^{-1} - \mathcal{D})^{-1} > 0 \tag{2.54b}$$
$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}[\vartheta - \vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta]\mathcal{R}^{-1} \tag{2.54c}$$

with $\mathcal{D} = \begin{bmatrix} (N_1 - Z_1 N_2)^T X_1 + X_1(N_1 - Z_1 N_2) & (N_1 - Z_1 N_2)^T X_1 \\ (*) & 0 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$ and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$ and $\mathcal{C}_r$ are full rank matrices such that $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$.

Proof. Let $V(\varphi(t)) = \varphi(t)^T X \varphi(t)$ be a Lyapunov function, thus its derivative along the trajectory of observer error system (2.51) is :

$$\dot{V}(\varphi(t)) = \varphi(t)^T[(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X + X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)]\varphi(t) \tag{2.55}$$

From Section 1.7.1, the asymptotic stability of observer error system (2.51) is guaranteed if and only if $\dot{V}(\varphi(t)) < 0$. This leads to the following LMI :

$$(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X + X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) < 0 \tag{2.56}$$

which can be rewritten as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{2.57}$$

where $\mathcal{X} = X\mathbb{Y}$, $\mathcal{D} = X\mathbb{A}_1 + \mathbb{A}_1^T X$, $\mathcal{C} = \mathbb{A}_2$ and $\mathcal{B} = -I$. According to Skelton et al. (1998) (see the elimination lemma of Section 1.5) inequality (2.57) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{2.58}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} N_3^{T\perp} & 0 \end{bmatrix}$. By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and W_1 inequality (2.58) becomes :

$$N_3^{T\perp}(N_1^T X_1 + X_1 N_1 - N_2^T W_1^T - W_1 N_2) N_3^{T\perp T} < 0. \tag{2.59}$$

From the elimination lemma if condition (2.58) is satisfied, then the parameter matrix $\mathbb{Y}$ is obtained as in (2.53) and (2.54). □

2.5.1 Particular cases

In this section we consider two particular cases of our results.

•*Proportional observer*

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_1 x(t) \end{aligned}$$

with the PO :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + F_a y(t) + Ju(t) \\ \hat{x}(t) &= P\zeta(t) + Q_a y(t) \end{aligned}$$

and the error dynamics (2.51) becomes :

$$\begin{aligned} \dot{\varepsilon}(t) &= (\bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2)\varepsilon(t) \\ e(t) &= \bar{\mathbb{P}}\varepsilon(t) \end{aligned}$$

where $\bar{\mathbb{A}}_1 = N_1 - Z_1 N_2$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{P}} = P_1$ and $\bar{\mathbb{Y}} = Y_1$. Consequently, matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.1 become :

$$\mathcal{D} = X(N_1 - Z_1 N_2) + (N_1 - Z_1 N_2)^T X,\ \mathcal{C} = N_3,\ \mathcal{B} = -I \text{ and } \mathcal{X} = X\bar{\mathbb{Y}}.$$

Matrices Σ and Ω are defined as $\Sigma = \begin{bmatrix} R \\ C_1 \end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C_1 \end{bmatrix}$.

•*Proportional-integral observer*

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_1 x(t) \end{aligned}$$

with the PIO :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \\ \dot{v}(t) &= y(t) - C_1\hat{x}(t) \\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \end{aligned}$$

and the error dynamics (2.51) becomes :

$$\dot{\varphi}(t) = (\bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2)\varphi(t)$$
$$e(t) = \bar{\bar{\mathbb{P}}}\varphi(t)$$

where $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix} N_1 - Z_1N_2 & 0 \\ -C_1P_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\bar{\bar{\mathbb{P}}} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} Y_1 & H \end{bmatrix}$.

Consequently matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.1 become :

$$\mathcal{D} = \begin{bmatrix} X_1(N_1 - Z_1N_2) + (N_1 - Z_1N_2)^TX_1 - X_1C_1P_1 - (C_1P_1)^TX_1 & (N_1 - Z_1N_2)^TX_1 - (C_1P_1^T)X_2 \\ (*) & 0 \end{bmatrix},$$

$\mathcal{C} = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathcal{B} = -I$ and $\mathcal{X} = X\bar{\bar{\mathbb{Y}}}$, such that $\begin{bmatrix} Y_1 & H \end{bmatrix} = \left(X \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^+ \mathcal{X}$.

2.5.2 Numerical example

In order to illustrate the results obtained, consider the following descriptor system described by (2.49), where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

Considering matrix $E^{\perp} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, we can verify Assumptions 2.1 and 2.2

$$\mathsf{rank} \begin{bmatrix} E \\ \hline E^{\perp}A \\ C_1 \end{bmatrix} = 4 \quad \text{and} \quad \mathsf{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 4$$

Generalized dynamic observer

For the GDO we have chosen matrix $R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, such that $\mathsf{rank}(\Sigma) = 4$.

By using YALMIP toolbox, we solve the LMI (2.52) to find matrices X and Z_1

$$X = \begin{bmatrix} 25.23 & 10 & 25.23 & 10 \\ 10 & 25.23 & 10 & 25.23 \\ 25.23 & 10 & 40.46 & 10 \\ 10 & 25.23 & 10 & 40.46 \end{bmatrix} \text{ and}$$

$$Z_1 = \begin{bmatrix} 50.96 & 43.55 & 0 & 0 & -122.18 & 19.77 & -51.86 \\ 52.45 & -57.73 & 0 & 0 & 78.08 & 20.3 & -89.84 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 9 & 7 & 7 & 3 & 9 \\ 7 & 3 & 9 & 9 & 6 & 4 & 9 \\ 6 & 3 & 9 & 2 & 8 & 1 & 0 \\ 6 & 3 & 9 & 2 & 7 & 4 & 9 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.01$, and by solving (2.53) and (2.54) we get :

$$\mathbb{Y} = \begin{bmatrix} -4.4 & 4.19 & 0.25 & 0.49 & 0.15 & 17.79 & -9.98 \\ -17.20 & 17.65 & 0.25 & 0.75 & 0.09 & 11.23 & -3.42 \\ 4.53 & -4.27 & 0 & -0.33 & 0.06 & -14.54 & 6.71 \\ 10.93 & -11 & 0 & -0.46 & 0.06 & -7.98 & 0.15 \end{bmatrix}.$$

Finally, we can get all the matrices of the observer as :

$$N = \begin{bmatrix} -4.42 & -13.01 \\ 25.93 & -8.92 \end{bmatrix},\ S = \begin{bmatrix} -4.4 & 4.4 \\ -10.97 & 10.97 \end{bmatrix},\ H = \begin{bmatrix} 17.79 & -9.98 \\ 11.23 & -3.42 \end{bmatrix},\ L = \begin{bmatrix} -14.54 & 6.71 \\ -7.98 & 0.15 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ F = \begin{bmatrix} -267.95 & 203.32 & -410.12 \\ 1687.84 & -1686.39 & 1086.07 \end{bmatrix},\ M = \begin{bmatrix} -290.86 & 290.86 & -139.3 \\ -724.74 & 724.74 & -347.09 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -31.6 & 31.6 & -31.38 \\ 31.6 & -30.6 & 31.38 \end{bmatrix}$$

In order to provide a comparison of the GDO with the PIO and PO, these latter are also designed.

Proportional observer

Consider matrices $R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 2 & 6 & 3 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_2 \times 0.1$ the following PO matrices are obtained :

$$N = \begin{bmatrix} -2.34 & 2.69 \\ 0.95 & -1.51 \end{bmatrix},\ F_a = \begin{bmatrix} -0.17 & -16.86 \\ 0.85 & 8.5 \end{bmatrix},\ J = \begin{bmatrix} 1.77 \\ -1.27 \end{bmatrix},\ P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } Q_a = \begin{bmatrix} 0 & 0.55 \\ 0 & 1 \\ 0 & 1.47 \\ 1 & -1.47 \end{bmatrix}$$

Proportional-integral observer

By considering matrices $R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 7 & 3 & 9 \\ 9 & 7 & 8 & 7 & 6 & 2 & 9 \\ 5 & 4 & 9 & 2 & 8 & 4 & 7 \\ 6 & 2 & 9 & 4 & 8 & 2 & 7 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.01$ the following PIO matrices are obtained :

$$N = \begin{bmatrix} -0.87 & 1.14 \\ -0.96 & -2.89 \end{bmatrix},\ H = \begin{bmatrix} -3.86 & 0.08 \\ -0.48 & -4.06 \end{bmatrix},\ J = \begin{bmatrix} -2.13 \\ -1.94 \end{bmatrix},\ F = \begin{bmatrix} 2.87 & 0.23 & -0.92 \\ 5.68 & -2.77 & 0.13 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0.51 & -0.51 & -0.01 \\ 0 & 0 & 1 \\ -0.97 & 0.97 & -0.1 \\ 0.97 & 0.02 & 0.1 \end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0.15,\ 0,\ 0,\ 0,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0,\ 0]^T$.

To evaluate the performance of the observers an uncertainty $\wp(t)$ is added in the system matrix A, then we obtain the following matrix $(A+\wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

The results of simulation are depicted in Figures 2.1 - 2.10. Figures 2.1 and 2.2 show the input $u(t)$ and the uncertainty factor $\delta(t)$. Figures 2.3 - 2.10 show the system states and their estimations by the GDO, PO and PIO, also these figures show the estimation error for each observer.

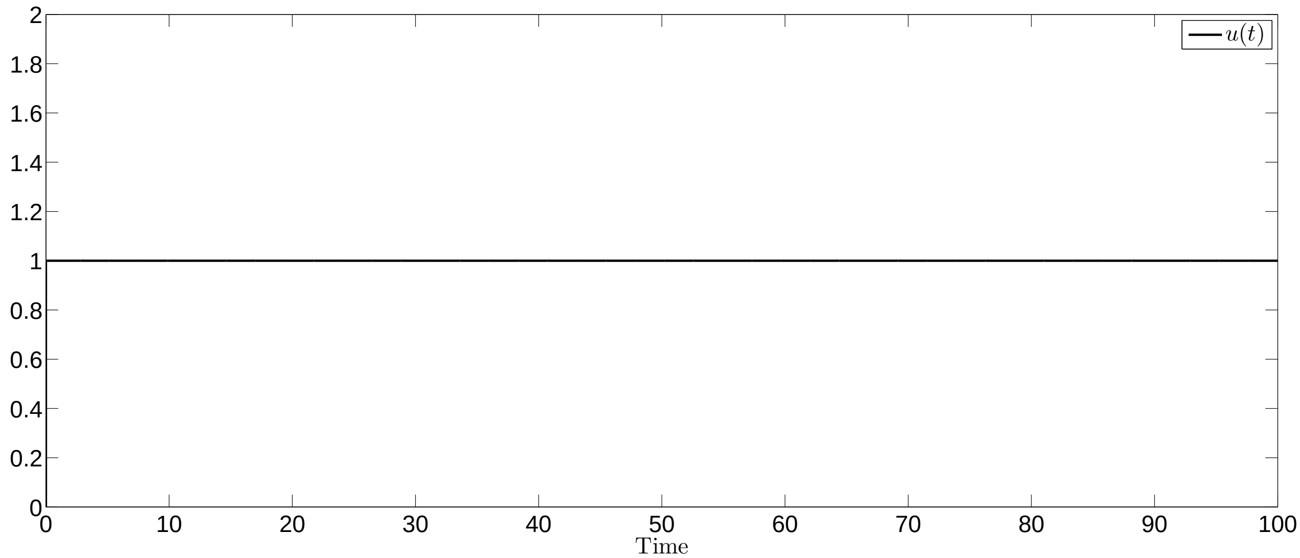

FIGURE 2.1 – Input $u(t)$.

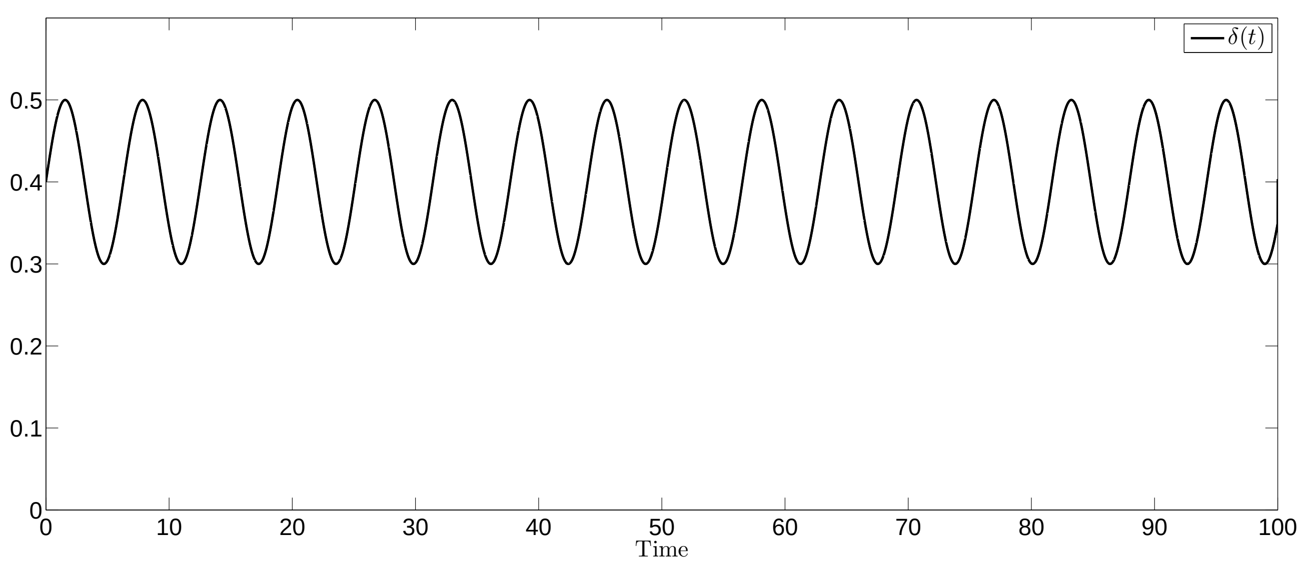

FIGURE 2.2 – Uncertainty factor $\delta(t)$.

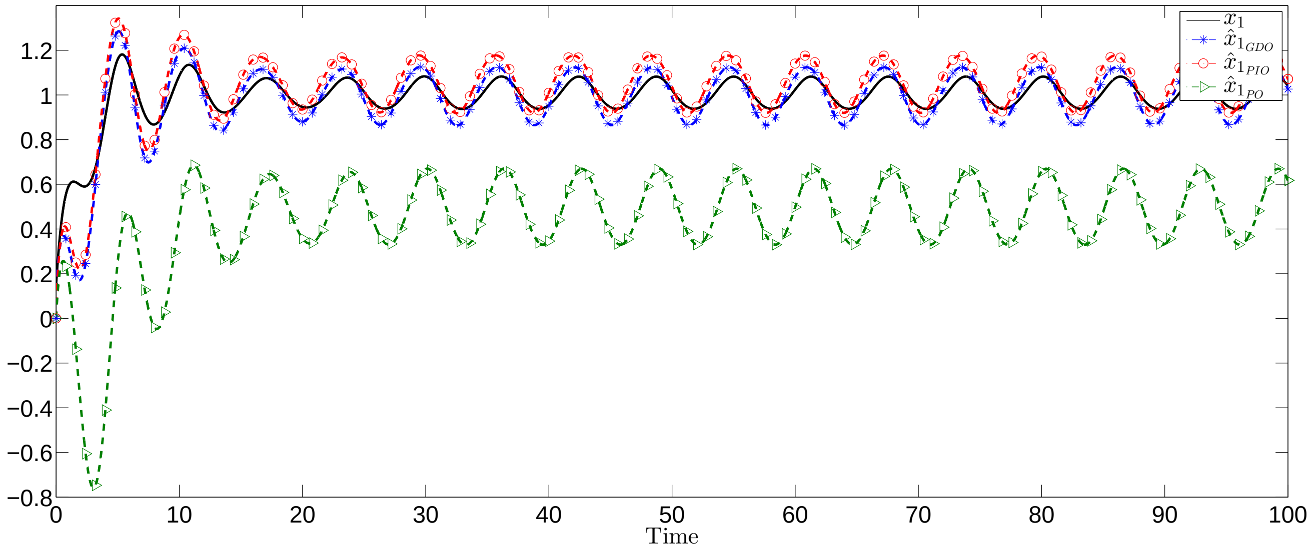

FIGURE 2.3 – Estimate of $x_1(t)$.

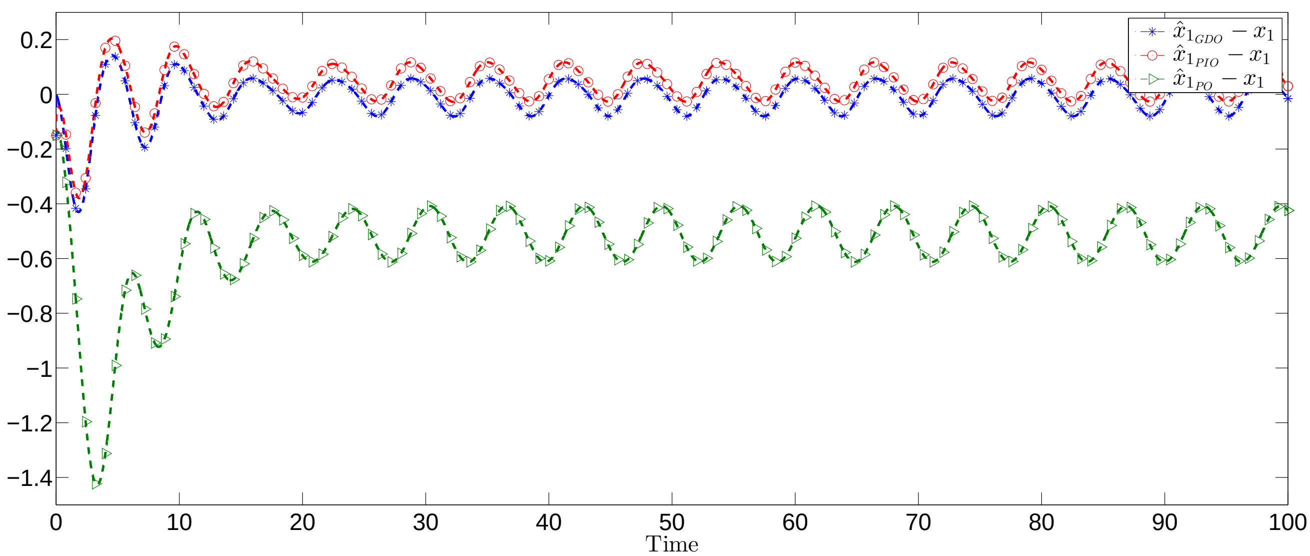

FIGURE 2.4 – Estimation error of $x_1(t)$.

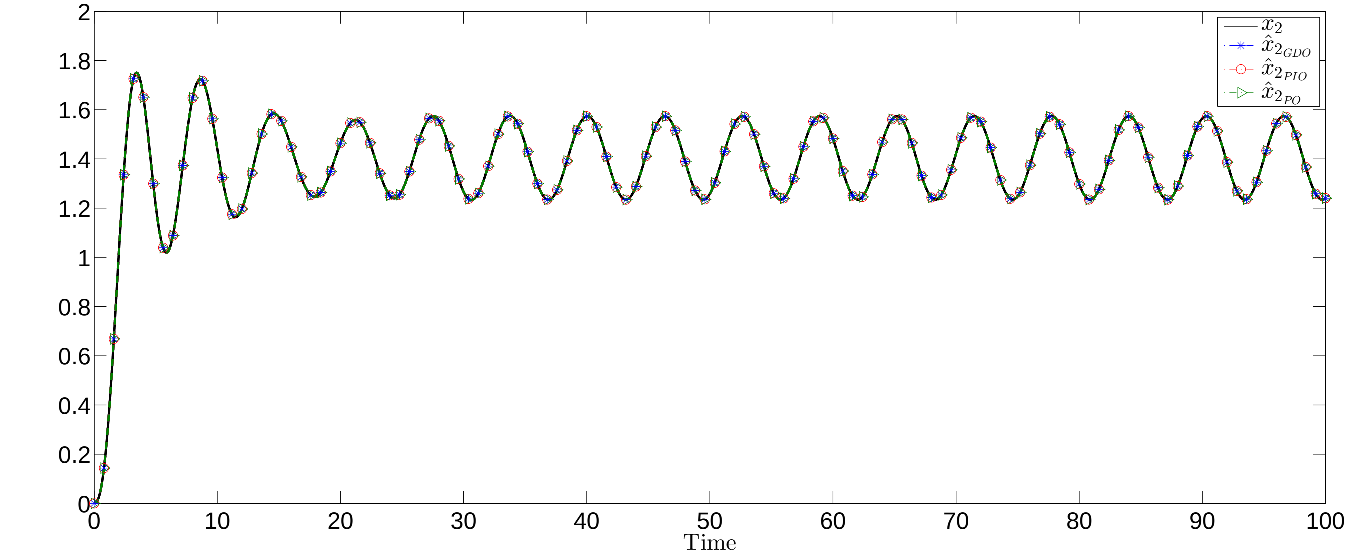

FIGURE 2.5 – Estimate of $x_2(t)$.

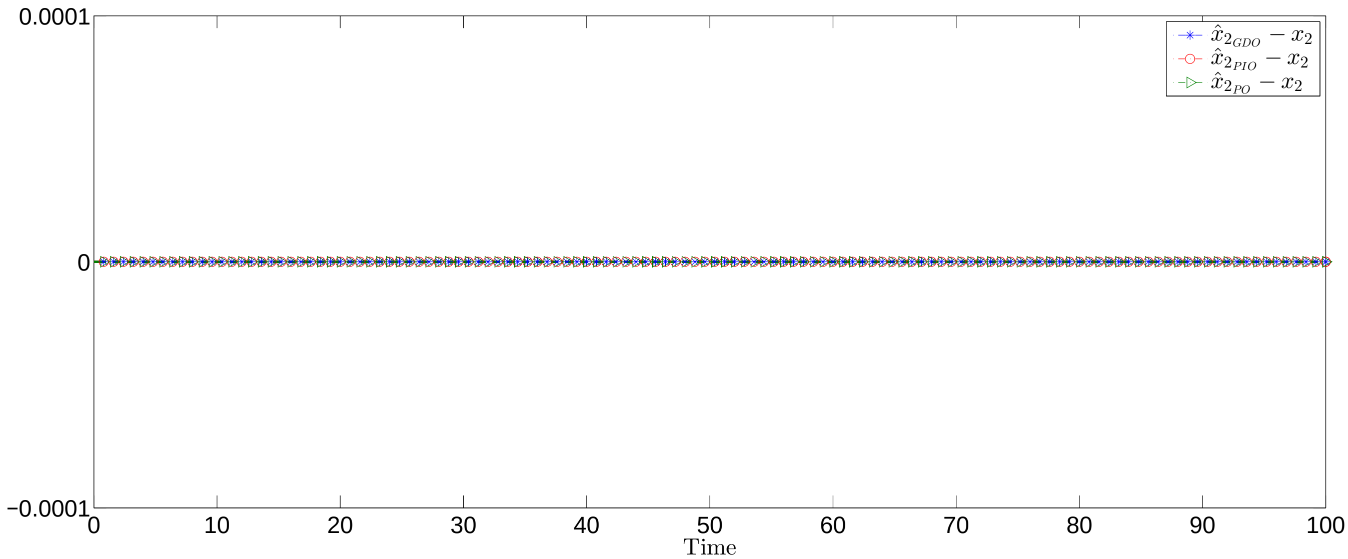

FIGURE 2.6 – Estimation error of $x_2(t)$.

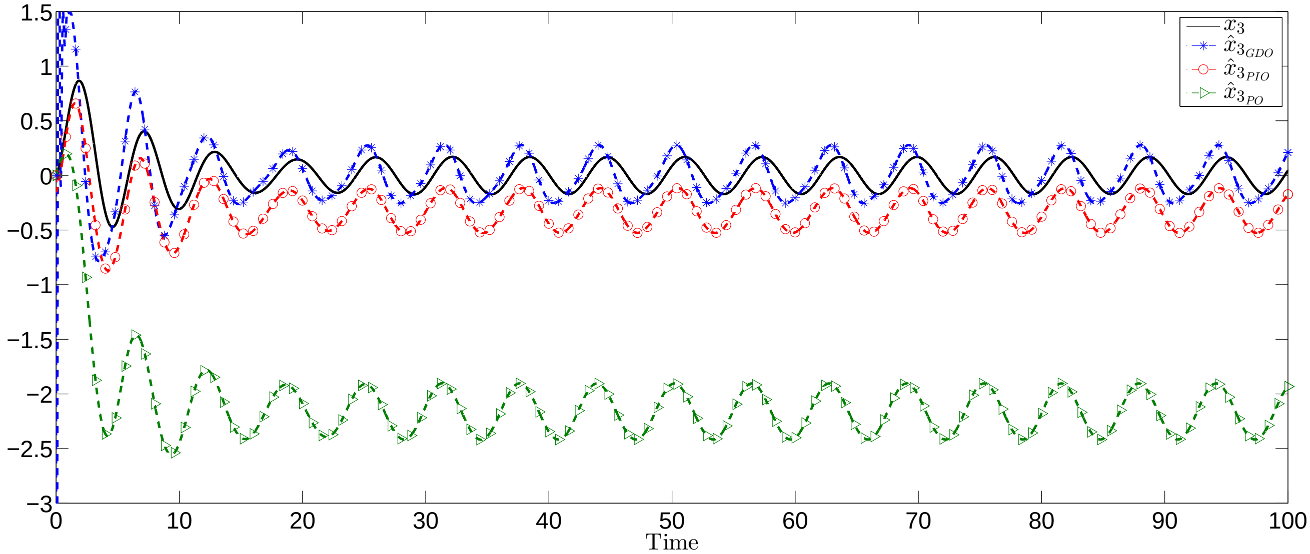

FIGURE 2.7 – Estimation of $x_3(t)$.

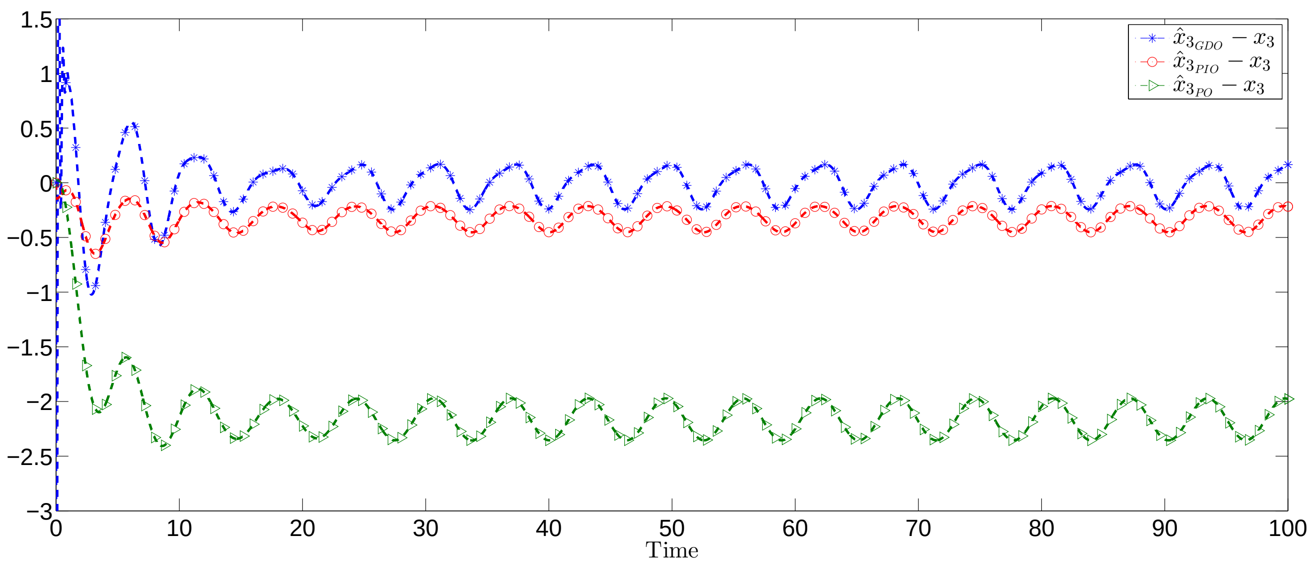

FIGURE 2.8 – Estimation error of $x_3(t)$.

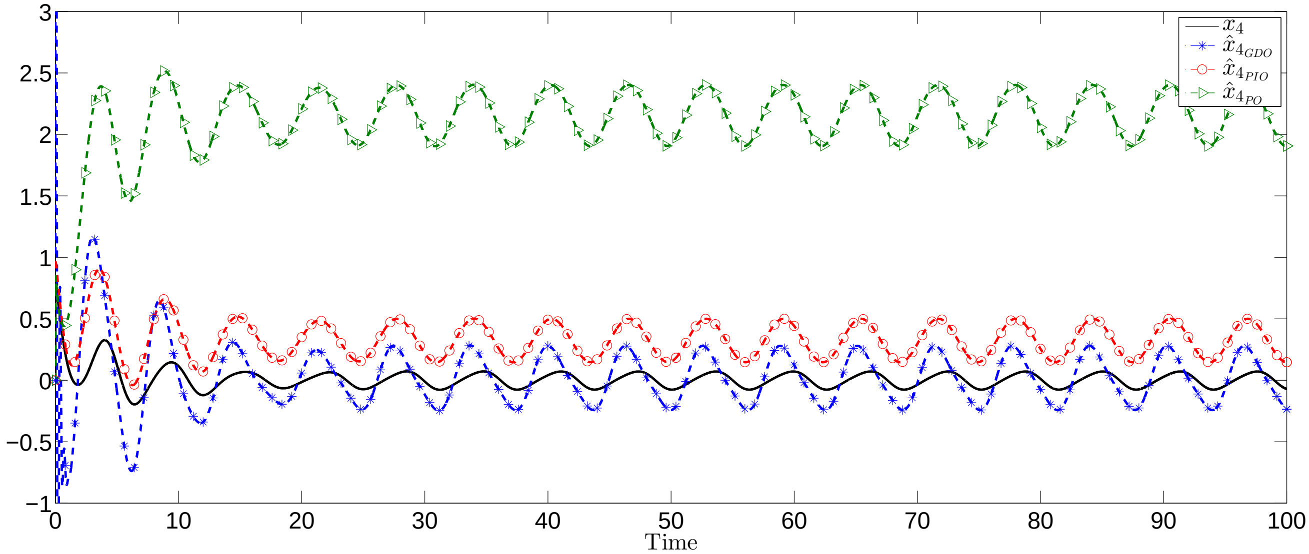

FIGURE 2.9 – Estimation of $x_4(t)$.

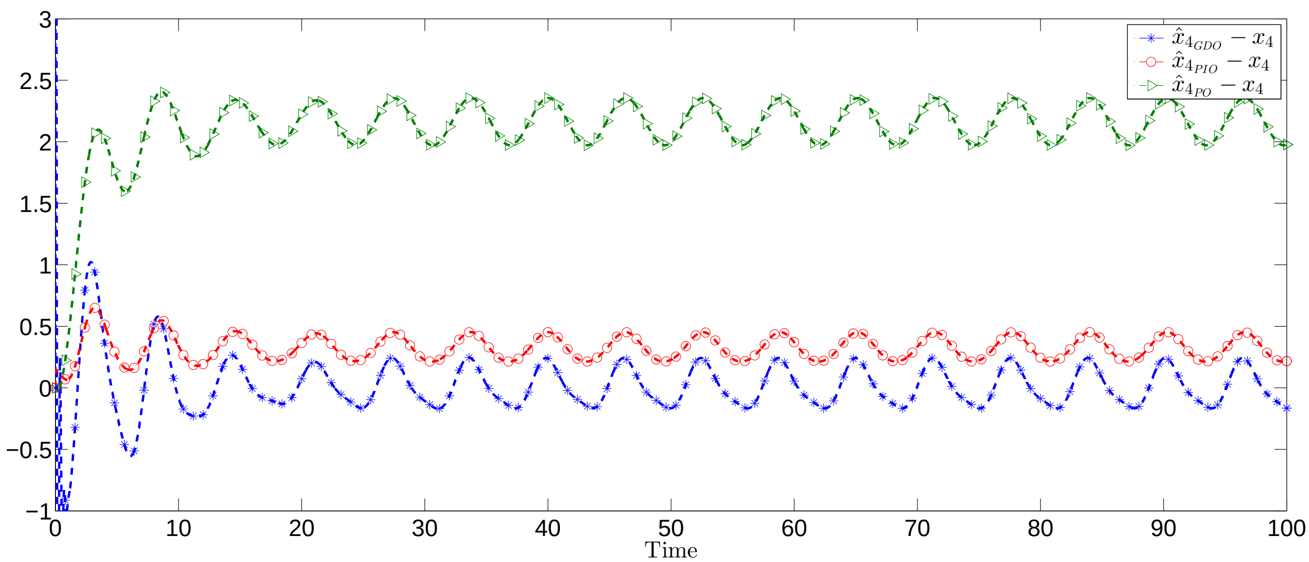

FIGURE 2.10 – Estimation error of $x_4(t)$.

From these results, we can see that the GDO has better estimation of the system states in presence of parametric uncertainties. After, is the PIO due to the integral action with an acceptable estimation, and finally is the PO, which in most of the states estimation presents a static error. In order to show the difference in the estimations the following table is presented, where the integral absolute error (IAE) is considered.

TABLE 2.1 – Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	5398.99	6675	54813.07
$x_2(t)$	0	0	0
$x_3(t)$	16699	32450.92	210982.44
$x_4(t)$	16699	32450.92	210982.44

2.6 H_∞ generalized dynamic observer design for disturbed descriptor systems, $w(t) \neq 0$

In this section we consider $w(t) \neq 0$, then we get system (2.1)

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1 x(t) + D_1 w(t) \end{aligned} \tag{2.60}$$

with the GDO :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{2.61a}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{2.61b}$$

$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{2.61c}$$

and the error dynamics (2.48)

$$\begin{aligned} \dot{\varphi}(t) &= (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) + (\mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2)w(t) \\ e(t) &= \mathbb{P}\varphi(t) + \mathbb{Q}w(t) \end{aligned} \tag{2.62}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{B}_1 = \begin{bmatrix} \mathcal{F}_{d_1} - T_1 D_1 + Z(\mathcal{T}_2 D_1 - \mathcal{F}_{d_2}) \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix}$,
$\mathbb{P} = \begin{bmatrix} P_1 - Y_3 N_3 & 0 \end{bmatrix}$, $\mathbb{Q} = \mathcal{Q}_{d_1} - Y_3 \mathcal{F}_{d_3}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

In this section we present a method for designing an H_∞ GDO given by (2.61). This design is obtained from the determination of matrices Z and $\mathbb{Y}$ such that the worst estimation error energy $\|e\|_2$ is minimum for all bounded energy disturbance $w(t)$. This problem is equivalent to $\|G_{we}\|_\infty < \gamma$, where G_{we} is the transfer function from the disturbance to the estimation error and γ is a given positive scalar. The solution to this problem is given by the following theorem.

Theorem 2.2. *Under Assumptions 2.1 and 2.2, there exists an H_∞ GDO (2.61) such that the error dynamics in (2.62) is stable and $\|G_{we}\|_\infty < \gamma$, if there exists a matrix $X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$, with $X_1 = X_1^T$ satisfying the following*

LMIs.

$$\mathcal{C}^{T\perp} \left[\begin{array}{cc|c|c} \Pi_1 & N_1^T X_1 - \mathcal{N}_2^T W_1^T & \Pi_2 & (P_1 - Y_3 N_3)^T \\ (*) & 0 & \Pi_2 & 0 \\ \hline (*) & (*) & -\gamma^2 I_{n_w} & (\mathcal{Q}_{d_1} - Y_3 \mathcal{F}_{d_3})^T \\ \hline (*) & 0 & (*) & -I_n \end{array}\right] \mathcal{C}^{T\perp T} < 0 \tag{2.63}$$

where

$$\Pi_1 = X_1 N_1 - W_1 \mathcal{N}_2 + N_1^T X_1 - \mathcal{N}_2^T W_1^T \tag{2.64a}$$

$$\Pi_2 = X_1(\mathcal{F}_{d_1} - T_1 D_1) + W_1(\mathcal{T}_2 D_1 - \mathcal{F}_{d_2}) \tag{2.64b}$$

and

$$\begin{bmatrix} -\gamma^2 I_{n_w} & (Q_{d_1} - Y_3 \mathcal{F}_{d_3})^T \\ (*) & -I_n \end{bmatrix} < 0 \tag{2.65}$$

In this case matrix $W_1 = X_1 Z$ and matrix $\mathbb{Y}$ is parameterized as follows :

$$\mathbb{Y} = X^{-1}(\mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+) \tag{2.66}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1} \mathcal{B}_l^T \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2} \mathcal{L} (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1/2} \tag{2.67a}$$

$$\vartheta = (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{2.67b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\vartheta - \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} \mathcal{C}_r \vartheta] \mathcal{B}_l \mathcal{R}^{-1} \tag{2.67c}$$

where $\mathcal{D} = \left[\begin{array}{cc|c|c} \Pi_1 & N_1^T X_1 - \mathcal{N}_2^T W_1^T & \Pi_2 & (P_1 - Y_3 N_3)^T \\ (*) & 0 & \Pi_2 & 0 \\ \hline (*) & (*) & -\gamma^2 I_{n_w} & (\mathcal{Q}_{d_1} - Y_3 \mathcal{F}_{d_3})^T \\ \hline (*) & 0 & (*) & -I_n \end{array}\right]$, $\mathcal{B} = \begin{bmatrix} -I_{q_0+q_1} \\ 0 \\ 0 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} & \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix} & 0 \end{bmatrix}$,
and matrices Π_1 and Π_2 are defined in equation (2.64), and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$.

Proof. The BRL of Lemma 1.2 guarantees that the observer error system (2.62) is stable and $\|G_{we}\|_\infty < \gamma$ if and only if there exists a matrix $X = X^T > 0$ such that :

$$\begin{bmatrix} (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X + X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) & X(\mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2) & \mathbb{P}^T \\ (*) & -\gamma^2 I_{n_w} & \mathbb{Q}^T \\ (*) & (*) & -I_n \end{bmatrix} < 0 \tag{2.68}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{2.69}$$

where $\mathcal{X} = X\mathbb{Y}$, $\mathcal{D} = \begin{bmatrix} \mathbb{A}_1^T X + X\mathbb{A}_1 & X\mathbb{B}_1 & \mathbb{P}^T \\ (*) & -\gamma^2 I_{n_w} & \mathbb{Q}^T \\ (*) & (*) & -I_n \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & \mathbb{B}_2 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} -I_{q_0+q_1} \\ 0 \\ 0 \end{bmatrix}$.

From the elimination lemma, the solvability conditions of inequality (2.69) are :

$$\mathcal{C}^{T\perp} \mathcal{D} \mathcal{C}^{T\perp T} < 0 \tag{2.70a}$$

$$\mathcal{B}^{\perp} \mathcal{D} \mathcal{B}^{\perp T} < 0 \tag{2.70b}$$

with $\mathcal{B}^\perp = \begin{bmatrix} 0 & I_{n_w} & 0 \\ 0 & 0 & I_n \end{bmatrix}$ and $\mathcal{C}^{T\perp} = \begin{bmatrix} \begin{bmatrix} \mathbb{A}_2^T \\ \mathbb{B}_2^T \end{bmatrix}^\perp & 0 \\ 0 & I_n \end{bmatrix}$. By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and W_1 the inequality (2.70a) becomes :

$$\mathcal{C}^{T\perp} \left[\begin{array}{cc|c|c} \Pi_1 & N_1^T X_1 - \mathcal{N}_2^T W_1^T & \Pi_2 & (P_1 - Y_3 N_3)^T \\ (*) & 0 & \Pi_2 & 0 \\ \hline (*) & (*) & -\gamma^2 I_{n_w} & (\mathcal{Q}_{d_1} - Y_3 \mathcal{F}_{d_3})^T \\ \hline (*) & 0 & (*) & -I_n \end{array}\right] \mathcal{C}^{T\perp T} < 0 \tag{2.71}$$

where matrices Π_1 and Π_2 are defined in equation (2.64), and by using matrices $\mathcal{B}$, $\mathcal{D}$ the inequality (2.70b) becomes :

$$\begin{bmatrix} -\gamma^2 I_{n_w} & (Q_{d_1} - Y_3 \mathcal{F}_{d_3})^T \\ (*) & -I_n \end{bmatrix} < 0. \tag{2.72}$$

From the elimination lemma if conditions (2.70a) and (2.70b) are satisfied, then parameter matrix $\mathbb{Y}$ is parameterized as in (2.66) and (2.67). □

2.6.1 Particular cases

•Proportional observer

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1 x(t) + D_1 w(t) \end{aligned}$$

with the PO :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + F_a y(t) + Ju(t) \\ \hat{x}(t) &= P\zeta(t) + Q_a y(t) \end{aligned}$$

and the error dynamics (2.62) becomes :

$$\begin{aligned} \dot{\varepsilon}(t) &= (\bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2)\varepsilon(t) + (\bar{\mathbb{B}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{B}}_2)w(t) \\ e(t) &= \bar{\mathbb{P}}\varepsilon(t) + \bar{\mathbb{Q}}w(t) \end{aligned}$$

where $\bar{\mathbb{A}}_1 = N_1 - Z\mathcal{N}_2$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{B}}_1 = \mathcal{F}_{d_1} - T_1 D_1 + Z(\mathcal{T}_2 D_1 - \mathcal{F}_{d_2})$, $\bar{\mathbb{B}}_2 = \mathcal{F}_{d_3}$, $\bar{\mathbb{P}} = P_1 - Y_3 N_3$, $\bar{\mathbb{Q}} = \mathcal{Q}_{d_1} - Y_3 \mathcal{F}_{d_3}$ and $\bar{\mathbb{Y}} = Y_1$. Consequently matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.2 become :

$$\mathcal{D} = \begin{bmatrix} (N_1 - Z\mathcal{N}_2)^T X + X(N_1 - Z\mathcal{N}_2) & X(\mathcal{F}_{d_1} - T_1 D_1 + Z(\mathcal{T}_2 D_1 - \mathcal{F}_{d_2})) & (P_1 - Y_3 N_3)^T \\ (*) & -\gamma^2 I_{n_w} & (\mathcal{Q}_{d_1} - Y_3 \mathcal{F}_{d_3})^T \\ (*) & (*) & -I_n \end{bmatrix},$$

$\mathcal{C} = \begin{bmatrix} N_3 & \mathcal{F}_{d_3} & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} -I_{q_0+q_1} \\ 0 \\ 0 \end{bmatrix}$, $\mathcal{X} = X\bar{\mathbb{Y}}$. Matrices Σ and Ω are defined as $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C \end{bmatrix}$.

•Proportional-integral observer

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1 x(t) + D_1 w(t) \end{aligned}$$

with the PIO :

$$\begin{aligned}
\dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix}-E^{\perp}Bu(t)\\ y(t)\end{bmatrix} + Ju(t)\\
\dot{v}(t) &= y(t) - C_1\hat{x}(t)\\
\hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix}-E^{\perp}Bu(t)\\ y(t)\end{bmatrix}
\end{aligned}$$

and the error dynamics (2.62) becomes :

$$\begin{aligned}
\dot{\varphi}(t) &= (\bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2)\varphi(t) + (\bar{\bar{\mathbb{B}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{B}}}_2)w(t)\\
e(t) &= \bar{\bar{\mathbb{P}}}\varphi(t) + \bar{\bar{\mathbb{Q}}}w(t)
\end{aligned}$$

$Y_3 = 0$ is taken for simplicity and matrices $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix}N_1 - Z\mathcal{N}_2 & 0\\ -C_1P_1 & 0\end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_1 = \begin{bmatrix}\mathcal{F}_{d_1} - T_1D_1 + Z(\mathcal{T}_2D_1 - \mathcal{F}_{d_2})\\ D_2 - C_1\mathcal{Q}_{d_1}\end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_2 = \begin{bmatrix}\mathcal{F}_{d_3}\\ 0\end{bmatrix}$, $\bar{\bar{\mathbb{P}}} = \begin{bmatrix}P_1 & 0\end{bmatrix}$, $\bar{\bar{\mathbb{Q}}} = \mathcal{Q}_{d_1}$ and $\mathbb{Y} = \begin{bmatrix}I\\0\end{bmatrix}\begin{bmatrix}Y_1 & H\end{bmatrix}$. Consequently matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.2 become :

$$\mathcal{D} = \left[\begin{array}{cc|c|c}
\Pi_1 & (N_1 - Z\mathcal{N}_2)^TX_1 - P_1^TC_1^TX_2 & \Pi_2 & P_1^T\\
(*) & 0 & \Pi_3 & 0\\ \hline
(*) & (*) & -\gamma^2 I_{n_w} & \mathcal{Q}_{d_1}^T\\ \hline
(*) & 0 & (*) & -I_n
\end{array}\right],$$

with

$$\begin{aligned}
\Pi_1 &= \left[(N_1 - Z\mathcal{N}_2)^T - P_1^TC_1^T\right]X_1 + X_1\left[(N_1 - Z\mathcal{N}_2) - C_1P_1\right]\\
\Pi_2 &= X_1\left[\mathcal{F}_{d_1} - T_1D_1 + D_2 - C_1\mathcal{Q}_{d_1} + Z(\mathcal{T}_2D_1 - \mathcal{F}_{d_2})\right]\\
\Pi_3 &= X_1\left[\mathcal{F}_{d_1} - T_1D_1 + Z(\mathcal{T}_2D_1 - \mathcal{F}_{d_2})\right] + X_2\left(D_2 - C_1\mathcal{Q}_{d_1}\right)
\end{aligned}$$

$$\mathcal{C} = \begin{bmatrix}\begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix} & \begin{bmatrix}\mathcal{F}_{d_3}\\ 0\end{bmatrix} & 0\end{bmatrix},\ \mathcal{B} = \begin{bmatrix}-I_{q_0+q_1}\\ 0\\ 0\end{bmatrix} \text{ and } \mathcal{X} = X\bar{\bar{\mathbb{Y}}}, \text{ such that } \begin{bmatrix}Y_1 & H\end{bmatrix} = \left(X\begin{bmatrix}I\\0\end{bmatrix}\right)^+\mathcal{X}.$$

2.6.2 Numerical example

In order to illustrate the results obtained, consider the following disturbed descriptor system described by (2.60), where

$$E = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\ A = \begin{bmatrix}-2.7 & 0 & 0.3\\ -0.2 & -3 & 0\\ -0.11 & 1.74 & -1\end{bmatrix},\ B = \begin{bmatrix}1\\0\\0\end{bmatrix},\ D = \begin{bmatrix}0.4\\0.3\\0.2\end{bmatrix},$$

$$C_1 = \begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix},\ D_1 = \begin{bmatrix}0.3\\0.5\end{bmatrix} \text{ and } x(t) = \begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}.$$

Considering matrix $E^{\perp} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$, we can verify Assumptions 2.1 and 2.2

$$\mathsf{rank}\left[\begin{array}{c}E\\ \hline E^{\perp}A\\ C_1\end{array}\right] = 3 \quad\text{and}\quad \mathsf{rank}\begin{bmatrix}sE - A\\ C_1\end{bmatrix} = 3$$

H_∞ Generalized dynamic observer

For the H_∞ GDO we have chosen matrix $R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then $\mathsf{rank}(\Sigma) = 4$. By fixing the value of $\gamma = 1$ and using YALMIP toolbox, we solve the LMIs (2.63) and (2.65) to find matrices X, Z and Y_3

$$X = \begin{bmatrix} 63.82 & 3.97 & 63.82 & 3.97 \\ 3.97 & 71.95 & 3.97 & 71.95 \\ 63.82 & 3.97 & 163.21 & 1.08 \\ 3.97 & 71.94 & 1.08 & 165.42 \end{bmatrix},$$

$$Z = \begin{bmatrix} -1.86 & -1.84 & -1.86 & -1.87 & -1.88 & -1.85 \\ -1.64 & -1.54 & -1.64 & -1.65 & -1.68 & -1.6 \end{bmatrix} \text{ and}$$

$$Y_3 = \begin{bmatrix} 0.17 & -15.25 & -12.29 & 7.37 & -14.97 \\ -0.03 & 0.06 & -0.26 & -0.19 & 0.07 \\ -0.04 & 1.16 & 0.64 & -0.76 & 1.15 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 & 9 \\ 3 & 2 & 6 & 3 & 3 & 5 & 4 \\ 6 & 1 & 4 & 2 & 6 & 2 & 6 \\ 9 & 5 & 2 & 7 & 4 & 3 & 2 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.0001$ and solving (2.66) and (2.67) we get :

$$\mathbb{Y} = \begin{bmatrix} 242.96 & -68.96 & -0.02 & -173.86 & -68.86 & 94.61 & -2.25 \\ -85.06 & 27.88 & 0.12 & 57.14 & 27.91 & 20.86 & 127.13 \\ -89.18 & 25.24 & 0.03 & 63.94 & 25.23 & -104.73 & -7.43 \\ 35.71 & -11.72 & -0.04 & -23.85 & -11.74 & -16.65 & -121.58 \end{bmatrix}.$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -243.63 & 65.3 \\ 84.36 & -31.81 \end{bmatrix},\ S = \begin{bmatrix} 89.18 & -25.23 \\ -35.66 & 11.75 \end{bmatrix},\ H = \begin{bmatrix} 94.61 & -2.25 \\ 20.86 & 127.13 \end{bmatrix},\ J = \begin{bmatrix} -0.01 \\ 0.02 \end{bmatrix},$$

$$L = \begin{bmatrix} -104.73 & -7.43 \\ -16.65 & -121.58 \end{bmatrix},\ F = \begin{bmatrix} 34.12 & 111.61 & -79.35 \\ -9.17 & -35.89 & 25.08 \end{bmatrix},\ P = \begin{bmatrix} -1.85 & 26.73 \\ 0.17 & 0.5 \\ 0.38 & -1.48 \end{bmatrix},$$

$$M = \begin{bmatrix} -11.57 & -41.31 & 29.74 \\ 4.98 & 14.74 & -9.75 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -3.23 & -11.31 & 8.08 \\ 0.1 & 0.34 & -0.24 \\ -0.34 & 1.29 & -0.64 \end{bmatrix}.$$

In order to provide a comparison of the GDO with the PIO and the PO, these latter are also designed.

H_∞ Proportional observer

By considering matrices $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 2 & 6 & 3 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$, $\mathcal{R} = I_2 \times 0.01$ and $\gamma = 1$ the following PO matrices are obtained :

$$N = \begin{bmatrix} -2.85 & 38.45 \\ -0.21 & -37.43 \end{bmatrix},\ F_a = \begin{bmatrix} -23.31 & 22.27 \\ 24.16 & -24.85 \end{bmatrix},\ J = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \\ 0 & -3.75 \end{bmatrix} \text{ and } Q_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2.5 & -1.5 \end{bmatrix}.$$

H_∞ Proportional-integral observer

By considering matrices $R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 & 9 \\ 3 & 2 & 6 & 3 & 3 & 5 & 4 \\ 6 & 1 & 4 & 2 & 6 & 2 & 6 \\ 9 & 5 & 2 & 7 & 8 & 4 & 9 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$,

$\mathcal{R} = I_4 \times 0.00001$ and $\gamma = 1$ the following PIO matrices are obtained :

$$N = \begin{bmatrix} -390.74 & 372.39 \\ 354.82 & -345.2 \end{bmatrix},\ H = \begin{bmatrix} -876.88 & -144.9 \\ -145 & -876.78 \end{bmatrix},\ J = \begin{bmatrix} 1.29 \\ 1.33 \end{bmatrix},\ P = \begin{bmatrix} 0.57 & 0.43 \\ -0.09 & 0.13 \\ -0.04 & 0.01 \end{bmatrix},$$

$$F = \begin{bmatrix} -75.5 & -70.3 & -4.82 \\ -7.24 & 175.82 & -182.66 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 2.84 & 10.22 & -7.38 \\ 0.49 & 0.72 & -0.23 \\ -0.31 & 0.14 & 0.55 \end{bmatrix}.$$

Simulation results

Note that the same value of γ is taken for the design of each observer. The initial conditions for the system are $x(0) = [0.1,\ 0,\ 0.2]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0]^T$.

To evaluate the performance of the observers an uncertainty $\wp(t)$ is added in the system matrix A, then we obtain the following matrix $(A + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The results of simulation are depicted in Figures 2.11 - 2.19. The input $u(t)$, the disturbance $w(t)$ and the uncertainty factor $\delta(t)$ are shown in Figures 2.11, 2.12 and 2.13, respectively. On Figures 2.14 - 2.19 they show the system semi-state and their estimations by the GDO, the PIO and the PO, and their respective error estimations.

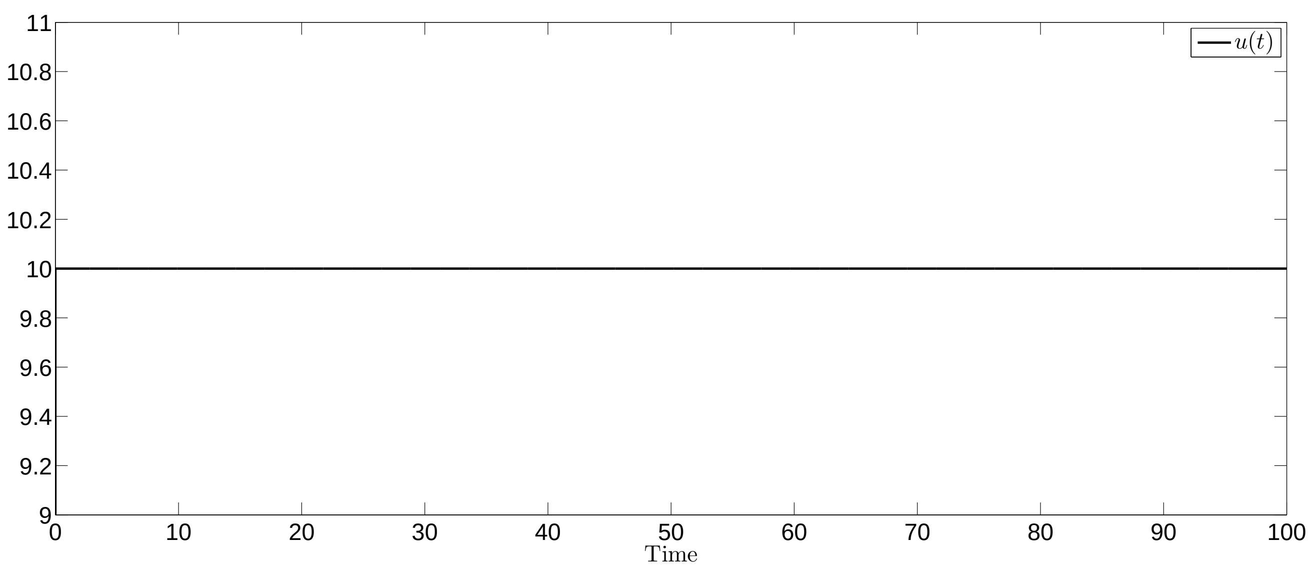

FIGURE 2.11 – H_∞ observers : Input $u(t)$.

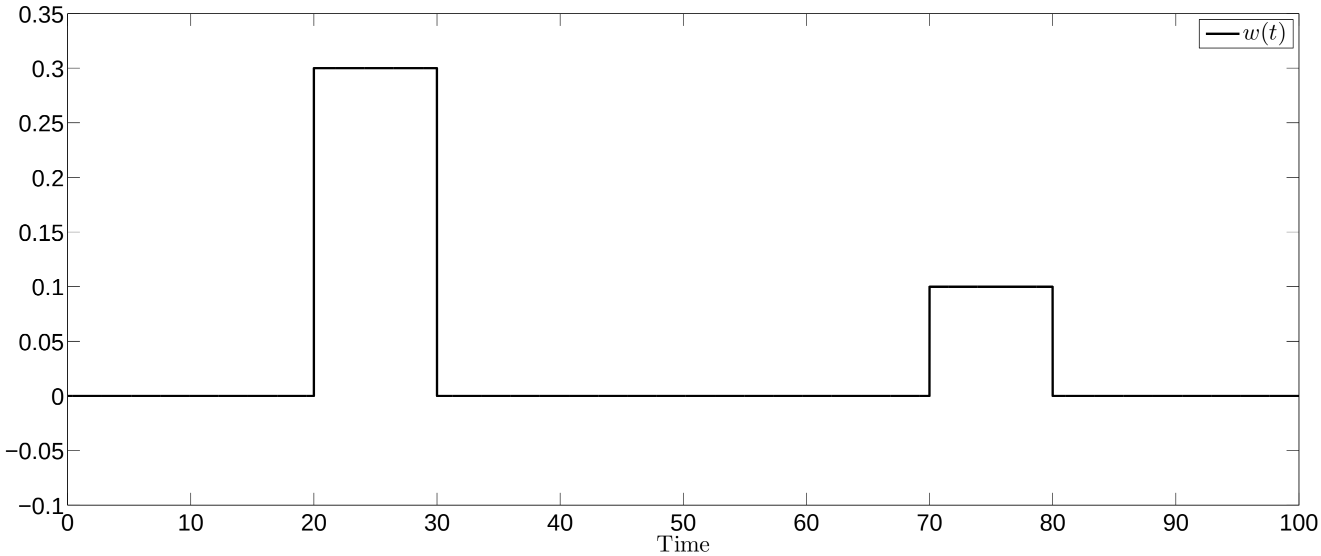

FIGURE 2.12 – H_∞ observers : Disturbance $w(t)$.

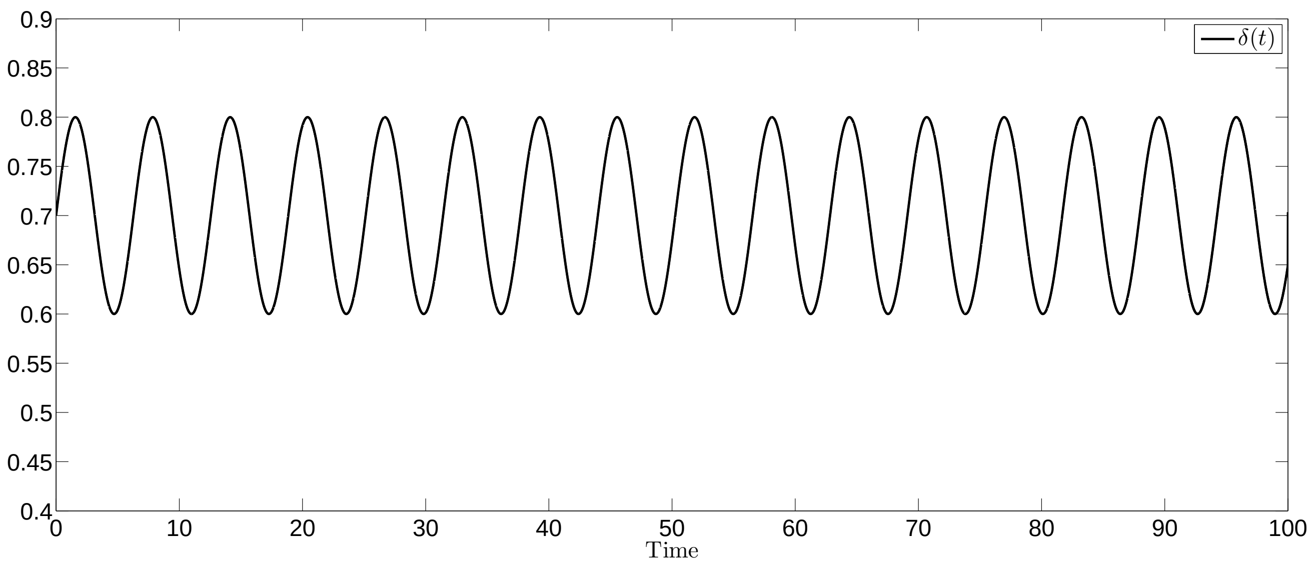

FIGURE 2.13 – H_∞ observers : Uncertainty factor $\delta(t)$.

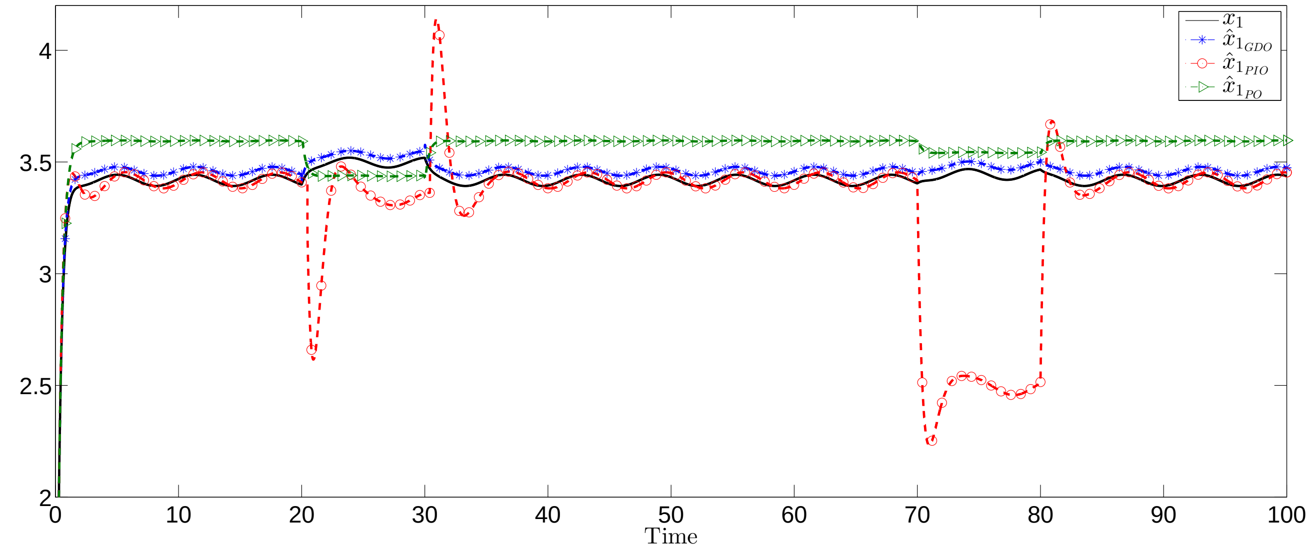

FIGURE 2.14 – H_∞ observers : Estimate of $x_1(t)$.

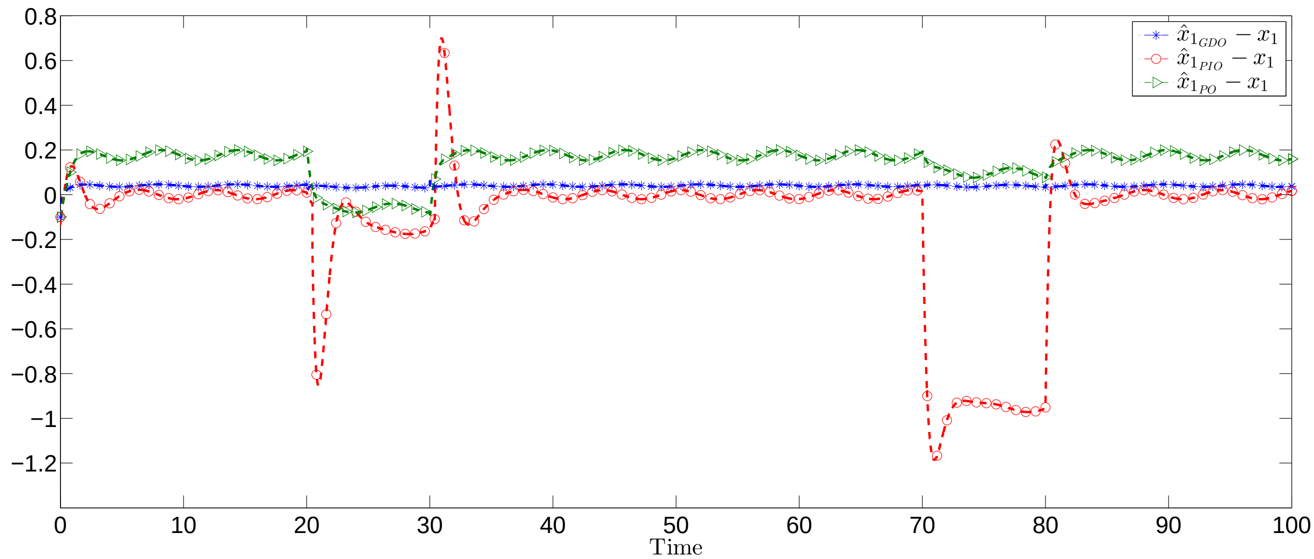

FIGURE 2.15 – H_∞ observers : Estimation error of $x_1(t)$

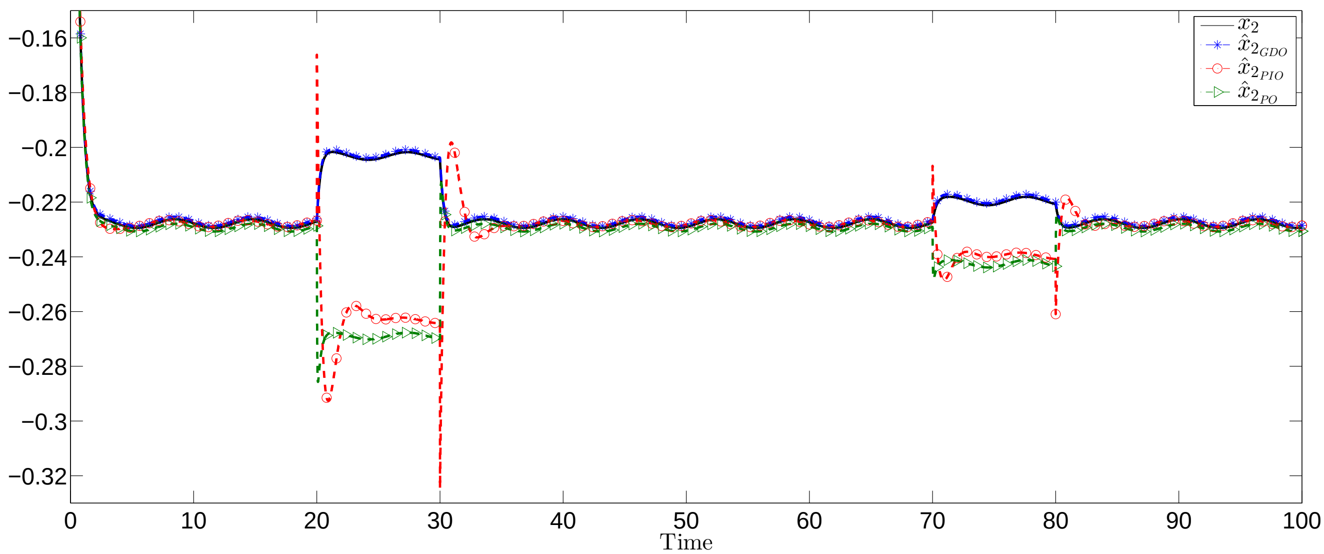

FIGURE 2.16 – H_∞ observers : Estimate of $x_2(t)$.

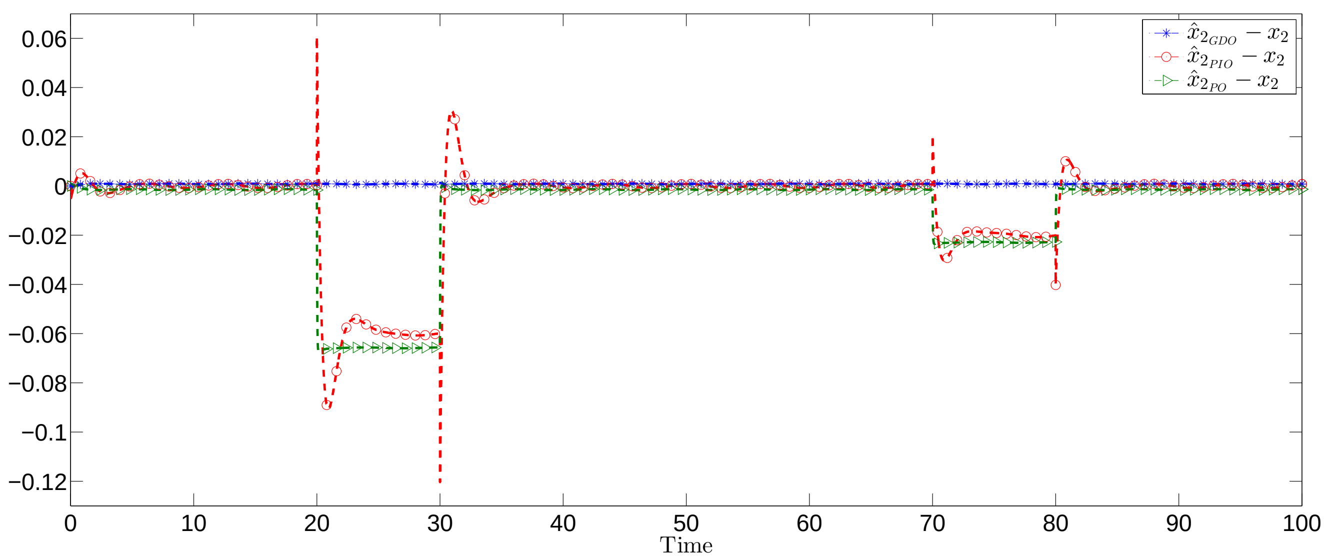

FIGURE 2.17 – H_∞ observers : Estimation error of $x_2(t)$.

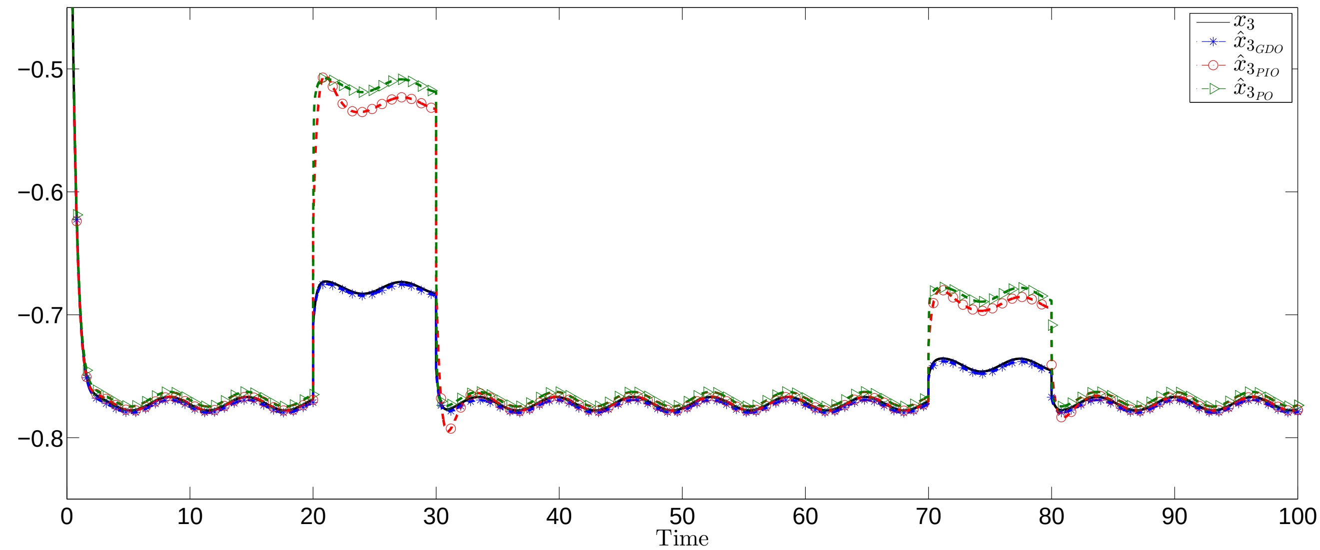

FIGURE 2.18 – H_∞ observers : Estimate of $x_3(t)$.

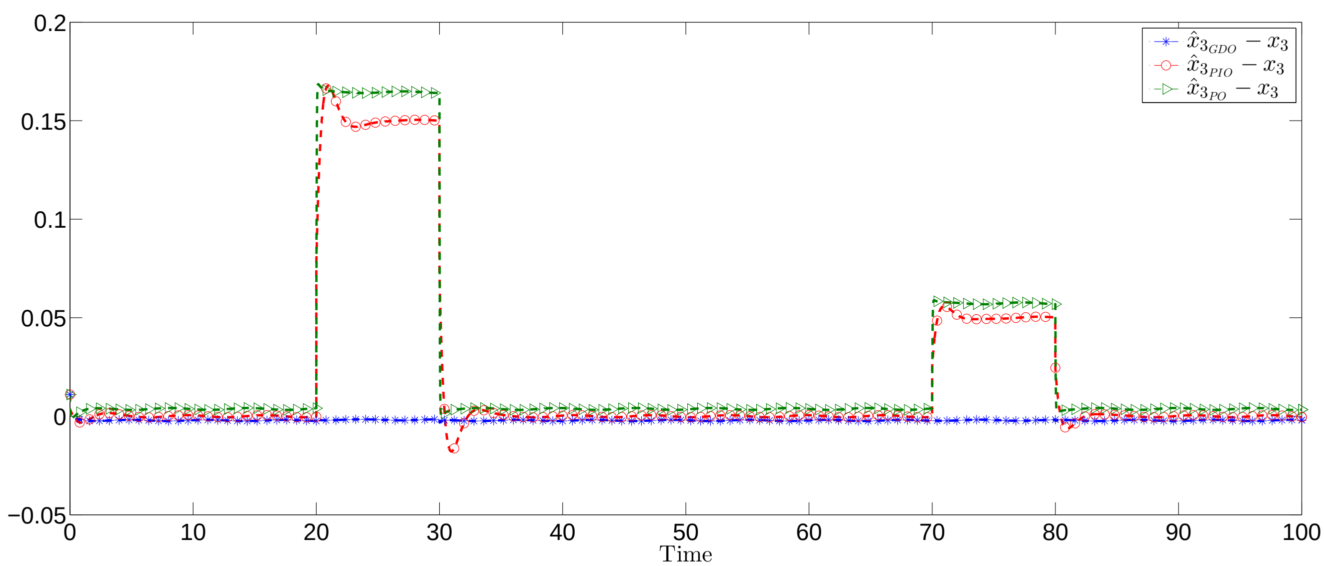

FIGURE 2.19 – H_∞ observers : Estimation error of $x_3(t)$.

From these results, we can see that the PO has a static error in their estimations, more evident in $x_1(t)$. The main difference between the GDO and the PIO is when the disturbance appears on the system, it is clear that the PIO has more error estimation than the GDO. The following table presents the IAE for the observers estimations.

TABLE 2.2 – H_∞ observers : Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	3981.52	14287.59	15463.76
$x_2(t)$	82.7	948.15	1004.23
$x_3(t)$	210.19	2082.05	2510.58

2.7 Generalized dynamic observer design for discrete-time descriptor systems, $w(t) = 0$

The system (2.1) can be written in its discrete-time form since $\dot{x}(t)$ in the continuous case represents $x(t+1)$ in the discrete case, such that the following discrete-time descriptor system with, $w(t) = 0$ is obtained :

$$\begin{aligned} Ex(t+1) &= Ax(t) + Bu(t) \\ y(t) &= C_1 x(t) \end{aligned} \tag{2.73}$$

with the GDO :

$$\zeta(t+1) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{2.74a}$$

$$v(t+1) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{2.74b}$$

$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{2.74c}$$

by considering the same assumption of $w(t) = 0$ the error dynamics (2.51) can be written in its discrete-time form as :

$$\begin{aligned} \varphi(t+1) &= (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) \\ e(t) &= \mathbb{P}\varphi(t) \end{aligned} \tag{2.75}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z_1 N_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$. $Y_3 = 0$ is taken for simplicity.

The following theorem gives the LMI conditions that allow the determination of all GDO matrices and guarantee the stability of matrix $(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)$.

Theorem 2.3. *Under Assumptions 2.1 and 2.2 there exist two parameter matrices $\mathbb{Y}$ and Z_1 such that the system (2.75) is asymptotically stable if there exist some symmetric positive definite matrices X_1 and X_2 and a matrix W_1 such that the following LMIs are satisfied*

$$\mathcal{C}^{T\perp}\left[\begin{array}{cc|cc} -X_1 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 \\ \hline X_1N_1 - W_1N_2 & 0 & -X_1 & (*) \\ X_1N_1 - W_1N_2 & 0 & -X_1 & -X_2 \end{array}\right]\mathcal{C}^{T\perp T} < 0 \tag{2.76}$$

and

$$X_2 - X_1 > 0. \tag{2.77}$$

In this case $W_1 = X_1 Z_1$ and matrix $\mathbb{Y}$ is parameterized as follows :

$$\mathbb{Y} = X^{-1}(\mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+) \tag{2.78}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}_l^T \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1/2} \tag{2.79a}$$

$$\vartheta = (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{2.79b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\vartheta - \vartheta \mathcal{C}_r^T(\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1}\mathcal{C}_r \vartheta]\mathcal{B}_l \mathcal{R}^{-1} \tag{2.79c}$$

with $\mathcal{D} = \left[\begin{array}{cc|cc} -X_1 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 \\ \hline X_1 N_1 - W_1 N_2 & 0 & -X_1 & (*) \\ X_1 N_1 - W_1 N_2 & 0 & -X_1 & -X_2 \end{array}\right]$, $\mathcal{B} = \begin{bmatrix} 0 \\ -I_{q_0+q_1} \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} & 0 \end{bmatrix}$, *and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$.*

Proof. Consider the following Lyapunov function

$$V(\varphi(t)) = \varphi(t)^T X \varphi(t) \tag{2.80}$$

with $X = \begin{bmatrix} X_1 & (*) \\ X_1 & X_2 \end{bmatrix} > 0$, with $X_1 = X_1^T > 0$. Now, variation of $V(\varphi(t))$ along the solution of (2.75) is :

$$\Delta V(\varphi(t)) = V(\varphi(t+1)) - V(\varphi(t)) \tag{2.81a}$$

$$= \varphi(t+1)^T X \varphi(t+1) - \varphi(t)^T X \varphi(t) \tag{2.81b}$$

$$= \varphi(t)^T (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) - \varphi(t)^T X \varphi(t) \tag{2.81c}$$

The inequality $\Delta V(\varphi(t)) < 0$ holds for all $\varphi(t) \neq 0$ if and only if

$$(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) - X < 0. \tag{2.82}$$

By using the Schur complement (see Lemma 1.4) in inequality (2.82) it gives :

$$\begin{bmatrix} -X & (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X \\ X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) & -X \end{bmatrix} < 0 \tag{2.83}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{2.84}$$

where $\mathcal{D} = \begin{bmatrix} -X & \mathbb{A}_1^T X \\ X\mathbb{A}_1 & -X \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 \\ -I_{q_0+q_1} \end{bmatrix}$ and $\mathcal{X} = X\mathbb{Y}$.

From the elimination lemma, the solvability conditions of inequality (2.84) are :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{2.85a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{2.85b}$$

with $\mathcal{B}^{\perp} = \begin{bmatrix} I_{q_0+q_1} & 0 \end{bmatrix}$ and $\mathcal{C}^{T\perp} = \begin{bmatrix} \begin{bmatrix} N_3^{T\perp} & 0 \end{bmatrix} & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$.

By using matrices $\mathcal{C}$, $\mathcal{D}$ and W_1 inequality (2.85a) becomes :

$$\mathcal{C}^{T\perp}\left[\begin{array}{cc|cc} -X_1 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 \\ \hline X_1N_1 - W_1N_2 & 0 & -X_1 & (*) \\ X_1N_1 - W_1N_2 & 0 & -X_1 & -X_2 \end{array}\right]\mathcal{C}^{T\perp T} < 0 \tag{2.86}$$

and using the definition of matrices $\mathcal{B}$ and $\mathcal{D}$ the inequality (2.85b) is $-X < 0$, by using the Schur complement in matrix X and since $X_1 = X_1^T$, we obtain :

$$X_2 - X_1 > 0. \tag{2.87}$$

From the elimination lemma if conditions (2.85a) and (2.85b) are satisfied, then the parameter matrix $\mathbb{Y}$ is obtained as in (2.78) and (2.79). □

2.7.1 Particular cases

•***Proportional observer***

Consider the following descriptor system :

$$\begin{aligned} Ex(t+1) &= Ax(t) + Bu(t) \\ y(t) &= C_1x(t) \end{aligned}$$

with the PO :

$$\begin{aligned} \zeta(t+1) &= N\zeta(t) + F_a y(t) + Ju(t) \\ \hat{x}(t) &= P\zeta(t) + Q_a y(t) \end{aligned}$$

and the error dynamics (2.75) becomes :

$$\begin{aligned} \varepsilon(t+1) &= (\bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2)\varepsilon(t) \\ e(t) &= \bar{\mathbb{P}}\varepsilon(t) \end{aligned}$$

where $\bar{\mathbb{A}}_1 = N_1 - Z_1N_2$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{P}} = P_1$ and $\bar{\mathbb{Y}} = Y_1$. Consequently, matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.3 become :

$$\mathcal{D} = \begin{bmatrix} -X & (N_1 - Z_1N_2)^TX \\ (*) & -X \end{bmatrix},\ \mathcal{C} = \begin{bmatrix} N_3 & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 \\ -I_{q_0} \end{bmatrix} \text{ and } \mathcal{X} = X\bar{\mathbb{Y}}.$$

Matrices Σ and Ω are defined as $\Sigma = \begin{bmatrix} R \\ C_1 \end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C_1 \end{bmatrix}$.

•***Proportional-integral observer***

Consider the following descriptor system :

$$\begin{aligned} Ex(t+1) &= Ax(t) + Bu(t) \\ y(t) &= C_1x(t) \end{aligned}$$

with the PIO :

$$\begin{aligned} \zeta(t+1) &= N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \\ v(t+1) &= y(t) - C_1\hat{x}(t) \\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \end{aligned}$$

and the error dynamics (2.75) becomes :

$$\varphi(t+1) = (\bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2)\varphi(t)$$
$$e(t) = \bar{\bar{\mathbb{P}}}\varphi(t)$$

where $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix} N_1 - Z_1N_2 & 0 \\ -C_1P_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\bar{\bar{\mathbb{P}}} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} Y_1 & H \end{bmatrix}$. Consequently matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.3 become :

$$\mathcal{D} = \left[\begin{array}{cc|cc} -X_1 & (*) & (*) & (*) \\ -X_1 & X_2 & 0 & 0 \\ \hline X_1(N_1 - Z_1N_2) - X_1C_1P_1 & 0 & -X_1 & (*) \\ X_1(N_1 - Z_1N_2) - X_2C_1P_1 & 0 & -X_1 & -X_2 \end{array}\right], \ \mathcal{C} = \begin{bmatrix} \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ -I_{q_0+q_1} \end{bmatrix},$$

and $\mathcal{X} = X\bar{\bar{\mathbb{Y}}}$, such that $\begin{bmatrix} Y_1 & H \end{bmatrix} = \left(X \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^+ \mathcal{X}$.

2.7.2 Numerical example

In order to illustrate the results obtained, consider the following descriptor system described by (2.73) where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0.7 & -0.4 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0.3 & 0 \\ -0.1 & 0.2 & 0 & 0.5 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

Considering matrix $E^{\perp} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, we can verify Assumptions 2.1 and 2.2

$$\operatorname{rank} \begin{bmatrix} E \\ \hline E^{\perp}A \\ C_1 \end{bmatrix} = 4 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 4$$

Generalized dynamic observer

For the GDO we have chosen matrix $R = \begin{bmatrix} 0.1 & 0 & 10 & 0.5 \\ 0 & 1 & 0 & 0.3 \end{bmatrix}$, such that $\operatorname{rank}(\Sigma) = 4$. By using YALMIP toolbox, we solve the LMIs (2.76) and (2.77) to find matrices X and Z_1

$$X = \begin{bmatrix} 968.64 & 1163.39 & 968.64 & 1163.39 \\ 1163.39 & 12875.59 & 1163.39 & 12875.59 \\ 968.64 & 1163.39 & 21675.79 & 1473.81 \\ 1163.39 & 12875.59 & 1473.81 & 34862.37 \end{bmatrix} \text{ and}$$

$$Z_1 = \begin{bmatrix} -23.25 & -16.84 & -16.69 & -23.39 & -22.05 & -37.78 & -24.12 \\ 0.38 & 0.03 & 0.02 & 0.39 & 0.31 & 1.35 & 0.43 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 8 & 2 & 9 & 7 & 3 & 9 & 2 \\ 9 & 3 & 7 & 5 & 9 & 7 & 3 \\ 9 & 2 & 8 & 5 & 8 & 6 & 1 \\ 9 & 3 & 8 & 5 & 1 & 5 & 1 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.0001$, and solving (2.78) and (2.79) we get :

$$\mathbb{Y} = \begin{bmatrix} 0.18 & 1.68 & 0.18 & -1.71 & -0.67 & -0.47 & -0.69 \\ 0.02 & 0.18 & 0.02 & -0.18 & -0.07 & -0.05 & -0.07 \\ 0 & 0 & 0 & 0 & 0 & -0.04 & -0.06 \\ -0.01 & -0.06 & -0.01 & 0.06 & 0.02 & -0.01 & -0.01 \end{bmatrix}$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -0.23 & -0.31 \\ -0.04 & 0.18 \end{bmatrix}, \ S = \begin{bmatrix} 0 & 0 \\ 0.01 & 0.06 \end{bmatrix}, \ H = \begin{bmatrix} -0.47 & -0.69 \\ -0.05 & -0.07 \end{bmatrix},$$

$$L = \begin{bmatrix} -0.04 & -0.06 \\ -0.01 & -0.01 \end{bmatrix}, \ F = \begin{bmatrix} 16.47 & 3.03 & 3.46 \\ -0.83 & 0.27 & 0.22 \end{bmatrix}, \ J = \begin{bmatrix} -28.99 \\ 1.46 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0.06 & -0.04 & -0.03 \end{bmatrix}, \ P = \begin{bmatrix} 0.04 & 2.35 \\ -0.05 & 0.5 \\ 0.1 & -0.04 \\ 0.02 & 0.18 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -9.28 & -0.47 & 4.64 \\ -0.29 & 0.24 & 0.14 \\ 0.72 & 0.45 & -0.36 \\ 0.17 & -0.12 & 0.91 \end{bmatrix}.$$

In order to provide a comparison of the GDO with the PIO and the PO, these last are also designed.

Proportional observer

By considering matrices $R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 8 & 2 & 9 & 7 \\ 9 & 3 & 7 & 5 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_2 \times 0.1$ the following PO matrices are obtained :

$$N = \begin{bmatrix} 0.51 & -0.11 \\ -0.07 & 0.28 \end{bmatrix}, \ F = \begin{bmatrix} 0.03 & 0.65 \\ 0.06 & 0.35 \end{bmatrix}, \ J = \begin{bmatrix} -0.16 \\ 0.05 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -0.41 & 0 \\ 0.31 & 0 \\ 0.69 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proportional-integral observer

By considering matrices $R = \begin{bmatrix} 0.1 & 0 & 10 & 0.5 \\ 0 & 1 & 0 & 0.3 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 8 & 2 & 9 & 7 & 3 & 9 & 9 \\ 9 & 3 & 7 & 5 & 9 & 2 & 3 \\ 9 & 2 & 8 & 5 & 3 & 6 & 1 \\ 9 & 3 & 8 & 0 & 1 & 5 & 1 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.0001$ the following PIO matrices are obtained :

$$N = \begin{bmatrix} 0.3 & 2.96 \\ -0.03 & -0.27 \end{bmatrix}, \ H = \begin{bmatrix} -1.31 & -1.86 \\ -0.06 & -0.09 \end{bmatrix}, \ J = \begin{bmatrix} -7.81 \\ 0.81 \end{bmatrix},$$

$$F = \begin{bmatrix} -7.55 & 1.22 & 3.78 \\ 0.82 & -0.12 & -0.41 \end{bmatrix}, \ P = \begin{bmatrix} 0.04 & 2.35 \\ -0.05 & 0.5 \\ 0.1 & -0.04 \\ 0.02 & 0.18 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -12.31 & 0.83 & 6.15 \\ -1.1 & 0.42 & 0.55 \\ 1.07 & 0.58 & -0.54 \\ -0.01 & 0 & 1.01 \end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0.1,\ 0,\ 0,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0,\ 0]^T$.

To evaluate the performance of the GDO an uncertainty $\wp(t)$ is added in the system matrix A, then we obtain the following matrix $(A + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The results of the simulation are depicted in Figures 2.20 - 2.29. Figures 2.20 and 2.21 show the input $u(t)$ and the uncertainty factor $\delta(t)$. Figures 2.22 - 2.29 show the system states and their estimations by the GDO, PO and PIO, also these figures show the estimation error for each observer.

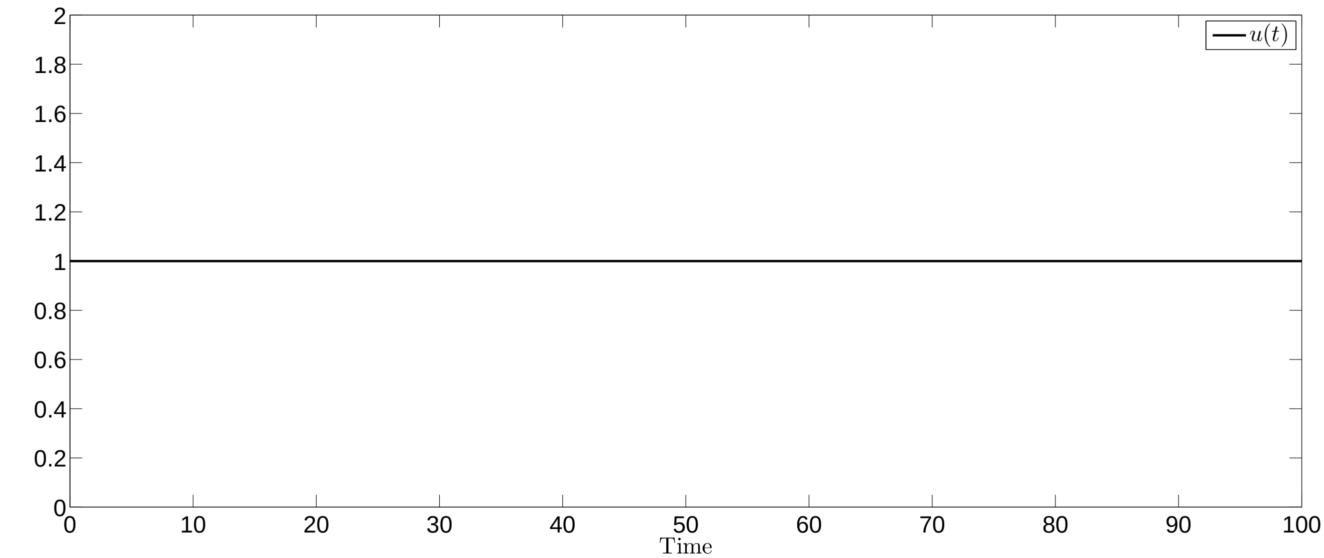

FIGURE 2.20 – Discrete-time observers : Input $u(t)$.

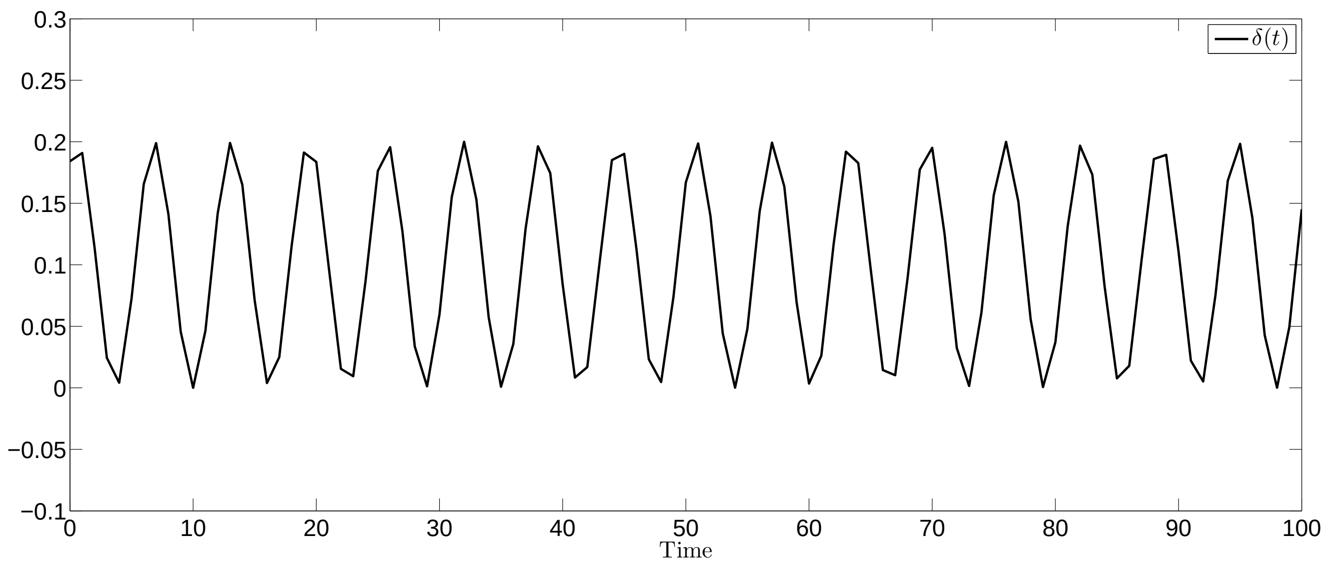

FIGURE 2.21 – Discrete-time observers : Uncertainty factor $\delta(t)$.

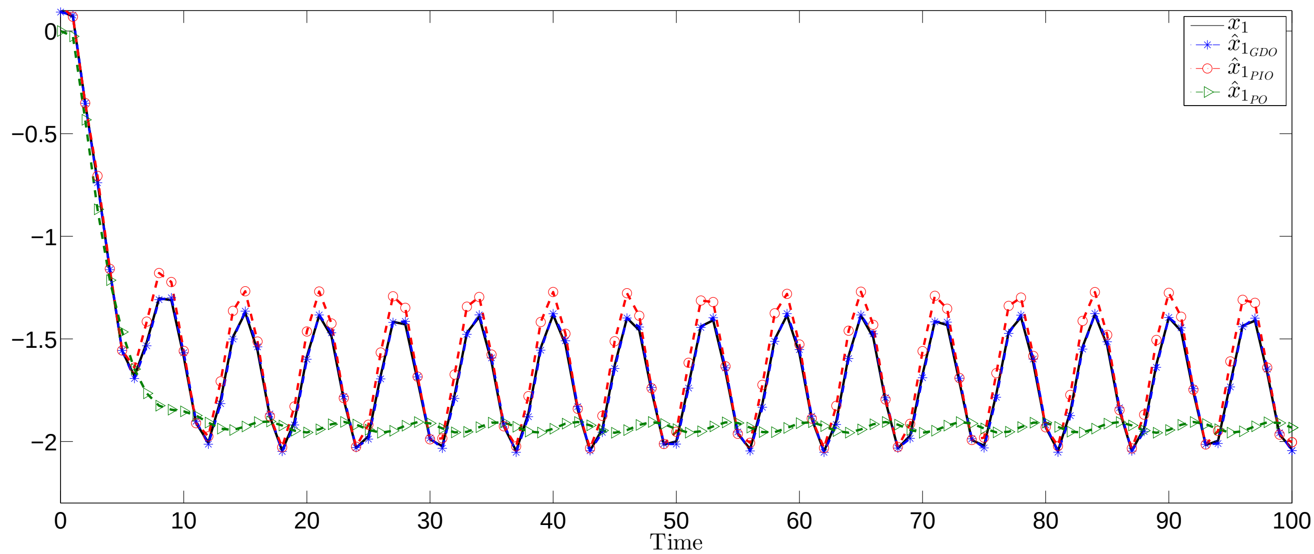

FIGURE 2.22 – Discrete-time observers : Estimate of $x_1(t)$

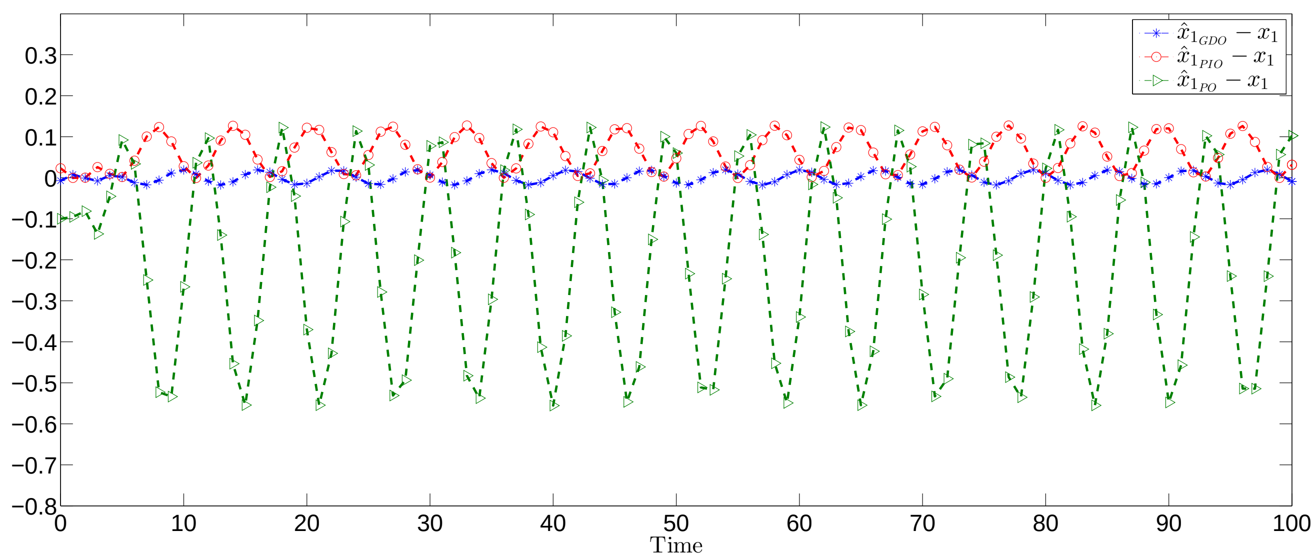

FIGURE 2.23 – Discrete-time observers : Estimation error of $x_1(t)$.

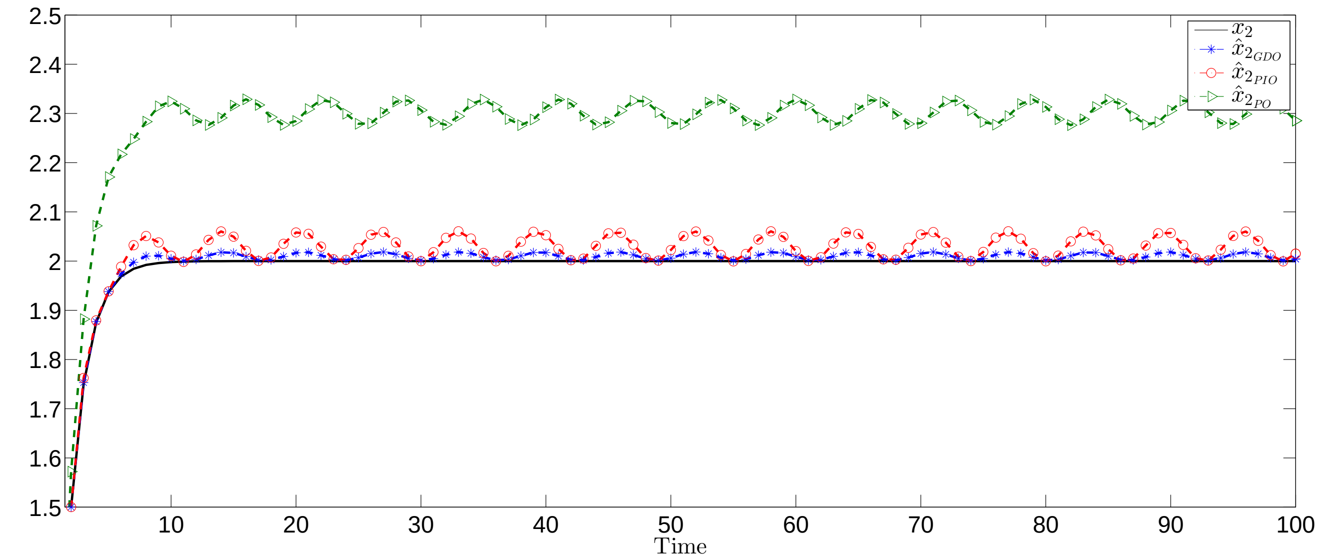

FIGURE 2.24 – Discrete-time observers : Estimate of $x_2(t)$.

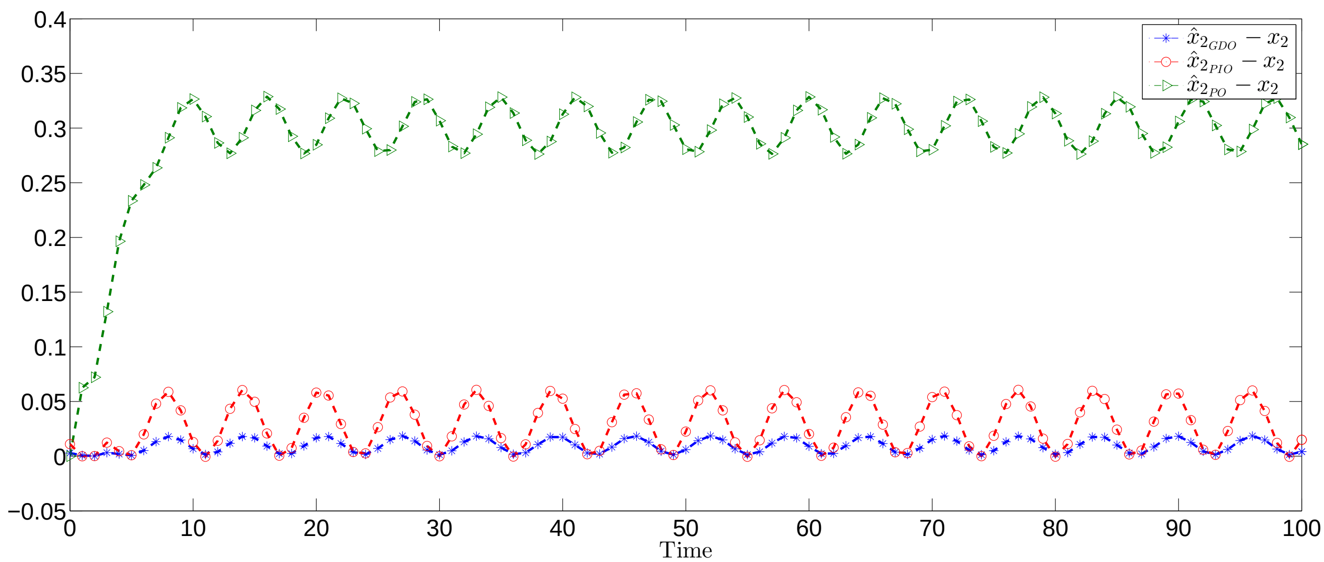

FIGURE 2.25 – Discrete-time observers : Estimation error of $x_2(t)$.

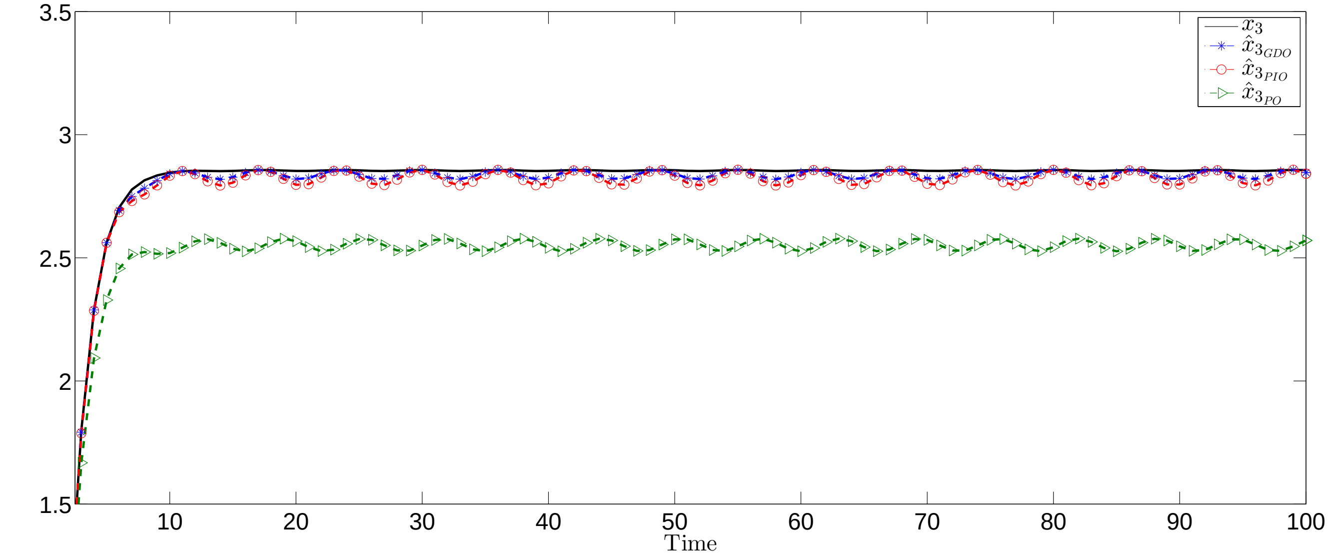

FIGURE 2.26 – Discrete-time observers : Estimate of $x_3(t)$.

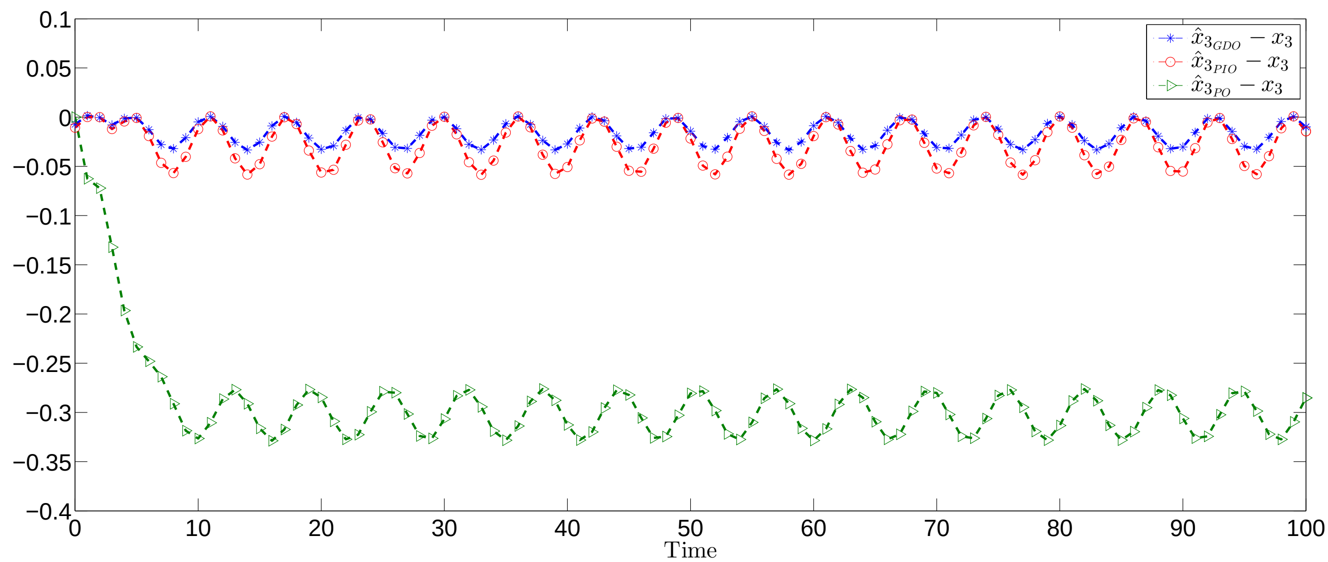

FIGURE 2.27 – Discrete-time observers : Estimation error of $x_3(t)$.

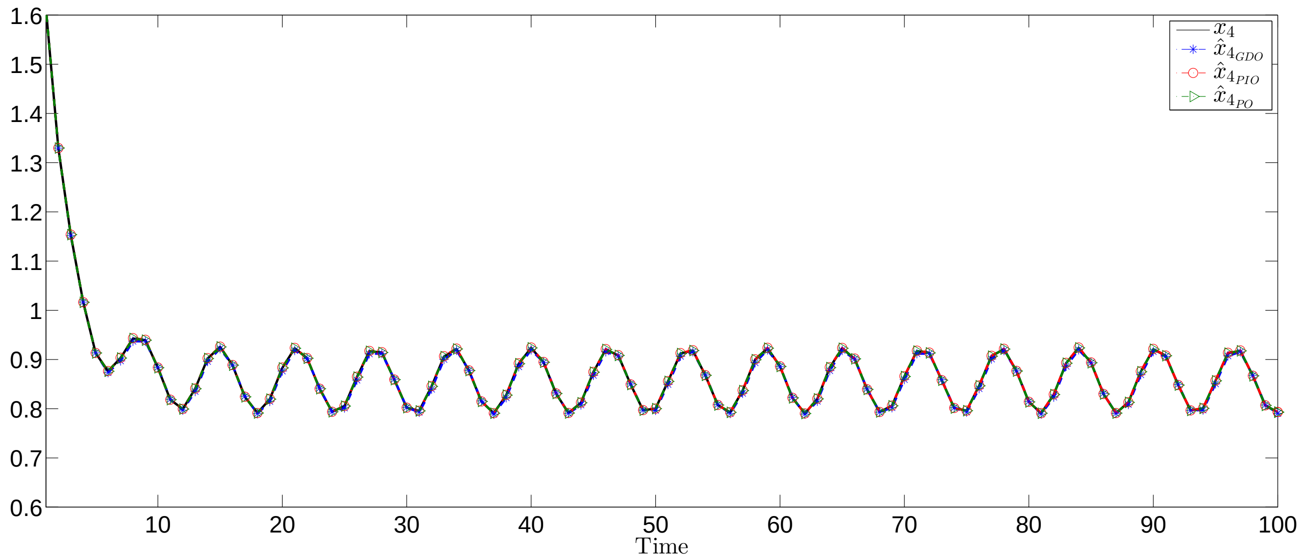

FIGURE 2.28 – Discrete-time observers : Estimate of $x_4(t)$.

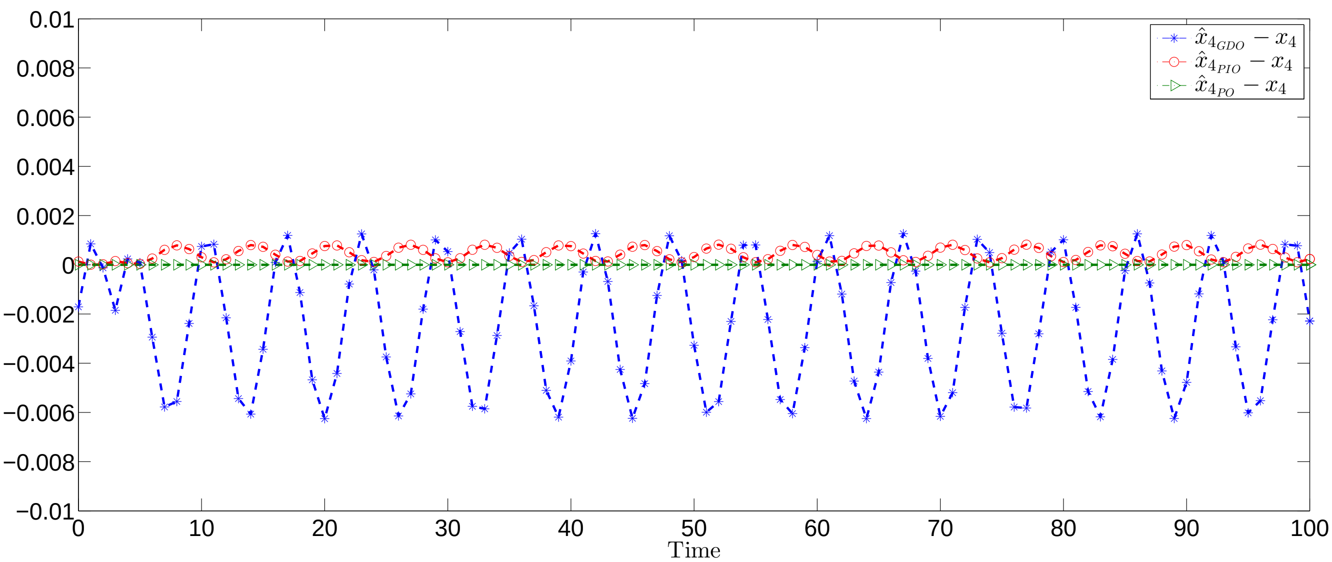

FIGURE 2.29 – Discrete-time observers : Estimation error of $x_4(t)$.

From these results, we can see that, in most of the cases the GDO presents better performance in presence of parameter uncertainties compared to the PIO and the PO. In order to make clear this difference the following table shows the error evaluation IAE.

TABLE 2.3 – Discrete-time observer : Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	1.15	6.07	25.35
$x_2(t)$	0.94	2.87	29.35
$x_3(t)$	1.57	2.77	29.35
$x_4(t)$	0.28	0.04	0

From these error evaluations, it is clearly that the GDO presents certain robustness in face to parameter uncertainties, even for the estimation of $x_4(t)$ the GDO has an acceptable performance.

2.8 H_∞ generalized dynamic observer design for discrete-time descriptor systems, $w(t) \neq 0$

In this section we consider $w(t) \neq 0$, then we get system (2.1) in its discrete-time form

$$\begin{aligned} Ex(t+1) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1 x(t) + D_1 w(t) \end{aligned} \tag{2.88}$$

with the GDO :

$$\zeta(t+1) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{2.89a}$$

$$v(t+1) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{2.89b}$$

$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \tag{2.89c}$$

and the observer error dynamics (2.62) in its discrete-time form as :

$$\begin{aligned} \varphi(t+1) &= (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) + (\mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2)w(t) \\ e(t) &= \mathbb{P}\varphi(t) + \mathbb{Q}w(t) \end{aligned} \tag{2.90}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{B}_1 = \begin{bmatrix} \mathcal{F}_{d_1} - T_1 D_1 + Z(\mathcal{T}_2 D_1 - \mathcal{F}_{d_2}) \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} P_1 - Y_3 N_3 & 0 \end{bmatrix}$, $\mathbb{Q} = \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

In this section we present a method for designing an H_∞ GDO given by (2.89). This design is obtained from the determination of the parameter matrices Y_3, Z and $\mathbb{Y}$ such that the estimation error energy $\|e\|_2$ is minimum for all bounded energy disturbance $w(t)$, this is equivalent to find those parameter matrices such that $\sup_{w\in\mathcal{L}_2-\{0\}} \dfrac{\|e\|_2}{\|w\|_2}$. This problem is equivalent to $\|G_{we}\|_\infty < \gamma$, where G_{we} is the transfer function from the disturbance to the estimation error, and γ is a given positive scalar. The solution to this problem is given by the following theorem.

Theorem 2.4. *Under Assumptions 2.1 and 2.2, there exists an H_∞ GDO (2.89) such that the error dynamics in (2.90) is stable and $\|T_{we}\|_\infty < \gamma$, if there exists a matrix $X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$ with $X_1 = X_1^T > 0$ satisfying the following LMIs.*

$$\mathcal{C}^{T\perp} \left[\begin{array}{cc|c|cc|c} -X_1 & (*) & 0 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I_{n_w} & (*) & (*) & (*) \\ \hline X_1N_1 - W_1\mathcal{N}_2 & 0 & \Pi_1 & -X_1 & (*) & 0 \\ X_1N_1 - W_1\mathcal{N}_2 & 0 & \Pi_1 & -X_1 & -X_2 & 0 \\ \hline P_1 - Y_3N_3 & 0 & \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3} & 0 & 0 & -I_n \end{array}\right] \mathcal{C}^{T\perp T} < 0 \tag{2.91}$$

where

$$\Pi_1 = X_1(\mathcal{F}_{d_1} - T_1D_1) - W_1(\mathcal{F}_{d_2} - \mathcal{T}_2D_1) \tag{2.92}$$

and

$$\left[\begin{array}{cc|c|c} -X_1 & (*) & 0 & (*) \\ -X_1 & -X_2 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I_{n_w} & (*) \\ \hline P_1 - Y_3N_3 & 0 & \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3} & -I_n \end{array}\right] < 0 \tag{2.93}$$

In this case matrix $W_1 = X_1 Z$ and matrix $\mathbb{Y}$ is parameterized as follows

$$\mathbb{Y} = X^{-1}(\mathcal{B}_r^+ \mathcal{K}\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+) \tag{2.94}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \tag{2.95a}$$

$$\vartheta = (\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{2.95b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\vartheta - \vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta]\mathcal{B}_l\mathcal{R}^{-1} \tag{2.95c}$$

where $\mathcal{D} = \left[\begin{array}{cc|c|cc|c} -X_1 & (*) & 0 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I_{n_w} & (*) & (*) & (*) \\ \hline X_1N_1 - W_1\mathcal{N}_2 & 0 & \Pi_1 & -X_1 & (*) & 0 \\ X_1N_1 - W_1\mathcal{N}_2 & 0 & \Pi_1 & -X_1 & -X_2 & 0 \\ \hline P_1 - Y_3N_3 & 0 & \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3} & 0 & 0 & -I_n \end{array}\right]$, $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I_{q_0+q_1} \\ 0 \end{bmatrix}$,

$\mathcal{C} = \begin{bmatrix} \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} & \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$, *and matrix Π_1 is defined in equation (2.92), and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$.*

Proof. The discrete-time BRL of Lemma 1.3 guarantees that the observer error system (2.90) is stable and $\|T_{we}\|_\infty < \gamma$, if and only if there exists a matrix $X = X^T > 0$ such that

$$\begin{bmatrix} \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 & \mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2 \\ \mathbb{P} & \mathbb{Q} \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I_{n_w} \end{bmatrix} \begin{bmatrix} \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 & \mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2 \\ \mathbb{P} & \mathbb{Q} \end{bmatrix} < \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I_{n_w} \end{bmatrix} \tag{2.96}$$

By using the Schur complement of Lemma 1.4 in the inequality (2.96) we get :

$$\begin{bmatrix} -X & 0 & (*) & (*) \\ 0 & -\gamma^2 I_{n_w} & (*) & (*) \\ \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 & \mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2 & -X^{-1} & 0 \\ \mathbb{P} & \mathbb{Q} & 0 & -I_n \end{bmatrix} < 0. \tag{2.97}$$

Now, premultiplying and postmultiplying the inequality (2.97) by $\begin{bmatrix} I_{q_0+q_1} & 0 & 0 & 0 \\ 0 & I_{n_w} & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}$ we obtain the following inequality

$$\begin{bmatrix} -X & 0 & (*) & (*) \\ 0 & -\gamma^2 I_{n_w} & (*) & (*) \\ X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) & X(\mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2) & -X & 0 \\ \mathbb{P} & \mathbb{Q} & 0 & -I_n \end{bmatrix} < 0. \tag{2.98}$$

Now, inequality (2.98) can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{2.99}$$

where $\mathcal{D} = \begin{bmatrix} -X & 0 & (*) & (*) \\ 0 & -\gamma^2 I_{n_w} & (*) & (*) \\ X\mathbb{A}_1 & X\mathbb{B}_1 & -X & 0 \\ \mathbb{P} & \mathbb{Q} & 0 & -I_n \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & \mathbb{B}_2 & 0 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I_{q_0+q_1} \\ 0 \end{bmatrix}$ and $\mathcal{X} = X\mathbb{Y}$. From the elimination lemma, the solvability conditions of inequality (2.99) are :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{2.100a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{2.100b}$$

with $\mathcal{B}^{\perp} = \begin{bmatrix} I_{q_0+q_1} & 0 & 0 & 0 \\ 0 & I_{n_w} & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}$ and $\mathcal{C}^{T\perp} = \begin{bmatrix} \begin{bmatrix} \mathbb{A}_2^T \\ \mathbb{B}_2^T \end{bmatrix}^{\perp} & 0 & 0 \\ 0 & I_{q_0+q_1} & 0 \\ 0 & 0 & I_n \end{bmatrix}$.

By using matrices $\mathcal{C}$, $\mathcal{D}$ and W_1 inequality (2.100a) becomes :

$$\mathcal{C}^{T\perp} \left[\begin{array}{cc|c|cc|c} -X_1 & (*) & 0 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I_{n_w} & (*) & (*) & (*) \\ \hline X_1N_1 - W_1\mathcal{N}_2 & 0 & \Pi_1 & -X_1 & (*) & 0 \\ X_1N_1 - W_1\mathcal{N}_2 & 0 & \Pi_1 & -X_1 & -X_2 & 0 \\ \hline P_1 - Y_3N_3 & 0 & \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3} & 0 & 0 & -I_n \end{array}\right] \mathcal{C}^{T\perp T} < 0 \tag{2.101}$$

where matrix Π_1 is defined in equation (2.92), and by using matrices $\mathcal{B}$ and $\mathcal{D}$ inequality (2.100b) becomes :

$$\left[\begin{array}{cc|c|c} -X_1 & (*) & 0 & (*) \\ -X_1 & -X_2 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I_{n_w} & (*) \\ \hline P_1 - Y_3N_3 & 0 & \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3} & -I_n \end{array}\right] < 0. \tag{2.102}$$

From the elimination lemma if conditions (2.100a) and (2.100b) are satisfied, then the parameter matrix $\mathbb{Y}$ is parameterized as in (2.94) and (2.95). □

2.8.1 Particular cases

•***Proportional observer***

Consider the following descriptor system :

$$\begin{aligned} Ex(t+1) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1x(t) + D_1w(t) \end{aligned}$$

with the PO :

$$\begin{aligned} \zeta(t+1) &= N\zeta(t) + F_a y(t) + Ju(t) \\ \hat{x}(t) &= P\zeta(t) + Q_a y(t) \end{aligned}$$

and the error dynamics (2.90) becomes :

$$\begin{aligned} \varepsilon(t+1) &= (\bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2)\varepsilon(t) + (\bar{\mathbb{B}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{B}}_2)\varphi(t) \\ e(t) &= \bar{\mathbb{P}}\varepsilon(t) + \mathbb{Q}w(t) \end{aligned}$$

where $\bar{\mathbb{A}}_1 = N_1 - Z\mathcal{N}_2$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{B}}_1 = \mathcal{F}_{d_1} - T_1 D_1 + Z(\mathcal{T}_2 D_1 - \mathcal{F}_{d_2})$, $\bar{\mathbb{B}}_2 = \mathcal{F}_{d_3}$, $\bar{\mathbb{P}} = P_1 - Y_3 N_3$, $\bar{\mathbb{Q}} = \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3}$ and $\bar{\mathbb{Y}} = Y_1$.

Consequently, matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.4 become :

$$\mathcal{D} = \begin{bmatrix} -X & 0 & (*) & (*) \\ 0 & -\gamma^2 I_{n_w} & (*) & (*) \\ X(N_1 - Z\mathcal{N}_2) & X[\mathcal{F}_{d_1} - T_1 D_1 + Z(\mathcal{T}_2 D_1 - \mathcal{F}_{d_2})] & -X & 0 \\ P_1 - Y_3 N_3 & \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3} & 0 & -I_n \end{bmatrix},$$

$\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I_{q_0} \\ 0 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} N_3 & \mathcal{F}_{d_3} & 0 & 0 \end{bmatrix}$ and $\mathcal{X} = X\bar{\mathbb{Y}}$. Matrices Σ and Ω are defines as $\Sigma = \begin{bmatrix} R \\ C_1 \end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C_1 \end{bmatrix}$.

•*Proportional-integral observer*

Consider the following descriptor system :

$$\begin{aligned} Ex(t+1) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1 x(t) + D_1 w(t) \end{aligned}$$

with the PIO :

$$\begin{aligned} \zeta(t+1) &= N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \\ v(t+1) &= y(t) - C_1\hat{x}(t) \\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \end{aligned}$$

and the error dynamics (2.90) becomes :

$$\begin{aligned} \varphi(t+1) &= (\bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2)\varphi(t) + (\bar{\mathbb{B}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{B}}_2)\varphi(t) \\ e(t) &= \bar{\bar{\mathbb{P}}}\varphi(t) + \bar{\bar{\mathbb{Q}}}w(t) \end{aligned}$$

$Y_3 = 0$ is taken for simplicity and matrices $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ -C_1 P_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_1 = \begin{bmatrix} \mathcal{F}_{d_1} - T_1 D_1 - Z(\mathcal{F}_{d_2} - \mathcal{T}_2 D_1) \\ D_1 - C_1\mathcal{Q}_{d_1} \end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_2 = \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix}$, $\bar{\bar{\mathbb{P}}} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{Q}}} = \mathcal{Q}_{d_1}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix} I \\ 0 \end{bmatrix}\begin{bmatrix} Y_1 & H \end{bmatrix}$.
Consequently matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 2.4 become :

$$\mathcal{D} = \left[\begin{array}{cc|c|cc|c} -X_1 & (*) & 0 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I_{n_w} & (*) & (*) & (*) \\ \hline X_1(N_1 - C_1 P_1 - Z\mathcal{N}_2) & 0 & \Pi_1 & -X_1 & (*) & 0 \\ X_1(N_1 - Z\mathcal{N}_2) - X_2 C_1 P_1 & 0 & \Pi_2 & -X_1 & -X_2 & 0 \\ \hline P_1 - Y_3 N_3 & 0 & \mathcal{Q}_{d_1} - Y_3\mathcal{F}_{d_3} & 0 & 0 & -I_n \end{array}\right],$$

with

$$\Pi_1 = X_1[\mathcal{F}_{d_1} - T_1D_1 - Z(\mathcal{F}_{d_2} - \mathcal{T}_2D_1) + D_2 - C_1\mathcal{Q}_{d_1}]$$
$$\Pi_2 = X_1[\mathcal{F}_{d_1} - T_1D_1 - Z(\mathcal{F}_{d_2} - \mathcal{T}_2D_1)] + X_2(D_2 - C_1\mathcal{Q}_{d_1})$$

$$\mathcal{C} = \begin{bmatrix}\begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1}\end{bmatrix} & \begin{bmatrix}\mathcal{F}_{d_3} \\ 0\end{bmatrix} & 0 & 0\end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I_{q_0+q_1} \\ 0\end{bmatrix} \text{ and } \mathcal{X} = X\bar{\bar{\mathbb{Y}}}, \text{ such that } \begin{bmatrix} Y_1 & H\end{bmatrix} = \left(X\begin{bmatrix} I \\ 0\end{bmatrix}\right)^+ \mathcal{X}.$$

2.8.2 Numerical example

In order to illustrate the results obtained, consider the following descriptor system described by (2.88) where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix},\ A = \begin{bmatrix} 0.7 & 1 & 0 \\ 0 & 0.5 & 0 \\ 0 & 1 & 0.3\end{bmatrix},\ B = \begin{bmatrix} 1 \\ -1 \\ 1\end{bmatrix},\ D = \begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0\end{bmatrix},\ D_1 = \begin{bmatrix} 0.1 \\ 0 \\ 0\end{bmatrix} \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t)\end{bmatrix}.$$

Considering $E^\perp = \begin{bmatrix} 0 & 0 & 1\end{bmatrix}$, we can verify Assumptions 2.1 and 2.2

$$\mathsf{rank}\begin{bmatrix} E \\ \hline E^\perp A \\ C_1\end{bmatrix} = 3 \quad \text{and} \quad \mathsf{rank}\begin{bmatrix} sE - A \\ C_1\end{bmatrix} = 3$$

H_∞ Generalized dynamic observer

For the H_∞ GDO we have chosen matrix $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1\end{bmatrix}$, then $\mathsf{rank}(\Sigma) = 3$. By fixing the value of $\gamma = 1.73$ and using YALMIP toolbox, we solve LMIs (2.91) and (2.93) to find matrices X, Z and Y_3

$$X = \begin{bmatrix} 2.84 & 0 & 0 & 2.84 & 0 & 0 \\ 0 & 2.43 & 0 & 0 & 2.43 & 0 \\ 0 & 0 & 2.43 & 0 & 0 & 2.43 \\ 2.84 & 0 & 0 & 7.35 & 0 & 0 \\ 0 & 2.43 & 0 & 0 & 6.95 & 0 \\ 0 & 0 & 2.43 & 0 & 0 & 6.95\end{bmatrix},$$

$$Z = \begin{bmatrix} -1.58 & -1.58 & -1.58 & -1.58 & -1.58 & -1.58 & -1.58 \\ -1.86 & -1.86 & -1.86 & -1.86 & -1.86 & -1.86 & -1.86 \\ -1.86 & -1.86 & -1.86 & -1.86 & -1.86 & -1.86 & -1.86\end{bmatrix} \text{ and}$$

$$Y_3 = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.29 & -0.12 & 0.22 & 0.22 & -0.17 & -0.34 \\ 0.1 & -0.12 & 0.37 & 0.06 & -0.28 & -0.11 & 0.17\end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 & 9 & 6 & 3 & 9 \\ 3 & 2 & 6 & 3 & 3 & 5 & 4 & 6 & 3 & 9 \\ 6 & 1 & 4 & 2 & 6 & 2 & 6 & 6 & 1 & 0 \\ 9 & 5 & 2 & 7 & 4 & 3 & 2 & 8 & 1 & 6 \\ 2 & 6 & 3 & 8 & 1 & 8 & 4 & 7 & 3 & 9 \\ 8 & 2 & 7 & 1 & 9 & 3 & 6 & 2 & 8 & 2 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_6 \times 0.1$ and solving (2.94) and (2.95) we get :

$$\mathbb{Y} = \begin{bmatrix} -0.85 & 2.57 & 1.69 & 1.12 & 3.99 & 1.96 & 3.76 & 1.14 & -0.03 & -0.03 \\ 1.46 & 0.75 & 2.01 & -0.41 & 2.62 & 1.74 & 1.03 & -0.06 & -0.16 & -0.16 \\ 2.03 & 1.35 & 0.94 & -0.19 & 2.02 & 0.81 & 2.34 & -0.06 & -0.16 & -0.16 \\ 1.55 & -0.26 & -0.35 & -0.09 & -1.11 & -0.08 & -0.89 & -0.21 & -0.07 & -0.07 \\ -0.22 & 0.75 & -0.33 & 0.87 & -0.75 & 0.71 & 0.32 & -0.02 & -0.05 & -0.05 \\ 0.44 & -0.13 & 0.56 & 0.19 & 0.68 & 0.18 & -0.01 & -0.02 & -0.05 & -0.05 \end{bmatrix}.$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} 0.7 & -0.08 & -0.01 \\ 0 & -0.04 & -0.11 \\ 0 & -0.05 & -0.10 \end{bmatrix}, \; S = \begin{bmatrix} 0 & 0.07 & -0.07 \\ 0 & 0.03 & -0.04 \\ 0 & 0.02 & -0.03 \end{bmatrix}, \; H = \begin{bmatrix} 1.14 & -0.03 & -0.03 \\ -0.06 & -0.16 & -0.16 \\ -0.06 & -0.16 & -0.16 \end{bmatrix},$$

$$J = \begin{bmatrix} 2.25 \\ 1.15 \\ 1.66 \end{bmatrix}, \; L = \begin{bmatrix} -0.21 & -0.07 & -0.07 \\ -0.02 & -0.05 & -0.05 \\ -0.02 & -0.05 & -0.05 \end{bmatrix}, \; F = \begin{bmatrix} 1.82 & -1.91 & 1.84 & -1.09 \\ 1.7 & -0.88 & 0.93 & -0.37 \\ 1.73 & -1.35 & 1.39 & -1.11 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; M = \begin{bmatrix} -0.21 & 0.62 & -0.55 & 0.73 \\ -0.19 & 0.36 & -0.30 & 0.48 \\ -0.14 & -0.19 & 0.23 & -0.09 \end{bmatrix} \text{ and}$$

$$Q = \begin{bmatrix} -0.06 & 0.63 & -0.61 & 0.34 \\ 0.03 & -0.34 & 0.33 & 0.63 \\ -0.06 & 0.66 & 0.36 & -0.29 \end{bmatrix}.$$

In order to provide a comparison of the GDO with the PIO and the PO, these last are also designed.

H_∞ Proportional observer

By considering matrices $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 \\ 3 & 2 & 6 & 3 & 3 & 5 \\ 6 & 1 & 4 & 2 & 6 & 2 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \end{bmatrix}$, $\mathcal{R} = I_3 \times 0.1$ and $\gamma = 1.73$ the following PO matrices are obtained :

$$N = \begin{bmatrix} 0.7 & 0.18 & -0.01 \\ 0 & 0.15 & -0.02 \\ 0 & 0.1 & -0.01 \end{bmatrix}, \; F_a = \begin{bmatrix} 6.83 & -6.56 & 8.37 \\ 3.98 & -3.75 & 4.65 \\ 4.02 & -3.79 & 4.41 \end{bmatrix}, \; J = \begin{bmatrix} 1.88 \\ 0.36 \\ 0.96 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & -0.21 & -0.23 \\ 0 & -0.01 & -0.2 \\ 0 & -0.19 & -0.03 \end{bmatrix} \text{ and } Q_a = \begin{bmatrix} -2.73 & 2.73 & -2.69 \\ -2.68 & 2.68 & -1.72 \\ -2.04 & 3.04 & -2.97 \end{bmatrix}.$$

H_∞ Proportional-integral observer

By considering matrices $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 & 9 & 6 & 3 & 9 \\ 3 & 2 & 6 & 3 & 3 & 5 & 4 & 6 & 3 & 9 \\ 6 & 1 & 4 & 2 & 6 & 2 & 6 & 6 & 1 & 0 \\ 9 & 5 & 2 & 7 & 4 & 3 & 2 & 8 & 1 & 6 \\ 2 & 6 & 3 & 8 & 1 & 8 & 4 & 7 & 3 & 9 \\ 8 & 2 & 7 & 1 & 9 & 3 & 6 & 2 & 8 & 2 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$, $\mathcal{R} = I_6 \times 10$ and $\gamma = 1.73$ the following PIO matrices are obtained :

$$N = \begin{bmatrix} 0.34 & 0.62 & 0.36 \\ -0.01 & 0.29 & 0.01 \\ 0.34 & 0.63 & 0.36 \end{bmatrix}, \; H = \begin{bmatrix} 0 & -0.04 & 0 \\ -0.02 & -0.04 & -0.01 \\ 0 & -0.04 & 0 \end{bmatrix}, \; P = \begin{bmatrix} 0.5 & -0.3 & 0.5 \\ 0 & -0.02 & 0 \\ 0 & -0.07 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 1.89 \\ 0.42 \\ 1.89 \end{bmatrix}, \; F = \begin{bmatrix} 0.7 & -0.7 & 0.76 & 0.11 \\ 0.38 & 0.05 & 0.12 & 0.54 \\ 0.76 & -1.26 & 1.29 & -0.49 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0.31 & -1.82 & 1.73 & -2.02 \\ 0.25 & -2.44 & 2.37 & -1.61 \\ 0.28 & -1.53 & 2.44 & -2.69 \end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0.1,\ 0,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0]^T$.

To evaluate the performance of the observers an uncertainty $\wp(t)$ is added in the system matrix A, then we obtain the following matrix $(A + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

The results of simulation are depicted in Figures 2.30 - 2.38. The input $u(t)$, the disturbance $w(t)$ and the uncertainty factor $\delta(t)$ are shown in Figures 2.30, 2.31 and 2.32, respectively. On Figures 2.33 - 2.38 the semi-states and their estimations by the GDO, the PIO and the PO are shown, with their respective errors estimations.

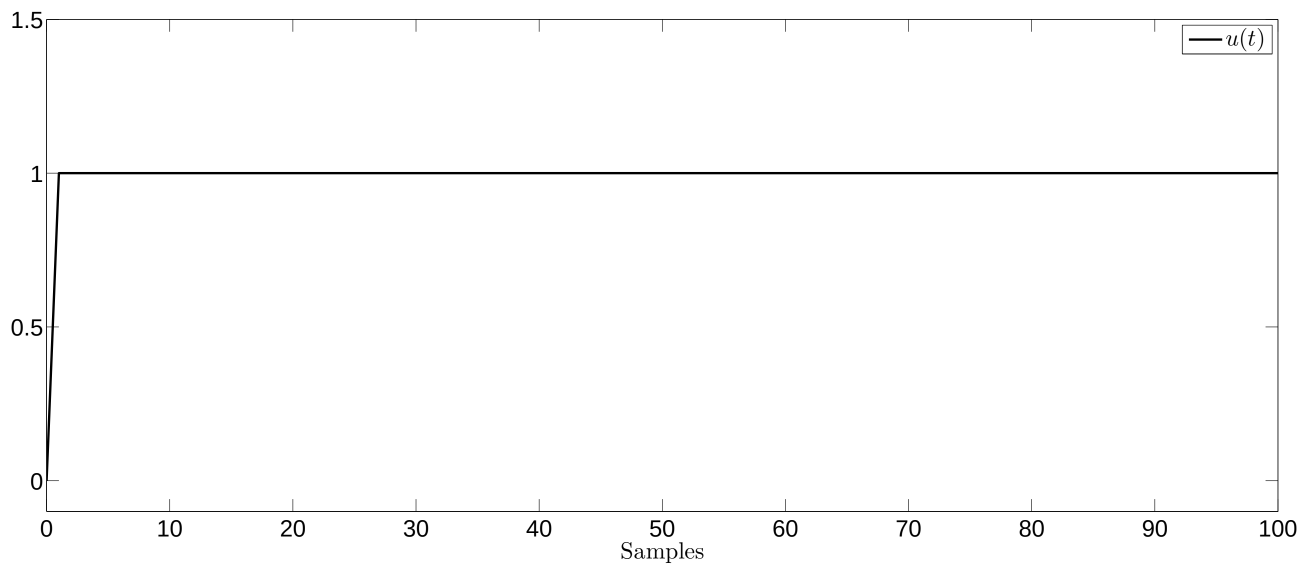

FIGURE 2.30 – H_∞ discrete-time observers : Input $u(t)$.

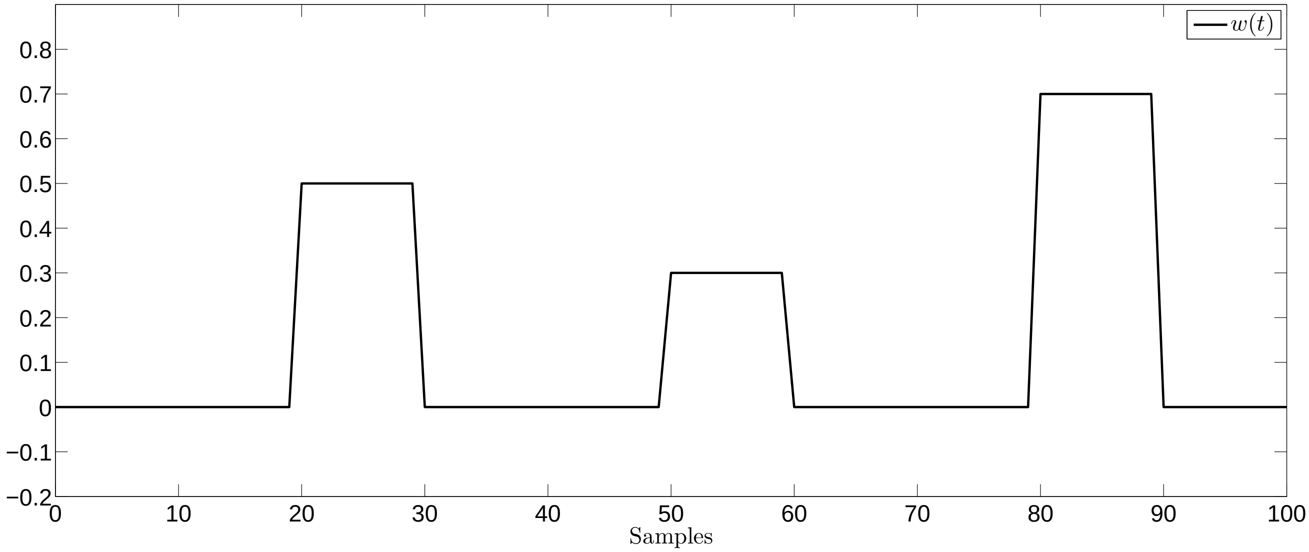

FIGURE 2.31 – H_∞ discrete-time observers : Disturbance $w(t)$.

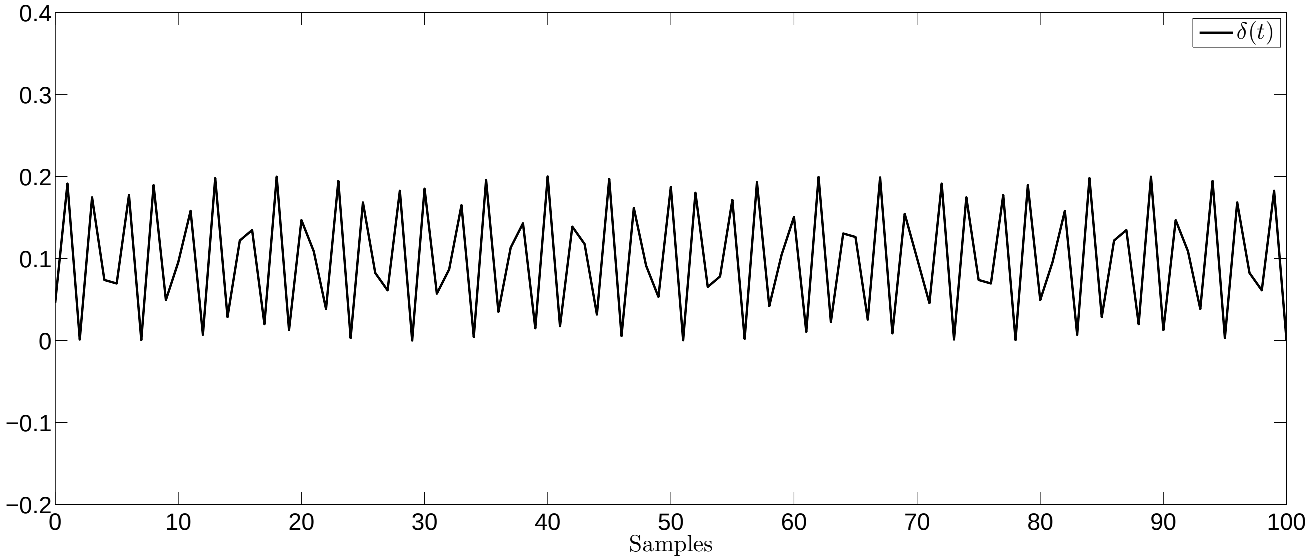

FIGURE 2.32 – H_∞ discrete-time observers : Uncertainty factor $\delta(t)$.

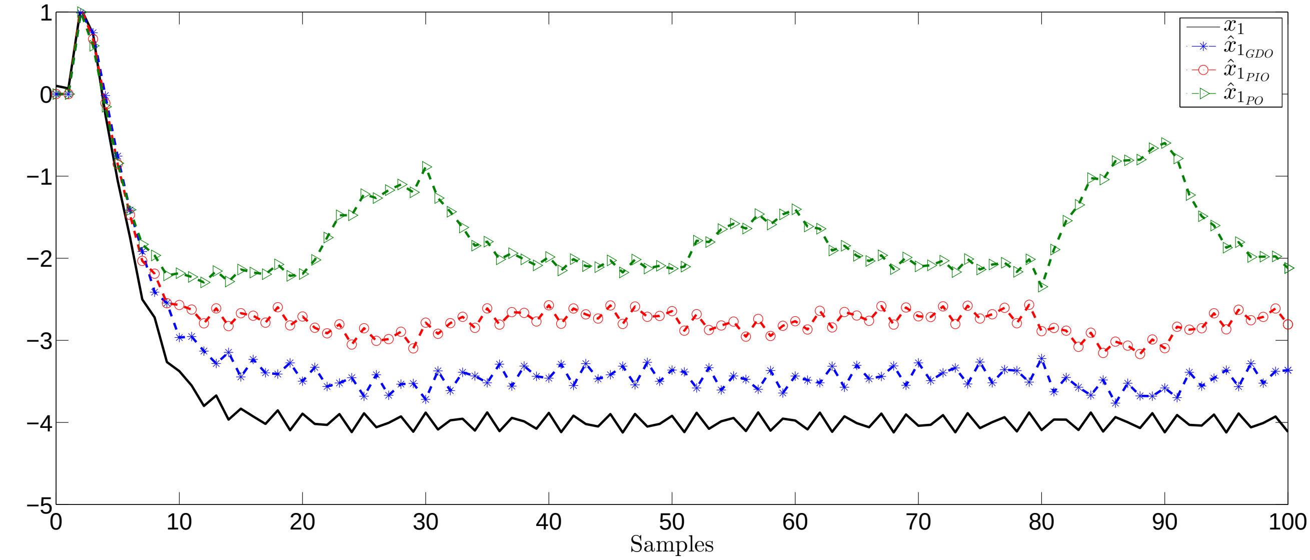

FIGURE 2.33 – H_∞ discrete-time observers : Estimate of $x_1(t)$.

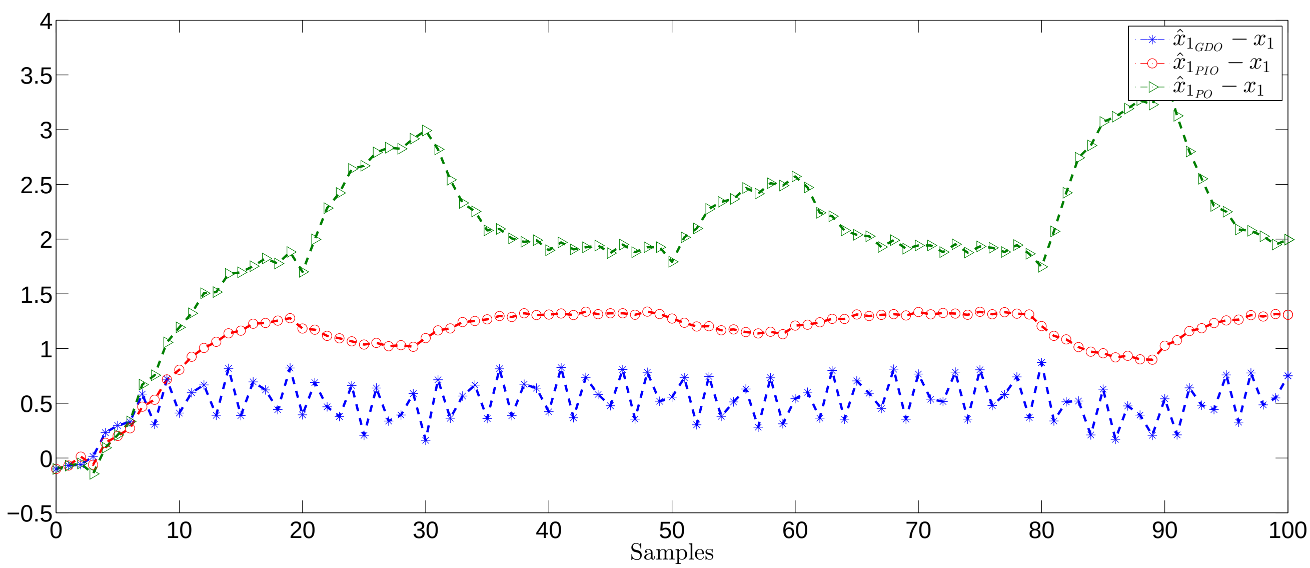

FIGURE 2.34 – H_∞ discrete-time observers : Estimation error of $x_1(t)$.

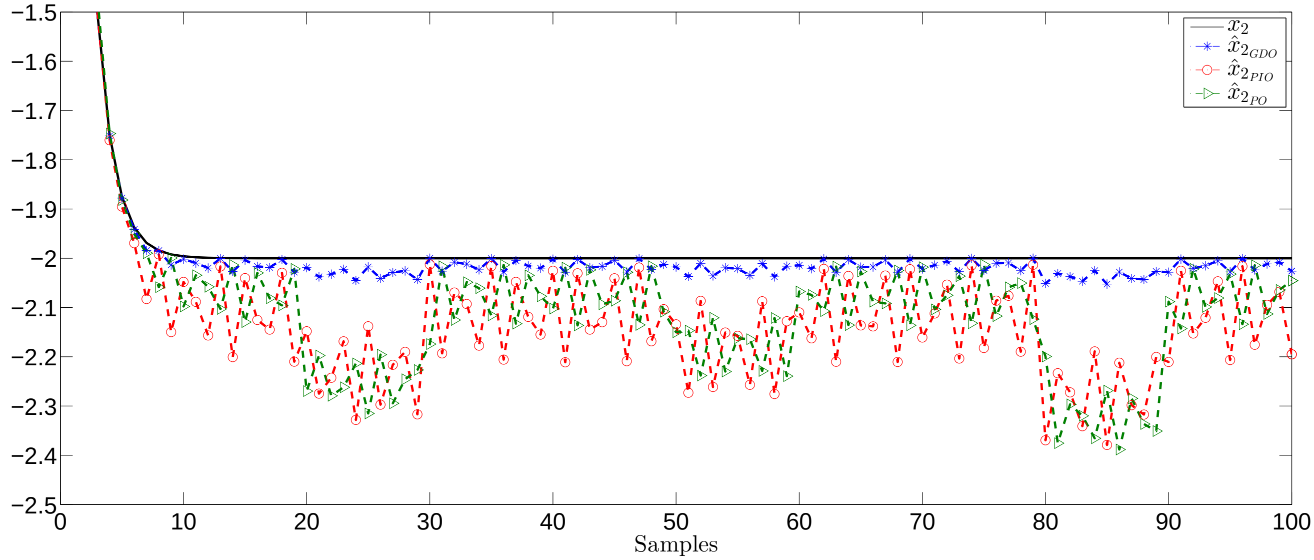

FIGURE 2.35 – H_∞ discrete-time observers : Estimate of $x_2(t)$.

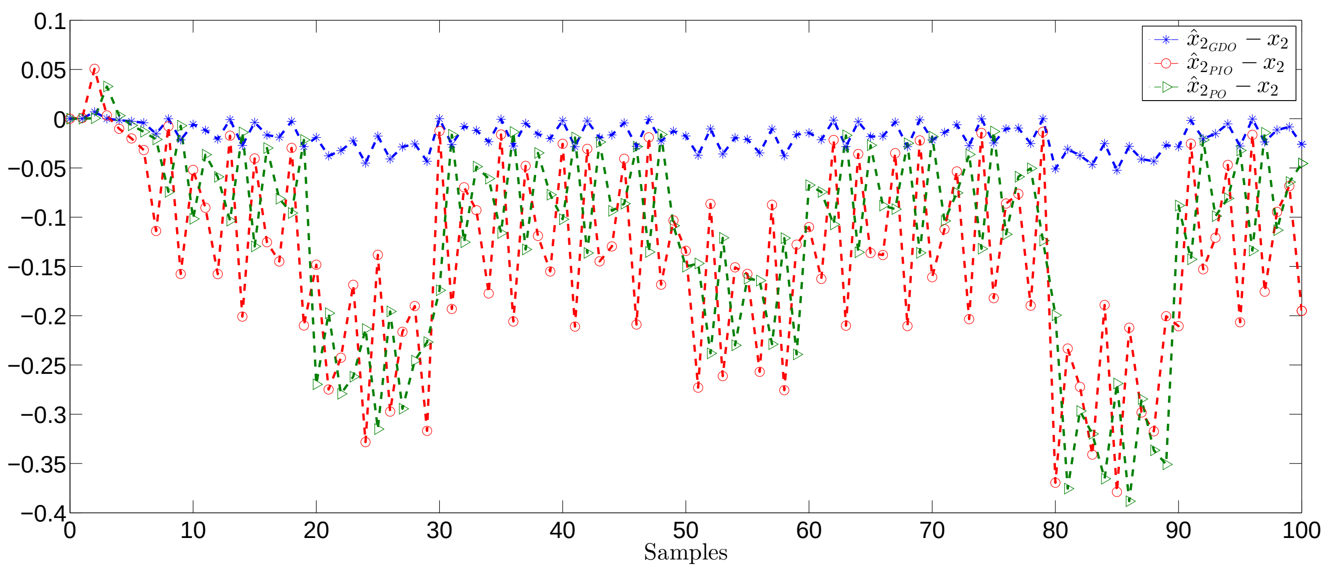

FIGURE 2.36 – H_∞ discrete-time observers : Estimation error of $x_2(t)$.

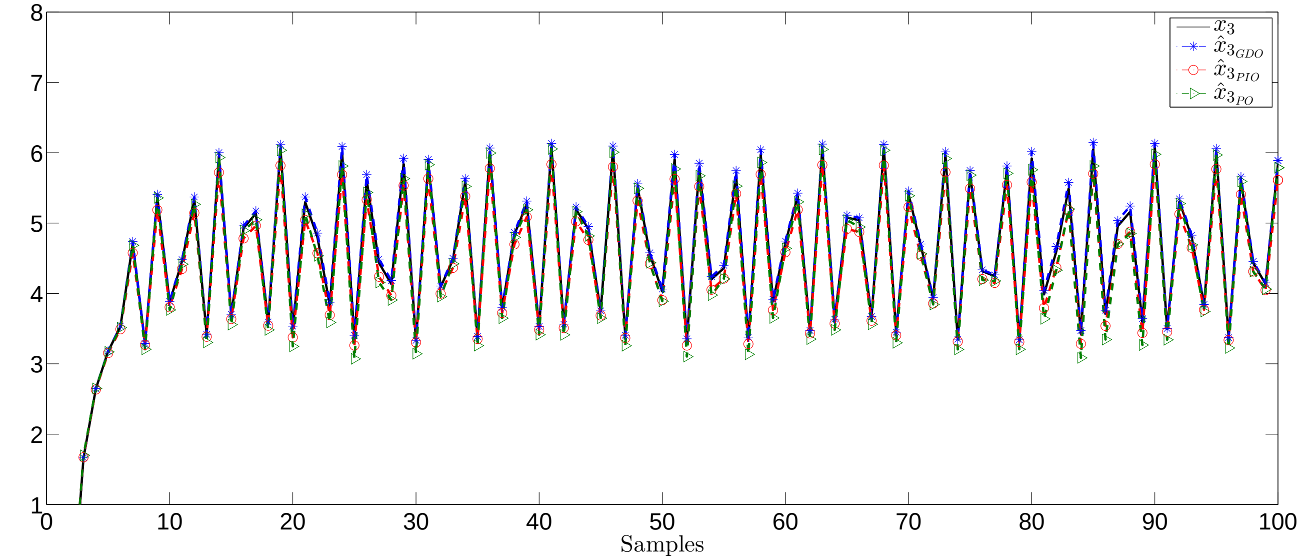

FIGURE 2.37 – H_∞ discrete-time observers : Estimate of $x_3(t)$.

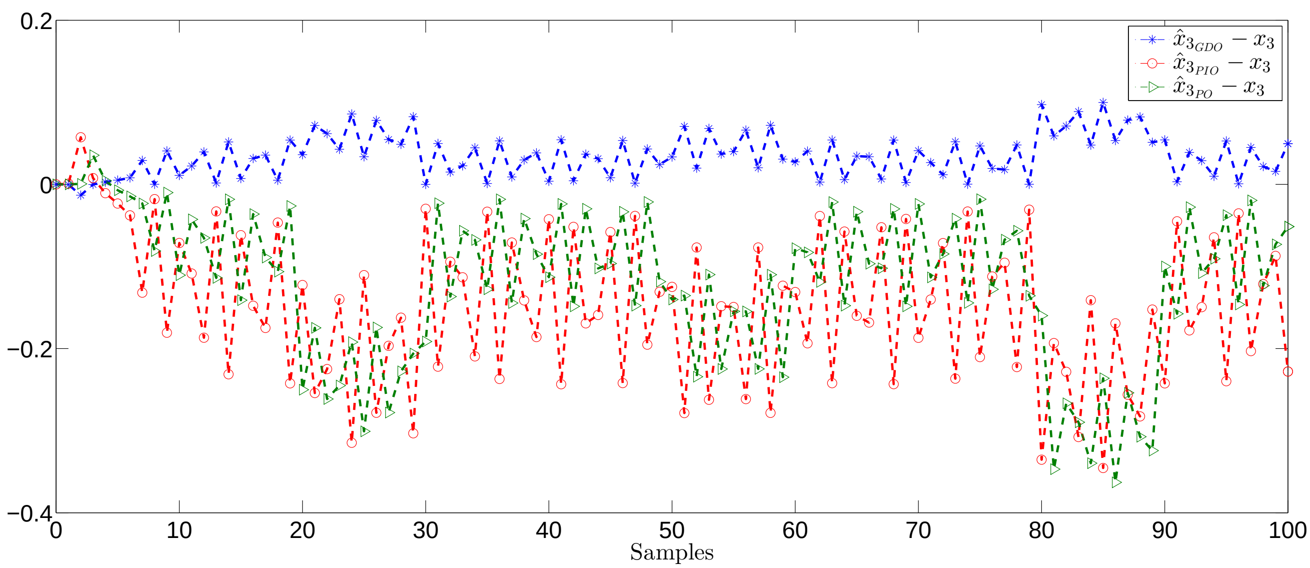

FIGURE 2.38 – H_∞ discrete-time observers : Estimation error of $x_3(t)$.

From these results, we can see that the GDO presents more robustness to parametric uncertainties in the measured states than the PIO and the PIO, and in the unmeasured state it presents less error estimation compared to the other. The following table shows the IAE to show the differences between the observers estimations.

TABLE 2.4 – H_∞ discrete-time observers : Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	51.53	111.27	204.79
$x_2(t)$	1.84	14.07	12.2
$x_3(t)$	3.51	15	12.18

From these error evaluations, we can see that, in general the GDO has the smaller values compared with the PIO and the PO.

2.9 Conclusions

In this chapter a method of synthesis of H_∞ observers with a new observer structure for descriptor systems was presented. This new approach was also developed for the continuous-time case and for the discrete-time case. The observer parameterization method is based on the solution of Sylvester equations. The observer matrices are obtained through the solution of LMIs. A numerical examples were presented to show the applicability of our approach, and the comparison with the PIO and the PO, as special cases of the GDO was also developed.

Chapter 3

Robust H_∞ generalized dynamic observer design

Contents

3.1 Introduction

In this chapter the Robust H_∞ GDO design for uncertain descriptor systems without or with disturbances is treated in Section 3.2. In Section 3.2.1 the class of uncertain descriptor system is presented. In Section 3.2.2 the problematic is posed through the stability of the observer error dynamics and the observer parameterization is boarded. In Section 3.2.3 the GDO design for uncertain descriptor systems free of disturbances is given. In section 3.2.4 the extension to H_∞ GDO is carried out. The H_∞ LPV GDO design for LPV descriptor systems without or with disturbances is presented in Section 3.3. In Section 3.3.1 the class of LPV descriptor systems is presented. In Section 3.3.2 the problematic and the observer parameterization are given. In Section 3.3.3 the LPV GDO for LPV descriptor systems without disturbances is boarded, and in Section 3.3.4 its extension to H_∞ LPV GDO is carried out.

Additionally, special cases of those designs as the PIO and the PO are developed to compare their results in simulation.

3.2 H_∞ generalized dynamic observer design for uncertain systems

In this section the robust H_∞ GDO design for uncertain descriptor systems with or without disturbances is treated.

3.2.1 Class of uncertain disturbed descriptor systems considered

Consider the following uncertain descriptor system :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Dw(t) \tag{3.1a}$$
$$y(t) = (C_1 + \Delta C(t))x(t) + D_1 w(t) \tag{3.1b}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector of bounded energy and $y(t) \in \mathbb{R}^{n_y}$ is the measured output. Matrices $E \in \mathbb{R}^{n_1 \times n}$, $A \in \mathbb{R}^{n_1 \times n}$, $C_1 \in \mathbb{R}^{n_y \times n_1}$, $D \in \mathbb{R}^{n_1 \times n_w}$ and $D_1 \in \mathbb{R}^{n_y \times n_w}$ are constant and known, and $\Delta A(t)$ and $\Delta C(t)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form :

$$\Delta A(t) = \mathcal{M}_1 \Gamma(t) \mathcal{G} \tag{3.2a}$$
$$\Delta C(t) = \mathcal{M}_2 \Gamma(t) \mathcal{G} \tag{3.2b}$$

where $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{G}$ are known real constant matrices and $\Gamma(t)$ is an unknown time-varying matrix satisfying

$$\Gamma(t)^T \Gamma(t) \leq I, \ \forall t \in [0, \infty) \tag{3.3}$$

as is shown in Section 1.3.1.

In the sequel we assume that

Assumption 3.1.

$$\operatorname{rank} \begin{bmatrix} E \\ E^\perp A \\ C_1 \end{bmatrix} = n.$$

Assumption 3.2.

$$\operatorname{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = n, \ \forall s \in \mathbb{C}^+, \ s \text{ finite.}$$

Assumption 3.3. *The descriptor system* (3.1) *with admissible uncertainties* $\Delta A(t)$ *is stable.*

3.2.2 Problem formulation

Consider the following GDO for system (3.1)

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) \tag{3.4a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + My(t) \tag{3.4b}$$
$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{3.4c}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. Matrices N, F, H, L, M, S, P and Q are unknown matrices of appropriate dimensions which must be determined such that $\hat{x}(t)$ asymptotically converges to $x(t)$.

Now, we can give the following lemma.

Lemma 3.1. *There exists an observer of the form* (3.4) *for the system* (3.1) *if the following two statements hold.*

1. *There exists a matrix T of appropriate dimension such that the following conditions are satisfied :*
   (a) $NTE + FC_1 - TA = 0$
   (b) $MC_1 + STE = 0$
   (c) $\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ C_1 \end{bmatrix} = I_n$.

2. *The matrix* $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ *is Hurwitz when* $w(t) = 0$ *and* $\Gamma(t) = 0$.

Proof. Let $T \in \mathbb{R}^{q_0 \times n_1}$ be a parameter matrix and define the error $\varepsilon(t) = \zeta(t) - TEx(t)$, then its derivative is given by

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) + (NTE + FC_1 - TA)\, x(t) + (F\Delta C(t) - T\Delta A(t))\, x(t) + (FD_1 - TD)\, w(t) \tag{3.5}$$

Now, by using the definition of $\varepsilon(t)$, equations (3.4b) and (3.4c) can be written as :

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + (STE + MC_1)\, x(t) + M\Delta C(t)x(t) + MD_1w(t) \tag{3.6}$$

$$\hat{x}(t) = P\varepsilon(t) + \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ C_1 \end{bmatrix} x(t) + Q\Delta C(t)x(t) + QD_1w(t) \tag{3.7}$$

If conditions $(a) - (c)$ of Lemma 3.1 are satisfied the following observer error dynamics is obtained from equations (3.5) and (3.6)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} F\Delta C(t) - T\Delta A(t) \\ M\Delta C(t) \end{bmatrix} x(t) + \begin{bmatrix} FD_1 - TD \\ MD_1 \end{bmatrix} w(t) \tag{3.8}$$

and from equation (3.7) we get :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon(t) + Q\Delta C(t)x(t) + QD_1w(t) \tag{3.9}$$

in this case if $w(t) = 0$, $\Gamma(t) = 0$ and matrix $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz, then $\lim_{t \to \infty} e(t) = 0$. □

Now, the parameterization of all solutions to the algebraic constraints $(a) - (c)$ of Lemma 3.1 is given.

In order to avoid the bilinearity of matrix F that is shown in Section 2.4.1.1 we have considered matrix $Z_1 = 0$, so that the parameterization used in this section is :

$$T = T_1 \tag{3.10}$$
$$N = N_1 - Y_1N_3 \tag{3.11}$$
$$F = \mathcal{F}_1 - Y_1\mathcal{F}_3 \tag{3.12}$$
$$S = -Y_2N_3 \tag{3.13}$$
$$M = -Y_2\mathcal{F}_3 \tag{3.14}$$
$$P = P_1 \tag{3.15}$$
$$Q = Q_1 \tag{3.16}$$

where $\mathcal{F}_1 = T_1A\Sigma^+ \begin{bmatrix} K_1 \\ I_{\varrho_1+n_y} \end{bmatrix}$, $\mathcal{F}_3 = (I_{q_0+\varrho_1+n_y} - \Sigma\Sigma^+) \begin{bmatrix} K_1 \\ I_{\varrho_1+n_y} \end{bmatrix}$ and matrices N_1, N_3, P_1 and Q_1 are defined in Lemma 2.5. $Y_3 = 0$ is taken for simplicity in matrices P and Q. Matrices Σ and Ω are defined as $\Sigma = \begin{bmatrix} R \\ C_1 \end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C_1 \end{bmatrix}$.

In order to study the stability of the observer, the observer error system (3.8) can be written as :

$$\dot{\varphi}(t) = \mathbb{A}\varphi(t) + \mathbb{F}\Gamma(t)\mathcal{G}x(t) + \mathbb{B}w(t) \tag{3.17}$$

where

$$\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 \tag{3.18}$$
$$\mathbb{F} = \mathbb{F}_1 - \mathbb{Y}\mathbb{F}_2 \tag{3.19}$$
$$\mathbb{B} = \mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2 \tag{3.20}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_1 = \begin{bmatrix} \mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1 \\ 0 \end{bmatrix}$, $\mathbb{F}_2 = \begin{bmatrix} \mathcal{F}_3\mathcal{M}_2 \\ 0 \end{bmatrix}$, $\mathbb{B}_1 = \begin{bmatrix} \mathcal{F}_1 D_1 - T_1 D \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathcal{F}_3 D_1 \\ 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

Now, bringing together (3.1a) and (3.17) we get :

$$\mathscr{E}\dot{\beta}(t) = \mathscr{A}\beta(t) + \mathscr{B}w(t) \tag{3.21}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A + \mathcal{M}_1\Gamma(t)\mathcal{G} & 0 \\ \mathbb{F}\Gamma(t)\mathcal{G} & \mathbb{A} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D \\ \mathbb{B} \end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varphi(t) \end{bmatrix}$.

Note that, from equation (3.21), $\beta(t)$ is stable if and only if the descriptor system (3.1) with admissible uncertainties $\Delta A(t)$ is stable.

3.2.3 Robust generalized dynamic observer design for uncertain descriptor systems, $w(t)$=0

In this section we consider $w(t) = 0$ and by replacing matrices $\Delta A(t)$ and $\Delta C(t)$ in system (3.1) it becomes :

$$E\dot{x}(t) = (A + \mathcal{M}_1\Gamma(t)\mathcal{G})x(t) \tag{3.22a}$$
$$y(t) = (C_1 + \mathcal{M}_2\Gamma(t)\mathcal{G})x(t) \tag{3.22b}$$

with the GDO :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) \tag{3.23a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + My(t) \tag{3.23b}$$
$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{3.23c}$$

and the error dynamics (3.17) becomes :

$$\dot{\varphi}(t) = \mathbb{A}\varphi(t) + \mathbb{F}\Gamma(t)\mathcal{G}x(t) \tag{3.24}$$

where matrices $\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2$ and $\mathbb{F} = \mathbb{F}_1 - \mathbb{Y}\mathbb{F}_2$ with $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_1 = \begin{bmatrix} \mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1 \\ 0 \end{bmatrix}$, $\mathbb{F}_2 = \begin{bmatrix} \mathcal{F}_3\mathcal{M}_2 \\ 0 \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$.

The following theorem shows the stability conditions of the GDO (3.23) for the descriptor system (3.22) in the form of LMIs.

Theorem 3.1. *Under Assumptions 3.1, 3.2 and 3.3 there exists a GDO* (3.23) *such that the error dynamics* (3.24) *is asymptotically stable if and only if there exists a symmetric positive matrix* $X_2 = \begin{bmatrix} X_{21} & X_{22} \\ X_{22}^T & X_{23} \end{bmatrix}$, *with* $X_{21} = X_{21}^T$, *and a nonsingular matrix* X_1 *such that the following LMIs are satisfied.*

$$E^T X_1 = X_1^T E \geq 0, \tag{3.25}$$

$$\mathcal{C}^{T\perp}\begin{bmatrix} A^TX_1+X_1A+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & \begin{bmatrix} N_1^TX_{21}+X_{21}N_1 & (*) \\ X_{22}^TN_1 & 0 \end{bmatrix} & (*) \\ 0 & 0 & \begin{bmatrix}(\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1)^TX_{21} & (\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1)^TX_{22}\end{bmatrix} & -\epsilon I \end{bmatrix}\mathcal{C}^{T\perp T}<0 \tag{3.26}$$

and

$$\begin{bmatrix} A^TX_1+X_1A+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 \\ 0 & 0 & -\epsilon I \end{bmatrix}<0. \tag{3.27}$$

with $\epsilon>0$, and matrix $\mathbb{Y}$ is parameterized as follows :

$$\mathbb{Y}=X_2^{-1}(\mathcal{B}_r^+\mathcal{K}\mathcal{C}_l^++\mathcal{Z}-\mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+) \tag{3.28}$$

where

$$\mathcal{K}=-\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}+\mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \tag{3.29a}$$

$$\vartheta=(\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T-\mathcal{D})^{-1}>0 \tag{3.29b}$$

$$\mathcal{S}=\mathcal{R}^{-1}-\mathcal{R}^{-1}\mathcal{B}_l^T[\vartheta-\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta]\mathcal{B}_l\mathcal{R}^{-1} \tag{3.29c}$$

where $\mathcal{D}=\begin{bmatrix} A^TX_1+X_1A & (*) & (*) & 0 & 0 & 0 \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 & 0 \\ \epsilon\mathcal{G} & 0 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1^TX_{21}+X_{21}N_1 & (*) & (*) \\ 0 & 0 & 0 & X_{22}N_1 & 0 & (*) \\ 0 & 0 & 0 & (\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1)^TX_{21} & (\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1)^TX_{22} & -\epsilon I \end{bmatrix}$, $\mathcal{B}=\begin{bmatrix}0\\0\\0\\0\\-I\\0\end{bmatrix}$,

$\mathcal{C}=\begin{bmatrix}0 & 0 & 0 & \begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix} & \begin{bmatrix}\mathcal{F}_3\mathcal{M}_2\\ 0\end{bmatrix}\end{bmatrix}$, *and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R}>0$ and $\|\mathcal{L}\|_2<1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C}=\mathcal{C}_l\mathcal{C}_r$ and $\mathcal{B}=\mathcal{B}_l\mathcal{B}_r$.*

Proof. Consider the following Lyapunov function

$$V(x(t),\varphi(t))=V_1(x(t))+V_2(\varphi(t)) \tag{3.30}$$

where $V_1(x(t))=x(t)^TE^TX_1x(t)$ with $E^TX_1=X_1^TE\geq 0$, and $V_2(\varphi(t))=\varphi(t)^TX_2\varphi(t)$ with $X_2=X_2^T>0$, and $X_2=\begin{bmatrix}X_{21} & X_{22}\\ X_{22}^T & X_{23}\end{bmatrix}$ with $X_{21}=X_{21}^T$.

So the derivative of $V_1(x(t))$ along the trajectory of system (3.22a) is :

$$\dot V_1(x(t))=x(t)^T\left(A^TX_1+X_1^TA\right)x(t)+2x(t)^TX_1^T\mathcal{M}_1\Gamma(t)\mathcal{G}x(t) \tag{3.31}$$

and the derivative of $V_2(\varphi(t))$ along the trajectory of observer error system (3.24) is :

$$\dot V_2(\varphi(t))=\varphi(t)^T(\mathbb{A}^TX_2+X_2\mathbb{A})\varphi(t)+2\varphi(t)^TX_2\mathbb{F}\Gamma(t)\mathcal{G}x(t) \tag{3.32}$$

Using Lemma 1.6 from Section 1.7.3, and since $\Gamma(t)^T\Gamma(t)\leq I$ the following inequalities can be formulated :

$$2x(t)^TX_1^T\mathcal{M}_1\Gamma(t)\mathcal{G}x(t)\leq\epsilon^{-1}x(t)^TX_1^T\mathcal{M}_1\mathcal{M}_1^TX_1x(t)+\epsilon x(t)^T\mathcal{G}^T\mathcal{G}x(t) \tag{3.33}$$

$$2\varphi(t)^TX_2\mathbb{F}\Gamma(t)\mathcal{G}x(t)\leq\epsilon^{-1}\varphi(t)^TX_2\mathbb{F}\mathbb{F}^TX_2\varphi(t)+\epsilon x(t)^T\mathcal{G}^T\mathcal{G}x(t) \tag{3.34}$$

with $\epsilon>0$. Thus,

$$\begin{aligned}\dot V(x(t),\varphi(t))\leq & x(t)^T(A^TX_1+X_1A+\epsilon^{-1}X_1^T\mathcal{M}_1\mathcal{M}_1^TX_1+2\epsilon\mathcal{G}^T\mathcal{G})x(t)+\\ & \varphi(t)^T(\mathbb{A}^TX_2+X_2\mathbb{A}+\epsilon^{-1}X_2\mathbb{F}\mathbb{F}^TX_2)\varphi(t)\end{aligned} \tag{3.35}$$

By applying the Schur complement to inequality (3.35) we obtain

$$\dot{V}(x(t),\varphi(t)) \leq \begin{bmatrix} x(t) \\ \varphi(t) \end{bmatrix}^T \left[\begin{array}{cc|cc} A^T X_1 + X_1 A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 \\ \hline 0 & 0 & \mathbb{A}^T X_2 + X_2 \mathbb{A} & (*) \\ 0 & 0 & \mathbb{F}^T X_2 & -\epsilon I \end{array}\right] \begin{bmatrix} x(t) \\ \varphi(t) \end{bmatrix} \tag{3.36}$$

The asymptotic stability of observer (3.23) is guaranteed if and only if $\dot{V}(x(t),\varphi(t)) < 0$. Which leads the following LMI :

$$\begin{bmatrix} A^T X_1 + X_1 A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & \mathbb{A}^T X_2 + X_2 \mathbb{A} & (*) \\ 0 & 0 & \mathbb{F}^T X_2 & -\epsilon I \end{bmatrix} < 0 \tag{3.37}$$

Now, replacing the form of matrices $\mathbb{A}$ and $\mathbb{F}$ from equations (3.18) and (3.19) we have :

$$\begin{bmatrix} A^T X_1 + X_1 A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X_2 + X_2(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) & (*) \\ 0 & 0 & (\mathbb{F}_1 - \mathbb{Y}\mathbb{F}_2)^T X_2 & -\epsilon I \end{bmatrix} < 0 \tag{3.38}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{3.39}$$

where $\mathcal{X} = X_2\mathbb{Y}$, $\mathcal{D} = \begin{bmatrix} A^T X_1 + X_1 A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & \mathbb{A}_1^T X_2 + X_2 \mathbb{A}_1 & (*) \\ 0 & 0 & \mathbb{F}_1^T X_2 & -\epsilon I \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} 0 & 0 & \mathbb{A}_2 & \mathbb{F}_2 \end{bmatrix}$, and $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \end{bmatrix}$.

Using the elimination lemma of Section 1.5, inequality (3.39) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{3.40a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{3.40b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \begin{bmatrix} \mathbb{A}_2^T \\ \mathbb{F}_2^T \end{bmatrix}^{\perp} \end{bmatrix}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (3.40a) becomes :

$$\mathcal{C}^{T\perp} \begin{bmatrix} A^T X_1 + X_1 A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & \begin{bmatrix} N_1^T X_{21} + X_{21} N_1 & (*) \\ X_{22}^T N_1 & 0 \end{bmatrix} & (*) \\ 0 & 0 & \begin{bmatrix} (\mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1)^T X_{21} & (\mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1)^T X_{22} \end{bmatrix} & -\epsilon I \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{3.41}$$

and by using $\mathcal{B}$ and $\mathcal{D}$ the inequality (3.40b) becomes :

$$\begin{bmatrix} A^T X_1 + X_1 A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 \\ 0 & 0 & -\epsilon I \end{bmatrix} < 0. \tag{3.42}$$

From the elimination lemma if conditions (3.40a) and (3.40b) are satisfied, parameter matrix $\mathbb{Y}$ is parameterized as in (3.28) and (3.29). $\square$

3.2.3.1 Particular cases

In this section we consider two particular cases of our results.

•Proportional observer

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= (A + \mathcal{M}_1\Gamma(t)\mathcal{G})x(t) \\ y(t) &= (C_1 + \mathcal{M}_2\Gamma(t)\mathcal{G})x(t) \end{aligned}$$

with the PO :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + F_a y(t) \\ \hat{x}(t) &= P\zeta(t) + Q_a y(t) \end{aligned}$$

and the observer error dynamics (3.24) becomes :

$$\dot{\varepsilon}(t) = (\bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2)\varepsilon(t) + (\bar{\mathbb{F}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{F}}_2)\Gamma(t)\mathcal{G}x(t)$$

where $\bar{\mathbb{A}}_1 = N_1$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{F}}_1 = \mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1$, $\bar{\mathbb{F}}_2 = \mathcal{F}_3\mathcal{M}_2$ and $\bar{\mathbb{Y}} = Y_1$. Consequently, matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 3.1 become :

$$\mathcal{D} = \begin{bmatrix} A^T X_1 + X_1 A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & N_1^T X_2 + X_2 N_1 & (*) \\ 0 & 0 & (\mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1)^T X_2 & -\epsilon I \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \end{bmatrix},$$

$\mathcal{C} = \begin{bmatrix} 0 & 0 & N_3 & \mathcal{F}_3\mathcal{M}_2 \end{bmatrix}$ and $\mathcal{X} = X_2\bar{\mathbb{Y}}$.

•Proportional-integral observer

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= (A + \mathcal{M}_1\Gamma(t)\mathcal{G})x(t) \\ y(t) &= (C_1 + \mathcal{M}_2\Gamma(t)\mathcal{G})x(t) \end{aligned}$$

with the PIO :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Hv(t) + Fy(t) \\ \dot{v}(t) &= y(t) - C_1\hat{x}(t) \\ \hat{x}(t) &= P\zeta(t) + Qy(t) \end{aligned}$$

and the error dynamics (3.24) becomes :

$$\dot{\varphi}(t) = (\bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2)\varphi(t) + (\bar{\bar{\mathbb{F}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{F}}}_2)\Gamma(t)\mathcal{G}x(t)$$

where $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix} N_1 & 0 \\ -C_1P_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_1 = \begin{bmatrix} \mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1 \\ \mathcal{M}_2 - C_1Q_1\mathcal{M}_2 \end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_2 = \begin{bmatrix} \mathcal{F}_3\mathcal{M}_2 \\ 0 \end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} Y_1 & H \end{bmatrix}$.

Consequently, matrices $\mathcal{D}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 3.1 become :

$$\mathcal{D} = \begin{bmatrix} A^TX_1 + X_1A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & 0 \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \Pi_1 & (*) & (*) \\ 0 & 0 & X_{22}^TN_1 - X_{23}C_1P_1 & 0 & (*) \\ 0 & 0 & \Pi_2 & \Pi_3 & -\epsilon I \end{bmatrix}$$

with

$$\begin{aligned} \Pi_1 &= N_1^TX_{21} - P_1^TC_1^TX_{22}^T + X_{21}N_1 - X_{22}C_1P_1 \\ \Pi_2 &= (\mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1)^TX_{21} + (\mathcal{M}_2 - C_1Q_1\mathcal{M}_2)^TX_{22}^T \\ \Pi_3 &= (\mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1)^TX_{22} + (\mathcal{M}_2 - C_1Q_1\mathcal{M}_2)^TX_{23} \end{aligned}$$

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix} & \begin{bmatrix} \mathcal{F}_3\mathcal{M}_2 \\ 0 \end{bmatrix} \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \end{bmatrix} \text{ and } \mathcal{X} = X_2\bar{\bar{\mathbb{Y}}}, \text{ such that } \begin{bmatrix} Y_1 & H \end{bmatrix} = \left(X_2 \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^+ \mathcal{X}.$$

3.2.3.2 Numerical example

In order to illustrate the results obtained, consider the following descriptor system described by (3.22) where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & 2 \\ 0 & -1 & -1 \end{bmatrix},\ \mathcal{M}_1 = \begin{bmatrix} 0.4 \\ 0 \\ 0.2 \end{bmatrix},\ \mathcal{G} = \begin{bmatrix} 0.1 & 0 & 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix},\ \mathcal{M}_2 = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Uncertainties $\Delta A(t)$ and $\Delta C(t)$ of system (3.22) are described by matrices $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{G}$.

Considering matrix $E^\perp = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, we can verify Assumptions 3.1 and 3.2

$$\operatorname{rank} \begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = 3 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 3$$

Generalized dynamic observer

For the GDO we have chosen matrix $R = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$ such that $\operatorname{rank}(\Sigma) = 3$. By fixing $\epsilon = 1$, and by using YALMIP toolbox, we solve the LMIs (3.25) - (3.27) to find matrices X_1 and X_2

$$X_1 = \begin{bmatrix} 0.65 & 0.08 & 0 \\ 0.08 & 0.48 & 0 \\ 1.5 & 0.34 & 0.68 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 1.27 & 0.15 & 0.42 & 0.42 \\ 0.15 & 0.51 & 0.42 & 0.42 \\ 0.42 & 0.42 & 1.27 & 0 \\ 0.42 & 0.42 & 0 & 1.27 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 \\ 3 & 2 & 6 & 3 & 3 & 5 \\ 6 & 1 & 4 & 2 & 6 & 2 \\ 9 & 5 & 2 & 7 & 4 & 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.1$, and by solving (3.28) and (3.29) we get :

$$\mathbb{Y} = \begin{bmatrix} 11.56 & 1.1 & 1.45 & -0.51 & 5.86 & 5.86 \\ 7.6 & 0.64 & 18.51 & -1.27 & 21.09 & 21.09 \\ -5.39 & 0.21 & -3.5 & 2.55 & -17.67 & -9.77 \\ -4.98 & 3.37 & -5.08 & 6.69 & -9.77 & -17.67 \end{bmatrix}.$$

Finally, we can get all the matrices of the observer as :

$$N = \begin{bmatrix} -11.5 & 0.67 \\ -7.63 & -1.67 \end{bmatrix}, \; S = \begin{bmatrix} 5.59 & 0 \\ 5.59 & 0 \end{bmatrix}, \; H = \begin{bmatrix} 5.86 & 5.86 \\ 21.09 & 21.09 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.19 & -0.19 \\ 0.19 & -0.19 \end{bmatrix}, \; L = \begin{bmatrix} -17.67 & -9.77 \\ -9.77 & -17.67 \end{bmatrix}, \; F = \begin{bmatrix} -0.3 & 0.37 \\ -0.2 & 0.33 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \\ 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ -0.33 & 0.33 \\ 1 & 0 \end{bmatrix}.$$

In order to provide a comparison of the GDO with the PIO and the PO, these last are also designed.

Proportional observer

Considering $\epsilon = 10$ and matrices $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 2 & 3 & 5 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$ and $\mathcal{R} = I_2 \times 0.001$ the following PO matrices are obtained :

$$N = \begin{bmatrix} -40.67 & 0.67 \\ -1.73 & -1.67 \end{bmatrix}, \; F_a = \begin{bmatrix} -12.78 & 13.45 \\ -0.02 & 1.35 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } Q_a = \begin{bmatrix} -0.83 & 0.83 \\ -0.33 & 0.33 \\ 1 & 0 \end{bmatrix}.$$

Proportional-integral observer

By considering $\epsilon = 1$ and matrices $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 \\ 3 & 2 & 6 & 3 & 3 & 5 \\ 6 & 1 & 4 & 2 & 6 & 2 \\ 9 & 5 & 2 & 7 & 4 & 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.01$ the following PIO matrices are obtained :

$$N = \begin{bmatrix} -53.98 & 0.67 \\ 26.75 & -1.67 \end{bmatrix}, \; H = \begin{bmatrix} -5.87 & -1.7 \\ 76.9 & 98.79 \end{bmatrix}, \; F = \begin{bmatrix} -17.22 & 17.88 \\ 9.47 & -8.14 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -0.83 & 0.83 \\ -0.33 & 0.33 \\ 1 & 0 \end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0,\ 0.7,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0]^T$.
To evaluate the performance of the observer an uncertainty $\wp(t)$ is added in the system matrix $A + \Delta A(t)$, then we obtain the following matrix $(A + \Delta A(t) + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}$.

The results of simulation are depicted in Figures 3.1 - 3.8. In Figures 3.1 and 3.2 the uncertainty factor $\delta(t)$ and the variation $\Gamma(t)$ are shown. Figures 3.3 - 3.8 show the system states and their estimations by the GDO, PO and PIO, also these figures show the estimation error for each observer.

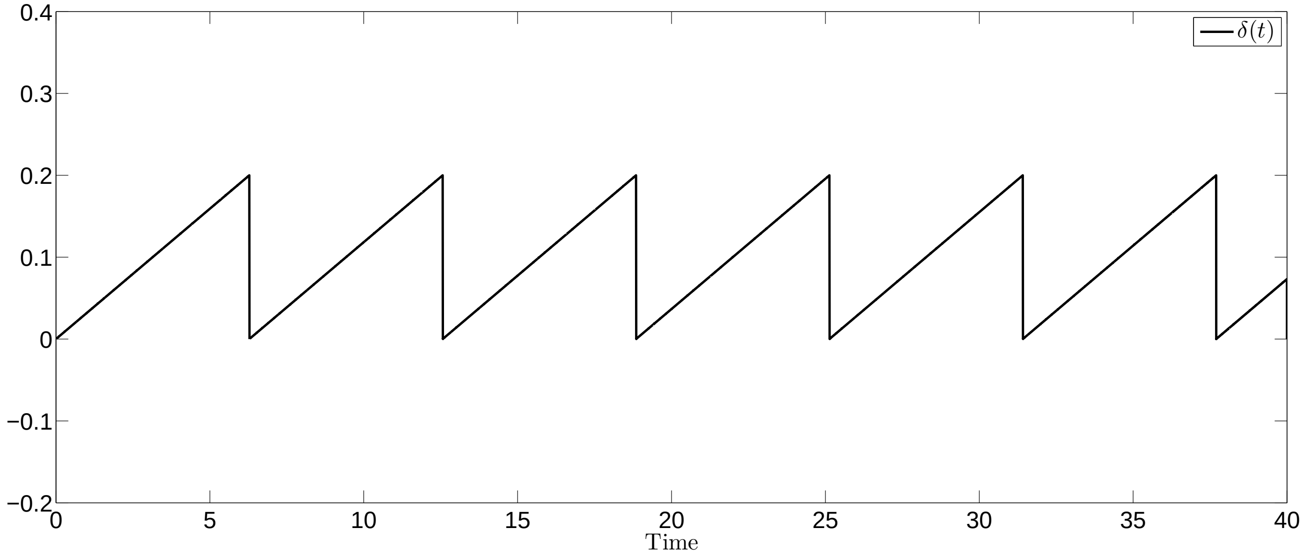

FIGURE 3.1 – Robust uncertain observers : Uncertainty factor $\delta(t)$.

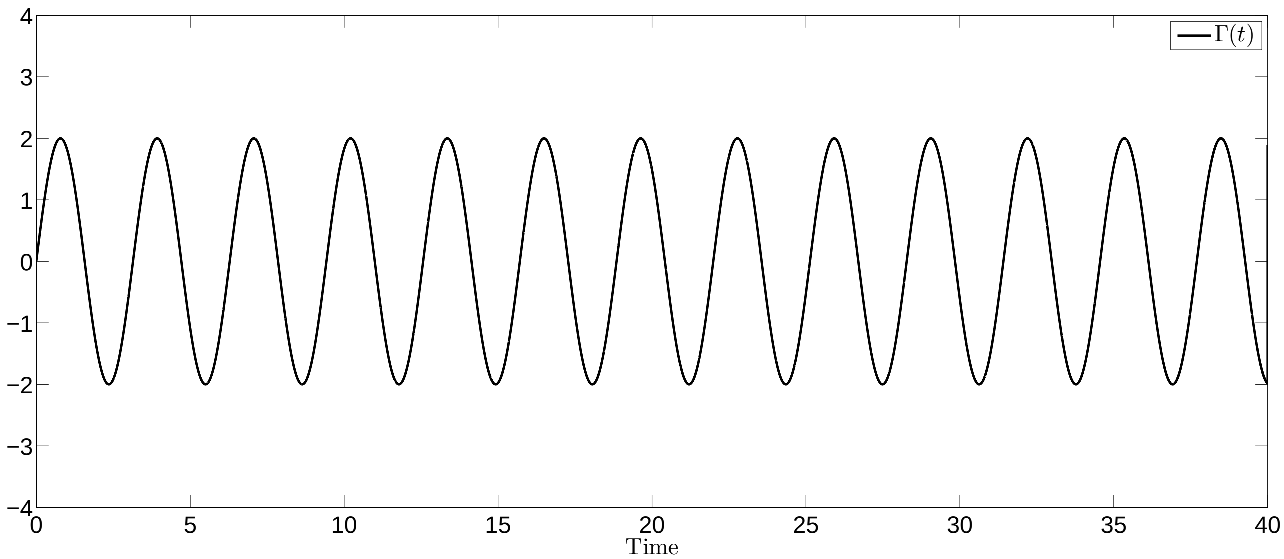

FIGURE 3.2 – Robust uncertain observers : Variation $\Gamma(t)$.

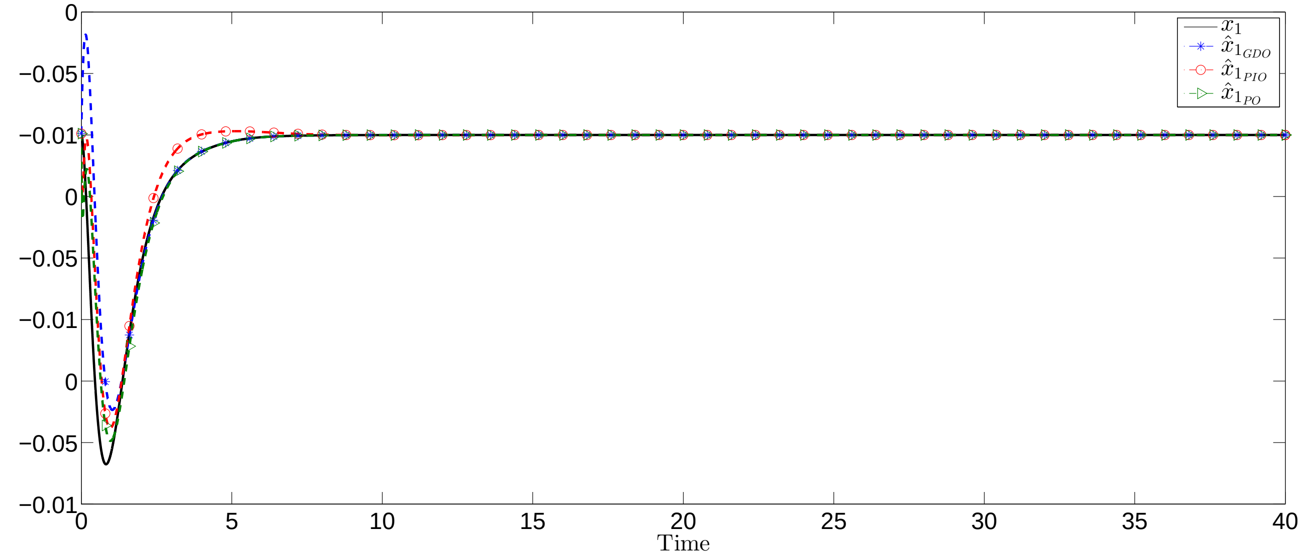

FIGURE 3.3 – Robust uncertain observers : Estimate of the position of $x_1(t)$.

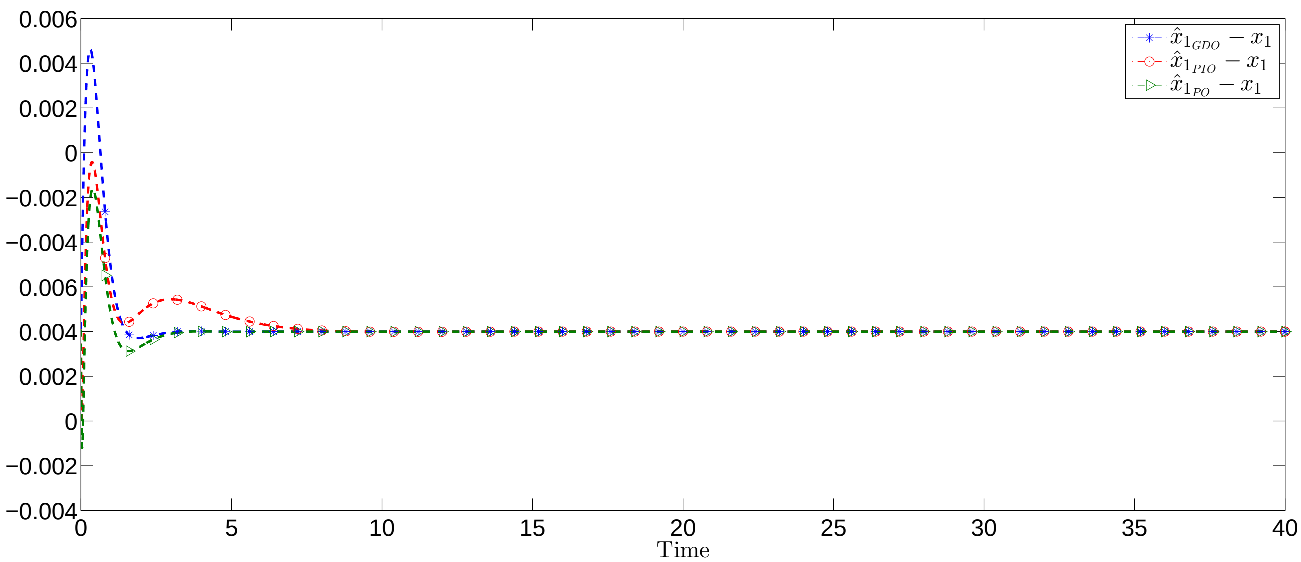

FIGURE 3.4 – Robust uncertain observers : Estimation error of $x_1(t)$.

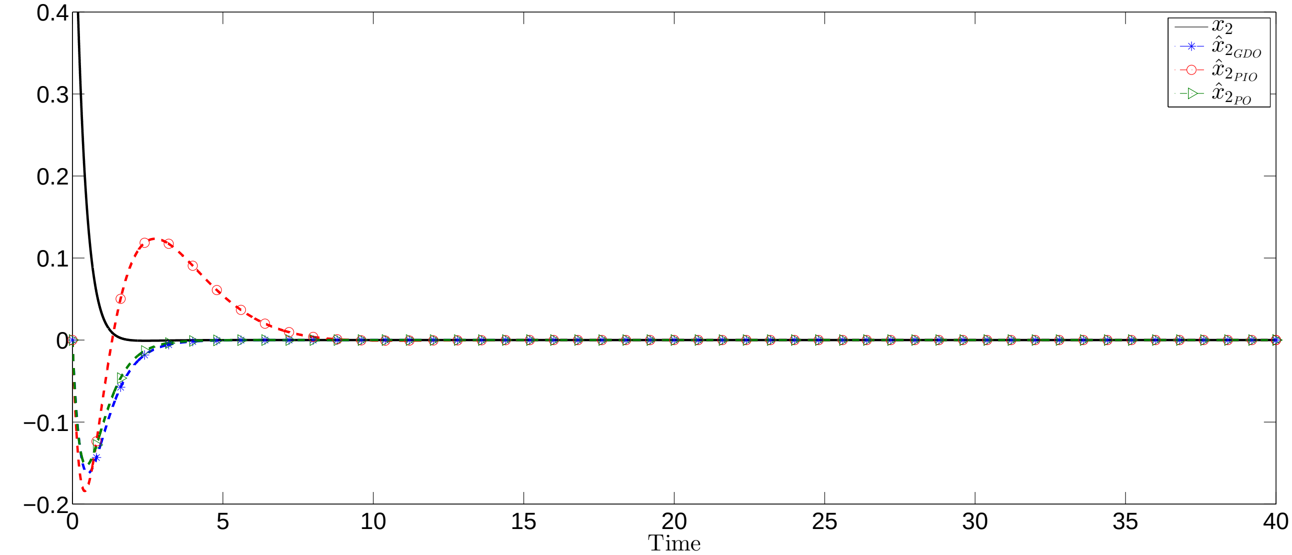

FIGURE 3.5 – Robust uncertain observers : Estimate of the position of $x_2(t)$.

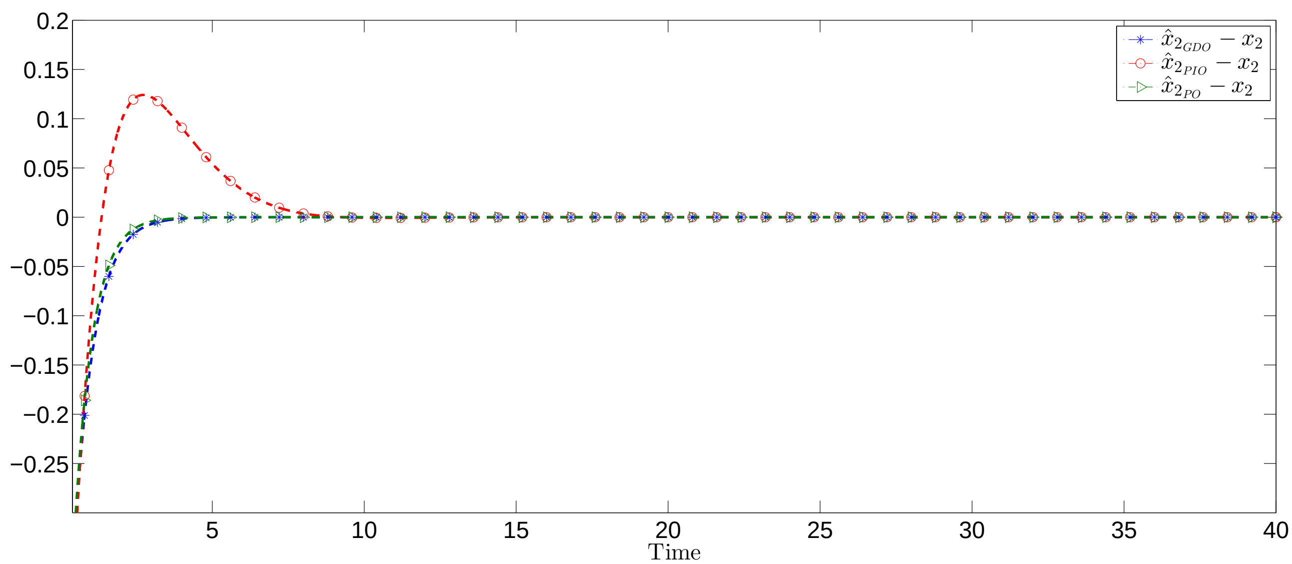

FIGURE 3.6 – Robust uncertain observers : Estimation error of $x_2(t)$.

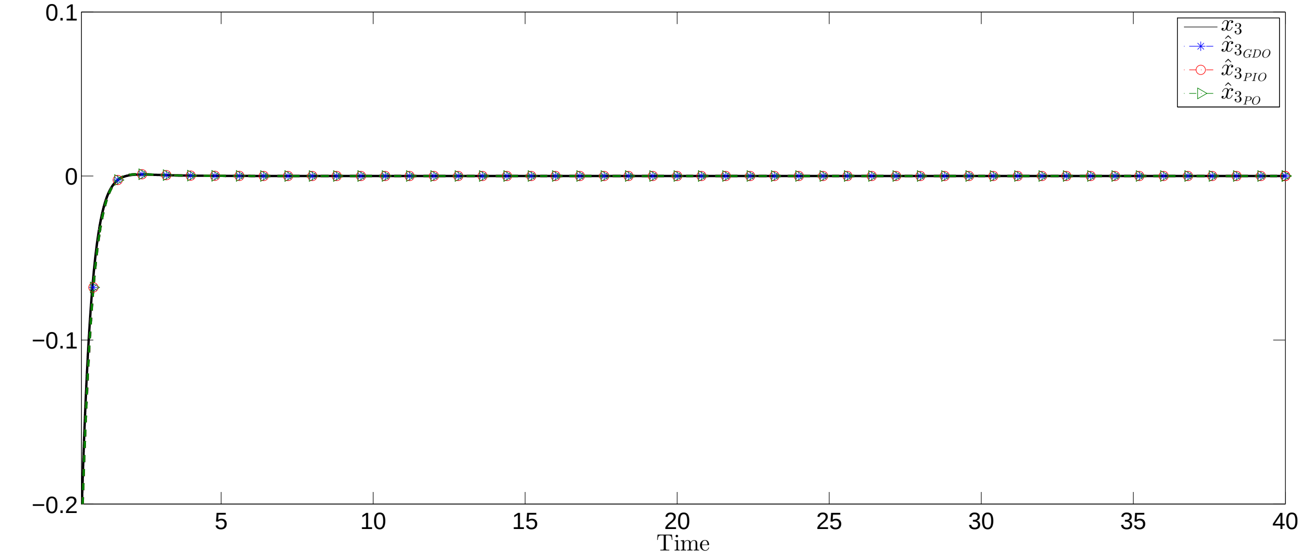

FIGURE 3.7 – Robust uncertain observers : Estimate of $x_3(t)$.

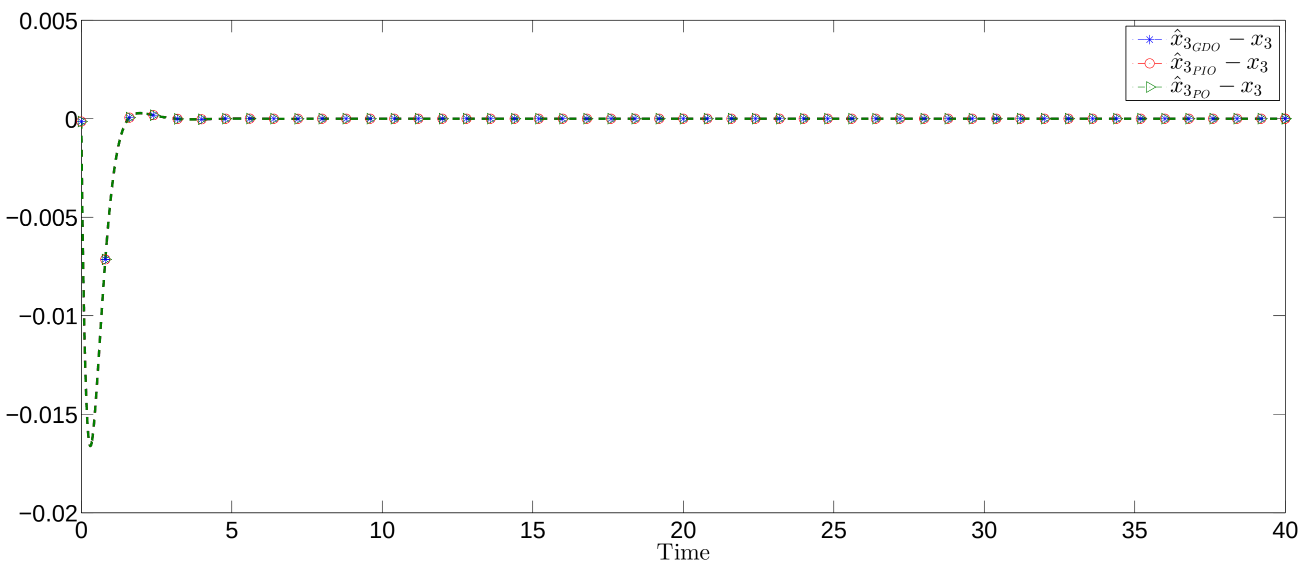

FIGURE 3.8 – Robust uncertain observers : Estimation error of $x_3(t)$.

From these results, we can see that the behavior between the GDO and the PO are almost similar. In order to highlight the difference in the estimations between the observers, the following table with the IAE is presented.

TABLE 3.1 – Robust uncertain observers : Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	4.51	4.82	2.42
$x_2(t)$	446.57	775.65	420
$x_3(t)$	11.92	11.92	11.92

From these error evaluations, we can see that, even the GDO does not have the smaller values, its convergence is faster than the other observers.

3.2.4 Robust H_∞ generalized dynamic observer design for uncertain disturbed descriptor systems, $w(t) \neq 0$

In this section we consider $w(t) \neq 0$, and by replacing $\Delta A(t)$ and $\Delta C(t)$ in system (3.1) it becomes :

$$E\dot{x}(t) = (A + \mathcal{M}_1\Gamma(t)\mathcal{G})x(t) + Dw(t) \tag{3.43a}$$
$$y(t) = (C_1 + \mathcal{M}_2\Gamma(t)\mathcal{G})x(t) + D_1w(t) \tag{3.43b}$$

with the GDO :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) \tag{3.44a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + My(t) \tag{3.44b}$$
$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{3.44c}$$

and the error dynamics (3.17)

$$\dot{\varphi}(t) = \mathbb{A}\varphi(t) + \mathbb{F}\Gamma(t)\mathcal{G}x(t) + \mathbb{B}w(t) \tag{3.45}$$

where $\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2$, $\mathbb{F} = \mathbb{F}_1 - \mathbb{Y}\mathbb{F}_2$, $\mathbb{B} = \mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2$ and with $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{B}_1 = \begin{bmatrix} \mathcal{F}_1D_1 - T_1D \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathcal{F}_3D_1 \\ 0 \end{bmatrix}$, $\mathbb{F}_1 = \begin{bmatrix} \mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1 \\ 0 \end{bmatrix}$, $\mathbb{F}_2 = \begin{bmatrix} \mathcal{F}_3\mathcal{M}_2 \\ 0 \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$.

The relation between the system (3.43a) and the error dynamics (3.45) is given by

$$\mathscr{E}\dot{\beta}(t) = \mathscr{A}\beta(t) + \mathscr{B}w(t) \tag{3.46}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A + \mathcal{M}_1\Gamma(t)\mathcal{G} & 0 \\ \mathbb{F}\Gamma(t)\mathcal{G} & \mathbb{A} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D \\ \mathbb{B} \end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varphi(t) \end{bmatrix}$.

The H_∞ GDO problem can thus formulated as finding the matrix $\mathbb{Y}$ such that the worst estimation error energy $\|\varphi(t)\|_2$ is minimum for all bounded energy disturbance $w(t)$. With this purpose, the following objective signal which only depends on the variable of estimation error $\varphi(t)$ is introduced :

$$z(t) = \mathcal{H}\varphi(t) = \begin{bmatrix} 0 & \mathcal{H} \end{bmatrix} \beta(t) \tag{3.47}$$

where $\mathcal{H}$ is a matrix of appropriate dimension arbitrarily chosen.

In other words $\|G_{wz}\|_\infty < \gamma$, where G_{wz} is the transfer function from the disturbance $w(t)$ to the function objective $z(t)$, and γ is a given positive scalar, or equivalently to the performance index :

$$J = \int_0^\infty \left[z(t)^T z(t) - \gamma^2 w(t)^T w(t) \right] dt. \tag{3.48}$$

The solution to this problem is given by the following theorem.

Theorem 3.2. *Under Assumptions 3.1, 3.2 and 3.3, there exists an H_∞ GDO (3.44) such that the error dynamics (3.46) is stable if and only if there exist a symmetric positive matrix $X_2 = \begin{bmatrix} X_{21} & X_{22} \\ X_{22}^T & X_{23} \end{bmatrix}$, with $X_{21} = X_{21}^T$, and a nonsingular matrix X_1 such that the following LMIs are satisfied.*

$$E^T X_1 = X_1^T E \geq 0, \tag{3.49}$$

$$\mathcal{C}^{T\perp} \begin{bmatrix} A^T X_1 + X_1 A & (*) & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 & 0 & 0 \\ \epsilon \mathcal{G} & 0 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{H}^T \mathcal{H} + \begin{bmatrix} N_1^T X_{21} + X_{21} N_1 & (*) \\ X_{22}^T N_1 & 0 \end{bmatrix} & (*) & (*) \\ 0 & 0 & 0 & \Pi_1 & -\epsilon I & 0 \\ D^T X_1 & 0 & 0 & \Pi_2 & 0 & -\gamma^2 I \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{3.50}$$

where

$$\Pi_1 = \begin{bmatrix} (\mathcal{F}_1 \mathcal{M}_2 - T_1 \mathcal{M}_1)^T X_{21} & (\mathcal{F}_1 \mathcal{M}_2 - T_1 \mathcal{M}_1)^T X_{22} \end{bmatrix} \tag{3.51a}$$

$$\Pi_2 = \begin{bmatrix} (\mathcal{F}_1 D_1 - T_1 D)^T X_{21} & (\mathcal{F}_1 D_1 - T_1 D)^T X_{22} \end{bmatrix} \tag{3.51b}$$

and

$$\begin{bmatrix} A^T X_1 + X_1 A & (*) & (*) & (*) \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 \\ \epsilon \mathcal{G} & 0 & -\epsilon I & 0 \\ D^T X_1 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{3.52}$$

Matrix $\mathbb{Y}$ is parameterized as follows :

$$\mathbb{Y} = X_2^{-1} (\mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+) \tag{3.53}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1} \mathcal{B}_l^T \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2} \mathcal{L} (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1/2} \tag{3.54a}$$

$$\vartheta = (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{3.54b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\vartheta - \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} \mathcal{C}_r \vartheta] \mathcal{B}_l \mathcal{R}^{-1} \tag{3.54c}$$

where $\mathcal{D} = \begin{bmatrix} A^T X_1 + X_1 A & (*) & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^T X_1 & -\epsilon I & 0 & 0 & 0 & 0 \\ \epsilon \mathcal{G} & 0 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{H}^T \mathcal{H} + \begin{bmatrix} N_1^T X_{21} + X_{21} N_1 & (*) \\ X_{22}^T N_1 & 0 \end{bmatrix} & (*) & (*) \\ 0 & 0 & 0 & \Pi_1 & -\epsilon I & 0 \\ D^T X_1 & 0 & 0 & \Pi_2 & 0 & -\gamma^2 I \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -I \\ 0 \\ 0 \end{bmatrix}$,

$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} & \begin{bmatrix} \mathcal{F}_3 \mathcal{M}_2 \\ 0 \end{bmatrix} & \begin{bmatrix} \mathcal{F}_1 D_1 \\ 0 \end{bmatrix} \end{bmatrix}$, *with matrices Π_1 and Π_2 defined in (3.51). Matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$.*

Proof. Consider the following Lyapunov function

$$V(x(t), \varphi(t)) = V_1(x(t)) + V_2(\varphi(t)) \tag{3.55}$$

where $V_1(x(t)) = x(t)^T E^T X_1 x(t)$ with $E^T X_1 = X_1^T E \geq 0$, and $V_2(\varphi(t)) = \varphi(t)^T X_2 \varphi(t)$ with $X_2 = X_2^T$, and $X_2 = \begin{bmatrix} X_{21} & X_{22} \\ X_{22}^T & X_{23} \end{bmatrix}$ with $X_{21} = X_{21}^T$.

Then, the derivative of $V_1(x(t))$ along the trajectory of system (3.43a) is :

$$\dot{V}_1(x(t)) = x(t)^T (A^T X_1 + X_1^T A) x(t) + 2x(t)^T X_1^T \mathcal{M}_1 \Gamma(t) \mathcal{G} x(t) + x(t)^T X_1 D w(t) + w(t)^T D^T X_1 x(t) \tag{3.56}$$

and the derivative of $V_2(\varphi(t))$ along the trajectory of observer error system (3.45) is :

$$\dot{V}_2(\varphi(t)) = \varphi(t)^T (\mathbb{A}^T X_2 + X_2 \mathbb{A}) \varphi(t) + 2\varphi(t)^T X_2 \mathbb{F} \Gamma(t) \mathcal{G} x(t) + \varphi(t)^T X_2 \mathbb{B} w(t) + w(t)^T \mathbb{B}^T X_2 \varphi(t) \tag{3.57}$$

Using Lemma 1.6 from Section 1.7.3, and since $\Gamma(t)^T\Gamma(t) \leq I$ the following inequalities can be formulated :

$$2x(t)^T X_1^T \mathcal{M}_1 \Gamma(t)\mathcal{G}x(t) \leq \epsilon^{-1}x(t)^T X_1^T \mathcal{M}_1\mathcal{M}_1^T X_1 x(t) + \epsilon x(t)^T \mathcal{G}^T\mathcal{G}x(t) \tag{3.58}$$

$$2\varphi(t)^T X_2 \mathbb{F}\Gamma(t)\mathcal{G}x(t) \leq \epsilon^{-1}\varphi(t)^T X_2\mathbb{F}\mathbb{F}^T X_2\varphi(t) + \epsilon x(t)^T \mathcal{G}^T\mathcal{G}x(t) \tag{3.59}$$

with $\epsilon > 0$. Thus,

$$\begin{aligned}\dot{V}(x(t),\varphi(t)) \leq\ & x(t)^T(A^TX_1 + X_1A + \epsilon^{-1}X_1^T\mathcal{M}_1\mathcal{M}_1^TX_1 + 2\epsilon\mathcal{G}^T\mathcal{G})x(t) + \\ & \varphi(t)^T(\mathbb{A}^TX_2 + X_2\mathbb{A} + \epsilon^{-1}X_2\mathbb{F}\mathbb{F}^TX_2)\varphi(t) + \varphi(t)^TX_2\mathbb{B}w(t) + \\ & w(t)^T\mathbb{B}^TX_2\varphi(t) + x(t)^TX_1Dw(t) + w(t)^TD^TX_1x(t)\end{aligned} \tag{3.60}$$

From equation (3.48) we get :

$$J < \int_0^\infty \left[z(t)^Tz(t) - \gamma^2 w(t)^Tw(t) + \dot{V}(t)\right]dt. \tag{3.61}$$

so a sufficient condition for $J < 0$ is that

$$z(t)^Tz(t) - \gamma^2w(t)^Tw(t) + \dot{V}(t) < 0, \quad \forall t \in [0,\infty). \tag{3.62}$$

Then, we have

$$\begin{aligned}z(t)^Tz(t) - \gamma^2w(t)^Tw(t) + \dot{V}(t) =\ & x(t)^T(A^TX_1 + X_1A + \epsilon^{-1}X_1^T\mathcal{M}_1\mathcal{M}_1^TX_1 + 2\epsilon\mathcal{G}^T\mathcal{G})x(t) + \\ & \varphi(t)^T(\mathcal{H}^T\mathcal{H} + \mathbb{A}^TX_2 + X_2\mathbb{A} + \epsilon^{-1}X_2\mathbb{F}\mathbb{F}^TX_2)\varphi(t) + \varphi(t)^TX_2\mathbb{B}w(t) + \\ & w(t)^T\mathbb{B}^TX_2\varphi(t) + x(t)^TX_1Dw(t) + w(t)^TD^TX_1x(t) - \gamma^2w(t)^Tw(t)\end{aligned} \tag{3.63}$$

By applying the Schur complement in inequality (3.63) gives

$$\begin{bmatrix} x(t) \\ \varphi(t) \\ w(t) \end{bmatrix}^T \left[\begin{array}{cc|cc|c} A^TX_1 + X_1A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 \\ \hline 0 & 0 & \mathcal{H}^T\mathcal{H} + \mathbb{A}^TX_2 + X_2\mathbb{A} & (*) & (*) \\ 0 & 0 & \mathbb{F}^TX_2 & -\epsilon I & 0 \\ \hline D^TX_1 & 0 & \mathbb{B}^TX_2 & 0 & -\gamma^2 I \end{array}\right] \begin{bmatrix} x(t) \\ \varphi(t) \\ w(t) \end{bmatrix} < 0 \tag{3.64}$$

The sufficient condition for satisfy (3.64) is :

$$\begin{bmatrix} A^TX_1 + X_1A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \mathcal{H}^T\mathcal{H} + \mathbb{A}^TX_2 + X_2\mathbb{A} & (*) & (*) \\ 0 & 0 & \mathbb{F}^TX_2 & -\epsilon I & 0 \\ D^TX_1 & 0 & \mathbb{B}^TX_2 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{3.65}$$

Now, replacing the form of matrices $\mathbb{A}$, $\mathbb{F}$ and $\mathbb{B}$ from equations (3.18) - (3.20) we get :

$$\begin{bmatrix} A^TX_1 + X_1A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \mathcal{H}^T\mathcal{H} + (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^TX_2 + X_2(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) & (*) & (*) \\ 0 & 0 & (\mathbb{F}_1 - \mathbb{Y}\mathbb{F}_1)^TX_2 & -\epsilon I & 0 \\ D^TX_1 & 0 & (\mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2)^TX_2 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{3.66}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{3.67}$$

where $\mathcal{X} = X_2\mathbb{Y}$, $\mathcal{D} = \begin{bmatrix} A^TX_1 + X_1A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \mathcal{H}^T\mathcal{H} + \mathbb{A}_1^TX_2 + X_2\mathbb{A}_1 & (*) & (*) \\ 0 & 0 & \mathbb{F}_1^TX_2 & -\epsilon I & 0 \\ D^TX_1 & 0 & \mathbb{B}_1^TX_2 & 0 & -\gamma^2 I \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \\ 0 \end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix} 0 & 0 & \mathbb{A}_2 & \mathbb{F}_2 & \mathbb{B}_2 \end{bmatrix}$.

Using the elimination lemma of Section 1.5, inequality (3.67) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{3.68a}$$
$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{3.68b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \begin{bmatrix}\mathbb{A}_2^T \\ \mathbb{F}_2^T \\ \mathbb{B}_2^T\end{bmatrix}^{\perp} \end{bmatrix}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$. By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (3.68a) becomes :

$$\mathcal{C}^{T\perp}\begin{bmatrix} A^TX_1+X_1A & (*) & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 & 0 \\ \epsilon\mathcal{G} & 0 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{H}^T\mathcal{H}+\begin{bmatrix} N_1^TX_{21}+X_{21}N_1 & (*) \\ X_{22}^TN_1 & 0\end{bmatrix} & (*) & (*) \\ 0 & 0 & 0 & \Pi_1 & -\epsilon I & 0 \\ D^TX_1 & 0 & 0 & \Pi_2 & 0 & -\gamma^2 I \end{bmatrix}\mathcal{C}^{T\perp T} < 0 \tag{3.69}$$

and by using $\mathcal{B}$ and $\mathcal{D}$ the inequality (3.68b) becomes :

$$\begin{bmatrix} A^TX_1+X_1A & (*) & (*) & (*) \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 \\ \epsilon\mathcal{G} & 0 & -\epsilon I & 0 \\ D^TX_1 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{3.70}$$

matrices Π_1 and Π_2 are defined in (3.51). From the elimination lemma if conditions (3.68a) and (3.68b) are satisfied, parameter matrix $\mathbb{Y}$ is parameterized as in (3.53) and (3.54). □

3.2.4.1 Particular cases

•Proportional observer

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= (A+\mathcal{M}_1\Gamma(t)\mathcal{G})x(t)+Dw(t) \\ y(t) &= (C_1+\mathcal{M}_2\Gamma(t)\mathcal{G})x(t)+D_1w(t) \end{aligned}$$

with the PO :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t)+F_ay(t) \\ \hat{x}(t) &= P\zeta(t)+Q_ay(t) \end{aligned}$$

and the observer error dynamics (3.45) becomes :

$$\dot{\varepsilon}(t) = (\bar{\mathbb{A}}_1-\bar{\mathbb{Y}}\bar{\mathbb{A}}_2)\varepsilon+(\bar{\mathbb{F}}_1-\bar{\mathbb{Y}}\bar{\mathbb{F}}_2)\Gamma(t)\mathcal{G}x(t)+(\bar{\mathbb{B}}_1-\bar{\mathbb{Y}}\bar{\mathbb{B}}_2)w(t)$$

where $\bar{\mathbb{A}}_1 = N_1$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{F}}_1 = \mathcal{F}_1\mathcal{M}_2 - T_1\mathcal{M}_1$, $\bar{\mathbb{F}}_2 = \mathcal{F}_3\mathcal{M}_2$, $\bar{\mathbb{B}}_1 = \mathcal{F}_1D_1 - T_1D$, $\bar{\mathbb{B}}_2 = \mathcal{F}_3D_1$ and $\bar{\mathbb{Y}} = Y_1$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{X}$ of Theorem 3.2 become :

$$\mathcal{D} = \begin{bmatrix} A^TX_1+X_1A+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & (*) \\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \mathcal{H}^T\mathcal{H}+N_1^TX_2+X_2N_1 & (*) & (*) \\ 0 & 0 & (\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1)^TX_2 & -\epsilon I & 0 \\ D^TX_1 & 0 & (\mathcal{F}_1D_1-T_1D)^TX_2 & 0 & -\gamma^2 I \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \\ 0 \end{bmatrix},$$

$\mathcal{C} = \begin{bmatrix} 0 & 0 & N_3 & \mathcal{F}_3\mathcal{M}_1 & \mathcal{F}_3D_1 \end{bmatrix}$ and $\mathcal{X} = X_2\bar{\mathbb{Y}}$.

•Proportional-integral observer

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= (A+\mathcal{M}_1\Gamma(t)\mathcal{G})x(t)+Dw(t)\\ y(t) &= (C_1+\mathcal{M}_2\Gamma(t)\mathcal{G})x(t)+D_1w(t)\end{aligned}$$

with the PIO :

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t)+Hv(t)+Fy(t)\\ \dot{v}(t) &= y(t)-C_1\hat{x}(t)\\ \hat{x}(t) &= P\zeta(t)+Qy(t)\end{aligned}$$

and the error dynamics (3.45) becomes :

$$\dot{\varphi}(t) = (\bar{\bar{\mathbb{A}}}_1-\bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2)\varphi(t)+(\bar{\bar{\mathbb{F}}}_1-\bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{F}}}_2)\Gamma(t)\mathcal{G}x(t)+(\bar{\bar{\mathbb{B}}}_1-\bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{B}}}_2)w(t) \tag{3.71}$$

where $\bar{\bar{\mathbb{A}}}_1=\begin{bmatrix}N_1 & 0\\ -C_1P_1 & 0\end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2=\begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_1=\begin{bmatrix}\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1\\ \mathcal{M}_2-C_1Q_1\mathcal{M}_2\end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_2=\begin{bmatrix}\mathcal{F}_3\mathcal{M}_2\\ 0\end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_1=\begin{bmatrix}\mathcal{F}_1D_1-T_1D\\ D_1-C_1Q_1D_1\end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_2=\begin{bmatrix}\mathcal{F}_3D_1\\ 0\end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}}=\begin{bmatrix}I\\ 0\end{bmatrix}\begin{bmatrix}Y_1 & H\end{bmatrix}$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{X}$ of Theorem 3.2 become :

$$\mathcal{D}=\begin{bmatrix}A^TX_1+X_1A+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & (*)\\ \mathcal{M}_1^TX_1 & -\epsilon I & 0 & 0 & 0\\ 0 & 0 & \Pi_1 & (*) & (*)\\ 0 & 0 & \Pi_2 & -\epsilon I & 0\\ D^TX_1 & 0 & \Pi_3 & 0 & -\gamma^2 I\end{bmatrix}$$

with

$$\begin{aligned}\Pi_1 &= \mathcal{H}^T\mathcal{H}+\begin{bmatrix}N_1^TX_{21}-P_1^TC_1^TX_{22}^T+X_{21}N_1-X_{22}C_1P_1 & (*)\\ X_{22}^TN_1-X_{23}C_1P_1 & 0\end{bmatrix},\\ \Pi_2 &= \begin{bmatrix}(\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1)^TX_{21}+(\mathcal{M}_2-C_1Q_1\mathcal{M}_2)^TX_{22}^T & (\mathcal{F}_1\mathcal{M}_2-T_1\mathcal{M}_1)^TX_{22}+(\mathcal{M}_2-C_1Q_1\mathcal{M}_2)^TX_{23}\end{bmatrix},\\ \Pi_3 &= \begin{bmatrix}(\mathcal{F}_1D_1-T_1D)^TX_{21}+(D_1-C_1Q_1D_1)^TX_{22}^T & (\mathcal{F}_1D_1-T_1D)^TX_{22}+(D_1-C_1Q_1D_1)^TX_{23}\end{bmatrix},\end{aligned}$$

$\mathcal{B}=\begin{bmatrix}0\\0\\-I\\0\\0\end{bmatrix}$, $\mathcal{C}=\begin{bmatrix}0 & 0 & \begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix} & \begin{bmatrix}\mathcal{F}_3\mathcal{M}_2\\ 0\end{bmatrix} & \begin{bmatrix}\mathcal{F}_3D_1\\ 0\end{bmatrix}\end{bmatrix}$ and $\mathcal{X}=X_2\bar{\bar{\mathbb{Y}}}$, such that $\begin{bmatrix}Y_2 & H\end{bmatrix}=\left(X_2\begin{bmatrix}I\\0\end{bmatrix}\right)^+\mathcal{X}$.

3.2.4.2 Numerical example

In order to illustrate the results obtained, consider the following descriptor systems described by (3.43) where

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\ A=\begin{bmatrix}-1&2&2\\1&-1&2\\0&-1&-1\end{bmatrix},\ \mathcal{M}_1=\begin{bmatrix}0.4\\0\\0.2\end{bmatrix},\ D=\begin{bmatrix}1\\0\\1\end{bmatrix},\ D_1=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathcal{G}=\begin{bmatrix}0.1 & 0 & 0.1\end{bmatrix},\ C_1=\begin{bmatrix}0&0&1\\1&0&1\end{bmatrix},\ \mathcal{M}_2=\begin{bmatrix}0.5\\0.3\end{bmatrix}\ \text{and}\ x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}.$$

Uncertainties $\Delta A(t)$ and $\Delta C(t)$ of system (3.43) are described by matrices $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{G}$.

Considering $E^\perp = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, we can verify Assumptions 3.1 and 3.2

$$\mathsf{rank}\begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = 3 \quad \text{and} \quad \mathsf{rank}\begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 3$$

H_∞ Generalized dynamic observer

For the H_∞ GDO we have chosen matrix $R = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$, such that $\mathsf{rank}(\Sigma) = 3$. By fixing $\epsilon = 1$ and $\gamma = 1.39$, and by using YALMIP toolbox, we solve the LMIs (3.49) - (3.52) to find matrices X_1 and X_2

$$X_1 = \begin{bmatrix} 0.27 & -0.14 & 0 \\ -0.14 & 0.38 & 0 \\ 0.58 & -0.05 & 0.46 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 1.08 & 0.08 & 0 & 0 \\ 0.08 & 0.71 & 0 & 0 \\ 0 & 0 & 1.08 & 0 \\ 0 & 0 & 0 & 1.08 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 \\ 3 & 2 & 6 & 3 & 3 & 5 \\ 6 & 1 & 4 & 2 & 6 & 2 \\ 9 & 5 & 2 & 7 & 4 & 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.1$ and solving (3.53) and (3.54) we get :

$$\mathbb{Y} = \begin{bmatrix} 10.26 & 2.59 & 0.32 & 2.54 & -0.84 & -0.84 \\ 0.49 & 2.55 & 8.44 & 4.23 & -0.35 & -0.35 \\ 1.11 & 0.93 & 3.71 & 2.3 & -10.16 & -0.88 \\ 1.59 & 4.64 & 1.85 & 7.17 & -0.88 & -10.16 \end{bmatrix}.$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -9.91 & 0.67 \\ -0.05 & -1.67 \end{bmatrix},\ S = \begin{bmatrix} -0.87 & 0 \\ -0.87 & 0 \end{bmatrix},\ H = \begin{bmatrix} -0.84 & -0.84 \\ -0.35 & -0.35 \end{bmatrix},$$

$$L = \begin{bmatrix} -10.16 & -0.88 \\ -0.88 & -10.16 \end{bmatrix},\ F = \begin{bmatrix} -0.25 & 0.32 \\ 0.05 & 0.08 \end{bmatrix},\ P = \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \\ 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} -0.03 & 0.03 \\ -0.03 & 0.03 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ -0.33 & 0.33 \\ 1 & 0 \end{bmatrix}.$$

In order to provide a comparison of the robust H_∞ GDO with the robust H_∞ PIO and the robust H_∞ PO, these last are also designed.

H_∞ Proportional observer

By considering matrices $R = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 2 & 6 & 3 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$, $\mathcal{R} = I_2 \times 0.0001$, $\epsilon = 1$ and $\gamma = 1.39$ the following PO matrices are obtained :

$$N = \begin{bmatrix} -6.13 & 2 \\ -4.19 & -0.33 \end{bmatrix},\ F_a = \begin{bmatrix} -2.46 & 4.46 \\ -2.3 & 4.97 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \\ -0.17 & 0 \end{bmatrix} \text{ and } Q_a = \begin{bmatrix} -0.5 & 0.5 \\ -0.17 & 0.17 \\ 0.83 & 0.17 \end{bmatrix}.$$

H_∞ Proportional-integral observer

By considering $R = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 \\ 3 & 2 & 6 & 3 & 3 & 5 \\ 6 & 1 & 4 & 2 & 6 & 2 \\ 9 & 5 & 2 & 7 & 4 & 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$, $\mathcal{R} = I_4 \times 0.01$ $\epsilon = 1$ and $\gamma = 1.39$ the following PIO matrices are obtained :

$$N = \begin{bmatrix} -1.91 & 0.67 \\ 0.74 & -1.67 \end{bmatrix},\ H = \begin{bmatrix} -1.08 & -0.78 \\ 2.15 & 3.88 \end{bmatrix},\ F = \begin{bmatrix} 0.01 & 0.05 \\ 0.08 & 0.05 \end{bmatrix}$$
$$P = \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \\ 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ -0.33 & 0.33 \\ 1 & 0 \end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0.1,\ 0.1,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0]^T$.

To evaluate the performance of the observers aun uncertainty $\wp(t)$ is added in the system matrix $A + \Delta A(t)$, then we obtain the matrix $(A + \Delta A(t) + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The results of simulation are depicted in Figures 3.9 - 3.17. The disturbance $w(t)$, the uncertainty factor $\delta(t)$ and the variation $\Gamma(t)$ are show in Figures 3.9, 3.10 and 3.11, respectively. Figures 3.12 - 3.17 show the system states and their estimations by the GDO, PO and PIO, also these figures show the error estimation for each observer.

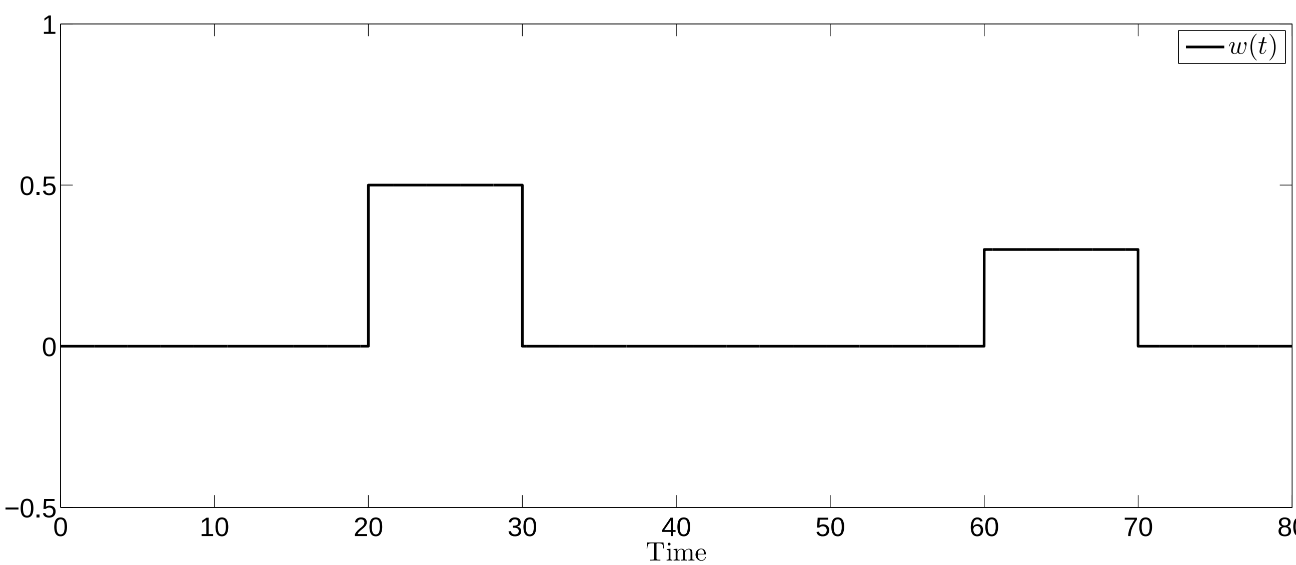

FIGURE 3.9 – Robust H_∞ uncertain observers : Disturbance $w(t)$.

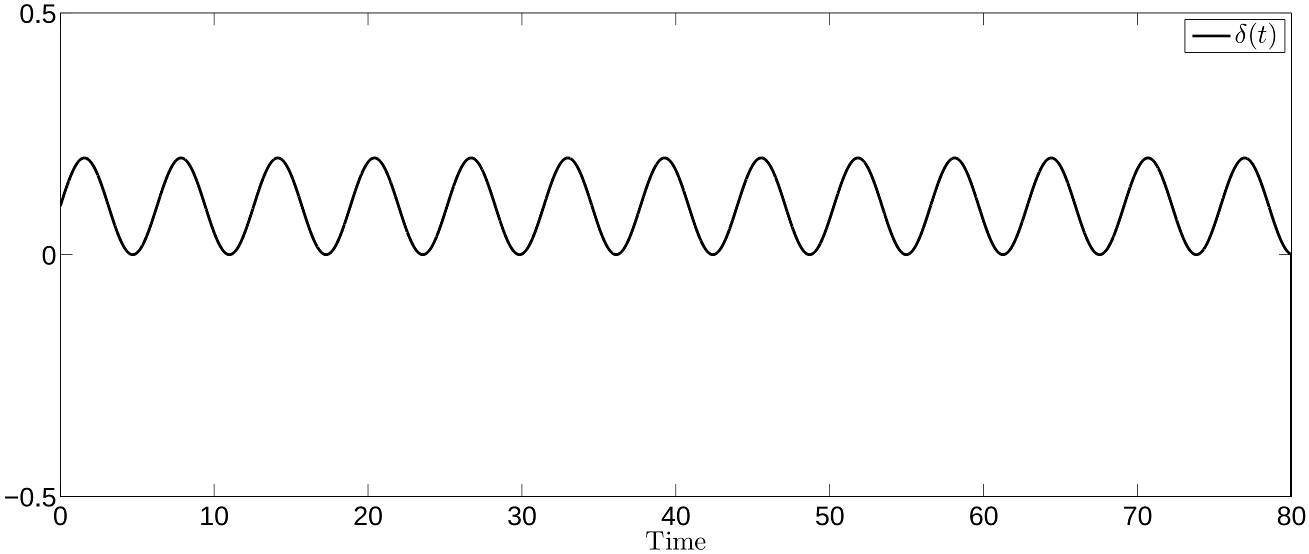

FIGURE 3.10 – Robust H_∞ uncertain observers : Uncertainty factor $\delta(t)$.

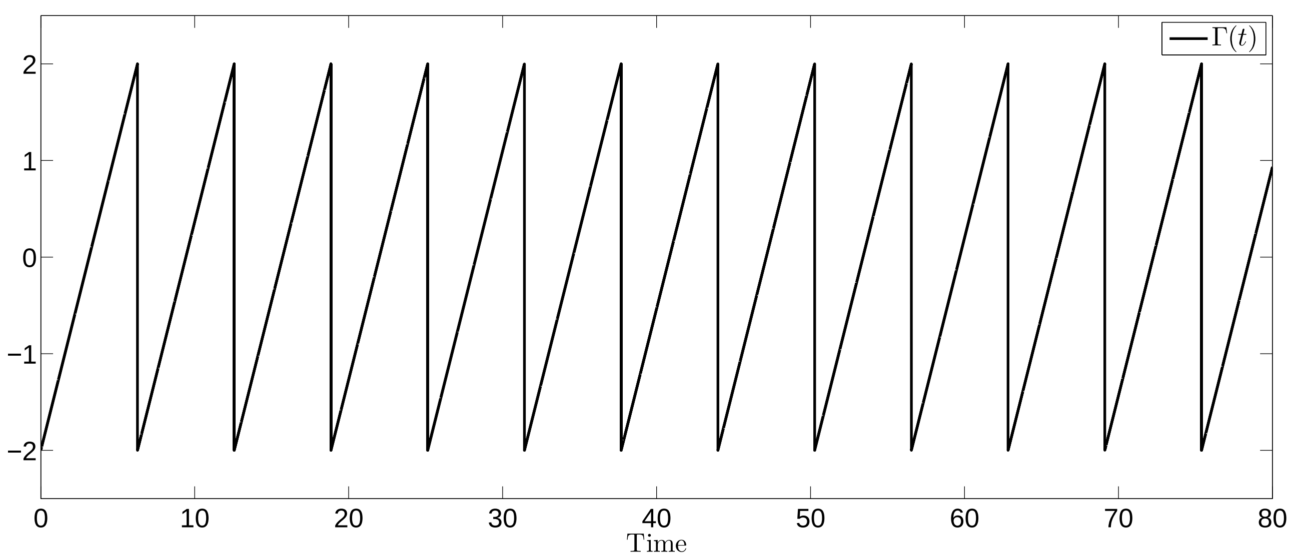

FIGURE 3.11 – Robust H_∞ uncertain observers : Variation $\Gamma(t)$.

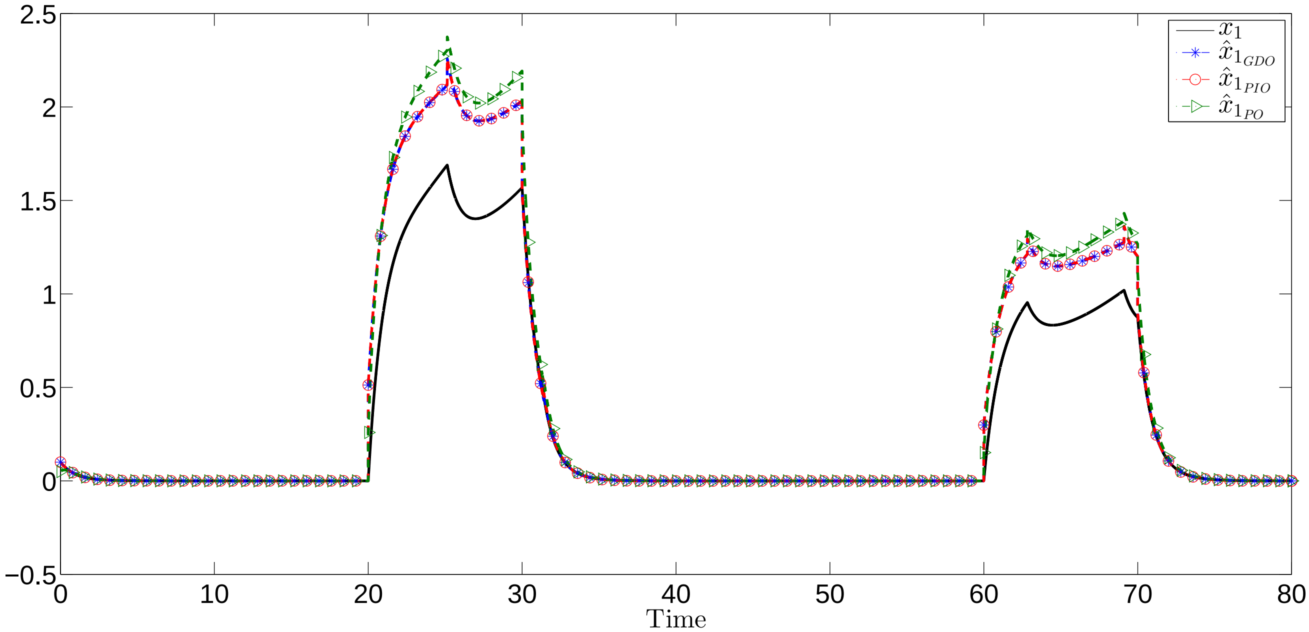

FIGURE 3.12 – Robust H_∞ uncertain observers : Estimate of $x_1(t)$.

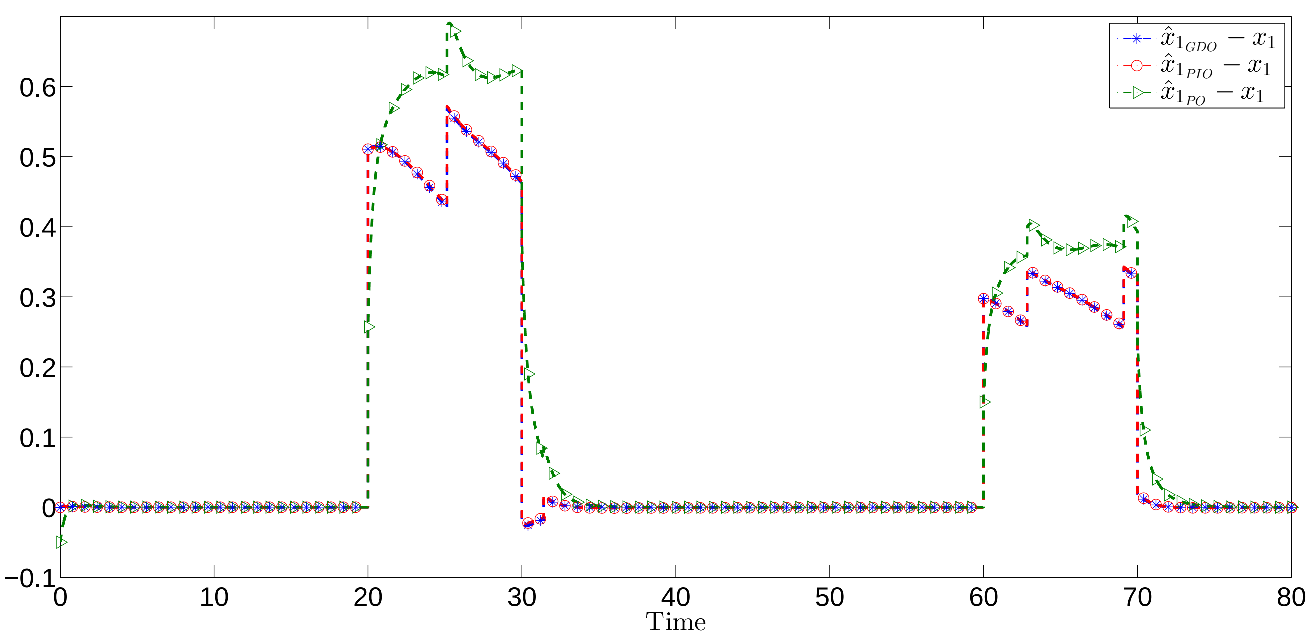

FIGURE 3.13 – Robust H_∞ uncertain observers : Estimation error of $x_1(t)$.

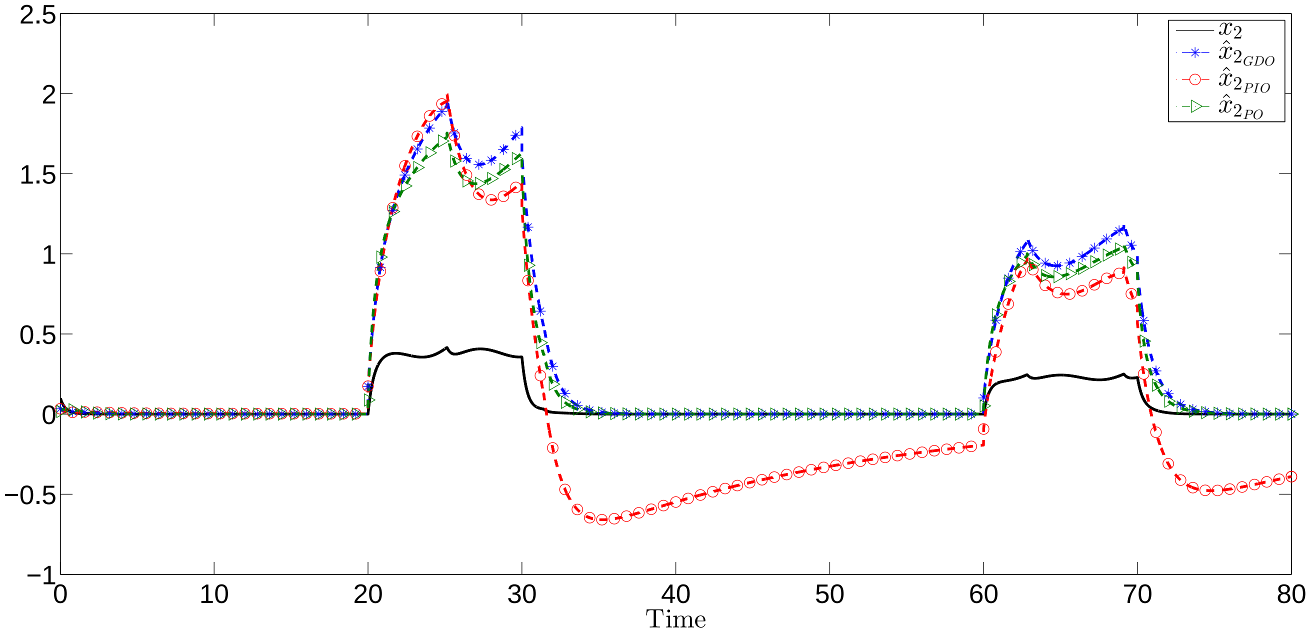

FIGURE 3.14 – Robust H_∞ uncertain observers : Estimate of $x_2(t)$.

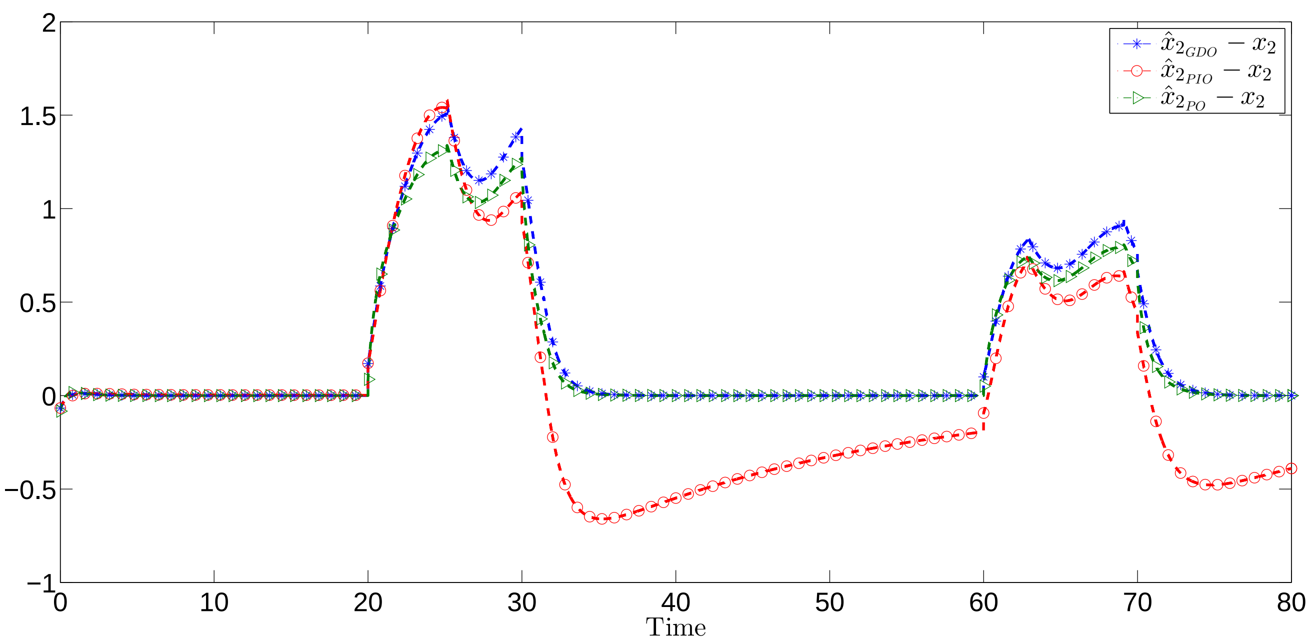

FIGURE 3.15 – Robust H_∞ uncertain observers : Estimation error of $x_2(t)$.

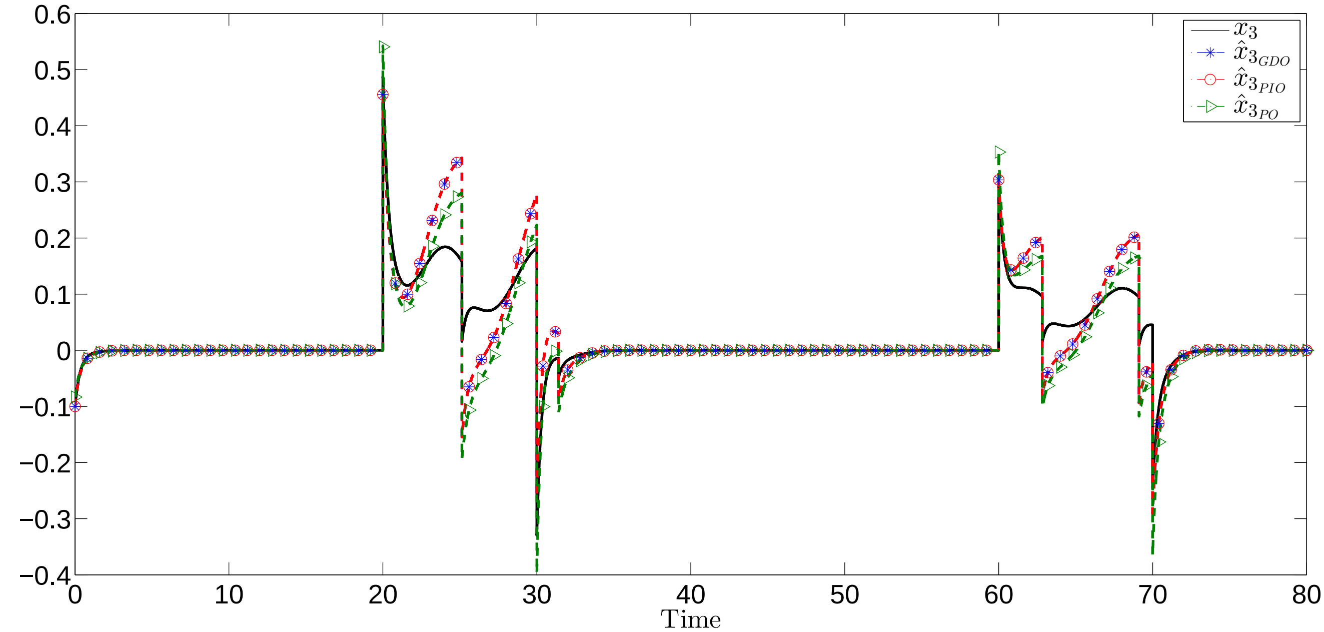

FIGURE 3.16 – Robust H_∞ uncertain observers : Estimate of $x_3(t)$.

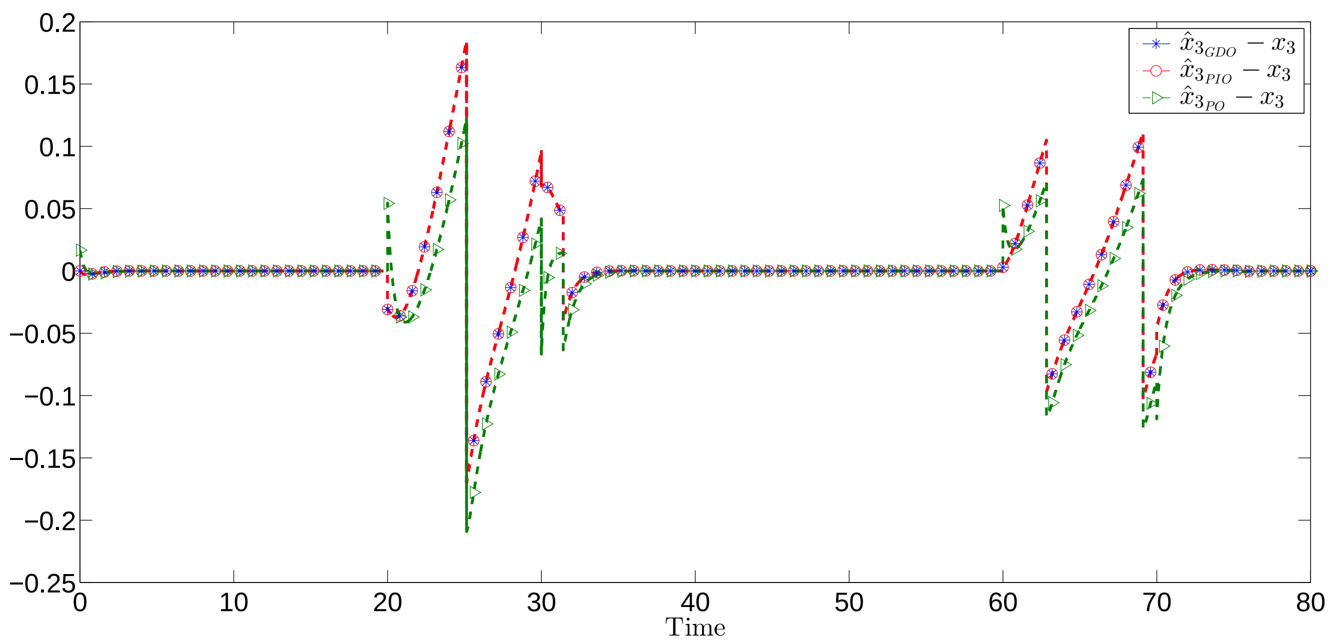

FIGURE 3.17 – Robust H_∞ uncertain observers : Estimation error of $x_3(t)$.

From these results, we can see that the behavior between the GDO and the PIO are similar. In order to highlight the difference in the estimations between the observers, the following table with the IAE is presented.

TABLE 3.2 – Robust H_∞ uncertain observers : Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	8008.3	8057.96	10103.16
$x_2(t)$	21386.44	32393.6	18961.28
$x_3(t)$	1317.32	1317.32	1247.56

From these error evaluations, we can see that, the GDO has the smaller value for the estimation of the unmeasured state, while the others state estimation are closer for the GDO and PO.

3.3 H_∞ generalized dynamic observer design for LPV systems

In this section the H_∞ LPV GDO design for LPV descriptor systems with or without disturbances is presented.

3.3.1 Class of disturbed LPV descriptor systems considered

Consider the LPV descriptor system described by :

$$\begin{aligned} E\dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) + Dw(t) \\ y(t) &= C_1x(t) + D_1w(t) \end{aligned} \tag{3.72}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ the input vector, $w(t) \in \mathbb{R}^{n_w}$ the disturbance vector and $y(t) \in \mathbb{R}^{n_y}$ represents the measured outputs vector. Matrix $E \in \mathbb{R}^{n_1 \times n}$, let $\mathsf{rank}(E) = r < n$. Matrices $A(\rho(t)) \in \mathbb{R}^{n_1 \times n}$, $B(\rho(t)) \in \mathbb{R}^{n_1 \times m}$, $D \in \mathbb{R}^{n_1 \times n_y}$, $C_1 \in \mathbb{R}^{n_y \times n_1}$ and $D_1 \in \mathbb{R}^{n_y \times n_w}$ are known matrices. $\rho(t) = \{\rho_1(t), \dots, \rho_j(t)\}$ is the vector of j variant parameters.

Considering that the parameter ρ varies in a convex polytope of τ vertices, where each vertex corresponds to the extreme values of ρ_k, $\forall k \in [1, \dots, j]$. Under this consideration, the structure of the LPV descriptor system (3.72) is :

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{\tau} \sigma_i(\rho(t))(A_ix(t) + B_iu(t)) + Dw(t) \\ y(t) &= C_1x(t) + D_1w(t) \end{aligned} \tag{3.73}$$

where

$$\sum_{i=1}^{\tau} \sigma_i(\rho(t)) = 1,\ 0 \le \sigma_i(\rho(t)) \le 1 \tag{3.74}$$

$\forall i \in [1, \dots, \tau]$ where $\tau = 2^j$. $\sigma_i(\rho(t)) = \sigma(\overline{\rho}_i,\ \underline{\rho}_i,\ \rho_i(t),\ t)$ ($\overline{\rho}_i$ and $\underline{\rho}_i$ represent the maximum and the minimum value of ρ_i respectively).

In the sequel we assume that :

Assumption 3.4.

$$\mathsf{rank} \begin{bmatrix} E \\ E^{\perp}A_i \\ C_1 \end{bmatrix} = n,\ \forall i \in [1, \dots, \tau]$$

Assumption 3.5.

$$\mathsf{rank} \begin{bmatrix} sE - A_i \\ C_1 \end{bmatrix} = n,\ \forall s \in \mathbb{C}^+,\ s \textit{ finite, and } \forall i \in [1, \dots, \tau]$$

3.3.2 Problem formulation

Consider the following GDO for system (3.73)

$$\dot{\zeta}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(N_i\zeta(t) + H_i v(t) + F_i y(t) + J_i u(t)) \tag{3.75a}$$

$$\dot{v}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(S_i\zeta(t) + L_i v(t) + M_i y(t)) \tag{3.75b}$$

$$\hat{x}(t) = \zeta(t) + Q_y(t) \tag{3.75c}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. Matrices N_i, H_i, F_i, S_i, L_i, M_i and Q are unknown matrices of appropriate dimensions.

Now, we can give the following lemma.

Lemma 3.2. *There exists an observer of the form* (3.75) *for the system* (3.73) *if the matrix* $\begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix}$ *is Hurwitz* $\forall i \in [1, \dots, \tau]$ *when* $w(t) = 0$*, and if there exists a matrix* T *of appropriate dimension such that the following conditions are satisfied*
(a) $N_iTE + F_iC_1 - TA_i = 0$
(b) $J_i = TB_i$
(c) $S_iTE + M_iC_1 = 0$
(d) $TE + QC_1 = I_n$

Proof. Let $T \in \mathbb{R}^{q_0 \times n_1}$ be a parameter matrix and define the error $\varepsilon(t) = \zeta(t) - TEx(t)$, then its derivative is given by

$$\dot{\varepsilon}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(N_i\varepsilon + H_i v(t) + (N_iTE + F_iC_1 - TA_i)x(t) + (J_i - TB_i)u(t) + (F_iD_1 - TD)w(t)) \tag{3.76}$$

By using the definition of $\varepsilon(t)$, equations (3.75b) and (3.75c) can be written as :

$$\dot{v}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(S_i\varepsilon(t) + L_i v(t) + (S_iTE + M_iC_1)x(t) + M_iD_1w(t)) \tag{3.77}$$

$$\hat{x}(t) = \varepsilon(t) + (TE + QC_1)x(t) + QD_1w(t) \tag{3.78}$$

Now, if conditions $(a) - (d)$ of Lemma 3.2 are satisfied the following observer error dynamics is obtained from (3.76) and (3.77)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \sum_{i=1}^{\tau} \sigma_i(\rho(t)) \left(\begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} F_iD_1 - TD \\ M_iD_1 \end{bmatrix} w(t) \right) \tag{3.79}$$

and from equation (3.78) we have :

$$\hat{x}(t) - x(t) = e(t) = \varepsilon(t) + QD_1w(t) \tag{3.80}$$

in this case if $w(t) = 0$ and matrix $\begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix}$ is Hurwitz, then $\lim_{t\to\infty} e(t) = 0$. □

Now, the parameterization of all solutions to the algebraic constraints $(a) - (d)$ of Lemma 3.2 are given.

Condition (d) of Lemma 3.2 can be written as

$$\begin{bmatrix} T & Q \end{bmatrix} \Sigma = I_n \tag{3.81}$$

where $\Sigma = \begin{bmatrix} E \\ C \end{bmatrix}$. The necessary and sufficient conditions for (3.81) to have solution is

$$\operatorname{rank} \begin{bmatrix} E \\ C_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} E \\ C_1 \\ I_n \end{bmatrix} = n \tag{3.82}$$

since (3.82) is satisfied, the solution to (3.81) is given by :

$$\begin{bmatrix} T & Q \end{bmatrix} = \Sigma^+ \tag{3.83}$$

where Σ^+ is any generalized inverse of Σ (see Definition 2.1). Equation (3.83) is equivalent to :

$$T = \Sigma^+ \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix} \tag{3.84}$$

$$Q = \Sigma^+ \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix} \tag{3.85}$$

Now, by taking TE from the condition (d) of Lemma 3.2 and replacing it in condition (a) we get :

$$N_i = TA_i + K_iC_1 \tag{3.86}$$

where $K_i = N_iQ - F_i$.

On the other hand, by replacing TE in condition (c) of Lemma 3.2 we get :

$$S_i = Z_iC_1 \tag{3.87}$$

where $Z_i = S_iQ - M_i$.

From the definition of K_i in equation (3.86) we can obtain :

$$F_i = TA_iQ + K_i(C_1Q - I) \tag{3.88}$$

and from the definition of Z_i in equation (3.87) we get :

$$M_i = Z_i(CQ - I). \tag{3.89}$$

In order to study the observer stability, the observer error dynamics (3.79) and (3.80) can be writing as :

$$\dot{\varphi}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))((\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)\varphi(t) + (\mathbb{B}_{1i} + \mathbb{Y}_i\mathbb{B}_2)w(t)) \tag{3.90a}$$

$$e(t) = \mathbb{P}\varphi(t) + \mathbb{Q}w(t) \tag{3.90b}$$

where $\mathbb{A}_{1i} = \begin{bmatrix} TA_i & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} C_1 & 0 \\ 0 & I_{q_1} \end{bmatrix}$, $\mathbb{B}_{1i} = \begin{bmatrix} TA_iQD_1 - TD \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} (C_1Q - I)D_1 \\ 0 \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} I & 0 \end{bmatrix}$, $\mathbb{Q} = QD_2$, $\mathbb{Y}_i = \begin{bmatrix} K_i & H_i \\ Z_i & H_i \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

3.3.3 Generalized dynamic observer design for LPV descriptor systems, $w(t) = 0$

In this section we consider $w(t) = 0$, then system (3.73) becomes :

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{\tau} \sigma_i(\rho(t))(A_ix(t) + B_iu(t)) \\ y(t) &= C_1x(t) \end{aligned} \tag{3.91}$$

and the GDO

$$\dot{\zeta}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(N_i\zeta(t) + H_iv(t) + F_iy(t) + J_iu(t)) \tag{3.92a}$$

$$\dot{v}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(S_i\zeta(t) + L_iv(t) + M_iy(t)) \tag{3.92b}$$

$$\hat{x}(t) = \zeta(t) + Q_y(t) \tag{3.92c}$$

and the error dynamics (3.90a) becomes :

$$\dot{\varphi}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))((\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)\varphi(t)) \tag{3.93}$$

where $\mathbb{A}_{1i} = \begin{bmatrix} TA_i & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} C_1 & 0 \\ 0 & I_{q_1} \end{bmatrix}$, $\mathbb{Y}_i = \begin{bmatrix} K_i & H_i \\ Z_i & H_i \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

The following theorem shows the stability conditions of the GDO (3.92) for the descriptor system (3.91) in form of LMIs.

Theorem 3.3. *Under Assumptions 3.4 and 3.5, there exists a parameter matrix $\mathbb{Y}_i$ such that the error dynamics (3.93) is asymptotically stable if and only if there exists a symmetric positive matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$, with $X_1 = X_1^T$, such that the following LMI is satisfied.*

$$\mathcal{C}^{T\perp} \begin{bmatrix} (TA_i)^T X_1 + X_1(TA_i) & (*) \\ X_2^T(TA_i) & 0 \end{bmatrix} \mathcal{C}^{T\perp T} < 0. \tag{3.94}$$

The matrix $\mathbb{Y}_i$ is parameterized as follows :

$$\mathbb{Y}_i = X^{-1}(\mathcal{K}_i\mathcal{C}_l^+ - \mathcal{Z}(I - \mathcal{C}_l\mathcal{C}_l^+)) \tag{3.95}$$

where

$$\mathcal{K}_i = \mathcal{R}^{-1}\vartheta_i\mathcal{C}_r^T(\mathcal{C}_r\vartheta_i\mathcal{C}_r^T)^{-1} + \mathcal{S}_i^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta_i\mathcal{C}_r^T)^{-1/2} \tag{3.96a}$$

$$\vartheta_i = (\mathcal{R}^{-1} - \mathcal{D}_i)^{-1} > 0 \tag{3.96b}$$

$$\mathcal{S}_i = \mathcal{R}^{-1} - \mathcal{R}^{-1}[\vartheta_i - \vartheta_i\mathcal{C}_r^T(\mathcal{C}_r\vartheta_i\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta_i]\mathcal{R}^{-1} \tag{3.96c}$$

where $\mathcal{D}_i = \begin{bmatrix} (TA_i)^T X_1 + X_1(TA_i) & (*) \\ X_2^T(TA_i) & 0 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix}$, and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$.

Proof. Consider the following Lyapunov function

$$V(\varphi(t)) = \varphi(t)^T X \varphi(t) > 0 \tag{3.97}$$

with $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0$ and $X_1 = X_1^T$. Its detivative along the trajectory of (3.93) is given by

$$\dot{V}(\varphi(t)) = \varphi(t)^T[(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)^T X + X(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)]\varphi(t) < 0 \tag{3.98}$$

From Section 1.7.1, the asymptotic stability of observer (3.92) is guaranteed if and only if $\dot{V}(\varphi(t)) < 0$. This leads to the following LMI :

$$(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)^T X + X(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2) < 0 \tag{3.99}$$

which can be written as :

$$\mathcal{B}\mathcal{X}_i\mathcal{C} + (\mathcal{B}\mathcal{X}_i\mathcal{C})^T + \mathcal{D}_i < 0 \tag{3.100}$$

where $\mathcal{X} = X\mathbb{Y}_i$, $\mathcal{B} = I$, $\mathcal{C} = \mathbb{A}_2$ and $\mathcal{D}_i = \mathbb{A}_{1i}^T X + X\mathbb{A}_{1i}$. Using the elimination lemma of Section 1.5 and since $\mathcal{B}^\perp = 0$, inequality (3.100) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}_i\mathcal{C}^{T\perp T} < 0 \tag{3.101}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix}^{T\perp}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}_i$ and X the inequality (3.101) becomes :

$$\mathcal{C}^{T\perp} \begin{bmatrix} (TA_i)^T X_1 + X_1(TA_i) & (*) \\ X_2^T(TA_i) & 0 \end{bmatrix} \mathcal{C}^{T\perp T} < 0. \tag{3.102}$$

From the elimination lemma if condition (3.101) is satisfied, the parameter matrix $\mathbb{Y}_i$ is parameterized as in (3.95) and (3.96). □

3.3.3.1 Particular cases

In this section we consider two particular cases of our results.

•Proportional observer

Consider the following LPV descriptor system :

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{\tau}\sigma_i(\rho(t))(A_ix(t)+B_iu(t)) \\ y(t) &= C_1x(t) \end{aligned}$$

with the PO :

$$\begin{aligned} \dot{\zeta}(t) &= \sum_{i=1}^{\tau}\sigma_i(\rho(t))(N_i\zeta(t)+F_iy(t)+J_iu(t)) \\ \hat{x}(t) &= \zeta(t)+Q_y(t) \end{aligned}$$

the observer error dynamics (3.93) becomes :

$$\dot{\varepsilon}(t) = \sum_{i=1}^{\tau}\sigma_i(\rho(t))(\bar{\mathbb{A}}_{1i}+\bar{\mathbb{Y}}_i\bar{\mathbb{A}}_2)\varepsilon(t) \tag{3.103}$$

where $\bar{\mathbb{A}}_{1i}=TA_i$, $\bar{\mathbb{A}}_2=C_1$ and $\bar{\mathbb{Y}}_i=K_i$. Consequently, matrices $\mathcal{D}_i$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}_i$ of Theorem 3.3 become :

$$\mathcal{D}_i=(TA_i)^TX+X(TA_i),\ \mathcal{C}=C_1,\ \mathcal{B}=I \text{ and } \mathcal{X}_i=X\bar{\mathbb{Y}}_i.$$

•Proportional-integral observer

Consider the following LPV descriptor system :

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{\tau}\sigma_i(\rho(t))(A_ix(t)+B_iu(t)) \\ y(t) &= C_1x(t) \end{aligned}$$

with the PIO :

$$\begin{aligned} \dot{\zeta}(t) &= \sum_{i=1}^{\tau}\sigma_i(\rho(t))(N_i\zeta(t)+H_iv(t)+F_iy(t)+J_iu(t)) \\ \dot{v}(t) &= y(t)-C_1\hat{x}(t) \\ \hat{x}(t) &= \zeta(t)+Q_y(t) \end{aligned}$$

and the observer error dynamics (3.93) becomes :

$$\dot{\varphi}(t) = \sum_{i=1}^{\tau}\sigma_i(\rho(t))(\bar{\bar{\mathbb{A}}}_{1i}-\bar{\bar{\mathbb{Y}}}_i\bar{\bar{\mathbb{A}}}_2)\varphi(t) \tag{3.104}$$

where $\bar{\bar{\mathbb{A}}}_{1i}=\begin{bmatrix}TA_i & 0\\ -C_1 & 0\end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2=\begin{bmatrix}C_1 & 0\\ 0 & I\end{bmatrix}$ and $\bar{\bar{Y}}_i=\begin{bmatrix}I\\0\end{bmatrix}\begin{bmatrix}K_i & H_i\end{bmatrix}$. Consequently, matrices $\mathcal{D}_i$, $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{X}_i$ of Theorem 3.3 become :

$$\mathcal{D}_i=\begin{bmatrix}(TA_i)^TX_1+X_1(TA_i)-C_1^TX_2^T-X_2C_1 & (*)\\ X_2^T(TA_i)-X_3C_1 & 0\end{bmatrix},\ \mathcal{C}=\begin{bmatrix}C_1 & 0\\ 0 & I\end{bmatrix},\ \mathcal{B}=I \text{ and } \mathcal{X}_i=X\bar{\bar{Y}}_i$$

3.3.3.2 Numerical example

In order to illustrate the results obtained, consider the following LPV descriptor system described by (3.91) where

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\ A(\rho(t))=\begin{bmatrix}-2.7&\rho(t)&0.3\\-0.2&-\rho(t)&0\\-0.11+\rho(t)&1.74&-1\end{bmatrix},\ B(\rho(t))=\begin{bmatrix}\rho(t)\\0\\1\end{bmatrix}\text{ and } C=\begin{bmatrix}0&0&1\end{bmatrix}$$

In this case the polytope has 2 vertices corresponding to the extreme values of parameter $\rho(t)$, which varies from 1 to 3.

The parameters $\sigma_i(\rho(t))$ are :

$$\sigma_1(\rho(t))=\frac{\overline{\rho}-\rho(t)}{\overline{\rho}-\underline{\rho}}=\frac{3-\rho(t)}{2}$$

$$\sigma_2(\rho(t))=\frac{\rho(t)-\underline{\rho}}{\overline{\rho}-\underline{\rho}}=\frac{\rho(t)-1}{2}$$

Considering $E^{\perp}=\begin{bmatrix}0&0&1\end{bmatrix}$, we can verify Assumptions 3.4 and 3.5

$$\mathsf{rank}\begin{bmatrix}E\\E^{\perp}A_i\\C_1\end{bmatrix}=3\quad\text{and}\quad\mathsf{rank}\begin{bmatrix}sE-A_i\\C_1\end{bmatrix}=3,\ \forall i\in[1,2]$$

Generalized dynamic observer

By using YALMIP toolbox we have solved the LMI (3.94) for the GDO to find matrix X

$$X=\begin{bmatrix}4.81&0.87&0&0&0&0\\0.87&4.92&0&0&0&0\\0&0&24.29&0&0&0\\0&0&0&24.29&0&0\\0&0&0&0&24.29&0\\0&0&0&0&0&24.29\end{bmatrix}.$$

Now, considering matrices $\mathcal{Z}=\begin{bmatrix}34&92&81&93\\45&12&65&23\\3&54&23&5\\12&6&23&7\\67&23&67&4\\2&74&11&6\end{bmatrix}$, $\mathcal{L}=\begin{bmatrix}0.01&0.01&0.01&0.01\\0.01&0.01&0.01&0.01\\0.01&0.01&0.01&0.01\\0.01&0.01&0.01&0.01\\0.01&0.01&0.01&0.01\\0.01&0.01&0.01&0.01\end{bmatrix}$, and $\mathcal{R}=I_6\times 0.001$ and by solving (3.95) and (3.96) we get :

$$\mathbb{Y}_1=\begin{bmatrix}0.02&0.31&0.31&0.31\\-0.27&0.22&0.22&0.22\\-40.76&0.41&0.41&0.41\\0.41&-40.76&0.41&0.41\\0.41&0.41&-40.76&0.41\\0.41&0.41&0.41&-40.76\end{bmatrix}\text{ and }\mathbb{Y}_2=\begin{bmatrix}0&0.29&0.29&0.29\\-0.18&0.31&0.31&0.31\\-40.76&0.41&0.41&0.41\\0.41&-40.76&0.41&0.41\\0.41&0.41&-40.76&0.41\\0.41&0.41&0.41&-40.76\end{bmatrix}.$$

Finally, we can compute all the matrices of the observer as :

$$N_1=\begin{bmatrix}-2.7&1&0.32\\-0.64&-1.87&0.23\\0&0&-40.76\end{bmatrix},\ N_2=\begin{bmatrix}-2.7&3&0.3\\-1.64&-3.87&0.32\\0&0&-40.76\end{bmatrix},\ S_1=S_2=\begin{bmatrix}0&0&0.41\\0&0&0.41\\0&0&0.41\end{bmatrix},$$

$$J_1=\begin{bmatrix}1\\-0.5\\0\end{bmatrix},\ J_2=\begin{bmatrix}3\\-0.5\\0\end{bmatrix},\ H_1=\begin{bmatrix}0.31&0.31&0.31\\0.22&0.22&0.22\\0.41&0.41&0.41\end{bmatrix},\ H_2=\begin{bmatrix}0.29&0.29&0.29\\0.31&0.31&0.31\\0.41&0.41&0.41\end{bmatrix},$$

$$L_1 = L_2 = \begin{bmatrix} -40.76 & 0.41 & 0.41 \\ 0.41 & -40.76 & 0.41 \\ 0.41 & 0.41 & -40.76 \end{bmatrix},\ F_1 = F_2 = \begin{bmatrix} 0.3 \\ 0.5 \\ 0 \end{bmatrix},\ M_1 = M_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In order to provide a comparison of the GDO with the PIO and the PO, these latter are also designed.

Proportional observer

By considering matrices $\mathcal{Z} = \begin{bmatrix} 8 \\ 9 \\ 9 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ and $\mathcal{R} = I_3 \times 10$ the following PO matrices are obtained :

$$N_1 = \begin{bmatrix} -2.7 & 1 & 0.31 \\ -0.64 & -1.87 & 0.48 \\ 0 & 0 & -0.06 \end{bmatrix},\ N_2 = \begin{bmatrix} -2.7 & 3 & 0.31 \\ -1.64 & -3.87 & 0.50 \\ 0 & 0 & -0.06 \end{bmatrix},$$

$$F_{a1} = F_{a2} = \begin{bmatrix} 0.3 \\ 0.5 \\ 0 \end{bmatrix},\ J_1 = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix},\ J_2 = \begin{bmatrix} 3 \\ -0.5 \\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Proportional-integral observer

By considering $\mathcal{Z} = \begin{bmatrix} 8 & 2 \\ 9 & 3 \\ 9 & 2 \\ 9 & 3 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.001$ the following PIO matrices are obtained :

$$N_1 = \begin{bmatrix} -2.7 & 1 & 3.18 \\ -0.64 & -1.87 & 2.23 \\ 0 & 0 & -37 \end{bmatrix},\ N_2 = \begin{bmatrix} -2.7 & 3 & 2.93 \\ -1.64 & -3.87 & 3.13 \\ 0 & 0 & -37 \end{bmatrix},\ H_1 = \begin{bmatrix} 3.17 \\ 2.22 \\ 4.17 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 2.92 \\ 3.11 \\ 4.17 \end{bmatrix},\ F_1 = F_2 = \begin{bmatrix} 0.3 \\ 0.5 \\ 0 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix},\ J_2 = \begin{bmatrix} 3 \\ -0.5 \\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Simulation results

The initial conditions for the system are $x(0) = [0.1,\ 0.1,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0,\ 0]^T$, $v(0)_{GDO} = [0]$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0,\ 0]^T$, $v(0)_{PIO} = [0]$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0]^T$, and for the PO are $\zeta(0)_{PO} = [0,\ 0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0]^T$.

To evaluate the performance of the observers an uncertainty $\wp(t)$ is added in the system matrix A_i, then we obtain the matrix $(A_i + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

The results of the simulation are depicted in Figures 3.18 - 3.27. Figures 3.18 and 3.19 show the input $u(t)$ and the uncertainty factor $\delta(t)$. Figures 3.20 and 3.21 show the variation of parameter $\rho(t)$ and the weighting functions of each model. Figures 3.22 - 3.27 show the system states and their estimations by the GDO, PIO and PO, also these show the estimation error of each observer.

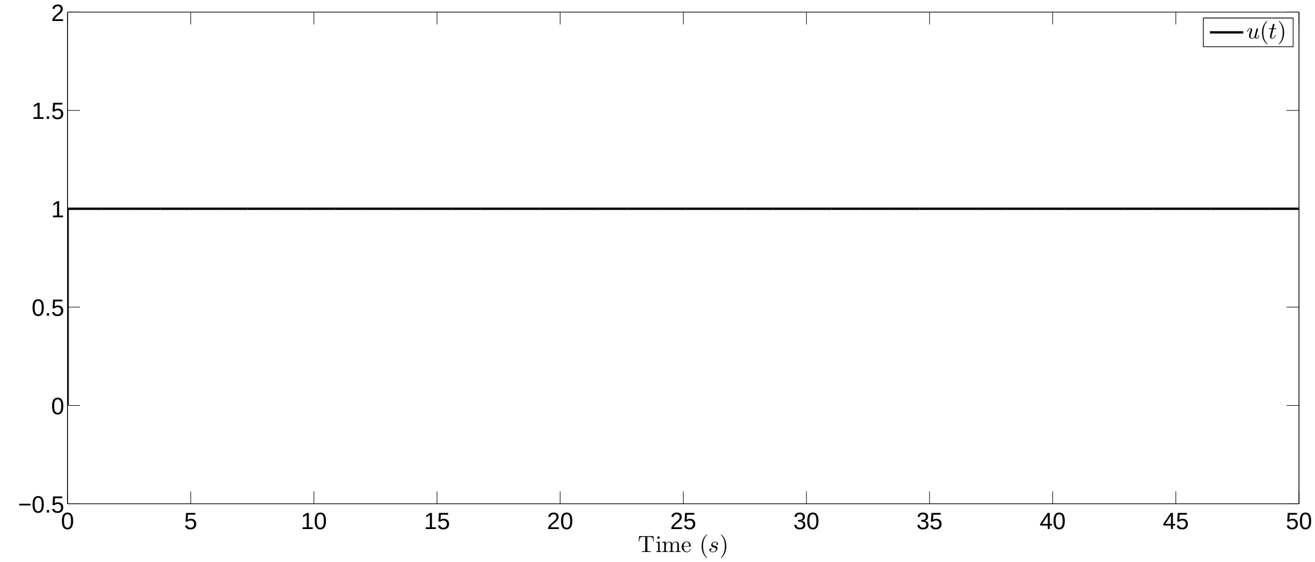

FIGURE 3.18 – LPV observers : Input $u(t)$.

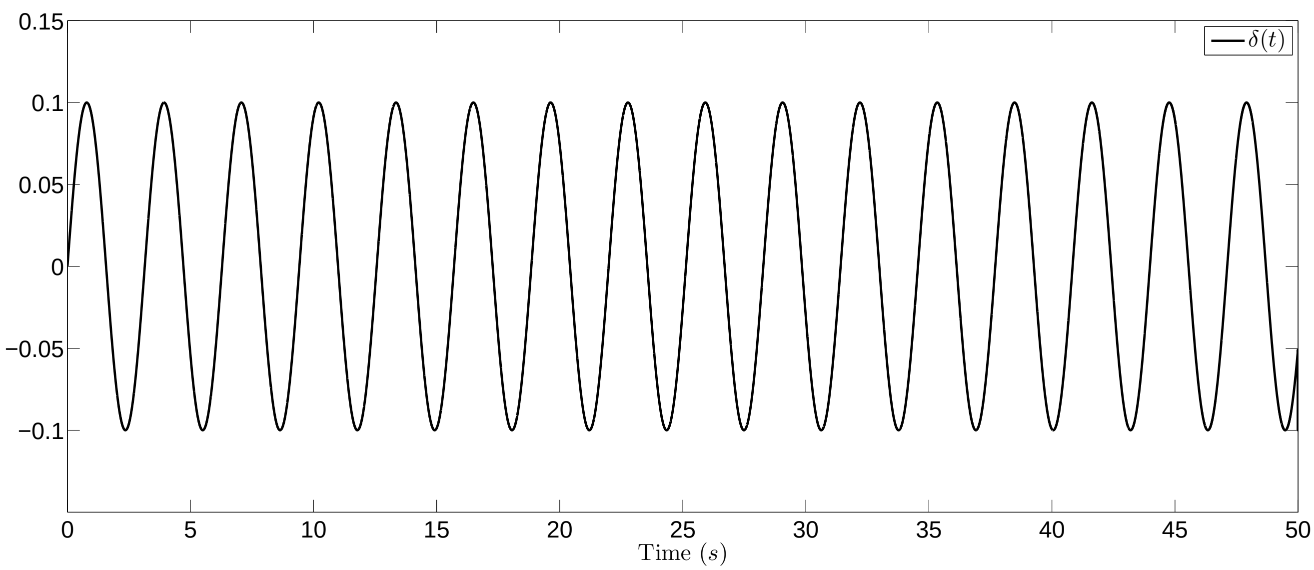

FIGURE 3.19 – LPV observers : Uncertainty factor $\delta(t)$.

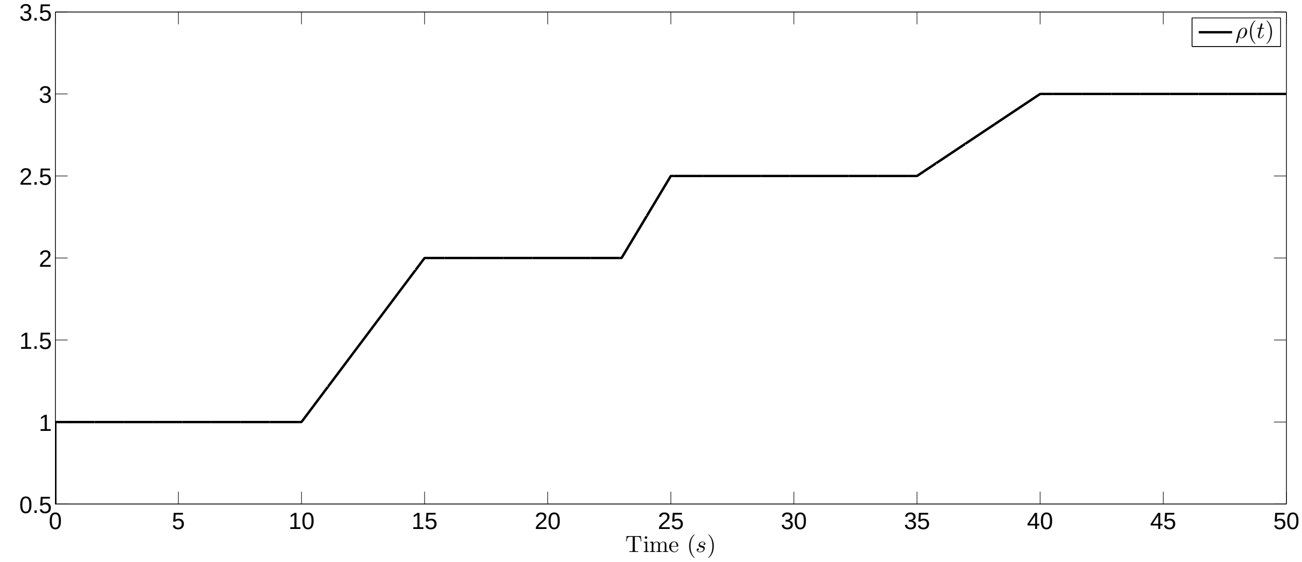

FIGURE 3.20 – LPV observers : Parameter variant $\rho(t)$.

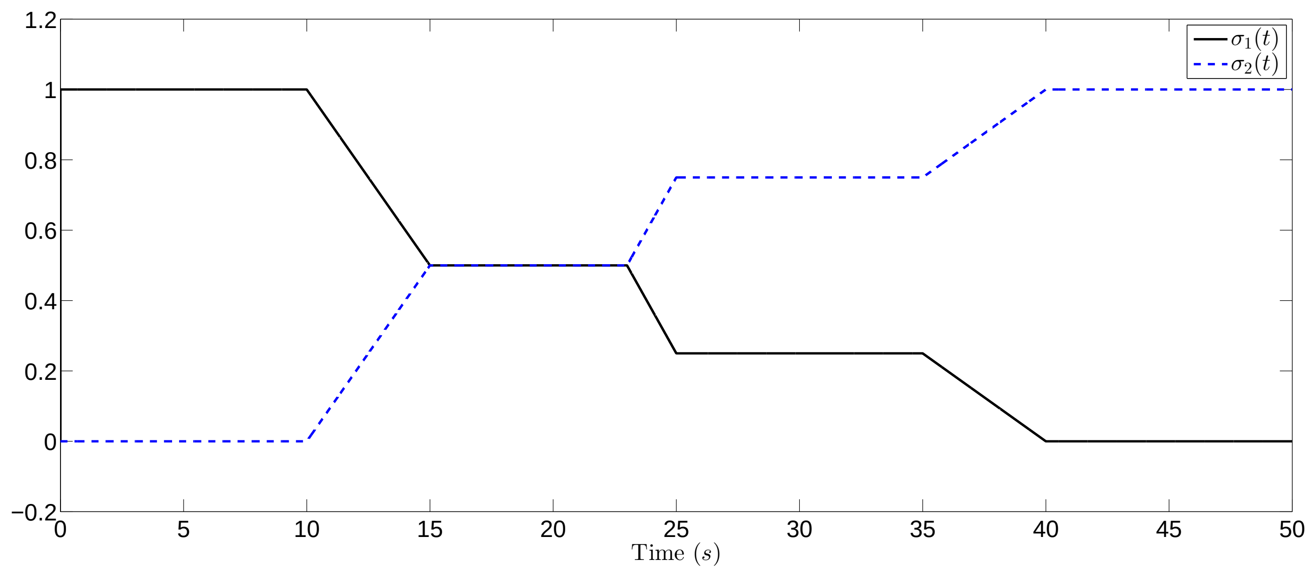

FIGURE 3.21 – LPV observers : Weighting functions $\sigma_1(t)$ and $\sigma_2(t)$.

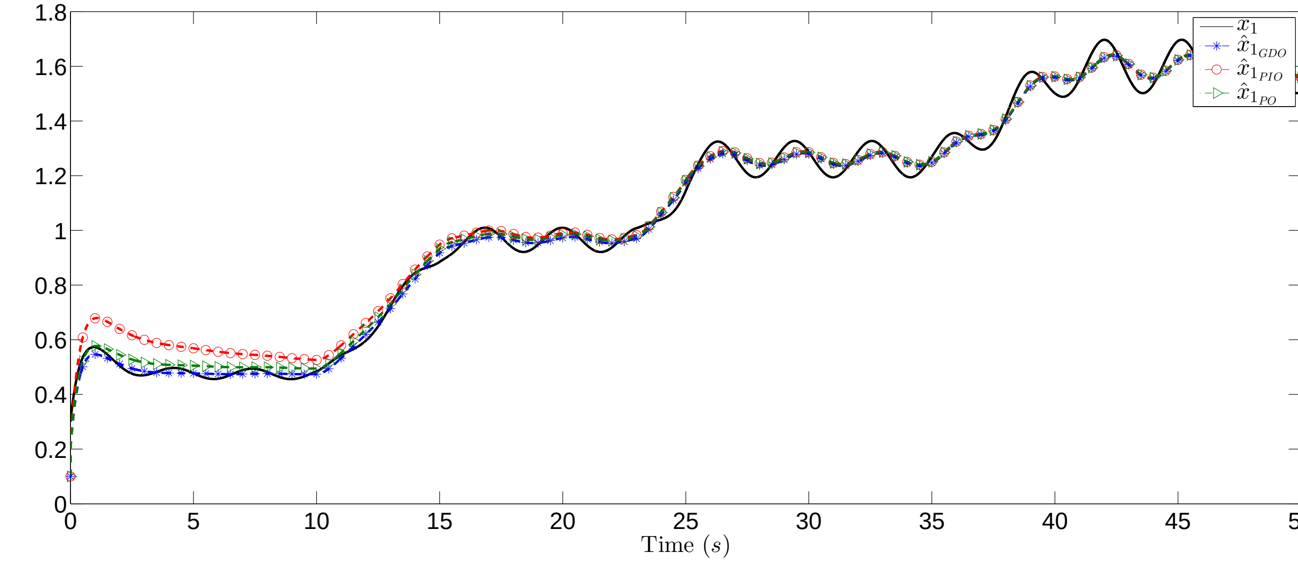

FIGURE 3.22 – LPV observers : Estimation of $x_1(t)$.

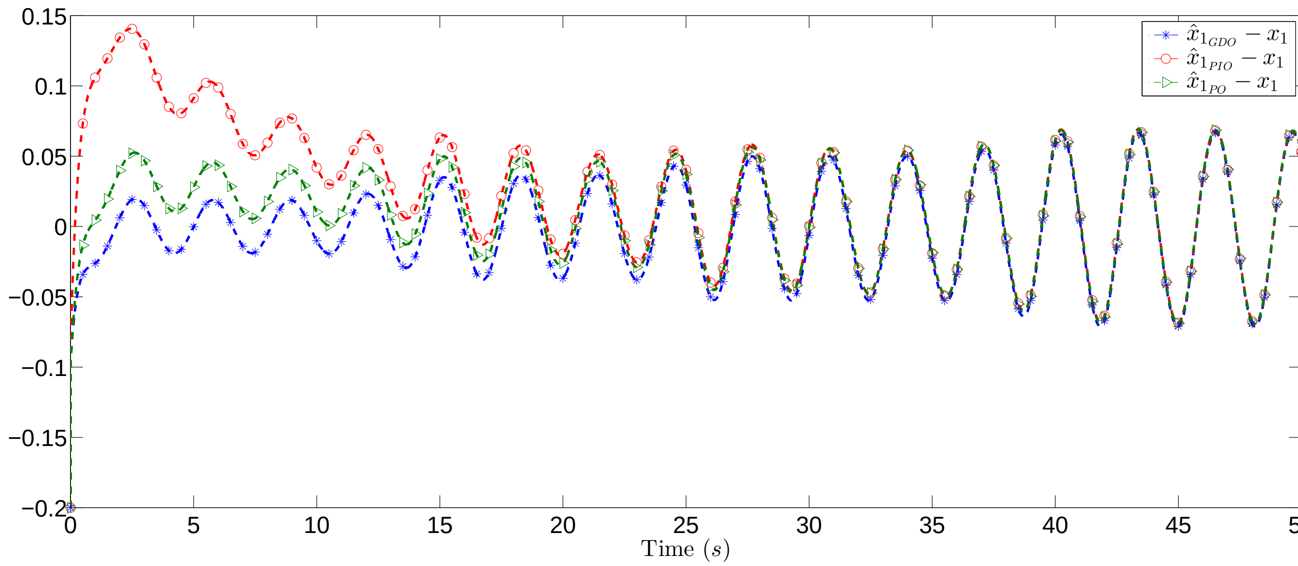

FIGURE 3.23 – LPV observers : Estimation error of $x_1(t)$.

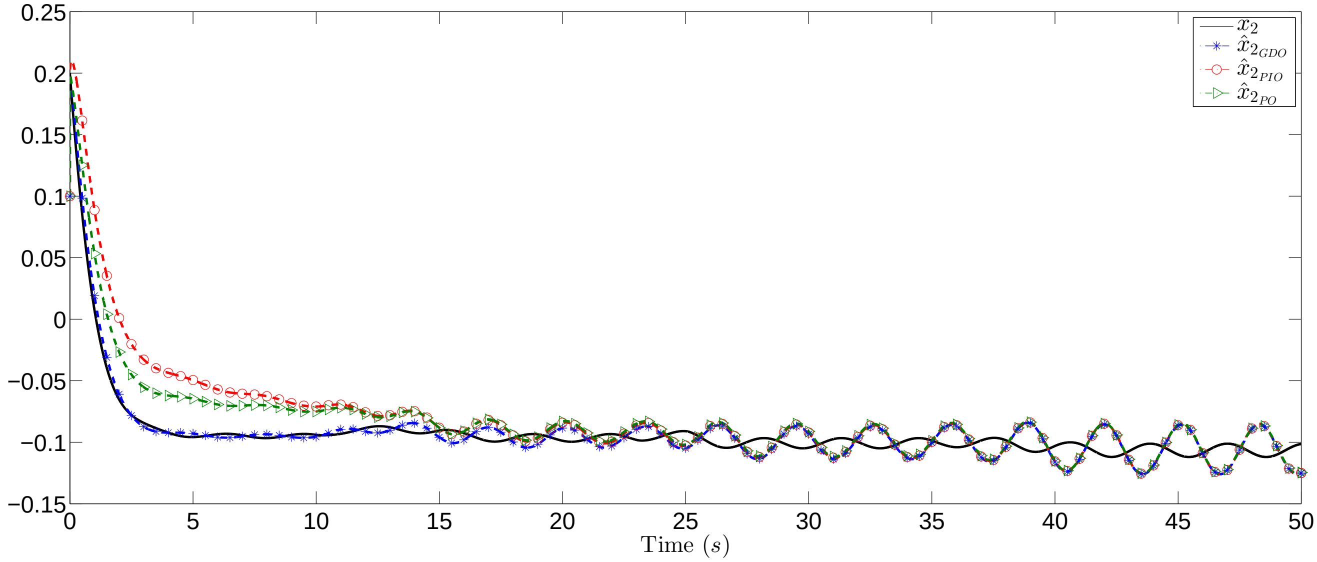

FIGURE 3.24 – LPV observers : Estimation of $x_2(t)$.

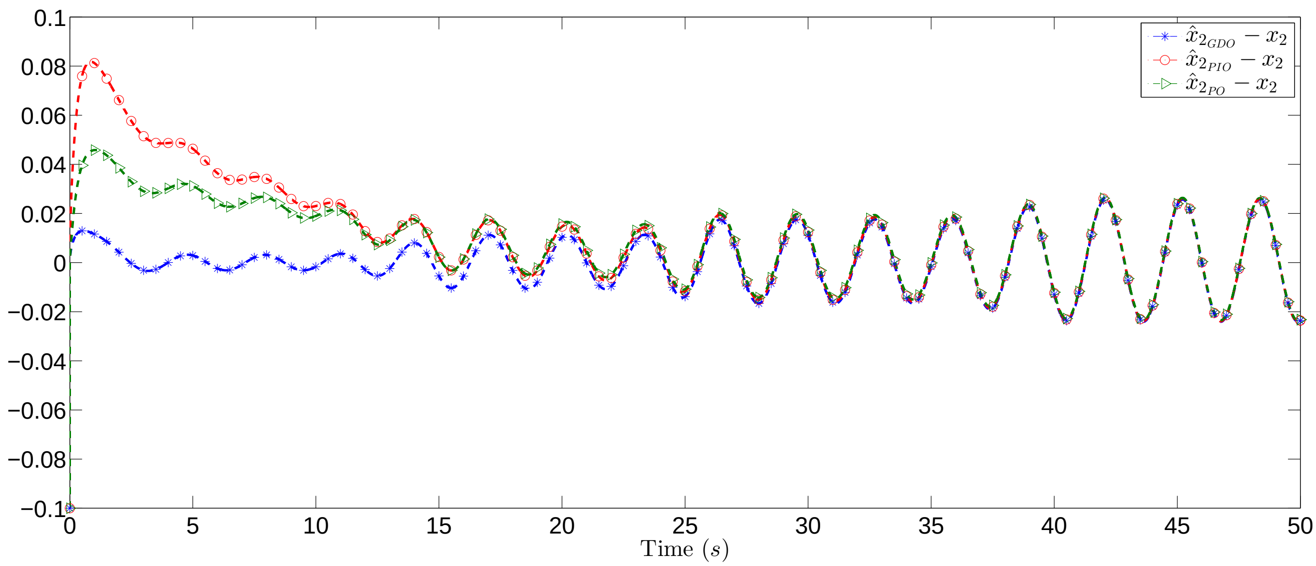

FIGURE 3.25 – LPV observers : Estimation error of $x_2(t)$.

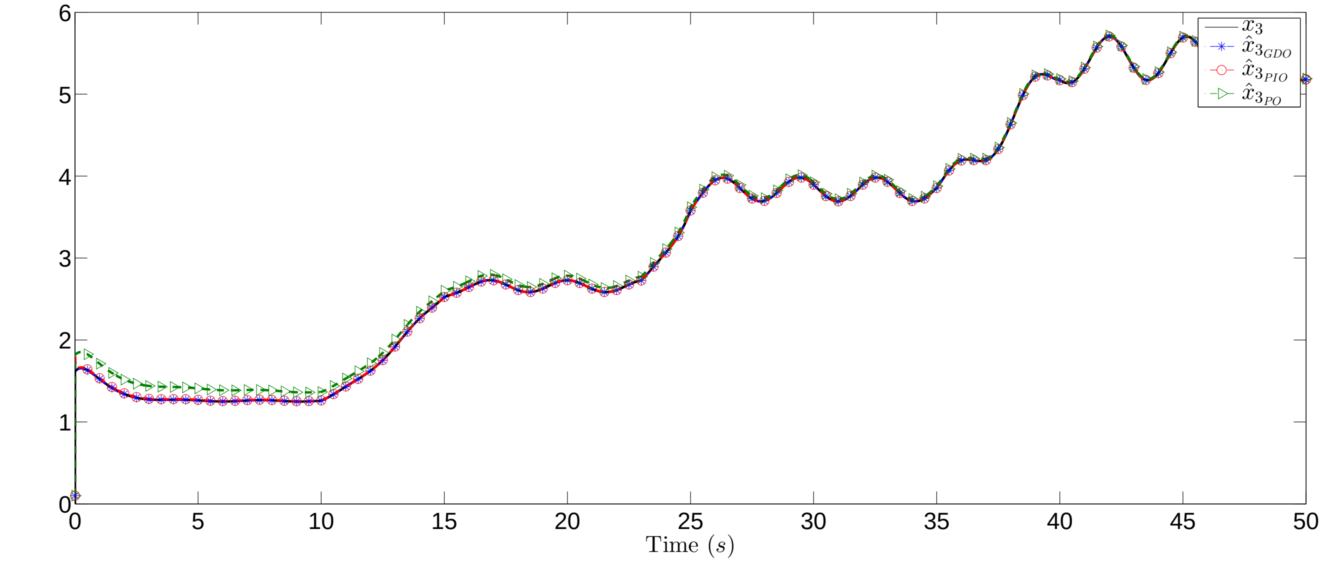

FIGURE 3.26 – LPV observers : Estimate of $x_3(t)$.

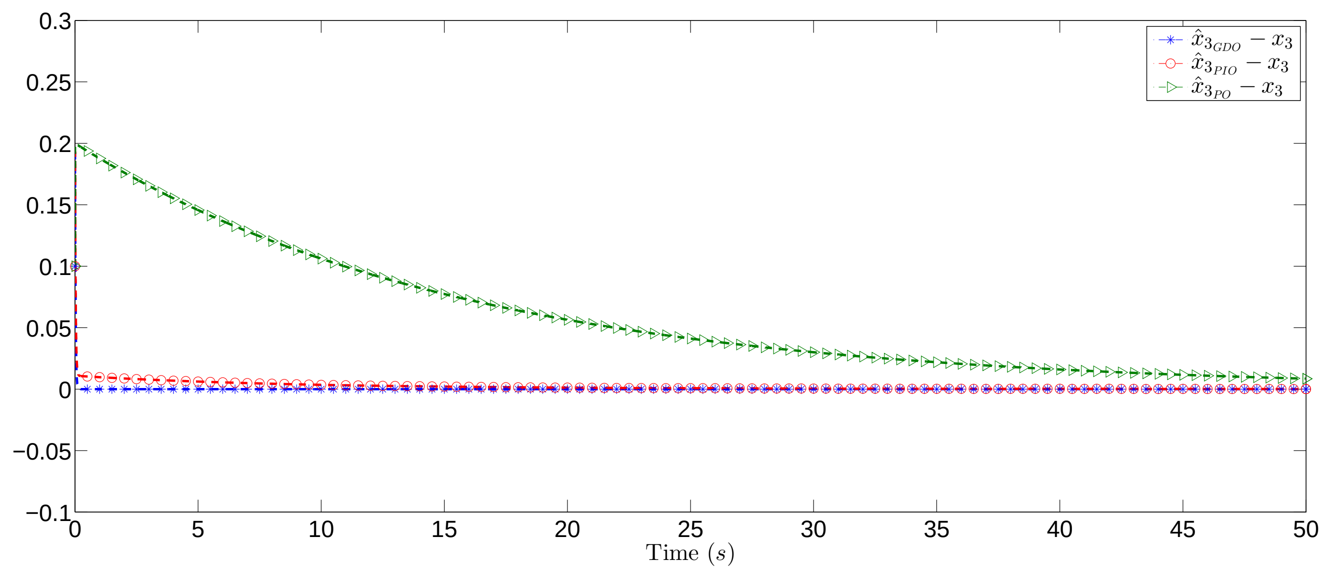

FIGURE 3.27 – LPV observers : Estimation error of $x_3(t)$.

From these results, we can see that the behavior of the estimated states are almost similar, however the main difference is in the transient state. In order to highlight this difference the following table is presented, where the IAE is presented.

TABLE 3.3 – LPV observers : Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	1432.03	2306.01	1584.42
$x_2(t)$	456.65	947.53	772.35
$x_3(t)$	4.88	104.59	3028.48

From these error evaluations, we can see that the GDO has smaller values compared with the PIO and the PO.

3.3.4 H_∞ generalized dynamic observer design for LPV disturbed descriptor systems, $w(t) \neq 0$

In this section we consider $w(t) \neq 0$, then we get system (3.73) :

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{\tau} \sigma_i(\rho(t))(A_i x(t) + B_i u(t)) + Dw(t) \\ y(t) &= C_1 x(t) + D_1 w(t) \end{aligned} \tag{3.105}$$

with the GDO :

$$\dot{\zeta}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(N_i\zeta(t) + H_i v(t) + F_i y(t) + J_i u(t)) \tag{3.106a}$$

$$\dot{v}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(S_i\zeta(t) + L_i v(t) + M_i y(t)) \tag{3.106b}$$

$$\hat{x}(t) = \zeta(t) + Q_y(t) \tag{3.106c}$$

and the observer error dynamics (3.90) :

$$\dot{\varphi}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))((\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)\varphi(t) + (\mathbb{B}_{1i} + \mathbb{Y}_i\mathbb{B}_2)w(t)) \tag{3.107a}$$

$$e(t) = \mathbb{P}\varphi(t) + \mathbb{Q}w(t) \tag{3.107b}$$

where $\mathbb{A}_{1i} = \begin{bmatrix} TA_i & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} C_1 & 0 \\ 0 & I_{q_1} \end{bmatrix}$, $\mathbb{B}_{1i} = \begin{bmatrix} TA_iQD_1 - TD \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} (C_1Q - I)D_1 \\ 0 \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} I & 0 \end{bmatrix}$, $\mathbb{Q} = QD_1$, $\mathbb{Y}_i = \begin{bmatrix} K_i & H_i \\ Z_i & H_i \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

In this section the observer matrices most be determined such that $\hat{x}(t)$ converges asymptotically to $x(t)$ for $w(t) = 0$, and for $w(t) \neq 0$ we must satisfy $\sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|e\|_2}{\|w\|_2} < \gamma$, where γ is a given positive scalar. The solution to this problem is given by the following theorem.

Theorem 3.4. *Under Assumptions 3.4 and 3.5, there exists an H_∞ GDO* (3.106) *such that the error dynamics* (3.107) *is stable and $\sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|e\|_2}{\|w\|_2} < \gamma$, if and only if there exists a matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0$, with $X_1 = X_1^T$, such that the following LMIs are satisfied.*

$$\mathcal{C}^{T\perp} \begin{bmatrix} (TA_i)^T X_1 + X_1(TA_i) & (TA_i)^T X_2 & X_1(TA_iQD_1 - TD) & I \\ X_2^T(TA_1) & 0 & X_2^T(TA_iQD_1 - TD) & 0 \\ (*) & (*) & -\gamma^2 I_{n_w} & (QD_1)^T \\ I & 0 & (*) & -I_n \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{3.108}$$

and

$$\begin{bmatrix} -\gamma^2 I_{n_w} & (QD_1)^T \\ (*) & -I_n \end{bmatrix} < 0. \tag{3.109}$$

Then, matrix $\mathbb{Y}_i$ *is parameterized as*

$$\mathbb{Y}_i = X^{-1}(\mathcal{B}_r^+\mathcal{K}_i\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+) \tag{3.110}$$

where

$$\mathcal{K}_i = -\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta_i\mathcal{C}_r^T(\mathcal{C}_r\vartheta_i\mathcal{C}_r^T)^{-1} + \mathcal{S}_i^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta_i\mathcal{C}_r^T)^{-1/2} \tag{3.111a}$$

$$\vartheta_i = (\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T - \mathcal{D}_i)^{-1} > 0 \tag{3.111b}$$

$$\mathcal{S}_i = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\vartheta_i - \vartheta_i\mathcal{C}_r^T(\mathcal{C}_r\vartheta_i\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta_i]\mathcal{B}_l\mathcal{R}^{-1} \tag{3.111c}$$

where $D_i = \begin{bmatrix} (TA_i)^TX_1 + X_1(TA_i) & (TA_i)^TX_2 & X_1(TA_iQD_1 - TD) & I \\ X_2^T(TA_1) & 0 & X_2^T(TA_iQD_1 - TD) & 0 \\ (*) & (*) & -\gamma^2 I_{n_w} & (QD_1)^T \\ I & 0 & (*) & -I_n \end{bmatrix}$,

$\mathcal{C} = \begin{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} (C_1Q - I)D_1 \\ 0 \end{bmatrix} & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} I_{q_0+q_1} \\ 0 \\ 0 \end{bmatrix}$, *and matrices* $\mathcal{L}$, $\mathcal{R}$ *and* $\mathcal{Z}$ *are arbitrary matrices of appropriate dimensions satisfying* $\mathcal{R} > 0$ *and* $\|\mathcal{L}\|_2 < 1$. *Matrices* $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ *and* $\mathcal{B}_r$ *are any full rank matrices such that* $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ *and* $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$.

Proof. Consider the Lyapunov function

$$V(\varphi(t)) = \varphi(t)^TX\varphi(t) \tag{3.112}$$

where $X = X^T > 0$, then the derivative along the trajectory of system (3.107) is

$$\dot{V}(\varphi(t)) = \varphi(t)^T\left[(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)^TX + X(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)\right]\varphi(t) + \varphi(t)^TX(\mathbb{B}_{1i} + \mathbb{Y}\mathbb{B}_2)w(t) + w(t)^T(\mathbb{B}_{1i} + \mathbb{Y}\mathbb{B}_2)^TX\varphi(t) \tag{3.113}$$

To satisfy $\sup\limits_{w\in\mathcal{L}_2-\{0\}} \dfrac{\|e\|_2}{\|w\|_2} < \gamma$ we get :

$$\int_0^\infty e(t)^Te(t) - \gamma^2 w(t)^Tw(t) + \dot{V}(\varphi(t)) < 0 \tag{3.114}$$

Replacing $e(t)$ from (3.107) and $\dot{V}(\varphi(t))$ from (3.113), we have that inequality (3.114) is equivalent to

$$\begin{aligned} &\varphi(t)^T\left[(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)^TX + X(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)\right]\varphi(t) + \varphi(t)^T\left[X(\mathbb{B}_{1i} + \mathbb{Y}\mathbb{B}_2) + \mathbb{P}^T\mathbb{Q}\right]w(t) + \\ &w(t)^T\left[(\mathbb{B}_{1i} + \mathbb{Y}\mathbb{B}_2)^TX + \mathbb{Q}^T\mathbb{P}\right]\varphi(t) + w(t)(\mathbb{Q}^T\mathbb{Q} - \gamma^2 I)w(t) \quad < 0 \end{aligned} \tag{3.115}$$

So, by applying the Schur complement to (3.115) we get :

$$\begin{bmatrix} (\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2)^TX + X(\mathbb{A}_{1i} + \mathbb{Y}_i\mathbb{A}_2) & (*) & (*) \\ (\mathbb{B}_{1i} + \mathbb{Y}_i\mathbb{B}_2)^TX & -\gamma^2 I_{n_w} & (*) \\ \mathbb{P} & \mathbb{Q} & -I_n \end{bmatrix} < 0 \tag{3.116}$$

which can be written as :

$$\mathcal{B}\mathcal{X}_i\mathcal{C} + (\mathcal{B}\mathcal{X}_i\mathcal{C})^T + \mathcal{D}_i < 0 \tag{3.117}$$

where $\mathcal{X}_i = X\mathbb{Y}_i$, $D_i = \begin{bmatrix} \mathbb{A}_{1i}X + X\mathbb{A}_{1i} & (*) & (*) \\ \mathbb{B}_{1i}^TX & -\gamma^2 I_{n_w} & (*) \\ \mathbb{P} & \mathbb{Q} & -I_n \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & \mathbb{B}_2 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} I_{q_0+q_1} \\ 0 \\ 0 \end{bmatrix}$.

From the elimination lemma of Section 1.5, the solvability conditions of inequality (3.117) are :

$$\mathcal{C}^{T\perp}\mathcal{D}_i\mathcal{C}^{T\perp T} < 0 \tag{3.118a}$$

$$\mathcal{B}^{\perp}\mathcal{D}_i\mathcal{B}^{\perp T} < 0 \tag{3.118b}$$

with $\mathcal{B}^{\perp} = \begin{bmatrix} 0 & I_{n_w} & 0 \\ 0 & 0 & I_n \end{bmatrix}$ and $\mathcal{C}^{T\perp} = \begin{bmatrix} \begin{bmatrix} \mathbb{A}_2^T \\ \mathbb{B}_2^T \end{bmatrix}^{\perp} & 0 \\ 0 & I_n \end{bmatrix}$. By using the definition of matrices $\mathcal{C}$ and $\mathcal{D}_i$ the inequality (3.118a) becomes :

$$\mathcal{C}^{t\perp}\begin{bmatrix}(TA_i)^TX_1+X_1(TA_i) & (TA_i)^TX_2 & X_1(TA_iQD_1-TD) & I\\ X_2^T(TA_1) & 0 & X_2^T(TA_iQD_1-TD) & 0\\ (*) & (*) & -\gamma^2 I_{n_w} & (QD_1)^T\\ I & 0 & (*) & -I_n\end{bmatrix}\mathcal{C}^{T\perp T}<0 \tag{3.119}$$

and by using the definition of matrices $\mathcal{B}$ and $\mathcal{D}_i$ the inequality (3.118b) becomes :

$$\begin{bmatrix}-\gamma^2 I_{n_w} & (QD_1)^T\\ (*) & -I_n\end{bmatrix}<0. \tag{3.120}$$

From the elimination lemma, if conditions (3.118a) and (3.118b) are satisfied, then parameter matrix $\mathbb{Y}_i$ is parameterized as in (3.110) and (3.111). □

3.3.4.1 Particular cases

•Proportional observer

Consider the following descriptor system :

$$\begin{aligned}E\dot{x}(t)&=\sum_{i=1}^{\tau}\sigma_i(\rho(t))(A_ix(t)+B_iu(t))+Dw(t)\\ y(t)&=C_1x(t)+D_1w(t)\end{aligned}$$

with the PO :

$$\begin{aligned}\dot{\zeta}(t)&=\sum_{i=1}^{\tau}\sigma_i(\rho(t))(N_i\zeta(t)+F_iy(t)+J_iu(t))\\ \hat{x}(t)&=\zeta(t)+Q_y(t)\end{aligned}$$

and the observer error dynamics (3.107) becomes :

$$\begin{aligned}\dot{\varepsilon}(t)&=\sum_{i=1}^{\tau}\sigma_i(\rho(t))(\bar{\mathbb{A}}_{1i}+\bar{\mathbb{Y}}_i\bar{\mathbb{A}}_2)\varepsilon(t)+(\bar{\mathbb{B}}_{1i}+\bar{\mathbb{Y}}_i\bar{\mathbb{B}}_2)w(t)\\ e(t)&=\bar{\mathbb{P}}\varepsilon(t)+\bar{\mathbb{Q}}w(t)\end{aligned}$$

where $\bar{\mathbb{A}}_{1i}=TA_i$, $\bar{\mathbb{A}}_2=C_1$, $\bar{\mathbb{B}}_{1i}=TA_iQD_1-TD$, $\bar{\mathbb{B}}_2=(C_1Q-I)D_1$, $\bar{\mathbb{P}}=I_{q_0}$, $\bar{\mathbb{Q}}=QD_1$ and $\bar{\mathbb{Y}}_i=K_i$. Consequently, matrices $\mathcal{D}_i$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{X}_i$ of Theorem 3.4 become :

$$D_i=\begin{bmatrix}(TA_i)^TX+X(TA_i) & X(TA_iQD_1-TD) & I_{q_0}\\ (*) & -\gamma^2 I_{n_w} & (QD_1)^T\\ (*) & (*) & -I_n\end{bmatrix},\ \mathcal{B}=\begin{bmatrix}I_{q_0}\\ 0\\ 0\end{bmatrix},\ \mathcal{C}=\begin{bmatrix}C_1 & (C_1Q-I)D_1 & 0\end{bmatrix}$$

and $\mathcal{X}_i=X\bar{\mathbb{Y}}_i$.

•Proportional-integral observer

Consider the following descriptor system :

$$\begin{aligned}E\dot{x}(t)&=\sum_{i=1}^{\tau}\sigma_i(\rho(t))(A_ix(t)+B_iu(t))+Dw(t)\\ y(t)&=C_1x(t)+D_1w(t)\end{aligned}$$

with the PIO :

$$\begin{aligned}\dot{\zeta}(t)&=\sum_{i=1}^{\tau}\sigma_i(\rho(t))(N_i\zeta(t)+H_iv(t)+F_iy(t)+J_iu(t))\\ \dot{v}(t)&=y(t)-C_1\hat{x}(t)\\ \hat{x}(t)&=\zeta(t)+Qy(t)\end{aligned}$$

and the observer error dynamics (3.107) becomes :

$$\dot{\varphi}(t) = \sum_{i=1}^{\tau} \sigma_i(\rho(t))(\bar{\bar{\mathbb{A}}}_{1i} + \bar{\bar{\mathbb{Y}}}_i \bar{\bar{\mathbb{A}}}_2)\varphi(t) + (\bar{\bar{\mathbb{B}}}_{1i} + \bar{\bar{\mathbb{Y}}}_i \bar{\bar{\mathbb{B}}}_2)w(t)$$
$$e(t) = \bar{\bar{\mathbb{P}}}\varphi(t) + \bar{\bar{\mathbb{Q}}}w(t)$$

where $\bar{\bar{\mathbb{A}}}_{1i} = \begin{bmatrix} TA_i & 0 \\ -C_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} C_1 & 0 \\ 0 & I_{q_1} \end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_{1i} = \begin{bmatrix} TA_iQD_1 - TD \\ -(C_1Q - I)D_1 \end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_2 = \begin{bmatrix} (C_1Q - I)D_1 \\ 0 \end{bmatrix}$, $\bar{\bar{\mathbb{P}}} = \begin{bmatrix} I & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{Q}}} = QD_1$ and $\bar{\bar{\mathbb{Y}}}_i = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} K_i & H_i \end{bmatrix}$. Consequently, matrices $\mathcal{D}_i$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{X}_i$ of Theorem 3.4 become :

$$D_i = \begin{bmatrix} (TA_i)^T X_1 + X_1(TA_i) - X_2C_1 - C_1^T X_2^T & (TA_i)^T X_2 - C_1^T X_3 & \Pi_1 & I \\ (*) & 0 & \Pi_2 & 0 \\ (*) & (*) & -\gamma^2 I_{n_w} & (QD_1)^T \\ I & 0 & (*) & -I_n \end{bmatrix},$$

where

$$\Pi_1 = X_1(TA_iQD_1 - TD) - X_2(C_1Q - I)D_1$$
$$\Pi_2 = X_2^T(TA_iQD_1 - TD) - X_3(C_1Q - I)D_1$$

$$\mathcal{B} = \begin{bmatrix} I_{q_0+q_1} \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} (C_1Q - I)D_1 \\ 0 \end{bmatrix} & 0 \end{bmatrix} \text{ and } \mathcal{X}_i = X\bar{\bar{\mathbb{Y}}}_i, \text{ such that } \begin{bmatrix} K_i & H_i \end{bmatrix} = \left(X \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^+ \mathcal{X}.$$

3.3.4.2 Numerical example

In order to illustrate the results obtained, consider the following LPV descriptor system described by (3.105)where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A(\rho(t)) = \begin{bmatrix} -2.7 & \rho(t) & 0.3 \\ -0.2 & -\rho(t) & 0 \\ -0.11 + \rho(t) & 1.74 & -1 \end{bmatrix}, \ B(\rho(t)) = \begin{bmatrix} \rho(t) \\ 0 \\ 1 \end{bmatrix}, \ D = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \text{ and } D =_1 = 0.2$$

In this case the polytope has 2 vertices corresponding to the extreme values of parameter $\rho(t)$, which varies from 1 to 3.

The parameters $\sigma_i(\rho(t))$ are :

$$\sigma_1(\rho(t)) = \frac{\overline{\rho} - \rho(t)}{\overline{\rho} - \underline{\rho}} = \frac{3 - \rho(t)}{2}$$

$$\sigma_2(\rho(t)) = \frac{\rho(t) - \underline{\rho}}{\overline{\rho} - \underline{\rho}} = \frac{\rho(t) - 1}{2}$$

Considering $E^\perp = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, we can verify Assumptions 3.4 and 3.5

$$\operatorname{rank} \begin{bmatrix} E \\ \hline E^\perp A_i \\ C_1 \end{bmatrix} = 3 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} sE - A_i \\ C_1 \end{bmatrix} = 3, \ \forall i \in [1, 2]$$

Generalized dynamic observer

For the GDO we have fixed $\gamma = 1.41$ and using YALMIP toolbox, we solve the LMIs (3.108) - (3.109) to find matrix X

$$X = \begin{bmatrix} 0.59 & 0.18 & 0 & 0 & 0 & 0 \\ 0.18 & 0.97 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.72 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.72 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.72 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.72 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 8 & 2 & 9 & 7 \\ 9 & 3 & 7 & 5 \\ 9 & 2 & 8 & 5 \\ 9 & 3 & 8 & 5 \\ 8 & 3 & 3 & 9 \\ 9 & 1 & 4 & 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_6 \times 0.01$ and solving (3.110) and (3.111) we get :

$$\mathbb{Y}_1 = \begin{bmatrix} 2.28 & 2.59 & 2.59 & 2.59 \\ 0.35 & 0.33 & 0.33 & 0.33 \\ -52.39 & 5.82 & 5.82 & 5.82 \\ 5.79 & -52.36 & 5.82 & 5.82 \\ 5.79 & 5.82 & -52.36 & 5.82 \\ 5.79 & 5.82 & 5.82 & -52.36 \end{bmatrix} \text{ and}$$

$$\mathbb{Y}_2 = \begin{bmatrix} 1.48 & 1.79 & 1.79 & 1.79 \\ 1.43 & 1.42 & 1.42 & 1.42 \\ -52.39 & 5.82 & 5.82 & 5.82 \\ 5.79 & -52.36 & 5.82 & 5.82 \\ 5.79 & 5.82 & -52.36 & 5.82 \\ 5.79 & 5.82 & 5.82 & -52.36 \end{bmatrix}.$$

Finally, we compute all the matrices of the observer as :

$$N_1 = \begin{bmatrix} -2.7 & 1 & 2.58 \\ -0.2 & -1 & 0.35 \\ 0 & 0 & -52.39 \end{bmatrix}, \; N_2 = \begin{bmatrix} -2.7 & 3 & 1.78 \\ -0.2 & -3 & 1.43 \\ 0 & 0 & -52.39 \end{bmatrix}, \; F_1 = F_2 = \begin{bmatrix} 0.3 \\ 0 \\ 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 2.59 & 2.59 & 2.59 \\ 0.33 & 0.33 & 0.33 \\ 5.82 & 5.82 & 5.82 \end{bmatrix}, \; H_2 = \begin{bmatrix} 1.79 & 1.79 & 1.79 \\ 1.42 & 1.42 & 1.42 \\ 5.82 & 5.82 & 5.82 \end{bmatrix},$$

$$M_1 = M_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; S_1 = S_2 = \begin{bmatrix} 0 & 0 & 5.79 \\ 0 & 0 & 5.79 \\ 0 & 0 & 5.79 \end{bmatrix},$$

$$L_1 = L_2 = \begin{bmatrix} -52.36 & 5.82 & 5.82 \\ 5.82 & -52.36 & 5.82 \\ 5.82 & 5.82 & -52.36 \end{bmatrix}, \; J_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } J_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

In order to provide a comparison of the GDO with the PIO and the PO, these last are also designed.

Proportional observer

By considering matrices $\mathcal{Z} = \begin{bmatrix} 8 \\ 9 \\ 9 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$, $\mathcal{R} = I_3 \times 0.1$ and $\gamma = 1.41$ the following PO matrices are obtained :

$$N_1 = \begin{bmatrix} -2.7 & 1 & 0.78 \\ -0.2 & -1 & 0.12 \\ 0 & 0 & -5.17 \end{bmatrix}, \; N_2 = \begin{bmatrix} -2.7 & 3 & 0.57 \\ -0.2 & -3 & 0.41 \\ 0 & 0 & -5.17 \end{bmatrix}, \; F_1 = F_2 = \begin{bmatrix} 0.3 \\ 0 \\ 0 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; J_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Proportional-integral observer

By considering $\mathcal{Z} = \begin{bmatrix} 8 & 2 \\ 9 & 3 \\ 9 & 2 \\ 9 & 3 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$, $\mathcal{R} = I_4 \times 0.1$ and $\gamma = 1.41$ the following PIO matrices are obtained :

$$N_1 = \begin{bmatrix} -2.7 & 1 & 0.85 \\ -0.2 & -1 & 0.12 \\ 0 & 0 & -5.13 \end{bmatrix},\ N_2 = \begin{bmatrix} -2.7 & 3 & 0.85 \\ -0.2 & -3 & 0.12 \\ 0 & 0 & -5.13 \end{bmatrix},\ F_1 = F_2 = \begin{bmatrix} 0.3 \\ 0 \\ 0 \end{bmatrix},$$

$$H_1 = H_2 = \begin{bmatrix} 0.84 \\ 0.11 \\ 0.62 \end{bmatrix},\ J_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ J_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0.1,\ 0.1,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0,\ 0]^T$, $v(0)_{GDO} = [0]$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0,\ 0]^T$, $v(0)_{PIO} = [0]$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0]^T$.

To evaluate the performance of the GDO an uncertainty $\wp(t)$ is added in the system matrix A_i, then we obtain the following matrix $(A_i + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

The results of the simulation are depicted in Figures 3.28 - 3.38. Figures 3.28, 3.29 and 3.30 show the input $u(t)$, the disturbance $w(t)$ and the uncertainty factor $\delta(t)$. Figures 3.31 and 3.32 shows the variation of the parameter $\rho(t)$ and the weighting functions of each model. Figures 3.33 - 3.38 show the system states and their estimations by the GDO, PO and PIO, also these figures show the estimation error for each observer.

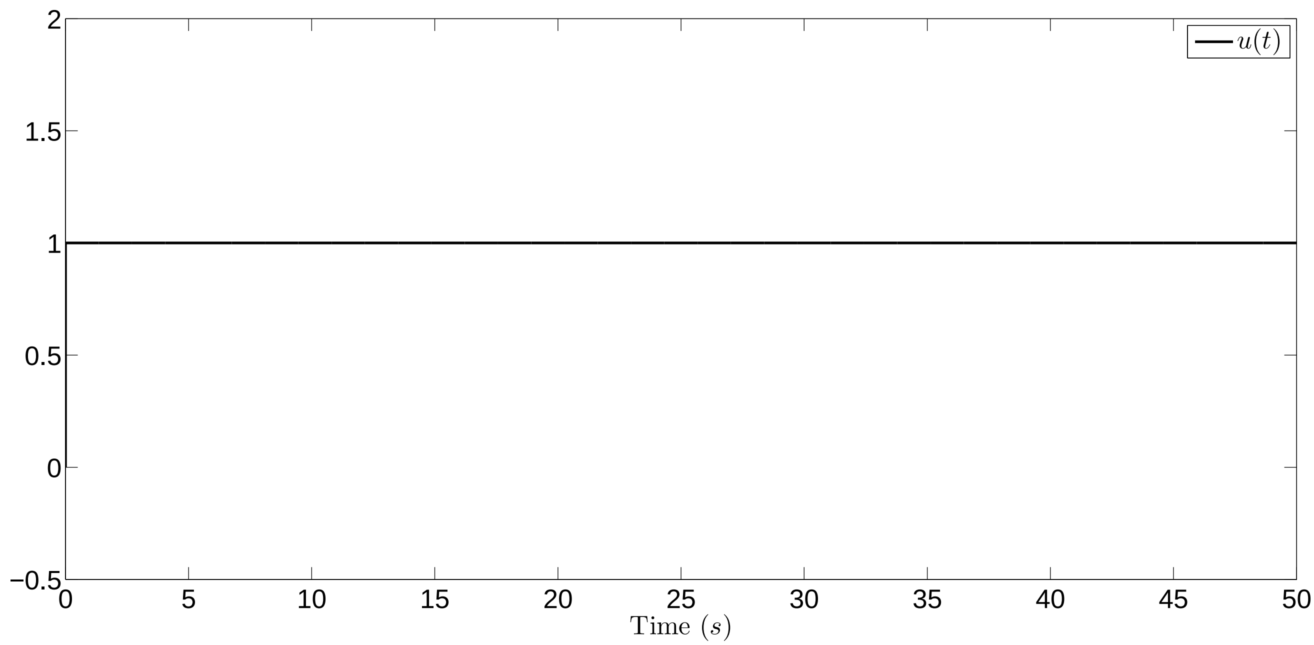

FIGURE 3.28 – H_∞ LPV observers : Input force $u(t)$.

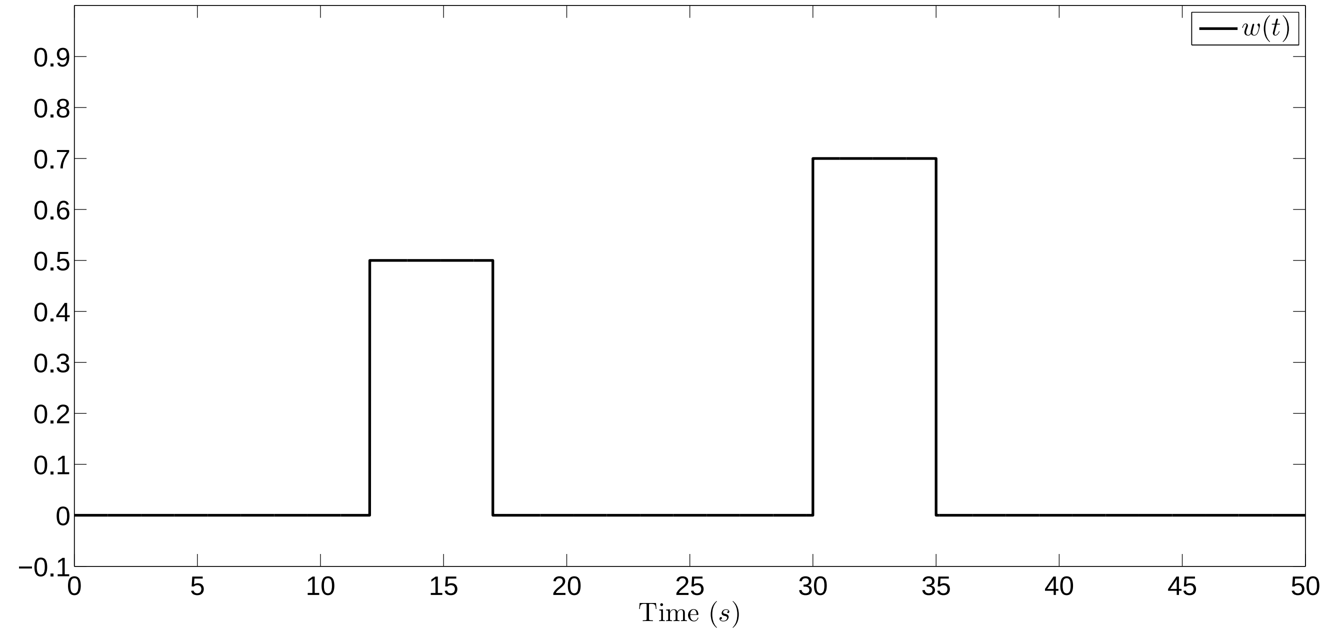

FIGURE 3.29 – H_∞ LPV observers : Disturbance force $w(t)$.

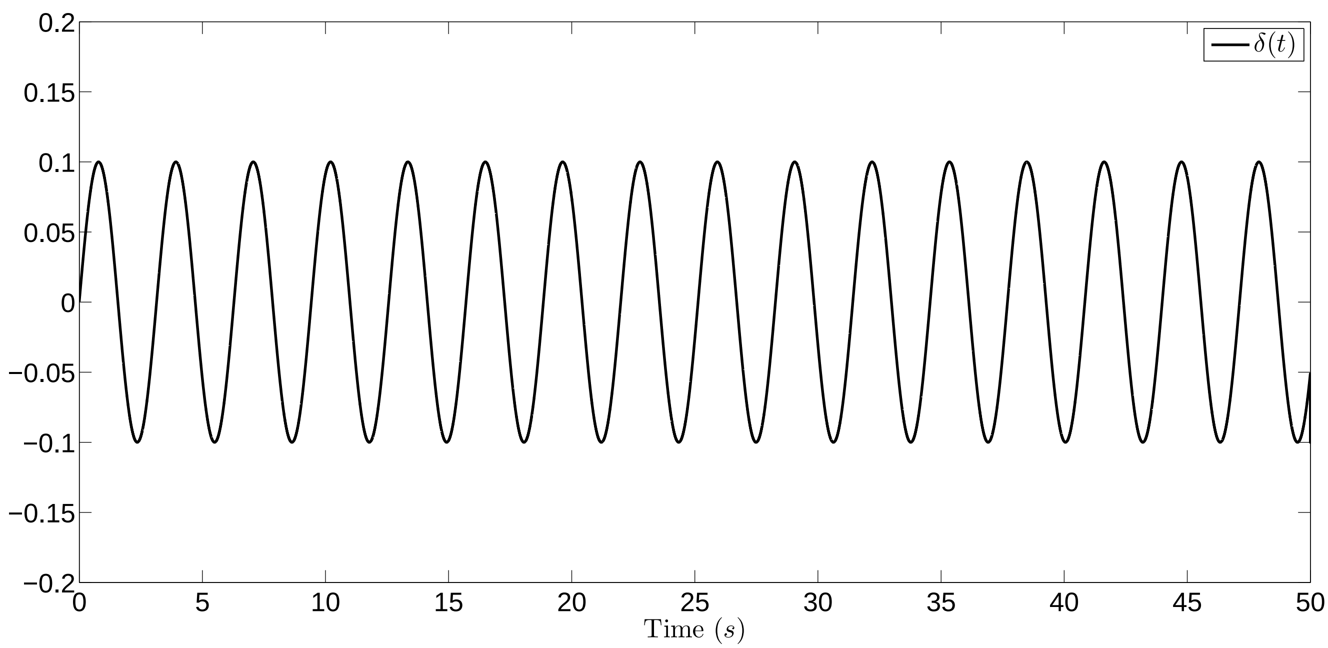

FIGURE 3.30 – H_∞ LPV observers : Uncertainty factor $\delta(t)$.

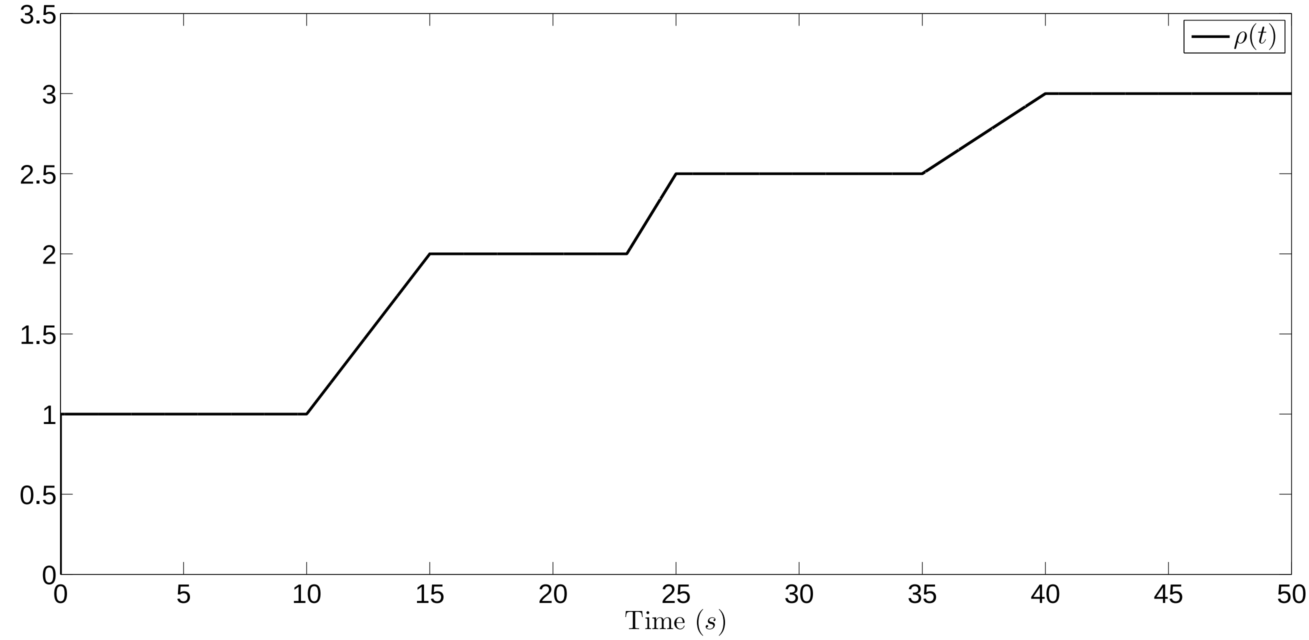

FIGURE 3.31 – H_∞ LPV observers : Parameter variant $\rho(t)$.

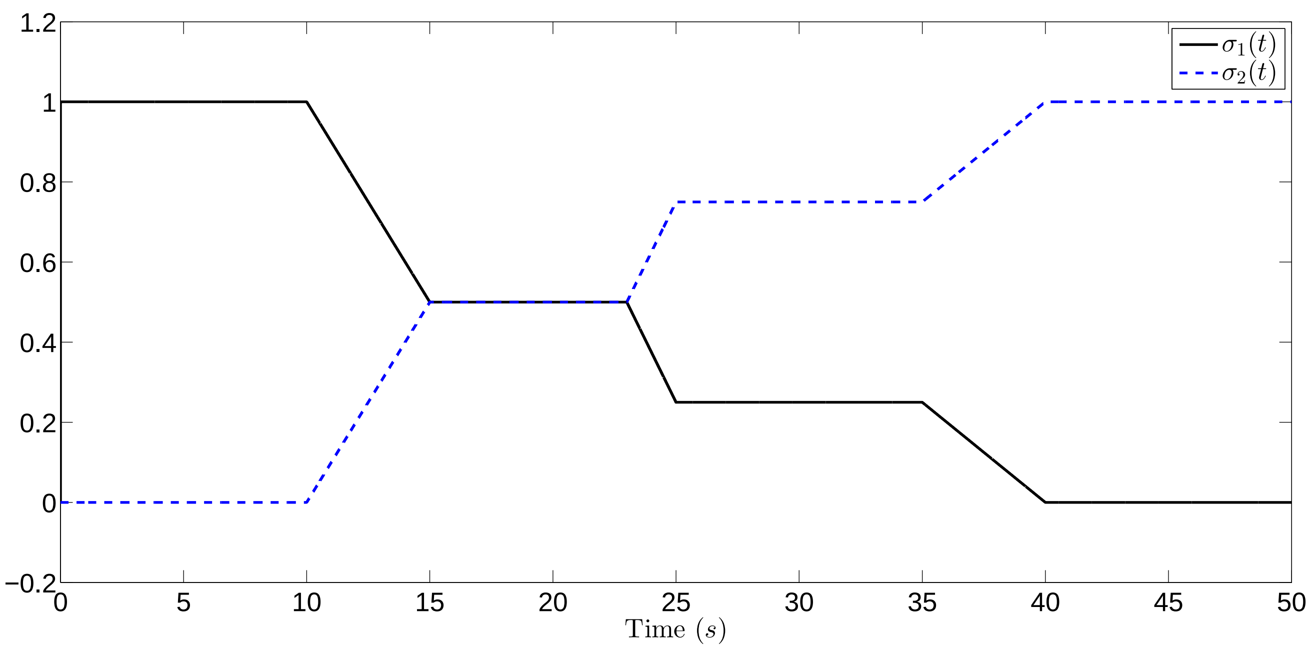

FIGURE 3.32 – H_∞ LPV observers : Weighting functions $\sigma_1(t)$ and $\sigma_2(t)$.

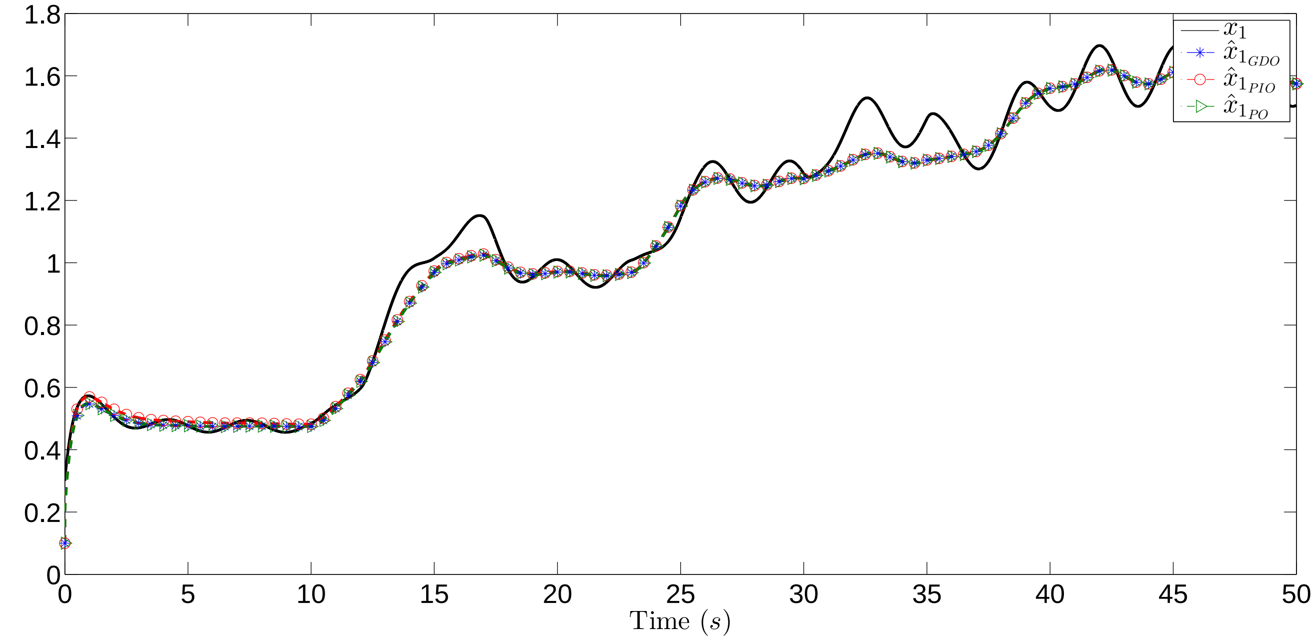

FIGURE 3.33 – H_∞ LPV observers : Estimate of $x_1(t)$.

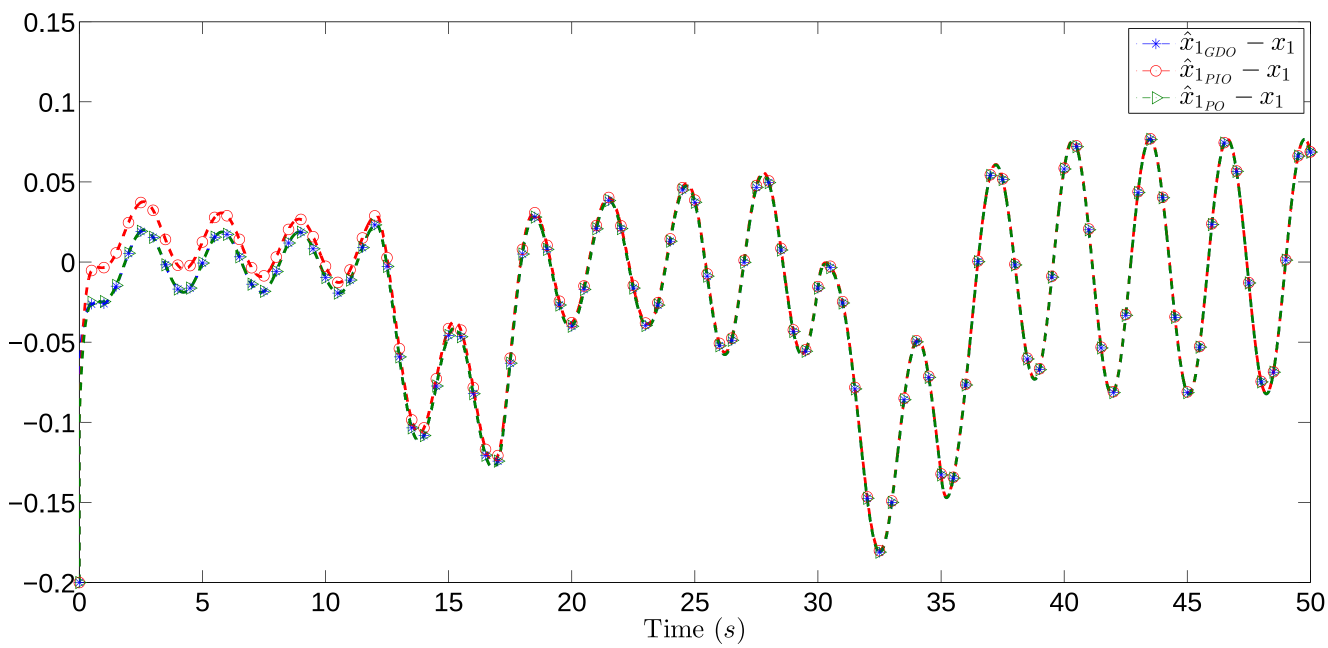

FIGURE 3.34 – H_∞ LPV observers : Estimation error of $x_1(t)$.

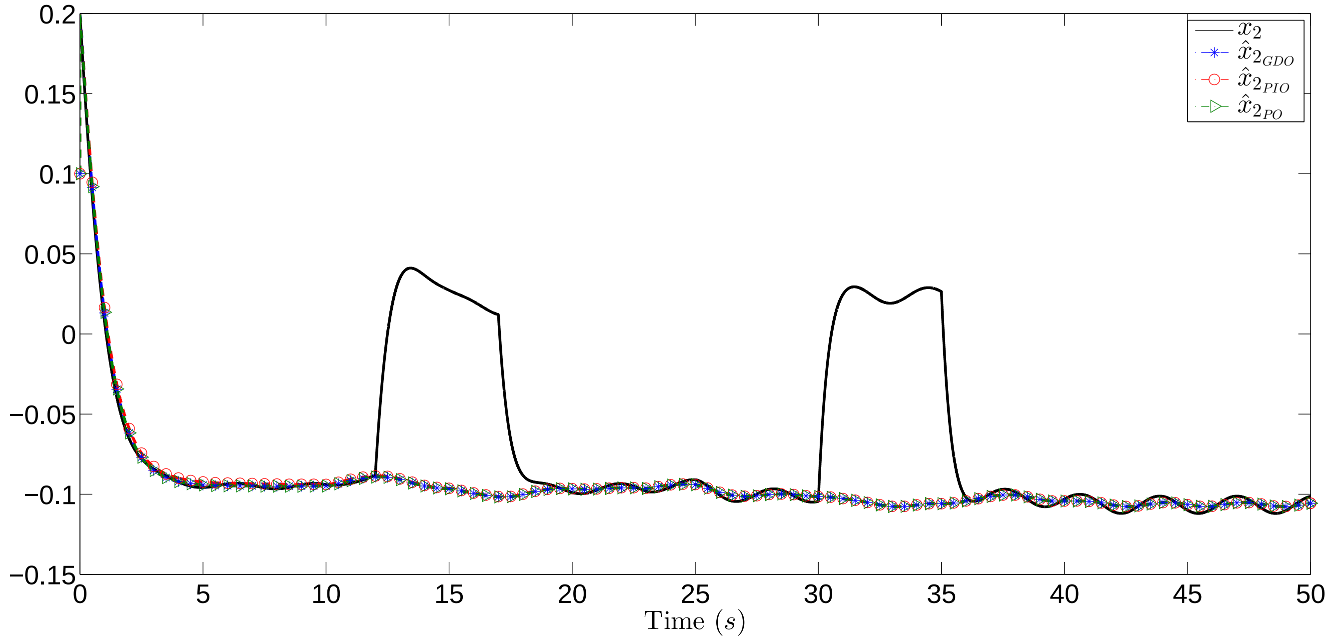

FIGURE 3.35 – H_∞ LPV observers : Estimate of $x_2(t)$.

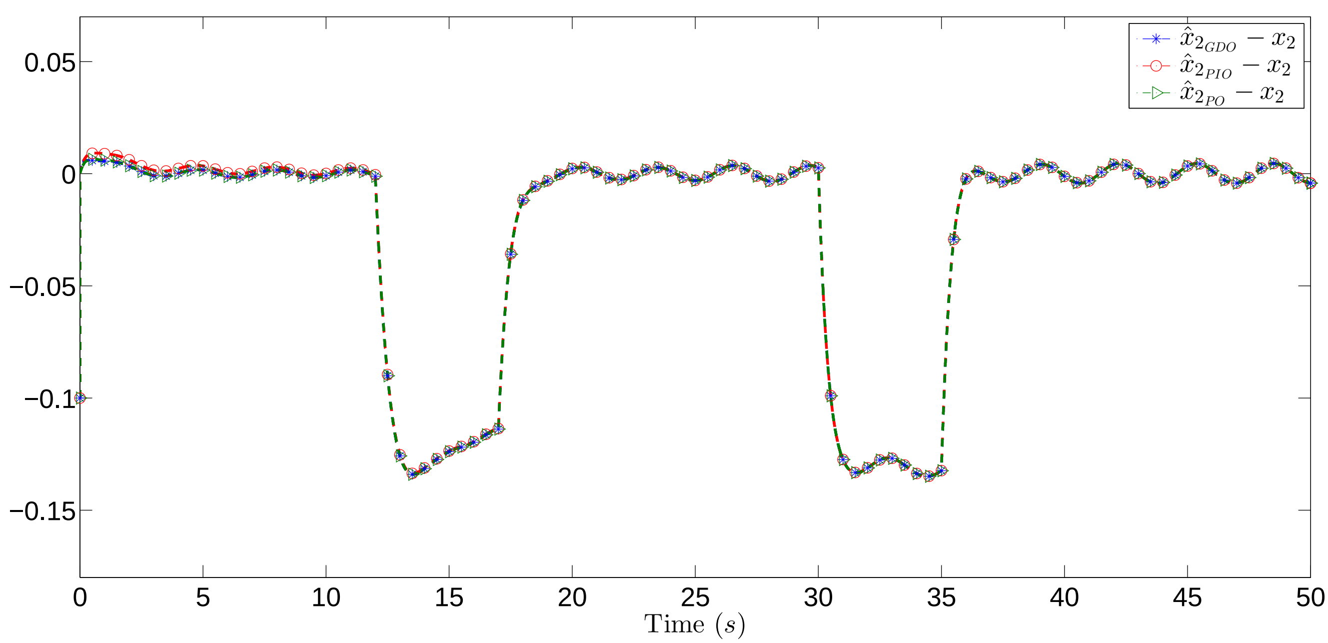

FIGURE 3.36 – H_∞ LPV observers : Estimation error of $x_2(t)$.

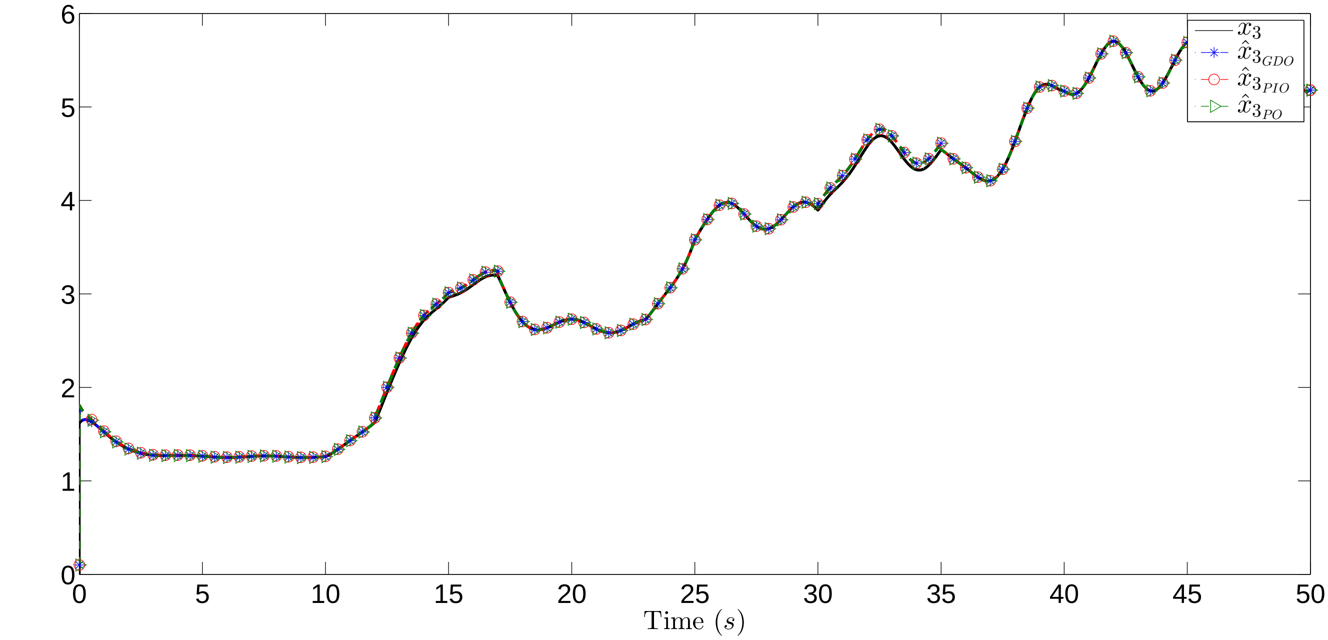

FIGURE 3.37 – H_∞ LPV observers : Estimate of $x_3(t)$.

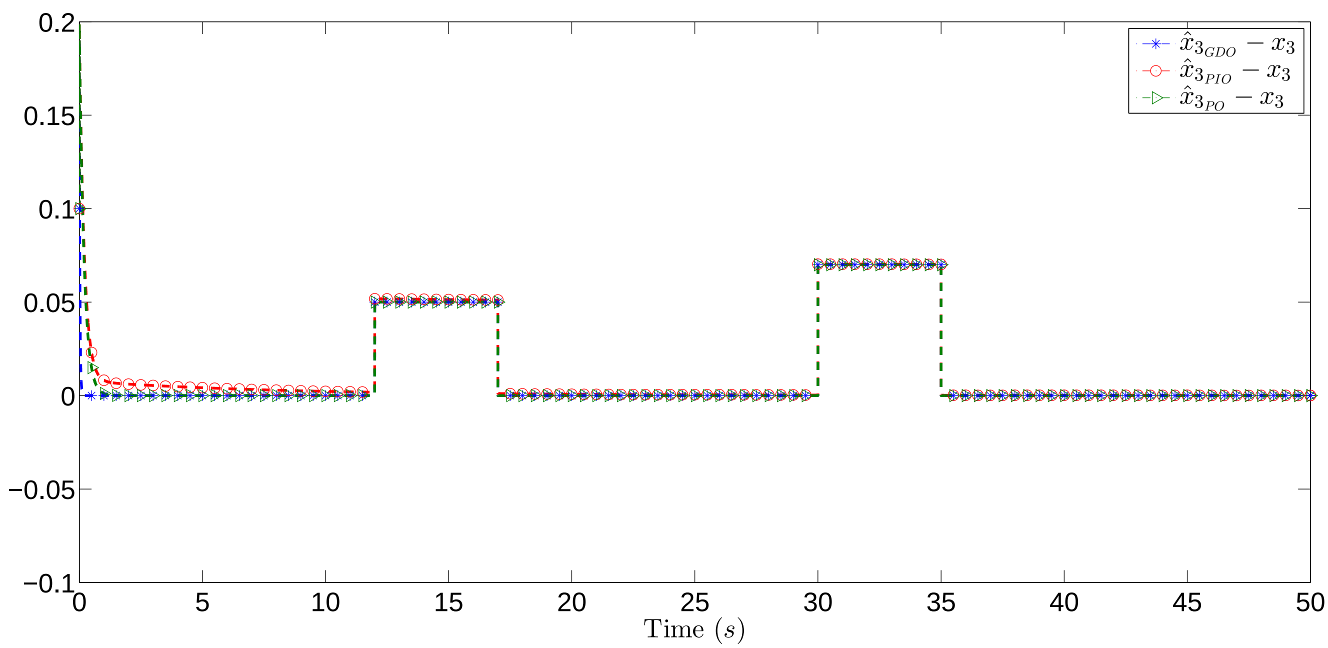

FIGURE 3.38 – H_∞ LPV observers : Estimation error of $x_3(t)$.

From these results, we can see that in general the three observers have almost the same behavior. In order to highlight the differences between the observer estimates the following table is presented.

TABLE 3.4 – H_∞ LPV observers : Error evaluation IAE.

State \ Observer	GDO	PIO	PO
$x_1(t)$	2191.87	2179.96	2194.5
$x_2(t)$	1363.57	1373.97	1363.73
$x_3(t)$	604.88	705.06	638.66

From these error evaluations, we can see that the GDO has most of the smaller values, while its estimation of the state $x_1(t)$ still been acceptable.

3.4 Conclusions

In this chapter two methods of synthesis of robust H_∞ observer for uncertain descriptor systems were presented. The first one, taking the uncertain as a time-variant function with norm bounded. And the second one, taking the uncertain as a parametric variation, which leads a system with LPV form. The stability of the observers are proved through the solution of LMIs. A numerical examples were presented to show the performance of the GDO, also the comparison with the PIO and the PO was carried out.

Chapter 4

H_∞ generalized dynamic observer-based control design

Contents

4.1 Introduction

In this chapter the H_∞ GDO-based control design for disturbed descriptor systems without or with disturbances is presented in Section 4.2. Section 4.2.1 presents the class of disturbed descriptor systems considered. In Section 4.2.2 the problematic is posed through the analysis of stability of the closed-loop between the system and the observer. In Section 4.2.3 the GDO-based control design for descriptor systems free of disturbances is given, and in Section 4.2.4 the extension to H_∞ GDO-based control is carried out.

Also in this chapter the robust H_∞ GDO-based control for uncertain descriptor systems without or with disturbances is presented. In Section 4.3.1 the class of uncertain descriptor systems considered is presented. In Section 4.3.3 the problematic is given.

In Section the robust H_∞ GDO-based control design for uncertain descriptor systems without disturbances is treated, and in Section 4.3.4 its extension to robust H_∞ GDO-based control is carried out.

4.2 H_∞ generalized dynamic observer-based control design for disturbed systems

In this section the H_∞ GDO-based control design for descriptor systems without or with disturbances is treated.

4.2.1 Class of disturbed descriptor systems considered

Consider the following descriptor system :

$$E\dot{x}(t) = Ax(t) + Bu(t) + Dw(t) \tag{4.1a}$$
$$y(t) = C_1 x(t) + D_1 w(t) \tag{4.1b}$$
$$z(t) = C_2 x(t) + D_2 w(t) \tag{4.1c}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector, $y(t) \in \mathbb{R}^{n_y}$ represents the measured output and $z(t) \in \mathbb{R}^s$ is the controlled output. Matrix $E \in \mathbb{R}^{n\times n}$ is singular. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times n_w}$, $C_1 \in \mathbb{R}^{n_y\times n}$, $D_1 \in \mathbb{R}^{n_y\times n_w}$, $C_2 \in \mathbb{R}^{s\times n}$ and $D_2 \in \mathbb{R}^{s\times n_w}$. Let $\mathsf{rank}(E) = \varrho < n$ and let $E^\perp \in \mathbb{R}^{\varrho_1\times n}$ be a full row rank matrix such that $E^\perp E = 0$, in this case $\varrho_1 = n - \varrho$.

In the sequel we assume that

Assumption 4.1.

$$\mathsf{rank}\begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = n.$$

Assumption 4.2.

$$\mathsf{rank}\begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = n,\ \forall s \in \mathbb{C}^+,\ s \textit{ finite}.$$

Assumption 4.3.

$$\mathsf{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n,\ \forall s \in \mathbb{C}^+,\ s \textit{ finite}.$$

See definitions from Section 1.6.

4.2.2 Problem formulation

Consider the following GDO-based control proposed for system (4.1)

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{4.2a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.2b}$$
$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.2c}$$
$$u(t) = -\kappa\hat{x}(t) \tag{4.2d}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer-based control, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. Matrices N, F, J, H, L, M, S, P, Q and κ are unknown matrices of appropriate dimensions which must be determined such that the obtained closed-loop between the system (4.1) and the observer (4.2) is stable.

Now, by taking the results of Lemma 2.3 the following observer error dynamics is obtained

$$\begin{aligned}\begin{bmatrix}\dot\varepsilon(t)\\ \dot v(t)\end{bmatrix} &= \begin{bmatrix}N & H\\ S & L\end{bmatrix}\begin{bmatrix}\varepsilon(t)\\ v(t)\end{bmatrix} + \begin{bmatrix}F\begin{bmatrix}E^{\perp}D\\ D_1\end{bmatrix} - TD\\ M\begin{bmatrix}E^{\perp}D\\ D_1\end{bmatrix}\end{bmatrix} w(t)\\ e(t) &= P\varepsilon(t) + Q\begin{bmatrix}E^{\perp}D\\ D_1\end{bmatrix} w(t)\end{aligned} \tag{4.3}$$

From the system (4.1) and the observer error (4.3) the following closed-loop system is obtained :

$$E\dot x(t) = (A - B\kappa)x(t) - B\kappa P\varepsilon(t) + \left(D - B\kappa Q\begin{bmatrix}E^{\perp}D\\ D_1\end{bmatrix}\right) w(t) \tag{4.4a}$$

$$\dot\varepsilon(t) = N\varepsilon(t) + Hv(t) + \left(F\begin{bmatrix}E^{\perp}D\\ D_1\end{bmatrix} - TD\right) w(t) \tag{4.4b}$$

$$\dot v(t) = S\varepsilon(t) + Lv(t) + M\begin{bmatrix}E^{\perp}D\\ D_1\end{bmatrix} w(t) \tag{4.4c}$$

$$z(t) = C_2x(t) + D_2w(t) \tag{4.4d}$$

By considering the observer parameterization of Section 2.4.1.1 the closed-loop (4.4) can be writing as :

$$\begin{aligned}\mathscr{E}\dot\beta(t) &= \mathscr{A}\beta(t) + \mathscr{B}w(t)\\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t)\end{aligned} \tag{4.5}$$

where $\mathscr{E} = \begin{bmatrix}E & 0\\ 0 & I_{q_0+q_1}\end{bmatrix}$, $\mathscr{A} = \begin{bmatrix}A - B\kappa & \begin{bmatrix}-B\kappa P_1 & 0\end{bmatrix}\\ 0 & \mathbb{A}\end{bmatrix}$, $\mathscr{B} = \begin{bmatrix}D - B\kappa \mathcal{Q}_{d_1}\\ \mathbb{B}\end{bmatrix}$, $\mathscr{C} = \begin{bmatrix}C_2 & 0 & 0\end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix}x(t)\\ \varepsilon(t)\\ v(t)\end{bmatrix}$. In this case $Y_3 = 0$ is taken for simplicity, and the robust parameterization is used, such that $w(t) \neq 0$.

Matrices $\mathbb{A}$ and $\mathbb{B}$ have the following form :

$$\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 \tag{4.6}$$

$$\mathbb{B} = \mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2 \tag{4.7}$$

where $\mathbb{A}_1 = \begin{bmatrix}N_1 - Z\mathcal{N}_2 & 0\\ 0 & 0\end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}$, $\mathbb{B}_1 = \begin{bmatrix}\mathcal{F}_{d_1} - T_1D + Z(\mathcal{T}_2D - \mathcal{F}_{d_2})\\ 0\end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix}\mathcal{F}_{d_3}\\ 0\end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix}Y_1 & H\\ Y_2 & L\end{bmatrix}$.

The problem is then reduced to find the parameter matrices of the observer-based control (4.2) such that the controlled output $z(t)$ converges asymptotically to zero for $w(t) = 0$.

4.2.3 Generalized dynamic observer-based control design for descriptor systems, $w(t) = 0$

In this section we consider $w(t) = 0$, then system (4.1) becomes :

$$E\dot x(t) = Ax(t) + Bu(t) \tag{4.8a}$$

$$y(t) = C_1x(t) \tag{4.8b}$$

$$z(t) = C_2x(t) \tag{4.8c}$$

with the GDO-based control :

$$\dot\zeta(t) = N\zeta(t) + Hv(t) + F\begin{bmatrix}-E^{\perp}Bu(t)\\ y(t)\end{bmatrix} + Ju(t) \tag{4.9a}$$

$$\dot v(t) = S\zeta(t) + Lv(t) + M\begin{bmatrix}-E^{\perp}Bu(t)\\ y(t)\end{bmatrix} \tag{4.9b}$$

$$\hat x(t) = P\zeta(t) + Q\begin{bmatrix}-E^{\perp}Bu(t)\\ y(t)\end{bmatrix} \tag{4.9c}$$

$$u(t) = -\kappa\hat x(t) \tag{4.9d}$$

and the closed-loop (4.5) becomes :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) \\ z(t) &= \mathscr{C}\beta(t) \end{aligned} \tag{4.10}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix} -B\kappa P_1 & 0 \end{bmatrix} \\ 0 & \mathbb{A} \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 & 0 \end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$.

With matrix $\mathbb{A}$ with the form :

$$\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 \tag{4.11}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z_1 N_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$. Since the system is free of disturbances, we have used the parameterization of Lemma 2.5.

The following theorem gives the conditions for the existence of the GDO-based control such that the closed-loop system (4.10) is admissible.

Theorem 4.1. *Under Assumptions 4.1, 4.2 and 4.3 there exist parameter matrices Z_1, κ and $\mathbb{Y}$ such that the closed-loop system (4.10) is admissible if there exists a symmetric positive definite matrix $X_2 = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix}$, with $X_{21} = X_{21}^T$, and a nonsingular matrix X_1 such that the following LMIs are satisfied.*

$$X_1 E^T = E X_1^T \geq 0, \tag{4.12}$$

$$\mathcal{C}^{T\perp} \begin{bmatrix} X_1 A^T + A X_1 & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \Pi_1 \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{4.13}$$

and

$$\begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} \Pi_1 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} < 0. \tag{4.14}$$

with

$$\Pi_1 = \begin{bmatrix} N_1^T X_{21} - N_2^T X_Z^T + X_{21} N_1 - X_Z N_2 & (*) \\ X_{21} N_1 - X_Z N_2 & 0 \end{bmatrix} - \begin{bmatrix} N_3^T & 0 \\ 0 & -I_{q_1} \end{bmatrix} X_Y^T - X_Y \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}. \tag{4.15}$$

In this case matrices $X_Z = X_{21} Z_1$ and $X_Y = X_2 \mathbb{Y}$, and matrix κ is parameterized as follows :

$$\kappa = (\mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+)^T \tag{4.16}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1} \mathcal{B}_l^T \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2} \mathcal{L} (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1/2} \tag{4.17a}$$

$$\vartheta = (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{4.17b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\vartheta - \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} \mathcal{C}_r \vartheta] \mathcal{B}_l \mathcal{R}^{-1} \tag{4.17c}$$

with $\mathcal{D} = \begin{bmatrix} X_1 A^T + A X_1 & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \Pi_1 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} -B^T & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \end{bmatrix}$, matrix Π_1 is defined in (4.15). Matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$.

Proof. From Lemma 1.1 of admissibility, the closed-loop system (4.10) is admissible if and only if there exists a nonsingular matrix $\bar{X}$ such that

$$\mathscr{E}^T \bar{X} = \bar{X}^T \mathscr{E} \geq 0 \tag{4.18}$$

and

$$\mathscr{A}^T \bar{X} + \bar{X}^T \mathscr{A} < 0 \tag{4.19}$$

are satisfied.

Now, let $\bar{X} = \begin{bmatrix} \bar{X}_1 & 0 \\ 0 & X_2 \end{bmatrix}$ with $X_2 = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix}$. By using matrix $\mathscr{E}$ from (4.10), the equation (4.18) becomes :

$$E^T \bar{X}_1 = \bar{X}_1^T E \geq 0. \tag{4.20}$$

Pre-multiplying the inequality (4.20) by $\bar{X}_1^{-T}$ and post-multiplying it by $\bar{X}_1^{-1}$ we obtain

$$X_1 E^T = E X_1^T \geq 0 \tag{4.21}$$

where $X_1 = \bar{X}_1^{-T}$.

On the other hand, by inserting matrix $\mathscr{A}$ from (4.10) into (4.19) we obtain the following matrix inequality :

$$\begin{bmatrix} (A - B\kappa)^T \bar{X}_1 + \bar{X}_1^T (A - B\kappa) & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix} \bar{X}_1 & \mathbb{A}^T X_2 + X_2 \mathbb{A} \end{bmatrix} < 0 \tag{4.22}$$

pre-multiplying the inequality (4.22) by $\begin{bmatrix} \bar{X}_1^{-T} & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$ and post-multiplying it by $\begin{bmatrix} \bar{X}_1^{-1} & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$ we get :

$$\begin{bmatrix} X_1 (A - B\kappa)^T + (A - B\kappa) X_1^T & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix} & \mathbb{A}^T X_2 + X_2 \mathbb{A} \end{bmatrix} < 0 \tag{4.23}$$

where $X_1 = \bar{X}_1^{-T}$.

Now, replacing matrix $\mathbb{A}$ from (4.11) we have :

$$\begin{bmatrix} X_1 (A - B\kappa)^T + (A - B\kappa) X_1^T & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix} & (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X_2 + X_2 (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) \end{bmatrix} < 0 \tag{4.24}$$

which can be written as :

$$\mathcal{B}\kappa^T \mathcal{C} + (\mathcal{B}\kappa^T \mathcal{C})^T + \mathcal{D} < 0 \tag{4.25}$$

where $\mathcal{D} = \begin{bmatrix} X_1 A^T + A X_1 & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \mathbb{A}_1^T X_2 + X_2 \mathbb{A}_1 - \mathbb{A}_2^T X_Y^T - X_Y \mathbb{A}_2 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} -B^T & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \end{bmatrix}$.

Using the elimination lemma of Section 1.5 inequality (4.25) is equivalent to :

$$\mathcal{C}^{T\perp} \mathcal{D} \mathcal{C}^{T\perp T} < 0 \tag{4.26a}$$

$$\mathcal{B}^{\perp} \mathcal{D} \mathcal{B}^{\perp T} < 0 \tag{4.26b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} -B^{\perp} & 0 \\ 0 & I \end{bmatrix}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} 0 & \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} \end{bmatrix}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (4.26a) becomes :

$$\mathcal{C}^{T\perp} \begin{bmatrix} X_1 A^T + A X_1 & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \Pi_1 \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{4.27}$$

and by using $\mathcal{B}$ and $\mathcal{D}$ the inequality (4.26b) becomes :

$$\begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} \Pi_1 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} < 0 \tag{4.28}$$

matrix Π_1 is defined in (4.15). From the elimination lemma, if conditions (4.26a) and (4.26b) are satisfied, the parameter matrix κ is parameterized as in (4.16) and (4.17). □

4.2.3.1 Particular cases

In this section two particular cases of our results are presented.

•Proportional observer-based control

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_1 x(t) \\ z(t) &= C_2 x(t) \end{aligned}$$

with the PO-based control

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Fy_a(t) + Ju(t) \\ \hat{x}(t) &= P\zeta(t) + Q_a y(t) \\ u(t) &= -\kappa\hat{x}(t) \end{aligned}$$

and the closed-loop (4.10) becomes :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) \\ z(t) &= \mathscr{C}\beta(t) \end{aligned}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & B\kappa P_1 \\ 0 & \bar{\mathbb{A}} \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 \end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix}$.

With matrix $\bar{\mathbb{A}} = \bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2$ where $\bar{\mathbb{A}}_1 = N_1 - Z_1 N_2$, $\bar{\mathbb{A}}_2 = N_3$ and $\bar{\mathbb{Y}} = Y_1$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.1 become :

$$\mathcal{D} = \begin{bmatrix} X_1 A^T + A X_1^T & 0 \\ 0 & \Pi_1 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} X_1 \\ P_1^T \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} -B^T & 0 \end{bmatrix}$$

where $\Pi_1 = N_1^T X_2 + X_2 N_1 - N_2^T X_Z^T - X_Z N_2 - N_3^T X_Y^T - X_Y N_3$. Matrices Σ and Ω are defined as $\Sigma = \begin{bmatrix} R \\ C_1 \end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C_1 \end{bmatrix}$.

•Proportional-integral observer-based control

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_1 x(t) \\ z(t) &= C_2 x(t) \end{aligned}$$

with the PIO-based control

$$\begin{aligned}
\dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix}-E^{\perp}Bu(t)\\ y(t)\end{bmatrix} + Ju(t)\\
\dot{v}(t) &= y(t) - C_1\hat{x}(t)\\
\hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix}-E^{\perp}Bu(t)\\ y(t)\end{bmatrix}\\
u(t) &= -\kappa\hat{x}(t)
\end{aligned}$$

and the closed-loop (4.10) becomes :

$$\begin{aligned}
\mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t)\\
z(t) &= \mathscr{C}\beta(t)
\end{aligned}$$

where $\mathscr{E} = \begin{bmatrix}E & 0\\ 0 & I_{q_0+q_1}\end{bmatrix}$, $\mathscr{A} = \begin{bmatrix}A - B\kappa & \begin{bmatrix}-B\kappa P_1 & 0\end{bmatrix}\\ \begin{bmatrix}0\\0\end{bmatrix} & \bar{\bar{\mathbb{A}}}\end{bmatrix}$, $\mathscr{C} = \begin{bmatrix}C_2 & 0 & 0\end{bmatrix}$ and $\beta(t) = \begin{bmatrix}x(t)\\ \varepsilon(t)\\ v(t)\end{bmatrix}$.

With matrix $\bar{\bar{\mathbb{A}}} = \bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2$ where $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix}N_1 - Z_1N_2 & 0\\ -C_1P_1 & 0\end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix}I\\0\end{bmatrix}\begin{bmatrix}Y_1 & H\end{bmatrix}$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.1 become :

$$\mathcal{D} = \begin{bmatrix}X_1A^T + AX_1^T & \begin{bmatrix}0 & 0\end{bmatrix}\\ \begin{bmatrix}0\\0\end{bmatrix} & \Pi_1\end{bmatrix},\ \mathcal{B} = \begin{bmatrix}X_1\\ \begin{bmatrix}P_1^T\\ 0\end{bmatrix}\end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix}-B^T\begin{bmatrix}0 & 0\end{bmatrix}\end{bmatrix}$$

where

$$\Pi_1 = \begin{bmatrix}X_{21}(N_1 - C_1P_1) + (N_1 - C_1P_1)^TX_{21} - X_ZN_2 - N_2^TX_Z^T & (*)\\ X_{21}N_1 - X_{22}C_1P_1 - X_ZN_2 & 0\end{bmatrix} - \begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}^TX_Y^T - X_Y\begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}.$$

4.2.3.2 Numerical example

In order to illustrate the results obtained, consider the following descriptor system described by (4.8) where

$$E = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\ A = \begin{bmatrix}2.7 & 0 & 0.3\\ -0.2 & -3 & 0\\ -0.1 & 1.7 & -1\end{bmatrix},\ B = \begin{bmatrix}1\\0\\0\end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ and } x(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}.$$

First of all, we would like to show that the system above is unstable by using Definition 1.3

$$\mathsf{eig}(E,\ A) = [2.65,\ -2.98]$$

Considering $E^{\perp} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$ we can verify Assumptions 4.1, 4.2 and 4.3

$$\mathsf{rank}\begin{bmatrix}E\\ \hline E^{\perp}A\\ C_1\end{bmatrix} = 3,\ \mathsf{rank}\begin{bmatrix}sE - A\\ C_1\end{bmatrix} = 3 \text{ and } \mathsf{rank}\begin{bmatrix}sE - A & B\end{bmatrix} = 3.$$

Generalized dynamic observer

For the GDO we have chosen matrix $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, such that $\mathsf{rank}(\Sigma) = 3$.

By using YALMIP toolbox, we solve the LMIs (4.12) - (4.14) to find matrices X_1, X_2, Z_1 and $\mathbb{Y}$

$$X_1 = \begin{bmatrix} 261.21 & 7.29 & 5.12 \\ 7.29 & 0.27 & 18.25 \\ 0 & 0 & 52.41 \end{bmatrix},\ X_2 = \begin{bmatrix} 36.84 & 0 & 36.84 & 0 \\ 0 & 36.84 & 0 & 36.84 \\ 36.84 & 0 & 110.51 & 0 \\ 0 & 36.84 & 0 & 110.51 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} -0.19 & -2.6 & 0 & -3.1 & -0.67 & -0.58 \\ 0.01 & 1.51 & 0 & 2.61 & 0.9 & -0.06 \end{bmatrix} \text{ and } \mathbb{Y} = \begin{bmatrix} -1.44 & 0.26 & 0 & 0.31 & -0.28 & 0.89 & 0.1 \\ 0.34 & 2.64 & 0 & -0.42 & -2.45 & -1.5 & -1.13 \\ 0.25 & 0.12 & 0 & -0.01 & -0.01 & -0.72 & -0.1 \\ -0.06 & -0.64 & 0 & 0.01 & 0.45 & 0.16 & -0.05 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ and $\mathcal{R} = I_3 \times 0.1$ and by solving (4.16) and (4.17) we get :

$$\kappa = \begin{bmatrix} 148.49 & 17.89 & 42.41 \end{bmatrix}.$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -1.23 & 0.07 \\ -1.57 & -1.9 \end{bmatrix},\ S = \begin{bmatrix} -0.13 & -0.06 \\ 0.04 & 0.54 \end{bmatrix},\ H = \begin{bmatrix} 0.89 & 0.1 \\ -1.5 & -1.13 \end{bmatrix},$$

$$L = \begin{bmatrix} -0.72 & -0.1 \\ 0.16 & -0.05 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1.82 & 0.02 \\ 1.31 & -0.67 & 1.3 \end{bmatrix},$$

$$J = \begin{bmatrix} 0.05 \\ -0.04 \end{bmatrix},\ M = \begin{bmatrix} -0.04 & 0.2 & -0.04 \\ -0.22 & 0.29 & -0.22 \end{bmatrix},$$

$$P = \begin{bmatrix} 8.5 & -5 \\ 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -3.47 & -2.52 & -3.47 \\ 0.26 & 0.05 & 0.26 \\ -0.22 & 0.34 & 0.78 \end{bmatrix}.$$

In order to provide a comparison of the GDO-based control with the PIO-based control and the PO-based control, these last are also designed.

Proportional observer-based control

By considering matrices $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ and $\mathcal{R} = I_3 \times 0.1$ the following PO-based control matrices are obtained :

$$\kappa = \begin{bmatrix} 54.66 & 129.66 & 18.84 \end{bmatrix},$$

$$N = \begin{bmatrix} -0.52 & 0.14 \\ -0.72 & -0.52 \end{bmatrix},\ F_a = \begin{bmatrix} -2.31 & 0.94 \\ 0.43 & -13.49 \end{bmatrix},\ J = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ P = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Q_a = \begin{bmatrix} -9.22 & 0 \\ 1.12 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proportional-integral observer-based control

By considering matrices $R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ and $\mathcal{R} = I_3 \times 0.01$ the following PIO-based control matrices are obtained :

$$\kappa = \begin{bmatrix} 62.9 & 229.09 & 19.7 \end{bmatrix},$$

$$N = \begin{bmatrix} -1.93 & 0.24 \\ -2 & -0.39 \end{bmatrix},\ H = \begin{bmatrix} 0.23 & 0.63 \\ -1.71 & -1.95 \end{bmatrix},\ J = \begin{bmatrix} 0.43 \\ 0.32 \end{bmatrix},\ F = \begin{bmatrix} 1.75 & -10.45 & -0.85 \\ 5.12 & -12.41 & 2.26 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.3 & 0.71 \\ 0.17 & -0.14 \\ 0.12 & -0.15 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -6.46 & 0.99 & -6.46 \\ 0.29 & 0.41 & 0.29 \\ 0.06 & 0.28 & 1.06 \end{bmatrix}$$

Simulation results

The initial conditions for the system are $x(0) = [0.1,\ 0,\ 0]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0]^T$.

To evaluate the performance of the GDO-based control an uncertainty $\wp(t)$ is added in the system matrix A, then we obtain matrix $(A + \wp(t))$, where $\wp(t) = \delta(t) \times \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0 & 0.3 & 0.1 \\ 0 & 0.2 & 0 \end{bmatrix}$.

The results of the simulation are depicted in Figures 4.1 - 4.4. Figure 4.1 shows the uncertainty factor $\delta(t)$. Figure 4.2 shows the control input $u(t)$ provided by the GDO, PIO and PO. Figures 4.3 and 4.4 show the controlled output by the GDO, PIO and PO.

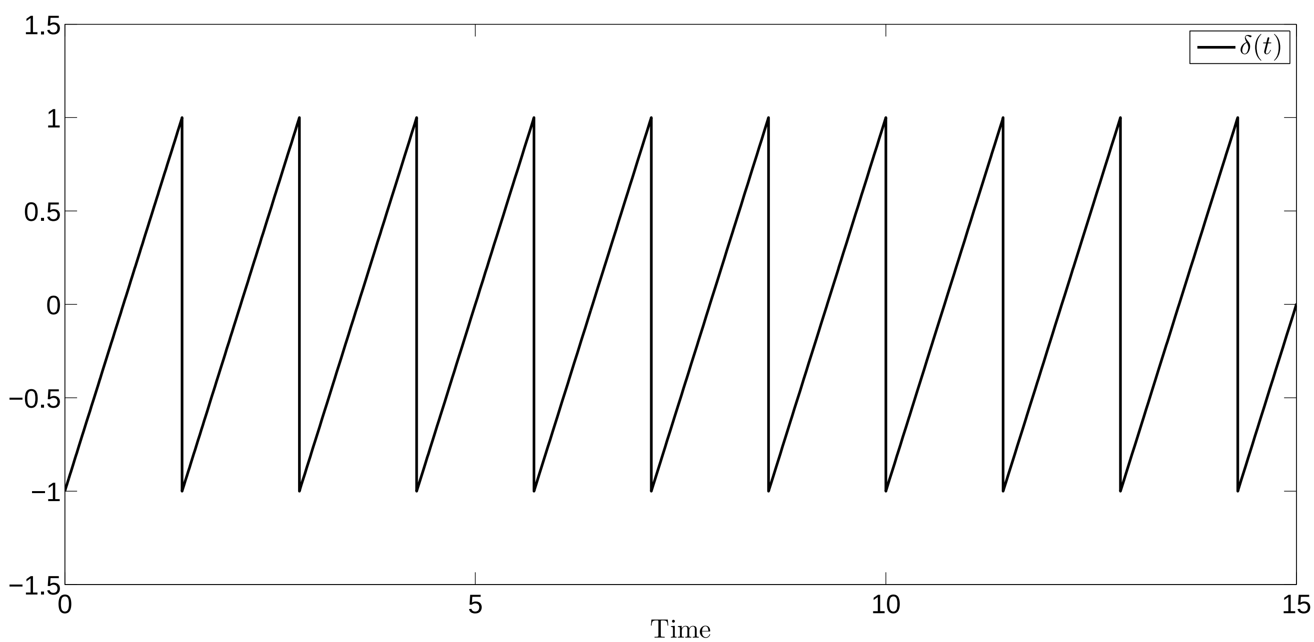

FIGURE 4.1 – Observers-based control : Uncertainty factor $\delta(t)$.

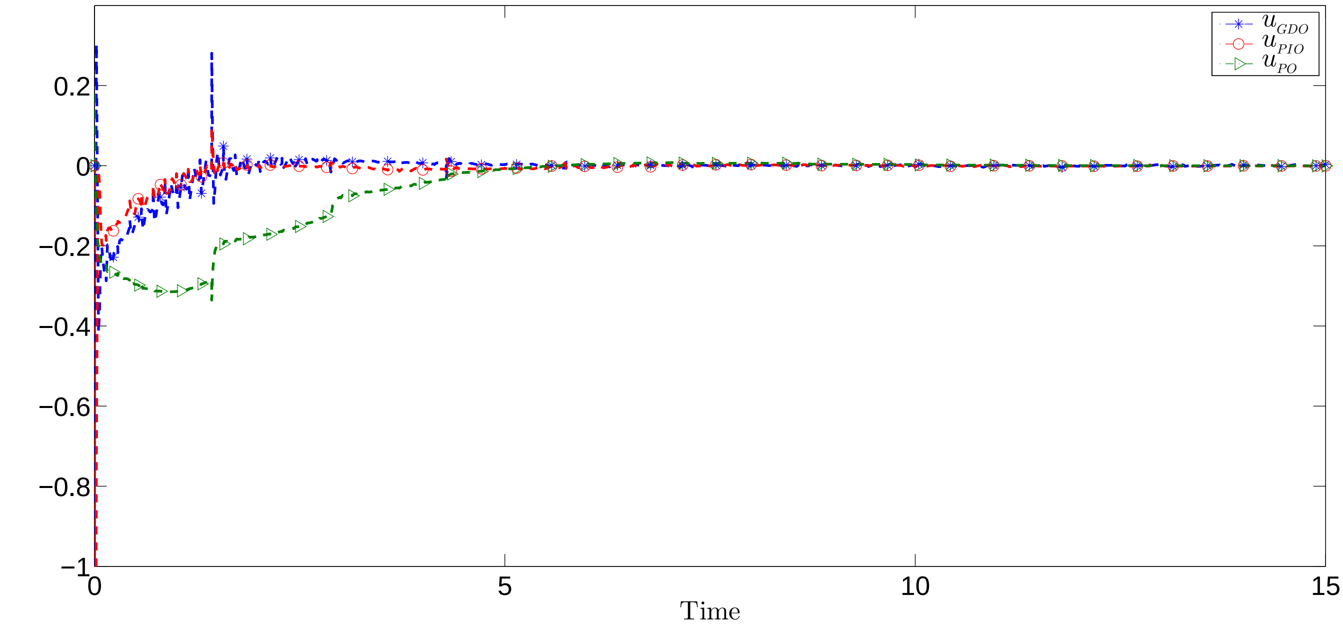

FIGURE 4.2 – Observers-based control : Control input $u(t)$.

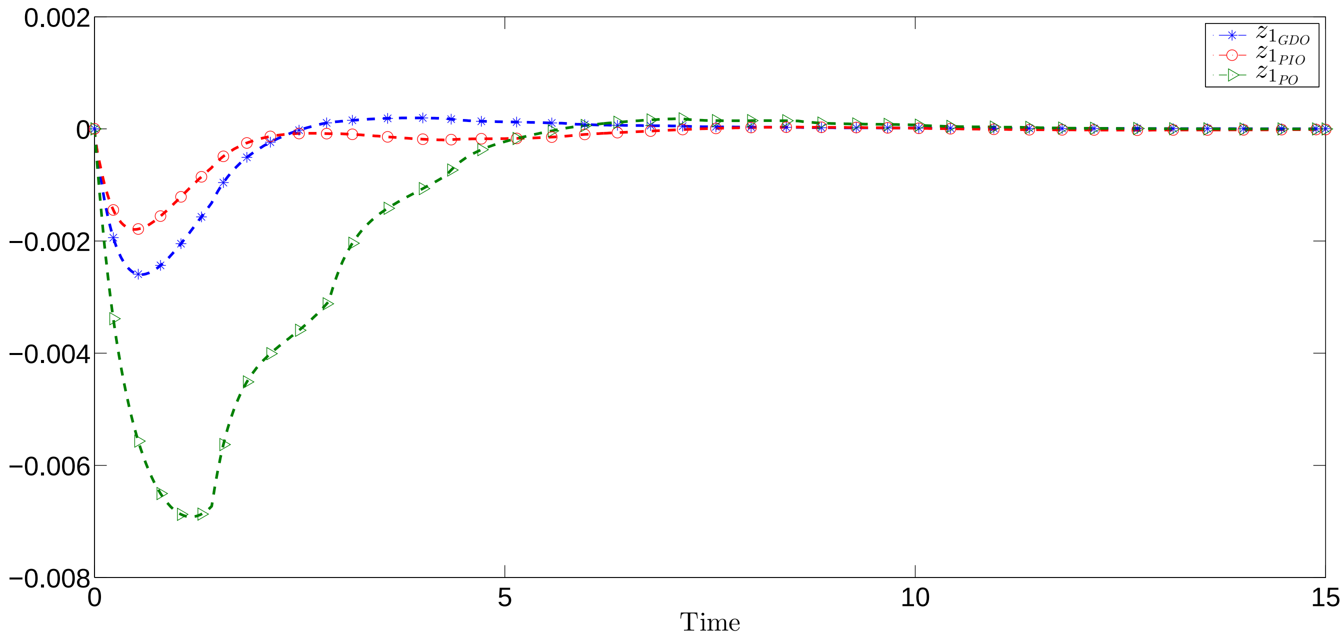

FIGURE 4.3 – Observers-based control : Controlled output $z_1(t)$.

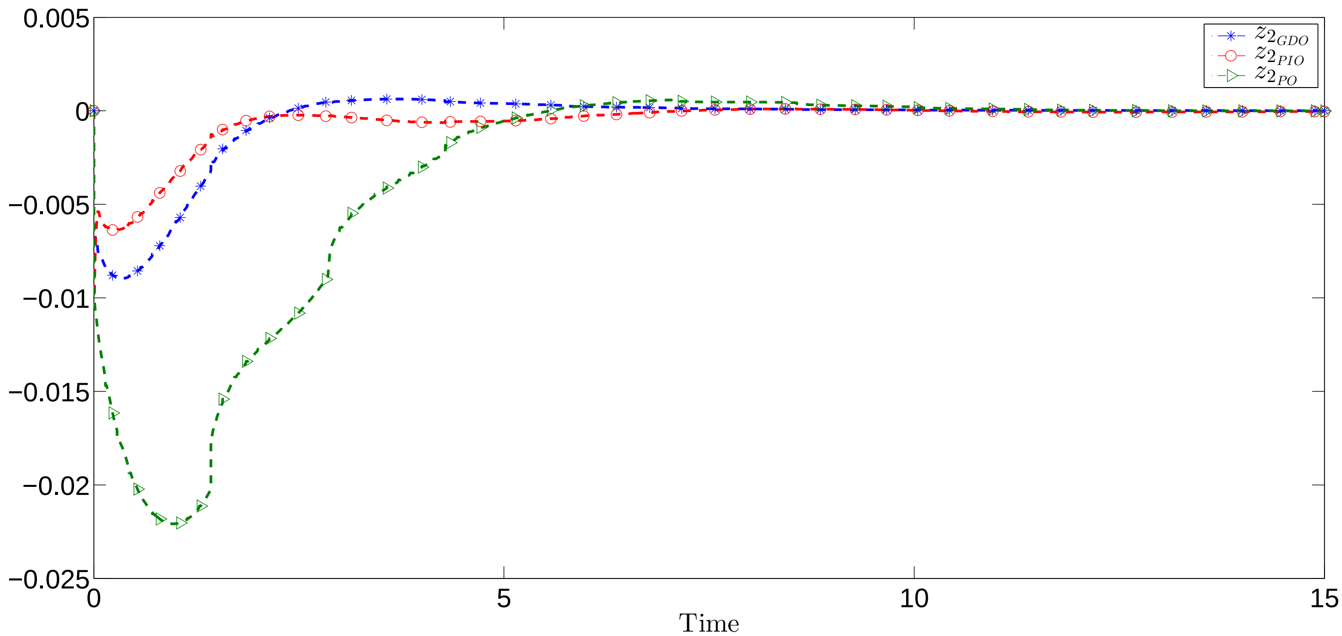

FIGURE 4.4 – Observers-based control : Controlled output $z_2(t)$.

From these results, we can see that the GDO-based control stabilize faster the system than the PIO and the PO, and the control input of the GDO has almost the same shape as this of the PIO, however the GDO control is faster than the PIO.

4.2.4 H_∞ generalized dynamic observer-based control design for disturbed descriptor systems, $w(t) \neq 0$

In this case we consider $w(t) \neq 0$, then we get system (4.1) :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) && (4.29a)\\ y(t) &= C_1x(t) + D_1w(t) && (4.29b)\\ z(t) &= C_2x(t) + D_2w(t) && (4.29c) \end{aligned}$$

with the GDO-based control :

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Ju(t) && (4.30a)\\ \dot{v}(t) &= S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} && (4.30b)\\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} && (4.30c)\\ u(t) &= -\kappa\hat{x}(t) && (4.30d) \end{aligned}$$

and the closed-loop (4.5) :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{B}w(t) \\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t) \end{aligned} \qquad (4.31)$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix} -B\kappa P_1 & 0 \end{bmatrix} \\ 0 & \mathbb{A} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D - B\kappa\mathcal{Q}_{d_1} \\ \mathbb{B} \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 & 0 \end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$.

Matrices $\mathbb{A}$ and $\mathbb{B}$ as :

$$\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 \quad (4.32)$$

$$\mathbb{B} = \mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2 \quad (4.33)$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{B}_1 = \begin{bmatrix} \mathcal{F}_{d_1} - T_1 D + Z(\mathcal{T}_2 D - \mathcal{F}_{d_2}) \\ 0 \end{bmatrix}$, $\mathbb{B}_2 = \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$.

The problem is then reduced to find parameter matrices of the observer-based control (4.31) such that for a prescribed scalar $\gamma > 0$, we ensure that the controlled output $z(t)$ converge asymptotically to zero in closed-loop for $w(t) = 0$, and for $w(t) \neq 0$ we must guarantee that the performance index

$$G = \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t)]dt \quad (4.34)$$

is negative $\forall w(t) \in \mathcal{L}_2[0,\infty)$, or equivalently $\|G_{wz}\|_\infty < \gamma$, where G_{wz} is the transfer function from the disturbance to the controlled output and γ is a given positive scalar. The solution to this problem is given in the following theorem.

Theorem 4.2. *Under Assumptions 4.1, 4.2 and 4.3 there exist parameter matrices Z, κ and $\mathbb{Y}$ such that the closed-loop system (4.31) is admissible and $\|G_{wz}\|_\infty < \gamma$ if there exists a symmetric positive definite matrix $X_2 \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix}$, with $X_{21} = X_{21}^T$, and a nonsingular matrix X_1 such that the following LMIs are satisfied.*

$$X_1 E^T = E X_1^T \geq 0, \quad (4.35)$$

$$\mathcal{C}^{T\perp} \begin{bmatrix} X_1 A^T + A X_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ D^T & \Pi_2^T & -\gamma^2 I_{n_w} & (*) \\ C_2 X_1^T & 0 & D_2 & -I_s \end{bmatrix} \mathcal{C}^{T\perp T} < 0, \quad (4.36)$$

and

$$\begin{bmatrix} \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} \Pi_1 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & (*) & 0 \\ \mathcal{Q}_{d_1}^{T\perp} \Pi_2^T \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & -\gamma^2 \mathcal{Q}_{d_1}^{T\perp} \mathcal{Q}_{d_1}^{T\perp T} & (*) \\ 0 & D_2 \mathcal{Q}_{d_1}^{T\perp T} & -I_s \end{bmatrix} < 0 \quad (4.37)$$

with

$$\Pi_1 = \begin{bmatrix} X_{21} N_1 - X_Z \mathcal{N}_2 + N_1^T X_{21} - \mathcal{N}_2^T X_Z^T & (*) \\ X_{21} N_1 - X_Z \mathcal{N}_2 & 0 \end{bmatrix} - X_Y \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} - \begin{bmatrix} N_3^T & 0 \\ 0 & -I_{q_1} \end{bmatrix} X_Y^T \quad (4.38a)$$

$$\Pi_2 = \begin{bmatrix} X_{21}(\mathcal{F}_{d_1} - T_1 D) + X_Z(\mathcal{T}_2 D - \mathcal{F}_{d_2}) \\ X_{21}(\mathcal{F}_{d_1} - T_1 D) + X_Z(\mathcal{T}_2 D - \mathcal{F}_{d_2}) \end{bmatrix} - X_Y \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix} \quad (4.38b)$$

In this case $X_Z = X_{21} Z$ and $X_Y = X_2 \mathbb{Y}$, and matrix κ is parameterized as follows :

$$\kappa = (\mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+)^T \quad (4.39)$$

where

$$\mathcal{K} = -\mathcal{R}^{-1} \mathcal{B}_l^T \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2} \mathcal{L} (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1/2} \quad (4.40a)$$

$$\vartheta = (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \quad (4.40b)$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\vartheta - \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} \mathcal{C}_r \vartheta] \mathcal{B}_l \mathcal{R}^{-1} \quad (4.40c)$$

with $\mathcal{D} = \begin{bmatrix} X_1 A^T + A X_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ D^T & \Pi_2^T & -\gamma^2 I_{n_w} & (*) \\ C_2 X_1^T & 0 & D_2 & -I_s \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} -B^T & 0 & 0 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \\ \mathcal{Q}_{d_1}^T \\ 0 \end{bmatrix}$, *matrices Π_1 and Π_2 are defined in (4.38). Matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$.*

Proof. The BRL of Lemma 1.2 guaranteed that the system (4.31) is admissible and $\|G_{wz}\|_\infty < \gamma$ if there exist a nonsingular matrix $\bar{X}$ such that

$$\mathscr{E}^T\bar{X} = \bar{X}^T\mathscr{E} \geq 0 \tag{4.41}$$

and

$$\begin{bmatrix} \mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A} & (*) & (*) \\ \mathscr{B}^T\bar{X} & -\gamma^2 I_{n_w} & (*) \\ \mathscr{C} & \mathscr{D} & -I_s \end{bmatrix} < 0 \tag{4.42}$$

are satisfied.

Let $\bar{X} = \begin{bmatrix} \bar{X}_1 & 0 \\ 0 & X_2 \end{bmatrix}$ with $X_2 = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix}$. By using matrix $\mathscr{E}$ from (4.31) in the equation (4.41) we obtain :

$$E^T\bar{X}_1 = \bar{X}_1^T E \geq 0 \tag{4.43}$$

pre-multiplying the inequality (4.43) by X_1^{-T} and post-multiplying it by X_1^{-1} we obtain

$$X_1E^T = EX_1^T \geq 0 \tag{4.44}$$

where $X_1 = \bar{X}_1^{-T}$.

On the other hand, by replacing the matrices of system (4.31) in inequality (4.42) we obtain :

$$\begin{bmatrix} (A - B\kappa)^T\bar{X}_1 + \bar{X}_1^T(A - B\kappa) & (*) & (*) & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix}\bar{X}_1 & \mathbb{A}^TX_2 + X_2\mathbb{A} & (*) & 0 \\ (D - B\kappa\mathcal{Q}_{d_1})^T\bar{X}_1 & \mathbb{B}^TX_2 & -\gamma^2 I_{n_w} & (*) \\ C_2 & 0 & D_2 & -I_s \end{bmatrix} < 0 \tag{4.45}$$

By pre-multiplying the inequality (4.45) by $\begin{bmatrix} \bar{X}_1^{-T} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and post-multiplying it by $\begin{bmatrix} \bar{X}_1^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ we get :

$$\begin{bmatrix} X_1(A - B\kappa)^T + (A - B\kappa)X_1^T & (*) & (*) & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix} & \mathbb{A}^TX_2 + X_2\mathbb{A} & (*) & 0 \\ (D - B\kappa\mathcal{Q}_{d_1})^T & \mathbb{B}^TX_2 & -\gamma^2 I_{n_w} & (*) \\ C_2X_1 & 0 & D_2 & -I_s \end{bmatrix} < 0 \tag{4.46}$$

where $X_1 = \bar{X}_1^{-T}$.

Now, replacing matrices $\mathbb{A}$ and $\mathbb{B}$ from (4.32) and (4.33) we have :

$$\begin{bmatrix} X_1(A - B\kappa)^T + (A - B\kappa)X_1^T & (*) & (*) & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix} & (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^TX_2 + X_2(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) & (*) & 0 \\ (D - B\kappa\mathcal{Q}_{d_1})^T & (\mathbb{B}_1 - \mathbb{Y}\mathbb{B}_2)^TX_2 & -\gamma^2 I_{n_w} & (*) \\ C_2X_1 & 0 & D_2 & -I_s \end{bmatrix} < 0 \tag{4.47}$$

which can be written as :

$$\mathcal{B}\kappa^T\mathcal{C} + (\mathcal{B}\kappa^T\mathcal{C})^T + \mathcal{D} < 0 \tag{4.48}$$

where $\mathcal{D} = \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \mathbb{A}_1^TX_2 + X_2\mathbb{A}_1 - \mathbb{A}_2^TX_Y^T - X_Y\mathbb{A}_2 & (*) & 0 \\ D^T & \mathbb{B}_1^TX_2 - \mathbb{B}_2^TX_Y^T & -\gamma^2 I_{n_w} & (*) \\ C_2X_1 & 0 & D_2 & -I_s \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \\ \mathcal{Q}_{d_1}^T \\ 0 \end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix} -B^T & \begin{bmatrix} 0 & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$.

Using the elimination lemma of Section 1.5 inequality (4.48) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{4.49a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{4.49b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} -B^{\perp} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} 0 & \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} & 0 & 0 \\ 0 & 0 & \mathcal{Q}_{d_1}^{T\perp} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (4.49a) becomes :

$$\mathcal{C}^{T\perp} \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ D^T & \Pi_2^T & -\gamma^2 I_{n_w} & (*) \\ C_2X_1^T & 0 & D_2 & -I_s \end{bmatrix} \mathcal{C}^{T\perp T} < 0, \tag{4.50}$$

and by using $\mathcal{B}$ and $\mathcal{D}$ the inequality (4.49b) becomes :

$$\begin{bmatrix} \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} \Pi_1 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & (*) & 0 \\ \mathcal{Q}_{d_1}^{T\perp}\Pi_2^T \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & -\gamma^2 \mathcal{Q}_{d_1}^{T\perp}\mathcal{Q}_{d_1}^{T\perp T} & (*) \\ 0 & D_2\mathcal{Q}_{d_1}^{T\perp T} & -I_s \end{bmatrix} < 0 \tag{4.51}$$

where matrices Π_1 and Π_2 are defined in (4.38). From the elimination lemma, if conditions (4.49a) and (4.49b) are satisfied, the parameter matrix κ is parameterized as in (4.39) and (4.40). □

4.2.4.1 Particular cases

In this section two particular cases of our results are presented.

•Proportional observer-based control

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1x(t) + D_1w(t) \\ z(t) &= C_2x(t) + D_2w(t) \end{aligned}$$

with the PO-based control

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Fy_a(t) + Ju(t) \\ \hat{x}(t) &= P\zeta(t) + Q_ay(t) \\ u(t) &= -\kappa\hat{x}(t) \end{aligned}$$

and the closed-loop (4.31) becomes :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{B}w(t) \\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t) \end{aligned}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q0} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & -B\kappa P_1 \\ 0 & \bar{\mathbb{A}} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D - B\kappa\mathcal{Q}_{d_1} \\ \bar{\mathbb{B}} \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 \end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix}$. With matrices $\bar{\mathbb{A}} = \bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2$ and $\bar{\mathbb{B}} = \bar{\mathbb{B}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{B}}_2$ where $\bar{\mathbb{A}}_1 = N_1 - Z\mathcal{N}_2$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{B}}_1 = \mathcal{F}_{d_1} - T_1D + Z(\mathcal{T}_2D - \mathcal{F}_{d_2})$, $\bar{\mathbb{B}}_2 = \mathcal{F}_{d_3}$ and $\bar{\mathbb{Y}} = Y_1$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.2 become :

$$\mathcal{D} = \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ D^T & \Pi_2^T & -\gamma^2 I_{n_w} & (*) \\ C_2X_1^T & 0 & D_2 & -I_s \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} X_1 \\ P_1^T \\ \mathcal{Q}_{d_1}^T \\ 0 \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} -B^T & 0 & 0 & 0 \end{bmatrix}$$

where

$$\Pi_1 = (N_1 - Z\mathcal{N}_2)^T X_2 + X_2(N_1 - Z\mathcal{N}_2) - N_3^T X_Y^T - X_Y N_3$$
$$\Pi_2 = X_2(\mathcal{F}_{d_1} - T_1D + Z(\mathcal{T}_2D - \mathcal{F}_{d_2})) - X_Y\mathcal{F}_{d_3}$$

Matrices Σ and Ω are defined as $\Sigma = \begin{bmatrix} R \\ C_1 \end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C_1 \end{bmatrix}$.

•*Proportional-integral observer-based control*

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= C_1x(t) + D_1w(t) \\ z(t) &= C_2x(t) + D_2w(t) \end{aligned}$$

with the PIO-based control

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \\ \dot{v}(t) &= y(t) - C_1\hat{x}(t) \\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \\ u(t) &= -\kappa\hat{x}(t) \end{aligned}$$

and the closed-loop (4.31) becomes :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{B}w(t) \\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t) \end{aligned}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix} -B\kappa P_1 & 0 \end{bmatrix} \\ 0 & \bar{\bar{\mathbb{A}}} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D - B\kappa\mathcal{Q}_{d_1} \\ \bar{\bar{\mathbb{B}}} \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 & 0 \end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$.

With matrices $\bar{\bar{\mathbb{A}}} = \bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2$ and $\bar{\bar{\mathbb{B}}} = \bar{\bar{\mathbb{B}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{B}}}_2$, where $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ -C_1P_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_1 = \begin{bmatrix} \mathcal{F}_{d_1} - T_1D + Z(\mathcal{T}_2D - \mathcal{F}_{d_2}) \\ D_1 - C_1\mathcal{Q}_{d_1} \end{bmatrix}$, $\bar{\bar{\mathbb{B}}}_2 = \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix} I \\ 0 \end{bmatrix}\begin{bmatrix} Y_1 & H \end{bmatrix}$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.2 become :

$$\mathcal{D} = \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ D^T & \Pi_2^T & -\gamma^2 I_{n_w} & (*) \\ C_2X_1^T & 0 & D_2 & -I_s \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \\ \mathcal{Q}_{d_1}^T \\ 0 \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} -B^T & \begin{bmatrix} 0 & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

where

$$\Pi_1 = \begin{bmatrix} X_{21}(N_1 - Z\mathcal{N}_2) + (N_1 - Z\mathcal{N}_2)^T X_{21} - X_{21}C_1P_1 - P_1^T C_1^T X_{21} & (*) \\ X_{21}(N_1 - Z\mathcal{N}_2) - X_{22}C_1P_1 & 0 \end{bmatrix} -$$

$$\begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}^T X_Y^T - X_Y \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} X_{21}(\mathcal{F}_{d_1} - T_1 D + Z(\mathcal{T}_2 D - \mathcal{F}_{d_2})) + X_{21}(D_1 - C_1\mathcal{Q}_{d_1}) \\ X_{21}(\mathcal{F}_{d_1} - T_1 D + Z(\mathcal{T}_2 D - \mathcal{F}_{d_2})) + X_{22}(D_1 - C_1\mathcal{Q}_{d_1}) \end{bmatrix} - X_Y \begin{bmatrix} \mathcal{F}_{d_3} \\ 0 \end{bmatrix}$$

4.2.4.2 Numerical example

Consider the following descriptor system described by (4.29), where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},\ D_2 = 0.2 \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

By using the Definition 1.3 we show that the system above is unstable

$$\mathsf{eig}(E,\ A) = [0.39,\ -0.7 + 1.43i,\ -0.7 - 1.43i]$$

Considering $E^\perp = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ we can verify Assumptions 4.1, 4.2 and 4.3

$$\mathsf{rank} \begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = 4,\ \mathsf{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 4 \text{ and } \mathsf{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = 4.$$

Generalized dynamic observer

For the GDO we have chosen matrix $R = \begin{bmatrix} 4 & 0 & 2 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix}$, then $\mathsf{rank}(\Sigma) = 4$.

By fixing $\gamma = 2.5$ and using YALMIP toolbox, we solve the LMIs (4.35) - (4.37) to find matrices X_1, X_2, Z and $\mathbb{Y}$

$$X_1 = \begin{bmatrix} 2.44 & 0.34 & 0.6 & 0 \\ 0.34 & 1.3 & -0.65 & -0.39 \\ 0.6 & -0.651.54 & -0.25 & \\ 0 & 0 & 0 & -0.68 \end{bmatrix},\ X_2 = \begin{bmatrix} 1.31 & 0.07 & 1.31 & 0.07 \\ 0.07 & 0.86 & 0.07 & 0.86 \\ 1.31 & 0.07 & 3.11 & 0.08 \\ 0.07 & 0.86 & 0.08 & 2.58 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & -9.91 & -33.56 & 27.07 & -47.88 & -2.99 & -13.41 \\ 0 & 18.93 & -28.44 & 8.5 & -54.2 & 8.64 & -0.19 \end{bmatrix} \text{ and } \mathbb{Y} = \begin{bmatrix} 2.44 & 0.34 & 0.6 & 0 \\ 0.34 & 1.3 & -0.65 & 0 \\ 0.6 & -0.65 & 1.54 & 0 \\ 0 & 0 & 0 & 2.06 \end{bmatrix}.$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.4 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.4$ and solving (4.39) and (4.40) we get :

$$\kappa = \begin{bmatrix} 5.76 & -4.69 & -4.29 & -1.12 \end{bmatrix}.$$

Finally, we compute all the matrices of the observer-based control as :

$$N = \begin{bmatrix} -0.65 & 0.25 \\ 0.36 & -1.06 \end{bmatrix},\ S = \begin{bmatrix} 0.12 & -0.23 \\ 0.04 & -0.08 \end{bmatrix},\ H = \begin{bmatrix} 1.14 & -0.58 \\ 0.8 & 2.61 \end{bmatrix},\ J = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

$$F=\begin{bmatrix}-20.83 & -5.9 & -5.53\\ -6.15 & -0.49 & -3.87\end{bmatrix},\ L=\begin{bmatrix}-0.82 & 0.22\\ -0.27 & -1.26\end{bmatrix},\ P=\begin{bmatrix}0.17 & 0.15\\ -0.07 & 0.15\\ 0.15 & -0.29\\ -0.05 & 0.11\end{bmatrix},$$

$$M=\begin{bmatrix}-0.44 & 0.8 & 0.44\\ -0.16 & 0.29 & 0.16\end{bmatrix}\text{ and } Q=\begin{bmatrix}0.11 & -0.68 & -0.11\\ 0.27 & 0.5 & -0.27\\ 0.45 & 0 & -0.45\\ 0.21 & -0.37 & 0.79\end{bmatrix}.$$

Proportional observer-based control

By considering matrices $R=\begin{bmatrix}5 & 0 & 2 & 0\\ 2 & 3 & 0 & 5\end{bmatrix}$, $\mathcal{Z}=\begin{bmatrix}2\\3\\1\\8\end{bmatrix}$, $\mathcal{L}=\begin{bmatrix}0.1\\0.1\\0.1\\0.1\end{bmatrix}$ and $\mathcal{R}=I_4\times 0.1$ and $\gamma=2.5$ the following PO-based control is obtained :

$$\kappa=\begin{bmatrix}3.66 & -1.08 & -0.65 & 0.92\end{bmatrix},$$

$$N=\begin{bmatrix}-0.62 & 0.56\\ 0 & -0.76\end{bmatrix},\ F_a=\begin{bmatrix}-7.08 & -4.24\\ -2.34 & -3.49\end{bmatrix},\ J=\begin{bmatrix}5\\2\end{bmatrix},\ P=\begin{bmatrix}0 & 0.5\\ 0 & 0\\ 0.5 & -1.25\\ 0 & 0\end{bmatrix}\text{ and } Q_a=\begin{bmatrix}-0.75 & 0\\ 1 & 0\\ 1.88 & 0\\ 0 & 1\end{bmatrix}.$$

Proportional-integral observer-based control

By considering matrices $R=\begin{bmatrix}1 & 1 & 0 & 1\\ 0 & 1 & 0 & 1\end{bmatrix}$, $\mathcal{Z}=\begin{bmatrix}2\\3\\1\\8\end{bmatrix}$, $\mathcal{L}=\begin{bmatrix}0.1\\0.1\\0.1\\0.1\end{bmatrix}$ and $\mathcal{R}=I_4\times 0.001$ and $\gamma=2.5$ the following PIO-based control is obtained :

$$\kappa=\begin{bmatrix}2.93 & -2.28 & -2.08 & 0.19\end{bmatrix},$$

$$N=\begin{bmatrix}-1.09 & -2.64\\ 0.3 & -0.44\end{bmatrix},\ H=\begin{bmatrix}0.04 & -0.18\\ 2.97 & -0.21\end{bmatrix},\ P=\begin{bmatrix}1 & -0.67\\ 0 & 0.33\\ 0 & -0.67\\ 0 & 0.33\end{bmatrix},\ J=\begin{bmatrix}1\\0\end{bmatrix},$$

$$F=\begin{bmatrix}-13.68 & 1.56 & -2.5\\ 2.81 & -1 & -0.59\end{bmatrix}\text{ and } Q=\begin{bmatrix}1.14 & -3.58 & -1.14\\ 0.1 & 0.89 & -0.1\\ 0.79 & -0.78 & -0.79\\ 0.1 & -0.11 & 0.89\end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0)=[0,\ 0.1,\ 0,\ 0.1]^T$, for the GDO are $\zeta(0)_{GDO}=[0,\ 0]^T$, $v(0)_{GDO}=[0,\ 0]^T$ and $\hat{x}(0)_{GDO}=[0,\ 0,\ 0]^T$, for the PIO are $\zeta(0)_{PIO}=[0,\ 0]^T$, $v(0)_{PIO}=[0,\ 0]^T$ and $\hat{x}(0)_{PIO}=[0,\ 0,\ 0]^T$ and for the PO are $\zeta(0)_{PO}=[0,\ 0]^T$ and $\hat{x}(0)_{PO}=[0,\ 0,\ 0]^T$.

To evaluate the performance of the control an uncertainty $\wp(t)$ is added in the system matrix A, then we obtain the matrix $(A+\wp(t))$, where $\wp(t)=\delta(t)\times\begin{bmatrix}0.3 & 0 & 0.5 & 0\\ 0 & 0.2 & 0 & 0\\ 0.1 & 0 & 0.3 & 0\\ 0 & 0.2 & 0.2 & 0\end{bmatrix}$.

The results of simulation are depicted in Figures 4.5 - 4.8. Figures 4.5 and 4.6 show the uncertainty factor $\delta(t)$ and the disturbance $w(t)$. Figure 4.7 shows the control input $u(t)$ provided by the GDO, PIO and PO. Figure 4.8 shows the controlled output by the GDO, PIO and PO.

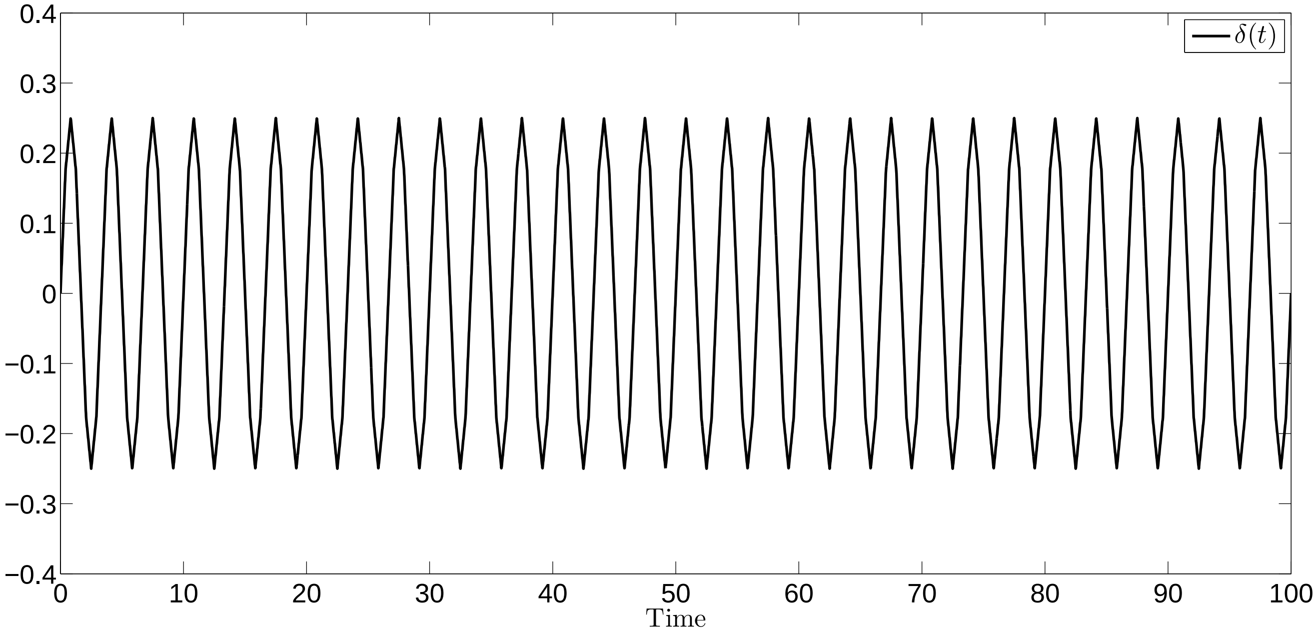

FIGURE 4.5 – H_∞ observers-based control : Uncertainty factor $\delta(t)$.

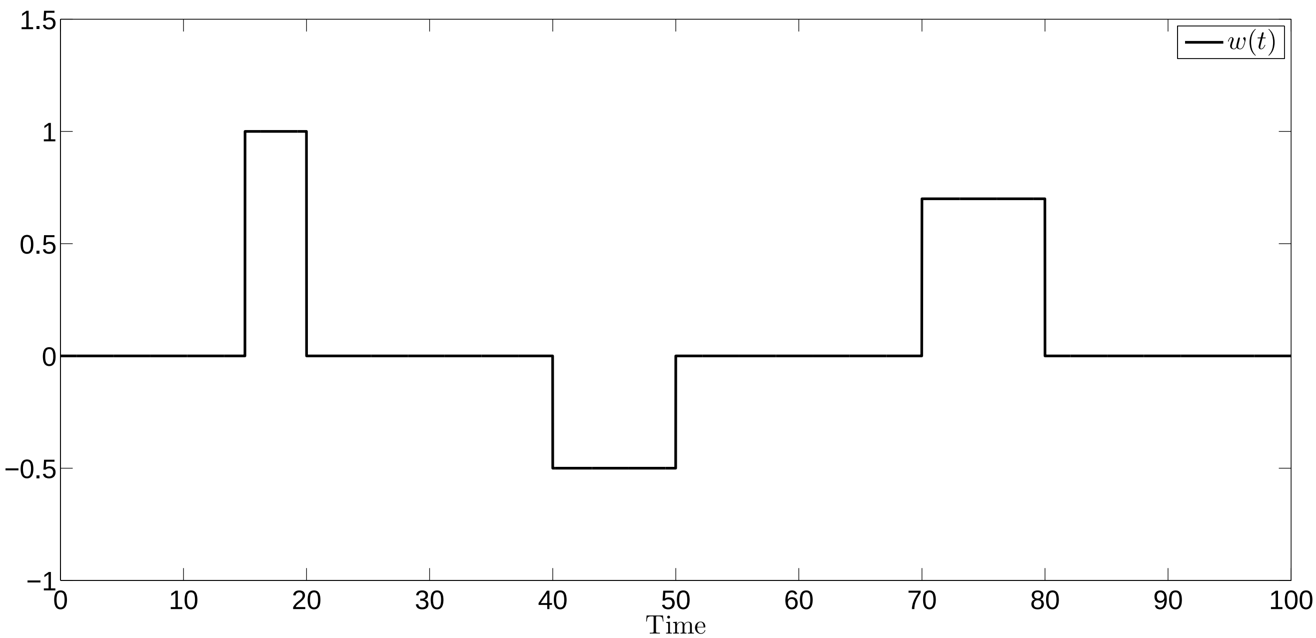

FIGURE 4.6 – H_∞ observers-based control : Disturbance $w(t)$.

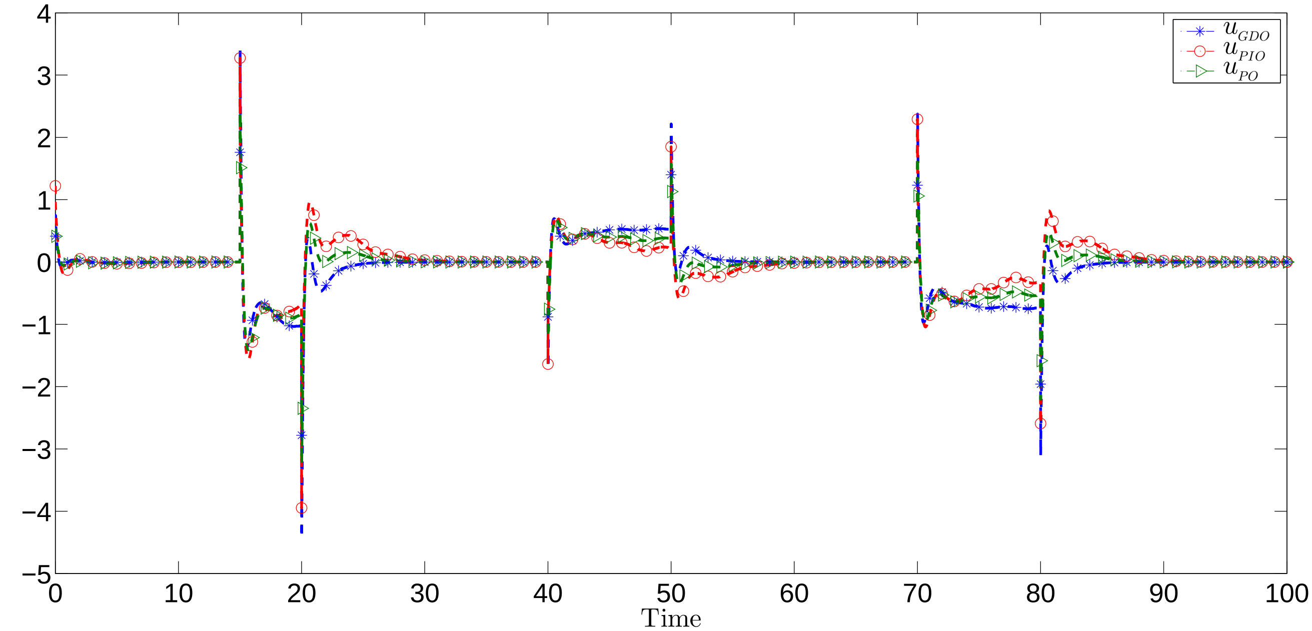

FIGURE 4.7 – H_∞ observers-based control : Control input $u(t)$.

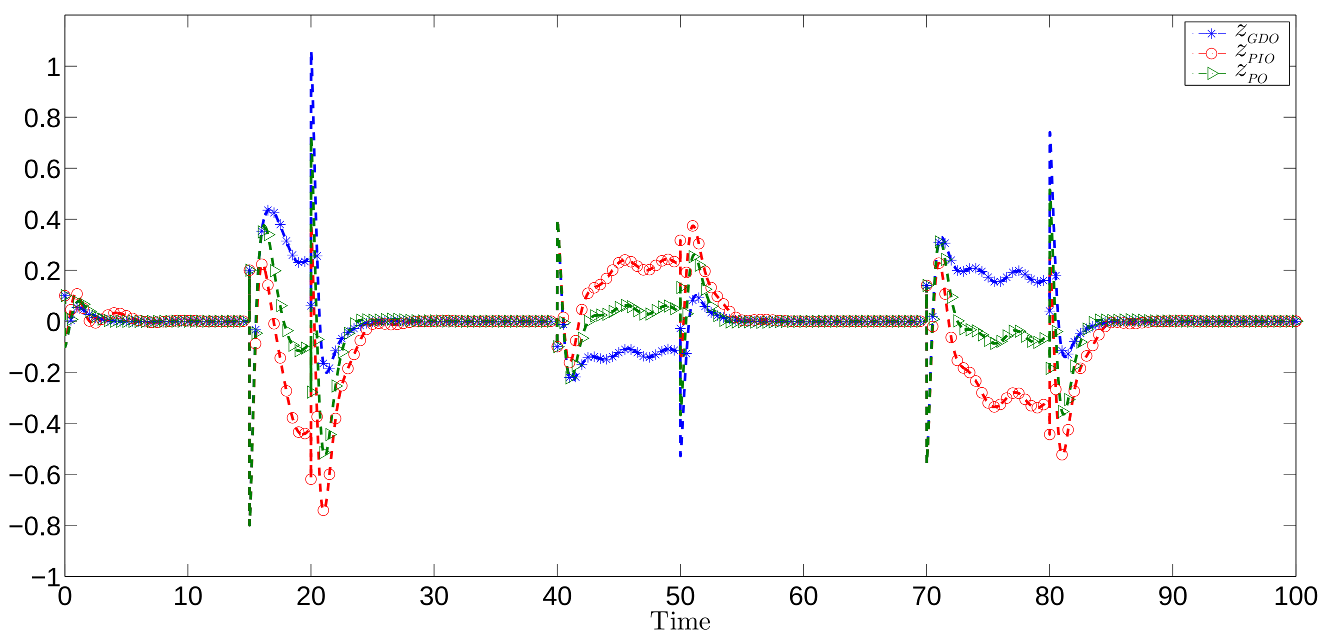

FIGURE 4.8 – H_∞ observers-based control : Controlled output $z(t)$.

From these results, we can see that all the observers achieved stabilize the output. However, just after the disturbance the GDO stabilize the output faster than the PIO and PO

4.3 Robust H_∞ generalized dynamic observer-based control design for uncertain systems

In this section the robust H_∞ GDO-based control designs for uncertain descriptor systems without or with disturbances are presented.

4.3.1 Class of uncertain disturbed descriptor systems considered

Consider the following uncertain descriptor system :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Bu(t) + Dw(t) \tag{4.52a}$$
$$y(t) = (C_1 + \Delta C(t))x(t) + D_1 w(t) \tag{4.52b}$$
$$z(t) = C_2 x(t) + D_2 w(t) \tag{4.52c}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector of bounded energy, $y(t) \in \mathbb{R}^{n_y}$ is the measured output and $z(t) \in \mathbb{R}^s$ is the controlled output. Matrices $E \in \mathbb{R}^{n_1 \times n}$, $A \in \mathbb{R}^{n_1 \times n}$, $B \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{n \times n_w}$, $C_1 \in \mathbb{R}^{n_y \times n_1}$, $D_1 \in \mathbb{R}^{n_y \times n_w}$, $C_2 \in \mathbb{R}^{s \times n}$ and $D_2 \in \mathbb{R}^{s \times n_w}$ are constant and known. Let $\mathsf{rank}(E) = \varrho < n$ and let $E^\perp \in \mathbb{R}^{\varrho_1 \times n}$ be a full row rank matrix such that $E^\perp E = 0$, in this case $\varrho_1 = n - \varrho$.

Matrices $\Delta A(t)$ and $\Delta C(t)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form :

$$\Delta A(t) = \mathcal{M}_1 \Gamma(t) \mathcal{G} \tag{4.53a}$$
$$\Delta C(t) = \mathcal{M}_2 \Gamma(t) \mathcal{G} \tag{4.53b}$$

where $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{G}$ are known real constant matrices and $\Gamma(t)$ is an unknown time-varying matrix satisfying

$$\Gamma(t)^T \Gamma(t) \leq I, \ \forall t \in [0, \infty). \tag{4.54}$$

In the sequel we assume that

Assumption 4.4.

$$\mathsf{rank} \begin{bmatrix} E \\ E^\perp A \\ C_1 \end{bmatrix} = n.$$

Assumption 4.5.

$$\mathsf{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = n, \ \forall s \in \mathbb{C}^+, \ s \textit{ finite}.$$

Assumption 4.6.

$$\mathsf{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = n, \ \forall s \in \mathbb{C}^+, \ s \textit{ finite}.$$

4.3.2 Problem formulation

Consider the following GDO-based control for system (4.52)

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{4.55a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + M \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.55b}$$
$$\hat{x}(t) = P\zeta(t) + Q \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.55c}$$
$$u(t) = -\kappa \hat{x}(t) \tag{4.55d}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer-based control, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. Matrices N, F, J, H, L, M, S, P, Q and κ are unknown matrices of appropriate dimensions which must be determined such that the closed-loop obtained between the system (4.52) and the observer-based control (4.55) is stable.
Now, we can give the following lemma.

Lemma 4.1. *There exists an observer-based control of the form* (4.55) *for the system* (4.52) *if the following statements hold.*

1. *There exists a matrix T of appropriate dimension such that the following conditions are satisfied :*

   (a) $NTE + F\begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} - TA = 0$

   (b) $J = TB$

   (c) $M\begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} + STE = 0$

   (d) $\begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \end{bmatrix} = I_n$

2. *The matrix $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz when $w(t) = 0$ and $\Gamma(t) = 0$.*

Proof. Let $T \in \mathbb{R}^{q_0 \times n_1}$ be a parameter matrix and define the error $\varepsilon(t) = \zeta(t) - TEx(t)$, then its derivative is given by :

$$\begin{aligned} \dot{\varepsilon}(t) = {} & N\varepsilon(t) + Hv(t) + \left(NTE + F\begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} - TA\right)x(t) + \left(F\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix} - T\Delta A(t)\right)x(t) + \\ & (J - TB)u(t) + \left(F\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - TD\right)w(t) \end{aligned} \tag{4.56}$$

By using the definition of $\varepsilon(t)$, equations (4.55b) and (4.55c) can be written as :

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + \left(M\begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} + STE\right)x(t) + M\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix}x(t) + M\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}w(t) \tag{4.57}$$

$$\hat{x}(t) = P\varepsilon(t) + \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \end{bmatrix}x(t) + Q\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix}x(t) + Q\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}w(t) \tag{4.58}$$

If conditions $(a) - (d)$ of Lemma 4.1 are satisfied the following observer error dynamics is obtained from equations (4.56) and (4.57)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix}\begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} F\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix} - T\Delta A(t) \\ M\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix} \end{bmatrix}x(t) + \begin{bmatrix} F\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - TD \\ M\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \end{bmatrix}w(t) \tag{4.59}$$

and from equation (4.58) we get :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon(t) + Q\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix}x(t) + Q\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}w(t) \tag{4.60}$$

in this case if $w(t) = 0$, $\Gamma(t) = 0$ and matrix $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz, then $\lim_{t\to\infty} e(t) = 0$. □

From the system (4.52) and the observer error (4.59) the following closed-loop system is obtained :

$$E\dot{x}(t) = (A - B\kappa)x(t) + \left(\Delta A(t) - B\kappa Q\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix}\right)x(t) - B\kappa P\varepsilon(t) + \left(D - B\kappa Q\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}\right)w(t) \tag{4.61a}$$

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) + \left(F\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix} - T\Delta A(t)\right)x(t) + \left(F\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - TD\right)w(t) \tag{4.61b}$$

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + M\begin{bmatrix} E^\perp \Delta A(t) \\ \Delta C(t) \end{bmatrix}x(t) + M\begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix}w(t) \tag{4.61c}$$

$$z(t) = C_2x(t) + D_2w(t) \tag{4.61d}$$

By considering the observer parameterization of Section 3.2.2 of equations (3.10) - (3.16), where $Z_1 = 0$ and $Y_3 = 0$, and the form of matrices $\Delta A(t)$ and $\Delta C(t)$ the closed-loop (4.61) can be written as :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) + \mathscr{B}w(t) \\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t) \end{aligned} \tag{4.62}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix} -B\kappa P_1 & 0 \end{bmatrix} \\ 0 & \mathbb{A} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D - B\kappa Q_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \\ \mathbb{F}_d \end{bmatrix}$, $\mathscr{F}_1 = \begin{bmatrix} \mathcal{M}_1 - B\kappa Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ \mathbb{F}_m \end{bmatrix}$,

$\mathscr{F}_2 = \begin{bmatrix} \mathcal{G} & 0 & 0 \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 & 0 \end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$.

Matrices $\mathbb{A}$, $\mathbb{F}_m$ and $\mathbb{F}_d$ have the following form :

$$\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 \tag{4.63}$$

$$\mathbb{F}_m = \mathbb{F}_{m1} - \mathbb{Y}\mathbb{F}_{m2} \tag{4.64}$$

$$\mathbb{F}_d = \mathbb{F}_{d1} - \mathbb{Y}\mathbb{F}_{d2} \tag{4.65}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_{m1} = \begin{bmatrix} F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1\mathcal{M}_1 \\ 0 \end{bmatrix}$, $\mathbb{F}_{m2} = \begin{bmatrix} F_3 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ 0 \end{bmatrix}$, $\mathbb{F}_{d1} = \begin{bmatrix} F_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - T_1 D \\ 0 \end{bmatrix}$,

$\mathbb{F}_{d2} = \begin{bmatrix} F_3 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \\ 0 \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$.

The problem is then reduced to find the parameter matrices of the observer-based control (4.55) such that the controlled output $z(t)$ converges asymptotically to zero for $w(t) = 0$ and $\Gamma(t) = 0$.

4.3.3 Robust generalized dynamic observer-based control design for uncertain descriptor systems, $w(t) = 0$

In this section we consider $w(t) = 0$, then system (4.52) becomes :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Bu(t) \tag{4.66a}$$

$$y(t) = (C_1 + \Delta C(t))x(t) \tag{4.66b}$$

$$z(t) = C_2 x(t) \tag{4.66c}$$

with the GDO-based control :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{4.67a}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.67b}$$

$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.67c}$$

$$u(t) = -\kappa\hat{x}(t) \tag{4.67d}$$

and the closed-loop (4.62) becomes :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) \\ z(t) &= \mathscr{C}\beta(t) \end{aligned} \tag{4.68}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix} -B\kappa P_1 & 0 \end{bmatrix} \\ 0 & \mathbb{A} \end{bmatrix}$, $\mathscr{F}_1 = \begin{bmatrix} \mathcal{M}_1 - B\kappa Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ \mathbb{F}_m \end{bmatrix}$, $\mathscr{F}_2 = \begin{bmatrix} \mathcal{G} & 0 & 0 \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 & 0 \end{bmatrix}$

and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$. Matrices $\mathbb{A}$ and $\mathbb{F}_m$ have the following form :

$$\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 \tag{4.69}$$
$$\mathbb{F}_m = \mathbb{F}_{m1} - \mathbb{Y}\mathbb{F}_{m2} \tag{4.70}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_{m1} = \begin{bmatrix} F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1\mathcal{M}_1 \\ 0 \end{bmatrix}$, $\mathbb{F}_{m2} = \begin{bmatrix} F_3 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ 0 \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$. The following theorem gives the conditions for the existence of the GDO-based control such that the closed-loop system (4.68) is stable.

Theorem 4.3. *Under Assumptions 4.4, 4.5 and 4.6 there exist parameter matrices κ and $\mathbb{Y}$ such that the closed-loop (4.68) is stable if there exists a symmetric positive definite matrix $X_2 = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix}$, with $X_{21} = X_{21}^T$, and a nonsingular matrix X_1 such that the following LMIs are satisfied.*

$$X_1 E^T = E X_1^T \geq 0, \tag{4.71}$$

$$\mathcal{C}^{T\perp} \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ \mathcal{M}_1^T & \Pi_2 & -\epsilon I & 0 \\ \epsilon \mathcal{G} X_1^T & 0 & 0 & -\epsilon I \end{bmatrix} \mathcal{C}^{T\perp T} < 0, \tag{4.72}$$

and

$$\begin{bmatrix} \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^\perp \Pi_1 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & (*) & 0 \\ \left(Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^{T\perp} \Pi_2 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & -\epsilon \left(Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^{T\perp} \left(Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^{T\perp T} & 0 \\ 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{4.73}$$

with

$$\Pi_1 = \begin{bmatrix} X_{21}N_1 + N_1^T X_{21} & (*) \\ X_{21}N_1 & 0 \end{bmatrix} - \begin{bmatrix} N_3 & 0 \\ 0 & 0 \end{bmatrix}^T X_Y^T - X_Y \begin{bmatrix} N_3 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.74a}$$

$$\Pi_2 = \begin{bmatrix} \left(F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1\mathcal{M}_1\right)^T X_{21} & \left(F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1\mathcal{M}_1\right)^T X_{21} \end{bmatrix} - \begin{bmatrix} F_3 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ 0 \end{bmatrix}^T X_Y^T \tag{4.74b}$$

In this case $X_Y = X_2\mathbb{Y}$, and matrix κ is parameterized as follows :

$$\kappa = (\mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+)^T \tag{4.75}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}_l^T \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1/2} \tag{4.76a}$$
$$\vartheta = (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{4.76b}$$
$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T [\vartheta - \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} \mathcal{C}_r \vartheta] \mathcal{B}_l \mathcal{R}^{-1} \tag{4.76c}$$

with $\mathcal{D} = \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ \mathcal{M}_1^T & \Pi_2 & -\epsilon I & 0 \\ \epsilon \mathcal{G} X_1^T & 0 & 0 & -\epsilon I \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} -B & 0 & 0 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \\ \left(Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^T \\ 0 \end{bmatrix}$, matrices Π_1 and Π_2 are defined in (4.74). Matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$.

Proof. Consider the following Lyapunov function

$$V(\beta(t)) = \beta(t)^T \mathscr{E}^T \bar{X} \beta(t) \geq 0 \tag{4.77}$$

where

$$\mathscr{E}^T\bar{X} = \bar{X}^T\mathscr{E} \geq 0 \tag{4.78}$$

with $\bar{X} = \begin{bmatrix} \bar{X}_1 & 0 \\ 0 & X_2 \end{bmatrix}$ and $X_2 = X_2^T = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix} > 0$.

By using matrix $\mathscr{E}$ from (4.68), the equation (4.78) becomes :

$$E^T\bar{X}_1 = \bar{X}_1^T E \geq 0 \tag{4.79}$$

pre-multiplying the inequality (4.79) by $\bar{X}_1^{-T}$ and post-multiplying it by $\bar{X}_1^{-1}$ we obtain

$$X_1 E^T = E X_1^T \geq 0 \tag{4.80}$$

where $X_1 = \bar{X}_1^{-T}$.

Now, the derivative of $V(\beta(t))$ along the trajectory of system (4.68) is :

$$\dot{V}(\beta(t)) = \beta(t)^T(\mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A})\beta(t) + 2\beta(t)^T\bar{X}^T\mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) \tag{4.81}$$

Using Lemma 1.6 from Section 1.7.3, and since $\Gamma(t)^T\Gamma(t) \leq I$ the following inequality can be formulated :

$$2\beta(t)^T\bar{X}^T\mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) \leq \epsilon^{-1}\beta(t)^T\bar{X}^T\mathscr{F}_1\mathscr{F}_1^T\bar{X}\beta(t) + \epsilon\beta(t)^T\mathscr{F}_2^T\mathscr{F}_2\beta(t) \tag{4.82}$$

with $\epsilon > 0$. Thus,

$$\dot{V}(\beta(t)) \leq \beta(t)^T(\mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A} + \epsilon^{-1}\bar{X}^T\mathscr{F}_1\mathscr{F}_1^T\bar{X} + \epsilon\mathscr{F}_2^T\mathscr{F}_2)\beta(t) \tag{4.83}$$

By applying the Schur complement to inequality (4.83) we get :

$$\dot{V}(\beta(t)) \leq \beta(t)^T \begin{bmatrix} \mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A} & \begin{bmatrix} \bar{X}^T\mathscr{F}_1 & \epsilon\mathscr{F}_2^T \end{bmatrix} \\ \begin{bmatrix} \mathscr{F}_1^T\bar{X} \\ \epsilon\mathscr{F}_2 \end{bmatrix} & -\epsilon I \end{bmatrix} \beta(t) \tag{4.84}$$

The asymptotic stability of system (4.68) is guaranteed if and only if $\dot{V}(\beta(t)) < 0$. Which leads to the following LMI :

$$\begin{bmatrix} \mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A} & \begin{bmatrix} \bar{X}^T\mathscr{F}_1 & \epsilon\mathscr{F}_2^T \end{bmatrix} \\ \begin{bmatrix} \mathscr{F}_1^T\bar{X} \\ \epsilon\mathscr{F}_2 \end{bmatrix} & -\epsilon I \end{bmatrix} < 0 \tag{4.85}$$

Now, inserting the form of matrices $\mathscr{A}$, $\mathscr{F}_1$, $\mathscr{F}_2$ and $\bar{X}$ we have :

$$\begin{bmatrix} (A - B\kappa)^T\bar{X}_1 + \bar{X}_1^T(A - B\kappa) & (*) & (*) & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix}\bar{X}_1 & X_2\mathbb{A} + \mathbb{A}^T X_2 & (*) & 0 \\ \left(\mathcal{M}_1 - B\kappa Q_1 \begin{bmatrix} E^\perp\mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^T \bar{X}_1 & \mathbb{F}_m^T X_2 & -\epsilon I & 0 \\ \epsilon\mathcal{G} & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{4.86}$$

pre-multiplying the inequality (4.86) by $\begin{bmatrix} \bar{X}_1^{-T} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and post-multiplying it by $\begin{bmatrix} \bar{X}_1^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ we get :

$$\begin{bmatrix} X_1(A - B\kappa)^T + (A - B\kappa)X_1^T & (*) & (*) & (*) \\ \begin{bmatrix} (-B\kappa P_1)^T \\ 0 \end{bmatrix} & X_2\mathbb{A} + \mathbb{A}^T X_2 & (*) & 0 \\ \left(\mathcal{M}_1 - B\kappa Q_1 \begin{bmatrix} E^\perp\mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^T & \mathbb{F}_m^T X_2 & -\epsilon I & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{4.87}$$

where $X_1 = \bar{X}_1^{-T}$.

Replacing the form of matrices $\mathbb{A}$ and $\mathbb{F}_m$ we have :

$$\begin{bmatrix} X_1(A-B\kappa)^T+(A-B\kappa)X_1^T & (*) & (*) & (*) \\ \begin{bmatrix}(-B\kappa P_1)^T\\ 0\end{bmatrix} & X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)+(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2 & (*) & 0 \\ \left(\mathcal{M}_1-B\kappa Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^T & (\mathbb{F}_{m1}-\mathbb{Y}\mathbb{F}_{m2})^TX_2 & -\epsilon I & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I\end{bmatrix}<0 \tag{4.88}$$

which can be written as :

$$\mathcal{B}\kappa^T\mathcal{C}+(\mathcal{B}\kappa^T\mathcal{C})^T+\mathcal{D}<0 \tag{4.89}$$

where $\mathcal{D}=\begin{bmatrix} X_1A^T+AX_1^T & 0 & (*) & (*) \\ 0 & X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)+(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2 & (*) & 0 \\ \mathcal{M}_1^T & (\mathbb{F}_{m1}-\mathbb{Y}\mathbb{F}_{m2})^TX_2 & -\epsilon I & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I\end{bmatrix}$, $\mathcal{B}=\begin{bmatrix} X_1 \\ \begin{bmatrix}P_1^T\\ 0\end{bmatrix} \\ \left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^T \\ 0\end{bmatrix}$ and $\mathcal{C}=\begin{bmatrix}-B & 0 & 0 & 0\end{bmatrix}$.

Using the elimination lemma of Section 1.5, inequality (4.89) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T}<0 \tag{4.90a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T}<0 \tag{4.90b}$$

with $\mathcal{C}^{T\perp}=\begin{bmatrix}-B^\perp & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}$ and $\mathcal{B}^\perp=\begin{bmatrix} 0 & \begin{bmatrix}P_1^T\\ 0\end{bmatrix}^\perp & 0 & 0 \\ 0 & 0 & \left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^{T\perp} & 0 \\ 0 & 0 & 0 & I\end{bmatrix}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (4.90a) becomes :

$$\mathcal{C}^{T\perp}\begin{bmatrix} X_1A^T+AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ \mathcal{M}_1^T & \Pi_2 & -\epsilon I & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I\end{bmatrix}\mathcal{C}^{T\perp T}<0, \tag{4.91}$$

and by using the definition of matrices $\mathcal{B}$ and $\mathcal{D}$ the inequality 4.90b becomes :

$$\begin{bmatrix} \begin{bmatrix}P_1^T\\ 0\end{bmatrix}^\perp\Pi_1\begin{bmatrix}P_1^T\\ 0\end{bmatrix}^{\perp T} & (*) & 0 \\ \left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^{T\perp}\Pi_2\begin{bmatrix}P_1^T\\ 0\end{bmatrix}^{\perp T} & -\epsilon\left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^{T\perp}\left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^{T\perp T} & 0 \\ 0 & 0 & -\epsilon I\end{bmatrix}<0 \tag{4.92}$$

matrices Π_1 and Π_2 are defined in (4.74). From the elimination lemma, if conditions (4.90a) and (4.90b) are satisfied, the parameter matrix κ is parameterized as in (4.75) and (4.76). □

4.3.3.1 Particular cases

•***Proportional observer-based control***

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= (A+\Delta A(t))x(t)+Bu(t) \\ y(t) &= (C_1+\Delta C(t))x(t) \\ z(t) &= C_2x(t)\end{aligned}$$

with the PO-based control :

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + F_a y(t) + Ju(t)\\ \hat{x}(t) &= P\zeta(t) + Q_a y(t)\\ u(t) &= -\kappa\hat{x}(t)\end{aligned}$$

and the closed-loop (4.68) becomes :

$$\begin{aligned}\mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t)\\ z(t) &= \mathscr{C}\beta(t)\end{aligned} \tag{4.93}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0}\end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & -B\kappa P_1 \\ 0 & \bar{\mathbb{A}}\end{bmatrix}$, $\mathscr{F}_1 = \begin{bmatrix}\mathcal{M}_1 - B\kappa Q_1\mathcal{M}_2 \\ \bar{\mathbb{F}}_m\end{bmatrix}$, $\mathscr{F}_2 = \begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0\end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t)\end{bmatrix}$. Matrices $\bar{\mathbb{A}} = \bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2$ and $\bar{\mathbb{F}}_m = \bar{\mathbb{F}}_{m1} - \bar{\mathbb{Y}}\bar{\mathbb{F}}_{m2}$ where $\bar{\mathbb{A}}_1 = N_1$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{F}}_{m1} = F_1\mathcal{M}_2 - T_1\mathcal{M}_1$, $\bar{\mathbb{F}}_{m2} = F_3\mathcal{M}_2$ and $\bar{\mathbb{Y}} = Y_1$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.3 become :

$$\mathcal{D} = \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ \mathcal{M}_1^T & \Pi_2 & -\epsilon I & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I\end{bmatrix},\ \mathcal{C} = \begin{bmatrix}-B & 0 & 0 & 0\end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} X_1 \\ P_1^T \\ (Q_1\mathcal{M}_2)^T \\ 0\end{bmatrix}$$

where

$$\begin{aligned}\Pi_1 &= N_1^TX_2 + X_2N_1 - X_YN_3N_3^TX_Y^T\\ \Pi_2 &= (F_1\mathcal{M}_2 - T_1\mathcal{M}_1)^T X_2 - \mathcal{M}_2^TF_3^TX_Y^T\end{aligned}$$

Matrices Σ and Ω become $\Sigma = \begin{bmatrix} R \\ C_1\end{bmatrix}$ and $\Omega = \begin{bmatrix} E \\ C_1\end{bmatrix}$.

•*Proportional-integral observer-based control*

Consider the following descriptor system :

$$\begin{aligned}E\dot{x}(t) &= (A + \Delta A(t))x(t) + Bu(t)\\ y(t) &= (C_1 + \Delta C(t))x(t)\\ z(t) &= C_2x(t)\end{aligned}$$

with the PIO-based control :

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix}-E^\perp Bu(t) \\ y(t)\end{bmatrix} + Ju(t)\\ \dot{v}(t) &= y(t) - C_1\hat{x}(t)\\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix}-E^\perp Bu(t) \\ y(t)\end{bmatrix}\\ u(t) &= -\kappa\hat{x}(t)\end{aligned}$$

and the closed-loop (4.68) becomes :

$$\begin{aligned}\mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t)\\ z(t) &= \mathscr{C}\beta(t)\end{aligned} \tag{4.94}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0}\end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix}-B\kappa P_1 & 0\end{bmatrix} \\ 0 & \bar{\bar{\mathbb{A}}}\end{bmatrix}$, $\mathscr{F}_1 = \begin{bmatrix}\mathcal{M}_1 - B\kappa Q_1\begin{bmatrix}E^\perp\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix} \\ \bar{\bar{\mathbb{F}}}_m\end{bmatrix}$, $\mathscr{F}_2 = \begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}$, $\mathscr{C} = \begin{bmatrix}C_2 & 0 & 0\end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t)\end{bmatrix}$. Matrices $\bar{\bar{\mathbb{A}}} = \bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2$ and $\bar{\bar{\mathbb{F}}}_m = \bar{\bar{\mathbb{F}}}_{m1} - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{F}}}_{m2}$ where $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix} N_1 & 0 \\ -C_1P_1 & 0\end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1}\end{bmatrix}$,

$\bar{\bar{\mathbb{F}}}_{m1} = \begin{bmatrix} F_1\begin{bmatrix}E^\perp\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix} - T_1\mathcal{M}_1 \\ \mathcal{M}_2 - C_1Q_1\begin{bmatrix}E^\perp\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix}\end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_{m2} = \begin{bmatrix} F_3\begin{bmatrix}E^\perp\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix} \\ 0\end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix} I \\ 0\end{bmatrix}\begin{bmatrix}Y_1 & H\end{bmatrix}$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.3 become :

$$\mathcal{D} = \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 \\ \mathcal{M}_1^T & \Pi_2^T & -\epsilon I & 0 \\ \epsilon \mathcal{G} X_1^T & 0 & 0 & -\epsilon I \end{bmatrix}, \; \mathcal{C} = \begin{bmatrix} -B & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \\ \left(Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \right)^T \\ 0 \end{bmatrix}$$

where

$$\Pi_1 = \begin{bmatrix} X_{21}N_1 - X_{21}C_1P_1 + N_1^TX_{21} - P_1^TC_1^TX_{21} & (*) \\ X_{21}N_1 - X_{22}C_1P_1 & 0 \end{bmatrix} - X_Y \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} - \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}^T X_Y^T$$

$$\Pi_2 = \begin{bmatrix} X_{21}\left(F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1\mathcal{M}_1 \right) + X_{21}\left(\mathcal{M}_2 - C_1Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \right) \\ X_{21}\left(F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1\mathcal{M}_1 \right) + X_{22}\left(\mathcal{M}_2 - C_1Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \right) \end{bmatrix} - X_Y \begin{bmatrix} F_3 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ 0 \end{bmatrix}$$

4.3.3.2 Numerical example

Consider the following descriptor system described by (4.66), where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \mathcal{M}_1 = \begin{bmatrix} 0.1 \\ 0.7 \\ 0.2 \\ 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; \mathcal{M}_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \; \mathcal{G} = \begin{bmatrix} 0.1 & 0 & 0.2 & 0.1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

By using Definition 1.3, we show that the system above is unstable when $\Gamma(t) = 0$

$$\mathsf{eig}(E,\ A) = [0.39,\ -0.7 + 1.43i,\ -0.7 - 1.43i]$$

Considering $E^\perp = \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}$ we can verify Assumptions 4.1, 4.2 and 4.3 when $\Gamma(t) = 0$

$$\mathsf{rank} \begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = 4, \; \mathsf{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 4 \text{ and } \mathsf{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = 4.$$

Generalized dynamic observer

For the GDO-based control we have chosen matrix $R = \begin{bmatrix} 4 & 0 & 2 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix}$, then $\mathsf{rank}(\Sigma) = 4$.

By using YALMIP toolbox, we solve the LMIs (4.71) - (4.73) to find matrices X_1, X_2 and $\mathbb{Y}$

$$X_1 = \begin{bmatrix} 54.92 & 11.58 & 2.29 & 0 \\ 11.58 & 49.24 & -8.52 & -40.44 \\ 2.29 & -8.52 & 54.73 & -24.9 \\ 0 & 0 & 0 & -27.15 \end{bmatrix}, \; X_2 = \begin{bmatrix} 44.56 & -27.47 & 44.56 & -27.47 \\ -27.47 & 29.29 & -27.47 & 29.29 \\ 44.56 & -27.47 & 104.2 & -29.16 \\ -27.47 & 29.29 & -29.16 & 85.06 \end{bmatrix} \text{ and}$$

$$\mathbb{Y} = \begin{bmatrix} -1.51 & 1.53 & 0.94 & -3.4 & -3.04 & 1.54 & -0.05 \\ -1.9 & 2.55 & 1.93 & -7.62 & -6.3 & 1.34 & 2.48 \\ 0.22 & -0.12 & -0.01 & -0.07 & 0 & -0.65 & 0.36 \\ 0.06 & -0.05 & 0.02 & -0.04 & 0.01 & -0.18 & -1.1 \end{bmatrix}$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix}2\\3\\1\\8\end{bmatrix}$, $\mathcal{L} = \begin{bmatrix}0.4\\0.4\\0.4\\0.4\end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.1$ and solving (4.75) and (4.76) we get :

$$\kappa = \begin{bmatrix}265.42 & -96.81 & -41.545 & -26.47\end{bmatrix}$$

Finally, we compute all the matrices of the observer-based control as :

$$N = \begin{bmatrix}0.74 & -2.06\\1.5 & -3\end{bmatrix},\ S = \begin{bmatrix}0.07 & -0.14\\0.15 & -0.3\end{bmatrix} \times 10^{-2},\ H = \begin{bmatrix}1.54 & -0.05\\1.34 & 2.48\end{bmatrix},\ J = \begin{bmatrix}4\\2\end{bmatrix},\ F = \begin{bmatrix}0.59 & 1.67 & 0.83\\-2.15 & 5.58 & 1.15\end{bmatrix},$$

$$L = \begin{bmatrix}-0.65 & 0.36\\-0.18 & -1.1\end{bmatrix},\ P = \begin{bmatrix}0.17 & 0.15\\-0.07 & 0.15\\0.15 & -0.29\\-0.05 & 0.11\end{bmatrix},\ M = \begin{bmatrix}-0.24 & 0.39 & 0.24\\-0.51 & 0.85 & 0.51\end{bmatrix} \times 10^{-2} \text{ and } Q = \begin{bmatrix}-0.1 & 0 & 0.1\\0.25 & 0.58 & -0.25\\0.5 & -0.16 & -0.5\\0.2 & -0.31 & 0.81\end{bmatrix}.$$

Proportional observer-based control

By considering matrices $R = \begin{bmatrix}8 & 0 & 3 & 0\\5 & 2 & 0 & 4\end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix}2\\3\\1\\8\end{bmatrix}$, $\mathcal{L} = \begin{bmatrix}0.4\\0.4\\0.4\\0.4\end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.1$ the following PO-based control is obtained :

$$\kappa = \begin{bmatrix}480.76 & -570.26 & 400.45 & -83.66\end{bmatrix},$$

$$N = \begin{bmatrix}2.67 & -4.87\\2 & -3.4\end{bmatrix},\ F_a = \begin{bmatrix}4.87 & 3\\3.4 & 0\end{bmatrix},\ J = \begin{bmatrix}8\\5\end{bmatrix},\ P = \begin{bmatrix}0 & 0.2\\0 & 0\\0.33 & -0.53\\0 & 0\end{bmatrix} \text{ and } Q_a = \begin{bmatrix}-0.2 & 0\\1 & 0\\0.53 & 0\\0 & 1\end{bmatrix}.$$

Proportional-integral observer-based control

By considering matrices $R = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix}2\\3\\1\\8\end{bmatrix}$, $\mathcal{L} = \begin{bmatrix}0.4\\0.4\\0.4\\0.4\end{bmatrix}$ and $\mathcal{R} = I_4$ the following PIO-based control is obtained :

$$\kappa = \begin{bmatrix}89.23 & -36.73 & -12.54 & -15.39\end{bmatrix},$$

$$N = \begin{bmatrix}-0.57 & -1\\0.14 & -0.97\end{bmatrix},\ H = \begin{bmatrix}-0.01 & -0.26\\0.44 & -0.06\end{bmatrix},\ J = \begin{bmatrix}1\\0\end{bmatrix},\ F = \begin{bmatrix}0.84 & -0.63 & -0.12\\-0.23 & 0.53 & -0.2\end{bmatrix},$$

$$P = \begin{bmatrix}1 & 0.67\\0 & 0.33\\0 & -0.67\\0 & 0.33\end{bmatrix} \text{ and } Q = \begin{bmatrix}-0.43 & 0.38 & 0.43\\0.14 & 0.76 & -0.14\\0.71 & -0.52 & -0.71\\0.14 & -0.24 & 0.86\end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0,\ 0.1,\ 0,\ 0.1]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]$, $v(0)_{GDO} = [0,\ 0]$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0,\ 0]$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]$, $v(0)_{PIO} = [0,\ 0]$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0,\ 0]$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0,\ 0]$.

To evaluate the performance of the controllers an uncertainty $\wp(t)$ is added in the system matrix $A + \Delta A(t)$, then we obtain matrix $(A + \Delta A(t) + \wp(t))$ where $\wp(t) = \delta(t) \times \begin{bmatrix}0.3 & 0 & 0.2 & 0\\0.1 & 0.2 & 0.3 & 0\\0 & 0 & 0.1 & 0\\0.2 & 0.4 & 0 & 0\end{bmatrix}$.

The results of simulation are depicted in Figures 4.9 - 4.12. Figures 4.9 and 4.10 show the uncertainty factor $\delta(t)$ and the variation $\Gamma(t)$. Figure 4.11 shows the control input $u(t)$ provided by the GDO, PIO and PO. Figure 4.12 shows the controlled output by the GDO, PIO and PO.

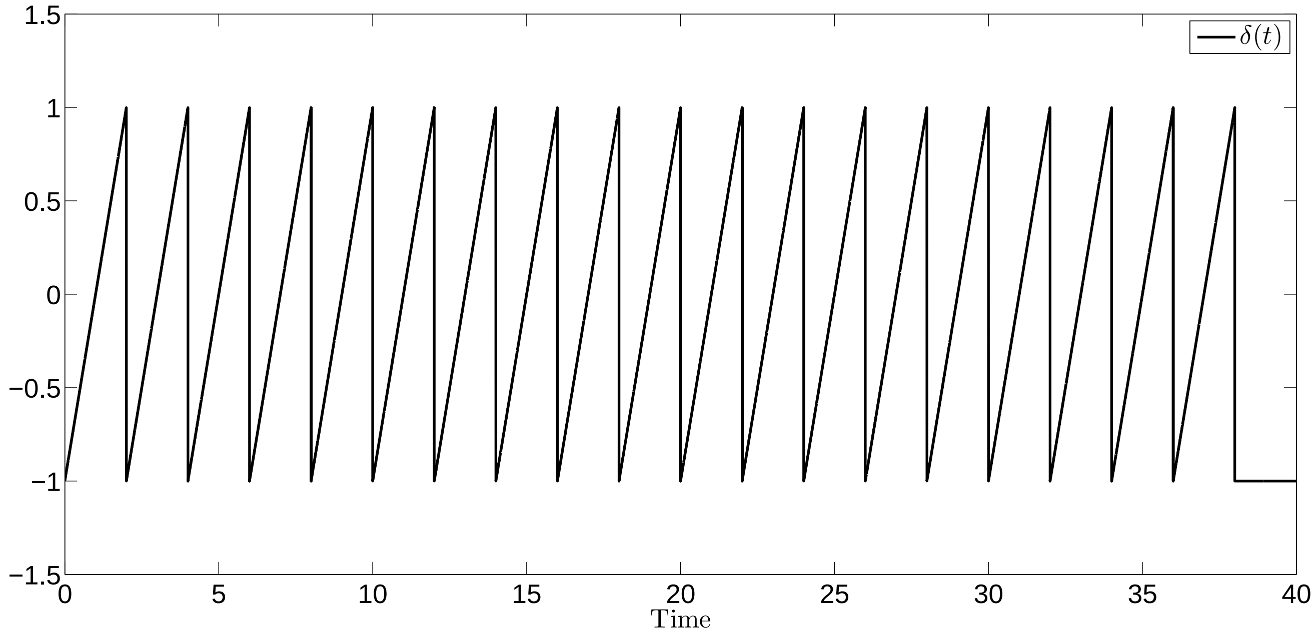

FIGURE 4.9 – Robust observers-based control : Uncertainty factor $\delta(t)$.

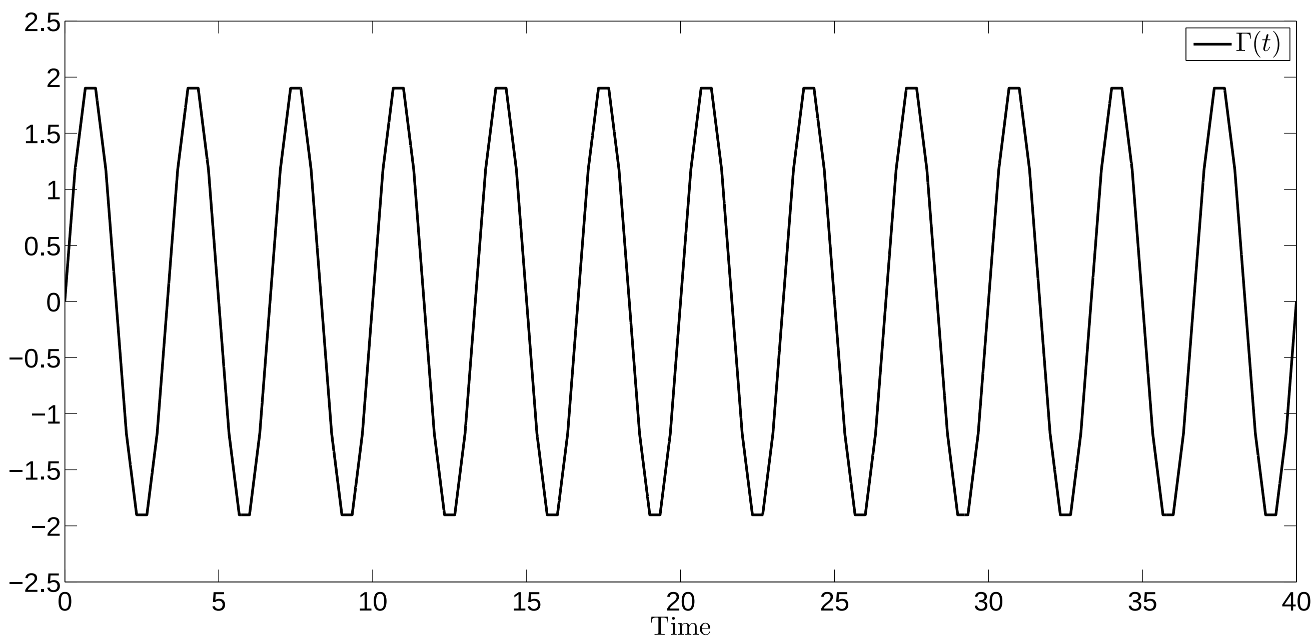

FIGURE 4.10 – Robust observers-based control : Variation $\Gamma(t)$.

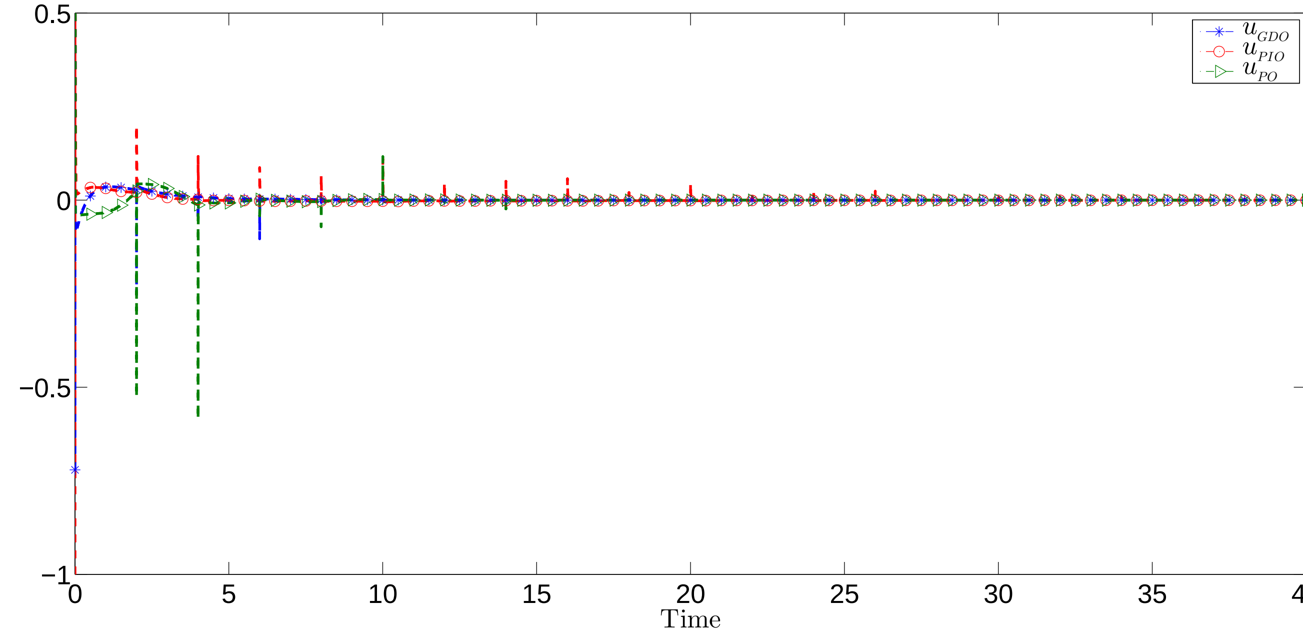

FIGURE 4.11 – Robust observers-based control : Control input $u(t)$.

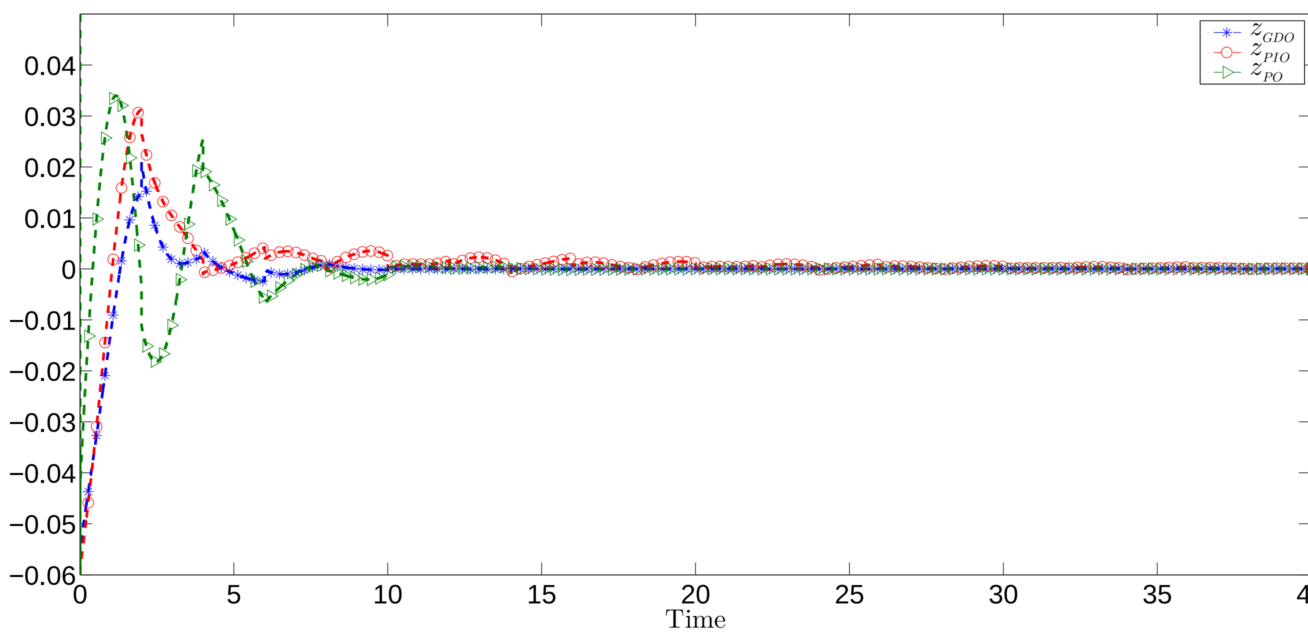

FIGURE 4.12 – Robust observers-based control : Controlled output $z(t)$.

From these results, we can see that the GDO-based control stabilize faster the system than the PIO and the PO.

4.3.4 Robust H_∞ generalized dynamic observer-based control design for uncertain disturbed descriptor systems, $w(t) \neq 0$

In this section we consider $w(t) \neq 0$, such that we get system (4.52) :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Bu(t) + Dw(t) \tag{4.95a}$$
$$y(t) = (C_1 + \Delta C(t))x(t) + D_1 w(t) \tag{4.95b}$$
$$z(t) = C_2 x(t) + D_2 w(t) \tag{4.95c}$$

with the GDO-based control :

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) \tag{4.96a}$$
$$\dot{v}(t) = S\zeta(t) + Lv(t) + M\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.96b}$$
$$\hat{x}(t) = P\zeta(t) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \tag{4.96c}$$
$$u(t) = -\kappa\hat{x}(t) \tag{4.96d}$$

and the closed-loop (4.62) :

$$\begin{aligned} \mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) + \mathscr{B}w(t) \\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t) \end{aligned} \tag{4.97}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix} -B\kappa P_1 & 0 \end{bmatrix} \\ 0 & \mathbb{A} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D - B\kappa Q \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \\ \mathbb{F}_d \end{bmatrix}$, $\mathscr{F}_1 = \begin{bmatrix} \mathcal{M}_1 - B\kappa Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ \mathbb{F}_m \end{bmatrix}$, $\mathscr{F}_2 = \begin{bmatrix} \mathcal{G} & 0 & 0 \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 & 0 \end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$.

Matrices $\mathbb{A}$, $\mathbb{F}_m$ and $\mathbb{F}_d$ have the following form :

$$\mathbb{A} = \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 \tag{4.98}$$
$$\mathbb{F}_m = \mathbb{F}_{m1} - \mathbb{Y}\mathbb{F}_{m2} \tag{4.99}$$
$$\mathbb{F}_d = \mathbb{F}_{d1} - \mathbb{Y}\mathbb{F}_{d2} \tag{4.100}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_{m1} = \begin{bmatrix} F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1\mathcal{M}_1 \\ 0 \end{bmatrix}$, $\mathbb{F}_{m2} = \begin{bmatrix} F_3 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ 0 \end{bmatrix}$, $\mathbb{F}_{d1} = \begin{bmatrix} F_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - T_1 D \\ 0 \end{bmatrix}$, $\mathbb{F}_{d2} = \begin{bmatrix} F_3 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \\ 0 \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$.

The problem of design an H_∞ GDO-based control is reduced to find matrices $\mathbb{Y}$ and κ such that for a prescribed scalar $\gamma > 0$, we ensure that the controlled output $z(t)$ converge asymptotically to zero in closed-loop for $w(t) = 0$, and for $w(t) \neq 0$ we must guarantee that the performance index

$$J = \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t)]dt \tag{4.101}$$

is negative $\forall w(t) \in \mathcal{L}_2[0, \infty)$. The solution to this problem is given by the following theorem.

Theorem 4.4. *Under Assumptions 4.4, 4.5 and 4.6 there exists an H_∞ GDO-based control* (4.96) *such that the closed-loop* (4.97) *is stable if and only if there exist a symmetric positive matrix $X_2 = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix}$, with $X_{21} = X_{21}^T$, and a nonsingular matrix X_1 such that the following LMIs are satisfied.*

$$E^T X_1 = X_1^T E \geq 0, \tag{4.102}$$

$$\mathcal{C}^{T\perp}\begin{bmatrix} X_1A^T+AX_1^T & 0 & (*) & (*) & (*) & (*)\\ 0 & \Pi_1 & (*) & 0 & (*) & 0\\ \mathcal{M}_1^T & \Pi_2^T & -\epsilon I & 0 & 0 & 0\\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I & 0 & 0\\ D^T & \Pi_3^T & 0 & 0 & -\gamma^2 I & (*)\\ C_2X_1^T & 0 & 0 & 0 & D_2 & -I\end{bmatrix}\mathcal{C}^{T\perp T}<0, \tag{4.103}$$

and

$$\begin{bmatrix} \begin{bmatrix}P_1^T\\0\end{bmatrix}^{\perp}\Pi_1\begin{bmatrix}P_1^T\\0\end{bmatrix}^{\perp T} & (*) & (*) & 0\\ \left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^{T\perp}\Pi_2^T\begin{bmatrix}P_1^T\\0\end{bmatrix}^{\perp T} & -\epsilon\Pi_4 & 0 & 0\\ \left(Q_1\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}\right)^{T\perp}\Pi_3^T\begin{bmatrix}P_1^T\\0\end{bmatrix}^{\perp T} & 0 & -\gamma^2\left(Q_1\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}\right)^{T\perp}\left(Q_1\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}\right)^{T\perp T} & (*)\\ 0 & 0 & D_2\left(Q_1\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}\right)^{T\perp T} & -I\end{bmatrix}<0 \tag{4.104}$$

where

$$\Pi_1=\begin{bmatrix}X_{21}N_1+N_1^TX_{21} & (*)\\ X_{21}N_1 & 0\end{bmatrix}-X_Y\begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}-\begin{bmatrix}N_3 & 0\\ 0 & -I_{q_1}\end{bmatrix}^TX_Y^T \tag{4.105a}$$

$$\Pi_2=\begin{bmatrix}X_{21}\left(F_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}-T_1\mathcal{M}_1\right)\\ X_{21}\left(F_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}-T_1\mathcal{M}_1\right)\end{bmatrix}-X_Y\begin{bmatrix}F_3\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\\ 0\end{bmatrix} \tag{4.105b}$$

$$\Pi_3=\begin{bmatrix}X_{21}\left(F_1\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}-T_1D\right)\\ X_{21}\left(F_1\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}-T_1D\right)\end{bmatrix}-X_Y\begin{bmatrix}F_3\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}\\ 0\end{bmatrix} \tag{4.105c}$$

$$\Pi_4=\left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^{T\perp}\left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^{T\perp T}. \tag{4.105d}$$

In this case $X_Y=X_2\mathbb{Y}$, and matrix κ is parameterized as follows :

$$\kappa=(\mathcal{B}_r^+\mathcal{K}\mathcal{C}_l^++\mathcal{Z}-\mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+)^T \tag{4.106}$$

where

$$\mathcal{K}=-\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}+\mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \tag{4.107a}$$

$$\vartheta=(\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T-\mathcal{D})^{-1}>0 \tag{4.107b}$$

$$\mathcal{S}=\mathcal{R}^{-1}-\mathcal{R}^{-1}\mathcal{B}_l^T[\vartheta-\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta]\mathcal{B}_l\mathcal{R}^{-1} \tag{4.107c}$$

where $\mathcal{D}=\begin{bmatrix} X_1A^T+AX_1^T & 0 & (*) & (*) & (*) & (*)\\ 0 & \Pi_1 & (*) & 0 & (*) & 0\\ \mathcal{M}_1^T & \Pi_2^T & -\epsilon I & 0 & 0 & 0\\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I & 0 & 0\\ D^T & \Pi_3^T & 0 & 0 & -\gamma^2 I & (*)\\ C_2X_1^T & 0 & 0 & 0 & D_2 & -I\end{bmatrix}$, $\mathcal{B}=\begin{bmatrix}X_1\\ \begin{bmatrix}P_1^T\\0\end{bmatrix}\\ \left(Q_1\begin{bmatrix}E^\perp\mathcal{M}_1\\ \mathcal{M}_2\end{bmatrix}\right)^T\\ 0\\ \left(Q_1\begin{bmatrix}E^\perp D\\ D_1\end{bmatrix}\right)^T\\ 0\end{bmatrix}$ *and* $\mathcal{C}=\begin{bmatrix}-B & 0 & 0 & 0 & 0 & 0\end{bmatrix}$*, matrices* Π_1*,* Π_2*,* Π_3 *and* Π_4 *are defined in* (4.105)*. Matrices* $\mathcal{L}$*,* $\mathcal{R}$ *and* $\mathcal{Z}$ *are arbitrary matrices of appropriate dimensions satisfying* $\mathcal{R}>0$ *and* $\|\mathcal{L}\|_2<1$*. Matrices* $\mathcal{C}_l$*,* $\mathcal{C}_r$*,* $\mathcal{B}_l$ *and* $\mathcal{B}_r$ *are any full rank matrices such that* $\mathcal{C}=\mathcal{C}_l\mathcal{C}_r$ *and* $\mathcal{B}=\mathcal{B}_l\mathcal{B}_r$*.*

Proof. Consider the following Lyapunov function

$$V(\beta(t))=\beta(t)^T\mathscr{E}^T\bar{X}\beta(t)\geq 0 \tag{4.108}$$

where

$$\mathscr{E}^T\bar{X} = \bar{X}^T\mathscr{E} \geq 0 \tag{4.109}$$

with $\bar{X} = \begin{bmatrix} \bar{X}_1 & 0 \\ 0 & X_2 \end{bmatrix}$ and $X_2 = X_2^T = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix} > 0.$

By using matrix $\mathscr{E}$ from (4.97), the equation (4.109) becomes :

$$E^T\bar{X}_1 = \bar{X}_1^TE \geq 0 \tag{4.110}$$

pre-multiplying the inequality (4.110) by $\bar{X}_1^{-T}$ and post-multiplying it by $\bar{X}_1^{-1}$ we get :

$$X_1E^T = EX_1^T \geq 0 \tag{4.111}$$

where $X_1 = \bar{X}_1^{-T}$.

Now, the derivate of $V(\beta(t))$ along the trajectory of the closed-loop (4.97) is :

$$\dot{V}(\beta(t)) = \beta(t)^T(\mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A})\beta(t) + 2\beta(t)^T\bar{X}^T\mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) + w(t)^T\mathscr{B}^T\bar{X}\beta(t) + \beta(t)^T\bar{X}^T\mathscr{B}w(t) \tag{4.112}$$

Using Lemma 1.6 from Section 1.7.3, and since $\Gamma(t)^T\Gamma(t) \leq I$ the following inequality can be formulated :

$$2\beta(t)^T\bar{X}^T\mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) \leq \epsilon^{-1}\beta(t)^T\bar{X}^T\mathscr{F}_1\mathscr{F}_1^T\bar{X}\beta(t) + \epsilon\beta(t)^T\mathscr{F}_2^T\mathscr{F}_2\beta(t) \tag{4.113}$$

with $\epsilon > 0$. Thus,

$$\dot{V}(\beta(t)) \leq \beta(t)^T(\mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A} + \epsilon^{-1}\bar{X}^T\mathscr{F}_1\mathscr{F}_1^T\bar{X} + \epsilon\mathscr{F}_2^T\mathscr{F}_2)\beta(t) + w(t)^T\mathscr{B}^T\bar{X}\beta(t) + \beta(t)^T\bar{X}^T\mathscr{B}w(t) \tag{4.114}$$

From equation (4.101) we get

$$J \leq \int_0^\infty [z(t)^Tz(t) - \gamma^2w(t)^Tw(t) + \dot{V}(\beta(t))]dt \tag{4.115}$$

so a sufficient condition for $J \leq 0$ is that

$$z(t)^Tz(t) - \gamma^2w(t)^Tw(t) + \dot{V}(\beta(t)) \leq 0, \ \forall t \in [0, \infty) \tag{4.116}$$

Then, we have

$$\begin{aligned} z(t)^Tz(t) - \gamma^2w(t)^Tw(t) + \dot{V}(\beta(t)) \leq\ & \beta(t)^T(\mathscr{A}^T\bar{X} + \bar{X}\mathscr{A} + \epsilon^{-1}\bar{X}^T\mathscr{F}_1\mathscr{F}_1^T\bar{X} + \epsilon\mathscr{F}_2^T\mathscr{F}_2 + \mathscr{C}^T\mathscr{C})\beta(t) + \\ & \beta(t)^T(\mathscr{C}^T\mathscr{D} + \bar{X}^T\mathscr{B})w(t) + w(t)^T(\mathscr{D}^T\mathscr{C} + \mathscr{B}^T\bar{X})\beta(t) + \\ & w(t)^T(\mathscr{D}^T\mathscr{D} - \gamma^2I)w(t) \end{aligned} \tag{4.117}$$

By applying the Schur complement of Lemma 1.4 in inequality (4.117) we get :

$$\begin{bmatrix} \beta(t) \\ w(t) \end{bmatrix}^T \left[\begin{array}{cc|c} \mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A} + \mathscr{C}^T\mathscr{C} & (*) & (*) \\ \begin{bmatrix} \mathscr{F}_1^T\bar{X} \\ \epsilon\mathscr{F}_2 \end{bmatrix} & -\epsilon I & 0 \\ \hline \mathscr{D}^T\mathscr{C} + \mathscr{B}^T\bar{X} & 0 & \mathscr{D}^T\mathscr{D} - \gamma^2 I \end{array}\right] \begin{bmatrix} \beta(t) \\ w(t) \end{bmatrix} < 0 \tag{4.118}$$

The asymptotic stability of system (4.97) is guaranteed if and only if inequality (4.118) is verified, and by applying the Schur complement to (4.118) we get the following LMI :

$$\begin{bmatrix} \mathscr{A}^T\bar{X} + \bar{X}^T\mathscr{A} & (*) & (*) & (*) \\ \begin{bmatrix} \mathscr{F}_1^T\bar{X} \\ \epsilon\mathscr{F}_2 \end{bmatrix} & -\epsilon I & 0 & 0 \\ \mathscr{B}^T\bar{X} & 0 & -\gamma^2 I & (*) \\ \mathscr{C} & 0 & \mathscr{D} & -I \end{bmatrix} < 0 \tag{4.119}$$

Inserting the form of matrices $\mathscr{A}$, $\mathscr{B}$, $\mathscr{F}_1$, $\mathscr{F}_2$ and $\bar{X}$ we have :

$$\begin{bmatrix} \bar{X}_1^T(A-B\kappa)+(A-B\kappa)^T\bar{X}_1 & (*) & (*) & (*) & (*) & (*) \\ \begin{bmatrix}(-B\kappa P_1)^T \\ 0\end{bmatrix}\bar{X}_1 & X_2\mathbb{A}+\mathbb{A}^TX_2 & (*) & 0 & (*) & 0 \\ \left(\mathcal{M}_1-B\kappa Q_1\begin{bmatrix}E^{\perp}\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix}\right)^T\bar{X}_1 & \mathbb{F}_m^TX_2 & -\epsilon I & 0 & 0 & 0 \\ \epsilon\mathcal{G} & 0 & 0 & -\epsilon I & 0 & 0 \\ \left(D-B\kappa Q_1\begin{bmatrix}E^{\perp}D \\ D_1\end{bmatrix}\right)^T\bar{X}_1 & \mathbb{F}_d^TX_2 & 0 & 0 & -\gamma^2 I & (*) \\ C_2 & 0 & 0 & 0 & D_2 & -I \end{bmatrix} < 0 \tag{4.120}$$

pre-multiplying the inequality (4.120) by $\begin{bmatrix} \bar{X}_1^{-T} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$ and post-multiplying it by $\begin{bmatrix} \bar{X}_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$ we get :

$$\begin{bmatrix} (A-B\kappa)X_1^T+X_1(A-B\kappa)^T & (*) & (*) & (*) & (*) & (*) \\ \begin{bmatrix}(-B\kappa P_1)^T \\ 0\end{bmatrix} & X_2\mathbb{A}+\mathbb{A}^TX_2 & (*) & 0 & (*) & 0 \\ \left(\mathcal{M}_1-B\kappa Q_1\begin{bmatrix}E^{\perp}\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix}\right)^T & \mathbb{F}_m^TX_2 & -\epsilon I & 0 & 0 & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I & 0 & 0 \\ \left(D-B\kappa Q_1\begin{bmatrix}E^{\perp}D \\ D_1\end{bmatrix}\right)^T & \mathbb{F}_d^TX_2 & 0 & 0 & -\gamma^2 I & (*) \\ C_2X_1^T & 0 & 0 & 0 & D_2 & -I \end{bmatrix} < 0 \tag{4.121}$$

where $X_1 = \bar{X}_1^{-T}$.

Replacing the form of matrices $\mathbb{A}$, $\mathbb{F}_m$ and $\mathbb{F}_d$ we have :

$$\begin{bmatrix} (A-B\kappa)X_1^T+X_1(A-B\kappa)^T & (*) & (*) & (*) & (*) & (*) \\ \begin{bmatrix}(-B\kappa P_1)^T \\ 0\end{bmatrix} & X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)+(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2 & (*) & 0 & (*) & 0 \\ \left(\mathcal{M}_1-B\kappa Q_1\begin{bmatrix}E^{\perp}\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix}\right)^T & (\mathbb{F}_{m1}-\mathbb{Y}\mathbb{F}_{m2})^TX_2 & -\epsilon I & 0 & 0 & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I & 0 & 0 \\ \left(D-B\kappa Q_1\begin{bmatrix}E^{\perp}D \\ D_1\end{bmatrix}\right)^T & (\mathbb{F}_{d1}-\mathbb{Y}\mathbb{F}_{d2})^TX_2 & 0 & 0 & -\gamma^2 I & (*) \\ C_2X_1^T & 0 & 0 & 0 & D_2 & -I \end{bmatrix} < 0 \tag{4.122}$$

which can be written as :

$$\mathcal{B}\kappa^T\mathcal{C}+(\mathcal{B}\kappa^T\mathcal{C})^T+\mathcal{D}<0 \tag{4.123}$$

where $\mathcal{D} = \begin{bmatrix} X_1A^T+AX_1^T & 0 & (*) & (*) & (*) & (*) \\ 0 & X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)+(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2 & (*) & 0 & (*) & 0 \\ \mathcal{M}_1^T & (\mathbb{F}_{m1}-\mathbb{Y}\mathbb{F}_{m2})^TX_2 & -\epsilon I & 0 & 0 & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I & 0 & 0 \\ D^T & (\mathbb{F}_{d1}-\mathbb{Y}\mathbb{F}_{d2})^TX_2 & 0 & 0 & -\gamma^2 I & (*) \\ C_2X_1^T & 0 & 0 & 0 & D_2 & -I \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix}P_1^T \\ 0\end{bmatrix} \\ \left(Q_1\begin{bmatrix}E^{\perp}\mathcal{M}_1 \\ \mathcal{M}_2\end{bmatrix}\right)^T \\ 0 \\ \left(Q_1\begin{bmatrix}E^{\perp}D \\ D_1\end{bmatrix}\right)^T \\ 0 \end{bmatrix}$ and

$\mathcal{C} = \begin{bmatrix}-B & 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

Using the elimination lemma of Section 1.5, inequality (4.123) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{4.124a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{4.124b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} -B^{\perp} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} 0 & \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(Q_1 \begin{bmatrix} E^{\perp}\mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^{T\perp} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (4.124a) is :

$$\mathcal{C}^{T\perp} \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 & (*) & 0 \\ \mathcal{M}_1^T & \Pi_2^T & -\epsilon I & 0 & 0 & 0 \\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I & 0 & 0 \\ D^T & \Pi_3^T & 0 & 0 & -\gamma^2 I & (*) \\ C_2X_1^T & 0 & 0 & 0 & D_2 & -I \end{bmatrix} \mathcal{C}^{T\perp T} < 0, \tag{4.125}$$

and by using the definition of matrices $\mathcal{B}$ and $\mathcal{D}$ the inequality (4.124b) is :

$$\begin{bmatrix} \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} \Pi_1 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & (*) & 0 & (*) & 0 \\ \left(Q_1 \begin{bmatrix} E^{\perp}\mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^{T\perp} \Pi_2^T \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & -\epsilon\Pi_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp} \Pi_3^T \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & 0 & 0 & -\gamma^2 \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp} \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp T} & (*) \\ 0 & 0 & 0 & D_2 \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp T} & -I \end{bmatrix} < 0 \tag{4.126}$$

now, removing an uncontrollable subspace, we get :

$$\begin{bmatrix} \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp} \Pi_1 \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & (*) & (*) & 0 \\ \left(Q_1 \begin{bmatrix} E^{\perp}\mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right)^{T\perp} \Pi_2^T \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & -\epsilon\Pi_4 & 0 & 0 \\ \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp} \Pi_3^T \begin{bmatrix} P_1^T \\ 0 \end{bmatrix}^{\perp T} & 0 & -\gamma^2 \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp} \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp T} & (*) \\ 0 & 0 & D_2 \left(Q_1 \begin{bmatrix} E^{\perp}D \\ D_1 \end{bmatrix}\right)^{T\perp T} & -I \end{bmatrix} < 0 \tag{4.127}$$

where matrices Π_1, Π_2, Π_3 and Π_4 are defined in (4.105). From the elimination lemma, if conditions (4.124a) and (4.124b) are satisfied, the matrix κ can be obtained from (4.106) and (4.107). □

4.3.4.1 Particular cases

•***Proportional observer-based control***

Consider the following descriptor system :

$$\begin{aligned} E\dot{x}(t) &= (A + \Delta A(t))x(t) + Bu(t) + Dw(t) \\ y(t) &= (C_1 + \Delta C(t))x(t) + D_1w(t) \\ z(t) &= C_2x(t) + D_2w(t) \end{aligned}$$

with the PO-based control :

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + F_a y(t) + Ju(t)\\ \hat{x}(t) &= P\zeta(t) + Q_a y(t)\\ u(t) &= -\kappa\hat{x}(t)\end{aligned}$$

and the closed-loop (4.97) becomes :

$$\begin{aligned}\mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) + \mathscr{B}w(t)\\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t)\end{aligned}$$

where $\mathscr{E} = \begin{bmatrix} E & 0\\ 0 & I_{q_0}\end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & -B\kappa P_1\\ 0 & \bar{\mathbb{A}}\end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D - B\kappa Q_1 D_1\\ \bar{\mathbb{F}}_d\end{bmatrix}$, $\mathscr{F}_1 = \begin{bmatrix}\mathcal{M}_1 - B\kappa Q_1\mathcal{M}_2\\ \bar{\mathbb{F}}_m\end{bmatrix}$, $\mathscr{F}_2 = \begin{bmatrix}\mathcal{G} & 0\end{bmatrix}$, $\mathscr{C} = \begin{bmatrix}C_2 & 0\end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix}x(t)\\ \varepsilon(t)\end{bmatrix}$. Matrices $\bar{\mathbb{A}} = \bar{\mathbb{A}}_1 - \bar{\mathbb{Y}}\bar{\mathbb{A}}_2$, $\bar{\mathbb{F}}_m = \bar{\mathbb{F}}_{m1} - \bar{\mathbb{Y}}\bar{\mathbb{F}}_{m2}$ and $\bar{\mathbb{F}}_d = \bar{\mathbb{F}}_{d1} - \bar{\mathbb{Y}}\bar{\mathbb{F}}_{d2}$ where $\bar{\mathbb{A}}_1 = N_1$, $\bar{\mathbb{A}}_2 = N_3$, $\bar{\mathbb{F}}_{m1} = F_1\mathcal{M}_2 - T_1\mathcal{M}_1$, $\bar{\mathbb{F}}_{m2} = F_3\mathcal{M}_2$, $\bar{\mathbb{F}}_{d1} = F_1 D_1 - T_1 D$, $\bar{\mathbb{F}}_{d2} = F_3 D_1$ and $\bar{\mathbb{Y}} = Y_1$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.4 become :

$$\mathcal{D} = \begin{bmatrix} X_1A^T + AX_1^T & 0 & (*) & (*) & (*) & (*)\\ 0 & \Pi_1 & (*) & 0 & (*) & 0\\ \mathcal{M}_1^T & \Pi_2^T & -\epsilon I & 0 & 0 & 0\\ \epsilon\mathcal{G}X_1^T & 0 & 0 & -\epsilon I & 0 & 0\\ D^T & \Pi_3^T & 0 & 0 & -\gamma^2 I & (*)\\ C_2X_1^T & 0 & 0 & 0 & D_2 & -I\end{bmatrix},\ \mathcal{B} = \begin{bmatrix} X_1\\ P_1^T\\ (Q_1\mathcal{M}_2)^T\\ 0\\ (Q_1D_1)^T\\ 0\end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix}-B & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

where

$$\begin{aligned}\Pi_1 &= X_2N_1 + N_1^TX_2 - X_YN_3 - N_3^TX_Y^T\\ \Pi_2 &= X_2(F_1\mathcal{M}_2 - T_1\mathcal{M}_1) - X_YF_3\mathcal{M}_3\\ \Pi_3 &= X_2(F_1D_1 - T_1D) - X_YF_3D_1\end{aligned}$$

Matrices Σ and Ω become $\Sigma = \begin{bmatrix}R\\ C_1\end{bmatrix}$ and $\Omega = \begin{bmatrix}E\\ C_1\end{bmatrix}$.

•*Proportional-integral observer-based control*

Consider the following descriptor system :

$$\begin{aligned}E\dot{x}(t) &= (A + \Delta A(t))x(t) + Bu(t) + Dw(t)\\ y(t) &= (C_1 + \Delta C(t))x(t) + D_1w(t)\\ z(t) &= C_2x(t) + D_2w(t)\end{aligned}$$

with the PIO-based control :

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + Hv(t) + F\begin{bmatrix}-E^\perp Bu(t)\\ y(t)\end{bmatrix} + Ju(t)\\ \dot{v}(t) &= y(t) - C\hat{x}(t)\\ \hat{x}(t) &= P\zeta(t) + Q\begin{bmatrix}-E^\perp Bu(t)\\ y(t)\end{bmatrix}\\ u(t) &= -\kappa\hat{x}(t)\end{aligned}$$

and the closed-loop (4.97) becomes :

$$\begin{aligned}\mathscr{E}\dot{\beta}(t) &= \mathscr{A}\beta(t) + \mathscr{F}_1\Gamma(t)\mathscr{F}_2\beta(t) + \mathscr{B}w(t)\\ z(t) &= \mathscr{C}\beta(t) + \mathscr{D}w(t)\end{aligned}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A - B\kappa & \begin{bmatrix} -B\kappa P_1 & 0 \end{bmatrix} \\ 0 & \bar{\bar{\mathbb{A}}} \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} D - B\kappa \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \\ \bar{\bar{\mathbb{F}}}_d \end{bmatrix}$, $\mathscr{F}_1 = \begin{bmatrix} \mathcal{M}_1 - B\kappa Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ \bar{\bar{\mathbb{F}}}_m \end{bmatrix}$,

$\mathscr{F}_2 = \begin{bmatrix} \mathcal{G} & 0 & 0 \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} C_2 & 0 & 0 \end{bmatrix}$, $\mathscr{D} = D_2$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$. Matrices $\bar{\bar{\mathbb{A}}} = \bar{\bar{\mathbb{A}}}_1 - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{A}}}_2$, $\bar{\bar{\mathbb{F}}}_m = \bar{\bar{\mathbb{F}}}_{m1} - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{F}}}_{m2}$ and

$\bar{\bar{\mathbb{F}}}_d = \bar{\bar{\mathbb{F}}}_{d1} - \bar{\bar{\mathbb{Y}}}\bar{\bar{\mathbb{F}}}_{d2}$ where $\bar{\bar{\mathbb{A}}}_1 = \begin{bmatrix} N_1 & 0 \\ -C_1 P_1 & 0 \end{bmatrix}$, $\bar{\bar{\mathbb{A}}}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_{m1} = \begin{bmatrix} F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1 \mathcal{M}_1 \\ \mathcal{M}_2 - C_1 Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_{m2} = \begin{bmatrix} F_3 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ 0 \end{bmatrix}$,

$\bar{\bar{\mathbb{F}}}_{d1} = \begin{bmatrix} F_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - T_1 D \\ D_1 - C_1 Q_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \end{bmatrix}$, $\bar{\bar{\mathbb{F}}}_{d2} = \begin{bmatrix} F_3 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \\ 0 \end{bmatrix}$ and $\bar{\bar{\mathbb{Y}}} = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} Y_1 & H \end{bmatrix}$.

Consequently, matrices $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ of Theorem 4.4 become :

$$\mathcal{D} = \begin{bmatrix} X_1 A^T + A X_1^T & 0 & (*) & (*) & (*) & (*) \\ 0 & \Pi_1 & (*) & 0 & (*) & 0 \\ \mathcal{M}_1^T & \Pi_2^T & -\epsilon I & 0 & 0 & 0 \\ \epsilon \mathcal{G} X_1^T & 0 & 0 & -\epsilon I & 0 & 0 \\ D^T & \Pi_3^T & 0 & 0 & -\gamma^2 I & (*) \\ C_2 X_1^T & 0 & 0 & 0 & D_2 & -I \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} X_1 \\ \begin{bmatrix} P_1^T \\ 0 \end{bmatrix} \\ \left(Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \right)^T \\ 0 \\ \left(Q_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \right)^T \\ 0 \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} -B & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\Pi_1 = \begin{bmatrix} X_{21} N_1 + N_1^T X_{21} - X_{21} C_1 P_1 - P_1^T C_1^T X_{21} & (*) \\ X_{21} N_1 - X_{22} C_1 P_1 & 0 \end{bmatrix} - X_Y \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} - \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}^T X_Y^T$$

$$\Pi_2 = \begin{bmatrix} X_{21} \left(F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1 \mathcal{M}_1 \right) + X_{21} \left(\mathcal{M}_2 - C_1 Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \right) \\ X_{21} \left(F_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} - T_1 \mathcal{M}_1 \right) + X_{22} \left(\mathcal{M}_2 - C_1 Q_1 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \right) \end{bmatrix} - X_Y \begin{bmatrix} F_3 \begin{bmatrix} E^\perp \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \\ 0 \end{bmatrix}$$

$$\Pi_3 = \begin{bmatrix} X_{21} \left(F_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - T_1 D \right) + X_{21} \left(D_1 - C_1 Q_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \right) \\ X_{21} \left(F_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} - T_1 D \right) + X_{22} \left(D_1 - C_1 Q_1 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \right) \end{bmatrix} - X_Y \begin{bmatrix} F_3 \begin{bmatrix} E^\perp D \\ D_1 \end{bmatrix} \\ 0 \end{bmatrix}.$$

4.3.4.2 Numerical example

Consider the following descriptor system described by (4.95), where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ D = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathcal{M}_1 = \begin{bmatrix} 0.1 \\ 0.7 \\ 0.2 \\ 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathcal{M}_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \ \mathcal{G} = \begin{bmatrix} 0.1 & 0 & 0.2 & 0.1 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \ D_2 = 0.2 \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

By using Definition 1.3, we show that the system above is unstable when $\Gamma(t) = 0$

$$\mathsf{eig}(E,\ A) = [0.39,\ -0.7 + 1.43i,\ -0.7 - 1.43i]$$

Considering $E^\perp = \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}$ we can verify Assumptions 4.1, 4.2 and 4.3 when $\Gamma(t) = 0$

$$\mathsf{rank}\begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = 4,\ \mathsf{rank}\begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 4 \text{ and } \mathsf{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = 4.$$

Generalized dynamic observer

For the GDO-based control we have chosen matrix $R = \begin{bmatrix} 7 & 3 & 6 & 2 \\ 6 & 1 & 8 & 4 \end{bmatrix}$, then $\mathsf{rank}(\Sigma) = 4$.

By fixing $\gamma = 2.5$, $\epsilon = 10$ and using YALMIP toolbox, we solve the LMIs (4.102) - (4.104) to find matrices X_1, X_2 and $\mathbb{Y}$

$$X_1 = \begin{bmatrix} 5124.12 & 7.48 & 4.34 & 0 \\ 7.48 & 2.51 & -0.36 & -0.64 \\ 4.34 & -0.36 & 2.49 & -1.75 \\ 0 & 0 & 0 & -0.87 \end{bmatrix},\ X_2 = \begin{bmatrix} 6.74 & -6.91 & 6.74 & -6.91 \\ -6.91 & 7.31 & -6.91 & 7.31 \\ 6.74 & -6.91 & 96.73 & -6.47 \\ -6.91 & 7.31 & -6.47 & 97.6 \end{bmatrix} \text{ and}$$

$$\mathbb{Y} = \begin{bmatrix} -52592.29 & -15685.89 & 4256.43 & -4389.14 & 3774.25 & -2.41 & 3.1 \\ -50477.55 & -15049.83 & 4084.18 & -4210.05 & 3625.18 & -2.75 & 3.2 \\ 61.44 & 18.65 & -4.95 & 5.16 & -4.24 & -0.53 & 0 \\ 62.7 & 18.34 & -5.06 & 5.18 & -4.67 & 0.06 & -0.53 \end{bmatrix}$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ and $\mathcal{R} = I_4 \times 0.1$ and solving (4.106) and (4.107) we get :

$$\kappa = \begin{bmatrix} 35.79 & -70.72 & -42.67 & 22.07 \end{bmatrix},$$

Finally, we compute all the matrices of the observer-based control as :

$$N = \begin{bmatrix} -3.9 & 3.98 \\ -2.58 & 2.49 \end{bmatrix},\ S = \begin{bmatrix} 0.01 & -0.02 \\ -0.02 & 0.02 \end{bmatrix},\ H = \begin{bmatrix} -2.41 & 3.1 \\ -2.75 & 3.2 \end{bmatrix},\ J = \begin{bmatrix} 7 \\ 6 \end{bmatrix},$$

$$F = \begin{bmatrix} 2.33 & 3.34 & 0.95 \\ 2.18 & 1.39 & 2.25 \end{bmatrix},\ L = \begin{bmatrix} -0.53 & 0 \\ 0.06 & -0.53 \end{bmatrix},\ M = \begin{bmatrix} 0.03 & -0.05 & -0.03 \\ -0.03 & 0.07 & 0.03 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.27 & -0.15 \\ 0.13 & -0.15 \\ -0.22 & 0.26 \\ 0.02 & -0.02 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -0.22 & -0.01 & 0.22 \\ 0.24 & 0.55 & -0.24 \\ 0.57 & -0.2 & -0.57 \\ 0.03 & -0.06 & 0.97 \end{bmatrix}.$$

Proportional observer-based control

By considering matrices $R = \begin{bmatrix} 4 & 0 & 2 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix}$ $\mathcal{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$, $\mathcal{R} = I_4 \times 1$ and $\gamma = 2.5$ and $\epsilon = 10$ the following PO-based control is obtained :

$$\kappa = \begin{bmatrix} 0.88 & -1.1 & -0.69 & 0.67 \end{bmatrix},$$

$$N = \begin{bmatrix} 2 & -5 \\ 1.75 & -4.25 \end{bmatrix},\ F_a = \begin{bmatrix} 7.5 & 2 \\ 6.37 & 0 \end{bmatrix},\ J = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 0.5 & -1 \\ 0 & 0 \end{bmatrix} \text{ and } Q_a = \begin{bmatrix} -0.75 & 0 \\ 1 & 0 \\ 1.5 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proportional-integral observer-based control

By considering matrices $R = \begin{bmatrix} 5 & 3 & 2 & 1 \\ 2 & 4 & 3 & 2 \end{bmatrix}$, $\mathcal{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.4 \end{bmatrix}$, $\mathcal{R} = I_4 \times 0.1$ and $\gamma = 2.5$ and $\epsilon = 10$ the following PIO-based control is obtained :

$$\kappa = \begin{bmatrix} 54.67 & -177.56 & -123.46 & 2.4 \end{bmatrix},$$

$$N = \begin{bmatrix} -31.8 & 78.71 \\ -11 & 26.51 \end{bmatrix}, \; H = \begin{bmatrix} -458.15 & -132.09 \\ -161.28 & -57.73 \end{bmatrix}, \; F = \begin{bmatrix} -50.14 & -1.31 & 50.86 \\ -15.65 & -1.43 & 16.65 \end{bmatrix}, \; J = \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.26 & -0.15 \\ -0.03 & 0.09 \\ -0.13 & 0.33 \\ 0.03 & -0.09 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -0.03 & -0.04 & 0.03 \\ -0.06 & 1 & 0.06 \\ 0.77 & -0.98 & -0.77 \\ 0.06 & 0 & 0.94 \end{bmatrix}.$$

Simulation results

The initial conditions for the system are $x(0) = [0,\ 0.1,\ 0,\ 0.1]^T$, for the GDO are $\zeta(0)_{GDO} = [0,\ 0]^T$, $v(0)_{GDO} = [0,\ 0]^T$ and $\hat{x}(0)_{GDO} = [0,\ 0,\ 0,\ ]^T$, for the PIO are $\zeta(0)_{PIO} = [0,\ 0]^T$, $v(0)_{PIO} = [0,\ 0]^T$ and $\hat{x}(0)_{PIO} = [0,\ 0,\ 0,\ ]^T$ and for the PO are $\zeta(0)_{PO} = [0,\ 0]^T$ and $\hat{x}(0)_{PO} = [0,\ 0,\ 0,\ ]^T$.

To evaluate the performance of the controllers an uncertainty $\wp(t)$ is added in the system matrix $A + \Delta A(t)$, then we obtain matrix $(A + \Delta A + \wp(t))$ where $\wp(t) = \delta(t) \times \begin{bmatrix} 0.3 & 0 & 0.2 & 0 \\ 0.1 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0.2 & 0.4 & 0 & 0 \end{bmatrix}$.

The results of simulation are depicted in Figures 4.13 - 4.17. Figures 4.13, 4.14 and 4.15 show the uncertainty factor $\delta(t)$, the variation $\Gamma(t)$ and the disturbance $w(t)$. Figure 4.16 shows the control input $u(t)$ provided by the GDO, PIO and PO. Figure 4.17 shows the controlled output by the GDO, PIO and PO.

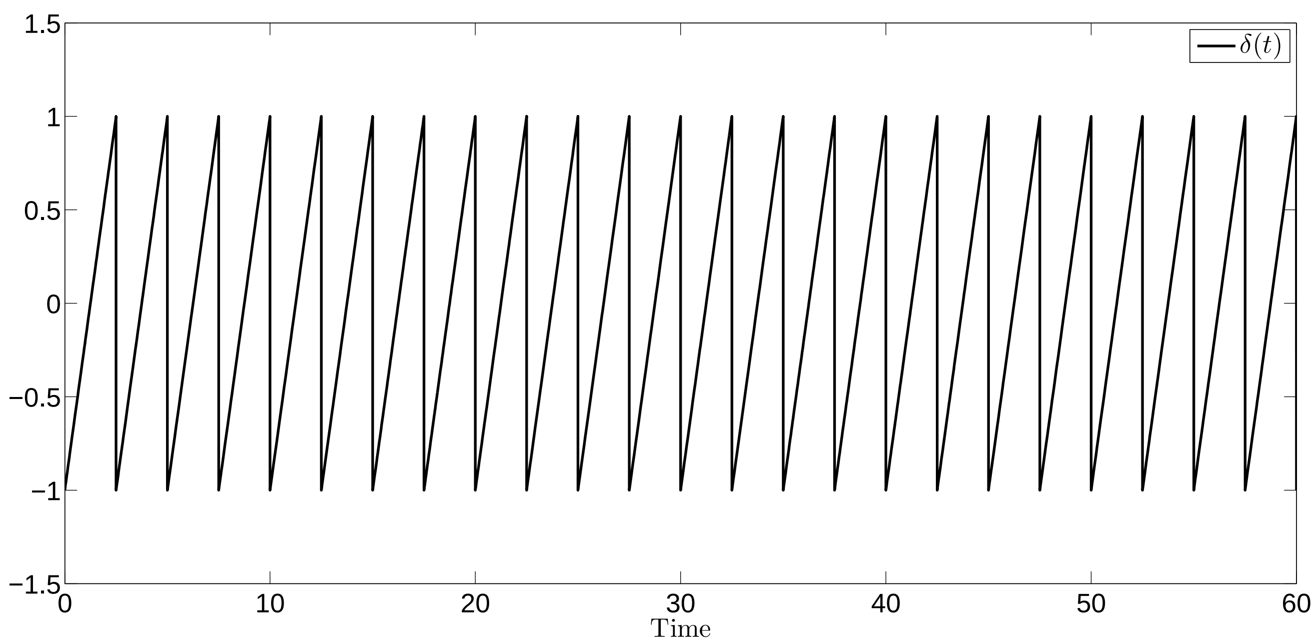

FIGURE 4.13 – H_∞ robust observers-based control : Uncertainty factor $\delta(t)$.

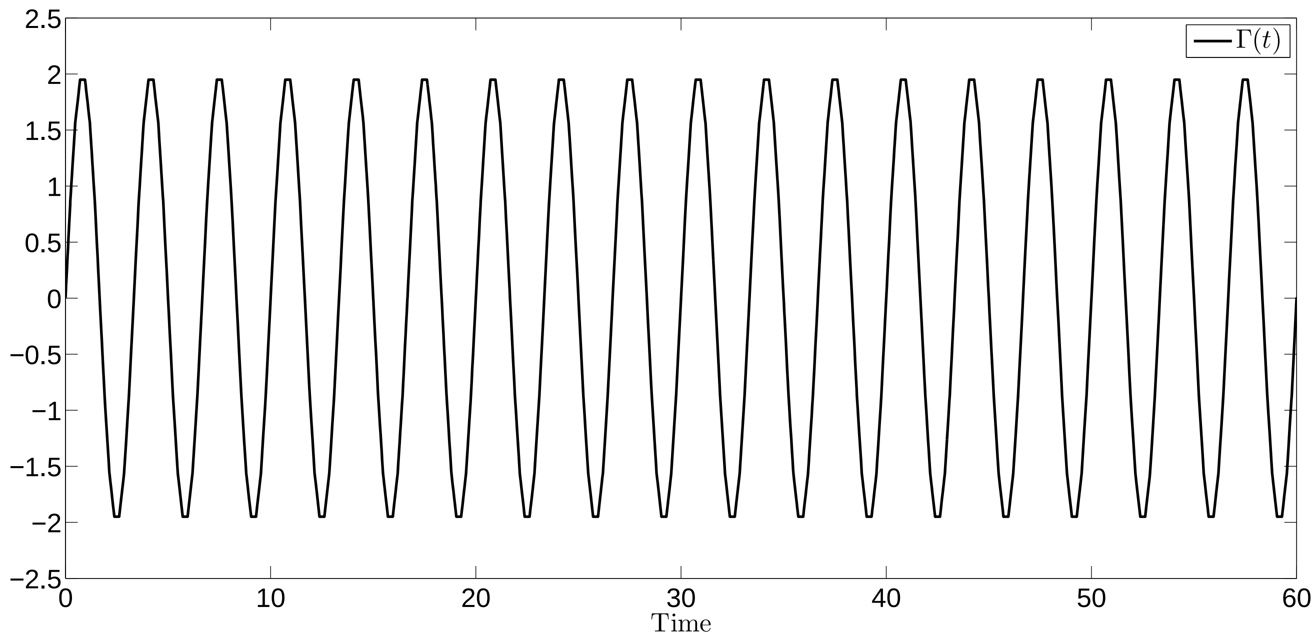

FIGURE 4.14 – H_∞ robust observers-based control : Variation $\Gamma(t)$.

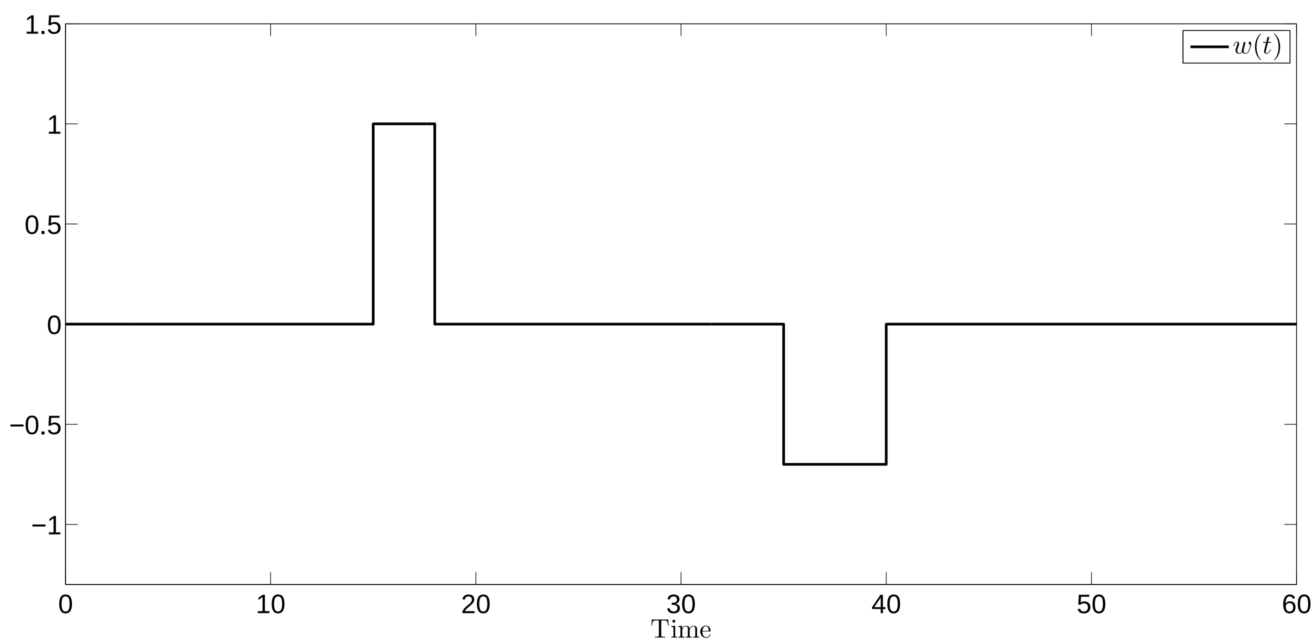

FIGURE 4.15 – H_∞ robust observers-based control : Disturbance $w(t)$.

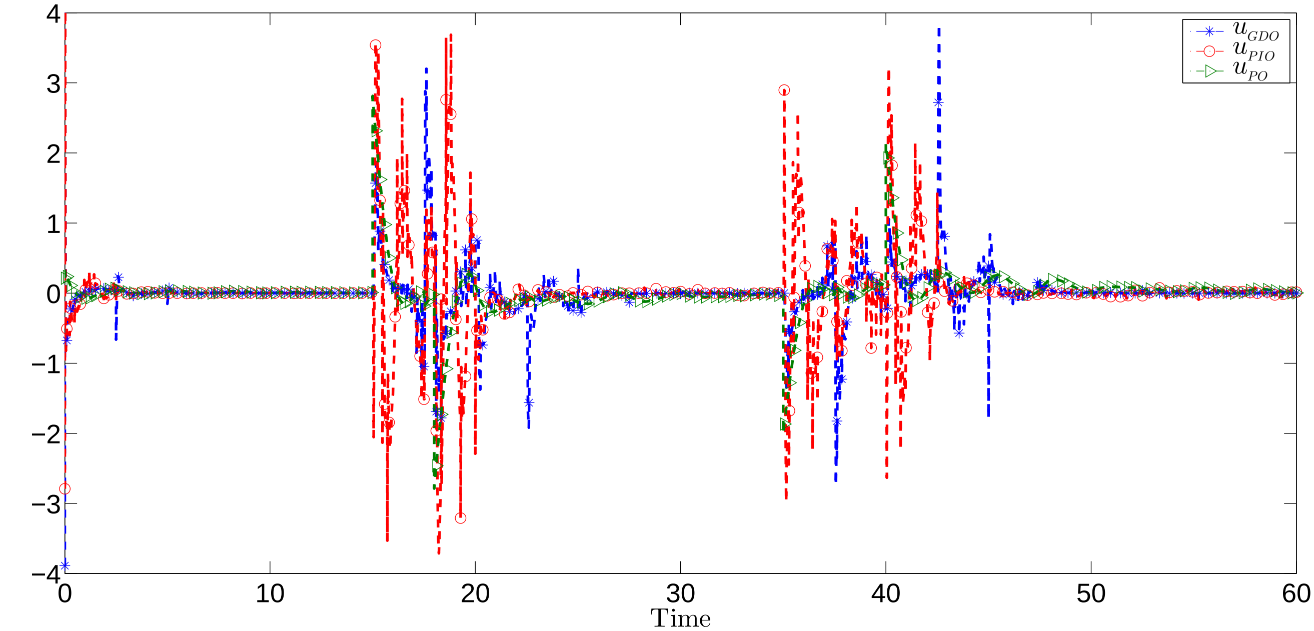

FIGURE 4.16 – H_∞ robust observers-based control : Control input $u(t)$.

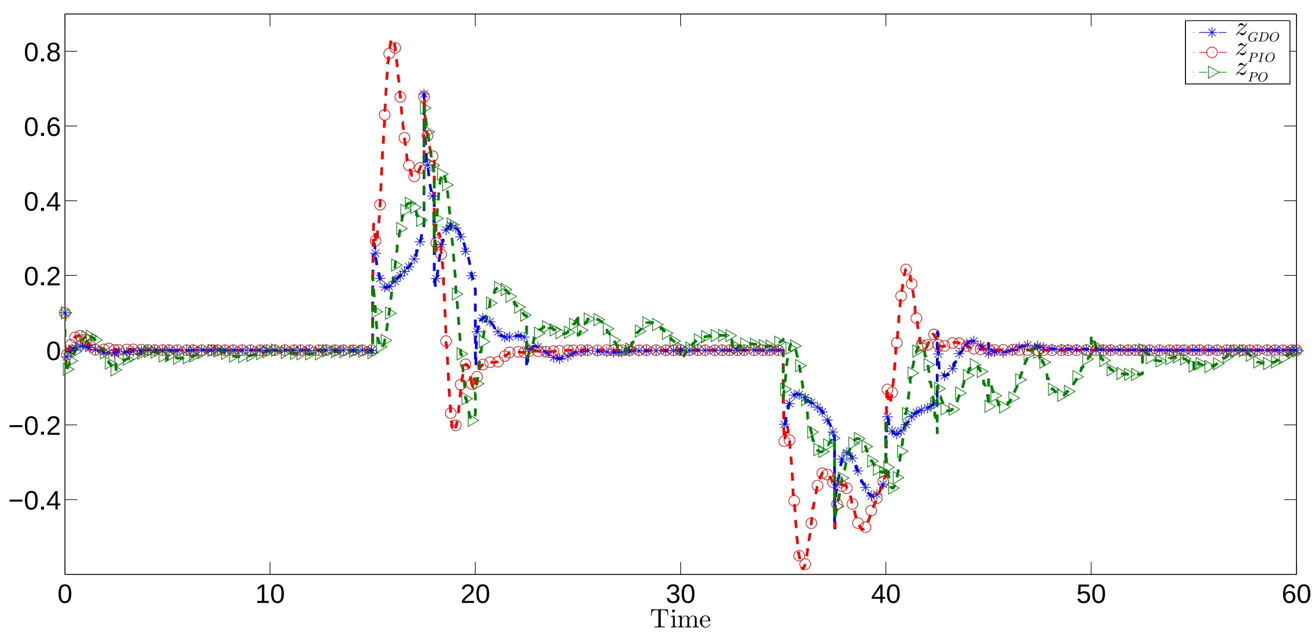

FIGURE 4.17 – H_∞ robust observers-based control : Controlled output $z(t)$.

From these results, we can see that all controllers achieve stabilize the system when $w(t) = 0$, but during the disturbance the GDO presents less variation than the PIO and PO, and after the disturbance the GDO and PIO achieve stabilize the system at the same time while the PO keeps variations. Concerning the control law of control, we can see that the PIO has more variations than the GDO.

4.4 Conclusion

In this chapter the controller design for descriptor system was presented. The observer-based control use the estimates provided by the observer to generate a control law to stabilize systems that normally are unstable. Two classes of descriptor systems were considered. The first controller was for disturbed descriptor systems, and the second one was for uncertain descriptor systems, where the uncertainty was taken as a time-variant function with norm bounded.

Chapter 5

Generalized dynamic observer-based fault diagnosis

Contents

5.1 Introduction

For centuries, the only way to know about faults and their location was through human intervention, either by observing changes of shape, color, listening to sounds unusual, touching to feel heat or vibrations and so on. Later, measuring devices were introduced, which provided more exact information about important physical variables. However, these devices (sensors) are also susceptible to faults, raising the dilemma of false alarms.

In general, faults are deviations form the normal behavior in the system or its instrumentation. There are many reasons for the appearance of faults. To name a few examples :

– wrong design, wrong assembling,

– wrong operation, missing maintenance,
– ageing, corrosion, wear during normal operation, etc.

Fault diagnosis scheme implement the following tasks :

– Fault detection, the indication that something is going wrong in the monitored system.
– Fault isolation, the determination of the exact location of the fault (the component which is faulty).
– Fault estimation, the determination of the magnitude of the fault.

The fault diagnosis scheme consists in the determination of the fault type as much detail as possible, as the fault size, location and time of detection.

Figure 5.1 summarizes the fault diagnosis scheme that is implemented for uncertain descriptor systems, where two observers are implemented. The first one deals with the tasks of fault detection and fault isolation through the residual generation. The advantage of the approach proposed is to obtain independent residuals, i.e. each residual respond just to one fault, in such way we can isolate multiple faults.

The second observer deals with simultaneous state and fault estimation by using an adaptive observer. With this we complete the scheme of fault diagnosis.

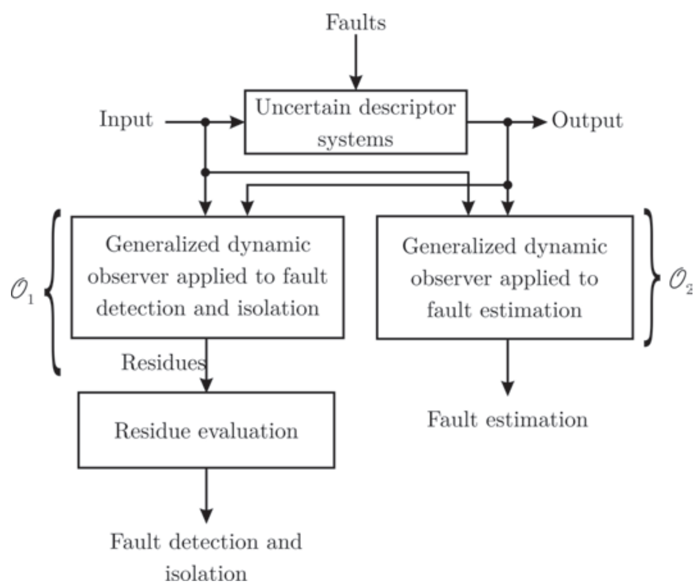

FIGURE 5.1 – Diagnostic scheme applied.

In this chapter the GDO-based fault diagnosis is designed for descriptor systems and for uncertain descriptor systems. The full scheme of fault diagnosis includes the tasks of fault detection, isolation and estimation, so that, we have proposed two observers based of the GDO structure to deal with fault diagnosis. In Section 5.2 fault diagnosis for descriptor systems is treated. Section 5.2.2 deals with fault detection and isolation (FDI), while Section 5.2.3 deals with fault estimation (FE).

In Section 5.3 fault diagnosis for uncertain descriptor systems is presented. In the same way, diagnosis task is divided in fault detection and isolation in Section 5.3.2 and fault estimation in Section 5.3.3.

5.2 Generalized dynamic observer design for descriptor systems with application to fault diagnosis

In this section the GDO is applied to fault diagnosis in descriptor systems. The fault diagnosis scheme is addressed through the design of two observers with specific tasks. The first observer deals with the tasks of fault detection and isolation, and the second with the task of fault estimation.

5.2.1 Class of descriptor systems considered

Consider the following descriptor system with actuator faults :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Gf(t) \\ y(t) &= C_1 x(t) \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t)$ is the input, $f(t) \in \mathbb{R}^{n_f}$ is the fault vector and $y(t) \in \mathbb{R}^{n_y}$ represents the measured output vector. Matrices $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $G \in \mathbb{R}^{n\times n_f}$ and $C_1 \in \mathbb{R}^{n_y\times n}$. Let $\mathsf{rank}(E) = \varrho < n$ and $E^\perp \in \mathbb{R}^{\varrho\times n}$ be a full row rank matrix such that $E^\perp \begin{bmatrix} E & G \end{bmatrix} = 0$, in this case $\varrho_1 = n - \varrho$.

In the sequel we assume that

Assumption 5.1.

$$\mathsf{rank} \begin{bmatrix} E \\ \hline E^\perp A \\ C_1 \end{bmatrix} = n.$$

Assumption 5.2.

$$\mathsf{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = n, \ \forall s \in \mathbb{C}^+, \ s \textit{ finite.}$$

5.2.2 Generalized dynamic observer-based fault detection and isolation

In this section the GDO is apply to FDI.

5.2.2.1 Problem formulation

Consider the following GDO for FDI for system (5.1)

$$\mathcal{O}_1 := \begin{cases} \dot{\zeta}(t) = N\zeta(t) + Hv(t) + F \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) & (5.2a) \\ \dot{v}(t) = S\zeta(t) + Lv(t) + M \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix} & (5.2b) \\ \hat{x}(t) = P\zeta(t) + Q \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix} & (5.2c) \\ r(t) = W(C_1\hat{x} - y(t)) & (5.2d) \end{cases}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector, $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ and $r(t) \in \mathbb{R}^{n_f}$ is the residual vector. Matrices N, F, H, L, M, S, P, Q and W are unknown matrices of appropriate dimensions.

Now, we can give the following lemma.

Lemma 5.1. *There exists a fault isolation observer of the form* (5.2) *for the system* (5.1) *if the following two statements hold.*

1. *There exists a matrix T of appropriate dimension such that the following conditions are satisfied :*
   (a) $NTE + F \begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} - TA = 0$
   (b) $J = TB$
   (c) $M \begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} + STE = 0$
   (d) $\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ \hline E^\perp A \\ C_1 \end{bmatrix} = I_n$
2. *The matrix $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz when $f(t) = 0$.*

Proof. Let $T \in \mathbb{R}^{q\times n}$ be a parameter matrix and define $\varepsilon(t) = \zeta(t) - TEx(t)$, then its derivative is given by :

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) + TGf(t) + \left(NTE + F \begin{bmatrix} E^\perp A \\ C_1 \end{bmatrix} - TA \right) x(t) + (J - TB)u(t) \tag{5.3}$$

by using the definition of $\varepsilon(t)$, equations (5.2b) and (5.2c) can be written as :

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + \left(M\begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} + STE\right)x(t) \tag{5.4}$$

$$\hat{x}(t) = P\varepsilon + \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} TE \\ E^{\perp}A \\ C_1 \end{bmatrix}x(t) \tag{5.5}$$

Now, if conditions $(a)-(d)$ of Lemma 5.1 are satisfied the following observer error dynamics is obtained from (5.3) and (5.4)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix}\begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} -TG \\ 0 \end{bmatrix} f(t) \tag{5.6}$$

and from equation (5.5) we get :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon \tag{5.7}$$

so that equation (5.2d) becomes :

$$r(t) = WC_1P\varepsilon \tag{5.8}$$

in this case if $f(t) = 0$ and $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz, then $\lim_{t\to\infty} e(t) = 0$. □

Now, all the parameterization of all solutions to the algebraic constraints $(a)-(d)$ of Lemma 5.1 are given.

By considering the parameterization of Lemma 2.5, let $\bar{T}_2 = T_2G$ and $Z_1 = Z(I_{n+n_y} - \bar{T}_2\bar{T}_2^+)$, where Z is an arbitrary matrix of appropriate dimension, so that TG, from the observer error system (5.6) becomes :

$$TG = T_1G \tag{5.9}$$

In the same way, we obtain the following expressions form matrices T, K, N and F

$$T = T_1 - Z\mathcal{T}_2 \tag{5.10}$$
$$K = K_1 - Z\mathcal{K}_2 \tag{5.11}$$
$$N = N_1 - Z\mathcal{N}_2 - Y_1N_3 \tag{5.12}$$
$$F = F_1 - Z\mathcal{F}_2 - Y_1F_3 \tag{5.13}$$

where $\mathcal{T}_2 = (I_{n+n_y} - \bar{T}_2\bar{T}_2^+)T_2$, $\mathcal{K}_2 = (I_{n+n_y} - \bar{T}_2\bar{T}_2^+)K_2$, $\mathcal{N}_2 = (I_{n+n_y} - \bar{T}_2\bar{T}_2^+)N_2$, $\mathcal{F}_2 = (I_{n+n_y} - \bar{T}_2\bar{T}_2^+)F_2$ and matrices T_1, T_2, K_1, K_2, N_1, N_2, N_3, F_1, F_2 and F_3 are defined in Section 2.4.1.

From Lemma 2.5 and considering $Y_3 = 0$ we get :

$$S = -Y_2N_3 \tag{5.14}$$
$$M = -Y_2F_3 \tag{5.15}$$
$$P = P_1 \tag{5.16}$$
$$Q = Q_1 \tag{5.17}$$

where matrices Q_1 and P_1 are defined in Section 2.4.1.

To study the stability of the observer, the observer error system (5.6) and (5.8) can be written as :

$$\begin{aligned} \dot{\varphi}(t) &= (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) + \mathbb{F}_1 f(t) \\ r(t) &= W\mathbb{C}_1\varphi(t) \end{aligned} \tag{5.18}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_1 = \begin{bmatrix} -T_1G \\ 0 \end{bmatrix}$, $\mathbb{C}_1 = \begin{bmatrix} C_1P_1 & 0 \end{bmatrix}$ and $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$.

The objective of fault detection is build a residuals to show the presence of faults into a system. The mathematical definition of a residual is :

$$\lim_{t\to\infty} r(t) = 0 \text{ for } f(t) = 0$$
$$r(t) \neq 0 \text{ for } f(t) \neq 0$$

The fault isolation objective is to locate the fault so, we propose design structural residuals by making the transfer function from faults $f(t)$ to residuals $r(t)$ equal to a diagonal, i.e. each residue is only sensitive to one fault and insensitive to the rest of them, so that we can locate simultaneous faults.

Taking Laplace transformation of (5.18), we obtain

$$G_{rf}(s) = \left[\begin{array}{c|c} \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 & \mathbb{F}_1 \\ \hline W\mathbb{C}_1 & 0 \end{array}\right] \tag{5.19}$$

The objective is to render $G_{rf}(s)$ diagonal, i.e.

$$G_{rf}(s) = \mathsf{diag}(g_1(s), \ldots, g_{n_f}(s)) \tag{5.20}$$

while the stability of the observer is guaranteed.

5.2.2.2 Fault detection and isolation based on a generalized observer design

Assumption 5.3. *The transfer function $G_{rf}(s)$ can be diagonalized if and only if $(\mathbb{C}_1\mathbb{F}_1)$ has full column rank i.e., $n_y \geq n_f$.*

Assumption 5.3 is also called output separability condition (White and Speyer, 1987). To isolate n_f faults in system (5.1) the rank of $(\mathbb{C}_1\mathbb{F}_1)$ must be n_f, which in turn requires n_y measured outputs.

The following theorem shows how to design an observer of the form (5.2) to perform FDI.

Theorem 5.1. *Consider that $n_y \geq n_f$ and let*

$$\Lambda = \mathsf{diag}(\lambda_1, \ldots, \lambda_{n_f}) \in \mathbb{R}^{n_f \times n_f}, \ \lambda_i < 0, \tag{5.21}$$

$$\Psi = \mathsf{diag}(\psi_1, \ldots, \psi_{n_f}) \in \mathbb{R}^{n_f \times n_f}, \ |\psi_i| > 0, \tag{5.22}$$

$\forall_i \in [1, \ldots, n_f]$, *be given. Then there exist matrices $\mathbb{Y}$ and W such that :*

$$(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1 = \mathbb{F}_1\Lambda \tag{5.23}$$

$$W\mathbb{C}_1\mathbb{F}_1 = \Psi \tag{5.24}$$

If $(\mathbb{A}_2\mathbb{F}_1)$ has full column rank, the matrices $\mathbb{Y}$ and W are given by :

$$\mathbb{Y} = (\mathbb{A}_1\mathbb{F}_1 - \mathbb{F}_1\Lambda)(\mathbb{A}_2\mathbb{F}_1)^+ - \widetilde{Z}(I - (\mathbb{A}_2\mathbb{F}_1)(\mathbb{A}_2\mathbb{F}_1)^+) \tag{5.25}$$

$$W = \Psi(\mathbb{C}_1\mathbb{F}_1)^+ \tag{5.26}$$

where $\widetilde{Z}$ is an arbitrary matrix of appropriate dimension. Finally, if there exist matrices $\mathbb{Y}$ and W satisfying equations (5.23) *and* (5.24)*, then*

$$G_{rf}(s) = \left[\begin{array}{c|c} \Lambda & I \\ \hline \Psi & 0 \end{array}\right] \tag{5.27a}$$

$$= \mathsf{diag}\left(\frac{\psi_1}{s-\lambda_1}, \cdots, \frac{\psi_{n_f}}{s-\lambda_{n_f}}\right). \tag{5.27b}$$

Proof. Since $\mathbb{F}_1$ has full column rank there exists a matrix completion $\mathbb{F}_1^{\perp} \in \mathbb{R}^{(q_0+q_1)\times(q_0+q_1-n_f)}$ such that $\widetilde{\mathbb{F}} = \begin{bmatrix}\mathbb{F}_1 & \mathbb{F}_1^{\perp}\end{bmatrix} \in \mathbb{R}^{(q_0+q_1)\times(q_0+q_1)}$ is nonsingular. Let $\widetilde{\mathbb{F}}^{-1} = \begin{bmatrix}\widetilde{\mathbb{F}}_1 & \widetilde{\mathbb{F}}_2\end{bmatrix}^T$ with $\widetilde{\mathbb{F}}_1 \in \mathbb{R}^{(q_0+q_1)\times n_f}$. Then we obtain :

$$G_{rf}(s) = \left[\begin{array}{c|c} \widetilde{\mathbb{F}}^{-1}(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\widetilde{\mathbb{F}} & \widetilde{\mathbb{F}}^{-1}\mathbb{F}_1 \\ \hline W\mathbb{C}_1\widetilde{\mathbb{F}} & 0 \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix}\widetilde{\mathbb{F}}_1^T \\ \widetilde{\mathbb{F}}_2^T\end{bmatrix}(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\begin{bmatrix}\mathbb{F}_1 & \mathbb{F}_1^{\perp}\end{bmatrix} & \begin{bmatrix}\widetilde{\mathbb{F}}_1^T \\ \widetilde{\mathbb{F}}_2^T\end{bmatrix}\mathbb{F}_1 \\ \hline W\mathbb{C}_1\begin{bmatrix}\mathbb{F}_1 & \mathbb{F}_1^{\perp}\end{bmatrix} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \widetilde{\mathbb{F}}_1^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1 & \widetilde{\mathbb{F}}_1^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1^{\perp} & I \\ \widetilde{\mathbb{F}}_2^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1 & \widetilde{\mathbb{F}}_2^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1^{\perp} & 0 \\ \hline W\mathbb{C}_1\mathbb{F}_1 & W\mathbb{C}_1\mathbb{F}_1^{\perp} & 0 \end{array}\right]$$

consider $\begin{bmatrix}\widetilde{\mathbb{F}}_1 & \widetilde{\mathbb{F}}_2\end{bmatrix}^T \begin{bmatrix}\mathbb{F}_1 & \mathbb{F}_1^{\perp}\end{bmatrix} = I$, then we have

$$G_{rf}(s) = \left[\begin{array}{cc|c} \widetilde{\mathbb{F}}_1^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1 & \widetilde{\mathbb{F}}_1^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1^{\perp} & I \\ 0 & \widetilde{\mathbb{F}}_2^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1^{\perp} & 0 \\ \hline W\mathbb{C}_1\mathbb{F}_1 & W\mathbb{C}_1\mathbb{F}_1^{\perp} & 0 \end{array}\right] \tag{5.28}$$

now, removing an uncontrollable subspace, we get :

$$G_{rf}(s) = \left[\begin{array}{c|c} \Lambda & I \\ \hline \Psi & 0 \end{array}\right] = \mathsf{diag}\left(\frac{\psi_1}{s-\lambda_1}, \cdots, \frac{\psi_{n_f}}{s-\lambda_{n_f}}\right) \tag{5.29}$$

From equation (5.29) we find that

$$(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1 = \mathbb{F}_1\Lambda \tag{5.30}$$

$$W\mathbb{C}_1\mathbb{F}_1 = \Psi \tag{5.31}$$

and from (5.30) the general form of $\mathbb{Y}$ is given by :

$$\mathbb{Y} = (\mathbb{A}_1\mathbb{F}_1 - \mathbb{F}_1\Lambda)(\mathbb{A}_2\mathbb{F}_1)^+ - \widetilde{Z}(I - (\mathbb{A}_2\mathbb{F}_1)(\mathbb{A}_2\mathbb{F}_1)^+) \tag{5.32}$$

where $\widetilde{Z}$ is an arbitrary matrix of appropriate dimension.

And the particular form of W is :

$$W = \Psi(\mathbb{C}_1\mathbb{F}_1)^+ \tag{5.33}$$

□

Replacing equations (5.32) and (5.33) in (5.18), we obtain :

$$\dot{\varphi}(t) = \underbrace{[\mathbb{A}_1 - (\mathbb{A}_1\mathbb{F}_1 - \mathbb{F}_1\Lambda)(\mathbb{A}_2\mathbb{F}_1)^+\mathbb{A}_2}_{\bar{\mathbb{A}}_1} + \widetilde{Z}\underbrace{(I - (\mathbb{A}_2\mathbb{F}_1)(\mathbb{A}_2\mathbb{F}_1)^+)\mathbb{A}_2]}_{\bar{\mathbb{A}}_2}\varphi(t) + \mathbb{F}_1 f(t) \tag{5.34a}$$

$$r(t) = \underbrace{\Psi(\mathbb{C}_1\mathbb{F}_1)^+\mathbb{C}_1}_{\bar{\mathbb{C}}_1}\varphi(t) \tag{5.34b}$$

where

$$\bar{\mathbb{A}}_1 = \begin{bmatrix} N_1 + (T_1G\Lambda - N_1T_1G)(N_3T_1G)^+N_3 - Z(\mathcal{N}_2 - \mathcal{N}_2T_1G(N_3T_1G)^+N_3) & 0 \\ 0 & 0 \end{bmatrix} \tag{5.35}$$

$$\bar{\mathbb{A}}_2 = \begin{bmatrix} N_3 - N_3T_1G(N_3T_1G)^+N_3 & 0 \\ 0 & -I \end{bmatrix} \tag{5.36}$$

$$\bar{\mathbb{C}}_1 = -\Psi\begin{bmatrix}(C_1P_1T_1G)^+C_1P_1 & 0\end{bmatrix} \tag{5.37}$$

The following theorem gives the LMI conditions that allow the determination of the observer matrices such that the observer error system (5.34) is stable.

Theorem 5.2. *Under Assumptions 5.1 and 5.2 there exists a GDO* (5.2) *such that the error dynamics* (5.34) *is stable if and only if there exists a matrix* $X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$, *with* $X_1 = X_1^T > 0$, *satisfying the following LMIs.*

$$\mathcal{C}^{T\perp}\begin{bmatrix} \Pi_1 & (N_1 + (T_1G\Lambda - N_1T_1G)(N_3T_1G)^+N_3)^TX_1 - (\mathcal{N}_2 - \mathcal{N}_2T_1G(N_3T_1G)^+N_3)^TW_1^T \\ (*) & 0 \end{bmatrix}\mathcal{C}^{T\perp T} < 0 \tag{5.38}$$

where

$$\begin{aligned}\Pi_1 = {} & (N_1 + (T_1G\Lambda - N_1T_1G)(N_3T_1G)^+N_3)^TX_1 + X_1(N_1 + (T_1G\Lambda - N_1T_1G)(N_3T_1G)^+N_3) - \\ & (\mathcal{N}_2 - \mathcal{N}_2T_1G(N_3T_1G)^+N_3)^TW_1^T - W_1(\mathcal{N}_2 - \mathcal{N}_2T_1G(N_3T_1G)^+N_3)\end{aligned} \tag{5.39}$$

In this case matrix $W_1 = X_1Z$ *and matrix* $\widetilde{Z}$ *is parameterized as follows :*

$$\tilde{Z} = X^{-1}(\mathcal{K}\mathcal{C}_l^+ + \mathcal{Z}(I - \mathcal{C}_l\mathcal{C}_l^+)) \tag{5.40}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1}\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \tag{5.41a}$$

$$\vartheta = (\mathcal{R}^{-1} - \mathcal{D})^{-1} > 0 \tag{5.41b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}[\vartheta - \vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta]\mathcal{R}^{-1} \tag{5.41c}$$

where $\mathcal{D} = \begin{bmatrix} \Pi_1 & (N_1 + (T_1G\Lambda - N_1T_1G)(N_3T_1G)^+N_3)^TX_1 - (\mathcal{N}_2 - \mathcal{N}_2T_1G(N_3T_1G)^+N_3)^TW_1^T \\ (*) & 0 \end{bmatrix}$, $\mathcal{B} = I_{q_0+q_1}$,
$\mathcal{C} = \begin{bmatrix} N_3 - N_3T_1G(N_3T_1G)^+N_3 & 0 \\ 0 & -I \end{bmatrix}$, *matrix* Π_1 *is defined in equation* (5.39), *and matrices* $\mathcal{L}$, $\mathcal{R}$ *and* $\mathcal{Z}$ *are arbitrary matrices of appropriate dimensions satisfying* $\mathcal{R} > 0$ *and* $\|\mathcal{L}\|_2 < 1$. *Matrices* $\mathcal{C}_l$ *and* $\mathcal{C}_r$ *are any full rank matrices such that* $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$.

Proof. Consider a matrix $X = X^T > 0$ such that

$$(\bar{\mathbb{A}}_1 + \widetilde{Z}\bar{\mathbb{A}}_2)^TX + X(\bar{\mathbb{A}}_1 + \widetilde{Z}\bar{\mathbb{A}}_2) < 0 \tag{5.42}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{5.43}$$

where $\mathcal{X} = X\widetilde{Z}$, $\mathcal{D} = \bar{\mathbb{A}}_1^TX + X\bar{\mathbb{A}}_1$, $\mathcal{C} = \bar{\mathbb{A}}_2$ and $\mathcal{B} = I_{q_0+q_1}$.

From the elimination lemma of Section 1.5 and since $\mathcal{B}^\perp = 0$ the solvability condition of inequality (5.43) is :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{5.44}$$

with $\mathcal{C}^{T\perp} = \bar{\mathbb{A}}_2^{T\perp}$. By using the definition of matrices $\mathcal{C}$ and $\mathcal{D}$ the inequality (5.44) becomes :

$$\mathcal{C}^{T\perp}\begin{bmatrix} \Pi_1 & (N_1 + (T_1G\Lambda - N_1T_1G)(N_3T_1G)^+N_3)^TX_1 - (\mathcal{N}_2 - \mathcal{N}_2T_1G(N_3T_1G)^+N_3)^TW_1^T \\ (*) & 0 \end{bmatrix}\mathcal{C}^{T\perp T} < 0 \tag{5.45}$$

where matrix Π_1, is defined in (5.39). From the elimination lemma if condition (5.44) is satisfied, then parameter matrix $\widetilde{Z}$ is parameterized as in (5.40) and (5.41). □

5.2.3 Generalized dynamic observer-based for simultaneous estimation of the state and faults

In this section the GDO is apply to simultaneous estimation of state and fault.

5.2.3.1 Problem formulation

Consider the following GDO for simultaneous estimation of state and fault for system (5.1)

$$\mathcal{O}_2 := \begin{cases} \dot{\zeta}(t) = N(\zeta(t) + TG\hat{f}(t)) + Hv(t) + F\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Ju(t) + TG\dot{\hat{f}}(t) & (5.46a) \\ \dot{v}(t) = S(\zeta(t) + TG\hat{f}(t)) + Lv(t) + M\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} & (5.46b) \\ \hat{x}(t) = P(\zeta(t) + TG\hat{f}(t)) + Q\begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} & (5.46c) \\ \dot{\hat{f}}(t) = \Phi(C_1\hat{x}(t) - y(t)) & (5.46d) \end{cases}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector, $\hat{x} \in \mathbb{R}^n$ is the estimate of $x(t)$ and $\hat{f}(t)$ is the estimate of $f(t)$. Matrices N, F, J, H, L, M, S, P, Q, T and Φ are unknown matrices of appropriate dimensions.

As can we see the observer has a particular structure, because we want to express the estimation error in terms of the fault estimation error, this fact will be show in the subsequent.

Lemma 5.2. *There exists an observer of the form* (5.46) *for the system* (5.1) *if the following two statements hold.*

1. *There exists a matrix T of appropriate dimension such that the following conditions are satisfied :*
   (a) $NTE + F \begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} - TA = 0$
   (b) $J = TB$
   (c) $M \begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} + STE = 0$
   (d) $\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ \hline E^{\perp}A \\ C_1 \end{bmatrix} = I_n$
2. *The matrix $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz when $f(t) = 0$.*

Proof. Let $T \in \mathbb{R}^{q \times n}$ be a parameter matrix and define

$$\varepsilon(t) = \zeta(t) - TEx(t) + TGf(t) \tag{5.47}$$

then its derivative is given by :

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) + (NTG + TG)\widetilde{f}(t) + \left(NTE + F \begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} - TA\right) x(t) + (J - TB)u(t) \tag{5.48}$$

where $\dot{f}(t) = 0$ and $\widetilde{f}(t) = \hat{f}(t) - f(t)$.

By using equation (5.47), equations (5.46b) and (5.46c) can be written as :

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + STG\widetilde{f}(t) + \left(M \begin{bmatrix} E^{\perp}A \\ C_1 \end{bmatrix} + STE\right) x(t) \tag{5.49}$$

$$\hat{x}(t) = P\varepsilon(t) + PTG\widetilde{f}(t) + \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ \hline E^{\perp}A \\ C_1 \end{bmatrix} x(t) \tag{5.50}$$

since $\dot{f}(t) = 0$ the derivative of $\widetilde{f}(t)$ is given by :

$$\dot{\widetilde{f}}(t) = \Phi C_1 P\varepsilon(t) + \Phi C_1 PTG\widetilde{f}(t) \tag{5.51}$$

Now, if conditions $(a) - (d)$ of Lemma 5.2 are satisfied the following observer error dynamics is obtained from (5.48) and (5.49)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} NTG + TG \\ STG \end{bmatrix} \widetilde{f}(t) \tag{5.52}$$

and from (5.50) we get :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon(t) + PTG\widetilde{f}(t) \tag{5.53}$$

in this case if $f(t) = 0$ and $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz, then $\lim_{t \to \infty} e(t) = 0$. □

By considering the parameterization of equations (5.10) - (5.17) the observer error dynamics (5.52) can be written as :

$$\dot{\varphi}(t) = (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) + (\mathbb{F}_1 - \mathbb{Y}\mathbb{F}_2)\widetilde{f}(t) \tag{5.54}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z\mathcal{N}_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_1 = \begin{bmatrix} N_1 T_1 G + T_1 G - Z\mathcal{N}_2 T_1 G \\ 0 \end{bmatrix}$, $\mathbb{F}_2 = \begin{bmatrix} N_3 T_1 G \\ 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

By putting together the observer error dynamics (5.54) and equation (5.51) we get :

$$\dot{\beta}(t) = \mathscr{A}\beta(t) \tag{5.55}$$

where $\mathscr{A} = \begin{bmatrix} \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 & \mathbb{F}_1 - \mathbb{Y}\mathbb{F}_2 \\ \mathbb{C}_1 & \mathbb{D}_1 \end{bmatrix}$ and $\beta(t) = \begin{bmatrix} \varphi(t) \\ \widetilde{f}(t) \end{bmatrix}$, with $\mathbb{C}_1 = \begin{bmatrix} \Phi C_1 P_1 & 0 \end{bmatrix}$ and $\mathbb{D}_1 = \Phi C_1 P_1 T_1 G$.

The observer design is obtained from the determination of matrices $\mathbb{Y}$, Φ and Z such that the system (5.55) is stable.

5.2.3.2 Simultaneous estimation of the state and faults based on a generalized observer design

The following theorem gives the stability conditions for the system (5.55) in the form of LMIs such that we can get the simultaneous estimation of state and fault.

Theorem 5.3. *Under Assumptions 5.1 and 5.2 there exist parameter matrices $\mathbb{Y}$, Φ and Z such that the system (5.55) is asymptotically stable if there exists a matrix $X_1 = \begin{bmatrix} X_{11} & X_{11} \\ X_{11} & X_{12} \end{bmatrix} > 0$, with $X_{11} = X_{11}^T$, and a matrix $X_2 > 0$ such that the following LMIs are satisfied.*

$$\mathcal{C}^{T\perp} \begin{bmatrix} \Pi_1 & N_1^T X_{11} - \mathcal{N}_2^T W_1^T & \Pi_2 \\ (*) & 0 & X_1(N_1 T_1 G + T_1 G) - W_1 \mathcal{N}_2 T_1 G \\ (*) & (*) & X_2 \Phi C_1 P_1 T_1 G + G^T T_1^T P_1^T C_1^T \Phi^T X_2 \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{5.56}$$

where

$$\Pi_1 = X_{11} N_1 + N_1^T X_{11} + W_1 \mathcal{N}_2 - \mathcal{N}_2 W_1^T \tag{5.57a}$$

$$\Pi_2 = X_1(N_1 T_1 G + T_1 G) - W_1 \mathcal{N}_2 T_1 G + P_1^T C_1^T \Phi^T X_2 \tag{5.57b}$$

and

$$X_2 \Phi C_1 P_1 T_1 G + G^T T_1^T P_1^T C_1^T \Phi^T X_2 < 0. \tag{5.58}$$

In this case matrix $W_1 = X_{11} Z$ and matrix $\mathbb{Y}$ is parameterized as follows :

$$\mathbb{Y} = X_1^{-1}(\mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+) \tag{5.59}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1} \mathcal{B}_l^T \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2} \mathcal{L} (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1/2} \tag{5.60a}$$

$$\vartheta = (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{5.60b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\vartheta - \vartheta \mathcal{C}_r^T (\mathcal{C}_r \vartheta \mathcal{C}_r^T)^{-1} \mathcal{C}_r \vartheta] \mathcal{B}_l \mathcal{R}^{-1} \tag{5.60c}$$

where $\mathcal{D} = \begin{bmatrix} \Pi_1 & N_1^T X_{11} - \mathcal{N}_2^T W_1^T & \Pi_2 \\ (*) & 0 & X_1(N_1 T_1 G + T_1 G) - W_1 \mathcal{N}_2 T_1 G \\ (*) & (*) & X_2 \Phi C_1 P_1 T_1 G + G^T T_1^T P_1^T C_1^T \Phi^T X_2 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} -I_{q_0+q_1} \\ 0 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} & \begin{bmatrix} N_3 T_1 G \\ 0 \end{bmatrix} \end{bmatrix}$, *matrices Π_1 and Π_2 are defined in* (5.57)*, and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$.*

Proof. Consider the following Lyapunov function

$$V(\beta(t)) = \beta(t)^T X \beta(t) \tag{5.61}$$

with symmetric matrix $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$. Then its derivative along the trajectory of (5.55) is :

$$\dot{V}(\beta(t)) = \beta(t)^T (\mathscr{A}^T X + X\mathscr{A}) \beta(t) \tag{5.62}$$

The asymptotic stability of system (5.55) is guaranteed if and only if $\dot{V}(\beta(t)) < 0$. This leads the following LMI :

$$\mathscr{A}^T X + X\mathscr{A} < 0. \tag{5.63}$$

By inserting the form of $\mathscr{A}$ and X, we obtain the following inequality :

$$\begin{bmatrix} X_1(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) + (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X_1 & X_1(\mathbb{F}_1 - \mathbb{Y}\mathbb{F}_2) + \mathbb{C}_1^T X_2 \\ (*) & X_2\mathbb{D}_1 + \mathbb{D}_1^T X_2 \end{bmatrix} < 0 \tag{5.64}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{5.65}$$

where $\mathcal{X} = X_1\mathbb{Y}$, $\mathcal{D} = \begin{bmatrix} X_1\mathbb{A}_1 + \mathbb{A}_1^T X_1 & X_1\mathbb{F}_1 + \mathbb{C}_1^T X_2 \\ (*) & X_2\mathbb{D}_1 + \mathbb{D}_1^T X_2 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} -I_{q_0+q_1} \\ 0 \end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & \mathbb{B}_2 \end{bmatrix}$. Using the elimination lemma, inequality (5.65) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{5.66a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{5.66b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} \mathbb{A}_2^T \\ \mathbb{B}_2^T \end{bmatrix}^{\perp}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} 0 & I \end{bmatrix}$. By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_1 the inequality (5.66a) becomes :

$$\mathcal{C}^{T\perp} \begin{bmatrix} \Pi_1 & N_1^T X_{11} - \mathcal{N}_2^T W_1^T & \Pi_2 \\ (*) & 0 & X_1(N_1T_1G + T_1G) - W_1\mathcal{N}_2T_1G \\ (*) & (*) & X_2\Phi C_1P_1T_1G + G^T T_1^T P_1^T C_1^T \Phi^T X_2 \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{5.67}$$

and by using $\mathcal{B}$ and $\mathcal{D}$ the inequality (5.66b) becomes :

$$X_2\Phi C_1P_1T_1G + G^T T_1^T P_1^T C_1^T \Phi^T X_2 < 0 \tag{5.68}$$

From the elimination lemma if conditions (5.66a) and (5.66b) are satisfied, the parameter matrix $\mathbb{Y}$ is parameterized as in (5.59) and (5.60). □

5.2.4 Numerical example

In order to illustrate the results obtained, consider the following descriptor system described by (5.1) where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -2.7 & 0 & 0.3 \\ -0.2 & -3 & 0 \\ -0.11 & 1.74 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}, \; G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Considering $E^{\perp} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ such that $E^{\perp}\begin{bmatrix} E & G \end{bmatrix} = 0$, we can verify Assumptions 5.1 and 5.2

$$\mathsf{rank} \begin{bmatrix} E \\ E^{\perp}A \\ C_1 \end{bmatrix} = 3 \text{ and } \mathsf{rank} \begin{bmatrix} sE - A \\ C_1 \end{bmatrix} = 3$$

Observer for FDI computation.

For the GDO apply to FDI we have chosen matrix $R = I_3$, such that $\mathsf{rank}(\Sigma) = 3$.

From Theorem 5.1 and since $n_y \geq n_f$, we have chosen matrices $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ and $\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to get matrix W as :

$$W = \begin{bmatrix} 2.12 & 5.85 \\ -6.05 & -4.97 \end{bmatrix}$$

and by using YALMIP toolbox, we solve the LMI (5.38) to find matrices X and Z

$$X = \begin{bmatrix} 5.22 & -1.96 & 0.15 & 5.22 & -1.96 & 0.15 \\ -1.96 & 11.16 & 1.67 & -1.96 & 11.16 & 1.67 \\ 0.15 & 1.67 & 11.86 & 0.15 & 1.67 & 11.86 \\ 5.22 & -1.96 & 0.15 & 20.83 & 0 & 0 \\ -1.96 & 11.16 & 1.67 & 0 & 20.83 & 0 \\ 0.15 & 1.67 & 11.86 & 0 & 0 & 20.83 \end{bmatrix} \text{ and}$$

$$Z = \begin{bmatrix} -182.04 & 64.68 & -11.11 & -11.58 & 126.48 & -65.09 \\ -23.83 & 3.9 & -4.16 & 0.48 & 13.9 & -8.34 \\ 87.05 & -72.45 & -3.25 & 23.72 & -84.65 & 32.74 \end{bmatrix}$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 9 & 3 & 2 & 1 & 8 & 9 & 9 & 0 & 3 \\ 9 & 4 & 1 & 3 & 8 & 7 & 8 & 2 & 8 \\ 9 & 2 & 8 & 4 & 1 & 7 & 7 & 4 & 2 \\ 9 & 4 & 4 & 5 & 7 & 8 & 3 & 8 & 3 \\ 9 & 1 & 8 & 4 & 7 & 2 & 8 & 4 & 1 \\ 9 & 4 & 8 & 2 & 8 & 4 & 9 & 3 & 8 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_6$, and solving (5.40) and (5.41) we get :

$$\widetilde{Z} = \begin{bmatrix} 2.28 & 0.39 & 0.79 & 0.18 & 1.85 & 1.91 & -0.15 & 0.12 & 0.05 \\ 1.46 & 0.21 & 0.3 & 0.59 & 0.7 & 1.19 & -0.06 & -0.2 & -0.07 \\ 0.35 & 0.28 & -0.1 & 0.3 & -0.67 & 1.09 & -0.02 & -0.11 & -0.17 \\ -0.01 & 0.12 & 0.01 & 0.24 & -0.05 & 0.02 & 0.08 & -0.04 & -0.03 \\ -0.24 & 0.07 & 0.09 & -0.2 & 0.26 & -0.38 & 0.01 & 0.18 & 0.05 \\ 0.05 & 0.08 & 0.29 & -0.17 & 0.74 & -0.49 & 0.01 & 0.08 & 0.14 \end{bmatrix}$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -2.92 & -3.19 & -3.15 \\ -0.92 & -2.33 & -1.58 \\ 0.15 & 0.45 & -0.81 \end{bmatrix}, \ H = \begin{bmatrix} 0.15 & -0.12 & -0.05 \\ 0.06 & 0.2 & 0.07 \\ 0.02 & 0.11 & 0.17 \end{bmatrix}, \ F = \begin{bmatrix} 12.18 & -0.97 & 2.23 \\ 4.19 & 0.1 & -0.1 \\ 2.87 & 0.42 & -0.88 \end{bmatrix},$$

$$J = \begin{bmatrix} 11.71 \\ 4.23 \\ 3.07 \end{bmatrix}, \ S = \begin{bmatrix} 0.49 & -0.77 & 1.33 \\ -1.49 & 2.33 & -4.02 \\ 2.53 & -3.97 & 6.84 \end{bmatrix} \times 10^{-2}, \ L = \begin{bmatrix} -7.6 & 4.47 & 2.62 \\ -1.48 & -17.91 & -4.9 \\ -1.28 & -8.04 & -14.55 \end{bmatrix} \times 10^{-2},$$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ P = \begin{bmatrix} 0.59 & -0.05 & -0.19 \\ -0.05 & 0.31 & 0.15 \\ -0.19 & 0.15 & 0.37 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.07 & 0.56 & -0.49 \\ 0.49 & 0.16 & 0.33 \\ -0.09 & 0.24 & 0.67 \end{bmatrix} \text{ and } W = \begin{bmatrix} 2.12 & 5.85 \\ -6.05 & -4.97 \end{bmatrix}.$$

Observer for FE computation.

For the GDO apply to FE we have chosen the same matrices $E^{\perp}$ and R as the observer for FDI.

From Theorem 5.3, since X_2 is linked to Φ we have chosen $X_2 = I_2$, and by using YALMIP toolbox, we have solved LMIs (5.56) - (5.58) to find matrices X_1, Z and Φ.

$$X_1 = \begin{bmatrix} 21.21 & -2.12 & 6.72 & 21.21 & -2.12 & 6.72 \\ -2.12 & 41.23 & -0.07 & -2.12 & 41.23 & -0.07 \\ 6.72 & -0.07 & 40.17 & 6.72 & -0.07 & 40.17 \\ 21.21 & -2.12 & 6.72 & 50 & 0 & 0 \\ -2.12 & 41.23 & -0.07 & 0 & 50 & 0 \\ 6.72 & -0.07 & 40.17 & 0 & 0 & 50 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 9.77 & 0 & 0 & 0 \\ 0 & 0 & 1.92 & 0 & 0 & 0 \\ 0 & 0 & -1.32 & 0 & 0 & 0 \end{bmatrix} \text{ and } \Phi = \begin{bmatrix} -238.94 & -161.63 \\ -161.63 & -323.92 \end{bmatrix}$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 7 & 6 & 3 & 9 \\ 3 & 2 & 6 & 3 & 3 & 4 & 6 & 3 & 9 \\ 6 & 1 & 4 & 2 & 2 & 6 & 6 & 1 & 0 \\ 9 & 5 & 2 & 4 & 3 & 2 & 8 & 1 & 6 \\ 2 & 6 & 8 & 1 & 8 & 4 & 7 & 3 & 9 \\ 8 & 7 & 1 & 9 & 3 & 6 & 2 & 8 & 2 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_6 \times 0.01$, and solving (5.59) and (5.60) we get :

$$\mathbb{Y} = \begin{bmatrix} -0.18 & -8.64 & -4.11 & 4.89 & 1.13 & 9.24 & 4.87 & -1.13 & 2.43 \\ 0.47 & 9.51 & -5.77 & -5.29 & -1.35 & 0.89 & -1.03 & 12.13 & -0.62 \\ 5.98 & -0.86 & 14.73 & -0.79 & -6.22 & -7.7 & 2.49 & -0.67 & 12.25 \\ -0.62 & 4.37 & -0.53 & -2.16 & 0.45 & -2.86 & -4.63 & 0.89 & -2.91 \\ -0.37 & -7.99 & 4.67 & 4.53 & 1.34 & -0.25 & 0.87 & -12.25 & 0.43 \\ -4.72 & 2.12 & -11.32 & 0.09 & 4.99 & 4.96 & -2.85 & 0.51 & -12.37 \end{bmatrix}$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -1.21 & 5.13 & 5.8 \\ -0.37 & -10.63 & 5.49 \\ -5.71 & 0.66 & -14.85 \end{bmatrix},\ H = \begin{bmatrix} 4.87 & -1.13 & 2.43 \\ -1.03 & 12.13 & -0.62 \\ 2.49 & -0.67 & 12.25 \end{bmatrix},\ F = \begin{bmatrix} -11.48 & 1.09 & -2.6 \\ -1.17 & 1.11 & -0.39 \\ 2.48 & 0.25 & 0.82 \end{bmatrix},$$

$$J = \begin{bmatrix} -9.16 \\ -1.84 \\ 1.1438 \end{bmatrix},\ T = \begin{bmatrix} 0.65 & -0.1 & -9.77 \\ -0.1 & 0.34 & -1.92 \\ -0.3 & 0.24 & 1.32 \end{bmatrix},\ S = \begin{bmatrix} 0.77 & -4.31 & 0.32 \\ 0.43 & 8.12 & -4.32 \\ 5.03 & -1.64 & 11.07 \end{bmatrix},$$

$$L = \begin{bmatrix} -4.63 & 0.89 & -2.91 \\ 0.87 & -12.25 & 0.43 \\ -2.85 & 0.51 & -12.37 \end{bmatrix},\ M = \begin{bmatrix} 0.85 & -0.73 & 1.59 \\ -1 & -0.88 & -0.12 \\ -0.9 & -0.24 & -0.66 \end{bmatrix},\ P = \begin{bmatrix} 0.59 & -0.05 & -0.19 \\ -0.05 & 0.31 & 0.15 \\ -0.19 & 0.15 & 0.37 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.07 & 0.56 & -0.49 \\ 0.49 & 0.16 & 0.33 \\ -0.09 & 0.24 & 0.67 \end{bmatrix} \text{ and } \Phi = \begin{bmatrix} -238.94 & -161.63 \\ -161.63 & -323.92 \end{bmatrix}.$$

Simulation results

The results of simulation are depicted in Figures 5.2 - 5.10. The input $u(t)$ of the system is considered constant as $u(t) = 1$. Figures 5.2 and 5.3 show the faults that are present in the system. Figures 5.4 and 5.5 give the residuals where each fault can be easily distinguished from the other.

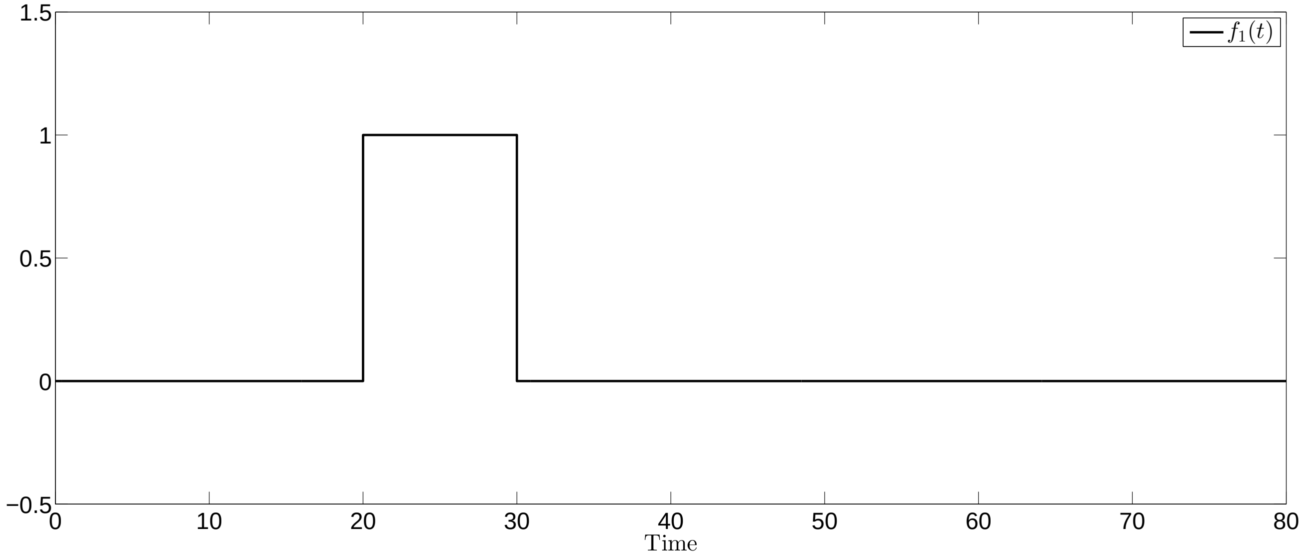

FIGURE 5.2 – Fault diagnosis : Actuator fault $f_1(t)$.

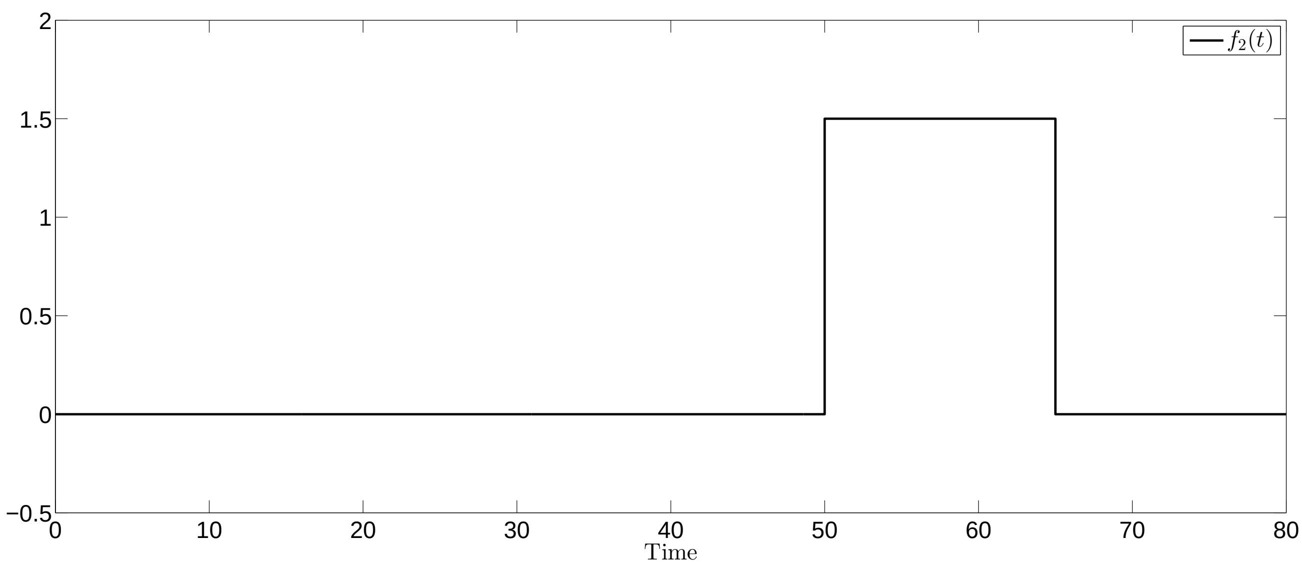

FIGURE 5.3 – Fault diagnosis : Actuator fault $f_2(t)$.

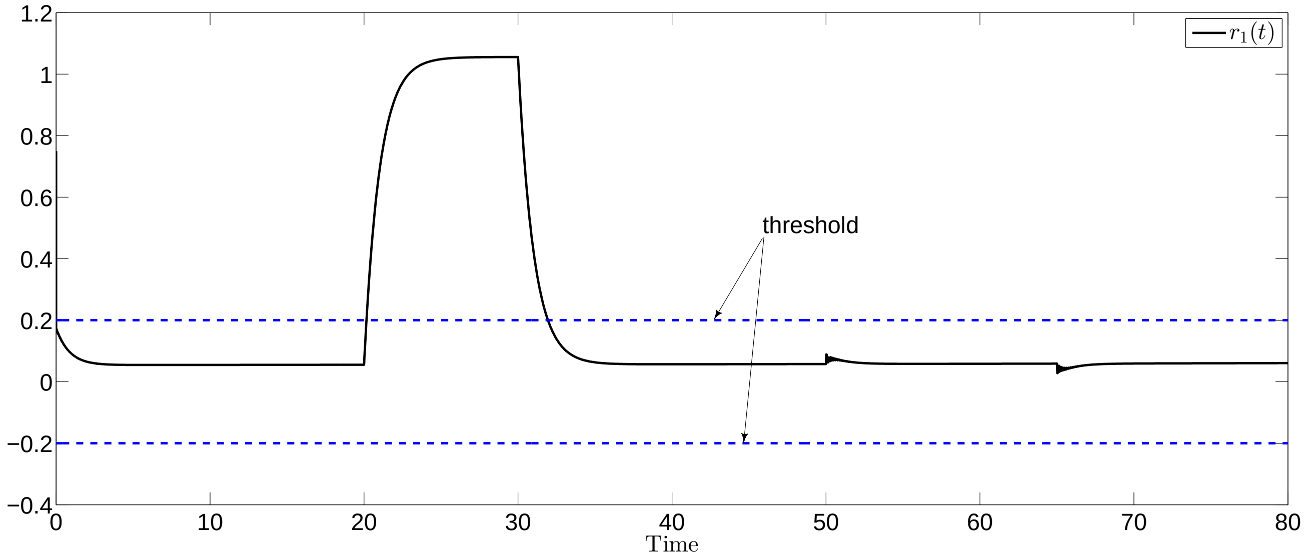

FIGURE 5.4 – Fault diagnosis : Residual $r_1(t)$.

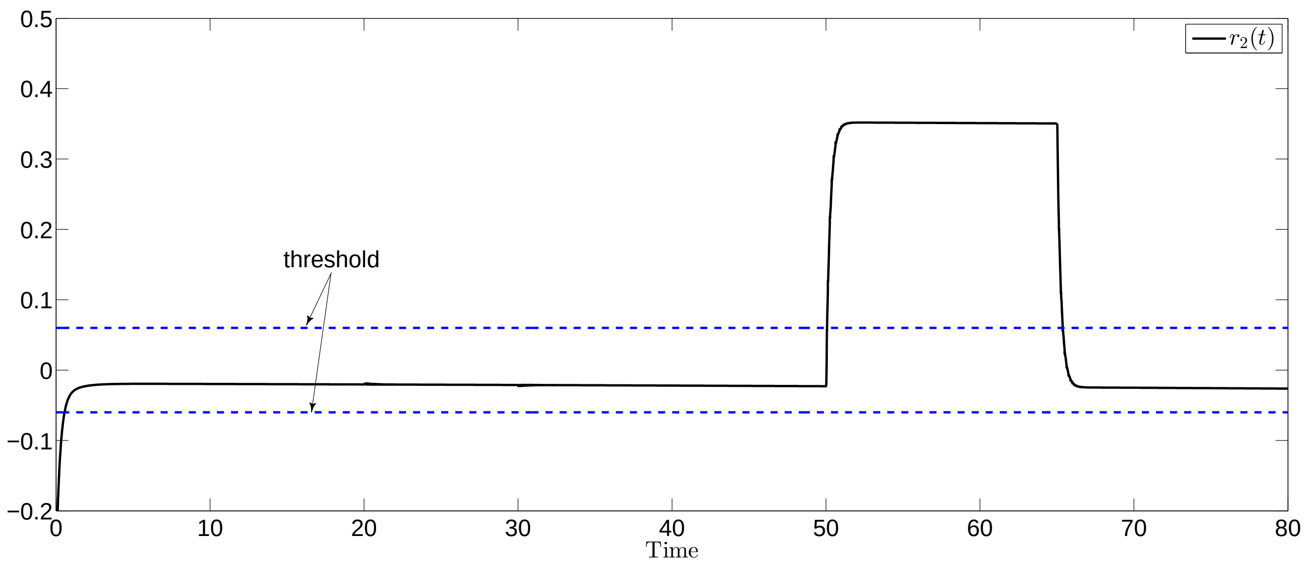

FIGURE 5.5 – Fault diagnosis : Residual $r_2(t)$.

Once the residuals were generated, the next step is the evaluation of residuals by assigning a symptom.

symptom		
1	if	residual > threshold
0	if	residual < threshold

In this case the threshold chosen is 0.2 for $r_1(t)$ and 0.06 for $r_2(t)$. Now, we can generate the following signature table :

TABLE 5.1 – Fault isolation : Residual evaluation.

Residue \ Time	$20.15 < t < 31.94$	$50.07 < t < 65.37$	other time
$r_1(t)$	1	0	0
$r_2(t)$	0	1	0

From Table 5.1 we can see that the signature for represent the presence of fault $f_1(t)$ is different from the one for represent the fault $f_2(t)$, so we can isolate each fault.

Figures 5.6 - 5.8 show the state estimation, and Figures 5.9 and 5.10 show the fault estimation.

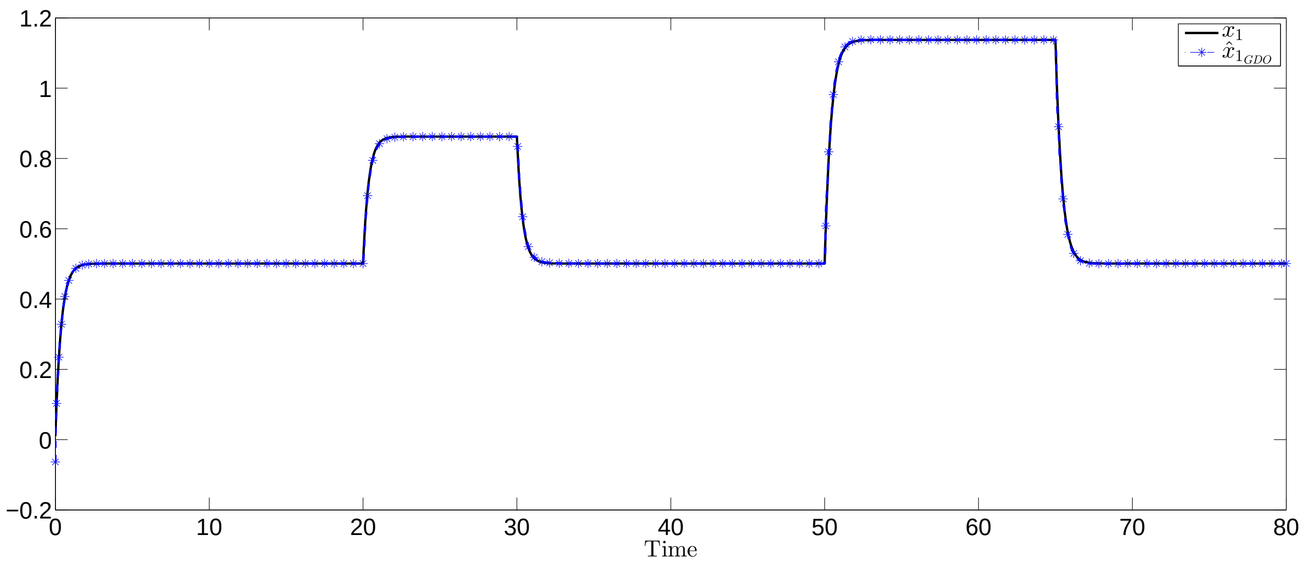

FIGURE 5.6 – Fault diagnosis : Estimation of $x_1(t)$.

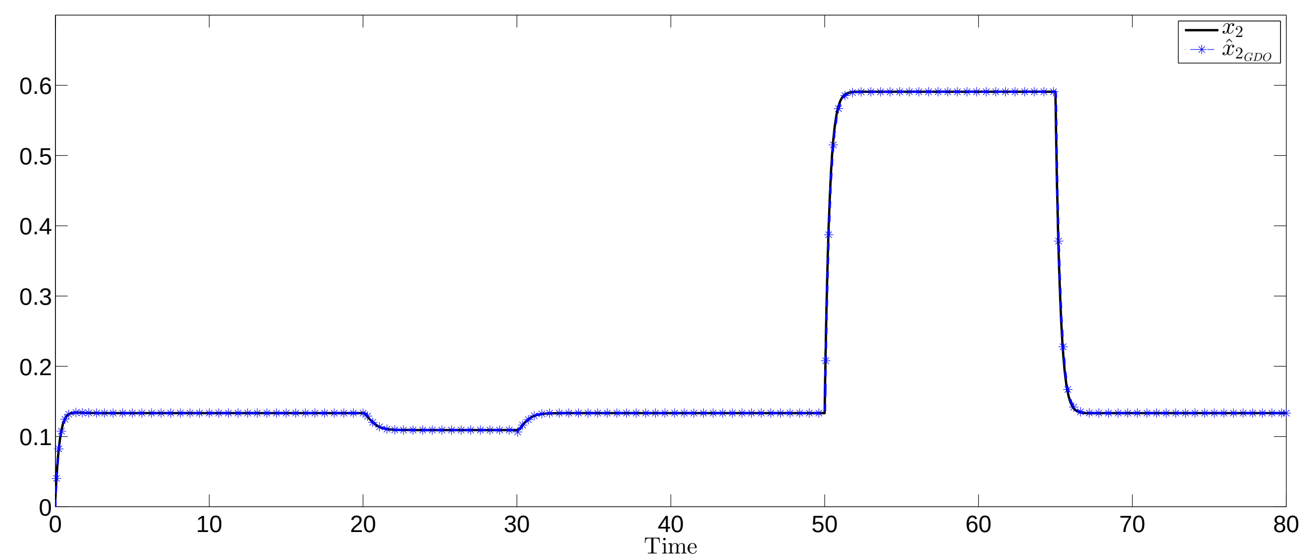

FIGURE 5.7 – Fault diagnosis : Estimation of $x_2(t)$.

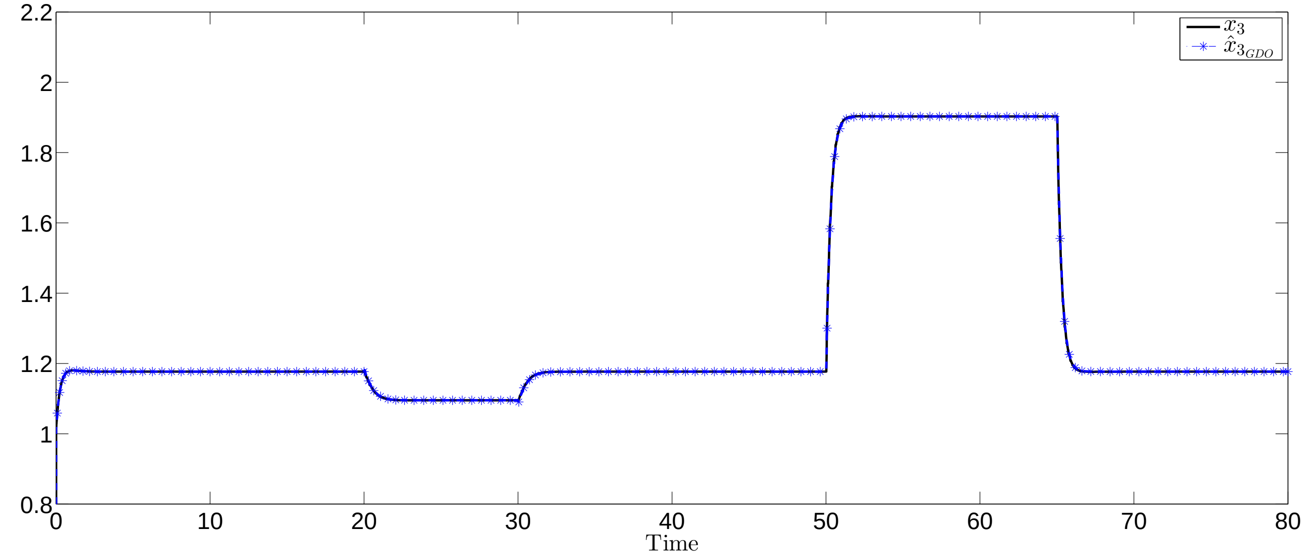

FIGURE 5.8 – Fault diagnosis : Estimation of $x_3(t)$.

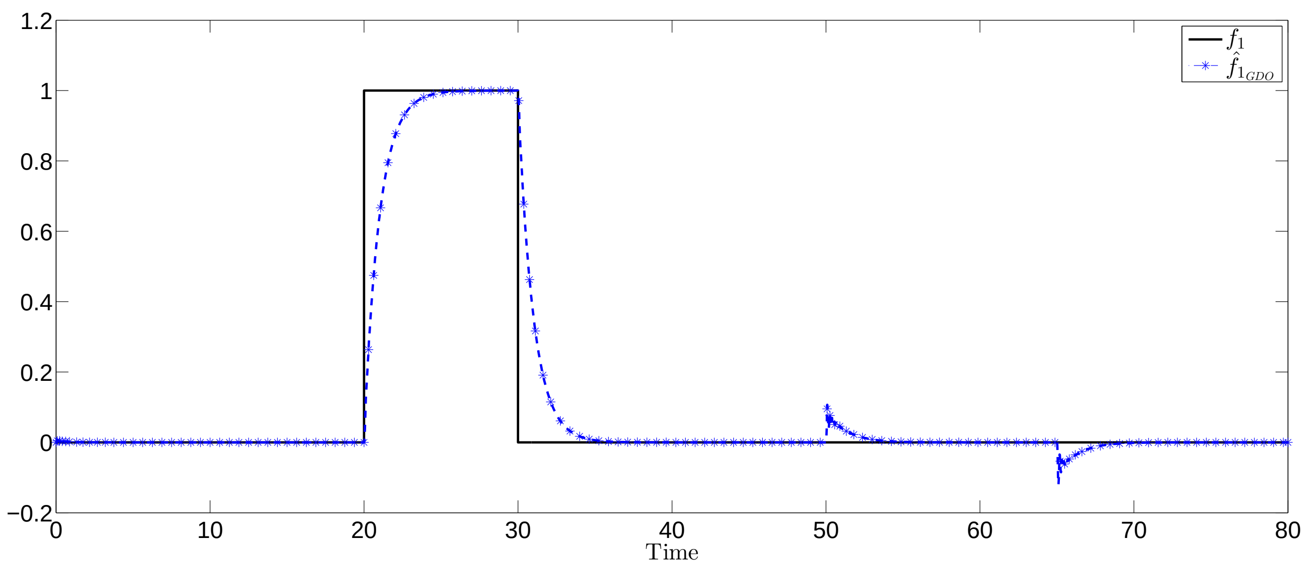

FIGURE 5.9 – Fault diagnosis : Estimation of $f_1(t)$.

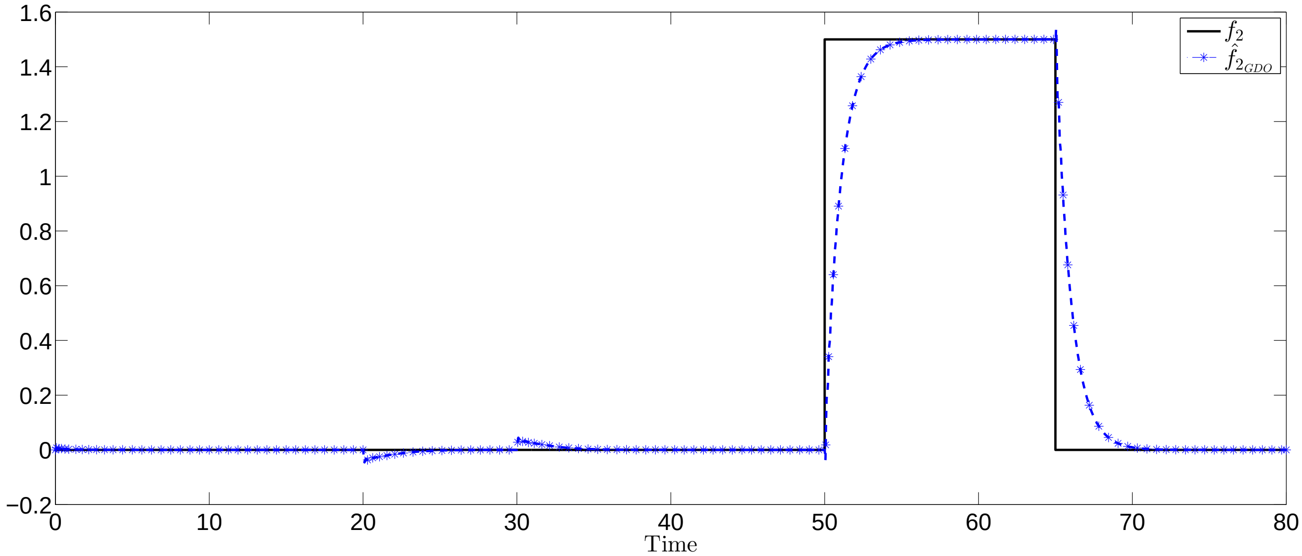

FIGURE 5.10 – Fault diagnosis : Estimation of $f_2(t)$.

From these results, we achieve detect, isolate and estimate faults in a descriptor system which complete the task of fault diagnosis.

5.3 Robust generalized dynamic observer design for uncertain descriptor systems with application to fault diagnosis

In this section the GDO is apply to fault diagnosis in uncertain descriptor systems. The fault diagnosis scheme is divided in two parts. The first part deals with fault detection and isolation, and the second on deals with simultaneous estimation of state and faults.

5.3.1 Class of uncertain descriptor systems considered

Consider the following uncertain descriptor system with actuator faults :

$$E\dot{x}(t) = (A + \Delta A(t))x(t) + Gf(t) \tag{5.69a}$$
$$y(t) = C_1 x(t) \tag{5.69b}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $f(t) \in \mathbb{R}^{n_f}$ is the fault vector and $y(t) \in \mathbb{R}^{n_y}$ represents the measured output vector. Matrices $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n_f}$ and $C_1 \in \mathbb{R}^{n_y\times n}$ are constant and known, and $\Delta A(t)$ is an unknown matrix representing time-varying parameter uncertainties, and is assumed to be of the form :

$$\Delta A(t) = \mathcal{M}\Gamma(t)\mathcal{G} \tag{5.70}$$

where $\mathcal{M}$ and $\mathcal{G}$ are known real constant matrices and $\Gamma(t)$ is an unknown time-varying matrix satisfying

$$\Gamma(t)^T\Gamma(t) \leq I, \ \forall t \in [0, \infty) \tag{5.71}$$

In the sequel we assume that

Assumption 5.4.

$$\mathsf{rank} \begin{bmatrix} E \\ E^{\perp}A \\ C_1 \end{bmatrix} = n.$$

Assumption 5.5.

$$\operatorname{rank}\begin{bmatrix} sE-A \\ C_1 \end{bmatrix} = n, \ \forall s \in \mathbb{C}^+, \ s \text{ finite.}$$

Assumption 5.6. *The uncertain descriptor system* (5.69) *with admissible uncertainties* $\Delta A(t)$ *is stable.*

5.3.2 Fault detection and isolation based on a robust generalized dynamic observer

In this section the GDO is apply to FDI.

5.3.2.1 Problem formulation

Consider the following GDO for FDI for system (5.69)

$$\mathcal{O}_1 := \begin{cases} \dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) & (5.72a) \\ \dot{v}(t) = S\zeta(t) + Lv(t) + My(t) & (5.72b) \\ \hat{x}(t) = P\zeta(t) + Qy(t) & (5.72c) \\ r(t) = W(C_1\hat{x}(t) - y(t)) & (5.72d) \end{cases}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector, $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ and $r(t) \in \mathbb{R}^{n_f}$ is the residual vector. Matrices N, F, H, S, L, M, P, Q and W are unknown matrices of appropriate dimensions.

Now, we can give the following lemma.

Lemma 5.3. *There exists an observer for fault isolation of the form* (5.72) *for the system* (5.69) *if the following two statements hold.*

1. *There exists a matrix T of appropriate dimension such that the following conditions are satisfied :*
   (a) $NTE + FC_1 - TA = 0$
   (b) $MC_1 + STE = 0$
   (c) $\begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} TE \\ C_1 \end{bmatrix} = I_n$
2. *The matrix* $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ *is Hurwitz when* $\Gamma(t) = 0$ *and* $f(t) = 0$.

Proof. Let $T \in \mathbb{R}^{q_0 \times n}$ be a parameter matrix and define the error $\varepsilon(t) = \zeta(t) - TEx(t)$, then its derivative is given by

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) + (NTE + FC_1 - TA)x(t) - T\Delta A(t)x(t) - TGf(t) \tag{5.73}$$

By using the definition of $\varepsilon(t)$, equations (5.72b) and (5.72c) can be written as :

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + (STE + MC_1)x(t) \tag{5.74}$$

$$\hat{x}(t) = P\varepsilon(t) + \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} TE \\ C_1 \end{bmatrix} x(t) \tag{5.75}$$

If conditions $(a) - (c)$ of Lemma 5.3 are satisfied the following observer error dynamics is obtained from (5.73) and (5.74)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix}\begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} -T\Delta A(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} -TG \\ 0 \end{bmatrix} f(t) \tag{5.76}$$

and from equation (5.75) we get :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon(t) \tag{5.77}$$

so that equation (5.72d) becomes :

$$r(t) = WC_1P\varepsilon(t) \tag{5.78}$$

in this case if $\Gamma(t) = 0$, $f(t) = 0$ and matrix $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz, then $\lim\limits_{t\to\infty} e(t) = 0$. ☐

Considering the parameterization of Lemma 2.5 but with $Z_1 = 0$ we get :

$$T = T_1 \tag{5.79}$$

$$K = K_1 \tag{5.80}$$

$$N = N_1 - Y_1N_3 \tag{5.81}$$

$$F = F_1 - Y_1 F_3 \tag{5.82}$$
$$S = -Y_2 N_3 \tag{5.83}$$
$$M = -Y_2 F_3 \tag{5.84}$$
$$P = P_1 \tag{5.85}$$
$$Q = Q_1 \tag{5.86}$$

where matrices T_1, K_1, N_1, N_3, F_1, F_3, P_1 and Q_1 are defined in Section 2.4.1 with matrices $\Sigma = \begin{bmatrix} E \\ C_1 \end{bmatrix}$ and $\Omega = \begin{bmatrix} R \\ C_1 \end{bmatrix}$.

So, the observer error system of equations (5.76) and (5.78) can be written as :

$$\begin{aligned} \dot{\varphi}(t) &= (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(t) + \mathbb{G}_1\Gamma(t)\mathcal{G}x(t) + \mathbb{F}_1 f(t) \\ r(t) &= W\mathbb{C}_1\varphi(t) \end{aligned} \tag{5.87}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{G}_1 = \begin{bmatrix} -T_1\mathcal{M} \\ 0 \end{bmatrix}$, $\mathbb{F}_1 = \begin{bmatrix} -T_1 G \\ 0 \end{bmatrix}$, $\mathbb{C}_1 = \begin{bmatrix} C_1 P_1 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$.

Taking Laplace transformation of (5.87), where

$$\begin{bmatrix} G_{rf}(s) & G_{rx}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2 & \mathbb{F}_1 & \mathbb{G}_1\Gamma(s)\mathcal{G} \\ \hline W\mathbb{C}_1 & 0 & 0 \end{array}\right] \tag{5.88}$$

The objective is to render $G_{rf}(s)$ diagonal, i.e.

$$G_{rf}(s) = \mathsf{diag}(g_1(s), \ldots, g_{n_f}(s)) \tag{5.89}$$

while the stability of the observer (5.72) is guaranteed.

5.3.2.2 Fault detection and isolation based on a robust observer design

Assumption 5.7. *The transfer function $G_{rf}(s)$ can be diagonalized if and only if $(\mathbb{C}_1\mathbb{F}_1)$ has full column rank, i.e. $n_y \geq n_f$.*

The following theorem shows the way to obtain the transfer function $G_{rf}(s)$ diagonal.

Theorem 5.4. *Consider that $n_y \geq n_f$ and let*

$$\Lambda = \mathsf{diag}(\lambda_1, \ldots, \lambda_{n_f}) \in \mathbb{R}^{n_f \times n_f},\ \lambda_i < 0, \tag{5.90}$$
$$\Psi = \mathsf{diag}(\psi_1, \ldots, \psi_{n_f}) \in \mathbb{R}^{n_f \times n_f}, |\psi_i| > 0. \tag{5.91}$$

$\forall i \in [1, \ldots, n_f]$ be given. Then there exist matrices $\mathbb{Y}$ and W such that :

$$(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\mathbb{F}_1 = \mathbb{F}_1\Lambda \tag{5.92}$$
$$W\mathbb{C}_1\mathbb{F}_1 = \Psi \tag{5.93}$$

If $(\mathbb{A}_2\mathbb{F}_1)$ has full column rank, the matrices $\mathbb{Y}$ and W are given by :

$$\mathbb{Y} = (\mathbb{A}_1\mathbb{F}_1 - \mathbb{F}_1\Lambda)(\mathbb{A}_2\mathbb{F}_1)^+ - \tilde{Z}(I - (\mathbb{A}_2\mathbb{F}_1)(\mathbb{A}_2\mathbb{F}_1)^+) \tag{5.94}$$
$$W = \Psi(\mathbb{C}_1\mathbb{F}_1)^+ \tag{5.95}$$

where $\widetilde{Z}$ is an arbitrary matrix of appropriate dimension. Finally, if there exists matrices $\mathbb{Y}$ and W satisfying (5.92) and (5.93), then

$$G_{rf}(s) = \left[\begin{array}{c|c} \Lambda & I \\ \hline \Psi & 0 \end{array}\right] \tag{5.96a}$$
$$= \mathsf{diag}\left(\frac{\psi_1}{s - \lambda_1}, \cdots, \frac{\psi_{n_f}}{s - \lambda_{n_f}}\right). \tag{5.96b}$$

The proof this lemma is the same as this of Theorem 5.1.

Now, replacing (5.94) and (5.95) in the observer error system (5.87), we obtain :

$$\dot{\varphi}(t) = \underbrace{[\mathbb{A}_1 - (\mathbb{A}_1\mathbb{F}_1 - \mathbb{F}_1\Lambda)(\mathbb{A}_2\mathbb{F}_1)^{+}\mathbb{A}_2}_{\bar{\mathbb{A}}_1} + \widetilde{Z}\underbrace{(I - (\mathbb{A}_2\mathbb{F}_1)(\mathbb{A}_2\mathbb{F}_1)^{+})\mathbb{A}_2]}_{\bar{\mathbb{A}}_2}\varphi(t) + \mathbb{G}_1\Gamma(t)\mathcal{G}x(t) + \mathbb{F}_1 f(t) \tag{5.97a}$$

$$r(t) = \underbrace{\Psi(\mathbb{C}_1\mathbb{F}_1)^{+}\mathbb{C}_1}_{\bar{\mathbb{C}}_1}\varphi(t) \tag{5.97b}$$

where

$$\bar{\mathbb{A}}_1 = \begin{bmatrix} N_1 + (T_1G\Lambda - N_1T_1G)(N_3T_1G)^{+}N_3 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.98}$$

$$\bar{\mathbb{A}}_2 = \begin{bmatrix} N_3 - N_3T_1G(N_3T_1G)^{+}N_3 & 0 \\ 0 & -I \end{bmatrix} \tag{5.99}$$

$$\bar{\mathbb{C}}_1 = -\Psi\begin{bmatrix}(C_1P_1T_1G)^{+}C_1P_1 & 0\end{bmatrix} \tag{5.100}$$

Bringing together (5.69a) and (5.97a) we get :

$$\mathscr{E}\dot{\beta}(t) = \mathscr{A}\beta + \mathscr{F}f(t) \tag{5.101}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 \\ 0 & I_{q_0+q_1} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A + \mathcal{M}\Gamma(t)\mathcal{G} & 0 \\ \mathbb{G}_1\Gamma(t)\mathcal{G} & \bar{\mathbb{A}}_1 + \widetilde{Z}\bar{\mathbb{A}}_2 \end{bmatrix}$, $\mathscr{F} = \begin{bmatrix} G \\ \mathbb{F}_1 \end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varphi(t) \end{bmatrix}$.

The following theorem gives the LMI conditions that allow the determination of the observer matrices such that the system (5.101) is stable.

Theorem 5.5. *Under Assumptions 5.4, 5.5 and 5.6 there exists a GDO (5.72) such that the system (5.101) is stable if there exists a matrix $X_2 = \begin{bmatrix} X_{21} & X_{21} \\ X_{21} & X_{22} \end{bmatrix} > 0$, with $X_{21} = X_{21}^T$, and a nonsingular matrix X_1 satisfying the following LMIs.*

$$E^TX_1 = X_1^TE \geq 0, \tag{5.102}$$

$$\mathcal{C}^{T\perp}\begin{bmatrix} A^TX_1 + X_1^TA + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & \Pi_1 & (*) \\ 0 & 0 & \begin{bmatrix}-\mathcal{M}^TT_1^TX_{21} & -\mathcal{M}^TT_1^TX_{21}\end{bmatrix} & -\epsilon I \end{bmatrix}\mathcal{C}^{T\perp T} < 0 \tag{5.103}$$

with

$$\Pi_1 = \begin{bmatrix} X_{21}N_1 + N_1^TX_{21} & (*) \\ X_{21} & 0 \end{bmatrix} \tag{5.104}$$

and

$$\begin{bmatrix} A^TX_1 + X_1^TA + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 \\ 0 & 0 & -\epsilon I \end{bmatrix} < 0. \tag{5.105}$$

with $\epsilon > 0$, and matrix $\widetilde{Z}$ is parameterized as follows :

$$\widetilde{Z} = X_2^{-1}(\mathcal{B}_r^{+}\mathcal{K}\mathcal{C}_l^{+} + \mathcal{Z} - \mathcal{B}_r^{+}\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^{+}) \tag{5.106}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \tag{5.107a}$$

$$\vartheta = (\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \tag{5.107b}$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\vartheta - \vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta]\mathcal{B}_l\mathcal{R}^{-1} \tag{5.107c}$$

where $\mathcal{D} = \begin{bmatrix} A^TX_1 + X_1A^T + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & \Pi_1 & (*) \\ 0 & 0 & \begin{bmatrix}-\mathcal{M}^TT_1^TX_{21} & -\mathcal{M}^TT_1^TX_{22}\end{bmatrix} & -\epsilon I \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}$,

$\mathcal{C} = \begin{bmatrix} 0 & 0 & \begin{bmatrix} N_3 - N_3T_1G(N_3T_1G)^+N_3 & 0 \\ 0 & -I \end{bmatrix} & 0 \end{bmatrix}$, *matrix* Π_1 *is defined in* (5.104) *and matrices* $\mathcal{L}$, $\mathcal{R}$ *and* $\mathcal{Z}$ *are arbitrary matrices of appropriate dimensions satisfying* $\mathcal{R} > 0$ *and* $\|\mathcal{L}\|_2 < 1$. *Matrices* $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ *and* $\mathcal{B}_r$ *are any full rank matrices such that* $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ *and* $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$.

Proof. Consider a matrix $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$ such that

$$\mathscr{E}^T X = X^T \mathscr{E} \geq 0 \tag{5.108}$$

and

$$\mathscr{A}^T X + X^T \mathscr{A} < 0 \tag{5.109}$$

By replacing $\mathscr{E}$ and X in inequality (5.108) we get :

$$E^T X_1 = X_1^T E \geq 0 \tag{5.110}$$

and $X_2 = X_2^T > 0$.

Now, replacing $\mathscr{A}$ and X in inequality (5.109) we obtain :

$$\begin{bmatrix} A^T X_1 + X_1^T A + X_1^T \mathcal{M}\Gamma(t)\mathcal{G} + \mathcal{G}^T\Gamma(t)^T\mathcal{G}^T X_1 & \mathbb{G}_1^T\Gamma(t)^T\mathcal{G}X_2 \\ (*) & X_2(\bar{\mathbb{A}}_1 + \widetilde{Z}\bar{\mathbb{A}}_2) + (\bar{\mathbb{A}}_1 + \widetilde{Z}\bar{\mathbb{A}}_2)^T X_2 \end{bmatrix} < 0 \tag{5.111}$$

which can be written as :

$$\begin{gathered} \begin{bmatrix} A^T X_1 + X_1^T A & 0 \\ 0 & X_2(\bar{\mathbb{A}}_2 + \widetilde{Z}\bar{\mathbb{A}}_2) + (\bar{\mathbb{A}}_2 + \widetilde{Z}\bar{\mathbb{A}}_2)^T X_2 \end{bmatrix} + \\ \left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)\left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)^T + \left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)\left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)^T < 0 \end{gathered} \tag{5.112}$$

Using Lemma 1.6 from Section 1.7.3, and since $\Gamma(t)^T\Gamma(t) \leq I$ the following inequalities can be formulated :

$$\left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)\left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)^T \leq \begin{bmatrix} \epsilon^{-1}X_1^T\mathcal{M}\mathcal{M}^T X_1 + \epsilon\mathcal{G}^T\mathcal{G} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.113}$$

$$\left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)\left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0\end{bmatrix}\right)^T \leq \begin{bmatrix} \epsilon\mathcal{G}^T\mathcal{G} & 0 \\ 0 & \epsilon^{-1}X_2\mathbb{G}_1\mathbb{G}_1^T X_2 \end{bmatrix} \tag{5.114}$$

with $\epsilon > 0$. So that, inequality (5.112) becomes :

$$\begin{bmatrix} A^T X_1 + X_1^T A + 2\epsilon\mathcal{G}^T\mathcal{G} + \epsilon^{-1}X_1^T\mathcal{M}\mathcal{M}^T X_1 & 0 \\ 0 & X_2(\bar{\mathbb{A}}_1 + \widetilde{Z}\bar{\mathbb{A}}_2) + (\bar{\mathbb{A}}_1 + \widetilde{Z}\bar{\mathbb{A}}_2)^T X_2 + \epsilon^{-1}X_2\mathbb{G}_1\mathbb{G}_1^T X_2 \end{bmatrix} < 0 \tag{5.115}$$

By applying the Schur complement to inequality (5.115) gives

$$\begin{bmatrix} A^T X_1 + X_1^T A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & X_2(\bar{\mathbb{A}}_2 + \widetilde{Z}\bar{\mathbb{A}}_2) + (\bar{\mathbb{A}}_2 + \widetilde{Z}\bar{\mathbb{A}}_2)^T X_2 & (*) \\ 0 & 0 & \mathbb{G}_1^T X_2 & -\epsilon I \end{bmatrix} < 0 \tag{5.116}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \tag{5.117}$$

where $\mathcal{X} = X_2\widetilde{Z}$, $\mathcal{D} = \begin{bmatrix} A^T X_1 + X_1^T A + 2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & X_2\bar{\mathbb{A}}_1 + \bar{\mathbb{A}}_1^T X_2 & (*) \\ 0 & 0 & \mathbb{G}_1^T X_2 & -\epsilon I \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} 0 & 0 & \bar{\mathbb{A}}_2 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}$.

Using the elimination lemma, inequality (5.117) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{5.118a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{5.118b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \bar{\mathbb{A}}_2^{T\perp} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (5.118a) becomes :

$$\mathcal{C}^{T\perp} \begin{bmatrix} A^T X_1 + X_1^T A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}^T X_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & \Pi_1 & (*) \\ 0 & 0 & \begin{bmatrix} -\mathcal{M}^T T_1^T X_{21} & -\mathcal{M}^T T_1^T X_{22} \end{bmatrix} & -\epsilon I \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{5.119}$$

and by using $\mathcal{B}$ and $\mathcal{D}$ the inequality (5.118b) becomes :

$$\begin{bmatrix} A^T X_1 + X_1^T A + 2\epsilon \mathcal{G}^T \mathcal{G} & (*) & 0 \\ \mathcal{M}^T X_1 & -\epsilon I & 0 \\ 0 & 0 & -\epsilon I \end{bmatrix} < 0. \tag{5.120}$$

matrix Π_1 is defined in (5.104). From the elimination lemma if conditions (5.118a) and (5.118b) are satisfied, then matrix $\widetilde{Z}$ is parameterized as in (5.106) and (5.107). □

5.3.3 Simultaneous estimation of the state and faults based on a robust generalized dynamic observer

In this section the GDO is apply to simultaneous estimation of state and fault.

5.3.3.1 Problem formulation

Consider the following GDO for simultaneous estimation of state and fault for system (5.69)

$$\mathcal{O}_2 := \begin{cases} \dot{\zeta}(t) = N(\zeta(t) + TG\hat{f}(t)) + Hv(t) + Fy(t) + TG\dot{\hat{f}}(t) & (5.121a) \\ \dot{v}(t) = S(\zeta(t) + TG\hat{f}(t)) + Lv(t) + My(t) & (5.121b) \\ \hat{x}(t) = P(\zeta(t) + TG\hat{f}(t)) + Qy(t) & (5.121c) \\ \dot{\hat{f}}(t) = \Phi(C_1\hat{x}(t) - y(t)) & (5.121d) \end{cases}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector, $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ and $\hat{f}(t)$ is the estimate of $f(t)$. Matrices N, F, H, L, M, S, P, Q, T and Φ are unknown matrices of appropriate dimensions.

Now, we can give the following lemma.

Lemma 5.4. *There exists an observer of the form* (5.121) *for the system* (5.69) *if the following two statements hold.*

1. *There exists a matrix T of appropriate dimension such that the following conditions are satisfied :*
   (a) $NTE + FC_1 - TA = 0$
   (b) $MC_1 + STE = 0$
   (c) $\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ C_1 \end{bmatrix} = I_n$
2. *The matrix* $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ *is Hurwitz when* $\Gamma(t) = 0$ *and* $f(t) = 0$.

Proof. Let $T \in \mathbb{R}^{q \times n}$ be a parameter matrix and define

$$\varepsilon(t) = \zeta(t) - TEx(t) + TGf(t) \tag{5.122}$$

then its derivative is given by :

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Hv(t) + (NTG + TG)\widetilde{f}(t) + (NTE + FC_1 - TA)\,x(t) - T\Delta A(t)x(t) \tag{5.123}$$

where $\dot{f}(t) = 0$ and $\widetilde{f}(t) = \hat{f}(t) - f(t)$.

By using the definition of $\varepsilon(t)$, equations (5.121b) and (5.121c) can be written as :

$$\dot{v}(t) = S\varepsilon(t) + Lv(t) + STG\widetilde{f}(t) + (MC_1 + STE)x(t) \tag{5.124}$$

$$\hat{x}(t) = P\varepsilon(t) + PTG\widetilde{f}(t) + \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} TE \\ C_1 \end{bmatrix} x(t) \tag{5.125}$$

since $\dot{f}(t) = 0$ the derivative of $\widetilde{f}(t)$ is given by :

$$\dot{\widetilde{f}}(t) = \Phi C_1 P\varepsilon(t) + \Phi C_1 PTG\widetilde{f}(t) \tag{5.126}$$

Now, if conditions $(a) - (c)$ of Lemma 5.4 are satisfied the following observer error dynamics is obtained from (5.123) and (5.124)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} -T\Delta A(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} NTG + TG \\ STG \end{bmatrix} \widetilde{f}(t) \tag{5.127}$$

and from (5.125) we get :

$$\hat{x}(t) - x(t) = e(t) = P\varepsilon(t) + PTG\widetilde{f}(t) \tag{5.128}$$

in this case if $\Gamma(t) = 0$ and $f(t) = 0$ and $\begin{bmatrix} N & H \\ S & L \end{bmatrix}$ is Hurwitz, them $\lim_{t\to\infty} e(t) = 0$. □

By considering the parameterization of equations (5.79) - (5.86) the observer error dynamics (5.127) can be written as :

$$\dot{\varphi}(t) = (\mathbb{A}_1 - \mathbb{YA})\varphi(t) + \mathbb{G}_1\Gamma(t)\mathcal{G}x(t) + (\mathbb{F}_1 - \mathbb{YF}_2)\widetilde{f}(t) \tag{5.129}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{G}_1 = \begin{bmatrix} -T_1\mathcal{M} \\ 0 \end{bmatrix}$, $\mathbb{F}_1 = \begin{bmatrix} N_1T_1G + T_1G \\ 0 \end{bmatrix}$, $\mathbb{F}_2 = \begin{bmatrix} N_3T_1G \\ 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & H \\ Y_2 & L \end{bmatrix}$ and $\varphi(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$. By putting together equation (5.69a), the observer error dynamics (5.129) and equation (5.126) we get :

$$\mathscr{E}\dot{\beta}(t) = \mathscr{A}\beta(t) + \mathscr{B}f(t) \tag{5.130}$$

where $\mathscr{E} = \begin{bmatrix} E & 0 & 0 \\ 0 & I_{q_1+q_0} & 0 \\ 0 & 0 & I_{n_f} \end{bmatrix}$, $\mathscr{A} = \begin{bmatrix} A + \mathcal{M}\Gamma(t)\mathcal{G} & 0 & 0 \\ \mathbb{G}_1\Gamma(t)\mathcal{G} & \mathbb{A}_1 - \mathbb{YA}_2 & \mathbb{F}_1 - \mathbb{YF}_2 \\ 0 & \mathbb{C}_1 & \mathbb{D}_1 \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} G \\ 0 \\ 0 \end{bmatrix}$ and $\beta(t) = \begin{bmatrix} x(t) \\ \varphi(t) \\ \widetilde{f}(t) \end{bmatrix}$, with $\mathbb{C}_1 = \begin{bmatrix} \Phi C_1 P_1 & 0 \end{bmatrix}$ and $\mathbb{D}_1 = \Phi C_1 P_1 T_1 G$.

The observer design is obtained from the determination of matrices $\mathbb{Y}$ and Φ such that the system (5.130) is stable.

5.3.3.2 Simultaneous estimation of state and fault based on robust observer design

The following theorem gives the stability conditions for system (5.130) in the form of LMIs such that we can get the simultaneous estimation of state and faults.

Theorem 5.6. *Under Assumptions 5.4, 5.5 and 5.6 there exist parameter matrices $\mathbb{Y}$ and Φ such that system (5.130) is stable if and only if there exist matrices X_1, $X_2 = \begin{bmatrix} X_{21} & X_{22} \\ X_{22}^T & X_{23} \end{bmatrix} > 0$, with $X_{21} = X_{21}^T$, and $X_3 = X_3^T$ satisfying the following LMIs.*

$$E^T X_1 = X_1^T E \geq 0, \tag{5.131}$$

$$\mathcal{C}^{T\perp}\begin{bmatrix} A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \begin{bmatrix} N_1^TX_{21}+X_{21}N_1 & (*) \\ X_{22}^TN_1 & 0 \end{bmatrix} & (*) & (*) \\ 0 & 0 & \begin{bmatrix} -\mathcal{M}^TT_1^TX_{21} & -\mathcal{M}^TT_1^TX_{22} \end{bmatrix} & -\epsilon I & 0 \\ 0 & 0 & \Pi_1 & 0 & \Pi_2 \end{bmatrix}\mathcal{C}^{T\perp T} < 0 \tag{5.132}$$

where

$$\Pi_1 = \begin{bmatrix} X_3\Phi C_1P_1+(N_1T_1G+T_1G)^TX_{21} & (N_1T_1G+T_1G)^TX_{22} \end{bmatrix} \tag{5.133a}$$

$$\Pi_2 = X_3\Phi C_1P_1T_1G+G^TT_1^TP_1^TC_1^T\Phi^TX_3 \tag{5.133b}$$

and

$$\begin{bmatrix} A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & -\epsilon I & 0 \\ 0 & 0 & 0 & \Pi_2 \end{bmatrix} < 0 \tag{5.134}$$

with $\epsilon > 0$, and matrix $\mathbb{Y}$ is parameterized as follows :

$$\mathbb{Y} = X_2^{-1}(\mathcal{B}_r^+\mathcal{K}\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+) \tag{5.135}$$

where

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}+\mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \tag{5.136a}$$

$$\vartheta = (\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T-\mathcal{D})^{-1} > 0 \tag{5.136b}$$

$$\mathcal{S} = \mathcal{R}^{-1}-\mathcal{R}^{-1}\mathcal{B}_l^T[\vartheta-\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta]\mathcal{B}_l\mathcal{R}^{-1} \tag{5.136c}$$

where $\mathcal{D} = \begin{bmatrix} A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \begin{bmatrix} N_1^TX_{21}+X_{21}N_1 & (*) \\ X_{22}^TN_1 & 0 \end{bmatrix} & (*) & (*) \\ 0 & 0 & \begin{bmatrix} -\mathcal{M}^TT_1^TX_{21} & -\mathcal{M}^TT_1^TX_{22} \end{bmatrix} & -\epsilon I & 0 \\ 0 & 0 & \Pi_1 & 0 & \Pi_2 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \\ 0 \end{bmatrix}$,
$\mathcal{C} = \begin{bmatrix} 0 & 0 & \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} & 0 & \begin{bmatrix} N_3T_1G \\ 0 \end{bmatrix} \end{bmatrix}$, *matrices Π_1 and Π_2 are defined in (5.133) and matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\|_2 < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$ and $\mathcal{B}_r$ are any full rank matrices such that $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$.*

Proof. Consider a matrix $X = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{bmatrix}$ such that

$$\mathscr{E}^TX = X^T\mathscr{E} \geq 0 \tag{5.137}$$

and

$$\mathscr{A}^TX + X^T\mathscr{A} < 0 \tag{5.138}$$

Replacing $\mathscr{E}$ and X in inequality (5.137) we obtain :

$$E^TX_1 = X_1^TE \geq 0 \tag{5.139}$$

and $X_2 = X_2^T > 0$ and $X_3 = X_3^T > 0$.

Now, replacing $\mathscr{A}$ and X in inequality (5.138) we get :

$$\begin{bmatrix} (A+\mathcal{M}\Gamma(t)\mathcal{G})^TX_1+X_1^T(A+\mathcal{M}\Gamma(t)\mathcal{G}) & (\mathbb{G}_1\Gamma(t)\mathcal{G})^TX_2 & 0 \\ (*) & (\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2+X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2) & \mathbb{C}_1^TX_3+X_2(\mathbb{F}_1-\mathbb{Y}\mathbb{F}_2) \\ 0 & (*) & X_3\mathbb{D}_1+\mathbb{D}_1^TX_3 \end{bmatrix} < 0 \tag{5.140}$$

which can be written as :

$$\begin{bmatrix} A^TX_1+X_1^TA & 0 & 0 \\ 0 & (\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2+X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2) & \mathbb{C}_1^TX_3+X_2(\mathbb{F}_1-\mathbb{Y}\mathbb{F}_2) \\ 0 & (*) & X_3\mathbb{D}_1+\mathbb{D}_1^TX_3 \end{bmatrix}+ \left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)^T + \left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)^T < 0 \tag{5.141}$$

Using Lemma 1.6 from Section 1.7.3, and since $\Gamma(t)^T\Gamma(t) \leq I$ the following inequalities can be formulated :

$$\left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix} X_1^T\mathcal{M} \\ 0 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)^T \leq \begin{bmatrix} \epsilon^{-1}X_1^T\mathcal{M}\mathcal{M}X_1+\epsilon\mathcal{G}^T\mathcal{G} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.142}$$

$$\left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix} 0 \\ X_2\mathbb{G}_1 \\ 0 \end{bmatrix}\Gamma(t)\begin{bmatrix}\mathcal{G} & 0 & 0\end{bmatrix}\right)^T \leq \begin{bmatrix} \epsilon\mathcal{G}^T\mathcal{G} & 0 & 0 \\ 0 & \epsilon^{-1}X_2\mathbb{G}_1\mathbb{G}_1^TX_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.143}$$

with $\epsilon > 0$. So that inequality (5.141) becomes :

$$\begin{bmatrix} \Pi_a & 0 & 0 \\ 0 & (\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2+X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)+\epsilon X_2\mathbb{G}_1\mathbb{G}_1^TX_2 & \mathbb{C}_1^TX_3+X_2(\mathbb{F}_1-\mathbb{Y}\mathbb{F}_2) \\ 0 & (*) & X_3\mathbb{D}_1+\mathbb{D}_1^TX_3 \end{bmatrix} < 0 \tag{5.144}$$

where

$$\Pi_a = A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G}+\epsilon^{-1}X_1^T\mathcal{M}\mathcal{M}^TX_1 \tag{5.145}$$

By applying the Schur complement to inequality (5.144) gives

$$\begin{bmatrix} A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & 0 \\ \mathcal{M}X_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & (\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX_2+X_2(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2) & (*) & (*) \\ 0 & 0 & \mathbb{G}_1^TX_2 & -\epsilon I & 0 \\ 0 & 0 & X_3\mathbb{C}_1+(\mathbb{F}_1-\mathbb{Y}\mathbb{F}_2)^TX_2 & 0 & X_3\mathbb{D}_1+\mathbb{D}_1^TX_3 \end{bmatrix} < 0 \tag{5.146}$$

which can be written as :

$$\mathcal{B}\mathcal{X}\mathcal{C}+(\mathcal{B}\mathcal{X}\mathcal{C})^T\mathcal{D} < 0 \tag{5.147}$$

where $\mathcal{X} = X_2\mathbb{Y}$, $\mathcal{D} = \begin{bmatrix} A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & 0 \\ \mathcal{M}X_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \mathbb{A}_1^TX_2+X_2\mathbb{A}_1 & (*) & (*) \\ 0 & 0 & \mathbb{G}_1^TX_2 & -\epsilon I & 0 \\ 0 & 0 & X_3\mathbb{C}_1+\mathbb{F}_1^TX_2 & 0 & X_3\mathbb{D}_1+\mathbb{D}_1^TX_3 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \\ 0 \end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix} 0 & 0 & \mathbb{A}_2 & 0 & \mathbb{F}_2 \end{bmatrix}$.

Using the elimination lemma of Section 1.5, inequality (5.147) is equivalent to :

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \tag{5.148a}$$

$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \tag{5.148b}$$

with $\mathcal{C}^{T\perp} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & \mathbb{A}_2^{T\perp} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & \mathbb{F}_2^{T\perp} \end{bmatrix}$ and $\mathcal{B}^{\perp} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$.

By using the definition of matrices $\mathcal{C}$, $\mathcal{D}$ and X_2 the inequality (5.148a) becomes :

$$\mathcal{C}^{T\perp}\begin{bmatrix} A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 & 0 & 0 \\ 0 & 0 & \begin{bmatrix} N_1^TX_{21}+X_{21}N_1 & (*) \\ X_{22}^TN_1 & 0 \end{bmatrix} & (*) & (*) \\ 0 & 0 & \begin{bmatrix} -\mathcal{M}^TT_1^TX_{21} & -\mathcal{M}^TT_1^TX_{22} \end{bmatrix} & -\epsilon I & 0 \\ 0 & 0 & \Pi_1 & 0 & \Pi_2 \end{bmatrix}\mathcal{C}^{T\perp T}<0 \tag{5.149}$$

and by using $\mathcal{B}$ and $\mathcal{D}$ the inequality (5.148b) becomes :

$$\begin{bmatrix} A^TX_1+X_1^TA+2\epsilon\mathcal{G}^T\mathcal{G} & (*) & 0 & 0 \\ \mathcal{M}^TX_1 & -\epsilon I & 0 & 0 \\ 0 & 0 & -\epsilon I & 0 \\ 0 & 0 & 0 & \Pi_2 \end{bmatrix}<0 \tag{5.150}$$

matrices Π_1 and Π_2 are defined in (5.133). From the elimination lemma if conditions (5.148a) and (5.148b) are satisfied, the parameter matrix $\mathbb{Y}$ is parameterized as in (5.135) and (5.136). □

5.3.4 Numerical example

In order to illustrate the results obtained, consider the following uncertain descriptor system described by (5.69) where

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A=\begin{bmatrix} -1.7 & 0 & 0.3 \\ -2 & -3 & 1 \\ -1 & 1.74 & -1 \end{bmatrix},\ G=\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},\ C_1=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\mathcal{M}=\begin{bmatrix} 0.3 \\ 0.7 \\ 0.4 \end{bmatrix},\ \mathcal{G}=\begin{bmatrix} 0.6 & 0.9 & 0.3 \end{bmatrix} \text{ and } x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Considering $E^{\perp}=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ such that $E^{\perp}\begin{bmatrix} E & G \end{bmatrix}=0$, we can verify Assumptions 5.1 and 5.2

$$\mathsf{rank}\begin{bmatrix} E \\ \hline E^{\perp}A \\ C_1 \end{bmatrix}=3 \text{ and } \mathsf{rank}\begin{bmatrix} sE-A \\ C_1 \end{bmatrix}=3$$

Robust observer for FDI computation.

For the robust GDO we have consider matrix $R=\begin{bmatrix} 7 & 2 & 9 \\ 6 & 3 & 9 \\ 6 & 2 & 8 \end{bmatrix}$ such that $\mathsf{rank}(\Sigma)=3$.

From Theorem 5.4 and since $n_y \geq n_f$, we have consider matrices $\Lambda=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\Psi=\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ to get matrix W as :

$$W=\begin{bmatrix} 1.72 & 3.44 & 10.64 \\ 2.8 & 5.61 & -21.45 \end{bmatrix}$$

and by using YALMIP toolbox we solve LMIs (5.102) - (5.105) to find matrices X_1 and X_2

$$X_1=\begin{bmatrix} 8.47 & -4.99 & 0 \\ -4.99 & 4.87 & 0 \\ 7.97 & -14.47 & 5.88 \end{bmatrix} \text{ and } X_2=\begin{bmatrix} 250.69 & 254.36 & -269.45 & 250.69 & 254.36 & -269.45 \\ 254.36 & 384.1 & -194.9 & 254.36 & 384.1 & -194.9 \\ -269.45 & -194.9 & 704.88 & -269.45 & -194.9 & 704.88 \\ 250.69 & 254.36 & -269.45 & 1114.61 & 246.52 & -322.3 \\ 254.36 & 384.1 & -194.9 & 246.52 & 1274.12 & -209.17 \\ -269.45 & -194.9 & 704.88 & -322.3 & -209.17 & 1489.88 \end{bmatrix}$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 9 & 6 & 3 & 9 \\ 3 & 2 & 6 & 3 & 5 & 4 & 6 & 3 & 9 \\ 6 & 1 & 4 & 6 & 2 & 6 & 6 & 1 & 0 \\ 9 & 5 & 7 & 4 & 3 & 2 & 8 & 1 & 6 \\ 2 & 3 & 8 & 1 & 8 & 4 & 7 & 3 & 9 \\ 2 & 7 & 1 & 9 & 3 & 6 & 2 & 8 & 2 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_6 \times 0.001$, and solving (5.106) and (5.107) we get :

$$\widetilde{Z} = \begin{bmatrix} -9.14 & -3.02 & 13.72 & 0.07 & 0.14 & 0.17 & -1.35 & -0.07 & -4.71 \\ 5.74 & 1.9 & -8.59 & -0.02 & -0.06 & -0.07 & -0.51 & -1.99 & 2.48 \\ -0.68 & -0.23 & 1.07 & 0.03 & 0.04 & 0.05 & -0.41 & -0.45 & -2.59 \\ 0.55 & 0.18 & -0.8 & 0 & -0.01 & -0.01 & 1.13 & 0.01 & 0.18 \\ -0.14 & -0.04 & 0.2 & 0 & 0 & 0 & 0.08 & 1.26 & -0.1 \\ -0.44 & -0.14 & 0.65 & 0 & 0 & 0 & 0.07 & 0.06 & 1.33 \end{bmatrix}$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -9.84 & -2.95 & 13.27 \\ 5.83 & 0.94 & -8.74 \\ -1.15 & -0.38 & 0.72 \end{bmatrix},\ H = \begin{bmatrix} 1.35 & 0.07 & 4.71 \\ 0.51 & 1.99 & -2.48 \\ 0.41 & 0.45 & 2.59 \end{bmatrix},\ F = \begin{bmatrix} 2.5 & -5 & 3.5 \\ 1.61 & -3.9 & 2.89 \\ 2.02 & -4.2 & 2.97 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.54 & 0.18 & -0.81 \\ -0.14 & -0.04 & 0.2 \\ -0.43 & -0.14 & 0.65 \end{bmatrix},\ L = \begin{bmatrix} -1.13 & -0.01 & -0.18 \\ -0.08 & -1.26 & 0.1 \\ -0.07 & -0.06 & -1.33 \end{bmatrix},\ M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.2 & -0.21 & 0.08 \\ -0.22 & 0.27 & -0.09 \\ -0.04 & 0.12 & 0 \end{bmatrix},\ Q = \begin{bmatrix} 0.51 & 0.18 & -0.18 \\ 0.36 & -0.37 & 0.37 \\ -0.25 & 0.62 & 0.38 \end{bmatrix} \text{ and } W = \begin{bmatrix} 1.72 & 3.44 & 10.64 \\ 2.8 & 5.61 & -21.45 \end{bmatrix}$$

Robust observer for FE computation.

For the GDO apply to FE we have chosen the same matrix R as the observer for FDI.

From Theorem 5.6, since X_3 is linked to Φ we have chosen $X_3 = I_2$, and by using YALMIP toolbox we have solved LMIs (5.131) -(5.134) to find matrices X_1, X_2 and Φ

$$X_1 = \begin{bmatrix} 1043.34 & -612.29 & 0 \\ -612.29 & 603.56 & 0 \\ -222.43 & 414.22 & 50.21 \end{bmatrix},\ X_2 = \begin{bmatrix} 514.17 & -419.33 & -181.43 & 0 & 0 & 0 \\ -419.33 & 763.11 & -253.65 & 0 & 0 & 0 \\ -181.43 & -253.65 & 469.22 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$\Phi = \begin{bmatrix} -422.45 & -1216.35 & -3321.22 \\ 117.45 & -531.18 & 1966.42 \end{bmatrix}$$

Now, considering matrices $\mathcal{Z} = \begin{bmatrix} 2 & 3 & 1 & 4 & 9 & 9 & 6 & 3 & 9 \\ 3 & 2 & 6 & 3 & 5 & 4 & 6 & 3 & 9 \\ 6 & 1 & 4 & 6 & 2 & 6 & 6 & 1 & 0 \\ 9 & 5 & 7 & 4 & 3 & 2 & 8 & 1 & 6 \\ 2 & 3 & 8 & 1 & 8 & 4 & 7 & 3 & 9 \\ 2 & 7 & 1 & 9 & 3 & 6 & 2 & 8 & 2 \end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ and $\mathcal{R} = I_6 \times 0.0001$, and solving (5.135) and (5.136) we get :

$$\mathbb{Y} = \begin{bmatrix} 6.93 & -0.02 & -5.05 & -7.91 & -16.32 & 9.27 & -5.85 & -5.85 & -5.85 \\ -3.34 & 9.65 & -2.96 & -8.16 & -16.74 & -3.93 & -5.01 & -5.01 & -5.01 \\ -6.44 & -1.29 & 11.46 & -7.18 & -14.8 & 5.35 & -5.19 & -5.19 & -5.19 \\ -62.45 & 99.52 & 91.59 & -570.48 & -1145.96 & 266.65 & -11000 & -1000 & -1000 \\ -66.86 & 98.8 & 88.49 & -574.67 & -1143.33 & 269.03 & -1000 & -11000 & -1000 \\ -69.6 & 100.35 & 87.01 & -565.91 & -1146.82 & 272.06 & -1000 & -1000 & -11000 \end{bmatrix}$$

Finally, we compute all the matrices of the observer as :

$$N = \begin{bmatrix} -9.1 & 3.78 & 4.44 \\ 1.61 & -6.59 & 2.49 \\ 4.61 & 4.48 & -11.97 \end{bmatrix},\ H = \begin{bmatrix} -5.85 & -5.85 & -5.85 \\ -5.01 & -5.01 & -5.01 \\ -5.19 & -5.19 & -5.19 \end{bmatrix},\ F = \begin{bmatrix} 8.53 & 5.72 & -7.22 \\ 6.08 & 2.12 & -3.13 \\ 7.11 & 4.39 & -5.62 \end{bmatrix},$$

$$T = \begin{bmatrix} 5 & -3 & 0 \\ 4.12 & -2.25 & 0 \\ 4.25 & -2.5 & 0 \end{bmatrix},\ S = \begin{bmatrix} 70.09 & -92.33 & -84.9 \\ 70.09 & -92.33 & -84.9 \\ 70.09 & -92.33 & -84.9 \end{bmatrix},\ L = \begin{bmatrix} -11000 & -1000 & -1000 \\ -1000 & -11000 & -1000 \\ -1000 & -1000 & -11000 \end{bmatrix},$$

$$M = \begin{bmatrix} 391.25 & 600.98 & -600.98 \\ 391.25 & 600.98 & -600.98 \\ 391.25 & 600.98 & -600.98 \end{bmatrix},\ P = \begin{bmatrix} 0.2 & -0.21 & 0.08 \\ -0.22 & 0.27 & -0.09 \\ -0.04 & 0.12 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.51 & 0.18 & -0.18 \\ 0.36 & -0.37 & 0.37 \\ -0.25 & 0.62 & 0.38 \end{bmatrix} \text{ and } \Phi = \begin{bmatrix} -422.45 & -1216.35 & -3321.22 \\ 117.45 & -531.18 & 1966.42 \end{bmatrix}$$

Simulation results

The results of simulation are depicted in figures 5.11 - 5.20. Figure 5.11 shows the parameter variation $\Gamma(t)$. Figures 5.12 and 5.13 show the faults in the system, as can we see these faults occur simultaneously in an interval of time. Figures 5.14 and 5.15 give the residuals where each fault can be easily distinguished from the other.

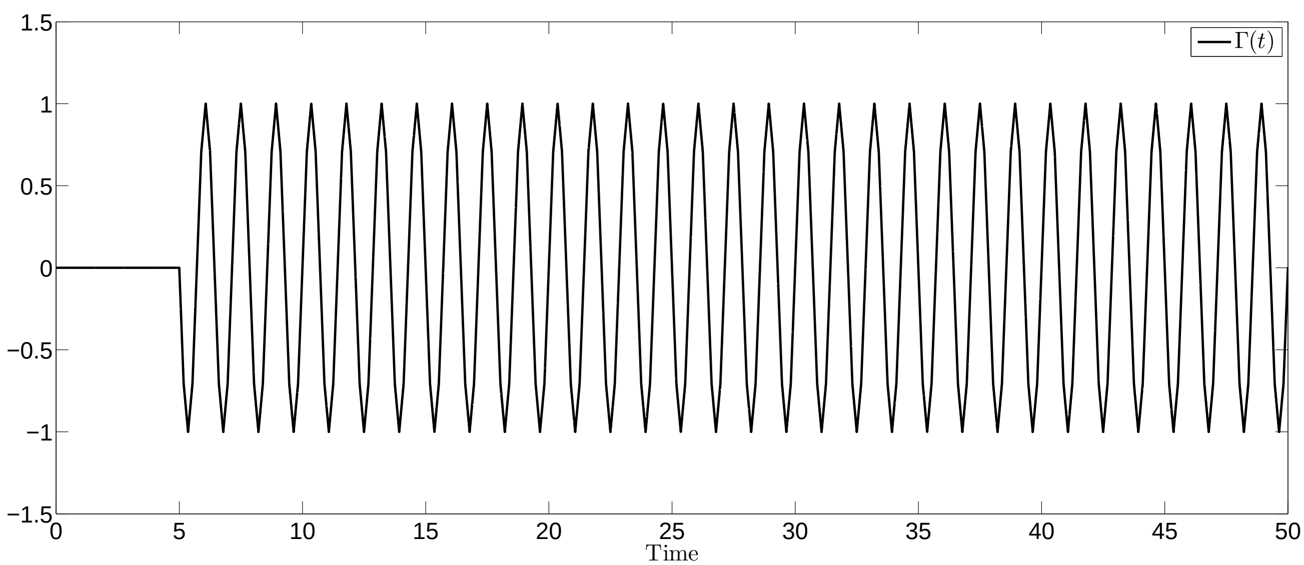

FIGURE 5.11 – Robust fault diagnosis : Variation $\Gamma(t)$.

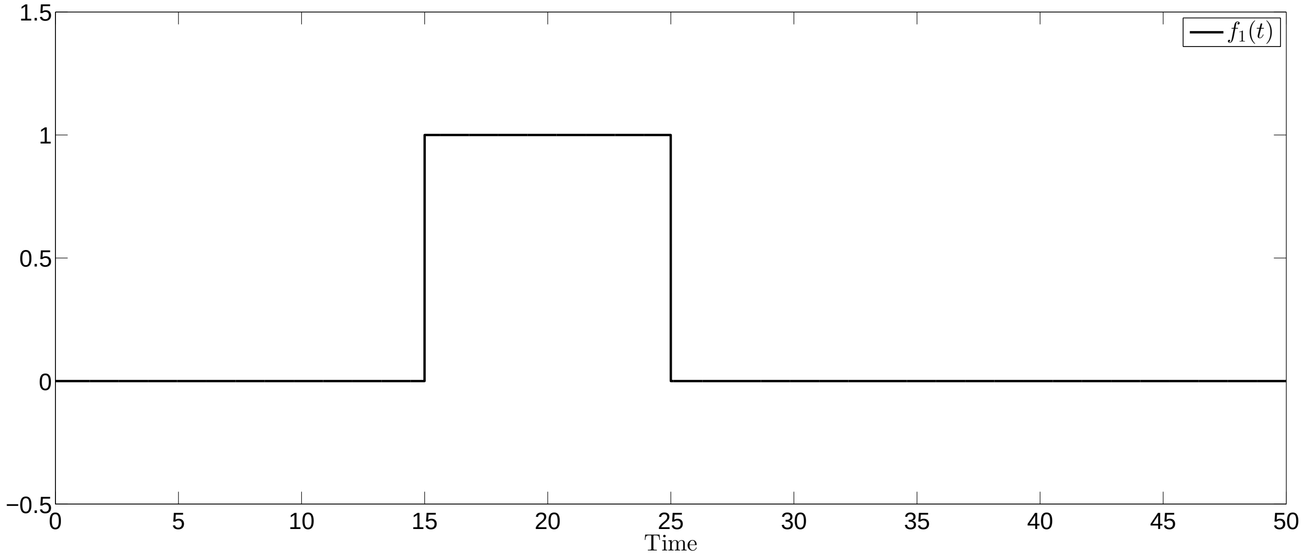

FIGURE 5.12 – Robust fault diagnosis : Actuator fault $f_1(t)$.

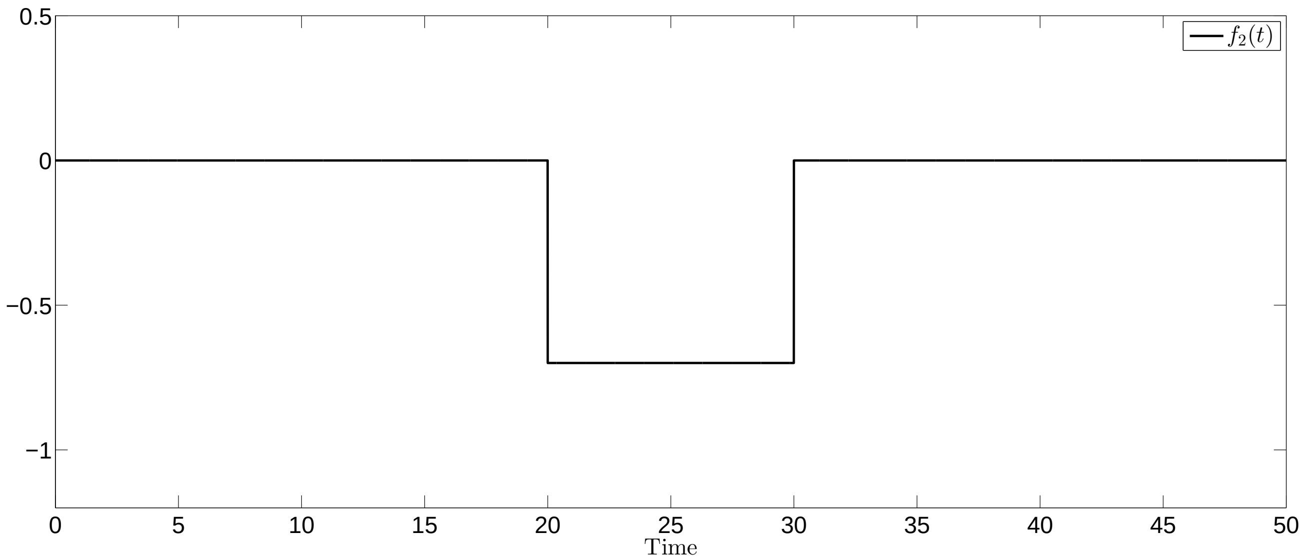

FIGURE 5.13 – Robust fault diagnosis : Actuator fault $f_2(t)$.

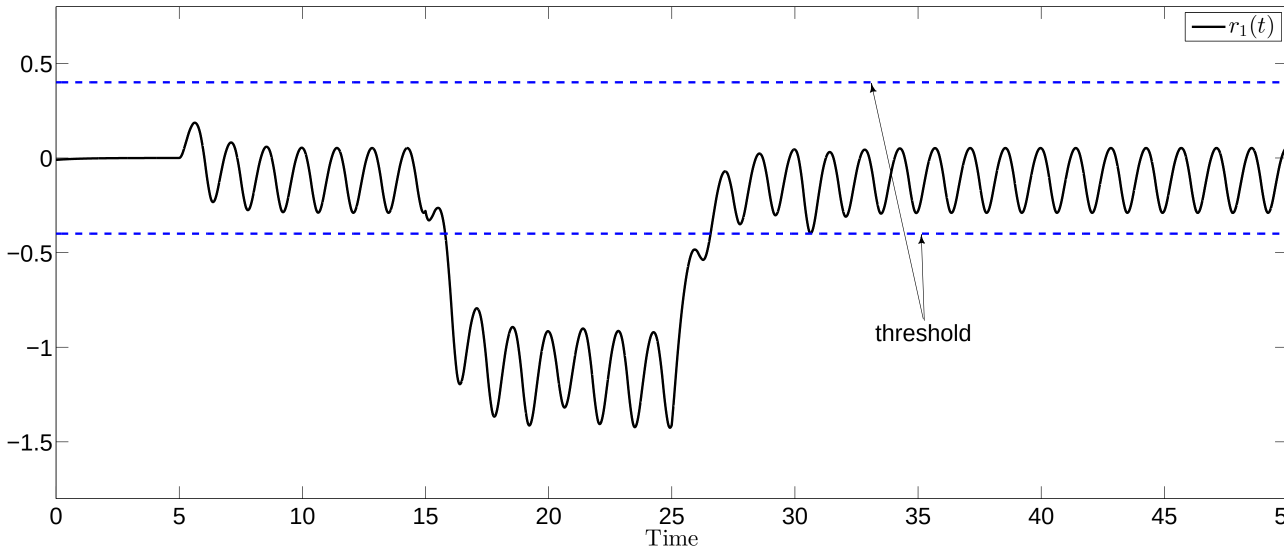

FIGURE 5.14 – Robust fault diagnosis : Residual $r_1(t)$.

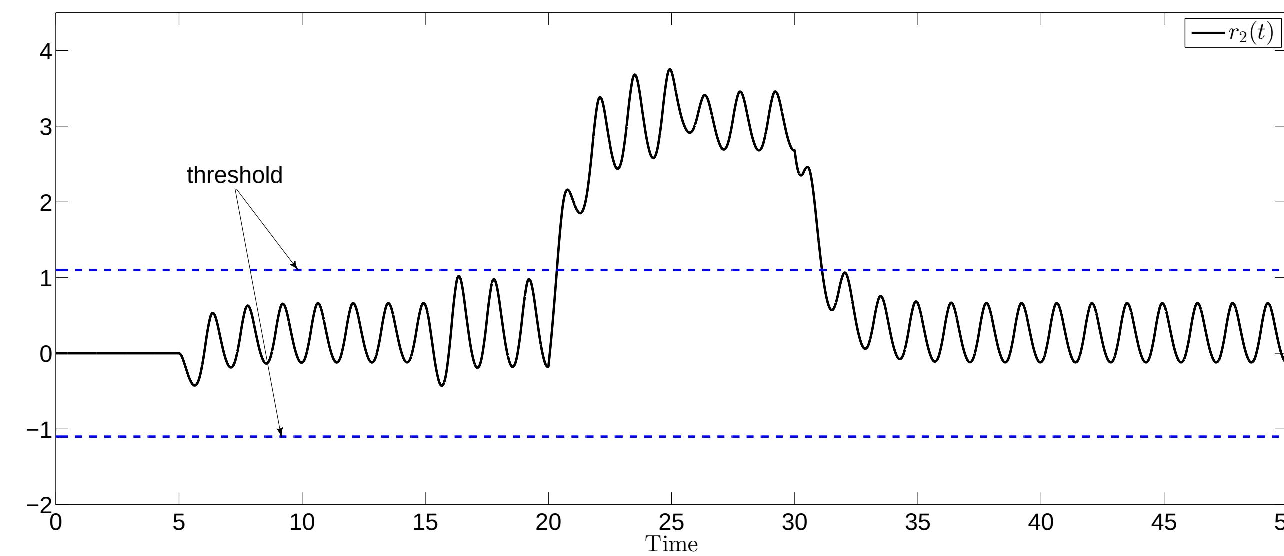

FIGURE 5.15 – Robust fault diagnosis : Residual $r_2(t)$.

Once the residuals where generated, the evaluation of residuals is done by choosing a threshold of 0.4 for $r_1(t)$ and of 1.1 for $r(t)$, such that we get the following signature table :

TABLE 5.2 – Robust fault isolation : Residual evaluation.

Residue \ Time	$15.83 < t < 20.34$	$20.35 < t < 26.58$	$26.59 < t < 31.1$	other time
$r_1(t)$	1	1	0	0
$r_2(t)$	0	1	1	0

From table 5.2 we can see that the signature for each fault, including simultaneous faults are different between them.

Figures 5.16 - 5.18 show the state estimation, and Figures 5.19 and 5.20 show the fault estimation.

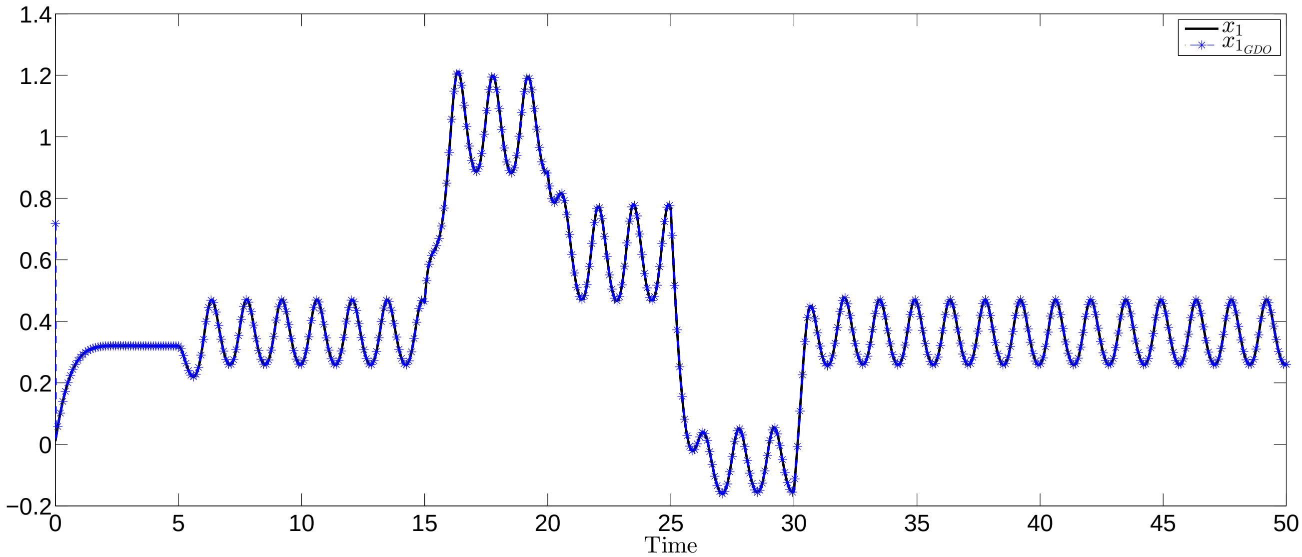

FIGURE 5.16 – Robust fault diagnosis : Estimation of $x_1(t)$.

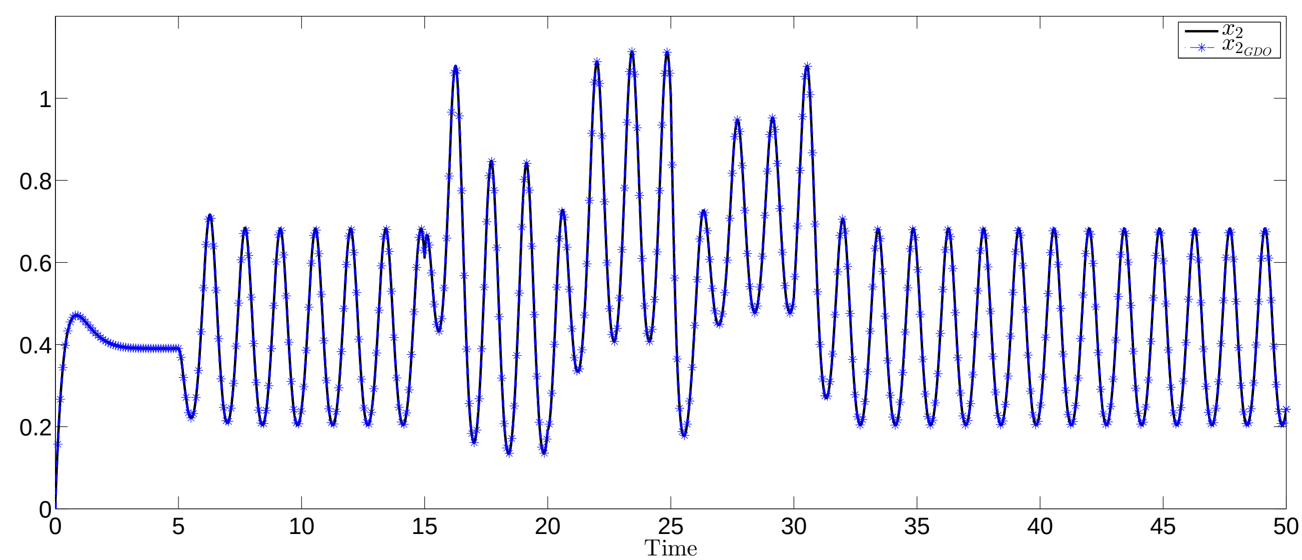

FIGURE 5.17 – Robust fault diagnosis : Estimation of $x_2(t)$.

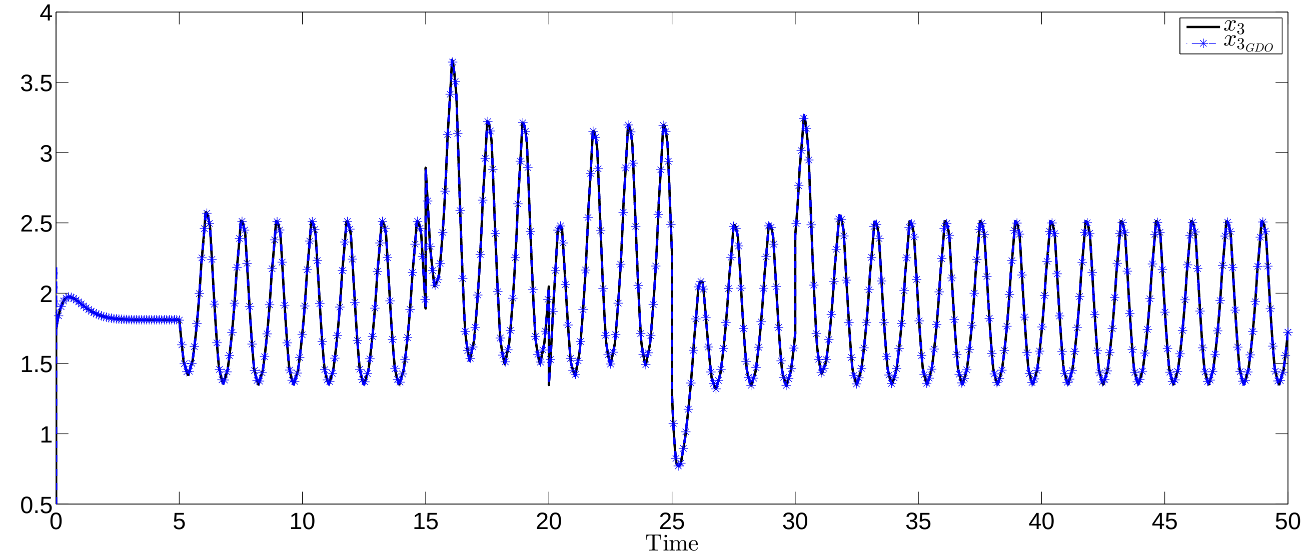

FIGURE 5.18 – Robust fault diagnosis : Estimation of $x_3(t)$.

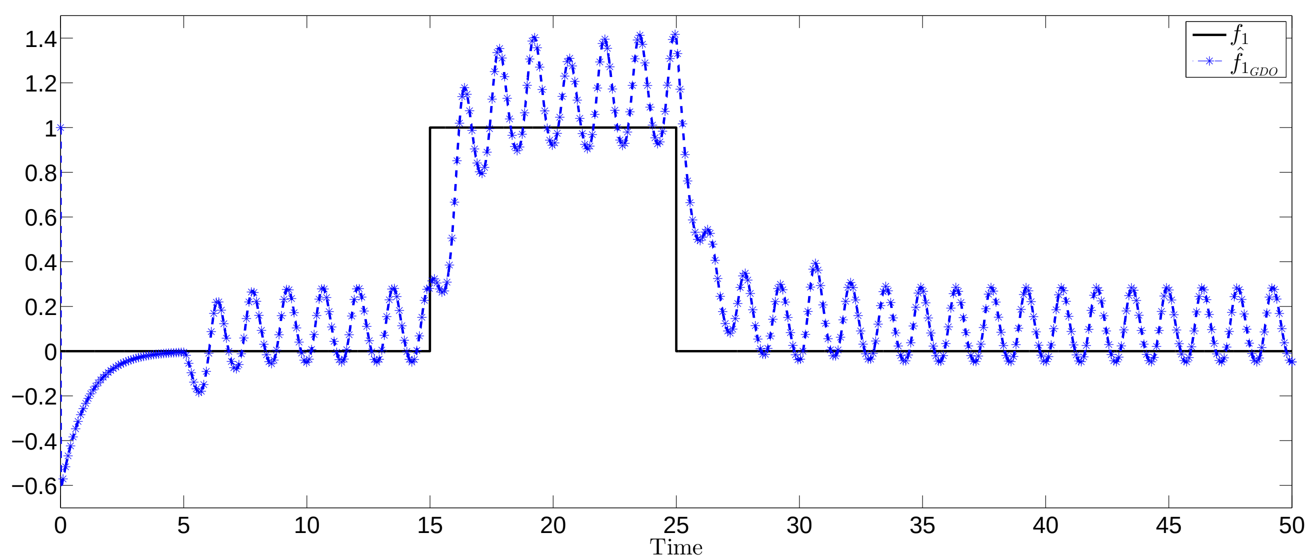

FIGURE 5.19 – Robust fault diagnosis : Estimation of $f_1(t)$.

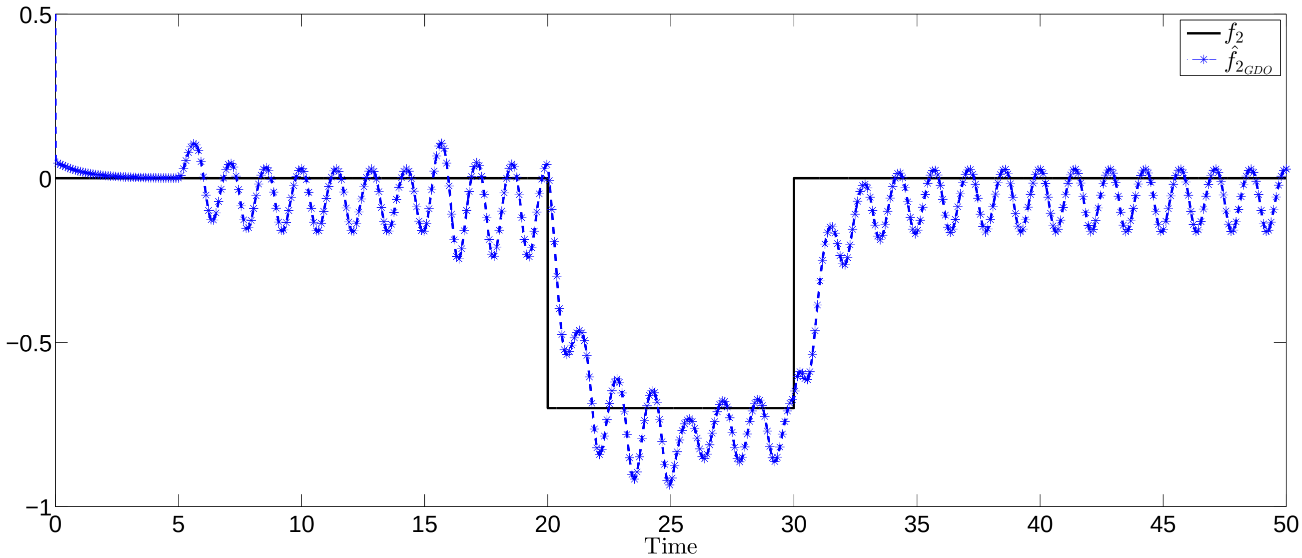

FIGURE 5.20 – Robust fault diagnosis : Estimation of $f_2(t)$.

From these results, we can see that the fault isolation is achieved with our method, while the fault estimation has an acceptable behavior, since it can be seen that the fault estimation is not robust to parameter uncertainties.

5.4 Conclusion

In this chapter fault diagnosis for descriptor systems and for uncertain descriptor systems were presented. The fault diagnosis scheme was treated thought two observers with specific tasks, the first observer was responsible for fault detection and isolation, in specific a diagonal transfer function from the residual to the fault was proposed to isolate multiple faults, i.e. each residual represents just the presence of one fault. The second observer dealt with simultaneous estimation of state and faults, which completes the fault diagnosis scheme.

Conclusions and perspectives

The work presented in this thesis makes a contribution to solving the problems of state estimation, control and fault diagnosis for uncertain descriptor systems, these class of systems are considered as a generalization of dynamical systems.

Descriptor systems have been studied in different domains, such as identification, control, estimation and fault diagnosis. With respect to estimation for linear descriptor systems, there exists numerous works in POs and PIOs design. Due to that, in this thesis we presented an alternative of state estimation more general than the PO and the PIO, named as GDO. The opportunity of introduce a new observer that generalized the existing ones and its utility in applications of control and fault diagnosis, are interests that have motivated our research into the design of observers for uncertain descriptor systems.

This chapter summarizes the work presented in this thesis to review the main conclusions. Before boarding the problem of state estimation, descriptor systems were presented citing some practical examples where this type of representation is used. A literature review on the design of observers for linear descriptor systems presenting some works in observers design for this class of systems is showed, to emphasize the differences between the observers reported in the literature and the observer that we propose.

In Chapter 2, two methods of observers synthesis for disturbed descriptor systems were presented. The first method develops the GDO for disturbed descriptor systems in continuous-time case, the necessary conditions for stability and convergence of the estimated state to the real ones are given in terms of LMIs. The H_∞ approach was considered in the observer design to guarantee stability of the estimation error, disturbance attenuation, and also minimize the error between the estimate state and the real state. The second method develops the GDO for disturbed descriptor systems in discrete-time case. Particular cases of our observer as the PO and the PIO are also designed, and in the sections of simulations results a comparison between them was made to show that the GDO has robustness to parameter uncertainties that were not considered initially in the model.

Two classes of disturbed descriptor systems with parametric variations are addressed in Chapter 3. The first one are the uncertain disturbed descriptor systems, where the uncertainty is unknown and it is assumed bounded, the second class are the disturbed LPV descriptor systems, where the parameter varying is known and measurable, so that the variation occurs in a polytope, and the vertices of the polytope are the variation limits of the parameter. In this chapter the GDO is designed using the H_∞ approach when the systems are subject to disturbances in the system and in the output. Simulation of numerical examples were proposed to illustrate the effectiveness of our methods. Additional parameter variation was taken into account in simulation to show that the GDO guarantee robustness in state estimation compared to the PO and the PIO.

In Chapter 4, the observers proposed are used for control purposes in the sense of stabilization. In general, two types of systems were considered. The first one was the disturbed descriptor systems, using the H_∞ approach to minimize the effect of the disturbance in the controlled output. The second type was the uncertain disturbed descriptor systems. In the section of simulation results it is considered that these systems are initially unstable. The GDO estimates the states trough the measurements of the input and output of the system and by the feedback of the estimated states weighted by a gain, the stabilization of the system is achieved. A comparison with the controllers made by the PO and the PIO observers were also presented to highlight the advantages of our observer in face to parametric uncertainties.

Chapter 5 treated the problem of fault diagnosis, which was divided in two parts. The first part, was devoted to fault detection and isolation, a particular objective was fixed in this part to obtain residuals independent, i.e. each residual represents just the presence of one fault, so that we can isolate multiple faults since the signature created in the evaluation of residuals is always different. The second part was dedicated to simultaneous state and fault estimation. In this part an adaptive observer based on the structure of the GDO was designed to achieve the objective of estimate faults. These approaches were developed for two types of systems : descriptor systems and for uncertain descriptor systems.

Analyzing the previous summaries of each chapter we can conclude that the methods developed in this theses fulfill the objective described in Section 2. It was demonstrated also that the new observer structure presented by the GDO achieves better performance than the PO and the PIO, in the same way we prove that the GDO proposed is an useful tool for control and fault diagnosis purposes. Additionally most of the methods presented in this thesis were submitted or published in conferences or journal.

The results proposed in this thesis offer many opportunities for future work. Some of these open fields of research are presented below :

- Apply the GDO to nonlinear descriptor systems in order to deal with more complex systems that approximate the real processes.
- Apply the GDO to descriptor systems with a more general class of uncertainties as the Integral Quadratic Constraints (IQC).
- Apply the GDO for LPV descriptor systems with unmeasurable parameters, i.e. the parameter that defines the presence of each local model is unknown.
- Develop residuals insensitive to disturbances in the scheme of fault detection and isolation.
- Develop the scheme of fault detection an isolation presented in Chapter 5 for discrete-time descriptor systems.
- Develop the scheme of fault diagnosis for descriptor systems with multiplicative and parametric faults.
- Apply the fault diagnosis scheme proposed in Chapter 5, to fault tolerant control in descriptor systems.
- Implement the approaches presented in this thesis in a real process that use the descriptor representation.

In these chapter some future works were described, however these are not the only problems that remain open for future investigations.

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