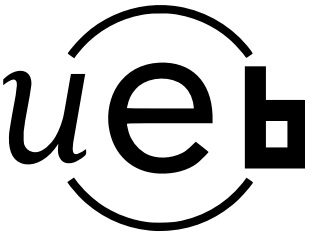
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# Stabilité de couches limites et d'ondes solitaires en mécanique des fluides

**Thèse soutenue à Rennes**
**le 8 juillet 2014**

devant le jury composé de :

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# Stabilité de couches limites et d'ondes solitaires en mécanique des fluides

Matthew Paddick

Thèse de doctorat préparée à l'Université de Rennes 1
sous la direction du Professeur Frédéric Rousset

Soutenue à Rennes, le mardi 8 juillet 2014.

*"Look into the pool, the ripples never come back."*
La mécanique des fluides selon Genesis, *Ripples*, 1976.

*For Nannie*

## Avant-propos

Que ces quatre années ont été riches !

Riches en mathématiques évidemment. La présente thèse permettra au lecteur, j'espère, d'entrevoir cette richesse. L'analyse des équations aux dérivées partielles contient une quantité et une variété séduisantes - en tous cas, qui m'ont séduit - de modèles et de problèmes dont je n'ai étudié bien sûr qu'une infime portion.

Je suis reconnaissant en premier lieu envers Frédéric Rousset pour l'occasion qu'il a proposé de travailler dans ce domaine si riche. Sa disponibilité, sa patience et son exigence ont été des alliées sans faille pour l'élaboration et l'aboutissement des résultats présentés ici. Sous sa direction, ces quatre années se seront déroulées dans une ambiance agréable et studieuse.

Je remercie ensuite les rapporteurs, Didier Bresch et Mark Williams, pour leurs remarques et questions sur ce tapuscrit. Je suis honoré qu'ils aient accepté de participer à l'évaluation de ce travail, et de même pour Christophe Cheverry, Marius Paicu et Franck Sueur, membres du jury.

En 2010, j'ai donc commencé à fréquenter de façon assidue le 2è étage de la Tour de Maths, croisé continuellement les professeurs et doctorants en analyse, et partagé avec plaisir le bureau 212 avec Hamdi. J'ai aussi commencé à enseigner de façon régulière, ce qui m'a amené à travailler avec des membres d'équipes variées. La liste de ces collègues est bien trop longue pour donner des noms (c'est quasiment toute la tour si on ajoute les enseignants pré-doctorat !), mais j'ai une pensée pour chacun d'eux et les moments passés avec eux entre ces murs, et je leur souhaite le meilleur pour l'avenir.

Et pour les démarches, que ce soit pour la préparation de la soutenance ou pour constituer une liste de candidats pour le Conseil d'UFR, les personnels administratifs ont toujours été enthousiastes et efficaces.

Mais cela faisait déjà un moment que je fréquentais Rennes : j'y suis arrivé en 2004. Au cours des 10 années passées ici, il y a eu les mathématiques, mais aussi un groupe d'amis intarissables en amusement, chaleur et soutien. Je voudrais nommer les amis de 10 ans, ceux rencontrés dès la toute première année à l'université, et qui ont donc été des constantes (à peu près) de ma vie à Rennes : Max, Ben et Nico. Et il y a les amis gardés depuis plus longtemps encore, à commencer par Laura J, toujours là pour m'écouter dans les moments durs. Et avec Yann, Sylvain et Judicaël, ce sont quatre promos consécutives du collège de La Gacilly qui envoient chacune (au moins) un élève jusqu'au doctorat ! J'ai également une pensée pour les relations tissées plus récemment, mais là, impensable de dresser de liste exhaustive...

Enfin, sans Maud, ces quatre années auraient été très différentes.

A présent, je m'apprête à quitter la Bretagne, région qui accueillait la famille Paddick d'Angleterre il y a exactement 20 ans. Cela fait longtemps que je me destine aux mathématiques, et ma famille, Mum, Dad, Rebecca et Laura, m'ont soutenu indéfectiblement dans cette voie.

*Thank you for everything.*

A Rennes, le 1$^{er}$ juillet 2014.

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# Résumé en français

## 1 Les équations de Navier-Stokes

### 1.1 Fluides compressibles et incompressibles

Les équations étudiées dans cette thèse ont pour point commun de faire partie d'une grande famille de modèles de la mécanique des fluides : les équations de Navier-Stokes. On considère un fluide évoluant dans un domaine $\Omega$, éventuellement à bord rigide et lisse, de $\mathbb{R}^d$, avec $d \in \{2, 3\}$. Le fluide est caractérisé en un point $X \in \Omega$ et à un temps $t \geq 0$ par :

- sa densité $\rho(t, X)$, scalaire,
- sa vitesse $u(t, X)$, un champ de vecteurs de $\mathbb{R}^d$,
- et son énergie interne $e(t, X)$, scalaire.

Il peut être soumis à une force (champ de vecteurs) $F(t, X)$ donnée. Le modèle Navier-Stokes complet, régissant le mouvement de ce fluide, est alors issu des lois de la mécanique classique. Tenant compte de la conservation de trois quantités physiques, il comprend trois équations aux dérivées partielles :

- une pour la conservation de la masse,

$$\partial_t \rho + \operatorname{div}\,(\rho u) = 0, \tag{1}$$

- une autre pour la conservation du moment $\rho u$,

$$\partial_t(\rho u) + \operatorname{div}\,(\rho u \otimes u) = \rho F + \operatorname{div}\,\Sigma, \tag{2}$$

- et une dernière pour la conservation de l'énergie totale $E = e + \frac{|u|^2}{2}$, en utilisant la loi de Fick pour le flux de chaleur,

$$\partial_t(\rho E) + \operatorname{div}\,(\rho E u) = \rho u \cdot F + \operatorname{div}\,(\Sigma u + \kappa \nabla \theta). \tag{3}$$

Le champ de vecteurs $(\rho, u, e) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^{d+2}$ est l'inconnue du système, tandis que le scalaire $\theta$, la température du fluide, est liée à l'énergie interne $e$ par une relation donnée (souvent, on prend $e$ proportionnelle à $\theta$). Le nombre $\kappa \geq 0$ est le coefficient de conduction thermique, et

la matrice $\Sigma \in \mathcal{M}_d$ est le tenseur des contraintes. Ce tenseur des contraintes prend en compte deux phénomènes internes au fluide qui sont les effets de pression et de viscosité : il s'écrit

$$\Sigma = 2\mu Su + \lambda_0 \text{div } u \, \mathbb{I}_d - P\mathbb{I}_d, \tag{4}$$

avec $\mathbb{I}_d$ la matrice identité, $Su = \frac{1}{2}(\nabla u + (\nabla u)^t)$ le gradient symétrique, $\mu \geq 0$ le coefficient de viscosité dynamique, et $\lambda := \mu + \lambda_0 \geq 0$ le coefficient de viscosité volumique. Ces coefficients peuvent être des constantes fixées ou être des fonctions de $(\rho, \theta)$. Enfin, $P$, la pression, est une fonction positive de $(\rho, \theta)$. Par exemple, l'hypothèse que l'on travaille avec un gaz parfait mène à choisir $P(\rho, \theta) = k\rho\theta$. Notons que lorsque $\mu$ et $\lambda$ sont constants, la divergence du tenseur des contraintes, qui apparaît dans l'équation (1), s'écrit

$$\text{div } \Sigma = \mu\Delta u + \lambda\nabla \text{div } u - \nabla P.$$

Lorsque les coefficients $\mu,\ \lambda,\ \kappa$ sont nuls, le modèle est non-visqueux et on parle d'équation d'Euler compressible.

Le lecteur pourra trouver une modélisation physique approfondie de ces équations dans le livre de G. Batchelor [8]. Pour compléter ces équations, on prescrit une condition initiale et des conditions aux limites. Nous évoquerons précisément les conditions au bord dans la section suivante.

Dans les travaux qui suivent, on travaille à entropie constante - le fluide est dit *isentropique*. L'entropie étant une fonction de $(\rho, \theta)$, l'hypothèse d'isentropie élimine l'équation (3), et on considèrera dans cette thèse des lois de pression dépendant uniquement de $\rho$ ou de $(\rho, \nabla\rho)$ (tenseur de Korteweg, voir dernière partie du résumé). Considérons dans la suite de cette partie le cas d'une loi de pression barotrope : $P$ est une fonction puissance $\gamma > 1$ de $\rho$. Dans ce cadre, on sait depuis les années 1960 et les travaux de J. Nash, puis V. Solonnikov, qu'il existe des solutions fortes localement en temps [89, 113] ; R. Danchin a, plus tard, obtenu la globalité des solutions dans les espaces de Besov critiques lorsque la condition initiale est petite [25, 21], ainsi que des raffinements du résultat d'existence locale de Solonnikov [28] (on pourra aussi consulter l'article [29] de Danchin, qui passe en revue l'état de l'art sur les solutions fortes de Navier-Stokes isentropique). La preuve de l'existence globale de solutions faibles aura attendu les années 1990, et les travaux de P-L. Lions (voir [73] ou le livre [75]) qui ont étendu à toute dimension le résultat d'existence en 1D de D. Hoff et D. Serre [57].

On peut s'intéresser à une approximation de ce modèle pour des fluides dont les vitesses typiques sont subsoniques. Dans le système adimensionné (cf. la sous-section suivante), on introduit le nombre de Mach, $\eta > 0$ appelé à tendre vers 0, et on suppose que $u(t, X) = \eta v(\eta t, X)$. Dans ce changement d'échelle (en anglais *scaling*), on pose $\rho = r(\eta t, X)$ et $(\mu, \lambda) = \eta(\nu, \tilde{\lambda})$, et le système (1)-(2) devient

$$\left\{ \begin{array}{rcl} \partial_t r + \text{div } (rv) & = & 0 \\ \partial_t(rv) + \text{div } (rv \otimes v) & = & rF + \nu\Delta v + \tilde{\lambda}\nabla \text{div } v - \frac{1}{\eta^2}\nabla(P(r)). \end{array} \right. \tag{5}$$

La seconde équation suggère que $r = \rho_0 + \mathcal{O}(\eta^2)$, et, en supposant $\rho_0 = 1$, la première équation tend, formellement, lorsque $\eta$ tend vers 0, vers div $v = 0$ : une condition dite d'incompressibilité. La limite en faible nombre de Mach est donc appelée *limite incompressible*, et le système limite de (5) est l'équation de Navier-Stokes incompressible, que nous serons aussi amenés à étudier :

$$\left\{ \begin{array}{rcl} \text{div } v & = & 0 \\ \partial_t v + \text{div } (v \otimes v) & = & F + \nu \Delta v - \nabla p, \end{array} \right. \tag{6}$$

où le scalaire $p(t, X)$ est la pression cinétique du fluide et $\nu$ la viscosité cinématique. Les inconnues de cette équation limite sont la vitesse $v$ et la pression $p$. Les premiers résultats d'existence globale de solutions faibles de (6) datent des années 1930, avec l'article fondateur de J. Leray [71] (on pourra aussi consulter le premier tome de P-L. Lions [74]). Si l'existence globale de solutions fortes de Navier-Stokes incompressible est acquise en 2D ($d = 2$, voir par exemple [70]), le même problème en 3D, dans tout l'espace $\mathbb{R}^3$ ou dans le cadre périodique $\mathbb{T}^3$, est encore ouvert, et constitue l'un des Problèmes du Millénaire du Clay Institute [20]. A ce jour, des résultats d'existence globale de solutions fortes en 3D ne sont donnés que pour des données petites, à commencer par H. Fujita et T. Kato dans les années 1960 [40] (voir aussi [67]).

Les premiers résultats sur la limite incompressible sont intervenus dans les années 1980, et portaient sur l'équation d'Euler compressible ($\mu = \lambda = 0$) : S. Klainerman et A. Majda [65, 66], S. Schochet [110], S. Ukai [119] (voir aussi le livre de Majda [79]). S'agissant de Navier-Stokes compressible, la convergence des solutions faibles de Lions vers les solutions faibles de Leray de (6) lorsque le nombre de Mach tend vers 0 a été obtenue par P-L. Lions et N. Masmoudi, peu de temps après les travaux sur l'existence globale de solutions faibles [76]. Le lecteur pourra aussi voir les travaux de B. Desjardins et E. Grenier [32] (avec Lions et Masmoudi) et [30] ; dans ce dernier, on trouve le problème de la limite incompressible traité avec des estimations de Strichartz. S'agissant des solutions fortes, des résultats de convergence ont été obtenus par R. Danchin dans le cas isentropique (espace complet [27], cas périodique [26], voir aussi [29]), et par T. Alazard sur le système complet [2, 3].

## 1.2 La limite non-visqueuse et couches limites

Les travaux de la présente thèse traitent majoritairement de la limite non-visqueuse dans les équations de Navier-Stokes, en régimes incompressible et compressible. Il s'agit de choisir des coefficients de viscosité qui tendent vers 0 ; l'équation limite de Navier-Stokes est alors l'équation d'Euler.

Dans le cas incompressible, on définit comme fonction de la viscosité le nombre de Reynolds $Re$, une quantité physique adimensionnelle ; il s'obtient en divisant le produit d'une vitesse $U$ et d'une longueur $L$ caractéristiques du système par la viscosité cinématique du fluide $\nu$. Il dépend donc des propriétés intrinsèques du fluide mais aussi du domaine et des vitesses typiques en jeu. Ainsi, en changeant d'échelle en divisant les variables et les inconnues par les quantités caractéristiques correspondantes, l'équation devient adimensionnée et le coefficient de viscosité dans (6) est égal à $1/Re$. Les écoulements avec des nombres de Reynolds élevés exhibent souvent

des comportements turbulents, avec une vorticité importante, tandis que ceux avec des nombres de Reynolds faibles sont laminaires [8]. L'étude de la limite non-visqueuse est donc motivée physiquement par les écoulements à grand nombre de Reynolds, pour savoir si l'équation d'Euler est une bonne approximation de l'équation de Navier-Stokes avec viscosité faible, et pour connaître le rôle de la turbulence dans cette transition vers le modèle non-visqueux. On observe des nombres de Reynolds élevés, de l'ordre de $10^6$ et plus, lorsqu'on considère, par exemple, le mouvement de l'eau dans le référentiel d'un animal marin, la vitesse caractéristique $U$ étant la vitesse de déplacement de l'animal, la longueur $L$ la taille de l'animal et le temps caractéristique $L/U$ (utilisé pour l'adimensionnement de l'équation). On obtient ainsi $Re = UL/\nu_{\text{eau}} = 3.10^8$ pour une baleine bleue nageant à 10 m/s [120]. Dans le cas compressible, la limite non-visqueuse permet d'obtenir des "bonnes" solutions faibles de l'équation d'Euler compressible lorsque la donnée initiale comporte un saut (engendrant donc un choc) ; voir par exemple le chapitre 4 de [39], où un tenseur des contraintes de type Korteweg, que nous présentons en section 4, est considéré.

La limite non-visqueuse pose des problèmes mathématiques du fait que l'équation change de nature : l'équation de Navier-Stokes est de type hyperbolique-parabolique d'ordre 2, alors que l'équation d'Euler est de type hyperbolique d'ordre 1, et la question de la limite non-visqueuse se pose autant en régime compressible que dans le modèle incompressible. S'agissant du modèle incompressible, le problème est résolu dans l'espace, $\Omega = \mathbb{R}^d$ ou $\mathbb{T}^d$, pour les solutions régulières (voir par exemple le livre de A. Majda et A. Bertozzi [80]). Le contrôle de la vorticité joue un rôle essentiel dans la limite non-visqueuse, ainsi les choses sont moins claires lorsque les données sont peu régulières - citons par exemple le cas des espaces critiques [55] ou celui des poches de vorticité [82]. Aussi, l'extraction de sous-suites faiblement convergentes dans un $L^p$ ($L^2$ faible ou $L^\infty$ faible-$*$), par exemple dans le cas où on aurait une borne uniforme seulement $L^p$ sur une famille de solutions, ne permet pas toujours d'obtenir des solutions d'Euler - déjà, la limite d'une suite faiblement convergente de solutions de l'équation d'Euler ne vérifie pas toujours l'équation d'Euler ([80] chapitre 12 en 3D, [22] en 2D).

Le problème de la limite non-visqueuse est d'autant plus difficile lorsque $\Omega$ a un bord. L'hypothèse d'un bord rigide imperméable mène à écrire une condition de non-pénétration : le fluide ne peut que glisser le long du bord, soit, si $X \in \partial\Omega$ et $\vec{n}(X)$ est la normale unitaire extérieure à $\partial\Omega$ au point $X$, on a

$$u \cdot \vec{n}|_{\partial\Omega} = 0. \tag{7}$$

Cette condition au bord vaut pour les équations d'Euler et pour Navier-Stokes, mais, ces dernières étant d'ordre deux, une condition supplémentaire est requise sur la vitesse tangentielle, $u_\tau = u + (u \cdot \vec{n})\vec{n}$. Le plus souvent, on choisit une condition de Dirichlet,

$$u_\tau|_{\partial\Omega} = 0 \;\Rightarrow\; u|_{\partial\Omega} = 0, \tag{8}$$

qui signifie que le fluide ne peut pas glisser le long du bord. Ceci suppose qu'il n'y a aucune interaction entre le fluide et le bord, hypothèse longuement débattue, et dès le XIX$^{\text{ème}}$ siècle,

H. Navier a proposé une autre condition, de type mixte, dont nous discuterons tout de suite. Toutefois, les observations physiques confirment que la condition de Dirichlet est adéquate pour décrire des écoulements à l'échelle macroscopique dans les conditions normales (discussion dans [8], p.149).

En examinant les interactions microscopiques entre le fluide et le bord rigide, H. Navier a proposé en 1823 dans son "Mémoire sur les lois du mouvement des fluides" [90] que la vitesse tangentielle du fluide au bord du domaine soit reliée au tenseur des contraintes comme suit,

$$[(\Sigma\vec{n} + au)|_{\partial\Omega}]_\tau = 0, \tag{9}$$

avec $a > 0$, permettant au fluide de glisser le long du bord (on parle aussi de condition de glissement, ou, en anglais, de *slip condition*, par opposition au *no-slip condition* de Dirichlet). Comme pour les conditions de Dirichlet, la condition de Navier est validée par des observations, mais dans des cas assez précis : aux parois des vaisseaux sanguins capillaires, par lesquels se font les échanges moléculaires entre les tissus et le sang [94], ou à l'interface de deux fluides immiscibles [97]. Mathématiquement, la condition de Navier apparaît en homogénéïsant un bord rugueux (voir [62], [1] pour les hauts nombres de Reynolds), et peut être dérivée rigoureusement d'un modèle microscopique de Boltzmann avec condition de réflexion de Maxwell [86].

Ainsi, les solutions de Navier-Stokes sont contraints de satisfaire une condition au bord dont les solutions de l'équation d'Euler se passent. Le fait que la limite non visqueuse de Navier-Stokes est bien Euler dans des cas sans bord mène à proposer un ansatz pour les solutions de Navier-Stokes de la forme "solution d'Euler + *couche limite*". Une couche limite est une fonction localisée près du bord et a pour principal but de rattraper la deuxième condition au bord, la première, celle d'annulation de la vitesse normale, étant commune à Euler et à Navier-Stokes. Considérons dans la suite de cette partie le demi plan, $X = (x, y) \in \Omega = \mathbb{R} \times \mathbb{R}^{+*}$, et regardons l'équation de Navier-Stokes 2D incompressible avec condition de Dirichlet au bord. Un ansatz pour une solution $u^\nu$ de (6)-(8) avec $F = 0$, proposé par L. Prandtl [96], s'écrit alors

$$u^\nu(t, x, y) = u^E(t, x, y) + u^B\left(t, x, \frac{y}{\sqrt{\nu}}\right), \tag{10}$$

où $u^E$ est une solution de l'équation d'Euler incompressible

$$\left\{\begin{array}{rcl} \text{div } u^E & = & 0 \\ \partial_t u^E + u^E \cdot \nabla u^E + \nabla P^E & = & 0 \\ u_2^E|_{y=0} & = & 0 \end{array}\right.$$

et $u^B$ une couche limite (*boundary layer* en anglais, d'où la notation avec l'exposant $B$), une fonction rapidement décroissante en $y$, agissant à courte portée du bord et permettant au tout $u^\nu$ de vérifier la condition de Dirichlet (8). L'ansatz laisse apparaître que les dérivées par rapport à la variable normale, $y$, sont difficiles à contrôler près du bord lorsque $\nu$ tend vers 0. Les difficultés

sont confirmées si on effectue le changement d'échelle suivant : $u^\nu = (\tilde{u}, \sqrt{\nu}\tilde{v})$ en les variables $(t, x, y' = \nu^{-1/2}y)$. Ceci mène à considérer que le comportement de la solution de Navier-Stokes au voisinage du bord est régi par l'équation de Prandtl :

$$\left\{ \begin{array}{rcl} \partial_t \tilde{u} + \tilde{u}\partial_x \tilde{u} + \tilde{v}\partial_{y'} \tilde{u} - \partial^2_{y'y'} \tilde{u} & = & (\partial_t u^E + u^E \cdot \nabla u^E)_1|_{y=0} \\ \partial_x \tilde{u} + \partial_{y'} \tilde{v} & = & 0 \\ (\tilde{u}, \tilde{v})|_{y=0} & = & 0 \\ \lim_{y\to+\infty} \tilde{u} & = & u^E|_{y=0}. \end{array} \right. \tag{11}$$

Or ce système est problématique, car si l'équation est bien posée dans certains cas, D. Gérard-Varet et E. Dormy ont montré qu'elle est mal posée dans les espaces de Sobolev [41]. Parmi les cas où le caractère bien posé a été prouvé, il y a le cas des données monotones par O. Oleĭnik dans les années 1960 (voir [92] ou [91]) - étendu à un cadre BV par Z. Xin et L. Zhang [127] -, le cas de données analytiques (M. Sammartino et R. Caflisch [108]) ou dans la classe de Gevrey $G^{7/4}$ (D. Gérard-Varet et N. Masmoudi [43]).

Puisque le comportement des couches limites est étroitement liée à cette équation de Prandtl, la résolution de la limite non-visqueuse pour l'équation de Navier-Stokes incompressible dans un domaine à bord avec condition de Dirichlet homogène est l'un des défis majeurs de la mécanique des fluides théorique. Les travaux effectués jusque là ont débouché sur des résultats tantôt positifs, tantôt négatifs quant à la stabilité des couches limites. A la suite de leur travail sur Prandtl analytique, Sammartino et Caflisch ont prouvé que l'ansatz (10) était valable dans ce cadre [109]. Par ailleurs, Y. Guo et T. Nguyen ont obtenu dans [53], en conséquence des résultats négatifs de Gérard-Varet et Dormy et de Gérard-Varet et Nguyen [44] sur l'équation de Prandtl, la non-validité du même développement asymptotique lorsque la couche limite a une structure de profil de cisaillement (voir section 2). Mais cette instabilité peut également avoir lieu lorsque l'équation de Prandtl est bien posée, et c'est le sens d'un théorème de E. Grenier dans [47]. La mécanique à l'oeuvre dans ce résultat, qui utilise aussi des profils de cisaillement, est la création de vorticité dans la couche limite, montrant que la question du caractère bien posé de l'équation des couches limites n'est pas seule en cause dans la difficulté du problème. Il y a aussi en jeu une échelle rapide dans la variable tangentielle $x$, qui n'apparaît pas dans l'ansatz de Prandtl (10). Nous discuterons plus amplement du résultat de Grenier dans la partie suivante.

En fin de compte, le théorème le plus général sur la convergence vers Euler des solutions de Navier-Stokes avec une condition de Dirichlet homogène est probablement celui de T. Kato, qui date des années 1980 et qui donne une condition nécessaire et suffisante pour la convergence en termes de contrôle de la norme $L^2$ des dérivées de $u^\nu$ sur un voisinage de taille $\nu$ du bord [63]. On pourra consulter les travaux de R. Temam et X. Wang [117] et de X. Wang [121] pour des améliorations de ce critère ; notons aussi que F. Sueur donne dans [116] une adaptation de ce critère au système compressible isentropique. Enfin, s'agissant de conditions de Dirichlet non-homogènes non-caractéristiques dans le cas incompressible, nous renvoyons à un autre article de R. Temam et X. Wang [118].

Nous renvoyons aux parties 2 et 3 pour l'état de l'art sur la limite non-visqueuse avec la condition de Navier ou dans le régime compressible.

### 1.3 Panorama des travaux de la thèse

Dans le cadre incompressible ci-dessus et dans le cas des conditions de Dirichlet, des résultats sur l'instabilité des couches limites pour des données initiales sous forme de profil de cisaillement, sont dûs à E. Grenier [47]. Nous montrons le théorème d'instabilité dans $L^\infty$ analogue pour Navier-Stokes avec une condition de Navier, ainsi qu'un résultat de convergence dans $L^2([0,T]\times\Omega)$ des solutions de Navier-Stokes vers Euler. Ces résultats ne sont pas contradictoires, et nous exhiberons un exemple dans le cas $x$-périodique où l'instabilité et la convergence $L^2$ ont lieu simultanément. Ensuite, dans la section 3, nous aborderons la limite non-visqueuse de l'équation de Navier-Stokes barotrope (modèle compressible) dans le demi-espace. Grâce à des estimations d'énergie dans les espaces de Sobolev conormaux, nous obtenons la stabilité $L^2$ et $L^\infty$ des couches limites dans ce cadre.

La dernière section aborde un problème différent : celui de la stabilité d'ondes solitaires dans l'équation d'Euler-Korteweg 2D ($\Omega=\mathbb{R}^2$). On considèrera le modèle compressible non visqueux ($\mu=\lambda=0$) avec une pression dépendant de $\rho$ et de $\nabla\rho$,

$$P(\rho,\nabla\rho)=P_0(\rho)+\frac{1}{2}(\rho K'(\rho)-K(\rho))|\nabla\rho|^2,$$

avec $K(\rho)>0$ un coefficient de capillarité. En écrivant comme ci-dessus $(x,y)\in\mathbb{R}^2$, l'onde solitaire est une solution de la forme $Q_c(t,x)=(\rho_c(t,x),u_c(t,x),0)$, et on se posera la question de savoir si les perturbations transverses de $Q_c$, c'est-à-dire avec des données initiales dépendant de $(x,y)$, restent proches de $Q_c$. Après avoir étudié le spectre de l'opérateur correspondant à l'équation linéarisée, nous verrons que nous pourrons adapter l'argument d'instabilité non-linéaire de Grenier à cette question.

## 2 Sur la limite non-visqueuse de l'équation de Navier-Stokes incompressible 2D avec conditions de Navier

*Les résultats de cette section sont développés dans le chapitre 1.*

### 2.1 Stabilité et instabilité

On considère la limite lorsque la viscosité $\varepsilon$ tend vers 0 de solutions de l'équation de Navier-Stokes incompressible 2D avec $X=(x,y)\in\Omega=\mathbb{R}\times\mathbb{R}^{+*}$,

$$\left\{\begin{array}{rcl}\partial_t u+(u\cdot\nabla)u-\varepsilon\Delta u+\nabla p &=& 0\\ \text{div } u &=& 0\\ u_2|_{y=0} &=& 0\end{array}\right. \tag{12}$$

avec, au bord, la condition de non-pénétration (3ème ligne ci-dessus) et une condition de Navier de la forme

$$\partial_y u_1|_{y=0} = \frac{2a}{\sqrt{\varepsilon}} u_1|_{y=0}, \tag{13}$$

avec $a > 0$. Cette condition exprime un comportement de type couche limite, c'est-à-dire que $u \sim u^B(t, x, \varepsilon^{-1/2} y)$ près du bord, avec $u^B$ satisfaisant une condition de Navier uniforme en $\varepsilon$. Formellement, lorsque $\varepsilon$ tend vers 0, cette condition s'approche d'une condition de Dirichlet homogène, et on peut se demander si l'ansatz (10) est valide pour les solutions de (12)-(13). L'objectif du travail effectué dans ce cadre a été d'une part d'étudier la convergence $L^2$ des solutions faibles de Navier-Stokes vers des solutions fortes de l'équation d'Euler

$$\left\{ \begin{array}{rcl} \partial_t v + (v \cdot \nabla) v + \nabla q &=& 0 \\ \text{div } v &=& 0 \\ v_2|_{y=0} &=& 0 \end{array} \right. \tag{14}$$

(moyennant régularité de la condition initiale limite), et d'autre part de démontrer qu'il se produit une instabilité non-linéaire fondée sur l'instabilité spectrale de l'équation d'Euler pour certains profils, comme dans le cas Dirichlet traité par E. Grenier [47].

Nombre de travaux ont conclu à la convergence dans $L^2$ de solutions faibles de Navier-Stokes vers des solutions fortes d'Euler, lorsque la condition initiale limite permet l'existence de ces dernières, et lorsque la condition au bord est de type Navier. Pour commencer, le problème avec $a = 0$, c'est-à-dire la condition de Neumann, a été résolu par C. Bardos dans les années 1970 [6]. Considérons une condition de Navier de la forme

$$\partial_y u_1|_{y=0} = \frac{2a}{\varepsilon^\beta} u_1|_{y=0} \tag{15}$$

et $a,\ \beta \in \mathbb{R}$ (même si la condition n'est pas physiquement justifiée pour $a < 0$, nous arrivons à obtenir des résultats), et une condition initiale $u|_{t=0} = u_0$ avec $u_0 \in L^2(\Omega)$.

Dans le cas $\beta = 0$, de nombreux résultats ont été obtenus dans [23, 78, 64]. Pour notre résultat de convergence, nous nous sommes inspirés de la méthode de D. Iftimie et G. Planas dans [60], où un résultat de convergence $L^2$ est démontré. On y effectue une estimation d'énergie sur l'équation satisfaite par $w = u^\varepsilon - v$, avec $u^\varepsilon$ solution de Leray de (12)-(15) avec $\beta = 0$, et $v$ solution de (14) avec $v|_{t=0} \in H^s(\Omega)$ pour un $s > 2$, et on utilise les bornes uniformes sur $v$ pour boucler. Cette méthode suppose donc que nous avons une solution de l'équation d'Euler, chose montrée par Bardos dans [6] ; ceci diffère de l'article [6] lui-même et du travail présenté dans la partie suivante, où la solution du problème non-visqueux est construite par passage à la limite. Remarquons qu'en 2D, nous avons l'avantage de l'unicité des solutions globales.

Cette approche se généralise à la condition de Navier (15), et permet d'obtenir une plage de puissances $\beta$ sur laquelle la convergence $L^2$ vers Euler a lieu.

**Theorème 2.1.** *[93] Soit $(u_0^\varepsilon)_{\varepsilon>0}$ une suite de $L^2(\Omega)$ qui converge dans $L^2$ vers $v_0$. On suppose que $v_0 \in H^s(\Omega)$, pour un certain $s > 2$. Alors, pour tout $T > 0$, la suite $(u^\varepsilon)$ de solutions de (12)-(15) avec condition initiale $u_0^\varepsilon$ converge dans $L^\infty([0,T], L^2(\Omega))$ vers $v$, solution classique de l'équation d'Euler avec $v|_{t=0} = v_0$, à condition que*

- *soit $a > 0$ et $\beta < 1$,*
- *soit $a < 0$ et $\beta \leq 1/2$.*

Ceci est à comparer avec la plage trouvée par X-P. Wang, Y-G. Wang et Z. Xin dans [122], qui était $\beta < 1/2$ sans condition de signe sur $a$, mais avec une viscosité anisotrope : $\varepsilon(\partial_{xx}^2 + \partial_{yy}^2)$ est remplacé par $\sqrt{\varepsilon}\partial_{xx}^2 + \varepsilon\partial_{yy}^2$. Notamment, ici, l'échelle des couches limites, $\beta = 1/2$, est traitée. De surcroît, ce résultat s'étend aux dimensions supérieures pour des données limites plus régulières ($v_0 \in H^s$ avec $s > d/2 + 1$ en dimension $d$), et la convergence a lieu sur un intervalle éventuellement fini, l'existence globale d'une solution forte d'Euler n'étant plus assurée. Pour le démontrer, on doit cependant adapter les estimations d'énergie, $u^\varepsilon$ n'ayant plus la bonne intégrabilité en espace pour servir de fonction test : on part alors de l'inégalité d'énergie incluse dans la définition de solution faible de Navier-Stokes.

Dans le cas de données initiales pour l'équation limite moins régulières, C. Bardos, F. Golse et L. Paillard ont récemment montré que si une suite $(u^\varepsilon)$ converge dans $L^\infty([0,T], L^2)$ faible-$*$, elle converge vers une solution dissipative de l'équation d'Euler [7]. Ce résultat vaut pour (15) avec $\beta < 1$.

On revient au cas $\beta = 1/2$ (condition (13)). Parallèlement à cette convergence, il se produit une instabilité dans $L^\infty$ que nous présentons maintenant. On choisit des conditions initiales sous la forme de *profils de cisaillement* : $u_{\mathrm{sh}}(x,y) = (u_{\mathrm{sh}}(y), 0)$, où $u_{\mathrm{sh}}$, notation désignant indifféremment le vecteur ou sa première composante (et 'sh' comme dans l'anglais *shear flow*), est de classe $\mathcal{C}^\infty$, bornée et satisfait une condition de Navier. Associé à une pression constante, un tel champ de vecteur est une solution stationnaire de l'équation d'Euler.

On s'intéresse dans un premier temps à l'instabilité linéaire pour l'équation d'Euler de ces profils, c'est-à-dire aux valeurs propres de l'équation d'Euler linéarisée autour de $u_{\mathrm{sh}}$. En cherchant des solutions sous la forme

$$u(t,x,y) = \begin{pmatrix} e^{ik(x-ct)}\psi'(y) \\ -ike^{ik(x-ct)}\psi(y) \end{pmatrix},$$

et en prenant le rotationnel de l'équation, on arrive à l'équation différentielle ordinaire satisfaite par la fonction courant $\psi$ : l'équation de Rayleigh

$$(u_{\mathrm{sh}} - c)(\partial_{yy}^2 - k^2)\psi - u_{\mathrm{sh}}''\psi = 0. \tag{16}$$

Un profil $u_{\mathrm{sh}}$ est dit linéairement instable s'il existe $\lambda = -ikc$ de partie réelle positive ($\Leftrightarrow \mathrm{Im}(c) > 0$) et $\psi \in H_0^1(\mathbb{R}^+)$ non triviale qui soit solution de (16) ; la solution correspondante de l'équation d'Euler linéarisée est alors exponentiellement croissante en temps. Une analyse du spectre de

cette équation est faite dans la section 4 de [47]. On y trouve notamment que, pour qu'il y ait instabilité, le profil $u_{\rm sh}$ doit comporter un point d'inflexion $y_0 > 0$, chose connue depuis les travaux du Lord Rayleigh à la fin du XIX$^{\text{ème}}$ siècle [115]. Si, de plus, $-u''_{\rm sh}(y)/(u_{\rm sh}(y) - u_{\rm sh}(y_0))$ est positive et continue sur $\mathbb{R}^+$, on prend $c = u_{\rm sh}(y_0)$ et on divise (16) par $u_{\rm sh}(y) - u_{\rm sh}(y_0)$ pour transformer l'équation de Rayleigh en un problème de Sturm-Liouville. Alors, en utilisant un résultat de Z. Lin [72], on démontre dans la Proposition 4.1 du chapitre 1 que pour tout $\delta > 0$, la fonction

$$u_\delta(y) = \tanh(y - \delta) + \zeta$$

avec $\zeta$ choisi pour que $u_\delta$ satisfasse la condition de Navier, constitue un profil instable, fournissant ainsi un exemple explicite et simple de profil linéairement instable et lisse.

C'est à partir de ces profils linéairement instables pour l'équation d'Euler que l'on peut obtenir des solutions instables de l'équation de Navier-Stokes.

**Theorème 2.2.** *[93] Soit $u_{\rm sh}$ un profil de cisaillement linéairement instable pour l'équation d'Euler. En posant $Y = \varepsilon^{-1/2} y$, on génère un profil couche limite solution de l'équation de Navier-Stokes (avec pression constante) en résolvant l'équation de la chaleur :*

$$\left\{ \begin{array}{rcl} \partial_t \overline{u_{\rm sh}}(t,Y) - \partial^2_{YY} \overline{u_{\rm sh}}(t,Y) & = & 0 \\ \overline{u_{\rm sh}}(0,Y) & = & u_{\rm sh}(Y) \\ \partial_Y \overline{u_{\rm sh}}(t,0) & = & 2a \overline{u_{\rm sh}}(t,0), \end{array} \right.$$

*la condition au bord étant précisément la condition de Navier (13). Alors, pour tout $n \in \mathbb{N}^*$, il existe $\delta_0$ et $\varepsilon_0$ positifs tels que pour tout $0 < \varepsilon < \varepsilon_0$, il existe une donnée initiale $u_0^\varepsilon$ avec*

$$\left\| u_0^\varepsilon - u_{\rm sh}\left(\frac{y}{\sqrt{\varepsilon}}\right) \right\|_{H^s(\Omega)} \leq C\varepsilon^n$$

*pour un $s > 0$, telle que la solution $u^\varepsilon$ de (12)-(13) avec $u^\varepsilon|_{t=0} = u_0^\varepsilon$ satisfait, à un temps $T^\varepsilon \sim n \ln(\varepsilon^{-1})\sqrt{\varepsilon}$,*

$$\left\| u^\varepsilon(T^\varepsilon, x, y) - \overline{u_{\rm sh}}\left(T^\varepsilon, \frac{y}{\sqrt{\varepsilon}}\right) \right\|_{L^\infty(\Omega)} \geq \delta_0. \tag{17}$$

Remarquons que $T^\varepsilon$ tend vers 0, ainsi les solutions de Navier-Stokes sont amplifiées en temps arbitrairement court.

Ce résultat fait écho à celui obtenu par E. Grenier dans [47] avec la condition de Dirichlet, sauf que dans (17), la norme $L^\infty$ est minorée par $\varepsilon^{1/4}\delta_0$, ce qui ne garantit pas que la couche limite est amplifiée jusqu'à atteindre $\mathcal{O}(1)$. La raison réside dans le développement asymptotique des solutions approchées de l'équation réechelonnée. Nous l'expliquons à la fin de la preuve ci-après.

On note que les Théorèmes 2.1 et 2.2 ne se contredisent pas. Cependant, tels qu'ils sont énoncés, les deux théorèmes de cette partie ne peuvent être satisfaits simultanément dans $\Omega$, puisqu'un profil de cisaillement ne peut être dans $L^2(\mathbb{R}\times\mathbb{R}^+)$. Mais, avec des différences dans les preuves qui sont purement techniques, ils sont aussi valables dans le domaine $x$-périodique $\Omega_0 = \mathbb{T}\times\mathbb{R}^{+*}$ (voir [47], Grenier y fait principalement ses démonstrations dans le cas périodique). Dans ces conditions, si on trouve un profil de cisaillement $u_{\rm sh}(y) \in L^2(\mathbb{R}^+)$ qui soit instable pour l'équation d'Euler, nous aurons $u_{\rm sh}(\varepsilon^{-1/2}y) \in L^2(\Omega_0)$ converge vers 0 dans $L^2$ lorsque $\varepsilon$ tend vers 0, ainsi les deux phénomènes, convergence $L^2$ et instabilité non-linéaire dans $L^\infty$, auraient lieu simultanément. Dans la dernière partie du chapitre 1, nous construisons un exemple de profil instable $L^2$ à partir de la tangente hyperbolique en la tronquant avec une fonction exponentiellement décroissante, de sorte que le théorème d'instabilité linéaire de Z. Lin [72] s'applique toujours. Cet exemple montre que les deux théorèmes peuvent être satisfaits par une même famille de données initiales.

Un problème ouvert que l'on pourra regarder dans ce cadre est l'étude de l'équation de Prandtl (11) avec les conditions de Navier, et de relier cette étude avec les résultats obtenus ici. La théorie pour Prandtl dans ce cas n'est en effet pas aussi avancée qu'avec la condition de Dirichlet. Une autre prépublication récente [34] a montré que l'équation de Prandtl avec une condition mixte de Robin est bien posée dans le cadre analytique, un résultat analogue à celui de Sammartino et Caflisch dans [108]. Mais l'analogue du résultat d'Oleĭnik pour des données monotones n'est, à notre connaissance, pas démontré. Dans ce cas, nous constatons que la transformée de Crocco ne semble plus adaptée ; en effet, les nouvelles variables sont $(t, x, \eta = \tilde{u}(t, x, y))$, avec $\tilde{u}$ monotone en $y$, et le bord du domaine transformé pour $\eta$, qui est $\eta = 0$ lorsqu'on utilise la condition de Dirichlet, est inconnu lorsqu'un autre type de condition au bord est prescrit. Toutefois, N. Masmoudi et T-K. Wong ont récemment revisité la preuve du résultat d'Oleĭnik en utilisant des estimations d'énergie [87]. Il serait intéressant de tester cette approche sur d'autres conditions au bord.

## 2.2 Stratégie de preuve du Theorème 2.2

La démonstration du Theorème 2.2 suit un argument de type "instabilité linéaire implique instabilité non-linéaire", mis au point par E. Grenier dans [47]. Il s'agit de construire un développement WKB dont le premier terme est la solution de référence $\bar{u}$ et le suivant est une solution instable de l'équation linéarisée autour de $\bar{u}$. On construit par récurrence une solution approchée $u^{\rm ap}$ de la solution exacte de l'équation non-linéaire, $u$, avec assez de termes pour assurer que la norme $L^2$ de $u^{\rm ap} - u$ soit petite assez longtemps pour voir $u^{\rm ap} - \bar{u}$ amplifié jusqu'à l'ordre $\mathcal{O}(1)$ par l'instabilité spectrale. Cette stratégie a permis de démontrer l'instabilité non-linéaire de profils de cisaillement pour l'équation d'Euler [47], ainsi que l'instabilité de couches limites d'Ekman pour les fluides en rotation (Navier-Stokes avec force de Coriolis, voir B. Desjardins et E. Grenier [31] ; voir aussi F. Rousset [100], N. Masmoudi et F. Rousset [83] pour des résultats de stabilité dans ce contexte).

Nous commençons par rééchelonner le système, en se plaçant dans les variables rapides $(t, x, y) = \varepsilon^{-1/2}(t, x, y)$ (on n'introduit pas de nouvelle notation pour ces variables rééchelonnées). L'équation (12) devient alors un nouveau système de Navier-Stokes incompressible avec une viscosité $\sqrt{\varepsilon}$, et (13) devient une condition de Navier indépendante de $\varepsilon$. On construira une solution approchée de Navier-Stokes en partant d'une donnée initiale $\varepsilon^n$-proche de $u_{\mathrm{sh}}$. Pour un entier $N$ à choisir, et en posant $\delta = \varepsilon^n$, on propose un développement WKB,

$$u^{\mathrm{ap}}(t) = \overline{u_{\mathrm{sh}}}(\sqrt{\varepsilon}t) + \sum_{j=1}^{N} \delta^j U_j(t). \tag{18}$$

Chaque $U_j$ résout une équation de Navier-Stokes linéarisée, or nous voulons partir d'une solution instable de l'équation d'Euler : l'idée est d'écrire $U_j$ sous forme de termes intérieurs (qui résolvent la partie d'ordre 1 de l'équation) + couches limites. Comme nous sommes en présence d'une condition de Navier indépendante de $\varepsilon$, nous pouvons nous inspirer d'un théorème de D. Iftimie et F. Sueur [61], qui annonce que l'ansatz pour une condition de Navier est meilleur que (10), avec une amplitude plus petite pour la couche limite,

$$u^{\varepsilon}(t, x, y) \sim u^E(t, x, y) + \varepsilon^{1/4} u^B\left(t, x, \frac{y}{\varepsilon^{1/4}}\right). \tag{19}$$

Ainsi, on propose le développement asymptotique suivant pour $U_j$ : en notant $u^i$ les termes intérieurs et $u^b$ les couches limites, on écrit

$$U_j(t, x, y) = \sum_{m=0}^{8n-1} \varepsilon^{m/8} \left(u^{i,j,m}(t, x, y) + \varepsilon^{1/4} u^{b,j,m}\left(t, x, \frac{y}{\varepsilon^{1/4}}\right)\right). \tag{20}$$

On construit alors $u^{i,1,0}$ sous la forme d'un paquet d'ondes,

$$u^{i,1,0}(t, x, y) = \int_{\mathbb{R}} \varphi(k) e^{ikx + \lambda(k)t} v(k, y)\, dk.$$

La fonction $\varphi$ est paire, lisse et à support compact de sorte $k \in \mathrm{supp}(\varphi)$ implique qu'il existe $\lambda(k)$ de partie réelle positive et $v(k) \in H^1_0(\mathbb{R})$ non nulle de sorte que $e^{ikx+\lambda(k)t} v(k, y)$ soit solution de l'équation d'Euler linéarisée autour de $u_{\mathrm{sh}}$ ; autrement dit, le paquet d'ondes ne contient que des modes instables. De plus, on peut choisir les modes les plus instables : il existe $k_0 > 0$ tel que $\sup_{k\in\mathbb{R}} \mathrm{Re}(\lambda(k)) = \mathrm{Re}(\lambda(k_0)) := \sigma_0$, et on choisit $\varphi$ de sorte que $k_0$ soit dans l'intérieur de son support, et que, pour tout $k \in \mathrm{supp}(\varphi)$, $\mathrm{Re}(\lambda(k)) > 3\sigma_0/4$.

Le premier terme $u^{i,1,0}$ étant choisi, on peut construire les autres termes par récurrence : les termes intérieurs $u^{i,j,m}$ satisfont des équations d'Euler linéarisées autour de $u_{\mathrm{sh}}$ avec des termes sources que l'on contrôle, tandis que les couches limites $u^{b,j,m}$ résolvent des problèmes de Stokes avec des conditions de Neumann inhomogènes au bord. On montre alors que

$$\|U_j(t)\|_{H^1} \leq C_j (1 + t)^{1/4} \frac{e^{j\sigma_0 t}}{(1 + t)^{j/2}}. \tag{21}$$

D'une part, on montre cette borne pour les termes intérieurs à l'aide d'une estimation de résolvante pour l'équation d'Euler linéarisée : si le terme source est majoré par $e^{\gamma t}/(1+t)^{\alpha}$ avec $\gamma > \sigma_0$ et $\alpha > 0$, alors il en va de même pour la solution de l'équation (Théorème 3.1 du chapitre 1). D'autre part, les termes de couches limites sont estimés en utilisant la fonction de Green de l'équation de la chaleur. De plus, $U_1$ étant sous la forme d'un paquet d'ondes, et $k_0$ étant un point critique de la fonction $k \mapsto \mathrm{Re}(\lambda(k))$, un équivalent de $|u^{i,1,0}(t,X)|$ par la méthode de Laplace permet d'obtenir une minoration de $\|U_1(t)\|_{L^2(\omega(t))}$, avec $\omega(t) \subset \Omega$ borné bien choisi :

$$\|U_1(t)\|_{L^2(\omega(t))} \geq C_1(1+t)^{1/4}\frac{e^{\sigma_0 t}}{(1+t)^{1/2}}. \tag{22}$$

Nous obtenons que $u^{\mathrm{ap}}$ est une solution de l'équation de Navier-Stokes avec un terme source $R^{\mathrm{ap}}$, qui contient essentiellement les interactions non-linéaires $U_{j_1} \cdot \nabla U_{j_2}$ avec $j_1 + j_2 \geq N+1$,

$$\partial_t u^{\mathrm{ap}} + (u^{\mathrm{ap}} \cdot \nabla)u^{\mathrm{ap}} - \varepsilon \Delta u^{\mathrm{ap}} + p^{\mathrm{ap}} = R^{\mathrm{ap}}.$$

En choisissant $t \leq T_0^{\varepsilon}$, de sorte que

$$\frac{\delta e^{\sigma_0 T_0^{\varepsilon}}}{\sqrt{1+T_0^{\varepsilon}}} = 1$$

(remarquons que $T_0^{\varepsilon} \sim -n\ln(\varepsilon)$), on montre qu'il existe $\tau_0 > 0$, indépendant de $\delta$, permettant d'avoir

$$\|R^{\mathrm{ap}}(t)\|_{L^2} \leq C(1+t)^{1/4}\frac{\delta^{N+1}e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/2}} \tag{23}$$

pour tout $t \leq T_0^{\varepsilon} - \tau_0$. Une estimation d'énergie rapide sur l'équation satisfaite par $w = u^{\varepsilon} - u^{\mathrm{ap}}$, où $u^{\varepsilon}$ est la solution exacte de l'équation de Navier-Stokes avec la condition initiale $u^{\varepsilon}(0) = u^{\mathrm{ap}}(0)$, donne

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 \leq \left(2\|\nabla u^{\mathrm{ap}}\|_{L^\infty} + \frac{1}{2}\right)\|w(t)\|_{L^2}^2 + C\|R^{\mathrm{ap}}(t)\|_{L^2}^2. \tag{24}$$

C'est ici qu'il faut choisir $N$, l'ordre du développement WKB (18) : il existe $\tau_1 \geq \tau_0$, indépendant de $\delta$, qui permet d'avoir $\|\nabla u^{\mathrm{ap}}(t)\|_{L^\infty} \leq \|\nabla \overline{u_{\mathrm{sh}}}(t)\|_{L^\infty} + 1/4$ pour $t \leq T_0^{\varepsilon} - \tau_1$, et on choisit alors $N$ de sorte que $2N\sigma_0 > 2(\|\nabla \overline{u_{\mathrm{sh}}}(t)\|_{L^\infty} + 1)$ pour tout $t \leq T_0^{\varepsilon} - \tau_1$. Une variante du lemme de Grönwall assure alors que $w$ est petit : pour $t \leq T_0^{\varepsilon} - \tau_1$,

$$\|w(t)\|_{L^2(\Omega)} \leq C(1+t)^{1/4}\frac{\delta^{N+1}e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/2}}. \tag{25}$$

On combine ensuite (22) et (21) pour écrire que

$$\|u^{\mathrm{ap}}(t) - \overline{u_{\mathrm{sh}}}(t)\|_{L^2(\omega(t))} \geq \frac{C_1}{2}(1+t)^{1/4}\frac{\delta e^{\sigma_0 t}}{(1+t)^{1/2}}$$

pour $t \leq T_0^\varepsilon - \tau_2$, avec $\tau_2 \geq \tau_1$ toujours indépendant de $\delta$, et nous en sommes presque à la conclusion. L'inégalité triangulaire donne

$$\begin{aligned}\|u^\varepsilon(t) - \overline{u_{\rm sh}}(t)\|_{L^2(\omega(t))} &\geq \|u^{\rm ap}(t) - \overline{u_{\rm sh}}(t)\|_{L^2(\omega(t))} - \|u^\varepsilon(t) - u^{\rm ap}(t)\|_{L^2(\Omega)} \\ &\geq (1+t)^{1/4}\left[\frac{C_1}{2}\frac{\delta e^{\sigma_0 t}}{(1+t)^{1/2}} - C\frac{\delta^{N+1}e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/2}}\right] \\ &\geq (1+t)^{1/4}\frac{C_1}{4}\frac{\delta e^{\sigma_0 t}}{(1+t)^{1/2}}\end{aligned}$$

pour $t \leq T_0^\varepsilon - \tau_3$, soit, en fixant $T > \tau_3$ et $\varepsilon$ assez petit pour que $T^\varepsilon = T_0^\varepsilon - T > 0$, on a

$$\|u^\varepsilon(T^\varepsilon) - \overline{u_{\rm sh}}(T^\varepsilon)\|_{L^2(\omega(T^\varepsilon))} \geq (1+T^\varepsilon)\frac{C_1 e^{-\sigma_0 T}}{4}.$$

Le domaine borné $\omega(T^\varepsilon)$ est choisi de mesure $\sqrt{1+T^\varepsilon}$, ce qui donne la minoration $L^\infty$ voulue par $\delta_0 = \frac{C_1}{4}e^{-\sigma_0 T}$, indépendent de $\delta$. Le temps d'instabilité est bien $\mathcal{O}(-n\ln(\varepsilon)\sqrt{\varepsilon})$ dans l'échelle des variables lentes d'origine. □

*Remarque :* l'amplitude $\mathcal{O}(\varepsilon^{1/4})$ des couches limites dans (19) permet de compenser la dérivation par rapport à $y$, et ceci joue un rôle central dans l'obtention de l'instabilité. Dans le cas des conditions de Dirichlet, l'amplitude des couches limites étant $\mathcal{O}(1)$, comme on voit dans (10), les dérivées ont une amplitude $\mathcal{O}(\varepsilon^{-1/4})$. On doit alors choisir des temps inférieurs à $T_1^\varepsilon \sim -(n-1/4)\ln(\varepsilon)$, ce qui mène à minorer $\|u^\varepsilon - \overline{u_{\rm sh}}\|_{L^\infty}$ par $\varepsilon^{1/4}\delta_0$. Toutefois, la norme $L^\infty$ de la vorticité en $T^\epsilon$ tend vers l'infini lorsque $\varepsilon$ tend vers 0.

Pour arriver à l'instabilité non-linéaire identique à la nôtre avec les conditions de Dirichlet, l'instabilité linéaire pour l'équation d'Euler ne suffit donc pas. Ainsi, E. Grenier, Y. Guo et T. Nguyen ont travaillé sur l'équation d'Orr-Sommerfeld (Rayleigh visqueux), qui correspond à la recherche de solutions qui s'écrivent comme gradient de fonctions courants, de l'équation de Navier-Stokes linéarisée autour d'un profil de cisaillement. Leurs résultats figurent dans la prépublication [49].

# 3 Sur la limite non-visqueuse de l'équation de Navier-Stokes compressible 3D avec conditions de Navier

*Les résultats de cette section sont développés dans le chapitre 2.*

## 3.1 Cadre fonctionnel et résultats

On considère dans cette partie le système Navier-Stokes isentropique (compressible)

$$\partial_t \rho + \text{div}\,(\rho u) = 0 \tag{26}$$

$$\partial_t(\rho u) + \text{div}\,(\rho u \otimes u) = \text{div}\,\Sigma + \rho F, \tag{27}$$

avec $F$ une force régulière donnée, et $\Sigma$ le tenseur des contraintes

$$\Sigma = 2\varepsilon\mu(\rho)Su + (\varepsilon\lambda_0(\rho)\text{div } u - P(\rho))\mathbb{I}_3, \tag{28}$$

dans lequel $\mu$ et $\lambda_0$ sont des fonctions régulières de $\rho$, telles que $\mu > 0$ et $\lambda := \mu + \lambda_0 > 0$. La pression est donnée par la loi barotrope $P(\rho) = \frac{k}{\gamma}\rho^\gamma$, avec $\gamma > 1$ et $k > 0$. Cette fois, on se place dans le demi-espace 3D, $\Omega = \mathbb{R}^2 \times \mathbb{R}^{+*}$, avec les conditions (7) et (9) au bord, dans lesquelles le coefficient de glissement sera pris de la forme $\varepsilon a$, avec $a \in \mathbb{R}$.

On décompose, dans toute cette partie, $X \in \Omega$ sous la forme $X = (x, y, z)$ avec $(x, y) \in \mathbb{R}^2$ et $z > 0$. La normale à $\partial\Omega$ étant constante, égale à $\vec{n} = (0, 0, -1)$, on étend à tout l'espace la notion de composante tangentielle : ainsi la vitesse tangentielle sera $u_\tau(t, X) = (u_1, u_2)(t, X)$ pour tout $X \in \bar{\Omega}$. Ainsi, les conditions au bord s'écrivent :

$$u_3|_{z=0} = 0 \tag{29}$$

$$(\mu(\rho)\partial_z u_\tau)|_{z=0} = 2a u_\tau|_{z=0}. \tag{30}$$

Vu qu'on va s'intéresser à des solutions fortes, et pour éviter l'écriture de conditions de compatibilité à un ordre élevé, on impose que le fluide soit au repos pour des temps négatifs, $(\rho, u)(t, X) = (1, 0)$ et $F(t, X) = 0$ pour $t < 0$, et que $F$ soit régulière au voisinage de $t = 0$. On s'intéresse à nouveau à la limite lorsque $\varepsilon$ tend vers 0 des solutions de (26)-(27)-(29)-(30).

Ce système possède une structure hyperbolique-parabolique quasi-linéaire symétrisable, type de système sur lequel on a un grand nombre résultats de stabilité de couches limites dans le cas où le bord est non-caractéristique ; évoquons par exemple le fascicule de G. Métivier et K. Zumbrun [88] (qui étend les résultats de [46, 48]) et le papier de F. Ancona et S. Bianchini [4], ce dernier traitant de la stabilité dans la classe BV. Des analyses de couches limites appliquées au système Navier-Stokes compressible ou à la magnétohydrodynamique figurent dans [101] par F. Rousset quand le bord est caractéristique, et dans [52] par O. Guès, G. Métivier, M. Williams et K. Zumbrun dans le cas non-caractéristique.

Assez peu de résultats sont connus sur le système Navier-Stokes isentropique lorsque les coefficients de viscosité sont variables. On n'a pas, comme dans le cas à coefficients constants avec P-L. Lions [73] ou E. Feireisl, A. Novotný et H. Petzeltová [38], des théorèmes d'existence globale de solutions faibles en général, mais seulement dans des cas très précis - D. Bresch, B. Desjardins et D. Gérard-Varet montrent dans [18] (basé sur [16, 19, 17] sur les modèles de Korteweg ou d'eau peu profonde) l'existence de solutions faibles sous la contrainte $\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho))$, lorsqu'une force de traînée est considérée ($F = F(u) = r_0|u|u$).

Il y a également peu de résultats sur la limite non-visqueuse dans ce cadre. Le système limite est l'équation d'Euler isentropique

$$\left\{ \begin{array}{rcl} \partial_t\rho + \text{div } (\rho u) & = & 0 \\ \partial_t(\rho u) + \text{div } (\rho u \otimes u) & = & -k\rho^{\gamma-1}\nabla\rho + \rho F. \\ u_3|_{z=0} & = & 0 \end{array} \right. \tag{31}$$

Dans le cadre des coefficients de viscosité constants, F. Sueur a montré dans [116] un résultat de convergence $L^2$ analogue au premier point du Theorème 2.1 (avec le coefficient de glissement dépendant de $\varepsilon$), tandis que Y-G. Wang et M. Williams ont construit dans [123] un développement WKB (similairement à ce qui se fait pour le Theorème 2.2) pour obtenir la convergence $L^\infty$. L'un des outils utilisés dans cet article est la *dérivée conormale*. Tandis que les dérivées partielles habituelles peuvent être associées aux vecteurs de la base canonique, les dérivées conormales découlent, elles, de champs de vecteurs, pouvant dépendre de $X \in \Omega$, qui sont tangents au bord. Dans le cas de notre domaine à bord plat, nous choisissons les dérivées conormales suivantes :

$$Z_0 = \partial_t,\ Z_{1,2} = \partial_{x,y},\ Z_3 = \phi(z)\partial_z,$$

avec $\phi(z) = z/(1+z)$. Comme $\phi(0) = 0$, $Z_3$ correspond bien à un champ de vecteurs tangent au bord.

A partir de ces dérivées conormales, on peut définir les espaces de Sobolev conormaux, qui seront le cadre des travaux présentés dans cette partie. Ces espaces ont servi dans l'étude de systèmes hyperboliques à bord caractéristique (par exemple [98, 51, 111]), mais également dans des problèmes de couches limites comme l'article [123] évoqué juste avant, ou [85] que nous évoquerons sous peu. En effet, en reprenant un ansatz comme (10) en 3D,

$$u^\varepsilon(t,x,y,z) = u^E(t,y,z) + u^B(t,x,y,\varepsilon^{-1/2}z),$$

la dérivée normale de la couche limite a une amplitude en $\mathcal{O}(\varepsilon^{-1/2})$, alors que sa dérivée conormale est d'amplitude $\mathcal{O}(1)$ sur un voisinage de taille $\sqrt{\varepsilon}$ du bord.

L'espace de Sobolev conormal $W^{m,p}_{co}([0,T]\times\Omega)$ au sens d'O.Guès [51] est l'ensemble des fonctions $f \in L^p([0,T]\times\Omega)$ telles que, pour tout multi-indice $\alpha$ de longueur inférieure ou égale à $m$, $Z^\alpha f \in L^p([0,T]\times\Omega)$, avec la notation $Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$. Remarquons que dans notre cadre, la dérivée temporelle $Z_0 = \partial_t$, qui n'est pas tangente au bord de $[0,T]$, est considérée comme conormale. Nous utiliserons $W^{1,\infty}_{co}([0,T]\times\Omega)$, $W^{m,2}_{co}([0,T]\times\Omega) := H^m_{co}([0,T]\times\Omega)$ et un sous-espace de ce dernier

$$X^m_T = \{f(t,X) \mid \forall 0 \le k \le m,\ \partial_t^k f \in L^\infty([0,T], H^{m-k}_{co}(\Omega))\}.$$

L'espace $H^m_{co}(\Omega)$ est l'espace de Sobolev conormal sur $\Omega$, qui ne tient compte que des dérivées conormales spatiales $Z_1$, $Z_2$ et $Z_3$.

L'utilisation d'estimations d'énergie dans les espaces de Sobolev conormaux permet d'obtenir des résultats d'existence locale sur des temps uniformes en $\varepsilon$ sans passer par le calcul de solutions approchées (méthode qu'utilisent Wang et Williams [123]). Elle permet aussi de résoudre le problème de la limite non-visqueuse dans ce cadre sans avoir, au préalable, l'existence d'une solution de l'équation d'Euler compressible - celle-ci se construit par passage à la limite, comme dans [6] pour l'équation d'Euler incompressible (passage à la limite non-visqueuse dans Navier-Stokes avec condition de Neumann). Cet outil a permis à N. Masmoudi et F. Rousset d'obtenir

les résultats analogues sur l'équation de Navier-Stokes incompressible avec condition de Navier [85], ainsi que l'équation de Navier-Stokes incompressible à surface libre [84].

Enonçons précisément les résultats. A $t$ fixé, on introduit les notations

$$\|f(t)\|_{m,p}^2 = \sum_{|\alpha|\leq m} \|Z^\alpha f(t)\|_{L^p}^2 \quad \text{et} \quad |||f|||_{m,p,T} = \sup_{t\in[0,T]} \|f(t)\|_{m,p}\,.$$

On omet l'indice $p$ lorsque $p = 2$. Soit $E_T^m$ l'ensemble des champs de vecteurs de $\mathbb{R}^4$ sous la forme $V = (r, v)(t, X)$ avec $r$ scalaire, tels que :

$$\mathcal{E}_m(T,V) := |||V|||_{m,T}^2 + |||\partial_z v_\tau|||_{m-1,T}^2 + \int_0^T \|\partial_z v_3(s)\|_{m-1}^2 + \|\partial_z r(s)\|_{m-1}^2 \; ds$$
$$+ |||\partial_z v_\tau|||_{1,\infty,T}^2 + \int_0^T \|\partial_z r(s)\|_{1,\infty}^2 + \|\partial_t\partial_z r(s)\|_{1,\infty}^2 \; ds < +\infty.$$

Nécessiter un contrôle $W_{co}^{1,\infty}$ des dérivées est habituel dans les systèmes à bord caractéristique, *cf* [51]. Nous obtenons alors les résultats suivants.

**Theorème 3.1.** *Soient $m \geq 7$, et on suppose que, pour tout $T > 0$,*

$$\sup_{t\in[-T,T]} \|F(t)\|_m^2 + \|\nabla F(t)\|_{m-1}^2 + \|\nabla F(t)\|_{2,\infty}^2 < +\infty.$$

*Les coefficients de viscosité sont des fonctions de classe $\mathcal{C}^m$ bornées et positives. Le coefficient de glissement a n'a pas de signe imposé.*

- *Existence uniforme de solutions. Pour tout $\varepsilon_0 > 0$, il existe $T^*$ tel que, pour tout $0 < \varepsilon < \varepsilon_0$, le système de Navier-Stokes isentropique (26)-(27)-(29)-(30) admet une unique solution $U = (\rho - 1, u)$ dans $E_{T^*}^m$. De plus, il existe $c > 0$ tel que $\rho(t,x) > c$ pour $t \in [0, T^*]$, $x \in \Omega$ ; autrement dit, sur un temps indépendant de $\varepsilon$, il n'y a pas de vide qui se crée.*
- *Limite non-visqueuse. Lorsque $\varepsilon$ tend vers 0, la famille $(U^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ ainsi constituée de solutions de Navier-Stokes converge dans $L^2([0,T^*]\times\Omega)$ et dans $L^\infty([0,T^*]\times\Omega)$ vers l'unique solution dans $E_{T^*}^m$ de l'équation d'Euler isentropique (31).*

Comme nous avons déjà dit, il s'agit résultats analogues à ceux de Masmoudi et Rousset [85, 84] qui sont obtenus avec la même tactique de preuve. Il est possible que cette même approche permette d'obtenir des résultats identiques sur le système de Navier-Stokes complet, dans lequel on tient compte de l'équation sur l'énergie (3) dans laquelle on suppose $e = C\theta$, permettant de remplacer cette équation par l'équation sur la température,

$$\rho\partial_t\theta + (\rho u\cdot\nabla)\theta - \varepsilon\kappa\Delta\theta - \Sigma : \nabla u + P\text{div } u = 0,$$

où $A : B$ désigne le produit contracté de deux matrices $A$ et $B$, $A : B = \sum_{i,j} A_{i,j}B_{i,j}$. Mais, si on peut obtenir des estimations sur $\partial_z P$ comme dans le cas isentropique, il est difficile d'obtenir

une autre combinaison de $\partial_z \rho$ et $\partial_z \theta$ à cause de leur couplage avec $\partial_z u_3$. Dans ce cas, on ne peut que faire des estimations sur le système couplé comme à la première étape de la preuve du Theorème 3.1. Ce travail n'est pas achevé, mais nous avons les estimations d'énergie $H^{m-1}_{co}$ grâce à un prolongement par symétrie (paire/impaire) de $\partial_z U$ à l'espace tout entier, après relèvement de la condition de Navier pour les composantes tangentielles de $\partial_z u$. L'estimation $W^{1,\infty}_{co}$ restante passe vraisemblablement par l'étude de la fonction de Green du système, sur le modèle de [77]. Pour des avancées sur le système complet, on renvoie le lecteur aux articles de F. Huang, Y. Wang et T. Yang (convergence vers des solutions de Riemann d'Euler, [59]) et le travail récent d'E. Feireisl et A. Novotný sur le système Navier-Stokes-Fourier [37], qui est un résultat de convergence "faible-fort", du même type que notre Theorème 2.1 dans le cadre incompressible, ou que le résultat de F. Sueur [116] dans le cas isentropique.

## 3.2 Stratégie de preuve du Theorème 3.1

Rappelons que $U = (\rho - 1, u)$. La clé de la démonstration du Theorème 3.1 est l'estimation suivante.

**Theorème 3.2.** *Sous les conditions du Theorème 3.1, il existe $\varepsilon_0 > 0$, $T > 0$ et une constante $C$ tels que, pour tout $0 < \varepsilon \leq \varepsilon_0$, $\mathcal{E}_m(T, U) + |||\partial_z u_3|||^2_{1,\infty,T} \leq C$.*

Cette estimation en poche, le théorème se démontre rapidement. Un résultat d'existence locale (sur des temps qui dépendent de $\varepsilon$) de solutions fortes pour le système Navier-Stokes compressible, comme dans la section 4.1 de [123], et un argument de *bootstrap* avec l'estimation uniforme ci-dessus permettent de rendre le temps d'existence indépendant de $\varepsilon$. Pour la limite non-visqueuse, on obtient par un argument de compacité (voir Proposition 2.2 et conséquences dans le chapitre 2) la convergence faible globale et la convergence forte locale dans $L^2$ de $U^\varepsilon$ vers un champ de vecteurs $V$, avec $V + (1, 0)$ solution de l'équation d'Euler. La convergence forte découle de la convergence de l'énergie classique de $U^\varepsilon$

$$\int_\Omega \rho^\varepsilon \frac{|u^\varepsilon|^2}{2} + \frac{P(\rho^\varepsilon) - k/\gamma - k(\rho^\varepsilon - 1)}{\gamma - 1}\, dx$$

vers celle de $V$.

Le gros du travail est cependant la démonstration de l'estimation *a priori* du Theorème 3.2. Il s'agit de montrer qu'il existe une fonction $Q$ telle que

$$\mathcal{E}_m(t, U) + |||\partial_z u_3|||^2_{1,\infty,t} \leq M_0 + (t + \varepsilon) Q(\mathcal{E}_m(t, U) + M_F(t)), \tag{32}$$

où $M_F(t)$ dépend de $F$ et $M_0$ des valeurs en $t = 0$ (sous les hypothèses du Theorème 3.1, $M_0 = 0$). Ceci se fait en bornant chaque morceau du membre de gauche de (32) individuellement. Les bornes sur les termes qui sont $L^2$ en espace sont obtenues par des estimations d'énergie (multiplication de l'équation satisfaite par chaque terme par sa solution et intégration), dans lesquelles il faut contrôler les commutateurs entre les dérivées conormales et les opérateurs

différentiels en jeu. Les normes $L^\infty$ en espace se contrôlent avec différentes techniques selon la structure de l'équation du terme considéré (principe du maximum, formule de Duhamel), mais dans tous les cas, on utilisera une inégalité de Sobolev valide pour les espaces conormaux,

$$|||f|||^2_{\infty,T} \leq C \int_0^T \|f(t)\|_3^2 + \|\partial_z f(t)\|_2^2 \ dt,$$

pour borner les commutateurs. En combinant ces bornes, on ferme l'estimation (32) pour $m \geq 7$, limitation dûe essentiellement aux indices qui apparaissent lors des applications de l'inégalité de Sobolev. Nous expliquons à présent les étapes de la preuve, suivies des principales idées pour traiter les commutateurs qui y apparaissent.

Les étapes et techniques. Pour alléger les notations, considérons dans cette introduction que les coefficients de viscosité sont constants. Durant le processus d'estimation, on suppose que $\rho(t, X)$ est uniformément borné par au-dessous par une constante $0 < c < 1$ sur le temps d'étude.

On commence par appliquer aux équations (26)-(27) une dérivée conormale $Z^\alpha$ d'ordre $m$, et réécrit le résultat sous la forme d'un système dont la partie hyperbolique est symétrisable,

$$A_0(\rho)\partial_t Z^\alpha U + \sum_{j=1}^{3} A_j \partial_{x_j} Z^\alpha U - \varepsilon(0, \mu\Delta Z^\alpha u + \lambda\nabla \text{div } (Z^\alpha u)) = (0, Z^\alpha(\rho F)) - \mathcal{C}^\alpha, \qquad (33)$$

où les $A_j$ sont des matrices $4 \times 4$ qui dépendent de $U$, et $\mathcal{C}^\alpha$ contient des commutateurs dont nous discuterons après - on y trouve, par exemple,

$$[Z^\alpha, A_3\partial_z]U = Z^\alpha(A_3\partial_z U) - A_3\partial_z Z^\alpha U. \qquad (34)$$

La multiplication par la matrice diagonale définie positive $D = \text{diag}(k\rho^{\gamma-2}, \mathbb{I}_3)$ donne des matrices $DA_j$ symétriques. En prenant alors le produit scalaire de $D\cdot(33)$ par $Z^\alpha U$, et en remarquant que $DA_0$ est supposé uniformément borné par en-dessous, nous arrivons à une estimation d'énergie par intégrations par parties. On utilise ensuite l'inégalité d'injection de Sobolev pour traiter les normes $L^\infty$ (sauf celles des dérivées par rapport à $z$) dans cette estimation.

Pour estimer $\partial_z U$, nous travaillons composante par composante, obtenant pour chacune des estimations $H^{m-1}_{co}$ et $W^{1,\infty}_{co}$. Les estimations dans $X^{m-1}_T$ de la dérivée des composantes tangentielles de $u$ s'obtiennent par simple estimation d'énergie, une fois effectuée le changement d'inconnue $W = \mu(\text{rot } u)_\tau - 2au^\perp_\tau$, avec rot $u$ le rotationnel de $u$ et $u^\perp_\tau = (-u_2, u_1)$. Ce passage par la vorticité simplifie les intégrations par parties, puisque $Z^\alpha W|_{z=0} = 0$ pour tout $\alpha \in \mathbb{N}^4$ d'après la condition de Navier (30), et permet de voir qu'une dérivée normale n'entraîne pas de perte de dérivée conormale supplémentaire (dans [51] par exemple, une dérivée normale sur les composantes tangentielles "coûte" deux dérivées conormales). Pour l'estimation $W^{1,\infty}_{co}$, on utilise un principe du maximum pour l'équation de transport-diffusion $a\partial_t f + b\cdot\nabla f - \mu\Delta f = g$.

Les estimations sur la dérivée de la composante normale de $u$ s'obtiennent en remarquant que, d'après l'équation (26),

$$\rho\partial_z u_3 = -\partial_t\rho - u\cdot\nabla\rho - \rho(\partial_x u_1 + \partial_y u_2),$$

où, du fait de l'annulation de $u_3$ au bord (inégalité de Hardy, utilisée dans le contrôle de couches limite dans de nombreux contextes, par exemple [81]), toutes les dérivées dans le membre de droite sont conormales. En faisant boucler l'estimation $H^{m-1}_{co}$ avec celle dans $W^{1,\infty}_{co}$, on se ramène à une inégalité de type (32) pour $\|\partial_z u_3\|_{H^{m-1}_{co}([0,T]\times\Omega)}$.

Il reste à traiter $\partial_z\rho$. La dérivée par rapport à $z$ de l'équation (26) contenant $\partial_{zz}u_3$, on la multiplie par $\varepsilon$ pour pouvoir remplacer $\varepsilon\partial_{zz}u_3$ par son expression dans la troisième composante de (27). On se ramène ainsi à une équation du type

$$\varepsilon(\partial_t + u\cdot\nabla)R + \rho P'(\rho)R = h,$$

avec $R = \partial_z\rho$ et $\rho P'(\rho) > cP'(c) > 0$. On fait l'estimation d'énergie conormale standard (appliquer $Z^\alpha$, mettre de côté les commutateurs et multiplier par $Z^\alpha R$) sur cette équation pour obtenir les bornes $H^{m-1}_{co}$, tandis que pour les bornes $W^{1,\infty}_{co}$ (qui sont $L^2$ en temps), on considère $R$ le long du flot de $u$ pour écarter le terme de transport, et on applique la formule de Duhamel de l'EDO $\varepsilon f' + \tilde\rho P'(\tilde\rho)f = \tilde h$, où $\tilde\rho$ est $\rho$ suivant le flot de $u$. Maintenant qu'on a (32) moyennant notre hypothèse de borne uniforme sur $\rho$, nous concluons la preuve du Theorème 3.2 avec un argument de *bootstrap* pour d'une part récupérer la borne sur $\rho$, et d'autre part obtenir l'uniformité en $\varepsilon$ du temps sur lequel l'estimation est valide.

Les commutateurs. Le commutateur donné en exemple dans (34) montre les deux phénomènes de non-commutation qui ont lieu avec les dérivées conormales. D'une part, comme les dérivées habituelles, elles ne commutent pas avec la multiplication par une fonction. Si $\alpha$ est de longueur $m$, $f \in H^m_{co}([0,T]\times\Omega)$ et $g$ telle que, pour chaque $j$, $Z_j g \in H^{m-1}_{co}([0,T]\times\Omega)$, la formule de Leibniz et l'estimation douce dans les espaces de Sobolev conormaux (Proposition 3.1 du chapitre 2) donnent

$$\int_0^T \|[Z^\alpha, g]f(t)\|_0^2\ dt \le C\sum_{j=0}^{3}\int_0^T |||Z_j g|||^2_{\infty,T}\,\|f(t)\|^2_{m-1} + |||f|||^2_{\infty,T}\,\|Z_j g(t)\|^2_{m-1}\ dt. \tag{35}$$

D'autre part, la dérivée conormale $Z_3 = \phi(z)\partial_z$ ne commute pas avec $\partial_z$ ; en effet, $[Z_3, \partial_z] = -\phi'(z)\partial_z$. Ainsi, si $f$ est telle que $\partial_z f \in H^{m-1}_{co}([0,T]\times\Omega)$ et $\alpha$ est de longueur $m$,

$$\int_0^T \|[Z^\alpha, \partial_z]f(t)\|_0^2\ dt \le C\int_0^T \|\partial_z f(t)\|^2_{m-1}\ dt. \tag{36}$$

Pour la plupart des commutateurs avec des fonctions et/ou $\partial_z$, ces inégalités suffisent. Mais à l'étape de l'estimation d'énergie de $W \sim \partial_z u_\tau$, on voit apparaître le commutateur $[Z^\alpha, u_3\partial_z]W$,

avec $\alpha$ de longueur $m-1$. La combinaison directe de (35) et de (36) mène à l'apparition de la norme $H^{m-2}_{co}$ de $\partial_{zz}u_\tau$, que nous ne pouvons borner indépendamment de $\varepsilon$. Cependant, grâce à la condition de non-pénétration (29), on peut utiliser une inégalité de Hardy sur le commutateur $[Z_3, u_3]\partial_z W$, ce qui permet de ne pas perdre de dérivée normale supplémentaire, au prix de l'apparition de la norme $W^{1,\infty}_{co}$ de $\partial_z u_3$ (Proposition 3.7 du chapitre 2).

Enfin, $Z_3$ ne commute pas avec $\partial_{zz}$, qui apparaît dans les termes d'ordre deux. Par exemple, $[Z_3, \partial_{zz}] = -2\phi'(z)\partial_{zz} - \phi''(z)\partial_z$. Comme nous l'avons dit, il n'est pas souhaitable d'avoir des normes de $\partial_{zz}U$, mais là où il y a un commutateur avec $\partial_{zz}$, il y a toujours un facteur $\varepsilon$, donc la combinaison de l'application de (36) sur les termes à une dérivée normale avec le remplacement de $\varepsilon\partial_{zz}u$ par son expression dans (27) permet le plus souvent de gérer ces commutateurs.

# 4 Sur l'instabilité transverse d'ondes solitaires de l'équation d'Euler-Korteweg 2D

*Les résultats de cette section sont développés dans le chapitre 3.*

## 4.1 Le modèle Euler-Korteweg

Dans cette dernière partie, on s'intéresse à un modèle d'Euler (sans viscosité) isentropique, avec une pression qui dépend de $\rho$ et de $\nabla\rho$. En effet, nous considérons au sein d'un fluide les effets de capillarité interne. La manifestation la plus reconnaissable du phénomène de capillarité est le ménisque ; dans un tube étroit (un tube à essais par exemple), la surface de l'eau n'est pas plate, elle remonte un peu la paroi ([8] p.67). La prise en compte de la capillarité à l'intérieur du fluide est justifiée lorsque l'on considère un fluide comportant plusieurs phases dont les interfaces sont diffus, tels les mélanges liquide-vapeur [14]. Etudiée à la fin du XIX$^{\text{ème}}$ siècle par J. van der Waals et D. Korteweg (voir par exemple [68]), la capillarité est modélisée par l'énergie libre

$$F(\rho, \nabla\rho) = F_0(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2$$

dont on dérive la pression,

$$P(\rho, \nabla\rho) = \rho\frac{\partial F}{\partial\rho} - F = P_0(\rho) + \frac{1}{2}(\rho K'(\rho) - K(\rho))|\nabla\rho|^2. \tag{37}$$

La fonction $F_0$ est donnée, et permet d'obtenir la partie standard de la pression $P_0$ (par exemple, une loi barotrope comme dans la section précédente), tandis que $K$ est le coefficient de capillarité, dépendant de $\rho$. En posant $P_0'(\rho) = \rho g_0'(\rho)$, on va étudier le système suivant dans le plan $\mathbb{R}^2$ :

$$\partial_t\rho + \text{div }(\rho u) = 0 \tag{38}$$

$$\partial_t u + (u\cdot\nabla)u + \nabla(g_0(\rho)) = \nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right). \tag{39}$$

On s'intéresse aux *ondes solitaires* de ce système : avec la notation $X = (x, y) \in \mathbb{R}^2$, ce sont des solutions sous forme d'onde progressive avec une vitesse de propagation $c \in \mathbb{R}$ :

$$Q_c(t, x) = (\rho_c(x - ct), u_c(x - ct), 0). \tag{40}$$

Un exemple de phénomène naturel modélisé par une onde solitaire est le mascaret, qui consiste d'une vague unique issue d'une marée montante qui remonte un fleuve. Parmi les premiers à avoir étudié les ondes solitaires de la mécanique des fluides et les problèmes de stabilité qui leur sont associés, que nous détaillons dans la partie suivante, figurent J. Boussinesq lors de la deuxième moitié du XIX$^{\text{ème}}$ siècle [15]. De nombreux systèmes admettent des solutions sous forme d'onde solitaire ; nous en citerons dans la section suivante.

Pour Euler-Korteweg, les propriétés hamiltoniennes de l'équation assurent l'existence de telles solutions [11], avec n'importe quelle vitesse $c$ et de deux sortes. Toutes admettent des limites $Q_\pm$ en $\pm\infty$, et on distingue les solitons, vérifiant $Q_+ = Q_-$, et les coudes (en anglais *kink*), aux limites différentes à l'infini. Ces derniers doivent toutefois satisfaire une propriété de type Rankine-Hugoniot. On trouvera des propriétés fines sur ces ondes solitaires dans l'article de S. Benzoni-Gavage, R. Danchin, S. Descombes et D. Jamet [14].

Considérons un soliton, satisfaisant donc

$$\lim_{z \to \pm\infty} Q_c(z) = Q_\infty = (\rho_\infty, u_\infty, 0),$$

avec $\rho_\infty > 0$ et supposons que

$$\rho_\infty g_0'(\rho_\infty) > (u_\infty - c)^2, \tag{41}$$

ce qui permet d'interpréter le point $Q_\infty$ comme un équilibre point-selle de l'équation différentielle ordinaire satisfaite par $Q_c$. Nous nous intéresserons aux perturbations de $Q_c$. L'existence locale en 2D de solutions fortes avec des données initiales proches de $Q_c$ est assurée par l'article de S. Benzoni, R. Danchin et S. Descombes [13] ; il repose sur le changement de variables $w = \sqrt{K/\rho}\,\nabla\rho$ et $z = u + iw$, et la réécriture de (39) sous la forme d'une équation de type Schrödinger

$$\partial_t z + u \cdot \nabla z + i\nabla z \cdot w + i\nabla(\mathcal{A}(\rho)\mathrm{div}\ z) = \mathcal{Q}(\rho),$$

pour deux fonctions $\mathcal{A}$ et $\mathcal{Q}$ que l'on n'explicite pas. Les auteurs effectuent ensuite des estimations d'énergie à poids sur cette équation, estimations qui sont plus simples lorsque la vitesse $u$ est potentielle, c'est-à-dire qu'elle s'écrit comme $u = \nabla\varphi$. Notre résultat sera valable dans le cas potentiel, en utilisant les estimations de [13] - les obtenir différemment en travaillant sur l'équation satisfaite par $(\rho, \varphi)$ est un travail en cours.

## 4.2 Instabilité transverse de solitons

La notion de stabilité habituelle est celle de Lyapounov : si la donnée initiale est proche de $Q_c$, alors la solution reste proche de $Q_c(\cdot - ct)$ en temps long. Mais cette notion est insatisfaisante pour

les ondes solitaires. Considérons, en effet, deux ondes progressives de vitesses $c \neq c'$ proches, et prenons $Q_c$ proche pour une certaine norme de $Q_{c'}$. Du fait des vitesses de propagation différentes, ces ondes vont s'éloigner l'une de l'autre, et $Q_c$ ne sera pas stable pour la norme choisie. Pourtant, les profils n'évoluant pas, si on les réaligne, en translatant $Q_c(x-ct)$ de $(c'-c)t$ (ce qui revient à faire un changement de variable, ainsi on compare $Q_c(x-c't)$ et $Q_{c'}(x-c't)$), on observe deux profils proches ; en fait, on retrouve la même image qu'en $t=0$.

La bonne notion de stabilité pour les ondes solitaires est donc celle de *stabilité orbitale*, ou de stabilité modulo les translations, qui ne modifient pas le profil de l'onde. On définit clairement ainsi : une onde solitaire $Q_c$ est orbitalement stable pour la norme $\|\cdot\|$ si, pour tout $\varepsilon > 0$ et $t > 0$, il existe $\delta > 0$ telle que pour toute donnée initiale $(\rho_0, u_0)$ satisfaisant $\|(\rho_0, u_0) - Q_c\| < \delta$, on ait que la solution $(\rho, u)$ de (38)-(39) issue de $(\rho_0, u_0)$ vérifie

$$\inf_{a\in\mathbb{R}} \|(\rho,u)(t) - Q_c(\cdot - ct - a)\| < \varepsilon.$$

On distingue deux cas pour les ondes solitaires, selon la dimension de la perturbation effectuée sur la donnée initiale, chacun décomposé en problème linéaire (spectre de l'équation linéarisée autour du profil, comme nous l'avions vu sur Navier-Stokes dans la partie 2.1), et problème non-linéaire. D'une part, il y a le cas où l'onde solitaire est soumise à des perturbations 1D, à savoir

$$(\rho, u) = Q_c(t,x) + (r(t,x), v_1(t,x), 0).$$

Pour les ondes solitaires d'Euler-Korteweg, une condition suffisante pour la stabilité orbitale des perturbations 1D est montrée par Benzoni *et al.* [14], en application directe d'un résultat abstrait de M. Grillakis, J. Shatah et W. Strauss [50] sur les systèmes hamiltoniens. Celle-ci est complétée par une condition suffisante d'instabilité linéaire par S. Benzoni [12].

D'autre part, on a le problème de la stabilité transverse, où on s'autorise des perturbations dans la direction transverse $y$ et dans la composante transverse (dans laquelle $Q_c$ est constante) :

$$(\rho, u) = Q_c(t,x) + (r(t,x,y), v_1(t,x,y), v_2(t,x,y)).$$

Dans le dernier article cité, S. Benzoni montre que les profils stables en 1D sont linéairement transversalement instables en utilisant la fonction d'Evans de l'équation linéarisée autour de $Q_c$. Parallèlement, F. Rousset et N. Tzvetkov [104] ont montré l'instabilité linéaire transverse pour des solitons, grâce à un critère abstrait mais simple, qui s'applique également à d'autres modèles (KP-I, dont l'onde solitaire résout l'équation de Korteweg-de Vries ; Gross-Pitaevskii). Nous allons utiliser cette instabilité linéaire pour montrer l'instabilité transverse non-linéaire des mêmes solitons.

Le passage de l'instabilité linéaire à l'instabilité non-linéaire se fera par le même argument que dans le Theorème 2.2. La méthode d'E. Grenier utilise en effet uniquement les modes les plus instables (un nombre fini dans le cadre périodique, un paquet d'ondes localisé en fréquence sur $\mathbb{R}$) pour construire une solution approchée qui exhibe l'instabilité, ainsi elle a pu être appliquée

aux problèmes de stabilité de couches limites, comme nous l'avons vu, mais également aux problèmes de stabilité d'ondes solitaires. F. Rousset et N. Tzvetkov ont ainsi obtenu l'instabilité non-linéaire d'ondes solitaires dans bon nombre de modèles : KP-I et Schrödinger non-linéaire [103], des équations hamiltoniennes dont Boussinesq et Zakharov-Kuznetsov [102], et *water-waves* à surface libre [105].

Nous obtenons le résultat analogue pour Euler-Korteweg.

**Theorème 4.1.** *Soit $Q_c$ un soliton de l'équation d'Euler-Korteweg dont la limite en l'infini vérifie (41). Alors il existe $\delta_0$ et $\varepsilon_0 > 0$ tels que pour tout $0 < \varepsilon < \varepsilon_0$, il existe une condition initiale $U_0 = (\rho_0, u_0)$ avec*

$$\|U_0 - Q_c\|_{H^s(\mathbb{R}^2)} \leq \varepsilon$$

*pour un $s > 0$, telle que la solution $U = (\rho, u)$ de (38)-(39) avec $U|_{t=0} = U_0$ satisfait, à un temps $T^\varepsilon \sim \ln(\varepsilon^{-1})$,*

$$\inf_{a \in \mathbb{R}} \|U(T^\varepsilon) - Q_c(\cdot - cT^\varepsilon - a)\|_{L^2(\mathbb{R}^2)} \geq \delta_0.$$

Dans la preuve, nous utilisons l'instabilité linéaire du profil et une estimée de résolvante de l'équation linéarisée pour construire une solution approchée $U^{\mathrm{ap}}$ avec les bonnes propriétés pour induire l'instabilité non-linéaire par l'argument de Grenier vu en partie 2 de ce résumé. Une partie délicate de l'argument est de s'assurer, par le biais d'estimations d'énergie sur $U - U^{\mathrm{ap}}$, que la solution existe assez longtemps pour observer l'instabilité. L'article [13] ne donne en effet que l'existence locale de solutions autour de $Q_c$, ainsi qu'un critère d'explosion, mais il est important de souligner que l'instabilité n'est pas reliée à un phénomène d'explosion (on l'observe même dans des cas où les solutions sont globales).

La preuve de l'instabilité s'étend sans difficulté au cas périodique en $y$, où on étudie (38)-(39) sur $\mathbb{R} \times \mathbb{T}_L$, avec $\mathbb{T}_L$ le tore de période $L > 0$. Il existe alors une longueur critique $L_0 > 0$ au-dessus de laquelle l'instabilité a lieu. En revanche, de façon analogue à ce qu'il se passe pour KP-I dans [103], en-dessous de cette longueur critique, on perd l'instabilité linéaire. Il serait alors intéressant de déterminer si on a, lorsque $L < L_0$, stabilité orbitale transverse (pour KP-I, c'est le cas, voir Rousset et Tzvetkov [106]).

# Chapitre 1

# Stabilité et instabilité des couches limites de Navier dans le modèle incompressible 2D

**Résumé.** *On étudie la limite non-visqueuse pour l'équation de Navier-Stokes incompressible sur le demi-plan, avec des conditions de Navier au bord qui dépendent de la viscosité. D'une part, on démontre la convergence $L^2$ de solutions de Leray vers la solution forte de l'équation d'Euler lorsque la condition initiale limite est régulière. D'autre part, nous montrons l'instabilité non-linéaire dans $L^\infty$ et dans $\dot{H}^s$ avec $s > 1$ de développements WKB autour de profils de cisaillements linéairement instables pour l'équation d'Euler. Ces résultats ne sont pas contradictoires, et, dans un domaine périodique, nous donnons un exemple explicite de profil permettant aux deux théorèmes d'être réalisés simultanément.*

Ces résultats font l'objet d'un article publié dans le journal Differential and Integral Equations, [93].

# 1 Introduction

## 1.1 The inviscid limit problem

We consider the two-dimensional incompressible Navier-Stokes equation on the half-plane $\Omega = \mathbb{R}\times]0,+\infty[$ with viscosity $\varepsilon > 0$ :

$$(NS(\varepsilon)) : \quad \begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon &= 0 \quad \text{in } \mathbb{R}^+ \times \Omega \\ \text{div } u^\varepsilon &= 0 \quad \text{in } \mathbb{R}^+ \times \Omega \\ u^\varepsilon|_{t=0} &= u_0^\varepsilon \quad \text{in } \Omega. \end{cases}$$

$u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the two-dimensional fluid speed and $p^\varepsilon$ the kinematic pressure. We add two boundary conditions : first, the standard non-penetration condition for the component of $u^\varepsilon$ normal to the boundary

$$u^\varepsilon \cdot \vec{n} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega,$$

where $\vec{n} = (0,-1)$, and a Navier condition that depends on the viscosity $\varepsilon$ and that describes the tangential part of the fluid's speed on the boundary,

$$(Su^\varepsilon \cdot \vec{n} + a^\varepsilon u^\varepsilon)_\tau = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega,$$

where $a^\varepsilon > 0$, and $Su^\varepsilon = \frac{1}{2}(\nabla u^\varepsilon + {}^t\nabla u^\varepsilon)$.

Unlike the homogeneous Dirichlet (no-slip) boundary condition, $u|_{y=0} = 0$, the Navier (slip) condition, introduced in the first half of the XIX$^{\text{th}}$ century by H.Navier himself ([90]), allows the fluid to slide along the boundary. Physically, this is more accurate than the no-slip condition when one takes into account interaction at the boundary. The slip condition is therefore used to model, for example, blood flow in capillary vessels that are a few microns wide, and where molecular exchanges with the neighbouring cells take place (see [94]). One also encounters a Navier condition when homogenising the no-slip condition on rough or porous walls (see [62] and [42]). Mathematically, it is shown in [86] that the Navier condition can be derived by taking the limit when the mean-free path goes to zero of renormalised solutions of the Boltzmann equation with a Maxwell reflection boundary condition.

This chapter will deal with the inviscid limit problem for $(NS(\varepsilon))$, *i.e.* we study the behaviour of a family of solutions of $(NS(\varepsilon))$ with a Navier boundary condition, $(u^\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$, relative to $v$, a solution of the incompressible Euler equation

$$(E) : \quad \begin{cases} \partial_t v + v \cdot \nabla v + \nabla q &= 0 \\ \text{div } v &= 0 \\ v|_{t=0} &= v_0 \\ v_2|_{y=0} &= 0 \end{cases}$$

when the viscosity $\varepsilon$ goes to 0.

The inviscid limit problem is well understood in the whole space or with periodic boundary conditions (see [80]). However, in domains with boundaries, a lack of compactness appears due to the presence of boundary layers, making the problem considerably more difficult. In recent years, much progress has been made in the case of non-characteristic boundaries. For incompressible fluids, these appear in the case of injection or succion boundary conditions ([118]). For related results for compressible models and general hyperbolic-parabolic systems, we refer to [88]. In this situation the boundary layer is of size $\varepsilon$ and has an amplitude $\mathcal{O}(1)$, and sharp stability and instability conditions can be shown (see for example [31] and [100] for the study of the stability or instability of the Ekman layer in rotating fluids).

On the half-plane, the inviscid limit problem remains mainly unsolved in the case of Navier-Stokes equations with the homogeneous Dirichlet boundary condition. Formally, one expects the boundary layer to be of size $\sqrt{\varepsilon}$ and hence to write a solution $u^D$ as

$$u^D(t,x,y) = u^i(t,x,y) + u^b(t,x,\varepsilon^{-1/2}y), \tag{1.1}$$

in which $u^i$ solves $(E)$ and $u^b$ is a corrective term. A major difficulty resides in the existence of such an expansion, as, when one formally rescales the Navier-Stokes equation with $(u_1,u_2) = (\tilde{u},\sqrt{\varepsilon}\tilde{v})$ in the variables $(t,x,y' = \varepsilon^{-1/2}y)$, one gets that the behaviour of the solution near the boundary is governed by the Prandtl equation :

$$\left\{\begin{array}{rcl} \partial_t\tilde{u} + \tilde{u}\partial_x\tilde{u} + \tilde{v}\partial_{y'}\tilde{u} - \partial^2_{y'y'}\tilde{u} &=& (\partial_t u^i + u^i\cdot\nabla u^i)_1|_{y=0} \text{ in } \Omega \\ \partial_x\tilde{u} + \partial_{y'}\tilde{v} &=& 0 \text{ in } \Omega \\ (\tilde{u},\tilde{v})|_{y=0} &=& 0 \\ \lim_{y\to+\infty}\tilde{u} &=& u^i|_{y=0}. \end{array}\right.$$

While the Prandtl system is well-posed with monotonous initial conditions in the strip $[0,L]\times\mathbb{R}^+$ (a fact known since the 60s, [92]), it has recently been proved by D.Gérard-Varet and E.Dormy in [41] that the Prandtl equation is ill-posed in Sobolev spaces on $\mathbb{T}\times\mathbb{R}^+$. This shows that the ansatz (1.1) is only formal in general, and E.Grenier showed in [47] that, even when such an ansatz can be justified, *e.g.* in analytic framework (see [107]), it may not be valid in $H^1$, and instability occurs on the derivatives in $L^\infty$. The instability phenomenon is linked to the linear instability of shear flows for the Euler equation.

For the Navier-Stokes equation with a Navier boundary condition, when $a^\varepsilon$ is a fixed number independent of $\varepsilon$, the inviscid limit problem is solved in $L^2$ framework, see [6], [23], [64], [78], [60]. Moreover, asympotic expansions of the form

$$u^N(t,x,y) = u^i(t,x,y) + \sqrt{\varepsilon}u^b(t,x,\varepsilon^{-1/2}y),$$

with the amplitude of the boundary layer much smaller than in (1.1), are rigorously justified in [61] and uniform conormal estimates in agreement with this behaviour are obtained in [85] and [45]. We also refer to [124], [9] and [10] for the study of special cases in 3D.

Here, we shall be interested in the case where the Navier condition is as above with $a^\varepsilon = a\varepsilon^{-1/2}$, $a > 0$ constant. Denoting the two components of the fluid's speed by $u_1^\varepsilon$ and $u_2^\varepsilon$, the boundary conditions translate as the following :

$$\begin{array}{rrl} (NP): & u_2^\varepsilon(t,x,0) & = 0 \\ (NC(a,\varepsilon)): & \frac{1}{2}\partial_y u_1^\varepsilon(t,x,0) & = \frac{a}{\sqrt{\varepsilon}} u_1^\varepsilon(t,x,0). \end{array}$$

Our motivation for this is to understand the transition between the unstable Dirichlet case and the stable Navier case studied in [61]. Note that for $\varepsilon$ small, our slip condition seems to approximate the no-slip Dirichlet condition, and we expect the formal asymptotic behaviour of the solution to replicate the ansatz of the Dirichlet case, *i.e.*

$$u^N(t,x,y) = u^i(t,x,y) + u^b(t,x,\varepsilon^{-1/2}y),$$

hence we should observe some instability.

**Vocabulary and summary of the results.** As of now, the term "solution of the Navier-Stokes equation $(NS(a,\varepsilon))$" will designate a solution of $(NS(\varepsilon))$ satisfying the boundary conditions $(NP)$ and $(NC(a,\varepsilon))$. We obtain two results : firstly, convergence in $L^2$ of every sequence of Leray solutions of $(NS(a,\varepsilon))$ (the full definition of which is given at the beginning of section 2) to a smooth solution of $(E)$ when $\varepsilon$ goes to zero and the initial conditions are $H^s$ for some $s > 2$, and secondly, we prove that there exist boundary layer profiles such that the WKB expansion, which is also of the form (1.1) in our case, is unstable in $L^\infty$.

## 1.2 The $L^2$ stability result

We show that $u^\varepsilon$ converges to $v$ in $L^2(\Omega)$, extending the result obtained by D.Iftimie and G.Planas in [60] for a constant Navier boundary condition, *i.e.* $(Su\cdot\vec{n} + au)_\tau = 0$ with $a > 0$ not depending on the viscosity.

**Theorem 1.1.** *Let $u^\varepsilon$ be the Leray solution of the Navier-Stokes equation $(NS(a,\varepsilon))$, and $v$ be the solution of the Euler equation $(E)$. We assume that the initial condition $v_0$ is in $H^s(\Omega)$ for some $s > 2$. If $u_0^\varepsilon$ converges to $v_0$ in $L^2(\Omega)$ as $\varepsilon$ goes to 0, then for every $T > 0$,*

$$\sup_{t\in[0,T]} \|u^\varepsilon(t) - v(t)\|_{L^2(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

The proof, based on classical energy estimates, is as in [60]. This method also allows us to expose a convergence rate. Providing $\|u^\varepsilon(0) - v(0)\|_{L^2} = \mathcal{O}(\varepsilon^{1/4})$, we get that $\|u^\varepsilon(t) - v(t)\|_{L^2}$ is also $\mathcal{O}(\varepsilon^{1/4})$.

We single out $a^\varepsilon = a\varepsilon^{-1/2}$ because this is the case that will provide the nonlinear instability in the next paragraph, but we can extend the proof of this theorem to a whole family of Navier boundary conditions, of the type

$$(NC(a,\varepsilon,\beta)): \quad \partial_y u_1^\varepsilon(t,x,0) = \frac{2a}{\varepsilon^\beta} u_1^\varepsilon(t,x,0),$$

whatever the sign of $a$ (although the physical meaning of $a<0$ is not well understood). Precisely, we have convergence in the following cases :

**Theorem 1.2.** *Let $u^\varepsilon$ be the Leray solution of $(NS(a,\varepsilon,\beta))$, the Navier-Stokes system in which the boundary condition $(NC(a,\varepsilon))$ has been replaced by $(NC(a,\varepsilon,\beta))$. Let $v$ and $v_0$ be as in Theorem 1.1, and $u_0^\varepsilon$ converge to $v_0$ in $L^2$ as $\varepsilon \to 0$. Then we have convergence of $u^\varepsilon$ to $v$ in the same sense as Theorem 1.1 if :*

*- either $a>0$ and $\beta<1$,*

*- or $a<0$ and $\beta \leq 1/2$.*

*In both cases, the convergence rate is $\mathcal{O}(\varepsilon^{(1-\beta)/2})$.*

This extends a recent result by X-P.Wang, Y-G.Wang and Z.Xin in [122], in which they proved convergence for $\beta<1/2$, regardless of the sign of $a$. Notably, we add the case $\beta=1/2$, which seems critical when $a<0$. Also, when $a>0$, we can go further than $\beta=1/2$, and all the way up to $\beta<1$. The convergence rate is in agreement with other works ([60], [45]).

We also point out that these results can be adapted to higher dimensions, for smoother limit data ($v_0 \in H^s$ for $s>d/2+1$ in $d$ dimensions), and for as long as a strong solution to the Euler equation exists. We will explain in the proof how to get the starting energy estimate in higher dimensions.

## 1.3 The nonlinear instability result

In [47], E.Grenier proved that linear instability implies nonlinear instability for the Euler equation, and for some viscous boundary layers ([31] with B.Desjardins). It is this argument that we shall use to get our instability result, and we shall use a shear flow that is linearly unstable for the Euler equation as a starting point :

**Definition.** *A shear flow is a 2D smooth vector field written as $u_{\rm sh} = (u_{\rm sh}(y),0)$ (the vector field and its first component are indifferently called $u_{\rm sh}$). Note that a shear flow, associated with a constant pressure, is automatically a stationary solution of the Euler equation. The shear flow $u_{\rm sh}$ is linearly unstable if there exist $k \in \mathbb{R}$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda)>0$, and a function $\Psi \in H_0^1(\mathbb{R}^+)$ such that*

$$u(t,x,y) = \begin{pmatrix} e^{\lambda t}e^{ikx}\Psi'(y) \\ -ike^{\lambda t}e^{ikx}\Psi(y) \end{pmatrix}$$

*is a solution of the Euler equation linearised around* $u_{\rm sh}$*,*

$$(EL): \quad \left\{ \begin{array}{rcl} \partial_t u + u_{\rm sh}\cdot\nabla u + u\cdot\nabla u_{\rm sh} + \nabla p &=& 0 \\ \text{div } u &=& 0 \end{array} \right. \quad \text{in } \mathbb{R}^+\times\Omega,$$

*with* $(NP)$ *condition on the boundary.*

**Theorem 1.3.** *Let* $u_{\rm sh}$ *be a linearly unstable shear flow for the Euler equation. Setting* $Y = \varepsilon^{-1/2}y$*, we generate a time-dependent boundary layer* $\overline{u_{\rm sh}}$ *as solution of the heat equation*

$$\left\{ \begin{array}{rcl} \partial_t\overline{u_{\rm sh}}(t,Y) - \partial^2_{YY}\overline{u_{\rm sh}}(t,Y) &=& 0 \\ \overline{u_{\rm sh}}(0,Y) &=& u_{\rm sh}(Y) \\ \frac{1}{2}\partial_Y\overline{u_{\rm sh}}(t,0) &=& a\overline{u_{\rm sh}}(t,0), \end{array} \right.$$

*the boundary condition being dictated by* $(NC(a,\varepsilon))$*,* $a>0$*;* $\overline{u_{\rm sh}}(t,y/\sqrt{\varepsilon})$ *therefore solves the Navier-Stokes system (with constant pressure). Then, for any* $n\in\mathbb{N}^*$*, there exist* $\delta_0$ *and* $\varepsilon_0>0$ *such that for any* $0<\varepsilon<\varepsilon_0$*, there exists a solution* $u^\varepsilon$ *of* $(NS(a,\varepsilon))$ *with initial data* $u_0^\varepsilon$ *such that*

$$\left\| u_0^\varepsilon - u_{\rm sh}\left(\frac{y}{\sqrt{\varepsilon}}\right)\right\|_{H^s(\Omega)} \leq C\varepsilon^n$$

*for some* $s>0$*, that satisfies the following : at a time* $T^\varepsilon\sim n\ln(\varepsilon^{-1})\sqrt{\varepsilon}$*,*

$$\left\| u^\varepsilon(T^\varepsilon,x,y) - \overline{u_{\rm sh}}\left(T^\varepsilon,\frac{y}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty(\Omega)} \geq \delta_0. \tag{1.2}$$

*Moreover, for* $\sigma>1$*,*

$$\left\| u^\varepsilon(T^\varepsilon,x,y) - \overline{u_{\rm sh}}\left(T^\varepsilon,\frac{y}{\sqrt{\varepsilon}}\right)\right\|_{\dot{H}^\sigma} \overset{\varepsilon\to 0}{\longrightarrow} +\infty.$$

Note that the solutions to the Navier-Stokes equation that we consider in this theorem are $L^2(\Omega)$ perturbations of $\overline{u_{\rm sh}}$, which is itself not necessarily in $L^2(\Omega)$. However, we point out that, like the equivalent statement for Dirichlet boundary conditions in [47], a similar result holds in the domain $\Omega' = \mathbb{T}\times]0,+\infty[$, the proof being slightly easier, and in this case, we can have shear flows that are square-integrable, and then we expect both our main results to occur simultaneously : the instability of WKB expansions is not an obstruction to $L^2$-convergence. In the final section, we will provide an explicit example of a profile which will lead to this situation.

What Theorem 1.3 provides is a nonlinear instability result in $L^\infty$ in the usual sense for WKB expansions of the form (1.1) : we show that when the internal layer $u^i$ is linearly unstable for the Euler equation, then the WKB expansion is unstable, in the sense that arbitrarily small perturbations yield instantaneous amplification on $[0,T^\varepsilon]$ to reach an $\mathcal{O}(1)$ amplitude. Note that $T^\varepsilon$ converges to 0 as $\varepsilon$ goes to 0.

Our result differs from the equivalent statement in [47] : in (1.2), one obtains $\delta_0\varepsilon^{1/4}$ with the Dirichlet condition instead of $\delta_0$ here, and therefore, the result with Dirichlet conditions is not a full instability result in the sense that it does not guarantee that perturbations reach an $\mathcal{O}(1)$ amplitude. This is due to the Prandtl boundary layers, the amplitude of which is the same as that of the internal layer. Indeed, after rescaling $(t,x,y)\mapsto\varepsilon^{-1/2}(t,x,y)$ (these new coordinates are indifferently denoted $(t,x,y)$), a solution of $(NS(\sqrt{\varepsilon}))$ with homogeneous Dirichlet boundary conditions can formally be written as (1.1). In our case of a Navier condition depending on $\sqrt{\varepsilon}$, the rescaled boundary condition no longer depends on the viscosity, so we will construct an approximate solution written as

$$u^\varepsilon(t,x,y)\overset{\varepsilon\to 0}{\sim}\overline{u_{\rm sh}}(y)+\varepsilon^n\left(u^i(t,x,y)+\varepsilon^{1/4}u^b\left(t,x,\frac{y}{\varepsilon^{1/4}}\right)\right),$$

in which the factor $\varepsilon^{1/4}$ in front of the boundary layer neutralises the one that appears when differentiating with respect to $y$. This $\mathcal{O}(\varepsilon^{1/4})$ amplitude of the boundary layers is inspired by the asymptotic expansion shown by D.Iftimie and F.Sueur in [61]. In the Dirichlet case, this compensation does not take place, and considering times that are $\mathcal{O}\left(\left(n-\frac{1}{4}\right)\ln(\varepsilon^{-1})\right)$ leads to a weaker form of instability.

The result is also different when $a=0$ : it is shown in [6] that, in this case, instability does not occur.

The proof of Theorem 1.3 will follow Grenier's approach, which mainly relies on the construction of a WKB approximate solution to $(NS(a,\varepsilon))$ starting with a linearly unstable solution to the Euler equation, and the use of a resolvent estimate on the linearised Euler equation (our Theorem 3.1). The strength of the method comes from the fact that the resolvent estimate only needs to be shown on a subspace that does not have to be dense or of infinite dimension - typically spaces of functions with one or a finite number of Fourier modes. This has allowed it to be used in other circumstances than the study of boundary layers : F.Rousset and N.Tzvetkov used it to prove transverse nonlinear instability of solitary waves for KdV and water-wave equations (see [103] and [105]).

An example of smooth, linearly unstable boundary-layer profile, which can be considered as a model case, will be given after proof of Theorem 1.3.

**Organisation of the chapter.** In the next section, we prove the $L^2$ convergence results. In section 3, we first prove the key resolvent estimate which will lead us to the proof of Theorem 1.3. In the final section, we provide an explicit profile that fit the instability result, from which we deduce a profile which will satisfy both theorems in the periodic setting.

# 2 $L^2$ stability

## 2.1 Proof of Theorem 1.1

We remind the reader of the notion of Leray solution to the Navier-Stokes system :

**Definition.** *$u^\varepsilon : \mathbb{R}^+ \times \Omega \to \mathbb{R}^2$ is a Leray solution of $(NS(a,\varepsilon))$ if :*

1. *$u^\varepsilon \in \mathcal{C}_w(\mathbb{R}^+, L^2_\sigma) \cap L^2([0,T], H^1_\sigma)$ for every $T > 0$, where, if $E(\Omega)$ is a functional space on $\Omega$, $E_\sigma$ designates the space of divergence-free vector fields, tangent to the boundary and belonging to $E(\Omega)$.*
2. *$u^\varepsilon$ is a weak solution to $(NS(a,\varepsilon))$ in the following sense : we have*

$$-\int_{\mathbb{R}^+}\int_\Omega u^\varepsilon \cdot \partial_t \varphi + 2a\sqrt{\varepsilon}\int_{\mathbb{R}^+}\int_{\partial\Omega} u^\varepsilon \cdot \varphi + 2\varepsilon \int_{\mathbb{R}^+}\int_\Omega Su^\varepsilon : S\varphi$$
$$-\int_{\mathbb{R}^+}\int_\Omega (u^\varepsilon \cdot \nabla\varphi)\cdot u^\varepsilon = \int_\Omega u^\varepsilon(0)\cdot\varphi(0) \tag{1.3}$$

   *for every $\varphi \in H^1(\mathbb{R}^+, H^1_\sigma)$, where $A : B = \sum A_{i,j}B_{i,j}$ is the contracted product of two same-sized matrices $A$ and $B$.*
3. *For every $t \geq 0$, $u^\varepsilon$ satisfies the following energy estimate :*

$$\|u^\varepsilon(t)\|^2_{L^2(\Omega)} + 4a\sqrt{\varepsilon}\int_0^t\int_{\partial\Omega}|u^\varepsilon|^2 + 4\varepsilon\int_0^t\int_\Omega |Su^\varepsilon|^2 \leq \|u^\varepsilon(0)\|^2_{L^2(\Omega)}\,. \tag{1.4}$$

For $\varepsilon > 0$ fixed, such solutions are known to exist and are global in time. Moreover, in 2D, we have uniqueness. Likewise, the existence and uniqueness of a classical global-in-time solution $v$ to the Euler equation $(E)$ when $v_0 \in H^s(\Omega)$, $s > 2$, are well-known, and for every $T > 0$, $v \in L^\infty([0,T], H^s(\Omega))$ (see [6] or [80] and references therein).

Integration by parts leads to the following : let $f,\ g \in H^2_\sigma$, with $f$ satisfying $(NC(a,\varepsilon))$. Then

$$-\int_\Omega \Delta f \cdot g = \frac{2a}{\sqrt{\varepsilon}}\int_{\partial\Omega} f\cdot g + 2\int_\Omega Sf : Sg. \tag{1.5}$$

This allows us to confirm that the weak formulation (1.3) contains the Navier-Stokes equation, the initial condition and the Navier boundary condition ($(NP)$ and the divergence-free conditions being given by the choice of the spaces in point 1 of the definition).

Let $w^\varepsilon = u^\varepsilon - v$, which solves the equation

$$\partial_t w^\varepsilon + u^\varepsilon\cdot\nabla w^\varepsilon + w^\varepsilon\cdot\nabla v - \varepsilon\Delta u^\varepsilon + \nabla(p^\varepsilon - q) = 0. \tag{1.6}$$

For $t > 0$, $w^\varepsilon$ satisfies the inequality

$$\frac{1}{2}\|w^\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega (w^\varepsilon\cdot\nabla v)\cdot w^\varepsilon + 2a\sqrt{\varepsilon}\int_0^t\int_{\partial\Omega} u^\varepsilon\cdot w^\varepsilon + 2\varepsilon\int_0^t\int_\Omega Su^\varepsilon : Sw^\varepsilon$$

$$\leq \frac{1}{2} \left\| w^\varepsilon(0) \right\|^2_{L^2(\Omega)}, \tag{1.7}$$

which can be seen as (1.6) multiplied by $w^\varepsilon$, integrated by parts in space and integrated in time. This direct computation is only possible because we are working in 2D and $w^\varepsilon$ has the right regularity to be used as a test function, but (1.7) also holds in 3D or higher ; one starts by writing the energy inequality (1.4), then get three energy equalities : $(I)$ by multiplying the Euler equation by $v$, $(II)$ by again multiplying $(E)$ by $u^\varepsilon$, and $(III)$ by multiplying the Navier-Stokes equation by $v$. One then gets (1.7) by doing $(1.4) + (I) - (II) - (III)$ and using (1.5). This idea appears, for example, in proofs of Kato-type theorems ([63], [117], [121]).

For every $x,\ y \in \mathbb{R}^2$, let $z = x - y$. We have

$$2(x \cdot z) = 2 \left| z + \frac{y}{2} \right|^2 - \frac{1}{2}|y|^2 \tag{1.8}$$

and the same goes for the contracted matrix product. So, from (1.7) we get

$$\begin{aligned} &\left\| w^\varepsilon(t) \right\|^2_{L^2} + 4a\sqrt{\varepsilon} \int_0^t \int_{\partial\Omega} \left| w^\varepsilon + \frac{v}{2} \right|^2 + 4\varepsilon \int_0^t \int_\Omega \left| S\left( w^\varepsilon + \frac{v}{2} \right) \right|^2 \\ &\qquad \leq \left\| w^\varepsilon(0) \right\|^2_{L^2} - 2 \int_0^t \int_\Omega (w^\varepsilon \cdot \nabla v) \cdot w^\varepsilon + a\sqrt{\varepsilon} \int_0^t \int_{\partial\Omega} |v|^2 + \varepsilon \int_0^t \int_\Omega |Sv|^2. \end{aligned} \tag{1.9}$$

The left side is greater than $\left\| w^\varepsilon(t) \right\|^2_{L^2}$, and we estimate each term on the right-hand side as follows :

- $\int_\Omega (w^\varepsilon \cdot \nabla v) \cdot w^\varepsilon \leq \left\| \nabla v \right\|_{L^\infty} \left\| w^\varepsilon \right\|^2_{L^2}$ because $v(\tau) \in H^s(\Omega)$ for every $\tau \geq 0$ and for a certain $s > 2$, so $v(\tau) \in \mathcal{C}^1(\overline{\Omega}) \cap H^1(\Omega)$, thus $\nabla v(\tau) \in L^\infty(\Omega)$ ;
- $\int_{\partial\Omega} |v|^2 \leq C_\gamma \left\| v \right\|^2_{H^1}$ for a certain $C_\gamma > 0$ ;
- $\left\| Sv \right\|^2_{L^2} \leq \left\| v \right\|^2_{H^1}$ given that $Sv = \frac{1}{2}(\nabla v + {}^t \nabla v)$ ;

so (1.9) becomes

$$\left\| w^\varepsilon(t) \right\|^2_{L^2} \leq \left\| w^\varepsilon(0) \right\|_{L^2} + \varepsilon \left( \frac{aC_\gamma}{\sqrt{\varepsilon}} + 1 \right) \int_0^t \left\| v \right\|^2_{H^1} \ d\tau + 2 \int_0^t \left\| \nabla v \right\|_{L^\infty} \left\| w^\varepsilon \right\|^2_{L^2} \ d\tau,$$

to which we apply Grönwall's lemma, and we obtain

$$\left\| w^\varepsilon(t) \right\|^2_{L^2} \leq \left[ \left\| w^\varepsilon(0) \right\|_{L^2} + \varepsilon \left( \frac{aC_\gamma}{\sqrt{\varepsilon}} + 1 \right) \int_0^t \left\| v \right\|^2_{H^1} \ d\tau \right] \exp\left( 2 \int_0^t \left\| \nabla v \right\|_{L^\infty} \right). \tag{1.10}$$

Now fix $T > 0$, and take the supremum on the right-hand side for $t \in [0, T]$ to get a uniform-in-time bound on $\left\| w^\varepsilon(t) \right\|_{L^2}$, thanks to which we conclude that

$$\sup_{t \in [0,T]} \left\| w^\varepsilon(t) \right\|_{L^2(\Omega)} = \underset{\varepsilon \to 0}{\mathcal{O}} \left( \max(\left\| w^\varepsilon(0) \right\|_{L^2}, \varepsilon^{1/4}) \right). \qquad \square$$

## 2.2 Extension of the result (Theorem 1.2)

First, considering $a > 0$, the entire proof of Theorem 1.1 can be rewritten with $\varepsilon^{-\beta}$ replacing $\varepsilon^{-1/2}$ in the Navier condition, therefore replacing $\sqrt{\varepsilon}$ by $\varepsilon^{1-\beta}$ in the boundary terms of (1.9). As a result, (1.10) becomes

$$\|w^\varepsilon(t)\|_{L^2} \leq \left[\|w^\varepsilon(0)\|_{L^2} + \varepsilon^{1-\beta} C_v \left(aC_\gamma + \varepsilon^\beta\right) T\right] \exp\left(2C_v T\right),$$

in which $C_v$ depends on norms of $v$, and the right-hand side clearly converges when $\beta < 1$. However we cannot conclude when $\beta \geq 1$, because the coefficient $\varepsilon^{1-\beta} a C_\gamma$ no longer goes to zero as $\varepsilon \to 0$.

We can also try rewriting the same proof with $a < 0$ and the $(NC(a, \varepsilon, \beta))$ boundary condition. Equality (1.7) becomes

$$\frac{1}{2}\|w^\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega (w^\varepsilon \cdot \nabla v) \cdot w^\varepsilon + 2\varepsilon \int_0^t \int_\Omega Su^\varepsilon : Sw^\varepsilon$$
$$\leq \frac{1}{2}\|w^\varepsilon(0)\|^2_{L^2(\Omega)} - 2a\varepsilon^{1-\beta} \int_0^t \int_{\partial\Omega} u^\varepsilon \cdot w^\varepsilon,$$

in which $-a > 0$, thus the final boundary term cannot be ignored as in the positive case. Instead of equality (1.8), we use the more obvious $x \cdot z = |z|^2 + y \cdot z$ when $z = x - y$, so we get, after standard manipulations on the terms involving $v$ :

$$\frac{1}{2}\|w^\varepsilon(t)\|^2_{L^2} + 2\varepsilon \int_0^t \|Sw^\varepsilon\|^2_{L^2} \leq \frac{1}{2}\|w^\varepsilon(0)\|^2_{L^2} + \int_0^t \|\nabla v\|_{L^\infty} \|w^\varepsilon\|^2_{L^2}$$
$$+(\varepsilon + |a|\varepsilon^{1-\beta}) \int_0^t \|v\|^2_{H^1} + \varepsilon \int_0^t \|Sw^\varepsilon\|^2_{L^2} + 2|a|\varepsilon^{1-\beta} \int_0^t \int_{\partial\Omega} |w^\varepsilon|^2.$$

We must deal with the troublesome final term $\Theta = 2|a|\varepsilon^{1-\beta}\|w^\varepsilon\|^2_{L^2(\partial\Omega)}$. In what follows, $C$ is a constant that can change from one line to the next, but that never depends on $\varepsilon$, $\beta$ or $\eta$, the latter being a parameter yet to appear.

– We start by using the trace theorem :

$$\Theta \leq C\varepsilon^{1-\beta}(\|w^\varepsilon\|_{L^2} \|\nabla w^\varepsilon\|_{L^2} + \|w^\varepsilon\|^2_{L^2});$$

– we now use Young's inequality with a parameter $\eta$ to be chosen in a moment :

$$\Theta \leq C\varepsilon^{1-\beta}\eta \|\nabla w^\varepsilon\|^2_{L^2} + C\varepsilon^{1-\beta}(\eta^{-1} + 1)\|w^\varepsilon\|^2_{L^2};$$

– and finally, we use Korn's inequality, which in the half-space reads the following : for $f \in H^1_\sigma$, $\|\nabla f\|^2_{L^2} \leq 2\|Sf\|^2_{L^2}$. So, we get

$$\Theta \leq C\varepsilon^{1-\beta}\eta \|Sw^\varepsilon\|^2_{L^2} + C\varepsilon^{1-\beta}(\eta^{-1} + 1)\|w^\varepsilon\|^2_{L^2}.$$

Thus we have made $\|Sw^\varepsilon\|_{L^2}^2$ appear - we therefore choose $\eta = \frac{1}{2C}\varepsilon^\beta$ so that this term is absorbed by $\varepsilon \|Sw^\varepsilon\|_{L^2}^2$ on the left-hand side. This leaves us with

$$\|w^\varepsilon(t)\|_{L^2}^2 \leq \|w^\varepsilon(0)\|_{L^2}^2 + \int_0^t (\|\nabla v\|_{L^\infty} + C\varepsilon^{1-\beta}(\eta^{-1}+1)) \|w^\varepsilon\|_{L^2}^2 + C\varepsilon^{1-\beta}(1+\varepsilon^\beta) \int_0^t \|v\|_{H^1}^2 ,$$

which, after applying the Grönwall lemma and choosing $t \leq T$ as in the proof above, leads to

$$\|w^\varepsilon(t)\|_{L^2}^2 \leq \left[\|w^\varepsilon(0)\|_{L^2}^2 + C_T(\varepsilon + \varepsilon^{1-\beta})\right] e^{C_T(1+\varepsilon^{1-\beta}+\varepsilon^{1-\beta}\eta^{-1})}$$

for $C_T$ uniform in $t$, depending on norms of $v$, but not depending on $\varepsilon$ or $\eta$. Noting that $\eta^{-1} \sim \varepsilon^{-\beta}$, we have, for $\varepsilon < 1$,

$$\|w^\varepsilon(t)\|_{L^2}^2 \leq \left[\|w^\varepsilon(0)\|_{L^2}^2 + C_T\varepsilon^{1-\beta}\right] e^{C_T\varepsilon^{1-2\beta}},$$

which converges to zero if $\beta \leq 1/2$, because $e^{C\varepsilon^{1-2\beta}}$ is unbounded for $\beta > 1/2$. Notice the convergence rate in the first factor : $\mathcal{O}\left(\max(\|w^\varepsilon(0)\|_{L^2}^2, \varepsilon^{1-\beta})\right)$. □

# 3 Nonlinear instability

As mentioned in the Introduction, a resolvent estimate on the Euler equation linearised around a shear flow $u_{\text{sh}}$ will be required ; we will begin by stating and proving it. To do so, we will consider individual Fourier modes and stream functions, and use the spectral theory of the linearised Euler equation in this setting with a fixed wave number, which is a one-dimensional problem. Then we shall prove Theorem 1.3 by constructing an asymptotic expansion of solutions of $(NS(\varepsilon))$ around a linearly unstable shear profile $u_{\text{sh}}$, as in [47]. The terms in this WKB expansion will be wave packets, $U^j = \int_{\mathbb{R}} \varphi^j(k) V^j(k;t,y) e^{ikx}\, dk$, where each $\varphi^j$ is smooth and compactly supported, thus $U^j$ is an $H^\infty$ function in the $x$-direction. We will choose $\varphi^1$ to be located around the most unstable wave number. Estimates on the important and unstable first term and the resolvent estimate will then give us the wanted instability. An example of linearly unstable shear flow will be given in the final paragraph.

## 3.1 Preliminary results on the linearised Euler equation

### 3.1.1 Linear instability of the Euler equation

Fix a wave number $k$, and consider the space $V_k$ of Fourier modes written as $u(t,x,y) = v(t,y)e^{ikx}$, with $v(t,\cdot) \in H^s$ for every $s > 0$. We set $\sigma(k)$ the highest real part of complex numbers $\lambda$ that are unstable eigenvalues of the Euler equation linearised around $u_{\text{sh}}$ on $V_k$ ; for each wave number $k$, it is finite, and $\sigma$ is an even analytic function of $k$. In turn, the function $\sigma$ has a maximum $\sigma_0 > 0$ (for details, see the study of the Rayleigh equation in [47], paragraph 4). We assume that this maximum is nondegenerate.

We write fields in $V_k$ that are tangent to the boundary and divergence-free by using stream functions :

$$u(t,x,y) = e^{ikx}(\partial_y \Psi(t,y), -ik\Psi(t,y)) = \nabla^\perp(e^{ikx}\Psi(t,y)) \tag{1.11}$$

with $\Psi(t,0) = 0$. For $u \in V_k$, we use the norm $\|u\|_l = \sqrt{\|u\|^2_{H^l_k} + \|\text{rot } u\|^2_{H^l_k}}$, where the $H^l_k$ norms are expressed as

$$\|u\|^2_{H^l_k} = \sum_{m=0}^{l} \|\nabla^m_k v\|^2_{L^2}$$

with $\nabla^m_k \Psi := (k^{m_1}\partial^{m_2}_y \Psi(y))_{m_1+m_2=m}$.

From now on in the linear study, $u$ is in $V_k$ and written as in (1.11). Note that we have $\|u\|_{H^l_k} = \|\nabla_k \Psi\|_{H^l_k}$. As rot $u = \partial_x u_2 - \partial_y u_1 = -\Delta(e^{ikx}\Psi(t,y)) = (k^2 - \partial^2_{yy})\Psi$, standard elliptic regularity (see [69] for example) gives us

$$\|\nabla u(t)\|_{L^2_k} = \left\|\nabla^2_k \Psi\right\|_{L^2} \sim \|\text{rot } u(t)\|_{L^2_k}. \tag{1.12}$$

Using this, and standard properties of Sobolev spaces, we get that if $u \in V_k$ and $u' \in V_{k'}$, then $u \cdot \nabla u' \in V_{k+k'}$ and

$$\left\|u \cdot \nabla u'\right\|_l \leq C \|u\|_{l+2} \left\|u'\right\|_{l+2}. \tag{1.13}$$

**Theorem 3.1.** *Let $\lambda' > \sigma_0$, and $w(t,x,y) \in V_k$ such that*

$$\|w(t)\|_l \leq C_w \frac{e^{\lambda' t}}{(1+t)^\alpha} \tag{1.14}$$

*for every $l \geq 2$ and for some $\alpha \geq 0$. Consider the linearised Euler equation with source term*

$$(ELS): \quad \begin{cases} \partial_t u + u_{\text{sh}} \cdot \nabla u + u \cdot \nabla u_{\text{sh}} + \nabla p &= w \\ \text{div } u &= 0 \\ u|_{t=0} &= u_0 \in V_k \\ u_2|_{y=0} &= u_{0,b} = e^{ikx} v_{0,b}(t) \end{cases}$$

*with $u_{0,b}$ satisfying the estimate $|u_{0,b}(t)| \leq C_w(1+t)^{-\alpha} e^{\lambda' t}$, and likewise for $\partial_t u_{0,b}$, and such that $u_{0,b}(0,x) = u_0(x)$. Then the solution of this system satisfies the estimate*

$$\|u(t)\|_{l-2} \leq C(1+t)^{-\alpha} e^{\lambda' t}$$

*for $t > 0$, with $u(t) \in V_k$, and $C$, which depends on $w$, $u_{\text{sh}}$, $u_0$, $u_{0,b}$, $l$ and $k$, and is locally bounded in the parameter $k$.*

We show the result for functions with a single Fourier mode, but it extends to wave packets

$$U(t,x,y) = \int_{\mathbb{R}} \varphi(k) u(k;t,x,y)\, dk,$$

with $u(k) \in V_k$ and $\varphi$ a smooth compactly supported function such that, for every $k \in \mathrm{supp}(\varphi)$, $w(k)$ satisfies (1.14) (see [47] for example).

*Proof :* we adapt the arguments used to prove the resolvent estimate for the KdV equation in [103]. Set

$$u(t,x,y) = e^{ikx} \begin{pmatrix} \partial_y \Psi(t,y) \\ -ik\Psi(t,y) \end{pmatrix} \quad \text{and} \quad w(t,x,y) = e^{ikx}\tilde{w}(t,y)$$

and note that $\mathrm{rot}\ u = (k^2 - \partial_{yy}^2)\Psi := B_k\Psi$, where, for $k \neq 0$, the differential operator $B_k : H^{l+2}(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+) \to H^l(\mathbb{R}^+)$ is invertible.

We first examine the $\|\mathrm{rot}\ u\|_{H_k^l}$ part of $\|u\|_l$. Taking the vorticity (rotational) of $(ELS)$, we obtain the equation

$$\partial_t B_k\Psi + ik(u_{\mathrm{sh}}B_k\Psi + u_{\mathrm{sh}}''\Psi) = \mathrm{rot}_k\ \tilde{w}, \tag{1.15}$$

where $\mathrm{rot}_k\ \tilde{w} = ik\tilde{w}_2 - \partial_y\tilde{w}_1$. We then set

$$\Psi_0(t,y) = \Psi(t,y) - e^{-\mu t}\Psi(0,y) - e^{-\mu y}\Psi(t,0) + e^{-\mu(t+y)}\Psi(0,0)$$

with an arbitrary $\mu > 0$, thus $\Psi_0(0,y) = 0$ and $\Psi_0(t,0) = 0$. $\Psi_0$ solves

$$\partial_t B_k\Psi_0 + ik(u_{\mathrm{sh}}B_k\Psi_0 + u_{\mathrm{sh}}''\Psi_0) = F_0.$$

We will not give the detailed expression of $F_0$, but we point out that integrating the $L^2$ hermitian dot-product of $F_0$ by $\Psi_0$ by parts leads to

$$|(F_0|\Psi_0)| \leq C\left(\|f_0\|_{L^2}^2 + \|\nabla_k\Psi_0\|_{L^2}^2\right)$$

where $f_0$ contains terms depending on the data $w$, $u_0$ and $u_{0,b}$, and $\|\nabla_k\Psi_0\|_{L^2}$ is the norm of the velocity. More precisely, $f_0 = \Phi_1 + \Phi_2 e^{-\mu t}$, where $\Phi_1$ contains $\mathrm{rot}_k\ \tilde{w}$ and terms with $y = 0$ (namely $u_{0,b}$ and $\partial_t u_{0,b}$), while $\Phi_2$ contains terms with $t = 0$.

Using the Laplace transform in time, $\mathcal{L}_t g(z,y) = \int_0^{+\infty} e^{-zs}g(s,y)\,ds$, and writing $\tilde{\Psi} = \mathcal{L}_t\Psi_0$ and $F = \mathcal{L}_t F_0$, we turn this differential equation into the eigenvalue problem :

$$zB_k\tilde{\Psi} + ik(u_{\mathrm{sh}}B_k\tilde{\Psi} + u_{\mathrm{sh}}''\tilde{\Psi}) = F. \tag{1.16}$$

We choose $\gamma_0 \in ]\sigma_0, \lambda'[$, and set $z = \gamma_0 + i\tau$. $\gamma_0$ being fixed, we abbreviate $\tilde{\Psi}(z) = \tilde{\Psi}(\tau)$. When $\tau$ evolves in $\mathbb{R}$, we get the following estimates :

**Lemma 3.2.** *Let $l \geq 0$, and $\tilde{\Psi}$ solve (1.16) with $\tilde{\Psi}(\tau,0) = 0$, and $F$ verifying*

$$|(F|\tilde{\Psi})| \leq C\left(||\nabla_k\tilde{\Psi}||_{L^2}^2 + \|f\|_{L^2}^2\right), \tag{1.17}$$

*where $f = \mathcal{L}_t f_0$ depends on the data. There exists $C$ depending on $k$, $l$ and $u_{\mathrm{sh}}$, locally bounded in the parameter $k$, such that*

$$||B_k\tilde{\Psi}(\tau)||_{H^l}^2 \leq C\,\|f(\tau)\|_{H^{l+2}}^2 \quad \textit{and} \quad ||\nabla_k\tilde{\Psi}||_{H^l}^2 \leq C\,\|f\|_{H^{l+2}}^2\,.$$

Note that the lemma provides estimates on both the vorticity and the velocity. We prove this lemma in the next sub-paragraph.

By Parseval's equality and the above lemma, we have

$$\begin{aligned}\int_0^{+\infty} e^{-2\gamma_0 t}\,\|B_k\Psi_0\|_{H^l}^2\ dt &= \int_{-\infty}^{+\infty} ||B_k\tilde{\Psi}(\tau)||_{H^l}^2\ d\tau\\ &\le C\int_{-\infty}^{+\infty}\|f(\tau)\|_{H^{l+2}}^2\ d\tau\\ &\le C\int_0^{+\infty} e^{-2\gamma_0 t}\,\|\Phi_1(t)\|_{H^{l+2}}^2\ dt\\ &\qquad + \int_0^{+\infty} e^{-2(\gamma_0+\mu)t}\,\|\Phi_2\|_{H^{l+2}}^2\ dt\\ &\le C\int_0^{+\infty} e^{-2\gamma_0 t}(\|\Phi_1(t)\|_{H^{l+2}}^2 + \|\Phi_2\|_{H^{l+2}}^2)\ dt.\end{aligned}$$

Replacing $\Phi_1$ by $\Phi_1\mathbf{1}_{[0,T]\times\mathbb{R}^+}$ for some $T>0$ does not affect the solution on $[0,T]\times\mathbb{R}^+$, so (1.14) gives us

$$\int_0^T e^{-2\gamma_0 t}\,\|B_k\Psi_0\|_{H^l}^2\ dt \le C\int_0^T \frac{e^{2(\lambda'-\gamma_0)t}}{(1+t)^\alpha}\ dt,$$

where $C$ depends on $\|u(0)\|_l$, $\gamma_0$, $l$ and $k$, and is a locally bounded function of $k$. Noticing that

$$\|\Psi_0(t)\|_{H^l} \ge \|\Psi(t)\|_{H^l} - \|u_0\|_{H_k^l} - \left\|e^{-\mu y}\right\|_{H^l}(|u_{0,b}(t)| + |\Psi(0,0)|),$$

we get

$$\int_0^T e^{-2\gamma_0 t}\,\|B_k\Psi\|_{H^l}^2\ dt \le C\int_0^T \frac{e^{2(\lambda'-\gamma_0)t}}{(1+t)^\alpha}\ dt \le C\frac{e^{2(\lambda'-\gamma_0)T}}{(1+T)^\alpha} \tag{1.18}$$

(the last inequality is obtained by integrating by parts).

Using the same procedure, estimate (1.18) also holds for the $\|u\|_{H_k^l}$ part of $\|u\|_l$ :

$$\int_0^T e^{-2\gamma_0 t}(\|k\Psi\|_{H^l}^2 + \|\partial_y\Psi\|_{H^l}^2)\ dt \le C\frac{e^{2(\lambda'-\gamma_0)T}}{(1+T)^\alpha}. \tag{1.19}$$

A quick energy estimate on (1.15) gives us

$$\frac{1}{2}\partial_t\,\|B_k\Psi(t)\|_{H^l}^2 \le C(\|B_k\Psi(t)\|_{H^l}^2 + \|u(t)\|_{H_k^l}^2 + \|\mathrm{rot}\ w(t)\|_{H_k^l}^2).$$

Multiply this by $e^{-2\gamma_0 t}$, and using the hypothesis on $w$, we get

$$\partial_t(e^{-2\gamma_0 t}\,\|B_k\Psi\|_{H^l}^2) \le C\left(e^{-2\gamma_0 t}\,\|B_k\Psi\|_{H^l}^2 + e^{-2\gamma_0 t}\,\|u\|_{H_k^l}^2 + \frac{e^{2(\lambda'-\gamma_0)t}}{(1+t)^\alpha}\right).$$

Finally, we integrate between 0 and $T$, and use (1.18) and (1.19) to we get the desired estimate : $\|\text{rot } u(t)\|_{H^l_k} \leq C(1+t)^{-\alpha}e^{\lambda' t}$.

Thanks to (1.12), it only remains to estimate $\|u\|_{L^2_k}$. We simply multiply the linearised Euler equation $(ELS)$ by $u$, and get

$$\frac{1}{2}\partial_t \|u\|^2_{L^2_k} \leq C(\|u\|^2_{L^2_k} + \|\nabla u\|^2_{L^2_k} + \|w\|^2_{L^2_k}),$$

where $C$ depends only on $u_{\text{sh}}$. Multiplying by $e^{-2\gamma_0 t}$, integrating in time and combining (1.12), (1.18) and (1.19), we get $\|u(t)\|_{L^2_k} \leq C(1+t)^{-\alpha}e^{\lambda' t}$. □

### 3.1.2 Proof of the resolvent lemma (Lemma 3.2)

First note that the estimate is obvious when $k = 0$. We must then prove the estimate for $k \neq 0$ and then show that the constant $C$ is a locally bounded function of $k$ as $k \to 0$. Therefore, we consider $|k| \leq K$, and divide the proof in two parts : with $z = \gamma_0 + i\tau$ and $\tau \in \mathbb{R}$, we show the estimate for $|\tau| \geq M$, with $M$ to be chosen independent of $l$, and then for $|\tau| \leq M$, using the notion of exponential dichotomy (see [24]).

**Step 1 : $|\tau|$ large.** We start with an $L^2$ estimate. Set $\Theta = B_k\tilde{\Psi}$. The imaginary part of the dot-product of (1.16) with $\Theta$ leads to

$$|\tau| \|\Theta\|^2_{L^2} \leq (C_{sh}|k| + 1) \|\Theta\|^2_{L^2} + \frac{1}{2}\|F\|^2_{L^2} + C_{sh}||\nabla_k\tilde{\Psi}||^2_{L^2}, \tag{1.20}$$

noting that $||k\tilde{\Psi}||^2_{L^2} \leq ||\nabla_k\tilde{\Psi}||^2_{L^2}$. To deal with this final term, we consider the real part of the dot-product of (1.16) and $\tilde{\Psi}$ : we integrate $(\partial^2_{yy}\tilde{\Psi}|\tilde{\Psi})$ by parts, notice that $(u_s''\tilde{\Psi}|\tilde{\Psi})$ is real and use inequality (1.17) as well as Young's inequality with parameter $\gamma_0$, to get

$$\gamma_0||\nabla_k\tilde{\Psi}||^2_{L^2} \leq \left[C_{sh}|k| + \frac{\gamma_0}{2}\right] ||\nabla_k\tilde{\Psi}||^2_{L^2} + C\|f\|^2_{H^1},$$

having noticed that $\|F\|^2_{L^2} \leq \|f\|^2_{H^1}$. Let $k_0 > 0$ be such that $C_{sh}k_0 + \frac{\gamma_0}{2} = \frac{3\gamma_0}{4}$. When $|k| \leq k_0$, we can absorb the first term on the right-hand side to get $||\nabla_k\tilde{\Psi}||^2_{L^2} \leq C\|f\|^2_{H^1}$, with $C$ depending only on $\gamma_0$ and $u_{\text{sh}}$. This allows us to conclude from estimate (1.20) :

$$h(\tau,k)\|\Theta\|^2_{L^2} = (|\tau| - C_{sh}|k| - 1)\|\Theta\|^2_{L^2} \leq C\|f\|^2_{H^1}.$$

By choosing $M > 0$ such that $h(M, k_0) \geq 1$, we have the desired uniform estimate for $|\tau| \geq M$ and $|k| \leq k_0$. In particular, we have the desired local boundedness of the constant $C$ when $|k| \to 0$.

When $K \geq |k| \geq k_0$, things are easier, as we have uniform elliptic regularity for the operator $B_k$. Consider the imaginary part of the $L^2$ dot-product of equation (1.16) with $\Theta = B_k\tilde{\Psi}$ :

thanks to the elliptic regularity bound $\|\Psi\|_{H^2} \leq \max(1, k_0^{-2}) \|B_k\Psi\|_{L^2}$, we have

$$|\tau| \|\Theta\|_{L^2}^2 \leq \left[C_{sh}|k| + \frac{1}{2}\right] \|\Theta\|_{L^2}^2 + \frac{1}{2} \|F\|_{L^2}^2 ,$$

so we can write once again $h(\tau, k) \|\Theta\|_{L^2}^2 \leq C \|f\|_{H^1}^2$, with $C$ depending on $u_{\rm sh}$ and $k_0$. Choosing $M$ so that $h(M, K) \geq 1$ (remember that $|k|$ is assumed to be bounded), we have, for $|\tau| \geq M$, $\|\Theta\|_{L^2} \leq C \|f\|_{H^1} \leq C \|f\|_{L^2}$. This ends the proof of the $L^2$ estimates for $|\tau|$ large.

$H^l$ estimates are easily obtained by induction : examine the dot-product of $\partial^l$(1.16) by $\partial^l\Theta$, and use Young's inequality and the induction hypothesis $\|\Theta\|_{H^{l-1}} \leq C \|F\|_{H^{l-1}}$ to get $\frac{\gamma_0}{2} \left\|\partial^l\Theta\right\|_{L^2}^2 \leq C \|F\|_{H^l}^2$, with $C$ depending on $\gamma_0$, $|k|$ and $l$.

**Step 2 : $|\tau|$ small.** We seek a solution to (1.16) written as $\tilde{\Psi} = \Psi_1 + \Psi_2$, where

$$zB_k\Psi_1 = F, \tag{1.21}$$

$$\text{and} \quad (z + iku_{\rm sh})B_k\Psi_2 + iku''_{\rm sh}\Psi_2 = -ik(u_{\rm sh}B_k\Psi_1 + u''_{\rm sh}\Psi_1) := kG \tag{1.22}$$

Immediately, we get $\|B_k\Psi_1\|_{H^l} \leq \frac{1}{\gamma_0} \|F\|_{H^l}$, and, by multiplying (1.21) by $\Psi_1$, we get

$$\frac{\gamma_0}{2} \|\nabla_k\Psi_1\|_{L^2}^2 \leq C \|f\|_{L^2}^2 ,$$

(it is proved similarly to (1.17), but also using Young's inequality) and likewise with $H^l$ norms, so a $\|\Psi_2\|_{H^l} \leq C \|kG\|_{H^l}$ estimate suffices to prove the result, as

$$\|kG\|_{H^l} \leq c|k| \|B_k\Psi_1\|_{H^l} + c' \|\nabla_k\Psi_1\|_{H^l} .$$

First rewrite (1.22) as an ordinary differential system :

$$\partial_y U(y) = A(k, z; y)U(y) + H(y) \tag{1.23}$$

with $U = (\Psi_2, \partial_y\Psi_2)$, $H = (0, -(z + iku_s)^{-1}kG)$ (we have $|H| \leq \gamma_0^{-1}|kG|$), and

$$A(k, z; y) = \begin{pmatrix} 0 & 1 \\ k^2 + \frac{iku''_{\rm sh}(y)}{z+iku_{\rm sh}(y)} & 0 \end{pmatrix} = A_\infty(k) + B(k, z; y),$$

where $A_\infty(k) = \lim_{y\to+\infty} A(k, z; y)$, and $B(k, z; y) = \mathcal{O}(e^{-\eta y})$ for some $\eta > 0$ since $u''_{\rm sh}$ decays exponentially. The eigenvalues of $A_\infty(k)$ are real, so by the roughness of exponential dichotomy [24], the system $\partial_y U = A(k, z)U$ has an exponential dichotomy on $\mathbb{R}^+$ ; this means that if $T(k, z; y, y')$ is the fundamental solution of this last equation with $T(k, z; y, y) = I_2$, there exist a projection $P(k, z; y)$, verifying

$$T(k, z; y, y')P(k, z; y') = P(k, z; y)T(k, z; y, y'),$$

and positive constants $C$ and $\alpha$, with all of these depending smoothly on $(k,z)$, such that for any $\xi \in \mathbb{C}^2$,

$$\begin{array}{rcll} |T(k,z;y,y')P(k,z;y')\xi| & \leq & C(k,z)e^{-\alpha(y-y')}|\xi| & \text{if } y \geq y' \geq 0 \\ |T(k,z;y,y')(I-P(k,z;y'))\xi| & \leq & C(k,z)e^{\alpha(y-y')}|\xi| & \text{if } y' \geq y \geq 0. \end{array}$$

For any $0 < \rho < \rho'$, $C(k,z)$ is uniformly bounded in the set

$$\mathcal{K}_{\rho,\rho'} = \{(k, \gamma_0 + i\tau) \mid |k| \in [\rho, \rho'] \text{ and } |\tau| \leq M\},$$

but we would like $C(k,z)$ to be bounded in $\mathcal{K}_{0,\rho'}$ for a certain $\rho'$.

To get that $C(k,z)$ is bounded near $k = 0$, we follow the proof of the persistence of ordinary dichotomies as done in [24]. $A_\infty(k)$ cannot be uniformly diagonalised near $k = 0$, because the basis of diagonalisation for $k \neq 0$ is $\begin{pmatrix} 1 & 1 \\ -|k| & |k| \end{pmatrix}$. Instead, we change to the basis $(v_1(k), v_2(k)) = \begin{pmatrix} 1 & 0 \\ -|k| & 1 \end{pmatrix}$. $v_1$ is an eigenvector of $A_\infty(k)$, with eigenvalue $-|k|$. $A_\infty(k)$ is therefore triangular in this basis, and $v_1$ spans a space corresponding to that of exponentially decreasing solutions of the equation $\partial U = A_\infty(k)U$. Setting $T_\infty$ the fundamental solution of that equation, there exists a projection $\Pi$ such that

$$\begin{array}{rcll} |T_\infty(k;y,y')(I-\Pi(k;y'))\xi| & \leq & e^{-|k|(y-y')}|\xi| & \text{if } y \geq y' \geq 0 \\ |T_\infty(k;y,y')\Pi(k;y')\xi| & \leq & e^{|k|(y-y')}|\xi| & \text{if } y' \geq y \geq 0. \end{array} \tag{1.24}$$

Consider $E(k)$, the Banach space of functions $V$ such that

$$\|V\|_{E(k)} := ||Ve^{|k|y}||_{\mathcal{C}^0} < +\infty,$$

and $S$ the linear mapping defined by

$$SV(y) = \int_0^y T_\infty(k;y,y')(I-\Pi(k;y'))B(k,z;y')V(y')\,dy'$$

$$- \int_y^{+\infty} T_\infty(k;y,y')\Pi(k;y')B(k,z;y')V(y')\,dy'.$$

Remember that $|B(k,z;y)| \leq b|k|e^{-\eta y}$. Using (1.24), we get that if $V \in E(k)$, then $SV \in E(k)$ with the estimate

$$|SV(y)e^{|k|y}| \leq \|V\|_{E(k)} \left( \int_0^{+\infty} b|k|e^{-\eta y'}\,dy' \right) \leq \frac{1}{2}\|V\|_{E(k)} \tag{1.25}$$

when $|k| \leq \rho'$ small enough. So $S$ is a contracting endomorphism of $E(k)$.

Let $U_\infty(k) \in E(k)$ be a solution of $\partial U = A_\infty(k)U$. By the Duhamel formula, a bounded solution of the equation $\partial U = AU$ is a fixed point of the affine transform $\tilde{S} = S + U_\infty$ ; thanks to (1.25), Picard's fixed point theorem allows us to conclude that such a solution $U$ exists in $E(k)$. Finally, we must uniformly bound $\|U\|_{E(k)}$. Choosing $U_\infty(k) = (e^{-|k|y}, -|k|e^{-|k|y})$, the decreasing eigenfunction of $A_\infty$, (1.25) gives us the wanted bound.

The end of the proof is the same as in [103] ; we provide it for completeness. Note that for $k \neq 0$, $I - P(k,z) : f \mapsto [y \mapsto (I - P(k,z;y))f(y)]$ is the projection on the subspace of solutions that go to 0 when $y \to +\infty$ ; let us define $Q(k,z)$ the projection on the subspace of solutions of which the first component vanishes at $y = 0$. As the linearised Euler equation does not have an eigenvalue with real part $\gamma_0$, necessarily we have

$$\mathcal{R}(I - P(k,z;0)) \cap \mathcal{R}(Q(k,z;0)) = \{0\}, \tag{1.26}$$

where we denote $\mathcal{R}(A)$ the range of a matrix $A$, and we define a basis of solutions $(e_1(k,z), e_2(k,z))$, with $e_1 \in \mathcal{R}(I - P(k,z;0))$ and $e_2 \in \mathcal{R}(Q(k,z;0))$. Define new projections

$$P'(k,z) = (e_1(k,z), e_2(k,z)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (e_1(k,z), e_2(k,z))^{-1},$$

and $P'(k,z;y) = T(k,z;y,0)P'(k,z)$, so that we have both $\mathcal{R}(P'(k,z;y)) = \mathcal{R}(Q(k,z;y))$ and $\mathcal{R}(I - P'(k,z;y)) = \mathcal{R}(I - P(k,z;y))$. We also have the estimates

$$\begin{array}{rcll} |T(k,z;y,y')P'(k,z;y')\xi| & \leq & C'(k,z)e^{-\alpha(y-y')}|\xi| & \text{if } y \geq y' \geq 0 \\ |T(k,z;y,y')(I - P'(k,z;y'))\xi| & \leq & C'(k,z)e^{\alpha(y-y')}|\xi| & \text{if } y' \geq y \geq 0. \end{array} \tag{1.27}$$

Again, we must check that $C'(k,z)$ is bounded in $\mathcal{K}_{0,\rho}$. To do so, we point out that the projections $Q(k,z)$ and $I - P(k,z)$ can be continued up to $k = 0$ : we have that $\mathcal{R}(Q(0,z))$ is the subspace of solutions to the equation $z\Psi'' = 0$ with $\Psi(0) = 0$, and $\mathcal{R}(I - P(0,z))$ is the subspace of bounded solutions to $z\Psi'' = 0$ (constants). As the only bounded solution of $z\Psi'' = 0$ with $\Psi(0) = 0$ is $\Psi \equiv 0$, (1.26) is true up to $k = 0$, so $C'(k,z)$ is bounded for $k$ near 0.

Finally, by Duhamel's formula, a bounded solution of (1.23) is, for fixed $(k,z)$,

$$U(y) = \int_0^y T(y,y')P'(y')H(y')\ dy' - \int_y^{+\infty} T(y,y')(I - P(y'))H(y')\ dy',$$

as the only solution of $\partial_y U = A(k,z)U$ with $U_1 \in H_0^1(\mathbb{R}^+)$ is zero. So, by using (1.27) and standard convolution estimates, we have $\|U\|_{L^2} \leq C(k,z)\,\|H\|_{L^2}$. To get estimates on the $L^2$ norms of the derivatives, just differentiate the equation $l$ times, notice that it is the same type of system and conclude by induction. □

### 3.2 Proof of the instability result

Recall that $u^\varepsilon$ is a solution of $(NS(a,\varepsilon))$. First and foremost, as the initial profile depends on $\varepsilon^{-1/2}y$, we rescale the variables $(t,x,y) \to \varepsilon^{1/2}(t,x,y)$, and these new variables will be indifferently noted $(t,x,y)$. From now on, the faster variable will be $Y = \varepsilon^{-1/4}y$. The system we get after rescaling is

$$(NS') : \quad \begin{cases} (NS(\sqrt{\varepsilon})) : & \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \sqrt{\varepsilon}\Delta u^\varepsilon + \nabla p^\varepsilon &=& 0 \\ & \text{div } u^\varepsilon &=& 0 \\ & u^\varepsilon(0,x,y) &=& u_0^\varepsilon(x,y) \\ (NP) : & u_2^\varepsilon(t,x,0) &=& 0 \\ (NC(a)) : & \frac{1}{2}\partial_y u_1^\varepsilon(t,x,0) &=& a u_1^\varepsilon(t,x,0) \end{cases}$$

with the initial condition $u_0^\varepsilon$ close to a linearly unstable shear flow $u_{\text{sh}}$. The Navier boundary condition no longer depends on the viscosity, and we will show the following asymptotic expansion :

$$u^\varepsilon(t,x,y) \sim \overline{u_{\text{sh}}}(t,y) + u^i(t,x,y) + \varepsilon^{1/4}u^b(t,x,Y), \tag{1.28}$$

where $u^i$ solves the Euler equation and $u^b$ is the boundary layer.

Starting with a linearly unstable solution to the Euler equation, we will construct an approximate solution to the Navier-Stokes system $(NS')$ above that will allow us to prove the instability inequality (1.2). The approximate solution is described in the following proposition.

**Proposition 3.3.** *For given integers $n$ and $N$, there exist a time $T_0^\varepsilon$, depending on $\varepsilon$ but not converging to zero, a constant $C_R > 0$ and a family $((U^j,P^j))_{j\in\{1,\cdots,N\}}$ of functions such that the vector field*

$$u^{ap}(t,x,y) = \overline{u_{\text{sh}}}(t,y) + \sum_{j=1}^{N} \varepsilon^{jn} U^j(t,x,y)$$

*is divergence-free and tangent to the boundary, and the pair $(u^{ap},p^{ap})$, with $p^{ap}$ also written as a WKB expansion, $p^{ap} = \sum_{j=1}^N \varepsilon^{jn}P^j$, solves the Navier-Stokes equation approximately, in the sense that $\|u^{ap}(0) - u_{\text{sh}}\|_{L^2(\Omega)} \le C\varepsilon^n$, and*

$$\partial_t u^{ap} + u^{ap}\cdot\nabla u^{ap} - \sqrt{\varepsilon}\Delta u^{ap} + \nabla p^{ap} = R,$$

*with $R$ satisfying the following growth bound on the time interval $[0,T_0^\varepsilon]$ :*

$$\|R(t)\|_{L^2} \le \frac{C_R\varepsilon^{(N+1)n}}{(1+t)^{(N+1)/2}}(1+t)^{1/4}e^{(N+1)\sigma_0 t}. \tag{1.29}$$

*Moreover, there exist constants $C_j$, $j \in \{1,\cdots,N\}$ such that the individual components of $u^{ap}$ satisfy the following inequalities on $[0,T_0^\varepsilon]$ :*

$$\left\|U^j(t)\right\|_{H^1} \le C_j(1+t)^{1/4}\frac{e^{j\sigma_0 t}}{(1+t)^{j/2}}, \tag{1.30}$$

$$\left\|U^j(t)\right\|_{L^\infty} \le \frac{C_j e^{j\sigma_0 t}}{(1+t)^{j/2}}, \tag{1.31}$$

*and there exists* $C_1' > 0$ *and a bounded domain* $\Omega_A(t) \subset \Omega$*, whose measure is of order* $\sqrt{1+t}$*, on which*

$$\left\|U^1(t)\right\|_{L^2(\Omega_A(t))} \geq C_1'(1+t)^{1/4}\frac{e^{\sigma_0 t}}{\sqrt{1+t}}. \tag{1.32}$$

This proposition will be proved over the next two sections : the first to explain the construction of $u^{ap}$, in which the role of $u^\sharp$ will be given in particular, and the second to prove the estimates. The final estimate in Proposition 3.3 will allow us to get a lower bound on the $L^\infty$ norm of $U^1$, which will be crucial to prove the instability in the final paragraph.

### 3.2.1 Proof of Proposition 3.3, part 1 Building an approximate solution

For now, fix $N$ large and arbitrary (it will be chosen in section 3.2.3). Each $U^j$ will be written as a wave packet ; letting $k_0$ be such that $\sigma(k_0) = \sigma_0$, and setting $\varphi^j$ as compactly supported smooth functions of $k$ with $k_0 \in \mathrm{supp}(\varphi)$, we write

$$U^j(t,x,y) = \int_{\mathbb{R}} \varphi^j(k) V^j(k;t,x,y)\ dk,$$

with the $V^j(k)$ being in the previously-defined $V_k$ space. We shall write these wave packets more precisely in the next paragraph.

The equations that the $U^j$ are supposed to solve are Navier-Stokes equations linearised around $\overline{u_{\mathrm{sh}}}$ :

$$\partial_t U^j + \overline{u_{\mathrm{sh}}}\cdot\nabla U^j + U^j\cdot\nabla\overline{u_{\mathrm{sh}}} - \sqrt{\varepsilon}\Delta U^j + \nabla P^j + \sum_{j_1+j_2=j} U^{j_1}\cdot\nabla U^{j_2} = 0. \tag{1.33}$$

We will not necessarily have div $U^j = 0$ individually, but in total div $u^{ap}$ must be zero. We again only solve (1.33) approximately, by writing a sub-expansion

$$U^j = \sum_{m=0}^{8n-1} \varepsilon^{m/8} u^{i,8(j-1)n+m}(t,x,y) + \varepsilon^{(m+2)/8} u^{b,8(j-1)n+m}(t,x,Y), \tag{1.34}$$

and likewise for the pressure - there is a gap of $\varepsilon^{1/4}$ between corresponding internal and boundary layers, which is inspired by the asymptotic expansion (1.28). We therefore solve (1.33) with an error $\varepsilon^n\mathcal{E}^j$, which is taken into account in the equation on $U^{j+1}$, so we solve, at each level,

$$\partial_t U^j + \overline{u_{\mathrm{sh}}}\cdot\nabla U^j + U^j\cdot\nabla\overline{u_{\mathrm{sh}}} - \sqrt{\varepsilon}\Delta U^j + \nabla P^j = -\mathcal{E}^{j-1} - \sum_{j_1+j_2=j} U^{j_1}\cdot\nabla U^{j_2}.$$

In the final term $U^N$, we will need to construct more than $8n$ terms, to ensure that the terms in the error $\mathcal{E}^N$ will have the right growth. This also means that the final error can be as small

as we want, of order $\varepsilon^{n+l_0}$, by choosing $l_0$ large. $U^N$ will also contain a corrective term $u^\sharp$ to ensure that div $u^{ap} = 0$ and $u_2^{ap}|_{y=0} = 0$.

Writing $q = 8(j-1)n + m$, we must now understand the equations that the $u^{i,q}$ and $u^{b,q}$ solve to get estimates on these functions.

We plug ansatz (1.34) into each equation of the linearised Navier-Stokes system, and consider the equations obtained when asking terms of a same order $\varepsilon^{n+q/8}$ to cancel out. We get that the internal layers $u^{i,q}$, which solve only the Euler part of the $(NS(\sqrt{\varepsilon}))$ equation, are solutions of linear Euler systems

$$(EL(q)): \quad \begin{cases} (E_q): & \partial_t u^{i,q} + u_{\rm sh}\cdot\nabla u^{i,q} + u^{i,q}\cdot\nabla u_{\rm sh} + r^{i,q} + \nabla p^{i,q} = 0 \\ & \text{div } u^{i,q} = 0 \\ (P_q): & u_2^{i,q}(t,x,0) + u_2^{b,q-2}(t,x,0) = 0 \end{cases}$$

where we linearise around $u_{\rm sh}$ instead of $\overline{u_{\rm sh}}$, because we have a better understanding of the linear equation around time-independent profiles, and

$$r^{i,q} = \frac{\overline{u_{\rm sh}} - u_{\rm sh}}{\varepsilon^{1/8}}\cdot\nabla u^{i,q-1} + u^{i,q-1}\cdot\nabla\left(\frac{\overline{u_{\rm sh}} - u_{\rm sh}}{\varepsilon^{1/8}}\right) - \Delta u^{i,q-4} + \sum_{8n+q_1+q_2=q} u^{i,q_1}\cdot\nabla u^{i,q_2}.$$

We also get that the boundary layers each solve a Stokes problem with a Neumann boundary condition

$$(S(q)): \quad \begin{cases} (S_q): & \partial_t u^{b,q} - \partial_{YY}^2 u^{b,q} + r^{b,q} + \nabla p^{b,q} = 0 \\ (D_q): & \partial_Y u_2^{b,q} + \partial_x u_1^{b,q-2} = 0 \\ (N_q): & \left(\frac{1}{2}\partial_Y u_1^{b,q} + \frac{1}{2}\partial_y u_1^{i,q} - a u_1^{i,q} - a u_1^{b,q-2}\right)(t,x,0) = 0 \\ (V_q): & \lim_{Y\to+\infty} u_2^{b,q}(t,x,Y) = 0. \end{cases}$$

We do not give any detail on $r^{b,q}$ other than it depends on $\varepsilon^{-1/8}\overline{u_{\rm sh}}$ and terms $u^{i,m}$ and $u^{b,m}$ with $m < q$.

In both systems, initial conditions are chosen to be compatible with the boundary conditions, and rapidly decreasing functions of $y$ (like $v(0,x,0)e^{-y}$); therefore $U^j|_{t=0}$ is $\mathcal{O}(\varepsilon^n)$. For low values of $q$, any term with a negative index is of course ignored.

Now that we know the equations that each layer is supposed to solve, we can construct the approximate solution by induction. As announced, we start by choosing $u^{i,0}$ as an unstable solution of the linear Euler equation $(E_1)$ with homogeneous non-penetration condition. We write

$$u^{i,0}(t,x,y) = \int_{\mathbb{R}} \varphi^1(k) v^{i,0}(k;t,x,y)\, dk,$$

for a compactly supported function $\varphi^1$ to be chosen a little later, and we choose the most unstable mode for each $k$, *i.e.*

$$v^{i,0}(k;t,x,y) = e^{\lambda(k)t}\tilde{v}(k;y)e^{ikx},$$

with $\mathrm{Re}(\lambda(k)) = \sigma(k)$. Then, to solve $(S(0))$, $(D_0)$ and the limit condition allow us to choose $u_2^{b,0} = 0$, then $p^{b,0} = 0$ and $u_1^{b,0}$ solves a heat equation.

Having built $u^{i,m}$ and $u^{b,m}$ for $m < q$, $u^{i,q}$ is the solution of the linear Euler system $(E(q))$. Now we build $u^{b,q}$ by taking the equations of the Stokes system $(S(q))$ one after another. First, $(D_q)$ is the equation relative to the divergence-free condition : integrating it between $Y$ and $+\infty$ (to satisfy $(V_q)$), we get an expression of the normal component of $u^{b,q}$ that goes to 0 as $Y \to +\infty$ ; the rapid-decrease property will follow from the induction below, and nothing guarantees that we have $u_2^{b,q}(t,x,0) = 0$, so this is where the boundary condition $(P_q)$ in the Euler system comes from. Then, notice that thanks to the structure of $\overline{u_{\mathrm{sh}}}$, the normal part of the Stokes equation $(S_q)$ depends only on $u_2^{b,q}$ and $p^{b,q}$ (and of course previous terms), so we get $p^{b,q}$ by the same method. Finally, the first component of $(S_q)$ is now a heat equation, with the Neumann boundary condition $(N_q)$. We continue the construction up to $M = 8nN + 8l_0$ ; all terms with $q \geq 8nN$ are part of $U^N$.

We now consider a corrective term $\varepsilon^{n+l_0} u^\sharp(t,x,y)$ (which is part of $U^N$ as mentioned above) such that

$$\begin{aligned} \mathrm{div}\ u^\sharp &= -\partial_x v_1^\sharp & (1.35)\\ \text{and}\quad u_2^\sharp(t,x,0) &= -v_2^\sharp(t,x,0), & (1.36) \end{aligned}$$

where $v^\sharp = u^{b,M-1} + \varepsilon^{1/8} u^{b,M}$. This ensures that $u^{ap}$ is divergence-free and tangent to the boundary, although each $U^j$, taken individually, is not. Note that, because of $(D_q)$, $\int_\Omega \partial_x v_1^\sharp = \int_{\partial\Omega} v_2^\sharp = 0$, so the equation is consistent with the divergence theorem.

To construct a solution, we decompose

$$u^\sharp(x,y) = u^{\sharp,1}(x,y) + u^{\sharp,2}(x,\varepsilon^{-1/4}y),$$

so that $u^{\sharp,1}$ is divergence-free with $u_2^{\sharp,1}(t,x,0) = -v_2^\sharp(t,x,0)$. We simply set

$$u_2^{\sharp,1}(t,x,y) = -v_2^\sharp(t,x,0)\chi(y)$$

with $\chi$ a smooth function supported in $[0,1]$ with $\chi(0) = 1$, and get $u_1^{\sharp,1}$ by integrating the divergence-free condition and using $(D_M)$ and $(D_{M-1})$ :

$$\begin{aligned} u_1^{\sharp,1}(t,x,y) &= \chi'(y) \int_x^{+\infty} v_2^\sharp(t,\xi,0)\ d\xi \\ &= -\chi'(y) \int_x^{+\infty} \int_0^{+\infty} \partial_x (u_1^{b,M-3} + \varepsilon^{1/8} u_1^{b,M-2})(t,\xi,Y)\ d\xi\ dY \\ &= \chi'(y) \int_0^{+\infty} u_1^{b,M-3}(t,x,Y) + \varepsilon^{1/8} u_1^{b,M-2}(t,x,Y)\ dY. \end{aligned}$$

Meanwhile, $u^{\sharp,2}$ will be tangent to the boundary and div $u^{\sharp,2} = -\partial_x v_1^\sharp$. This equation can be solved classically (as in the introduction of [5] for example), by decomposing $u^{\sharp,2} = \nabla\Psi^\sharp + \nabla^\perp\Phi^\sharp$, and solving a Poisson equation. Thus, we have

$$\left\|u^{\sharp,2}\right\|_{H^s} \leq C \left\|v^\sharp\right\|_{H^s}. \tag{1.37}$$

### 3.2.2 Proof of Proposition 3.3, part 2 Estimates on the approximate solution

The important estimates are those on the first term $U^1$, or more precisely, those on the wave packet of unstable modes of the Euler equation

$$u^{i,0}(t,x,y) = \int_{\mathbb{R}} \varphi^1(k) e^{ikx} e^{\lambda(k)t} \tilde{v}(k;y)\ dk,$$

with $\varphi^1$ supported in $I = ]-k_0-\eta, -k_0+\eta[\cup]k_0-\eta, k_0+\eta[$, where $\eta$ is small enough to have $I \subset \{k \mid \sigma(k) > 0\}$, and to allow us to apply Theorem 3.1 in what follows. Also, $k_0$ must be the only critical point of $\sigma$ in $]k_0-\eta, k_0+\eta[$. By Parseval's equality, we have

$$\left\|u^{i,0}(t)\right\|_{H^s}^2 = \int_{\mathbb{R}} \varphi^1(k)^2 \left\|\tilde{v}(k)\right\|_{H^s}^2 e^{2\sigma(k)t}\ dk$$

so, as $\pm k_0$ is the only nondegenerate maximum of $\sigma$ in each sub-interval of $I$, using the Laplace method, we get $\left\|u^{i,0}(t)\right\|_{H^s}^2 \sim t^{-1/2} e^{2\sigma_0 t}$ when $t \to +\infty$, so we have, given that $\left\|u^{i,0}(t)\right\|_{H^s}$ is bounded near $t = 0$,

$$C' \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \leq \left\|u^{i,0}(t)\right\|_{H^s} \leq C \frac{e^{\sigma_0 t}}{(1+t)^{1/4}}. \tag{1.38}$$

This allows us to estimate the other terms $u^{i,q}$ and $u^{b,q}$; this is done by induction. Fix $j$, and suppose that the internal layers $u^{i,q}$ with $0 \leq q < j$ satisfy

$$\left\|u^{i,q}(t)\right\|_{H^s} \leq c_q \frac{\exp\left[\sigma_0\left(1+\frac{q}{8n}\right)t\right]}{(1+t)^{1/4(1+q/8n)}}. \tag{1.39}$$

As $e^{\sigma_0 \alpha t}(1+t)^{-\alpha} \overset{t\to+\infty}{\longrightarrow} +\infty$, we have

$$\left\|u^{i,q}(t)\right\|_{H^s} \leq C_j \frac{\exp\left[\sigma_0\left(1+\frac{j}{8n}\right)t\right]}{(1+t)^{1/4(1+j/8n)}}. \tag{1.40}$$

Then, with energy estimates on the heat equation with the Robin boundary condition induced by the Navier condition, we get

$$\left\|\frac{\overline{u_{\mathrm{sh}}}(\sqrt{\varepsilon}t) - u_{\mathrm{sh}}}{\varepsilon^{1/8}}\right\|_{H^s} \leq \frac{e^{C\sqrt{\varepsilon}t} - 1}{\varepsilon^{1/8}},$$

so $\varepsilon^{-1/8}(\overline{u_{\rm sh}}(\sqrt{\varepsilon}t)-u_{\rm sh})$ is bounded in $H^s$ uniformly for $t\leq\varepsilon^{-1/32}$. Therefore, the remainder $r^{i,j}$ in the Euler equation $(E_j)$ verifies (1.40), as does $\partial_t r^{i,j}$, so $u^{i,j}$ also satisfies (1.40) by Theorem 3.1. Note that, because of our rescaling, we are studying the stability of $\overline{u_{\rm sh}}(\sqrt{\varepsilon}t)$.

Now we consider the boundary layer $u^{b,j}$. Suppose that $u^{b,q}$, $q<j$, are rapidly decreasing functions in the $Y$-variable and satisfy (1.39). By construction, $u_2^{b,j}$ and $p^{b,j}$ are rapidly decreasing functions in the $Y$-variable. Then, as $\varepsilon^{-1/8}|\overline{u_{\rm sh}}(\sqrt{\varepsilon}t,\varepsilon^{1/4}Y)|$ is also uniformly bounded in $\varepsilon$ for $t\leq\varepsilon^{-1/32}$, $Y\in\mathbb{R}^+$ and $\varepsilon<\varepsilon_0$ small enough, we get, by using the Green function of the heat equation with Neumann boundary conditions (see [114]), that $u^{b,j}$ is a rapidly decreasing function of $Y$ satisfying

$$\left\|u^{b,j}\right\|_{H^s}\leq C\sqrt{1+t}\frac{e^{\sigma_0(1+(j-1)/8n)t}}{(1+t)^{1/4[1+(j-1)/8n]}}\leq C_j\frac{\exp\left[\sigma_0\left(1+\frac{j}{8n}\right)t\right]}{(1+t)^{1/4(1+j/8n)}}. \tag{1.41}$$

Now, we estimate the main expansion terms, starting with $U^1$. Overall, after operating a change of variables in the boundary layer terms, we have

$$\left\|\varepsilon^n U^1(t)\right\|_{H^1}\leq C_1\sum_{j=0}^{8n-1}\frac{\varepsilon^{n+j/8}}{(1+t)^{1/4(1+j/8n)}}\exp\left[\sigma_0\left(1+\frac{j}{8n}\right)t\right]. \tag{1.42}$$

We will be interested in times $t=T_0^\varepsilon-\tau$, with $T_0^\varepsilon$ such that

$$\frac{\varepsilon^n e^{\sigma_0 T_0^\varepsilon}}{\sqrt{1+T_0^\varepsilon}}=1.$$

The important manoeuvre here is to write $t=T_0^\varepsilon-\tau$, so that

$$\frac{\varepsilon^n e^{\sigma_0 t}}{\sqrt{1+t}}=e^{-\sigma_0\tau}\sqrt{\frac{1+T_0^\varepsilon}{1+T_0^\varepsilon-\tau}}:=K_0^\varepsilon(\tau)e^{-\sigma_0\tau}$$

with $K_0^\varepsilon(\tau)\in]1,2]$ for $\varepsilon$ small enough, and with $0\leq\tau\leq\tau_0$ where $\tau_0$ does not depend on $\varepsilon$. As $T_0^\varepsilon\sim c\ln(\varepsilon^{-1})$, we can choose $\varepsilon<\varepsilon_0$ so that $T_0^\varepsilon-\tau_0>0$. $\tau_0$, and therefore $\varepsilon_0$, will be chosen at the end of the proof in the next paragraph.

We show how this manoeuvre works in detail. Writing $t=T_0^\varepsilon-\tau$, (1.42) becomes

$$\begin{aligned}\left\|\varepsilon^n U^1(T_0^\varepsilon-\tau)\right\|_{H^1} &\leq (1+T_0^\varepsilon-\tau)^{1/4}C_1K_0^\varepsilon(\tau)e^{-\sigma_0\tau}\sum_{j=0}^{8n-1}(K_0^\varepsilon(\tau)e^{-\sigma_0\tau})^{j/8n}\\ &\leq (1+T_0^\varepsilon-\tau)^{1/4}16nC_1K_0^\varepsilon(\tau)e^{-\sigma_0\tau}.\end{aligned}$$

Thus, by returning to the $t$-variable, for $t\leq T_0^\varepsilon$, we get (1.30) for $U^1$ :

$$\left\|U^1(t)\right\|_{H^1}\leq(1+t)^{1/4}\frac{C_1e^{\sigma_0 t}}{\sqrt{1+t}}.$$

This estimate means that $\left\|U^1(t)\right\|_{H^1}$ behaves like $\left\|u^{i,0}(t)\right\|_{H^1}$.

In Theorem 1.3, we want an estimate from below in $L^\infty$, so we will work a little more on $U^1$ in order to get a key under-estimate of a local $L^2$ norm, which will be (1.32). If $\alpha = \mathrm{Im}(\lambda'(k_0))$ and $\beta = -\sigma''(k_0) > 0$, then

$$|u^{i,0}(t,x,y)| \sim e^{\sigma_0 t}\left|\int_I \varphi^1(k)\tilde{v}(k,y)\exp(i(x+\alpha t)(k-k_0) - \beta t(k-k_0)^2)\ dk\right|.$$

But, writing $\kappa = k - k_0$, we have (by factorising)

$$\int_{\mathbb{R}} \exp(i(x+\alpha t)\kappa - \beta t\kappa^2)\ d\kappa = \frac{c}{\sqrt{t}}\exp\left(\frac{-(x+\alpha t)^2}{4t}\right)$$

with $c$ complex, so there exists $C$ such that

$$|u^{i,0}(t,x,y)| \geq \frac{Ce^{\sigma_0 t}}{\sqrt{1+t}}\exp\left(\frac{-(x+\alpha t)^2}{4t}\right).$$

Integrating $|u^{i,0}(t)|^2$ on the bounded domain

$$\Omega_A(t) = \{(x,y) \mid y \leq A \text{ and } |x+\alpha t| \leq A\sqrt{1+t}\}$$

for $A$ large enough, and using the fact that $u^{i,0}$ is the dominant term in $U^1$, we get (1.32) :

$$\left\|U^1(t)\right\|_{L^2(\Omega_A(t))} \geq C_1'(1+t)^{1/4}\frac{e^{\sigma_0 t}}{\sqrt{1+t}}.$$

As the measure of $\Omega_A(t)$ is $A^2\sqrt{1+t}$, this gives us an under-estimate of the norm $\left\|U^1(t)\right\|_{L^\infty}$. In fact, $\left\|U^1(t)\right\|_{L^\infty} \sim C(1+t)^{-1/2}e^{\sigma_0 t}$.

Now we specify the structure of $U^j$ to get $L^\infty$ and $H^1$ estimates : we write

$$U^j(t,x,y) = \int_I \cdots \int_I V^j(k_1,\cdots,k_j;t,y)e^{ik_1 x}\cdots e^{ik_j x}\ dk_1\cdots dk_j$$

with $|V^j(k_1,\cdots,k_j;t,y)| \leq C_j\exp[(\sigma(k_1)+\cdots+\sigma(k_j))t]$, which gives us

$$\begin{aligned}\left\|U^j(t)\right\|_{L^\infty} &\leq C_j\int_I\cdots\int_I \exp[(\sigma(k_1)+\cdots+\sigma(k_j))t]\ dk_1\cdots dk_j \\ &\leq C_j e^{j\sigma_0 t}\prod_{m=1}^{j}\int_{\mathbb{R}}\exp(-\beta(k_m-k_0)^2 t)\ dk_m \leq \frac{C_j e^{j\sigma_0 t}}{(1+t)^{j/2}},\end{aligned}$$

which is (1.31), thanks to the second-order Taylor inequality

$$\sigma(k) \leq \sigma_0 - \beta(k-k_0)^2. \tag{1.43}$$

For an $H^1$ estimate, we rewrite $U^j$ more precisely as

$$U^j(t,x,y) = \int_{jk\in jI} \int_{k_1+\cdots+k_j=jk} V^j(k_1,\cdots,k_j;t,y)e^{ijkx}\ dk_1\cdots dk_j$$

with $jI = I+\cdots+I$. As, for a given $k$, the value of $k_j$ is imposed by the other variables, Parseval's equality yields

$$\left\|U^j(t)\right\|_{H^1}^2 = \int_{jk\in jI} \left\| \int_{k_1+\cdots+k_j=jk} V^j(k_1,\cdots,k_j;t)\ dk_1\cdots dk_{j-1}\right\|_{H^1}^2 dk := \int_{jk\in jI} \mathcal{N}(k,t)^2\ dk.$$

Using (1.43), and noticing that

$$\sum_{m=1}^{j}(k_m-k_0)^2 = j(k-k_0)^2 + \sum_{m=1}^{j}(k-k_m)^2,$$

we have

$$\mathcal{N}(t)^2 \le Ce^{2(j\sigma_0 - j\beta(k-k_0)^2)t} \int_{\sum_{m=1}^j k_m = jk} \exp(-2\beta\sum_{m=1}^{j}(k_m-k)^2 t)\ dk_1\cdots dk_{j-1}.$$

Remembering that $k_j = jk - \sum_{m=1}^{j-1} k_m$, we integrate these gaussian functions, so

$$\left\|U^j(t)\right\|_{H^1}^2 \le \int_{\mathbb{R}} \frac{Ce^{2j\sigma_0 t - 2j\beta(k-k_0)^2 t}}{t^{j-1}}\ dk.$$

Integrating this final gaussian function, and taking into account the boundedness of $\left\|U^j(t)\right\|_{H^1}$ near $t=0$, we get (1.30) :

$$\left\|U^j(t)\right\|_{H^1} \le C_j(1+t)^{1/4}\frac{e^{j\sigma_0 t}}{(1+t)^{j/2}}.$$

Finally, to end the proof of Proposition 3.3, we need to prove that $u^{ap}$ is indeed an approximate solution to the Navier-Stokes equation, in that the error $R$ satisfies estimate (1.29). This error, $R = R^1 + R^\sharp$, contains two types of terms. Firstly, $R^1$ is made up of the terms $\varepsilon^{(j+l)n}U^j\cdot\nabla U^l$ with $j+l>N$, in other words terms from the nonlinearity of the Navier-Stokes equation, so by using (1.13), we have the estimate

$$\left\|R^1(t)\right\|_{L^2} \le C(1+t)^{1/4}\varepsilon^{(N+1)n}\sum_{j=N+1}^{2N}\frac{e^{j\sigma_0 t}}{(1+t)^{j/2}}.$$

Replacing $t$ by $T_0^\varepsilon - \tau$, we get

$$\begin{aligned}\left\|R^1(T_0^\varepsilon-\tau)\right\|_{L^2} &\leq C(1+T_0^\varepsilon-\tau)^{1/4}\cdot K_0^\varepsilon(\tau)^{2N}\cdot\sum_{j=N+1}^{2N}e^{-j\sigma_0\tau}\\ &\leq C(1+T_0^\varepsilon-\tau)^{1/4}\cdot 2^{N-1}K_0^\varepsilon(\tau)^{N+1}\cdot Ne^{-j(N+1)\tau},\end{aligned}$$

and so, for $t\in[T_0^\varepsilon-\tau_0,T_0^\varepsilon]$,

$$\left\|R^1(t)\right\|_{L^2}\leq\frac{C\varepsilon^{(N+1)n}}{(1+t)^{(N+1)/2}}(1+t)^{1/4}e^{(N+1)\sigma_0 t}. \tag{1.44}$$

The constant $C$ is then adjusted so that (1.44) is true on $[0,T_0^\varepsilon]$.

Secondly, in $R^\sharp$, we find terms involving the corrector term $\varepsilon^{(N+1)n+l_0}u^\sharp$ and the error terms coming from the laplacian (the ones that do not appear in any of the equations $(E_q)$ or $(S_q)$) also appear. Specifically, we need estimates on the final four layers of $U^N$, so we set

$$\begin{aligned}w^\sharp(t,x,y) &= w^{i,\sharp}(t,x,y)+\varepsilon^{1/4}w^{b,\sharp}(t,x,\varepsilon^{-1/4}y)\\ &= \varepsilon^{(N+1)n+l_0}\sum_{q=0}^{3}\left(\varepsilon^{-q/8}u^{i,M-q}+\varepsilon^{1/4-q/8}u^{b,M-q}(t,x,\varepsilon^{-1/4}y)\right),\end{aligned}$$

and we need to get estimates for $\left\|w^\sharp\right\|_{H^1}+\varepsilon^{1/2}\left\|w^\sharp\right\|_{H^2}$. Indeed, given our construction of $u^{\sharp,1}$ and (1.37), we have, due to the boundary-layer scale of $w^{b,\sharp}$,

$$\left\|\varepsilon^{(N+1)n+l_0}u^\sharp\right\|_{H^s}\leq C\varepsilon^{3/8-s/4}\left\|w^{b,\sharp}\right\|_{H^s}.$$

We now note that, for any $l_0\geq 1$, the terms in $w^{i,\sharp}$ satisfy (1.40) and those in $w^{b,\sharp}$ satisfy (1.41), both with $1+\frac{j}{8n}>N+1$, so we have that $w^\sharp$, and therefore $R^\sharp(t)$, satisfies (1.44) for $t\in[0,T_0^\varepsilon]$, and therefore, we have (1.29).

### 3.2.3 Proof of Theorem 1.3 : instability

For this paragraph, we denote $\delta=\varepsilon^n$. The order $N$ of the main expansion will be chosen in this section. Let $u^\varepsilon$ be an exact solution to the $(NS')$ system with initial condition $u^{ap}(0)$. We have $\|u^\varepsilon(0)-u_{\rm sh}\|_{L^2}\leq\varepsilon^n$. The aim is to get a lower bound of $\|u^\varepsilon-\overline{u_{\rm sh}}\|_{L^\infty}$, so we will look for a local $L^2$ lower bound.

Let us first estimate $\|u^{ap}-\overline{u_{\rm sh}}\|_{L^2(\Omega_A(t))}$ : using (1.30) and (1.32), we have

$$\|u^{ap}(t)-\overline{u_{\rm sh}}(t)\|_{L^2(\Omega_A(t))}\geq(1+t)^{1/4}\left(\frac{C_0\delta}{(1+t)^{1/2}}e^{\sigma_0 t}-\sum_{j=2}^{N}\frac{C_j\delta^j}{(1+t)^{j/2}}e^{j\sigma_0 t}\right).$$

Writing $t = T_0^\varepsilon - \tau$, we get

$$\|u^{ap} - \overline{u_{\rm sh}}\|_{L^2(\Omega_A(T_0^\varepsilon-\tau))} \geq (1+T_0^\varepsilon-\tau)^{1/4} K_0^\varepsilon(\tau)\left(C_0 e^{-\sigma_0\tau} - C_0' e^{-2\sigma_0\tau}\right)$$

with $C_0' = N2^N \max_{1\leq j\leq N} C_j$. There exists $\tau_1 > 0$ such that

$$C_0 e^{-\sigma_0\tau} - C_0' e^{-2\sigma_0\tau} \geq \frac{C_0}{2} e^{-\sigma_0\tau}$$

for $\tau \geq \tau_1$. So, choosing $\tau_0 > \tau_1$, we have, for $T_0^\varepsilon - \tau_0 \leq t \leq T_0^\varepsilon - \tau_1$,

$$\|u^{ap}(t) - \overline{u_{\rm sh}}(t)\|_{L^2(\Omega_A(t))} \geq (1+t)^{1/4} \frac{C_0\delta}{2(1+t)^{1/2}} e^{\sigma_0 t}. \tag{1.45}$$

Note that $\tau_1$ does not depend on $\delta$.

Now we estimate $\|w(t)\|_{L^2(\Omega)}$, where $w = u^\varepsilon - u^{ap}$ and $q = p^\varepsilon - p^{ap}$. The pair $(w,q)$ solves the system

$$\begin{cases} \partial_t w + w\cdot\nabla w - \sqrt{\varepsilon}\Delta w + \nabla q &= -u^{ap}\cdot\nabla w - w\cdot\nabla u^{ap} + R \\ \text{div } w &= 0 \\ (\partial_y w_1 - 2a w_1)(t,x,0) &= r(t,x,0) \\ w_2(t,x,0) &= 0 \end{cases}$$

with $w|_{t=0} = 0$. Note that there is an error in the Navier boundary condition :

$$r(t,x,0) = \varepsilon^{(N+1)n+l_0}(\partial_y u_1^\sharp - 2a u_1^\sharp - 2a v_1^\sharp)(t,x,0).$$

We compute the $L^2$ dot-product of this equation and $w$ ; integrating by parts and using the Young inequality with a parameter $\rho_a$ depending on $a$ on the boundary term $\int_{\partial\Omega} r w_1$ (to absorb $\int_{\partial\Omega}|w_1|^2$), and the Young inequality again on the source term $\int_\Omega Rw$, we get the energy estimate

$$\begin{aligned} \partial_t \|w\|_{L^2}^2 + 2\sqrt{\varepsilon}\|\nabla w\|_{L^2}^2 + 2a\sqrt{\varepsilon}\int_{\partial\Omega}|w_1|^2 \\ \leq (2\|\nabla u^{ap}\|_{L^\infty} + \frac{1}{2})\|w\|_{L^2}^2 + \frac{1}{2\rho_a}\sqrt{\varepsilon}\int_{\partial\Omega}|r|^2 + \frac{1}{2}\|R\|_{L^2}^2 . \end{aligned}$$

With the trace theorem, we re-use the estimates on $u^\sharp$ and $v^\sharp$ to notice that the $H^1$ norm of the Navier error $r$ has the same behaviour as the $L^2$ norm of $R$, so there exists $C_a > 0$ such that

$$\partial_t \|w\|_{L^2}^2 + 2\sqrt{\varepsilon}\left(\|\nabla w\|_{L^2}^2 + a\int_{\partial\Omega}|w_1|^2\right) \leq (2\|\nabla u^{ap}\|_{L^\infty} + \frac{1}{2})\|w\|_{L^2}^2 + C_a\|R\|_{L^2}^2 .$$

An $L^\infty$ estimate on $\nabla u^{ap}$ is crucial here : as each $U^j$ can be decomposed into two parts $U^j = U^{i,j}(t,x,y) + \varepsilon^{1/4}U^{b,j}(t,x,Y)$, the first order derivatives of $U^j$ verify (1.31)-type estimates, since $\varepsilon^{1/4}\nabla(U^{b,j}(t,x,\varepsilon^{-1/4}y)) = (\nabla U^{b,j})(t,x,Y)$, so

$$\|\nabla u^{ap}(t)\|_{L^\infty} \leq \|\overline{u_{\rm sh}}(t)\|_{L^\infty} + \sum_{j=1}^N \frac{C_j\delta^j e^{j\sigma_0 t}}{(1+t)^{j/2}}.$$

Using the $t = T_0^\varepsilon - \tau$ manoeuvre, the sum on the right is smaller than $\frac{1}{4}$ for $t \leq T_0^\varepsilon - \tau_2$, with $\tau_2 \geq \tau_1$ independent of $\delta$.

Here we choose $N$, the order of the main expansion, to be such that, for $t \geq 0$, $2N\sigma_0 > 2(\|\nabla \overline{u_{\rm sh}}(t)\|_{L^\infty} + 1)$. With (1.29) and this choice of $N$, the energy estimate becomes

$$\partial_t \|w(t)\|_{L^2}^2 \leq 2N\sigma_0 \|w(t)\|_{L^2}^2 + \frac{C_R \delta^{N+1}}{(1+t)^{(N+1)/2}} (1+t)^{1/4} e^{(N+1)\sigma_0 t}.$$

Therefore, $u^\varepsilon(t)$ is an $L^2$ perturbation of $\overline{u_{\rm sh}}(t)$, as we have the estimate

$$\|w(t)\|_{L^2(\Omega)} \leq \frac{C_0' \delta^{N+1}}{(1+t)^{(N+1)/2}} (1+t)^{1/4} e^{(N+1)\sigma_0 t}, \tag{1.46}$$

by applying the following lemma, which is obtained using a variant of Grönwall's lemma from [95] and integration by parts :

**Lemma 3.4.** *Let $\varphi$ be a function such that*

$$\partial_t \varphi(t) \leq \lambda \varphi(t) + C \frac{e^{\mu t}}{(1+t)^\alpha}$$

*for every $t \geq 0$, and for parameters $\mu > \lambda \geq 0$ and $\alpha > 1$, then*

$$\varphi(t) \leq C' \frac{e^{\mu t}}{(1+t)^\alpha}$$

*with $C'$ depending on $\varphi(0)$, $\lambda$, $\mu$ and $C$.*

With (1.45) and (1.46), we can now conclude :

$$\begin{aligned} \|u^\varepsilon(t) - \overline{u_{\rm sh}}(t)\|_{L^2(\Omega_A(t))} &\geq \|u^{ap}(t) - \overline{u_{\rm sh}}(t)\|_{L^2(\Omega_A(t))} - \|w(t)\|_{L^2(\Omega)} \\ &\geq (1+t)^{1/4} \left[ \frac{C_0 \delta e^{\sigma_0 t}}{2(1+t)^{1/2}} - \frac{C_0' \delta^{N+1} e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/2}} \right]. \end{aligned}$$

Writing $t = T_0^\varepsilon - \tau$ for the last time, we have

$$\frac{C_0 \delta e^{\sigma_0 t}}{2(1+t)^{1/2}} - \frac{C_0' \delta^{N+1} e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/2}} \geq \frac{C_0 \delta e^{\sigma_0 t}}{4(1+t)^{1/2}}$$

for $t \leq T_0^\varepsilon - \tau_3$, with $\tau_3 \geq \tau_2$ independent of $\delta$. Setting $T^\varepsilon = T_0^\varepsilon - T$, with $\tau_3 < T < \tau_0$ fixed (here we can finally fix $\tau_0$ and $\varepsilon_0$), we have

$$\|u^\varepsilon(T^\varepsilon) - \overline{u_{\rm sh}}(T^\varepsilon)\|_{L^2(\Omega_A(T^\varepsilon))} \geq (1+T^\varepsilon)^{1/4} \frac{C_0}{4} e^{-\sigma_0 T} := (1+T^\varepsilon)^{1/4} \delta_0,$$

where $\delta_0$ does not depend on $\varepsilon$. Thus, $\|u^\varepsilon(T^\varepsilon) - \overline{u_{\rm sh}}(T^\varepsilon)\|_{L^\infty} \geq C\delta_0$.

Also, as $\dot{H}^s \hookrightarrow L^\infty$ for $s > 1$, we have $\|u^\varepsilon(T^\varepsilon) - \overline{u_{\rm sh}}(T^\varepsilon)\|_{\dot{H}^s} \geq C\delta_0$. Remember that we are still using the fast variables, so, returning to the original scale of time and space (again without changing notation), we get (1.2) and

$$\left\|u^\varepsilon(T^\varepsilon, x, y) - \overline{u_{\rm sh}}(T^\varepsilon, \varepsilon^{-1/2}y)\right\|_{\dot{H}^s} \geq C\delta_0\varepsilon^{(1-s)/2} \overset{\varepsilon\to 0}{\longrightarrow} +\infty.$$

Note that in the original scale of time, $T^\varepsilon = \mathcal{O}(n\sqrt{\varepsilon}\ln(\varepsilon^{-1}))$. □

*Remark :* we can also use the Sobolev embedding $\dot{H}^s \hookrightarrow L^{p^*}$ for $0 < s < 1$, with $p^* = \frac{2}{1-s}$, and the Hölder inequality to get a result similar to Grenier's in the Dirichlet case :

$$\left\|u^\varepsilon(T^\varepsilon, x, y) - \overline{u_{\rm sh}}\left(T^\varepsilon, \frac{y}{\sqrt{\varepsilon}}\right)\right\|_{\dot{H}^s(\Omega)} \geq \delta_0(s)\varepsilon^{(1-s)/4},$$

where $\delta_0(s)$ is a constant depending on $s$, $\delta_0$ in Theorem 1.3 and $A$, the parameter in the bounded domain $\Omega_A(t)$. This gives a little information on the transition between stability in $L^2$ and instability in $L^\infty$.

# 4 An example of unstable profile

Examples of piecewise-linear flows that are linearly unstable for the Euler equation are given in [35] and [47], the latter stating that close-enough regularisations of these are smooth unstable shear flows. We wish to provide an explicit example of smooth linearly unstable profile, which is the object of Proposition 4.1 below. From this example, we easily deduce the expression of a smooth unstable profile that fits Theorem 1.3, and we can also derive an unstable profile that tends to 0 exponentially. This final example fits the hypotheses of both of our main theorems in the periodic case, meaning that the convergence in $L^2$ and the instability in $L^\infty$ can be simultaneous.

## 4.1 The model case : a hyperbolic tangent

**Proposition 4.1.** *For every $\delta > 0$ and $\zeta \in \mathbb{R}$, the shear flow $(u_\delta, 0)$ with*

$$u_\delta(y) = \tanh(y - \delta) + \zeta$$

*is a linearly unstable for the Euler equation.*

*Proof :* writing $\lambda = -ikc$ for $\mathrm{Re}(\lambda) > 0$ and $k \neq 0$, and taking the curl of the Euler equation linearised around $u_\delta$, we obtain a linear second-order differential equation for $\Psi$ : the Rayleigh equation

$$(R(c,k)) : \quad (u_\delta - c)(\partial^2_{yy} - k^2)\Psi - u''_\delta\Psi = 0$$

with the conditions $\Psi(0) = 0$ and $\lim_{y\to+\infty} \Psi(y) = 0$. Our problem is now finding $c$ in the complex upper-half-plane, and real numbers $k$ such that $(R(c,k))$ has a solution in $H^1_0(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$.

It is well known that if the Rayleigh equation has an unstable solution, then $u_\delta$ must have an inflection point in $]0,+\infty[$ (Rayleigh's theorem, [115]), and $u_\delta$ has exactly one inflection point, $y_0 = \delta$, with inflection value $u_0 = u_\delta(y_0) = \zeta$. Also, the function

$$K_\delta(y) = \frac{-u''_\delta(y)}{u_\delta(y) - u_0} = \frac{-(\tanh(y-\delta))''}{\tanh(y-\delta)} = 2(1 - \tanh(y-\delta)^2)$$

has a limit when $y \to y_0$, and is a positive continuous function on $\mathbb{R}^+$, vanishing at infinity. Thus, choosing $c = u_0$, we can divide $(R(c,k))$ by $u_{\rm sh} - u_0$ (which is not usually possible when $c$ is in the range of $u_\delta$), and we have a Sturm-Liouville problem

$$-\Psi''(y) - K_\delta(y)\Psi(y) = -k^2\Psi(y),$$

in which the square of the wave number $k$ intervenes as an eigenvalue of the operator $S_\delta = -\partial^2 - K_\delta$. We shall therefore use the following result by Z.Lin (Theorem 1.5 in [72]) :

**Theorem 4.2.** *[72] Let $U$ be a $\mathcal{C}^2$ profile which has a limit $l$ as $y \to +\infty$, and an inflection point $y_0$ such that, writing $u_0 = U(y_0)$, the function $K(y) = \frac{-U''(y)}{U(y)-u_0}$ is a positive continuous function which goes to zero as $y \to +\infty$. We assume that $U$ takes the value $l$ only a finite number of times. If the lowest eigenvalue of the operator $S = -\partial^2 - K$, defined on the Sobolev space $H^1_0(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$, is strictly negative, then, letting $-\mu^2$ be the lowest eigenvalue, for every $k \in (0,\mu)$, there exists $c(k)$ such that the Rayleigh equation $(R(c(k),k))$ has an unstable solution.*

The result is first shown on a finite interval ; one proves that eigenfunctions of $S$ with negative eigenvalues are limits of unstable solutions, and this is done by using the Picard fixed point theorem. A compactness argument is then used to get the result on the half-line. We detail the proof of this theorem no further.

In the case of $u_\delta$, it remains to show that $S_\delta = -\partial^2 - K_\delta$ has a negative eigenvalue. We hope to use a standard Sturm-Liouville argument, and notice that $v_\delta(y) = \tanh(y-\delta)$ is a solution to the equation

$$-\varphi''(y) - K_\delta(y)\varphi(y) = 0 \tag{1.47}$$

that has exactly one zero in $]0,+\infty[$. Since $K_\delta$ decreases exponentially to 0, the multiplication by $K_\delta$ is a compact perturbation of $-\partial^2$, so by Weyl's theorem (see [99]), the essential spectrum of $S_\delta$ is $\mathbb{R}^+$. But that means trouble, because the 0 in (1.47) corresponds to the lowest point in the essential spectrum, and the one zero of $v_\delta$ means that $S_\delta$ has between 0 and 3 negative eigenvalues, according to the classical Sturm-Liouville analysis shown in [36], chapter XIII.

So we must use a different tool to determine that $S_\delta$ has a strictly negative eigenvalue. Fix $\delta$ and let $Q_\delta$ be the quadratic form associated with $S_\delta$ : for $u \in H_0^1(\mathbb{R}^+)$, we define

$$Q_\delta(u) = \int_0^{+\infty} |u'(y)|^2 - K_\delta(y)|u(y)|^2 \, dy.$$

Let $v_\eta(y) = \tanh(y-\eta)$ for $\eta > 0$, and $\chi$ be a [0,2]-supported smooth function, with $\chi(y) = 1$ for $y \leq 1$. For any $n \in \mathbb{N}^*$, we define

$$w_\eta^n(y) = \begin{cases} 0 & \text{if } y \leq \eta \\ v_\eta(y) & \text{if } y \in [\eta, \delta] \\ v_\eta(y)\chi\left(\frac{y}{n}\right) & \text{if } y \geq \delta. \end{cases}$$

$w_\eta^n$ is equal to $v_\eta$ on $[\delta, n]$, continuous on $\mathbb{R}^+$, and is in $H_0^1(\mathbb{R}^+)$, and since $v_\eta'$ and $\sqrt{K_\delta} v_\eta$ are square-integrable on $\mathbb{R}^+$, we have

$$\lim_{n \to +\infty} Q_\delta(w_\eta^n) = \int_\eta^{+\infty} |v_\eta'|^2 - K_\delta |v_\eta|^2 := Q(\eta).$$

By integrating by parts, we have $Q(\delta) = 0$, and $Q$ is a differentiable function of $\eta$, so we look at $\partial_\eta Q(\delta)$. Defining $K_\eta = -v_\eta''/v_\eta$, we have

$$\partial_\eta Q(\eta) = \int_\eta^{+\infty} -2v_\eta'' v_\eta' + 2K_\delta v_\eta v_\eta' \, dy - |v_\eta'(\eta)|^2 = \int_\delta^{+\infty} 2v_\eta v_\eta'(K_\eta + K_\delta) \, dy - 1.$$

Thus, $\partial_\eta Q(\delta) = 8 \int_\delta^{+\infty} v_\delta(y) v_\delta'(y)^2 \, dy - 1$, and a change of variables $z = v_\delta(y+\delta)$ yields

$$\partial_\eta Q(\delta) = 8 \int_0^1 z(1-z^2) \, dz - 1 = 1 > 0.$$

So there exists $\eta_0$ such that, for $\eta \in ]\eta_0, \delta[$, $Q(\eta) < 0$, and therefore, for a given $\eta$ in that interval, there exists $n$ large enough such that $w_\eta^n$ verifies $Q_\delta(w_\eta^n) < 0$.

We have proved that the lowest point of the spectrum of $S_\delta$ is negative, and $u_\delta$ satisfies all the hypotheses of Theorem 4.2. Proposition 4.1 is proved. □

*Remark :* $S_\delta$ has in fact exactly one negative eigenvalue. Indeed, remember that $v_\delta(y) = \tanh(y-\delta)$ is a bounded continuous solution of $-\varphi'' - K_\delta \varphi = 0$ that vanishes only at $y_0 = \delta$, and that $Q_\delta(u) = \int_0^{+\infty} |u'|^2 - K_\delta |u|^2$ is the quadratic form on $H_0^1(\mathbb{R}^+)$ associated with the operator $S_\delta$. We shall show that $Q_\delta \geq 0$ on $F_\delta = \{u \in H_0^1(\mathbb{R}^+) \mid u(\delta) = 0\}$. A function $u \in F_\delta$ can be written as $u = v_\delta w$, with $w \in H_0^1(\mathbb{R}^+)$. Let us take $w \in \mathcal{C}_C^\infty(\mathbb{R}^+)$, with $\mathrm{supp}(w) \subset [h, y_0 - h] \cup [y_0 + h, +\infty[$ for a certain $h > 0$. Replacing $u$ by $v_\delta w$ in $Q(u)$, we get

$$-u'' - K_\delta u = -v_\delta'' w - 2v_\delta' w' - v_\delta w'' - K v_\delta w = -2v_\delta' w' - v_\delta w''$$

so, using $Q_\delta(u) = \int_0^{+\infty} S_\delta u \cdot u$, we have $Q_\delta(v_\delta w) = \int_0^{+\infty} -2v_\delta v_\delta' ww' - v_\delta^2 ww''$. Integrating by parts, we get the factorisation $Q_\delta(v_\delta w) = \int_0^{+\infty} v_\delta^2 w'^2$, *i.e.*

$$Q_\delta(u) = \int_0^{+\infty} v_\delta(y)^2 \left| \left( \frac{u(y)}{v_\delta(y)} \right)' \right|^2 dy \quad \text{for all } u \in F_\delta \tag{1.48}$$

as $\mathcal{C}_C^\infty((0,\delta) \cup (\delta, +\infty[)$ is dense in $F_\delta$. We now have $Q_\delta \geq 0$ on $F_\delta$, which is a hyperplane of $H_0^1(\mathbb{R}^+)$, thus $Q_\delta$ is negative only on a subspace of dimension 1 : $S_\delta$ has only one negative eigenvalue.

We can now give an example of unstable shear flow for Theorem 1.3 : for a fixed positive $a$, it is one of the linearly unstable profiles for the Euler equation, $u_\delta(y) = \tanh(y-\delta)+\zeta$, with any $\delta > 0$ and $\zeta$ to be chosen such that $u_\delta$ satisfies the rescaled Navier condition, $\frac{1}{2}\partial_y u_\delta(0) = a u_\delta(0)$. Setting $X = \tanh(\delta)$, we simply have

$$\zeta = X + \frac{1}{2a}(1 - X^2) > 0.$$

## 4.2 An $L^2$ example and simultaneous realisation of the results

The idea is to "truncate" the hyperbolic tangent in a way that Theorem 4.2 can still be applied. The starting point will therefore be a profile of the form

$$u_\delta^\nu(y) = u_\delta(y)\xi(\nu y),$$

where $\xi$ is $\mathcal{C}^\infty$, such that $\xi(y) = 1$ for $y \leq 1$, decreasing for $y > 1$ and goes to zero exponentially at infinity (compactly-supported will not do). For $\nu > 0$ small enough, we will have $\xi(\nu y) = 1$ for $y \in [0, 2n_0]$, where $n_0$ is such that $Q_\delta(w_\eta^{n_0}) < 0$, as shown in the previous paragraph. Given that the Sturm-Liouville potential for the truncated profile will be equal to $K_\delta$ on the support of $w_\eta^{n_0}$, the quadratic forms for $w_\eta^{n_0}$ coincide and we will easily have that the lower bound on the spectrum of the corresponding Sturm-Liouville operator will be negative.

We must, however, be careful as to how we truncate the hyperbolic tangent. Indeed, to be able to apply Z.Lin's result, we must have that

$$K_\delta^\nu = \frac{-(u_\delta^\nu)''}{u_\delta^\nu - \zeta}$$

is a positive (this is why a compactly-supported $\xi$ isn't convenient) and continuous function. This is not guaranteed because

$$u_\delta^\nu - \zeta = \tanh(y-\delta)\xi(\nu y) - \zeta(1 - \xi(\nu y))$$

vanishes a second time, for $\nu y > 1$. Given the flatness of the hyperbolic tangent when $y$ is large, it is reasonable to assume that it does not vanish a subsequent time, and that $u_\delta^\nu$ is decreasing

for $\nu y \geq 5/4$. We must therefore construct $\xi$ so that this zero of the denominator of $K_\delta^\nu$ coincides with the second inflection point of $u_\delta^\nu$, which is generated by the truncation function $\xi$.

Set $\nu > 0$ small enough so that $\xi(\nu y) = 1$ on $[0, 2n_0]$. We can assume that $n_0$ is large enough so that we have the inequality

$$1 - \nu^2 < \tanh(y - \delta) < 1$$

for $y \geq 2n_0$. This will allow us to get estimates on where the denominator will vanish. Indeed, we have

$$u_\delta^\nu(y) - \zeta \leq \xi(\nu y) - \zeta(1 - \xi(\nu y)) \tag{1.49}$$

$$\text{and } u_\delta^\nu(y) - \zeta \geq (1 - \nu^2)\xi(\nu y) - \zeta(1 - \xi(\nu y)). \tag{1.50}$$

The right-hand-sides of (1.49) and (1.50) are zero for a certain $y_0$ such that

$$\frac{\zeta}{1+\zeta} < \xi(\nu y_0) < \frac{\zeta}{1+\zeta-\nu}.$$

Let us arbitrarily set $\xi(3/2) = \frac{\zeta}{1+\zeta}$ ; as $\xi$ is decreasing, the denominator therefore vanishes at $\nu y_0 = \frac{3}{2} - h(\nu)$, with $h(\nu) > 0$ going to zero as $\nu$ goes to zero. Indeed, impose that $\xi'$ has uniform-in-$\nu$ bounds on $[5/4, 3/2]$ ($\zeta$ is a fixed number that does not depend on $\nu$, so this is reasonable) and that $\xi' < 0$ on this interval. In this case, if $\nu$ is small enough so that $\frac{\zeta}{1+\zeta-\nu} < \xi(5/4)$, we get an estimate of $h(\nu)$ from first-order Taylor inequalities for $\xi$ :

$$|h(\nu)| \leq C(\xi(3/2 - h) - \xi(3/2)) \leq C\zeta \left( \frac{1}{1+\zeta-\nu} - \frac{1}{1+\zeta} \right) \leq C\nu.$$

Then, by setting $\xi'(3/2 - h) = -p$ for a fixed $p > 0$ and by detailing the expression of $(u_\delta^\nu)''(y)$, we can get the value of $\xi''(3/2 - h)$ that cancels out the numerator of $K_\delta^\nu$ :

$$\xi''(\nu y_0) = \frac{\xi(\nu y_0)}{\zeta} \left[ 2p(1 - \tanh^2(y_0 - \delta)) + 2\xi(\nu y_0) \tanh(y_0 - \delta)(1 - \tanh^2(y_0 - \delta)) \right].$$

As $u_\delta^\nu$ is $\mathcal{C}^\infty$, we can then prescribe a third-derivative value that ensures that $K_\delta^\nu$ has a positive limit at $\nu y_0 = 3/2 - h$ ($\xi^{(3)}(\nu y_0) < -2p\zeta$ for example). We have therefore constructed a function $\xi$ such that $K_\delta^\nu$ is defined, continuous and positive for $\nu y \leq 3/2$. We then set, for $z \geq 2$,

$$\xi(z) = \frac{e^{-\gamma z}}{u_\delta(z/\nu)},$$

so that $u_\delta^\nu(y) = e^{-\gamma \nu y}$ for $\nu y \geq 2$, thus

$$K_\delta^\nu(y) = \frac{-\gamma^2 e^{-\gamma \nu y}}{e^{-\gamma \nu y} - \zeta} > 0$$

for $\nu y \geq 2$. We finally choose $\gamma$ large enough so that a $\mathcal{C}^\infty$ attachment to the first part of $\xi$ (for $\nu y \leq 3/2$) can be done while maintaining $\xi$ decreasing, and on which the convexity and the sign of $u_\delta^\nu - \zeta$ remain the same. As $\zeta$ is independent of $\nu$ and $\xi(3/2)$ is a fixed function of $\zeta$, $\gamma$ can be chosen independent of $\nu$.

Thus, for each $0 < \nu < \nu_0$, we can construct a function $\xi^\nu$, element of a family $(\xi^\nu)_{\nu>0}$ which is uniformly bounded in $\mathcal{C}^3$ as $\nu \to 0$, so that $K_\delta^\nu$ is continuous and positive, equal to $K_\delta$ on $[0, 2n_0]$ so that $Q_\delta^\nu(w_\eta^{n_0}) = Q_\delta(w_\eta^{n_0}) < 0$, where, naturally,

$$Q_\delta^\nu(u) = \int_0^{+\infty} |u'(y)|^2 - K_\delta^\nu(y)|u(y)|^2 \, dy$$

is the quadratic form associated with the operator $-\partial_y^2 - K_\delta^\nu$ defined on $H_0^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$, proving that this operator has negative eigenvalues. We can therefore apply Theorem 4.2 once again, and conclude with the following.

**Corollary 4.3.** *For any $0 < \nu < \nu_0$ and $\delta > 0$, there exist $\zeta > 0$ and a positive, non-increasing $\mathcal{C}^\infty$ function $\xi$ such that $\xi(y) = 1$ for $y \in [0, 1]$ and $\xi(y) \overset{y\to+\infty}{\longrightarrow} 0$ at an exponential rate, such that $u_\delta^\nu(y) = (\tanh(y - \delta) + \zeta)\xi(\nu y)$ has two inflection points, and is an unstable profile for Theorem 1.3.*

As it exponentially goes to zero at infinity, we have $u_\delta^\nu \in L^2(\mathbb{R}^+)$, and therefore $u_s(y) = (u_\delta^\nu(\varepsilon^{-1/2}y), 0)$ is in $L^2(\Omega')$, where $\Omega' = \mathbb{T} \times \mathbb{R}^+$. As both Theorems 1.1 and 1.3 are true in $\Omega'$ (see [47] for the minor technical differences in the proof of the instability theorem), we get that not only are our two main results not contradictory, but, that in the periodic case, there are families of initial conditions that allow both to be satisfied simultaneously.

# Chapitre 2

# Existence et régularité conormale uniformes, et limite non-visqueuse pour l'équation de Navier-Stokes isentropique 3D

**Résumé.** *Nous obtenons l'existence locale de solutions de l'équation de Navier-Stokes compressible isentropique dans le demi-espace avec condition de Navier au bord, dans des espaces de Sobolev conormaux. Le temps d'existence ne dépend pas des paramètres de viscosité lorsque ceux-ci tendent vers 0, et nous montrons que ces solutions convergent fortement dans $L^2$ vers des solutions de l'équation d'Euler compressible lorsque l'on passe à la limite non-visqueuse.*

# 1 Introduction

## 1.1 The isentropic compressible Navier-Stokes system and the inviscid limit problem

We consider the motion of a compressible fluid in the half-space $\Omega = \mathbb{R}^2 \times \mathbb{R}^+$. The density of the fluid is a scalar function $\rho(t,x)$ with $t$ being the time variable and $x \in \Omega$, its velocity is $u(t) : \Omega \to \mathbb{R}^3$ : these will be the unknowns of the system. The temperature of the fluid, $\theta$, is constant throughout the chapter, and the pressure $P$ will follow an isentropic law : $P = P(\rho) = \frac{k}{\gamma}\rho^\gamma$, with $k > 0$ and the adiabatic constant $\gamma > 1$. The motion of the fluid is governed by the isentropic compressible Navier-Stokes system, which consists of two conservation laws :

– conservation of mass

$$\partial_t \rho + \text{div } (\rho u) = 0, \tag{2.1}$$

– and conservation of momentum

$$\rho(\partial_t u + u \cdot \nabla u) - \text{div } \Sigma = \rho F, \tag{2.2}$$

with $F(t,x)$ a force term. To avoid technical complications with compatibility conditions, we consider that the force is smooth and that fluid is at rest for negative times : $F(t,x) = 0$ and $(\rho, u)(t,x) = (1,0)$ for $t < 0$.

In these equations, $\Sigma$ is the internal stress tensor, $\Sigma = 2\varepsilon\mu Su + (\varepsilon\lambda_0 \text{div } u - P(\rho))I_3$, in which $Su = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric gradient, and $I_3$ is the $3 \times 3$ identity matrix. $\varepsilon > 0$ will be small, while the parameters $\mu > 0$ (dynamic viscosity) and $\lambda_0 > 0$ (bulk viscosity) will be given regular functions of $\rho$. We will most often use the viscosity parameters $\mu$ and $\lambda := \lambda_0 + \mu$ and write

$$\text{div } \Sigma = \varepsilon(\mu\Delta u + \lambda \nabla \text{div } u) - \nabla(P(\rho)) + \varepsilon\sigma(\nabla\rho, \nabla u).$$

The final term $\sigma$ is non-zero only when $\lambda'(\rho)$ and $\mu'(\rho)$ are non-zero ($\lambda$, $\mu$ not constant).

We shall not go into detailed historics about this isentropic system ; a survey of existence and regularity results up to 1998 was put together by B.Desjardins and C-K.Lin, [33]. We shall quickly cite the emblematic result on the subject : the global existence theorem for weak solutions, as proved by P-L.Lions in the 1990s ([73] or [75]), a result that has since been improved, for instance, by E.Feireisl, A.Novotný and H.Petzeltová, [38], and extended to some cases with density-dependent (variable, therefore) viscosity coefficients by D.Bresch, B.Desjardins and D.Gérard-Varet [18]. We also refer to the local strong existence theory, initiated in the 1970s by V.A.Solonnikov [113] and improved on by R.Danchin (for instance, see survey [29]). In this framework, blowup can occur [125].

In this chapter, we will aim to obtain local-in-time existence of solutions to the isentropic Navier-Stokes system in conormal Sobolev spaces, with a lower bound for the time of existence that does not depend on the viscosity parameters when these are small. We will also be interested in the inviscid limit problem. As we are on the half-space, we will have some boundary conditions.

**Notation.** *Throughout the chapter, we will use the notation $x = (y, z)$, with $y \in \mathbb{R}^2$ and $z \in \mathbb{R}^+$ ; the boundary is therefore the set $\{x = (y, z) \in \mathbb{R}^3 \mid z = 0\}$.*

*Moreover, for a vector field $v(x)$, the tangential part to the boundary is, for $x$ on the boundary, $v_\tau(x) = v(x) + (v(x) \cdot \vec{n}(x))\vec{n}(x)$, where $\vec{n}(x)$ is the outer normal vector to $\partial\Omega$ at point $x$. As $\vec{n}(x) \equiv (0, -1)$, we extend the notation to all $\Omega$ : $v_\tau(x) = (v_1(x), v_2(x), 0)$.*

The boundary conditions on $u$ are the standard non-penetration condition on the boundary,

$$u \cdot \vec{n}(t, y, 0) = u_3(t, y, 0) = 0, \tag{2.3}$$

and the Navier (slip) boundary condition $\left(\frac{1}{\varepsilon}\Sigma\vec{n} + au\right)_\tau = 0$ in which $a \in \mathbb{R}$. In our case, with a flat boundary, the Navier condition can be rewritten as

$$\mu(\rho(t, y, 0))\partial_z u_\tau(t, y, 0) = 2au_\tau(t, y, 0) \tag{2.4}$$

for $t > 0$ and $y \in \mathbb{R}^2$. As opposed to the Dirichlet or no-slip condition, which, in our setting, would be $u|_{z=0} = 0$, the Navier condition, proposed by H.Navier himself in the XIX$^{\text{th}}$ century [90], allows the fluid to slip along the boundary, and this occurs wherever interaction at the boundary is non negligeable. For instance, the slip phenomenon can be observed on the contact line of two immiscible flows [97], and in capillary blood vessels, which are the microscopic, tissue-irrigating vessels where molecular exchanges with the neighbouring cells take place [94]. It also appears when homogenising rough and porous boundaries ([62] and [42]), and can be derived mathematically from a Boltzmann microscopic model with a Maxwell reflection boundary condition [86]. To be physically pertinent, the slip coefficient $a$ should be chosen positive, but we will see that our results do not technically require $a$ to have a specific sign.

We also impose the limit condition

$$U(t, x) := (\rho(t, x) - 1, u(t, x)) \overset{|x|\to+\infty}{\to} 0 \tag{2.5}$$

so that $U(t) \in L^2(\Omega)$.

Formally, taking $\varepsilon = 0$ leads to the compressible Euler equations

$$\left\{ \begin{array}{rcl} \partial_t\rho + \text{div } (\rho u) & = & 0 \\ \rho\partial_t u + \rho u \cdot \nabla u + \nabla P(\rho) & = & \rho F. \end{array} \right. \tag{2.6}$$

The Euler equation is of order one, so it only requires one boundary condition equation, which is (2.3). This leads to the appearance of boundary layers : if the solutions of low-viscosity Navier-Stokes equations are expected to behave like a solution of the Euler equation far away enough from the boundary, solutions of Navier-Stokes are still required to satisfy a second boundary condition, whereas the reference solution of the Euler equation is not. A typical boundary layer expansion for solutions to Navier-Stokes will read

$$u^\varepsilon(t, y, z) = u^E(t, y, z) + V\left(t, y, \frac{z}{\sqrt{\varepsilon}}\right), \tag{2.7}$$

with $u^E$ solving the Euler equation, and $V$, acting on a shorter scale, picking up the boundary condition.

The inviscid limit problem is a major challenge for mathematicians, whether one considers compressible or incompressible fluids. We remind the reader of the (non-forced) incompressible system :

$$\left\{ \begin{array}{rcl} \text{div } u & = & 0 \\ \partial_t u + u \cdot \nabla u - \tilde{\mu} \Delta u + \nabla q & = & 0 \\ u|_{t=0} & = & u_0, \end{array} \right.$$

in which $\tilde{\mu} = \frac{\mu}{\rho}$, with $\rho$ constant, is the kinematic viscosity and $q$ is the kinematic pressure. One of the major issues in this setting with Dirichlet boundary conditions is the ill-posedness of the Prandtl equation, which appears when performing boundary-layer rescaling (see [41]). Regarding the inviscid limit results on the incompressible system with Navier boundary conditions, the problem is solved in $L^2$ framework in 2D (see [6], [23], [64]), and convergence of weak solutions of the Navier-Stokes equation towards a strong solution of the Euler equation, when the limit initial condition is regular enough, has been obtained for a range of Navier slip coefficients of the form $a = a' \tilde{\mu}^{-\beta}$ : starting with D.Iftimie and G.Planas [60] ($\beta = 0$), [122] ($\beta < 1/2$) and [93] ($\beta < 1$ for positive slip coefficients and $\beta \leq 1/2$ regardless of sign) have extended the range of numbers $\beta$ for which convergence occurs. C.Bardos, F.Golse and L.Paillard recently obtained weak convergence results of Leray solutions of Navier-Stokes towards dissipative solutions of the Euler equation [7].

The solutions to incompressible Navier-Stokes equations with a Navier boundary condition have a better asymptotic expansion than (2.7) ; D.Iftimie and F.Sueur showed in [61] that the boundary layer $V$ has a smaller amplitude :

$$u^\varepsilon(t, y, z) = u^E(t, y, z) + \sqrt{\varepsilon} V(t, y, \varepsilon^{-1/2} z). \tag{2.8}$$

The existence of strong solutions and the corresponding inviscid limit problem in 3D, showing behaviour in agreement with this ansatz, have been solved by N.Masmoudi and F.Rousset [85]. Their approach is based on energy estimates in conormal Sobolev spaces, and the same technique has allowed them to prove similar results for the corresponding free-boundary system [84], and will also be put to use here. We shall also see that the formal behaviour of solutions to the isentropic system replicates (2.8). Finally, we refer the reader to [124] and [9] for other studies in 3D.

On the 3D isentropic Navier-Stokes system with Navier boundary conditions that we are interested in, D.Hoff obtained global solutions with intermediate regularity (more regularity than weak solutions *à-la*-Lions, but not classical solutions) in 2005 [56]. Y-G.Wang and M.Williams recently justified a WKB expansion for the isentropic system [123], and, on a domain of $\mathbb{R}^3$, a result by F.Sueur, [116], features convergence of weak solutions to a strong solution of the Euler equation when the initial condition is smooth, and for slip coefficients that can depend on $\varepsilon$, such as $a/\varepsilon^\beta$ with $\beta < 1$. Advances on the full compressible Navier-Stokes equations, with an extra equation on the internal energy, temperature or entropy, include F.Huang, Y.Wang and

T.Yang [59], and Feireisl and Novotný [37]. We also refer to [126] for results on the linearised 2D system, [101] and [52] for boundary layer analysis with characteristic and non-characteristic boundary conditions respectively, and [88], [4] for results on more general parabolic-hyperbolic systems.

### 1.2 Functional setting : conormal Sobolev spaces

The functional setting used in this chapter will be that of conormal Sobolev spaces. Introduced in the mid-60s [58], these spaces have been used to work on hyperbolic systems with characteristic boundaries (see, for example, [98], [51], [111]). Such spaces on a domain $\Omega$, which has a boundary, are constructed by differentiating functions following a finite set of generators of vector fields that are tangent to the boundary of $\Omega$. Namely, in the case of the half-space, we can choose

$$Z_{1,2} = \partial_{y_1,y_2},\ Z_3 = \phi(z)\partial_z,$$

with $\phi$ a smooth, positive, bounded function of $\mathbb{R}^+$ such that $\phi(0) = 0$ and $\phi'(0) \neq 0$ - typically, consider $\phi(z) = \frac{z}{1+z}$. In this setting, if we consider a boundary-layer expansion of the form (2.7), we see that $Z^3(V(\varepsilon^{-1/2}z)) = \varepsilon^{-1/2}\phi(z)\partial_z V(\varepsilon^{-1/2}z)$ is of amplitude $\mathcal{O}(1)$ in a neighbourhood of the boundary of size $\sqrt{\varepsilon}$, thus the conormal setting is the only one in which we can expect uniform bounds on the derivatives.

The conormal Sobolev space on $\Omega$, $W^{m,p}_{co}(\Omega)$, is the set of functions $f(x) \in L^p(\Omega)$ such that the conormal derivatives of order at most $m$ of $f$ are also in $L^p(\Omega)$. However, to the above set of conormal derivatives, we add $Z_0 = \partial_t$ for functions that depend on $(t, x)$. This is a difference with the conormal spaces used by Masmoudi and Rousset in [85] ; indeed, in the incompressible case, one can use the equation div $u = 0$ to replace $\partial_z u_3$ by the conormal combination $\partial_{y_1} u_1 + \partial_{y_2} u_2$, whereas our conservation of mass equation, (2.1), involves $\partial_t \rho$. We thus introduce the conormal Sobolev spaces on $[0, T] \times \Omega$ in the sense of O.Guès [51], for a set time $T$. For $\alpha \in \mathbb{N}^4$, we write $Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$, and $|\alpha| = \sum_{i=0}^3 \alpha_i$ : the conormal Sobolev space $W^{m,p}_{co}([0, T] \times \Omega)$ is the set of functions $f : [0, T] \times \Omega \to \mathbb{R}^d$ such that $Z^\alpha f \in L^p([0, T] \times \Omega)$, for every $\alpha$ with $|\alpha| \leq m$. We will only use $p = +\infty$ and $p = 2$, with the notation $H^m_{co} = W^{m,2}_{co}$. We therefore have

$$H^m_{co}([0, T] \times \Omega) = \{f(t, x) \mid \forall\ 0 \leq k \leq m,\ \partial_t^k f \in L^2([0, T], H^{m-k}_{co}(\Omega))\}.$$

The notation is slightly abusive ; indeed, $W^{m,p}_{co}([0, T] \times \Omega)$ is not, strictly speaking, a conormal Sobolev space on $[0, T] \times \Omega$ since $Z_0 = \partial_t$ is not tangent to the boundary of $[0, T]$.

In our *a priori* estimation process, we will be interested in the following space :

$$X^m_T(\Omega) = \{f(t, x) \mid \forall\ 0 \leq k \leq m,\ \partial_t^k f \in L^\infty([0, T], H^{m-k}_{co}(\Omega)\}.$$

This is more restrictive than asking for $f \in H^m_{co}([0, T] \times \Omega)$. For a set $t \geq 0$, we introduce the semi-norms

$$\|f(t)\|_m^2 = \sum_{|\alpha| \leq m} \|Z^\alpha f(t)\|^2_{L^2(\Omega)} \quad \text{and} \quad \|f(t)\|_{m,\infty} = \sum_{|\alpha| \leq m} \|Z^\alpha f(t)\|_\infty .$$

Note that these semi-norms coincide with the $H^m_{co}(\Omega)$ and $W^{m,\infty}_{co}(\Omega)$ norms if $f$ is stationary. Based on these semi-norms, we construct two norms on $X^m_T$ and $W^{m,\infty}_{co}([0,T]\times\Omega)$ respectively, which are essentially $L^\infty$-in-time norms,

$$|||f|||_{m,T} := \sup_{t\in[0,T]} \|f(t)\|_m \quad \text{and} \quad |||f|||_{m,\infty,T} := \sup_{t\in[0,T]} \|f(t)\|_{m,\infty},$$

the latter of which coincides with the $W^{m,\infty}_{co}([0,T]\times\Omega)$ norm, and $L^2$-in-time norms

$$\int_0^T \|f(t)\|^2_m \; dt \quad \text{and} \quad \int_0^T \|f(t)\|^2_{m,\infty} \; dt,$$

the former of which is the natural norm for $H^m_{co}([0,T]\times\Omega)$. We will prefer not to abbreviate these last norms, as the forms we have given will make the *a priori* estimation process clearer.

We add the following abbreviations : $\|f(t)\|_\infty := \|f(t)\|_{0,\infty}$, $|||f|||_{\infty,T} := |||f|||_{0,\infty,T}$, and we denote by $|||f|||_{\mathrm{Lip},T}$ the standard Lipschitz norm on $[0,T]\times\Omega$.

### 1.3 Results and proof strategy

We introduce the notation $U(t,x) = (\rho - 1, u)(t,x)$, and will consider the class of solutions satisfying the following property :

$$\mathcal{E}_m(T,U) := |||U|||^2_{m,T} + |||\partial_z u_\tau|||^2_{m-1,T} + \int_0^T \|\partial_z u_3(s)\|^2_{m-1} + \|\partial_z \rho(s)\|^2_{m-1} \; ds$$

$$+ |||\partial_z u_\tau|||^2_{1,\infty,T} + \int_0^T \|\partial_z \rho(s)\|^2_{1,\infty} + \|\partial_t \partial_z \rho(s)\|^2_{1,\infty} \; ds < +\infty. \quad (2.9)$$

In terms of our functional setting in the previous section, if we have $U \in X^m_T$ and $\partial_z U \in X^{m-1}_T \cap W^{2,\infty}_{co}([0,T]\times\Omega)$ (with $L^\infty$ in time norms only), then $\mathcal{E}_m(T,U)$ is also finite.

Note that in $\mathcal{E}_m(T,U)$ we only have control of $L^2$-in-time norms on the derivatives of $u_3$ and $\rho$. Simply using $|f(t,x)|^2 = |f(0,x)|^2 + \int_0^t \partial_t(|f(s,x)|^2)\; ds$, we get that

$$|||\partial_z \rho|||^2_{1,\infty,T} \leq \|\partial_z \rho(0)\|^2_{1,\infty} + C\int_0^t \|\partial_z \rho(s)\|^2_{1,\infty} + \|\partial_t \partial_z \rho(s)\|^2_{1,\infty} \; ds, \quad (2.10)$$

thus the final integral in $\mathcal{E}_m(T,U)$ acts to control the $W^{1,\infty}_{co}$ norm of $\partial_z \rho$. Requiring control of the $W^{1,\infty}_{co}$ norm of $\nabla U$ is typical of characteristic hyperbolic problems, see [51].

The force must be smooth on $\mathbb{R}\times\Omega$, hence we introduce the notation :

$$\mathcal{N}_m(T,F) = \sup_{t\in[-T,T]} \|F(t)\|^2_m + \|\nabla F(t)\|^2_{m-1} + \|\nabla F(t)\|^2_{2,\infty}.$$

**Theorem 1.1.** *Uniform existence of solutions to the Navier-Stokes system.*
*Let $m \geq 7$, $F = (0, F)$ be such that $\mathcal{N}_m(t, F) < +\infty$ for any $t > 0$, and $\mu$ and $\lambda$ are positive and bounded $\mathcal{C}^m$ functions of $\rho$. Then, for $\varepsilon_0 > 0$, there exists $T > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there is a unique $U^\varepsilon$ satisfying $\mathcal{E}_m(T, \cdot) < +\infty$, solution to (2.1)-(2.2)-(2.3)-(2.4)-(2.5), the isentropic compressible Navier-Stokes system on $(0, T) \times \Omega$ with Navier boundary conditions. Moreover, there is no vacuum on this time interval : there exists $c > 0$ such that $\rho(t, x) \geq c$ for $t \in [0, T]$ and $x \in \Omega$.*

**Theorem 1.2.** *Inviscid limit.*
*Under the same conditions as above, the family $(U^\varepsilon = (\rho^\varepsilon - 1, u^\varepsilon))_{0<\varepsilon<\varepsilon_0}$ of solutions to the Navier-Stokes system converges in $L^2([0, T] \times \Omega)$ and $L^\infty([0, T] \times \Omega)$, towards $V = (\rho - 1, u)$, the unique solution to the isentropic compressible Euler system, (2.6)-(2.3)-(2.5), that satisfies $\mathcal{E}_m(T, V) < +\infty$.*

Note that there are no restrictions on the viscosity parameters other than positiveness and sufficient regularity ($\mathcal{C}^m$). It seems physically justified to ask $\mu(\rho)$ and $\lambda(\rho)$ to be increasing with the density, but, like the sign of the slip coefficient, the signs of $\mu'$ and $\lambda'$ do no intervene technically.

Finally, we expect that our results are also valid in any domain of $\mathbb{R}^3$ with a $\mathcal{C}^{m'}$ boundary, for $m'$ large enough, locally characterised by equations of type $z = \psi(y)$. As shown in [85], the differences are technical, and extra difficulties arise only because the normal vector to the boundary is no longer constant.

Our results are obtained by classical arguments once a uniform estimate is shown. The key bound is the following.

**Theorem 1.3.** *Uniform energy bound.*
*Let $m \geq 7$ and $M_0 > 0$. We assume that $\mathcal{N}_m(t, F) < +\infty$ for every $t > 0$, and that the initial value of $U$ satisfies $\mathcal{N}_m(0, U) \leq M_0$. Then, there exist $\varepsilon_0 > 0$, $T > 0$ and a positive increasing function $Q : \mathbb{R}^+ \to \mathbb{R}^+$, with $Q(z) \geq z$, such that, for $\varepsilon \leq \varepsilon_0$ and $t \leq T$,*

$$\mathcal{E}_m(t, U) + |||\partial_z u_3|||^2_{1,\infty,t} + \varepsilon \int_0^t \|\nabla u(s)\|_m^2 + \left\|\nabla^2 u_\tau(s)\right\|_{m-1}^2 \, ds \leq Q(2M_0) \tag{2.11}$$

Let us outline the proof of Theorem 1.3. It is proved by showing that the left-hand side of (2.11) is bounded by

$$Q(M_0) + (t + \varepsilon) Q(\mathcal{E}_m(t, U) + \mathcal{N}_m(t, F)).$$

The energy function we consider, $\mathcal{E}_m$, contains $L^2$ in space norms with many derivatives, on the first line of (2.9), and conormal-Lipschitz norms on the second line. The first type of term is dealt with by performing energy estimates, in which we will have to control the commutators between the conormal derivatives and the operators that appear in the equation. For the $L^\infty$

norms, we will widely use an anisotropic Sobolev embedding theorem (Theorem 3.3), which is the main contributor to the restriction on $m$. For the terms in the second line of (2.9), a maximum principle will provide us with bounds on $|||\partial_z u_\tau|||_{1,\infty,T}$, while the Duhamel formula for the equation satisfied by $\partial_z \rho$, which is obtained by combining $\varepsilon \times$ (2.1) with the third component of (2.2), will give us the bounds for the $W^{1,\infty}_{co}$ norms on $\partial_z \rho$. A bootstrap argument closes the proof.

Note that, in the context of Theorems 1.1 and 1.2, $M_0$ can be arbitrarily small, as $\mathcal{N}_m(0, U) = 0$. We will be able to prove the energy estimate taking initial conditions into account ; this allows one to extend our results to less regular force terms or different initial values, providing the compatibility conditions yield uniform bounds on the norms of $U$ at $t = 0$.

**Organisation of the chapter.** In the next section, we prove Theorems 1.1 and 1.2, assuming Theorem 1.3. The remaining sections will all be dedicated to proving this uniform estimate. Starting with some important commutator estimates in section 3, we then proceed to prove the bound, looking at each component of $U = (\rho - 1, u)$ separately, and getting the required $H^m_{co}$ and $L^\infty$ bounds on each of them : $U$ in section 4, the normal derivative of $u_\tau$ in section 5, $\partial_z u_3$ in section 6, and the normal derivative of the density in section 7. We conclude the proof of Theorem 1.3 in section 8.

## 2 Proof of Theorems 1.1 and 1.2

In this section, we assume the uniform estimate in Theorem 1.3. To obtain Theorem 1.1, we can use a classical fixed-point iteration method to get existence of solutions for any fixed $\varepsilon > 0$ on a time interval $[0, T^\varepsilon]$ depending on the viscosity (see [123], section 4.1). A bootstrap argument with the uniform bounds then yields a uniform existence time.

We will further detail the proof of Theorem 1.2. We first get local convergence by using standard compactness argument.

**Proposition 2.1.** *Conormal compact embedding theorem.*
*Let $(U_n)_{n\in\mathbb{N}}$ be a bounded sequence of $H^m_{co}([0,T]\times\Omega)$, such that the sequence $(\nabla U_n)_n$ is bounded in $H^{m-1}_{co}([0,T]\times\Omega)$. Then we can extract a sub-sequence $(U_{n_j})_{j\in\mathbb{N}}$ such that, for every $\alpha \in \mathbb{N}^4$ with $|\alpha| \leq m-1$, $Z^\alpha U_{n_j}$ converges in $L^2_{loc}([0,T]\times\Omega)$ - we will say that $(U_n)$ is locally compact in $H^{m-1}_{co}([0,T]\times\Omega)$.*

The bound on the gradient of $U_n$ is crucial here. Other constructions of conormal Sobolev spaces, such as in [51] and [112], treat normal derivation as worth two conormal derivations, thus their $H^m_{co}$ spaces locally and compactly embed in $H^{m-2}_{co}$. The proof of this proposition is similar to that in [112].

Let $U^\varepsilon = (\rho^\varepsilon - 1, u^\varepsilon)$ be the solution to (2.1)-(2.2)-(2.3)-(2.4)-(2.5) with viscosity coefficient $\varepsilon$, given by Theorem 1.1. The energy estimate in Theorem 1.3 tells us that the family

$(\mathcal{E}_m(T,U^\varepsilon))_{0<\varepsilon<\varepsilon_0}$ is bounded, so we can immediately state that the sequence $(U^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is locally compact in $H^{m-1}_{co}([0,T]\times\Omega)$ by Proposition 2.1. As $H^{m-1}_{co}([0,T]\times\Omega)\hookrightarrow\mathcal{C}([0,T],L^2(\Omega))$, we can consider a sequence $\varepsilon_n \overset{n\to+\infty}{\to} 0$ such that $U^{\varepsilon_n}$ converges locally in $H^{m-1}_{co}([0,T]\times\Omega)$ and in $\mathcal{C}([0,T],L^2(\Omega))$ towards a function $V=(\rho-1,u)$, which is easily seen to be a weak solution to the compressible Euler system. Thanks to the uniform bounds, we see that $V$ has the same regularity as $U^\varepsilon$, and in particular $V$ is Lipschitz-class, which yields uniqueness of solutions for the Euler equation in the space of functions satisfying $\mathcal{E}_m(T,V)<+\infty$. The whole sequence then converges towards $V$, strongly and locally in $H^{m-1}_{co}([0,T]\times\Omega)$, and $U^\varepsilon(t)$ converges weakly in $L^2([0,T]\times\Omega)$ towards $V(t)$.

We now prove strong convergence in $L^2$. We start with the classical energy inequality for the Navier-Stokes system, as in [76],

$$\left|E(t,U^\varepsilon)-E(0,U^\varepsilon)-\int_0^t\int_\Omega \rho^\varepsilon F(s)\cdot u^\varepsilon(s)\;dx\;ds\right|\leq C\varepsilon\int_0^t\|u^\varepsilon(s)\|^2_{H^1(\Omega)}\;ds, \tag{2.12}$$

$$\text{where}\quad E(t,U^\varepsilon)=\int_\Omega\frac{1}{2}\rho^\varepsilon(t)|u^\varepsilon(t)|^2+\frac{k}{\gamma(\gamma-1)}\int_\Omega(\rho^\varepsilon)^\gamma(t)-1-\gamma(\rho^\varepsilon(t)-1)\;dx.$$

Note that $E(0,U^\varepsilon)=0$ for every $\varepsilon$. As $\|u^\varepsilon\|_{H^1([0,T]\times\Omega)}$ is bounded by Theorem 1.3 and $\rho^\varepsilon u^\varepsilon$ converges weakly towards $\rho u$ in $L^2$ (as that is the case in the sense of distributions by local strong convergence), we get that, for $t\in[0,T]$,

$$E(t,U^\varepsilon)\overset{\varepsilon\to0}{\longrightarrow}\int_0^t\int_\Omega\rho(s,x)F(s,x)\cdot u(s,x)\;dx\;ds.$$

But $\int_{[0,t]\times\Omega}\rho F\cdot u$ is equal to $E(t,V)$, as (2.12) with $\varepsilon=0$ yields the energy equality for the compressible Euler equation. So, we have $E(t,U^\varepsilon)\overset{\varepsilon\to0}{\longrightarrow}E(t,V)$.

We get $L^2$ convergence of the density by using the strict convexity of $z\mapsto z^\gamma$ for $z>0$ and $\gamma>1$. On one hand, we have

$$\rho^\varepsilon|u^\varepsilon|^2-\rho|u|^2-2\sqrt{\rho}u\cdot(\sqrt{\rho^\varepsilon}u^\varepsilon-\sqrt{\rho}u)=|\sqrt{\rho^\varepsilon}u^\varepsilon-\sqrt{\rho}u|^2,$$

and on the other hand, there exists $c>0$ such that $\rho(t,x)>c>0$, thus the Taylor expansion of $z\mapsto z^\gamma$ yields

$$(\rho^\varepsilon)^\gamma-\rho^\gamma-\gamma\rho^{\gamma-1}(\rho^\varepsilon-\rho)\geq c\gamma(\gamma-1)(\rho^\varepsilon-\rho)^2.$$

Adding the two together with the coefficients that appear in $E$, and taking the integral on $\Omega$, we get

$$E(t,U^\varepsilon)-E(t,V)-R(t,U^\varepsilon,V)\geq c_0\left(\left\|(\sqrt{\rho^\varepsilon}u^\varepsilon-\sqrt{\rho}u)(t)\right\|^2_{L^2(\Omega)}+\|(\rho^\varepsilon-\rho)(t)\|^2_{L^2(\Omega)}\right) \tag{2.13}$$

where $R(t,U^\varepsilon,V)=\int_\Omega\sqrt{\rho}u\cdot(\sqrt{\rho^\varepsilon}u^\varepsilon-\sqrt{\rho}u)+\frac{k}{\gamma-1}(\rho^\varepsilon-\rho)(\rho^{\gamma-1}-1)\;dx$, which converges to zero. Indeed, the uniform boundedness of the energy means that $\left\|\sqrt{\rho^\varepsilon(t)}u^\varepsilon(t)\right\|_{L^2}$ is bounded,

so we can extract a weakly converging sub-sequence in $L^2(\Omega)$, $\sqrt{\rho^{\varepsilon_n}(t)}u^{\varepsilon_n}(t)$, whose limit is necessarily $\sqrt{\rho(t)}u(t)$, thus the whole sequence converges weakly in $L^2(\Omega)$. So the first term of $R(t, U^\varepsilon, V)$ goes to zero. Likewise, we have $\rho^\varepsilon(t) \rightharpoonup \rho(t)$ in $L^2(\Omega)$, and $\rho^{\gamma-1} - 1$ is seen to be in $L^2(\Omega)$ by use of the order-one Taylor expansion of $z \mapsto z^{\gamma-1}$ at $z = 1$, so the second term also converges to zero.

Moreover, $E(t, U^\varepsilon) - E(t, V) \overset{\varepsilon\to 0}{\longrightarrow} 0$, so (2.13) gives us the global, strong $L^2$ convergence of $\rho^\varepsilon$ towards $\rho$. Now, by remarking that

$$\left| \left\| (\sqrt{\rho^\varepsilon} u^\varepsilon)(t) \right\|^2_{L^2(\Omega)} - \| (\sqrt{\rho} u^\varepsilon)(t) \|^2_{L^2(\Omega)} \right| \leq \|u^\varepsilon(t)\|_{L^\infty} \|u^\varepsilon(t)\|_{L^2} \|\rho^\varepsilon(t) - \rho(t)\|_{L^2}\,, \tag{2.14}$$

we get that $\lim_{\varepsilon\to 0} \|(\sqrt{\rho} u^\varepsilon)(t)\|_{L^2(\Omega)} = \lim_{\varepsilon\to 0} \left\|(\sqrt{\rho^\varepsilon} u^\varepsilon)(t)\right\|_{L^2(\Omega)} = \|\sqrt{\rho}(t)u(t)\|_{L^2(\Omega)}$, as $u^\varepsilon$ is uniformly bounded in $L^2$ and $L^\infty$. So $u^\varepsilon$ converges towards $u$ in $L^2([0,T] \times \Omega, dt\ \rho dx)$, and $\rho dx$ is an equivalent measure to the Lebesgue measure, hence we conclude that $U^\varepsilon$ converges towards $V$ in $L^2([0,T] \times \Omega)$. $L^\infty$ convergence is obtained by using the Sobolev embedding inequality and the uniform bounds. □

# 3 Preliminary properties of conormal derivatives

We begin this section by reminding the reader of some important properties of our functional setting, and that will be used throughout the *a priori* estimation process.

**Proposition 3.1.** *Generalised Sobolev-Gagliardo-Nirenberg inequality (or tame estimate). [51]*
*There exists a constant $C(T)$, which does not blow up as $T \to 0$, such that, for $f, g \in L^\infty([0,T] \times \Omega) \cap H^m_{co}([0,T] \times \Omega)$, and $\alpha_1,\ \alpha_2 \in \mathbb{N}^4$ such that $|\alpha_1| + |\alpha_2| = m$,*

$$\int_0^T \|(Z^{\alpha_1} f Z^{\alpha_2} g)(s)\|_0^2\ ds \leq C(T) \left[ |||f|||^2_{\infty,T} \int_0^T \|g(s)\|_m^2\ ds + |||g|||^2_{\infty,T} \int_0^T \|f(s)\|_m^2\ ds \right].$$

**Proposition 3.2.** *Trace inequality.*
*If $f \in L^2([0,T] \times \Omega)$ and $\nabla f \in L^2([0,T] \times \Omega)$, then*

$$\int_0^T \int_{\partial\Omega} |f(t,y,0)|^2\ dt\ dy \leq C \int_0^T \|f(t)\|_0 \|\partial_z f(t)\|_0\ dt$$

**Proposition 3.3.** *Anisotropic Sobolev embedding theorem.*
*If $f \in H^3_{co}([0,T] \times \Omega)$, $\nabla f \in H^2_{co}([0,T] \times \Omega)$, then $f \in L^\infty([0,T] \times \Omega)$ and*

$$|||f|||^2_{\infty,T} \leq C \left( \|f(0)\|_2^2 + \|\partial_z f(0)\|_1^2 + \int_0^T \|f(t)\|_3^2 + \|\partial_z f(t)\|_2^2\ dt \right)$$

This last theorem is a direct application of the $H^m_{co}(\Omega)$ Sobolev embedding, used in [85] and [84] : for a given $t$, we have

$$\|f(t)\|^2_\infty \leq C(\|f(t)\|^2_{H^2_{co}(\Omega)} + \|\nabla f(t)\|^2_{H^1_{co}(\Omega)}) \leq C(\|f(t)\|^2_2 + \|\partial_z f(t)\|^2_1).$$

We combine this with the following property : for $f \in H^{m+1}_{co}([0,T]\times\Omega)$,

$$|||f|||^2_{m,t} \leq \|f(0)\|^2_m + C\int_0^t \|f(s)\|^2_{m+1}\ ds. \tag{2.15}$$

This is shown by writing $n(t) = n(0) + \int_0^t n'(s)\ ds$ for $n(s) = \|f(s)\|^2_m$ ; given that

$$n'(s) = 2\sum_{|\alpha|\leq m}\int_\Omega (\partial_t Z^\alpha f(s,x))(Z^\alpha f(s,x))\ dx,$$

we easily see that $n'(s) \leq 2\|f(s)\|^2_{m+1}$. Morally, the bound (2.15) means that we can exchange an $L^\infty$-in-time norm for an $L^2$-in-time norm for the cost of one conormal derivative, similarly to (2.10).

We will now show the important commutator properties we will need.

### 3.1 Commuting with $\partial_z$

The first technical key to the proof in the subsequent parts of the chapter is how to estimate the commutators that will appear when applying $Z^\alpha$ to equations (2.1), (2.2). A lot of the commutators are trivial, since all $Z_i$'s commute with $\partial_t$, $\partial_{y_1}$ and $\partial_{y_2}$, but $Z_3$ does not commute with $\partial_z$. Specifically, we have

$$[Z_3, \partial_z] = \phi(z)\partial_{zz} - \partial_z(\phi(z)\partial_z) = -\phi'(z)\partial_z \tag{2.16}$$

Likewise, we can observe the commutator with $\partial_{zz}$, which will come from the div $\Sigma$ term of (2.2). This time, we have

$$[Z_3, \partial_{zz}] = -2\phi'\partial_{zz} - \phi''\partial_z. \tag{2.17}$$

When commutating with a higher order operator, $Z_3^m$ for $m > 1$, we show the following :

**Proposition 3.4.** *(a) For $m \geq 1$, there exist two families of bounded functions $(\varphi_{\beta,m})_{0\leq\beta<m}$ and $(\varphi^{\beta,m})_{0\leq\beta<m}$, such that*

$$[Z_3^m, \partial_z] = \sum_{\beta=0}^{m-1}\varphi_{\beta,m}(z)Z_3^\beta\partial_z = \sum_{\beta=0}^{m-1}\varphi^{\beta,m}(z)\partial_z Z_3^\beta \tag{2.18}$$

*(b) For $m \geq 1$, there exist four families of bounded functions $(\psi_{1,\beta,m})$, $(\psi_{2,\beta,m})$, $(\psi^{1,\beta,m})$ and $(\psi^{2,\beta,m})$, for $0 \leq \beta < m$, such that*

$$\begin{aligned}[Z_3^m, \partial_{zz}] &= \sum_{\beta=0}^{m-1} \psi_{1,\beta,m}(z) Z_3^\beta \partial_z + \psi_{2,\beta,m}(z) Z_3^\beta \partial_{zz} \\ &= \sum_{\beta=0}^{m-1} \psi^{1,\beta,m}(z) \partial_z Z_3^\beta + \psi^{2,\beta,m}(z) \partial_{zz} Z_3^\beta.\end{aligned}$$

In practice, we can therefore choose to place the normal derivatives as the first or last derivative to be applied in all the terms of the commutator. We can deduce from the proposition the basic estimate for the commutators with $\partial_z$ :

**Corollary 3.5.** *For any $f \in H^m_{co}([0,T] \times \Omega)$ such that $\partial_z f \in H^{m-1}_{co}([0,T] \times \Omega)$, and for any $|\alpha| \leq m$,*

$$\int_0^T \|[Z^\alpha, \partial_z] f(t)\|_0^2 \ dt \leq C \int_0^T \|\partial_z f(t)\|_{m-1}^2 \ dt$$

We will deal with the commutators with $\partial_{zz}$ directly in context : they often appear with a factor $\varepsilon$, and equation (2.2) will allow us to substitute the difficult terms.

*Proof of Proposition 3.4* : equation (2.16) shows the case $m = 1$, and we continue by induction. Let us just explain the case $m = 2$ to show the mechanism ; the rest is left to the reader. We have

$$\begin{aligned}[Z_3^2, \partial_z] &= Z_3[Z_3, \partial_z] + [Z_3, \partial_z] Z_3 \\ &= -\phi\phi'' \partial_z - \phi'(Z_3 \partial_z + \partial_z Z_3).\end{aligned}$$

Using (2.16), we can write the second part of the last line as either $\phi'(z)(2Z_3\partial_z + \phi'\partial_z)$ or $\phi'(z)(2\partial_z Z_3 - \phi'\partial_z)$, which proves the proposition for $m = 2$.

The proof of (b) is also an elementary induction. □

## 3.2 Commuting with a function

**Notation.** *For $\alpha,\ \beta \in \mathbb{N}^4$, we write $\beta \leq \alpha$ iff, for every $i$, $\beta_i \leq \alpha_i$.*

**Proposition 3.6.** *Let $\alpha \in \mathbb{N}^4$, $|\alpha| = m > 0$, be fixed, and $f \in H^m_{co}([0,T] \times \Omega) \cap L^\infty([0,T] \times \Omega)$ and $g \in L^\infty$ such that $\partial_t g,\ \nabla g \in H^{m-1}_{co}([0,T] \times \Omega) \cap L^\infty([0,T] \times \Omega)$. Then we have the following*

*inequality :*

$$\int_0^T \|[Z^\alpha, g]f\|_0^2 \, dt \leq C\sum_{j=0}^{3}\int_0^T |||Z_j g|||_{\infty,T}^2 \, \|f\|_{m-1}^2 + |||f|||_{\infty,T}^2 \, \|Z_j g\|_{m-1}^2 \, dt, \quad (2.19)$$
$$\leq C\int_0^T |||g|||_{1,\infty,T}^2 \, \|f(s)\|_{m-1}^2 + |||f|||_{\infty,T}^2 \, \|g(s)\|_m^2 \, dt,$$

*if, moreover,* $g \in L^2([0,T]\times\Omega)$.

This proposition is easily proved, using the Leibniz formula and Proposition 3.1. We will require formulation (2.19) whenever $g \notin L^2$ (for example when $g = \rho$).

We prove one more commutator estimate, which we will need when estimating the normal derivatives of $u$ in the conormal spaces (section 5), as directly using the above would lead to $\|\partial_{zz}u\|_{m-2}$ appearing, which we cannot estimate uniformly in $\varepsilon$ by using the equation.

**Proposition 3.7.** *Let* $f$, $g$, $\alpha$ *be as in Proposition 3.6, with* $g$ *scalar such that* $g|_{z=0} = 0$. *Then there exists* $C$ *which does not depend on* $T$ *such that*

$$\int_0^T \|[Z^\alpha, g\partial_z]f\|_0^2 \, dt \leq C\int_0^T (|||g|||_{\mathrm{Lip},T}^2 + |||\nabla g|||_{1,\infty,T}^2)\, \|f(s)\|_m^2 + |||f|||_{1,\infty,T}^2 \, \|\nabla g(s)\|_{m-1}^2 \, dt$$

*Proof :* we decompose the commutator as $[Z^\alpha, g\cdot\nabla]f = [Z^\alpha, g]\partial_z f + g[Z^\alpha, \partial_z]f$, and start by taking a closer look at the second term. By Proposition 3.4, it is equal to a sum of terms of the form $\varphi^\beta(z)g_3\partial_z Z^\beta f$, with $\beta \leq \alpha$, $\beta \neq \alpha$. As $g|_{z=0} = 0$, we have

$$|g(t,x)| \leq \phi(z)\, |||\partial_z g|||_{\infty,T}\,, \quad (2.20)$$

so, as $Z_3 = \phi(z)\partial_z$,

$$\|g[Z^\alpha, \partial_z]f\|_0 \leq C\, |||\nabla g|||_{\infty,T}\, \|\phi(z)\partial_z f\|_{m-1} \leq C\, |||g|||_{\mathrm{Lip},T}\, \|f\|_m\,. \quad (2.21)$$

Now we look at the first term of the decomposition. We prove that

$$\int_0^T \|[Z^\alpha, g]\partial_z f\|_0^2 \leq C\int_0^T (|||g|||_{\mathrm{Lip},T}^2 + |||\nabla g|||_{1,\infty,T}^2)\, \|f\|_m^2 + |||f|||_{1,\infty,T}^2 \, \|\nabla g\|_{m-1}^2 \, dt, \quad (2.22)$$

which would end the proof of the proposition.

We can write $[Z^\alpha, g]\partial_z f$ as a sum, on $\beta$ and $\delta$, of terms of the form $\varphi^\delta(z)Z^\beta g\partial_z Z^\delta f$, where $\beta \leq \alpha$ and $|\beta| > 0$, and $\delta \leq (\alpha - \beta)$. The idea is to insert $\frac{1}{\phi(z)} \times \phi(z)$, thus we have to estimate

$$\left\| \frac{1}{\phi(z)}(Z^\beta g)(Z_3 Z^\delta f) \right\|_0 .$$

This will obviously be done by using the tame estimate, but we also have to deal with the first factor : as $\phi^{-1}Z^\beta g = Z^\beta(\phi^{-1}g) - [Z^\beta, \phi^{-1}]g$. We need to write this last term more explicitly.

**Lemma 3.8.** *For every $b \in \mathbb{N}$, $\phi Z_3^b(\phi^{-1})$ is smooth and bounded, then there exists a family of smooth bounded functions $(\sigma_p)_{0\leq p\leq b}$ such that for every $f \in H^m_{co}$,*

$$[Z_3^m, \phi^{-1}]f = \sum_{p=0}^{m} \sigma_p(z) Z_3^p(\phi^{-1} f)$$

*Proof :* as $\phi$ only depends on $z$, we are only interested in $[Z_3^m, \phi^{-1}]$ for $m \geq 1$. The Leibniz formula yields that the commutator is a linear combination of terms written as $(Z_3^b(\phi^{-1}))(Z^{m-b}f)$, for $0 < b \leq m$. In the case of $\phi(z) = \frac{z}{1+z}$, we notice that $Z_3^b(\phi^{-1})$ has the same following properties as $\phi^{-1}$ : it is smooth on $]0, +\infty[$, bounded at infinity and has the same blow-up rate at $z = 0$ as $\phi^{-1}$ (blows up like $z^{-1}$). So, for each $b$, $\phi Z_3^b(\phi^{-1})$ is a bounded function on $[0, +\infty[$, and if we write

$$(Z_3^b(\phi^{-1}))(Z^{m-b}f) = \phi Z_3^b(\phi^{-1}) \left[ Z^{m-b}\left(\frac{1}{\phi} f\right) - [Z^{m-b}, \phi^{-1}]f \right],$$

we have $m - b < m$, and we can reiterate the process. As $m$ is fixed, we conclude the proof with a finite number of iterations. □

Applying the lemma, we have that, in any norm,

$$\left\| \phi^{-1} Z^\beta g \right\| \leq C \sum_{\beta' \leq \beta} \left\| Z^{\beta'}(\phi^{-1} g) \right\| \tag{2.23}$$

With that, the tame estimate gives us

$$\int_0^T \left\| \phi^{-1}(Z^\beta g)(Z_3 Z^\delta f) \right\|_0^2 \, dt \leq C \int_0^T |||Z_3 f|||^2_{\infty,T} \sum_{j=0}^{3} \left\| Z_j \left( \frac{g(s)}{\phi} \right) \right\|^2_{m-2}$$
$$+ \left( \left|\left|\left| \phi^{-1} g \right|\right|\right|^2_{\infty,T} + \sum_{j=0}^{3} \left|\left|\left| Z_j(\phi^{-1} g) \right|\right|\right|^2_{\infty,T} \right) \|Z_3 f(s)\|^2_{m-1} \, dt$$

and the $\left|\left|\left| \phi^{-1} g_3 \right|\right|\right|_{\infty,T}$ comes from the terms in (2.23) with $\beta' = 0$.

It remains to deal with the terms involving $\phi^{-1} g$, and the key fact here is that $Z^\beta g|_{z=0} = 0$. We easily have by (2.20),

$$\left\| \phi^{-1} g \right\|_\infty \leq \|g\|_{\mathrm{Lip}} \quad \text{and} \quad \left\| Z_j(\phi^{-1} g) \right\|_\infty \leq C \|\nabla g\|_{1,\infty}, \tag{2.24}$$

as $Z_3(\phi^{-1} g) = \phi^{-1} Z_3 g + \phi' \phi^{-1} g$, with $\phi'$ bounded, and likewise we can use the Hardy inequality to get

$$\left\| Z_j(\phi^{-1} g) \right\|_{m-2} \leq C \|\partial_z g\|_{m-1},$$

which ends the proof of (2.22). □

# 4 Proof of Theorem 1.3, part I *A priori* estimates on $U$

**Assumption 4.1.** *Throughout the a priori estimation process, we will assume that there is no vacuum : there exists* $0 < c_0 < 1$ *such that* $c_0 \leq \rho(t,x)$ *for* $x \in \Omega$ *and* $t \leq T^*$. *Also, as* $u_3|_{z=0} = 0$, *for any* $\delta > 0$, *we assume that there exists* $z_\delta > 0$, *independent of* $\varepsilon$, *such that* $|u_3(t,x)| \leq \delta$ *for* $x \in F_\delta := \mathbb{R}^2 \times [0, z_\delta]$ *and* $t \leq T^*$.

*After getting the estimates, we will show in section 8 that the final bounds actually imply these two properties on* $[0,T^*]$, *with* $T^*$ *such that* $\mathcal{E}_m(T^*,U) \leq M$, *and prove by a bootstrap argument that* $T^*$ *does not depend on* $\varepsilon$.

**Notation.** *As of now,* $0 < c < 1$ *will designate a small constant,* $C > 1$ *a large constant, and* $Q(z)$ *a positive increasing function of* $\mathbb{R}^+$, *with polynomial growth and* $Q(z) \geq z$. *All three can change from one line to the next, and can depend on any of the system's parameters (constants* $a$, $k$ *and* $\gamma$, *or bounds of viscosity functions* $\mu(\rho)$ *and* $\lambda(\rho)$ *- we will often omit the dependence on* $\rho$*), on the order of derivation* $m$, *on* $T^*$ *or on* $\varepsilon_0$.

## 4.1 Conormal energy estimates

We start with energy estimates on $U = (\rho - 1, u)$ and its conormal derivatives : this will estimate the $|||U|||_m$ part of $\mathcal{E}_m(t,U)$, and it will uncover other terms of $\mathcal{E}_m(t,U)$ that we will need to estimate later.

We will be able to estimate $\rho - 1$ and $u$ simultaneously by symmetrising the order-one part of the Navier-Stokes system (2.1), (2.2). We rewrite it as

$$A_0(\rho)\partial_t U + \sum_{j=1}^{3} A_j(U)\partial_{x_j} U - \varepsilon(0, \mu(\rho)\Delta u + \lambda(\rho)\nabla \text{div } u)^t = (0, \rho F - \varepsilon\sigma(\nabla U))^t, \tag{2.25}$$

where $\lambda = \mu + \lambda_0 > 0$, $\sigma$ has the following expression

$$\sigma(\nabla\rho, \nabla u) = 2Su \cdot \nabla(\mu(\rho)) + \text{div } u\nabla(\lambda_0(\rho)) = 2\mu'(\rho)Su \cdot \nabla\rho + \lambda'(\rho)\text{div } u\nabla\rho.$$

The matrices $A_j(U)$ are : $A_0(\rho) = \text{diag}(1, \rho, \rho, \rho)$,

$$A_1(U) = \begin{pmatrix} u_1 & \rho & 0 & 0 \\ k\rho^{\gamma-1} & \rho u_1 & 0 & 0 \\ 0 & 0 & \rho u_1 & 0 \\ 0 & 0 & 0 & \rho u_1 \end{pmatrix}, \; A_2(U) = \begin{pmatrix} u_2 & 0 & \rho & 0 \\ 0 & \rho u_2 & 0 & 0 \\ k\rho^{\gamma-1} & 0 & \rho u_2 & 0 \\ 0 & 0 & 0 & \rho u_2 \end{pmatrix}$$

$$\text{and } A_3(U) = \begin{pmatrix} u_3 & 0 & 0 & \rho \\ 0 & \rho u_3 & 0 & 0 \\ 0 & 0 & \rho u_3 & 0 \\ k\rho^{\gamma-1} & 0 & 0 & \rho u_3 \end{pmatrix}.$$

This system is symmetrisable : multiplying these matrices on the left by the positive diagonal matrix $D(\rho) = \mathrm{diag}(k\rho^{\gamma-2}, 1, 1, 1)$, we get :

$$DA_0\partial_t U + \sum_{j=1}^{3} DA_j\partial_{x_j} U - \varepsilon(0, \mu\Delta u + \lambda\nabla \mathrm{div}\ u)^t = (0, \rho F - \varepsilon\sigma(\nabla U))^t \tag{2.26}$$

$DA_0(\rho)$ is symmetric, so

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega DA_0 U\cdot U\right) = \int_\Omega DA_0\partial_t U\cdot U + \frac{1}{2}\int_\Omega \partial_t(DA_0)U\cdot U.$$

Integrating this in time between 0 and $t$, and since $\rho$ is uniformly bounded from below by $c_0$ and from above by $c_1$, there exist $0 < c < C$ such that

$$c\,\|U(t)\|_0^2 \le C\,\|U(0)\|_0^2 + C\,|||\partial_t DA_0|||_{\infty,t}\int_0^t \|U(s)\|_0^2 + \int_0^t\int_\Omega DA_0\partial_t U(s)\cdot U(s)\ ds \tag{2.27}$$

We replace $DA_0\partial_t U$ by its expression in (2.26), and we use integration by parts on the integrals with order-one derivatives of $U$ :

$$\int_\Omega DA_j\partial_{x_j} U\cdot U = -\frac{1}{2}\int_\Omega \partial_{x_j}(DA_j)U\cdot U. \tag{2.28}$$

Indeed, $DA_j(U)$ is symmetric, so $DA_jU\cdot\partial_{x_j}U = U\cdot DA_j\partial_{x_j}U$, and we notice that, for each $j$, $DA_jU\cdot U = (3\gamma P + \rho|u|^2)u_j$, which means that there is no boundary term (at $z=0$ for $j=3$) when we integrate by parts, which gives us

$$\begin{aligned} c\,\|U(t)\|_0^2 &\le C\,\|U(0)\|_0^2 + C\left(\sum_{j=0}^{3}|||DA_j|||_{\mathrm{Lip},t}\right)\int_0^t \|U(s)\|_0^2\ ds \\ &\quad +\varepsilon\int_0^t\int_\Omega \mu\Delta u\cdot u + \lambda\nabla\mathrm{div}\ u\cdot u + \sigma\cdot u\ ds + \int_0^t\int_\Omega \rho F\cdot u\ ds \\ &\le C\,\|U(0)\|_0^2 + C(1+|||U|||_{\mathrm{Lip},t}^2)\int_0^t \|U(s)\|_0^2\ ds \\ &\quad +\varepsilon\int_0^t\int_\Omega \mu\Delta u\cdot u + \lambda\nabla\mathrm{div}\ u\cdot u + \sigma\cdot u\ ds + \int_0^t\int_\Omega \rho F\cdot u\ ds \end{aligned} \tag{2.29}$$

We use integration by parts and the Navier boundary condition (2.4) on the order-two derivatives :

$$\begin{aligned} \int_\Omega \mu\Delta u\cdot u + \lambda\nabla\mathrm{div}\ u\cdot u &= -\int_\Omega(\mu|\nabla u|^2 + \lambda|\mathrm{div}\ u|^2) - \int_{z=0} 2a|u_\tau|^2 \\ &\quad -\int_\Omega\left(\nabla(\mu(\rho))\cdot\nabla u\cdot u + \mathrm{div}\ u\ \nabla(\lambda(\rho))\cdot u\right), \end{aligned} \tag{2.30}$$

with the notation $\int_{z=0} f = \int_{\mathbb{R}^2} f(y,0)\,dy$. The first term of (2.30) is moved the left-hand side of (2.29), as is the second if $a > 0$. However, when $a < 0$, we use the trace theorem, Proposition 3.2, followed by Young's inequality, $\nu\zeta \leq \frac{\eta}{2}\nu^2 + \frac{1}{2\eta}\zeta^2$ for any $(\nu,\zeta) \in \mathbb{R}^2$ and $\eta > 0$, with an adequate parameter $\eta$ (same as in [93]). Young's inequality is also used on the term containing the derivatives of $\mu$ and $\lambda$ ($\int_\Omega \sigma \cdot u$ is bounded the same way), which turns (2.30) into

$$\int_\Omega (\mu\Delta u \cdot u + \lambda\nabla\mathrm{div}\ u \cdot u) \leq \frac{c(1 + |||\rho|||_{\mathrm{Lip},t})}{\eta} \|u\|_0^2 - (c - (|a|+1)\eta) \|\nabla u\|_0^2, \tag{2.31}$$

so we choose $\eta$ so that $c - (|a|+1)\eta = c/2$, which allows us to move the $\|\nabla u\|_0^2$ term to the left-hand side and absorb it with the first term of (2.30) that we moved there earlier.

Combining (2.29) and (2.31), and the assumption that $\rho$ is uniformly bounded, there exist $0 < c < C$ such that

$$c\left[\|U(t)\|_0^2 + \varepsilon\int_0^t \|\nabla u\|_0^2\right] \leq C\left(\|U(0)\|_0^2 + (1 + |||U|||_{\mathrm{Lip},t}^2)\int_0^t \|U(s)\|_0^2 + \|F(s)\|_0^2\ ds\right),$$

as $\|\nabla(DA_j)\|_\infty \leq C\|U\|_\infty \|\nabla U\|_\infty$.

We have shall now show the following, higher order estimate.

**Proposition 4.2.** *For every $m \geq 0$,*

$$c\left[\|U(t)\|_m^2 + \varepsilon\int_0^t \|\nabla u(s)\|_m^2\ ds\right]$$
$$\leq C\left[\|U(0)\|_m^2 + (1 + |||U|||_{\mathrm{Lip},t}^2 + |||F|||_\infty^2)\int_0^t \|U\|_m^2 + \|\nabla U\|_{m-1}^2 + \|F\|_m^2\ ds\right].$$

*Proof :* higher-order estimates work exactly the same way as above, only we will have to estimate the commutators between $Z^\alpha$, with $|\alpha| \leq m$, and the operators in the equation. We apply $Z^\alpha$ to equation (2.25), and isolate the highest-order terms as follows :

$$A_0\partial_t Z^\alpha U + \sum_{j=1}^3 A_j\partial_{x_j}(Z^\alpha U) - \varepsilon(0, \mathrm{div}\ Z^\alpha(\mu\nabla u) + \lambda\nabla\mathrm{div}\ (Z^\alpha u)) = (0, Z^\alpha(\rho F + \varepsilon\tilde{\sigma})) - \mathcal{C}^\alpha.$$

$\mathcal{C}^\alpha$ contains the commutators :

$$\mathcal{C}^\alpha = [Z^\alpha, A_0\partial_t]U + \sum_{j=1}^3 [Z^\alpha, A_j\partial_{x_j}]U - \varepsilon(0, [Z^\alpha, \mathrm{div}\ ](\mu\nabla u) + [Z^\alpha, \lambda(\rho)\nabla\mathrm{div}\ ]u).$$

Notice the peculiar formulation for the laplacian term. This allows us to avoid difficult commutator terms on the boundary when $\mu$ is not constant ; indeed, the Navier boundary condition

for $Z^\alpha u$ is $Z^\alpha(\mu\partial_z u_\tau) = 2aZ^\alpha u_\tau$, thus, when we multiply the equation by $Z^\alpha u$ and integrate by parts, we can use the boundary condition immediately. $\tilde{\sigma}$ is the equivalent of $\sigma$, the remainder term for the order two part of the equation, for this formulation.

We then multiply by the matrix $D$, which is uniformly bounded in $L^\infty$, and can repeat the above to obtain :

$$c\left[\|Z^\alpha U(t)\|_0^2 + \varepsilon\int_0^t\left(\|Z^\alpha\nabla u\|_0^2 + \int_{z=0}|Z^\alpha u|^2\right)\right] \leq C\|Z^\alpha U(0)\|_0^2$$
$$+C\int_0^t(1+|||U|||^2_{\mathrm{Lip},t}+|||F|||_{\infty,t})\|U\|_m^2+\varepsilon\|\nabla U\|_{m-1}^2+\|F\|_m^2\ ds$$
$$+C\int_0^t\int_\Omega|(\varepsilon Z^\alpha\sigma+\mathcal{C}^\alpha)\cdot Z^\alpha U|+|\mathcal{I}_\nabla^\alpha|\ ds. \tag{2.32}$$

The term $\mathcal{I}_\nabla^\alpha$ contains extra commutators arising from the following integration by parts :

$$\begin{aligned}-\int_\Omega \mathrm{div}\ Z^\alpha(\mu\nabla u)\cdot Z^\alpha u &= \int_{\partial\Omega}Z^\alpha(\mu\partial_z u)\cdot Z^\alpha u+\int_\Omega Z^\alpha(\mu\nabla u):\nabla Z^\alpha u\\ &= \int_{\partial\Omega}2a|Z^\alpha u_\tau|^2+\int_\Omega\mu|Z^\alpha\nabla u|^2\\ &\quad+\int_\Omega([Z^\alpha,\mu]\nabla u):Z^\alpha\nabla u-Z^\alpha(\mu\nabla u):([Z^\alpha,\nabla]u),\end{aligned}$$

so $\mathcal{I}_\nabla^\alpha$ is the final integral multiplied by $\varepsilon$. Here, we have used the contracted matrix product : $A:B=\sum_{i,j}a_{i,j}b_{i,j}$.

To deal with the commutator terms, we will use the tools shown in section 3. Let us first look at $\mathcal{I}_\nabla^\alpha$. In the first term, we have

$$\varepsilon\left|\int_\Omega([Z^\alpha,\mu]\nabla u):Z^\alpha\nabla u\right|\leq\|[Z^\alpha,\mu]\nabla u\|_0\times\varepsilon\|Z^\alpha\nabla u\|_0.$$

By estimate (2.19) in Proposition 3.6, the integral in time of the square of the first norm is easily bounded, whereas the second can be moved to the left-hand side (absorbed) by using Young's inequality with an adequate parameter $\eta$ to have

$$\left|\int_0^t\varepsilon\|\nabla u\|_m\|[Z^\alpha,\mu]\nabla u\|_0\right|\leq C\varepsilon(1+|||U|||^2_{\mathrm{Lip},t})\int_0^t(\|\rho-1\|_m^2+\|\nabla u\|_{m-1}^2)+\frac{c}{4}\varepsilon\int_0^t\|Z^\alpha\nabla u\|_0^2. \tag{2.33}$$

The second term of $\mathcal{I}_\nabla^\alpha$ is dealt with in exactly the same fashion, with Corollary 3.5 controlling the side of the product containing the commutator, and Young's inequality allowing to absorb the other.

Now to the commutators in $\mathcal{C}^\alpha$ relative to the order-one part of the equation. We notice that, for $j \in \{0,1,2,3\}$,

$$[Z^\alpha, A_j\partial_{x_j}]U = [Z^\alpha, A_j]\partial_{x_j}U + A_j[Z^\alpha, \partial_{x_j}]U,$$

where $x_0$ is understood to be $t$. The first term is estimated using inequality (2.19) of Proposition 3.6 (we cannot write $\|A_j\|_m$ because $\rho \notin L^2$), and the second, which is either 0, if $j \neq 3$, or $A_3[Z^\alpha, \partial_z]U$, is estimated using Corollary 3.5, so estimating this commutator yields

$$\int_\Omega |A_3[Z^\alpha, \partial_z]U \cdot Z^\alpha U| \leq C(1 + |||U|||^2_{\mathrm{Lip},t}) \|\nabla U\|^2_{m-1} + \|U\|^2_m . \tag{2.34}$$

The commutators on the order-two part of the equation can be split into terms of two types as follows : commutators on the viscosity parameters (functions of $\rho$) and commutators on the differential operators, for instance

$$[Z^\alpha, \lambda\nabla\mathrm{div}\ ] = [Z^\alpha, \lambda]\nabla\mathrm{div}\ + \lambda[Z^\alpha, \nabla\mathrm{div}\ ]. \tag{2.35}$$

We begin with the first term. Notice that we can write this commutator as a sum of the following, using Propositions 3.6 and 3.4 :

$$\begin{aligned} I_{i,j} &:= \int_0^t \int_\Omega \varepsilon[Z^\alpha, \lambda]\partial_{x_i,x_j}u \cdot Z^\alpha u \\ &= \varepsilon \sum_{|\beta|+|\delta|\leq m, |\delta|<m} \int_0^t \int_\Omega \psi_{\beta,\delta}(z) Z^\beta(\lambda(\rho))\ \partial_{x_i} Z^\delta \partial_{x_j} u \cdot Z^\alpha u\ ds \end{aligned} \tag{2.36}$$

The $\psi_{\beta,\delta}$ are $\mathcal{C}^\infty$ bounded functions of $z$. If $|\beta| < m$, we can integrate by parts on the $x_i$ variable :

$$I_{i,j} \leq \sum_{|\beta|+|\delta|\leq m} c_{\beta,\delta}\varepsilon \int_0^t \int_\Omega |\partial_{x_i} Z^\beta(\lambda(\rho))\ Z^\delta \partial_{x_j} u \cdot Z^\alpha u + Z^\beta(\lambda(\rho))\ Z^\delta \partial_{x_j} u \cdot \partial_{x_i} Z^\alpha u|\ ds. \tag{2.37}$$

We use Young's inequality, then we use the tame estimate on the left of the scalar products to get

$$\begin{aligned} I_{i,j} \leq \frac{c\varepsilon}{2} \int_0^t \|\nabla u\|^2_m\ ds + \varepsilon Q(|||\rho|||^2_{\mathrm{Lip},t}) \int_0^t \|\nabla u\|^2_{m-1} + \|u\|^2_m\ ds \\ + \varepsilon\, |||\nabla u|||^2_{\infty,t}\, Q\left( \int_0^t \|\nabla\rho\|^2_{m-1}\ ds \right). \end{aligned}$$

This allows us to absorb the first term. Note that we have used the tame estimate on the pair $(Z\rho, \partial_{x_j}u)$ rather than $(\rho, \partial_{x_j}u)$ on the second term of (2.37) to get $H^{m-1}_{co}$ norms of $\nabla u$, which is controlled by induction on $m$.

There is a slight difference when $\beta = \alpha$ in (2.36). In this case, $\delta = 0$, but after integrating by parts, we would need to control $\|\nabla\rho\|_m$. So we cannot integrate by parts and must estimate

$J_{i,j} := \varepsilon \int_0^t \int_\Omega Z^\alpha(\lambda(\rho))\ \partial_{x_i,x_j} u \cdot u$ directly. This is not a problem if $(i,j) \neq (3,3)$ : we bound $J_{i,j}$ by

$$J_{i,j} \leq \varepsilon \int_0^t |||u|||_\infty^2 \, \|\nabla u\|_1^2 \, Q\left(\|\rho - 1\|_m^2\right)\ ds.$$

At this stage, if $m = 1$, we use Young's inequality to absorb $\varepsilon \int_0^t \|\nabla u\|_1^2$, and if $m \geq 2$, we can use Proposition 4.2 with $m = 1$. When $(i,j) = (3,3)$, we instead replace $\varepsilon \partial_{zz} u$ by using the equation. We have

$$\varepsilon(\mu \partial_{zz} u_\tau, (\mu + \lambda)\partial_{zz} u_3) = \rho \partial_t u + (\rho u \cdot \nabla u) + \nabla P - \rho F - \varepsilon v_2,$$

where order-two derivatives appear in $v_2$. The terms that appear are therefore either controlled in $L^\infty$ by $\|U\|_{\text{Lip}}$, or, in the case of $v_2$, dealt with using Young's inequality as above. The term $\int_\Omega \varepsilon Z^\alpha \sigma \cdot Z^\alpha u$ in (2.32) is dealt with the same techniques as these commutators (the terms it contains ressemble those in (2.37)).

For the remaining commutator in (2.35) (the estimation of the last term $[Z^\alpha, \text{div }](\mu \nabla u)$ follows the same lines), we assume that $\alpha_3 > 0$, as these commutators are non-zero only if $\alpha_3 > 0$. The terms of the commutator $[Z^\alpha, \nabla \text{div }]u$ of the form $[Z^\alpha, \nabla_\tau \partial_z] u_3$ (the others are either trivial or contain $\partial_{zz}$), can then be estimated with Corollary 3.5 :

$$\int_\Omega \varepsilon\lambda |[Z^\alpha, \partial_z] \nabla_\tau u_3 \cdot Z^\alpha u_\tau| \leq C\varepsilon \, \|\partial_z u\|_m \, \|u\|_m \leq C\varepsilon \, \|u\|_m^2 + \frac{c}{4}\varepsilon \, \|\partial_z u\|_m^2 \,,$$

again using Young's inequality to allow us to absorb $\varepsilon \int_0^t \|\partial_z u\|_m^2$. On to the remaining term $\varepsilon [Z^\alpha, \partial_{zz}] u$. By Proposition 3.4, there exist two families of functions, $(\varphi_\beta)_{\beta \leq \alpha}$ and $(\psi^\beta)_{\beta \leq \alpha}$, such that

$$[Z^\alpha, \partial_{zz}] u = \sum_{\beta \leq \alpha,\ \beta \neq \alpha} \varphi_\beta(z) Z^\beta \partial_z u + \psi^\beta(z) \partial_{zz} Z^\beta u$$

The first part of the right-hand side is simply bounded by $\|\partial_z u\|_{m-1}$, so it only remains to control $J := \varepsilon \int_\Omega |\partial_{zz} Z^\beta u \cdot Z^\alpha u|$. As $\alpha_3 > 0$, we have $Z^\alpha u = 0$ on the boundary, so integrating by parts and again using Young's inequality provides us with

$$J = \varepsilon \int_\Omega |\partial_z Z^\beta u \cdot \partial_z Z^\alpha u| \leq C\varepsilon \, \|\partial_z u\|_{m-1} \, \|\partial_z u\|_m \leq C\varepsilon \, \|\partial_z u\|_{m-1}^2 + \frac{c}{4}\varepsilon \, \|\partial_z u\|_m^2 \,.$$

Combining this last inequality with (2.32), (2.33) and (2.34), then summing for all $|\alpha| \leq m$, we get the result. □

## 4.2 $L^\infty$ estimates

With conormal energy estimates, we obtain inequalities for any order of derivation $m$, and these inequalities contain $L^\infty$ or Lipschitz norms of $U$, and, as we shall see soon, $W^{1,\infty}_{co}$ norms

of $\nabla U$. These need to be controlled, and the goal of this control is to get the conormal energy estimates to be closed for $m$ large enough, by using the Sobolev embedding inequality in Proposition 3.3. Putting the $L^\infty$ norm of the normal derivative $\partial_z U$ to one side for now (this will be dealt with in the next sections), we can apply Proposition 3.3 to $|||U|||_{1,\infty,t}$, and (2.15) again on the terms involving the derivatives of $\rho$ and $u_3$.

**Proposition 4.3.** *We have the following bound for* $|||U|||_{1,\infty,t}$ *:*

$$\begin{aligned} |||U|||^2_{1,\infty,t} \;\leq\; & C(\|U(0)\|_4^2 + \|\partial_z U(0)\|_4^2) \\ & + Ct\left(|||U|||^2_{5,t} + |||\partial_z u_\tau|||^2_{4,t} + \int_0^t \|\partial_z u_3\|_5^2 + \|\partial_z \rho\|_5^2 \; ds\right) \end{aligned}$$

We can inject this into Proposition 4.2. We widely use the Young inequality to separate product terms, and we get the following.

**Proposition 4.4.** *For every* $m \geq 6$*, there exists a constant* $C > 1$ *and a positive increasing function* $Q : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\begin{aligned} \|U(t)\|_m^2 \;\leq\; & C\,\|U(0)\|_m^2 + tQ(|||U|||^2_{m,t} + |||\partial_z u_\tau|||^2_{m-1,t} + |||F|||^2_{m,t} + |||\nabla F|||^2_{m-1,t}) \\ & + C\,|||\partial_z U|||^2_{\infty,t} \int_0^t \|\partial_z \rho(s)\|_{m-1}^2 + \|\partial_z u_3(s)\|_{m-1}^2 \; ds. \end{aligned}$$

*Thus, under the conditions of Assumption 4.1, we have, for* $t \leq T^*$*,*

$$|||U|||^2_{m,t} + \varepsilon \int_0^t \|\nabla u\|_m^2 \leq M_0 + Q(M + M_F)\left(t + |||\partial_z U|||^2_{\infty,t} \int_0^t \|\partial_z \rho(s)\|_{m-1}^2 + \|\partial_z u_3(s)\|_{m-1}^2 \; ds\right).$$

# 5 Proof of Theorem 1.3, part II *A priori* estimates on $\partial_z u_\tau$

## 5.1 Conormal energy estimates

In this section, we get estimates on the tangential components of $\partial_z u$. We will perform conormal energy estimates on the first two components of the equation of the vorticity

$$\omega = \text{rot } u = \begin{pmatrix} \partial_{y_2} u_3 - \partial_z u_2 \\ \partial_z u_1 - \partial_{y_1} u_3 \\ \partial_{y_1} u_2 - \partial_{y_2} u_1 \end{pmatrix}.$$

By applying derivations to equation (2.2), we get that $w = \omega_\tau$ solves the following equation :

$$\rho \partial_t w + \rho u \cdot \nabla w - \varepsilon \mu \Delta w = \mathcal{M},$$

where $\mathcal{M} = -\rho\omega \cdot \nabla u + \rho(\text{div } u)\omega + \text{rot } (\rho F) + \mathcal{M}^I + \varepsilon\mathcal{M}^{II}$, in which $\mathcal{M}^I$ is a sum of terms written as $\partial_{x_i}\rho(\partial_t u_j + u \cdot \nabla u_j)e_j$ and $\mathcal{M}^{II}$ is a sum of terms written as $\kappa'(\rho)\partial_{x_j}\rho(\partial^2_{x_k,x_l}u_j)e_j$, with $i,j,k,l \in \{1,2,3\}$, $i \neq j$, $(e_1,e_2,e_3)$ is the canonical basis of $\mathbb{R}^3$, and $\kappa$ is either $\lambda$ or $\mu$. The terms in $\mathcal{M}^{II}$ come from the derivation of the laplacian terms, but also from rot $\sigma$, in which there are no terms with two derivatives on $\rho$ thanks to rot $\nabla = 0$. But the boundary condition on $\omega_\tau$, according to (2.4) and the non-penetration condition $u_3|_{z=0} = 0$, is

$$\mu(\rho)\omega_\tau|_{z=0} = 2au_\tau^\perp|_{z=0},$$

where for $v = (v_1, v_2) \in \mathbb{R}^2$, $v^\perp = (-v_2, v_1)$, which makes integrations by parts difficult. So we introduce a modified vorticity :

$$W = \omega_\tau - \frac{2a}{\mu(\rho)}u_\tau^\perp$$

By the tame estimate, we can easily link conormal Sobolev norms of $W$ with those of $\omega_\tau$ and $\partial_z u_\tau$. The modified vorticity satisfies $W = 0$ on the boundary, and solves the equation

$$\rho\partial_t W + \rho u \cdot \nabla W - \varepsilon\mu\Delta W = H, \tag{2.38}$$

$$\begin{aligned}\text{with} \quad H = {} & 2a[(\rho\partial_t + \rho u \cdot \nabla - \varepsilon\mu\Delta), \mu^{-1}]u_\tau^\perp - 2a\varepsilon\mu^{-1}\lambda\nabla_\tau^\perp \text{div } u \\ & +2ak\mu^{-1}\rho^{\gamma-1}\nabla_\tau^\perp\rho - 2a\mu^{-1}\rho F_\tau^\perp + \mathcal{M}_\tau.\end{aligned}$$

We now prove the following.

**Proposition 5.1.** *The modified vorticity $W$ satisfies the following conormal energy estimate for every $m \geq 3$ :*

$$\begin{aligned}& c\left[\|W(t)\|^2_{m-1} + \varepsilon\int_0^t \|\nabla W(s)\|^2_{m-1}\ ds\right] \leq C\left[\|W(0)\|^2_{m-1} + \|U(0)\|^2_m\right] \\ & +Q(1+|||U|||^2_{\text{Lip},t}+|||\nabla U|||^2_{1,\infty,t}+|||\nabla F|||^2_{\infty,t})\int_0^t(\|U\|^2_m+\|\partial_z U\|^2_{m-1}+\|F\|^2_m+\|\nabla F_\tau\|^2_{m-1}),\end{aligned} \tag{2.39}$$

*for some positive increasing function $Q(z)$.*

*Proof :* we repeat the reasoning of section 4.1 on the above equation. We start by taking the $L^2$ scalar product of (2.38) with $W$ and integrate by parts to get

$$\begin{aligned}c\|W(t)\|^2_0 + \varepsilon\mu\int_0^t \|\nabla W\|^2_0 \leq {} & C\|W(0)\|^2_0 \\ & +C\int_0^t |||\text{div } (\rho u)|||_{\infty,t}\|W\|^2_0 + \int_\Omega H \cdot W\ ds.\end{aligned} \tag{2.40}$$

We notice that, thanks to the compressibility equation (2.1), $|||\text{div } (\rho u)|||_{\infty,t}$ can be replaced by $|||\partial_t\rho|||_{\infty,t} \leq |||\rho|||_{1,\infty,t}$. It remains to estimate $\int_\Omega H \cdot W$. The only terms we need this scalar

product form for are the order-two terms in $\mathcal{M}^{II}$, and $\varepsilon \int_\Omega \mu^{-1}(\Delta\mu)u_\tau \cdot W$, which comes from the commutator $\varepsilon\mu[\Delta, \mu^{-1}]u_\tau$. After integrating $\int_\Omega \mu^{-1}(\Delta\mu)u_\tau \cdot W$ by parts, we can simply bound $\mu$ and $\nabla\mu$ in $L^\infty$ and use Young's inequality on the terms involving $u$ and $W$. The term $\int_\Omega \mathcal{M}^{II} \cdot W$ is bounded by integration by parts on the variable $x_k$ with $k \neq 3$ and the use of Young's inequality with an adequate to absorb $\|\nabla W\|_0$ whenever it appears, while the terms containing $\partial^2_{x_k,x_i}\rho$ are controlled by bounding the order-two part on $\rho$ in $L^\infty$ (thus getting $\|\nabla\rho\|_{1,\infty}$). When $(k,l) = (3,3)$, we use (2.2) to replace $\varepsilon\partial_{zz}u$. For example,

$$\varepsilon\mu\partial_{zz}u_1 = \rho\partial_t u_1 + (\rho u \cdot \nabla)u_1 + \partial_{y_1}P(\rho) - \rho F_1 - \varepsilon[\mu(\partial_{y_1y_1} + \partial_{y_2y_2})u_1 + \lambda\partial_{y_1}\text{div } u],$$

and the only difficulty is to control $\varepsilon \int_0^t \|\partial_z u_1\|_1^2$, but this is given by Proposition 4.2.

We can use the Cauchy-Schwarz inequality and handle the norm of the rest of $H$ as follows :

- the other commutators with $\mu^{-1}$ are controlled by $Q(1 + \|U\|_{\text{Lip}})(\|U\|_1 + \|W\|_0)$,
- $\left\|\rho^{\gamma-1}\nabla_\tau^\perp\rho\right\|_0 \leq C\|U\|_1$ as $\rho$ is assumed to be uniformly bounded,
- $\|\rho\omega \cdot \nabla u_\tau + \rho(\text{div } u)\omega\|_0 \leq C\|\nabla u\|_\infty (\|W\|_0 + \|u\|_0)$,
- $\mathcal{M}^I$ is bounded using $\|\partial_{x_i}\rho(\partial_t u_j + u \cdot \nabla u_j)\|_0 \leq |||\nabla\rho|||_{\infty,t}(\|u\|_1 + |||\nabla u|||_{\infty,t}\|u\|_0)$,
- and finally, we have $\varepsilon\|\nabla_\tau \text{div } u\|_0\|W\|_0 \leq \varepsilon(\|\nabla u\|_1^2 + \|W\|_0^2)$ and use Proposition 4.2 to control $\varepsilon\|\nabla u\|_1$,

$$\varepsilon\int_0^t \|\nabla u\|_1^2 \leq C\left(\|U(0)\|_1^2 + (1 + |||U|||^2_{\text{Lip},t})\int_0^t (\|U(s)\|_1^2 + \|\nabla U(s)\|_0^2)\, ds\right).$$

We thus get the final $L^2$ estimate on $W$ :

$$c\left[\|W(t)\|_0^2 + \varepsilon\int_0^t \|\nabla W(s)\|_0^2\, ds\right] \leq C(\|W(0)\|_0^2 + \|U(0)\|_1^2)$$

$$+Q(1 + |||U|||^2_{\text{Lip},t} + |||\nabla U|||^2_{1,\infty,t})\int_0^t \|U\|_1^2 + \|\partial_z U\|_0^2 + \|F\|_1^2 + \|\nabla F\|_0^2\, ds.$$

Now we move on to the $H^{m-1}_{co}$ estimate, which will also follow the same pattern as the previous section. We apply $Z^\alpha$ to (2.38), for $|\alpha| \leq m-1$ and isolate the maximum order terms :

$$\rho\partial_t Z^\alpha W + \rho u \cdot \nabla Z^\alpha W - \varepsilon\mu\Delta Z^\alpha W = Z^\alpha H - \mathcal{C}^\alpha_W,$$

where $\mathcal{C}^\alpha_W = [Z^\alpha, \rho]\partial_t W + [Z^\alpha, \rho u \cdot \nabla]W - \varepsilon[Z^\alpha, \mu\Delta]W$ contains the commutators. Multiplying the above equation by $Z^\alpha W$ in $L^2$, we have

$$c\left[\|Z^\alpha W(t)\|_0^2 + \varepsilon\int_0^t \|Z^\alpha\nabla W\|_0^2\right] \leq C\left[\|Z^\alpha W(0)\|_0^2 + |||\rho|||_{1,\infty,t}\int_0^t \|W\|^2_{m-1}\right]$$

$$+C\varepsilon\int_0^t \|\nabla W\|^2_{m-2}\, ds + C\int_0^t \|H\|^2_{m-1} + \int_\Omega |\mathcal{C}^\alpha_W \cdot Z^\alpha W|\, ds. \tag{2.41}$$

The terms on the second line of this inequality are the ones we need to control. First, $\varepsilon \int_0^t \|\nabla W\|^2_{m-2}$, which comes from changing $\nabla Z^\alpha W$ into $Z^\alpha \nabla W$ (using Proposition 3.4) in the integration by parts on the laplacian, is dealt with by induction, by using (2.39) at a lower order. Then, $\|H\|_{m-1}$ is easily estimated as above, also using the tame estimate in Proposition 3.1 :

$$\begin{aligned}\int_0^t \|H\|^2_{m-1} \;\leq\; & Q(|||U|||^2_{\infty,t} + |||\nabla U|||^2_{1,\infty,t} + |||\nabla F|||^2_{\infty,t}) \\ & \times \int_0^t (\|U\|^2_m + \|\partial_z U\|^2_{m-1} + \|F\|^2_m + \|\nabla F_\tau\|^2_{m-1}) + \int_0^t \varepsilon \|\nabla u\|^2_m \,.\end{aligned}$$

Using Proposition 4.2 to cover the worst term $\varepsilon \int_0^t \|\nabla u\|^2_m$, we have

$$\begin{aligned}\int_0^t \|H\|^2_{m-1} \;\leq\; & C\,\|U(0)\|^2_m + Q(|||U|||^2_{\infty,t} + |||\nabla U|||^2_{1,\infty,t} + |||\nabla F|||^2_{\infty,t}) \\ & \times \int_0^t \|U\|^2_m + \|\partial_z U\|^2_{m-1} + \|F\|^2_m + \|\nabla F\|^2_{m-1} \,. \qquad (2.42)\end{aligned}$$

We finally need to estimate the commutators in $\|\mathcal{C}^\alpha_W\|_0$. The first and last terms of $\mathcal{C}^\alpha_W$ are easily estimated as in the previous section,

$$\int_0^t \|[Z^\alpha, \rho]\partial_t W\|^2_0 \leq C \int_0^t (|||\rho|||^2_{\infty,t} + |||\partial_t W|||^2_{\infty,t})(\|W\|^2_{m-1} + \|U\|^2_{m-1}) \qquad (2.43)$$

by Proposition 3.6, and again using the decomposition in Proposition 3.4 (b) and integrating by parts, we have

$$\varepsilon \int_\Omega |[Z^\alpha, \mu\Delta]W \cdot Z^\alpha W| \leq \frac{1 + |||\rho|||_{\mathrm{Lip},t}}{c} \varepsilon \|\partial_z W\|^2_{m-2} + \frac{c}{4} \varepsilon \|\partial_z W\|^2_{m-1} \,, \qquad (2.44)$$

in which the first term is estimated by induction (using (2.39) at order $m-2$) and the second is absorbed by the left-hand side of (2.41). Finally, instead of applying the commutator results we have used so far to $\int_0^t \|[Z^\alpha, \rho u \cdot \nabla]W\|^2_0$, which would yield a $\left\|\partial^2_{zz} U\right\|_{m-2}$ term that we do not expect to control uniformly in $\varepsilon$, we use Proposition 3.7 :

$$\begin{aligned}\int_0^t \|[Z^\alpha, \rho u \cdot \nabla]W\|^2_0 \; ds \leq & \int_0^t Q(|||U|||^2_{\mathrm{Lip},t} + |||\nabla U|||^2_{1,\infty,t}) \, \|W(s)\|^2_{m-1} \; ds \\ & + \int_0^t Q(|||\nabla U|||^2_{1,\infty,t}) \, \|\nabla U(s)\|^2_{m-1} \; ds.\end{aligned}$$

This finishes off the proof of Proposition 5.1. □

## 5.2 $L^\infty$ estimates

To deal with $\|\partial_z u_\tau\|_{1,\infty}$, we examine $\omega_\tau$. The result is :

**Proposition 5.2.** *There exists a positive, increasing function on $\mathbb{R}^+$, $Q$, such that*

$$\|\omega_\tau(t)\|_{1,\infty}^2 \leq Q(\|U(0)\|_6^2 + \|\omega_\tau(0)\|_5^2 + \|\omega_\tau(0)\|_{1,\infty}^2)$$
$$+Q(|||U|||_{\mathrm{Lip},t}^2 + |||\nabla U|||_{1,\infty,t}^2 + \mathcal{N}_6(t,F)) \int_0^t 1 + \|U(s)\|_6^2 + \|\omega_\tau(s)\|_6^2 + \|\partial_z \rho(s)\|_5^2 \ ds.$$

We can now deduce the following update of Proposition 4.4.

**Corollary 5.3.** *Under the conditions of Assumption 4.1, for $m \geq 7$ and $t \leq T^*$,*

$$|||U|||_{m,t}^2 + |||\nabla u_\tau|||_{m-1,t}^2 + |||\nabla u_\tau|||_{1,\infty,t}^2 + \varepsilon \int_0^t \|\nabla u\|_m^2 + \|\nabla W\|_{m-1}^2 \ ds$$
$$\leq Q(M_0) + Q(M + M_F) \left( t + |||\partial_z(\rho, u_3)|||_{1,\infty,t}^2 \int_0^t \|\partial_z \rho\|_{m-1}^2 + \|\partial_z u_3\|_{m-1}^2 \ ds \right).$$

The first tool to prove Proposition 5.2 is the following version of the maximum principle.

**Proposition 5.4.** *Consider $X$ a Lipschitz-class solution to the following hyperbolic-parabolic system on the half-space $\Omega \subset \mathbb{R}^3$ :*

$$\begin{cases} a\partial_t X + b \cdot \nabla X - \varepsilon \mu(a) \Delta X &= G & \text{in } \Omega \\ X &= h & \text{on } \partial\Omega \\ X|_{t=0} &= X_0 \in L^\infty(\Omega), \end{cases} \tag{2.45}$$

*in which $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$, $a$ is a scalar function, bounded and positive uniformly in time ($\inf_{(t,x)} a(t,x) > c > 0$), $b : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$ is tangent to the boundary, $G : \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$ and $h : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^d$ are in $L^\infty$ and $\mu$ is a positive regular function of $a$. We assume that $a$ and $b$ satisfy $\partial_t a + \operatorname{div} b = 0$. Then we have, for $t \in [0,T]$,*

$$\|X(t)\|_\infty \leq \|X(0)\|_\infty + |||h|||_{\infty,T} + \frac{1}{\inf a} \int_0^t \|G(s)\|_\infty + \varepsilon \left\|\mu'(a(s)) \nabla a(s) \cdot \nabla X(s)\right\|_\infty \ ds.$$

Note that when $\mu$ is constant, the term involving $\nabla(\mu(a))$ vanishes on the right-hand side. We can then apply the result to the modified vorticity $W$ instead of $\omega$, which has the advantage of satisfying a homogeneous boundary condition. In the non-constant case however, if we consider $W$, we cannot deal with the term $\partial_{zz}\rho$ which emerges from the commutator $[\Delta, \mu^{-1}]$, so we

will use the proposition on $\omega_\tau$. Having $\nabla X$ in the right-hand side looks disastrous, but in the context of the equation to which we will apply this result, the factor $\varepsilon$ will allow us to deal with it.

*Proof :* we look for a function $g(t)$ that will control $\|X(t)\|_\infty$. To do so, we perform energy estimates on the functions $(X-g)_+$ and $(X+g)_-$ ($g$ will indifferently designate the scalar function $g$ and the vector $g\mathbf{1} \in \mathbb{R}^d$), with the convention that, for a given scalar function $f$, $f = f_+ + f_-$ (so $f_- \leq 0$).

We concentrate on $(X-g)_+$, the procedure on $(X+g)_-$ being identical. We want to find a function $g(t)$ that satisfies

$$\|(X(0)-g(0))_+\|_0 = 0 \quad \text{and} \quad \frac{d}{dt}\|(X(t)-g(t))_+\|_0^2 \leq 0, \ t>0;$$

such a function $g$ is an upper bound for $\|X_+\|_\infty$. Choosing $g(0) = \|X(0)\|_\infty$, we examine

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\int_\Omega a|(X(t)-g(t))_+|^2\right) &= \int_\Omega a\partial_t X\cdot(X-g)_+ - \int_\Omega a\partial_t g\cdot(X-g)_+ \\
&\quad + \frac{1}{2}\int_\Omega \partial_t a|(X-g)_+|^2 \\
&= \int_\Omega (-b\cdot\nabla X + \varepsilon\mu\Delta X + G)\cdot(X-g)_+ \\
&\quad + \int_\Omega -a\partial_t g\cdot(X-g)_+ + \frac{1}{2}\partial_t a|(X-g)_+|^2 \qquad (2.46)
\end{aligned}$$

Because of the scalar product with $(X-g)_+$, which is zero wherever $X-g\leq 0$, we can replace $\nabla X$ by $\nabla((X-g)_+)$ and $\Delta X$ by $\Delta((X-g)_+)$. Then, integration by parts gives us

$$-\int_\Omega b\cdot\nabla X\cdot(X-g)_+ = \frac{1}{2}\int_\Omega \text{div } b|(X-g)_+|^2 = -\frac{1}{2}\int_\Omega \partial_t a|(X-g)_+|^2,$$

by the first line of (2.45), so this term cancels out with the final term of (2.46), and it remains to look at the term involving the laplacian. When integrating by parts, we need to guarantee that $(X-g)_+$ vanishes on the boundary. We therefore impose the condition

$$g(t) \geq |||h|||_{L^\infty(\partial\Omega),T} \qquad (2.47)$$

and write

$$\begin{aligned}
\varepsilon\mu\int_\Omega \Delta X\cdot(X-g)_+ &= -\varepsilon\mu\|\nabla((X-g)_+)\|_0^2 - \varepsilon\int_\Omega \nabla(\mu(a))\cdot\nabla X\cdot(X-g)_+ \\
&\leq -\varepsilon\int_\Omega \mu'(a)\nabla a\cdot\nabla X\cdot(X-g)_+
\end{aligned}$$

Therefore, setting $\tilde{G} = G - \varepsilon\mu'(a)\nabla X\cdot\nabla a$, of (2.46) there only remains

$$\frac{d}{dt}\left(\int_\Omega a|(X(t)-g(t))_+|^2\right) \leq 2\int_\Omega (\tilde{G} - a\partial_t g)\cdot(X-g)_+,$$

which is negative if, for each $j \in \{1, \cdots, p\}$, $\tilde{G}_j - a\partial_t g \leq 0$ on $\Omega$. Integrating this leads to $g(t) \geq g(0) + \int_0^t \frac{\tilde{G}_j(s,x)}{a(s,x)}\, ds$ for every $x \in \Omega$.

Identical estimates on $(X+g)_-$ lead to $g(t) \geq g(0) - \int_0^t \frac{\tilde{G}_j(s,x)}{a(s,x)}\, ds$, so, also taking into account (2.47), we choose

$$g(t) = \|X(0)\|_\infty + |||h|||_{L^\infty(\partial\Omega),T} + \int_0^t \left\| \frac{\tilde{G}(s)}{a(s)} \right\|_\infty ds,$$

and this controls $\|X(t)\|_\infty$ as desired. □

*Proof of Proposition 5.2 :* we immediately apply Proposition 5.4 to $w = \omega_\tau$ and $Z_j w$, to get

$$\|w(t)\|_{1,\infty} \leq \|w(0)\|_{1,\infty} + 2a \left|||\mu^{-1} u_\tau|||_{1,\infty,t} + \int_0^t \left\| \frac{\mathcal{M}_\tau(s)}{\rho(s)} \right\|_{1,\infty} + \sum_{j=0}^{3} \left\| \mathcal{C}_w^j(s) \right\|_\infty ds,$$

where $\mathcal{C}_w^j = [Z_j, \rho]\partial_t w + [Z_j, \rho u \cdot \nabla] w - \varepsilon [Z_j, \mu\Delta] w$ are the commutator terms that appear in the equations on $Z_j w$. Thus, by using the Cauchy-Schwarz and Young inequalities,

$$\|w(t)\|_{1,\infty}^2 \leq \|w(0)\|_{1,\infty}^2 + 4a^2 \left|||\mu u_\tau|||_{1,\infty,t}^2 + t \int_0^t \left\| \frac{\mathcal{M}_\tau(s)}{\rho(s)} \right\|_{1,\infty}^2 + \sum_{j=0}^{3} \left\| \mathcal{C}_w^j(s) \right\|_\infty^2 ds. \quad (2.48)$$

Note that a factor $t$ has been extracted in front of the source terms and commutators, so we can be satisfied with fairly crude bounds on these. The $W_{co}^{1,\infty}$ term stemming from the boundary is easily dealt with using Proposition 3.3.

Let us start with the control of the commutators. Given that, by (2.17),

$$\varepsilon\mu [Z_3, \partial_{zz}] w = -2\varepsilon\mu\phi' \partial_{zz} w - \varepsilon\mu\phi'' \partial_z w,$$

we replace $\varepsilon\mu\partial_{zz} w$ by its expression in the equation. So we write, using the notation $\Delta_\tau = \partial_{y_1 y_1} + \partial_{y_2 y_2}$,

$$\begin{aligned} \varepsilon^2\mu^2 \|[Z_3, \partial_{zz}] w\|_\infty^2 &\leq C\varepsilon^2 \|\partial_z w\|_\infty^2 + C \|\rho\partial_t w + \rho u \cdot \nabla w - \varepsilon\mu\Delta_\tau w - \mathcal{M}_\tau\|_\infty^2 \\ &\leq C\varepsilon^2 \|\partial_z w\|_\infty^2 + C(\|\rho\|_{1,\infty}^2 + \|u\|_{\mathrm{Lip}}^2 \|w\|_{1,\infty}^2) \\ &\qquad + C\varepsilon^2 \|w\|_{2,\infty} + C \|\mathcal{M}_\tau\|_\infty^2, \end{aligned}$$

by using the fact that $\|u_3 \partial_z w\|_\infty \leq C \|\partial_z u_3\|_\infty \|Z_3 w\|_\infty$. We see that $\left\|\mathcal{C}_w^3\right\|_\infty$ is bounded, among other terms, by $\|\mathcal{M}_\tau\|_\infty$ ; we'll examine the source term last. In this commutator, it remains

to control $J := \int_0^t \varepsilon^2(\|\partial_z w\|_\infty^2 + \|w\|_{2,\infty}^2)$. On both parts of this term, we start by applying Proposition 3.3, the Sobolev embedding theorem, to get

$$J \leq C \int_0^t \varepsilon^2(\|\partial_{zz} w\|_3^2 + \|\partial_z w\|_5^2 + \|w\|_5^2).$$

The second term, $\int_0^t \varepsilon^2 \|\partial_z w\|_5^2$, is controlled by using Proposition 5.1, and, like above, the first term, $\int_0^t \|\varepsilon \partial_{zz} w\|_3^2$, is controlled by replacing $\varepsilon \partial_{zz} w$ by its expression in (2.38), and by using the tame estimate :

$$\begin{aligned}\int_0^t \|\varepsilon \partial_{zz} w\|_3^2 \;\; &\leq \;\; C \int_0^t (|||\rho|||_{\infty,t}^2 + |||\rho u|||_{\infty,t}^2) \, \|w\|_4^2 + |||w|||_{1,\infty,t}^2 \, \|U\|_3^2 \\ &\qquad + \|\rho u_3 \partial_z w\|_3 + \|w\|_5^2 + \|\mathcal{M}_\tau\|_3^2 \; ds\end{aligned}$$

The source term norm $\int_0^t \|\mathcal{M}_\tau\|_3^2 \; ds$ is given by (2.42), and $\int_0^t \|\rho u_3 \partial_z w\|_3^2 \; ds$ is controlled in the same way as is done in Proposition 3.7 :

$$\begin{aligned}\int_0^t \|\rho u_3 \partial_z w\|_3^2 \;\; &= \;\; \int_0^t \left\| \frac{\rho u_3}{\phi} Z_3 w \right\|_3^2 \\ &\leq \;\; C \int_0^t |||w|||_{1,\infty,t}^2 \, \|\partial_z(\rho u)\|_4^2 + (\|\rho u\|_{\text{Lip}}^2 + \|\nabla(\rho u)\|_{1,\infty}^2) \, \|w\|_4^2\end{aligned}$$

The commutator $\varepsilon[Z_3, \mu]\Delta w = Z_3(\mu(\rho))\Delta w$ is controlled by replacing $\varepsilon \partial_{zz} w$ by its expression. Therefore, in total, we have

$$\begin{aligned}\varepsilon^2 \int_0^t \|[Z_3, \mu\Delta] w\|_\infty^2 \leq C[\|U(0)\|_6^2 + \|w(0)\|_5^2] + C \int_0^t \|\mathcal{M}_\tau(s)\|_\infty^2 \; ds \\ + C \int_0^t Q(|||U|||_{\text{Lip},t}^2 + |||\nabla U|||_{1,\infty,t}^2)(1 + \|U\|_6^2 + \|w\|_5^2 + \|\partial_z \rho\|_5^2) \; ds + \mathcal{N}_6(t, F) \quad (2.49)\end{aligned}$$

It is now straightforward to notice that the whole commutator satisfies (2.49), the main tool to bound $\|[Z_j, \rho]\partial_t w + [Z_j, \rho u \cdot \nabla] w\|_\infty$ being property (2.24), as used in Proposition 3.7.

Moving on to the source term, we now focus on the terms in $\varepsilon \mathcal{M}^{II}$ which arise when the viscosity coefficients are not constant. First, we examine terms involving one normal derivative on $u_1$ or $u_2$. These are linked to $\nabla w$, which we split into two parts :

$$\begin{aligned}\varepsilon \nabla w \;\; &= \;\; \varepsilon \nabla((\text{rot } u)_\tau) = \varepsilon \nabla(\partial_z u_\tau^\perp - \nabla_\tau^\perp u_3) \\ &= \;\; \varepsilon \begin{pmatrix} \nabla_\tau \\ \partial_z \end{pmatrix} \partial_z u_\tau^\perp - \varepsilon \nabla(\nabla_\tau^\perp u_3)\end{aligned}$$

We are interested in the $L^2$-in-time $W_{co}^{1,\infty}$ norm of this. On the terms with only one normal derivative, which are the second and the $\nabla_\tau$ components of the first, we apply the Sobolev

embedding, Proposition 3.3, leading to having to bound $\|\varepsilon\partial_{zz}u_j\|_5^2$ for a certain $j$ ; we once again can do so by replacing $\partial_{zz}u_j$ by its expression in (2.2). The remains of the first term, $\partial_{zz}u_\tau$, are no more difficult : having replaced $\varepsilon\partial_{zz}u_\tau$ using (2.2), all the subsequent terms are dealt with easily using the Sobolev embedding in most places, including on $\varepsilon\lambda\nabla_\tau\partial_z u_3$, which comes from the $\nabla$div term, and another replacement of $\varepsilon\partial_{zz}u_3$ completes the estimate.

In the remaining terms of $\varepsilon\mathcal{M}^{II}$, terms with two normal derivatives on $u$ can appear : we then replace them using the equation. Terms with two conormal derivatives are dealt with by using the Sobolev embedding inequality.

Estimating the $L^2$-in-time conormal-Lipschitz norm of the rest of the source term, $\mathcal{M}_\tau$, which, we remind the reader, is the tangential component of $-\rho\omega\cdot\nabla u+\rho(\text{div } u)\omega+\text{rot }(\rho F)+\mathcal{M}^I$, is straight-forward. □

# 6 Proof of Theorem 1.3, part III *A priori* estimates on $\partial_z u_3$

To begin with, we bound $\int_0^t\|\partial_z u_3(s)\|_{m-1}^2\ ds$, which will essentially only need the tame estimate in Proposition 3.1. The compressibility equation (2.1) gives us

$$\rho Z^\alpha\partial_z u_3=-Z^\alpha\partial_t\rho-Z^\alpha(u\cdot\nabla\rho)-Z^\alpha(\rho\text{div}_\tau u_\tau)-[Z^\alpha,\rho]\partial_z u_3,$$

with $|\alpha|\leq m-1$ and introducing the notation $\text{div}_\tau u_\tau=\partial_{y_1}u_1+\partial_{y_2}u_2$. Integrating the square of this equality in time and space, we quickly get

$$\begin{aligned}c\int_0^t\|Z^\alpha\partial_z u_3(s)\|_0^2\ ds&\leq C(1+|||U|||_{\text{Lip},t}^2)\int_0^t\|U(s)\|_m^2\ ds\\&+C\,|||u_3|||_{\infty,t}^2\int_0^t\|\partial_z\rho(s)\|_{m-1}^2\ ds+C\,|||\rho|||_{1,\infty,t}^2\int_0^t\|\partial_z u_3(s)\|_{m-2}^2\ ds,\end{aligned}$$

using Proposition 3.6 on the commutator term. With this, we can perform an induction on $m>0$ and conclude that there exists a positive increasing polynomial function $Q$ such that

$$\int_0^t\|\partial_z u_3(s)\|_{m-1}^2\ ds\leq Q(1+|||U|||_{\text{Lip},t}^2)\int_0^t\|U(s)\|_m^2+\|\partial_z\rho(s)\|_{m-1}^2\ ds.$$

Essentially, this means that $H_{co}^{m-1}([0,T]\times\Omega)$ norms of $\partial_z u_3$ can be replaced by the same norms of $\partial_z\rho$ and $W_{co}^{1,\infty}$ norms of $\partial_z u_3$, which, in turn, will be bounded by $H_{co}^{m_0}$ norms for a certain $m_0$, so we need to extract a small parameter, in this case $t$, to close the estimate. We already have

$$\int_0^t\|\partial_z u_3(s)\|_{m-1}^2\ ds\leq Q(1+|||U|||_{\text{Lip},t}^2)\left(t\,|||U|||_{m,t}+\int_0^t\|\partial_z\rho(s)\|_{m-1}^2\ ds\right),\tag{2.50}$$

so we need the estimates on $\|\partial_z u_3\|_{1,\infty}$ and $\|\partial_z \rho\|_{1,\infty}$ to yield a small parameter, $t$ or $\varepsilon$, as we cannot use (2.15) on the $L^2$-in-time norm present here (by Proposition 4.3, $\|U\|_{1,\infty}$ already yields the desired parameter).

For now, we focus on $\|\partial_z u_3\|_{1,\infty}$. Simply reading (2.1), we have

$$|||\partial_z u_3|||^2_{\infty,t} \leq |||\partial_t \rho + \mathrm{div}_\tau(\rho u_\tau)|||^2_{\infty,t} + |||u_3|||^2_{\infty,t} \, |||\partial_z \rho|||^2_{\infty,t} \, ,$$

to which we can apply (2.10), and we get

$$\begin{aligned} |||\partial_z u_3|||^2_{\infty,t} &\leq Q(1 + \|U(0)\|^2_{\mathrm{Lip}}) + \int_0^t Q(1 + \|U(s)\|^2_{2,\infty} + \|\partial_z \rho(s)\|^2_{1,\infty}) \, ds \\ &\leq Q(1 + \|U(0)\|^2_{\mathrm{Lip}}) + tQ(1 + |||U|||^2_{5,t} + |||\partial_z U|||^2_{4,t} + |||\partial_z \rho|||^2_{1,\infty,t}) \quad (2.51) \end{aligned}$$

We can now do the same for $\|\partial_z u_3\|_{1,\infty}$, bearing in mind that there is a commutator, $[Z_j, \rho]\partial_z u_3 = (Z_j \rho)\partial_z u_3$, and that we do not want to lose derivatives on $\|\partial_z \rho\|_{1,\infty}$ this time :

$$\begin{aligned} |||\partial_z u_3|||^2_{1,\infty,t} &\leq |||\partial_t \rho + \mathrm{div}_\tau(\rho u_\tau)|||^2_{1,\infty,t} + |||u_3|||^2_{1,\infty,t} \, |||\partial_z \rho|||^2_{1,\infty,t} + |||\rho|||^2_{1,\infty} \, |||\partial_z u_3|||^2_{\infty} \\ &\leq Q(1 + \|U(0)\|^2_{2,\infty}) + \int_0^t \|\partial_t \rho + \mathrm{div}_\tau(\rho u_\tau)\|^2_{2,\infty} \\ &\qquad + |||\partial_z \rho|||^2_{1,\infty,t} \left( \|u_3(0)\|^2_{1,\infty} + \int_0^t \|u_3(s)\|_{2,\infty} \, ds \right) \\ &\qquad + |||\partial_z u_3|||^2_{\infty,t} \left( \|\rho(0)\|^2_{1,\infty} + \int_0^t \|\rho(s)\|^2_{2,\infty} \, ds \right) \\ &\leq Q(1 + \|U(0)\|^2_{\mathrm{Lip}} + \|\nabla U(0)\|^2_{1,\infty}) + C \, |||\partial_z \rho|||^4_{1,\infty,t} \\ &\qquad + tQ(|||U|||^2_{6,t} + |||\partial_z U|||^2_{5,t} + |||\partial_z \rho|||^2_{1,\infty,t}), \end{aligned}$$

by (2.51), which, by applying (2.15) provides us with the following.

**Proposition 6.1.** *(a) There exists an increasing, positive polynomial function $Q$ on $\mathbb{R}^+$ such that*

$$\begin{aligned} |||\partial_z u_3|||^2_{1,\infty,t} &\leq Q(1 + \|U(0)\|^2_{\mathrm{Lip}} + \|\nabla U(0)\|^2_{1,\infty}) + Q(|||\partial_z \rho|||^2_{1,\infty,t}) \\ &\quad + tQ\left(1 + |||U|||^2_{6,t} + |||\nabla u_\tau|||^2_{5,t} + |||\partial_z \rho|||^2_{1,\infty,t} + \int_0^t \|\partial_z u_3(s)\|^2_6 + \|\partial_z \rho(s)\|^2_6 \, ds\right) \end{aligned} \quad (2.52)$$

*(b) Combining (2.50) and (2.52), we get that, for $m \geq 7$,*

$$\int_0^t \|\partial_z u_3(s)\|^2_{m-1} \, ds \leq Q(1 + \|U(0)\|^2_{\mathrm{Lip}} + \|\nabla U(0)\|^2_{1,\infty})$$

$$+tQ\left(1+|||U|||^2_{m,t}+|||\nabla u_\tau|||^2_{m-1,t}+|||\nabla u_\tau|||^2_{1,\infty,t}+\int_0^t \|\partial_z u_3(s)\|^2_{m-1}\ ds\right)$$
$$+Q\left(|||\partial_z\rho|||^2_{1,\infty,t}+\int_0^t \|\partial_z\rho(s)\|^2_{m-1}\ ds\right) \tag{2.53}$$

As a result, we can update Corollary 5.3. Set

$$\begin{aligned}\tilde{E}^\varepsilon_m(t,U)^2 &= |||U|||^2_m+|||\nabla u_\tau|||^2_{m-1,t}+\int_0^t \|\partial_z u_3(s)\|^2_{m-1}\ ds+|||\nabla u|||^2_{1,\infty,t}\\ &\quad +\varepsilon\int_0^t \|\nabla u\|^2_m+\|\nabla W\|^2_{m-1}\ ds.\end{aligned}$$

This includes the terms of $\mathcal{E}_m(t,U)$ except the ones involving $\partial_z\rho$, plus the $W^{1,\infty}_{co}$ norm of $\partial_z u_3$ and the gradient terms from the energy estimates. The combination of Corollary 5.3 and Proposition 6.1 (b) gives us the following.

**Corollary 6.2.** *Under the conditions of Assumption 4.1, for $m\geq 7$ and $t\leq T^*$,*

$$\tilde{E}^\varepsilon_m(t,U)\leq Q(M_0)+Q(M+M_F)\left(t+Q\left(|||\partial_z\rho|||^2_{1,\infty,t}+\int_0^t \|\partial_z\rho(s)\|^2_{m-1}\ ds\right)\right)$$

We see on this estimate that it remains to look at the normal derivative of the density.

# 7 Proof of Theorem 1.3, part IV *A priori* estimates on $\partial_z\rho$

## 7.1 Conormal energy estimates

In this section, we examine $R:=\partial_z\rho$. The equation satisfied by $R$ is given by the differentiation of (2.1) :

$$\partial_t R+u\cdot\nabla R+R(\text{div } u+\partial_z u_3)+\rho\partial_{zz}u_3=-\rho\partial_z\text{div}_\tau u_\tau-\partial_z u_\tau\cdot\nabla_\tau\rho.$$

A very problematic term appears in this equation : $\partial_{zz}u_3$. The idea is to multiply the equation by $l\varepsilon:=(\mu+\lambda)\varepsilon$, in order to replace $l\varepsilon\partial_{zz}u_3$ by its expression in (2.2). This brings us to

$$l\varepsilon(\partial_t R+u\cdot\nabla R)+\gamma P(\rho)R=h, \tag{2.54}$$

where we remind the reader that $\gamma P=k\rho^\gamma$, and $h$ will be treated as a source term :

$$\begin{aligned}h &= \varepsilon[\rho(\mu\Delta_\tau u_3+\lambda\partial_z\text{div}_\tau u_\tau+\sigma(\nabla U)_3)\\ &\quad -l\partial_z u_\tau\cdot\nabla_\tau\rho-l(\text{div } u+\partial_z u_3)R]-\rho^2(\partial_t u_3+u\cdot\nabla u_3)+\rho^2F_3.\end{aligned}$$

In these estimates, we will strongly use Assumption 4.1 : for $t \leq T^*$, we have $\mathcal{E}_m(t,U) \leq M$, $\rho(t,x) \geq c_0 > 0$ and $|u_3(t,x)|$ is uniformly bounded near the boundary on the same time interval. This will greatly simplify the presentation.

**Proposition 7.1.** *Under the conditions of Assumption 4.1, for $m \geq 6$, there exists a positive, increasing function on $\mathbb{R}^+$, $Q$, such that, for $t \leq T^*$ and $\varepsilon \leq \varepsilon_0$,*

$$\int_0^t \|R(s)\|_{m-1}^2 \, ds \leq Q(\|R(0)\|_{m-1}^2 + \|U(0)\|_m^2) + (t+\varepsilon)Q(M+M_F) + Q(|||R|||_{1,\infty,t}^2).$$

*Proof :* we start with the $L^2$ estimate. We multiply (2.54) by $R$ and integrate in space, and, as usual, we integrate the term containing $u \cdot \nabla R$ by parts and get

$$\frac{l\varepsilon}{2}\frac{d}{dt}(\|R\|_0^2) + \gamma \int_\Omega PR^2 = \int_\Omega \left( \frac{l\varepsilon}{2} R^2 \text{div } u + \frac{l'(\rho)\varepsilon}{2} R^2 u \cdot \nabla \rho + hR \right). \tag{2.55}$$

The coefficient $l$ and the pressure $P$ are bounded from below, so there exist $0 < c < 1 < C$ such that the left-hand side, integrated in time, is greater than

$$c\left( \varepsilon \|R(t)\|_0^2 + \int_0^t \|R(s)\|_0^2 \, ds \right) - C\|R(0)\|_0^2 .$$

Then, using the Young inequality on the right-hand side of (2.55) with the parameter $\eta = \frac{c}{2}$, we get

$$c\varepsilon \|R(t)\|_0^2 + \frac{c}{2}\int_0^t \|R(s)\|_0^2 \, ds \leq C\|R(0)\|_0^2 + C\int_0^t \varepsilon \, |||U|||_{\text{Lip},t} \|R(s)\|_0^2 + \|h(s)\|_0^2 \, ds.$$

It only remains to bound $\int_0^t \|h\|_0^2$, and this is simple :

$$\int_0^t \|h(s)\|_0^2 \, ds \leq C\varepsilon^2 \int_0^t |||\nabla u|||_{\infty,t}^2 \|R(s)\|_0^2 + \|\nabla u(s)\|_1^2 + \|\rho(s)-1\|_1^2 \, ds$$
$$+ C\int_0^t (1 + \|\partial_z u_3\|_\infty^2)\|u_3\|_1^2 + \|F\|_0^2 \, ds,$$

in which $\varepsilon \int_0^t \|\nabla u(s)\|_1$ is bounded using Proposition 4.2, so we have proved the estimate on $R$ for $m = 1$.

We now move to the $H_{co}^{m-1}$ estimates, and once again, the main focus will be the commutators. Given $\alpha$ of length $m-1$, the equation on $Z^\alpha R$, with $|\alpha| < m$, is

$$l\varepsilon(\partial_t Z^\alpha R + u \cdot \nabla Z^\alpha R) + \gamma P Z^\alpha R = Z^\alpha h - \mathcal{C}_R^\alpha,$$

with $\mathcal{C}_R^\alpha = l\varepsilon[Z^\alpha, u\cdot\nabla]R + \gamma[Z^\alpha, P]R$. Repeating the above procedure, we reach the estimate

$$c\varepsilon \|Z^\alpha R(t)\|_0^2 + c\int_0^t \|Z^\alpha R(s)\|_0^2 \, ds \le C\|R(0)\|_{m-1}^2$$
$$+C\int_0^t \varepsilon |||U|||_{\mathrm{Lip},t} \|R(s)\|_{m-1}^2 + \|h(s)\|_{m-1}^2 + \|\mathcal{C}_R^\alpha(s)\|_0^2 \, ds. \tag{2.56}$$

Controlling $\int_0^t \|h\|_{m-1}^2$ is immediate, except for the term $\rho^2 u_3 \partial_z u_3$, which figures in $\rho^2 u\cdot\nabla u_3$. We start by replacing $\rho\partial_z u_3$ by using (2.1),

$$\rho^2 u_3\partial_z u_3 = -\rho u_3(\partial_t\rho + \rho \mathrm{div}_\tau u_\tau + u_\tau\cdot\nabla_\tau\rho + u_3\partial_z\rho),$$

and the problematic term here is clearly $\rho u_3^2\partial_z\rho$. Directly using the tame estimate basically yields $\int_0^t \|R\|_{m-1}^2$, but there is no factor $\varepsilon$ in this term to allow us to hope to absorb it. However, we have a factor $u_3$, and we use the assumption that, for a given $\delta>0$, there exist $z_\delta>0$ which does not depend on $\varepsilon$, such that $|u_3(x)|<\delta$ for $x$ in the strip $\omega_\delta = \mathbb{R}^2\times[0,z_\delta]$. Now, we split the $H_{co}^{m-1}$ norm of $\rho u_3^2 R$ into two parts,

$$\int_0^t \left\|\rho u_3^2 R(s)\right\|_{m-1}^2 \, ds = \int_0^t \left\|\rho u_3^2 R(s)\right\|_{m-1,\omega_\delta}^2 + \left\|\rho u_3^2 R(s)\right\|_{m-1,\Omega\setminus\omega_\delta}^2 \, ds.$$

Here, for $\omega\subset\Omega$, we set $\|f(s)\|_{m-1,\omega}$ as a sort of semi-norm of $f(s)$ restricted to $\omega$ :

$$\|f(s)\|_{m-1,\omega}^2 = \sum_{|\beta|\le m-1} \left\|(Z^\beta f)(s)|_\omega\right\|_0^2$$

We apply the tame estimate to both norms :

$$\begin{aligned}\int_0^t \left\|\rho u_3^2 R\right\|_{m-1}^2 \;\le\; C\int_0^t &|||\rho u_3^2|||_{L^\infty(\omega_\delta),t}^2 \|R\|_{m-1}^2 + |||R|||_{\infty,t}^2 \left\|\rho u_3^2\right\|_{m-1}^2 \\ &+ |||\rho u_3^2|||_{\infty,t}^2 \|\partial_z\rho\|_{m-1,\Omega\setminus\omega_\delta}^2 \, ds.\end{aligned}$$

The two key terms are the first and the last. To deal with the last term, we use Proposition 3.4 to write

$$\|\partial_z\rho\|_{m-1,\Omega\setminus\omega_\delta}^2 \le C\sum_{|\beta|<m} \left\|\partial_z Z^\beta\rho\right\|_0^2,$$

and we note that, for $z\ge z_\delta$, $\phi(z)\ge\phi(z_\delta)$, therefore

$$|\partial_z Z^\beta\rho(x)| \le \frac{1}{\phi(z_\delta)}|Z_3 Z^\beta\rho(x)|$$

for $x\neq\omega_\delta$. This means that conormal derivatives are equivalent to standard derivatives away from the boundary, thus, $\|\partial_z\rho\|_{m-1,\Omega\setminus\omega_\delta}^2 \le \|\rho-1\|_{H_{co}^m(\Omega)}^2$. The first term is led by $\left\|\rho u_3^2\right\|_{L^\infty(\omega_\delta)}^2$,

which, given the boundedness of $\rho$ and the properties of $u_3$ on $\omega_\delta$, is bounded by $c_1^2\delta^4$, which is smaller that $\frac{c}{2}$ if $\delta$ is small enough, where $c$ is the coefficient on the left-hand side of (2.56) : we therefore choose $\delta$ so that this term can be absorbed.

The other terms in $h$ are straight-forward, as we can use Proposition 4.2 on the $\varepsilon^2 \int_0^t \|\nabla u\|_m^2$ term that comes from the order-two terms of $h$ - and the factor $\varepsilon^2$ is essential, as it leaves a factor $\varepsilon$ which will allow us to close the complete estimate for $\varepsilon$ and $t$ small. In total, we get

$$\int_0^t \|h(s)\|_{m-1}^2 \, ds \leq C \|U(0)\|_m^2 + \left(\varepsilon Q(1 + |||U|||_{\mathrm{Lip},t}^2) + \frac{c}{2}\right) \int_0^t \|R(s)\|_{m-1}^2 \, ds$$
$$+C \int_0^t Q(1 + |||U|||_{\mathrm{Lip},t}^2 + |||\nabla u|||_{1,\infty,t}^2)(\|U(s)\|_m^2 + \|\nabla u_\tau(s)\|_{m-1}^2 + \|F(s)\|_{m-1}^2) \, ds \tag{2.57}$$

The estimation of the commutators is also mostly straight-forward. $[Z^\alpha, u \cdot \nabla]R$ can be controlled by using Proposition 3.7, bearing in mind that there is a factor $\varepsilon$ in front of it :

$$\varepsilon^2 \int_0^t \|[Z^\alpha, u \cdot \nabla]R\|_0^2 \leq C\varepsilon \int_0^T (|||u|||_{\mathrm{Lip},t}^2 + |||\nabla u|||_{1,\infty,t}^2) \|R(s)\|_{m-1}^2 \, ds$$
$$+C\varepsilon \int_0^t |||R|||_{1,\infty,t}^2 \|\nabla u(s)\|_{m-1}^2 + |||R|||_{\infty,t}^2 \|u(s)\|_m^2 \, ds.$$

The other commutator, $[Z^\alpha, P]$ does not have a factor $\varepsilon$, so we need to gain a derivative on $R$ by using Proposition 3.6,

$$\int_0^t \|[Z^\alpha, \rho^\gamma]R\|_0^2 \, ds \leq C \, |||\rho|||_{1,\infty,t}^{2\gamma} \int_0^t \|R(s)\|_{m-2}^2 \, ds + |||R|||_{\infty,t}^2 \int_0^t \|(\rho - 1)(s)\|_{m-1}^2 \, ds,$$

and use (2.15) to extract a factor $t$ in the first term. This leaves us with an isolated $t \, |||\rho|||_{1,\infty,t}$, to which we apply the anistropic Sobolev embedding, Proposition 3.3. Thus,

$$t \, |||\rho|||_{1,\infty,t}^2 \leq Ct \left(1 + \|R(0)\|_4^2 + |||U|||_{4,t}^2 + \int_0^t \|R(s)\|_5^2 \, ds\right)$$

Note that it is this last inequality that restricts us to $m - 1 \geq 5$. We conclude the proof of Proposition 7.1 by combining (2.56), (2.57) and these bounds on the commutators. □

## 7.2 $L^\infty$ estimates

As stated in the introduction, we will control $L^\infty$ norms of $R$ with $L^2$-in-time bounds by virtue of (2.10),

$$|||R|||_{1,\infty,t}^2 \leq \|R(0)\|_{1,\infty}^2 + C \int_0^t \|\partial_t R(s)\|_{1,\infty}^2 + \|R(s)\|_{1,\infty}^2 \, ds. \tag{2.58}$$

Let us define $Y_t$ the space of functions $f$ satisfying $\|f\|_{Y_t}^2 := \int_0^t \|f(s)\|_{1,\infty}^2 + \|\partial_t f(s)\|_{1,\infty}^2 \, ds$ finite.

**Proposition 7.2.** *Under the conditions of Assumption 4.1, there exists an increasing, positive function $Q$ on $\mathbb{R}^+$ such that, for $t \leq T^*$ and $\varepsilon \leq \varepsilon_0$,*

$$\|R\|_{Y_t}^2 \leq Q(M_0) + (t+\varepsilon)Q(M+M_F)$$

This ends the proof of Theorem 1.3 providing we can pick up Assumption 4.1.

*Proof :* the main tool in this proof will be the Duhamel formula for the ordinary differential equation $\varepsilon f' + \tilde{p} f = \tilde{h}$. We reach this ODE by considering $R$ along the characteristics of the transport equation $\partial_t R + u \cdot \nabla R = 0$, in other words

$$f(t,x) = R^X(t,x) := R(t, X(t,x)),$$

where $X(t,x)$ satisfies $\partial_t X(t,x) = u(t, X(t,x))$ and $X(0,x) = x$. Thus, we have the identity $\partial_t(R^X) = (\partial_t R + u \cdot \nabla R)^X$, so (2.54) becomes, denoting $l^X = l(\rho^X)$,

$$l^X \varepsilon \partial_t(R^X) + \gamma P^X R^X = h^X.$$

For the higher-order estimates, it is important to apply the conormal derivatives first, then follow the flow of $u$. So, the equation we are interested in is

$$l^X \varepsilon \partial_t((Z^\alpha R)^X) + \gamma P^X (Z^\alpha R)^X = (Z^\alpha h)^X - (\mathcal{C}_R^\alpha)^X,$$

with $\mathcal{C}_R^\alpha = l\varepsilon[Z^\alpha, u \cdot \nabla]R + \gamma[Z^\alpha, P]R$ as in the previous paragraph, and $\alpha$ is either of length $\leq 1$, or of length 2 with $\alpha_0 \geq 1$ (these are the $\alpha$ that intervene in $Y_t$).

To lighten the load, we introduce the following notations : $g_\alpha = Z^\alpha h - \mathcal{C}_R^\alpha$,

$$j(s',s,x) = \int_s^{s'} \frac{\gamma}{l^X(\sigma)\varepsilon} P^X(\sigma,x)\, d\sigma \ , \text{ and } J(s,x) = \int_0^s \frac{g_\alpha^X(\sigma,x)}{l^X(\sigma)\varepsilon} e^{-j(s,\sigma,x)}\, d\sigma.$$

The Duhamel formula for this equation then reads :

$$(Z^\alpha R)^X(s,x) = Z^\alpha R(0,x) e^{-j(s,0,x)} + J(s,x).$$

We integrate the square of this equality in time between 0 and $t$, which yields

$$\int_0^t [(Z^\alpha R)^X(s,x)]^2\, ds \leq C \left[ \|Z^\alpha R(0)\|_\infty^2 \int_0^t e^{-2j(s,0,x)}\, ds + \int_0^t J^2(s,x)\, ds \right]. \tag{2.59}$$

Again, we use the uniform bounds on $\rho$ to get, for any $s,\ s' \in \mathbb{R}^+$, $x \in \Omega$,

$$j(s',s,x) \geq \int_s^{s'} \frac{c}{\varepsilon} = \frac{c(s'-s)}{\varepsilon}. \tag{2.60}$$

Thus,

$$\int_0^t e^{-cj(s,0,x)}\ ds \leq C\varepsilon, \tag{2.61}$$

which deals with the first term of the right-hand side of (2.59). For the second term, we use (2.60) again and write the integral to reveal an $L^2$ norm of a convolution in the time variable :

$$\begin{aligned}
\int_0^t J^2(s,x)\ ds &\leq \int_0^t \left[\int_{\mathbb{R}} |g_\alpha^X|(\sigma,x)\mathbf{1}_{(0,t)}(\sigma)\frac{e^{-c\varepsilon^{-1}(s-\sigma)}}{\varepsilon}\mathbf{1}_{(0,t)}(s-\sigma)\ d\sigma\right]^2 ds \\
&\leq \left\| \left(g_\alpha^X(\cdot,x)\mathbf{1}_{(0,t)}\right) * \left(\varepsilon^{-1}e^{-c\varepsilon^{-1}(\cdot)}\mathbf{1}_{(0,t)}\right)\right\|^2_{L^2(0,t)} \\
&\leq \left\|g_\alpha^X(\cdot,x)\right\|^2_{L^2(0,t)} \left\|\varepsilon^{-1}e^{-c\varepsilon^{-1}(\cdot)}\right\|^2_{L^1(0,t)} \\
&\leq C\int_0^t \|g_\alpha(s)\|^2_\infty\ ds,
\end{aligned}$$

by (2.61) and standard convolution inequalities.

So we now only need to control the $Y_t$ norm of $h$ and the $L^\infty$ norm of the commutators. We remind the reader that

$$\begin{aligned}
h &= \varepsilon[\rho(\mu\Delta_\tau u_3 + \lambda\partial_z \mathrm{div}_\tau u_\tau + \sigma(\nabla U)_3) \\
&\qquad -l\partial_z u_\tau\cdot\nabla_\tau\rho - l(\mathrm{div}\ u + \partial_z u_3)R] - \rho^2(\partial_t u_3 + u\cdot\nabla u_3) + \rho F_3.
\end{aligned}$$

The starting point here is to notice that, for two functions $f$ and $g$,

$$\|fg\|_{2,\infty} \leq C(\|f\|_{2,\infty}\|g\|_\infty + \|f\|_{1,\infty}\|g\|_{2,\infty}). \tag{2.62}$$

In the case of $h_1 := \rho(\mu\Delta_\tau u_3 + \lambda\partial_z \mathrm{div}_\tau u_\tau)$, $f = \rho$, so

$$\|h_1\|^2_{2,\infty} \leq C(\|\rho\|^2_{2,\infty}(\|u\|^2_{2,\infty} + \|\nabla u_\tau\|^2_{1,\infty}) + \|\rho\|_{1,\infty}(\|u\|^2_{4,\infty} + \|\nabla u_\tau\|^2_{3,\infty})),$$

in which we apply the Sobolev inequality to all terms except $\|\nabla u_\tau\|^2_{1,\infty}$ (it is part of the total quantity we wish to bound), and use the Young inequality to split the products up. In the process, we obtain $(\varepsilon^2\int_0^t \|\partial_z\nabla u_\tau(s)\|^2_4\ ds)^2$, which is controlled by using Proposition 5.1. Therefore, in total, using all the estimates available to us,

$$\int_0^t \varepsilon^2\|h_1\|^2_{2,\infty}\ ds \leq \varepsilon Q(M_0) + \varepsilon\int_0^t Q(1 + \|U\|^2_7 + \|\nabla u_\tau\|^2_6 + \|R\|^2_6 + \|\nabla u_\tau\|^2_{1,\infty} + \|R\|^2_{1,\infty})\ ds,$$

$$\text{thus}\quad \int_0^t \varepsilon^2\|h_1\|^2_{2,\infty}\ ds \leq Q(M_0) + C\varepsilon(t+1)Q(M + M_F) + C\varepsilon t\,|||R|||^2_{1,\infty,t}\,. \tag{2.63}$$

Likewise, setting $h_2 := \partial_z u_\tau \cdot \nabla_\tau \rho + (\text{div } u + \partial_z u_3)R + \sigma_3 = \partial_z u \cdot \nabla \rho + R\text{div } u + \sigma_3$, we use (2.62) again, applied to the space $Y_t$ instead of $W^{2,\infty}_{co}$, and we get, after applying the Sobolev embedding to all the terms except $\|\nabla u_\tau\|^2_{1,\infty}$, and $\|R\|^2_{Y_t}$, to get

$$\varepsilon^2 \|h_2\|_{Y_t} \leq Q(M_0) + \varepsilon(t+1)Q(M+M_F) + C\varepsilon^2 \|R\|^2_{Y_t} + C\varepsilon^2 \int_0^t \|\partial_{zz} u_3\|^2_6 \ ds.$$

We have once again used Proposition 5.1 to control the $H^4_{co}([0,T]\times\Omega)$ norm of $\varepsilon\partial_z \nabla u_\tau$, and we have used (2.15) to get an $L^2$-in-time norm of $\partial_z u_3$ (which can then be controlled using Proposition 4.2) and $\partial_{zz} u_3$. Thus, we replace $\varepsilon\partial_{zz} u_3$ by its expression in (2.2),

$$l\varepsilon\partial_{zz} u_3 = \rho\partial_t u_3 + \rho u \cdot \nabla u_3 - \mu\varepsilon\Delta_\tau u_3 - \lambda\varepsilon\partial_z \text{div}_\tau u_\tau + R,$$

use the tame estimate, take the supremum inside the integral and extract a factor $t$, and this yields

$$\varepsilon^2 \|h_2\|^2_{Y_t} \leq Q(M_0) + \varepsilon(t+1)Q(M+M_F) + C\varepsilon \|R\|^2_{Y_t}. \tag{2.64}$$

In the source term $h$, it remains to examine $h_3 := \rho^2(\partial_t u_3 + u \cdot \nabla u_3 + F_3)$. The difference here is that there is no factor $\varepsilon$ ready to provide us with a small parameter. Instead, we once again extract $t$ from the integral to use as the small parameter. By simply using the Sobolev embedding inequality and breaking it down almost completely with the tame estimate, we get

$$\begin{aligned}
\int_0^t \|h_3(s)\|^2_{2,\infty} \ ds &\leq Q(M_0) + C\int_0^t \|h_3(s)\|^2_5 + \|\partial_z h_3(s)\|^2_4 \ ds \\
&\leq Q(M_0) + \int_0^t Q(1 + |||U|||^2_{1,\infty,t} + |||\partial_z U|||^2_{1,\infty,t} + M_F) \\
&\qquad \times \left(1 + \|U\|^2_6 + \|\partial_z U\|^2_5 + \|u_3\partial_{zz} u_3\|^2_4 \ ds\right).
\end{aligned} \tag{2.65}$$

The only product we do not split using the tame estimates and the Young inequality is $u_3\partial_{zz} u_3$, which appears in $\partial_z h_3$. Indeed, we need the factor $u_3$ to be able to compensate for the two $z$-derivatives. We have already used in (2.65) the fact that

$$|||u_3\partial_{zz} u_3|||^2_{\infty,t} \leq |||\partial_z u_3|||^4_{1,\infty,t},$$

due to (2.20), and $\|u_3\partial_{zz} u_3\|_3$ is dealt with in same way as in the proof of (2.22) in Proposition 3.7 : multiply and divide by $\phi$, which means we are actually looking at $\left\|\phi^{-1}u_3 Z_3 \partial_z u_3\right\|_3$, and use the Hardy inequality to get that this quantity satisfies (2.22), with $g = u_3$ and $f = \partial_z u_3$, so

$$\int_0^t \|h_3(s)\|^2_{2,\infty} \ ds \leq Q(M_0) + Q(1 + |||U|||^2_{1,\infty,t} + |||\partial_z U|||^2_{1,\infty,t}) \int_0^t \|U\|^2_6 + \|\partial_z U\|^2_5 \ ds.$$

Now we extract $t$ and use (2.15) to get $L^2$-in-time norms on $R$ and $\partial_z u_3$, so

$$\int_0^t \|h_3(s)\|^2_{2,\infty} \ ds \leq Q(M_0) + tQ(M+M_F). \tag{2.66}$$

Finally, we examine the $L^\infty$ norms of the commutators, $\mathcal{C}_R^\alpha$, with $|\alpha| = 1$, or $|\alpha| = 2$ and $\alpha_0 \geq 1$. We begin with the case $|\alpha| = 1$, so $Z^\alpha = Z_j$ for a certain $j$, and

$$\mathcal{C}_R^\alpha = l\varepsilon[Z_j, u \cdot \nabla]R + \gamma[Z_j, P(\rho)]R = l\varepsilon(Z_j u) \cdot \nabla R + l\varepsilon u_3 [Z_j, \partial_z]R + \gamma(Z_j P)R,$$

in which the second term is either 0 ($j \neq 3$) or $-l\varepsilon\phi' u_3 \partial_z R$. Bounding this is straight-forward, using (2.24) along the way :

$$\int_0^t \|\mathcal{C}_R^\alpha(s)\|_\infty^2 \, ds \leq Ct(1+\varepsilon^2) \, |||\nabla u|||_{1,\infty,t}^2 \, |||R|||_{1,\infty,t}^2 \,. \tag{2.67}$$

The second case, $|\alpha| = 2$ and $\alpha_0 > 0$ is also simple. We can write $Z^\alpha = Z_j \partial_t$ for a certain $j$, so

$$\begin{aligned} \mathcal{C}_R^\alpha \;=\; & l\varepsilon \left( (Z_j \partial_t u) \cdot \nabla R + (Z_j u) \cdot \nabla(\partial_t R) + (\partial_t u) \cdot \nabla(Z_j R) + (\partial_t u_3)[Z_j, \partial_z]R \right. \\ & \left. + u_3 [Z_j, \partial_z] \partial_t R) + \gamma((Z_j \partial_t P)R + (Z_j P)(\partial_t R) + (\partial_t P)(Z_j R)) \right) \end{aligned}$$

The $L^\infty$ norm of this is, for most terms, bounded using only the Sobolev embedding and/or (2.24), wherever $u_3 \partial_z$ or conormal derivatives of $u_3 \partial_z$ appear. One can then take the supremum in time inside the integral and integrate to get a factor $t$. For instance, the last term satisfies

$$\begin{aligned} \|(\partial_t r)(Z_j R)\|_\infty^2 \;&\leq\; C(\|u_\tau\|_{2,\infty}^2 + \|\partial_z u_3\|_{1,\infty}^2 + \|\rho\|_{1,\infty}^2) \, \|R\|_{1,\infty}^2 \\ &\leq\; C(1 + \|U\|_5^2 + \|\nabla u_\tau\|_4^2 + \|R\|_3^2 + \|\partial_z u_3\|_{1,\infty}^2) \, \|R\|_{1,\infty}^2 \,, \end{aligned}$$

then we apply Proposition 6.1 (a) on $\|\partial_z u_3\|_{1,\infty}$. Two specificities do appear though in the terms containing $u_3 \partial_z R$ and the like.

- The term $l\varepsilon u_3 [Z_3, \partial_z] \partial_t R = -l\varepsilon\phi' u_3 \partial_{zt} R$ leads to two conormal derivatives on $R$. We cannot extract a factor $t$ from the integral here, since we want $L^2$-in-time norms of $\|\partial_t R\|_{1,\infty}$. However, we do have a factor $l\varepsilon$, which will act as the small parameter. So,

$$\int_0^t \left\| l\varepsilon\phi' u_3 \partial_{zt} R \right\|_\infty^2 \, ds \leq C\varepsilon^2 \, |||\partial_z u_3|||_{\infty,t}^2 \int_0^t \|\partial_t R(s)\|_{1,\infty}^2 \, ds.$$

- The term $l\varepsilon(Z_j \partial_t u) \cdot \nabla R$ contains $\varepsilon(Z_j \partial_t u_3)\partial_z R$, which, after using (2.24), leads to a term with two conormal derivatives on $\varepsilon \partial_z u_3$ (and no more than one on $R$, which is therefore not problematic), so we re-use the trick we used when estimating $\varepsilon h_2$ : we use the Sobolev inequality on the term with two conormal derivatives, to get

$$\int_0^t \varepsilon^2 \, \|\partial_z u_3\|_{2,\infty}^2 \leq M_0 + C \int_0^t \varepsilon^2 \, \|\partial_z u_3\|_5^2 + \|\varepsilon \partial_{zz} u_3\|_5^2 \,,$$

  and we replace $\varepsilon \partial_{zz} u_3$ by its expression in (2.2), use the tame estimate, take the supremum inside the integral and extract a factor $t$.

In total, combining the above consideration with inequalities (2.63) to (2.67), we conclude that

$$\int_0^t \|g_\alpha(s)\|_\infty^2 \, ds \leq Q(M_0) + (t+\varepsilon) Q(M + M_F),$$

which ends the proof of the estimate. □

# 8 Proof of Theorem 1.3, part V Conclusion

Let us consider times $t \leq T^*$, where

$$T^* = \sup\{t \leq T_0 \mid \mathcal{E}_m(t, U) \leq Q(2M_0)\}.$$

$T^*$ depends *a priori* on $\varepsilon$. With this, we pick up the uniform boundedness from below of $\rho$ and the uniform smallness of $u_3$ near the boundary that we have so far assumed.

– Let $\rho(0, x) \geq c_0$. (2.1) provides us with the differential equation

$$\partial_t \rho^X(t, x) = -(\rho \text{div } u)^X(t, x),$$

where $f^X$ is once again $f$ following the flow of $u$. Thus,

$$\rho^X(t, x) = \rho(0, x) \exp\left(-\int_0^t \text{div } u(s, x)\, ds\right),$$

and div $u$ is uniformly bounded on $[0, T^*]$, so

$$|\rho(t, x)| \geq c_0 e^{-Q(2M_0)T^*} := c_0^*,$$

for $t \leq T^*$, and $c_0^* > 0$ can be chosen independent of $\varepsilon$ (if $T^* = +\infty$ for some $\varepsilon$, we work with $t \leq T_0^* = \min(T_0, T^*)$ with $T_0$ independent of $\varepsilon$). The density $\rho$ is therefore uniformly bounded from below on $[0, T^*]$.

– For $t \leq T^*$, we can repeat the proof of (2.51),

$$\|\partial_z u_3\|_\infty^2 \leq Q(1 + \|U(0)\|_{\text{Lip}}^2) + tQ(1 + |||U|||_{4,t}^2 + |||\partial_z U|||_{3,t}^2 + |||\partial_z \rho|||_{1,\infty,t}^2),$$

which is only a reading of (2.1) combined with the Sobolev embedding and property (2.10), hence $\partial_z u_3(t)$ has a uniform bound for $t \leq T^*$, thus $|u_3(t, z)| \leq M' z$ for some $M'$ when $t \leq T^*$. As a result, we get uniform smallness of $u_3$ near the boundary : $|u_3(t, z)| \leq M'\delta$ when $z < \delta$.

For $t \leq T*$, we have the bounds required to make the *a priori* estimation process valid, so, for $t \leq T^*$, we have

$$\mathcal{E}_m(t, U) \leq Q(M_0) + (t + \varepsilon)Q(Q(2M_0) + M_F).$$

For $\varepsilon \leq \varepsilon_0$ small enough, we see that the right-hand side is smaller than $Q(2M_0)$ for $t \leq T' \leq T^*$, with $T'$ depending on $\varepsilon_0$ but not on $\varepsilon$. Thus $T^*$ can be chosen independent of $\varepsilon$, and the theorem is proved. □

# Chapitre 3

# Instabilité transverse non-linéaire d'ondes solitaires de l'équation d'Euler-Korteweg 2D

**Résumé.** *On montre que l'instabilité linéaire d'ondes solitaires dans le modèle d'Euler-Korteweg 2D pour fluides avec capillarité interne implique leur instabilité transverse non-linéaire.*

# 1 Introduction

We consider the motion of a compressible, inviscid and isentropic planar fluid, in which internal capillarity is taken into account. This phenomenon occurs for example at diffuse interfaces in liquid-vapour mixes [14]. In this model, the free energy of the fluid depends on both the density of the fluid, the scalar function $\rho$, and its derivatives $\nabla\rho$ in the following way :

$$F(\rho, \nabla\rho) = F_0(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2,$$

with $K$ and $F_0$ two given smooth, positive functions. We then derive the pressure from the free energy,

$$P(\rho, \nabla\rho) = \rho\frac{\partial F}{\partial\rho} - F = P_0(\rho) + \frac{1}{2}(\rho K'(\rho) - K(\rho))|\nabla\rho|^2,$$

in which $P_0$ is the standard part of the pressure. Let $g_0(\rho)$ be the bulk chemical potential of the fluid, so that $\rho g_0'(\rho) = P_0'(\rho)$. Then the principles of classical mechanics yield the Euler-Korteweg equation that we will study :

$$\left\{\begin{array}{rcl} \partial_t\rho + \text{div }(\rho u) & = & 0 \\ \partial_t u + (u\cdot\nabla)u & = & \nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 - g_0(\rho)\right) \\ (\rho, u)|_{t=0} & = & (\rho_0, u_0). \end{array}\right. \tag{3.1}$$

The variables are $t \in \mathbb{R}^+$ and $(x, y) \in \mathbb{R}^2$ ; as is standard, the operators $\nabla$, div and $\Delta$ contain only derivatives with respect to the space variables $x$ and $y$. The unknowns of equation (3.1) are the density $\rho$ and the velocity vector field $u : \mathbb{R}^+\times\mathbb{R}^2 \to \mathbb{R}^2$. The scalar functions $g_0$ and $K$ are given.

In this chapter, we will be interested in the transverse stability of solitary wave solutions of (3.1). These are 1D travelling waves written as

$$Q_c(t, x) = \left(\begin{array}{c} \rho_c(t, x) \\ u_c(t, x) \end{array}\right) = Q_c(x - ct),$$

with $u_c$ scalar (not a 2D vector field). Based on a remark by T.Benjamin [11], S.Benzoni, R.Danchin, S.Descombes and D.Jamet showed in [14] that the hamiltonian structure of the system led to the existence of solitary wave solutions, for every $c \in \mathbb{R}$, of two kinds, related to the homoclinic or heteroclinic nature of the connecting orbits. In the homoclinic case, the wave has identical endstates, and we can write

$$\lim_{|z|\to+\infty} Q_c(z) = Q_{c,\infty} = (\rho_{c,\infty}, u_{c,\infty}),$$

and such a travelling wave solution is called a soliton. We will be studying this type of solitary wave. In the heteroclinic case, the endstates are different, but must nonetheless satisfy a Rankine-Hugoniot-type condition. See [14] for more on these solutions, which are called kinks.

From now on, we set $c$ and $Q_c$ is a soliton, whose endstate, which we also set (thus we drop the index $c$), satisfies

$$\rho_\infty g_0'(\rho_\infty) > (u_\infty - c)^2, \tag{3.2}$$

which means that $Q_\infty$ is a saddle point for the hamiltonian ODE satisfied by $Q_c$. Under this condition, we have that $\rho_c'$ vanishes only once.

The standard Lyapunov stability notion is that if solutions of equation (3.1) have initial conditions close to $Q_c$, they remain close to $Q_c(x-ct)$ at all times. But this notion is not satisfactory in describing the stability of travelling waves. Indeed, let $c' \neq c$ be close to $c$; we then have that the profile $Q_{c'}$ is close to $Q_c$, but, as the speeds are different, $Q_{c'}(x-c't)$ and $Q_c(x-ct)$ drift apart, despite their profiles remaining very similar. To see this, for a given $t$, compare $Q_{c'}(x-c't)$ with the translated profile $Q_c(x-c't) = Q_c(x-ct+(c-c')t)$. The correct notion of stability therefore stems from taking the difference of solutions with all the translated versions of $Q_c(x-ct)$. A travelling wave solution will be considered stable if it is orbitally stable : for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|(\rho_0, u_0) - Q_c\| \leq \delta$, then

$$\sup_{t\in\mathbb{R}^+} \inf_{a\in\mathbb{R}} \|(\rho(t), u(t)) - Q_c(\cdot - a - ct)\| \leq \varepsilon.$$

The problem of orbital stability can be divided into two parts, depending on the type of perturbation we consider. 1D perturbations are perturbations of $Q_c$ that depend only on $x$ and satisfy $u_2(t,x) = 0$. The stability problem associated with these perturbations has been in part dealt with by S.Benzoni *et al.* in [14], and improved upon by Benzoni in [12]. A sufficient condition for orbital stability was obtained in the first paper using an argument by M.Grillakis, J.Shatah and W.Strauss [50], while the second article adds a sufficient condition for linear instability.

The question of transverse stability deals with perturbations that also depend on the transverse variable $y$ and have a 2D vector field for the velocity. So far, Benzoni in [12] and F.Rousset and N.Tzvetkov in [104] have proved linear instability. This occurs when the linearised equation around $Q_c$ has eigenvalues with positive real part (see part 2 of the chapter). On one hand, Benzoni used Evans functions computations to get that orbitally stable solitons are transversally linearly unstable. On the other hand, Rousset and Tzvetkov applied an abstract, but simple, criterion for instability in linearised PDEs with a hamiltonian structure in the case where the endstate of the soliton satisfies (3.2). We recall this criterion in Theorem 2.2. This criterion was applied to other equations with solitary waves in the same article : KP-I and Gross-Pitaevskii.

The result of this chapter is that the spectral instability mentioned above implies nonlinear instability of Euler-Korteweg solitons.

**Theorem 1.1.** *Let $Q_c(x-ct)$ be a soliton solution to (3.1) such that the endstate $Q_\infty$ satisfies (3.2). Then there exist $\delta_0$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there exists an initial condition $U_0 = (\rho_0, u_0)$ with*

$$\|U_0 - Q_c\|_{H^s(\mathbb{R}^2)} \leq \varepsilon$$

*for some $s > 0$, such that, for every $a \in \mathbb{R}$, the solution $U = (\rho, u)$ of (3.1) with this initial condition satisfies, at a time $T^\varepsilon \sim \ln(\varepsilon^{-1})$,*

$$\|U(T^\varepsilon) - Q_c(\cdot - cT^\varepsilon - a)\|_{L^2(\mathbb{R}^2)} \geq \delta_0.$$

The proof relies on an argument originally by E.Grenier [47], in which one constructs an approximate solution $U^{\mathrm{ap}}$ to the equation based on a WBK expansion starting with the reference solution and unstable eigenmodes. An energy estimate on $U - U^{\mathrm{ap}}$ shows that, if there are enough terms in the expansion, this difference is small, thus, for times under $T^\varepsilon$, the linear instability is dominant. Primarily used to obtain nonlinear instability of boundary layers in numerous settings (unstable Euler shear flows and Prandtl layers [47], Ekman layers for rotating fluids [31, 100, 83], Ekman-Hartman layers in MHD [31], Navier-Stokes with a boundary-layer-scale slip condition [93]), the method relies on a compact set of unstable wave numbers, thus the idea has been transposed to showing transverse nonlinear instability of solitary waves, when these can be shown to be linearly unstable. F.Rousset and N.Tzvetkov have thus obtained nonlinear instability of solitary waves in many models : KP-I and NLS [103], multiple hamiltonian models including generalised KP-I and the Boussinesq equations [102], and the free-surface water-waves equation [105].

Similarly to the KP-I equation [103], we expect our result to extend to the $y$-periodic framework $\mathbb{R} \times \mathbb{T}_L$, where $\mathbb{T}_L$ is the torus of length $L > 0$, in the following way : there exists a critical period $L_0 > 0$ such that if $L > L_0$, we have Theorem 1.1. The proof is identical, and limited to above a critical period due to the loss, below this critical period, of linear instability (to be studied). Rousset and Tzvetkov also proved that, for KP-I, below the critical period, there is orbital stability [106]. It would be worth looking at what happens to the Euler-Korteweg solitons in this situation.

**Outline of the proof.** The proof of Theorem 1.1 is in two parts. First, in section 2, we build on Rousset and Tzvetkov's linear instability theorem, deriving an important resolvent estimate for the Euler-Korteweg equation, linearised around $Q_c$. This will allow us, in section 3, to build an approximate solution $U^{\mathrm{ap}}$ with the appropriate behaviour of being predominantly unstable for $t \sim T^\varepsilon$. An energy estimate on $U - U^{\mathrm{ap}}$ will then be used ; this is particularly important ensure that the time of existence of $U$ is large enough to get the desired amplification. Indeed, we only have local existence for solutions to (3.1), as shown by Benzoni, Danchin and Descombes in [13]. This article also provides a blow-up criterion, but the instability phenomenon is not related to the blow-up if it occurs ; indeed, the mechanism is also observed on systems that have global solutions (most of the previous examples, whether they concern boundary layer or solitary wave instability, fall in this category). Finally, combining the two will lead to the instability result.

## 2 Linear analysis

In this part, we consider the linearised problem about $(\rho_c, u_c)$ to obtain an essential ingredient in order to apply Grenier's method.

Considering that $u$ is potential, we write the system on $(\rho, \varphi)$, where $u = \nabla\varphi$ :

$$\left\{ \begin{array}{rcl} \partial_t\rho + \nabla\varphi\cdot\nabla\rho + \rho\Delta\varphi & = & 0 \\ \partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 & = & K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 - g_0(\rho), \end{array} \right.$$

which we linearise around $(\rho_c, u_c)$. Having changed the space variable from $x$ to $x - ct$ (which turns the solitary wave into a stationary solution), we are interested in

$$\left\{ \begin{array}{rcl} \partial_t\rho & = & (c\partial_x - u_c\partial_x - u_c')\rho - (\rho_c'\partial_x + \rho_c\Delta)\varphi \\ \partial_t\varphi & = & (c\partial_x - u_c\partial_x)\varphi + (K(\rho_c)\Delta + K'(\rho_c)\rho_c'\partial_x - m)\rho, \end{array} \right. \tag{3.3}$$

with $m = K'(\rho_c)\rho_c'' + K'(\rho_c)(\rho_c')^2 - g_0'(\rho_c)$. We abbreviate the system by defining two operators

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} -\partial_x(K(\rho_c)\partial_x) - K(\rho_c)\partial_{yy}^2 - m & (u_c - c)\partial_x \\ -\partial_x((u_c - c)\cdot) & -\partial_x(\rho_c\partial_x) - \rho_c\partial_{yy}^2 \end{pmatrix},$$

thus the system (3.3) can be written synthetically as $\partial_t V = J\mathcal{L}V$, where $V = (\rho, \varphi)$.

The first part of the linear analysis involves finding unstable eigenmodes for (3.3). These are non-trivial solutions to the equation that can be written as $V(t, x, y) = e^{\sigma t}e^{iky}v(x)$ for $k \neq 0$ and $\mathrm{Re}(\sigma) > 0$. This form inspires us to write (3.3) using the Fourier transform on the transverse variable $y$ : the equation $\partial_t V = J\mathcal{L}V$ becomes $\sigma v = JL(k)v$ with

$$L(k) = \begin{pmatrix} -\partial_x(K(\rho_c)\partial_x) + K(\rho_c)k^2 - m & (u_c - c)\partial_x \\ -\partial_x((u_c - c)\cdot) & -\partial_x(\rho_c\partial_x) + \rho_c k^2 \end{pmatrix}.$$

We begin by examining the existence of eigenmodes and the behaviour of $\sigma$ depending on $k$, and we follow up with an important resolvent estimate for $JL(k)$.

**Proposition 2.1.** *Properties of the linearised equation.*

*(a) The linearised equation is unstable, that is there exist eigenmodes written as $V(t, x, y) = e^{\sigma t}e^{iky}v(x)$, with $v \in H^2(\mathbb{R})$ and $\mathrm{Re}(\sigma) > 0$, that solve (3.3). For each $k$, the dimension of the subspace of unstable solutions of $\sigma v = JL(k)v$ is at most 1.*

*The instability is localised in the transverse Fourier space : there exists $k_{\max} > 0$ such that, for $|k| \geq k_{\max}$, eigenvalues necessarily satisfy $\mathrm{Re}(\sigma) \leq 0$. Let $\sigma(k)$ be the eigenvalue of $JL(k)$ with highest real part. Then the function $k \mapsto \mathrm{Re}(\sigma(k))$ has a global maximum $\sigma_0 > 0$ at a certain $k_0 > 0$.*

*(b) If* $V(t,x,y) = e^{iky}U(t,x)$*, we define the following semi-norm for* $U$ *:*

$$\|U\|^2_{X^j_k} = \|U_1\|^2_{H^{j+1}(\mathbb{R})} + \|\partial_x U_2\|^2_{H^j(\mathbb{R})} + |k|^2 \|U\|^2_{H^j(\mathbb{R})}.$$

*It is essentially the* $H^j$ *norm of* $|k|U$ *plus the* $H^{j+1}$ *norm of* $U$*, omitting the* $L^2$ *norm of* $U_2$*.*

*Set* $\gamma > \sigma_0$*,* $n > 0$*,* $s \in \mathbb{N}$ *and let* $U$ *solve*

$$\partial_t U(t,x) = JL(k)U(t,x) + F(t,x,k), \tag{3.4}$$

*with* $U(0,x) = 0$*, and,* $F$ *satisfying, for every* $j \leq s$*,*

$$\left\|\partial_t^{s-j} F(k,t)\right\|_{H^{j+1}(\mathbb{R})} \leq M_s \frac{e^{\gamma t}}{(1+t)^n}, \tag{3.5}$$

*uniformly for* $|k| \leq k_{\max}$ *(*$M_s$ *does not depend on* $k$*). Then* $U$ *satisfies a similar estimate : for every* $j \leq s$*, and uniformly for* $|k| \leq k_{\max}$*, we have*

$$\left\|\partial_t^{s-j} U(t)\right\|_{X^j_k} \leq C_s (1+t)^{-n} e^{\gamma t}. \tag{3.6}$$

**Consequences :** under the hypotheses of part (b), a quick energy estimate on the equation of $U_2$ yields that $U_2(t) \in L^2(\mathbb{R}^2)$ (as $\varphi(0) = 0$), and this $L^2$ norm also satisfies (3.6). We will therefore consider in the final parts that the result is valid in $H^s$, for any $s \geq 0$.

By the Parseval equality, this result also implies identical $H^s$ bounds for $(\rho, u)$ when written as $(\rho, u) = (\rho, \nabla\varphi)$ and $(\rho, \varphi) = \int_{\mathbb{R}} f(y) e^{iky} U(t,x)\, dk$, with $f \in \mathcal{C}_0^\infty(\mathbb{R})$. Indeed, norms of $|k|^2 U$ can be replaced, using the equation, by derivatives on $x$ and $t$ that satisfy (3.6).

## 2.1 Proof of Proposition 2.1 (a) Properties of eigenmodes

### 2.1.1 Existence of unstable eigenmodes.

The existence of unstable eigenmodes was shown by F.Rousset and N.Tzvetkov [104] using a general criterion for detecting transverse linear instability of solitary waves. We have seen that equation (3.3) for functions written as $V(t,x,y) = e^{\sigma t} e^{iky} v(x)$ becomes an eigenvalue problem, that is

$$\sigma v = JL(k)v \tag{3.7}$$

with $J$ a skew-symmetric matrix and $\mathcal{L}$ a self-adjoint differential operator on $(L^2(\mathbb{R}^2))^2$ whose domain is seen to be $(H^2(\mathbb{R}^2))^2$. We have the following property for such systems.

**Theorem 2.2.** *(Rousset and Tzvetkov, [104])*

*If* $L$ *has the following properties,*

- *(H1) there exists* $k_{\max} > 0$ *and* $\alpha > 0$ *such that* $L(k) \geq \alpha \mathrm{Id}$ *for* $|k| \geq k_{\max}$ *;*
- *(H2) for every* $k \neq 0$*, the essential spectrum of* $L(k)$ *is included in* $[\alpha_k, +\infty[$ *with* $\alpha_k > 0$ *;*
- *(H3)* $L'(k)$ *is a positive operator ;*
- *(H4) the spectrum of* $L(0)$ *consists of one isolated negative eigenvalue* $-\lambda$ *and a subset of* $\mathbb{R}^+$ *;*

*then there exist* $\sigma > 0$ *and* $k \neq 0$ *such that (3.7) has a non-trivial solution, and, for every unstable wavenumber* $k$*, such an eigenvalue* $\sigma$ *is unique.*

This is shown by finding $k' > 0$ such that $L(k')$ has a one-dimensional kernel, and by using the Lyapunov-Schmidt method in the vicinity of this point ; we do not detail the proof of this theorem. Proof that the linearised Euler-Korteweg system satisfies the hypotheses of this theorem was also done in [104], but we shall briefly recall this, as it contains some useful arguments for the subsequent points of Proposition 2.1 (a).

(H1) : using Young's inequality, $ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$, with $\delta = \frac{K(\rho_c)}{2}$, we quickly get that

$$(L(k)v, v) \geq \int_{\mathbb{R}} \frac{K(\rho_c)}{2}|\partial_x v_1|^2 + \left(k^2 - \frac{1}{2}\right)|v_1|^2 + \rho_c|\partial_x v_2|^2 + \left(k^2 - \frac{1}{K(\rho_c)}\right)|v_2|^2, \qquad (3.8)$$

which is greater than $\alpha \|U\|_{L^2}^2$ for $|k|$ large enough (remember that $\rho_c$ and $K(\rho_c)$ are positive).

(H2) : as $\lim_{x\to\pm\infty} Q_c(x) = Q_\infty = (\rho_\infty, u_\infty)$, with standard arguments [54], and using that $L(k)$ is self-adjoint, the essential spectrum of $L(k)$ is given by that of

$$L_\infty(k) = \begin{pmatrix} K(\rho_\infty)(-\partial_{xx}^2 + k^2) + g_0'(\rho_\infty) & (u_\infty - c)\partial_x \\ (c - u_\infty)\partial_x & \rho_\infty(-\partial_{xx}^2 + k^2) \end{pmatrix},$$

whose essential spectrum we can determine by using the Fourier transform in the $x$-variable, and explicitly writing the eigenvalues $\mu(\xi, k)$. We get that these are positive when $k \neq 0$. The essential spectrum of $L(k)$ is equal to that of $L_\infty(k)$, so (H2) is verified.

(H3) : we easily have $L'(k) = \mathrm{diag}(2kK(\rho_c), 2k\rho_c)$.

(H4) : computations lead to the following algebraic lemma and its application to $L(0)$.

**Lemma 2.3.** *Let* $L$ *be a symmetric operator on a Hilbert space such that*

$$L = \begin{pmatrix} L_1 & A \\ A^* & L_2 \end{pmatrix}$$

*with* $L_2$ *invertible. Then, we can write*

$$(Lv, v) = \left((L - AL_2^{-1}A^*)v_1, v_1\right) + \left(L_2(v_2 + L_2^{-1}A^*v_1), v_2 + L_2^{-1}A^*v_1\right).$$

As a result, we can write

$$(L(0)v,v) = (Mv_1,v_1) + \int_{\mathbb{R}} \rho_c \left| \partial_x v_2 + \frac{1}{\rho_c}(u_c - c)v_1 \right|^2 \, dx,$$

with $M = -\partial_x(K(\rho_c)\partial_x \cdot) - m - \frac{(u_c-c)^2}{\rho_c}$. $M$ is a second-order differential operator on which we can perform Sturm-Liouville analysis [36]. First, the essential spectrum of $M$ is included in $[\alpha, +\infty[$ with $\alpha > 0$; indeed $M$ is a perturbation of $M_\infty = -K(\rho_\infty)\partial^2_{xx} + g_0'(\rho_\infty) - \frac{(u_\infty-c)^2}{\rho_\infty}$, whose essential spectrum is positive under the assumption that $\rho_\infty g_0'(\rho_\infty) > (u_\infty - c)^2$. Next, the function $\rho_c'$ is in the kernel of $M$, and it has one zero, so by Sturm-Liouville theory, $M$ has a unique negative eigenvalue associated with an eigenfunction $U_1^-$. Setting $U_2^-$ such that $\partial_x v_2^- = \frac{-1}{\rho_c}(u_c - c)v_1^-$, we have a generalised (the second component is not in $L^2$) eigenfunction for $L(0)$. By using $H^2$ approximations of $U_2^-$, we see that $(L(0)v,v)$ can be negative with $U \in H^2$, confirming that $L(0)$ has one negative eigenvalue.

### 2.1.2 Localisation of instability and boundedness of unstable eigenvalues.

We start be taking the real part of the $L^2$ scalar product of (3.7) and $L(k)U$ : we get that $\mathrm{Re}(\sigma)(L(k)U,U) = 0$, as $J$ is skew-symmetric. The operator $L(k)$ satisfies (H1) of Theorem 2.2, so, if $\mathrm{Re}(\sigma) > 0$, we must have $0 = (L(k)U,U) \geq \alpha \|U\|^2_{L^2}$ for $|k| \geq k_{\max}$. Thus, the only function satisfying $\sigma U = JL(k)U$ with $\mathrm{Re}(\sigma) > 0$ and $|k|$ large is $U = 0$; there are no unstable eigenfunctions for $|k|$ large.

In order to get the boundedness of the unstable eigenvalues, we decompose $L(k)$ as follows : $L(k) = L_0(k) + L_1$ with

$$L_0(k) := \begin{pmatrix} -\partial_x(K(\rho_c)\partial_x) + K(\rho_c)k^2 - m_0 & 0 \\ 0 & -\partial_x(\rho_c\partial_x) + \rho_c k^2 \end{pmatrix},$$

where $-m_0(x) = \max\left(-m(x), \frac{1}{2}g_0'(\rho_\infty)\right)$. We compute the scalar product of (3.7) and $L_0(k)v$, and take the real part, which gives us

$$\mathrm{Re}(\sigma)(L_0(k)v,v) = \mathrm{Re}(JL_1v, L_0(k)v). \tag{3.9}$$

It is quickly noticed that there exists $\alpha > 0$ such that

$$\mathrm{Re}(\sigma)(L_0(k)v,v) \geq \alpha \mathrm{Re}(\sigma)\left(\|\partial_x v\|^2_{L^2} + k^2\|v\|^2_{L^2} + \|v_1\|^2_{L^2}\right). \tag{3.10}$$

We shall now bound $|\mathrm{Re}(JL_1v, L_0(k)v)|$ by the same norms as on the right. Note that

$$JL_1 = \begin{pmatrix} -\partial_x((u_c - c)\cdot) & 0 \\ m - m_0 & -(u_c - c)\partial_x \end{pmatrix}.$$

Computing the scalar product directly, and bounding the terms involving $\rho_c$ and $u_c$ in $L^\infty$, we get

$$\begin{aligned}|\mathrm{Re}(JL_1v, L_0(k)v)| \quad \leq \quad & C\left[(1+k^2)(|\mathrm{Re}(v_1, \partial_x v_1)| + |\mathrm{Re}(\partial_x v_2, v_2)|) + |\mathrm{Re}(v_1, \partial_x v_2)|\right. \\ & + \|\partial_x v_1\|_{L^2}^2 + k^2 \|v_1\|_{L^2}^2 + \|\partial_x v_2\|_{L^2}^2 \\ & \left. + |(v_1, \partial_{xx} v_1)| + |(v_1, \partial_{xx}^2 v_2)| + |(|k|v_1, |k|v_2)|\right]\end{aligned}$$

The top line vanishes by integration by parts, and, also by integrating by parts, the first two terms in the final line are repeats of terms in the second line. Finally, by using Young's inequality, the last term is bounded by $k^2 \|v\|_{L^2}^2$, thus proving that, combining with (3.9) and (3.10), there exists $C > 0$ such that

$$\mathrm{Re}(\sigma)\left(\|\partial_x v\|_{L^2}^2 + k^2 \|v\|_{L^2}^2 + \|v_1\|_{L^2}^2\right) \leq C\left(\|\partial_x v\|_{L^2}^2 + k^2 \|v\|_{L^2}^2 + \|v_1\|_{L^2}^2\right),$$

which implies that $\mathrm{Re}(\sigma)$ cannot be unbounded when positive. This ends the proof of part (a) of Proposition 2.1.

## 2.2 Proof of Proposition 2.1 (b) Resolvent estimate

The proof of part (b) is split in two. First, we get the result for $s = 0$; we bound the $X_k^0$ norm of $U$ by similar norms of $F$ by using the Laplace transform and spectral arguments. The case $s > 0$ is then obtained by induction on $s$, the number of total (time and space) derivatives.

### 2.2.1 The case $s = 0$

The proof of (3.6) at $s = 0$ relies on the Laplace transform, and is similar to the resolvent estimate proofs in [103] and [93]. Let $\sigma_0 < \gamma_0 < \gamma$. For $f(t)$, we denote by $\tilde{f}(\tau)$ the following Laplace transform :

$$\tilde{f}(\tau) := \Lambda_t f(\gamma_0 + i\tau) = \int_0^{+\infty} \exp(-(\gamma_0 + i\tau)t) f(t)\, dt.$$

Using the Laplace transform turns equation (3.4), $\partial_t U = JL(k)U + F$, into an eigenvalue problem :

$$(\gamma_0 + i\tau)\tilde{U}(\tau) = JL(k)\tilde{U}(\tau) + \tilde{F}(\tau). \tag{3.11}$$

As $\gamma_0 > \sigma_0$, $\gamma_0 + i\tau$ is not in the spectrum of $JL(k)$. Indeed, we can use the strategy employed to prove that hypothesis (H2) is satisfied to show that the essential spectrum of $JL(k)$ is embedded in $i\mathbb{R}$. Once again using the argument from [54], we can examine the spectrum of the Fourier transform in $x$ of $JL_\infty(k)$,

$$\mathcal{F}_x(JL_\infty)(\xi, k) = \begin{pmatrix} -i(u_\infty - c)\xi & \rho_\infty(\xi^2 + k^2) \\ -K(\rho_\infty)(\xi^2 + k^2) - g_0'(\rho_\infty) & -i(u_\infty - c)\xi \end{pmatrix},$$

which are the solutions of the equation, depending on $(\xi, k)$,

$$X^2 - 2i\xi(u_\infty - c)X + \rho_\infty K(\rho_\infty)(\xi^2 + k^2)^2 + \rho_\infty g_0'(\rho_\infty)k^2 + (\rho_\infty g_0'(\rho_\infty) - (u_\infty - c)^2)\xi^2 = 0.$$

Using the positiveness of $\rho_\infty$, $K(\rho_\infty)$ and condition (3.2), we get that the discriminant of this equation is negative for $(\xi, k) \neq 0$, and clearly the only eigenvalue at $(\xi, k) = (0, 0)$ is zero, so the essential spectrum of $JL(k)$ is imaginary.

As $\gamma_0 + i\tau$ is not in the spectrum of $JL(k)$ for any $\tau \in \mathbb{R}$, the norm of the resolvent $((\gamma_0 + i\tau)\mathrm{Id} - JL(k))^{-1}$ is uniformly bounded for $(\tau, k)$ in any compact subset of $\mathbb{R}^2$. It remains to show that, for $|k| \leq k_{\max}$, there exists the following bound for $|\tau|$ large.

**Lemma 2.4.** *If $\tilde{U}$ solves (3.11), then there exists $C,\ M > 0$ such that, for $|\tau| \geq M$,*

$$||\tilde{U}(\tau)||_{X_k^0} \leq C||\tilde{F}(\tau)||_{H^1}. \tag{3.12}$$

*Proof :* we consider the scalar product of (3.11) with $L(k)\tilde{U}$, and write

$$(\gamma_0 + i\tau)(L(k)\tilde{U}, \tilde{U}) = (\tilde{F}, L(k)\tilde{U}). \tag{3.13}$$

Note that $(L(k)\tilde{U}, \tilde{U}) = (L(0)\tilde{U}, \tilde{U}) + K(\rho_c)k^2||\tilde{U}_1||_{L^2}^2 + \rho_c k^2||\tilde{U}_2||_{L^2}^2$. Let us concentrate on the term $(L(0)\tilde{U}, \tilde{U})$. Using Lemma 2.3, we know that it is equal to

$$(L(0)\tilde{U}, \tilde{U}) = (M\tilde{U}_1, \tilde{U}_1) + \int_{\mathbb{R}} \rho_c \left| \partial_x \tilde{U}_2 + \frac{1}{\rho_c}(u_c - c)\tilde{U}_1 \right|^2 dx,$$

The operator $M$, which we remind the reader is equal to $-\partial_x(K(\rho_c)\partial_x \cdot) - m - \frac{(u_c - c)^2}{\rho_c}$, and whose quadratic form is defined on $H^1$, has one simple negative eigenvalue, as well as a one-dimensional kernel containing $\rho_c'$.

Recall $U^-$ the generalised eigenfunction corresponding to the negative eigenvalue of $L(0)$. We do not have $U_2^- \in L^2$, so we consider $U^- = (U_1^-, 0)$. We denote $U^0 = (\rho_c', 0)$, which is in the kernel of $M \otimes \mathrm{Id}$. We renormalise $U^-$ and $U^0$ so that their $L^2$ norms are equal to 1. Let $U^+$ be orthogonal to $U^-$ and $U^0$ ; we show that

$$(L(0)U^+, U^+) \geq \eta \left\| U^+ \right\|_{X_0^0} \tag{3.14}$$

for some $\eta > 0$. First, as $U_1^+$ is not in the kernel or the negative eigenspace of $M$, we have $(MU_1^+, U_1^+) \geq \alpha \left\| U_1^+ \right\|_{H^1}^2$, since the essential spectrum of $M$ is included in $[\alpha, +\infty[$. Thus, we already have

$$(L(0)U^+, U^+) \geq \alpha \left\| U_1^+ \right\|_{H^1}^2. \tag{3.15}$$

But this does not suffice to get the $X_0^0$ norm. Using the fact that $\rho_c$ is positive, there is a positive $\beta$ such that

$$\begin{aligned}\int_{\mathbb{R}} \rho_c \left|\partial_x U_2^+ + \frac{1}{\rho_c}(u_c - c)U_1^+\right|^2 \, dx &\geq \beta \left\|\partial_x U_2^+ + \frac{1}{\rho_c}(u_c - c)U_1^+\right\|_{L^2}^2 \\ &\geq \beta \left\|\partial_x U_2^+\right\|_{L^2}^2 + C\beta \left\|U_1^+\right\|_{L^2}^2 - 2C|(\partial_x U_2^+, U_1^+)|\end{aligned}$$

for some $C > 0$. We use Young's inequality on the final term with $\delta = \beta/2C$, thus there exists $C' \in \mathbb{R}$ such that

$$(L(0)U^+, U^+) \geq (\alpha + C') \left\|U_1^+\right\|_{H^1}^2 + \frac{\beta}{2} \left\|\partial_x U_2^+\right\|_{L^2}^2 .$$

If perchance $C'$ is negative, we add $\frac{|C'|}{\alpha} \times (3.15)$ to the above, and obtain that there does indeed exist $\eta > 0$ such that we have (3.14), and $(L(k)U^+, U^+) \geq \eta \left\|U^+\right\|_{X_k^0}^2$.

We now write the orthogonal decomposition in $L^2$ of the first component, $\tilde{U}_1 = aU_1^- + bU_1^0 + U_1^+$, and replace in (3.13). The eigenfunctions $U^-$ and $U^0$ are fixed, so their $H^1$ norms are given constants. On the right-hand side, using integration by parts and basic estimates including Young's inequality, we have

$$|(\tilde{F}, L(k)\tilde{U})| \leq \frac{\eta}{4} \left\|U^+\right\|_{X_k^0}^2 + C(||\tilde{F}||_{H^1}^2 + a^2 + b^2),$$

while on the left-hand side, we have

$$(L(k)\tilde{U}, \tilde{U}) \geq \eta(|k|^2||U^+||_{L^2}^2 + \left\|U^+\right\|_{X_0^0}^2) - C(a^2 + (|a| + |b|) \left\|U^+\right\|_{X_0^0}^2).$$

Taking the real part of (3.13) and moving the negative part of the above to the right-hand side and once again applying Young's inequality to absorb $\|U^+\|_{X_0^0}$, we get

$$\left\|U^+\right\|_{X_k^0}^2 \leq C(||\tilde{F}||_{H^1}^2 + a^2 + b^2). \tag{3.16}$$

To finish off, we take the dot product of (3.11) with $U^-$ and $U^0$. We quickly get

$$(\gamma_0 + i\tau)a = -(\tilde{U}, L(k)JU^-) + (\tilde{F}, U^-) \quad \text{and} \quad (\gamma_0 + i\tau)b = -(\tilde{U}, L(k)JU^0) + (\tilde{F}, U^0).$$

Integrating by parts as usual, we get

$$a^2 + b^2 \leq \frac{C}{\gamma_0^2 + |\tau|^2}(a^2 + b^2 + \left\|U^+\right\|_{X_k^0}^2 + ||\tilde{F}||_{H^1}^2).$$

We see that if $|\tau|$ is large enough, $a^2 + b^2$ can be absorbed. Combining with (3.16), we get the result. □

For the end of the proof of Proposition 2.1 (b), we start by using the Parseval equality in the following,

$$\int_0^T e^{-2\gamma_0 t} \|U(t)\|_{X_k^0}^2 \, dt = \int_0^T ||\tilde{U}(t)||_{X_k^0}^2 \, dt \leq C \int_0^T ||\tilde{F}(t)||_{H^1} \, dt = C \int_0^T e^{-2\gamma_0 t} \|F(t)\|_{H^1} \, dt.$$

Now recall assumption (3.5) : we have

$$\int_0^T e^{-2\gamma_0 t} \|U(t)\|_{X_k^0}^2 \, dt \leq C M_0 \int_0^T \frac{e^{2(\gamma-\gamma_0)t}}{(1+t)^\alpha} \, dt \leq C_0 \frac{e^{2(\gamma-\gamma_0)T}}{(1+T)^n}.$$

We inject this in the energy estimate on (3.4), that is

$$\frac{d}{dt}(\|U(t)\|_{X_k^0}^2) \leq C(\|U(t)\|_{X_k^0}^2 + \|F(t)\|_{H^1}^2),$$

and multiply the result by $e^{-2\gamma_0 t}$, integrate in time and we get the result. □

### 2.2.2 The induction for $s > 0$

The extension of Proposition 2.1 (b) to every $s \geq 0$ will follow the lines of the similar result on the linearised water-waves equation in [105] ; it is done with a double induction, double in the sense that one is embedded in the other.

The first induction is on $s$, the total number of derivatives. Set $s > 0$, and we assume that, for every $s' < s$ and $j \leq s'$, we have (3.6), that is

$$\left\| \partial_t^{s'-j} U(t) \right\|_{X_k^j} \leq C_{s'} \frac{e^{\gamma t}}{(1+t)^n}.$$

To get the wanted result, we must prove that, for every $0 \leq j \leq s$,

$$\|U\|_{\dot{X}_k}^2 := \left\| \partial_t^{s-j} \partial_x^j U \right\|_{\dot{H}^1}^2 + |k|^2 \left\| \partial_t^{s-j} \partial_x^j U \right\|_{L^2}^2 \leq C_s \frac{e^{2\gamma t}}{(1+t)^{2n}}, \tag{3.17}$$

where $\dot{H}^1$ is the usual homogeneous Sobolev norm on $\mathbb{R}$. This is done by induction on the number of space derivatives, $j$. Note that the $\dot{X}_k$ norm (semi-norm if $k = 0$) defined here is a sort of homogeneous Sobolev norm expressed in the Fourier space. Morally, at rank $s$ of the induction, we must get bounds for the $L^2$ norms of terms involving $s + 1$ derivatives.

Starting with $j = 0$, we are interested in the $X_k^0$ norm of $\partial_t^s U$. Simply differentiate the equation, (3.4), $s$ times with respect to $t$, and notice that $W(t) = \partial_t^s U(t) - \partial_t^s U(0)$ satisfies $\partial_t W = JL(k)W + G$, with $W|_{t=0} = 0$ and $\|G(t)\|_{H^{s+1}} \leq 2M_s(1+t)^{-n} e^{\gamma t}$ : we can re-use the case $s = 0$ shown above.

Now, let $j > 0$. To lighten the notations, we will write $V_{s,j} = \partial_t^{s-j}\partial_x^j V$. Note that we want to control the $\dot{X}_k$ norm of $U_{s,j}$, which means $s+1$ derivatives in total, of which $j+1$ space or Fourier derivatives. We apply $\partial_t^{s-j}\partial_x^j$ to the equation. This time, the derivatives do not commute with $JL(k)$, hence we consider

$$\partial_t U_{s,j} = JL(k)U_{s,j} + J[\partial_x^j, L(k)]U_{s-j,0} + F_{s,j} := JM_{s,j}(k)U + F_{s,j}.$$

We take the real part of the scalar product of this equation with $M_{s,j}(k)U$, which yields

$$\frac{1}{2}\frac{d}{dt}(U_{s,j}, L(k)U_{s,j}) = -\mathrm{Re}(U_{s+1,j}, [\partial_x^j, L(k)]U_{s-j,0}) + \mathrm{Re}(F_{s,j}, M_{s,j}(k)U).$$

To bound the second part of the right-hand side, we look more closely at the commutator term in $M(k)U$. We notice that there exist two sets of $L^\infty$ matrices $(m_i^1, m_i^2)_{0\leq i\leq j+1}$ such that

$$[\partial_x^j, L(k)]\partial_t^{s-j}U = \left(\sum_{i=0}^{j+1} m_i^1(x)U_{s-j+i,i}\right) + \left(\sum_{i=0}^{j-1} m_i^2(x)k^2 U_{s-j+i,i}\right). \tag{3.18}$$

We notice that all the terms, *except* the one with $i = j+1$, have a total of $s$ derivatives or less, and thus, using $k^2 \leq k_{\max}|k|$, they are controlled by our induction hypothesis. Integrating the terms of $(F_{s,j}, L(k)U_{s,j})$ involving $j+2$ space derivatives and using assumption (3.5) and Young's inequality with a parameter $\eta$ to be chosen later, we obtain that the right-hand side is bounded by

$$|(F_{s,j}, M_{s,j}(k)U)| \leq \frac{\eta}{2}\|U_{s,j}\|_{\dot{X}_k}^2 + C\frac{e^{2\gamma t}}{(1+t)^{2n}}.$$

To deal with the first term of the right-hand side, we notice that $U_{s+1,j}$ has $s+1$ derivatives, of which $j$ space derivatives, hence the $L^2$ norm of $U_{s+1,j}$ falls under our induction hypothesis. We can thus use Young's inequality and use decomposition (3.18) once again, and get

$$\frac{1}{2}\frac{d}{dt}(U_{s,j}, L(k)U_{s,j}) \leq \eta\|U_{s,j}\|_{\dot{X}_k}^2 + C\frac{e^{2\gamma t}}{(1+t)^{2n}}.$$

Finally, we integrate this in time, and recall (3.8) from the verification of the (H1) hypothesis of Theorem 2.2, which says that

$$(L(k)U_{s,j}, U_{s,j}) \geq \eta'\|U_{s,j}\|_{\dot{X}_k}^2 - C\|U_{s-1,j-1}\|_{L^2}^2,$$

in which the final term can be moved to the right-hand side and controlled by the induction hypothesis. Note that we can take the $L^2$ norm of $U_{s-1,j-1}$ since the number of time derivatives is preserved (we do not encounter the second component of $U$ which is not assumed to be in $L^2$). In total, we therefore have

$$\|U_{s,j}(T)\|_{\dot{X}_k}^2 \leq \frac{\eta}{\eta'}\int_0^T \|U_{s,j}(t)\|_{\dot{X}_k}^2\, dt + C\frac{e^{2\gamma t}}{(1+t)^{2n}}.$$

We choose $\eta$ in the Young inequalities above so that $\eta/\eta' \leq 2\gamma$, and the Grönwall lemma gives us (3.17) for the couple $(s,j)$. Both inductions are now complete. □

# 3 Nonlinear instability

In this part, $U = (\rho, u)$. Obtaining Theorem 1.1 relies on the construction of an approximate solution $U^{\mathrm{ap}}$ built around unstable eigenmodes of the linearised equation. In our case, this construction is classical and we will not write all the details of the calculations (see also, for instance, [47, 31, 102, 105]). Energy estimates must then be obtained on $U - U^{\mathrm{ap}}$ to ensure that the approximate solution is close enough to the exact solution for long enough to see the difference between $U^{\mathrm{ap}}$ and $Q_c$ reach an amplitude $\mathcal{O}(1)$. This is the more delicate part, as we remind the reader that only local existence is guaranteed by [13], thus we must also ensure that the solution $U$ still exists when the instability appears. The closing argument is standard for this method.

## 3.1 Construction and properties of the approximate solution

For a whole number $N$ independent of $\varepsilon$ to be chosen later, we will set

$$U^{\mathrm{ap}} = Q_c(t,x) + \sum_{j=1}^{N} \varepsilon^j U_j(t,x,y),$$

with $u_j$ potential : we set $V^{\mathrm{ap}} = (\rho^{\mathrm{ap}}, \varphi^{\mathrm{ap}})$, $V_j = (\rho_j, \varphi_j)$. This is expected to solve the Euler-Korteweg system leaving an error of order $\varepsilon^{N+1}$, as follows,

$$\left\{ \begin{array}{rcl} \partial_t \rho^{\mathrm{ap}} + \mathrm{div}\,(\rho^{\mathrm{ap}} \nabla \varphi^{\mathrm{ap}}) & = & \varepsilon^{N+1} R_1^{\mathrm{ap}} \\ \partial_t \varphi^{\mathrm{ap}} + \frac{1}{2}|\nabla \varphi^{\mathrm{ap}}|^2 + g_0(\rho^{\mathrm{ap}}) & = & K(\rho^{\mathrm{ap}})\Delta \rho^{\mathrm{ap}} + \frac{1}{2}K'(\rho^{\mathrm{ap}})|\nabla \rho^{\mathrm{ap}}|^2 + \varepsilon^{N+1} R_2^{\mathrm{ap}}, \end{array} \right.$$

thus, expanding $V^{\mathrm{ap}}$ in these equations and isolating the terms of order $\varepsilon$, we see that $V_j$ solves the linearised Euler-Korteweg equation around $Q_c$ with a source term :

$$\partial_t V_j = J\mathcal{L} V_j + R_j, \tag{3.19}$$

with $R_j$ containing nonlinear interaction terms between the $V_n$ with $n < j$, but with the sum on indices in each interaction term equal to $j$. For instance, in the equation on $\rho_2$, we will see $\mathrm{div}\,(\rho_1 \nabla \varphi_1)$ appear in $R_2$. In the equation on $\varphi_j$, we notice that there will be quadratic terms like the one we have mentioned, but also cubic interaction terms, in the sense that they involve the $V_{n_1}$, $V_{n_2}$ and $V_{n_3}$ with $n_1 + n_2 + n_3 = j$. Likewise, $R^{\mathrm{ap}}$ contains all the interaction terms whose sum of indices is greater than $N$. We do not detail the remainders any further.

Thus, we can construct $V_1$ as a wavepacket of unstable eigenmodes of the linearised equation. Recall that $k_0 > 0$ is the global maximum for the function

$$\tilde{\sigma} : k \mapsto \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(JL(k))\},$$

where $\sigma(JL(k))$ is the spectrum of the operator $JL(k)$. We assume that this maximum is non-degenerate (the method is easily adapted to the degenerate case, see [105]). The function $\tilde{\sigma}$ is continuous, hence we can define

$$V_1(t,x,y) = \int_{\mathbb{R}} f_1(k) e^{iky} e^{\sigma(k)t} v_1(k,x)\ dk,$$

with $f_1(k)$ smooth, even, equal to 1 in the vicinity of $k_0$ and supported in the set $\{k \mid \tilde{\sigma}(k) > 3\sigma_0/4\}$, and $w(t,x) = e^{\sigma(k)t} v_1(k,x)$ solving $\partial_t w = JL(k)w$. For any $s \geq 0$, we consider the quantity

$$\sum_{s'=0}^{s} \int_{\mathbb{R}} f_1^2(k)|k|^{2s'} C_{s,s'} e^{2\tilde{\sigma}(k)t}\ dk = \sum_{s'=0}^{s} \int_{\mathbb{R}} f_1^2(k)|k|^{2s'} \left\|v_1(k,t)\right\|^2_{H^{s-s'}(\mathbb{R})}\ dk = \left\|V_1(t)\right\|^2_{H^s(\mathbb{R}^2)},$$

the second equality being the Parseval equality. We use the Laplace method around the critical point $k_0$ to get that $\|V_1(t)\|^2_{H^s} \underset{t\to+\infty}{\sim} t^{-1/2} e^{2\sigma_0 t}$, so

$$\frac{1}{C_{1,s}} \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \leq \left\|V_1(t)\right\|_{H^s(\mathbb{R}^2)} \leq C_{1,s} \frac{e^{\sigma_0 t}}{(1+t)^{1/4}}. \tag{3.20}$$

We get estimates on $V_j$ by induction. Assume that $I = \mathrm{supp}(f_1)$ is made up of two separate intervals around $\pm k_0$, and we set

$$V_j(t,x,y) = \int_I \cdots \int_I v_j(k_1,\cdots,k_j;t,x) e^{ik_1 y} \cdots e^{ik_j y}\ dk_1 \cdots dk_j.$$

Assuming that, for every $n < j$,

$$\left\|v_n(k_1,\cdots,k_i;t)\right\|_{H^s} \leq C_n \exp[n(\tilde{\sigma}(k_1)+\cdots+\tilde{\sigma}(k_n))t], \tag{3.21}$$

we get that $v_j$ solves the linearised Fourier-transformed equation

$$\partial_t v_j(k_1,\cdots,k_j) = JL(k_1+\cdots+k_j)v(k_1,\cdots,k_j) + r_j(k_1,\cdots,k_j),$$

in which $\|r_j(k_1,\cdots,k_j)\|_{H^s} \leq C\exp[n(\tilde{\sigma}(k_1)+\cdots+\tilde{\sigma}(k_j))t]$ by the structure of the remainder (combination of products between $v_n$'s with $n<j$) and (3.21). We set $v_j|_{t=0} = 0$. Then, since, for $k \in I$, $\tilde{\sigma}(k) > 3\sigma_0/4$, the sum in the exponential is greater than $\sigma_0$, we can apply Proposition 2.1 (b) to get that $v_j(k_1,\cdots,k_j;t)$ satisfies (3.21). We then use Parseval's equality and the Taylor expansion of $\tilde{\sigma}$ around the critical point $k_0$ to write that, for some $\beta > 0$,

$$\begin{aligned}
\left\|V_j(t,x,y)\right\|^2_{H^s} &= \int_{jk\in jI} \left\| \int_{k_1+\cdots+k_j=jk} v_j(k_1,\cdots,k_j;t,x) e^{ijky}\ dk_1\cdots dk_{j-1} \right\|^2_{H^s} dk \\
&\leq \int_{jk\in jI} C e^{2(j\sigma_0 - j\beta(k-k_0)^2)t} \int_{\sum_{m=1}^j k_m = jk} e^{-2\beta\sum_{m=1}^j (k_m-k)^2 t}\ dk_1\cdots dk_{j-1}\ dk
\end{aligned}$$

Integrating these gaussian functions (remembering that $k_j = jk - \sum_{n=1}^{j-1} k_n$), and we get the desired inequality : for every $j \leq 1$,

$$\|V_j(t)\|_{H^s(\mathbb{R}^2)} \leq C_j \frac{e^{j\sigma_0 t}}{(1+t)^{j/4}}. \tag{3.22}$$

We now take a look at the remainder of the equation on $V^{\rm ap}$, $R^{\rm ap}$, which contains the interaction terms of the equation whose sum of indices is greater than $N$. Similarly to our proof of (3.22), we have

$$\left\|\varepsilon^{N+1} R^{\rm ap}(t)\right\|_{H^s} \leq \sum_{j=N+1}^{3N} C_j \frac{\varepsilon^j e^{j\sigma_0 t}}{(1+t)^{j/4}}, \tag{3.23}$$

and, in what follows, we will be interested in times for which the first term of this sum is dominant. Indeed, we set $T_\varepsilon^*$ such that

$$\frac{\varepsilon e^{\sigma_0 T_\varepsilon^*}}{(1+T_\varepsilon^*)^{1/4}} = \kappa,$$

for $0 < \kappa < 1$ to be chosen later. For times smaller than $T_\varepsilon^*$, the smaller powers of $\varepsilon(1+t)^{-1/2}e^{\sigma_0 t}$ are dominant. Indeed, set $t = T_\varepsilon^* - \tau$, and replace in (3.23). We have

$$\begin{aligned}\left\|\varepsilon^{N+1} R^{\rm ap}(T_\varepsilon^* - \tau)\right\|_{H^s} &\leq \left(\max_{j\in\{N+1,\cdots,3N\}} C_j\right) \sum_{j=N+1}^{3N} \kappa^j e^{-j\sigma_0\tau} \\ &\leq C_R \kappa^{N+1} e^{-(N+1)\sigma_0\tau},\end{aligned}$$

which, returning to the original time variable $t$, gives us, for $t \leq T_\varepsilon^*$,

$$\left\|\varepsilon^{N+1} R^{\rm ap}(t)\right\|_{H^s} \leq C_R \frac{\varepsilon^{N+1} e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/4}}. \tag{3.24}$$

## 3.2 Getting the instability

If $U$ is the solution of the Euler-Korteweg system (3.1) with the initial condition $U(0) = U^{\rm ap}(0)$, we will observe the instability by studying

$$\|U(t) - Q_c(t)\|_{L^2(\mathbb{R}^2)} \geq \|U^{\rm ap}(t) - Q_c(t)\|_{L^2(\mathbb{R}^2)} - \|U(t) - U^{\rm ap}(t)\|_{L^2(\mathbb{R}^2)}.$$

On one hand, we have $U^{\rm ap} - Q_c = \sum_{j=1}^N \varepsilon^j U_j$, and

$$\begin{aligned}\left\|\sum_{j=1}^N \varepsilon^j U_j\right\|_{L^2(\mathbb{R}^2)} &\geq \|\varepsilon U_1(t)\|_{L^2(\mathbb{R}^2)} - \sum_{j=2}^N \left\|\varepsilon^j U_j\right\|_{L^2(\mathbb{R}^2)} \\ &\geq C_1' \frac{\varepsilon e^{\sigma_0 t}}{(1+t)^{1/4}} - \sum_{j=2}^N C_j \frac{\varepsilon^j e^{j\sigma_0 t}}{(1+t)^{j/4}}\end{aligned}$$

by (3.20) and (3.22). Taking times smaller than $T_\varepsilon^*$, we can consider that the sum on the right behaves like $\varepsilon^2(1+t)^{-1}e^{2\sigma_0 t}$, and, replacing $t$ by $T_\varepsilon^* - \tau$, we write

$$\begin{aligned}\|(U^{\rm ap} - Q_c)(T_\varepsilon^* - \tau)\|_{L^2} &\geq \kappa\left[C_1' e^{-\sigma_0\tau} - \kappa C_2' e^{-2\sigma_0\tau}\right] \\ &\geq \kappa C_1' e^{-\sigma_0\tau}(1 - \frac{\kappa C_2'}{C_1'} e^{-\sigma_0\tau}).\end{aligned}$$

We notice that, for a given $C > 0$, there exists $\tau_C > 0$ such that, for $\tau \geq \tau_C$, $1 - Ce^{-\sigma_0\tau} \geq 1/2$, so we set $\tau_1 > 0$, independent of $\varepsilon$, such that, for $\tau \geq \tau_1$,

$$\|(U^{\rm ap} - Q_c)(T_\varepsilon^* - \tau)\|_{L^2(\mathbb{R}^2)} \geq \frac{\kappa C_1'}{2} e^{-\sigma_0\tau}. \tag{3.25}$$

On the other hand, we will use energy estimates to ensure that $\|U(t) - U^{\rm ap}(t)\|_{L^2(\mathbb{R}^2)}$ is small. We will readily use those shown by S.Benzoni, R.Danchin and S.Descombes in [13], which are obtained by considering the equation on $(G, z) = (G, u + iw)$, with $G$ a primitive of the function $\rho \mapsto \sqrt{K(\rho)/\rho}$ and $w = \nabla(G(\rho))$. The equation on $z$ is a Schrödinger-type equation, written as

$$\partial_t z + u \cdot \nabla z + i\nabla z \cdot w + i\nabla(\mathcal{A}(\rho)\text{div } z) = \mathcal{Q}(\rho),$$

with functions $\mathcal{A}$ and $\mathcal{Q}$ that we do not detail, while $G$ satisfies

$$\partial_t G + (u \cdot \nabla)G + \mathcal{A}(\rho)\text{div } u = 0.$$

The approximate solution satisfies a similar system with a remainder term which we will denote $\mathcal{R} = (\mathcal{R}_G, \mathcal{R}_z)$. We use tildes to designate the difference between the exact and approximate terms, *e.g.* $\tilde{u} = u - u^{\rm ap}$, $\tilde{\mathcal{A}} = \mathcal{A}(\rho) - \mathcal{A}(\rho^{\rm ap})$. The difference $(\tilde{G}, \tilde{z})$ satisfies the equation

$$\left\{\begin{array}{l} \partial_t \tilde{G} + u \cdot \nabla \tilde{G} + \tilde{u} \cdot \nabla G^{\rm ap} + \mathcal{A}(\rho)\text{div } \tilde{u} + \tilde{A}\text{div } u^{\rm ap} = -\mathcal{R}_G \\ \partial_t \tilde{z} + u \cdot \nabla \tilde{z} + \tilde{u} \cdot \nabla z^{\rm ap} + i\nabla \tilde{z} \cdot w + i\nabla z^{\rm ap} \cdot \tilde{w} \\ \qquad\qquad + i\nabla(\mathcal{A}(\rho)\text{div } \tilde{z}) + i\nabla(\tilde{\mathcal{A}}\text{div } z^{\rm ap}) = \tilde{Q} - \mathcal{R}_z. \end{array}\right. \tag{3.26}$$

The energy estimates in $H^s$ on this equation involve multiplying by an adequately chosen gauge $\psi_s$. In the potential case, which is our case, this gauge is $\mathcal{A}(\rho)^{s/2}$ (whereas in the non-potential case, a additional gauge is needed), and the weighted norm $\|\psi_s \Lambda^s \cdot\|_{L^2}$, in which $\Lambda^s$ is the standard Fourier multiplier for $s$ derivatives, is equivalent to the standard $H^s$ norm. We send the reader to article [13] for details, in particular part 3 where the reasons for the simpler norm is explained, and part 6 where energy estimates on the difference between a solution and a reference solution ((3.26) without the source term $\mathcal{R}$) are obtained.

Using the $H^s$ norms of $Q_c - Q_\infty$ as constants, the energy estimate reads

$$\frac{d}{dt}\|\tilde{z}\|_{H^s}^2 \leq C\|\tilde{z}\|_{H^s}(\|\tilde{G}\|_{L^2} + \|\tilde{z}\|_{H^s})(1 + \|\nabla z^{\rm ap}\|_{H^{s+1}})(1 + \|\nabla z^{\rm ap}\|_{H^{s-1}} + \|\tilde{z}\|_{H^s}) + C\|\mathcal{R}\|_{H^s}^2.$$

We rewrite this as follows : let $W = U - U^{\rm ap}$ ; there exists a polynomial function $Q$ such that, for $s > 0$ large enough (to handle the $L^\infty$ norms by Sobolev embedding),

$$\|W(t)\|_{H^s}^2 \leq \int_0^t Q(\|U^{\rm ap}(z) - Q_\infty\|_{H^s} + \|U(z)\|_{H^s}) \, \|W(z)\|_{H^s}^2 + \|R^{\rm ap}(z)\|_{H^s}^2 \; dz.$$

We will now choose $N$ to get the right growth in time for $W$, as well as $\kappa$ to get the existence up to $T_\varepsilon^*$ of the exact solution $U$. First of all, in the same way that we get (3.25), we note that

$$\begin{aligned} \|U^{\rm ap}(t) - Q_\infty\|_{H^s} &\leq \|Q - Q_\infty\|_{H^s} + \sum_{j=1}^N C_j \frac{\varepsilon^j e^{j\sigma_0 t}}{(1+t)^{j/4}} \\ &\leq \|Q - Q_\infty\|_{H^s} + \kappa \end{aligned}$$

when $t \leq T_\varepsilon^* - \tau_2$, with $\tau_2 \geq \tau_1$ independent of $\varepsilon$. We consider times $t \leq T_W$ so that $\|W(t)\|_{H^s} \leq 1$ and $\rho(t,x,y) > 0$ (no vacuum on the exact solution), and choose $N$ so that, for $t \leq T_W$,

$$2N\sigma_0 > Q(\|Q - Q_\infty\|_{H^s} + 1 + \kappa).$$

A variant of the Grönwall inequality (from [95], given as Lemma 3.4 in chapter 1) then provides us with

$$\|W(t)\|_{H^s}^2 \leq C \frac{\varepsilon^{N+1} e^{2(N+1)\sigma_0 t}}{(1+t)^{(N+1)/2}} \tag{3.27}$$

for $t \leq T_W$. Now, take $t = T_\varepsilon^* - \tau$ : we notice that the right-hand side is smaller than $C(N)\kappa^{2(N+1)}$, which is therefore smaller than $\kappa$ if $\kappa < 1$ is small enough. We now choose $\kappa$ so that $2\kappa < \min \rho_c$, to ensure that there is no vacuum. We therefore have $T_W \geq T_\varepsilon^*$ by a bootstrap argument. So, (3.27) is valid for $t = T_\varepsilon^* - \tau$ with $\tau \geq \tau_2$, and we have

$$\|W(T_\varepsilon^* - \tau)\|_{H^s} \leq C_0' \kappa^{N+1} e^{-(N+1)\sigma_0 \tau}. \tag{3.28}$$

Recall that $f_1$, which is the localising function for the unstable term $U_1$, is supported in $I$ which is made up of two closed intervals that do not contain 0. Set $f$ a smooth function such that $f(k) = 1$ for $k \in I$ and $f = 0$ in the vicinity of 0, and let us define $\Pi$, a Fourier projector on frequencies in $I$, by

$$\mathcal{F}_y(\Pi u)(x,k) = f(k)(\mathcal{F}_y u)(x,k).$$

As $U(0,x,y) = Q_c(x) + \varepsilon U_1(0,x,y)$, $\Pi U|_{t=0} = U_0$. Moreover, for any $a \in \mathbb{R}$, the difference $Q_c(x - ct - a) - Q_c(x - ct)$ does not depend on $y$, hence $\Pi(Q_c(\cdot - ct - a) - Q_c(\cdot - ct)) = 0$. We can now combine (3.25) and (3.28) to show the instability : we have, for any $a$,

$$\begin{aligned} \|U(t) - Q_c(\cdot - a - ct)\|_{L^2(\mathbb{R}^2)} \, |_{t=T_\varepsilon^* - \tau} &\geq \|\Pi(U(t) - Q_c(\cdot - ct)\|_{L^2(\mathbb{R}^2)} \, |_{t=T_\varepsilon^* - \tau} \\ &\geq \frac{\kappa C_1'}{2} e^{-\sigma_0 \tau} \left[ 1 - \frac{2\kappa^N C_0'}{C_1'} e^{-N\sigma_0 \tau} \right], \end{aligned}$$

For $\tau \geq \tau_3 \geq \tau_2$, we have the last exponential on the right smaller than $1/2$, and, as a result, letting $\tau' \geq \tau_3$ be fixed, independent of $\varepsilon$,

$$\left\|(U - Q_c)(T_\varepsilon^* - \tau')\right\|_{L^2} \geq \frac{\kappa C_1'}{4} e^{-\sigma_0 \tau'} := \delta_0.$$

The number $\delta_0$ is fixed, positive and does not depend on $\varepsilon$ : Theorem 1.1 is proved. $\square$

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## Résumé

La présente thèse traite de deux questions de stabilité en mécanique des fluides.

Les deux premiers résultats de la thèse sont consacrés au problème de la limite non-visqueuse pour les équations de Navier-Stokes. Il s'agit de déterminer si une famille de solutions de Navier-Stokes dans un demi-espace avec une condition de Navier au bord converge vers une solution du modèle non visqueux, l'équation d'Euler, lorsque les paramètres de viscosité tendent vers zéro.

Dans un premier temps, on considère le modèle incompressible 2D. Nous obtenons la convergence dans $L^2$ des solutions faibles de Navier-Stokes vers une solution forte d'Euler, et une instabilité dans $L^\infty$ en temps très court pour certaines données initiales qui sont des solutions stationnaires de l'équation d'Euler. Ces résultats ne sont pas contradictoires, et on construit un exemple de donnée initiale permettant de voir se réaliser les deux phénomènes simultanément dans le cadre périodique.

Dans un second temps, on s'intéresse au modèle compressible isentropique (température constante) en 3D. On démontre l'existence de solutions dans des espaces de Sobolev conormaux sur un temps qui ne dépend pas de la viscosité lorsque celle-ci devient très petite, et on obtient la convergence forte de ces solutions vers une solution de l'équation d'Euler sur ce temps uniforme par des arguments de compacité.

Le troisième résultat de cette thèse traite d'un problème de stabilité d'ondes solitaires. Précisément, on considère un fluide isentropique et non visqueux avec capillarité interne, régi par le modèle d'Euler-Korteweg, et on montre l'instabilité transverse non-linéaire de solitons, c'est-à-dire que des perturbations 2D initialement petites d'une solution sous forme d'onde progressive 1D peuvent s'éloigner de manière importante de celle-ci.

## Abstract

This thesis deals with a couple of stability problems in fluid mechanics.

In the first two parts, we work on the inviscid limit problem for Navier-Stokes equations. We look to show whether or not a sequence of solutions to Navier-Stokes in a half-space with a Navier slip condition on the boundary converges towards a solution of the inviscid model, the Euler equation, when the viscosity parameters vanish.

First, we consider the 2D incompressible model. We obtain convergence in $L^2$ of weak solutions of Navier-Stokes towards a strong solution of Euler, as well as the instability in $L^\infty$ in a very short time of some initial data chosen as stationary solutions to the Euler equation. These results are not contradictory, and we construct initial data that allows both phenomena to occur simultaneously in the periodic setting.

Second, we look at the 3D isentropic (constant temperature) compressible equations. We show that solutions exist in conormal Sobolev spaces for a time that does not depend on the viscosity when this is small, and we get strong convergence towards a solution of the Euler equation on this uniform time of existence by compactness arguments.

In the third part of the thesis, we work on a solitary wave stability problem. To be precise, we consider an isentropic, compressible, inviscid fluid with internal capillarity, governed by the Euler-Korteweg equations, and we show the transverse nonlinear instability of solitons, that is that initially small 2D perturbations of a 1D travelling wave solution can end up far from it.