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Glossary

General notations & functional spaces

$d \geqslant 1$	Dimension of the space domain.
$T > 0$	Maximal time of the study.
$]0, T[$	Time interval of the study.
D	A bounded Lipschitz domain of $\mathbb{R}^d$ with a Lipschitz boundary if $d \geqslant 2$.
∂D	The Lipschitz boundary of D for $d \geqslant 2$.
Σ	The space product $]0, T[\times\partial D$.
Ω	The probability space.
P	The probability measure defined on Ω.
E	The expectation, *i.e.* the integral on Ω with respect to the measure P.
$\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$	A countably generated filtration with $\mathcal{F}_0$ containing the negligible sets.
$(\Omega, \mathcal{F}, P)$	The classical Wiener space.
$\mathcal{P}_T$	The predictable σ-field generated by the sets $\{0\} \times \mathcal{F}_0$ and $]s, t] \times \mathcal{A}$, $\mathcal{A} \in \mathcal{F}_s$.
w	A standart adapted one-dimensional continuous Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with $w_0 = 0$, also denoted $\{w_t, \mathcal{F}_t, 0 \leqslant t \leqslant T\}$.
X	A separable Banach space.
$N_w^2(0, T, X)$	The separable Banach space of the predictable X-valued processes endowed with the norm $\|\phi\|_T := \left[E \int_0^T \|\phi\|_X^2 ds\right]^{\frac{1}{2}}$.

$\mathcal{D}(\mathbb{R}^d)$	The space of $\mathcal{C}^\infty(\mathbb{R}^d)$-functions with compact support in $\mathbb{R}^d$.
$\mathcal{D}(D)$	The space of $\mathcal{C}^\infty(D)$-functions with compact support in D.
$H^1(\mathbb{R}^d)$	The closure of $\mathcal{D}(\mathbb{R}^d)$.
$H^1_0(D)$	The closure of $\mathcal{D}(D)$.
$BV(D)$	The space of integrable function with bounded variation on D.
G	A subset of $\mathbb{R}^d$.
$\mathcal{D}(G)$	The restriction to G of $\mathcal{D}(\mathbb{R}^d)$ functions u such that $\mathrm{supp} u \cap G$ is compact.
$\int_0^\cdot \varphi(s)dw(s)$	The Itô integral of $\varphi \in N^2_w(0,T,X)$ belonging to $\mathcal{C}\big([0,T];L^2(\Omega,X)\big)$.
$\mathrm{supp}(\phi)$	The support of any function ϕ.
$\epsilon > 0$	Parameter of the parabolic regularizations.
k	An arbitrary real.
n,m,l	Positive integers.
$0 < \tilde{\delta} < \delta$	Real parameters.
s,t	Points of the time interval $]0,T[$.
x,y	Elements of $\mathbb{R}^d$.
α,β	Reals of the interval $]0,1[$.
p	For convenience, equal to (t,x,α).
q	For convenience, equal to (s,y,β).

Time discretization

N	A positive integer.
n,k	Arbitrary integers of $\{0,\dots,N\}$.
$\Delta t, \Delta$	The step of the time discretization equal to $\frac{T}{N}$.
t_n	Point of the time discretization equal to $n\Delta t = n\Delta$.
w^n	The Brownian motion at time t_n, $w(t_n)$.

Chapter I

Q	The product space $]0,T[\times D$.
u_0	The initial condition of Problems (1.1) and (1.4) in $H^1_0(D)$.
f	A bi-Lipschitz-continuous function defined on $\mathbb{R}$, increasing with $f(0)=0$.
h	An element of $\mathcal{N}^2_w(0,T,H^1_0(D))$.
$\mathcal{H}$	A Lipschitz-continuous mapping with $\mathcal{H}(0)=0$ defined over $H^1_0(D)$.
a	The bilinear application $(u,v)\in H^1_0(D)^2\mapsto\int_D\nabla u\nabla v dx\in\mathbb{R}$.
u^n	The element of $L^2\big((\Omega,\mathcal{F}_{t_n}),H^1_0(D)\big)$ used for the approximation of u.
h^n	The element of $L^2\big((\Omega,\mathcal{F}_{t_n}),H^1_0(D)\big)$ defined by $\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}h(s)ds$.
B_n	The element of $L^2\big((\Omega,\mathcal{F}_{t_n}),H^1_0(D)\big)$ defined by $\displaystyle\sum_{k=0}^{n-1}(w^{k+1}-w^k)h^k$.
U^{n+1}	For convenience, $u^{n+1}-\displaystyle\sum_{k=0}^{n}(w^{k+1}-w^k)h^k$, for any n in $\mathbb{N}$.
$u^{\Delta t}$	The simple function defined by $\displaystyle\sum_{k=0}^{N-1}u^{k+1}1_{[t_k,t_{k+1}[}$.
$\widetilde{u}^{\Delta t}$	The affine function defined by $\displaystyle\sum_{k=0}^{N-1}\left[\frac{u^{k+1}-u^k}{\Delta t}(.-t_k)+x_k\right]1_{[t_k,t_{k+1}[}$.
$h^{\Delta t}$	The element of $\mathcal{N}^2_w\left(0,T,H^1_0(D)\right)$ defined by $\displaystyle\sum_{k=0}^{N-1}h^{k+1}1_{[t_k,t_{k+1}[}$.
$h_{\Delta t}$	The element of $\mathcal{N}^2_w\left(0,T,H^1_0(D)\right)$ defined by $\displaystyle\sum_{k=0}^{N-1}h^k1_{[t_k,t_{k+1}[}=h^{\Delta t}(.-\Delta t)$.
$B^{\Delta t}$	The simple function defined by $\displaystyle\sum_{k=0}^{N-1}\sum_{n=0}^{k}(w^{n+1}-w^n)h^n1_{[t_k,t_{k+1}[}$.
$\widetilde{B}^{\Delta t}$	The affine function defined by $\displaystyle\sum_{k=0}^{N-1}\left[\frac{w^{k+1}-w^k}{\Delta t}h^k(.-t_k)+B_k\right]1_{[t_k,t_{k+1}[}$.
$(.,.)$	The scalar product in $L^2(D)$.
$\Vert.\Vert$	The norm in $L^2(D)$, $\Vert.\Vert_{L^2(D)}$ associated to the scalar product $(.,.)$.
h_n	A regularization of h in $\mathcal{N}^2_w(0,T,\mathcal{D}(D))$, for any n in $\mathbb{N}$.
u_n	The solution of Problem (1.1) with the regularization h_n, for any n in $\mathbb{N}$.
U_n	For convenience, $u_n-\displaystyle\int_0^t h_n dw$, for any n in $\mathbb{N}$.

Chapter II

H	A Hilbert space, in addition separable in the stochastic case.
H'	The dual space of H.
V	A reflexive separable Banach space.
V'	The dual space of V.
$(.,.)$	The scalar product in H.
$\lvert . \rvert$	The norm in H.
$\langle .,. \rangle$	The dual product $V' - V$.
$\lVert . \rVert$	The norm in V.
u_0	The initial condition for Problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ in V.
F	A continuous, Gâteaux-differentiable and proper convex function defined from H to $\mathbb{R}$.
f	The operator of Problem $(\mathcal{P}_1)$ defined from H to H' by $f = \partial F$ and in addition strongly monotone for Problem $(\mathcal{P}_2.)$
J	A continuous, Gâteaux-differentiable and proper convex function defined from V to $\mathbb{R}$ and in addition satisfying $J(u) > 0 = J(0)$ if $u \neq 0$ for Problem $(\mathcal{P}_2)$.
A	The operator of Problem $(\mathcal{P}_1)$ defined from V to V' by $A = \partial J$ and in addition linear for Problem $(\mathcal{P}_2)$.
g	An element of $L^2(0,T,H)$ associated to Problem $(\mathcal{P}_1)$.
h	An element of $\mathcal{N}_w^2(0,T,V)$ associated to Problem $(\mathcal{P}_2)$.
$\mathcal{H}$	A Lipschitz-continuous mapping over V associated to Problem $(\mathcal{P}_{\mathcal{H}})$.
δ	A positive real parameter.
φ_1	An operator defined over V by $\varphi_1(u) = \delta\lvert u\rvert^2 + J(u)$.
φ_2	An operator defined over V by $\varphi_2(u) = F(u) + J(u)$.
φ_3	An operator defined from V to $\mathbb{R}$ by $\varphi_3(u) = \Delta t F(\frac{u-u^n}{\Delta t}) + J(u) - (g^{n+1}, u)$.
$W^{1,p,q}(0,T,V,H)$	The space of functions $u \in L^p(0,T,V)$ with $\partial_t u \in L^q(0,T,H)$.
g^n	An approximation of g defined by $\dfrac{1}{\Delta t}\displaystyle\int_{t_n}^{t_{n+1}} g(s)ds$.
$(.,.)_A$	A scalar product in V defined by $\langle A.,. \rangle$.
$\lVert . \rVert_A$	A norm in V, defined by $\sqrt{\langle A.,. \rangle}$.

Chapter III

Q	The product space $]0,T[\times\mathbb{R}^d$.
$\vec{\mathbf{f}}=(f_1,..,f_d)$	A Lipschitz-continuous function defined from $\mathbb{R}$ to $\mathbb{R}^d$ with $\vec{\mathbf{f}}(0)=\mathbf{0}$.
h	A Lipschitz-continuous function defined from $\mathbb{R}$ to $\mathbb{R}$ with $h(0)=0$.
u_0	The initial condition of Problem (0.1) in $L^2(\mathbb{R}^d)$.
sgn_0	The sign function defined by $\mathrm{sgn}_0(x)=\frac{x}{\lvert x\rvert}$ if $x\neq 0$ and 0 otherwise.
$\mathcal{E}$	The set of $C^{2,1}(\mathbb{R})$ nonnegative convex approximations of the absolute-value function vanishing in zero.
η,η_δ	Members of $\mathcal{E}$.
η'_δ	A Lipschitz-continuous approximation of the sign function with $\eta'_\delta(0)=0$.
η''_δ	A Lipschitz-continuous function with $\mathrm{supp}\eta''_\delta\subset]\text{-}\delta,\delta[$ and $\lVert\eta''_\delta\rVert_\infty\leqslant\frac{c}{\delta}$, $c>0$.
F(.,.)	The function defined on $\mathbb{R}^2$ by $F(a,b)=\mathrm{sgn}_0(a-b)[\vec{\mathbf{f}}(a)-\vec{\mathbf{f}}(b)]$.
$F^\eta(.,.)$	The function defined on $\mathbb{R}^2$ by $F^\eta(a,b)=\int_b^a\eta'(\sigma-b)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma$.
u_0^ϵ	The initial condition of Problem (2.1) in $\mathcal{D}(\mathbb{R}^d)$.
u_ϵ	The viscous solution of Problem (2.1).
u	Any entropy solution of Problem (0.1).
$\mathbf{u},\hat{\mathbf{u}}$	Any measure-valued entropy solutions of Problem (0.1).
$\mu_{\eta,k}$	The distribution in $\mathbb{R}^{d+1}$ defined for any real k and smooth function η p.63.
K	A compact set of $\mathbb{R}^d$.
φ	For the local Kato inequality, a positive function of $\mathcal{D}([0,T]\times\mathbb{R}^d)$ with $\mathrm{supp}\varphi(t,.)\subset K$.
ρ_n	An usual mollifier sequence in $\mathbb{R}$ with $\mathrm{supp}\rho_n\subset[\text{-}\frac{2}{n},0]$.
ρ_m	An usual mollifier sequence in $\mathbb{R}^d$ with $\mathrm{supp}\rho_m\subset\mathcal{B}(0,\frac{1}{m})$.
ρ_l	An usual mollifier sequence in $\mathbb{R}$ with $\mathrm{supp}\rho_l\subset[\text{-}\frac{1}{l},\frac{1}{l}]$.
$\mathcal{B}_k^l$	For convenience, equal to $\rho_l(u_\epsilon(s,y)-k)$.
$\mathcal{A}_k^l$	For convenience, equal to $\rho_l(\hat{\mathbf{u}}(p)-k)$.

Chapter IV

$\vec{\mathbf{f}} = (f_1, .., f_d)$	A Lipschitz-continuous function defined from $\mathbb{R}$ to $\mathbb{R}^d$ with $\vec{\mathbf{f}}(0) = \mathbf{0}$.
h	A Lipschitz-continuous function defined on $\mathbb{R}$ with $h(0) = 0$.
u_0	The initial condition of Problem (0.1) in $L^2(D)$.
$(.)^+$	The positive-part function defined by $(x)^+ = 0$ if $x \leqslant 0$, $(x)^+ = x$ otherwise.
$\mathcal{E}^+$	The set of $C^{2,1}(\mathbb{R})$ nonnegative convex approximations of the positive-part function vanishing in zero.
$\mathcal{E}^-$	The set of functions η such that $\check{\eta} \in \mathcal{E}^+$.
η_δ	A typical element of $\mathcal{E}^+$ defined p.108.
η	An element of $\mathcal{E}^+ \cup \mathcal{E}^-$.
$\check{\eta}$	The function defined by $\check{\eta}(x) = \eta(-x)$ for any real x.
$\mathbb{A}^+$	The set $\{(k, \varphi, \eta) \in \mathbb{R} \times \mathcal{D}^+(\mathbb{R}^{d+1}) \times \mathcal{E}^+,\ k < 0 \Rightarrow \varphi \in \mathcal{D}^+([0,T] \times D)\}$.
$\mathbb{A}^-$	The set $\{(k, \varphi, \eta),\ (-k, \varphi, \check{\eta}) \in \mathbb{A}^+\}$.
$\mathbb{A}$	The union of sets $\mathbb{A}^+ \cup \mathbb{A}^-$.
sgn_0^+	The function defined by $\mathrm{sgn}_0^+(x) = 1$ if $x > 0$ and 0 else.
sgn_0^-	The function defined by $\mathrm{sgn}_0^-(x) = -\mathrm{sgn}_0^+(-x)$.
sgn_0	The function defined by $\mathrm{sgn}_0 = \mathrm{sgn}_0^+ + \mathrm{sgn}_0^-$.
F(.,.)	The function defined on $\mathbb{R}^2$ by $F(a,b) = \mathrm{sgn}_0(a-b)[\vec{\mathbf{f}}(a) - \vec{\mathbf{f}}(b)]$.
$F^{+(-)}(.,.)$	The function defined on $\mathbb{R}^2$ by $F^{+(-)}(a,b) = \int_b^a \mathrm{sgn}_0^{+(-)}(\sigma - b)\vec{\mathbf{f}}'(\sigma)\, \mathrm{d}\sigma$.
$F^\eta(.,.)$	The function defined on $\mathbb{R}^2$ by $F^\eta(a,b) = \int_b^a \eta'(\sigma - b)\vec{\mathbf{f}}'(\sigma)\, \mathrm{d}\sigma$.
$\mu_{\eta,k}$	The distribution in $\mathbb{R}^{d+1}$ defined for any real k and smooth function η p.110.
u_0^ϵ	The initial condition of Problem (2.3) in $\mathcal{D}(D)$.
u_ϵ	The viscous solution of Problem (2.3).
u	Any entropy solution of Problem (0.1).
$\mathbf{u}, \hat{\mathbf{u}}$	Any measure-valued entropy solutions of Problem (0.1).
$(\mathcal{B}_i)_{i=0,\cdots,k}$	Balls of the covering of $\overline{D}$.
B	Equal to $\mathcal{B}_i$ for some i in $\{0, \cdots, k\}$.

φ	For the Local Kato inequality, a function of $\mathcal{D}^+([0,T[\times D)$. For the Global Kato inequality, a function of $\mathcal{D}^+([0,T[\times\mathbb{R}^d)$, with $\text{supp}\varphi(t,.)\subset B$.
ρ_n	An usual mollifier sequence in $\mathbb{R}$ with $\text{supp}\rho_n\subset[-\frac{2}{n},0]$.
ρ_m	A shifted mollifier sequence in $\mathbb{R}^d$.
ρ_l	An usual mollifier sequence in $\mathbb{R}$ with $\text{supp}\rho_l\subset[-\frac{2}{l},0]$.
$\theta_m(y)$	The sequence defined for $y\in\mathbb{R}^d$ by $\displaystyle\int_D\rho_m(x-y)dx$.
$\sigma_n(s)$	The sequence defined for $s\in\mathbb{R}$ by $\displaystyle\int_0^T\rho_n(t-s)dt$.
$\psi^k_{\delta,\widetilde{\delta}}(x)$	For convenience, equal to $\eta_\delta(k-\eta_{\widetilde{\delta}}(x))+\eta_\delta(-x)$ for the first inequality (3.3) and to $\eta_\delta(\eta_{\widetilde{\delta}}(x)-k)$ for the second inequality (3.4).
$\widetilde{\mathcal{B}}^l_k$	For convenience, equal to $\rho_l(\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k)$.
$\check{\mathcal{A}}^l_k$	For convenience, equal to $\rho_l(k-\hat{\mathbf{u}}(p))$.

Chapter V

$\mathcal{E}$, η, F^η, $\mu_{\eta,k}$	See notations of Chapter IV.
Q	The space product $]0,T[\times D$.
A	Any P-measurable set.
$\Vert.\Vert$	The usual norm in $L^1(D)$, $\Vert.\Vert_{L^1(D)}$.
$\vec{\mathbf{f}}$	A Lipschitz-continuous function defined from $\mathbb{R}$ to $\mathbb{R}^d$ with $\vec{\mathbf{f}}(0)=\mathbf{0}$.
h	A Lipschitz-continuous function defined on $\mathbb{R}$ with $h(0)=0$.
$K_{\vec{\mathbf{f}}}$	The Lipschitz constant of $\vec{\mathbf{f}}$.
K_h	The Lipschitz constant of h.
$M>0$	A real such that $\text{supp}h\subset[-M,M]$.
u_0	The initial condition of Problem (1.1) in $L^\infty(D)\cap BV(D)$.
M_1	The constant equal to $max(M,\Vert u_0\Vert_{L^\infty D)})$.
M_2	The constant equal to $e^{cT}e^{K_{\vec{\mathbf{f}}}T}$, where c only depends on the geometry of ∂D.
$TV_x(u_0)$	The total variation of u_0 in D.

n	Any integer of $\{0,\ldots,N-1\}$.
x	An element of D.
t,s	Points of the time interval $[0,T]$.
$R(t,s)$	The operator which takes a number $\overline{u}$ to the solution u at time t of the o.d.e $du = h(u)dw$ and $u(t=s) = \overline{u}$, $s \leqslant t$.
$S(t-s)$	The operator which takes a function $u(x,s)$ to the weak solution u at time t of the conservation law $\partial_t u + \operatorname{div}\vec{\mathbf{f}}(u) = 0$ and $u(t=s) = u(x,s)$.
$(u^n)_{n\in\mathbb{N}}$	The sequence defined by $u^0(x) = u_0(x)$ and $u^{n+1}(x) = S(\Delta)R(t_{n+1},t_n)u^n(x)$.
$u^{\Delta}(t,x)$	The function equal to $u^n(x)$ if $t = t_n$ and to $R(t,t_n)u^n(x)$ if $t \in]t_n,t_{n+1}[$.
$u_-^{n+1}(x)$	For convenience, equal to $R(t_{n+1},t_n)u^n(x)$.
$\widetilde{u}(t,x)$	For convenience, equal to $S(t-t_n)R(t_{n+1},t_n)u^n(x) = S(t-t_n)u_-^{n+1}(x)$.

Introduction générale

Dans cette thèse, notre but est d'étudier des équations aux dérivées partielles (EDP) auxquelles nous introduisons l'aléatoire *via* une différentielle stochastique de type bruit blanc gaussien. En d'autres termes, l'aléatoire s'immisce dans l'EDP par le biais d'une intégrale stochastique, et se propage jusqu'à la solution qui est désormais un processus stochastique. On parle alors de perturbation stochastique, d'EDP bruitée, d'EDP stochastique ou encore de forçage aléatoire ou stochastique. L'étude de cette classe d'équations repose sur une théorie à la frontière entre l'analyse des EDP et les probabilités (calcul stochastique). L'enjeu dans ce travail étant, partant d'un bagage mathématique orienté analyse des EDP, d'adapter les techniques habituellement utilisées dans le cas déterministe (*i.e.* sans terme de probabilité) au cas des EDP stochastiques.

Dans ce travail, nous nous intéressons au bruitage d'EDP par l'introduction d'un objet stochastique particulier : l'intégrale d'Itô. Les problèmes étudiés dépendent alors d'une nouvelle variable (l'aléatoire) qui nous oblige à revisiter nos réflexes d'analystes d'EDP déterministes. Par exemple, du fait de cette dépendance en l'aléa, il devient délicat d'utiliser des résultats classiques de compacité dans les espaces de Sobolev à valeurs vectorielles. Cela s'avère relativement compliqué lorsque l'on considère des bruits multiplicatifs, *i.e.* lorsque la solution apparaît dans le terme de bruit. Nous verrons dans les différents chapitres les méthodes alternatives utilisées pour contourner ces difficultés. Dans la littérature, nous trouvons plusieurs motivations pour porter un intérêt particulier à l'étude de tels problèmes. Citons quelques idées conductrices. La première, que citent G. Díaz et J-I. Díaz [34] ou encore C. Prévôt et M. Röckner [64] est de rendre l'EDP stochastique pour obtenir des informations sur sa version déterministe. Il existe en effet des cas où sans l'ajout d'un bruit, on ne sait montrer l'unicité d'une solution et en étudier la stabilité (Díaz-Díaz [34]), ou en exhiber certaines caractéristiques (Prévôt-Röckner [64]). Est alors ajouté à l'EDP un bruit multiplié par un paramètre ϵ voué à tendre vers 0. Une deuxième motivation est d'intégrer de l'incertitude dans le modèle utilisé dans l'idée de mieux mimer la réalité que le modèle est censé représenter. La nature étant complexe et de nombreux phénomènes encore mal compris, un modèle purement déterministe est souvent incomplet pour bien décrire une situation réelle, soit par ce qu'il n'existe pas encore de modèle exact associé au phénomène à décrire, soit par ce qu'il manque des informations/données comme on peut le rencontrer lors de la modélisation d'un réservoir souterrain [43]. L'introduction d'un terme stochastique peut également s'avérer pertinent pour prendre en compte de manière efficace des phénomènes ayant lieu à des échelles non représentées [72]. Le choix d'ajouter un bruit compenserait ou du moins incorporerait au modèle toutes ces incertitudes. Un exemple est donné par A. Debussche, N. Glatt, R. Temam et M. Ziane [31]. Les auteurs s'intéressent à des modèles en océanographie et introduisent une perturbation stochastique pour tenir compte des propriétés de radiation des nuages dont la connaissance est incomplète ou incertaine. Notons également que certains phénomènes que l'on souhaite décrire par des modèles sont intrinsèquement non déterministes. De ce fait le choix d'un modèle stochastique semble plus cohérent. Une autre idée que l'on peut trouver dans la même référence [31] et qui rejoint la première : l'impossibilité de résoudre numériquement certains problèmes. L'introduction du stochastique dans le modèle s'avère alors utile pour fermer le système et pouvoir procéder aux calculs souhaités, on parle de “stochastic parametrization”. Cela parait déroutant à première vue de simplifier la résolution d'un problème en lui ajoutant un objet d'une complexité certaine. Enfin le stochastique peut être aussi directement incorporé dans le modèle. Citons l'exemple de M. Viot [80], où l'auteur s'intéresse à l'évolution de la fréquence d'apparition d'un gène à l'intérieur d'une population. Le bruit apparaît dans l'EDP de modélisation et traduit les effets aléatoires de transmission des caractères génétiques d'une génération à l'autre. Citons enfin le livre de Carmona-Rozovskii [21] où les auteurs rassemblent les perspectives données par l'étude d'EDP stochastiques dans différents domaines. Les domaines d'application sont nombreux : en physique, en chimie, en biologie (neurophysiologie, génétique des populations) et même en économie (finance). Dans notre cas, certaines équations étudiées (modèle de Barenblatt et lois de conservation) sont issues de l'étude d'écoulements multiphasiques en milieux poreux, dans le cadre de la modélisation de réservoirs.

Dans cette thèse, nous étudierons plusieurs familles d'EDP stochastiques. Nous considèrerons pour cela, un bruit blanc en temps Gaussien de type mouvement Brownien, noté w, adapté, continu, de dimension un et à valeurs réelles. Notons que ce choix n'est pas une restriction, ainsi un bruit de dimension finie pourrait être considéré dans les résultats proposés. Nous commencerons par l'étude d'un problème parabolique en domaine borné. Cela nous permettra d'introduire une méthode d'existence de solution reposant sur un schéma de semi-discrétisation en temps, ainsi que les principaux outils d'analyse stochastique utilisés dans ce mémoire. Ensuite nous regarderons ce genre d'équations dans un cadre plus abstrait. Puis, nous nous intéresserons à des problèmes de type hyperbolique. Nous étudierons le problème de Cauchy pour une loi de conservation stochastique, puis nous regarderons le problème de Dirichlet pour cette même loi. Suite à cela, nous considèrerons un schéma numérique pour approcher les solutions de lois de conservation stochastiques dans le cadre monodimensionnel. Enfin, nous proposerons quelques simulations numériques de solutions entropiques de l'équation de Burgers avec ou sans bruit, pour différentes conditions initiales.

Tout au long de la rédaction, nous mettrons en avant les difficultés rencontrées dans l'application des techniques déterministes, et les méthodes alternatives utilisées. Par souci de clarté, nous avons fait le choix de répéter dans chaque chapitre les notations et de rappeler aux endroits concernés les résultats utilisés plutôt que de référencer aux chapitres précédents. On se propose dans l'organisation de ce manuscrit de reprendre les articles élaborés en collaboration avec J. Giacomoni, G. Vallet et P. Wittbold, sous une forme plus détaillée et en version anglaise [11], [12], [14] et [13].

Dans le **Chapitre I**, nous nous intéresserons à une perturbation stochastique du modèle physique introduit par G. I. Barenblatt [10] traduisant la filtration d'un fluide élastique traversant un milieu poreux elasto-plastique. Nous étudierons le problème parabolique :

$$\begin{cases} f\left(\partial_t(u - \int_0^t h dw)\right) - \Delta u = 0 \text{ dans }]0,T[\times D \times \Omega, \\ u = 0 \text{ sur }]0,T[\times \partial D \times \Omega, \\ u(0,x) = u_0(x) \text{ dans } D, \end{cases}$$

où $f : \mathbb{R} \to \mathbb{R}$ est une fonction lipschitzienne strictement croissante, h un processus prévisible à valeur dans $L^2(D)$, $\int_0^t h dw$ l'intégrale d'Itô du processus h, $T > 0$, D un ouvert borné de $\mathbb{R}^d$ et $(\Omega, \mathcal{F}, P)$ un espace de probabilités. Cette étude fait partie d'un travail avec J. Giacomoni et G. Vallet [11]. La rédaction proposée ici sera une version plus détaillée que celle de l'article. Nous exhiberons un résultat d'existence et d'unicité d'une solution dans $\mathcal{N}^2_w(0,T,H^1_0(D))$, obtenue par un schéma de semi-discrétisation en temps. De plus nous montrerons que cette solution vérifie une inégalité d'énergie qui nous permettra d'étendre notre résultat au cas multiplicatif *i.e* lorsque la donnée dans l'intégrale d'Itô dépend de la solution. Nous considérons $\mathcal{H} : H_0^1(D) \to H_0^1(D)$ lipschitzienne, nulle en zéro et le problème :

$$\begin{cases} f\left(\partial_t(u - \int_0^t \mathcal{H}(u)dw)\right) - \Delta u = 0 \text{ dans }]0,T[\times D \times \Omega, \\ u = 0 \text{ sur }]0,T[\times \partial D, \\ u(0,x) = u_0(x) \text{ dans } D. \end{cases}$$

Dans le **Chapitre II**, une version abstraite des équations de Barenblatt sera considérée, issue d'un travail avec G. Vallet [12], rejoignant les travaux effectués par A. Bensoussan et R. Temam [16] sur des équations d'évolution non linéaires monotones à entrées stochastiques. Nous considèrerons V et H deux espaces de Hilbert séparables formant le triplet de Gelfand-Lions $V \hookrightarrow H \equiv H' \hookrightarrow V'$, l'opérateur $f : H \to H'$, demicontinu, maximal et fortement monotone, $A : V \to V'$ le sous différentiel linéaire d'une fonction $J : V \to \mathbb{R}$ continue, Gâteaux-différentiable, convexe et propre, et g dans $L^2(0,T,H)$. Notons que J vérifie aussi un principe de coercivité sur V, avec 0 pour minimum uniquement atteint en 0. Nous nous intéresserons à l'existence et l'unicité de solution pour les problèmes :

$$(P) : \begin{cases} f(\partial_t u) + Au &= g, \\ u(t=0) &= u_0. \end{cases} \qquad \text{et} \qquad (P_s) : \begin{cases} f\left(\partial_t(u - \int_0^t hdw)\right) + Au &= 0, \\ u(t=0) &= u_0. \end{cases}$$

où (P_s) correspond à une perturbation stochastique de (P) par un bruit additif. Notre démarche sera la suivante : nous commencerons par montrer dans le cas déterministe l'existence d'une unique solution dans $W^{1,\infty}(0,T,V,H)$, en utilisant les outils classiques sur les opérateurs pseudo-monotones. Puis, nous étendrons ce résultat au cas stochastique, cherchant la solution dans $\mathcal{N}_w^2(0,T,V)$. La difficulté étant de récupérer la mesurabilité de la solution par rapport à la variable de probabilité, puis la propriété de prévisibilité (une mesurabilité plus difficile à obtenir) nécessaire puisque nous voudrons par la suite également un résultat dans le cas d'un bruit multiplicatif lorsque $\mathcal{H} : V \to V$ est lipschitzienne :

$$\begin{cases} f\left(\partial_t(u - \int_0^t \mathcal{H}(u)dw)\right) + Au &= 0, \\ u(t=0) &= u_0 \in V. \end{cases}$$

Dans le **Chapitre III**, nous présenterons l'étude faite avec G. Vallet et P. Wittbold [14] d'une perturbation stochastique du problème de Cauchy pour une loi de conservation :

$$\begin{cases} du - \operatorname{div} \vec{\mathbf{f}}(u)dt = h(u)dw \text{ dans }]0,T[\times\mathbb{R}^d \times \Omega, \\ u(0,x) = u_0(x) \in L^2(\mathbb{R}^d), \end{cases} \tag{0.1}$$

avec $d \geqslant 1$, $T > 0$, $\vec{\mathbf{f}} = (f_1, .., f_d) : \mathbb{R} \to \mathbb{R}^d$ lipschitzienne, $f_i(0) = 0$, $\forall i = 1, ..., d$ et $h : \mathbb{R} \to \mathbb{R}$ lipschitzienne avec $h(0) = 0$.
Nous chercherons à montrer l'existence et l'unicité d'une solution pour le Problème (0.1). En particulier nous nous attacherons à retrouver le résultat d'existence et d'unicité obtenu par A. Debussche et J. Vovelle dans [32]. Remarquons que, même dans le cas déterministe une solution faible d'une loi de conservation n'est pas unique en général. On se doit d'introduire la notion de solution entropique afin d'isoler la solution physiquement admissible. L'existence reposera sur une méthode de viscosité artificielle. En utilisant le concept de solution mesure et de formulation entropique de type Kruzhkov, nous prouverons l'unicité.
Nous étudierons dans un premier temps l'équation parabolique suivante :

$$\begin{cases} du_\epsilon - [\epsilon \Delta u_\epsilon + \operatorname{div} \vec{\mathbf{f}}(u_\epsilon)]dt = h(u_\epsilon)dw \text{ dans }]0,T[\times\mathbb{R}^d \times \Omega, \\ u_\epsilon(0,x) = u_0^\epsilon(x) \in \mathcal{D}(\mathbb{R}^d). \end{cases}$$

Cherchant la solution faible u_ϵ dans $\mathcal{N}_w^2(0,T,H_0^1(\mathbb{R}^d))$, le résultat d'existence et d'unicité est donné par exemple par G. Vallet dans [75], les techniques utilisées pour l'étude de ce problème parabolique sont semblables à celles introduites dans le Chapitre I. En appliquant la formule d'Itô au processus u_ϵ, nous obtiendrons une formulation entropique dite "visqueuse", puis par passage à la limite sur le paramètre ϵ, et par le biais de la théorie des mesures de Young, nous montrerons l'existence d'une solution mesure entropique $\mathbf{u}$ dans $\mathcal{N}_w^2\big(0,T,L^2(\mathbb{R}^d\times]0,1[)\big)$ et $L^\infty\big(]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[)\big)$.
En revisitant la méthode de dédoublement des variables de Kruzhkov, nous obtiendrons l'unicité de cette solution mesure et aussi l'existence et l'unicité d'une solution faible entropique. Dans ce chapitre, un effort particulier sera fait pour présenter l'adaptation des techniques incluant les entropies de Kruzhkov, techniques alourdies par l'ajout de la variable aléatoire et la gestion de l'intégrale d'Itô.

Dans le **Chapitre IV**, nous étudierons le problème introduit au Chapitre III en domaine borné, avec des conditions de Dirichlet homogènes au bord. Il s'agit d'une étude faite avec G. Vallet et P. Wittbold [13] sur une perturbation stochastique du problème de Dirichlet pour une loi de conservation scalaire :

$$\left\{\begin{array}{rcl} du - \operatorname{div}\vec{\mathbf{f}}(u)dt & = & h(u)dw \text{ dans }]0,T[\times D\times\Omega, \\ ''u & = & 0'' \text{ sur } \Sigma\times\Omega, \\ u(0,x) & = & u_0(x)\in L^2(D), \end{array}\right. \tag{0.2}$$

avec $d \geqslant 1$, $T > 0$, $\vec{\mathbf{f}} = (f_1,..,f_d) : \mathbb{R}\to\mathbb{R}^d$ lipschitzienne, $f_i(0) = 0$, $\forall i = 1,...,d$, $h:\mathbb{R}\to\mathbb{R}$ lipschitzienne avec $h(0) = 0$, D est un domaine borné de $\mathbb{R}^d$ de bord lipschitzien lorsque $d \geqslant 2$ et $\Sigma =]0,T[\times\partial D$.
Notre but dans ce chapitre sera de montrer que le Problème (0.2) admet une unique solution. Rappelons que les solutions faibles de ce type de problème ne sont pas régulières, ainsi les traces de telles solutions doivent être considérées avec précaution, dans un sens faible. Nous choisirons d'utiliser la manière dont J. Carillo [23] interprète la condition de bord pour de tels problèmes. Le résultat d'existence sera une adaptation de celui utilisé au Chapitre III et ne sera pas redémontré. Dans ce chapitre, la difficulté principale sera d'adapter la méthode de dédoublement des variables incluant les semi-entropies de Kruzhkov.

Dans le **Chapitre V**, nous considèrerons un schéma numérique pour approcher la solution faible entropique du Problème (0.2) étudié au Chapitre IV. Pour des raisons techniques, nous ferons les hypothèses supplémentaires suivantes : la donnée initiale u_0 sera dans $L^\infty(D)\cap BV(D)$ et il existera $M > 0$ tel que le support de la fonction h soit inclus dans $[-M,M]$. Le schéma introduit reposera sur une méthode de splitting proposée par H. Holden et N.H. Risebro [49]. Le résultat de convergence de la suite approximant la solution faible entropique du Problème (0.2) sera obtenu en utilisant les résultats théoriques du Chapitre IV. Enfin, nous terminerons par des simulations réalisées sous le logiciel *Scilab* pour l'équation de Burgers stochastique dans le cas monodimensionnel :

$$du + f(u)_x dt = h(u)dw \text{ dans } \Omega\times]-1,1[\times]0,1[,$$

où $f(u)=\frac{u^2}{2}$, $h:\mathbb{R}\to\mathbb{R}$ est à support compact dans $]0,1[$ et définie par :

$$h(x)=\begin{cases} 2e^{\frac{1}{|2x-1|^2-1}} & \text{if } 0<x<1 \\ 0 & \text{sinon.}\end{cases}$$

Nous testerons avec différentes données initiales :

$$u_1^0(x)=\begin{cases} -\frac{1}{2} & \text{si } x<0 \\ 0 & \text{si } x=0 \\ \frac{1}{2} & \text{sinon.}\end{cases} \qquad u_2^0(x)=\begin{cases} \frac{1}{2} & \text{si } x<0 \\ 0 & \text{si } x=0 \\ -\frac{1}{2} & \text{sinon.}\end{cases} \qquad \begin{aligned} u_3^0(x) &= 1-\frac{2}{\pi}\arctan(x). \\ \\ u_4^0(x) &= -\sin(\pi x).\end{aligned}$$

Enfin, ce document se termine par une section dévouée à la présentation de perspectives de l'étude faite dans ce travail, ainsi que par des annexes reliées aux Chapitres I, II et III où nous détaillons certains calculs et citons quelques résultats utilisés.

Chapter I

On the Stochastic Barenblatt Equation

In this chapter, we are interested in a non linear stochastic partial differential equation: the Barenblatt equation with homogeneous Dirichlet boundary conditions. This study is a part of a joint work with J.Giacomoni and G. Vallet [11]. In this paper we first investigated the following quasilinear parabolic problem of Barenblatt type:

$$\begin{cases} f(\cdot,\partial_t u) - \Delta_p u - \epsilon\Delta(\partial_t u) &= g \text{ in } Q =]0,T[\times D, \\ u &= 0 \text{ on }]0,T[\times\partial D, \\ u(0,x) &= u_0(x) \in W_0^{1,p}(D), \end{cases} \tag{P_t}$$

where D is a bounded domain with Lipschitz boundary, denoted by Γ in $\mathbb{R}^d$ with $d \geq 1$, $\frac{2d}{d+2} < p < \infty$, $\epsilon \geq 0$, $0 < T < +\infty$, $u_0 \in W_0^{1,p}(D)$ and f is a Carathéodory function which satisfies $f(x,0) = 0$ and suitable growth assumptions and $g \in L^2(Q)$, and when $\epsilon = 0$, f satisfies additionnaly monotonicity assumptions. We looked for *weak* solutions of Problem (P_t), and discussed the following issues: uniqueness and regularity of solutions. Next, we focused on the nondegenerate case $p = 2$ and $\epsilon = 0$. In a second part we were concerned with the following Barenblatt equation involving a stochastic perturbation:

$$\begin{cases} f\Big(\partial_t(u - \int_0^t h dw)\Big) - \Delta u &= 0 \text{ in }]0,T[\times D \times \Omega, \\ u &= 0 \text{ on }]0,T[\times\partial D \times \Omega, \\ u(0,.) &= u_0 \in H_0^1(D), \end{cases} \tag{S_t}$$

where $\int_0^t h dw$ denotes the Itô integral of h, $f : \mathbb{R} \to \mathbb{R}$ is an increasing Lipschitz-continuous function, $(\Omega, \mathcal{F}, P)$ is the probability space. We were able to discuss the existence and the uniqueness of weak solutions even in the case where h depends on u. This is the aim of this section, giving in detail sketches of this proof.

1 Introduction

1.1 Former results

These classes of Barenblatt equations were originally considered by Barenblatt in [10]. In Kamin-Peletier-Vázquez [52], the authors establish the existence of self-similar solutions of a Barenblatt equation arising in porous media models. The existence of self-similar solutions for a class of quasilinear degenerate Barenblatt equations related to porous media problems were further investigated in Hulshof-Vázquez [50] and in Igbida [51]. Barenblatt type problems appear also in a wide variety of situations in Physics, Biology and Engineering. In particular, in Colli-Luterotti-Schimperna-Stefanelli [28] a pseudo-parabolic Barenblatt equation motivated by an irreversible phase change model is studied and in Ptashnyk [65] the analysis of similar kind of equations is used for reaction-diffusion with absorption problems in Biochemistry. In the context of constrained stratigraphic problems in Geology, the study of Barenblatt equations were recently revisited by different authors. In this regard, we can quote the contributions Antontsev-Gagneux-Luce-Vallet [3], [4], [5], [6], Vallet [75] and for related problems with stochastic coefficients Adimurthi-Seam-Vallet [2]. Finally, in Díaz-Díaz [33] and in Ha [47] the existence of solutions to a class of homogeneous quasilinear Barenblatt equations is established by means of monotone methods for m-accretive operators.
As far as we know, there has not been any publications on stochastic perturbations of these classes of Barenblatt equations. This subject is developed in the following study.

1.2 Content of the study

Our aim is to propose a result of existence and uniqueness of a solution for Problem (S_t), particularly when the function h depends on the solution u (multiplicative noise). In a first step, we investigate the problem with an additive noise. The approach is the following: we use an implicit time discretization scheme to show the existence of a solution. It relies on studying properties of the approximate solution. Because of the random variable, classical results of compactness doesn't hold, and the purpose of this chapter is to present alternative methods. In addition, one shows that the mild solution depends continuously on the data. Thus, the uniqueness result follows immediately using this continuous dependence. By the way of a fixed-point theorem, we are able to extend our result of existence and uniqueness to the multiplicative case.
The chapter is organized as follows. After giving assumptions on the data, the definition of a solution and the main results of this chapter, we find in Section 2 an implicit time discretization

of Problem (S_t), where we investigate properties of the approximate solution. Using this study, we are able to show in Section 3 the existence and uniqueness of the solution of Problem (S_t), even in the multiplicative case.

1.3 Assumptions and main results

We consider the following formal stochastic partial differential equation:

$$\begin{cases} f\Big(\partial_t[u-\int_0^t hdw]\Big)-\Delta u &= 0 \text{ in }]0,T[\times D\times\Omega,\\ u(0,.) &= u_0\in H_0^1(D), \end{cases} \tag{1.1}$$

with homogeneous Dirichlet boundary conditions. In the sequel, one assumes that D is a bounded Lipschitz domain of $\mathbb{R}^d$, T a positive number, and denote $Q=D\times]0,T[$. We consider a standard adapted one dimensional continuous Brownian motion $w=\{w_t,\mathcal{F}_t,0\leqslant t\leqslant T\}$ defined on a complete probability space $(\Omega,\mathcal{F},P)$ with a countably generated σ-field $\mathcal{F}$, such that $w_0=0$, and $\mathcal{F}_0$ contains the negligible sets (see [30], [64] for further informations on stochastic analysis).
Let us assume in the sequel that

- f is a bi-Lipschitz-continuous function, increasing and such that $f(0)=0$,
- h is in $\mathcal{N}_w^2(0,T,H_0^1(D))$.

Remark 1.1 *As mentioned by* Da Prato-Zabczyk *[30] p.94 considering X a separable Banach space, one has*

$$\mathcal{N}_w^2(0,T,X):=\{\Phi:[0,T]\times\Omega\mapsto X|\ \Phi \textit{ is predictable and } \|\Phi\|_T<\infty\}.$$

In details, $\Phi:[0,T]\times\Omega\mapsto X$ is predictable means that Φ is measurable with respect to the field P_T which is generated by elements of the form $1_{]s,t]}\times F_s$ and $\{0\}\times F_0$ where $0\leqslant s<t, F_s\in\mathcal{F}_s$ and $F_0\in\mathcal{F}_0$.
Endowed with the norm $\|\Phi\|_T:=\left[E\int_0^T\|\Phi\|_X^2ds\right]^{\frac{1}{2}}$, $\mathcal{N}_w^2(0,T,X)$ is a separable Banach space. Moreover, if X is a separable Hilbert space, $\mathcal{N}_w^2(0,T,X)$ is also a separable Hilbert space.

Our aim is now to give a result of existence and uniqueness of the variational solution of the above-mentioned problem. Let us fix in what sense such a solution is understood.

Definition 1.2 *Any function u of $\mathcal{N}_w^2(0,T,H_0^1(D))$ such that $\partial_t[u-\int_0^t hdw]\in L^2(\Omega\times Q)$ is a solution to our stochastic problem (1.1) if t-almost everywhere in $]0,T[$, P-almost surely in Ω, the variational formulation holds*

$$\forall v\in H_0^1(D),\ \int_D f\Big(\partial_t[u-\int_0^t hdw]\Big)v+\nabla u.\nabla v dx=0$$

with $u(0,.)=u_0\in H_0^1(D)$.

Remark 1.3 ***Sense of the initial condition***
The unique solution of this chapter is in $L^2\big(\Omega,\mathcal{C}([0,T],L^2(D))\big)$. Particularly, it satisfies the initial condition in the following sense:

$$P\text{-}a.s,\qquad u(t=0,.)=\lim_{t\to 0}u(t,.)\ \text{ in } L^2(D).\tag{1.2}$$

Indeed, using the regularity of h, $u-\int_0^{\cdot} hdw\in L^2(]0,T[\times\Omega,H_0^1(D))$. Moreover, as $\partial_t(u-\int_0^{\cdot} hdw)\in L^2(]0,T[\times\Omega,L^2(D))$, one gets $u-\int_0^{\cdot} hdw\in L^2\big(\Omega,W(0,T,H_0^1(D),L^2(D))^{\dagger}\big)$. Particularly, as $L^2\big(\Omega,W(0,T,H_0^1(D),L^2(D))\big)\hookrightarrow L^2\big(\Omega,\mathcal{C}([0,T],L^2(D))\big)$, $u-\int_0^{\cdot} hdw$ is in $L^2\left(\Omega,\mathcal{C}([0,T],L^2(D))\right)$. As the Itô integral of an $N_w^2(0,T,L^2(D))$ process is a continuous square integrable $L^2(D)$-valued martingale ([30]), $\int_0^{\cdot} hdw$ is also in $L^2\left(\Omega,\mathcal{C}([0,T],L^2(D))\right)$. Thus, u is in $L^2\big(\Omega,\mathcal{C}([0,T],L^2(D))\big)$ and (1.2) has a sense. Notice also that $u\in\mathcal{C}([0,T],L^2(\Omega\times D))$.

Remark 1.4 *With a suitable regularity of the domain D, one can show by elliptic regularity that the solution u of (1.1) in the sense of Definition 1.2 is also in $L^2\big(]0,T[\times\Omega,H_0^1(D)\cap H^2(D)\big)$.*

The results we want to prove in the sequel are the following :

Theorem 1.5 *Under the above hypothesis, there exists a unique solution to the Problem (1.1) in the sense of Definition 1.2.*

Moreover, the solution depends continuously on the data, this is stated in the following result.

$^{\dagger}W(0,T,H_0^1(D),L^2(D))$ denotes the space of functions $u\in L^2(0,T,H_0^1(D))$ with $\partial_t u\in L^2(0,T,L^2(D))$.

Proposition 1.6 *Consider h and $\hat{h}$ in $\mathcal{N}^2_w(0,T,H^1_0(D))$ and u, $\hat{u}$ the associated solutions. Then for all t in $[0,T]$ with $Q_t =]0,t[\times D$, we have the energy inequality:*

$$\begin{aligned} & E\int_{Q_t} |\partial_t(U-\hat{U})|^2 dxdt + \frac{1}{2}E\|\nabla(u-\hat{u})(t)\|^2_{L^2(D)} \\ \leqslant \ & \frac{1}{2}\|\nabla(u-\hat{u})(0)\|^2_{L^2(D)} + E\int_{Q_t}|\nabla(h-\hat{h})|^2 dxdt, \end{aligned} \tag{1.3}$$

where $U = u - \int_0^t h dw$ and $\hat{U} = \hat{u} - \int_0^t \hat{h} dw$.

Theorem 1.7 *Assume that $\mathcal{H} : H^1_0(D) \to H^1_0(D)$ is a Lipschitz-continuous mapping with $\mathcal{H}(0) = 0$. Then, there exists a unique solution in $\mathcal{N}^2_w(0,T,H^1_0(D))$ to the problem*

$$\begin{cases} f\Big(\partial_t[u - \int_0^t \mathcal{H}(u)dw]\Big) - \Delta u & = & 0 \quad \text{in }]0,T[\times D\times\Omega, \\ u(0,.) & = & u_0 \ \in H^1_0(D). \end{cases}$$

2 Study of the implicit time discretization

Let us first introduce the way we approximate Problem (1.1). It relies on a discretization of the time interval $[0,T]$. We thus construct approximate functions: a simple and an affine one. The study consists then in obtaining first variational formulation satisfied by such functions. Then we look for *a priori* estimates which allow us to discuss about convergences. Moreover, suitable properties of such approximations are conserved at the limit (particularly the predictability). We consider a positive integer N and denote by $\Delta t = \frac{T}{N}$, $t_k = k\Delta t$ and $w_k = w(t_k)$ $\forall k \in [0,N]$.

2.1 Existence of the approximate sequence

Let us begin with a useful lemma. For the sake of conciseness, we refer the interested reader to Appendix G Section 1 for the proof of this classical result.

Lemma 2.1 *Consider a positive integer n and ϖ in $L^2\big((\Omega,\mathcal{F}_{t_n}),H^1_0(D)\big)$. Then, there exists a unique u in $L^2\big((\Omega,\mathcal{F}_{t_n}),H^1_0(D)\big)$ such that P-almost surely in Ω*

$$\forall v \in H^1_0(D), \int_D f(u)vdx + \Delta t\int_D \nabla u\nabla v dx + \int_D \nabla\varpi\nabla v dx = 0. \tag{2.1}$$

Remark 2.2 *Notice that thanks to the separability of $H^1_0(D)$, the formulation (2.1) is satisfied equivalently for any $v \in H^1_0(D)$ and P-almost surely.*

For convenience, we define the application

$$\begin{array}{rcl} a: \ H_0^1(D)^2 & \to & \mathbb{R} \\ (u,v) & \mapsto & \displaystyle\int_D \nabla u \nabla v dx. \end{array}$$

Using this lemma, the result of this section is the following.

Proposition 2.3 *For given u^n and h^n in $L^2\big((\Omega,\mathcal{F}_{t_n}),H_0^1(D)\big)$, there exists a unique u^{n+1} in $L^2\big((\Omega,\mathcal{F}_{t_{n+1}}),H_0^1(D)\big)$ such that P-almost surely in Ω,*

$$\forall v \in H_0^1(D), \int_D f\Big(\frac{u^{n+1}-u^n}{\Delta t} - h^n\frac{w^{n+1}-w^n}{\Delta t}\Big)v dx + a(u^{n+1},v) = 0. \tag{2.2}$$

Proof. Considering u^n and h^n in $L^2\big((\Omega,\mathcal{F}_{t_n}),H_0^1(D)\big)$, we use Lemma 2.1 with $\varpi = u^n + (w^{n+1}-w^n)h^n$ in $L^2\big((\Omega,\mathcal{F}_{t_{n+1}}),H_0^1(D)\big)$. Thus, there exists u in $L^2\big((\Omega,\mathcal{F}_{t_{n+1}}),H_0^1(D)\big)$ solution of (2.1). Consequently, there exists a unique u^{n+1} in $L^2\big((\Omega,\mathcal{F}_{t_{n+1}}),H_0^1(D)\big)$ solution of our Problem (2.2), with $u = \dfrac{u^{n+1}-u^n}{\Delta t} - h^n\dfrac{w^{n+1}-w^n}{\Delta t}$. □

2.2 Notations and computations

We consider X a separable Banach space. We define the following simple and respectively affine functions:

Definition 2.4 *For any sequence $(x_n)_n \subset X$, let us denote*

$$\begin{array}{rcl} x^{\Delta t} & = & \displaystyle\sum_{k=0}^{N-1} x_{k+1} 1_{[t_k,t_{k+1}[}, \\ \widetilde{x}^{\Delta t} & = & \displaystyle\sum_{k=0}^{N-1} \Big[\frac{x_{k+1}-x_k}{\Delta t}[.-t_k] + x_k\Big] 1_{[t_k,t_{k+1}[}. \end{array}$$

And, elementary computations yield

$$\begin{array}{rcl} \|x^{\Delta t}\|^2_{L^2(0,T;X)} = \Delta t \displaystyle\sum_{k=1}^{N} \|x_k\|^2_X & ; & \|\widetilde{x}^{\Delta t}\|^2_{L^2(0,T;X)} \leqslant C\Delta t \displaystyle\sum_{k=0}^{N} \|x_k\|^2_X; \\ \|x^{\Delta t} - \widetilde{x}^{\Delta t}\|^2_{L^2(0,T;X)} & = & \Delta t \displaystyle\sum_{k=0}^{N-1} \|x_{k+1}-x_k\|^2_X; \\ \|\dfrac{\partial \widetilde{x}^{\Delta t}}{\partial t}\|^2_{L^2(0,T;X)} & = & \dfrac{1}{\Delta t} \displaystyle\sum_{k=0}^{N-1} \|x_{k+1}-x_k\|^2_X; \\ \|x^{\Delta t}\|_{L^\infty(0,T;X)} = \displaystyle\max_{k=1,..,N} \|x_k\|_X & ; & \|\widetilde{x}^{\Delta t}\|_{L^\infty(0,T;X)} = \displaystyle\max_{k=0,..,N} \|x_k\|_X, \end{array}$$

where C is a positive constant.

Remark 2.5 *Detail of calculations is given in Appendix G Section 2.*

For any positive integers n, k, and any $h^k \in L^2\big((\Omega, \mathcal{F}_{t_k}), X\big)$, denote by:

$$B_n := \sum_{k=0}^{n-1} (w^{k+1} - w^k) h^k.$$

Remark 2.6 $B_n = \int_0^{t_n} h_{\Delta t}(s) dw(s)$, *where* $h_{\Delta t} = \sum_{k=0}^{n-1} h^k 1_{[t_k, t_{k+1}[} = h^{\Delta t}(. - \Delta t)$.
Indeed, as h^k *is* $\mathcal{F}_{t_k}$*-measurable, one has*

$$B_n = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} h^k dw(s) = \int_0^{t_n} \sum_{k=0}^{n-1} h^k 1_{[t_k, t_{k+1}[}(s) dw(s) = \int_0^{t_n} h_{\Delta t}(s) dw(s).$$

Now, for $h \in \mathcal{N}_w^2(0, T, X)$ with the convention that $h(s) = 0$ if $s < 0$, consider for any n in $\{1, ..., N\}$, $h^n := \dfrac{1}{\Delta t} \int_{t_{n-1}}^{t_n} h(s) ds$. Let us state some results on such a sequence, useful for the next sub-sections.

Remark 2.7 *As* h *is predictable,* h^n *is* $\mathcal{F}_{t_n}$*-measurable for all* n *in* $\{1, ..., N\}$.

Lemma 2.8 *Consider* X *a separable Hilbert space,* $h \in \mathcal{N}_w^2(0, T, X)$*, for any* n *in* $\{1, ..., N\}$*,* $h^n := \dfrac{1}{\Delta t} \int_{t_{n-1}}^{t_n} h(s) ds$*, then there exists a constant* $C \geqslant 0$ *such that*

$$E \sum_{j=0}^{n} \|h^j\|_X^2 \leqslant \frac{C}{\Delta t}.$$

Proof. Recall that

$$\begin{aligned}
\|h^j\|_X^2 &= \|\frac{1}{\Delta t} \int_{t_{j-1}}^{t_j} h(s) ds\|_X^2, \\
&\leqslant \frac{1}{\Delta t^2} \Big(\int_{t_{j-1}}^{t_j} \|h(s)\|_X ds \Big)^2 \\
&\leqslant \frac{1}{\Delta t^2} \times \Delta t \int_{t_{j-1}}^{t_j} \|h(s)\|_X^2 ds
\end{aligned}$$

And, for $n \in \{1, ..., N\}$:

$$E \sum_{j=0}^{n} \|h^j\|_X^2 \leqslant \frac{1}{\Delta t} E \sum_{j=0}^{n} \int_{t_{j-1}}^{t_j} \|h(s)\|_X^2 ds \leqslant \frac{1}{\Delta t} \|h\|^2_{L^2(0,T,L^2(\Omega,X))} \leqslant \frac{C}{\Delta t}.$$

□

And now a result of convergence mentioned by Simon [69], Lemma 12 p.52:

Lemma 2.9 *Consider X a separable Hilbert space, $h \in \mathcal{N}_w^2(0,T,X)$ and for any n in $\{1,\dots,N\}$ denote $h^n := \dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n} h(s)ds$. Then*

$$h_{\Delta t} \to h \text{ in } \mathcal{N}_w^2(0,T,X).$$

With such a choice of h^n, we also have the following result:

Proposition 2.10 *Consider X a separable Hilbert space, $h \in \mathcal{N}_w^2(0,T,X)$ and for any n in $\{1,...,N\}$ denote $h^n := \dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n} h(s)ds$. Then:*

$$B^{\Delta t}, \widetilde{B}^{\Delta t} \to \int_0^{\cdot} hdw \text{ in } L^2(0,T,L^2(\Omega,X)).$$

Proof.

$$\begin{aligned}
& E\int_0^T \|\widetilde{B}^{\Delta t} - \int_0^{\cdot} hdw\|_X^2 ds \\
= \;& E\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \|\frac{(w^{k+1}-w^k)}{\Delta t}h^k(s-t_k) + B_k - \int_0^s h(\sigma)dw(\sigma)\|_X^2 ds \\
\leqslant \;& 2E\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} |\frac{w^{k+1}-w^k}{\Delta t}|^2(s-t_k)^2\|h^k\|_X^2 + \|B_k - \int_0^s h(\sigma)dw(\sigma)\|_X^2 ds \\
\leqslant \;& \underbrace{C\Delta t^2 E\sum_{k=0}^{n-1}\|h^k\|_X^2}_{\leqslant C\Delta t \text{ (Lemma 2.8)}} + CE\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \|\int_0^{t_k}(h_{\Delta t}-h)(\sigma)dw(\sigma)\|_X^2 + \|\int_{t_k}^s h(\sigma)dw(\sigma)\|_X^2 ds \\
\leqslant \;& C\Delta t + CE\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left[\int_0^{t_k}\|(h_{\Delta t}-h)(\sigma)\|_X^2 d\sigma + \int_{t_k}^s \|h(\sigma)\|_X^2 d\sigma\right] ds \text{ (Itô isometry)} \\
\leqslant \;& C\Delta t + C\|h_{\Delta t} - h\|^2_{\mathcal{N}_w^2(0,T,X)} + C\Delta t\|h\|_{\mathcal{N}_w^2(0,T,X)} \\
\to \;& 0.
\end{aligned}$$

and so

$$\widetilde{B}^{\Delta t} \to \int_0^{\cdot} hdw \text{ in } L^2(0,T,L^2(\Omega,X)).$$

Moreover

$$\begin{aligned}\|B^{\Delta t}-\widetilde{B}^{\Delta t}\|^2_{L^2(0,T,L^2(\Omega,X))} &= \Delta t E\sum_{k=0}^{N-1}\|B_{k+1}-B_k\|_X^2\\ &= \Delta t E\sum_{k=0}^{N-1}\|(w^{k+1}-w^k)h^k\|_X^2\\ &= \Delta t^2 E\sum_{k=0}^{N-1}\|h^k\|_X^2\\ &\leqslant C\Delta t \text{ thanks to Lemma 2.8.}\end{aligned}$$

And $B^{\Delta t}\to\int_0^\cdot h dw$ in $L^2(0,T,L^2(\Omega,X))$. □

2.3 *A priori* estimates

Let us denote in the sequel $\|.\|=\|.\|_{L^2(D)}$, $(.,.)$ the scalar product in $L^2(D)$, and define for any n in $\{1,...,N\}$ $h^n:=\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}h(s)ds$.
One gets, t-almost everywhere in $]0,T[$ and P-almost surely in Ω, the discretization

$$\forall v\in H_0^1(D),\ \int_D f(\partial_t[\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t}])v+\nabla u^{\Delta t}.\nabla v dx=0. \tag{2.3}$$

The aim here is to obtain boundedness results for the sequences $\widetilde{u}^{\Delta t}$ and $u^{\Delta t}$.

Proposition 2.11 *A constant C exists such that*

$$\begin{aligned}\|\partial_t[\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t}]\|_{L^2(\Omega\times Q)} &\leq C, &(2.4)\\ \|\nabla\widetilde{u}^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega\times D))},\|\nabla u^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega\times D))} &\leq C, &(2.5)\\ \|\nabla(\widetilde{u}^{\Delta t}-u^{\Delta t})\|_{L^2(\Omega\times Q)} &\leq C\Delta t, &(2.6)\\ \|\nabla(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})\|_{L^2(\Omega\times Q)} &\leq C. &(2.7)\end{aligned}$$

Proof. The variational formulation (2.2) with the particular test function
$v=\dfrac{u^{n+1}-u^n}{\Delta t}-\dfrac{w^{n+1}-w^n}{\Delta t}h^n$ gives us

$$\begin{aligned}&\int_D f\Big(\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\Big)\Big(\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\Big)\,dx\\ &+a\Big(u^{n+1},\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\Big)=0.\end{aligned}$$

Then

$$\begin{aligned} & c\|\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\|^2+a(u^{n+1},\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n) \\ \leq\ & 0. \end{aligned}$$

Moreover

$$\begin{aligned} & c\|\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\|^2+a(u^{n+1},\frac{u^{n+1}-u^n}{\Delta t}) \\ & -a(u^{n+1}-u^n,\frac{w^{n+1}-w^n}{\Delta t}h^n)-a(u^n,\frac{w^{n+1}-w^n}{\Delta t}h^n) \\ \leq\ & 0, \end{aligned}$$

and,

$$a(u^{n+1},\frac{u^{n+1}-u^n}{\Delta t})=\frac{1}{2\Delta t}[\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2+\|\nabla(u^{n+1}-u^n)\|^2].$$

Finally, for all $\alpha>0$, one gets

$$\begin{aligned} & c\|\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\|^2+\frac{1}{2\Delta t}[\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2+\|\nabla(u^{n+1}-u^n)\|^2] \\ \leq\ & \frac{\alpha}{2\Delta t}\|\nabla(u^{n+1}-u^n)\|^2+\frac{|w^{n+1}-w^n|^2}{2\alpha\Delta t}\|\nabla h^n\|^2-a(u^n,\frac{w^{n+1}-w^n}{\Delta t}h^n). \end{aligned}$$

Then, since u^n and h^n are $\mathcal{F}_{t_n}$-measurable

$$\begin{aligned} & cE\|\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\|^2+\frac{1}{2\Delta t}E\left[\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2+\|\nabla(u^{n+1}-u^n)\|^2\right] \\ \leq\ & \frac{\alpha}{2\Delta t}E\|\nabla(u^{n+1}-u^n)\|^2+\frac{1}{2\alpha}E\|\nabla h^n\|^2. \end{aligned}$$

In this way

$$\begin{aligned} & c\Delta tE\|\frac{u^{n+1}-u^n}{\Delta t}-\frac{w^{n+1}-w^n}{\Delta t}h^n\|^2+\frac{1}{2}E[\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2]+\frac{1-\alpha}{2}E\|\nabla(u^{n+1}-u^n)\|^2 \\ \leq\ & \frac{\Delta t}{2\alpha}E\|\nabla h^n\|^2 \end{aligned}$$

and,

$$
\begin{aligned}
& 2c\Delta t\sum_{k=0}^{n} E\|\frac{u^{k+1}-u^k}{\Delta t}-\frac{w^{k+1}-w^k}{\Delta t}h^k\|^2+E\|\nabla u^{n+1}\|^2+(1-\alpha)\sum_{k=0}^{n} E\|\nabla(u^{k+1}-u^k)\|^2 \\
\le\ & \frac{\Delta t}{\alpha}\sum_{k=0}^{n} E\|\nabla h^k\|^2+\|\nabla u^0\|^2 \\
\le\ & C,\ \text{thanks to Lemma 2.8.}
\end{aligned}
$$

Consequently, by taking $\alpha = 1/2$ for example, we obtain

$$
E\|\nabla u^{n+1}\|^2 \le C \tag{2.8}
$$

$$
\Delta t\sum_{k=0}^{n} E\|\frac{u^{k+1}-u^k}{\Delta t}-\frac{w^{k+1}-w^k}{\Delta t}h^k\|^2+\frac{1}{2}\sum_{k=0}^{n} E\|\nabla(u^{k+1}-u^k)\|^2 \le C, \tag{2.9}
$$

and we get the desired estimations:

$$
\begin{aligned}
\|\partial_t[\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t}]\|_{L^2(\Omega\times Q)} &\le C,\ \text{thanks to (2.9)},\\
\|\nabla\widetilde{u}^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega\times D))}, \|\nabla u^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega\times D))} &\le C,\ \text{thanks to (2.8)},\\
\|\nabla(\widetilde{u}^{\Delta t}-u^{\Delta t})\|_{L^2(\Omega\times Q)} &\le C\Delta t,\ \text{thanks to (2.9)}.
\end{aligned}
$$

Finally let us show that $\|\nabla(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})\|_{L^2(\Omega\times Q)} \leqslant C$. Thanks to (2.5), $\|\nabla\widetilde{u}^{\Delta t}\|_{L^2(\Omega\times Q)}\leqslant C$, and so, it remains to show that $\|\nabla\widetilde{B}^{\Delta t}\|_{L^2(\Omega\times Q)} \leqslant C$. One has

$$
\begin{aligned}
\|\nabla\widetilde{B}^{\Delta t}\|^2_{L^2(\Omega\times Q)} &\leqslant \Delta t\sum_{k=0}^{n} E\|\nabla B_k\|^2_{L^2(D)}\\
&= \Delta t\sum_{k=0}^{n} E\int_D\left[\sum_{j=0}^{k}(w^{j+1}-w^j)\nabla h^j\right]^2 dx\\
&= \Delta t\sum_{k=0}^{n}\sum_{j=0}^{k} E\int_D\left[(w^{j+1}-w^j)\nabla h^j\right]^2 dx\\
&= \Delta t\sum_{k=0}^{n}\sum_{j=0}^{k}\int_D \underbrace{E\left[(w^{j+1}-w^j)^2\right]}_{=\Delta t} E(\nabla h^j)^2 dx\\
&= \Delta t\sum_{k=0}^{n} \underbrace{\Delta t E\sum_{j=0}^{k}\|h^j\|^2_{H^1_0(D)}}_{\leqslant C \text{ thanks to Lemma 2.8}}.
\end{aligned}
$$

and the result holds. □

2.4 At the limit

Here we would like to pass to the limit in (2.3) with respect to Δt. For technical reasons, in a first step, we have to take h in $\mathcal{N}^2_w(0,T,H^1_0(D)\cap H^2(D))$.
In the next part, we will see how to get back a solution with the weaker hypothesis $h \in \mathcal{N}^2_w(0,T,H^1_0(D))$.

Proposition 2.12 *Up to subsequences denoted in the same way, there exists u in $\mathcal{N}^2_w(0,T,H^1_0(D))$ weak limit of $\widetilde{u}^{\Delta t}$ and $u^{\Delta t}$ and there exists χ in $L^2(\Omega\times Q)$ weak limit of $f(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t}))$ such that*

$$\begin{array}{ll}
(i) & \nabla\widetilde{u}^{\Delta t}, \nabla u^{\Delta t} \overset{*}{\rightharpoonup} \nabla u \text{ in } L^\infty(0,T,L^2(\Omega\times D)),\\
(ii) & f(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})) \rightharpoonup \chi \text{ in } L^2(\Omega\times Q),\\
(iii) & \widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t} \rightharpoonup u - \int_0^{\cdot} h dw \text{ in } L^2\big(\Omega, W(0,T,H^1_0(D),L^2(D))^{\dagger}\big),\\
(iv) & u_0 = u(0,.) \text{ in } H^1_0(D).
\end{array}$$

Proof.
(i) By using (2.5), there exists u and $\widetilde{u}$ in $L^2(]0,T[\times\Omega, H^1_0(D))$ such that, up to subsequences denoted in the same way, we have

$$\nabla\widetilde{u}^{\Delta t} \overset{*}{\rightharpoonup} \nabla u \text{ and } \nabla u^{\Delta t} \overset{*}{\rightharpoonup} \nabla\widetilde{u} \text{ in } L^\infty(0,T,L^2(\Omega\times D)),$$

thus in $L^2(\Omega\times Q)$. Finally, thanks to (2.6), $u=\widetilde{u}$. Moreover, up to a subsequence, $\nabla u^{\Delta t}(.-\Delta t) \rightharpoonup \nabla u$ in $L^2(\Omega\times Q)$ (see Appendix G). As $\nabla u^{\Delta t}(.-\Delta t)$ is in the Hilbert space $\mathcal{N}^2_w(0,T,L^2(D))$ equipped with the norm of $L^2(\Omega\times Q)$, u is also in $\mathcal{N}^2_w(0,T,H^1_0(D))$.

(ii) In the same way, as f is a Lipschitz-continuous function and thanks to (2.4), the sequence $f(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t}))$ is bounded in $L^2(\Omega\times Q)$ and there exists χ in the same space such that, up to a subsequence

$$f(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})) \rightharpoonup \chi \text{ in } L^2(\Omega\times Q).$$

(iii) Thanks to (2.4) and (2.7), there exists ζ in $L^2\big(\Omega, W(0,T,H^1_0(D),L^2(D))\big)$ such that, up to a subsequence, $\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t} \rightharpoonup \zeta$ in the same space. As $\widetilde{u}^{\Delta t} \rightharpoonup u$ in $L^2(\Omega\times Q)$, and $\widetilde{B}^{\Delta t} \to \int_0^{\cdot} hdw$ in $L^2(\Omega\times Q)$ thanks to Proposition 2.10, one gets $\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t} \rightharpoonup u-\int_0^{\cdot} hdw$ in $L^2(\Omega\times Q)$. And by uniqueness of the limit, $\zeta = u-\int_0^{\cdot} hdw$. To conclude, we use the continuity of the time derivative

† $W(0,T,H^1_0(D),L^2(D))$ denotes the space of functions $u\in L^2(0,T,H^1_0(D))$ with $\partial_t u \in L^2(0,T,L^2(D))$.

operator

$$\partial_t : L^2\big(\Omega, W(0,T,H_0^1(D),L^2(D))\big) \to L^2(\Omega\times Q),$$

and $\partial_t(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}) \rightharpoonup \partial_t(u - \int_0^{\cdot} h dw)$ in $L^2(\Omega\times Q)$.

(iv) As $W\big(0,T,H_0^1(D),L^2(D)\big) \hookrightarrow \mathcal{C}(0,T,L^2(D))$ continuously,
$\widetilde{u}^{\Delta t}(0) - \widetilde{B}^{\Delta t}(0) \rightharpoonup (u - \int_0^{\cdot} h dw)(0)$ in $L^2(D)$ and so $u_0 = u(0,.) \in H_0^1(D)$. □

Now we would like to identify the weak limit χ in $L^2(\Omega\times Q)$ of $f(\partial_t(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}))$. We consider our discrete Equation (2.2) for any n in $\mathbb{N}$ with the test function $\dfrac{U^{n+1}-U^n}{\Delta t}$ where $U^{n+1} = u^{n+1} - \sum_{k=0}^{n}(w^{k+1}-w^k)h^k$. One gets:

$$\begin{aligned} 0 \;=\; & \int_D f\left(\frac{U^{n+1}-U^n}{\Delta t}\right)\frac{U^{n+1}-U^n}{\Delta t}dx \\ & + a(U^{n+1}, \frac{U^{n+1}-U^n}{\Delta t}) + \sum_{k=0}^{n}(w^{k+1}-w^k)\int_D \Delta h^k \frac{U^{n+1}-U^n}{\Delta t}dx \end{aligned}$$

and

$$\begin{aligned} & \Delta t\int_D f\left(\frac{U^{n+1}-U^n}{\Delta t}\right)\frac{U^{n+1}-U^n}{\Delta t}dx + \frac{1}{2}\|\nabla U^{n+1}\|^2 - \frac{1}{2}\|\nabla U^n\|^2 \\ \le\; & -\Delta t\int_D \Delta B^{n+1}\frac{U^{n+1}-U^n}{\Delta t}dx. \end{aligned}$$

By adding from $k=0$ to $n-1$, we obtain

$$\begin{aligned} & \Delta t\sum_{k=0}^{n-1}\int_D f\left(\frac{U^{k+1}-U^k}{\Delta t}\right)\frac{U^{k+1}-U^k}{\Delta t}dx + \frac{1}{2}\sum_{k=0}^{n-1}\left(\|\nabla U^{k+1}\|^2 - \|\nabla U^k\|^2\right) \\ \le\; & -\Delta t\sum_{k=0}^{n-1}\int_D \Delta B^{k+1}\frac{U^{k+1}-U^k}{\Delta t}dx. \end{aligned}$$

$$\begin{aligned} \Leftrightarrow\quad & \int_Q f\big(\partial_t(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t})\big)\partial_t(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t})dtdx + \frac{1}{2}\big(\|\nabla U^n\|^2 - \|\nabla U^0\|^2\big) \\ & \le -\int_Q \Delta B^{\Delta t}\partial_t(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t})dtdx. \end{aligned}$$

We notice that $\nabla\widetilde{U}^{\Delta t}(T) = \nabla U^N$, and finally

$$\begin{aligned}&\int_Q f\left(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})\right)\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})dtdx+\frac{1}{2}\|\nabla\widetilde{U}^{\Delta t}(T)\|^2-\frac{1}{2}\|\nabla U^0\|^2\\ \leq\ &-\int_Q \Delta B^{\Delta t}\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})dtdx.\end{aligned}$$

By passing to the superior limit in

$$\begin{aligned}&E\int_Q f\left(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})\right)\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})dtdx+\frac{1}{2}E\|\nabla\widetilde{U}^{\Delta t}(T)\|^2-\frac{1}{2}\|\nabla U^0\|^2\\ \leqslant\ &-E\int_Q \Delta B^{\Delta t}\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})dtdx\end{aligned} \tag{2.10}$$

one gets the following result.

Proposition 2.13 *Up to a subsequence*

$$f(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t}))\rightharpoonup f(\partial_t(u-\int_0^{\cdot} hdw)) \text{ in } L^2(\Omega\times Q).$$

Proof. Passing to the superior limit in (2.10), we have

$$\begin{aligned}&\limsup_{\Delta t\to 0} E\int_Q f(\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t}))\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})dtdx+\frac{1}{2}\liminf_{\Delta t\to 0} E\|\nabla\widetilde{U}^{\Delta t}(T)\|^2\\ \leqslant\ &-E\int_Q\int_0^t \Delta hdw\partial_t(u-\int_0^t hdw)dtdx+\frac{1}{2}E\|\nabla U^0\|^2.\end{aligned} \tag{2.11}$$

Indeed, $\partial_t(\widetilde{u}^{\Delta t}-\widetilde{B}^{\Delta t})\rightharpoonup\partial_t(u-\int_0^t hdw)$ in $L^2(\Omega\times Q)$ and $B^{\Delta t}\to\int_0^{\cdot} hdw$ in $L^2\big(]0,T[\times\Omega, H_0^1(D)\cap H^2(D)\big)$ and so, by continuity of the Laplace operator

$$\Delta : L^2\big(]0,T[\times\Omega, H_0^1(D)\cap H^2(D)\big)\to L^2(\Omega\times Q),$$

$\Delta B^{\Delta t}\to\Delta\int_0^{\cdot} hdw$ in $L^2(\Omega\times Q)$, and $\Delta\int_0^{\cdot} hdw=\int_0^{\cdot}\Delta hdw$ (see for example Prévôt-Röckner [64] Lemma 2.4.1 p.35). In addition, as for all t in [0,T], $\widetilde{U}^{\Delta t}(t)\rightharpoonup U(t)$ in $L^2(\Omega, H_0^1(D))$, one has

$$\liminf_{\Delta t\to 0} E\|\nabla\widetilde{U}^{\Delta t}(T)\|^2\geqslant E\|\nabla U(T)\|^2.$$

P-almost surely, $U := u - \int_0^t hdw$ satisfies the heat equation

$$\left\{\begin{array}{rcl} \partial_t U - \Delta U & = & g, \\ U(0,.) & = & u_0 \in H_0^1(D), \end{array}\right.$$

where $g = \partial_t U - \chi + \int_0^t \Delta h dw$. Thus, one has the following energy equality (see Brezis [19] Theorem X.11 p.220 for example):
for any $t \in [0,T]$, by denoting $Q_t =]0,t[\times D$

$$\int_{Q_t} \chi \partial_t U dt dx + \frac{1}{2}\|\nabla U(t)\|^2 = -\int_{Q_t}\int_0^s \Delta h dw \partial_t U ds dx + \frac{1}{2}\|\nabla u_0\|^2,$$

and by taking the expectation :

$$E\int_{Q_t} \chi \partial_t U dt dx + \frac{1}{2}E\|\nabla U(t)\|^2 = -E\int_{Q_t}\int_0^s \Delta h dw \partial_t U ds dx + \frac{1}{2}E\|\nabla u_0\|^2.$$

Then, Lebesgue's theorem yields the continuity of $t \in [0,T] \mapsto E\|\nabla U(t)\|^2$, and as $\int_0^t hdw$ is a continuous martingale, it is continuous, and $t \in [0,T] \mapsto E\|\nabla u(t)\|^2$ is continuous too.
In this way, by replacing $-E\int_Q\int_0^t \Delta h dw \partial_t(u - \int_0^t hdw) dt dx + \frac{1}{2}E\|\nabla u^0\|^2$ in (2.11) we finally have

$$\limsup_{\Delta t \to 0} E\int_Q f\left(\partial_t(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t})\right)\partial_t(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}) dt dx \le E\int_Q \chi \partial_t U dt dx.$$

As $f : \mathbb{R} \to \mathbb{R}$ is monotone, the operator $A_f : u \in L^2(\Omega \times Q) \mapsto f(u) \in L^2(\Omega \times Q)$ is monotone and as f is a Lipschitz-continuous function, A_f is continuous and so A_f is a maximal operator and one gets $\chi = f(\partial_t(u - \int_0^t hdw))$ (see Lions [59] p.172). $\square$

3 Existence and uniqueness of the solution

With the study done in the previous section, we are able to show the result of existence and uniqueness of a solution for Problem (1.1) stated in Theorem 1.5. Let us begin with the uniqueness result, thus we will present the existence one, and finally the extension to the multiplicative case.

3.1 Uniqueness

We consider h in $\mathcal{N}^2_w(0,T,H^1_0(D))$, u, $\hat{u}$ two solutions of our stochastic problem

$$\begin{cases} \int_D f\Big(\partial_t[u-\int_0^t hdw]\Big)vdx + a(u,v) &= 0,\\ u(0,.) &= u_0. \end{cases}$$

Keeping the notation $U = u - \int_0^t hdw$, $\hat{U} = \hat{u} - \int_0^t hdw$, one has

$$\begin{cases} f(\partial_t U) - f(\partial_t \hat{U}) - \Delta(U-\hat{U}) &= 0,\\ (U-\hat{U})(0) &= 0. \end{cases}$$

This means that $W := U - \hat{U}$ is the solution of the heat equation

$$\begin{cases} \partial_t W - \Delta W &= \partial_t(U-\hat{U}) - \big[f(\partial_t U) - f(\partial_t \hat{U})\big],\\ W(0) &= 0. \end{cases}$$

And as previously one has the energy equality with $t \in [0,T] \mapsto E\|\nabla W(t)\|^2$ continuous:

$$\frac{1}{2}E\|\nabla W(T)\|^2_{L^2(D)} - \frac{1}{2}\|\nabla W_0\|^2_{L^2(D)} = -E\int_Q \big[f(\partial_t U) - f(\partial_t \hat{U})\big]\partial_t W dxdt.$$

And there exists $c>0$ such that

$$cE\int_Q |\partial_t(U-\hat{U})|^2 dxdt + \frac{1}{2}E\|\nabla(U-\hat{U})(T)\|^2_{L^2(D)} = 0,$$

and finally $u = \hat{u}$.

3.2 Existence

Considering h in $\mathcal{N}^2_w(0,T,H^1_0(D)\cap H^2(D))$, thanks to the previous sections, for all v in $H^1_0(D)$, α in $L^2(0,T)$ and β in $L^\infty(\Omega)$, the following variational equation holds

$$\int_{\Omega\times(0,T)}\int_D f(\partial_t(u-\int_0^t hdw))v\alpha\beta dxdtdP + \int_{\Omega\times(0,T)}\int_D \nabla u\nabla v\alpha\beta dxdtdP = 0.$$

Therefore, for any v in $H_0^1(D)$, t a.e in $]0,T[$ and P-a.s in Ω

$$\int_D f(\partial_t(u - \int_0^t hdw))vdx + \int_D \nabla u \nabla v dx = 0.$$

Since $H_0^1(D)$ is separable, one gets that t-a.e in $]0,T[$ and P-a.s in Ω, for any v in $H_0^1(D)$

$$\int_D f(\partial_t(u - \int_0^t hdw))vdx + \int_D \nabla u \nabla v dx = 0.$$

Using Proposition 2.12, $u(0,.) = u_0 \in H_0^1(D)$, and we have the existence of a solution u in the sense of Definition 1.2.

Let us treat the case $h \in \mathcal{N}_w^2(0,T,H_0^1(D))$, stated by Theorem 1.5. We decide to approach h by a sequence $(h_n)_n \subset \mathcal{N}_w^2(0,T,\mathcal{D}(D))$. Then, from the previous proof, there exists u_m and u_n in $\mathcal{N}_w^2(0,T,H_0^1(D))$, such that $\partial_t(u_m - \int_0^t h_m dw)$ and $\partial_t(u_n - \int_0^t h_n dw)$ are in $L^2(\Omega\times Q)$ and satisfy, for a common initial condition

$$\begin{aligned} f(\partial_t[u_m - \int_0^t h_m dw]) - \Delta u_m &= 0, \\ f(\partial_t[u_n - \int_0^t h_n dw]) - \Delta u_n &= 0. \end{aligned}$$

With $U_n = u_n - \displaystyle\int_0^t h_n dw$ and $U_m = u_m - \displaystyle\int_0^t h_m dw$,

$$f(\partial_t(U_n)) - f(\partial_t(U_m)) - \Delta(u_n - u_m) = 0.$$

Then, by taking the test function $v = \dfrac{(U_n - U_m)(t) - (U_n - U_m)(t-\Delta t)}{\Delta t}$, we get

$$\begin{aligned} &\int_D [f(\partial_t U_n) - f(\partial_t U_m)]\frac{(U_n - U_m)(t) - (U_n - U_m)(t-\Delta t)}{\Delta t}dx \\ &-\frac{1}{\Delta t}a\Big(u_n(t) - u_m(t), \int_{t-\Delta t}^t (h_n - h_m)dw\Big) \\ &+\frac{1}{\Delta t}a\Big(u_n(t) - u_m(t), u_n(t) - u_m(t) - u_n(t-\Delta t) - u_m(t-\Delta t)\Big) \\ =\ & 0. \end{aligned}$$

Moreover, by noticing that

$$
\begin{aligned}
a\big(u_n - u_m, u_n - u_m - (u_n - u_m)(t-\Delta t)\big) &= \frac{1}{2}\Big[\|\nabla(u_n - u_m)\|^2 - \|\nabla(u_n - u_m)(t-\Delta t)\|^2 \\
&\quad + \|\nabla\left[(u_n - u_m) - (u_n - u_m)(t-\Delta t)\right]\|^2\Big],
\end{aligned}
$$

we obtain

$$
\begin{aligned}
0 &= \int_D \left[f(\partial_t U_n) - f(\partial_t U_m)\right] \frac{(U_n - U_m)(t) - (U_n - U_m)(t-\Delta t)}{\Delta t} dx \\
&+ \frac{1}{2\Delta t}\Big[\|\nabla(u_n - u_m)(t)\|^2 - \|\nabla(u_n - u_m)(t-\Delta t)\|^2 \\
&+ \|\nabla[(u_n - u_m)(t) - (u_n - u_m)(t-\Delta t)]\|^2\Big] \\
&- \frac{1}{\Delta t} a\left((u_n - u_m)(t) - (u_n - u_m)(t-\Delta t), \int_{t-\Delta t}^{t}(h_n - h_m)dw\right) \\
&- \frac{1}{\Delta t} a\left((u_n - u_m)(t-\Delta t), \int_{t-\Delta t}^{t}(h_n - h_m)dw\right).
\end{aligned}
$$

And

$$
\begin{aligned}
0 &= \int_D \left[f(\partial_t U_n) - f(\partial_t U_m)\right] \frac{(U_n - U_m)(t) - (U_n - U_m)(t-\Delta t)}{\Delta t} dx \\
&+ \frac{1}{2\Delta t}\Big[\|\nabla(u_n - u_m)(t)\|^2 - \|\nabla(u_n - u_m)(t-\Delta t)\|^2\Big] \\
&+ \frac{1}{4\Delta t}\Big[\|\nabla[(u_n - u_m)(t) - (u_n - u_m)(t-\Delta t)]\|^2 - 4\|\nabla \int_{t-\Delta t}^{t}(h_n - h_m)dw\|^2 \\
&+ \|\nabla\big((u_n - u_m)(t) - (u_n - u_m)(t-\Delta t) - 2\int_{t-\Delta t}^{t}(h_n - h_m)dw\big)\|^2\Big] \\
&- \frac{1}{\Delta t} a\left((u_n - u_m)(t-\Delta t), \int_{t-\Delta t}^{t}(h_n - h_m)dw\right).
\end{aligned}
$$

By taking the expectation, the integral from Δt to T, and using properties of the Brownian motion, one gets

$$
\begin{aligned}
&\int_{\Delta t}^{T} E \int_D [f(\partial_t U_n) - f(\partial_t U_m)] \frac{(U_n - U_m)(t) - (U_n - U_m)(t-\Delta t)}{\Delta t} dx dt \\
&+ \frac{1}{2\Delta t} \int_{\Delta t}^{T} E\left[\|\nabla(u_n - u_m)(t)\|^2 - \|\nabla(u_n - u_m)(t-\Delta t)\|^2\right] dt \\
\leq &\int_{\Delta t}^{T} E \frac{1}{\Delta t}\|\nabla \int_{t-\Delta t}^{t}(h_n - h_m)dw\|^2 dt = \int_{\Delta t}^{T} E \frac{1}{\Delta t} \int_{t-\Delta t}^{t} \|\nabla(h_n - h_m)\|^2 ds dt
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&\int_{\Delta t}^{T} E\int_D[f(\partial_t U_n)-f(\partial_t U_m)]\frac{(U_n-U_m)(t)-(U_n-U_m)(t-\Delta t)}{\Delta t}dxdt\\
&+\frac{1}{2\Delta t}\int_{\Delta t}^{T}E\|\nabla(u_n-u_m)(t)\|^2dt-\frac{1}{2\Delta t}E\int_0^{\Delta t}\|\nabla(u_n-u_m)(t)\|^2dt\\
\leq\ & E\int_Q|\nabla(h_n-h_m)|^2dxdt.
\end{aligned}
$$

We pass to the limit on Δt to get

$$
\begin{aligned}
& E\int_Q[f(\partial_t U_n)-f(\partial_t U_m)]\partial_t(U_n-U_m)dxdt+\frac{1}{2}E\|\nabla(u_n-u_m)(T)\|^2\\
\leq\ & E\int_Q|\nabla(h_n-h_m)|^2dxdt.
\end{aligned}
$$

As f is a bi-Lipschitz-continuous function, there exists a constant $c>0$ such that

$$
\begin{aligned}
& cE\int_Q|\partial_t(U_n-U_m)|^2dxdt+\frac{1}{2}E\|\nabla(u_n-u_m)(T)\|^2\\
\leq\ & E\int_Q|\nabla(h_n-h_m)|^2dxdt. \qquad (3.1)
\end{aligned}
$$

Moreover, by denoting $Q_t=]0,t[\times D$, one has for all $t\in[0,T]$

$$
\begin{aligned}
E\int_{Q_t}|\partial_t(U_m-U_n)|^2dxdt+\frac{1}{2}E\|\nabla(u_m-u_n)(t)\|^2_{L^2(D)} &\leqslant E\int_{Q_t}|\nabla(h_m-h_n)|^2dxdt,\\
&\leqslant E\int_Q|\nabla(h_m-h_n)|^2dxdt. \qquad (3.2)
\end{aligned}
$$

Then, as $(h_n)_n$ is a Cauchy sequence in $\mathcal{N}^2_w(0,T,H^1_0(D))$ for $\epsilon>0$,

$$
E\int_Q|\partial_t(U_m-U_n)|^2dxdt\leqslant C\epsilon \quad \text{and} \quad \sup_{t\in[0,T]}E\|\nabla(u_m-u_n)(t)\|^2_{L^2(D)}\leqslant C\epsilon.
$$

Finally, $(u_n)_n$ and $(U_n)_n$ are also Cauchy sequences respectively in $L^\infty(0,T,L^2(\Omega,H^1_0(D)))$ and $L^2\big(\Omega,W(0,T,H^1_0(D),L^2(D))\big)$. As mentioned by Da Prato-Zabczyk [30], $\mathcal{N}^2_w(0,T,H^1_0(D))$ is complete, and there exists u in $\mathcal{N}^2_w(0,T,H^1_0(D))$ such that

$$
\begin{aligned}
u_n\to u \quad & \text{in} \quad \mathcal{N}^2_w(0,T,H^1_0(D))\\
\partial_t U_n\to\partial_t[u-\int_0^t hdw] \quad & \text{in} \quad L^2(\Omega\times Q).
\end{aligned}
$$

Thus, we get for all v in $H_0^1(D)$

$$\int_D f(\partial_t(u - \int_0^t hdw))vdx + \int_D \nabla u \nabla v dx = 0,$$

and we have the existence result when $h \in \mathcal{N}_w^2(0,T,H_0^1(D))$, as announced in Theorem 1.5.

Let us show the continuous dependence of the solution with respect to the data, stated in Proposition 1.6. We use the same arguments as previously. Consider h and $\hat{h}$ in $\mathcal{N}_w^2(0,T,H_0^1(D))$, denote by u, $\hat{u}$ the associated solutions in $\mathcal{N}_w^2(0,T,H_0^1(D))$, h_n and $\hat{h}_n$ regularizations of h and $\hat{h}$. Then there exists u_n and $\hat{u}_n$ in $\mathcal{N}_w^2(0,T,H_0^1(D))$ satisfying

$$\begin{aligned} f\Big(\partial_t(u_n - \int_0^t h_n dw)\Big) - \Delta u_n &= 0, \\ f\Big(\partial_t(\hat{u}_n - \int_0^t \hat{h}_n dw)\Big) - \Delta \hat{u}_n &= 0. \end{aligned}$$

Moreover, $t \in [0,T] \mapsto E\|\nabla(u_n - \hat{u}_n)(t)\|^2$ is continuous and (3.2) still holds:
for all $t \in [0,T]$,

$$E\int_{Q_t} \partial_t[U_n - \hat{U}_n]^2 dxdt + \frac{1}{2}E\|\nabla(u_n - \hat{u}_n)(t)\|^2_{L^2(D)} \leqslant E\int_{Q_t} |\nabla(h_n - \hat{h}_n)|^2 dxdt, \tag{3.3}$$

with $U_n = u_n - \int_0^t h_n dw$ and $\hat{U}_n = \hat{u}_n - \int_0^t \hat{h}_n dw$. Thanks to a Cauchy sequence argument and the uniqueness of the solution of our stochastic problem, $u_n \to u$ and $\hat{u}_n \to \hat{u}$, both in $\mathcal{N}_w^2(0,T,H_0^1(D)) \cap \mathcal{C}\big(0,T,L^2(\Omega,H_0^1(D))\big)$. Passing to the limit in (3.3) one gets the announced result.

3.3 Multiplicative case

One is able to extend the existence an uniqueness result in the case of multiplicative perturbation:

$$\begin{cases} f\Big(\partial_t[u - \int_0^t \mathcal{H}(u)dw]\Big) - \Delta u = 0 & \text{in }]0,T[\times D \times \Omega, \\ u(0,.) = u_0 & \in H_0^1(D). \end{cases}$$

i.e. when the function $\mathcal{H}$ depends on the solution u. Let us explicit this result, stated in

Theorem 1.7. Consider $\mathcal{H} : H_0^1(D) \mapsto H_0^1(D)$ a Lipschitz-continuous mapping with $\mathcal{H}(0) = 0$, and assume the same hypothesis as previously for the other data. Consider the application

$$\begin{aligned} T : \mathcal{N}_w^2(0,T,H_0^1(D)) &\to \mathcal{N}_w^2(0,T,H_0^1(D)) \\ S &\mapsto u_s \end{aligned}$$

where u_S is the solution of the additive problem

$$\begin{cases} f\Big(\partial_t(u_S - \int_0^t \mathcal{H}(S)dw)\Big) - \Delta u_S = 0 \text{ in }]0,T[\times D \times \Omega \\ u(0,.) = u_0, \end{cases}$$

and $u_0 \in H_0^1(D)$. The existence of a such u_S is given by Theorem 1.5. For S and $\hat{S}$ in $\mathcal{N}_w^2(0,T,H_0^1(D))$, one has for a common initial condition

$$f\Big(\partial_t(u_S - \int_0^t \mathcal{H}(S)dw)\Big) - f\Big(\partial_t(u_{\hat{S}} - \int_0^t \mathcal{H}(\hat{S})dw)\Big) = \Delta(u_S - u_{\hat{S}}).$$

By denoting $U = u_S - \int_0^t \mathcal{H}(S)dw$, $\hat{U} = u_{\hat{S}} - \int_0^t \mathcal{H}(\hat{S})dw$, and considering (1.3), one gets for all $t \in [0,T]$

$$\begin{aligned} & cE\int_0^t \|\partial_t(U-\hat{U})\|^2 dt + \frac{1}{2}E\|\nabla(u-\hat{u})(t)\|^2 \\ \le\ & \frac{1}{2}E\|\nabla(u_0-\hat{u}_0)\|^2 + E\int_0^t \|\nabla\big(\mathcal{H}(S)-\mathcal{H}(\hat{S})\big)(s)\|^2 ds \\ \le\ & CE\int_0^t \|\nabla\big(S-\hat{S})(s)\big)\|^2 ds. \end{aligned}$$

Then, for any positive α

$$\begin{aligned} & E\int_0^T e^{-\alpha t}\|\nabla(u-\hat{u})(t)\|^2 dt \\ \le\ & CE\int_0^T e^{-\alpha t}\int_0^t \|\nabla\big(S-\hat{S}\big)(s)\|^2 ds dt. \\ =\ & C\times\frac{1}{\alpha}E\int_0^T e^{-\alpha t}\|\nabla(S-\hat{S})(t)\|^2 dt \underbrace{-E\Big(\frac{1}{\alpha}e^{-\alpha T}\int_0^T \|\nabla(S-\hat{S})(t)\|^2 dt\Big)}_{\leqslant 0} \\ \le\ & \frac{C}{\alpha}E\int_0^T e^{-\alpha t}\|\nabla(S-\hat{S})(t)\|^2 dt. \end{aligned}$$

Finally

$$E\int_0^T e^{-\alpha t}\|T(S)-T(\hat{S})\|^2_{H_0^1(D)}dt \le \frac{C}{\alpha}E\int_0^T e^{-\alpha t}\|\nabla(S-\hat{S})(t)\|^2 dt.$$

Since the exponential weight in time provides an equivalent norm in $\mathcal{N}_w^2(0,T,H_0^1(D))$, if $\alpha > C$, T is a contractive mapping, it has a unique fixed-point and the result holds.

Chapter II

On abstract Barenblatt equations

In this chapter we are interested in abstract problems of Barenblatt's type. This study is published in a joint work with G. Vallet [12]. In a first part, we investigate the problem $f(\partial_t u) + Au = g$ where f and A are maximal monotone operators and by assuming that A derives from a potential J. With general assumptions on the operators, we prove the existence of a solution. In a second part, we examine a stochastic version of the above problem: $f[\partial_t(u - \int_0^t hdw)] + Au = 0$, with some restrictive assumptions on the data due principally to the framework of the Itô integral.

1 Introduction

We are interested in the deterministic and the stochastic abstract problems of Barenblatt's type:

$$(\mathcal{P}_1) : \begin{cases} f(\partial_t u) + Au &= g, \\ u(t=0) &= u_0. \end{cases} \qquad \text{and} \qquad (\mathcal{P}_2) : \begin{cases} f\left(\partial_t(u - \int_0^t hdw)\right) + Au &= 0, \\ u(t=0) &= u_0. \end{cases}$$

1.1 Former results

In the deterministic case, such a problem has been investigated by Díaz-Díaz [33] where the authors were interested in the asymptotic behavior of the solution of the problem $\partial_t u - \Delta\beta(u) = 0$ where β is a maximal monotone graph in $\mathbb{R}^2$. The essential tool was to consider the "dual" problem $\partial_t v + \beta(-\Delta v) = 0$ of type $(\mathcal{P}_1)$: $f(\partial_t v) - \Delta v = 0$ where $f = [-\beta(-\cdot)]^{-1}$. The study of such a problem was based on the work of Ha [47] where the author was interested in the existence of solutions to a class of quasilinear Barenblatt equations of type $f \in \partial_t u + \beta A(u)$ when A is assumed to be a m-accretive operator in $L^\infty(\Omega)$.

Then, more recently, Schimperna-Segatti-Stefanelli [67] have been interested in the differential inclusion: $f \in \alpha(\partial_t u) - div(b(x, \nabla u)) + W'(u)$ where, among other things, $\alpha \subset \mathbb{R}^2$ was a maximal monotone graph, W a λ-convex function and $f \in W^{1,1}(0, T, H)$. Presented in an abstract way, this work was in connection with the modeling of phase change phenomena and gas flow in porous media.
Concerning such kind of modelings, problems of type $(\mathcal{P}_1)$ were originally considered by Barenblatt [10] in the theory of fluids in elasto-plastic porous medium. Written in following way: $F(\partial_t u) - \Delta u = 0$ where $F(x) = x + \gamma|x|$ $(0 < |\gamma| < 1)$, existence of regular and self-similar solutions have been investigated by Kamin-Peletier-Vázquez [52]. Formal solutions given by expansions of a suitable new variable $\chi = \chi(t, x)$ is also proposed in Chen-Cheng [26] concerning nonlinear diffusive process with a non-conservative mass.
For nonlinear operators A, the existence of self-similar solutions has been proposed by Hulshof-Vázquez [50] for the so called "modified porous medium equation" : $F(\partial_t u) - \Delta u^m = 0$. For the "modified p-Laplace equation": $F(\partial_t u) - \Delta_p u = 0$, a result of existence of self-similar solutions has been proposed by Igbida [51], and the existence of weak solutions by Bauzet-Giacomoni-Vallet [11].
A first approach of the stochastic case has been proposed by Adimurthi-Seam-Vallet [2] concerning the existence of a solution to the stochastic pseudoparabolic Barenblatt problem: $f\left(\partial_t (u - \int_0^t h dw)\right) - \Delta u - \epsilon \Delta \partial_t u = 0$ when $\epsilon > 0$. Then, Bauzet-Giacomoni-Vallet [11] has envisaged the case $\epsilon = 0$, where strong solutions are considered, see also Díaz-Langa-Valero [35].

1.2 Content of the study

In the present work, we propose to extend the previous cited results concerning $(\mathcal{P}_1)$ by weakening the assumptions on the data and we propose to study the abstract stochastic parabolic-Barenblatt problem $(\mathcal{P}_2)$ with additive noise, then with a multiplicative one.

This chapter is organized as follows. In the first part, H is a Hilbert space and V is a reflexive separable Banach space such that V is embedded in H with a dense and compact injection and one will identify H with its dual space H'. One denotes by $(.,.)$, resp. $|.|$, the scalar product of H, resp. the norm in H, by $\langle .,. \rangle$ the dual product $V' - V$ and by $\|.\|$ the norm in V. $f : H \to H' \equiv H$ and $A : V \to V'$ are maximal monotone operators, A derives from a potential J, and general assumptions are made to prove the existence of a solution. In particular, we assume neither strong monotonicity for f, nor a control from bellow of $J(u)$ by a power of the

norm of u in V. This allows us to apply our results in the case of Orlicz spaces (See ADAMS [1], Chap. VIII, p. 227 *sqq.*) when the problem allows easily a control of the modulus given by the N-function, rather than the Luxembourg norm. One can cite for example the case of the Musielak–Orlicz spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ spaces (See DIENING-HARJULEHTO-HÄSTÖ-RUZICKA [36]).
In the second part, we will be interested in stochastic problems. Because of the theory of the stochastic integration, V needs to be a Hilbert space and for technical reasons, one assumes that $(u,v) \mapsto \langle Au, v\rangle$ is a scalar product whose associated norm is equivalent to the one of V. Using results of the deterministic case, we show existence and uniqueness of a solution.

1.3 Notations, assumptions and main results

Denote by $(\mathcal{P}_1)$ the following problem

$$(\mathcal{P}_1) : \begin{cases} f(\partial_t u) + Au = g, \\ u(t=0) = u_0, \end{cases}$$

and assume that

- H is a Hilbert space and V is a reflexive separable Banach space such that $V \hookrightarrow H$ with a dense and compact injection. Thus, one has the classical Gelfand-Lions triplet: $V \hookrightarrow H \equiv H' \hookrightarrow V'$.

- $f : H \to H' \equiv H$ is a demicontinuous (univoque) maximal monotone operator.

  Remark 1.1 *This is the case for example if f is the subdifferential ∂F of a continuous, Gâteaux-differentiable and proper convex function $F : H \to \mathbb{R}$.*

  Assume moreover that

  - $\exists \alpha > 0, \lambda \in \mathbb{R},\ \forall x \in H,\ (f(x), x) \geq \alpha|x|^2 - \lambda.$
  - $\exists C_1, C_2 \geq 0,\ \forall x \in H,\ |f(x)| \leq C_1|x| + C_2.$

- $J : V \to \mathbb{R}$ is a continuous, Gâteaux-differentiable and proper convex function. Since J can be defined modulo a constant value, assuming that $J(0) = 0$ does not affect the generality. One denotes by A its subdifferential $\partial J : V \to V'$. We recall that it is a demicontinuous (univoque) maximal monotone operator.
  One assumes moreover that

- J is bounded above on bounded subsets of V (therefore, A maps bounded subsets of V into bounded nonempty subsets of V' (*e.g.* [18] Prop. 4.1.25),
- Either $\exists \delta > 0$, $\varphi_1 : u \mapsto \delta|u|^2 + J(u)$ is coercive over V in the sense: $\frac{\delta|u_n|^2+J(u_n)}{\|u_n\|}$ goes to $+\infty$ if $\|u_n\|$ goes to $+\infty$.
  Or, f derives from the potential F (see Remark 1.1), and $\varphi_2 : u \mapsto F(u) + J(u)$ is coercive over V.

- $u_0 \in V$ and $g \in L^2(0,T,H)$.

The main results in that case are:

Theorem 1.2 *There exists $u \in W^{1,\infty,2}(0,T,V,H)^*$ solution of $(\mathcal{P}_1)$.*
Moreover, for a.e. t, $Au(t) = g(t) - f(\partial_t u(t)) \in H$, $J(u) \in W^{1,1}(0,T)$ and, for any t,

$$\int_0^t (f(\partial_t u), \partial_t u)\, ds + J(u(t)) = J(u_0) + \int_0^t (g, \partial_t u)\, ds,$$
$$\text{and} \qquad \alpha \int_0^t |\partial_t u|^2 ds + 2J(u(t)) \leq 2J(u_0) + \frac{1}{\alpha}\int_0^t |g|^2 ds + 2\lambda.$$

Corollary 1.3
If A is linear and f strictly monotone, then the solution is unique.

If A is linear and $J(0) < J(w)$ for any $w \neq 0$, then the solution is unique and it belongs to $C([0,T],V)$.
Moreover, the application $(u_0,g) \mapsto u$ is continuous from $V \times L^2(0,T,H)$ to $C([0,T],V)$.

If A is linear and f strongly monotone, the application $(u_0,g) \mapsto \partial_t u$ is continuous from $V \times L^2(0,T,H)$ to $L^2(0,T,H)$.

Denote by $(\mathcal{P}_2)$ the following problem:

$$(\mathcal{P}_2) : \begin{cases} f\left(\partial_t(u - \int_0^t h dw)\right) + Au = 0, \\ u(t=0) = u_0. \end{cases}$$

*$W^{1,p,q}(0,T,V,H)$ denotes the space of functions $u \in L^p(0,T,V)$ such that $\partial_t u \in L^q(0,T,H)$.

In addition to the above hypothesis, assume moreover that

- H and V are separable Hilbert spaces.
- $w = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on some complete probability space $(\Omega, \mathcal{F}, P)$, with the property that $w_0 = 0$.
- $A = \partial J$ is a linear operator, $J(u) > 0 = J(0)$ if $u \neq 0$ and f is strongly monotone.
- $u_0 \in V$ and $h \in N_w^2(0,T,V)$ where, for a separable Hilbert space X, $N_w^2(0,T,X)$ denotes the set of predictable processes of $L^2(]0,T[\times\Omega, X)$ (*Cf.* Prévôt-Röckner [64] for example).

The main results in that case are:

Theorem 1.4 *There exists a unique* $u \in N_w^2(0,T,V)$*, such that* $\partial_t(u - \int_0^t h dw) \in L^2\big(]0,T[\times\Omega, H\big)$*, solution of* $(\mathcal{P}_2)$*.*
Moreover, $u \in C([0,T], L^2(\Omega, V))$ *and, for any* $u_0, \hat{u}_0 \in V$*, any* $h, \hat{h} \in N_w^2(0,T,V)$ *and any* t*,*

$$\begin{aligned} & E\int_0^t \big(f(\partial_t U) - f(\partial_t \hat{U}), \partial_t[U(t) - \hat{U}(t)]\big)\, ds + E\|(u-\hat{u})(t)\|_A^2 \\ \leq \ & E\|u_0 - \hat{u}_0\|_A^2 + \int_0^t E\|h - \hat{h}\|_A^2 ds, \end{aligned}$$

where U *(resp.* $\hat{U}$*) denotes* $u - \int_0^t h dw$ *(resp.* $\hat{u} - \int_0^t \hat{h} dw$*).*

Corollary 1.5 *Assume that* $\mathcal{H} : V \to V$ *is a Lipschitz-continuous mapping. Then there exists a unique* $u \in N_w^2(0,T,V)$ *such that* $\partial_t[u - \int_0^t \mathcal{H}(u) dw] \in L^2(]0,T[\times\Omega, H)$ *solution of Problem*

$$(\mathcal{P}_{\mathcal{H}}) : f\Big(\partial_t[u - \int_0^t \mathcal{H}(u) dw]\Big) + Au = 0, \qquad \text{with } u(0,.) = u_0.$$

2 The deterministic case

The aim of this section is to prove Theorem 1.2 and Corollary 1.3. We propose to prove the existence of a solution by passing to the limit in a time discretization scheme, following what has been discussed in Chapter I. The content of the study will be appreciably the same: first showing existence of the approximate solution, finding *a priori* estimates and then passing to

the limit with respect to the time step parameter. Then we investigate the uniqueness of such a solution but under additional hypothesis on the data.
For any positive integers N and any $n \leq N$, we denote by

$$\Delta t = \frac{T}{N}, \quad t_n = n\Delta t \quad \text{and} \quad g^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} g(s)ds.$$

2.1 Existence of the approximate sequence

Lemma 2.1 *For any sequence $(g^n) \subset H$, there exists a sequence $(u^n) \subset V$ such that $u^0 = u_0$ and*

$$f(\frac{u^{n+1}-u^n}{\Delta t}) + Au^{n+1} = g^{n+1}. \tag{2.1}$$

Proof. Since $H' \hookrightarrow V'$, $M : u \in V \mapsto f(\frac{u-u^n}{\Delta t}) \in V'$ is a monotone operator.
If one denotes by S a bounded subset of V, then, for any $s \in S$,

$$\|M(s)\|_{V'} \leq C|f(\tfrac{s-u_n}{\Delta t})| \leq C|\tfrac{s-u_n}{\Delta t}| + C \leq C\|\tfrac{s-u_n}{\Delta t}\| + C$$

and M is a bounded operator.
If one considers that u_k converges weakly to u in V, then, it converges to u in H and Mu_n converges weakly to Mu in H since f is demicontinuous in H. Thus, for any $v \in V$, $\lim_k(Mu_k, u_k - v) = (Mu, u-v)$ and M is pseudomonotone in V.
For any $u \in V$, one has that

$$\begin{aligned}(f(\frac{u-u^n}{\Delta t}),u) &= \Delta t(f(\frac{u-u^n}{\Delta t}),\frac{u-u^n}{\Delta t}) + (f(\frac{u-u^n}{\Delta t}),u^n)\\ &\geq \alpha\Delta t|\frac{u-u^n}{\Delta t}|^2 - C(\Delta t)(|u|+c(u^n)) \geq \frac{\alpha}{\Delta t}|u|^2 - C(\Delta t)(|u|+c(u^n)).\end{aligned}$$

Thus, for small values of Δt, one gets that

$$\begin{aligned}\frac{(f(\frac{u-u^n}{\Delta t}),u) + \langle Au,u\rangle}{\|u\|} &\geq \frac{\frac{\alpha}{\Delta t}|u|^2 - C(\Delta t)(|u|+c(u^n)) + J(u) - J(0)}{\|u\|}\\ &\geq \frac{\delta|u|^2 + J(u)}{\|u\|} - C(\Delta t, u^n).\end{aligned}$$

Then, since by assumption φ_1 is coercive, the result of the lemma holds thanks to classical arguments on pseudomonotone operators (*Cf.* Showalter [68] : Cor. 7.1 p.84).
If one assumes that f derives form a potential F (see Remark 1.1), then, the convex function

$\varphi_3 : V \mapsto \mathbb{R}$, defined for any $u \in V$ by

$$\varphi_3(u) = \Delta t F(\frac{u-u^n}{\Delta t}) + J(u) - (g^{n+1}, u),$$

is continuous and Gâteaux-differentiable. Moreover,

$$\begin{aligned}
\langle d\varphi_3(u), v\rangle &= (f(\frac{u-u^n}{\Delta t}), v) + \langle Au, v\rangle - (g^{n+1}, v),\\
F(u) &= F(\Delta t\frac{u-u^n}{\Delta t} + u^n) \leq \Delta t F(\frac{u-u^n}{\Delta t}) + (1-\Delta t)F(\frac{u^n}{1-\Delta t})\\
&\text{and,}\\
\varphi_3(u) &\geq F(u) + J(u) - (g^{n+1}, u) - (1-\Delta t)F(\frac{u^n}{1-\Delta t})\\
&\geq F(u) + J(u) - |g^{n+1}|.|u| - C(\Delta t) = \left[\frac{F(u)+J(u)}{|u|} - |g^{n+1}|\right]|u| - C(\Delta t).
\end{aligned}$$

The coercivity of φ_2 yields the existence a critical point to φ_3 which corresponds to a solution u^{n+1} for the lemma (see Proposition 2.1.14 p.26 of Borwein-Vanderwerff [18]). □

Remark 2.2 *Note that if f, or A, is strictly monotone, then the solution is unique. Indeed, if u and $\hat{u}$ are two given solutions, one has*

$$\Delta t\left(f(\frac{u-u^n}{\Delta t}) - f(\frac{\hat{u}-u^n}{\Delta t}), \frac{u-u^n}{\Delta t} - \frac{\hat{u}-u^n}{\Delta t}\right) + \langle Au - A\hat{u}, u-\hat{u}\rangle = 0.$$

2.2 *A priori* estimates

Let us test Equation (2.1) with $v = \frac{u^{n+1}-u^n}{\Delta t}$. Then,

$$\left(f(\frac{u^{n+1}-u^n}{\Delta t}), \frac{u^{n+1}-u^n}{\Delta t}\right) + \left\langle Au^{n+1}, \frac{u^{n+1}-u^n}{\Delta t}\right\rangle = \left(g^{n+1}, \frac{u^{n+1}-u^n}{\Delta t}\right),$$

yields

$$\Delta t\frac{\alpha}{2}\left|\frac{u^{n+1}-u^n}{\Delta t}\right|^2 + J(u^{n+1}) \leq J(u^n) + \frac{\Delta t}{\alpha}|g^{n+1}|^2 + \lambda\Delta t.$$

Thus, there exists a constant C such that

$$\sum_{k=0}^{n} \Delta t \frac{\alpha}{2} \left| \frac{u^{k+1} - u^k}{\Delta t} \right|^2 + J(u^{n+1}) \leq J(u_0) + T\lambda + \frac{\Delta t}{\alpha} \sum_{k=0}^{n} |g^{k+1}|^2 \leq C,$$

and

Lemma 2.3 *There exists a constant C such that*

$$\left\| \partial_t \widetilde{u}^{\Delta t} \right\|_{L^2(0,T,H)} + \left\| J(u^{\Delta t}) \right\|_{L^\infty(0,T)} + \frac{1}{\Delta t^2} \| \widetilde{u}^{\Delta t} - u^{\Delta t} \|^2_{L^2(0,T,H)} \leq C.$$

Since, $\left| f\left(\frac{u^{k+1} - u^k}{\Delta t} \right) \right| \leq C_1 \left| \frac{u^{k+1} - u^k}{\Delta t} \right| + C_2$, one has that

Lemma 2.4 *There exists a constant C such that*

$$\left\| f\left(\partial_t \widetilde{u}^{\Delta t} \right) \right\|_{L^2(0,T,H)} \leq C.$$

If one assumes that f derives form a potential F (see Remark 1.1), then

$$F(u^{n+1}) \leq F(u_0) + |f(u^{n+1})|.|u_0 - u^{n+1}| \leq F(u_0) + [C_1|u^{n+1}| + C_2]|u_0 - u^{n+1}| \leq Cte$$

and there exists a constant C such that $\varphi_2(u^n) \leq C$. Since φ_1 (resp. φ_2) is coercive, this yields

Lemma 2.5 *There exists a constant C such that $\|u^{\Delta t}\|_{L^\infty(0,T,V)} + \|\widetilde{u}^{\Delta t}\|_{L^\infty(0,T,V)} \leq C$.*

Finally, since for any $v \in V$, $\langle Au^{n+1}, v \rangle = \left(g^{n+1} - f(\frac{u^{n+1} - u^n}{\Delta t}), v \right)$, one gets that $\sup\limits_{v \neq 0} \dfrac{\langle Au^{n+1}, v \rangle}{\|v\|} \leq C \left| g^{n+1} - f(\dfrac{u^{n+1} - u^n}{\Delta t}) \right|$ and

Lemma 2.6 *There exists a constant C such that $\|Au^{\Delta t}\|_{L^2(0,T,V')} \leq C$.*

2.3 At the limit

Let us recall that, by construction, almost everywhere in $]0,T[$, one has the discretization

$$\forall v \in V, \quad \left(f(\partial_t \widetilde{u}^{\Delta t}), v \right) + \langle Au^{\Delta t}, v \rangle = \left(g^{\Delta t}, v \right). \tag{2.2}$$

As $\widetilde{u}^{\Delta t}$ is bounded in $W^{1,\infty,2}(0,T,V,H)$, up to a subsequence denoted similarly, Simon's compactness argument ensures the existence of $u \in W^{1,\infty,2}(0,T,V,H)$ such that $\widetilde{u}^{\Delta t}$ converges

weakly to u in $L^\infty(0,T,V)$ weak-* and strongly in $C([0,T],H)$.
Moreover, $u^{\Delta t}$ converges to u in $L^\infty(0,T,V)$ weak-* and strongly in $L^2(0,T,H)$, and $\partial_t\widetilde{u}^{\Delta t}$ converges weakly to $\partial_t u$.
Concerning the nonlinear terms, one denotes by f_u and A_u the weak limits, respectively in $L^2(0,T,H)$ and $L^2(0,T,V')$, of $f(\partial_t\widetilde{u}^{\Delta t})$ and $Au^{\Delta t}$. Thus,

$$\int_0^T \langle Au^{\Delta t}, u^{\Delta t}-u\rangle dt = \int_0^T \left(g^{\Delta t} - f(\partial_t\widetilde{u}^{\Delta t}), u^{\Delta t}-u\right) dt \to 0 \quad \text{when } \Delta t \to 0,$$

and, $\displaystyle\int_0^T \langle Au^{\Delta t}, u^{\Delta t}\rangle dt \to \int_0^T \langle A_u, u\rangle dt$ when $\Delta t \to 0$.
By assumption, the application $u \in V \mapsto \langle Au, v\rangle$ is continuous. Thus, if $w:(0,T)\to V$ is a measurable function, Aw is a weak-* measurable one. Since by assumption V is a separable reflexive Banach space, Aw is firstly weakly measurable, then measurable.

Set $v \in L^\infty(0,T,V)$ and $|\lambda|\le 1$. By monotonicity of A and thanks to the previous convergence, one gets that $0 \le \lambda\int_0^T\langle A_u - A(u-\lambda v), v\rangle dt$.
For $t \in]0,T[$ a.e., one gets that $\|u(t)-\lambda v(t)\| \le C = \|u\|_{L^\infty(0,T,V)} + \|v\|_{L^\infty(0,T,V)}$.
Since J is bounded above on bounded subsets of V, $A(\overline{B}_V(0,C))$ is bounded (*e.g.* [18] Prop. 4.1.25 p.137) and M exists such that, t a.e. in $]0,T[$, $\|A(u(t)-\lambda v(t))\|_{V'} \le M$.
Since A is demi-continuous, $< A_u - A(u-\lambda v), v >$ converges to $< A_u - Au, v >$ when λ goes to 0. Then, Lebesgue's theorem yields the convergence of $\int_0^T\langle A_u - A(u-\lambda v), v\rangle dt$ to $\int_0^T\langle A_u - A(u), v\rangle dt$ when λ goes to 0 and one concludes that $0 = \int_0^T < A_u - A(u), v > dt$ and that $A_u = Au$.
By passing to the limit, one gets that $f_u + Au = g$ in $L^2(0,T,H)$, or similarly, that $\partial_t u + Au = h := g - f_u + \partial_t u$ where $h \in L^2(0,T,H)$ and $u_0 \in V$.
Then, thanks to Appendix H, for any t, the following equality holds:

$$\int_0^t (f_u, \partial_t u) ds + J(u(t)) = J(u_0) + \int_0^t (g, \partial_t u) ds.$$

Coming back to the discrete formulation, adding

$$\left(f(\frac{u^{n+1}-u^n}{\Delta t}), \frac{u^{n+1}-u^n}{\Delta t}\right) + < Au^{n+1}, \frac{u^{n+1}-u^n}{\Delta t} >= \left(g^{n+1}, \frac{u^{n+1}-u^n}{\Delta t}\right)$$

over n yields

$$\Delta t \sum_{n=0}^{N-1}\left(f(\frac{u^{n+1}-u^n}{\Delta t}), \frac{u^{n+1}-u^n}{\Delta t}\right) + J(u^N) \le J(u_0) + \Delta t\sum_{n=0}^{N-1}\left(g^{n+1}, \frac{u^{n+1}-u^n}{\Delta t}\right),$$

i.e.

$$\int_0^T (f(\partial_t \widetilde{u}^{\Delta t}), \partial_t \widetilde{u}^{\Delta t}) dt + J(\widetilde{u}^{\Delta t}(T)) \leq J(u_0) + \int_0^T (g^{\Delta t}, \partial_t \widetilde{u}^{\Delta t}) dt.$$

Since $\widetilde{u}^{\Delta t}$ converges to u in $C([0,T],H)$ and as $\widetilde{u}^{\Delta t}(T)$ is bounded in V, one gets that $\widetilde{u}^{\Delta t}(T)$ converges weakly to $u(T)$ in V (note that the same can be told for any t, *i.e.* $\widetilde{u}^{\Delta t}(t)$ converges weakly to $u(t)$ in V), and we get back the initial condition $u(t=0) = u_0$ in V. Thus,

$$\begin{aligned}
& \limsup_{\Delta t} \int_0^T (f(\partial_t \widetilde{u}^{\Delta t}), \partial_t \widetilde{u}^{\Delta t}) dt + J(u(T)) \\
\leq\ & \limsup_{\Delta t} \int_0^T (f(\partial_t \widetilde{u}^{\Delta t}), \partial_t \widetilde{u}^{\Delta t}) dt + \liminf_{\Delta t} J(\widetilde{u}^{\Delta t}(T)) \\
\leq\ & \limsup_{\Delta t} \left[\int_0^T (f(\partial_t \widetilde{u}^{\Delta t}), \partial_t \widetilde{u}^{\Delta t}) dt + J(\widetilde{u}^{\Delta t}(T)) \right] \\
\leq\ & J(u_0) + \int_0^T (g, \partial_t u) dt = \int_0^T (f_u, \partial_t u) ds + J(u(T)).
\end{aligned}$$

Then, an argument of Minty's type in $L^2(0,T,H)$, similar to one used above with A, leads to $f_u = f(\partial_t u)$ and to the existence of a solution. Note in particular that $Au = g - f(\partial_t u) \in H$ and

that, for any t,

$$\int_0^t (f(\partial_t u), \partial_t u) ds + J(u(t)) = J(u_0) + \int_0^t (g, \partial_t u) ds \tag{2.3}$$

and

$$\alpha \int_0^t |\partial_t u|^2 ds + 2J(u(t)) \leq 2J(u_0) + \frac{1}{\alpha} \int_0^t |g|^2 ds + 2\lambda. \tag{2.4}$$

2.4 If A is linear

If A is a linear operator and if u and $\hat{u}$ are given solutions associated to the initial conditions $u_0, \hat{u}_0$ and the right hand side members $g, \hat{g}$, one gets that

$$(f(\partial_t u) - f(\partial_t \hat{u}), v) + \langle A(u - \hat{u}), v \rangle = (g - \hat{g}, v), \quad (u - \hat{u})(t=0) = u_0 - \hat{u}_0,$$

i.e., by denoting $W = u - \hat{u}$,

$$(\partial_t W, v) + \langle AW, v \rangle = ([g - \hat{g}] - [f(\partial_t u) - f(\partial_t \hat{u})] + \partial_t W, v), \quad (u - \hat{u})(t=0) = u_0 - \hat{u}_0.$$

Then, thanks to Appendix H, for any t,

$$\int_0^t |\partial_t W|^2 ds + J(W(t)) = J(u_0 - \hat{u}_0) + \int_0^t ([g - \hat{g}] - [f(\partial_t u) - f(\partial_t \hat{u})] + \partial_t W, \partial_t W) ds,$$

and

$$\int_0^t (f(\partial_t u) - f(\partial_t \hat{u}), \partial_t W) ds + J(W(t)) = J(u_0 - \hat{u}_0) + \int_0^t (g - \hat{g}, \partial_t W) ds. \tag{2.5}$$

Since A is linear, $A(0) = 0$ and $0 \in \partial J(0)$, *i.e.* $J(0) = \min J$ (Prop. 4.1.8 p.130 [18] for example), and

Proposition 2.7 *If moreover A is a linear operator and assuming that either f is strictly monotone, or the optimal value of J is only satisfied at* 0*, then the solution is unique.*

If A is linear and $J(0) = 0$, then, for any $u, v \in V$, one gets that $J(u) = \frac{1}{2}\langle Au, u\rangle$, $\langle Au, v\rangle = \langle Av, u\rangle$ and $\|.\|_A : u \in V \mapsto \sqrt{\langle Au, u\rangle}$ is a norm on V associated to the scalar product $(u, v)_A \mapsto \langle Au, v\rangle$.
Note that assuming that $J(v) > 0$ if $v \neq 0$ yields that $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent norms over V. Indeed, the first inequality holds since A is bounded on the bounded sets, A is a continuous linear operator.
Assume that the second one doesn't hold. Then, there exists a sequence $(v_n) \in V$ such that $\|v_n\| = 1$, v_n converges weakly (resp. strongly) to a given v in V (resp. H) and $J(v_n) = 2\|v_n\|_A^2$ goes to 0. Since J is a continuous convex function, one gets that $0 = J(0) \leq J(v) \leq 0$, and since $J(v) > 0$ if $v \neq 0$, one concludes that $v = 0$.
As φ_1 is a bilinear coercive mapping, there exists a positive constant α such that, for any $u \in V$, $\varphi_1(u) = \delta|u|^2 + \|u\|_A^2 \geqslant \alpha\|u\|^2$. Since $\varphi_1(v_n)$ tends to 0, one has that v_n goes to 0 in V, one gets a contradiction and the norms are equivalent.

The solution u belongs to $W^{1,\infty,2}(0, T, V, H)$. Then, it belongs to $C_w([0, T], V)$, the V-valued scalar continuous functions. Since (2.3) yields the continuity of the norm, u is a V-valued continuous function.

Since (2.5) and (2.4) yield the existence of a positive constant $C = C(u_0, \hat{u}_0, g, \hat{g})$ such that, for any t,

$$\int_0^t (f(\partial_t u) - f(\partial_t \hat{u}), \partial_t(u - \hat{u}))\, ds + \frac{1}{2}\|(u - \hat{u})(t)\|_A^2 \quad \leq \quad J(u_0 - \hat{u}_0) + C\|g - \hat{g}\|_{L^2(0,T,H)},$$

one gets the continuity of the infinity-norm of the solution with respect to u_0 and g.

If f is assumed to be strongly monotone, then the time derivative of the solution is continuous with respect to u_0 and g in $L^2(0,T,H)$. This finishes the proof of the corollary.

3 The stochastic case

In this section, we are interested in the stochastic version of Barenblatt's equations. So, we need first to precise the sense we wish to give to the stochastic version of an equation with such a nonlinear term. For this, remark that the homogeneous deterministic equation writes : $\partial_t u \in f^{-1}(-Au)$. Then, the stochastic version of the problem would be: $du \in f^{-1}(-Au)dt + hdw$ where $w = \{w_t, \mathcal{F}_t; 0 \le t \le T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with the property that $w_0 = 0$; and $h \in \mathcal{N}_w^2(0,T,V)$, the set of predictable functions of $L^2\left(]0,T[\times\Omega, V\right)$ ([64] p.28 for example). Following Adimurthi-Seam-Vallet [2], Barbu [8] Section 44 p. 183, Vallet [77] or Vallet-Wittbold [78] for example, the equation can be understood in the following way:

$$\partial_t[u - \int_0^t hdw(s)] \in f^{-1}(-Au), \quad i.e. \quad f(\partial_t[u - \int_0^t hdw(s)]) + Au = 0,$$

where $\int_0^t hdw(s)$ denotes the Itô integration of h.
Then $U = u - \int_0^t hdw(s)$ is a solution to the random equation

$$f(\partial_t U) + A(t)U = 0 \quad \text{where } A(t)U = A[U + \int_0^t hdw(s)].$$

Since we are interested in strong solutions, standard argumentations do not suit and additional assumptions are needed.
In the sequel, A is assumed to be linear, $J(v) > J(0) = 0$ for any $v \neq 0$, and f strongly monotone. Thus, as explained in the end of the previous section, $J(u) = \frac{1}{2}\langle Au, u\rangle$ and $(.,.)_A : (u,v) \mapsto \langle Au, v\rangle$ is a scalar product. One denotes by $\|\cdot\|_A$ the associated norm; it is equivalent to the one of V.

Thanks to the continuity and the linearity of A (see Prévôt-Röckner [64] Lemma 2.4.1 p. 35 for example), the problem is equivalent to

$$(\tilde{\mathcal{P}}_1) \quad : \quad f(\partial_t U) + AU = -A\int_0^t hdw = -\int_0^t Ahdw, \quad U(t=0) = u_0,$$

where $Ah \in N_w^2(0,T,V')$.

Lemma 3.1 *There exists at most one solution of Problem (P_1).*

Indeed, if u and $\hat{u}$ are two solutions, associated with the same function h, if one denotes by $U = u - \int_0^t hdw$, $\hat{U} = \hat{u} - \int_0^t hdw$ and $W = u - \hat{u} = U - \hat{U}$, one gets:

$$\partial_t W + AW = G := \partial_t W + f(\partial_t \hat{U}) - f(\partial_t U), \qquad W(t=0) = 0.$$

Then, thanks to Appendix H, the following energy equality holds for any t:

$$\begin{aligned} &\int_0^t |\partial_t W|^2 ds + J(W(t)) = J(0) + \int_0^t (G, \partial_t W) ds, \\ i.e. \quad &\int_0^t \left(f(\partial_t U) - f(\partial_t \hat{U}), \partial_t [U - \hat{U}] \right) ds + J(W(t)) = 0, \end{aligned}$$

and the solution is unique.

We wish, in the sequel, to use the previous section. So, in a first step, we assume that $h \in N_w^2(0,T,V)$ and $Ah \in N_w^2(0,T,H)$.
Thus, P-a.s. in Ω, $\int_0^t Ahdw \in L^2(0,T,H)$ and there exists a unique solution to $(\tilde{\mathcal{P}}_1)$.
Moreover, the result of continuity of Corollary 1.3 ensures that $U \in N_w^2(0,T,V)$ thus $u \in N_w^2(0,T,V)$ as well, and that $\partial_t U \in L^2(\Omega\times]0,T[,H)$.
In particular, for any t,

$$\begin{aligned} \int_0^t (f(\partial_t U), \partial_t U) ds + \frac{1}{2}\|U(t)\|_A^2 &= \frac{1}{2}\|u_0\|_A^2 - \int_0^t \left(\int_0^s Ahdw, \partial_t U \right) ds, \\ \text{and} \quad \alpha \int_0^t |\partial_t U|^2 ds + \|U(t)\|_A^2 &\leq \|u_0\|_A^2 + \frac{1}{\alpha}\int_0^T \left| \int_0^s Ahdw \right|^2 ds + 2\lambda. \end{aligned}$$

The same corollary asserts that, P-a.s., $U \in C([0,T],V)$. Thus, for any fixed time t and any sequence $(t_n) \in [0,T]$ such that t_n converges to t, one gets that $\|U(t_n) - U(t)\|$ goes to 0 P-a.s. Moreover,

$$\begin{aligned} \|U(t_n) - U(t)\|^2 &\leq 2\|U(t_n)\|^2 + 2\|U(t)\|^2 \leq C(\|U(t_n)\|_A^2 + \|U(t)\|_A^2) \\ &\leq C[\|u_0\|_A^2 + \frac{1}{\alpha}\int_0^T \left| \int_0^s Ahdw \right|^2 ds + 2\lambda] \end{aligned}$$

Thanks to the above inequality, Lebesgue's theorem yields $E\|U(t_n) - U(t)\|^2$ goes to 0 and leads to the continuity of U from $[0,T]$ to $L^2(\Omega, V)$. Then, thanks to the properties of the Itô integral, it is the same for u.

Consider two solutions u and $\hat{u}$, associated with $U = u - \int_0^t h dw$ and $\hat{U} = \hat{u} - \int_0^t \hat{h} dw$ and with the initial conditions u_0 and $\hat{u}_0$. For convenience, set $W = u - \hat{u} - \int_0^t [h - \hat{h}] dw$ and $\overline{u} = u - \hat{u}$ and note that for any $t > \Delta t > 0$,

$$\begin{aligned}
&\big(f(\partial_t U) - f(\partial_t \hat{U}), W(t) - W(t-\Delta t)\big) + (\overline{u}(t), \overline{u}(t) - \overline{u}(t-\Delta t))_A \\
= \ & (\overline{u}(t), \int_{t-\Delta t}^t (h - \hat{h}) dw)_A.
\end{aligned}$$

Then, an integration from Δt to t gives

$$\begin{aligned}
&\int_{\Delta t}^t \left(f(\partial_t U) - f(\partial_t \hat{U}), \frac{W(s) - W(s-\Delta t)}{\Delta t} \right) ds + \frac{1}{2\Delta t} \int_{\Delta t}^t \|\overline{u}(s) - \overline{u}(s-\Delta t)\|_A^2 ds \\
&+ \frac{1}{2\Delta t} \int_{t-\Delta t}^t \|\overline{u}(s)\|_A^2 ds \\
\leq \ & \frac{1}{2\Delta t} \int_0^{\Delta t} \|\overline{u}(s)\|_A^2 ds + \frac{1}{\Delta t} \int_{\Delta t}^t \left(\overline{u}(s) - \overline{u}(s-\Delta t), \int_{s-\Delta t}^s (h - \hat{h}) dw \right)_A ds \\
&+ \frac{1}{\Delta t} \int_{\Delta t}^t \left(\overline{u}(s-\Delta t), \int_{s-\Delta t}^s (h - \hat{h}) dw \right)_A ds
\end{aligned}$$

and, by taking the expectation, the following inequalities hold

$$\begin{aligned}
&E \int_{\Delta t}^t \left(f(\partial_t U) - f(\partial_t \hat{U}), \frac{W(s) - W(s-\Delta t)}{\Delta t} \right) ds + \frac{1}{2\Delta t} E \int_{t-\Delta t}^t \|\overline{u}(s)\|_A^2 ds \\
\leq \ & \frac{1}{2\Delta t} E \int_0^{\Delta t} \|\overline{u}(s)\|_A^2 ds + \frac{1}{2\Delta t} \int_{\Delta t}^t E \| \int_{s-\Delta t}^s (h - \hat{h}) dw \|_A^2 ds \\
\leq \ & \frac{1}{2\Delta t} E \int_0^{\Delta t} \|\overline{u}(s)\|_A^2 ds + \frac{1}{2\Delta t} \int_{\Delta t}^t \int_{s-\Delta t}^s E \|(h - \hat{h})(\sigma)\|_A^2 d\sigma ds.
\end{aligned}$$

At the limit, one gets that for any t

$$\begin{aligned}
&E \int_0^t \big(f(\partial_t U) - f(\partial_t \hat{U}), \partial_t [U(t) - \hat{U}(t)] \big) ds + \frac{1}{2} E \|(u - \hat{u})(t)\|_A^2 \\
\leq \ & \frac{1}{2} E \|u_0 - \hat{u}_0\|_A^2 + \frac{1}{2} \int_0^t E \|h - \hat{h}\|_A^2 ds. \qquad (3.1)
\end{aligned}$$

Consider $h \in N_w^2(0,T,V)$ and $(h_n) \subset N_w^2(0,T,V)$ such that $Ah_n \in N_w^2(0,T,H)$ and (h_n) converges to h in $N_w^2(0,T,V)$. Thanks to the previous inequality, the sequence (u_n) of the corresponding solutions is a Cauchy sequence in $C([0,T], L^2(\Omega, V))$.
As the same kind of calculations leads to the boundedness of $\partial_t(u_n - \int_0^t h_n dw)$ in $L^2(]0,T[\times\Omega, H)$, the uniqueness of the possible limit-point for the weak convergence yields the

weak convergence of the sequence to $\partial_t(u - \int_0^t hdw)$ in $L^2(]0,T[\times\Omega, H)$. Moreover, up to a subsequence, $f\left(\partial_t(u_{n_k} - \int_0^t h_{n_k}dw)\right)$ converges weakly to a given element χ in $L^2(]0,T[\times\Omega, H)$. Using again (3.1), one gets that

$$\limsup_{n,m} E\int_0^T \left(f(\partial_t U^n) - f(\partial_t \hat{U}^m), \partial_t[U^n(t) - \hat{U}^m(t)]\right) dt \leq 0,$$

and thanks to the assumptions on f, one concludes that $\chi = f(\partial_t(u - \int_0^t hdw))$ (see Lemma 2.3 p.38 of Barbu [8] for example), that a solution exists and that (3.1) holds for any h and $\hat{h} \in N^2_w(0,T,V)$.

3.1 The multiplicative case

Assume that $\mathcal{H} : V \to V$ is a Lipschitz-continuous mapping. Then $\mathcal{H}(h) \in \mathcal{N}^2_w(0,T,V)$ if $h \in \mathcal{N}^2_w(0,T,V)$ (Prévôt-Röckner [64] Lemma 2.41 p.35), and the result of this section is the following:

Denote by $\Phi : \mathcal{N}^2_w(0,T,V) \to \mathcal{N}^2_w(0,T,V)$ the map defined for any $h \in \mathcal{N}^2_w(0,T,V)$ by $\Phi(h) = u$ where u is the solution of the Barenblatt's problem

$$f\left(\partial_t[u - \int_0^t \mathcal{H}(h)dw]\right) + Au = 0$$

for the initial condition $u(0,.) = u_0$. Then, u is a solution to Problem $(\mathcal{P}_{\mathcal{H}})$, if and only if, u is a fixed-point to Φ.
Then, for any positive α, (3.1) yields

$$\begin{aligned}\int_0^T e^{-\alpha t}E\|[\Phi(h) - \Phi(\hat{h})](t)\|^2_A dt &\leq 2\int_0^T e^{-\alpha t}\int_0^t E\|[\mathcal{H}(h) - \mathcal{H}(\hat{h})](s)\|^2 dsdt \\ &\leq \frac{C}{\alpha}\int_0^T e^{-\alpha s}E\|[h - \hat{h}](s)\|^2_A ds.\end{aligned}$$

Since the exponential weight in time provides an equivalent norm in $\mathcal{N}^2_w(0,T,V)$, if $\alpha > C$, Φ is a contractive mapping, it has a unique fixed-point and the result holds.

Chapter III

The Cauchy problem for a conservation law with a multiplicative stochastic perturbation

In this chapter, we are interested in the Cauchy problem for a multi-dimensional nonlinear conservation law with a multiplicative stochastic perturbation. This study is published in a joint work with G. Vallet and P. Wittbold [14]. We investigate the following problem:

$$du - \operatorname{div}\vec{\mathbf{f}}(u)dt = h(u)dw \quad \text{in }]0,T[\times\mathbb{R}^d\times\Omega, \tag{0.1}$$

with an initial condition u_0 and $d \geq 1$. In the sequel we assume that T is a positive number, $Q =]0,T[\times\mathbb{R}^d$ and that $w = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on the classical Wiener space $(\Omega, \mathcal{F}, P)$. These assumptions on w are made for convenience.

Let us assume that

H$_1$: $\vec{\mathbf{f}} = (f_1, .., f_d) : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz-continuous function and $f_i(0) = 0$, $\forall i = 1, ..., d$.

H$_2$: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$.

H$_3$: $u_0 \in L^2(\mathbb{R}^d)$.

Our aim is to prove a result of existence and uniqueness of the stochastic entropy solution to the above-mentioned problem.
Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. One needs to introduce the notion of entropy solution in order to discriminate the physical solution. Using the concept of measure-valued solutions and Kruzhkov's entropy formulation, a result of existence and uniqueness of entropy solution is proved.

1 Introduction

1.1 Former results

Many papers on the viscous parabolic Burgers type stochastic problem can be found in the literature. Let us mention, without exhaustiveness, Da Prato-Debussche-Temam [29], Da Prato-Zabczyk [30], Grecksch-Tudor [45] or Gyöngy-Nualart [46].
Only few papers have been devoted to the study of stochastic perturbation of nonlinear first-order hyperbolic problems. Most of them are interested in the Cauchy problem in the 1D case and/or in the case of additive noise (*i.e.*, with a right-hand side $h(t,x)dw$ independent of the solution u). Let us cite the paper of Holden-Risebro [49] where an operator splitting method is proposed to prove the existence of a weak solution to the Cauchy problem

$$du + f(u)_x dt = g(u)dw \qquad \text{in }]0,T[\times\mathbb{R}.$$

The convergence is obtained by using pathwise arguments.
In the paper of E-Khanin-Mazel-Sinai [38], the authors are interested in the invariant measures for the Burgers equation

$$du + \frac{1}{2}(u^2)_x dt = \left(\sum_{k\geq 0} F_k(x)dw_k\right)_x \qquad \text{in }]0,T[\times\mathbb{R}$$

with a periodic assumption in space. Existence and uniqueness of a stochastic entropy solution is proved thanks to a Hopf-Lax type formula for the corresponding stochastic Hamilton-Jacobi equation.
In the paper of Kim [53] a method of compensated compactness is used to prove, *via* vanishing viscosity approximation, existence of a stochastic weak entropy solution to the Cauchy problem

$$du + f(u)_x dt = h(t,x)dw \qquad \text{in }]0,T[\times\mathbb{R}.$$

A Kruzhkov-type method is used to prove the uniqueness.
In Vallet-Wittbold [78], the authors proposed to extend the result of Kim to the multi-dimensional Dirichlet problem for a nonlinear conservation law with additive noise

$$\begin{aligned} du + \operatorname{div}\vec{\mathbf{f}}(u)dt = h(t,x)dw \quad &\text{in} \quad]0,T[\times D \\ "u = 0 \quad &\text{on} \quad]0,T[\times\partial D" \end{aligned}$$

where D is a bounded domain in $\mathbb{R}^d$ ($d \geq 1$). As weak and entropy solutions are not smooth enough allowing for trace properties and, moreover, a Dirichlet condition can only be imposed on the free set of entering characteristics, the boundary condition has to be understood in an appropriate way. In Vallet-Wittbold [78] the authors followed the approach of J. Carrillo which consists in formulating the boundary condition implicitly *via* global integral entropy inequalities involving the semi-Kruzhkov entropies. Using the vanishing viscosity method and Young measure techniques the authors proved existence, and, *via* Kruzhkov doubling variables technique, the uniqueness of the stochastic entropy solution.
Feng-Nualart [41] proposed an extension of Kim's result in another direction, namely to the Cauchy problem in $\mathbb{R}^d$

$$du + \operatorname{div}\mathbf{F}(u)dt = \int_{z\in Z} \sigma(\cdot, u, z)dw(t, z)$$

with multiplicative noise. Regarding the random term on the right-hand side of the equation, Z is a metric space and $W(t, dz)$ is a space-time Gaussian white noise martingale random measure with respect to a filtration $\mathcal{F}_t$. The authors consider a flux function $\mathbf{F}$ of class $\mathcal{C}^2(\mathbb{R}; \mathbb{R}^d)$ such that its second derivatives have at most polynomial growth, and a random initial condition u_0 satisfying $E[\|u_0\|_p^p + \|u_0\|_2^p] < \infty$ for every $p \geqslant 1$. The dependence of the right-hand side on u leads to considerable new difficulties. Indeed, in the case of additive noise $h(t, x)dw$ the equation, which has to be understood in the following way

$$\partial_t \left[u - \int_0^t h dw(s) \right] - \operatorname{div}\vec{\mathbf{f}}(u) = 0 \qquad \textit{in } \mathcal{D}'(Q),$$

can be formulated, *via* the change of variable $v = u - \int_0^t h dw(s)$, as the random problem

$$\partial_t v - \operatorname{div}\vec{\mathbf{f}}(\omega, t, x, v) = 0$$

with a flux function $\vec{\mathbf{f}}(\omega, t, x, v) = \vec{\mathbf{f}}(v + \int_0^t h dw(s))$. In this equation, the stochastic variable ω, at least formally, only plays the role of a parameter and thus essentially deterministic techniques can be applied (though it is not possible to use exclusively pathwise arguments).
In presence of multiplicative noise, the problem $\partial_t \left[u - \int_0^t h(u)dw(s)\right] - \operatorname{div}\vec{\mathbf{f}}(u) = 0$ is nonlocal in time and, mainly due to the lack of regularity of u, a similar reduction is not possible. For this reason, Feng-Nualart introduced in [41] a notion of strong entropy solution in order to prove uniqueness of the entropy solution. Using the vanishing viscosity and compensated compactness arguments, the authors established existence of strong entropy solutions only in

the 1D case.
In the recent paper Chen-Ding-Karlsen [27] propose to revisit the work of J. Feng and D. Nualart. They prove under a "BV-bound" additional assumption on u_0 that the multidimensional stochastic problem is well-posed by using a uniform spatial BV-bound. They show the existence of strong stochastic entropy solutions in $L^p \cap BV$ and develop a "continuous dependence" theory for stochastic entropy solutions in BV, which can be used to derive an error estimate for the vanishing viscosity method.
Finally, let us mention the paper by Debussche-Vovelle [32] which gives the first complete well-posedness result for multi-dimensional scalar conservation laws driven by a general multiplicative noise:

$$du + \operatorname{div}(A(u))dt = \Phi(u)dW(t), \quad x \in \mathbb{T}^N, t \in (0,T),$$

which is considered on the N-dimensional torus $\mathbb{T}^N$. The flux function A is supposed to be of class $\mathcal{C}^2(\mathbb{R};\mathbb{R}^N)$ such that its derivatives have at most polynomial growth, and they assume that $u_0 \in L^\infty(\mathbb{T}^N)$. The authors consider a general noise W, assuming to be a cylindrical Wiener process. They use the kinetic formulation of the problem and prove existence and uniqueness of a kinetic solution.

1.2 Goal of the study

In the present work, we propose to prove a result of existence and uniqueness of a stochastic entropy solution, in the sense of Definition 2.2, to the Cauchy problem for the stochastic conservation law with multiplicative noise (0.1) in the d-dimensional case. Comparing with the previous authors, we consider here a time Gaussian white noise, assuming w to be a real-valued continuous Brownian motion of dimension 1, $\vec{\mathbf{f}}$ a Lipschitz-continuous function and u_0 the initial condition chosen deterministic and in $L^2(D)$. A method of artificial viscosity is proposed to prove the existence of a solution. The compactness properties used are based on the theory of Young measures and on measure-valued solutions. In particular, for the convergences, instead of path-wise arguments and adapted processes, we propose to use the topological properties of the L^2-type Lebesgue space when it is endowed with the predictable σ-algebra.
Then, an appropriate adaptation of Kruzhkov's doubling variables technique is proposed to prove that any stochastic entropy solution is equal to the one given by the artificial viscosity method. Then, the notion of strong entropy condition of J. Feng and D. Nualart does not

seem to be required to capture the noise-noise interaction. Thus, the entropy inequalities are to sufficient to prove the uniqueness via Kato-type inequalities. This yields the uniqueness of the measure-valued entropy solution, and, by standard arguments, this allows to deduce existence and uniqueness of the stochastic weak entropy solution.

1.3 Plan of the study

The chapter is organized as follows. In Section 2 we introduce the notion of stochastic entropy (resp. measure-valued entropy) solution for (0.1) and establish some basic properties of such solutions. In Section 3 existence of a measure-valued entropy solution for (0.1) is proved *via* a vanishing viscosity approximation. Section 4 is devoted to the proof of uniqueness and of a contraction principle for measure-valued solutions. As a by-product we deduce existence and uniqueness of the entropy solution of the Cauchy problem for (0.1). In the Appendix I we have collected several auxiliary results (e.g. on regularity of stochastic integrals with respect to parameters, some basic results from Young measure theory and, for the convenience of the reader, we have also included a proof of existence of weak solutions for the approximate viscous parabolic problem).

1.4 Notations and functional setting

In the sequel we denote by $H^1(\mathbb{R}^d)$ the usual Sobolev space. We recall that $H^1(\mathbb{R}^d)$ is also the closure of $\mathcal{D}(\mathbb{R}^d)$, the space of $C^\infty(\mathbb{R}^d)$-functions with compact support in $\mathbb{R}^d$. We denote by $H^{-1}(\mathbb{R}^d)$ the dual space of $H^1(\mathbb{R}^d)$ which is also the space of derivatives of order less than one of elements of $L^2(\mathbb{R}^d)$ in the common Gelfand-Lions identification $H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \equiv L^2(\mathbb{R}^d)' \hookrightarrow H^1(\mathbb{R}^d)'$.

For any positive M, denote by $Q_M =]0,T[\times B(0,M)$ where $B(0,M)$ is the bounded open ball in $\mathbb{R}^d$ of radius M.

In general, if $G \subset \mathbb{R}^k$, $\mathcal{D}(G)$ denotes the restriction to G of $\mathcal{D}(\mathbb{R}^k)$ functions u such that support$(u) \cap G$ is compact. Then, $\mathcal{D}^+(G)$ will denote the subset of nonnegative elements of $\mathcal{D}(G)$.

For a given separable Banach space X we denote by $N^2_w(0,T,X)$ the space of the predictable X-valued processes (cf. Da Prato-Zabczyk [30] p.94 or Prévôt-Röckner [64] p.28 for example). This space is the space $L^2(]0,T[\times\Omega, X)$ for the product measure $dt \otimes dP$

on $\mathcal{P}_T$, the predictable σ-field (*i.e.* the σ-field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $]s,t] \times A$ for any $A \in \mathcal{F}_s$).
If $X = L^2(\mathbb{R}^d)$, one gets that $N^2_w(0,T,L^2(\mathbb{R}^d)) \subset L^2(Q \times \Omega)$.

Consider $\mathcal{E}$ the set of any $C^{2,1}(\mathbb{R})$ nonnegative convex approximative function of the absolute-value function such that $\eta(0) = 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ (resp. -1) if $x > \delta$ (resp. $x < \delta$). Then, η'' has a compact support and η and η' are Lipschitz-continuous functions (see Figure III.1).

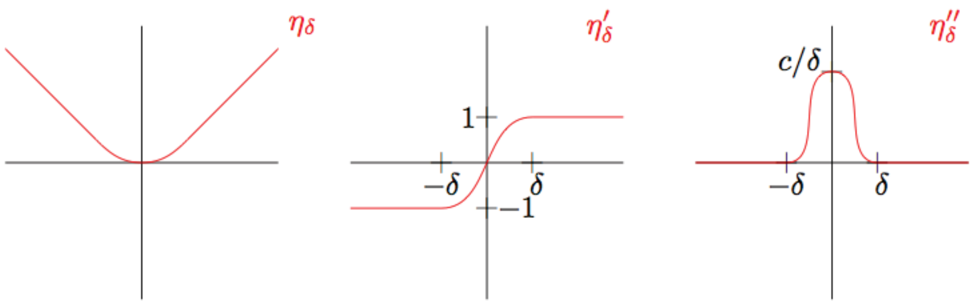


Figure III.1 – Kruzhkov smooth entropies

For convenience, denote by $\mathrm{sgn}_0(x) = \frac{x}{|x|}$ if $x \neq 0$ and 0 otherwise; $F(a,b) = \mathrm{sgn}_0(a-b)[\vec{\mathbf{f}}(a) - \vec{\mathbf{f}}(b)]$ and $F^\eta(a,b) = \displaystyle\int_b^a \eta'(\sigma - b)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma$. Note, in particular, that F and F^η are Lipschitz-continuous functions.

2 Entropy formulation

Let us analyze the viscous parabolic case in order to propose an entropy formulation. Assume that for any positive ϵ, u_ϵ is the solution of the stochastic nonlinear parabolic problem

$$\mathrm{d}u_\epsilon - \left[\epsilon \Delta u_\epsilon + \mathrm{div}(\vec{\mathbf{f}}(u_\epsilon))\right] \mathrm{d}t = h(u_\epsilon)dw \quad \text{in }]0,T[\times \mathbb{R}^d \times \Omega, \tag{2.1}$$

for a smooth initial condition $u_0^\epsilon \in \mathcal{D}(\mathbb{R}^d)$. See Appendix I Section 2 for further information. The idea to obtain the entropy formulation satisfied for every solution of (0.1) is to pass to the limit on ϵ in a "viscous entropy formulation" satisfied by any solution u_ϵ of (2.1).

Let us explain the way we obtain the "viscous entropy formulation". Consider φ in $\mathcal{D}^+(\bar{Q})$, k a real number and $\eta \in \mathcal{E}$. Since $\eta(u_\epsilon - k)\varphi \in L^2(0,T,H^1(\mathbb{R}^d))$ P-a.s., it is possible to apply the Itô formula to the operator $\Psi(t,u_\epsilon) := \int_{\mathbb{R}^d} \eta(u_\epsilon - k)\varphi \,\mathrm{dx}$ and thus we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \eta(u_\epsilon(T) - k)\varphi(T)\,\mathrm{dx}\\
=\ &\int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\,\mathrm{dx} + \int_Q \eta(u_\epsilon - k)\partial_t\varphi\,\mathrm{dx\,dt}\\
&-\epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi\,\mathrm{dx\,dt} - \epsilon\int_Q \eta''(u_\epsilon - k)\varphi\nabla u_\epsilon\nabla u_\epsilon\,\mathrm{dx\,dt}\\
&-\int_Q \eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\nabla\varphi\,\mathrm{dx\,dt} - \int_Q \eta''(u_\epsilon - k)\varphi\vec{\mathbf{f}}(u_\epsilon)\nabla u_\epsilon\,\mathrm{dx\,dt}\\
&+\int_0^T\int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\,\mathrm{dx}dw(t) + \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi\,\mathrm{dx\,dt}.
\end{aligned}$$

Indeed, consider the process

$$u_\epsilon(t) = u_\epsilon(0) + \int_0^t \epsilon\Delta u_\epsilon + \operatorname{div}\vec{\mathbf{f}}(u_\epsilon)ds + \int_0^t h(u_\epsilon)dw(s),$$

defined for all $t \in [0,T]$. As $\epsilon\Delta u_\epsilon + \operatorname{div}\vec{\mathbf{f}}(u_\epsilon)$ is in $L^2(Q\times\Omega)$ (see Appendix I Section 2), it's an $L^2(\mathbb{R}^d)$-valued process Bochner integrable on $[0,T]$ (the reader can find the sense of the Bochner integrability in Da Prato-Zabczyk [30] p.19). We consider

$$\begin{aligned}
\Psi : [0,T]\times L^2(\mathbb{R}^d) &\to \mathbb{R}\\
(t,v) &\mapsto \int_{\mathbb{R}^d} \eta(v-k)\varphi(t,x)dx,
\end{aligned}$$

Ψ and its partial derivatives Ψ_t, Ψ_v and Ψ_{vv} are uniformly continuous on bounded subsets of $[0,T]\times L^2(\mathbb{R}^d)$ thanks to the regularities of φ and η. Thus P-a.s and for all $t \in [0,T]$, one gets:

$$\begin{aligned}
\Psi(t,u_\epsilon(t)) &= \Psi(0,u_0^\epsilon) + \int_0^t \Psi_t(s,u_\epsilon)ds\\
&+\int_0^t < \Psi_v(s,u_\epsilon), \epsilon\Delta u_\epsilon + \operatorname{div}\vec{\mathbf{f}}(u_\epsilon) > ds\\
&+\int_0^t < \Psi_v(s,u_\epsilon), h(u_\epsilon) > dw(s)\\
&+\frac{1}{2}\int_0^t < \Psi_{vv}(s,u_\epsilon), h^2(u_\epsilon) > ds.
\end{aligned}$$

For all $\overline{v} \in L^2(\mathbb{R}^d)$ one has

$$\begin{aligned}
\Psi_t(s,u_\epsilon) &= \int_{\mathbb{R}^d} \eta(u_\epsilon - k)\partial_t\varphi(s,x)dx,\\
<\Psi_v(s,u_\epsilon),\overline{v}> &= \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)\varphi(s,x)\overline{v}dx,\\
<\Psi_{vv}(s,u_\epsilon),\overline{v}> &= \int_{\mathbb{R}^d} \eta''(u_\epsilon - k)\varphi(s,x)\overline{v}dx.
\end{aligned}$$

Since the support of η'' is compact, for any $i = 1,\dots,d$, $\mathbb{R} \ni r \mapsto \eta''(r-k)f_i(r)$ is a bounded continuous function. Then, thanks to the chain-rule for Sobolev functions

$$\begin{aligned}
-\int_Q \eta''(u_\epsilon - k)\varphi\vec{\mathbf{f}}(u_\epsilon)\nabla u_\epsilon\, dxdt &= -\int_Q \varphi \operatorname{div}[\int_0^{u_\epsilon} \eta''(\sigma-k)\vec{\mathbf{f}}(\sigma)d\sigma]dxdt\\
&= \int_Q \nabla\varphi[\int_0^{u_\epsilon} \eta''(\sigma-k)\vec{\mathbf{f}}(\sigma)d\sigma]dxdt,
\end{aligned}$$

and so

$$\begin{aligned}
0 \le\ & \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\, dx + \int_Q \eta(u_\epsilon - k)\partial_t\varphi dxdt\\
& -\epsilon\int_Q \eta''(u_\epsilon - k)|\nabla u_\epsilon|^2\varphi dxdt - \epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi\, dxdt\\
& -\int_Q \eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon)\nabla\varphi\, dxdt + \int_Q \nabla\varphi[\int_0^{u_\epsilon} \eta''(\sigma-k)\vec{\mathbf{f}}(\sigma)d\sigma]dxdt\\
& +\int_0^T\int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\, dxdw(t) + \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi dxdt.
\end{aligned}$$

And, as $\vec{\mathbf{f}}(0) = \vec{0}$, by integration by parts

$$-\eta'(u_\epsilon - k)\vec{\mathbf{f}}(u_\epsilon) + \int_0^{u_\epsilon} \eta''(\sigma-k)\vec{\mathbf{f}}(\sigma)d\sigma = -\int_0^{u_\epsilon} \eta'(\sigma-k)\vec{\mathbf{f}}'(\sigma)d\sigma,$$

one gets

$$\begin{aligned}
0 \le\ & \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\, dx + \int_Q \eta(u_\epsilon - k)\partial_t\varphi dxdt - \epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi\, dxdt\\
& -\epsilon\int_Q \eta''(u_\epsilon - k)|\nabla u_\epsilon|^2\varphi\, dxdt - \int_Q\int_0^{u_\epsilon} \eta'(\sigma-k)\vec{\mathbf{f}}'(\sigma)\, d\sigma\nabla\varphi\, dxdt\\
& +\int_Q \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\, dxdw(t) + \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi dxdt.
\end{aligned}$$

In particular, if φ is null on the boundary or if $\eta'(\mathbb{R}^-) = \{0\}$ with $k \geq 0$ then, as $\int_0^k \eta'(\sigma - k)\vec{\mathbf{f}}'(\sigma)\,d\sigma$ is constant with respect to x and

$$
\begin{aligned}
0 \quad \leq \quad & \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\ dx + \int_Q \eta(u_\epsilon - k)\partial_t\varphi dxdt - \epsilon\int_Q \eta''(u_\epsilon - k)|\nabla u_\epsilon|^2\varphi\ dxdt \\
& -\epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi\ dxdt - \int_Q\int_k^{u_\epsilon} \eta'(\sigma - k)\vec{\mathbf{f}}'(\sigma)\,d\sigma\nabla\varphi\ dxdt \\
& + \int_Q \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\ dxdw(t) + \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi dxdt.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0 \quad \leq \quad & \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\,\mathrm{dx} + \int_Q \eta(u_\epsilon - k)\partial_t\varphi\,\mathrm{dx\,dt} - \epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi\,\mathrm{dx\,dt} \\
& - \int_Q F^\eta(u_\epsilon, k)\nabla\varphi\,\mathrm{dx\,dt} + \int_0^T\int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\,\mathrm{dx}dw(t) \\
& + \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi\,\mathrm{dx\,dt}.
\end{aligned}
$$

Given $A \in \mathcal{F}$ a P-measurable set, and taking the expectation, one finally has

$$
\begin{aligned}
0 \quad \leq \quad & E\int_{\mathbb{R}^d} 1_A\eta(u_0^\epsilon - k)\varphi(0)\ dx + E\int_Q 1_A\eta(u_\epsilon - k)\partial_t\varphi dxdt \\
& -\epsilon E\int_Q 1_A\eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi\ dxdt - E\int_Q 1_A\int_k^{u_\epsilon} \eta'(\sigma - k)\vec{\mathbf{f}}'(\sigma)\,d\sigma\nabla\varphi\ dxdt \\
& + E\int_Q 1_A\eta'(u_\epsilon - k)h(u_\epsilon)\varphi\ dxdw(t) + \frac{1}{2}E\int_Q 1_A h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi dxdt. \qquad (2.2)
\end{aligned}
$$

Let us call this inequality the "viscous entropy formulation". Now let us assume that, as ϵ tends to 0, the approximate solutions u_ϵ converge in an appropriate sense to a function $u \in \mathcal{N}_w^2(0, T; L^2(\mathbb{R}^d))$ such that $\epsilon\int_Q 1_A\eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi dxdt$ tends to 0 (the convergence issue will be studied rigorously in Section 3). Then we may pass to the limit in the above inequality and obtain a family of entropy inequalities satisfied by the limit function u. This observation motivates the definition of entropy solution for the stochastic conservation law (0.1) we will give below.

For convenience, for any function u of $\mathcal{N}_w^2(0, T, L^2(\mathbb{R}^d))$, any real k and any regular function η, denote P-a.s. in Ω by $\mu_{\eta,k}$, the distribution in $\mathbb{R}^{d+1}$, defined by

$$
\begin{aligned}
\varphi \mapsto \mu_{\eta,k}(\varphi) \quad = \quad & \int_{\mathbb{R}^d} \eta(u_0 - k)\varphi(0)\,\mathrm{dx} + \int_Q \eta(u - k)\partial_t\varphi - F^\eta(u, k)\nabla\varphi\,\mathrm{dx\,dt} \\
& + \int_0^T\int_{\mathbb{R}^d} \eta'(u - k)h(u)\varphi\,\mathrm{dx}dw(t) + \frac{1}{2}\int_Q h^2(u)\eta''(u - k)\varphi\,\mathrm{dx\,dt}.
\end{aligned}
$$

Remark 2.1 *Thanks to the Itô-integration by parts formula applied to the process* $X(t) = \int_0^t \eta'(u-k)h(u)dw$ *and the application* $F(t,X(t)) = X(t)\varphi(t)$ *one has* $F_x(t,X(t)) = \varphi(t)$, $F_{xx}(t,X(t)) = F_t(t,X(t)) = 0$ *and*
$\varphi(T)X(T) - \varphi(0)X(0) = \int_0^T \varphi_t[X(0) + \int_0^t \eta'(u-k)h(u)dw]dt + \int_0^T \varphi(t)\eta'(u-k)h(u)dw.$
In this way:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(T,x)\int_0^T \eta'(u-k)h(u)dw(t)\,\mathrm{dx}\\
=&\int_0^T\int_{\mathbb{R}^d}\varphi_t(t,x)\int_0^t \eta'(u-k)h(u)dw(s)\,\mathrm{dx}\,\mathrm{dt} + \int_0^T\int_{\mathbb{R}^d}\eta'(u-k)h(u)\varphi\,\mathrm{dx}dw(t),
\end{aligned}$$

and thus, P-a.s. in Ω,

$$\begin{aligned}
\varphi \mapsto \mu_{\eta,k}(\varphi) \;=\;& \int_{\mathbb{R}^d}\eta(u_0-k)\varphi(0)\,\mathrm{dx} + \int_Q \eta(u-k)\partial_t\varphi - F^\eta(u,k)\nabla\varphi\,\mathrm{dx}\,\mathrm{dt}\\
&+\int_{\mathbb{R}^d}\varphi(T,x)\int_0^T\eta'(u-k)h(u)dw(t)\,\mathrm{dx} + \frac{1}{2}\int_Q h^2(u)\eta''(u-k)\varphi\,\mathrm{dx}\,\mathrm{dt}\\
&-\int_0^T\int_{\mathbb{R}^d}\varphi_t(t,x)\int_0^t\eta'(u-k)h(u)dw(s)\,\mathrm{dx}\,\mathrm{dt}.
\end{aligned}$$

From the preceding considerations, we are now naturally led to give the following definition.

Definition 2.2 ***Entropy solution***
A function u of $\mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d))\cap L^\infty(0,T,L^2(\Omega,L^2(\mathbb{R}^d)))$ is an entropy solution of the stochastic conservation law (0.1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$ if for any $\eta \in \mathcal{E}$ and any $(k,\varphi)\in\mathbb{R}\times\mathcal{D}^+([0,T]\times\mathbb{R}^d)$

$$0 \;\le\; \mu_{\eta,k}(\varphi), \qquad P-a.s.$$

For technical reasons we also need to consider a generalised notion of entropy solution. In fact, in a first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we are able to deduce the existence of an entropy solution in the sense of Definition 2.2.

Definition 2.3 ***Measure-valued entropy solution***
A function $u \in \mathcal{N}_w^2\left(0,T,L^2(\mathbb{R}^d\times]0,1[)\right)\cap L^\infty\left(]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[)\right)$ is a (Young) measure-valued entropy solution of (0.1) with the initial condition $u_0\in L^2(\mathbb{R}^d)$ if for any $\eta\in\mathcal{E}$ and any

$(k, \varphi) \in \mathbb{R} \times \mathcal{D}^+([0,T] \times \mathbb{R}^d)$

$$0 \quad \leq \quad \int_0^1 \mu_{\eta,k}(\varphi) d\alpha, \quad P - a.s.$$

Note that in this definition the measure $\mu_{\eta,k}$ also depends on α because u does.

Remark 2.4 ***On the place of the "P-a.s" in Definitions 2.2 and 2.3:***
If u is an entropy solution in the sense of Definition 2.2, then there exists a measurable set $\widetilde{\Omega} \subset \Omega$ of full measure such that for any $\omega \in \widetilde{\Omega}, 0 \leqslant \mu_{n,k}(\phi)$ for all $k \in Q$, for all $\phi \in \mathcal{D}^+(\mathbb{R}^{d+1})$. Indeed, denote by $\mathcal{A} = \{a_M;\ M \in \mathbb{N}\} \subset \mathcal{D}^+(\mathbb{R}^{d+1})$ a countable dense sub-family of $\mathcal{D}^+(\mathbb{R}^{d+1})$ for the topology of $H^r(\mathbb{R}^{d+1})$ for arbitrary sufficiently large r.
From the definition it follows that there exists $\tilde{\Omega}_M \subset \Omega$ such that $P(\Omega \backslash \tilde{\Omega}_M) = 0$ and, for any $\omega \in \tilde{\Omega}_M$: $\quad 0 \leq \mu_{\eta,k}(a_M)$ for all $k \in \mathbb{Q}$.
Now if $\tilde{\Omega} = \cap_M \tilde{\Omega}_M$, one gets that $P(\Omega \backslash \tilde{\Omega}) = 0$ and, for any $\omega \in \tilde{\Omega}$: $\quad 0 \leq \mu_{\eta,k}(\varphi)$ for all $k \in \mathbb{Q}$ and all $\varphi \in \mathcal{A}$.
Since by Remark 2.1 $\mu_{\eta,k}$ is a $H^r(\mathbb{R}^{d+1})$-continuous function, it follows that for any $\omega \in \tilde{\Omega}$: $0 \leq \mu_{\eta,k}(\varphi)$ for all $k \in \mathbb{Q}$, for all $\varphi \in D^+(\mathbb{R}^{d+1})$.
Thanks to Appendix I Section 1, $\mu_{\eta,k}$ is continuous with respect to k and by approximating any real number by a sequence of rational ones, for any $\omega \in \tilde{\Omega}$: $0 \leq \mu_{\eta,k}(\varphi)$ for all $k \in \mathbb{R}$, for all $\varphi \in D^+(\mathbb{R}^{d+1})$. Finally, P-a.s and for all $(k,\varphi) \in \mathbb{R} \times \mathcal{D}^+([0,T] \times \mathbb{R}^d)$, $0 \leq \mu_{\eta,k}(\varphi)$.
The same remark holds for a measure-valued entropy solution.

Remark 2.5 $\mathbf{L^\infty(]0,T[, L^2(\Omega \times \mathbb{R}^d))}$ ***regularity of the entropy solution:***
We will detail in Section 3 the proof of the existence of a measure-valued solution $\mathbf{u}$*. It relies on the approximation of (0.1) by the viscous parabolic stochastic problems (2.1).*
Since the sequence of solutions u_ϵ of (2.1) is bounded in $L^\infty(]0,T[, L^2(\Omega \times \mathbb{R}^d))$ (cf. Proposition 2.1), the compactness theorem of Prohorov (cf. Appendix I Section 3), ensures the existence of a Young measure limit $\mathbf{u}$*. Then, thanks to the a priori estimates and the compact support of the test-functions, one will be able to pass to the limit, in the sense of the Young measures in (2.2), and any limit-point of u_ϵ (where we keep the same notation u_ϵ for a subsequence) is an element of $L^\infty(]0,T[, L^2(\Omega \times \mathbb{R}^d \times]0,1[))$. In fact, according to* Balder *[7], for any positive Carathéodory function $\psi :]0,T[\times \mathbb{R}^d \times \Omega \times \mathbb{R} \to \mathbb{R}$, we have*

$$E \int_{Q \times]0,1[} \psi(., \mathbf{u}) \,\mathrm{dx}\,\mathrm{dt}\,\mathrm{d}\alpha \leq \liminf_{\epsilon} E \int_Q \psi(., u_\epsilon) \,\mathrm{dx}\,\mathrm{dt}.$$

Now, if we choose $\psi(t,x,\omega,\lambda) = \beta(t)|\lambda|^2$ with $\beta \in L^1(0,T)$, $\beta \geq 0$, we find

$$\begin{aligned}\int_0^T \beta(t)E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2\,\mathrm{dx}\,\mathrm{d}\alpha\,\mathrm{dt} &\leq \liminf_{\epsilon} E\int_Q \beta(t)|u_\epsilon|^2\,\mathrm{dx}\,\mathrm{dt}\\ &\leq \liminf_{\epsilon}\|u_\epsilon\|^2_{L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))}\int_0^T\beta(t)\,\mathrm{dt}\\ &\leq C\int_0^T\beta(t)\,\mathrm{dt}\end{aligned}$$

as u_ϵ is bounded in $L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))$.
As, for any $\beta\in L^1(0,T)$, $\beta=\beta^+-\beta^-$ and, moreover, $t\mapsto E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2\,\mathrm{dx}\,\mathrm{d}\alpha$ is a measurable function, we get

$$\begin{aligned}\left|\int_0^T \beta(t)E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2\,\mathrm{dx}\,\mathrm{d}\alpha\,\mathrm{dt}\right| &\leq \liminf_{\epsilon} E\int_Q \beta(t)|u_\epsilon|^2\,\mathrm{dx}\,\mathrm{dt}\\ &\leq \liminf_{\epsilon}\|u_\epsilon\|^2_{L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))}\|\beta\|_{L^1(0,T)}\\ &\leq C\|\beta\|_{L^1(0,T)}\end{aligned}$$

which implies that $E\int_{\mathbb{R}^d\times]0,1[}|\mathbf{u}|^2\,\mathrm{dx}\,\mathrm{d}\alpha$ is an element of the dual of $L^1(0,T)$, i.e., an element of $L^\infty(0,T)$, hence $\mathbf{u}\in L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[))$.
Therefore it is natural to include in the definition the condition that an entropy (resp. measure-valued entropy) solution belongs to $L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d))$ (resp. $L^\infty(]0,T[,L^2(\Omega\times\mathbb{R}^d\times]0,1[))$).

Let us also mention some results satisfied by any entropy solution of (0.1).

Proposition 2.6 *Any entropy solution of (0.1) is almost surely a weak solution, too.*

Proof. Indeed, following CARRILLO-WITTBOLD [24], P-a.s, for any positive $\varphi\in\mathcal{D}^+([0,T[\times\mathbb{R}^d)$, we have

$$\begin{aligned}\mu_{\eta,k}(\varphi) &= \int_Q\left\{(u-k)\partial_t\varphi-[\vec{\mathbf{f}}(u)-\vec{\mathbf{f}}(k)].\nabla\varphi\right\}\mathrm{dx}\,\mathrm{dt}+\int_{\mathbb{R}^d}(u_0-k)\varphi(0)\,\mathrm{dx} \quad (:=I_1)\\ &\quad+\int_Q[\eta(u-k)-u+k]\partial_t\varphi-[F^\eta(u,k)-\vec{\mathbf{f}}(u)+\vec{\mathbf{f}}(k)]\nabla\varphi\,\mathrm{dx}\,\mathrm{dt}\\ &\quad+\int_{\mathbb{R}^d}[\eta(u_0-k)-u_0+k]\varphi(0)\,\mathrm{dx} \quad (:=I_2)\\ &\quad-\int_0^T\int_{\mathbb{R}^d}\varphi_t(t,x)\eta'(u-k)h(u)\,\mathrm{dx}dw(t) \quad (:=I_3)\\ &\quad+\frac{1}{2}\int_Q h^2(u)\eta''(u-k)\varphi\,\mathrm{dx}\,\mathrm{dt}. \quad (:=I_4)\end{aligned}$$

Note that

$$I_1 = \int_Q \left\{ u\partial_t\varphi - \vec{\mathbf{f}}(u).\nabla\varphi \right\} \mathrm{dx}\,\mathrm{dt} + \int_{\mathbb{R}^d} u_0\varphi(0)\,\mathrm{dx}.$$

Since $\eta \in \mathcal{E}$, for $k < 0$, we get

$$\begin{aligned}
& |F^\eta(u,k) - f(u) + f(k)| \\
= & \left| \int_k^u \eta'(\sigma - k)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma - (f(u) - f(k)) \right| = \left| \int_k^u [\eta'(\sigma - k) - 1]\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma \right| \\
\leq & c(\vec{\mathbf{f}}') \left| \int_k^u [1 - \eta'(\sigma - k)]\,\mathrm{d}\sigma \right| \leq c(\vec{\mathbf{f}}') \left| \int_0^{u-k} [1 - \eta'(\sigma)]\,\mathrm{d}\sigma \right| \\
\leq & c(\vec{\mathbf{f}}')[\delta + 2(u-k)^-] \leq c(\vec{\mathbf{f}}')[\delta + 2u^-]
\end{aligned}$$

where δ is the parameter associated to η in the definition of the elements of $\mathcal{E}$.

Note that, as k goes to $-\infty$,

$$\eta(u-k) - (u-k) = \int_0^{u-k} \eta'(r) - 1\,\mathrm{dr} \to \int_0^\infty \eta'(r) - 1\,\mathrm{dr} = \int_0^\delta \eta'(r) - 1\,\mathrm{dr} = \eta(\delta) - \delta$$

Therefore $\lim_{\delta\to 0}\lim_{k\to-\infty} I_2 = 0$. Similarly, $\lim_{k\to-\infty} I_4 = 0$.
Moreover, I_3 converges to $-\int_0^T \int_{\mathbb{R}^d} \varphi_t(t) \int_0^t h(u)dw(\sigma)\,\mathrm{dx}\,\mathrm{dt}$ when $k \to -\infty$.

Thus, for any positive $\varphi \in \mathcal{D}([0,T[\times\mathbb{R}^d)$

$$0 \leq \int_Q \left\{ (u - \int_0^t h(u)dw(\sigma))\partial_t\varphi - \vec{\mathbf{f}}(u).\nabla\varphi \right\} \mathrm{dx}\,\mathrm{dt} + \int_{\mathbb{R}^d} u_0\varphi(0,.)\,\mathrm{dx}.$$

Since the opposite inequality can be proved by using $k - u$ instead of $u - k$ in I_1, passing to the limit when k goes to $+\infty$, we find that u is a solution in the sense of distributions. □

Proposition 2.7 *The unique solution obtained in this chapter satisfies the initial condition in the following sense: for any compact set* $K \subset \mathbb{R}^d$

$$\operatorname*{ess\,lim}_{t\to 0^+} E\int_K |u - u_0|\,\mathrm{dx} = 0.$$

Proof. By the existence proof, the solution u will be in $L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))$. Therefore, following Málek-Necas-Otto-Rokyta-Ruzicka [60] (see also Vallet [74]), if one considers any $k \in \mathbb{R}$ and any $\beta \in \mathcal{D}^+(\mathbb{R}^d)$, then, for any nonnegative α in $H^1(0,T)$, one has that

$$\begin{aligned}
0 &\leq \int_0^T \Big\{\alpha' E\int_{\mathbb{R}^d}\eta(u-k)\beta\,\mathrm{dx} + \alpha E\int_{\mathbb{R}^d}\frac{1}{2}h^2(u)\eta''(u-k)\beta - F^\eta(u,k).\nabla\beta\,\mathrm{dx}\Big\}\,\mathrm{dt}\\
&\quad +E\int_0^T\alpha\int_{\mathbb{R}^d}\beta\eta'(u-k)h(u)\,\mathrm{dx}dw(t) + \alpha(0)\int_{\mathbb{R}^d}\eta(u_0-k)\beta\,\mathrm{dx}\\
&= \int_0^T[\alpha'(t)A_{k,\beta}(t)+\alpha(t)B_{k,\beta}(t)]\,\mathrm{dt}+\alpha(0)C_{k,\beta},
\end{aligned}$$

where $A_{k,\beta}(t) = E\int_{\mathbb{R}^d}\eta(u-k)\beta\,\mathrm{dx}$, $B_{k,\beta}(t) = E\int_{\mathbb{R}^d}h^2(u)\eta''(u-k)\beta - F^\eta(u,k).\nabla\beta\,\mathrm{dx}$.

Therefore, $\mathbb{T} : \alpha \in \mathcal{D}^+(\mathbb{R}) \mapsto \int_0^T[\alpha'(t)A_{k,\beta}(t)+\alpha(t)B_{k,\beta}(t)]\,\mathrm{dt}+\alpha(0)C_{k,\beta}$ is a positive Radon measure on $\mathbb{R}$. Its restriction to $]0,T[$, denoted by $\mathbb{T}_{]0,T[}$, is a positive bounded Radon measure on $]0,T[$ and

$$|\mathbb{T}_{]0,T[}| \leq E\mu_{\eta,k}(1\otimes\beta) = \int_0^T B_{k,\beta}(t)\,\mathrm{dt} + C_{k,\beta} \leq C(\eta,\beta,\mathrm{supp}\beta,\|u\|_{L^2(Q\times\Omega)}).$$

In particular, $\psi : t \mapsto A_{k,\beta}(t) - \int_0^t B_{k,\beta}(s)\,\mathrm{ds}$ is a nonincreasing function of bounded variation on $[0,T]$. Thus, $\psi(0^+) = \operatorname{ess\,lim}_{t\to0^+}\psi(t)$ exists and

$$\psi(0^+) = \lim_{n\to\infty} n\int_0^{1/n}\psi(t)\,\mathrm{dt} = \lim_{n\to\infty}\int_0^T \alpha_n'\psi(t)\,\mathrm{dt},$$

where $\alpha_n(t) = \min(nt,1)^+$.
Since $\lim_{t\to0^+}\int_0^t B_{k,\beta}(s)\,\mathrm{ds} = 0$, $A_{k,\beta}(0^+) = \operatorname{ess\,lim}_{t\to0^+} A_{k,\beta}(t) = \psi(0^+)$ and

$$\begin{aligned}
0 \leq A_{k,\beta}(0^+) &= \lim_{n\to\infty}\int_0^T(\alpha_n-1)'[A_{k,\beta}(t) - \int_0^t B_{k,\beta}(s)\,\mathrm{ds}]\,\mathrm{dt}\\
&= \lim_{n\to\infty} -\int_0^T[(1-\alpha_n)'A_{k,\beta}(t) + B_{k,\beta}(t)(1-\alpha_n)]\,\mathrm{dt},\\
&= \lim_{n\to\infty}\big[-\mu_{\eta,k}[(1-\alpha_n)\otimes\beta]\big] + C_{k,\beta} \leq \int_{\mathbb{R}^d}\eta(u_0-k)\beta\,\mathrm{dx}.
\end{aligned}$$

Let us fix $K \subset \mathbb{R}^d$ a compact set and denote by $L^2_K(\mathbb{R}^d)$ the Hilbert-subspace of $L^2(\mathbb{R}^d)$ of functions with support in K. Thanks to the hypothesis on u, uniformly with respect to t, $\beta \mapsto A_{k,\beta}(t)$ is a continuous linear function on $L^2_K(\mathbb{R}^d)$, and thus a density argument leads to the existence, for any real k and any nonnegative β in $L^2_K(\mathbb{R}^d)$, of $\operatorname{ess\,lim}_{t\to0^+} A_{k,\beta}(t) = A_{k,\beta}(0^+)$ with, moreover, $A_{k,\beta}(0^+) \leq \int_{\mathbb{R}^d}\eta(u_0-k)\beta\,\mathrm{dx}$.
In order to keep essential limits, consider k in $\mathbb{Q}$. Then, if $w_n = \sum_{i=0}^n k_i 1_{B_i}$ is a simple function

with k_i in $\mathbb{Q}$, one gets that

$$A_{w_n,\beta}(t) = E\int_{\mathbb{R}^d}\eta(u(t)-w_n)\beta\,\mathrm{dx} = \sum_{i=0}^{n} E\int_{\mathbb{R}^d}\eta(u(t)-k_i)\beta 1_{B_i}\,\mathrm{dx} = \sum_{i=0}^{n} A_{k_i,\beta}1_{B_i}(t),$$

and $\operatorname{ess\,lim}_{t\to 0^+} A_{w_n,\beta}(t)$ exists with moreover $A_{w_n,\beta}(0^+) \le \int_{\mathbb{R}^d}\eta(u_0-w_n)\beta\,\mathrm{dx}$, for any nonnegative β in $L^2_K(\mathbb{R}^d)$ and any $\mathbb{Q}$-valued simple function w_n.
As any w of $L^2(\mathbb{R}^d)$ is a limit in $L^2(\mathbb{R}^d)$ of a sequence of such simple functions and since for w and $\hat{w}$ in $L^2(\mathbb{R}^d)$, $|A_{w,\beta}(t) - A_{\hat{w},\beta}(t)| \le \|w-\hat{w}\|_{L^2(\mathbb{R}^d)}\|\beta\|_{L^2(\mathbb{R}^d)}$, independently of t, the same argument of density leads to
$\operatorname{ess\,lim}_{t\to 0^+} A_{w,\beta}(t)$ exists with, moreover, $A_{w,\beta}(0^+) \le \int_{\mathbb{R}^d}\eta(u_0-w)\beta\,\mathrm{dx}$, for any nonnegative β in $L^2_K(\mathbb{R}^d)$ and any w in $L^2(\mathbb{R}^d)$.

Now, for $w = u_0$ and $\beta = 1_K$ this leads to: $\operatorname{ess\,lim}_{t\to 0^+} E\int_K \eta(u(t)-u_0)\,\mathrm{dx} = 0$.

Since it is possible to approximate the absolute value function from below by a nondecreasing sequence of functions in $\mathcal{E}$, the theorem of Dini assures us the uniform convergence of the sequence. Thus, $\operatorname{ess\,lim}_{t\to 0^+} E\int_K |u(t)-u_0|\,\mathrm{dx} = 0$. □

Remark 2.8 *Replacing* $\int_{\mathbb{R}^d}\cdots\,\mathrm{dx}$ *by* $\int_{\mathbb{R}^d\times]0,1[}\cdots\,\mathrm{dx}\,\mathrm{d}\alpha$ *in the preceding arguments, one can prove in the same way that* $\operatorname{ess\,lim}_{t\to 0^+} E\int_{K\times]0,1[}|u(t)-u_0|\,\mathrm{dx}\,\mathrm{d}\alpha = 0$ *for a measure-valued entropy solution.*

The main result of this chapter is

Theorem 2.9
Under assumptions $H_1 - H_2 - H_3$ there exists a unique measure-valued entropy solution in the sense of Definition 2.3.
Moreover, this solution is the unique entropy solution of (0.1), for any initial condition u_0 in $L^2(\mathbb{R}^d)$, in the sense of Definition 2.2.
If u_1, u_2 are entropy solutions of (0.1) corresponding to initial condition $u_{1,0}, u_{2,0}$ in $L^2(\mathbb{R}^d)$, respectively, then, for any $K > 0$ and any t

$$E\int_{B(0,K-\kappa t)}|u_1-u_2|\,\mathrm{dx} \le \int_{B(0,K)}|u_{1,0}-u_{2,0}|\,\mathrm{dx},$$

where $\kappa = \|\vec{\mathbf{f}}'\|_\infty$.

Remark 2.10 *Following* VALLET *[77] Section 6.1, if* $0 \le u_0 \le 1$ *and if* $supp h \subset [0,1]$*, then* $0 \le u \le 1$.

3 Existence of a measure-valued solution

The aim of this section is to prove

Theorem 3.1 *Under assumptions* $H_1 - H_2 - H_3$ *there exists a measure-valued entropy solution in the sense of Definition 2.3.*

The technique is based on the notion of narrow convergence of Young measures (or entropy processes) (cf. Appendix I Section 3). Then, thanks to the uniqueness result of the next section, we will be able to prove that the measure-valued solution is an entropy solution in the sense of Definition 2.2 and that the sequence of approximation proposed to prove the existence of the solution converges in $L^p(]0,T[\times\Omega, L^p_{loc}(\mathbb{R}^d))$ for any $1 \le p < 2$.

Recall that for any positive ϵ, there exists a unique weak solution u_ϵ of the stochastic parabolic equation

$$\partial_t[u_\epsilon - \int_0^t h(u_\epsilon)dw(s)] - \epsilon\Delta u_\epsilon - \operatorname{div}\vec{\mathbf{f}}(u_\epsilon) = 0,$$

associated with a regular initial condition u_0^ϵ, see Appendix I Section 2.
More precisely, there exists $u_\epsilon \in \mathcal{N}_w^2(0,T;H_0^1(\mathbb{R}^d)) \cap C([0,T],L^2(\Omega\times\mathbb{R}^d))$ with moreover Δu_ϵ, $\partial_t[u_\epsilon - \mathcal{K}] \in L^2\big(]0,T[\times\Omega;L^2(\mathbb{R}^d)\big)$ where $\mathcal{K} = \int_0^t h(u_\epsilon)dw(s)$, $u_\epsilon(t=0) = u_0^\epsilon$ and, a.s. in Ω, a.e. in $]0,T[$, for any v in $H_0^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\left[\partial_t[u_\epsilon - \int_0^t h(u_\epsilon)dw(s)]v + \epsilon\nabla u_\epsilon.\nabla v + \vec{\mathbf{f}}(u_\epsilon).\nabla v\right]\mathrm{dx} = 0. \tag{3.1}$$

In the same section the following uniform *a priori* estimate on the approximate solutions is proved:

Lemma 3.2 *Assume that* $(u_\epsilon^0)_\epsilon$ *is bounded in* $L^2(\mathbb{R}^d)$*. Then there exists a positive constant* C *such that*

$$||u_\epsilon||^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))} + \epsilon||u_\epsilon||^2_{L^2(]0,T[\times\Omega;H_0^1(\mathbb{R}^d))} \le C.$$

If, for some $p \ge 1$*,* $u_0^\epsilon \in L^{2p}(\mathbb{R}^d)$*, then* $u_\epsilon \in L^\infty(0,T,L^{2p}(\Omega\times\mathbb{R}^d))$.

Moreover, thanks to Section 2, u_ϵ satisfies the inequality (2.2) which can be written in the following way:
For any $\varphi \in \mathcal{D}^+(\bar{Q})$, any real k, any $\eta \in \mathcal{E}$ and any dP-measurable set A

$$\begin{aligned} 0 \quad \leq \quad & E\Big[1_A \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\,\mathrm{dx}\Big] - \epsilon E\Big[1_A \int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi\,\mathrm{dx}\,\mathrm{dt}\Big] \\ & + E\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi\,\mathrm{dx}dw(t)\Big] \\ & + E\Big[1_A \int_Q \eta(u_\epsilon - k)\partial_t\varphi - F^\eta(u_\epsilon, k)\nabla\varphi + \frac{1}{2}h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi\,\mathrm{dx}\,\mathrm{dt}\Big] \\ := \quad & \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 + \mathbb{I}_5 + \mathbb{I}_6. \end{aligned} \tag{3.2}$$

In order to obtain an entropy formulation for our Problem (0.1), we would like to pass to the limit in (3.2) with respect to ϵ. Because of the random variable, we are not able to use classical results of compactness. But the one given by the concept of Young measure is appropriate here, and the technique is based on the notion of entropy processes, we refer to Balder [7] but also to Eymard-Gallouët-Herbin [40].
Since u_ϵ is a bounded sequence in $\mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d))$ and thanks to the compact support of φ in $\mathbb{R}^d$, the associated Young measure sequence $\mathbf{u}_\epsilon$ converges (up to a subsequence still indexed in the same way) to an "entropy process" denoted by $\mathbf{u} \in L^\infty(0,T,L^2(\Omega \times \mathbb{R}^d \times]0,1[))$.

Remark 3.3 *Since (u_ϵ) is bounded in the Hilbert space $\mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d))$, by identification, one shows that $u_\epsilon \rightharpoonup \int_0^1 \mathbf{u}(.,\alpha)\,\mathrm{d}\alpha$ in the same space, and so $\int_0^1 \mathbf{u}(.,\alpha)\,\mathrm{d}\alpha$ is a predictable process. The interesting point is the measurability of $\mathbf{u}$ with respect to all its variables (t,x,ω,α). Revisiting the work of* Panov *[61] with the σ-field $\mathcal{P}_T \otimes L(\mathbb{R}^d)$, one shows that $\mathbf{u}$ is measurable for the σ-field $\mathcal{P}_T \otimes L(\mathbb{R}^d \times]0,1[)$. See Appendix I Section 3.3 for further information.*

As we choose the initial condition u_0^ϵ in $\mathcal{D}(\mathbb{R}^d)$ such that $u_0^\epsilon \to u_0$ in $L^2(\mathbb{R}^d)$ one is able to pass to the limit in the first term of (3.2) and the *a priori* estimate on $(\sqrt{\epsilon}\nabla u_\epsilon)$ yields that the second one tends to 0 with ϵ. By assumptions on η, all the others integrands in (3.2) are uniformly integrable and passing to the limit is possible in all the integrals.

Therefore at the limit one gets

$$\begin{aligned} 0 \;\leq\; & E\Big[1_A \int_{\mathbb{R}^d} \eta(u_0-k)\varphi(0)\,\mathrm{dx}\Big] \\ &+E\Big[1_A \int_0^T \int_{\mathbb{R}^d}\int_0^1 \eta'(\mathbf{u}(.,\alpha)-k)h(\mathbf{u}(.,\alpha))\varphi\,\mathrm{d}\alpha\,\mathrm{dx}dw(t)\Big] \\ &+E\Big[1_A \int_Q \int_0^1 [\eta(\mathbf{u}(.,\alpha)-k)\partial_t\varphi - F^\eta(\mathbf{u}(.,\alpha),k)\nabla\varphi]\,\mathrm{d}\alpha\,\mathrm{dx}\,\mathrm{dt}\Big] \\ &+E\Big[1_A \int_Q \int_0^1 \frac{1}{2}h^2(\mathbf{u}(.,\alpha))\eta''(\mathbf{u}(.,\alpha)-k)\varphi\,\mathrm{d}\alpha\,\mathrm{dx}\,\mathrm{dt}\Big]. \end{aligned}$$

Let us explicit this passage to the limit. We propose to study the terms $(\mathbb{I}_i)_{i\in\{1,\dots,6\}}$ separately.

- $\mathbb{I}_1 = E\displaystyle\int_{\mathbb{R}^d} 1_A\eta(u_0^\epsilon - k)\varphi(0)\;dx$. Since $u_0^\epsilon \underset{\epsilon\to 0}{\longrightarrow} u_0$ in $L^2(\mathbb{R}^d)$, and with the regularities of η and φ, one gets
$$\mathbb{I}_1 \underset{\epsilon\to 0}{\longrightarrow} E\int_{\mathbb{R}^d} 1_A\eta(u_0-k)\varphi(0)\;dx.$$

- $\mathbb{I}_2 = -\epsilon E\displaystyle\int_Q 1_A\eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi\;dxdt$. Thanks to Proposition 2.1 (see Appendix I Section 2), we know that $(\sqrt{\epsilon}\nabla u_\epsilon)_{\epsilon>0}$ is bounded in $L^2(\Omega\times Q)$. Moreover, as
$$|1_A\eta'(u_\epsilon - k)\nabla\varphi| \leqslant C$$
$$\mathbb{I}_2 = -\sqrt{\epsilon}E\int_Q 1_A\eta'(u_\epsilon-k)\sqrt{\epsilon}\nabla u_\epsilon\nabla\varphi\;dxdt,$$
one gets
$$\mathbb{I}_2 \underset{\epsilon\to 0}{\longrightarrow} 0.$$

- $\mathbb{I}_3 = E\displaystyle\int_0^T\int_{\mathbb{R}^d} 1_A\eta'(u_\epsilon - k)h(u_\epsilon)\varphi\;dxdw(t)$. Define $v_\epsilon = \eta'(u_\epsilon - k)h(u_\epsilon)\varphi$, which is bounded in $L^2(\Omega\times Q)$, and there exists v in $L^2(\Omega\times Q)$ such that $v_\epsilon \rightharpoonup v$ in the same space, and, for all χ in $L^2(\Omega\times Q)$,
$$E\int_Q v_\epsilon\chi(t,x,\omega)dxdt \underset{\epsilon\to 0}{\longrightarrow} E\int_Q v\chi dxdt.$$
A priori, u_ϵ is bounded in $L^2(\Omega\times Q)$, and so there exists $\mathbf{u}$ in $L^2\big(\Omega\times Q\times(0,1)\big)$ such that, up to a subsequence denoted the same way, one has $u_\epsilon \to u$ in the sense of the Young measures. Given the Carathéodory function $\Psi(t,x,\omega,\lambda) := \eta'(\lambda-k)h(\lambda)\varphi\chi(t,x,\omega)$, $\Psi(.,u_\epsilon)$ is uniformly integrable , then thanks to Saadoune-Valadier [66] for example, at the

limit one has:

$$E\int_Q v_\epsilon \chi(t,x,\omega)dxdt \underset{\epsilon\to 0}{\longrightarrow} E\int_{Q\times(0,1)} \eta'(\mathbf{u}(\alpha,.)-k)h(\mathbf{u}(\alpha,.))\varphi d\alpha\chi dxdt.$$

By uniqueness of the weak limit in $L^2(\Omega\times Q)$, $v=\int_0^1 \eta'(\mathbf{u}(\alpha,.)-k)h(\mathbf{u}(\alpha,.))\varphi d\alpha$. To conclude, we use the continuity of the stochastic integral (see Da Prato-Zabczyk [30] p.101) $I_t : v\mapsto \int_0^t v dw(s)$ from $L^2(\Omega\times Q)$ to $L^2(\Omega\times\mathbb{R}^d)$. As the integral is in addition linear, it is weakly continuous on the same spaces and $v_\epsilon \rightharpoonup v$ in $L^2(\Omega\times Q)$ implies that $I_t(v_\epsilon)\rightharpoonup I_t(v)$ in $L^2(\Omega\times\mathbb{R}^d)$. With $\chi = 1_A 1_{supp\varphi}$ one has,

$$\mathbb{I}_3 \underset{\epsilon\to 0}{\longrightarrow} E\int_0^T\int_{\mathbb{R}^d\times(0,1)} 1_A\eta'(\mathbf{u}(\alpha,.)-k)h(\mathbf{u}(\alpha,.))\varphi d\alpha dx dw(t).$$

- $\mathbb{I}_4 = E\int_Q 1_A\eta(u_\epsilon-k)\partial_t\varphi dxdt$. Consider the Carathéodory function $\Psi(.,u_\epsilon)=1_A\eta(u_\epsilon-k)\partial_t\varphi$, as it is bounded in $L^2(\Omega\times Q)$, one gets:

$$\mathbb{I}_4 \underset{\epsilon\to 0}{\longrightarrow} E\int_{Q\times(0,1)} 1_A\eta(\mathbf{u}(\alpha,.)-k)\partial_t\varphi d\alpha dxdt.$$

- $\mathbb{I}_5 = -E\int_Q 1_A\int_k^{u_\epsilon}\eta'(\sigma-k)\vec{\mathbf{f}}'(\sigma)\,d\sigma\nabla\varphi\ dxdt$. Define

$$g(u_\epsilon)=\int_k^{u_\epsilon}\eta'(\sigma-k)\vec{\mathbf{f}}'(\sigma)\,d\sigma,$$

and as $|\eta'(\sigma-k)\vec{\mathbf{f}}'(\sigma)|\leqslant M$, g is a Lipschitz-continuous function.
In this way

$$|\nabla\varphi||1_A||g(x)|\leqslant C[c|x|+\widetilde{C}],$$

and using Young measures theory, one gets

$$\mathbb{I}_5 \underset{\epsilon\to 0}{\longrightarrow} -E\int_{Q\times(0,1)}\nabla\varphi\int_0^{\mathbf{u}(\alpha,.)} 1_A\eta'(\sigma-k)\vec{\mathbf{f}}'(\sigma)d\sigma d\alpha dxdt.$$

- $\mathbb{I}_6 = \frac{1}{2}E\int_Q 1_A h^2(u_\epsilon)\eta''(u_\epsilon-k)\varphi dxdt$. By using the compact support of $\eta''(.-k)$, $h^2(u_\epsilon)\eta''(u_\epsilon-k)\leqslant C_k$, and $1_A h^2(u_\epsilon)\eta''(u_\epsilon-k)\varphi$ is bounded in $L^2(\Omega\times Q)$ in function of k, but k is a fixed real, and the Young measures theory gives us

$$\mathbb{I}_6 \xrightarrow[\epsilon\to 0]{} \frac{1}{2}E\int_{Q\times(0,1)} 1_A h^2(\mathbf{u}(\alpha,.))\eta''(\mathbf{u}(\alpha,.)-k)\varphi d\alpha dx dt.$$

Finally, at the limit one gets

$$\begin{aligned} 0 &\leqslant E\big[1_A\int_{\mathbb{R}^d}\eta(u_0-k)\varphi(0)dx\big] \\ &+ E\big[1_A\int_0^T\int_{\mathbb{R}^d\times(0,1)}\varphi\eta'(\mathbf{u}(\alpha,.)-k)h(\mathbf{u}(\alpha,.))d\alpha dx dw(t)\big] \\ &+ E\big[1_A\int_{Q\times(0,1)}\varphi_t\eta(\mathbf{u}(\alpha,.)-k)-\nabla\varphi F^\eta(\mathbf{u}(\alpha,.),k)d\alpha dx dt\big] \\ &+ E\big[1_A\int_{Q\times(0,1)}\frac{1}{2}h^2(\mathbf{u}(\alpha,.))\eta''(\mathbf{u}(\alpha,.)-k)]\varphi d\alpha dx dt\big]. \end{aligned}$$

A separability argument for the norm of $H^1(Q)$ yields the existence of a Young measure solution.

4 Uniqueness

The aim of this section is to prove

Theorem 4.1 *The solution given by Theorem 3.1 is the unique measure-valued entropy solution in the sense of Definition 2.3.*
Moreover, it is the unique entropy solution in the sense of Definition 2.2.

Remark 4.2 *We will also prove for entropy solutions a result of stability ("contraction principle") in L^1 (see Proposition 4.6 below).*

4.1 Local Kato inequality

Our purpose is to prove the following interior Kato inequality:

Proposition 4.3 *Let u_1, u_2 be Young measure valued entropy solutions to (0.1) with initial data $u_{1,0}, u_{2,0} \in L^2(\mathbb{R}^d)$, respectively. Then, for any nonnegative $H^1(\mathbb{R}^{d+1})$-function φ with* *compact support, it holds*

$$\begin{aligned} 0 \;\leq\; & \int_{\mathbb{R}^d} |u_{1,0} - u_{2,0}|\varphi(0)\,\mathrm{dx} + E\int_{Q\times]0,1[^2} \Big|u_1(t,x,\alpha) - u_2(t,x,\beta)\Big|\partial_t\varphi\,\mathrm{dx}\,\mathrm{dt}d\alpha d\beta \\ & -E\int_{Q\times]0,1[^2} F\Big(u_1(t,x,\alpha), u_2(t,x,\beta)\Big).\nabla\varphi\,\mathrm{dx}\,\mathrm{dt}d\alpha d\beta. \end{aligned}$$

Let us denote by $\mathbf{u}$ the Young measure entropy solution from the previous section (a limit point of (u_ϵ)) and $\hat{\mathbf{u}}$ any other admissible Young measure-valued entropy solution associated to two initial conditions u_0 and $\hat{u}_0$ in $L^2(\mathbb{R}^d)$, respectively. In a first step we will prove the local Kato inequality for $u_1 = \hat{\mathbf{u}}$ and $u_2 = \mathbf{u}$. In a second step (see Section 4.2), exploiting the finite propagation speed property for conservation laws with Lipschitz-continuous flux function, we will deduce from the local Kato inequality a global one and, in particular, obtain a local L^1-contraction principle (see Proposition 4.6 below). As a consequence (choosing $u_{1,0} = \hat{u}_0 = u_0 = u_{2,0}$) we deduce that $\mathbf{u} = \hat{\mathbf{u}}$ and thus any Young measure valued solution is obtained as the limit of solutions u_ϵ of viscous parabolic approximations to (0.1). Then it follows immediately that, in fact, the local Kato inequality also holds for any arbitrary pair of measure-valued entropy solutions.
The aim of this work is to use tools of the deterministic setting for studying stochastic PDE. For convenience, we propose in the following subsection to discuss about the way of obtaining the interior Kato inequality, advancing main difficulties we have met in the use of deterministic methods. Thus, in the next subsection, all the technical computations are presented.

4.1.1 Plan of the proof

We propose here to present stages of the proof of Proposition 4.3 , emphasizing on differences with the deterministic setting and advancing techniques developed to treat the stochastic terms. The main idea is to use Kruzhkov's doubling variables method. The approach is the following: we apply the usual techniques and advice when we meet difficulties. For convenience in the explanations, let us denote for all $u, u_\epsilon \in \mathcal{N}^2_w(0,T,L^2(\mathbb{R}^d))$, $k \in \mathbb{R}$, $\eta \in \mathcal{E}$ and $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$:

$$\begin{aligned} \mu_{u,\eta,k}(\psi) \;=\; & \int_{\mathbb{R}^d} \eta(u_0 - k)\psi(0)\,\mathrm{dx} + \int_Q \eta(u-k)\partial_t\psi - F^\eta(u,k)\nabla\psi\,\mathrm{dx}\,\mathrm{dt} \\ & + \int_Q \eta'(u-k)h(u)\psi\,\mathrm{dx}dw(t) + \frac{1}{2}\int_Q h^2(u)\eta''(u-k)\psi\,\mathrm{dx}\,\mathrm{dt}, \end{aligned}$$

and

$$\begin{aligned}\mu^{\epsilon}_{u_\epsilon,\eta,k}(\psi) \ := \ & \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\psi(0)\,\mathrm{dy} \\ & + \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\psi\,\mathrm{dy}dw(s) - \epsilon \int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\psi\,\mathrm{dy}\,\mathrm{ds} \\ & + \int_Q \eta(u_\epsilon - k)\partial_t\psi - F^\eta(u_\epsilon,k)\nabla\psi + \frac{1}{2}h^2(u_\epsilon)\eta''(u_\epsilon - k)\psi\,\mathrm{dy}\,\mathrm{ds}.\end{aligned}$$

We consider two measure-valued entropy solutions $\hat{\mathbf{u}}$, $\mathbf{u}$ and these inequalities P-a.s.:

$$0 \leqslant \int_0^1 \mu_{\hat{\mathbf{u}}(t,x,\alpha),\eta_\delta,k}(\psi)d\alpha \qquad ; \qquad 0 \leqslant \int_0^1 \mu_{\mathbf{u}(s,y,\beta),\eta_\delta,\hat{k}}(\psi)\,\mathrm{d}\beta, \tag{4.1}$$

where $\eta_\delta \in \mathcal{E}$, $k, \hat{k} \in \mathbb{R}$ and $\psi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$.
Notice that, comparing with the deterministic case, the stochastic perturbation of our conservation law brings new terms in the entropy inequalities, ones containing an Itô integral:

$$\begin{aligned}&\int_0^1 \int_Q \eta'_\delta(\hat{\mathbf{u}}(t,x,\alpha) - k)h(\hat{\mathbf{u}})\psi\,\mathrm{dx}dw(t)\,\mathrm{d}\alpha, \\ &\int_0^1 \int_Q \eta'_\delta(\mathbf{u}(s,y,\beta) - \hat{k})h(\mathbf{u})\psi\,\mathrm{dy}dw(s)\,\mathrm{d}\beta,\end{aligned} \tag{4.2}$$

and others containing the second derivative of η_δ:

$$\begin{aligned}&\frac{1}{2}\int_0^1 \int_Q h^2(\hat{\mathbf{u}}(t,x,\alpha))\eta''_\delta(\hat{\mathbf{u}} - k)\psi\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{d}\alpha, \\ &\frac{1}{2}\int_0^1 \int_Q h^2(\mathbf{u}(s,y,\beta))\eta''_\delta(\mathbf{u} - \hat{k})\psi\,\mathrm{ds}\,\mathrm{dy}\,\mathrm{d}\beta.\end{aligned} \tag{4.3}$$

Usually, we take in (4.1) $\hat{k} = \hat{\mathbf{u}}(t,x,\alpha)$, $k = \mathbf{u}(s,y,\beta)$, $\psi(t,x,s,y) = \varphi(s,y)\rho_m(x-y)\rho_n(t-s)$ with $\varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$, $supp\varphi(t,.) \subset K$ a compact set of $\mathbb{R}^d$, ρ_n and ρ_m the usual mollifier sequences in $\mathbb{R}$ and $\mathbb{R}^d$ respectively, with $supp\rho_n \subset [-\frac{2}{n}, 0]$. Then, we integrate with respect to (s,y,β) for the first inequality, with respect to (t,x,α) for the second one, we add those two news inequalities and pass to the limit on δ, n and m.
In our case, there is a problem with this technique when we treat the stochastic integrals (4.2). Indeed, because of the definition of the Itô integral, we require an $\mathcal{F}_t$-measurability for replacing k and an $\mathcal{F}_s$-measurability for replacing $\hat{k}$, whiches are not satisfied by $\hat{k} = \hat{\mathbf{u}}(t,x,\alpha)$ and $k = \mathbf{u}(s,y,\beta)$ because we ignore if $s > t$ or $s < t$ (recall that $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 \leqslant s \leqslant t$).
For this reason, we consider (4.1) with the same real k, and multiply by a kernel of convolution

$\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)$ the inequality coming from $\mu_{\mathbf{u},\eta_\delta,k}$ and integrate with respect to (t,x,α); multiply by $\rho_l(\mathbf{u}(s,y,\beta)-k)$ the inequality coming from $\mu_{\hat{\mathbf{u}},\eta_\delta,k}$ and integrate with respect to (s,y,β), we add those two inequalities, then integrate over k in $\mathbb{R}$ all the formulation and take the expectation, we get:

$$\begin{aligned}0 \leqslant\, & E\int_Q\int_{]0,1[^2}\int_{\mathbb{R}}\mu_{\hat{\mathbf{u}},\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(\mathbf{u}(s,y,\beta)-k)\,\mathrm{dk}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{ds}\,\mathrm{dy}\\ &+E\int_Q\int_{]0,1[^2}\int_{\mathbb{R}}\mu_{\mathbf{u},\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,\mathrm{dk}\,\mathrm{d}\beta\,\mathrm{d}\alpha\,\mathrm{dt}\,\mathrm{dx}.\end{aligned}$$

With a judicious order for the passage to the limit, we are able to avoid our initial difficulty. Indeed, we first pass to the limit on n, then we get the same time everywhere (t or s, it doesn't matter), and the problem of measurability with respect to the σ-field $\mathcal{P}_T$ is resolved. Then passing to the limit on l we get back that $\hat{\mathbf{u}}(t,x,\alpha)$ and $\mathbf{u}(s,y,\beta)$ replace k in our formulation, as we wished at the beginning. And we pass to the limit on δ and m.
The second delicate point appears with terms containing the second derivative of η_δ (4.3) when we want to pass to the limit on δ. Indeed, because of η_δ'', we are not able to identify the limit of those terms, all we can say is that the limit exists (Tanaka formula). The problem is that we need to know the limit to obtain the local Kato inequality. For this reason, we decide to consider a viscous regular solution u_ϵ instead of $\mathbf{u}$. Indeed, the suitable regularity of such a solution allows us to apply the Itô formula. Following the concept of J. Feng and D. Nualart for treating the stochastic term, the idea remains on combining terms containing η_δ'' with others coming from stochastic calculus. The passage to the limit on n and l on terms containing η_δ'' gives:

$$\begin{aligned}&\frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(\hat{\mathbf{u}})\eta_\delta''(\hat{\mathbf{u}}(s,x)-u_\epsilon(s,y))\rho_m(x-y)\,\mathrm{d}\alpha\varphi\,\mathrm{ds}\,\mathrm{dx}\,\mathrm{dy}\\ &+\frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(u_\epsilon)\eta_\delta''(u_\epsilon(s,y)-\hat{\mathbf{u}}(s,x,\alpha))\rho_m(x-y)\varphi\,\mathrm{d}\alpha\,\mathrm{ds}\,\mathrm{dx}\,\mathrm{dy}\\ :=\;& A+B.\end{aligned}$$

Moreover, thanks to the martingale property of the Itô integral, the stochastic terms coming from the entropy inequality of $\mathbf{\hat{u}}$ can be written like this

$$\begin{aligned}
& E\int_Q\int_{\mathbb{R}}\int_{s-2/n}^{s}\int_{\mathbb{R}^d}\int_0^1 \eta_\delta'(\mathbf{\hat{u}}-k)h(\mathbf{\hat{u}}\varphi\rho_m(x-y)\rho_n(t-s))\,\mathrm{d}\alpha\,\mathrm{dx}dw(t) \\
& \qquad\qquad \times\rho_l(u_\epsilon(s,y)-k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
= \; & E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta_\delta'(\mathbf{\hat{u}}-k)h(\mathbf{\hat{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx} \\
& \qquad\qquad \times[\rho_l(u_\epsilon(s,y)-k)-\rho_l(u_\epsilon(s-2/n,y)-k)]\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
:= \; & C_{n,l}.
\end{aligned}$$

Here the choice of u_ϵ instead of $\mathbf{u}$ is crucial. Indeed, the regularity of u_ϵ allows us to apply Itô's formula with $\mathrm{d}u_\epsilon = [\epsilon\Delta u_\epsilon + \operatorname{div}\vec{\mathbf{f}}(u_\epsilon)]\,\mathrm{dt} + h(u_\epsilon)dw = A_\epsilon\,\mathrm{dt} + h(u_\epsilon)dw$ and to get:

$$\begin{aligned}
& \rho_l(u_\epsilon(s,y)-k)-\rho_l(u_\epsilon(s-2/n,y)-k) \\
= \; & \int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma + \int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma) \\
& +\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l''(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma,
\end{aligned}$$

which wasn't possible with a measure-valued solution. Thus, by integration by parts with respect to the variable k, it comes:

$$\begin{aligned}
C_{n,l} \; = \; & -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta_\delta''(\mathbf{\hat{u}}-k)h(\mathbf{\hat{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx} \\
& \times\Big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma+\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma) \\
& +\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
\underset{n,l}{\longrightarrow} \; & -E\int_Q\int_{\mathbb{R}^d}\Big[\int_0^1\eta_\delta''(\mathbf{\hat{u}}(s,x,\alpha)-u_\epsilon(s,y))h(\mathbf{\hat{u}}(s,x,\alpha))h(u_\epsilon(s,y))\,\mathrm{d}\alpha\Big]\varphi\rho_m(x-y)\mathrm{dydxds} \\
:= \; & C.
\end{aligned}$$

Thus,

$$\begin{aligned} A+B+C &= \frac{1}{2}E\int_Q\int_{\mathbb{R}^d}\int_0^1 [h(\hat{\mathbf{u}})-h(u_\epsilon)]^2\eta''_\delta(u_\epsilon(s,y)-\hat{\mathbf{u}}(s,x,\alpha))\rho_m(x-y)\varphi \, \mathrm{d}\alpha \, \mathrm{dy} \, \mathrm{dx} \, \mathrm{ds} \\ &\underset{\delta}{\rightarrow} 0. \end{aligned}$$

In summary, this is the plan of the proof. By doing stochastic computations on the Itô integral and passing to the limit (with classical techniques) with respect to n, l, δ, ϵ, m in this order on

$$\begin{aligned} 0 \leqslant\; & E\int_Q\int_0^1\int_{\mathbb{R}} \mu_{\hat{\mathbf{u}},\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(u_\epsilon(s,y)-k) \, \mathrm{dk} \, \mathrm{d}\alpha \, \mathrm{ds} \, \mathrm{dy} \\ +& E\int_Q\int_0^1\int_{\mathbb{R}} \mu^\epsilon_{u_\epsilon,\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k) \, \mathrm{dk} \, \mathrm{d}\alpha \, \mathrm{dt} \, \mathrm{dx}, \end{aligned}$$

we finally obtain the local Kato inequality.

4.1.2 Details of the proof

Let us have a look in this subsection at the proof in detail. We denote by $\mathbf{u}$ the Young measure entropy solution from the previous section (a limit point of (u_ϵ)) and $\hat{\mathbf{u}}$ any other admissible Young measure-valued entropy solution associated to two initial conditions u_0 and $\hat{u}_0$ in $L^2(\mathbb{R}^d)$, respectively.

Consider φ in $\mathcal{D}^+([0,T]\times\mathbb{R}^d)$, $K \subset \mathbb{R}^d$ a compact set such that supp $\varphi(t,.) \subset K$ and denote by $G(t,x,s,y) = \varphi(s,y)\rho_m(x-y)\rho_n(t-s)$ where ρ_m and ρ_n denote the usual mollifier sequences in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, with supp$\rho_n \subset [-\frac{2}{n},0]$. Denote also by ρ_l a mollifier sequence in $\mathbb{R}$ and for convenience set $p=(t,x,\alpha)$.

Since $\hat{\mathbf{u}} = \hat{\mathbf{u}}(p)$ is a Young measure solution, by considering the test function G, multiplying the entropy formulation by $\mathcal{B}^l_k := \rho_l(u_\epsilon(s,y)-k)$ and integrating k over $\mathbb{R}$, we get, on the one hand, that

$$
\begin{aligned}
0 \;\leq\; & E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\eta(\hat{u}_0(x)-k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\mathcal{B}_k^l\mathrm{dkdyds}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\eta(\hat{\mathbf{u}}(p)-k)\partial_t\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\eta(\hat{\mathbf{u}}(p)-k)\varphi(s,y)\partial_t\rho_n(t-s)\rho_m(x-y)\,\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}}(p),k)\nabla_x\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}}(p),k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-k)\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\eta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\mathcal{B}_k^l\mathrm{dkdyds}\\
=:\; & I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}
$$

On the other hand, if one denotes $\mathcal{A}_k^l = \rho_l(\hat{\mathbf{u}}(p) - k)$, since u_ϵ is a viscous solution, one gets that

$$
\begin{aligned}
0 \;\le\; & E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\eta(u_0^\epsilon(y)-k)\varphi(0,y)\rho_n(t)\rho_m(x-y)\,\mathrm{dy}\int_0^1\mathcal{A}_k^l\mathrm{dkdp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\eta(u_\epsilon(s,y)-k)\partial_s\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\int_0^1\mathcal{A}_k^l\mathrm{dkdydsdp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\eta(u_\epsilon(s,y)-k)\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\int_0^1\mathcal{A}_k^l\mathrm{dkdydsdp}\\
&-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q\eta'(u_\epsilon(s,y)-k)\nabla_y u_\epsilon(s,y)\nabla_y\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\int_0^1\mathcal{A}_k^l\mathrm{dkdydsdp}\\
&-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q\eta'(u_\epsilon(s,y)-k)\nabla_y u_\epsilon(s,y)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\int_0^1\mathcal{A}_k^l\mathrm{dkdydsdp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^\eta(u_\epsilon(s,y),k)\nabla_y\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\int_0^1\mathcal{A}_k^l\mathrm{dkdydsdp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^\eta(u_\epsilon(s,y),k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\int_0^1\mathcal{A}_k^l\mathrm{dkdydsdp}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(s,y))\eta''(u_\epsilon(s,y)-k)\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\int_0^1\mathcal{A}_k^l\mathrm{dkdydsdp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\eta'(u_\epsilon(s,y)-k)h(u_\epsilon(s,y))\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}dw(s)\int_0^1\mathcal{A}_k^l\mathrm{dkdp}\\
&=: J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8+J_9.
\end{aligned}
$$

Summing up the preceding two inequalities, our aim is now to pass to the limit in the following order: 1. $n\to\infty$ (time), 2. $l\to\infty$, 3. $\eta\to|\cdot|$, 4. $\epsilon\to 0$, 5. $m\to\infty$ (space). In the following, as a uniform approximation of the absolute value function, we choose $\eta=\eta_\delta\in\mathcal{E}$ with $\eta_\delta'(r)=1$ for $r>\delta$, $=\sin(\frac{\pi}{2\delta}r)$ if $|r|\le\delta$ and $=-1$ for $r<-\delta$.

First let us consider

1. $I_1 + J_1 =$

$$
\begin{aligned}
& E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\eta(\hat{u}_0(x)-k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\rho_l(u_\epsilon(s,y)-k)\,\mathrm{dk\,dy\,ds}\\
&+E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\eta(u_0^\epsilon(y)-k)\varphi(0,y)\rho_m(x-y)\rho_n(t)\,\mathrm{dy}\int_0^1\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,\mathrm{dk\,dp}\\
=\;& E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\eta(\hat{u}_0(x)-k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\rho_l(u_\epsilon(s,y)-k)\,\mathrm{dk\,dy\,ds}
\end{aligned}
$$

as supp $\rho_n \subset [-\frac{2}{n}, 0]$. Therefore,

$$
\begin{aligned}
I_1 + J_1 \quad &\to_{n\to\infty} E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-k)\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\rho_l(u_0^\epsilon(y)-k)\,\mathrm{dk}\,\mathrm{dy} \\
&\to_{l\to\infty} E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_0^\epsilon(y))\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy} \\
&\to_{\eta\to|.|} E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\hat{u}_0(x)-u_0^\epsilon(y)|\,\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy} \\
&\to_{\epsilon\to 0} E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\hat{u}_0(x)-u_0(y)|\,\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy} \\
&\to_{m\to\infty} E\int_{\mathbb{R}^d} |\hat{u}_0(x)-u_0(x)|\,\varphi(0,x)\,\mathrm{dx}.
\end{aligned}
$$

For the convenience of the reader let us justify all passages to the limit in detail. As to the passage to the limit with $n\to\infty$, note that, by a simple change of variables,

$$
\begin{aligned}
\mathcal{A}_1 \;:=\;& E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\rho_l(u_\epsilon(s,y)-k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& -E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-k)\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\rho_l(u_0^\epsilon(y)-k)\,\mathrm{dk}\,\mathrm{dy} \\
=\;& E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_\epsilon(s,y)+k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& -E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_0^\epsilon(y)+k)\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy} \\
=\;& E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} \eta(\hat{u}_0(x)-u_\epsilon(s,y)+k)[\varphi(s,y)-\varphi(0,y)]\rho_n(-s) \\
& \qquad\qquad\qquad\qquad \times\rho_m(x-y)\,\mathrm{dx}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& +E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} [\eta(\hat{u}_0(x)-u_\epsilon(s,y)+k)-\eta(\hat{u}_0(x)-u_0^\epsilon(y)+k)]\varphi(0,y) \\
& \qquad\qquad\qquad\qquad \times\rho_n(-s)\rho_m(x-y)\rho_l(k)\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds},
\end{aligned}
$$

and thus,

$$
\begin{aligned}
|\mathcal{A}_1| &\leq \|\varphi_t\|_\infty E\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} 1_K(y)\eta(\hat{u}_0(x) - u_\epsilon(s,y) + k) s\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad + \|\eta'\|_\infty E\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u_\epsilon(s,y) - u_0^\epsilon(y)|\varphi(0,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
&\leq \frac{\|\varphi_t\|_\infty \|\eta'\|_\infty}{n} E\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} 1_K(y)\left[|\hat{u}_0(x) - u_\epsilon(s,y)| + |k|\right]\rho_n(-s) \\
&\qquad\qquad \times \rho_m(x-y)\,\mathrm{dx}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad + \|\eta'\|_\infty E\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u_\epsilon(s,y) - u_0^\epsilon(y)|\varphi(0,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
&\leq \frac{\|\varphi_t\|_\infty \|\eta'\|_\infty}{n} E\int_Q \int_{\mathbb{R}^d} 1_K(y)\left[|\hat{u}_0(x) - u_\epsilon(s,y)| + 1\right]\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad + \|\varphi\|_\infty \|\eta'\|_\infty E\int_0^T \int_K |u_\epsilon(s,y) - u_0^\epsilon(y)|\rho_n(-s)\,\mathrm{dy}\,\mathrm{ds} \\
&\leq \frac{\|\varphi_t\|_\infty \|\eta'\|_\infty}{n}\left[\|\hat{u}_0(x)\|_{L^1(\mathbb{R}^d)} + E\int_K \int_0^T |u_\epsilon(s,y)|\rho_n(-s)\,\mathrm{ds}\,\mathrm{dy} + \mathrm{Meas}(K)\right] \\
&\quad + \|\varphi\|_\infty \|\eta'\|_\infty E\int_0^T \int_K |u_\epsilon(s,y) - u_0^\epsilon(y)|\rho_n(-s)\,\mathrm{dy}\,\mathrm{ds} \\
&\to_{n\to\infty} \quad 0,
\end{aligned}
$$

as $u_\epsilon \in C([0,T], L^2(\Omega \times \mathbb{R}^d))$ with $u_\epsilon(0,\cdot) = u_0^\epsilon$.
Next, let us consider the passage to the limit with $l \to \infty$.

$$
\begin{aligned}
\mathcal{A}_2 \quad &:= \quad E\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \eta(\hat{u}_0(x) - k)\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\rho_l(u_0^\epsilon(y) - k)\,\mathrm{dk}\,\mathrm{dy} \\
&\qquad - E\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \eta(\hat{u}_0(x) - u_0^\epsilon(y))\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy} \\
&= \quad E\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} [\eta(\hat{u}_0(x) - k) - \eta(\hat{u}_0(x) - u_0^\epsilon(y))]\varphi(0,y)\rho_m(x-y)\,\mathrm{dx} \\
&\qquad\qquad \times \rho_l(u_0^\epsilon(y) - k)\,\mathrm{dk}\,\mathrm{dy}.
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{A}_2| \quad &\leq \quad \|\eta'\|_\infty E\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u_0^\epsilon(y)) - k|\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\rho_l(u_0^\epsilon(y) - k)\,\mathrm{dk}\,\mathrm{dy} \\
&\leq \quad \frac{\|\eta'\|_\infty}{l}\|\varphi\|_\infty \mathrm{Meas}(K) \\
&\to_{l\to\infty} \quad 0.
\end{aligned}
$$

As to the passage to the limit with $\eta = \eta_\delta \to |\cdot|$, note that

$$\begin{aligned}\mathcal{A}_3 &:= E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\eta(\hat{u}_0(x)-u_0^\epsilon(y))\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\\ &\quad -E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{u}_0(x)-u_0^\epsilon(y)|\,\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\\ &= E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\eta(\hat{u}_0(x)-u_0^\epsilon(y))-|\hat{u}_0(x)-u_0^\epsilon(y)|\right]\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}.\end{aligned}$$

As $|\eta(r)-|r||\leq\delta$ for any $r\in\mathbb{R}$, we have

$$|\mathcal{A}_3| \;\leq\; \delta\|\varphi\|_\infty \mathrm{Meas}(K)\xrightarrow[\delta\to 0]{} 0.$$

Finally, consider the passage to the limit with $\epsilon\to 0$. As

$$\begin{aligned}\mathcal{A}_4 &:= E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{u}_0(x)-u_0^\epsilon(y)|\,\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\\ &\quad -E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{u}_0(x)-u_0(y)|\,\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\\ &= E\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[|\hat{u}_0(x)-u_0^\epsilon(y)|-|\hat{u}_0(x)-u_0(y)|\right]\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy},\end{aligned}$$

by the reverse triangle inequality, we have

$$|\mathcal{A}_4| \;\leq\; E\int_K |u_0(y)-u_0^\epsilon(y)|\,\mathrm{dy}$$

which tends to 0 as $u_0^\epsilon\to u_0$ in $L^2(\mathbb{R}^d)$ as $\epsilon\to 0$.

2. As φ is a function of variables (s, y)

$$
\begin{aligned}
I_2 + J_2 \;=\; & E\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta(\hat{\mathbf{u}}(p) - k)\partial_t\varphi(s, y)\ \rho_n(t - s)\rho_m(x - y)\ \mathrm{dp} \\
& \qquad\qquad\qquad \times\rho_l(u_\epsilon(s, y) - k)\ \mathrm{dk\ dy\ ds} \\
& +E\int_Q \int_{\mathbb{R}} \int_Q \eta(u_\epsilon(s, y) - k)\partial_s\varphi(s, y)\ \rho_n(t - s)\rho_m(x - y) \\
& \qquad\qquad\qquad \times\int_0^1 \rho_l(\hat{\mathbf{u}}(t, x, \alpha) - k)\ \mathrm{dk\ dy\ ds\ dp} \\
=\; & E\int_Q \int_{\mathbb{R}} \int_Q \eta(u_\epsilon(s, y) - k)\partial_s\varphi(s, y)\ \rho_n(t - s)\rho_m(x - y) \\
& \qquad\qquad\qquad \times\int_0^1 \rho_l(\hat{\mathbf{u}}(t, x, \alpha) - k)\ \mathrm{dk\ dy\ ds\ dp} \\
\to_{n\to\infty}\; & E\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \eta(u_\epsilon(s, y) - k)\partial_s\varphi(s, y)\ \rho_m(x - y) \\
& \qquad\qquad\qquad \times\int_0^1 \rho_l(\hat{\mathbf{u}}(s, x, \alpha) - k)\ \mathrm{dk\ dy\ ds\ dx\ d}\alpha \\
\to_{l\to\infty}\; & E\int_Q \int_{\mathbb{R}^d} \int_0^1 \eta(u_\epsilon(s, y) - \hat{\mathbf{u}}(s, x, \alpha))\partial_s\varphi(s, y)\ \rho_m(x - y)\ \mathrm{dy\ dx\ ds\ d}\alpha \\
\to_{\eta\to|.|}\; & E\int_Q \int_{\mathbb{R}^d} \int_0^1 |u_\epsilon(s, y) - \hat{\mathbf{u}}(s, x, \alpha)|\,\partial_s\varphi(s, y)\ \rho_m(x - y)\ \mathrm{dy\ dx\ ds\ d}\alpha \\
\to_{\epsilon\to 0}\; & E\int_Q \int_{\mathbb{R}^d} \int_0^1 \int_0^1 |\mathbf{u}(s, y, \beta) - \hat{\mathbf{u}}(s, x, \alpha)|\,\partial_s\varphi(s, y) \\
& \qquad\qquad\qquad \times\rho_m(x - y)\ \mathrm{dy\ dx\ ds\ d}\alpha\ \mathrm{d}\beta \\
\to_{m\to\infty}\; & E\int_Q \int_0^1 \int_0^1 |\mathbf{u}(s, y, \beta) - \hat{\mathbf{u}}(s, y, \alpha)|\,\partial_s\varphi(s, y)\ \mathrm{dy\ ds\ d}\alpha\ \mathrm{d}\beta.
\end{aligned}
$$

The passages to the limit are similar to the previous ones, by using moreover the properties of convolution in $L^1(0, T, L^1(\Omega\times\mathbb{R}^d\times(0, 1)))$, the absolute continuity of the integral on the one hand, and the limit in the sense of Young measures with the Carathéodory-function $F(k, s, y) = \int_{\mathbb{R}^d}\int_0^1 |k - \hat{\mathbf{u}}(s, x, \alpha)|\,\partial_s\varphi(s, y)\ \rho_m(x - y)\ \mathrm{dx\ d}\alpha$ on the second hand.

3. $I_3 + J_3 =$

$$E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta(\hat{\mathbf{u}}(p)-k)\varphi(s,y)\,\partial_t\rho_n(t-s)\rho_m(x-y)\,\mathrm{d}p\rho_l(\underbrace{u_\epsilon(s,y)-k}_{=\tau})\,\mathrm{d}k\,\mathrm{d}y\,\mathrm{d}s$$
$$+E\int_Q\int_{\mathbb{R}}\int_Q \eta(u_\epsilon(s,y)-k)\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\int_0^1 \rho_l(\underbrace{\hat{\mathbf{u}}(p)-k}_{=-\sigma})\,\mathrm{d}k\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}p$$
$$= E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta(\hat{\mathbf{u}}(p)-u_\epsilon(s,y)+\tau)\varphi(s,y)\,\partial_t\rho_n(t-s)\rho_m(x-y)\mathrm{d}p\rho_l(\tau)\mathrm{d}\tau\mathrm{d}y\mathrm{d}s$$
$$+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta(u_\epsilon(s,y)-\hat{\mathbf{u}}(p)-\sigma)\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\mathrm{d}y\mathrm{d}s\rho_l(-\sigma)\mathrm{d}\sigma\mathrm{d}p$$
$$= 0$$

since η and ρ_l are even functions.

4. $J_4 + J_5 =$

$$-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q \eta'(u_\epsilon(s,y)-k)\nabla_y u_\epsilon(s,y)\nabla_y\varphi(s,y)\,\rho_n(t-s)\rho_m(x-y)\,\mathrm{d}y\,\mathrm{d}s$$
$$\times\int_0^1 \rho_l(\hat{\mathbf{u}}(p)-k)\,\mathrm{d}k\,\mathrm{d}p$$
$$-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q \eta'(u_\epsilon(s,y)-k)\nabla_y u_\epsilon(s,y)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s$$
$$\times\int_0^1 \rho_l(\hat{\mathbf{u}}(p)-k)\,\mathrm{d}k\,\mathrm{d}p$$
$$\to_{n\to\infty} -\epsilon E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 \eta'(u_\epsilon(t,y)-k)\nabla_y u_\epsilon(t,y)\nabla_y[\varphi\rho_m(x-y)]\mathcal{A}_k^l\,\mathrm{d}y\,\mathrm{d}k\,\mathrm{d}p$$
$$\to_{l\to\infty} -\epsilon E\int_Q\int_{\mathbb{R}^d}\int_0^1 \eta'(u_\epsilon(t,y)-\hat{\mathbf{u}}(t,x,\alpha))\nabla_y u_\epsilon(t,y)\nabla_y[\varphi(t,y)\rho_m(x-y)]\,\mathrm{d}y\,\mathrm{d}p$$
$$\to_{\eta\to|.|} -\epsilon E\int_Q\int_{\mathbb{R}^d}\int_0^1 \mathrm{sgn}_0\,(u_\epsilon(t,y)-\hat{\mathbf{u}}(t,x,\alpha))\,\nabla_y u_\epsilon(t,y)\nabla_y[\varphi(t,y)\rho_m(x-y)]\mathrm{d}y\mathrm{d}p$$
$$\to_{\epsilon\to 0} 0.$$

The passages to the limit with n and $l\to\infty$ follow by classical arguments for convolutions in the deterministic setting.The passage to the limit with $\eta\to|\cdot|$ follows from pointwise convergence and with Lebesgue's dominated convergence theorem. The convergence with $\epsilon\to 0$ follows from the *a priori* estimate given in Lemma 3.2 which implies that $\epsilon\nabla u_\epsilon$ converges to 0 in $L^2(]0,T[\times\Omega, L^2(\mathbb{R}^d))$.

5. Since φ is a function of variables (s, y)

$I_4 + J_6 =$

$$-E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}}(p),k)\nabla_x\varphi(s,y)\ \rho_m(x-y)\rho_n(t-s)\ \mathrm{dp}\mathcal{B}_k^l\ \mathrm{dk\ dy\ ds}$$

$$-E\int_Q\int_{\mathbb{R}}\int_Q F^\eta(u_\epsilon(s,y),k)\nabla_y\varphi(s,y)\ \rho_m(x-y)\rho_n(t-s)\ \mathrm{dy\ ds}\int_0^1\mathcal{A}_k^l\ \mathrm{dk\ dp}$$

$$= -E\int_Q\int_{\mathbb{R}}\int_Q F^\eta(u_\epsilon(s,y),k)\nabla_y\varphi(s,y)\ \rho_m(x-y)\rho_n(t-s)\ \mathrm{dy\ ds}\int_0^1\mathcal{A}_k^l\ \mathrm{dk\ dp},$$

and

$$I_4 + J_6 \to_{n\to\infty} -E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(u_\epsilon(t,y),k)\nabla_y\varphi(t,y)\ \rho_m(x-y)\mathcal{A}_k^l\ \mathrm{dy\ dk\ dp}$$

$$\to_{l\to\infty} -E\int_{\mathbb{R}^d}\int_Q\int_0^1 F^\eta(u_\epsilon(t,y),\hat{\mathbf{u}}(t,x,\alpha))\nabla_y\varphi(t,y)\ \rho_m(x-y)\ \mathrm{dy\ dp}$$

$$\to_{\eta\to|.|} -E\int_{\mathbb{R}^d}\int_Q\int_0^1 F(u_\epsilon(t,y),\hat{\mathbf{u}}(t,x,\alpha))\nabla_y\varphi(t,y)\ \rho_m(x-y)\ \mathrm{dy\ dp}$$

$$\to_{\epsilon\to 0} -E\int_{\mathbb{R}^d}\int_Q\int_0^1\int_0^1 F(\mathbf{u}(t,y,\beta),\hat{\mathbf{u}}(t,x,\alpha))\nabla\varphi(t,y)\ \rho_m(x-y)\mathrm{dydpd}\beta$$

$$\to_{m\to\infty} -E\int_Q\int_0^1\int_0^1 F(\mathbf{u}(t,x,\beta),\hat{\mathbf{u}}(t,x,\alpha))\nabla\varphi(t,x)\ \mathrm{dp\ d}\beta.$$

The passages to the limit are similar to the previous ones by noticing that $F^\eta(\cdot,k)$ is a Lipschitz-continuous function with Lipschitz constant $\|\vec{f}'\|_\infty$ and that $G(k,t,y) = \int_{\mathbb{R}^d}\int_0^1 F(k,\hat{\mathbf{u}}(t,x,\alpha))\nabla_y\varphi(t,y)\ \rho_m(x-y)\ \mathrm{dx\ d}\alpha$ is a Carathéodory-function.

6. Now let us consider $I_5 + J_7$:

As $\nabla_y \rho_m(x-y) = -\nabla_x \rho_m(x-y)$, using similar arguments as before (symmetry of F, *i.e.* $F(r,s) = F(s,r)$, the fact that for $\eta = \eta_\delta$: $|F^\eta(r,s) - F(r,s)| \leq \delta \|\vec{\mathbf{f}}'\|_\infty$ and Lipschitz continuity of F with respect to both of its variables), we get

$$|I_5 + J_7| =$$

$$\begin{aligned}
& \Big|E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}},k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi \,\mathrm{dp}\rho_l(u_\epsilon(s,y)-k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& +E\int_Q\int_{\mathbb{R}} F^\eta(u_\epsilon,k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,\mathrm{dk}\,\mathrm{dp}\Big| \\
= \;& \Big|E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}},u_\epsilon(s,y)-k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi\,\mathrm{dp}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& +E\int_Q\int_0^1\int_{\mathbb{R}} F^\eta(u_\epsilon,\hat{\mathbf{u}}(t,x,\alpha)-k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi\,\mathrm{dy}\,\mathrm{ds}\rho_l(k)\,\mathrm{dk}\,\mathrm{dp}\Big| \\
= \;& \Big|E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}},u_\epsilon(s,y)-k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi\,\mathrm{dp}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& -E\int_Q\int_0^1\int_{\mathbb{R}} F^\eta(u_\epsilon,\hat{\mathbf{u}}(t,x,\alpha)-k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi\,\mathrm{dy}\,\mathrm{ds}\rho_l(k)\,\mathrm{dk}\,\mathrm{dp}\Big| \\
= \;& \Big|E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 [F^\eta(\hat{\mathbf{u}},u_\epsilon(s,y)-k) - F^\eta(u_\epsilon,\hat{\mathbf{u}}(p)-k)]\nabla_x\rho_m(x-y) \\
& \qquad\qquad\qquad\qquad \times\rho_n(t-s)\varphi\,\mathrm{dp}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\Big|
\end{aligned}$$

and that $\lim_{\delta\to 0}\lim_{l\to\infty}\limsup_{n\to\infty}|I_5 + I_7| = 0$.

Let us now turn to the study of the integrals coming from the stochastic term. We start with the additional deterministic integrals coming from the Itô chain rule formula:

7. $I_6 + J_8 =$

$$\begin{aligned}
&\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}})\eta''(\hat{\mathbf{u}}-k)\rho_m(x-y)\rho_n(t-s)\varphi \,\mathrm{dp}\mathcal{B}_l^k \,\mathrm{dk\,dy\,ds}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon-k)\rho_m(x-y)\rho_n(t-s)\varphi\,\mathrm{dy\,ds}\int_0^1\mathcal{A}_l^k\,\mathrm{dk\,dp}
\end{aligned}$$

$$\begin{aligned}
\to_{n\to\infty}\quad &\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}})\eta''(\hat{\mathbf{u}}-k)\rho_m(x-y)\varphi\,\mathrm{dp}\rho_l(u_\epsilon(t,y)-k)\,\mathrm{dk\,dy}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} h^2(u_\epsilon)\eta''(u_\epsilon-k)\rho_m(x-y)\varphi\,\mathrm{dy}\int_0^1\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\mathrm{dk\,dp}
\end{aligned}$$

$$\begin{aligned}
\to_{l\to\infty}\quad &\frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(\hat{\mathbf{u}})\eta''(\hat{\mathbf{u}}-u_\epsilon(t,y))\rho_m(x-y)\varphi\,\mathrm{dp\,dy}\\
&+\frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(u_\epsilon)\eta''(u_\epsilon-\hat{\mathbf{u}}(t,x,\alpha))\rho_m(x-y)\varphi\,\mathrm{dy\,dp},
\end{aligned}$$

where the passages to the limit follow from standard arguments (properties of convolution, Lebesgue points), using, moreover, the fact that η'' is bounded and Lipschitz-continuous. Note that it is not possible to pass to the limit with $\eta \to |\cdot|$ in the preceding term $\lim_l \lim_n I_6 + I_8$, as we ignore the limit of η''.

Instead, we keep this term for the moment. We will combine it later on with corresponding integrals resulting from the stochastic integrals and show that the sum of these terms vanishes as $\eta = \eta_\delta \to |\cdot|$. For the convenience of the reader, let us just give the details of the arguments used to prove the above passages to the limit with n and then l to ∞.

- Limit as $n \to \infty$: By a simple change of variable we have

$$\begin{aligned}
\mathcal{A}_1 = & \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-k)\rho_m(x-y) \\
& \qquad\times\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\mathcal{B}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& -\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-k)\rho_m(x-y) \\
& \qquad\times\varphi(t,y)\rho_l(u_\epsilon(t,y)-k)\,\mathrm{dp}\,\mathrm{dk}\,\mathrm{dy} \\
& + \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(s,y))\eta''(u_\epsilon(s,y)-k)\rho_m(x-y) \\
& \qquad\times\rho_n(t-s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\mathcal{A}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
& -\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\rho_m(x-y) \\
& \qquad\times\varphi(t,y)\,\mathrm{dy}\int_0^1\mathcal{A}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
= & \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(s,y)+k)\rho_m(x-y) \\
& \qquad\times\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& -\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)\rho_m(x-y) \\
& \qquad\times\varphi(t,y)\rho_l(k)\,\mathrm{dp}\,\mathrm{dk}\,\mathrm{dy} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\left[h^2(u_\epsilon(s,y))\eta''(u_\epsilon(s,y)-k)-h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\right] \\
& \qquad\times\rho_n(t-s)\varphi(s,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\mathcal{A}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)[\varphi(s,y)-\varphi(t,y)]) \\
& \qquad\times\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\mathcal{A}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\varphi(t,y)[1-\int_0^T\rho_n(t-s)\,\mathrm{ds}] \\
& \qquad\times\rho_m(x-y)\,\mathrm{dy}\int_0^1\mathcal{A}_k^l\,\mathrm{dk}\,\mathrm{dp}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}_1 = & \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1[\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(s,y)+k)-\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)] \\
& \qquad\qquad \times h^2(\hat{\mathbf{u}}(p))\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)[\varphi(s,y)-\varphi(t,y)] \\
& \qquad\qquad \times h^2(\hat{\mathbf{u}}(p))\rho_m(x-y)\rho_n(t-s)\,\mathrm{dp}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& -\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-u_\epsilon(t,y)+k)\rho_m(x-y)\varphi(t,y) \\
& \qquad\qquad \times[1-\int_0^T\rho_n(t-s)\,\mathrm{ds}]\rho_l(k)\,\mathrm{dp}\,\mathrm{dk}\,\mathrm{dy} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\big[h^2(u_\epsilon(s,y))-h^2(u_\epsilon(t,y))\big]\eta''(u_\epsilon(s,y)-k)\rho_n(t-s) \\
& \qquad\qquad \times\varphi(s,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\rho_l(\hat{\mathbf{u}}(p)-k)\,\mathrm{dk}\,\mathrm{dp} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\big[\eta''(u_\epsilon(s,y)-k)-\eta''(u_\epsilon(t,y)-k)\big]h^2(u_\epsilon(t,y))\rho_n(t-s) \\
& \qquad\qquad \times\varphi(s,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\rho_l(\hat{\mathbf{u}}(p)-k)\,\mathrm{dk}\,\mathrm{dp} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)[\varphi(s,y)-\varphi(t,y)]\rho_n(t-s) \\
& \qquad\qquad \times\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\rho_l(\hat{\mathbf{u}}(p)-k)\,\mathrm{dk}\,\mathrm{dp} \\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y)-k)\varphi(t,y)[1-\int_0^T\rho_n(t-s)\,\mathrm{ds}] \\
& \qquad\qquad \times\rho_m(x-y)\,\mathrm{dy}\int_0^1\rho_l(\hat{\mathbf{u}}(p)-k)\,\mathrm{dk}\,\mathrm{dp}.
\end{aligned}
$$

Therefore, as η'' is bounded and Lipschitz-continuous, we have

$$
\begin{aligned}
|\mathcal{A}_1| \le\ & \frac{c(\eta'')}{2} E \int_Q \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \\
& \qquad \times \rho_m(x-y)\rho_n(t-s)\varphi(s,y) \,\mathrm{dp}\,\mathrm{dy}\,\mathrm{ds} \\
& + \frac{c(\varphi)}{2n} E \int_Q \int_{\mathbb{R}} \int_K \int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p) - u_\epsilon(t,y) + k) \\
& \qquad \times \rho_m(x-y) \,\mathrm{dp}\rho_l(k)\,\mathrm{dk}\,\mathrm{dy} \\
& + \frac{1}{2} E \int_{T-2/n}^T \int_K \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p) - u_\epsilon(t,y) + k) \\
& \qquad \times \rho_m(x-y)\varphi(t,y)\rho_l(k) \,\mathrm{dp}\,\mathrm{dk}\,\mathrm{dy} \\
& + \frac{\|\eta''\|_\infty}{2} E \int_Q \int_{\mathbb{R}} \int_Q \big[h^2(u_\epsilon(s,y)) - h^2(u_\epsilon(t,y))\big]\rho_n(t-s) \\
& \qquad \times \varphi(s,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds} \int_0^1 \rho_l(\hat{\mathbf{u}}(p) - k)\,\mathrm{dk}\,\mathrm{dp} \\
& + \frac{c(\eta'')}{2} E \int_Q \int_{\mathbb{R}} \int_Q \min[2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) h^2(u_\epsilon(t,y))\rho_n(t-s) \\
& \qquad \times \varphi(s,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds} \int_0^1 \rho_l(\hat{\mathbf{u}}(p) - k)\,\mathrm{dk}\,\mathrm{dp} \\
& + \frac{c(\varphi)}{2n} E \int_K \int_{\mathbb{R}} \int_Q h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y) - k)\rho_n(t-s) \\
& \qquad \times \rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds} \int_0^1 \rho_l(\hat{\mathbf{u}}(p) - k)\,\mathrm{dk}\,\mathrm{dp} \\
& + \frac{1}{2} E \int_{\mathbb{R}^d} \int_{T-2/n}^T \int_{\mathbb{R}} \int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\eta''(u_\epsilon(t,y) - k)\varphi(t,y) \\
& \qquad \times \rho_m(x-y)\,\mathrm{dy} \int_0^1 \rho_l(\hat{\mathbf{u}}(p) - k)\,\mathrm{dk}\,\mathrm{dp} \\
\le\ & \frac{c(\eta'')}{2} E \int_Q \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \\
& \qquad \times \rho_m(x-y)\rho_n(t-s)\varphi(s,y) \,\mathrm{dp}\,\mathrm{dy}\,\mathrm{ds} \\
& + \frac{c(\varphi,\eta'')}{2n} E \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{dp} \\
& + \frac{c(\varphi,\eta'')}{2} E \int_{T-2/n}^T \int_K \int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{dp} \\
& + \frac{c(\varphi)\|\eta''\|_\infty}{2} E \int_Q \int_0^T \big[h^2(u_\epsilon(s,y)) - h^2(u_\epsilon(t,y))\big]\rho_n(t-s)\,\mathrm{dy}\,\mathrm{ds}\,\mathrm{dt} \\
& + \frac{c(\varphi,\eta'')}{2} E \int_0^T \int_Q \min[2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) h^2(u_\epsilon(t,y)) \\
& \qquad\qquad \times \rho_n(t-s)\,\mathrm{dy}\,\mathrm{ds}\,\mathrm{dt} \\
& + \frac{c(\varphi)\|\eta''\|_\infty}{2n} E \int_Q h^2(u_\epsilon(t,y))\,\mathrm{dy}\,\mathrm{dt} \\
& + \frac{\|\eta''\|_\infty}{2} E \int_{T-2/n}^T \int_{\mathbb{R}^d} h^2(u_\epsilon(t,y))\varphi(t,y)\,\mathrm{dy}\,\mathrm{dt}
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{A}_1| \le \frac{c(\eta'', m, \varphi)}{2} & E \int_Q \int_K \int_0^T \int_0^1 h^2(\hat{\mathbf{u}}(p)) \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \\
& \qquad\qquad \times \rho_n(t-s) \,\mathrm{dp}\,\mathrm{dy}\,\mathrm{ds} \\
+ \frac{c(\varphi, \eta'')}{2} & E \int_0^T \int_Q \min[2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|]) h^2(u_\epsilon(t,y)) \\
& \qquad\qquad \times \rho_n(t-s) \,\mathrm{dy}\,\mathrm{ds}\,\mathrm{dt} \\
+ \frac{c(\varphi, \eta'')}{2} & E \int_Q \int_0^T \left[h^2(u_\epsilon(s,y)) - h^2(u_\epsilon(t,y))\right] \\
& \qquad\qquad \times \rho_n(t-s) \,\mathrm{dy}\,\mathrm{ds}\,\mathrm{dt} \\
+ \frac{c(\varphi, \eta'')}{2} & E \int_{T-2/n}^T \int_K \int_0^1 h^2(\hat{\mathbf{u}}(p)) + h^2(u_\epsilon(t,x)) \,\mathrm{dp} \\
+ \frac{c(\varphi, \eta'')}{2n} & E \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) + h^2(u_\epsilon(t,x)) \,\mathrm{dp}.
\end{aligned}
$$

Obviously, the last two integrals tend to 0 as $n \to \infty$. As to the first integral on the right, note that

$$
\begin{aligned}
& E \int_{\mathbb{R}^d} \int_K \int_0^1 \int_0^T \int_0^T h^2(\hat{\mathbf{u}}(p)) \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \rho_n(t-s) \mathrm{dpdsdy} \\
&= E \int_{\mathbb{R}^d} \int_K \int_0^1 \int_0^T h^2(\hat{\mathbf{u}}(p)) \int_0^T \min\left[2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|\right] \rho_n(t-s) \mathrm{dsdpdy} \\
&= E \int_{\mathbb{R}^d} \int_K \int_0^1 \int_0^T h^2(\hat{\mathbf{u}}(p)) A_n(t) \mathrm{dpdy},
\end{aligned}
$$

where $A_n(t) = \displaystyle\int_0^T \min(2\|\eta''\|_\infty, |u_\epsilon(s,y) - u_\epsilon(t,y)|) \rho_n(t-s) \,\mathrm{ds}$.

For almost all (ω, x, y, α) fixed, $s \mapsto u_\epsilon(s,.)$ and $t \mapsto \hat{\mathbf{u}}(t,.)$ are elements of $L^2(0,T)$. Therefore, $|A_n(t)| \le \displaystyle\int_0^T |u_\epsilon(s,y) - u_\epsilon(t,y)| \rho_n(t-s) \,\mathrm{ds}$ converges to 0 as $n \to \infty$ a.e. on $]0,T[$ (in every Lebesgue point), and thus $h^2(\hat{\mathbf{u}}(p)) A_n(t)$ tends to 0 for a.a. $t \in]0,T[$. As $|h^2(\hat{\mathbf{u}}(p)) A_n(t)| \le 2\|\eta''\|_\infty h^2(\hat{\mathbf{u}}(p))$, by Lebesgue's theorem, $\displaystyle\int_0^T h^2(\hat{\mathbf{u}}(p)) A_n(t) \,\mathrm{dt}$ tends to 0 as $n \to \infty$, for a.a. (ω, x, y, α).

Then, using the same estimate, by Lebesgue's dominated convergence theorem we may conclude that the integral over $\Omega \times \mathbb{R}^d \times K \times (0,1)$ converges to 0. The second integral on the right can be shown to converge to 0 by using the same type of arguments. Finally, note that the third integral tends to 0 by the properties of convolution in $L^1(0,T; L^1(\Omega \times \mathbb{R}^d))$.

- Limit as $l \to \infty$: Again by a simple change of variables we have

$$
\begin{aligned}
\mathcal{A}_2 := {} & \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(\hat{\mathbf{u}}(p)) \eta''(\hat{\mathbf{u}}(p) - k) \rho_l(u_\epsilon(t,y) - k) \,\mathrm{dk} \\
& \qquad \times \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\,\mathrm{dy} \\
& + \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} h^2(u_\epsilon(t,y)) \eta''(u_\epsilon(t,y) - k) \rho_m(x-y)\varphi(t,y) \,\mathrm{dy} \int_0^1 \mathcal{A}_k^l \,\mathrm{dk}\,\mathrm{dp} \\
& - \frac{1}{2} E \int_{\mathbb{R}^d} \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) \eta''(\hat{\mathbf{u}}(p) - u_\epsilon(t,y)) \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\,\mathrm{dy} \\
& - \frac{1}{2} E \int_{\mathbb{R}^d} \int_Q \int_0^1 h^2(u_\epsilon(t,y)) \eta''(u_\epsilon(t,y) - \hat{\mathbf{u}}(t,x,\alpha)) \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\,\mathrm{dy} \\
= {} & \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(\hat{\mathbf{u}}(p)) \eta''(\hat{\mathbf{u}}(p) - u_\epsilon(t,y) + k) \rho_l(k) \,\mathrm{dk} \\
& \qquad \times \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\,\mathrm{dy} \\
& + \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(u_\epsilon(t,y)) \eta''(u_\epsilon(t,y) - \hat{\mathbf{u}}(t,x,\alpha) + k) \rho_l(k) \,\mathrm{dk} \\
& \qquad \times \rho_m(x-y)\varphi(t,y) \,\mathrm{dy}\,\mathrm{dp} \\
& - \frac{1}{2} E \int_{\mathbb{R}^d} \int_Q \int_0^1 h^2(\hat{\mathbf{u}}(p)) \eta''(\hat{\mathbf{u}}(p) - u_\epsilon(t,y)) \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\,\mathrm{dy} \\
& - \frac{1}{2} E \int_{\mathbb{R}^d} \int_Q \int_0^1 h^2(u_\epsilon(t,y)) \eta''(u_\epsilon(t,y) - \hat{\mathbf{u}}(t,x,\alpha)) \rho_m(x-y)\varphi(t,y) \,\mathrm{dy}\,\mathrm{dp} \\
= {} & \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(\hat{\mathbf{u}}(p)) [\eta''(\hat{\mathbf{u}}(p) - u_\epsilon(t,y) + k) - \eta''(\hat{\mathbf{u}}(p) - u_\epsilon(t,y))] \\
& \qquad \times \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\rho_l(k) \,\mathrm{dk}\,\mathrm{dy} \\
& + \frac{1}{2} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 [\eta''(u_\epsilon(t,y) - \hat{\mathbf{u}}(t,x,\alpha) + k) - \eta''(u_\epsilon(t,y) - \hat{\mathbf{u}}(t,x,\alpha))] \\
& \qquad \times h^2(u_\epsilon(t,y)) \rho_m(x-y)\varphi(t,y) \,\mathrm{dy}\rho_l(k) \,\mathrm{dk}\,\mathrm{dp}.
\end{aligned}
$$

Using once again the Lipschitz continuity of η'' we obtain the estimate

$$
\begin{aligned}
|\mathcal{A}_2| \leq {} & \frac{c(\eta'')}{2l} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(\hat{\mathbf{u}}(p)) \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\rho_l(k) \,\mathrm{dk}\,\mathrm{dy} \\
& + \frac{c(\eta'')}{2l} E \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 h^2(u_\epsilon(t,y)) \rho_m(x-y)\varphi(t,y) \,\mathrm{dy}\rho_l(k) \,\mathrm{dk}\,\mathrm{dp} \\
\leq {} & \frac{c(\eta'')}{2l} E \int_Q \int_{\mathbb{R}^d} \int_0^1 [h^2(\hat{\mathbf{u}}(p)) + h^2(u_\epsilon(t,y))] \rho_m(x-y)\varphi(t,y) \,\mathrm{dp}\,\mathrm{dy} \\
& \to_{l\to\infty} 0.
\end{aligned}
$$

Now we come to the estimate of the most interesting part, the stochastic integrals.

8. $I_7 + J_9 =$

$$E\int_Q\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\mathcal{B}_k^l\,\mathrm{dk\,dy\,ds}$$
$$+E\int_Q\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}^d}\eta'(u_\epsilon-k)h(u_\epsilon)\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}dw(s)\int_0^1\mathcal{A}_k^l\,\mathrm{dk\,dp}$$

$$= E\int_Q\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\mathcal{B}_k^l\,\mathrm{dk\,dy\,ds}.$$

Indeed, since $\mathrm{supp}\rho_n \subset \mathbb{R}^-$

$$E\int_Q\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}^d}\eta'(u_\epsilon-k)h(u_\epsilon)\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}dw(s)\int_0^1\mathcal{A}_k^l\mathrm{dk\,dp}$$

$$= \int_Q\int_{\mathbb{R}}E\Big[\int_t^T\int_{\mathbb{R}^d}\eta'(u_\epsilon-k)h(u_\epsilon)\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}dw(s)\int_0^1\mathcal{A}_k^l\mathrm{d}\alpha\Big]\mathrm{dkdxdt}$$

$$= \quad 0,$$

since $\alpha(t) := \int_0^1 \rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\,\mathrm{d}\alpha$ is predictable and if one denotes by $\beta(s) := \int_{\mathbb{R}^d}\eta'(u_\epsilon-k)h(u_\epsilon)\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}$, one has that

$$\begin{aligned} E\Big[\alpha(t)\int_t^T\beta(s)dw(s)\Big] &= E\Big[\alpha(t)\int_0^T\beta(s)dw(s)\Big]-E\Big[\alpha(t)\int_0^t\beta(s)dw(s)\Big]\\ &= E\Big[\alpha(t)\int_0^t\beta(s)dw(s)\Big]-E\Big[\alpha(t)\int_0^t\beta(s)dw(s)\Big]\\ &= 0,\end{aligned}$$

using the martingale property of the Itô integral:

$$\begin{aligned} E\Big[\alpha(t)\int_0^T\beta(s)dw(s)\Big] &= E\Big[\alpha(t)E[\int_0^T\beta(s)dw(s)|F_t]\Big]\\ &= E\Big[\alpha(t)\int_0^t\beta(s)dw(s)\Big].\end{aligned}$$

Then, by the same type of argument with $\rho_l(u_\epsilon(s-2/n,y)-k)$ and $\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)\,\mathrm{dx}\,\mathrm{dw(t)}$, since supp $\rho_n(\cdot-s)\subset[s-2/n,s]$, we deduce

$$
\begin{aligned}
I_7+J_9 &= E\int_Q\int_{\mathbb{R}}\int_{s-2/n}^{s}\int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\\
&\qquad\qquad\qquad\qquad \times\rho_l(u_\epsilon(s,y)-k)\,\mathrm{dk\,dy\,ds}\\
&= E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx}\\
&\qquad\qquad\qquad\qquad \times[\rho_l(u_\epsilon(s,y)-k)-\rho_l(u_\epsilon(s-2/n,y)-k)]\,\mathrm{dkdyds}\\
&= E\int_Q\int_{\mathbb{R}} G_{n,m}(\omega,s,y,k)[\rho_l(u_\epsilon(s,y)-k)-\rho_l(u_\epsilon(s-2/n,y)-k)]\,\mathrm{dkdyds}
\end{aligned}
$$

where $y\mapsto G_{n,m}(\omega,s,y,k)=\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1\eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\mathrm{dx}$.

Remind at this point that u_ϵ is the solution of $u_\epsilon(0,\cdot)=u_0^\epsilon$ and

$$\mathrm{d}u_\epsilon=[\epsilon\Delta u_\epsilon+\ \mathrm{div}\vec{\mathbf{f}}(u_\epsilon)]\,\mathrm{dt}+h(u_\epsilon)dw=A_\epsilon\,\mathrm{dt}+h(u_\epsilon)dw.$$

Thanks to Appendix I Section 2, this solution is in $\mathcal{N}_w^2(0,T,H^2(\mathbb{R}^d))$ and

$$y\in\mathbb{R}^d\ a.e.,\ \mathrm{d}u_\epsilon(.,y)=A_\epsilon(.,y)\,\mathrm{dt}+h(u_\epsilon(.,y))dw.$$

Therefore, by Itô's formula

$$
\begin{aligned}
&\rho_l(u_\epsilon(s,y)-k)-\rho_l(u_\epsilon(s-2/n,y)-k)\\
=\ &\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma+\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma)\\
&+\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l''(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\\
=\ &-\frac{\partial}{\partial k}\Big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma+\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma)\\
&+\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big].
\end{aligned}
$$

Thus, using regularity of the stochastic integral with respect to parameters proposed by KUNITA [57] and reminded in Appendix I Section 1

$$
\begin{aligned}
I_7 + J_9 = {} & E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx} \\
& \qquad \times[\rho_l(u_\epsilon(s,y)-k)-\rho_l(u_\epsilon(s-2/n,y)-k)]\,\mathrm{dk\,dy\,ds} \\
= {} & -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx} \\
& \times\frac{\partial}{\partial k}\Big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma+\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma) \\
& +\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]\,\mathrm{dk\,dy\,ds} \\
= {} & E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\partial}{\partial k}\int_{s-2/n}^{s}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx} \\
& \times\Big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma+\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma) \\
& +\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]\,\mathrm{dk\,dy\,ds} \\
= {} & -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx} \\
& \times\Big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma+\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma) \\
& +\frac{1}{2}\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]\,\mathrm{dk\,dy\,ds} \\
& =: \mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3.
\end{aligned}
$$

We will prove that $\mathbb{I}_1$ and $\mathbb{I}_3$ tend to 0 as $n\to\infty$. To this end we estimate first $\mathbb{I}_1$. Using Cauchy-Schwarz inequality, Jensen inequality and the isometry property of the Itô integral, we find

$$
\begin{aligned}
|\mathbb{I}_1| &= \Big|\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} E\Big[\big[\int_{s-2/n}^{s}\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)\varphi\rho_m(x-y)dw(t)\big]\\
&\qquad\qquad\times\big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma\big]\Big]\mathrm{dxdkdyds}\Big|\\
&\le \int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big[E\big[\int_{s-2/n}^{s}\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)\varphi\rho_m(x-y)dw(t)\big]^2\Big]^{1/2}\\
&\qquad\qquad\times\Big[E\big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\,\mathrm{d}\sigma\big]^2\Big]^{1/2}\mathrm{dxdkdyds}\\
&\le \frac{\sqrt{2}}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big[E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n^2(t-s)\varphi^2\rho_m^2(x-y)\,\mathrm{dt}\Big]^{1/2}\\
&\qquad\qquad\times\Big[E\int_{s-\frac{2}{n}}^{s}\rho_l^2(u_\epsilon(\sigma,y)-k)A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\Big]^{1/2}\mathrm{dxdkdyds}\\
&\le C(\varphi)\frac{m^d nl}{\sqrt{n}}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}\int_{\mathbb{R}}\Big[E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,\mathrm{d}\alpha\,\mathrm{dt}\Big]^{1/2}\\
&\qquad\qquad\times\Big[E\int_{s-\frac{2}{n}}^{s}1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\Big]^{1/2}\mathrm{dxdkdyds}\\
&\le C(\varphi)m^d l\sqrt{n}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}\int_{\mathbb{R}}E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}(t,x)-k)\\
&\qquad\qquad\times h^2(\hat{\mathbf{u}}(t,x))\,\mathrm{d}\alpha\,\mathrm{dt}\,\mathrm{dk}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds}\\
&\quad +C(\varphi)l\sqrt{n}\int_{]0,T[}\int_K\int_{\mathbb{R}}E\int_{s-\frac{2}{n}}^{s}1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\le C(\varphi,\eta)m^d l\sqrt{n}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}E\int_{s-2/n}^{s}\int_0^1 h^2(\hat{\mathbf{u}}(t,x))\,\mathrm{d}\alpha\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds}\\
&\quad +C(\varphi)\sqrt{n}\int_{]0,T[}\int_K E\int_{s-\frac{2}{n}}^{s}A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy}\,\mathrm{ds}\\
&\le \frac{C(\varphi,\eta)m^d l}{\sqrt{n}}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}E\int_0^1 h^2(\hat{\mathbf{u}}(s,x))\,\mathrm{d}\alpha\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds}\\
&\quad +\frac{C(\varphi)}{\sqrt{n}}\int_{]0,T[}\int_K EA_\epsilon^2(s,y)\,\mathrm{dy}\,\mathrm{ds}\\
&\le \frac{C(\varphi,\eta)l}{\sqrt{n}}[\|h(\hat{\mathbf{u}})\|^2+\|A_\epsilon\|^2]\to_{n\to+\infty}0.
\end{aligned}
$$

Since, by Proposition 2.4, $u_\epsilon\in L^4(Q\times\Omega)$ if $u_0^\epsilon\in L^4(\mathbb{R}^d)$, we can also prove that $\lim_{n\to+\infty}\mathbb{I}_3=0$.

Indeed, using the same arguments as before, we have the following estimate on $\mathbb{I}_3$:

$$\begin{aligned}
|\mathbb{I}_3| &= \Big|\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d} E\Big[\big[\int_{s-2/n}^{s}\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)\varphi\rho_m(x-y)dw(t)\big] \\
&\qquad \times\big[\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\big]\Big]\,\mathrm{dx\,dk\,dy\,ds}\Big| \\
&\leq \int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big[E\big[\int_{s-2/n}^{s}\int_0^1 \eta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n(t-s)\varphi\rho_m(x-y)dw(t)\big]^2\Big]^{1/2} \\
&\qquad \times\Big[E\big[\int_{s-\frac{2}{n}}^{s}\rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\big]^2\Big]^{1/2}\,\mathrm{dx\,dk\,dy\,ds} \\
&\leq \frac{\sqrt{2}}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big[E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,\mathrm{d}\alpha\rho_n^2(t-s)\varphi^2\rho_m^2(x-y)\,\mathrm{dt}\Big]^{1/2} \\
&\qquad \times\Big[E\int_{s-\frac{2}{n}}^{s}\rho_l'^2(u_\epsilon(\sigma,y)-k)h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]^{1/2}\,\mathrm{dx\,dk\,dy\,ds} \\
&\leq C(\varphi,l)\frac{m^d n}{\sqrt{n}}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}\int_{\mathbb{R}}\Big[E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}-k)h^2(\hat{\mathbf{u}})\,\mathrm{d}\alpha\,\mathrm{dt}\Big]^{1/2} \\
&\qquad \times\Big[E\int_{s-\frac{2}{n}}^{s}1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]^{1/2}\,\mathrm{dk\,dy\,ds\,dx} \\
&\leq C(\varphi,l)m^d\sqrt{n}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}\int_{\mathbb{R}}E\int_{s-2/n}^{s}\int_0^1 \eta''^2(\hat{\mathbf{u}}(t,x)-k) \\
&\qquad \times h^2(\hat{\mathbf{u}}(t,x))\,\mathrm{dp\,dk\,dy\,ds} \\
&\quad +C(\varphi,l)\sqrt{n}\int_{]0,T[}\int_K\int_{\mathbb{R}}E\int_{s-\frac{2}{n}}^{s}1_{\{|u_\epsilon(\sigma,y)-k|<1/l\}}h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\,\mathrm{dk\,dy\,ds} \\
&\leq C(\varphi,\eta,l)m^d\sqrt{n}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}E\int_{s-2/n}^{s}\int_0^1 h^2(\hat{\mathbf{u}}(t,x))\,\mathrm{dp\,dy\,ds} \\
&\quad +C(\varphi,l)\sqrt{n}\int_{]0,T[}\int_K E\int_{s-\frac{2}{n}}^{s}h(u_\epsilon)^4(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy\,ds} \\
&\leq \frac{C(\varphi,\eta,l)m^d}{\sqrt{n}}\int_{]0,T[}\int_K\int_{\|x-y\|<1/m}E\int_0^1 h^2(\hat{\mathbf{u}}(s,x))\,\mathrm{d}\alpha\,\mathrm{dx\,dy\,ds} \\
&\quad +\frac{C(\varphi,l)}{\sqrt{n}}\int_{]0,T[}\int_K Eh(u_\epsilon)^4(s,y)\,\mathrm{dy\,ds} \\
&\leq \frac{C(\varphi,\eta,l)}{\sqrt{n}}[\|h(\hat{\mathbf{u}})\|^2+\|h(u_\epsilon)^2\|^2] \\
&\to_{n\to+\infty} 0.
\end{aligned}$$

It remains to consider $\mathbb{I}_2$. Using the properties of the stochastic Itô integral we find

$$
\begin{aligned}
\mathbb{I}_2 &= -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)dw(t)\varphi\rho_m(x-y)\,\mathrm{dx} \\
&\qquad\qquad \times\Big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma)\Big]\,\mathrm{dk\ dy\ ds} \\
&= -\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\Big[\big[\int_{s-2/n}^{s}\eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)dw(t)\big] \\
&\qquad\qquad \times\big[\int_{s-\frac{2}{n}}^{s}\rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))dw(\sigma)\big]\Big]\varphi\rho_m(x-y)\,\mathrm{dx\ dk\ dy\ ds} \\
&= -\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\Big[\int_{s-2/n}^{s}\eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\,\mathrm{dt}\Big] \\
&\qquad\qquad \times\varphi\rho_m(x-y)\,\mathrm{dx\ dk\ dy\ ds}
\end{aligned}
$$

which is now a deterministic term, and then it follows again by standard arguments that

$$
\begin{aligned}
\mathbb{I}_2 &\underset{n}{\rightarrow} -\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}E\Big[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\,\mathrm{dt}\Big] \\
&\qquad\qquad \times\varphi\rho_m(x-y)\,\mathrm{dx\ dk\ dy\ ds} \\
&\underset{l}{\rightarrow} -\int_Q\int_{\mathbb{R}^d}E\Big[\eta''(\hat{u}(t,x)-u_\epsilon(t,y))h(\hat{u}(t,x))h(u_\epsilon(t,y))\,\mathrm{dt}\Big]\varphi\rho_m(x-y)\mathrm{dxdyds}
\end{aligned}
$$

- limit as $n \to \infty$

$$
\begin{aligned}
\mathcal{A}_1 \quad := \quad & -\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\int_{s-2/n}^{s} \eta''(\hat{u}-k)h(\hat{u})\rho_n(t-s)\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\mathrm{dt}\Big] \\
& \qquad \times \varphi\rho_m(x-y)\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy}\ \mathrm{ds} \\
& + \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\ \mathrm{dt}\Big] \\
& \qquad \times \varphi\rho_m(x-y)\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
= \quad & -\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\Big]\rho_m(x-y) \\
& \qquad \times [\int_0^T \varphi\rho_n(t-s)\ \mathrm{ds}]\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
& + \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times \varphi\rho_m(x-y)\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy}\ \mathrm{dt} \\
= \quad & \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times [\varphi(t,y) - \int_0^T \varphi(s,y)\rho_n(t-s)\ \mathrm{ds}]\rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
= \quad & \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times \int_0^T [\varphi(t,y)-\varphi(s,y)]\rho_n(t-s)\ \mathrm{ds}\rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
& + \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times \varphi(t,y)\Big[1 - \int_0^T \rho_n(t-s)\ \mathrm{ds}\Big]\rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy}. \\
|\mathcal{A}_1| \quad \leq \quad & \frac{c(\varphi)}{n} \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times \rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
& + \int_{T-2/n}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-k)h(\hat{u})\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times \varphi(t,y)\rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
\leq \quad & \frac{c(\varphi)}{n} \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-u_\epsilon(t,y)+k)h(\hat{u})\rho_l(k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times \rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
& + \int_{T-2/n}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}-u_\epsilon(t,y)+k)h(\hat{u})\rho_l(k)h(u_\epsilon(t,y))\Big] \\
& \qquad \times \varphi(t,y)\rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dk}\ \mathrm{dy} \\
\leq \quad & \frac{c(\varphi,\eta'')}{n} \int_Q \int_{\mathbb{R}^d} E\Big[h(\hat{u})h(u_\epsilon(t,y))\Big]\rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dy} \\
& + \int_{T-2/n}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E\Big[h(\hat{u})h(u_\epsilon(t,y))\Big]\varphi(t,y)\rho_m(x-y)\ \mathrm{dt}\ \mathrm{dx}\ \mathrm{dy} \\
\to_{n\to\infty} \quad & 0.
\end{aligned}
$$

- limit as $l \to \infty$:

$$
\begin{aligned}
\mathcal{A}_2 &= -\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}(t,x)-k)h(\hat{u}(t,x))\rho_l(u_\epsilon(t,y)-k)h(u_\epsilon(t,y))\,\mathrm{dt}\Big] \\
&\qquad\qquad \times\varphi\rho_m(x-y)\,\mathrm{dx\,dk\,dy\,ds} \\
&\quad + \int_Q \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}(t,x)-u_\epsilon(t,y))h(\hat{u}(t,x))h(u_\epsilon(t,y))\,\mathrm{dt}\Big] \\
&\qquad\qquad \times\varphi\rho_m(x-y)\,\mathrm{dx\,dy\,ds} \\
&= -\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}(t,x)-u_\epsilon(t,y)+k)-\eta''(\hat{u}(t,x)-u_\epsilon(t,y))\Big] \\
&\qquad\qquad \times h(\hat{u}(t,x))\rho_l(k)h(u_\epsilon(t,y))\varphi\rho_m(x-y)\,\mathrm{dx\,dk\,dy\,ds},
\end{aligned}
$$

which can be estimated, using the Lipschitz continuity of η'',

$$
\begin{aligned}
|\mathcal{A}_2| &\leq \frac{c(\eta'')}{l}\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} Eh(\hat{u}(t,x))\rho_l(k)h(u_\epsilon(t,y))\varphi\rho_m(x-y)\,\mathrm{dx\,dk\,dy\,ds} \\
&\leq \frac{c(\eta'')}{l}\int_Q \int_{\mathbb{R}^d} Eh(\hat{u}(t,x))h(u_\epsilon(t,y))\varphi\rho_m(x-y)\,\mathrm{dx\,dy\,ds} \to_{l\to\infty} 0.
\end{aligned}
$$

Combining the preceding estimates yields that

$$
\begin{aligned}
&\lim_l \lim_n [I_6 + J_8 + I_7 + J_9] \\
= \;& -\int_Q \int_{\mathbb{R}^d} E\Big[\eta''(\hat{u}(t,x)-u_\epsilon(t,y))h(\hat{u}(t,x))h(u_\epsilon(t,y))\,\mathrm{dt}\Big]\varphi\rho_m(x-y)\,\mathrm{dy\,dx} \\
& + \frac{1}{2}E\int_Q \int_{\mathbb{R}^d} h^2(\hat{u})\eta''(\hat{u}-u_\epsilon(s,y))\rho_m(x-y)\varphi\,\mathrm{dx\,dy\,ds} \\
& + \frac{1}{2}E\int_Q \int_{\mathbb{R}^d} h^2(u_\epsilon)\eta''(u_\epsilon-\hat{u}(s,x))\rho_m(x-y)\varphi\,\mathrm{dy\,dx\,ds} \\
= \;& \frac{1}{2}E\int_Q \int_{\mathbb{R}^d} [h(\hat{u})-h(u_\epsilon)]^2\eta''(u_\epsilon-\hat{u}(s,x))\rho_m(x-y)\varphi\,\mathrm{dy\,dx\,ds} \\
\underset{\eta}{\to} \;& 0,
\end{aligned}
$$

for $\eta = \eta_\delta \in \mathcal{E}$, the approximation of the absolute value function as defined above, since supp $\eta'' \subset [-\delta, \delta]$, and $|\eta''| \leq \frac{2\pi}{\delta}$.

Indeed, choosing this sequence of entropies yields the estimate

$$\begin{array}{cl} & |E\displaystyle\int_Q\int_{\mathbb{R}^d}[h(\hat{u})-h(u_\epsilon)]^2\eta''(u_\epsilon-\hat{u}(s,x))\rho_m(x-y)\varphi\ \mathrm{dy\ dx\ ds}| \\ \leq & c(h)E\displaystyle\int_Q\int_{\mathbb{R}^d}[\hat{u}-u_\epsilon]^2\eta''(u_\epsilon-\hat{u}(s,x))\rho_m(x-y)\varphi\ \mathrm{dy\ dx\ ds} \\ \leq & c(h)\delta^2E\displaystyle\int_Q\int_{\mathbb{R}^d}\eta''(u_\epsilon-\hat{u}(s,x))\rho_m(x-y)\varphi\ \mathrm{dy\ dx\ ds} \\ \leq & c(h)\delta E\displaystyle\int_Q\int_{\mathbb{R}^d}\rho_m(x-y)\varphi\ \mathrm{dy\ dx\ ds} \\ \leq & c(h)\delta\displaystyle\int_Q\varphi\ \mathrm{dy\ ds} \\ \to_{\delta\to 0} & 0. \end{array}$$

Passing to the limits in $I_1+..+I_7+J_1+..+J_9$ successively with n, l, η, ϵ and m, we thus obtain

$$\begin{aligned} 0 \leq\ & E\int_{\mathbb{R}^d}|\hat{u}_0-u_0|\,\varphi(0)\ \mathrm{dx}+E\int_Q\int_0^1\int_0^1|\mathbf{u}(s,y,\beta)-\hat{\mathbf{u}}(s,y,\alpha)|\,\partial_s\varphi\ \mathrm{dy\ ds}\ \mathrm{d}\alpha\ \mathrm{d}\beta \\ & -E\int_Q\int_0^1\int_0^1F(\mathbf{u}(t,x,\beta),\hat{\mathbf{u}}(t,x,\alpha))\nabla\varphi\ \mathrm{dp}\ \mathrm{d}\beta. \end{aligned} \tag{4.4}$$

4.2 Uniqueness and stability

Proposition 4.4 *The measure-valued solution is unique. Moreover, it is the unique entropy solution.*

Proof. Note first that, thanks to a density argument, the Kato inequality still holds for any nonnegative test-function $\varphi\in H^1(Q)$ with a compact support.

Set $\kappa=\|f'\|_\infty$, $K>0$ and denote by ψ any nonincreasing regular function with $1_{]-\infty,K]}\leq\psi\leq 1_{]-\infty,K+1]}$. Considering $\varphi(t,x)=\psi(|x|+\kappa t)\gamma(t)$ in (4.4) leads to

$$\begin{aligned} 0 \leq\ & \gamma(0)E\int_{\mathbb{R}^d}|\hat{u}_0-u_0|\,\psi(|x|)\mathrm{dx}+E\int_Q\gamma'(t)\int_0^1\int_0^1|\mathbf{u}(t,x,\beta)-\hat{\mathbf{u}}(t,x,\alpha)|\,\psi(|x|+\kappa t)\mathrm{dpd}\beta \\ & +E\int_Q\int_0^1\int_0^1\kappa\psi'(|x|+\kappa t)\,|\mathbf{u}(t,x,\beta)-\hat{\mathbf{u}}(t,x,\alpha)|\,\gamma(t)\ \mathrm{dp}\ \mathrm{d}\beta \\ & -E\int_Q\int_0^1\int_0^1\psi'(|x|+\kappa t)F(\mathbf{u}(t,x,\beta),\hat{\mathbf{u}}(t,x,\alpha))\frac{x}{|x|}\gamma(t)\ \mathrm{dp}\ \mathrm{d}\beta. \end{aligned}$$

As ψ is a nonincreasing function, the choice of κ yields

$$-E\int_Q \gamma'(t)\int_0^1\int_0^1 |\mathbf{u}(t,x,\beta)-\hat{\mathbf{u}}(t,x,\alpha)|\,\psi(|x|+\kappa t)\mathrm{d}p\mathrm{d}\beta \;\le\; \gamma(0)E\int_{\mathbb{R}^d}|\hat{u}_0-u_0|\,\psi(|x|)\mathrm{d}x.$$

If $\hat{u}_0 = u_0$, $\gamma(t) = \frac{(T-t)^+}{T}$, for any positive R, fixing $K = R+\kappa T$ leads to

$$E\int_{B(0,R)}\int_0^1\int_0^1 |\mathbf{u}(t,x,\beta)-\hat{\mathbf{u}}(t,x,\alpha)|\ \mathrm{d}p\ \mathrm{d}\beta = 0.$$

This implies that, for any $R>0$, $\mathbf{u}(t,x,\beta) = \hat{\mathbf{u}}(t,x,\alpha)$ for almost any $x \in B(0,R)$, $t \in]0,T[$, $\omega \in \Omega$, $\alpha,\beta \in]0,1[$. Thus, on the one hand $\mathbf{u} = \hat{\mathbf{u}}$; on the other hand $\mathbf{u}(t,x,\alpha) = u(t,x)$ is independent of α, hence an entropy solution in the sense of Definition 2.2.

□

Remark 4.5 *Thanks to this proposition, let us mention that the viscous solution u_ϵ converges to u in $L^1(Q\times\Omega)$ thanks to the Young measure theory. Moreover, as u_ϵ is bounded in $L^2(Q\times\Omega)$, u_ϵ converges to u strongly in $L^p(]0,T[\times\Omega, L^p(D))$ for every $1 \leqslant p < 2$, using Vitali's theorem.*

Proposition 4.6 *Entropy solutions satisfy a "contraction principle": if u_1, u_2 are entropy solutions of (0.1) corresponding to initial data $u_{1,0}, u_{2,0} \in L^2(\mathbb{R}^d)$, respectively, then, for any positive K and time t*

$$E\int_{B(0,K-\kappa t)} |u_1-u_2|\ \mathrm{d}x \le \int_{B(0,K)} |u_{1,0}-u_{2,0}|\ \mathrm{d}x.$$

Proof. This is a consequence of the previous proof when passing to the limit when ψ converges to $1_{]-\infty,K]}$. □

Chapter IV

The Dirichlet problem for a conservation law with a multiplicative stochastic perturbation

As an extension of the study of Chapter III, we present here the results of a submitted joint work with G. Vallet and P. Wittbold [13]. We are interested in the formal stochastic nonlinear conservation law of type:

$$\mathrm{d}u - \operatorname{div}\vec{\mathbf{f}}(u)\,\mathrm{d}t = h(u)dw \text{ in }]0,T[\times D\times\Omega, \qquad "u=0" \text{ on } \Omega\times\Sigma, \tag{0.1}$$

with an initial condition $u(t=0)=u_0$ in D; here $D\subset\mathbb{R}^d$ is a bounded domain, with a Lipschitz boundary if $d\geq 2$; T is a positive number, $Q=]0,T[\times D$, $\Sigma=]0,T[\times\partial D$ and $w=\{w_t,\mathcal{F}_t; 0\leq t\leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion with $w_0=0$, defined on the classical Wiener space $(\Omega,\mathcal{F},P)$ (cf. Billingsley [17] p.80 *et sqq.* for example). These assumptions on w are made for convenience.
Let us assume that

H$_1$: $\vec{\mathbf{f}}=(f_1,..,f_d):\mathbb{R}\to\mathbb{R}^d$ is a Lipschitz-continuous function and $\vec{\mathbf{f}}(0)=0_{\mathbb{R}^d}$.

H$_2$: $h:\mathbb{R}\to\mathbb{R}$ is a Lipschitz-continuous function with $h(0)=0$.

H$_3$: $u_0\in L^2(D)$.

Our aim is to give a definition of a stochastic entropy solution to the above-mentioned problem and to prove a result of existence and uniqueness of such a solution.

1 Introduction

1.1 Former results

Many papers on parabolic stochastic problems can be found in the literature, but much less papers have been devoted to the study of stochastic perturbation of nonlinear first-order hyperbolic problems. Before 2008, most of them concern the Cauchy problem in the 1D case in the case of additive noise, except HOLDEN-RISEBRO [49] where the authors have used an operator splitting method to prove the existence of a weak solution to the Cauchy problem. They considered a multiplicative noise with a compact support to get L^∞ estimates and the convergence is obtained by using pathwise arguments.

E-KHANIN-MAZEL-SINAI [38] were interested in the invariant measures for the Burgers equation with additive noise. Existence and uniqueness of a stochastic entropy solution is proved thanks to a Hopf-Lax type formula for the corresponding stochastic Hamilton-Jacobi equation.

KIM [53] proposed a method of compensated compactness to prove, *via* vanishing viscosity approximation, the existence of a stochastic weak entropy solution to the Cauchy problem. A Kruzhkov-type method was used to prove the uniqueness.

VALLET-WITTBOLD [78] proposed to extend the result of Kim to the multi-dimensional Dirichlet problem with additive noise. The formulation of the Dirichlet boundary condition was inspired by the approach of J. Carrillo which consists in formulating the boundary condition implicitly *via* global integral entropy inequalities involving semi-Kruzhkov entropies. Using the vanishing viscosity method and Young measure techniques the authors proved the existence, and, *via* Kruzhkov doubling variables technique, the uniqueness of the stochastic entropy solution.

Concerning multiplicative noise, for the Cauchy problem, FENG-NUALART [41] introduced a notion of strong entropy solution in order to prove the uniqueness of the entropy solution. Using the vanishing viscosity and compensated compactness arguments, the authors established the existence of strong entropy solutions only in the 1D case.

In the recent paper, CHEN-DING-KARLSEN [27] proposed to revisit the work of FENG and NUALART. They proved that the multidimensional stochastic problem is well-posed by using a uniform spatial BV-bound. They proved the existence of strong stochastic entropy solutions in $L^p \cap BV$ and developed a "continuous dependence" theory for stochastic entropy solutions in BV, which can be used to derive an error estimate for the vanishing viscosity method.

Using a kinetic formulation, DEBUSSCHE-VOVELLE [32] proved a result of existence and uniqueness of the entropy solution to the problem posed in a d-dimensional torus and, by the way of the theory of Young measure-valued solutions, Bauzet-Vallet-Wittbold [14] proved a result of existence and uniqueness of the solution to the multidimensional Cauchy problem. Since the method consists in comparing a weak measure-valued entropy solution to a regular one (the viscous solution in our case) and not to a strong one, the authors could consider very general assumptions on the data.

1.2 Goal of the study

Unlike already published results, we consider here a boundary-value problem. Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. One needs to introduce the notion of entropy solution in order to discriminate the "physical admissible solution" (see Kruzhkov [56]).
We also recall that weak and entropy solutions are not smooth solutions, thus, traces have to be understood in a weak way. Moreover, it is not possible to impose the Dirichlet condition on the whole boundary of D, but only on a free set: the one corresponding to entering characteristics (without exhaustiveness, see for example Bardos-Leroux-Nédélec [9], Carrillo-Wittbold [24], Málek-Necas-Otto-Rokyta-Ruzicka[60], Panov [62], Vallet [74], Vasseur [79] and Vallet-Wittbold [78] in the stochastic case with additive noise). As an extension, we propose in this chapter to prove a result of existence and uniqueness of the stochastic entropy solution to the Dirichlet problem for the stochastic conservation law with multiplicative noise (0.1) in the d-dimensional case. A method of artificial viscosity is proposed to prove the existence of a solution. The compactness properties used are based on the theory of Young measures and on measure-valued solutions. An appropriate adaptation of Kruzhkov's doubling variables technique is proposed to prove the uniqueness of the measure-valued entropy solution. By using non-convex entropies for the regularized solution and then standard arguments, we are able to deduce existence (and uniqueness) of the stochastic weak entropy solutions.

1.3 Plan of the study

The chapter is organized as follows. In Section 2 we introduce the notion of stochastic entropy (resp. measure-valued entropy) solution for (0.1), in particular the way one has to consider the boundary conditions and establish some basic properties of such solutions. We mention the main result of this chapter: the existence and uniqueness of the entropy solution of Problem (0.1). We propose to recall briefly the way to obtain a result of existence of a measure-valued entropy solution. It relies on a vanishing viscosity approximation and the techniques can be adapted from those proposed in Chapter III. We also state properties satisfies by viscous and measure-valued solutions. Section 3 is devoted to the proof of the uniqueness and of a contraction principle for measure-valued solutions. As a by-product, we deduce the existence and the uniqueness of the entropy solution of the Dirichlet problem for (0.1).

1.4 Notations and functional setting

In the sequel we denote by $H^1(D)$ the usual Sobolev space and by $H_0^1(D)$ the closure of $\mathcal{D}(D)$, the space of $C^\infty(D)$-functions with compact support in D. We recall that since D is regular, $H_0^1(D)$ is also the kernel of the trace operator. We denote by $H^{-1}(D)$ the dual space of $H_0^1(D)$ which is also the space of derivatives of order less than one of elements of $L^2(D)$ in the common Gelfand-Lions identification $H_0^1(D) \hookrightarrow L^2(D) \equiv L^2(D)' \hookrightarrow H^{-1}(D)$.

In general, if $G \subset \mathbb{R}^k$, $\mathcal{D}(G)$ denotes the restriction to G of $\mathcal{D}(\mathbb{R}^k)$ functions u such that support$(u) \cap G$ is compact. Then, $\mathcal{D}^+(G)$ will denote the subset of nonnegative elements of $\mathcal{D}(G)$.

For a given separable Banach space X, we denote by $N_w^2(0,T,X)$ the space of the predictable X-valued processes (cf. [30] p.94 or [64] p.28 for example). This space is the space $L^2(]0,T[\times\Omega, X)$ for the product measure $\,\mathrm{dt}\otimes dP$ on $\mathcal{P}_T$, the predictable σ-field (*i.e.* the σ-field generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $]s,t]\times A$ for any $A\in\mathcal{F}_s$).

If $X = L^2(D)$, one gets that $N_w^2(0,T,L^2(D)) \subset L^2(Q\times\Omega)$.

We denote $\mathcal{E}^+$ the set of nonnegative convex functions η in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x \mapsto x^+$ such that $\eta(x) = 0$ if $x \le 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$. Then, η'' has a compact support and η and η' are Lipschitz-continuous functions.

A typical element of $\mathcal{E}^+$ is the function denoted by η_δ such that (see Figure IV.1)

$$\eta_\delta(0) = 0, \text{ and } \eta_\delta'(r) = \begin{cases} 1 & \text{if} \quad r > \delta \\ \frac{1+\sin(\frac{\pi}{2\delta}(2r-\delta))}{2} & \text{if} \quad 0 \le r \le \delta \\ 0 & \text{if} \quad r < 0. \end{cases}$$

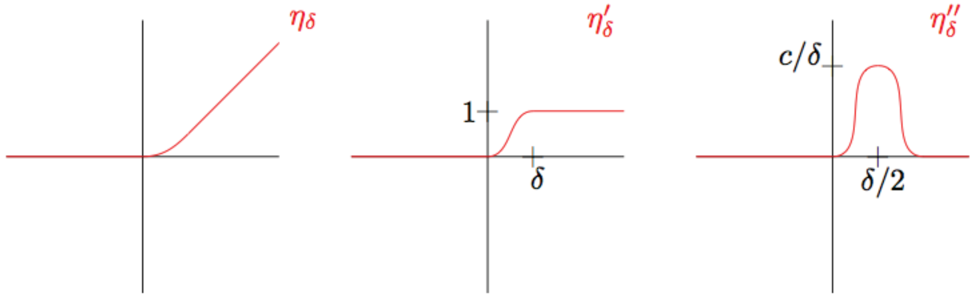


Figure IV.1 – semi-Kruzhkov entropies

Then, one denotes by $\mathcal{E}^-$ the set $\{\check{\eta} := \eta(-\cdot),\ \eta \in \mathcal{E}^+\}$; and, for the definition of the entropy inequality, one denotes by

$$\begin{aligned}\mathbb{A}^+ &= \{(k,\varphi,\eta) \in \mathbb{R} \times \mathcal{D}^+(\mathbb{R}^{d+1}) \times \mathcal{E}^+,\ k < 0 \Rightarrow \varphi \in \mathcal{D}^+([0,T] \times D)\},\\ \mathbb{A}^- &= \{(k,\varphi,\eta),\ (-k,\varphi,\check{\eta}) \in \mathbb{A}^+\} \quad \text{and} \quad \mathbb{A} = \mathbb{A}^+ \cup \mathbb{A}^-.\end{aligned}$$

And, for convenience, denote by:

$$\begin{aligned}&\mathrm{sgn}_0^+(x) = 1 \text{ if } x > 0 \text{ and } 0 \text{ else }; \ \mathrm{sgn}_0^-(x) = -\mathrm{sgn}_0^+(-x) \text{ and } \mathrm{sgn}_0 = \mathrm{sgn}_0^+ + \mathrm{sgn}_0^-\\ &F(a,b) = \mathrm{sgn}_0(a-b)[\vec{\mathbf{f}}(a) - \vec{\mathbf{f}}(b)] \ ; \ F^{+(-)}(a,b) = \mathrm{sgn}_0^{+(-)}(a-b)[\vec{\mathbf{f}}(a) - \vec{\mathbf{f}}(b)]\\ &\text{and for any } \eta \in \mathcal{E}^+ \cup \mathcal{E}^-,\ F^\eta(a,b) = \int_b^a \eta'(\sigma - b)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma.\end{aligned}$$

Note, in particular, that F and F^η are Lipschitz-continuous functions.

2 Entropy formulation and existence of a measure-valued solution

The aim of this section is to present the definition of an entropy solution, the properties implicitly satisfied by such a solution and the main result of the chapter. We will not give the details of the proofs of these properties since they are very close to the one presented in the Chapter III and Appendix I Section 2 and can be adapted straightforward (see also Vallet-Wittbold [78]) to a bounded domain. So, we invite the reader interested in the proofs to have a look at this references.

For any function u of $\mathcal{N}^2_w(0,T,L^2(D))$, any real k and any regular function η, denote P-a.s. in Ω by $\mu_{\eta,k}$, the distribution in $\mathbb{R}^{d+1}$, defined by

$$\begin{aligned}\varphi \mapsto \mu_{\eta,k}(\varphi) \quad &= \quad \int_D \eta(u_0-k)\varphi(0)\,\mathrm{dx} + \int_Q \eta(u-k)\partial_t\varphi - F^\eta(u,k)\nabla\varphi\,\mathrm{dx}\,\mathrm{dt} \\ &\quad + \int_0^T\int_D \eta'(u-k)h(u)\varphi\,\mathrm{dx}dw(t) + \frac{1}{2}\int_Q h^2(u)\eta''(u-k)\varphi\,\mathrm{dx}\,\mathrm{dt}.\end{aligned}$$

Now let us define the notion of entropy solution we consider.

Definition 2.1 ***Entropy solution***
A function u of $\mathcal{N}^2_w(0,T,L^2(D))$ is an entropy solution of the stochastic conservation law (0.1) with the initial condition $u_0 \in L^2(D)$ if $u \in L^\infty(0,T,L^2(\Omega,L^2(D)))$ and

$$\forall(k,\varphi,\eta)\in\mathbb{A},\quad 0 \le \mu_{\eta,k}(\varphi) \quad P-a.s. \tag{2.1}$$

For technical reasons, we also need to consider a generalized notion of entropy solution. In fact, in a first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we will be able to deduce the existence of an entropy solution in the sense of Definition 2.1.

Definition 2.2 ***Measure-valued entropy solution***
A function $u \in N^2_w[0,T,L^2(D\times]0,1[)] \cap L^\infty\,[]0,T[,L^2(\Omega\times D\times]0,1[)]$ is a (Young) measure-valued entropy solution of (0.1) with the initial data $u_0 \in L^2(D)$ if

$$\forall(k,\varphi,\eta)\in\mathbb{A},\quad 0 \le \int_0^1 \mu_{\eta,k}(\varphi)\,\mathrm{d}\alpha \quad P-a.s. \tag{2.2}$$

Remark 2.3 (Cf. Chapter III and Vallet-Wittbold **[78])**

- *Any limit point for the weak-* convergence in $L^\infty\left(]0,T[,L^2(\Omega\times D)\right)$ of a sequence of solutions to the approximate parabolic viscous stochastic problems will generate a Young measure limit in $L^\infty\left(]0,T[,L^2(\Omega\times D\times]0,1[)\right)$.*
- *If u is an entropy solution, then, P-a.s., $0 \le \mu_{\eta,k}(\varphi)$ holds for all k and φ such that $(k,\varphi,\eta)\in\mathbb{A}$. The same remark holds for a measure-valued entropy solution.*
- *Any entropy solution is, P-a.s., a solution to $\partial_t\left[u - \int_0^t h(u)dw(s)\right] - \mathit{div}\,\vec{\mathbf{f}}(u) = 0$ in the sense of distributions in Q.*

. *Since a solution u will be in $L^\infty(0,T,L^2(\Omega\times D))$, one gets that*

$$ess\lim_{t\to 0^+} E\int_D |u-u_0|\,\mathrm{dx} = 0.$$

Similarly, for any measure-valued solution $\mathbf{u}$, *one gets that*

$$ess\lim_{t\to 0^+} E\int_{D\times]0,1[} |\mathbf{u}-u_0|\,\mathrm{dx}\,\mathrm{d}\alpha = 0.$$

The main result of this chapter is:

Theorem 2.4
Under assumptions $H_1-H_2-H_3$ there exists a unique measure-valued entropy solution in the sense of Definition 2.3 and this solution is obtained by viscous approximation.
It is the unique entropy solution in the sense of Definition 2.1.
If u_1, u_2 are entropy solutions of (0.1) corresponding to initial data $u_{1,0},u_{2,0}\in L^2(D)$, respectively, then, for any t,

$$E\int_D (u_1-u_2)^+\,\mathrm{dx} \le \int_D (u_{1,0}-u_{2,0})^+\,\mathrm{dx}.$$

The technique to prove the result of existence is based on the notion of narrow convergence of Young measures (or entropy processes). Then, thanks to the uniqueness result of the next section, we will be able to prove that the measure-valued solution is an entropy solution in the sense of Definition 2.1, and that the sequence of approximation proposed to prove the existence of the solution converges in $L^p(]0,T[\times\Omega,L^p(D))$ for any $1\le p<2$.
Let us first state the result of existence of a measure-valued entropy solution. This result is based on a classical evanescent viscosity method and the proof of results stated in the sequel of this section can be adapted from Chapter III, see also Bauzet-Vallet-Wittbold [14].

Theorem 2.5 *Under assumptions $H_1-H_2-H_3$ there exists a measure-valued entropy solution in the sense of Definition 2.2.*

Indeed, for any positive ϵ, there exists a unique weak solution u_ϵ of the stochastic viscous parabolic equation:

$$\partial_t[u_\epsilon-\int_0^t h(u_\epsilon)dw(s)]-\epsilon\Delta u_\epsilon-\operatorname{div}\vec{\mathbf{f}}(u_\epsilon)=0, \tag{2.3}$$

associated with a regular initial condition u_0^ϵ.

Lemma 2.6 *More precisely, one has that*

- $u_\epsilon \in \mathcal{N}^2_w(0,T;H^1_0(\mathbb{R}^d)) \cap C([0,T],L^2(\Omega\times D))$,
- $\Delta u_\epsilon,\ \partial_t[u_\epsilon - \mathcal{K}] \in L^2\big(]0,T[\times\Omega;L^2(D)\big)$ *where* $\mathcal{K} = \int_0^t h(u_\epsilon)dw(s)$,
- *if* $(u^0_\epsilon)_\epsilon$ *is bounded in* $L^2(D)$, *then there exists a positive constant* C *such that*

$$||u_\epsilon||^2_{L^\infty[0,T;L^2(\Omega\times D)]} + \epsilon||u_\epsilon||^2_{L^2[]0,T[\times\Omega;H^1_0(D)]} \le C.$$

- *if, for some* $p\ge 1$, $u^\epsilon_0 \in L^{2p}(D)$, *then* $u_\epsilon \ \in L^\infty(0,T,L^{2p}(\Omega\times D))$.
- $\forall(\varphi,k,\eta)\in\mathbb{A},\ 0\le \mu_{\eta,k}(\varphi) - \epsilon\int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon\nabla\varphi \,\mathrm{dx}\,\mathrm{dt}, \quad P-a.s.$

Since u_ϵ is a bounded sequence in $\mathcal{N}^2_w(0,T,L^2(D))$, the associated Young measure sequence $\mathbf{u}_\epsilon$ converges (up to a subsequence still indexed in the same way) to a Young measure denoted by $\mathbf{u}$. Thanks to the *a priori* estimates and the compatibility of the Itô integration with respect to the weak convergence in $\mathcal{N}^2_w(0,T,L^2(D))$, one gets that this Young measure is a measure-valued entropy solution.

3 Uniqueness

The aim of this section is to prove

Theorem 3.1 *The solution given by Theorem 2.5 is the unique measure-valued entropy solution in the sense of Definition 2.2.*
Moreover, it is the unique entropy solution in the sense of Definition 2.1.

Remark 3.2 *We will also prove (Proposition 3.8) a comparison principle for entropy solutions.*

In order to prove the result stated in Theorem 3.1, one has to show the following global Kato inequality. The proof of this result is the principal part of the chapter:

Lemma 3.3 *Let* $\mathbf{u}$, $\hat{\mathbf{u}}$ *be Young measure-valued entropy solutions to (0.1) with initial data* $u_0,\hat u_0 \in L^2(D)$, *respectively, and assume that at least one of them is obtained by viscous approximation. Then, for any* $\mathcal{D}^+([0,T]\times\mathbb{R}^d)$*-function* φ, *one gets that*

$$\begin{aligned} 0 \ \le\ & \int_D (\hat u_0 - u_0)^+\varphi(0)\,\mathrm{dx} + E\int_{Q\times]0,1[^2}\Big(\hat{\mathbf{u}}(t,x,\beta) - \mathbf{u}(t,x,\alpha)\Big)^+\partial_t\varphi\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ & -E\int_{Q\times]0,1[^2} F^+\Big(\hat{\mathbf{u}}(t,x,\beta),\mathbf{u}(t,x,\alpha)\Big).\nabla\varphi\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \end{aligned}$$

Remark 3.4

- *It will follow in Section 3.3 the uniqueness of the entropy measure-valued solution by considering a function $\varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$ independent of x on D.*

- *It will also follow the fact that any measure-valued entropy solution is obtained by viscous approximation, that the global Kato inequality is satisfied for any couple of measure-valued entropy solutions and the existence and the uniqueness of the entropy solution.*

3.1 Plan of the proof

Let us explicit in this part, for the sake of convenience, sketches of the proof of Lemma 3.3, technical details will be given in Section 3.2.
We would like to show the Kato inequality for a function $\varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$. The idea is to use a partition of unity subordinate to a covering of $\overline{D}$ by balls $\mathcal{B}_i$, $i=1,\cdots,k$ satisfying $\mathcal{B}_0\cap\partial D=\emptyset$ and, for $i=1,\cdots,k$, $\mathcal{B}_i\subset\mathcal{B}'_i$ with $\mathcal{B}'_i\cap\partial D$ part of a Lipschitz graph. Let us denote by θ_i the associated functions to such a covering. If we are able to show that the Kato inequality holds for a function $\varphi\in\mathcal{D}^+([0,T]\times B)$ where $B:=\mathcal{B}_i$ for some i, thus the result will hold for any $\varphi\in\mathcal{D}^+([0,T]\times\mathbb{R}^d)$ using the covering. In order to do this, we follow the idea of Carrillo [22] in the deterministic setting. We have to obtain two inequalities: a local Kato inequality obtained for any function $\varphi\in\mathcal{D}^+([0,T]\times D)$:

$$\begin{aligned}0 \;\le\;& E\int_{Q\times]0,1[^2}\Big(\hat{\mathbf{u}}(t,x,\beta)-\mathbf{u}(t,x,\alpha)\Big)^+\partial_t\varphi\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &-E\int_{Q\times]0,1[^2}F^+\Big(\hat{\mathbf{u}}(t,x,\beta),\mathbf{u}(t,x,\alpha)\Big).\nabla\varphi\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &+\int_D(\hat{u}_0-u_0)^+\varphi(0)\,\mathrm{dx},\end{aligned}\tag{3.1}$$

and a global one satisfied by any function $\varphi\in\mathcal{D}^+([0,T]\times B)$:

$$\begin{aligned}0 \;\le\;& \int_D(\hat{u}_0-u_0)^+\varphi(0)\,\mathrm{dx}\\ &+E\int_Q\int_0^1\int_0^1(\hat{\mathbf{u}}(t,x,\alpha)-\mathbf{u}(t,x,\beta))^+\partial_t\varphi\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}\\ &-E\int_Q\int_0^1\int_0^1\mathrm{sgn}_0^+(\hat{\mathbf{u}}(t,x,\alpha)-\mathbf{u}(t,x,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(t,x,\alpha))-\vec{\mathbf{f}}(\mathbf{u}(t,x,\beta))]\nabla\varphi\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}\\ &+\lim_m\mathcal{L}(\varphi\theta_m)+\lim_m\widetilde{\mathcal{L}}(\varphi\theta_m),\end{aligned}\tag{3.2}$$

where we assume that $\mathbf{u}$ is any Young measure-valued entropy solution obtained by viscous approximation (as a limit point of u_ϵ) and that $\hat{\mathbf{u}}$ is any other admissible Young-measure-valued entropy solution with associated initial data u_0 and $\hat{u}_0$.
We will see in the proof of the global inequality that we will generate two linear operators $\mathcal{L}(\varphi\theta_m)$ and $\widetilde{\mathcal{L}}(\varphi\theta_m)$. The key point is that if we consider $\varphi \in \mathcal{D}^+([0,T[\times B)$, then $\varphi = \theta_n\varphi + (1-\theta_n)\varphi$ and $\theta_n\varphi \in \mathcal{D}^+([0,T[\times D)$ for n sufficiently large. Applying the local Kato inequality (3.1) with $\theta_n\varphi$ and the global one (3.2) with $(1-\theta_n)\varphi$, yields:

$$
\begin{aligned}
0 \quad \le \quad & \int_D (\hat{u}_0 - u_0)^+ \varphi(0)\,\mathrm{dx} \\
& + E\int_Q \int_0^1 \int_0^1 (\hat{\mathbf{u}}(t,x,\alpha) - \mathbf{u}(t,x,\beta))^+ \partial_t\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\
& - E\int_Q \int_0^1 \int_0^1 \mathrm{sgn}_0^+(\hat{\mathbf{u}}(t,x,\alpha) - \mathbf{u}(t,x,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(t,x,\alpha)) - \vec{\mathbf{f}}(\mathbf{u}(t,x,\beta))]\nabla\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\
& + \lim_m \mathcal{L}(\varphi(1-\theta_n)\theta_m) + \lim_m \widetilde{\mathcal{L}}(\varphi(1-\theta_n)\theta_m).
\end{aligned}
$$

As $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are linear operators and $\theta_n\theta_m = \theta_n$ if m is large (this is detailed in the next section), one gets that

$$
\begin{aligned}
& \lim_m \mathcal{L}(\varphi(1-\theta_n)\theta_m) + \lim_m \widetilde{\mathcal{L}}(\varphi(1-\theta_n)\theta_m) \\
= \; & \lim_m \mathcal{L}(\varphi\theta_m) - \mathcal{L}(\varphi\theta_n) + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m) - \widetilde{\mathcal{L}}(\varphi\theta_n)
\end{aligned}
$$

and $\lim_n \lim_m \mathcal{L}(\varphi(1-\theta_n)\theta_m) + \lim_n \lim_m \widetilde{\mathcal{L}}(\varphi(1-\theta_n)\theta_m) = 0$.
Thus, the global Kato inequality holds for any $\varphi \in \mathcal{D}^+([0,T[\times B)$, and by using a partition of unity, it holds for any $\varphi \in \mathcal{D}^+([0,T[\times\mathbb{R}^d)$.
Now let us explain how to obtain (3.1) and (3.2).

1. **Local Kato inequality (3.1)**: The proof is a straightforward adaptation of the local Kato inequality presented in Chapter III (see also Bauzet-Vallet-Wittbold [14]) for classical Kruzhkov's entropy and of the techniques used to prove the global Kato inequality of the next section.

2. **Global Kato inequality (3.2)**: In order to obtain the global Kato inequality, as in the deterministic case, we would like to show two "half" inequalities, one involving $(\hat{\mathbf{u}} - \mathbf{u}^+)^+$ and another one with $(-\mathbf{u} - (-\hat{\mathbf{u}})^+)^+$, as in Carrillo [22] . In the deterministic setting, one is able to use the symmetry of the role of $\hat{\mathbf{u}}$ and $\mathbf{u}$ in the first "half" inequality

and thus to deduce immediately the one for $(-\mathbf{u}-(-\hat{\mathbf{u}})^+)^+$, essentially because we compare two measure-valued solutions. But unfortunately, in our case the roles of $\mathbf{u}$ and $\hat{\mathbf{u}}$ are not symmetric and we can not simply replace $\mathbf{u}$ by $-\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}$ by $-\mathbf{u}$. This comes from the fact that we compare a measure-valued solution with one obtained by viscous approximation. This choice is motivated by the same constraints of regularity, measurability and control of stochastic integral imposed by the stochastic perturbation as presented in Chapter III Section 4.1.1. The arguments used in the proof are adapted to the situation where the solution $\mathbf{u}$ we compare with is obtained by viscous approximation and they do not work if it is not $\mathbf{u}$, but $\hat{\mathbf{u}}$ which is obtained by viscous approximation. The idea to obtain those inequalities is to follow techniques introduced in Chapter III, but considering Kruzhkov semi-entropies, applying the Itô formula with different functions and multiplying inequalities with suitable functions. Details are given in Section 3.2.2, here we only state the plan of the proof with the main inequalities we will show after.

- **A first inequality:** For any function $\varphi \in \mathcal{D}^+([0,T[\times\mathbb{R}^d)$ with $\operatorname{supp}\varphi(t,.) \subset B := \mathcal{B}_i$ for some $i \in \{1,\cdots,k\}$, we prove that

$$\begin{aligned}
0 \ \leq\ & \int_D (\hat{u}_0 - u_0^+)^+ \varphi(0,x)\,\mathrm{dx} + \lim_m \mathcal{L}(\varphi\theta_m) \\
&+E\int_Q\int_0^1\int_0^1 (\hat{\mathbf{u}}(.,\alpha)-\mathbf{u}^+(.,\beta))^+\,\partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \qquad (3.3)\\
&-E\int_Q\int_0^1\int_0^1 \operatorname{sgn}_0^+(\hat{\mathbf{u}}(.,\alpha)-\mathbf{u}^+(.,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(.,\alpha))-\vec{\mathbf{f}}(\mathbf{u}^+(.,\beta))]\nabla\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}(\varphi\theta_m) \ :=\ & E\int_D (u_0(y))^-\varphi(0,y)\theta_m(y)\,\mathrm{dy} \\
&+E\int_Q\int_0^1 (\mathbf{u}(s,y,\beta))^-\partial_s\varphi(s,y)\theta_m(y)\,\mathrm{ds\,dy\,d}\beta \\
&-E\int_Q\int_0^1 F^{(.)^-}(\mathbf{u}(s,y,\beta),0)\nabla_y[\varphi(s,y)\theta_m(y)]\,\mathrm{ds\,dy\,d}\beta.
\end{aligned}$$

⋆ **A second inequality:** For any function $\varphi \in \mathcal{D}^+([0,T[\times\mathbb{R}^d)$ with $\text{supp}\varphi(t,.) \subset B := \mathcal{B}_i$ for some $i \in \{1,\cdots,k\}$, we prove that

$$\begin{aligned}
0 \leq &\int_D (u_0 - \hat{u}_0^+)^+ \varphi(0)\,\text{dx} + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m)\\
&+E\int_Q\int_0^1\int_0^1 (\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}^+(t,x,\alpha))^+ \partial_t\varphi \,\text{d}\alpha\,\text{d}\beta\,\text{d(t,x)}, \qquad (3.4)\\
&-E\int_Q\int_0^1\int_0^1 \text{sgn}_0^+(\mathbf{u}(t,x,\beta)-\hat{\mathbf{u}}^+(t,x,\alpha))[\vec{\mathbf{f}}(\mathbf{u}(t,x,\beta))-\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(t,x,\alpha))]\nabla\varphi\,\text{d}\alpha\,\text{d}\beta\,\text{d(t,x)},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\mathcal{L}}(\varphi\theta_m) &= \int_D \hat{u}_0^-(x)\varphi(0,x)\theta_m(x)\,\text{dx}\\
&+E\int_Q\int_0^1 \hat{\mathbf{u}}^-(t,x,\alpha)\partial_t\varphi(t,x)\theta_m(x)\,\text{dt dx d}\alpha\\
&-E\int_Q\int_0^1 F^{(.)^-}(\hat{\mathbf{u}}(t,x,\alpha),0)\nabla_x(\varphi\theta_m)(x)\,\text{dt dx d}\alpha.
\end{aligned}$$

Note that $-\hat{\mathbf{u}}$, resp. $-\mathbf{u}$, is a measure-valued entropy solution of $dv = \text{div}\widetilde{\vec{\mathbf{f}}}(v)\,\text{dt} + \widetilde{h}(v)dw$ with $\widetilde{\vec{\mathbf{f}}}(x) = -\vec{\mathbf{f}}(-x)$, $\widetilde{h}(x) = -h(-x)$ and the initial condition $-\hat{u}_0$, resp. $-u_0$, and by assuming that $-\mathbf{u}$ is obtained as the limit of $-u_\epsilon$, solution of the viscous problem $dv = [\epsilon\Delta v + \text{div}\widetilde{\vec{\mathbf{f}}}(v)]\,\text{dt} + \widetilde{h}(v)dw$. Consequently, replacing $\hat{\mathbf{u}}$ by $-\hat{\mathbf{u}}$ and $\mathbf{u}$ by $-\mathbf{u}$ in the inequality (3.4), we get the estimate (where one denotes by $x^- = (-x)^+$)

$$\begin{aligned}
0 \leq &\int_D (-u_0 - \hat{u}_0^-)^+ \varphi(0)\,\text{dx} + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m)\\
&+E\int_Q\int_0^1\int_0^1 (-\mathbf{u}(.,\beta) - \hat{\mathbf{u}}^-(.,\alpha))^+ \partial_t\varphi\,\text{d}\alpha\,\text{d}\beta\,\text{d(t,x)}\\
&+E\int_Q\int_0^1\int_0^1 \text{sgn}_0^+(-\mathbf{u}(.,\beta)-\hat{\mathbf{u}}^-(.,\alpha))[\vec{\mathbf{f}}(\mathbf{u}(.,\beta))-\vec{\mathbf{f}}(-\hat{\mathbf{u}}^-(.,\alpha))]\nabla\varphi d\alpha\,\text{d}\beta\,\text{d(t,x)}.
\end{aligned}$$

As, for any a,b, $(b-a^+)^+ + (-a-b^-)^+ = (b-a)^+$ and
$-\text{sgn}_0^+(b-a)[\vec{\mathbf{f}}(b)-\vec{\mathbf{f}}(a)] = -\text{sgn}_0^+(b-a^+)[\vec{\mathbf{f}}(b)-\vec{\mathbf{f}}(a^+)]+\text{sgn}_0^+(-a-b^-)[\vec{\mathbf{f}}(a)-\vec{\mathbf{f}}(-b^-)]$,

we find that

$$\begin{aligned}0 \;\leq\; & \int_D (\hat{u}_0 - u_0)^+ \varphi(0)\,\mathrm{dx} + E\int_Q \int_0^1 \int_0^1 (\hat{\mathbf{u}}(.,\alpha) - \mathbf{u}(.,\beta))^+ \partial_t\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}(\mathrm{t},\mathrm{x}) \\ & \underline{- \int_D (\hat{u}_0 - u_0^+)^+ \varphi(0)\,\mathrm{dx} + E\int_Q \int_0^1 \int_0^1 (\hat{\mathbf{u}}(.,\alpha) - \mathbf{u}^+(.,\beta))^+ \partial_t\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}(\mathrm{t},\mathrm{x})} \\ & \underline{-E\int_Q \int_0^1 \int_0^1 \mathrm{sgn}_0^+ (\hat{\mathbf{u}}(.,\alpha) - \mathbf{u}^+(.,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(.,\alpha)) - \vec{\mathbf{f}}(\mathbf{u}^+(.,\beta))]\nabla\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}(\mathrm{t},\mathrm{x})} \\ & -E\int_Q \int_0^1 \int_0^1 \mathrm{sgn}_0^+ (\hat{\mathbf{u}}(.,\alpha) - \mathbf{u}(.,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(.,\alpha)) - \vec{\mathbf{f}}(\mathbf{u}(.,\beta))]\nabla\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}(\mathrm{t},\mathrm{x}) \\ & + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m).\end{aligned}$$

As the sum of the underlined terms is less or equal to $\lim_m \mathcal{L}(\varphi\theta_m)$ according to (3.3), one gets:

$$\begin{aligned}0 \;\leq\; & \int_D (\hat{u}_0 - u_0)^+ \varphi(0)\,\mathrm{dx} \\ & +E\int_Q \int_0^1 \int_0^1 (\hat{\mathbf{u}}(.,\alpha) - \mathbf{u}(.,\beta))^+ \partial_t\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}(\mathrm{t},\mathrm{x}) \\ & -E\int_Q \int_0^1 \int_0^1 \mathrm{sgn}_0^+ (\hat{\mathbf{u}}(.,\alpha) - \mathbf{u}(.,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(.,\alpha)) - \vec{\mathbf{f}}(\mathbf{u}(.,\beta))]\nabla\varphi \,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}(\mathrm{t},\mathrm{x}) \\ & + \lim_m \mathcal{L}(\varphi\theta_m) + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m),\end{aligned}$$

which is the desired inequality (3.2).

3.2 Details of the proof

3.2.1 Local Kato inequality (3.1)

In this section, we state some details of the proof of following local Kato inequality.

Lemma 3.5 *Let* $\mathbf{u}$, $\hat{\mathbf{u}}$ *be Young measure-valued entropy solutions to (0.1) with initial data* $u_0, \hat{u}_0 \in L^2(D)$, *respectively and assume that at least one of them is obtained by viscous approximation. Then, for any* $\mathcal{D}^+([0,T]\times D)$*-function* φ, *one has that*

$$\begin{aligned}0 \;\leq\; & E\int_{Q\times]0,1[^2} \Big(\hat{\mathbf{u}}(t,x,\beta) - \mathbf{u}(t,x,\alpha)\Big)^+ \partial_t\varphi \,\mathrm{dx}\,\mathrm{dt}\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ & -E\int_{Q\times]0,1[^2} F^+\Big(\hat{\mathbf{u}}(t,x,\beta), \mathbf{u}(t,x,\alpha)\Big).\nabla\varphi \,\mathrm{dx}\,\mathrm{dt}\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ & + \int_D (\hat{u}_0 - u_0)^+ \varphi(0)\,\mathrm{dx}.\end{aligned}$$

Proof. As said before, the proof of this lemma is an adaptation of techniques introduced in Chapter III. We only give below main ideas of the proof. Assume that $\mathbf{u}$ is a Young measure-valued entropy solution obtained by viscous approximation (as a limit point of u_ϵ) and that $\hat{\mathbf{u}}$ is any other admissible Young-measure-valued entropy solution with associated initial data u_0 and $\hat{u}_0$. Consider φ in $\mathcal{D}^+([0,T[\times\mathbb{R}^d)$ and denote by $G(t,x,s,y)=\varphi(s,y)\rho_m(x-y)\rho_n(t-s)$ where ρ_m and ρ_n denote the usual mollifier sequences in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, with $\mathrm{supp}\rho_n\subset[-\frac{2}{n},0]$ and m big enough in order G to vanish on the boundary, for both variables x and y.
Denote also by ρ_l a mollifier sequence in $\mathbb{R}$ and for convenience set $p=(t,x,\alpha)$.
Since $\hat{\mathbf{u}}(t,x,\alpha)$ is a Young measure-valued solution, by considering the entropy $\eta(.-k)=\eta_\delta(\cdot-k)$ and the test function G, multiplying the entropy formulation by $\mathcal{B}_k^l:=\rho_l(u_\epsilon(s,y)-k)$ and integrating k over $\mathbb{R}$, we get that

$$
\begin{aligned}
0 \;\le\;& E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\eta(\hat{u}_0(x)-k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\mathrm{dx}\mathcal{B}_k^l\mathrm{dkdyds}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\eta(\hat{\mathbf{u}}(p)-k)\varphi(s,y)\,\partial_t\rho_n(t-s)\rho_m(x-y)\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^\eta(\hat{\mathbf{u}}(p),k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''(\hat{\mathbf{u}}(p)-k)\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\mathrm{dp}\mathcal{B}_k^l\mathrm{dkdyds}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\eta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\mathrm{dx}dw(t)\mathcal{B}_k^l\mathrm{dkdyds}
\end{aligned}
$$

On the other hand, since u_ϵ is a viscous solution, the Itô formula with entropy $\check{\eta}(u_\epsilon-k)=\eta(k-u_\epsilon)$* and multiplying it by $\mathcal{A}_k^l:=\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)$, we get:

$$
\begin{aligned}
0 \;\le\;& E\int_Q\int_{\mathbb{R}}\int_Q\eta(k-u_\epsilon(s,y))\partial_s\varphi(s,y)\;\rho_n(t-s)\rho_m(x-y)\mathrm{dyds}\int_0^1\mathcal{A}_k^l)\mathrm{dkdp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\eta(k-u_\epsilon(s,y))\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\mathrm{dyds}\int_0^1\mathcal{A}_k^l\mathrm{dkdp}\\
&+\epsilon E\int_Q\int_{\mathbb{R}}\int_Q\eta'(k-u_\epsilon(s,y))\nabla_y u_\epsilon(s,y)\nabla_y\left[\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\right]\mathrm{dyds}\int_0^1\mathcal{A}_k^l\mathrm{dkdp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^{\check{\eta}}(u_\epsilon(s,y),k)\nabla_y\left[\varphi(s,y)\;\rho_m(x-y)\rho_n(t-s)\right]\mathrm{dyds}\int_0^1\mathcal{A}_k^l\mathrm{dkdp}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(s,y))\eta''(k-u_\epsilon(s,y))\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\mathrm{dyds}\int_0^1\mathcal{A}_k^l\mathrm{dkdp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q\eta'(k-u_\epsilon(s,y))h(u_\epsilon(s,y))\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\mathrm{dy}dw(s)\int_0^1\mathcal{A}_k^l\mathrm{dkdp}
\end{aligned}
$$

*One denotes $\check{\eta}(x)$ by $\eta(-x)$.

Summing up the preceding inequalities, we pass to the limit in the following over: n, l, η, ϵ and m. Let us mention that we choose of $\check{\eta}(u_\epsilon - k)$ in the second entropy inequality is motivated by the idea to get back a parity property in order to show that the sum of the two underlined terms vanishes. □

3.2.2 Global Kato inequality (3.2)

⋆ **A first inequality:**

To this end, assume in the sequel, without restriction, that $\mathbf{u}$ is obtained by viscous approximation and choose a partition of unity subordinate to a covering of $\overline{D}$ by balls $\mathcal{B}_i$, $i = 1, \cdots, k$ satisfying $\mathcal{B}_0 \cap \partial D = \varnothing$ and, for $i = 1, \cdots, k$, $\mathcal{B}_i \subset \mathcal{B}'_i$ with $\mathcal{B}'_i \cap \partial D$ part of a Lipschitz graph. Then, let:

φ in $\mathcal{D}^+([0,T[\times\mathbb{R}^d)$ with supp$\varphi(t,.) \subset B := \mathcal{B}_i$ for some $i \in \{1, \cdots, k\}$;
ρ_n a sequence of mollifiers in $\mathbb{R}$ with supp$\rho_n \subset [-2/n, 0]$;
ρ_m a shifted[†] sequence of mollifiers in $\mathbb{R}^d$ such that $y \mapsto \rho_m(x-y) \in \mathcal{D}(D)$ for all $x \in B \cap D$.

Note that for m big enough, $y \mapsto \varphi(s,y)\rho_m(x-y) \in \mathcal{D}(D)$ and that we have

$$\int_0^T \int_D \varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}\,\mathrm{dt} = \varphi(s,y)\theta_m(y)\sigma_n(s),$$

where $\theta_m(y) = \displaystyle\int_D \rho_m(x-y)\,\mathrm{dx}$ and $\sigma_n(s) = \displaystyle\int_0^T \rho_n(t-s)\,\mathrm{dt}$ are nonnegative, nondecreasing sequences bounded by 1.

In order to take $\varphi\rho_m\rho_n$ as a test function in the entropy inequality for $\hat{\mathbf{u}}(t,x,\alpha)$, we have to use the entropy $\eta_\delta(.-k)$ for nonnegative k and thus we multiply the entropy inequality by $\rho_l(\eta_{\widetilde{\delta}}(u_\epsilon) - k)$ $(0 < \widetilde{\delta} < \delta)$ with supp$\rho_l = [-\frac{2}{l}, 0]$ to force $k \geq 0$ to be nonnegative.

As now ρ_l is no longer even, we use the Itô formula for u_ϵ first with the regular nonlinear function $\psi^k_{\delta,\widetilde{\delta}}(x) = \eta_\delta(k - \eta_{\widetilde{\delta}}(x)) + \eta_\delta(-x)$ (this corresponds to testing the equation with the Lipschitz-continuous function $(\psi^k)'_{\delta,\widetilde{\delta}}(u_\epsilon) = -\eta'_\delta(k - \eta_{\widetilde{\delta}}(u_\epsilon))\eta'_{\widetilde{\delta}}(u_\epsilon) - \eta'_\delta(-u_\epsilon)$ and then multiply by $\rho_l(k - \hat{u})$.

[†]For every $i = 1, .., k$, depending on the local representation of the boundary of D in B'_i as the graph of a Lipschitz function, we can construct a vector $\eta_i \in \mathbb{R}^d$ such that the translated sequence of mollifiers $\rho_n(x-y) = \bar{\rho}_n(x - y - \frac{1}{n}\eta_i)$ satisfies that $y \mapsto \bar{\rho}_n(x - y - \frac{1}{n}\eta_i) \in \mathcal{D}(D)$ for all $x \in B = B_i$, where $\bar{\rho}_n$ denotes the standard mollifier sequence, see Carrilo [22] or Girault [44].

Remark 3.6

1. *The asymmetry of the multiplicative factors* $\rho_l(\eta_{\widetilde{\delta}}(u_\epsilon) - k)$ *and* $\rho_l(k - \hat{u})$ *is a consequence of the asymmetry of* ρ_l*; in a sense it compensates the asymmetry of* ρ_l *and permits to show that* $I_3 + J_3 = 0$ *(see below) also in the global case.*

2. *One has to work with a smooth approximation* $\eta_{\widetilde{\delta}}(u_\epsilon)$ *of* u_ϵ^+ *as due to the Itô calculus, the second derivative of the functions arises in the formulation.*

3. *One has to use* $\psi^k_{\delta,\widetilde{\delta}}$ *for the equation for* u_ϵ *with* $0 < \widetilde{\delta} < \delta$ *and pass to the limit first with* $\widetilde{\delta}$ *to* 0 *and then with* δ *to* 0*. It is not clear whether it is possible to pass to the limit if* $\widetilde{\delta} = \delta$*.*

4. *Note that* $\psi^k_{\delta,\widetilde{\delta}}(s)$ *converges, when* δ *and* $\widetilde{\delta}$ *go to* 0*, to* $(k - s^+)^+ + s^- = (k^+ - s)^+$ *which is convex in* s*, so the Kato inequality* $\Delta u j'(u) \leq \Delta j(u)$ *in* $\mathcal{D}'$ *can be applied to* $j(s) = (k^+ - s)^+$*. This is the reason why we test the equation for* u_ϵ *with* $(\psi^k)'_{\delta,\widetilde{\delta}}$ *and not only with* $-(\eta_\delta)'(k - \eta_{\widetilde{\delta}}(u_\epsilon))\eta'_{\widetilde{\delta}}(u_\epsilon)$ *which one would like to do at first glance.*

To simplify matters, denote $p=(t,x,\alpha)$, $q=(s,y,\beta)$ and $\widetilde{\mathcal{B}}_k^l := \rho_l(\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k)$

$$
\begin{aligned}
0 \quad \le \quad & E\int_Q\int_{\mathbb{R}}\int_D \eta_\delta(\hat{u}_0(x)-k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
& +E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(\hat{\mathbf{u}}(p)-k)\underbrace{\partial_t\varphi(s,y)}_{=0}\rho_n(t-s)\rho_m(x-y)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
& +E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(\hat{\mathbf{u}}(p)-k)\varphi(s,y)\,\partial_t\rho_n(t-s)\rho_m(x-y)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
& -E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\eta_\delta}(\hat{\mathbf{u}}(p),k)\underbrace{\nabla_x\varphi(s,y)}_{=0}\rho_m(x-y)\rho_n(t-s)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
& -E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\eta_\delta}(\hat{\mathbf{u}}(p),k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
& +\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta_\delta''(\hat{\mathbf{u}}(p)-k)\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
& +E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
& =: \quad I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}
$$

On the other hand, if one denotes $\check{\mathcal{A}}^l_k = \rho_l(k - \hat{\mathbf{u}}(p))$, since u_ϵ is a viscous solution, the Itô formula applied to $\displaystyle\int_D \psi^k_{\delta,\widetilde{\delta}}(u_\epsilon)\rho_n(t-s)\rho_m(x-y)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}$ yields:

$$
\begin{aligned}
0 \leq\ & E\int_Q\int_{\mathbb{R}}\int_D \eta_\delta[k-\eta_{\widetilde{\delta}}(u_0^\epsilon(y))]\varphi(0,y)\rho_n(t)\rho_m(x-y)\,\mathrm{dy}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&+E\int_Q\int_{\mathbb{R}}\int_D \check{\eta}_\delta(u_0^\epsilon(y))\varphi(0,y)\rho_n(t)\rho_m(x-y)\,\mathrm{dy}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[k-\eta_{\widetilde{\delta}}(u_\epsilon(s,y))]\partial_s\varphi(s,y)\ \rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q \check{\eta}_\delta(u_\epsilon(s,y))\partial_s\varphi(s,y)\ \rho_n(t-s)\rho_m(x-y)\ \mathrm{dy}\ \mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[k-\eta_{\widetilde{\delta}}(u_\epsilon(s,y))]\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q \check{\eta}_\delta(u_\epsilon(s,y))\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q \eta'_\delta(k-\eta_{\widetilde{\delta}}(u_\epsilon))\eta'_{\widetilde{\delta}}(u_\epsilon)\Delta_y u_\epsilon(s,y)\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q \eta'_\delta(-u_\epsilon)\Delta_y u_\epsilon(s,y)\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^{\eta_\delta[k-\eta_{\widetilde{\delta}}(.+k)]}(u_\epsilon(s,y),k)\nabla_y\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0)\nabla_y\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^{\eta_\delta[k-\eta_{\widetilde{\delta}}(.+k)]}(u_\epsilon(s,y),k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q[(\eta'_\delta(u_\epsilon(s,y)))^2\eta''_\delta(k-\eta_{\widetilde{\delta}}(u_\epsilon(s,y)))-\eta''_{\widetilde{\delta}}(u_\epsilon(s,y))\eta'_\delta(k-\eta_{\widetilde{\delta}}(u_\epsilon(s,y))]\\
&\qquad\qquad\times h^2(u_\epsilon(s,y))\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(s,y))\eta''_\delta(-u_\epsilon(s,y))\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q \eta'_\delta(k-\eta_{\widetilde{\delta}}(u_\epsilon))\eta'_{\widetilde{\delta}}(u_\epsilon)h(u_\epsilon(s,y))\varphi(s,y)\rho_n(t-s)\,\mathrm{dy}dw(s)\rho_m(x-y)\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q \eta'_\delta(-u_\epsilon)h(u_\epsilon(s,y))\varphi(s,y)\rho_n(t-s)\,\mathrm{dy}dw(s)\rho_m(x-y)\int_0^1 \check{\mathcal{A}}^l_k\,\mathrm{dk}\,\mathrm{dp}\\
&:= J_1+K_1+J_2+K_2+J_3+K_3+J_4+K_4+J_5+K_5+J_6+K_6+J_7+K_7+J_8+K_8.
\end{aligned}
$$

First note that, thanks to Fubini's theorem and since $\displaystyle\int_0^1 \int_{\mathbb{R}} \rho_l(k - \hat{\mathbf{u}}(p))\,\mathrm{dk}\,\mathrm{d}\alpha = 1$,

$$
\begin{aligned}
&K_1 + K_2 + \cdots + K_8 = \\
&\quad E \int_D \check{\eta}_\delta(u_0^\epsilon(y))\varphi(0,y) \int_Q \underbrace{\rho_n(t)}_{=0} \rho_m(x-y)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy} \\
&\quad + E \int_Q \check{\eta}_\delta(u_\epsilon(s,y))\partial_s\varphi(s,y) \int_Q \rho_n(t-s)\rho_m(x-y)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad + E \int_Q \check{\eta}_\delta(u_\epsilon(s,y))\varphi(s,y) \int_Q \partial_s\rho_n(t-s)\rho_m(x-y)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad - \epsilon E \int_Q \eta_\delta'(-u_\epsilon)\Delta_y u_\epsilon(s,y)\varphi(s,y) \int_Q \rho_n(t-s)\rho_m(x-y)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad - E \int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0)\nabla_y\varphi(s,y) \int_Q \rho_m(x-y)\rho_n(t-s)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad - E \int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0) \int_Q \nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad + \frac{1}{2} E \int_Q h^2(u_\epsilon(s,y))\eta_\delta''(-u_\epsilon(s,y)) \int_Q \rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad - E \int_Q \eta_\delta'(-u_\epsilon)h(u_\epsilon(s,y))\varphi(s,y) \int_Q \rho_m(x-y)\rho_n(t-s)\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dy} dw(s).
\end{aligned}
$$

Then, since $\int_Q \rho_n(t-s)\rho_m(x-y)\,\mathrm{dt}\,\mathrm{dx} = \sigma_n(s)\theta_m(y)$ with $\mathrm{supp}\rho_n \subset [-\frac{2}{n},0]$, one has that $\sigma_n(s)=1$ in $[\frac{2}{n},T]$ and σ_n is a nondecreasing sequence that converges to 1. Thus,

$$
\begin{aligned}
K_1 &= 0.\\
K_2 &= E\int_Q \check{\eta}_\delta(u_\epsilon(s,y))\partial_s\varphi(s,y)\sigma_n(s)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}\\
&\underset{n}{\rightarrow} E\int_Q \check{\eta}_\delta(u_\epsilon(s,y))\partial_s\varphi(s,y)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds} \underset{\delta}{\rightarrow} E\int_Q (u_\epsilon(s,y))^-\partial_s\varphi(s,y)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}\\
&\underset{\epsilon}{\rightarrow} E\int_Q\int_0^1 (\mathbf{u}(q))^-\partial_s\varphi(s,y)\theta_m(y)\,\mathrm{dq} \underset{m}{\rightarrow} E\int_Q\int_0^1 (\mathbf{u}(q))^-\partial_s\varphi(s,y)\,\mathrm{dq}.\\
K_3 &= E\int_Q \check{\eta}_\delta(u_\epsilon(s,y))\varphi(s,y)[\rho_n(-s)-\rho_n(T-s)]\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}\\
&= E\int_0^T \rho_n(-s)\int_D \check{\eta}_\delta(u_\epsilon(s,y))\varphi(s,y)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds} \underset{n}{\rightarrow} E\int_D \check{\eta}_\delta(u_0^\epsilon)\varphi(0,y)\theta_m(y)\,\mathrm{dy}\\
&\underset{\delta}{\rightarrow} E\int_D (u_0^\epsilon)^-\varphi(0,y)\theta_m(y)\,\mathrm{dy} \underset{\epsilon}{\rightarrow} E\int_D (u_0)^-\varphi(0,y)\theta_m(y)\,\mathrm{dy} \underset{m}{\rightarrow} E\int_D (u_0)^-\varphi(0,y)\,\mathrm{dy}.\\
K_4 &= -\epsilon E\int_Q \eta_\delta'(-u_\epsilon)\Delta_y u_\epsilon(s,y)\varphi(s,y)\theta_m(y)\sigma_n(s)\,\mathrm{dy}\,\mathrm{ds}\\
&\underset{n}{\rightarrow} -\epsilon E\int_Q \eta_\delta'(-u_\epsilon)\Delta_y u_\epsilon(s,y)\varphi(s,y)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}\\
&\underset{\delta}{\rightarrow} -\epsilon E\int_Q \mathrm{sgn}_0^-(u_\epsilon(s,y))\Delta_y u_\epsilon(s,y)\varphi(s,y)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}.
\end{aligned}
$$

$$
\begin{aligned}
K_5 &= -E\int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0)\nabla_y\varphi(s,y)\sigma_n(s)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}\\
&\quad \underset{n}{\rightarrow} -E\int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0)\nabla_y\varphi(s,y)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}\\
&\quad \underset{\delta}{\rightarrow} -E\int_Q F^{(.)^-}(u_\epsilon(s,y),0)\nabla_y\varphi(s,y)\theta_m(y)\,\mathrm{dy}\,\mathrm{ds}\\
&\quad \underset{\epsilon}{\rightarrow} -E\int_Q\int_0^1 F^{(.)^-}(\mathbf{u}(q),0)\nabla_y\varphi(s,y)\theta_m(y)\,\mathrm{dq}.\\
K_6 &= -E\int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0)\nabla_y\theta_m(y)\sigma_n(s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\\
&\quad \underset{n}{\rightarrow} -E\int_Q F^{\check{\eta}_\delta}(u_\epsilon(s,y),0)\nabla_y\theta_m(y)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\\
&\quad \underset{\delta}{\rightarrow} -E\int_Q F^{(.)^-}(u_\epsilon(s,y),0)\nabla_y\theta_m(y)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\\
&\quad \underset{\epsilon}{\rightarrow} -E\int_Q\int_0^1 F^{(.)^-}(\mathbf{u}(q),0)\nabla_y\theta_m(y)\varphi(s,y)\,\mathrm{dq}
\end{aligned}
$$

Since h is Lipschitz-continuous with $h(0) = 0$ and $0 \le \eta''_\delta(z) \le \dfrac{\pi}{2\delta} 1_{0<z<\delta}$, we have

$$\begin{aligned}
K_7 &= \frac{1}{2} E \int_Q h^2(u_\epsilon(s,y))\eta''_\delta(-u_\epsilon(s,y))\theta_m(y)\sigma_n(s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds} \\
&\underset{n}{\to} \frac{1}{2} E \int_Q h^2(u_\epsilon(s,y))\eta''_\delta(-u_\epsilon(s,y))\theta_m(y)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds} \underset{\delta}{\to} 0 \\
K_8 &= -E \int_Q \int_D \eta'_\delta(-u_\epsilon) h(u_\epsilon(s,y))\varphi(s,y)\sigma_n(s)\theta_m(y)\,\mathrm{dy} dw(s) = 0.
\end{aligned}$$

Thus, one is able to conclude that

$$\begin{aligned}
&\lim_\epsilon \lim_{\delta,\tilde{\delta}} \lim_n K_1 + K_2 + K_3 + K_5 + K_6 + K_7 + K_8 \\
= \;& E \int_D (u_0(y))^- \varphi(0,y)\theta_m(y)\,\mathrm{dy} + E \int_Q \int_0^1 (\mathbf{u}(q))^- \partial_s \varphi(s,y)\theta_m(y)\,\mathrm{dq} \\
&- E \int_Q \int_0^1 F^{(.)^-}(\mathbf{u}(q),0)\nabla_y[\varphi(s,y)\theta_m(y)]\,\mathrm{dq} \\
=: \;& \mathcal{L}(\varphi\theta_m).
\end{aligned}$$

Note that a proof, similar to the one of the above convergences, ensures that

$$\forall \psi \in \mathcal{D}([0,T[\times\mathbb{R}^d),\ \mathcal{L}(\psi) = \lim_\delta E\mu_{\check{\eta}_\delta,0}(\psi).$$

Then, since $\mathbf{u}$ is an entropy measure-valued solution, $\mathcal{L}$ is a nonnegative operator and since $0 \le \varphi\theta_m \le \varphi\theta_{m+1} \le \varphi$ in $\mathcal{D}([0,T[\times\mathbb{R}^d)$, one concludes that $\lim_m \mathcal{L}(\varphi\theta_m)$ exists on $[0,+\infty[$.

Now it remains to combine appropriately the integrals J_4 and K_4 to apply a Kato formula available with u_ϵ and the integrals I_i and J_k, almost as in the local case, to pass to the limit in the other terms. These limits are obtained *via* standard arguments: properties of convolution, Lebesgue points, continuity of translations in Lebesgue spaces, uniform-integrability and Carathéodory functions for the Young measures, as detailed in particular in Chapter III or in

Vallet-Wittbold [78].

$$\begin{aligned}
I_1 + J_1 &= E\int_Q\int_{\mathbb{R}}\int_D \eta_\delta(\hat{u}_0(x)-k)\varphi(s,y)\rho_n(-s)\rho_m(x-y)\,\mathrm{dx}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\underset{n}{\rightarrow} E\int_D\int_{\mathbb{R}}\int_D \eta_\delta(\hat{u}_0(x)-k)\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\rho_l(\eta_{\widetilde{\delta}}(u_0^\epsilon(y))-k)\,\mathrm{dk}\,\mathrm{dy}\\
&\underset{l}{\rightarrow} E\int_D\int_D \eta_\delta(\hat{u}_0(x)-u_0^\epsilon(y))\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\\
&\underset{\delta,\widetilde{\delta}}{\longrightarrow} E\int_D\int_D [\hat{u}_0(x)-(u_0^\epsilon(y))^+]^+\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\\
&\underset{\epsilon}{\rightarrow} E\int_D\int_D(\hat{u}_0(x)-u_0^+(y))^+\varphi(0,y)\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dy}\\
&\underset{m}{\rightarrow} E\int_D(\hat{u}_0-u_0^+)^+\varphi(0,.)\,\mathrm{dy}.
\end{aligned}$$

Similar calculations lead to

$$\begin{aligned}
J_2 &= E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[k-\eta_{\widetilde{\delta}}(u_\epsilon(s,y))]\partial_s\varphi(s,y)\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp}\\
&\underset{n,\dots,m}{\longrightarrow} E\int_Q\int_0^1\int_0^1[\hat{u}(s,y,\alpha)-(u(q))^+]^+\partial_s\varphi(s,y)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{dy}\,\mathrm{ds}.
\end{aligned}$$

$I_3 + J_3$

$$\begin{aligned}
&= E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(\mathbf{\hat{u}}(p)-k)\varphi(s,y)\partial_t\rho_n(t-s)\rho_m(x-y)\,\mathrm{dp}\rho_l(\underbrace{\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k}_{\tau})\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\quad +E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[k-\eta_{\widetilde{\delta}}(u_\epsilon(s,y))]\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\int_0^1\rho_l(\underbrace{k-\mathbf{\hat{u}}(p)}_{\tau})\,\mathrm{dk}\,\mathrm{dp}\\
&= E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(\mathbf{\hat{u}}(p)-\eta_{\widetilde{\delta}}(u_\epsilon(s,y))+\tau)\varphi(s,y)\underbrace{\partial_t\rho_n(t-s)}_{=-\partial_s\rho_n(t-s)}\rho_m(x-y)\,\mathrm{dp}\rho_l(\tau)\,\mathrm{d}\tau\,\mathrm{dy}\,\mathrm{ds}\\
&\quad +E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta[\tau+\mathbf{\hat{u}}(p)-\eta_{\widetilde{\delta}}(u_\epsilon(s,y))]\varphi(s,y)\partial_s\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds}\rho_l(\tau)\,\mathrm{d}\tau\,\mathrm{dp}\\
&= 0.
\end{aligned}$$

$$
\begin{aligned}
K_4 + J_4 \quad = \quad & -\epsilon E \int_Q \int_{\mathbb{R}} \int_Q \left[\eta_\delta'(k - \eta_{\tilde{\delta}}(u_\epsilon))\eta_{\tilde{\delta}}'(u_\epsilon) + \eta_\delta'(-u_\epsilon)\right] \Delta_y u_\epsilon(s,y)\varphi(s,y) \\
& \qquad \times\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds} \int_0^1 \rho_l(k - \hat{\mathbf{u}}(p))\,\mathrm{dk}\,\mathrm{dp} \\
= \quad & -\epsilon E \int_Q \int_{\mathbb{R}} \int_Q \left[\eta_\delta'(k - \eta_{\tilde{\delta}}(u_\epsilon))\eta_{\tilde{\delta}}'(u_\epsilon) + \eta_\delta'(-u_\epsilon)\right] \Delta_y u_\epsilon(s,y)\varphi(s,y) \\
& \qquad \times\rho_n(t-s)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{ds} \int_0^1 \rho_l(k - \hat{\mathbf{u}}(p))\,\mathrm{dk}\,\mathrm{dp} \\
\underset{n,l}{\longrightarrow} \quad & -\epsilon E \int_Q \int_0^1 \int_D \left[\eta_\delta'(\hat{\mathbf{u}}(p) - \eta_{\tilde{\delta}}(u_\epsilon))\eta_{\tilde{\delta}}'(u_\epsilon) + \eta_\delta'(-u_\epsilon)\right] \Delta_y u_\epsilon(t,y) \\
& \qquad \times\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp} \\
\underset{\tilde{\delta},\delta}{\longrightarrow} \quad & -\epsilon E \int_Q \int_0^1 \int_D \left[\mathrm{sgn}_0^+\left(\hat{\mathbf{u}}(p) - |u_\epsilon|\right)\mathrm{sgn}_0^+\left(u_\epsilon\right) + \mathrm{sgn}_0^+\left(-u_\epsilon\right)\right] \\
& \qquad \times\Delta_y u_\epsilon(t,y)\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp} \\
= \quad & -\epsilon E \int_Q \int_0^1 \int_D \left[\mathrm{sgn}_0^+\left(\hat{\mathbf{u}}^+(p) - u_\epsilon\right)\right] 1_{\{u_\epsilon \neq 0\}} \Delta_y u_\epsilon(t,y)\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp} \\
= \quad & -\epsilon E \int_Q \int_0^1 \int_D \left[\mathrm{sgn}_0^+\left(\hat{\mathbf{u}}^+(p) - u_\epsilon\right)\right] \Delta_y u_\epsilon(t,y)\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp} \\
& +\epsilon E \int_Q \int_0^1 \int_D \left[\mathrm{sgn}_0^+\left(\hat{\mathbf{u}}^+(p) - u_\epsilon\right)\right] 1_{\{u_\epsilon = 0\}} \Delta_y u_\epsilon(t,y)\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp}.
\end{aligned}
$$

Note that $y \mapsto \varphi(t,y)\rho_m(x-y)$ has compact support and that $y \mapsto u_\epsilon(t,y) \in H^2_{loc}(D)$. Thus, one concludes in a first step that $\nabla u_\epsilon = 0$ a.e. in $A := \{u_\epsilon = 0 \ a.e.\}$ and that $A \subset B := \{\nabla u_\epsilon = 0 \ a.e.\}$. Then, in a second one, using again the lemma of Saks, all the second order derivatives are null a.e. in A since they are already in B. Thus, $1_{\{u_\epsilon = 0\}}\Delta_y u_\epsilon(t,y)\varphi(t,y)\rho_m(x-y) = 0$ and

$$
\lim_{\delta,\tilde{\delta},l,n} K_4 + J_4 = -\epsilon E \int_Q \int_0^1 \int_D \mathrm{sgn}_0^+\left(\hat{\mathbf{u}}^+(p) - u_\epsilon\right) \Delta_y u_\epsilon(t,y)\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp}.
$$

Then, Kato's inequality (see Brezis-Ponce [20] for example) yields

$$
\begin{aligned}
\lim_{\delta,\tilde{\delta},l,n} K_4 + J_4 &= \epsilon E \int_Q \int_0^1 \int_D \mathrm{sgn}_0^+(\hat{\mathbf{u}}^+(p) - u_\epsilon) \Delta_y [\hat{\mathbf{u}}^+(p) - u_\epsilon(t,y)]\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp} \\
&\leq \epsilon E \int_Q \int_0^1 \int_D \Delta_y [\hat{\mathbf{u}}^+(p) - u_\epsilon(t,y)]^+ \varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp} \\
&= -\epsilon E \int_Q \int_0^1 \int_D \nabla_y [\hat{\mathbf{u}}^+(p) - u_\epsilon(t,y)]^+ \nabla_y \varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp}.
\end{aligned}
$$

Thus, $\limsup_{\epsilon} \lim_{\delta,\tilde{\delta},l,n} K_4 + J_4 \leq 0$.

$$\begin{aligned}
J_5 &= E\int_Q\int_{\mathbb{R}}\int_Q\int_k^{u_\epsilon(s,y)}\vec{\mathbf{f}}'(\sigma)\eta'_{\widetilde{\delta}}(\sigma)\eta'_\delta[k-\eta_{\widetilde{\delta}}(\sigma)]\,\mathrm{d}\sigma\nabla_y\varphi(s,y)\\
&\qquad\times\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk}\,\mathrm{dp}\\
&\underset{l,n}{\longrightarrow}\ E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}(p)}^{u_\epsilon(t,y)}\vec{\mathbf{f}}'(\sigma)\eta'_{\widetilde{\delta}}(\sigma)\eta'_\delta[\hat{\mathbf{u}}(p)-\eta_{\widetilde{\delta}}(\sigma)]\,\mathrm{d}\sigma\nabla_y\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp}\\
&\underset{\widetilde{\delta},\delta}{\longrightarrow}\ E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}(p)}^{u_\epsilon(t,y)}\vec{\mathbf{f}}'(\sigma)\mathrm{sgn}_0^+(\sigma)\,\mathrm{sgn}_0^+(\hat{\mathbf{u}}(p)-|\sigma|)\,\mathrm{d}\sigma\nabla_y\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp}\\
&= -E\int_Q\int_0^1\int_D F^+(\hat{\mathbf{u}}(p),u_\epsilon^+(t,y))\,\nabla_y\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{dp}\\
&\underset{\epsilon}{\longrightarrow} -E\int_Q\int_0^1\int_0^1\int_D F^+(\hat{\mathbf{u}}(p),\mathbf{u}^+(t,y,\beta))\,\nabla_y\varphi(t,y)\rho_m(x-y)\,\mathrm{dy}\,\mathrm{d}\beta\,\mathrm{dp}\\
&\underset{m}{\longrightarrow} -E\int_Q\int_0^1\int_0^1 F^+(\hat{\mathbf{u}}(p),\mathbf{u}^+(t,x,\beta))\,\nabla_x\varphi(t,x)\,\mathrm{d}\beta\,\mathrm{dp}.
\end{aligned}$$

$$\begin{aligned}
I_5+J_6 &= -E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\eta_\delta}(\hat{\mathbf{u}}(p),k)\nabla_x\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\quad -E\int_Q\int_{\mathbb{R}}\int_Q F^{\eta_\delta[k-\eta_{\widetilde{\delta}}(.+k)]}(u_\epsilon(s,y),k)\nabla_y\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp}\\
&\underset{l,n}{\longrightarrow}\ -E\int_Q\int_D\int_0^1\int_{\eta_{\widetilde{\delta}}(u_\epsilon(t,y))}^{\hat{\mathbf{u}}(p)}\eta'_\delta(\sigma-\eta_{\widetilde{\delta}}(u_\epsilon(t,y)))\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_x\rho_m(x-y)\varphi(t,y)\,\mathrm{dp}\,\mathrm{dy}\\
&\qquad +E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}(p)}^{u_\epsilon(t,y)}\eta'_{\widetilde{\delta}}(\sigma)\eta'_\delta(\hat{\mathbf{u}}(p)-\eta_{\widetilde{\delta}}(\sigma))\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_y\rho_m(x-y)\varphi(t,y)\,\mathrm{dy}\,\mathrm{dp}\\
&\underset{\widetilde{\delta},\delta}{\longrightarrow}\ -E\int_Q\int_D\int_0^1\int_{u_\epsilon^+(t,y)}^{\hat{\mathbf{u}}(p)}\mathrm{sgn}_0^+(\sigma-u_\epsilon^+(t,y))\,\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_x\rho_m(x-y)\varphi(t,y)\,\mathrm{dp}\,\mathrm{dy}\\
&\qquad +E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}(p)}^{u_\epsilon(t,y)}\mathrm{sgn}_0^+(\sigma)\,\mathrm{sgn}_0^+(\hat{\mathbf{u}}(p)-\sigma^+)\,\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_y\rho_m(x-y)\varphi(t,y)\,\mathrm{dy}\,\mathrm{dp}\\
&= -E\int_Q\int_D\int_0^1 F^+(\hat{\mathbf{u}}(p),u_\epsilon^+(t,y))\nabla_x\rho_m(x-y)\varphi(t,y)\,\mathrm{dp}\,\mathrm{dy}\\
&\qquad -E\int_Q\int_0^1\int_D F^+(\hat{\mathbf{u}}(p),u_\epsilon^+(t,y))\nabla_y\rho_m(x-y)\varphi(t,y)\,\mathrm{dy}\,\mathrm{dp}\\
&= 0 \qquad \text{since } \nabla_x\rho_m(x-y)=-\nabla_y\rho_m(x-y).
\end{aligned}$$

$$\begin{aligned} I_6+J_7 &= \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\rho_l(\eta_{\tilde{\delta}}(u_\epsilon(s,y))-k) \\ &\qquad \times\eta''_\delta(\hat{\mathbf{u}}(p)-k)\,\mathrm{dy\,ds\,dp\,dk} \\ &+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q h^2(u_\epsilon(s,y))\rho_m(x-y)\rho_n(t-s)\varphi(s,y)\int_0^1\rho_l(k-\hat{\mathbf{u}}(p)) \\ &\times\Big[\big(\eta'_\delta[u_\epsilon(s,y)]\big)^2\eta''_\delta\big(k-\eta_{\tilde{\delta}}[u_\epsilon(s,y)]\big)-\eta''_{\tilde{\delta}}[u_\epsilon(s,y)]\eta'_\delta\big(k-\eta_{\tilde{\delta}}[u_\epsilon(s,y)]\big)\Big]\mathrm{dy\,ds\,dk\,dp} \\ \underset{l,n}{\longrightarrow} &\ \frac{1}{2}E\int_Q\int_D\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta''_\delta(\hat{\mathbf{u}}(p)-\eta_{\tilde{\delta}}(u_\epsilon(t,y)))\rho_m(x-y)\varphi(t,y)\,\mathrm{dy\,dp} \\ &+\frac{1}{2}E\int_Q\int_0^1\int_D h^2(u_\epsilon(t,y))\rho_m(x-y)\varphi(t,y) \\ &\times\Big[\big(\eta'_\delta[u_\epsilon(t,y)]\big)^2\eta''_\delta\big(\hat{\mathbf{u}}(p)-\eta_{\tilde{\delta}}[u_\epsilon(t,y)]\big)-\eta''_{\tilde{\delta}}[u_\epsilon(t,y)]\eta'_\delta\big(\hat{\mathbf{u}}(p)-\eta_{\tilde{\delta}}[u_\epsilon(t,y)]\big)\Big]\mathrm{dy\,dp}. \end{aligned}$$

Since $\alpha(t)=\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))$ is predictable and if one denotes by

$$\beta(s)=\int_D\eta'_\delta(k-\eta_{\tilde{\delta}}(u_\epsilon))\eta'_{\tilde{\delta}}(u_\epsilon)h(u_\epsilon(s,y))\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy},$$

one has that:

$$E[\alpha(t)\int_t^T\beta(s)dw(s)]=E[\alpha(t)\int_0^T\beta(s)dw(s)]-E[\alpha(t)\int_0^t\beta(s)dw(s)]=0$$

as $E[\alpha(t)\int_0^T\beta(s)dw(s)]=E[\alpha(t)E(\int_0^T\beta(s)dw(s)|F_t)]=E[\alpha(t)\int_0^t\beta(s)dw(s)]$.
Then, by the same type of arguments with $\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s-\frac{2}{n},y))-k]$ we deduce:

$$\begin{aligned} I_7+J_8 &= E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1\eta'_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t) \\ &\qquad\times\rho_l(\eta_{\tilde{\delta}}(u_\epsilon(s,y))-k)\,\mathrm{dk\,dy\,ds} \\ &-E\int_Q\int_{\mathbb{R}}\int_t^T\int_D\eta'_\delta(k-\eta_{\tilde{\delta}}(u_\epsilon))\eta'_{\tilde{\delta}}(u_\epsilon)h(u_\epsilon(s,y))\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dy}dw(s) \\ &\qquad\times\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk\,dp} \\ &= E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1\eta'_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t) \\ &\qquad\times\Big[\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s,y))-k]-\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s-\frac{2}{n},y))-k]\Big]\,\mathrm{dk\,dy\,ds}. \end{aligned}$$

As $du_\epsilon = [\epsilon\Delta u_\epsilon + \operatorname{div}\vec{\mathbf{f}}(u_\epsilon)]\,\mathrm{dt} + h(u_\epsilon)dw = A_\epsilon\,\mathrm{dt} + h(u_\epsilon)dw$, by Itô's formula,

$$
\begin{aligned}
&\rho_l[\eta_{\tilde\delta}(u_\epsilon(s,y)) - k] - \rho_l[\eta_{\tilde\delta}(u_\epsilon(s-\frac{2}{n},y)) - k]\\
=\;& \int_{(s-\frac{2}{n})^+}^{s} \rho_l'[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde\delta}'(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\,\mathrm{d}\sigma\\
&+\int_{(s-\frac{2}{n})^+}^{s} \rho_l'[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde\delta}'(u_\epsilon(\sigma,y))h(u_\epsilon(\sigma,y))\,\mathrm{dw}(\sigma)\\
&+\frac{1}{2}\int_{(s-\frac{2}{n})^+}^{s}\Big[\rho_l''[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k][\eta_{\tilde\delta}'(u_\epsilon(\sigma,y))]^2 + \rho_l'[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde\delta}''(u_\epsilon(\sigma,y))\Big]\\
&\qquad\qquad\times h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\\
=\;& -\frac{\partial}{\partial k}\Big\{\int_{(s-\frac{2}{n})^+}^{s} \rho_l[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde\delta}'(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\,\mathrm{d}\sigma\\
&+\int_{(s-\frac{2}{n})^+}^{s} \rho_l[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde\delta}'(u_\epsilon(\sigma,y))h(u_\epsilon(\sigma,y))\,\mathrm{dw}(\sigma)\\
&+\frac{1}{2}\int_{(s-\frac{2}{n})^+}^{s}\Big[\rho_l'[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k][\eta_{\tilde\delta}'(u_\epsilon(\sigma,y))]^2 + \rho_l[\eta_{\tilde\delta}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde\delta}''(u_\epsilon(\sigma,y))\Big]\\
&\qquad\qquad\times h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big\}.
\end{aligned}
$$

Therefore, using regularity of the stochastic integral with respect to parameters proposed by Kunita [57] and reminded in Appendix I Section 1

$$
\begin{aligned}
I_7 + J_8 \;=\;& E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1 \eta_\delta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\\
&\times\Big[\rho_l[\eta_{\tilde\delta}(u_\epsilon(s,y)) - k] - \rho_l[\eta_{\tilde\delta}(u_\epsilon(s-\frac{2}{n},y)) - k]\Big]\,\mathrm{dk\ dy\ ds}\\
=\;& -E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1 \eta_\delta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\\
&\times\frac{\partial}{\partial k}\{\cdots\}\,\mathrm{dk\ dy\ ds}\\
=\;& -E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1 \eta_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\\
&\times\{\cdots\}\,\mathrm{dk\ dy\ ds}\\
=\;& L_1 + L_2 + L_3.
\end{aligned}
$$

Let us separately evaluate the limits of L_1 and L_3. Then, we will add L_2 to a previous term. Using principally Cauchy-Schwarz inequality and the Itô isometry, we get

$$
\begin{aligned}
|L_1| \leq & \int_Q\int_{\mathbb{R}}\int_D\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\int_0^1\eta_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)dw(t)\Big]^2\Big\}^{\frac{1}{2}} \\
& \times\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\,\mathrm{d}\sigma\Big]^2\Big\}^{\frac{1}{2}}\,\mathrm{dk\,dx\,dy\,ds} \\
\leq & \int_Q\int_{\mathbb{R}}\int_D\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\int_0^1\eta_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)dw(t)\Big]^2\Big\}^{\frac{1}{2}} \\
& \times\frac{\sqrt{2}}{\sqrt{n}}\Big\{E\int_{(s-\frac{2}{n})^+}^s\Big[\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\Big]^2\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\,\mathrm{dk\,dx\,dy\,ds} \\
\leq & \int_Q\int_{\mathbb{R}}\int_D\rho_m(x-y)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1[\eta_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\varphi(s,y)\rho_n(t-s)]^2\,\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}} \\
& \times\frac{\sqrt{2}}{\sqrt{n}}\Big\{E\int_{(s-\frac{2}{n})^+}^s\Big[\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))-k]\underbrace{\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))}_{0\leq\cdot\leq 1}A_\epsilon(\sigma,y)\Big]^2\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\,\mathrm{dk\,dx\,dy\,ds} \\
\leq & \quad \frac{Cnl}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\rho_m(x-y)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1[\eta_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))]^2\,\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}} \\
& \times\Big\{E\int_{(s-\frac{2}{n})^+}^s 1_{\{-\frac{2}{l}\leq\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))-k\leq 0\}}A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\,\mathrm{dk\,dx\,dy\,ds} \\
\leq & \quad Cl\sqrt{n}\int_Q\int_{\mathbb{R}}E\int_{(s-\frac{2}{n})^+}^s\int_0^1[\eta_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))]^2\,\mathrm{d}\alpha\,\mathrm{dt\,dk\,dx\,ds} \\
& +Cl\sqrt{n}\int_Q\int_{\mathbb{R}}E\int_{(s-\frac{2}{n})^+}^s 1_{\{-\frac{2}{l}\leq\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))-k\leq 0\}}A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dk\,dy\,ds} \\
\leq & \quad \frac{Cl\sqrt{n}}{\delta^2}\int_Q E\int_{(s-\frac{2}{n})^+}^s\int_0^1\int_{\mathbb{R}}1_{\{\hat{\mathbf{u}}(p)-\delta\leq k\leq\hat{\mathbf{u}}(p)\}}\,\mathrm{dk}h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt\,dx\,ds} \\
& +Cl\sqrt{n}\int_Q E\int_{(s-\frac{2}{n})^+}^s\int_{\mathbb{R}}1_{\{\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))\leq k\leq\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))+\frac{2}{l}\}}\,\mathrm{dk}A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy\,ds} \\
\leq & \quad \frac{Cl\sqrt{n}}{\delta}\int_Q E\int_{(s-\frac{2}{n})^+}^s\int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt\,dx\,ds}+C\sqrt{n}\int_Q E\int_{(s-\frac{2}{n})^+}^s A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy\,ds} \\
\leq & \quad \frac{Cl}{\delta\sqrt{n}}E\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt\,dx}+C\frac{1}{\sqrt{n}}E\int_Q A_\epsilon^2(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy} \\
\underset{n}{\rightarrow} & \quad 0.
\end{aligned}
$$

Moreover, as $u_0^\epsilon \in L^4(D)$ one also has $u_\epsilon \in L^4(\Omega\times Q)$. We get by similar calculations

$$\begin{aligned}
|L_3| &\leq \frac{1}{2}\Big|E\Big\{\int_Q\int_{\mathbb{R}}\int_D\int_{(s-\frac{2}{n})^+}^s\int_0^1 \eta''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)dw(t)\\
&\quad\times\int_{(s-\frac{2}{n})^+}^s\Big[\rho'_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k][\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))]^2+\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta''_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\Big]\\
&\quad\times h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\Big\}\Big|\\
&\leq \frac{1}{2}\int_Q\int_{\mathbb{R}}\int_D\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\int_0^1\eta''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_n(t-s)dw(t)\Big]^2\Big\}^{\frac{1}{2}}\\
&\quad\times\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\Big[\rho'_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k][\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))]^2+\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta''_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\Big]\\
&\quad\times h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]^2\Big\}^{\frac{1}{2}}\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\leq \frac{C}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1\big[\eta''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\varphi(s,y)\rho_n(t-s)\big]^2\,\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}}\\
&\quad\times\Big\{E\int_{(s-\frac{2}{n})^+}^s\Big[\rho'_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k][\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))]^2+\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta''_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\Big]^2\\
&\quad\times h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\rho_m(x-y)\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\leq \frac{Cn}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\rho_m(x-y)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1\frac{1}{\delta^2}1_{\{\hat{\mathbf{u}}(p)-\delta\leq k\leq\hat{\mathbf{u}}(p)\}}h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}}\\
&\quad\times\Big\{E\int_{(s-\frac{2}{n})^+}^s[l^4+\frac{l^2}{\widetilde{\delta}^2}]1_{\{\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\leq k\leq\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))+\frac{2}{l}\}}h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\leq \frac{Cnl^2}{\widetilde{\delta}\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\rho_m(x-y)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1 1_{\{\hat{\mathbf{u}}(p)-\delta\leq k\leq\hat{\mathbf{u}}(p)\}}h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt}\\
&\quad+E\int_{(s-\frac{2}{n})^+}^s 1_{\{\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\leq k\leq\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))+\frac{2}{l}\}}h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big\}\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\leq \frac{Cl^2}{\widetilde{\delta}\sqrt{n}}[E\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{dp}+E\int_Q h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\,\mathrm{dy}]
\end{aligned}$$

$\underset{n}{\rightarrow} 0$ as above.

Thus, thanks to Fubini's theorem and the properties of Itô's integral, one has
$\lim_n I_7 + J_8$

$$\begin{aligned}
&= -\lim_n \int_Q \int_{\mathbb{R}} \int_D E \int_{(s-\frac{2}{n})^+}^{s} \int_0^1 \eta''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)dw(t)\\
&\qquad \times \int_{(s-\frac{2}{n})^+}^{s} \rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y)) - k]\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))h(u_\epsilon(\sigma,y))\,\mathrm{dw}(\sigma)\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&= -\lim_n \int_Q \int_{\mathbb{R}} \int_D E \int_{(s-\frac{2}{n})^+}^{s} \int_0^1 \eta''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(s,y)\rho_m(x-y)\rho_n(t-s)\\
&\qquad \times\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(t,y)) - k]\eta'_{\widetilde{\delta}}(u_\epsilon(t,y))h(u_\epsilon(t,y))\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
&\underset{l}{\to} -E\int_Q\int_D\int_0^1 \eta''_\delta[\hat{\mathbf{u}}(p)-\eta_{\widetilde{\delta}}(u_\epsilon(t,y))]h(\hat{\mathbf{u}}(p))\varphi(t,y)\rho_m(x-y)\eta'_{\widetilde{\delta}}(u_\epsilon(t,y))h(u_\epsilon(t,y))\,\mathrm{dy}\,\mathrm{dp}
\end{aligned}$$

Therefore, we get

$\lim_{l,n} I_6 + J_7 + I_7 + J_8$

$$\begin{aligned}
&= \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(x-y)\varphi(t,y)\eta''_\delta(\hat{\mathbf{u}}(p) - \eta_{\widetilde{\delta}}(u_\epsilon(t,y)))\\
&\qquad \times\Big[h^2(\hat{\mathbf{u}}(p)) - 2h(\hat{\mathbf{u}}(p))\eta'_{\widetilde{\delta}}(u_\epsilon(t,y))h(u_\epsilon(t,y)) + \{\eta'_\delta[u_\epsilon(t,y)]h[u_\epsilon(t,y)]\}^2\Big]\,\mathrm{dy}\,\mathrm{dp}\\
&-\frac{1}{2}E\int_Q\int_0^1\int_D \eta''_{\widetilde{\delta}}(u_\epsilon(t,y))\eta'_\delta[\hat{\mathbf{u}}(p) - \eta_{\widetilde{\delta}}(u_\epsilon(t,y))]h^2(u_\epsilon(t,y))\rho_m(x-y)\varphi(t,y)\,\mathrm{dy}\,\mathrm{dp}\\
&\leq \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(x-y)\varphi(t,y)\eta''_\delta(\hat{\mathbf{u}}(p) - \eta_{\widetilde{\delta}}(u_\epsilon(t,y)))\\
&\qquad\qquad \times\Big\{h(\hat{\mathbf{u}}(p)) - \eta'_{\widetilde{\delta}}[u_\epsilon(t,y)]h[u_\epsilon(t,y)]\Big\}^2\,\mathrm{dy}\,\mathrm{dp}\\
&\underset{\widetilde{\delta}}{\to} \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(x-y)\varphi(t,y)\eta''_\delta(\hat{\mathbf{u}}(p) - u^+_\epsilon(t,y))\\
&\qquad\qquad \times\{h(\hat{\mathbf{u}}(p)) - \mathrm{sgn}_0^+(u_\epsilon(t,y))\,h[u_\epsilon(t,y)]\}^2\,\mathrm{dy}\,\mathrm{dp}\\
&= \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(x-y)\varphi(t,y)\eta''_\delta(\hat{\mathbf{u}}(p) - u^+_\epsilon(t,y))\,\{h(\hat{\mathbf{u}}(p)) - h[u^+_\epsilon(t,y)]\}^2\,\mathrm{dy}\,\mathrm{dp}\\
&\underset{\delta}{\to} 0,
\end{aligned}$$

and $\limsup_{\delta,\widetilde{\delta}} \lim_{l,n} I_6 + J_7 + I_7 + J_8 \leq 0$.

Combining all the estimates yields

$$
\begin{aligned}
0 \;\leq\; & \int_D (\hat{u}_0 - u_0^+)^+ \varphi(0,x)\,\mathrm{dx} + E\int_Q\int_0^1\int_0^1 (\hat{\mathbf{u}}(p) - \mathbf{u}^+(t,x,\beta))^+ \partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\
& - E\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+(\hat{\mathbf{u}}(p) - \mathbf{u}^+(t,x,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(p)) - \vec{\mathbf{f}}(\mathbf{u}^+(t,x,\beta))]\nabla\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\
& + \lim_m \mathcal{L}(\varphi\theta_m).
\end{aligned}
$$

⋆ **A second inequality:**

After this first half of the global Kato inequality: an inequality for $(\hat{\mathbf{u}} - \mathbf{u}^+)^+$, as in Carillo [22], one would like to get the second half, an inequality for $(-\mathbf{u} - (-\hat{\mathbf{u}})^+)^+$, by the same type of arguments. It is easy to see that $-\mathbf{u}$ resp. $-\hat{\mathbf{u}}$ are measure-valued entropy solutions of $dv = \mathrm{div}\,\widetilde{\vec{\mathbf{f}}}(v)\,\mathrm{dt} + \widetilde{h}(v)dw$ with $\widetilde{\vec{\mathbf{f}}}(x) = -\vec{\mathbf{f}}(-x)$, $\widetilde{h}(x) = -h(-x)$ and initial condition $-u_0$ resp. $-\tilde{u}_0$. But unfortunately, the roles of $\mathbf{u}$ and $\hat{\mathbf{u}}$ in the proof of the first inequality are not symmetric and we can not simply replace $\mathbf{u}$ by $-\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}$ by $-\mathbf{u}$ in the preceding proof.

The arguments used in the proof are adapted to the situation where the solution $\mathbf{u}$ we compare with is obtained by viscous approximation and they do not work if it is not $\mathbf{u}$, but $\hat{\mathbf{u}}$ which is obtained by viscous approximation. Therefore, we have to prove the second half of the global Kato inequality using different arguments. To this end, choose the same partition of unity, φ in $\mathcal{D}^+([0,T[\times\mathbb{R}^d)$ with $\mathrm{supp}\varphi \subset B := \mathcal{B}_i$ for some $i \in \{1,\cdots,k\}$, ρ_n with $\mathrm{supp}\rho_n \subset]-2/n,0[$ and the shifted sequence of mollifiers ρ_m in $\mathbb{R}^d$. But the test-function will be $\varphi(t,x)\rho_m(y-x)$, *i.e.* a permutation of variables (t,x) and (s,y) is considered in comparison with the first inequality.

Unlike the first inequality, we are not able to impose a sign for k (non-positive values here) in the entropy inequality for $\hat{\mathbf{u}}(p)$, thus we have to use a test-function $\varphi\rho_m\rho_n$ that vanishes on the boundary. Then, in order to get a positive part at the limit, we need an additional term, the one given by $\widetilde{\mathcal{L}}$ in (3.5). Then, we use the Itô formula for u_ϵ with the regular nonlinear function $\psi^k_{\delta,\widetilde{\delta}}(x) = \eta_\delta(\eta_{\widetilde{\delta}}(x) - k)$. This corresponds to testing the equation with the Lipschitz-continuous function $(\psi^k)'_{\delta,\widetilde{\delta}}(u_\epsilon) = \eta'_\delta(\eta_{\widetilde{\delta}}(u_\epsilon) - k)\eta'_{\widetilde{\delta}}(u_\epsilon)$ that vanishes on the boundary.

By denoting again $\widetilde{\mathcal{B}}_k^l := \rho_l(\eta_{\widetilde{\delta}}(u_\epsilon(s,y)) - k)$ one has:

$$\begin{aligned}
&0 \leq E\int_Q\int_{\mathbb{R}}\int_D \check{\eta}_\delta(\hat{u}_0(x)-k)\varphi(0,x)\rho_n(-s)\rho_m(y-x)\,\mathrm{dx}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk\,dy\,ds}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \check{\eta}_\delta(\hat{\mathbf{u}}(p)-k)\partial_t\varphi(t,x)\,\rho_n(t-s)\rho_m(y-x)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk\,dy\,ds}\\
&+E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \check{\eta}_\delta(\hat{\mathbf{u}}(p)-k)\varphi(t,x)\,\partial_t\rho_n(t-s)\rho_m(y-x)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk\,dy\,ds}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\check{\eta}_\delta}(\hat{\mathbf{u}}(p),k)\nabla_x\varphi(t,x)\,\rho_m(y-x)\rho_n(t-s)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk\,dy\,ds}\\
&-E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\check{\eta}_\delta}(\hat{\mathbf{u}}(p),k)\nabla_x\rho_m(y-x)\rho_n(t-s)\varphi(t,x)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk\,dy\,ds}\\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\check{\eta}_\delta''(\hat{\mathbf{u}}(p)-k)\rho_m(y-x)\rho_n(t-s)\varphi(t,x)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk\,dy\,ds}\\
&+E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1 \check{\eta}_\delta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)\,\mathrm{dx}dw(t)\widetilde{\mathcal{B}}_k^l\,\mathrm{dk\,dy\,ds}\\
&=: \quad I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}$$

Moreover, the entropy formulation, with $k=0$ and any regular non-negative φ, yields

$$0 \quad\leq\quad \int_D \eta(-\hat{u}_0)\varphi(0)\,\mathrm{dx} + E\int_Q\int_0^1 \eta(-\hat{\mathbf{u}})\partial_t\varphi - F^{\check{\eta}}(\hat{\mathbf{u}},0)\nabla\varphi + \frac{1}{2}h^2(\hat{\mathbf{u}})\eta''(-\hat{\mathbf{u}})\varphi\,\mathrm{dp}.$$

The limit when η goes to $(.)^+$, since $h(0)=0$, gives

$$0 \quad\leq\quad \int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx} + E\int_Q\int_0^1 \hat{\mathbf{u}}^-\partial_t\varphi - F^{(.)^-}(\hat{\mathbf{u}},0)\nabla\varphi\,\mathrm{dp}.$$

Thus, the operator $\widetilde{\mathcal{L}}$ denoted by

$$\widetilde{\mathcal{L}}(\varphi) \quad=\quad \int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx} + E\int_Q\int_0^1 \hat{\mathbf{u}}^-\partial_t\varphi - F^{(.)^-}(\hat{\mathbf{u}},0)\nabla\varphi\,\mathrm{dp} \tag{3.5}$$

is linear and non-negative over $\mathcal{D}([0,T[\times\mathbb{R}^d)$. Since $0 \leq \varphi\theta_m \leq \varphi\theta_{m+1} \leq \varphi$, $\widetilde{\mathcal{L}}(\varphi\theta_m)$ has a limit in $[0,+\infty[$ when $m \to +\infty$ such that :

$$\begin{aligned}
\lim_m \widetilde{\mathcal{L}}(\varphi\theta_m) &= \int_D \hat{u}_0^- \varphi(0)\,\mathrm{dx} + E\int_Q\int_0^1 \hat{\mathbf{u}}^- \partial_t\varphi\,\mathrm{dp} - E\int_Q\int_0^1 F^{(.)^-}(\hat{\mathbf{u}},0)\nabla\varphi\,\mathrm{dp} \\
&\quad - \lim_m E\int_Q\int_0^1 \varphi F^{(.)^-}(\hat{\mathbf{u}},0)\nabla\theta_m\,\mathrm{dp} \\
&= \widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3 + \lim_m \widetilde{I}_4.
\end{aligned}$$

On the other hand, denoting again $\check{\mathcal{A}}_k^l := \rho_l(k - \hat{\mathbf{u}})$, since u_ϵ is a viscous solution, the Itô formula applied to $\int_D \eta_\delta(\eta_{\widetilde{\delta}}(u_\epsilon) - k)\rho_n(t-s)\rho_m(y-x)\varphi(t,x)\,\mathrm{dy}\,\mathrm{ds}$ yields

$$
\begin{aligned}
0 \leq\ & E\int_Q\int_{\mathbb{R}}\int_D \eta_\delta[\eta_{\widetilde{\delta}}(u_0^\epsilon(y)) - k]\varphi(t,x)\underbrace{\rho_n(t)}_{=0}\rho_m(y-x)\,\mathrm{dy}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&+E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[\eta_{\widetilde{\delta}}(u_\epsilon(s,y)) - k]\underbrace{\partial_s\varphi(t,x)}_{=0}\rho_n(t-s)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&+E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[\eta_{\widetilde{\delta}}(u_\epsilon(s,y)) - k]\varphi(t,x)\partial_s\rho_n(t-s)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q \eta'_\delta(\eta_{\widetilde{\delta}}(u_\epsilon) - k)\eta'_{\widetilde{\delta}}(u_\epsilon)\nabla_y u_\epsilon(s,y)\underbrace{\nabla_y\varphi(t,x)}_{=0} \\
&\qquad\qquad\times\rho_n(t-s)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&-\epsilon E\int_Q\int_{\mathbb{R}}\int_Q [\eta''_\delta(\eta_{\widetilde{\delta}}(u_\epsilon) - k)[\eta'_{\widetilde{\delta}}(u_\epsilon)]^2 + \eta'_\delta(\eta_{\widetilde{\delta}}(u_\epsilon) - k)\eta''_{\widetilde{\delta}}(u_\epsilon)]|\nabla_y u_\epsilon(s,y)|^2\varphi(t,x) \\
&\qquad\qquad\times\rho_n(t-s)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^{\eta_\delta[\eta_{\widetilde{\delta}}(.)-k]}(u_\epsilon(s,y),k)\underbrace{\nabla_y\varphi(t,x)}_{=0}\rho_m(y-x)\rho_n(t-s)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&-E\int_Q\int_{\mathbb{R}}\int_Q F^{\eta_\delta[\eta_{\widetilde{\delta}}(.)-k]}(u_\epsilon(s,y),k)\nabla_y\rho_m(y-x)\rho_n(t-s)\varphi(t,x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q [(\eta'_{\widetilde{\delta}}(u_\epsilon(s,y)))^2\eta''_\delta(\eta_{\widetilde{\delta}}(u_\epsilon(s,y)) - k) + \eta''_{\widetilde{\delta}}(u_\epsilon(s,y))\eta'_\delta(\eta_{\widetilde{\delta}}(u_\epsilon(s,y) - k)] \\
&\qquad\qquad\times h^2(u_\epsilon(s,y))\rho_m(y-x)\rho_n(t-s)\varphi(t,x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
&+E\int_Q\int_{\mathbb{R}}\int_0^T\int_D \eta'_\delta(\eta_{\widetilde{\delta}}(u_\epsilon) - k)\eta'_{\widetilde{\delta}}(u_\epsilon)h(u_\epsilon(s,y))\rho_n(t-s)\,\mathrm{dy}dw(s) \\
&\qquad\qquad\times\varphi(t,x)\rho_m(y-x)\int_0^1 \check{\mathcal{A}}_k^l\,\mathrm{dk}\,\mathrm{dp} \\
=:\ & J_1 + J_2 + J_3 + J_{4,1} + J_{4,2} + J_5 + J_6 + J_7 + J_8.
\end{aligned}
$$

Now it remains to combine appropriately the integrals I_i and J_k to pass to the limit.

$$\begin{aligned}
J_1+I_1-\widetilde{I}_1 &= E\int_Q\int_{\mathbb{R}}\int_D \check{\eta}_\delta(\hat{u}_0(x)-k)\varphi(0,x)\rho_n(-s)\rho_m(y-x)\,\mathrm{dx}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}-\int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx}\\
&\underset{n}{\to} E\int_D\int_{\mathbb{R}}\int_D \check{\eta}_\delta(\hat{u}_0(x)-k)\varphi(0,x)\rho_m(y-x)\,\mathrm{dx}\rho_l(\eta_{\widetilde{\delta}}(u_0^\epsilon(y))-k)\,\mathrm{dk}\,\mathrm{dy}-\int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx}\\
&\underset{l}{\to} E\int_D\int_D \check{\eta}_\delta(\hat{u}_0(x)-\eta_{\widetilde{\delta}}(u_0^\epsilon(y)))\varphi(0,x)\rho_m(y-x)\,\mathrm{dx}\,\mathrm{dy}-\int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx}
\end{aligned}$$

$$\begin{aligned}
J_1+I_1-\widetilde{I}_1 \underset{\widetilde{\delta}}{\to}\ & E\int_D\int_D \check{\eta}_\delta(\hat{u}_0(x)-(u_0^\epsilon(y))^+)\varphi(0,x)\rho_m(y-x)\,\mathrm{dx}\,\mathrm{dy}-\int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx}\\
\underset{\delta}{\to}\ & E\int_D\int_D [(u_0^\epsilon(y))^+-\hat{u}_0(x)]^+\varphi(0,x)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{dx}-\int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx}\\
\underset{\epsilon}{\to}\ & E\int_D\int_D [(u_0(y))^+-\hat{u}_0(x)]^+\varphi(0,x)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{dx}-\int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx}\\
\underset{m}{\to}\ & E\int_D [(u_0)^+-\hat{u}_0]^+\varphi(0,.)\,\mathrm{dy}-\int_D \hat{u}_0^-\varphi(0)\,\mathrm{dx}.
\end{aligned}$$

Similar calculations lead to

$$\begin{aligned}
J_2+I_2-\widetilde{I}_2 &= -E\int_Q\int_0^1 \hat{\mathbf{u}}^-\partial_t\varphi\,\mathrm{dp}\\
&\quad +E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \check{\eta}_\delta(\hat{\mathbf{u}}(p)-k)\partial_t\varphi(t,x)\ \rho_n(t-s)\rho_m(y-x)\,\mathrm{dp}\widetilde{\mathcal{B}}_k^l\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
\xrightarrow[n,\cdots,m]{}\ & E\int_Q\int_0^1\int_0^1 [(u(t,x,\beta))^+-\hat{u}(p)]^+\partial_t\varphi(t,x)\ \mathrm{d}\alpha\ \mathrm{d}\beta\ \mathrm{dx}\ \mathrm{dt}-E\int_Q\int_0^1 \hat{\mathbf{u}}^-\partial_t\varphi\ \mathrm{dp}.
\end{aligned}$$

$$\begin{aligned}
I_3+J_3 &= E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \check{\eta}_\delta(\hat{\mathbf{u}}(p)-k)\varphi(t,x)\ \partial_t\rho_n(t-s)\rho_m(y-x)\ \mathrm{dp}\\
&\qquad\qquad \times\rho_l(\underbrace{\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k}_{\tau})\ \mathrm{dk}\ \mathrm{dy}\ \mathrm{ds}\\
&\quad +E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k]\varphi(t,x)\partial_s\rho_n(t-s)\rho_m(y-x)\ \mathrm{dy}\ \mathrm{ds}\\
&\qquad\qquad \times\int_0^1 \rho_l(\underbrace{k-\hat{\mathbf{u}}(p)}_{\tau})\ \mathrm{dk}\ \mathrm{dp}\\
&= E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-\hat{\mathbf{u}}(p)-\tau)\varphi(t,x)\,\underbrace{\partial_t\rho_n(t-s)}_{=-\partial_s\rho_n(t-s)}\rho_m(y-x)\,\mathrm{dp}\rho_l(\tau)\,\mathrm{d}\tau\,\mathrm{dy}\,\mathrm{ds}\\
&\quad +E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta[\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-\hat{\mathbf{u}}(p)-\tau]\varphi(t,x)\partial_s\rho_n(t-s)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1 \rho_l(\tau)\,\mathrm{d}\tau\,\mathrm{dp}\\
&= 0.
\end{aligned}$$

$$
\begin{aligned}
J_{4,1}+J_{4,2} \;=\; & -\epsilon E\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta'(\eta_{\widetilde{\delta}}(u_\epsilon)-k)\eta_{\widetilde{\delta}}'(u_\epsilon)\nabla_y u_\epsilon(s,y)\nabla_y\varphi(t,x) \\
& \qquad\qquad \times\rho_n(t-s)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk}\,\mathrm{dp} \\
& -\epsilon E\int_Q\int_{\mathbb{R}}\int_Q [\eta_\delta''(\eta_{\widetilde{\delta}}(u_\epsilon)-k)[\eta_{\widetilde{\delta}}'(u_\epsilon)]^2+\eta_\delta'(\eta_{\widetilde{\delta}}(u_\epsilon)-k)\eta_{\widetilde{\delta}}''(u_\epsilon)] \\
& \times|\nabla_y u_\epsilon(s,y)|^2\varphi(t,x)\rho_n(t-s)\rho_m(y-x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk}\,\mathrm{dp} \\
\leq\; & 0.
\end{aligned}
$$

$$
\begin{aligned}
J_5+I_4-\widetilde{I}_3 \;=\; & -E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\check{\eta}_\delta}(\hat{\mathbf{u}}(p),k)\nabla_x\varphi(t,x)\;\rho_m(y-x)\rho_n(t-s)\,\mathrm{dp} \\
& \qquad\qquad \times\rho_l(\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& +E\int_Q\int_0^1 F^{(.)^-}(\hat{\mathbf{u}},0)\nabla\varphi\,\mathrm{dp} \\
=\; & E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\int_k^{\hat{\mathbf{u}}(p)}\vec{\mathbf{f}}'(\sigma)\eta_\delta'(k-\sigma)\,\mathrm{d}\sigma\nabla_x\varphi(t,x)\;\rho_m(y-x)\rho_n(t-s)\,\mathrm{dp} \\
& \qquad\qquad \times\rho_l(\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
& -E\int_Q\int_0^1\vec{\mathbf{f}}(-\hat{\mathbf{u}}^-)\nabla\varphi\,\mathrm{dp} \\
\underset{l,n}{\longrightarrow}\; & E\int_D\int_Q\int_0^1\int_{\eta_{\widetilde{\delta}}(u_\epsilon(t,y))}^{\hat{\mathbf{u}}(p)}\vec{\mathbf{f}}'(\sigma)\eta_\delta'[\eta_{\widetilde{\delta}}(u_\epsilon(t,y))-\sigma]\,\mathrm{d}\sigma\nabla_x\varphi(t,x)\;\rho_m(y-x)\,\mathrm{dp}\,\mathrm{dy} \\
& -E\int_Q\int_0^1\vec{\mathbf{f}}(-\hat{\mathbf{u}}^-)\nabla\varphi\,\mathrm{dp} \\
\underset{\widetilde{\delta},\delta}{\longrightarrow}\; & E\int_D\int_Q\int_0^1\int_{u_\epsilon^+(t,y)}^{\hat{\mathbf{u}}(p)}\vec{\mathbf{f}}'(\sigma)\mathrm{sgn}_0^+[u_\epsilon^+(t,y)-\sigma]\,\mathrm{d}\sigma\nabla_x\varphi(t,x)\;\rho_m(y-x)\,\mathrm{dp}\,\mathrm{dy} \\
& -E\int_Q\int_0^1\vec{\mathbf{f}}(-\hat{\mathbf{u}}^-)\nabla\varphi\,\mathrm{dp} \\
=\; & E\int_D\int_Q\int_0^1\mathrm{sgn}_0^+[u_\epsilon^+(t,y)-\hat{\mathbf{u}}(p)][\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))-\vec{\mathbf{f}}(u_\epsilon^+(t,y))]\nabla_x\varphi(t,x)\rho_m(y-x)\,\mathrm{dp}\,\mathrm{dy} \\
& -E\int_Q\int_0^1\vec{\mathbf{f}}(-\hat{\mathbf{u}}^-)\nabla\varphi\,\mathrm{dp}.
\end{aligned}
$$

And if we denote $q=(t,y,\beta)$,

$$\begin{aligned}
\lim_{l,n,\widetilde{\delta},\delta} J_5+I_4-\widetilde{I}_3 \\
&\underset{\epsilon}{\rightarrow} -E\int_D\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+[\mathbf{u}^+(q)-\hat{\mathbf{u}}(p)][\vec{\mathbf{f}}(\mathbf{u}^+(q))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))]\nabla_x\varphi(t,x)\rho_m(y-x)\,\mathrm{dp}\,\mathrm{dy}\,\mathrm{d}\beta \\
&\qquad -E\int_Q\int_0^1 \vec{\mathbf{f}}(-\hat{\mathbf{u}}^-)\nabla\varphi\,\mathrm{dp} \\
&\underset{m}{\rightarrow} -E\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+[\mathbf{u}^+(t,x,\beta)-\hat{\mathbf{u}}(p)][\vec{\mathbf{f}}(\mathbf{u}^+(t,x,\beta))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))]\nabla_x\varphi(t,x)\,\mathrm{dp}\,\mathrm{d}\beta \\
&\qquad -E\int_Q\int_0^1 \vec{\mathbf{f}}(-\hat{\mathbf{u}}^-)\nabla\varphi\,\mathrm{dp}.
\end{aligned}$$

$$\begin{aligned}
I_5+J_6 &= -E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 F^{\check{\eta}_\delta}(\hat{\mathbf{u}}(p),k)\nabla_x\rho_m(y-x)\rho_n(t-s) \\
&\qquad \times\varphi(t,x)\,\mathrm{dp}\rho_l(\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad -E\int_Q\int_{\mathbb{R}}\int_Q\int_k^{u_\epsilon} \vec{\mathbf{f}}'(\sigma)\eta_\delta'[\eta_{\widetilde{\delta}}(\sigma)-k]\eta_{\widetilde{\delta}}'(\sigma)\,\mathrm{d}\sigma\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk} \\
&\qquad \times\nabla_y\rho_m(y-x)\rho_n(t-s)\varphi(t,x)\,\mathrm{dy}\,\mathrm{ds}\,\mathrm{dp} \\
&= E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1\int_k^{\hat{\mathbf{u}}(p)} \vec{\mathbf{f}}'(\sigma)\eta_\delta'(k-\sigma)\,\mathrm{d}\sigma\nabla_x\rho_m(y-x)\rho_n(t-s) \\
&\qquad \times\varphi(t,x)\,\mathrm{dp}\rho_l(\eta_{\widetilde{\delta}}(u_\epsilon(s,y))-k)\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds} \\
&\quad -E\int_Q\int_{\mathbb{R}}\int_Q\int_k^{u_\epsilon} \vec{\mathbf{f}}'(\sigma)\eta_\delta'[\eta_{\widetilde{\delta}}(\sigma)-k]\eta_{\widetilde{\delta}}'(\sigma)\,\mathrm{d}\sigma\nabla_y\rho_m(y-x)\rho_n(t-s) \\
&\qquad \times\varphi(t,x)\,\mathrm{dy}\,\mathrm{ds}\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk}\,\mathrm{dp} \\
&\underset{l,n}{\longrightarrow} E\int_Q\int_D\int_0^1\int_{\eta_{\widetilde{\delta}}(u_\epsilon(t,y))}^{\hat{\mathbf{u}}(p)} \eta_\delta'(\eta_{\widetilde{\delta}}(u_\epsilon(t,y))-\sigma)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy} \\
&\quad -E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}(p)}^{u_\epsilon(t,y)} \eta_{\widetilde{\delta}}'(\sigma)\eta_\delta'(\eta_{\widetilde{\delta}}(\sigma)-\hat{\mathbf{u}}(p))\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_y\rho_m(y-x)\varphi(t,x)\,\mathrm{dy}\,\mathrm{dp} \\
&\underset{\widetilde{\delta},\delta}{\longrightarrow} -E\int_Q\int_D\int_0^1\int_{u_\epsilon^+(t,y)}^{\hat{\mathbf{u}}(p)} \mathrm{sgn}_0^+\left(u_\epsilon^+(t,y)-\sigma\right)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy} \\
&\quad +E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}(p)}^{u_\epsilon(t,y)} \mathrm{sgn}_0^+(\sigma)\mathrm{sgn}_0^+\left(\sigma^+-\hat{\mathbf{u}}(p)\right)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_y\rho_m(y-x)\varphi(t,x)\,\mathrm{dy}\,\mathrm{dp}
\end{aligned}$$

$$
\begin{aligned}
&= -E\int_Q\int_D\int_0^1\int_{u_\epsilon^+(t,y)}^{\hat{\mathbf{u}}(p)} \operatorname{sgn}_0^+\left(u_\epsilon^+(t,y)-\sigma\right)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy}\\
&+E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}(p)}^{u_\epsilon^+(t,y)} \operatorname{sgn}_0^+(\sigma)\operatorname{sgn}_0^+\left(\sigma^+-\hat{\mathbf{u}}(p)\right)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_y\rho_m(y-x)\varphi(t,x)\,\mathrm{dy}\,\mathrm{dp}\\
&= E\int_Q\int_D\int_0^1 \operatorname{sgn}_0^+\left(u_\epsilon^+(t,y)-\hat{\mathbf{u}}\right)\left[\vec{\mathbf{f}}(u_\epsilon^+(t,y))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy}\\
&+E\int_Q\int_0^1\int_D\int_{\hat{\mathbf{u}}^+(p)}^{u_\epsilon^+(t,y)} \operatorname{sgn}_0^+\left(\sigma-\hat{\mathbf{u}}(p)\right)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma\nabla_y\rho_m(y-x)\varphi(t,y)\,\mathrm{dy}\,\mathrm{dp}\\
&= E\int_Q\int_D\int_0^1 \operatorname{sgn}_0^+\left(u_\epsilon^+(t,y)-\hat{\mathbf{u}}\right)\left[\vec{\mathbf{f}}(u_\epsilon^+(t,y))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy}\\
&+E\int_Q\int_0^1\int_D \operatorname{sgn}_0^+\left(u_\epsilon^+-\hat{\mathbf{u}}(p)\right)\left[\vec{\mathbf{f}}(u_\epsilon^+(t,y))-\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))\right]\nabla_y\rho_m(y-x)\varphi(t,x)\,\mathrm{dy}\,\mathrm{dp}\\
&=^{\dagger} E\int_Q\int_D\int_0^1 \operatorname{sgn}_0^+\left(u_\epsilon^+(t,y)-\hat{\mathbf{u}}\right)\left[\vec{\mathbf{f}}(u_\epsilon^+(t,y))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy}\\
&-E\int_Q\int_0^1\int_D \operatorname{sgn}_0^+\left(u_\epsilon^+-\hat{\mathbf{u}}(p)\right)\left[\vec{\mathbf{f}}(u_\epsilon^+(t,y))-\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))\right]\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dy}\,\mathrm{dp}\\
&= E\int_Q\int_D\int_0^1 \operatorname{sgn}_0^+\left(u_\epsilon^+(t,y)-\hat{\mathbf{u}}\right)\left[\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy}\\
&= E\int_Q\int_D\int_0^1 \left[\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\rho_m(y-x)\varphi(t,x)\,\mathrm{dp}\,\mathrm{dy}\\
&= E\int_Q\int_0^1 \left[\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\theta_m(x)\varphi(t,x)\,\mathrm{dp}.
\end{aligned}
$$

Note that $\operatorname{sgn}_0^+\left(u_\epsilon^+(t,y)-\hat{\mathbf{u}}\right)\left[\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right] = \vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p)) = 0$ when $\hat{\mathbf{u}} \geq 0$, *i.e.*, when $\operatorname{sgn}_0^+\left(u_\epsilon^+(t,y)-\hat{\mathbf{u}}\right) = 0$; then, added to $-\widetilde{I}_4$, we get

$$
\begin{aligned}
&E\int_Q\int_0^1\left[\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\theta_m(x)\varphi(t,x)\,\mathrm{dp} - E\int_Q\int_0^1 \varphi F^{(\cdot)^-}(\hat{\mathbf{u}},0)\nabla\theta_m\,\mathrm{dp}\\
&= E\int_Q\int_0^1\left[\vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))-\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))\right]\nabla_x\theta_m(x)\varphi(t,x)\,\mathrm{dp} + E\int_Q\int_0^1 \varphi\vec{\mathbf{f}}(-\hat{\mathbf{u}}^-)\nabla\theta_m\,\mathrm{dp} = 0.
\end{aligned}
$$

† since $\nabla_x\rho_m(y-x) = -\nabla_y\rho_m(y-x)$

$$I_6 + J_7 = \frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta_\delta''(k-\hat{\mathbf{u}}(p))\rho_m(y-x)\rho_n(t-s)\varphi(t,x)\,\mathrm{dp}$$
$$\times\rho_l(\eta_{\tilde{\delta}}(u_\epsilon(s,y))-k)\,\mathrm{dk\,dy\,ds}$$

$$+\frac{1}{2}E\int_Q\int_{\mathbb{R}}\int_Q[(\eta_\delta')^2(u_\epsilon(s,y))\eta_\delta''(\eta_{\tilde{\delta}}(u_\epsilon(s,y))-k)+\eta_{\tilde{\delta}}''(u_\epsilon(s,y))\eta_\delta'(\eta_{\tilde{\delta}}(u_\epsilon(s,y)-k)]$$
$$\times h^2(u_\epsilon(s,y))\rho_m(y-x)\rho_n(t-s)\varphi(t,x)\,\mathrm{dy\,ds}\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk\,dp}$$

$$\underset{l,n}{\longrightarrow}\frac{1}{2}E\int_Q\int_D\int_0^1 h^2(\hat{\mathbf{u}}(p))\eta_\delta''(\eta_{\tilde{\delta}}(u_\epsilon(t,y))-\hat{\mathbf{u}}(p))\rho_m(y-x)\varphi(t,x)\,\mathrm{dy\,dp}$$
$$+\frac{1}{2}E\int_Q\int_0^1\int_D h^2(u_\epsilon(t,y))\rho_m(y-x)\varphi(t,x)$$
$$\times\Big[(\eta_\delta')^2[u_\epsilon(t,y))]\eta_\delta''(\eta_{\tilde{\delta}}(u_\epsilon(t,y))-\hat{\mathbf{u}}(p)+\eta_{\tilde{\delta}}''(u_\epsilon(t,y))\eta_\delta'(\eta_{\tilde{\delta}}(u_\epsilon(t,y)-\hat{\mathbf{u}}(p))\Big]\,\mathrm{dy\,dp}$$

Thanks to the properties of the Itô integral, one has that

$$I_7 + J_8 = E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1\check{\eta}_\delta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)\,\mathrm{dx}dw(t)$$
$$\times\rho_l(\eta_{\tilde{\delta}}(u_\epsilon(s,y))-k)\,\mathrm{dk\,dy\,ds}$$

$$+E\int_Q\int_{\mathbb{R}}\int_t^T\int_D\eta_\delta'(\eta_{\tilde{\delta}}(u_\epsilon)-k)\eta_{\tilde{\delta}}'(u_\epsilon)h(u_\epsilon(s,y))\varphi(t,x)\rho_m(y-x)\rho_n(t-s)\,\mathrm{dy}dw(s)$$
$$\times\int_0^1\rho_l(k-\hat{\mathbf{u}}(p))\,\mathrm{dk\,dp}$$

$$= E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1\check{\eta}_\delta'(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)\,\mathrm{dx}dw(t)$$
$$\times\Big[\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s,y))-k]-\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s-\frac{2}{n},y))-k]\Big]\mathrm{dk\,dy\,ds}.$$

As $du_\epsilon = [\epsilon\Delta u_\epsilon + \operatorname{div}\vec{\mathbf{f}}(u_\epsilon)]\,\mathrm{dt} + h(u_\epsilon)dw = A_\epsilon\,\mathrm{dt} + h(u_\epsilon)dw$, by Itô's formula with $u_\epsilon(\sigma,y) = u_\epsilon$

$$\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s,y)) - k] - \rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s-\frac{2}{n},y)) - k]$$

$$\begin{aligned}
= {} & \int_{(s-\frac{2}{n})^+}^{s} \rho_l'[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\,\mathrm{d}\sigma \\
+ & \int_{(s-\frac{2}{n})^+}^{s} \rho_l'[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))h(u_\epsilon(\sigma,y))\,\mathrm{dw}(\sigma) \\
+ & \frac{1}{2}\int_{(s-\frac{2}{n})^+}^{s} \Big[\rho_l''[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k][\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))]^2 + \rho_l'[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde{\delta}}''(u_\epsilon(\sigma,y))\Big] \times h^2(u_\epsilon)\,\mathrm{d}\sigma \\
= {} & -\frac{\partial}{\partial k}\Bigg\{ \int_{(s-\frac{2}{n})^+}^{s} \rho_l[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\,\mathrm{d}\sigma \\
+ & \int_{(s-\frac{2}{n})^+}^{s} \rho_l[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))h(u_\epsilon(\sigma,y))\,\mathrm{dw}(\sigma) \\
+ & \frac{1}{2}\int_{(s-\frac{2}{n})^+}^{s} \Big[\rho_l'[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k][\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))]^2 + \rho_l[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y)) - k]\eta_{\tilde{\delta}}''(u_\epsilon(\sigma,y))\Big] \times h^2(u_\epsilon)\,\mathrm{d}\sigma\Bigg\} \\
= {} & -\frac{\partial}{\partial k}\{\dots\}
\end{aligned}$$

Therefore, using regularity of the stochastic integral with respect to parameters proposed by Kunita [57] and reminded in Appendix I Section 1

$$\begin{aligned}
I_7 + J_8 = {} & E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1 \check{\eta}_\delta'(\hat{\mathbf{u}}(p) - k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)\,\mathrm{dx}dw(t) \\
& \times\Big[\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s,y)) - k] - \rho_l[\eta_{\tilde{\delta}}(u_\epsilon(s-\frac{2}{n},y)) - k]\Big]\,\mathrm{dk\,dy\,ds} \\
= {} & -E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1 \check{\eta}_\delta'(\hat{\mathbf{u}}(p) - k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)\,\mathrm{dx}dw(t) \\
& \times\frac{\partial}{\partial k}\{\cdots\}\,\mathrm{dk\,dy\,ds} \\
= {} & -E\int_Q\int_{\mathbb{R}}\int_0^T\int_D\int_0^1 \check{\eta}_\delta''(\hat{\mathbf{u}}(p) - k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)\,\mathrm{dx}dw(t) \\
& \times\{\cdots\}\,\mathrm{dk\,dy\,ds} \\
= {} & L_1 + L_2 + L_3.
\end{aligned}$$

Let us separately evaluate the limits of L_1 and L_3, then add L_2 to a previous term.

Using principally Cauchy-Schwarz inequality and the Itô's isometry, we get

$$
\begin{aligned}
|L_1| \leq & \int_Q\int_{\mathbb{R}}\int_D\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\int_0^1\check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)dw(t)\Big]^2\Big\}^{\frac{1}{2}}\\
&\times\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\,\mathrm{d}\sigma\Big]^2\Big\}^{\frac{1}{2}}\mathrm{dk}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds}\\
\leq & \int_Q\int_{\mathbb{R}}\int_D\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\int_0^1\check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)dw(t)\Big]^2\Big\}^{\frac{1}{2}}\\
&\times\frac{\sqrt{2}}{\sqrt{n}}\Big\{E\int_{(s-\frac{2}{n})^+}^s\Big[\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))A_\epsilon(\sigma,y)\Big]^2\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\mathrm{dk}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds}\\
\leq & \int_Q\int_{\mathbb{R}}\int_D\rho_m(y-x)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1\Big[\check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\varphi(t,x)\rho_n(t-s)\Big]^2\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}}\\
&\times\frac{\sqrt{2}}{\sqrt{n}}\Big\{E\int_{(s-\frac{2}{n})^+}^s\Big[\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\underbrace{\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))}_{0\leq\cdot\leq1}A_\epsilon(\sigma,y)\Big]^2\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\mathrm{dk}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds}\\
\leq & \frac{Cnl}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\rho_m(y-x)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1\Big[\check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\Big]^2\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}}\\
&\times\Big\{E\int_{(s-\frac{2}{n})^+}^s 1_{\{-\frac{2}{l}\leq\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k\leq0\}}A^2_\epsilon(\sigma,y)\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\,\mathrm{dk}\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{ds}\\
\leq & Cl\sqrt{n}\int_Q\int_{\mathbb{R}}E\int_{(s-\frac{2}{n})^+}^s\int_0^1\Big[\check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\Big]^2\mathrm{d}\alpha\,\mathrm{dt}\,\mathrm{dk}\,\mathrm{dx}\,\mathrm{ds}\\
&+Cl\sqrt{n}\int_Q\int_{\mathbb{R}}E\int_{(s-\frac{2}{n})^+}^s 1_{\{-\frac{2}{l}\leq\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k\leq0\}}A^2_\epsilon(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
\leq & \frac{Cl\sqrt{n}}{\delta^2}\int_Q E\int_{(s-\frac{2}{n})^+}^s\int_0^1\int_{\mathbb{R}}1_{\{\hat{\mathbf{u}}(p)-\delta\leq k\leq\hat{\mathbf{u}}(p)\}}\,\mathrm{dk}h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{ds}\\
&+Cl\sqrt{n}\int_Q E\int_{(s-\frac{2}{n})^+}^s\int_{\mathbb{R}}1_{\{\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\leq k\leq\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))+\frac{2}{l}\}}\,\mathrm{dk}A^2_\epsilon(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy}\,\mathrm{ds}\\
\leq & \frac{Cl\sqrt{n}}{\delta}\int_Q E\int_{(s-\frac{2}{n})^+}^s\int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt}\,\mathrm{dx}\,\mathrm{ds}+C\sqrt{n}\int_Q E\int_{(s-\frac{2}{n})^+}^s A^2_\epsilon(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy}\,\mathrm{ds}\\
\leq & \frac{Cl}{\delta\sqrt{n}}E\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt}\,\mathrm{dx}+C\frac{1}{\sqrt{n}}E\int_Q A^2_\epsilon(\sigma,y)\,\mathrm{d}\sigma\,\mathrm{dy}\\
\underset{n}{\rightarrow}\ & 0.
\end{aligned}
$$

Moreover, as $u_0^\epsilon \in L^4(D)$ one also has $u_\epsilon \in L^4(\Omega \times Q)$. We get by similar calculations

$$\begin{aligned}
|L_3| \leq\ & \frac{1}{2}\Big|E\int_Q\int_{\mathbb{R}}\int_D\int_{(s-\frac{2}{n})^+}^s\int_0^1 \check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)dw(t)\\
&\times\int_{(s-\frac{2}{n})^+}^s\Big[\rho'_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k][\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))]^2+\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta''_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\Big]\\
&\times h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\Big|\\
\leq\ & \frac{1}{2}\int_Q\int_{\mathbb{R}}\int_D\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\int_0^1\check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_n(t-s)dw(t)\Big]^2\Big\}^{\frac{1}{2}}\\
&\times\Big\{E\Big[\int_{(s-\frac{2}{n})^+}^s\Big[\rho'_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k][\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))]^2\\
&+\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\check{\eta}''_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\Big]h^2(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big]^2\Big\}^{\frac{1}{2}}\rho_m(y-x)\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
\leq\ & \frac{C}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1\big[\check{\eta}''_\delta(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\varphi(t,x)\rho_n(t-s)\big]^2\,\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}}\\
&\times\Big\{E\int_{(s-\frac{2}{n})^+}^s\Big[\rho'_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k][\eta'_{\widetilde{\delta}}(u_\epsilon(\sigma,y))]^2\\
&+\rho_l[\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta''_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\Big]^2h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\rho_m(y-x)\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
\leq\ & \frac{Cn}{\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\rho_m(y-x)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1\frac{1}{\delta^2}1_{\{\hat{\mathbf{u}}(p)-\delta\leq k\leq\hat{\mathbf{u}}(p)\}}h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt}\Big\}^{\frac{1}{2}}\\
&\times\Big\{E\int_{(s-\frac{2}{n})^+}^s[l^4+\frac{l^2}{\widetilde{\delta}^2}]1_{\{\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\leq k\leq\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))+\frac{2}{l}\}}h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big\}^{\frac{1}{2}}\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
\leq\ & \frac{Cnl^2}{\widetilde{\delta}\sqrt{n}}\int_Q\int_{\mathbb{R}}\int_D\rho_m(y-x)\Big\{E\int_{(s-\frac{2}{n})^+}^s\int_0^1 1_{\{\hat{\mathbf{u}}(p)-\delta\leq k\leq\hat{\mathbf{u}}(p)\}}h^2(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\,\mathrm{dt}\\
&\qquad+E\int_{(s-\frac{2}{n})^+}^s 1_{\{\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))\leq k\leq\eta_{\widetilde{\delta}}(u_\epsilon(\sigma,y))+\frac{2}{l}\}}h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\Big\}\,\mathrm{dx}\,\mathrm{dk}\,\mathrm{dy}\,\mathrm{ds}\\
\leq\ & \frac{Cl^2}{\widetilde{\delta}\sqrt{n}}[E\int_Q\int_0^1 h^2(\hat{\mathbf{u}}(p))\,\mathrm{dp}+E\int_Q h^4(u_\epsilon(\sigma,y))\,\mathrm{d}\sigma\,\mathrm{dy}]
\end{aligned}$$

$\underset{n}{\rightarrow} 0$ as above.

Thus, thanks to Fubini's theorem and the properties of Itô's integral, one has

$$\lim_n I_7+J_8 = -\lim_n \int_Q\int_{\mathbb{R}}\int_D E\int_{(s-\frac{2}{n})^+}^{s}\int_0^1 \check{\eta}_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)dw(t)$$

$$\times\int_{(s-\frac{2}{n})^+}^{s}\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(\sigma,y))-k]\eta_{\tilde{\delta}}'(u_\epsilon(\sigma,y))h(u_\epsilon(\sigma,y))\,\mathrm{dw}(\sigma)\,\mathrm{dx\,dk\,dy\,ds}$$

$$= -\lim_n \int_Q\int_{\mathbb{R}}\int_D E\int_{(s-\frac{2}{n})^+}^{s}\int_0^1 \check{\eta}_\delta''(\hat{\mathbf{u}}(p)-k)h(\hat{\mathbf{u}}(p))\,\mathrm{d}\alpha\varphi(t,x)\rho_m(y-x)\rho_n(t-s)$$

$$\times\rho_l[\eta_{\tilde{\delta}}(u_\epsilon(t,y))-k]\eta_{\tilde{\delta}}'(u_\epsilon(t,y))h(u_\epsilon(t,y))\,\mathrm{dt\,dx\,dk\,dy\,ds}$$

$$\underset{l}{\to}\ -\int_Q\int_D E\int_0^1 \check{\eta}_\delta''[\hat{\mathbf{u}}(p)-\eta_{\tilde{\delta}}(u_\epsilon(t,y))]h(\hat{\mathbf{u}}(p))\varphi(t,x)\rho_m(y-x)$$

$$\times\eta_{\tilde{\delta}}'(u_\epsilon(t,y))h(u_\epsilon(t,y))\,\mathrm{dp\,dy}$$

$$= -\int_Q\int_D E\int_0^1 \eta_\delta''[\eta_{\tilde{\delta}}(u_\epsilon(t,y))-\hat{\mathbf{u}}(p)]h(\hat{\mathbf{u}}(p))\varphi(t,x)\rho_m(y-x)$$

$$\times\eta_{\tilde{\delta}}'(u_\epsilon(t,x))h(u_\epsilon(t,y))\,\mathrm{dp\,dy}.$$

Therefore, we get

$$\lim_{l,n} I_6+J_7+I_7+J_8$$

$$= \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(y-x)\varphi(t,x)\eta_\delta''(\eta_{\tilde{\delta}}(u_\epsilon(t,y))-\hat{\mathbf{u}}(p))$$

$$\times\left[h^2(\hat{\mathbf{u}}(p))-2h(\hat{\mathbf{u}}(p))\eta_{\tilde{\delta}}'(u_\epsilon(t,y))h(u_\epsilon(t,y))+\{\eta_\delta'[u_\epsilon(t,y)]h[u_\epsilon(t,y)]\}^2\right]\,\mathrm{dy\,dp}$$

$$+\frac{1}{2}E\int_Q\int_0^1\int_D \eta_{\tilde{\delta}}''(u_\epsilon(t,y))\eta_\delta'[\hat{\mathbf{u}}(p)-\eta_{\tilde{\delta}}(u_\epsilon(t,y))]h^2(u_\epsilon(t,y))\rho_m(y-x)\varphi(t,x)\,\mathrm{dy\,dp}$$

$$= \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(y-x)\varphi(t,x)\eta_\delta''(\eta_{\tilde{\delta}}(u_\epsilon(t,y))-\hat{\mathbf{u}}(p))$$

$$\times\left\{h(\hat{\mathbf{u}}(p))-\eta_{\tilde{\delta}}'[u_\epsilon(t,y)]h[u_\epsilon(t,y)]\right\}^2\,\mathrm{dy\,dp}$$

$$+\frac{1}{2}E\int_Q\int_0^1\int_D \eta_{\tilde{\delta}}''(u_\epsilon(t,y)-0)\eta_\delta'[\hat{\mathbf{u}}(p)-\eta_{\tilde{\delta}}(u_\epsilon(t,y))]$$

$$\times[h(u_\epsilon(t,y))-h(0)]^2\rho_m(y-x)\varphi(t,x)\,\mathrm{dy\,dp}$$

$$\underset{\tilde{\delta}}{\to}\ \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(y-x)\varphi(t,x)\eta_\delta''(u_\epsilon^+(t,y)-\hat{\mathbf{u}}(p))$$

$$\times\{h(\hat{\mathbf{u}}(p))-\mathrm{sgn}_0^+(u_\epsilon(t,y))\,h[u_\epsilon(t,y)]\}^2\,\mathrm{dy\,dp}$$

$$= \frac{1}{2}E\int_Q\int_D\int_0^1 \rho_m(y-x)\varphi(t,x)\eta_\delta''(u_\epsilon^+(t,y)-\hat{\mathbf{u}}(p))\,\{h(\hat{\mathbf{u}}(p))-h[u_\epsilon^+(t,y)]\}^2\,\mathrm{dy\,dp}$$

$$\underset{\delta}{\to}\ 0$$

and $\limsup_{\delta,\widetilde{\delta}} \lim_{l,n} I_6 + J_7 + I_7 + J_8 \le 0$. Combining all the estimates yields :

$$\begin{aligned}
0 \ \le\ & \int_D [(u_0^+ - \hat{u}_0)^+ - \hat{u}_0^-]\varphi(0,x)\,\mathrm{dx} - E\int_Q \int_0^1 \vec{\mathbf{f}}(-\hat{\mathbf{u}}^-(p))\nabla\varphi(t,x)\,\mathrm{dp} + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m)\\
&+E\int_Q\int_0^1\int_0^1 [(\mathbf{u}^+(t,x,\beta) - \hat{\mathbf{u}}(p))^+ - \hat{\mathbf{u}}^-]\partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}\\
&-E\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+(\mathbf{u}^+(t,x,\beta) - \hat{\mathbf{u}}(p))[\vec{\mathbf{f}}(\mathbf{u}^+(t,x,\beta)) - \vec{\mathbf{f}}(\hat{\mathbf{u}}(p))]\nabla\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)},
\end{aligned}$$

i.e.

$$\begin{aligned}
0 \ \le\ & \int_D (u_0 - \hat{u}_0^+)^+\,\varphi(0,x)\,\mathrm{dx} + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m)\\
&+E\int_Q\int_0^1\int_0^1 (\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}^+(p))^+\,\partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}\\
&-E\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+(\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}^+(p))[\vec{\mathbf{f}}(\mathbf{u}(t,x,\beta)) - \vec{\mathbf{f}}(\hat{\mathbf{u}}^+(p))]\nabla\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}.
\end{aligned}$$

Note that $-\hat{\mathbf{u}}$, resp. $-\mathbf{u}$, is a measure-valued entropy solution of $dv = \mathrm{div}\,\widetilde{\vec{\mathbf{f}}}(v)\,\mathrm{dt} + \widetilde{h}(v)dw$ with $\widetilde{\vec{\mathbf{f}}}(x) = -\vec{\mathbf{f}}(-x)$, $\widetilde{h}(x) = -h(-x)$ and the initial condition $-\hat{u}_0$, resp. $-u_0$, and by assuming that $-\mathbf{u}$ is obtained as the limit of $-u_\epsilon$, solution of the viscous problem $dv = [\epsilon\Delta v + \mathrm{div}\,\widetilde{\vec{\mathbf{f}}}(v)]\,\mathrm{dt} + \widetilde{h}(v)dw$. Consequently, replacing $\hat{\mathbf{u}}$ by $-\hat{\mathbf{u}}$ and $\mathbf{u}$ by $-\mathbf{u}$ in the above inequality, we get the estimate (where one denotes by $x^- = (-x)^+$)

$$\begin{aligned}
0 \ \le\ & \int_D (-u_0 - \hat{u}_0^-)^+\,\varphi(0,x)\,\mathrm{dx} + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m)\\
&+E\int_Q\int_0^1\int_0^1 (-\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}^-(p))^+\,\partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}\\
&+E\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+(-\mathbf{u}(t,x,\beta) - \hat{\mathbf{u}}^-(p))[\vec{\mathbf{f}}(\mathbf{u}(t,x,\beta)) - \vec{\mathbf{f}}(-\hat{\mathbf{u}}^-(p))]\nabla\varphi(t,x)d\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}.
\end{aligned}$$

As, for any a, b, $-\mathrm{sgn}_0^+(b-a)\,[\vec{\mathbf{f}}(b)-\vec{\mathbf{f}}(a)] = -\mathrm{sgn}_0^+(b-a^+)\,[\vec{\mathbf{f}}(b)-\vec{\mathbf{f}}(a^+)] + \mathrm{sgn}_0^+(-a-b^-)\,[\vec{\mathbf{f}}(a)-\vec{\mathbf{f}}(-b^-)]$ and $(b-a^+)^+ + (-a-b^-)^+ = (b-a)^+$, we find that

$$\begin{aligned} 0 \ \leq\ & \int_D (\hat{u}_0 - u_0)^+ \varphi(0,x)\,\mathrm{dx} \\ & + E\int_Q\int_0^1\int_0^1 (\hat{\mathbf{u}}(p)-\mathbf{u}(t,x,\beta))^+ \partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\ & - E\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+(\hat{\mathbf{u}}(p)-\mathbf{u}(t,x,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))-\vec{\mathbf{f}}(\mathbf{u}(t,x,\beta))]\nabla\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\ & + \lim_m \mathcal{L}(\varphi\theta_m) + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m). \end{aligned}$$

Now, let $\varphi \in \mathcal{D}^+([0,T[\times B)$, then $\varphi = \theta_n\varphi + (1-\theta_n)\varphi$ and $\theta_n\varphi \in \mathcal{D}^+([0,T[\times D)$ for n sufficiently large.
Then, applying the local Kato inequality with $\theta_n\varphi$ and the global one with $(1-\theta_n)\varphi$, yields

$$\begin{aligned} 0 \ \leq\ & \int_D (\hat{u}_0 - u_0)^+ \varphi(0,x)\,\mathrm{dx} \\ & + E\int_Q\int_0^1\int_0^1 (\hat{\mathbf{u}}(p)-\mathbf{u}(t,x,\beta))^+ \partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\ & - E\int_Q\int_0^1\int_0^1 \mathrm{sgn}_0^+(\hat{\mathbf{u}}(p)-\mathbf{u}(t,x,\beta))[\vec{\mathbf{f}}(\hat{\mathbf{u}}(p))-\vec{\mathbf{f}}(\mathbf{u}(t,x,\beta))]\nabla\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)} \\ & + \lim_m \mathcal{L}(\varphi(1-\theta_n)\theta_m) + \lim_m \widetilde{\mathcal{L}}(\varphi(1-\theta_n)\theta_m). \end{aligned}$$

As $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are linear operators and $\theta_n\theta_m = \theta_n$ if m is large, one gets that

$$\begin{aligned} & \lim_m \mathcal{L}(\varphi(1-\theta_n)\theta_m) + \lim_m \widetilde{\mathcal{L}}(\varphi(1-\theta_n)\theta_m) \\ = \ & \lim_m \mathcal{L}(\varphi\theta_m) - \mathcal{L}(\varphi\theta_n) + \lim_m \widetilde{\mathcal{L}}(\varphi\theta_m) - \widetilde{\mathcal{L}}(\varphi\theta_n) \end{aligned}$$

and $\lim_n \lim_m \mathcal{L}(\varphi(1-\theta_n)\theta_m) + \lim_n \lim_m \widetilde{\mathcal{L}}(\varphi(1-\theta_n)\theta_m) = 0$.

Thus, the global Kato inequality holds for any $\varphi \in \mathcal{D}^+([0,T[\times B)$, and by using a partition of unity, it holds for any $\varphi \in \mathcal{D}^+([0,T[\times\mathbb{R}^d)$.
Especially, if $\varphi \in \mathcal{D}^+([0,T[)$, we obtain

$$0 \ \leq\ \int_D (\hat{u}_0 - u_0)^+ \varphi(0,x)\,\mathrm{dx} + E\int_Q\int_0^1\int_0^1 (\hat{\mathbf{u}}(t,x,\alpha)-\mathbf{u}(t,x,\beta))^+ \partial_t\varphi(t,x)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d(t,x)}$$

and the Kato inequality holds.

3.3 Uniqueness and stability

Proposition 3.7 *The measure-valued entropy solution of (0.1) with initial data u_0 in $L^2(D)$ is unique. Moreover, it is the unique entropy solution and any entropy solution is obtained as the limit of a sequence of solutions to some viscous problems.*

Proof. If $\hat{u}_0 = u_0$, the sequel of the proof of the Kato inequality yields

$$E\int_0^T\int_D\int_0^1\int_0^1 (\hat{\mathbf{u}}(t,x,\alpha)-\mathbf{u}(t,x,\beta))^+ \, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d(t,x)} \le 0.$$

As mentioned above, though the roles played by $\hat{\mathbf{u}}$, $\mathbf{u}$ are not symmetric, it is easy to note that $-\hat{\mathbf{u}}$, resp. $-\mathbf{u}$, is a measure-valued entropy solution of $dv = \operatorname{div}\widetilde{\vec{\mathbf{f}}}(v)\,\mathrm{dt} + \widetilde{h}(v)dw$ with $\widetilde{\vec{\mathbf{f}}}(x) = -\vec{\mathbf{f}}(-x)$, $\widetilde{h}(x) = -h(-x)$ and the same initial condition $-u_0$. Moreover, if $\mathbf{u}$ is obtained by viscous approximation, then so is $-\mathbf{u}$. Consequently, by replacing $\hat{\mathbf{u}}$ by $-\hat{\mathbf{u}}$ and $\mathbf{u}$ by $-\mathbf{u}$ in the above inequality, we get the estimate

$$\begin{aligned} & E\int_0^T\int_D\int_0^1\int_0^1 (-\hat{\mathbf{u}}(p)+\mathbf{u}(t,x,\beta))^+ \, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d(t,x)} \le 0, \\ \textit{i.e.}\quad & E\int_0^T\int_D\int_0^1\int_0^1 (\hat{\mathbf{u}}(p)-\mathbf{u}(t,x,\beta))^- \, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d(t,x)} \le 0. \end{aligned}$$

This proves that $\hat{\mathbf{u}}(p) = \mathbf{u}(t,x,\beta)$ for a.e. $(t,x) \in Q$ and $(\alpha,\beta) \in]0,1[^2$ and the result holds. □

A direct consequence is the following contraction principle:

Proposition 3.8 *If u_1, u_2 are entropy solutions of (0.1) corresponding to initial data $u_{1,0}, u_{2,0} \in L^2(D)$, respectively, then, for any time t,*

$$E\int_D (u_1-u_2)^+ \,\mathrm{dx} \le \int_D (u_{1,0}-u_{2,0})^+ \,\mathrm{dx}.$$

Remark 3.9

. *As mentioned in* Bauzet-Vallet-Wittbold *[14], if one is concerned by the modeling of fluid flow in porous media and if u has to be a saturation, then one gets that $0 \le u \le 1$ as soon as it is the case for the initial condition u_0 and if the support of h is contained in $[0,1]$. Indeed, thanks to the Itô formula, this maximum principle is direct for the viscous solutions u_ϵ, then it is conserved at the limit for u.*

. *The operator $T : u_0 \mapsto u$ can be extended from $L^2(D)$ to $L^1(D)$.*

Chapter V

On a splitting method for a stochastic conservation law with Dirichlet boundary condition & numerical experiments

In this chapter, we present a numerical scheme for a first-order hyperbolic equation of nonlinear type perturbed by a multiplicative noise. The problem is set in a bounded domain of $\mathbb{R}^d$ and with homogeneous Dirichlet boundary condition. Using a splitting method, we are able to show the existence of an approximate solution. The result of convergence of such a sequence is based on the work of Bauzet-Vallet-Wittbold [13] presented in Chapter IV, where the authors used the concept of measure-valued solution and Kruzhkov's entropy formulation to show existence and uniqueness of the weak entropy solution. Then, we propose numerical experiments by applying this scheme to the stochastic Burgers' equation in the one-dimensional case.

1 Introduction

We are interested in the formal stochastic nonlinear conservation law of type:

$$\mathrm{d}u + \operatorname{div}\vec{\mathbf{f}}(u)\,\mathrm{d}t = h(u)dw \text{ in }]0,T[\times D\times\Omega, \tag{1.1}$$

for an initial condition u_0 and with homogeneous Dirichlet boundary condition.
One assumes that D is a bounded domain of $\mathbb{R}^d$ $(d \geqslant 1)$ with a Lipschitz boundary if $d \geqslant 2$, T a positive number, $Q =]0,T[\times D$ and that $w = \{w_t, \mathcal{F}_t; 0 \le t \le T\}$ is a 1-D standard adapted Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Moreover we suppose that :

H$_1$: $\vec{\mathbf{f}} : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz-continuous function with $\vec{\mathbf{f}}(0) = \vec{\mathbf{0}}$.

H_2: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$.

H_3: $u_0 \in L^\infty(D) \cap BV(D)^\dagger$.

H_4: There exists $M > 0$ such that $\text{supp}h \subset [-M, M]$.

1.1 Former results

Only few papers have been devoted to the study of numerical experiments for stochastic conservation laws. Let us cite the paper of Holden-Risebro [49] where an operator splitting method is proposed to prove the existence of pathwise weak solutions to the Cauchy problem

$$du + f(u)_x dt = g(u) dw \quad \text{in } \Omega \times]0, T[\times \mathbb{R}.$$

The operator-splitting approach has also been studied in [15] by Bensoussan-Glowinski-Raşcanu, where the authors are interested in approximating stochastic partial differential equations of parabolic type by some iterative schemes suggested by the Lie-Trotter product formulas. The convergence of the operator-splitting method is based on the continuity of the considered operator, which does not hold in our case.

Concerning the Cauchy problem for a conservation law with multiplicative noise, Feng-Nualart [41] introduced a notion of strong entropy solution in order to prove the uniqueness of the entropy solution. Using the vanishing viscosity and compensated compactness arguments, the authors established the existence of strong entropy solutions only in the 1D case.

In the recent paper of Chen-Ding-Karlsen [27], the authors proposed a generalization of the work of Feng-Nualart: they considered a multi-dimensional stochastic balance law:

$$\partial_t(t, \mathbf{x}) + \nabla.\mathbf{f}(u(t, \mathbf{x})) = \sigma(u(t, \mathbf{x}))\partial_t W(t), \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,$$

with initial data $u(0, \mathbf{x}) = u_0(\mathbf{x})$. They identified a class of nonlinear balance laws for which uniform spatial BV bound for vanishing viscosity approximations can be achieved. Moreover they established temporal equicontinuity in L^1 of the approximations, uniformly in the viscosity coefficient. In details, they proved that this stochastic problem is well-posed by using a uniform spatial BV-bound. They also proved the existence of strong stochastic entropy solutions in $L^p \cap BV$ and developed a "continuous dependence" theory for stochastic entropy solutions in BV, which can be used to derive an error estimate for the vanishing viscosity method. Various

†where $BV(D)$ denotes the set of integrable functions with bounded variation on D.

further generalizations of their results are discussed.
In the work of Kröeker-Rohde [55] the authors are interested in a method of handling the finite volume schemes for the approximate solution of the Cauchy problem for an hyperbolic balance law with random noise. For a class of monotone numerical fluxes they establish the pathwise convergence of a semi-discrete finite volume solution towards a stochastic entropy solution. The main tool is a stochastic version of the compensated compactness approach. It avoids the use of a maximum principle and total-variation estimates.

Using a kinetic formulation, Debussche-Vovelle [32] proved the first complete well-posedness result for multi-dimensional scalar conservation laws set in a d-dimensional torus and driven by a general multiplicative noise. As an extension of this work, in a recent paper Hofmanová [48] presents a Bhatnagar-Gross-Krook-like approximation to this problem. Using the stochastic characteristics method the author establishes the existence of an approximate solution and show its convergence to the kinetic solution of [32].

By the way of the theory of Young measure-valued solutions, Bauzet-Vallet-Wittbold [14] proved a result of existence and uniqueness of the solution to the multi-dimensional Cauchy problem in $L^2(\Omega\times Q)$. Since the method consists in comparing a weak measure-valued entropy solution to a regular one (the viscous solution in our case) and not to a strong one, the authors could consider very general assumptions on the data.

In Bauzet-Vallet-Wittbold [13] the authors investigated the Dirichlet Problem for equation (1.1) with the initial condition u_0 in $L^2(D)$ and under assumptions H_1 and H_2. They proved a result of existence and uniqueness of the stochastic entropy solution by using the concept of measure-valued solutions and Kruzhkov's semi-entropy formulations. In the present work, we will use their theoretical results.

1.2 Goal of the study

Our aim is to revisit and generalize the splitting method introduced by Holden-Risebro [49] for the same conservation law but in a bounded domain of $\mathbb{R}^d$ and prove that the pathwise weak solution they obtained is an entropy weak solution and that the whole sequence of approximation converges. The idea is to complete the work of Bauzet-Vallet-Wittbold [13] presented in Chapter IV by numerical experiments using their theoretical study. For technical reasons, we need to assume the additional hypothesis on the data H_3 and H_4.
The chapter is organized as follows. In Section 2, we recall for convenience the notion of stochastic entropy (resp. measure-valued entropy) solution for (1.1), in particular the way we consider the boundary conditions in Chapter IV, and our main result. In Section 3, we present

a splitting method for the stochastic conservation law (1.1) which allows us to construct an approximate solution. Then, we introduce an entropic formulation satisfied by such a sequence. Using Young measure compactness arguments, one shows that this approximate solution tends to a measure-valued entropy solution of (1.1). The theoretical study of (1.1) in Chapter IV allows us to conclude that this limit is the unique stochastic entropy solution of (1.1). Finally, in Section 4, we propose a numerical application with the stochastic Burgers' equation in the one-dimensional case. We introduce the scheme used and present simulations of solution obtained for different initial conditions by the free software *Scilab*.

Remark 1.1 *Recall that weak and entropy solutions are not smooth solutions, thus, trace has to be understood in a weak way. We followed the approach of* CARILLO *[23] which consists in formulating the boundary condition implicitly via global integral entropy inequalities involving the semi-Kruzhkov entropies.*

1.3 Notations

Consider $BV(D)$ the set of integrable functions with bounded variation on D endowed with the norm $\|v\|_{BV(D)} = \|v\|_{L^1(D)} + TV_x(v)$, where $TV_x(v)$ denotes the total variation of v on D (see EVANS-GARIEPY [39] for example).

For a given separable Banach space X we denote by $N^2_w(0,T,X)$ the space of the predictable X-valued processes. This space is $L^2(]0,T[\times\Omega,X)$ endowed with the product measure $dt\otimes dP$ and the predictable σ-field $\mathcal{P}_T$: *i.e.* the σ-field generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $]s,t]\times A$ for any $A\in\mathcal{F}_s$ (we refer the reader to DA PRATO-ZABCZYK [30]).

We denote by $\mathcal{E}^+$ the set of nonnegative convex functions η in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x\mapsto x^+$ such that $\eta(x)=0$ if $x\leq 0$ and that there exists $\delta>0$ such that $\eta'(x)=1$ if $x>\delta$.

Then, one denotes by $\mathcal{E}^-$ the set $\{\check{\eta}:=\eta(-\cdot),\ \eta\in\mathcal{E}^+\}$; and, for the definition of the entropy inequality, consider the sets

$$\begin{aligned}
\mathbb{A}^+ &= \{(k,\varphi,\eta)\in\mathbb{R}\times\mathcal{D}^+(\mathbb{R}^{d+1})\times\mathcal{E}^+,\ k<0\Rightarrow\varphi\in\mathcal{D}^+([0,T]\times D)\},\\
\mathbb{A}^- &= \{(k,\varphi,\eta),\ (-k,\varphi,\check{\eta})\in\mathbb{A}^+\},\\
\mathbb{A} &= \mathbb{A}^+\cup\mathbb{A}^-,
\end{aligned}$$

and the function $F^\eta(a,b) = \displaystyle\int_b^a \eta'(\sigma - b)\vec{\mathbf{f}}'(\sigma)\,\mathrm{d}\sigma$, defined for $\eta \in \mathcal{E}^+ \cup \mathcal{E}^-$.

2 Existence and uniqueness result

Let us recall the definitions and the result introduced in Chapter IV. For any function u of $\mathcal{N}_w^2(0,T,L^2(D))$, any real k and any regular function η, denote P-a.s. in Ω by $\mu_{\eta,k}$, the distribution in $\mathbb{R}^{d+1}$, defined by

$$\begin{aligned}\mu_{\eta,k}(\varphi) = &\int_D \eta(u_0 - k)\varphi(0)dx + \int_Q \eta(u-k)\partial_t\varphi + F^\eta(u,k)\nabla\varphi dxdt \\ &+ \int_0^T \int_D \eta'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q h^2(u)\eta''(u-k)\varphi dxdt.\end{aligned}$$

Definition 2.1 ***Entropy solution***
A function u of $\mathcal{N}_w^2(0,T,L^2(D))$ is an entropy solution of the stochastic conservation law (1.1) with the initial condition $u_0 \in L^2(D)$ if $u \in L^\infty(0,T,L^2(\Omega,L^2(D)))$ and

$$\forall (k,\varphi,\eta) \in \mathbb{A}, \quad 0 \le \mu_{\eta,k}(\varphi) \quad P-a.s. \tag{2.1}$$

Remark 2.2 *Any entropy solution is, P-a.s., a solution in the sense of distributions in Q to*

$$\partial_t\left[u - \int_0^t h(u)dw(s)\right] + \ div\, \vec{\mathbf{f}}(u) = 0.$$

For technical reasons, as in Chapter IV, we also need to consider a generalized notion of entropy solution. In fact, in a first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we will be able to deduce the existence of an entropy solution in the sense of Definition 2.1.

Definition 2.3 ***Measure-valued entropy solution***
A function $u \in N_w^2(0,T,L^2(D\times]0,1[)) \cap L^\infty(]0,T[,L^2(\Omega\times D\times]0,1[))$ is a (Young) measure-valued entropy solution of (1.1) with the initial condition $u_0 \in L^2(D)$ if

$$\forall (k,\varphi,\eta) \in \mathbb{A}, \quad 0 \le \int_0^1 \mu_{\eta,k}(\varphi)\,\mathrm{d}\alpha \quad P-a.s. \tag{2.2}$$

And the main result of Chapter IV is

Theorem 2.4
Under assumptions $H_1 - H_2 - H_3$ there exists a unique measure-valued entropy solution in the sense of Definition 2.3 and this solution is obtained by viscous approximation.
It is the unique entropy solution in the sense of Definition 2.1.
If u_1, u_2 are entropy solutions of (1.1) corresponding to initial conditions $u_{1,0}, u_{2,0} \in L^2(D)$, respectively, then, for any t in $[0,T]$

$$E \int_D (u_1(t) - u_2(t))^+ \,\mathrm{dx} \leq \int_D (u_{1,0} - u_{2,0})^+ \,\mathrm{dx}.$$

3 Splitting method

3.1 Introduction

Our aim is to approximate Problem (1.1) under the assumptions H_1 to H_5. As proposed by Holden-Risebro in [49], we introduce a method to split the effect of the source term, this technique allows us to construct a sequence to approximate the solution of (1.1). In few words, this approach is based on considering the equation in two parts, solving first a stochastic differential equation, and then using the obtained solution as an initial condition for a scalar hyperbolic conservation law without source term. As an extension of [49], we propose in this paper to generalize their estimates on the approximate sequence to the bounded d-dimensional case, in the idea of Chen-Ding-Karlsen [27] concerning BV estimates. Following the idea and notations introduced in [49] we define here two operators for s, $t \in [0,T]$.
Let $R(t,s)$ be the operator which takes a number $\overline{u}$ to the solution u at time t of the stochastic differential equation, $\forall t \in [s,T]$

$$\begin{cases} du(t) &= h(u)dw(t) \\ u(t=s) &= \overline{u}, \end{cases} \tag{3.1}$$

i.e $u(t) = R(t,s)\overline{u} = \overline{u} + \int_s^t h(u)dw$.
And $S(t-s)$ denotes the operator which takes an initial function $u(x,s)$ at time s to the weak entropy solution u at time t of the conservation law

$$\begin{cases} \partial_t u + \operatorname{div} \vec{\mathbf{f}}(u) &= 0 & \text{in }]0,T[\times D, \\ ''u &= 0'' & \text{on }]0,T[\times \partial D, \\ u(t=s) &= u(x,s), \end{cases} \tag{3.2}$$

i.e $u(x,t) = S(t-s)u(x,s)$.

Remark 3.1 *Let us precise that thanks to the assumptions on the data, both R and S are well-defined.*

Let us introduce for the sequel of the chapter, useful results of such operators.

Lemma 3.2 *Consider $s \in [0,T]$. Then P-a.s and for all $t \in [s,T]$, $R(t,s)$ will take $[-M,M]$ into itself and be the identity outside this interval, where $M > 0$ is defined in H_4.*

Proof. Consider the process u defined for all $t \in [s,T]$ by $u(t) = R(t,s)u(s)$. Applying the Itô formula to a regular function Ψ independent of the time variable t, vanishing in $[-M,M]$ and increasing outside this interval, one gets, P-a.s:

$$\begin{aligned}\Psi(u(t)) &= \Psi(u(s)) + \int_s^t \underbrace{\Psi_t(u(\sigma))}_{=0} d\sigma + \int_s^t \underbrace{\Psi_x(u(\sigma))h(u(\sigma))}_{=0} dw(\sigma) \\ &+\frac{1}{2}\int_s^t \underbrace{\Psi_{xx}(u(\sigma))h^2(u(\sigma))}_{=0} d\sigma, \qquad \forall t \in [s,T].\end{aligned}$$

Consider $\omega \in \tilde{\Omega}$, where $\tilde{\Omega}$ is a full measure subset of Ω and $t \in [s,T]$. Thus, if $u(s,\omega) \in [-M,M]$, $\Psi(u(s,\omega)) = 0 = \Psi(u(t,\omega))$ and $u(t,\omega) \in [-M,M]$. Else, $\Psi(u(t,\omega)) = \Psi(u(s,\omega))$, by injectivity of Ψ in $\mathbb{R} - [-M,M]$, $u(t,\omega) = u(s,\omega)$ and $R(t,s) = I_d$. □

Lemma 3.3 *Consider $s \in [0,T]$, $v_0 \in L^2(\Omega \times D)$ a $\mathcal{F}_s$-measurable process such that*

$$E[TV_x(v_0)] < \infty.$$

Define the process v for all $t \in [s,T]$ by $v(t) = R(s,t)v_0$. Thus for all $t \in [s,T]$

$$E\|v(t)\|_{BV(D)} \leqslant E\|v_0\|_{BV(D)}.$$

Remark 3.4 *Let us mention that using the lower semi continuity property and the positivity of the total variation TV_x on $L^1(D)$, for all v in $L^1(\Omega \times D)$, $E[TV_x(v)]$ has a sense.*

Proof. Consider $s \in [0,T]$ and let $v_0 \in L^2(\Omega \times D)$ be a $\mathcal{F}_s$-measurable process with $E[TV_x(v_0)] < \infty$. Define for all $t \in [s,T]$ $v(t) = R(s,t)v_0$ and consider η_δ a regular approximation of the absolute value function with η''_δ a mollifier sequence satisfying $\operatorname{supp}(\eta''_\delta) \subset [-\delta,\delta]$, $\delta > 0$. Applying

Itô's formula with the process v and the function η_δ, one gets by taking the integral over D and the expectation, for every $t \in [s,T]$

$$E\int_D \eta_\delta(v(t))dx = E\int_D \eta_\delta(v_0)dx + \frac{1}{2}E\int_D\int_s^t \eta_\delta''(v)h^2(v)d\sigma dx.$$

Passing to the limit on δ to 0 to get for every $t \in [s,T]$

$$E\|v(t)\|_{L^1(D)} = E\|v_0\|_{L^1(D)}.$$

Let us recall a classical result on approximation of BV functions in the deterministic setting, referring the reader to Evans-Gariepy [39]. For every $w \in BV(D)$, there exists an approximate sequence $(w_\epsilon)_\epsilon \subset \mathcal{C}^\infty(D) \cap BV(D)$ such that w_ϵ converges to w in $L^1(D)$. One is also able to assert the following inequalities, for every $\epsilon > 0$

$$\begin{aligned} \|w_0^\epsilon\|_{L^1(D)} &\leqslant \|w_0\|_{L^1(D)} + 1, \\ TV_x(w_0^\epsilon) &\leqslant TV_x(w_0) + 4\epsilon. \end{aligned} \tag{3.3}$$

For technical reasons in the present proof, one needs to work with Hilbert space, thus using the same notation we consider by a regularization process that w_0^ϵ is also in $W^{1,2}(D)^\dagger$ and satisfies, for every $\epsilon > 0$

$$\begin{aligned} \|w_0^\epsilon\|_{L^1(D)} &\leqslant \|w_0\|_{L^1(D)} + 1 + \epsilon, \\ TV_x(w_0^\epsilon) &\leqslant TV_x(w_0) + 5\epsilon. \end{aligned}$$

Notice that in our case, $v_0 \in L^1(\Omega \times D)$ and $E[TV_x(v_0)] < \infty$, thus P-a.s, $v_0 \in L^1(D) \cap BV(D)$. P-a.s, the deterministic regularization process holds and $v_0^\epsilon \to v_0$ P-a.s in $L^1(D)$. Then this convergence holds strongly in $L^1(\Omega \times D)$ using (3.3) and the dominated convergence theorem. By Remark 3.4, we finally have:

$$E[TV_x(v_0^\epsilon)] \leqslant E[TV_x(v_0)] + 5\epsilon. \tag{3.4}$$

Now we need estimate on $\partial_{x_i} v_\epsilon$ in order to obtain BV estimate for v. Let us define for all t in $[s,T]$ $v_\epsilon(t) = R(s,t)v_0^\epsilon$. Applying Itô's formula to the process $d(v_\epsilon - v) = [h(v_\epsilon) - h(v)]dw$ and

$^\dagger W^{1,2}(D)$ denotes the set of functions u in $L^2(D)$ such that $\partial_{x_i} u \in L^2(D)$, for all $i \in \{1,\ldots,d\}$.

the function η_δ, taking the integral over D and the expectation, we obtain for every $t \in [s,T]$

$$E\int_D \eta_\delta(v_\epsilon - v)(t)dx = E\int_D \eta_\delta(v_0^\epsilon - v_0)dx + \frac{1}{2}E\int_D\int_s^t \eta_\delta''(v_\epsilon - v)[h(v_\epsilon) - h(v)]^2 d\sigma dx.$$

Passing to the limit on δ to 0 to get for every $t \in [s,T]$

$$E\|(v_\epsilon - v)(t)\|_{L^1(D)} = E\|v_0^\epsilon - v_0\|_{L^1(D)}.$$

Thus, for every $t \in [s,T]$, $v_\epsilon(t) \to v(t)$ in $L^1(\Omega \times D)$.
As P-a.s and for all $t \in [0,T]$, $v_\epsilon(t) = v_\epsilon(0) + \int_0^t h(v_\epsilon)dw$ in $W^{1,2}(D)$, using the linear-continuity of the derivation operator $\partial_{x_i} : W^{1,2}(D) \to L^2(D)$ for all $i \in \{1,\dots,d\}$ and the chain-rule derivation formula, we get for all $i \in \{1,\dots,d\}$ $\partial_{x_i} v_\epsilon(0) = \partial_{x_i} v_0^\epsilon$ and:

$$\begin{aligned}\partial_{x_i} v_\epsilon(t) &= \partial_{x_i} v_\epsilon(0) + \partial_{x_i}\int_0^t h(v_\epsilon)dw \\ &= \partial_{x_i} v_\epsilon(0) + \int_0^t h'(v_\epsilon)\partial_{x_i} v_\epsilon dw, \text{ in } L^2(D).\end{aligned}$$

Applying Itô's formula with such a process and the function η_δ to get that, after taking the integral over D, the expectation and passing to the limit on δ, for all $t \in [s,T]$

$$E\int_D |\partial_{x_i} v_\epsilon| dx = E\int_D |\partial_{x_i} v_0^\epsilon| dx < \infty. \tag{3.5}$$

Thus, for all $t \in [s,T]$ and P-a.s, $v_\epsilon(t) \in BV(D)$. As $v_\epsilon(t) \to v(t)$ in $L^1(\Omega \times D)$, for a subsequence denoted in the same way, for all $t \in [s,T]$ and P-a.s, $v_\epsilon(t) \to v(t)$ in $L^1(D)$. According to Málek-Nečas-Otto-Rokyta-Růžička [60] p.36, we thus have for all $t \in [s,T]$ and P-a.s

$$TV_x(v(t)) \leqslant \liminf_\epsilon TV_x(v_\epsilon(t)).$$

Using again Remark 3.4, for all $t \in [s,T]$, $TV_x(v(t))$ is measurable with respect to the probability measure P. Consequently, taking the expectation, using Fatou's Lemma, (3.5) then (3.4), one gets that for every $t \in [s,T]$

$$E[TV_x(v(t))] \leqslant \liminf_\epsilon E[TV_x(v_\epsilon(t))] = \liminf_\epsilon E[TV_x(v_0^\epsilon)] \leqslant E[TV_x(v_0)],$$

and the result holds. □

From the general theory for scalar conservation law, let us now introduce properties satisfied by the operator S(.).

Lemma 3.5 *Let $u_0 \in L^\infty(D) \cap BV(D)$, $t > 0$, and $u(t) = S(t)u_0$. Then:*
i) For almost every $t > 0$,

$$\|u(t)\|_{L^\infty(D)} \leqslant \|u_0\|_{L^\infty(D)}.$$

ii) There exists a constant $C > 0$, such that for all $t_1, t_2 \in [0, T]$

$$\int_D |u(t_1, x) - u(t_2, x)|dx \leqslant C\|u_0\|_{BV(D)}|t_1 - t_2|.$$

iii) There exists a constant c depending only on the geometry of the boundary ∂D of D, such that for all $t \in [0, T]$

$$\|u(t, .)\|_{BV(D)} \leqslant (1 + ct)\|u_0\|_{BV(D)} e^{K_{\vec{\mathbf{f}}} t}$$

where $K_{\vec{\mathbf{f}}}$ denotes the Lipschitz constant of $\vec{\mathbf{f}}$.

Proof. These results are classical ones and the proof would be outside the scope of the present work, we refer the reader to Málek-Nečas-Otto-Rokyta-Růžička [60] but also to Gagneux-Madaune [42] for detailed explanations. These results are obtained by the study of viscous solutions. Let us mention the work of Peyroutet [63] which gives us precisely the expression and also the dependence of the constants introduced in this lemma. □

3.2 Construction of the approximate solution

Let us now explain the construction of the approximate solution as introduced in Holden-Risebro [49]. We consider a positive integer N, denote by $\Delta = \frac{T}{N}$ and split the time interval by denoting $t_n = n\Delta$, $n \in \{0, \ldots, N\}$ each point of the time discretization. For each step of discretization Δ, we consider the function

$$u^\Delta(t, x) = \begin{cases} u^n(x) & \text{if} \quad t = t_n \\ R(t, t_n)u^n(x) & \text{if} \quad t \in]t_n, t_{n+1}[, \end{cases}$$

where the sequence $(u^n)_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} u^0(x) &= u_0(x) \\ u^{n+1}(x) &= S(\Delta)R(t_{n+1},t_n)u^n(x). \end{cases}$$

For convenience in the sequel, let us introduce some notations.

<u>**Notations:**</u> $\forall n \in \{0,\dots,N-1\}$, $t \in [0,T]$ and $x \in D$:

- $u_-^{n+1}(x) := R(t_{n+1},t_n)u^n(x)$, $\forall n \in \{0,\dots,N-1\}$ and $x \in D$.
- $\widetilde{u}(t,x) := S(t-t_n)R(t_{n+1},t_n)u^n(x) = S(t-t_n)u_-^{n+1}(x)$.

Proposition 3.6 *(A priori estimate)*
There exists a constant M_1 independent of n and Δ such that P-a.s and for all $t \in [0,T]$

$$\|u^\Delta(t)\|_{L^\infty(D)} \leqslant M_1 := max(M, \|u_0\|_{L^\infty(D)}).$$

Proof. Let us mention that the construction of u^Δ is done by induction, so the proofs of the associated results also rely on inductive reasoning. Consider $n \in \{0,\dots,N-1\}$, and $u^{n+1} = S(\Delta)u_-^{n+1}$. Thanks to Lemma 3.5 i),

$$\|u^{n+1}\|_{L^\infty(D)} \leqslant \|u_-^{n+1}\|_{L^\infty(D)}, \ P\text{-a.s.}$$

Moreover, thanks to Lemma 3.2, P-a.s and $\forall t \in [t_n, t_{n+1}]$

$$\|R(t,t_n)u^n\|_{L^\infty(D)} \leqslant max(M, \|u^n\|_{L^\infty(D)})$$

and particularly for $t = t_{n+1}$, one has P-a.s

$$\begin{aligned} \|u_-^{n+1}\|_{L^\infty(D)} &= \|R(t_{n+1},t_n)u^n\|_{L^\infty(D)} \\ &\leqslant max(M, \|u^n\|_{L^\infty(D)}) \\ &\leqslant max(M, \|u^0\|_{L^\infty(D)}) := M_1. \end{aligned}$$

Notice that the construction of u^Δ is countable, so P-a.s, for all $t \in [0,T]$ and all possible discretization parameter $N \in \mathbb{N}^*$:

$$\|u^\Delta(t,.)\|_{L^\infty(D)} \leqslant M_1,$$

where M_1 does not depend on Δ and the result holds. □

Proposition 3.7 *(BV(D)-bound)*
There exists a constant M_2 such that for every $i \in \{0, \dots, N\}$:

$$E\|u^i\|_{BV(D)} \leqslant M_2\|u_0\|_{BV(D)}.$$

Proof. Consider $i \in \{0, \dots, N-1\}$. As $u^i = S(\Delta)u^i_-$, and $u^i_- = R(t_i, t_{i-1})u^{i-1}$, using Lemma 3.5 then Lemma 3.3 one gets

$$E\|u^i\|_{BV(D)} \leqslant (1+c\Delta)e^{K_{\vec{\mathbf{f}}}\Delta}E\|u^i_-\|_{BV(D)} \leqslant (1+c\Delta)e^{K_{\vec{\mathbf{f}}}\Delta}E\|u^{i-1}\|_{BV(D)},$$

a reasoning by induction gives us

$$E\|u^i\|_{BV(D)} \leqslant (1+c\Delta)^i e^{K_{\vec{\mathbf{f}}}\Delta\times i}E\|u_0\|_{BV(D)}.$$

Elementary calculations leads to $(1+c\Delta)^i \leqslant e^{c\Delta\times i} \leqslant e^{cT}$, thus $M_2 := e^{cT}e^{K_{\vec{\mathbf{f}}}T}$. □

Let us introduce a lemma on the increment of u^Δ, useful for the sequel.

Lemma 3.8 *Let $n \in \{1, \dots, N\}$ and consider $t \in [t_n, t_{n+1}[$. Then:*

$$E\int_D |u^\Delta(t_{n+1}, x) - u^\Delta(t, x)|dx \leqslant CM_2\Delta\|u_0\|_{BV(D)} + \widetilde{C}\sqrt{\Delta},$$

where C is defined in Lemma 3.5 ii), M_2 in Proposition 3.7 and $\widetilde{C}$ only depends on h, M_1 and $mes(D)$.

Proof. Let $n \in \{1, \dots, N\}$ and consider $t \in [t_n, t_{n+1}[$. For all $x \in D$,

$$\begin{aligned} u^\Delta(t,x) &= R(t,t_n)u^n(x) = u^n(x) + \int_{t_n}^t h(u^\Delta(\sigma))dw(\sigma) \\ u^\Delta(t_{n+1},x) &= u^{n+1}(x). \end{aligned}$$

Thus,

$$E\int_D |u^\Delta(t_{n+1},x)-u^\Delta(t,x)|dx \leqslant E\int_D |u^{n+1}(x)-u^n(x)|dx$$
$$+E\int_D |\int_{t_n}^t h(u^\Delta(\sigma))dw(\sigma)|dx.$$

$$\begin{aligned}
E\int_D |u^{n+1}(x)-u^n(x)|dx &\leqslant E\int_D |u^{n+1}(x)-u_-^{n+1}(x)|+|u_-^{n+1}(x)-u^n(x)|dx\\
&= E\int_D |S(\Delta)u_-^{n+1}(x)-u_-^{n+1}(x)|+|R(t_{n+1},t_n)u^n(x)-u^n(x)|dx\\
&\leqslant^{*} EC\Delta\|u_-^{n+1}\|_{BV(D)}+E\int_D |\int_{t_n}^{t_{n+1}} h(u^\Delta(s,x))dw(s)|dx\\
&\leqslant^{\dagger} CM_2\Delta\|u_0\|_{BV(D)}+E\int_D |\int_{t_n}^{t_{n+1}} h(u^\Delta(s,x))dw(s)|dx.
\end{aligned}$$

And it remains to show that

$$E\int_D |\int_{t_n}^t h(u^\Delta(s,x))dw(s)|dx + E\int_D |\int_{t_n}^{t_{n+1}} h(u^\Delta(s,x))dw(s)|dx \leqslant \widetilde{C}\sqrt{\Delta}.$$

Notice that $|t_n - t| \leqslant \Delta$, thus

$$\begin{aligned}
E\int_D |\int_{t_n}^t h(u^\Delta(s,x))dw(s)|dx &\leqslant^{\ddagger} \sqrt{mes(D)}\Big(E\int_D |\int_{t_n}^t h(u^\Delta(s,x))dw(s)|^2dx\Big)^{\frac{1}{2}}\\
&=^{\S} \sqrt{mes(D)}\Big(E\int_D \int_{t_n}^t h^2(u^\Delta(s,x))dsdx\Big)^{\frac{1}{2}}\\
&\leqslant \widetilde{C}'\sqrt{\Delta},
\end{aligned}$$

where $\widetilde{C}'$ only depends on $mes(D), M_1$ and C_h the Lipschitz constant of h. Similarly one shows that $E\int_D |\int_{t_n}^{t_{n+1}} h(u^\Delta(s,x)dw(s)| \leqslant \widetilde{C}'\sqrt{\Delta}$, and so $\widetilde{C} = 2\widetilde{C}'$. □

*Lemma 3.5 ii)
†Proposition 3.7
‡Cauchy-Schwartz inequality on $\Omega \times D$
§Itô isometry

3.3 Entropy formulation

We follow the idea of Peyroutet [63] for introducing the entropy formulation satisfied by the approximate solution. In order to do this, consider

$$\widetilde{u}(t,x) = S(t-t_n)u_-^{n+1}(x)$$

and write the entropy formulation satisfied by such a solution. In order to be compatible with the Definition 2.1, as in Chapter IV we consider boundary conditions in the way Carillo [23] introduced them. Using notations of Section 1.3, as a weak entropy solution of a conservation law, $\widetilde{u}$ satisfies the following condition, $\forall (k,\varphi,\eta) \in \mathbb{A}$:

$$\begin{aligned}\int_D \eta(\widetilde{u}(t_n)-k)\varphi(t_n)dx - \int_D \eta(\widetilde{u}(t_{n+1})-k)\varphi(t_{n+1})dx \\ + \int_D \int_{t_n}^{t_{n+1}} \eta(\widetilde{u}-k)\partial_t\varphi + F^\eta(\widetilde{u},k)\nabla\varphi dt dx \quad &\geqslant \quad 0. \end{aligned} \tag{3.6}$$

We would like to approximate this formulation. The idea is to introduce in (3.6) informations coming from the initial condition $\widetilde{u}(t_n)$. We consider $(k,\varphi,\eta) \in \mathbb{A}$ and denote for $s \in [t_n, t_{n+1}]$, $v(s) := R(s,t_n)u^n$ the solution in (t_n, t_{n+1}) of the stochastic differential equation

$$\left\{ \begin{array}{rcl} dv &=& h(v)dw \\ v(t=t_n) &=& u^n. \end{array} \right.$$

Applying the Itô formula to the process v and the regular function $\Psi(t,\lambda) = \eta(\lambda - k)$, one gets P-a.s:

$$\begin{aligned} \eta(v(t_{n+1})-k) &= \eta(v(t_n)-k) + \int_{t_n}^{t_{n+1}} \eta'(v(t)-k)h(v(t))dw(t) \\ &\quad + \frac{1}{2}\int_{t_n}^{t_{n+1}} \eta''(v(t)-k)h^2(v(t))dt. \end{aligned}$$

Remark that $v(t) = u^\Delta(t)$ for all $t \in [t_n, t_{n+1}[$ and $v(t_{n+1}) = \widetilde{u}(t_n)$, in this way, P-a.s:

$$\begin{aligned} &\int_D \eta(\widetilde{u}(t_n,x)-k)\varphi(t_n,x)dx - \int_D \eta(u^\Delta(t_n,x)-k)\varphi(t_n,x)dx \\ = &\int_D \int_{t_n}^{t_{n+1}} \eta'(u^\Delta(t,x)-k)h(u^\Delta(t,x))dw(t)\varphi(t_n,x)dx \\ &+ \frac{1}{2}\int_D \int_{t_n}^{t_{n+1}} \eta''(u^\Delta(t,x)-k)h^2(u^\Delta(t,x))dt\varphi(t_n,x)dx. \end{aligned}$$

Moreover,

$$\int_D \eta(\widetilde{u}(t_{n+1},x)-k)\varphi(t_{n+1},x)dx = \int_D \eta(u^\Delta(t_{n+1},x)-k)\varphi(t_{n+1},x)dx.$$

Thus one first gets, for any P-measurable set A

$$\begin{aligned}
& E\left(\int_D \eta(u^\Delta(t_n,x)-k)\varphi(t_n)dx1_A - \int_D \eta(u^\Delta(t_{n+1},x)-k)\varphi(t_{n+1},x)dx1_A\right)\\
& +E\left(\int_D\int_{t_n}^{t_{n+1}} \eta'(u^\Delta(t,x)-k)h(u^\Delta(t,x))dw(t)\varphi(t_n,x)dx1_A\right)\\
& +\frac{1}{2}E\left(\int_D\int_{t_n}^{t_{n+1}} \eta''(u^\Delta(t,x)-k)h^2(u^\Delta(t,x))dt\varphi(t_n,x)dx1_A\right)\\
& +E\left(\int_D\int_{t_n}^{t_{n+1}} \eta(\widetilde{u}(t,x)-k)\varphi_t(t,x)+F^\eta(\widetilde{u}(t,x),k)\nabla\varphi(t,x)dtdx1_A\right)\\
\geqslant\ & 0.
\end{aligned}$$

We propose to approximate $E(\int_D\int_{t_n}^{t_{n+1}} \eta(\widetilde{u}(t,x)-k)\varphi_t(t,x)dtdx1_A)$ by $E(\int_D\int_{t_n}^{t_{n+1}} \eta(u^{n+1}-k)\varphi_t(t,x)dtdx1_A)$ making an error only of order Δ^2. Indeed,

$$\begin{aligned}
& \left|E(\int_D\int_{t_n}^{t_{n+1}} \eta(\widetilde{u}(t)-k)\varphi_t dtdx - \int_D\int_{t_n}^{t_{n+1}} \eta(u^{n+1}-k)\varphi_t dtdx1_A)\right|\\
\leqslant\ & CE\int_D\int_{t_n}^{t_{n+1}} |\eta(\widetilde{u}(t)-k)-\eta(u^{n+1}-k)|.|\varphi_t|dtdx\\
\leqslant\ & C\|\varphi_t\|_\infty E\int_D\int_{t_n}^{t_{n+1}} |\widetilde{u}(t)-u^{n+1}|dtdx.
\end{aligned}$$

As

$$\begin{aligned}
E\int_{t_n}^{t_{n+1}}\int_D |\widetilde{u}(t)-u^{n+1}|dxdt &= E\int_{t_n}^{t_{n+1}} \|S(t-t_n)u_-^{n+1}-S(t_{n+1}-t_n)u_-^{n+1}\|_{L^1(D)}dt\\
&\leqslant^\dagger C\int_{t_n}^{t_{n+1}} |t-t_{n+1}|E\|u_-^{n+1}\|_{BV(D)}dt\\
&\leqslant^\ddagger C\int_{t_n}^{t_{n+1}} |t-t_{n+1}|\|u_0\|_{BV(D)}dt\\
&\leqslant C\|u_0\|_{BV(D)}\Delta^2,
\end{aligned}$$

In the same way, one shows by using the Lipschitz-continuity of $F^\eta(.,k)$ that $E(\int_D\int_{t_n}^{t_{n+1}} F^\eta(u^{n+1}(x),k)\nabla\varphi dtdx1_A)$ is an approximation of $E(\int_D\int_{t_n}^{t_{n+1}} F^\eta(\widetilde{u}(t,x),k)\nabla\varphi(t,x)dtdx1_A)$ also with an error of order Δ^2.

Finally we obtain by summing over n

$$
\begin{aligned}
&E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta(u^\Delta(t_{n+1},x)-k)\varphi_t(t,x)+F^\eta(u^\Delta(t_{n+1},x),k)\nabla\varphi(t,x)dtdx1_A\right)\\
&+E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta'(u^\Delta(t,x)-k)h(u^\Delta(t,x))dw(t)\varphi(t_n,x)dx1_A\right)\\
&+\frac{1}{2}E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta''(u^\Delta(t,x)-k)h^2(u^\Delta(t,x))dt\varphi(t_n,x)dx1_A\right)\\
&+mes(A)\int_D\eta(u_0(x)-k)\varphi(0,x)dx-E\left(\int_D\eta(u^\Delta(T,x)-k)\varphi(T,x)dx1_A\right)\\
&\geqslant-\widetilde{c}\Delta,
\end{aligned}
$$

where $\widetilde{c}\Delta$ tends to 0 when Δ does.

Remark 3.9 *For technical reasons, we keep the term $\int_D\eta(\widetilde{u}(t_{n+1})-k)\varphi(t_{n+1})dx$ in the entropy formulation (3.6), in order to vanish two by two terms when we do the summation over n with $\int_D\eta(\widetilde{u}(t_n)-k)\varphi(t_n)dx$. Then, last term of the sum: $E(\int_D\eta(u^\Delta(T)-k)\varphi(T)dx1_A)$ is nonnegative and we remove it from the inequality.*

3.4 Convergence of the approximate solution

Our aim is to pass to the limit with respect to Δ in the following inequality:

$$
\begin{aligned}
E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta'(u^\Delta(t,x)-k)h(u^\Delta(t,x))dw(t)\varphi(t_n,x)dx1_A\right) &:= I_1^\Delta\\
+\frac{1}{2}E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta''(u^\Delta(t,x)-k)h^2(u^\Delta(t,x))dt\varphi(t_n,x)dx1_A\right) &:= I_2^\Delta\\
+E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta(u^\Delta(t_{n+1},x)-k)\varphi_t(t,x)dtdx1_A\right) &:= I_3^\Delta\\
+E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}F^\eta(u^\Delta(t_{n+1},x),k)\nabla\varphi(t,x)dtdx1_A\right) &:= I_4^\Delta\\
+mes(A)\int_D\eta(u_0(x)-k)\varphi(0,x)dx &\geqslant-\widetilde{c}\Delta, \qquad (3.7)
\end{aligned}
$$

where A is a P-measurable set. We use the notion of narrow convergence of Young measures (or entropy processes). Then thanks to the uniqueness result of Section 2, we will be able

[†] Lemma 3.5 ii)

[‡] Proposition 3.7

to prove that the measure-valued limit is an entropy solution in the sense of Definition 2.1. Since (u^Δ) is a bounded sequence in $L^\infty(Q\times\Omega)$, the associated Young measure sequence (u^Δ) converges (up to a subsequence still indexed in the same way) to a Young measure denoted $\mathbf{u}\in L^\infty(Q\times\Omega\times]0,1[)$. Furthermore, according to Balder [7] and also to Valadier [73], for any Carathéodory function Ψ such that $\Psi(.,u^\Delta)$ is uniformly integrable:

$$E\int_Q \Psi(u^\Delta(t,x))dtdx \to E\int_Q\int_0^1 \Psi(\mathbf{u}(t,x,\alpha))d\alpha dtdx \text{ when } \Delta\to 0.$$

Moreover, revisiting the work of Panov [61] on the measurability of $\mathbf{u}$ with respect to all its variables, one shows that as u^Δ is a predictable process with values in $L^2(D)$, $\mathbf{u}$ is in $\mathcal{N}^2_w(0,T,L^2(D\times]0,1[))$. We refer the reader to the Appendix 2 Section 3 for detailed explanations to obtain this measurability.

Let us consider separately terms of (3.7) and analyze passage to the limit $\Delta\to 0$ for each term. In order to make the lecture more fluent, we omit the variable (t,x) when no confusion is possible.

1. $I_1^\Delta \to E\left(\int_D\int_0^T\int_0^1 \eta'(\mathbf{u}(\alpha)-k)h(\mathbf{u}(.,\alpha))d\alpha\varphi dw(t)dx1_A\right) := I_1.$

$$\begin{aligned}
|I_1^\Delta - I_1| &= |E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \eta'(u^\Delta-k)h(u^\Delta)[\varphi(t_n)-\varphi(t)]dw(t)dx1_A\right) \\
&\quad +E\left(\int_D\int_0^T\left[\eta'(u^\Delta-k)h(u^\Delta)-\int_0^1\eta'(\mathbf{u}(.,\alpha)-k)h(\mathbf{u}(.,\alpha))d\alpha\right]\varphi(t)dw(t)dx1_A\right)| \\
&:= |I_{1,1}^\Delta + I_{1,2}^\Delta|.
\end{aligned}$$

$$
\begin{aligned}
|I_{1,1}^{\Delta}| &= |E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta'(u^{\Delta}-k)h(u^{\Delta})[\varphi(t_n)-\varphi(t)]dw(t)dx1_A\right)| \\
&\leqslant^{*} C\sum_{n=0}^{N-1}\left[E\int_D\left(\int_{t_n}^{t_{n+1}}\eta'(u^{\Delta}-k)h(u^{\Delta})[\varphi(t_n)-\varphi(t)]dw(t)\right)^2 dx\right]^{\frac{1}{2}} \\
&=^{\dagger} C\sum_{n=0}^{N-1}\left[E\int_D\int_{t_n}^{t_{n+1}}\left[\eta'(u^{\Delta}-k)h(u^{\Delta})[\varphi(t_n)-\varphi(t)]\right]^2 dtdx\right]^{\frac{1}{2}} \\
&\leqslant^{\ddagger} C\sum_{n=0}^{N-1}\left[E\int_{t_n}^{t_{n+1}}\int_D[\eta'(u^{\Delta}-k)h(u^{\Delta})]^2\underbrace{(\varphi(t_n)-\varphi(t))^2}_{\leqslant C\Delta^2}dxdt\right]^{\frac{1}{2}} \\
&\leqslant C\sum_{n=0}^{N-1}\left[E\int_{t_n}^{t_{n+1}}mes(D)\times\Delta^2 dt\right]^{\frac{1}{2}} \\
&\leqslant C\sum_{n=0}^{N-1}\Delta^{\frac{3}{2}} \\
&= C\sqrt{\Delta}\to 0.
\end{aligned}
$$

Let us show that $I_{1,2}^{\Delta}\to 0$. Denote $v^{\Delta}=\eta'(u^{\Delta}-k)h(u^{\Delta})\varphi$. Thanks to Proposition 3.6, v^{Δ} is bounded in $L^2(Q\times\Omega)$ and there exists $v\in L^2(Q\times\Omega)$ such that $v^{\Delta}\rightharpoonup v$ in the same space. Moreover, $\Psi:(t,x,\omega,\lambda)\mapsto\eta'(\lambda-k)h(\lambda)\varphi(t,x)$, $(t,x,\omega,\lambda)\in Q\times\Omega\times\mathbb{R}$ is a Carathéodory function and $\Psi(.,u^{\Delta})$ is uniformly integrable as it is bounded in $L^2(Q\times\Omega)$. By identification, $v=\int_0^1\Psi(\mathbf{u}(.,\alpha))d\alpha$. Furthermore, for all $t\in[0,T]$,

$$
\begin{aligned}
I_t: L^2(Q\times\Omega) &\to L^2(D\times\Omega) \\
\overline{u} &\mapsto \int_0^t\overline{u}(t,x,\omega)dw(t)
\end{aligned}
$$

is a linear continuous function, and so it is a weakly continuous function from $L^2(Q\times\Omega)$

*Cauchy-Schwarz inequality on $\Omega\times D$

†Itô isometry

‡Jensen inequality

to $L^2(D\times\Omega)$. Consequently, $I_t(v^\Delta)\to I_t(v)$ in $L^2(D\times\Omega)$. In this manner,

$$\begin{aligned} & E\Big(\int_D\int_0^T \eta'(u^\Delta-k)h(u^\Delta)\varphi dw(t)dx1_A\Big)\\ \to\ & E\Big(\int_D\int_0^T\int_0^1 \eta'(\mathbf{u}(.,\alpha)-k)h(\mathbf{u}(.,\alpha))d\alpha\varphi dw(t)dx1_A\Big),\end{aligned}$$

and $|I^\Delta_{1,2}|\to 0$.

2. $I_2^\Delta \to \dfrac{1}{2}E\Big(\int_Q\int_0^1 \eta''(\mathbf{u}(.,\alpha)-k)h^2(\mathbf{u}(.,\alpha))d\alpha\varphi dtdx1_A\Big) := I_2.$

$$\begin{aligned} |I_2^\Delta - I_2| &= \frac{1}{2}|E\Big(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta''(u^\Delta-k)h^2(u^\Delta)[\varphi(t_n)-\varphi(t)]dtdx1_A\Big)\\ &\quad + E\Big(\int_Q\Big[\eta''(u^\Delta-k)h^2(u^\Delta)-\int_0^1\eta''(\mathbf{u}(.,\alpha)-k)h^2(\mathbf{u}(.,\alpha))d\alpha\Big]\varphi(t)dtdx1_A\Big)|\\ &:= \frac{1}{2}|I^\Delta_{2,1}+I^\Delta_{2,2}|.\end{aligned}$$

$$\begin{aligned} |I^\Delta_{2,1}| &\leqslant E\Big(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\Big|\eta''(u^\Delta-k)h^2(u^\Delta)[\varphi(t_n)-\varphi(t)]\Big|dtdx\Big)\\ &\leqslant C\sum_{n=0}^{N-1}\Delta^2\\ &\leqslant C\Delta\to 0.\end{aligned}$$

Note that $\Psi(t,x,\omega,\lambda)=\eta''(\lambda-k)h^2(\lambda)\varphi(t,x)1_A$ is a Carathéodory function such that $\Psi(.,u^\Delta)$ is uniformly integrable, thus $I^\Delta_{2,2}\to 0$ and the result holds.

3. $I_3^\Delta\to E\Big(\int_Q\int_0^1\eta(\mathbf{u}(.,\alpha)-k)d\alpha\varphi_t dtdx1_A\Big) := I_3.$

$$\begin{aligned} |I_3^\Delta - I_3| &\leqslant \Big|E\Big(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}[\eta(u^\Delta(t_{n+1})-k)-\eta(u^\Delta(t)-k)]\varphi_t dtdx1_A\Big)\Big|\\ &\quad +\Big|E\Big(\int_Q[\eta(u^\Delta-k)-\int_0^1\eta(\mathbf{u}(.,\alpha)-k)d\alpha]\varphi_t dtdx1_A\Big)\Big|\\ &:= |I^\Delta_{3,1}|+|I^\Delta_{3,2}|.\end{aligned}$$

$$
\begin{aligned}
|I_{3,1}^{\Delta}| &\leqslant E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}|\eta(u^{\Delta}(t_{n+1})-k)-\eta(u^{\Delta}(t)-k)|.|\varphi_t|dtdx1_A\right)\\
&\leqslant CE\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}|u^{\Delta}(t_{n+1})-u^{\Delta}(t)|dtdx.
\end{aligned}
$$

On the other hand, one shows in the proof of Lemma 3.8 that

$$
E\int_D|u^{\Delta}(t_{n+1})-u^{\Delta}(t)|dx\leqslant CM_2\Delta\|u_0\|_{BV(D)}+\widetilde{C}\sqrt{\Delta}
$$

where C is defined in Lemma 3.5 ii), M_2 in Proposition 3.7, $\widetilde{C}$ only depends on h, M_1 and D. Finally, $|I_{3,1}^{\Delta}|\to 0$. Let us now show that $I_{3,2}^{\Delta}\to 0$. We consider the Carathéodory function $\Psi(t,x,\omega,\lambda)=\eta(\lambda-k)\varphi_t(t,x)1_A$. As previously, $\Psi(.,u^{\Delta})$ is uniformly integrable and

$$
E\int_Q\eta(u^{\Delta}-k)\varphi_t1_Adtdx\to E\int_Q\int_0^1\eta(\mathbf{u}(.,\alpha)-k)d\alpha\varphi_t1_Adtdx.
$$

4. $I_4^{\Delta}\to E\left(\int_Q\int_0^1F^{\eta}(\mathbf{u}(.,\alpha),k)d\alpha\nabla\varphi dtdx1_A\right):=I_4.$

$$
\begin{aligned}
|I_4^{\Delta}-I_4| &\leqslant \left|E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\left[F^{\eta}(u^{\Delta}(t_{n+1}),k)-F^{\eta}(u^{\Delta}(t),k)\right]\nabla\varphi)dtdx1_A\right)\right|\\
&\quad+\left|E\left(\int_Q\left[F^{\eta}(u^{\Delta}(t),k)-\int_0^1F^{\eta}(\mathbf{u}(.,\alpha),k)d\alpha\right]\nabla\varphi dtdx1_A\right)\right|\\
&:= |I_{4,1}^{\Delta}|+|I_{4,2}^{\Delta}|.
\end{aligned}
$$

As previously, one shows that $|I_{4,1}^{\Delta}|\to 0$ using the Lipschitz-continuity of $F^{\eta}(.,k)$ for all $k\in\mathbb{R}$. And as $\Psi(t,x,\omega,\lambda)=\int_k^{\lambda}\eta'(\sigma-k)f'(\sigma)d\sigma\nabla\varphi 1_A$ is a Carathéodory function with $\Psi(.,u^{\Delta})$ uniformly integrable, one gets that $I_{4,2}^{\Delta}\to 0$ by the way of Young measure theory.

Finally, for all $(k,\varphi,\eta) \in \mathbb{A}$ and for any P-measurable set A:

$$\begin{aligned}
& E\Big(\int_D \int_0^T \int_0^1 \eta'(\mathbf{u}(\alpha)-k)h(\mathbf{u}(.,\alpha))d\alpha\varphi dw(t)dx 1_A\Big)\\
& +\frac{1}{2}E\Big(\int_Q \int_0^1 \eta''(\mathbf{u}(.,\alpha)-k)h^2(\mathbf{u}(.,\alpha))d\alpha\varphi dt dx 1_A\Big)\\
& +E\Big(\int_Q \int_0^1 [\eta(\mathbf{u}(.,\alpha)-k)\varphi_t + F^\eta(\mathbf{u}(.,\alpha),k)\nabla\varphi]d\alpha dt dx 1_A\Big)\\
& + mes(A)\int_D \eta(u_0(x)-k)\varphi(0,x)dx\\
\geqslant\ & 0.
\end{aligned}$$

Thus, $\mathbf{u}(.,\alpha)$ is a measure-valued entropy solution of (1.1) in the sense of Definition 2.3. Thanks to the main result of Chapter IV resumed in Theorem 2.4, any measure-valued entropy solution in the sense of Definition 2.3 is unique and is the unique entropy solution in the sense of Definition 2.1. In this way, our approximate sequence u^Δ of the stochastic conservation law (1.1) converges to the unique weak entropy solution u of such a problem.

Remark 3.10 *Let us mention that the approximate sequence u^Δ converges to u in $L^1(Q\times\Omega)$ thanks to the Young measure theory. Moreover, as u^Δ is bounded in $L^\infty(Q\times\Omega)$, u^Δ converges to u strongly in $L^p(Q\times\Omega)$ for every $1\leqslant p<\infty$, using Vitali's theorem.*

Remark 3.11 ***Extension in the $\mathbb{R}^d$-case***
Using the theoretical study of Chapter III on the Cauchy problem for Problem (1.1) setting in $\mathbb{R}^d$ instead of a bounded domain D, one is also able to propose a splitting method in the $\mathbb{R}^d$-case to approximate the stochastic weak entropy solution. Indeed, the book of Málek-Nečas-Otto-Rokyta-Růžička *[60] gives us necessary tools on scalar conservation laws in unbounded domain as in Proposition 3.5. Moreover, in order to manage integrals on $\mathbb{R}^d$ it suffices to argue as in* Holden-Risebro *[49] with compactly supported test functions.*

Remark 3.12 ***On the choice of the splitting method***
We admit that the splitting method is not a suitable choice for numerical approximation of PDE in general, as we have to make expensive BV hypothesis. Thus an outlook of this work would be to propose another way for approximating our stochastic conservation law, by the way of the finite volume method for example.

4 Numerical experiments

We propose here an application of this splitting method in the one-dimensional case to the stochastic Burgers' equation:

$$\partial_t(u - \int_0^{\cdot} h(u)dw) + f(u)_x = 0 \text{ in }]0,1[\times]-1,1[\times\Omega,$$

where $f(u) = \frac{u^2}{2}$, $h : \mathbb{R} \to \mathbb{R}$ has a compact support in]0,1[and is defined by

$$h(x) = \begin{cases} 2e^{\frac{1}{|2x-1|^2-1}} & \text{if } 0 < x < 1 \\ 0 & \text{else.} \end{cases} \tag{4.1}$$

Remark 1 *Let us mention that in this case, although the flux function $\vec{\mathbf{f}}$ is not Lipschitz-continuous but only locally Lipschitz-continuous in $\mathbb{R}$, we are in the theoretical framework presented in the previous section. Indeed, we work with solution explicitly bounded by a known constant denoted M_1, which only depends on h and u_0. The trick consists in truncating the flux function outsides $[-M_1, M_1]$.*

The scheme relies on finite volume method. We denote by Δ_t the time step of the discretization of the interval $[0,T]$, $\{t_0,\dots,t_n\}$ points of this discretization, Δ_x the space step for the interval $]-1,1[$ and $\{x_0,\dots,x_m\}$ the points of the associated discretization.
In a vector U_0, we put the initial condition, computed at every points of the space discretization $\{x_0,\dots,x_m\}$, $U_0 = (u_0(x_1),\dots,u_0(x_m))$. We denote U_j^i the approximate solution at time t_i and computed in the space point x_j of our Problem (1.1). We initialize the first time denoted t at 0, thus take $t = 0$.

1. With an Euler method (let us mention the book of Kloeden-Platen [54] for details), we solve the stochastic differential equation (3.1) with the initial condition u_0, over the time step Δ_t. The idea of such a method is to represent the time increment ΔW of the Brownian motion by generating a random variable following a normal distribution $N(0,\Delta_t)$. We thus create a vector denoted V containing solution of the SDE (3.1) at time $t = \Delta_t$ for every $x \in [-1,1]$:

$$V = U_0 + h(U_0) \times \Delta W.$$

2. We use vector V as an initial condition for solving the conservation law (3.2) over the

time step Δ_t. We use the **Godunov method** which seems to be a suitable choice for the Burgers' 1D-equation, particularly the way this scheme takes into account the boundary conditions and the behavior of the flux function. We refer the reader to the book of LEVEQUE [58]. We thus create a vector $U = (U_1, \dots, U_m)$ containing the solution of the stochastic Burgers' equation at time $t = t + \Delta_t$ for every $x \in [-1, 1]$. Let us give some details of this scheme. To compute the value U_i which corresponds to the solution of (1.1) at time $t + \Delta_t$ and at the space point x_i denoted $u(\Delta_t, x_i)$ we use the following formula:

$$U_i = V_i - \frac{\Delta t}{\Delta_x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}),$$

where

$$F_{i+\frac{1}{2}} = \begin{cases} \min\limits_{V_i \leqslant v \leqslant V_{i+1}} & f(v) \quad \text{if } V_i \leqslant V_{i+1} \\ \\ \max\limits_{V_{i+1} \leqslant v \leqslant V_i} & f(v) \quad \text{else,} \end{cases}$$

and

$$F_{i-\frac{1}{2}} = \begin{cases} \min\limits_{V_{i-1} \leqslant v \leqslant V_i} & f(v) \quad \text{if } V_{i-1} \leqslant V_i \\ \\ \max\limits_{V_i \leqslant v \leqslant V_{i-1}} & f(v) \quad \text{else.} \end{cases}$$

Remark 4.1 *We consider for space points required out of the interval* $[-1, 1]$ *("*V_{-1}*" and "*V_{m+1}*") the value* 0 *(Dirichlet boundary condition). This method is stable under the CFL condition* $\frac{\Delta_t}{\Delta_x}\|u_0\|_{L^\infty(D)} \leqslant 1$*, for the particular case of Burgers' equation.*

3. We repeat these steps, by using U as initial condition instead of U_0 to compute the solution at the next time $t = t + \Delta_t$, until reach the final time $T = 1$.

We implement simulations with different initial conditions: u_1^0, u_2^0, u_3^0 and u_4^0 defined below:

$$u_1^0(x) = \begin{cases} -\frac{1}{2} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{else.} \end{cases} \qquad u_2^0(x) = \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -\frac{1}{2} & \text{else.} \end{cases} \qquad \begin{aligned} u_3^0(x) &= 1 - \tfrac{2}{\pi}\arctan(x). \\ \\ u_4^0(x) &= -\sin(\pi x). \end{aligned}$$

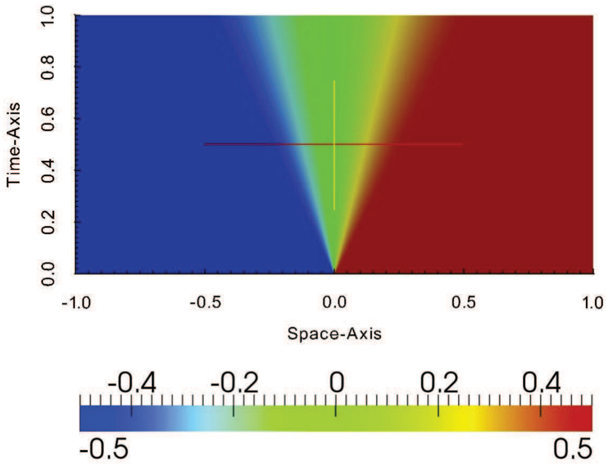


Figure V.1 – Burgers with u_1^0

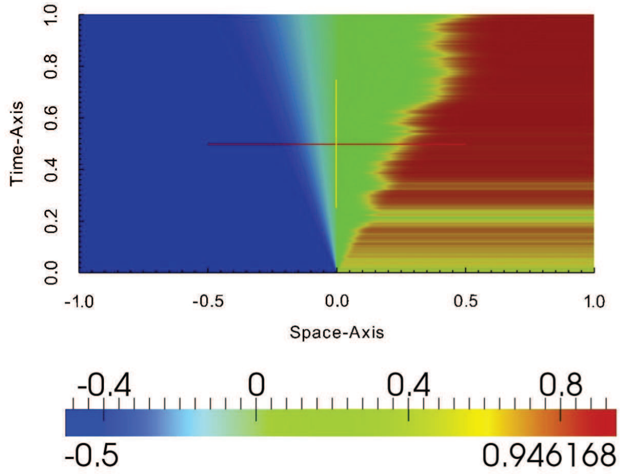


Figure V.2 – Stochastic Burgers with u_1^0

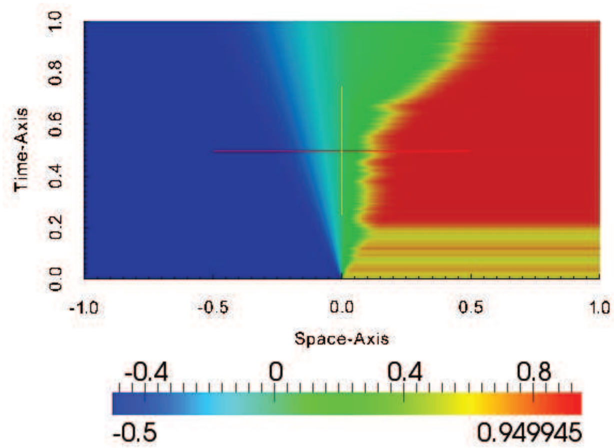


Figure V.3 – Stochastic Burgers with u_1^0

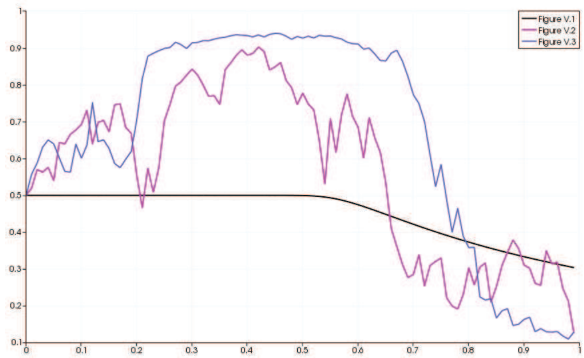


Figure V.4 – Pathwises for $x = 0.3$

To illustrate our proposal, we give, for each initial condition, a simulation of the solution in the deterministic case (i.e. when $h = 0$), and for the h given by (4.1), two sample path simulations. For these three simulations, we propose to highlight the time behavior of the solutions by drawing the pathwises of solutions at a given point x of the interval $[-1, 1]$. We get the following graphics with $\Delta_x = 0.002$ and $\Delta_t = 0.001$.These simulations have been implemented with the free software *Scilab*.

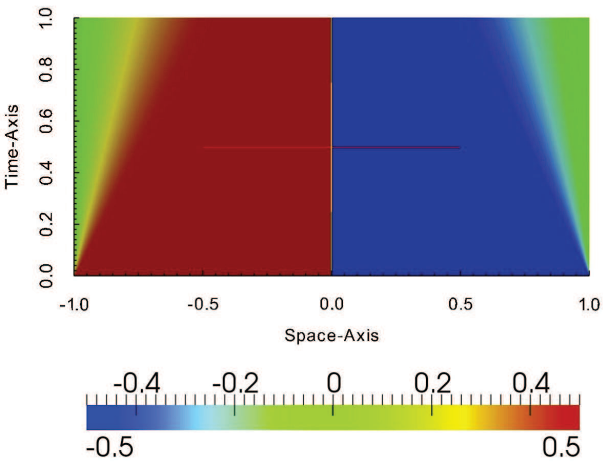


Figure V.5 – Burgers with u_2^0

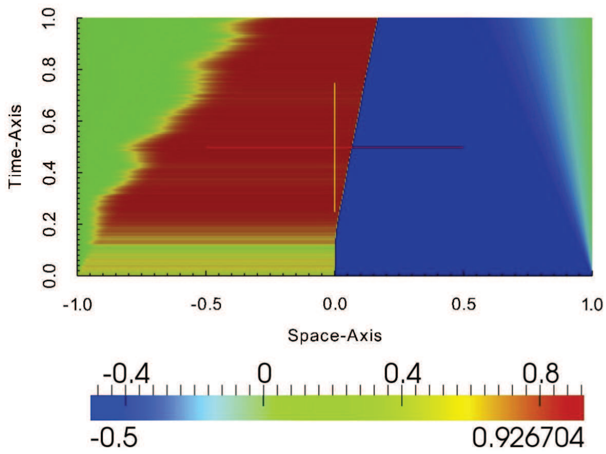


Figure V.6 – Stochastic Burgers with u_2^0

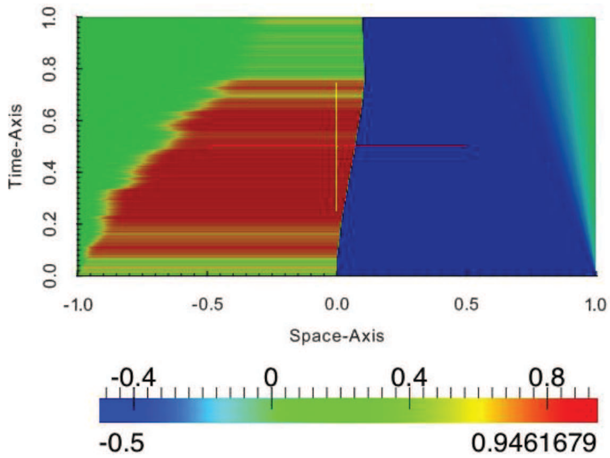


Figure V.7 – Stochastic Burgers with u_2^0

Figure V.8 – Pathwises for $x = -0.5$

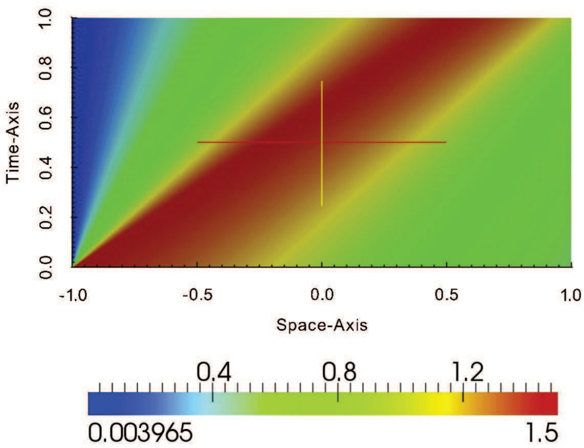


Figure V.9 – Burgers with u_3^0

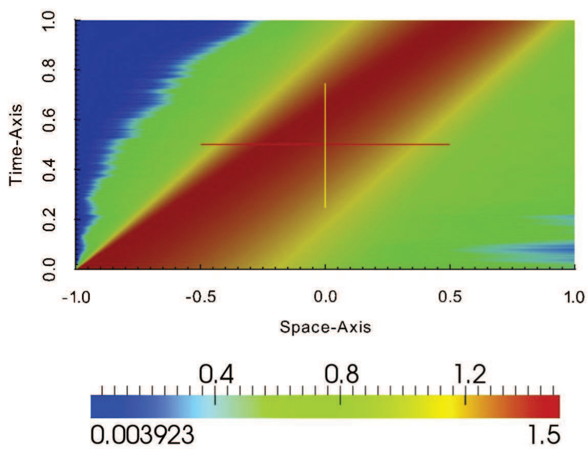


Figure V.10 – Stochastic Burgers with u_3^0

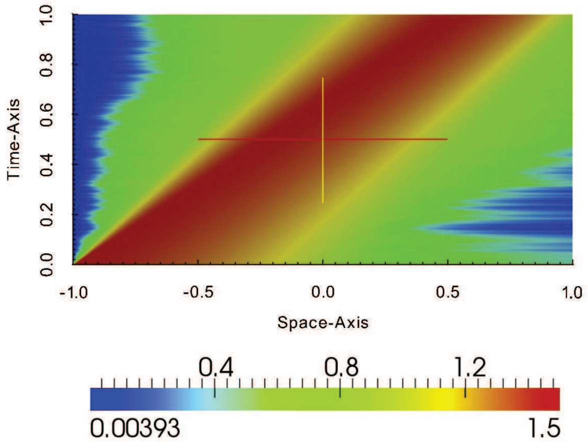


Figure V.11 – Stochastic Burgers with u_3^0

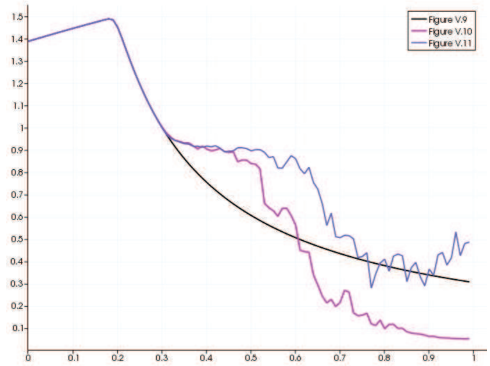


Figure V.12 – Pathwises for $x = -0.7$

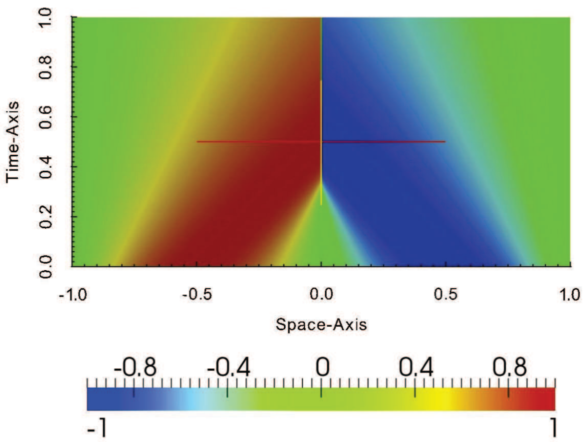


Figure V.13 – Burgers with u_4^0

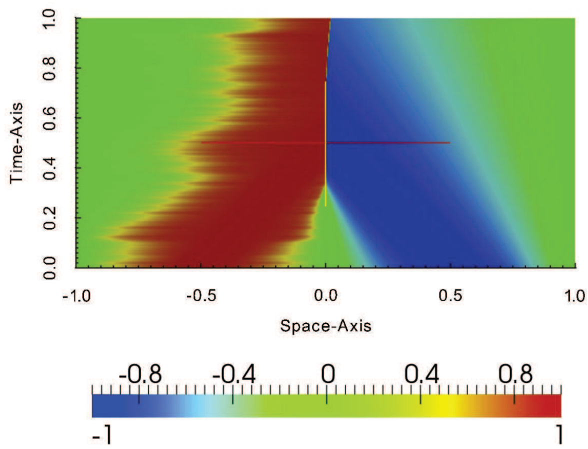


Figure V.14 – Stochastic Burgers with u_4^0

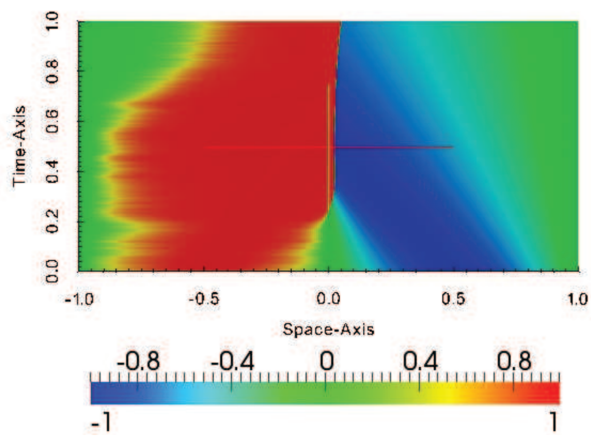


Figure V.15 – Stochastic Burgers with u_4^0

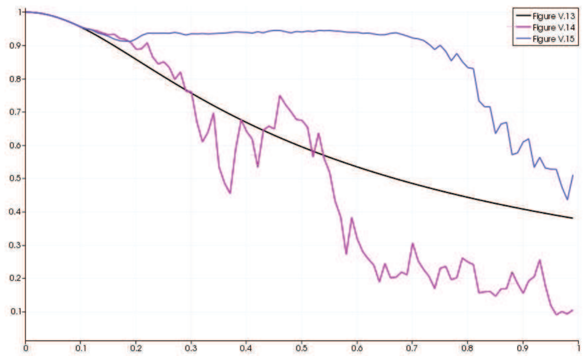


Figure V.16 – Pathwises for $x = -0.5$

Comments on these numerical experiments

To our knowledge, there are only few papers presenting numerical simulations of solutions of scalar conservation laws with multiplicative stochastic perturbation. Let us mention again the paper of Holden-Risebro [49] where the authors made simulations with data associated to oil-reservoirs models. As an illustration, we present in Appendix J our simulations of these models and get back those presented in [49]. In the paper of Kroker-Rhode [55], the authors are interested in numerical results on the Cauchy problem for a scalar conservation law with random noise. They use combination of finite volume methods and the Euler-Maruyama method. In the section devoted to numerical experiments, they aim to compare solutions of deterministic and stochastic Burgers' equations. In order to have exact expression of the solutions, the particular case of an additive perturbation is considered. Although the average of the stochastic solutions is close to the deterministic solution, it is not equal to it, and further numerical experiments indicate that it does not converge as the number of realizations increases.

Let us now analyze our simulations with Burgers type equations. In the deterministic case (Figure V.1, V.5, V.9 and V.13) we obtain expected characteristic curves (or characteristics) associated to the solution. The initial condition and the Dirichlet boundary condition are propagated in time along these characteristics. In the particular case of homogeneous Burgers' equations, it is a well-known fact that the characteristic lines are straight lines with slope-value depending on the initial and boundary conditions, thus this slope-value will change as long as the initial condition is not constant. Simulations with the piecewise-constant functions u_1^0 and u_2^0 present us the two following configurations:

1. **Shock wave:** (illustrated with Figure V.5 associated to u_2^0) This phenomenon appears when the characteristics intersect. This is due to the initial condition: the value u_L of u_2^0 when $x < 0$ is greater than the value u_R when $x > 0$. We have to create a shock and the slope-value of this shock for the weak solution is given by the average value $\frac{u_L+u_R}{2}$ (in the (t,x)-plane).

2. **Rarefaction wave:** (illustrated with Figure V.1 associated to u_0^1) Formally, this phenomenon appears when the characteristics do not cover the entire space (x,t) and so there are many weak solutions. This is due to the initial condition: the value of u_1^0 when $x < 0$ denoted u_L is smaller than the value when $x > 0$ denoted u_R. Instead of creating shocks, the physical admissible solution will be constructed by fulfilling the empty area defined

by $\{(t,x) : u_L t \leqslant x \leqslant u_R t\}$, using waves *i.e.* auto-similar functions of type $(t,x) \mapsto x/t$.

As there is no uniqueness of weak solutions, the entropy solution (*i.e.* the physical one) we are looking for will only have admissible shocks: the creation of shocks is allowed when $u_L > u_R$, where u_L is the value of the solution coming from the left, and u_R the one coming from the right. Then in the simulations associated to u_3^0 (Figure V.9) and u_4^0 (Figure V.13) we see these two behaviors: creation of rarefaction waves in the area of the (x,t)-plane which are empty and creation of an admissible shock wave when characteristics cross.

Now let us analyze the stochastic case. First, we warm the reader that scales are unfortunately not the same for simulations in the deterministic and in the stochastic case, so colors don't have the same values for all the simulations. As expected perturbations appear when the solution u takes values in the support of h (included in $]0,1[$). Visually, we get back an illustration of the stochastic version of characteristics studied by [38].

- **Simulations with** u_1^0: for $x > 0$, horizontal variations appear due to the fact that we consider a time-noise. We preserve rarefaction waves, and the behavior of these rarefaction waves is perturbed horizontally, but we don't have creation of a shock. Notice that we keep here a "rarefaction waves" configuration, as for a fixed $t \in [0,T]$, $u(t,x)$ stays less or equal to $u(t,y)$ for $x \leqslant y$, and so there is no competition of the characteristics also in the stochastic case, and we keep continuous solution.

- **Simulations with u_2^0:** as previously we notice perturbations appearing on the rarefaction waves. The important point is the behavior of the shock wave in Figure V.6 and V.7. In the deterministic case, note that the slope-value of the shock-wave is equal to 0, thus we get vertical line in the (x,t)-plane. In Figure V.6 and V.7 we notice that we have conservation of this phenomenon for small time and then the shock wave is modified. In Figure V.6, as the value coming from the left $u_L > 0$ is larger than the one coming from the right u_R, the slope-value will be positive and we have an increasing straight line. Figure V.7 gives us a nice modification of the shock wave which will be first an increasing straight-line up to time $t \approx 0.7$ as the slope-value is positive, and then will be a decreasing-line as the slope-value is negative.

- **Simulations with u_3^0 and u_4^0:** We get back the same kind of behaviors as these described previously. Let us mention that pathwises for a fixed x of Figure V.12 show that all the pathwises are the same as long as the perturbation does not act.

Work in progress & Outlooks

1 On the stochastic $p(t,x)$-Laplace equation

As an extension of the present work, an interested point is to have a look at the stochastic forcing of a nonlinear singular/degenerated parabolic problem of $p(t,x)$-Laplace type. In a joint work with G. Vallet, P. Wittbold and A. Zimmermann, we aim to give a result of existence and uniqueness of a solution to the following problem:

$$(P) : \begin{cases} \partial_t u - \Delta_{p(.)} u = h(u) dW & \text{in} \quad \Omega\times]0,T[\times D, \\ u = 0 & \text{on} \quad \Omega\times]0,T[\times\partial D, \\ u(t=0,.) = u_0(.) & \text{in} \quad \Omega\times D. \end{cases}$$

We consider here $D \subset \mathbb{R}^d$ a bounded domain with a Lipschitz-continuous boundary ∂D, $T > 0$ and denote by $Q_T =]0,T[\times D$. The initial condition $u_0 \in L^2(\mathrm{D})$ and homogeneous Dirichlet boundary conditions are required. The variable exponent is a measurable function $p : Q_T \to (1,\infty)$ satisfying

$$1 < p^- = \operatorname{ess}\inf_{(s,y)\in Q_T} p(s,y) \leq p(t,x) \leq p^+ = \operatorname{ess}\sup_{(s,y)\in Q_T} p(s,y) < \infty,$$

and $\Delta_{p(.)} u$ denotes the formal differential operator $\operatorname{div}\left[|\nabla u|^{p(t,x)-2}\nabla u\right]$, called the $p(.)$-Laplacian of u. We will assume that the variable exponent satisfies the following condition:
p is globally log-Hölder continuous, *i.e.* there exists a constant $c_{log} > 0$ such that

$$|p(t,x) - p(s,y)| \leq \frac{c_{log}}{\ln\left(e + \frac{1}{|(t,x)-(s,y)|}\right)}$$

is satisfied for all $(t,x),(s,y) \in Q_T$.
Note that this log-Hölder continuity of the variable exponent is crucial to have suitable properties of the functional spaces of the $p(.)$ operator's framework.

The idea to propose a result of existence and uniqueness for Problem (P) is to use the same kind of reasoning as in Chapter I. Our steps are the following ones: first, we consider a singular perturbation of Problem (P) with a "nice" function h independent of u and we obtain a stability result of the solution with respect to h; passing to the limit with respect to the singular perturbation, we prove that Problem (P) is well posed for an additive noise if h is a "nice" function, then we prove it for any h in $N^2_W(0,T,L^2(\mathrm{D}))$ by a density argument; in the last step, we solve Problem (P) for a multiplicative noise by a fixed-point argument.

2 The Cauchy problem for a degenerated hyperbolic-parabolic equation with a multiplicative stochastic perturbation

We would like to investigate on the Cauchy problem for an hyperbolic-parabolic degenerated equation with a multiplicative stochastic perturbation. As in the present work the idea is to adapt known method of the deterministic setting to the stochastic case. Thus, we consider the degenerate problem studied for instance by Vallet [76], with a multiplicative noise and in unbounded domain:

$$\begin{cases} \mathrm{du} - \mathrm{div}\,\vec{\mathbf{f}}(u)\,\mathrm{dt} - \Delta\phi(u)\,\mathrm{dt} = h(u)dw \text{ in } \Omega\times]0,T[,\times\mathbb{R}^d \\ u(0,.) = u_0 \in L^2(\mathbb{R}^d), \end{cases}$$

where we assume that $\vec{\mathbf{f}}$, ϕ and h are Lipschitz-continuous functions (and maybe be even $\phi \in \mathbf{C}^1(\mathbb{R})$ for convenience), $\vec{\mathbf{f}}(0) = \vec{\mathbf{0}}$ and $\phi(0) = h(0) = 0$. First we would like to investigate on the following regularized problem defined for $\epsilon > 0$

$$\begin{cases} \mathrm{du}_\epsilon - [\mathrm{div}\,\vec{\mathbf{f}}(u_\epsilon) + \Delta\phi(u_\epsilon)]\,\mathrm{dt} = h(u_\epsilon)\,\mathrm{dw} \text{ in } \Omega\times]0,T[\times\mathbb{R}^d, \\ u_\epsilon(0,.) = u_0^\epsilon \in \mathcal{D}(\mathbb{R}^d), \end{cases}$$

where ϕ_ϵ is the Lipschitz-continuous inversible and increasing function defined by $\phi_\epsilon = \phi + \epsilon Id$. The purpose is to show a result of existence and uniqueness of a viscous solution using an implicit semi-discretization in time and study the regularity and boundedness results of such a solution. Using the compactness argument of Young measure theory, the existence result of a measure-valued solution should not be too much difficult to obtain. Then following the plan of the uniqueness proof introduced in Chapter III we would like to compare viscous and measure-valued solution. The problem is that it seems that we need in the technique for controlling stochastic terms to apply the Itô formula, and so a suitable regularity for the solution u_ϵ is required, which doesn't seem to be obvious. An alternative idea would be to compare the measure-valued solution with the time-continuous affine viscous sequence $\widetilde{u}_\epsilon^{\Delta t}(t, x, \omega)$ constructed with the semi-discretization scheme (using notation of Chapter I).

Appendix G

On the stochastic Barenblatt equation: detailed results

1 Proof of Lemma 2.1

We consider the Hilbert space $L^2\big((\Omega,\mathcal{F}_{t_n}),H_0^1(D)\big)$ (see [68] p.103-104). Because of the smallness of the coefficient Δt on the Laplace operator, we are not able to use directly the fixed-point theorem of Banach. We define the application for $\lambda>0$

$$\begin{array}{rccc} J_\lambda: & L^2\big((\Omega,\mathcal{F}_{t_n}),L^2(D)\big) & \to & L^2\big((\Omega,\mathcal{F}_{t_n}),L^2(D)\big) \\ & v & \mapsto & v-\dfrac{\lambda}{\Delta t}f(v). \end{array}$$

Our aim is to solve the problem

$$\left\{\begin{array}{rclcl} -\Delta t\Delta u+f(u)-\Delta\varpi & = & 0 & \text{in} & D\times\Omega, \\ u & = & 0 & \text{on} & \partial D, \end{array}\right. \tag{E}$$

and we first consider the equivalent problem

$$\left\{\begin{array}{rclcl} -\lambda\Delta u+u-\dfrac{\lambda}{\Delta t}\Delta\varpi & = & J_\lambda(u) & \text{in} & D\times\Omega, \\ u & = & 0 & \text{on} & \partial D. \end{array}\right. \tag{E_λ}$$

We use the fixed-point theorem of Banach to solve (E_λ):

Let us introduce

$$\begin{array}{rrcl} T_\lambda : & L^2\big((\Omega,\mathcal{F}_{t_n}),L^2(D)\big) & \to & L^2\big((\Omega,\mathcal{F}_{t_n}),L^2(D)\big) \\ & S & \mapsto & u \end{array}$$

where u is the solution in $L^2\big((\Omega,\mathcal{F}_{t_n}),H_0^1(D)\big)$ of the linear problem

$$\left\{ \begin{array}{rcccl} -\lambda\Delta u + u - \dfrac{\lambda}{\Delta t}\Delta\varpi & = & J_\lambda(S) & \text{in} & D\times\Omega, \\ u & = & 0 & \text{on} & \partial D. \end{array} \right. \tag{P_λ}$$

T_λ is well defined using the Lax-Milgram theorem (see [19] p.84). Moreover, as f is an increasing Lipschitz-continuous function, by denoting K the Lipschitz coefficient of f, we show easily that J_λ is a contraction when $\lambda \leqslant K^{-1}$. In this way, using Poincaré's inequality, one gets that T_λ is a strict contraction. And then, there exists a unique u in $L^2\big((\Omega,\mathcal{F}_{t_n}),H_0^1(D)\big)$ solution of (E_λ) and in this way, (E) admits a unique solution in $L^2\big((\Omega,\mathcal{F}_{t_n}),H_0^1(D)\big)$.

2 Some calculations

In detail, we have:

1. $$\|x^{\Delta t}\|^2_{L^2(0,T;X)} = E\int_0^T \|x^{\Delta t}\|^2_X dt = E\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}\|x_k\|^2_X dt = \Delta t E\sum_{k=1}^{N}\|x_k\|^2_X.$$

2. $$\begin{aligned}\|\widetilde{x}^{\Delta t}\|^2_{L^2(0,T;X)} &= E\int_0^T \|\widetilde{x}^{\Delta t}\|^2_X dt\\ &= E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\|\frac{x_{k+1}-x_k}{\Delta t}(t-t_k)+x_k\|^2_X dt\\ &\leqslant E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}2\|\frac{x_{k+1}-x_k}{\Delta t}\|^2_X \underbrace{(t-t_k)^2}_{\leqslant \Delta t^2} dt + 2\Delta t\|x_k\|^2_X\\ &\leqslant E\sum_{k=0}^{N-1}2\Delta t\|x_{k+1}-x_k\|^2_X + 2\Delta t\|x_k\|^2_X\\ &\leqslant C\Delta t E\sum_{k=0}^{N}\|x_k\|^2_X.\end{aligned}$$

3. $$\begin{aligned}\|x^{\Delta t}-\widetilde{x}^{\Delta t}\|^2_{L^2(0,T;X)} &= E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\|x_{k+1}-\frac{x_{k+1}-x_k}{\Delta t}(t-t_k)+x_k\|^2_X dt\\ &= E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\|(x_{k+1}-x_k)(1-\frac{t-t_k}{\Delta t})\|^2_X dt\\ &= E\sum_{k=0}^{N-1}\|x_{k+1}-x_k\|^2_X\int_{t_k}^{t_{k+1}}(1-\frac{t-t_k}{\Delta t})^2 dt\\ &= E\sum_{k=0}^{N-1}\|x_{k+1}-x_k\|^2_X(-\Delta t)\left[\frac{1}{3}(1-\frac{t-t_k}{\Delta t})^3\right]_{t_k}^{t_{k+1}}\\ &= C\Delta t E\sum_{k=0}^{N-1}\|x_{k+1}-x_k\|^2_X.\end{aligned}$$

4. $$\|\frac{\partial\widetilde{x}^{\Delta t}}{\partial t}\|^2_{L^2(0,T;X)} = E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\frac{1}{\Delta t^2}\|x_{k+1}-x_k\|^2_X dt = \frac{1}{\Delta t}E\sum_{k=0}^{N-1}\|x_{k+1}-x_k\|^2_X.$$

5. $$\|x^{\Delta t}\|_{L^\infty(0,T;X)} = \max_{k=1,..,N}\|x_k\|_X.$$

6. $$\|\widetilde{x}^{\Delta t}\|_{L^\infty(0,T;X)} = \max_{k=0,..,N}\|x_k\|_X.$$

3 Convergence of $\nabla u^{\Delta t}(. - \Delta t)$

Let us show that

$$\nabla u^{\Delta t}(. - \Delta t) \rightharpoonup \nabla u \text{ in } L^2(\Omega \times Q).$$

One has

$$\begin{aligned}
\|\nabla u^{\Delta t}(. - \Delta t) - \nabla u^{\Delta t}\|^2_{L^2(\Omega\times Q)} &= E\int_0^T \int_D (\nabla u^{\Delta t}(t,x) - \nabla u^{\Delta t}(t - \Delta t, x))^2 dxdt \\
&= E\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_D (\nabla u^{\Delta t}(t,x) - \nabla u^{\Delta t}(t - \Delta t, x))^2 dxdt \\
&= E\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|\nabla u_k - \nabla u_{k-1}\|^2_{L^2(D)} dt \\
&= \Delta t E \sum_{k=1}^n \|\nabla u_k - \nabla u_{k-1}\|^2_{L^2(D)}
\end{aligned}$$

Let us recall that (2.9) p.27 gives us

$$\sum_{k=0}^n E\|\nabla(u^{k+1} - u^k)\|^2_{L^2(D)} \leq C$$

and so

$$\|\nabla u^{\Delta t}(. - \Delta t) - \nabla u^{\Delta t}\|^2_{L^2(\Omega\times Q)} \leqslant C\Delta t.$$

Moreover thanks to (2.8) p.27

$$\|\nabla u^{\Delta t}(. - \Delta t)\|_{L^2(\Omega\times Q)} \quad \leq \quad C,$$

and so, $\nabla u^{\Delta t}(. - \Delta t) \rightharpoonup \nabla u$ in $L^2(\Omega \times Q)$.

Appendix H

On abstract equations: the parabolic case

Let us consider the following nonlinear parabolic problem:

$$(P):\begin{cases}\partial_t u + Au = h \in L^2(0,T,H),\\ u(t=0)=u_0\in V.\end{cases}$$

It is a classical result that there exists a unique weak solution u and that this solution is the mild solution. With the hypothesis on the data, $u \in W^{1,\infty,2}(0,T,V,H)$ and for any t

$$\int_0^t |\partial_t u|^2 ds + J(u(t)) \le J(u_0) + \int_0^t (h,\partial_t u) ds.$$

Moreover, following [8] p.158 for example, one gets that for t in $]0,T[$ a.e., $u(t)\in D(A)$ and $J(u)\in W^{1,1}(0,T)$. By testing the equation with $u(.+\Delta t)-u$, one has that

$$(\partial_t u(s), u(s+\Delta t)-u(s)) + \langle Au(s), u(s+\Delta t)-u(s)\rangle = (h(s), u(s+\Delta t)-u(s))$$

and

$$(\partial_t u(s), u(s+\Delta t)-u(s)) + J(u(s+\Delta t)) \ge J(u(s)) + (h(s), u(s+\Delta t)-u(s))$$

By dividing by $\Delta t>0$ and integrating in time from 0 to $t-\Delta t$, the continuity of $s\mapsto J(u(s))$ yields

$$\int_0^t |\partial_t u|^2 ds + J(u(t)) \ge J(u_0) + \int_0^t (h,\partial_t u) ds.$$

In conclusion, for any t

$$\int_0^t |\partial_t u|^2 ds + J(u(t)) = J(u_0) + \int_0^t (h,\partial_t u) ds.$$

Appendix I

On the Cauchy problem for a stochastic conservation law: detailed results

1 Regularity of integrals with respect to parameters

Denote by Λ a set of parameters, a bounded domain of $\mathbb{R}^d$ for example.

Theorem 1.1 ([57] Theorem 7.6, p. 180) *Suppose $p \geq 2$ and $rp > d$. Let $f_s(\lambda)$, $\lambda \in \Lambda$, be a predictable C_b^r-valued process satisfying for any $|k| \leqslant r$, $\displaystyle\int_0^T \|D^k f_s\|_\infty^p \,\mathrm{dt} < \infty$ a.s..*
Then the real valued stochastic integral $\int_0^t f_s(\lambda) dw(s)$ with parameter λ has a modification $L_t(\lambda)$ which satisfies the following properties:

- *$L_t(\lambda)$ is continuous in (t, λ) and l-times continuously differentiable in λ where $l < r - d/p$.*
- *If $|k| < r - d/p$, then $D^k L_t(\Lambda)$ is continuous in (t, λ) and satisfies $D^k L_t(\Lambda) = \displaystyle\int_0^t D^k f_s(\lambda) dw(s)$ for any t and a.s..*

Let us fix r and denote

$$\begin{aligned}
f_n(y,s,k)(t) &= \int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx},\\
F_n(y,s,k) &= \int_0^T\int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t)\\
&= \int_{s-2/n}^s\int_{\mathbb{R}^d}\int_0^1 \eta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\,\mathrm{d}\alpha\varphi\rho_m(x-y)\rho_n(t-s)\,\mathrm{dx}dw(t).
\end{aligned}$$

Classical result of differentiation yields $f_n \in C^\infty(Q \times \mathbb{R})$ and $f_n(y,s,k)$ is a predictable C_b^r-valued process for any m.
For a multi-indice k, $|D^k f_n(s,y,k)(t)| \leq C(D^k\rho_m \otimes \rho_n \otimes \eta', \varphi)\displaystyle\int_0^1 |h(\hat{\mathbf{u}})|\,\mathrm{d}\alpha$ and one gets that

$E\int_0^T \|D^k f_n(s,y,k)(t)\|_\infty^2 \,\mathrm{dt} < \infty.$

Therefore, F_n is a regular function and $D^k F_n = \int_0^T D^k f_n(.)(\sigma) dw_\sigma$.

2 On the parabolic regularization

The following result is a classic one. One can refer to [30],[45], [64] and many others authors. For the sequel of our purpose, we propose to refer to [77] but for the sake of convenience we propose to redevelop the proofs.

Proposition 2.1
For any positive ϵ, there exists a unique $u_\epsilon \in L^2(]0,T[\times\Omega, H_0^1(\mathbb{R}^d))$, $L^2(\mathbb{R}^d)$-adapted process to the filtration, $\partial_t[u - \int_0^t h(u_\epsilon)dw] \in L^2(]0,T[\times\Omega, H^{-1}(\mathbb{R}^d))$ and weak solution of the stochastic nonlinear parabolic problem

$$du - [\epsilon\Delta u - \mathrm{div}(\vec{\mathbf{f}}(u))]\,\mathrm{dt} = h(u)dw \quad \text{in } \Omega\times\mathbb{R}^d\times]0,T[, \tag{2.1}$$

for the initial condition $u_0 \in L^2(\mathbb{R}^d)$.
Moreover, there exists a positive constant C such that

$$\forall\epsilon > 0, \quad ||u||^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))} + \epsilon||u||^2_{L^2(]0,T[\times\Omega;H_0^1(\mathbb{R}^d))} \leq C.$$

Proof. Following [77], we propose a result of existence of a solution based on an implicit time discretization. The scheme is the following one:
For given small positive parameter Δt and u_n in $L^2(\Omega, H_0^1(\mathbb{R}^d))$, $\mathcal{F}_{n\Delta t}$-measurable, find u_{n+1} in $L^2(\Omega, H_0^1(\mathbb{R}^d))$, $\mathcal{F}_{(n+1)\Delta t}$-measurable, such that P-a.s and for any v in $H_0^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\Big[(u_{n+1}-u_n)v + \Delta t\{\epsilon\nabla u_{n+1}.\nabla v + \vec{\mathbf{f}}(u_{n+1}).\nabla v\}\Big]\,\mathrm{dx} \;=\; (w_{n+1}-w_n)\int_{\mathbb{R}^d} h(u_n)v\,\mathrm{dx}.$$

Lemma 2.2 *If $\Delta t < \dfrac{2\epsilon}{\|\vec{\mathbf{f}}'\|_\infty^2}$, such a sequence (u_n) exists.*

Proof. [of Lemma]
Denote by $\mathbb{V} = L^2(\Omega, H^1(\mathbb{R}^d), \mathcal{F}_{(n+1)\Delta t}, P)$, $\mathbb{L} = L^2(\Omega, L^2(\mathbb{R}^d), \mathcal{F}_{(n+1)\Delta t}, P)$ and by T the appli-

cation, defined for any $S \in \mathbb{H}$, by $u = T(S)$ is the solution in $\mathbb{V}$ of the varational problem

$$\forall v \in \mathbb{V}, E\Big[\int_{\mathbb{R}^d}\Big[(u-u_n)v + \Delta t\{\epsilon\nabla u.\nabla v + \vec{\mathbf{f}}(S).\nabla v\}\Big]\mathrm{dx}\Big] = E\Big[(w_{n+1}-w_n)\int_{\mathbb{R}^d} h(u_n)v\mathrm{dx}\Big].$$

Thanks to the theorem of Lax-Milgram, T is a well-defined function. Moreover, for any $S_1, S_2 \in \mathbb{H}$, one has that

$$E\int_{\mathbb{R}^d}|u_1-u_2|^2\,\mathrm{dx} + \Delta t\epsilon E\int_{\mathbb{R}^d}|\nabla(u_1-u_2)|^2\,\mathrm{dx} \;=\; \Delta t E\int_{\mathbb{R}^d}(\vec{\mathbf{f}}(S_1)-\vec{\mathbf{f}}(S_2)).\nabla(u_1-u_2)\,\mathrm{dx},$$

and

$$E\int_{\mathbb{R}^d}|T(S_1)-T(S_2)|^2\mathrm{dx} + \Delta t\epsilon E\int_{\mathbb{R}^d}|\nabla(T(S_1)-T(S_2))|^2\mathrm{dx} \le \frac{\Delta t}{2\epsilon}E\int_{\mathbb{R}^d}(\vec{\mathbf{f}}(S_1)-\vec{\mathbf{f}}(S_2))^2\mathrm{dx}.$$

Thus, if $\Delta t < \dfrac{2\epsilon}{\|\vec{\mathbf{f}}'\|_\infty^2}$, T is a contractive mapping in $\mathbb{H}$ and the result holds*. □

Setting the test-function u_{n+1} yields

$$\begin{aligned}
&\frac{1}{2}E\int_{\mathbb{R}^d}[|u_{n+1}|^2-|u_n|^2+|u_{n+1}-u_n|^2]\mathrm{dx} + \Delta t\epsilon E\int_{\mathbb{R}^d}|\nabla u_{n+1}|^2\mathrm{dx} + \Delta t E\int_{\mathbb{R}^d}\vec{\mathbf{f}}(u_{n+1}).\nabla u_{n+1}\mathrm{dx}\\
&= E\Big[(w_{n+1}-w_n)\int_{\mathbb{R}^d}h(u_n)[u_{n+1}-u_n]\mathrm{dx}\Big] + E[(w_{n+1}-w_n)\int_{\mathbb{R}^d}h(u_n)u_n\mathrm{dx}].
\end{aligned}\tag{2.2}$$

Note that $\int_{\mathbb{R}^d}\vec{\mathbf{f}}(u).\nabla u\,\mathrm{dx} = 0$ for any $u \in D(\mathbb{R}^d)$, thus for any $u \in H^1(\mathbb{R}^d)$. Then

$$\begin{aligned}
&\frac{1}{2}E\int_{\mathbb{R}^d}[|u_{n+1}|^2-|u_n|^2+|u_{n+1}-u_n|^2]\,\mathrm{dx} + \Delta t\epsilon E\int_{\mathbb{R}^d}|\nabla u_{n+1}|^2\,\mathrm{dx}\\
\le\;& \Delta t E\int_{\mathbb{R}^d}h^2(u_n)\,\mathrm{dx} + \frac{1}{4}E\int_{\mathbb{R}^d}[u_{n+1}-u_n]^2\,\mathrm{dx},
\end{aligned}$$

and, if one denotes by $\|.\|$ the norm in $L^2(\mathbb{R}^d)$

$$\frac{1}{2}E\|u_n\|^2 + \frac{1}{4}\sum_{k=0}^{n-1}E\|u_{k+1}-u_k\|^2 + \Delta t\epsilon\sum_{k=0}^{n-1}E\|\nabla u_{k+1}\|^2 \le \frac{1}{2}\|u_0\|^2 + \Delta t\sum_{k=0}^{n-1}E\|h(u_n)\|^2,$$

The discrete Gronwall lemma asserts then that

$$\begin{aligned}
\frac{1}{2}E\|u_n\|^2 + \frac{1}{4}\sum_{k=0}^{n-1}E\|u_{k+1}-u_k\|^2 + \Delta t\epsilon\sum_{k=0}^{n-1}E\|\nabla u_{k+1}\|^2 \;&\le\; \frac{1}{2}\|u_0\|^2 + \|u_0\|^2\Delta t\|h'\|_\infty^2\sum_{k=0}^{n-1}e^{2\|h'\|_\infty^2 k\Delta t}\\
&\le\; C.
\end{aligned}$$

*The variational equality holds a.s and for all $v \in H^1(\mathbb{R}^d)$ since $H^1(\mathbb{R}^d)$ is separable.

This only gives an $L^\infty(0,T,L^2(\Omega\times\mathbb{R}^d))$ estimate on $u^{\Delta t}$ and an $L^2(\Omega\times Q)$ estimate on $\epsilon\nabla u^{\Delta t}$.

If $u_0\in H^1(\mathbb{R}^d)$, setting the test-function $v=u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)$ yields

$$\begin{aligned}
&\|u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)\|^2_{L^2(\mathbb{R}^d)}\\
&+\Delta t\epsilon\int_{\mathbb{R}^d}\nabla u_{n+1}.\nabla\left[u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)\right]\,\mathrm{dx}\\
=\;&\Delta t\int_{\mathbb{R}^d}\left[u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)\right]\vec{\mathbf{f}}'(u_{n+1}).\nabla u_{n+1}\,\mathrm{dx}\\
\leq\;&\frac{1}{2}\|u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)\|^2_{L^2(\mathbb{R}^d)}+\frac{1}{2}C(\vec{\mathbf{f}}')(\Delta t)^2\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}.
\end{aligned}$$

Since $E(w_{n+1}-w_n)\int_{\mathbb{R}^d}\nabla u_n.\nabla h(u_n)\,\mathrm{dx}=0$, one gets that

$$\begin{aligned}
&E\int_{\mathbb{R}^d}\nabla u_{n+1}.\nabla\left[u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)\right]\,\mathrm{dx}\\
=\;&\frac{1}{2}E\left[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}+\|\nabla(u_{n+1}-u_n)\|^2_{L^2(\mathbb{R}^d)^d}-\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}\right]\\
&-E(w_{n+1}-w_n)\int_{\mathbb{R}^d}\nabla[u_{n+1}-u_n].\nabla h(u_n)\,\mathrm{dx}\\
\geq\;&\frac{1}{2}E\left[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}+\frac{1}{2}\|\nabla(u_{n+1}-u_n)\|^2_{L^2(\mathbb{R}^d)^d}-\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}-2\Delta t\|\nabla h(u_n)\|^2_{L^2(\mathbb{R}^d)^d}\right].
\end{aligned}$$

And then

$$\begin{aligned}
&E\|u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)\|^2_{L^2(\mathbb{R}^d)}\\
&+\Delta t\epsilon E\left[\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}-\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}+\frac{1}{2}\|\nabla(u_{n+1}-u_n)\|^2_{L^2(\mathbb{R}^d)^d}\right]\\
\leq\;&2(\Delta t)^2\epsilon E\|\nabla h(u_n)\|^2_{L^2(\mathbb{R}^d)^d}+C(\vec{\mathbf{f}}')(\Delta t)^2E\|\nabla u_{n+1}\|^2_{L^2(\mathbb{R}^d)^d}.
\end{aligned}$$

Consequently, for any k,

$$\begin{aligned}
&\sum_{n=0}^{k}\Delta tE\|\frac{u_{n+1}-u_n-(w_{n+1}-w_n)h(u_n)}{\Delta t}\|^2_{L^2(\mathbb{R}^d)}+\epsilon E\|\nabla u_{k+1}\|^2_{L^2(\mathbb{R}^d)^d}\\
&+\frac{\epsilon}{2}E\sum_{n=0}^{k}\|\nabla(u_{n+1}-u_n)\|^2_{L^2(\mathbb{R}^d)^d}\\
\leq\;&C(\vec{\mathbf{f}}',h',\epsilon)\Delta t\sum_{n=0}^{k+1}E\|\nabla u_n\|^2_{L^2(\mathbb{R}^d)^d}+\epsilon E\|\nabla u_0\|^2_{L^2(\mathbb{R}^d)^d}\leq Cte.
\end{aligned}$$

Let us denote $u^{\Delta t}=\sum_{k=1}^{N}u_k1_{[(k-1)\Delta t,k\Delta t[}$, $\widetilde{u}^{\Delta t}=\sum_{k=1}^{N}\Big[\frac{u_k-u_{k-1}}{\Delta t}[t-(k-1)\Delta t]+u_{k-1}\Big]1_{[(k-1)\Delta t,k\Delta t[}$ and

$\widetilde{B}^{\Delta t} = \sum_{k=1}^{N} \Big[\frac{B_k - B_{k-1}}{\Delta t} [t - (k-1)\Delta t] + B_{k-1} \Big] 1_{[(k-1)\Delta t, k\Delta t[}$ with

$B_n = \sum_{k=0}^{n-1} (w^{k+1} - w^k) h(u^k) = \int_0^{n\Delta t} h(u^{\Delta t}(. - \Delta t)) dw$ †

One gets that $u^{\Delta t}$ and $\widetilde{u}^{\Delta t}$ are bounded in $L^\infty(0,T,L^2(\Omega,H^1(\mathbb{R}^d)))$, $\partial_t \big[\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}\big]$ is bounded in $L^2(0,T,L^2(\Omega,H^{-1}(\mathbb{R}^d)))$ and, if $u_0 \in H^1(\mathbb{R}^d)$, in $L^2(0,T,L^2(\Omega,L^2(\mathbb{R}^d)))$ and $\widetilde{u}^{\Delta t} - u^{\Delta t}$ converges to 0 in $L^2(0,T,L^2(\Omega,H^1(\mathbb{R}^d)))$.

Denote by u a limit point of $u^{\Delta t}$ and $\widetilde{u}^{\Delta t}$ for the weak-*convergence in $L^\infty(0,T,L^2(\Omega,H^1(\mathbb{R}^d)))$, h_u, respectively $\vec{\mathbf{f}}_u$, a limit point of $h(u^{\Delta t})$, resp. $\vec{\mathbf{f}}(u^{\Delta t})$, for the weak convergence in $L^2(0,T,L^2(\Omega,H^1(\mathbb{R}^d)))$.

Since $\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}$ converges weakly in $L^2(\Omega, W(0,T))$ where $W(0,T)$ denotes the set of $L^2(0,T,H^1(\mathbb{R}^d))$-functions w such that $\partial_t w \in L^2(0,T,H^{-1}(\mathbb{R}^d))$ with the common identification of $L^2(\mathbb{R}^d)$ with its dual space, $\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}$ converges weakly in $L^2(\Omega, C([0,T],L^2(\mathbb{R}^d)))$. Thus, for any $t \in [0,T]$, $(\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t})(t)$ converges weakly in $L^2(\Omega,L^2(\mathbb{R}^d))$.

Note that for $t \in [n\Delta t, (n+1)\Delta t[$, one has

$$\widetilde{B}^{\Delta t}(t) - \int_0^t h(u^{\Delta t}(s-\Delta t) dw(s) = (w^{n+1} - w^n) h(u^n) \frac{t - n\Delta t}{\Delta t} - (w(t) - w^n) h(u^n).$$

Then, thanks to the *a priori* estimates,

$$\begin{aligned} & E[\|(w^{n+1} - w^n) h(u^n) \frac{t - n\Delta t}{\Delta t} - (w(t) - w^n) h(u^n)\|^2] \\ = \quad & E[\|h(u^n)\|^2] \Big[\frac{(t - n\Delta t)^2}{\Delta t} - 2\frac{t - n\Delta t}{\Delta t}(t - n\Delta t) + (t - n\Delta t) \Big] \leq C\Delta t. \end{aligned}$$

Since $h(u^{\Delta t}(.-\Delta t))$, as $h(u^{\Delta t})$ converges weakly to some function h_u in $L^2(0,T,L^2(\Omega,L^2(\mathbb{R}^d)))$, thanks to the properties of the Itô integral, $\int_0^\cdot h(u^{\Delta t}(s-\Delta t) dw(s)$ converges weakly to $\int_0^\cdot h_u dw(s)$ in $C([0,T],L^2(\Omega,L^2(\mathbb{R}^d)))$, and $\widetilde{B}^{\Delta t}$ does the same.

Thus, the weak convergence of $\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}$ is toward $u - \int_0^\cdot h_u dw(s)$ and, for any t, $\widetilde{u}^{\Delta t}(t)$ converges weakly in $L^2(\Omega,L^2(\mathbb{R}^d))$ to $u(t)$.

Moreover, for any $v \in H^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \partial_t \big[\widetilde{u}^{\Delta t} - \widetilde{B}^{\Delta t}\big] v \, \mathrm{dx} + \epsilon \int_{\mathbb{R}^d} \nabla u^{\Delta t} \nabla v \, \mathrm{dx} + \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u^{\Delta t}) \nabla v \, \mathrm{dx} = 0$$

†We consider that $u^{\Delta t}(s) = u_0$ if $s < 0$.

and at the limit, $u(0,.) = u_0$ and,

$$< \partial_t \left[u - \int_0^t h_u dw(s) \right], v > + \epsilon \int_{\mathbb{R}^d} \nabla u \nabla v \text{ dx} + \int_{\mathbb{R}^d} \vec{\mathbf{f}}_u \nabla v \text{ dx} = 0,$$

with the remark that one has an integral over $\mathbb{R}^d$ instead of the duality bracket if $u_0 \in H^1(\mathbb{R}^d)$. Then, the Itô formula yields, for any positive c

$$\begin{aligned} & e^{-ct} E\|u(t)\|^2 + 2\epsilon \int_0^t e^{-cs} E\|\nabla u\|^2 \text{ ds} + 2\int_0^t E \int_{\mathbb{R}^d} e^{-cs} \vec{\mathbf{f}}_u \nabla u \text{ dx ds} \\ = \;& \|u_0\|^2 - c\int_0^t e^{-cs} E\|u(s)\|^2 \text{ ds} + \int_0^t e^{-cs} E\|h_u\|^2 \text{ ds.} \end{aligned} \tag{2.3}$$

From (2.2), one has, for any positive c and $n > 0$, that

$$\begin{aligned} & E \int_{\mathbb{R}^d} [e^{-cn\Delta t}|u_{n+1}|^2 - e^{-c(n-1)\Delta t}|u_n|^2] \text{ dx} + \Delta t 2\epsilon e^{-cn\Delta t} E \int_{\mathbb{R}^d} |\nabla u_{n+1}|^2 \text{ dx} \\ \leq \;& \Delta t e^{-cn\Delta t} E \int_{\mathbb{R}^d} h^2(u_n) \text{ dx} + [e^{-cn\Delta t} - e^{-c(n-1)\Delta t}] E \int_{\mathbb{R}^d} |u_n|^2 dx. \end{aligned}$$

Adding from 0 to k, one gets that

$$\begin{aligned} & e^{-ck\Delta t} E\|u_{k+1}\|^2 + \Delta t 2\epsilon \sum_{n=0}^{k} e^{-cn\Delta t} E\|\nabla u_{n+1}\|^2 \\ \leq \;& \|u_0\|^2 + \Delta t \sum_{n=0}^{k} e^{-cn\Delta t} E\|h(u_n)\|^2 - c\Delta t \sum_{n=1}^{k} e^{-c(n+1)\Delta t} E\|u_n\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} & e^{-ck\Delta t} E\|u_{k+1}\|^2 + 2\epsilon \int_0^{(k+1)\Delta t} e^{-cs} E\|\nabla u^{\Delta t}\|^2 \text{ ds} \\ \leq \;& \|u_0\|^2 + \Delta t\|h(u_0)\|^2 + \int_0^{k\Delta t} e^{-cs} E\|h(u^{\Delta t})\|^2 \text{ ds} - c e^{-c\Delta t} \int_0^{k\Delta t} e^{-cs} E\|u^{\Delta t}\|^2 \text{ ds.} \end{aligned}$$

For $t \in [k\Delta t, (k+1)\Delta t[$, we obtain

$$\begin{aligned} & e^{-ct} E\|u^{\Delta t}(t)\|^2 + 2\epsilon \int_0^t e^{-cs} E\|\nabla u_{\Delta t}\|^2 \text{ ds} \\ \leq \;& \|u_0\|^2 + \Delta t\|h(u_0)\|^2 + \int_0^t e^{-cs} E\|h(u^{\Delta t})\|^2 \text{ ds} - c e^{-c\Delta t} \int_0^{(t-\Delta t)^+} e^{-cs} E\|u^{\Delta t}\|^2 \text{ ds,} \end{aligned}$$

and, since $u^{\Delta t}$ is bounded in $L^\infty(0,T,L^2(\Omega,L^2(\mathbb{R}^d)))$, one gets that

$$\begin{aligned}
& e^{-ct}E\|u^{\Delta t}(t)\|^2 + 2\epsilon\int_0^t e^{-cs}E\|\nabla u^{\Delta t}\|^2\,\mathrm{ds} \\
\le\ & \|u_0\|^2 + \Delta t\|h(u_0)\|^2 + \int_0^t e^{-cs}E\|h(u^{\Delta t})\|^2\,\mathrm{ds} \\
& + ce^{-c\Delta t}\Big[\int_{(t-\Delta t)^+}^t e^{-cs}E\|u^{\Delta t}\|^2\,\mathrm{ds} - \int_0^t e^{-cs}E\|u^{\Delta t}\|^2\,\mathrm{ds}\Big] \\
\le\ & \|u_0\|^2 + C\Delta t + \int_0^t e^{-cs}E\|h(u^{\Delta t})\|^2\,\mathrm{ds} - ce^{-c\Delta t}\int_0^t e^{-cs}E\|u^{\Delta t}\|^2\,\mathrm{ds}.
\end{aligned}$$

As, for any v in $H^1(\mathbb{R}^d)$ $\int_{\mathbb{R}^d}\vec{\mathbf{f}}(v)\nabla v\,\mathrm{dx} = 0$, one has

$$\begin{aligned}
& e^{-ct}E\|u^{\Delta t}(t)\|^2 + 2\epsilon\int_0^t e^{-cs}E\|\nabla(u^{\Delta t}-u)\|^2\,\mathrm{ds} \\
& + 2\int_0^t e^{-cs}E\int_{\mathbb{R}^d}[\vec{\mathbf{f}}(u^{\Delta t}) - \vec{\mathbf{f}}(u)]\nabla[u^{\Delta t}-u]\,\mathrm{dx}\,\mathrm{ds} \\
& + 4\epsilon\int_0^t e^{-cs}E\int_{\mathbb{R}^d}\nabla u^{\Delta t}\nabla u\,\mathrm{dx}\,\mathrm{ds} - 2\epsilon\int_0^t e^{-cs}E\|\nabla u\|^2\,\mathrm{ds} \\
\le\ & \|u_0\|^2 + C\Delta t - 2\int_0^t e^{-cs}E\int_{\mathbb{R}^d}\vec{\mathbf{f}}(u^{\Delta t})\nabla u\,\mathrm{dx}\,\mathrm{ds} - 2\int_0^t e^{-cs}E\int_{\mathbb{R}^d}\vec{\mathbf{f}}(u)\nabla u^{\Delta t}\,\mathrm{dx}\,\mathrm{ds} \\
& + 2\int_0^t e^{-cs}E\int_{\mathbb{R}^d}h(u^{\Delta t})h(u)\,\mathrm{dx}\,\mathrm{ds} - \int_0^t e^{-cs}E\|h(u)\|^2\,\mathrm{ds} \\
& + \int_0^t e^{-cs}E\|h(u^{\Delta t}) - h(u)\|^2\,\mathrm{ds} - ce^{-c\Delta t}\int_0^t e^{-cs}E\|u^{\Delta t}-u\|^2\,\mathrm{ds} \\
& - 2ce^{-c\Delta t}\int_0^t e^{-cs}E\int_{\mathbb{R}^d}u^{\Delta t}u\,\mathrm{dx}\,\mathrm{ds} + ce^{-c\Delta t}\int_0^t e^{-cs}E\|u\|^2\,\mathrm{ds}.
\end{aligned}$$

Note that there exists $c = C(\vec{\mathbf{f}},h,\epsilon) > 0$ such that, for Δt small, one has that

$$\begin{aligned}
& -2\epsilon\int_0^t e^{-cs}E\|\nabla(u^{\Delta t}-u)\|^2\,\mathrm{ds} - 2\int_0^t e^{-cs}E\int_{\mathbb{R}^d}[\vec{\mathbf{f}}(u^{\Delta t}) - \vec{\mathbf{f}}(u)]\nabla[u^{\Delta t}-u]\,\mathrm{dx}\,\mathrm{ds} \\
& + \int_0^t e^{-cs}E\|h(u^{\Delta t}) - h(u)\|^2\,\mathrm{ds} - ce^{-c\Delta t}\int_0^t e^{-s}E\|u^{\Delta t}-u\|^2\,\mathrm{ds} \\
\le\ & -\epsilon\int_0^t e^{-cs}E\|\nabla(u^{\Delta t}-u)\|^2\,\mathrm{ds} + \frac{1}{\epsilon}\int_0^t e^{-cs}E\|\vec{\mathbf{f}}(u^{\Delta t}) - \vec{\mathbf{f}}(u)\|^2\,\mathrm{ds} \\
& + \int_0^t e^{-cs}E\|h(u^{\Delta t}) - h(u)\|^2\,\mathrm{ds} - ce^{-c\Delta t}\int_0^t e^{-cs}E\|u^{\Delta t}-u\|^2\,\mathrm{ds} \\
\le\ & -\epsilon\int_0^t e^{-cs}E\|\nabla(u^{\Delta t}-u)\|^2\,\mathrm{ds}.
\end{aligned}$$

Thus

$$
\begin{aligned}
& \int_0^T e^{-ct} E\|u^{\Delta t}(t)\|^2 \,\mathrm{dt} + \epsilon \int_0^T \int_0^t e^{-cs} E\|\nabla(u^{\Delta t} - u)\|^2 \,\mathrm{ds}\,\mathrm{dt} \\
\leq\ & T\|u_0\|^2 + C\Delta t \qquad (2.4)\\
& -2\int_0^T \int_0^t e^{-cs} E \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u^{\Delta t})\nabla u \,\mathrm{dx}\,\mathrm{ds}\,\mathrm{dt} - 2\int_0^T \int_0^t e^{-cs} E \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u)\nabla u^{\Delta t} \,\mathrm{dx}\,\mathrm{ds}\,\mathrm{dt} \\
& +2\int_0^T \int_0^t e^{-cs} E \int_{\mathbb{R}^d} h(u^{\Delta t})h(u) \,\mathrm{dx}\,\mathrm{ds}\,\mathrm{dt} - \int_0^T \int_0^t e^{-cs} E\|h(u)\|^2 \,\mathrm{ds}\,\mathrm{dt} \\
& -2ce^{-c\Delta t} \int_0^T \int_0^t e^{-cs} E \int_{\mathbb{R}^d} u^{\Delta t} u \,\mathrm{dx}\,\mathrm{ds}\,\mathrm{dt} + ce^{-c\Delta t} \int_0^T \int_0^t e^{-cs} E\|u\|^2 \,\mathrm{ds}\,\mathrm{dt} \\
& -4\epsilon \int_0^T \int_0^t e^{-cs} E \int_{\mathbb{R}^d} \nabla u^{\Delta t} \nabla u \,\mathrm{dx}\,\mathrm{ds}\,\mathrm{dt} + 2\epsilon \int_0^T \int_0^t e^{-cs} E\|\nabla u\|^2 \,\mathrm{ds}\,\mathrm{dt}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
& \limsup_{\Delta t} \int_0^T e^{-ct} E\|u^{\Delta t}(t)\|^2 \,\mathrm{dt} \\
\leq\ & \int_0^T \Big[\|u_0\|^2 - 2\int_0^t e^{-cs} E \int_{\mathbb{R}^d} \vec{\mathbf{f}}_u \nabla u \mathrm{dx}\mathrm{ds} - 2\epsilon \int_0^t e^{-cs} E\|\nabla u\|^2 \mathrm{ds} - c\int_0^t e^{-cs} E\|u\|^2 \mathrm{ds}\Big]\mathrm{dt} \\
& +2\int_0^T \int_0^t e^{-cs} E \int_{\mathbb{R}^d} h_u h(u) \mathrm{dx}\mathrm{ds}\mathrm{dt} - \int_0^T \int_0^t e^{-cs} E\|h(u)\|^2 \mathrm{ds}\mathrm{dt},
\end{aligned}
$$

and, thanks to (2.3),

$$
\limsup_{\Delta t} \int_0^T e^{-ct} E\|u^{\Delta t}(t)\|^2 \,\mathrm{dt} + \int_0^T \int_0^t e^{-cs} E\|h_u - h(u)\|^2 \,\mathrm{ds}\,\mathrm{dt} \leq \int_0^T e^{-ct} E\|u(t)\|^2 \,\mathrm{dt}.
$$

Thus, one gets that $h_u = h(u)$, $u^{\Delta t}$ converges to u in $L^2(]0,T[\times\Omega\times\mathbb{R}^d)$ and $\vec{\mathbf{f}}_u = \vec{\mathbf{f}}(u)$. This means that u is a solution.
Remark that it is a direct proof to show that it is unique.

Then, the stochastic energy asserts that (see for example Grecksch-Tudor [45] Th. 3.4 p.42):

$$
\begin{aligned}
& \|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \int_{\mathbb{R}^d} [\epsilon|\nabla u|^2 + \vec{\mathbf{f}}(u).\nabla u] \,\mathrm{dx}\,\mathrm{ds} \\
=\ & \|u(0)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \int_{\mathbb{R}^d} uh(u) \,\mathrm{dx}dw_s + \int_0^t \int_{\mathbb{R}^d} h^2(u) \,\mathrm{dx}\,\mathrm{ds}.
\end{aligned}
$$

Since $\int_0^t \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u_n).\nabla u_n \,\mathrm{dx}\,\mathrm{ds} = 0$, taking the expectation and using the lemma of Gronwall yield the result of Proposition 2.1. □

Corollary 2.3 *If, moreover $u_0 \in H_0^1(\mathbb{R}^d)$, then*

$$\partial_t[u - \int_0^t h(u)dw] \in L^2(]0,T[\times\Omega, L^2(\mathbb{R}^d)) \quad and \quad \Delta u \in L^2(]0,T[\times\Omega, L^2(\mathbb{R}^d)).$$

Proof. The first part of the corollary is in the previous proof.
Then, one gets that $-\epsilon\Delta u = \operatorname{div}\vec{\mathbf{f}}(u) - \partial_t(u - \int_0^t h(u)dw) \in L^2(]0,T[\times\Omega\times\mathbb{R}^d)$. □

Proposition 2.4 *If the initial condition $u_0 \in L^{2p}(\mathbb{R}^d)$, $p \geq 1$, then $u \in L^\infty(0,T,L^{2p}(\Omega\times\mathbb{R}^d))$ as well.*

Proof. For any positive k, denote by ϕ_k the even function such that

$$\phi_k(x) = \begin{cases} if\ 0 \leq x < k & x^{2p} \\ if\ k \leq x & p(2p-1)k^{2(p-1)}x^2 - 4p(p-1)k^{2p-1}x + (p-1)(2p-1)k^{2p} \end{cases}$$

ϕ_k is a C^2-convex function and ϕ_k' is a Lipschitz-continuous function with $\phi_k'(0) = 0$. Thus, for any positive x, one gets $0 \leq \phi_k'(x) \leq c(k)x$ and $0 \leq \phi_k(x) = \int_0^x \phi_k'(s)\,\mathrm{ds} \leq \frac{c(k)}{2}x^2$. This yields $E\int_{\mathbb{R}^d}\phi_k(u(t))\,\mathrm{dx} < \infty$.

Lemma 2.5 *For any $x \in \mathbb{R}$, one has $0 \leq x^2\phi_k''(x) \leq 2(2p-1)^2\phi_k(x)$.*

Proof.
If $|x| < k$, $x^2\phi_k''(x) = 2p(2p-1)x^{2p} \leq 2(2p-1)^2\phi_k(x)$ since $p \leq 2p-1$.
If $|x| \geq k$,

$$\begin{aligned} & x^2\phi_k''(x) - 2(2p-1)^2\phi_k(x) \\ =& 2p(2p-1)x^2k^{2(p-1)} - 2(2p-1)^2[p(2p-1)k^{2(p-1)}x^2 - 4p(p-1)k^{2p-1}x + (p-1)(2p-1)k^{2p}] \\ =& 8p^2(2p-1)(1-p)k^{2(p-1)}x^2 + 8(2p-1)^2p(p-1)k^{2p-1}x - 2(2p-1)^3(p-1)k^{2p} \\ =& 2(2p-1)(p-1)k^{2(p-1)}\Big[-4p^2x^2 + 4(2p-1)pkx - (2p-1)^2k^2\Big] \leq 0. \end{aligned}$$

□

Thanks to the Itô formula, one gets that

$$\begin{aligned} & E\int_{\mathbb{R}^d}\phi_k(u(t))\,\mathrm{dx} + \epsilon E\int_0^t\int_{\mathbb{R}^d}\phi_k''(u)|\nabla u|^2\,\mathrm{dx\,ds} + E\int_0^t\int_{\mathbb{R}^d}\vec{\mathbf{f}}(u).\nabla\phi_k'(u)\,\mathrm{dx\,ds} \\ =\ & E\int_{\mathbb{R}^d}\phi_k(u_0)\,\mathrm{dx} + \frac{1}{2}E\int_0^t\int_{\mathbb{R}^d}h^2(u)\phi_k''(u)\,\mathrm{dx\,ds}. \end{aligned}$$

Since $\phi_k'' \geq 0$ and $E\int_0^t \int_{\mathbb{R}^d} \vec{\mathbf{f}}(u).\nabla\phi_k'(u)\,\mathrm{dx}\,\mathrm{ds} = 0$, one has that

$$E\int_{\mathbb{R}^d} \phi_k(u(t))\,\mathrm{dx} \quad\leq\quad C + \frac{1}{2}E\int_0^t \int_{\mathbb{R}^d} h^2(u)\phi_k''(u)\,\mathrm{dx}\,\mathrm{ds}.$$

Then, assumptions on h and the previous lemma yield

$$E\int_{\mathbb{R}^d} \phi_k(u(t))\,\mathrm{dx} \quad\leq\quad C + C(h,p)E\int_0^t \int_{\mathbb{R}^d} \phi_k(u)\,\mathrm{dx}\,\mathrm{ds}.$$

Thanks to Gronwall's lemma, $E\displaystyle\int_{\mathbb{R}^d} \phi_k(u(t))\,\mathrm{dx}$ is bounded, independently of k and at the limit when k goes to infinity the theorem of Beppo Levi yields the proof. $\square$

3 A basic reminder of Young measures

3.1 In finite measure spaces

In this section we recall some basic facts on Young measures and refer to Balder [7], Castaing-Raynaud de Fitte-Valadier [25], Saadoune-Valadier [66] and Valadier [73] for an abstract setting on the convergence of Young measures; and to DiPerna [37], Eymard-Gallouët-Herbin [40], Panov [61], Szepessy [70] and Tartar [71] for an application to nonlinear PDE.

Consider the space $L^1(\Theta,\mu,\mathbb{R})$ where $(\Theta,\mathcal{F},\mu)$ is a measure space with a positive bounded measure μ.

For u in $L^1(\Theta,\mu,\mathbb{R})$, the Young measure associated with u is τ_u, the measure on $\Theta\times\mathbb{R}$ image of μ by $x \mapsto (x,u(x))$.

A general Young measure τ is a positive measure on $\Theta\times\mathbb{R}$ such that, for any A in $\mathcal{F}$, $\tau(A\times\mathbb{R}) = \mu(A)$.

A Young measure τ is described by its disintegration which is the unique family of probability measures on $\mathbb{R}$, $(\,\mathrm{d}\tau_x)_{x\in\Theta}$, such that for any $\tau-$measurable function ψ

$$x \mapsto \int_{\mathbb{R}} \psi(x,\lambda)\,\mathrm{d}\tau_x(\lambda) \text{ is } \mu-\text{measurable on } \Theta \text{ and}$$

$$\text{if } \psi\geq 0,\ \int_{\Theta\times\mathbb{R}} \psi\,\mathrm{d}\tau = \int_\Theta\int_{\mathbb{R}} \psi(x,\lambda)\,\mathrm{d}\tau_x(\lambda)\,\mu(\,\mathrm{dx})\ .$$

Therefore, if $\tau = \tau_u$ is the Young measure associated with the above function u, then $\tau_x = \delta_{u(x)}$, the Dirac mass at $u(x)$.

Another way to define Young measures on $\Theta\times\mathbb{R}$ is to consider $\widetilde{\mathbf{u}}$ the notion of entropy process proposed by EYMARD-GALLOUËT-HERBIN [40] or $\mathbf{u}$ the strong measure-valued solution proposed by PANOV [61]. For a Young measure τ on $\Theta\times\mathbb{R}$ and F_x the left-continuous repartition function of τ_x, the functions $\widetilde{\mathbf{u}}$ and $\mathbf{u}$ are defined in $\Theta\times]0,1[$ by

$$\widetilde{\mathbf{u}}(x,\alpha)=\sup\{t\in\mathbb{R},F_x(t)<\alpha\},\quad \mathbf{u}(x,\alpha)=\inf\{t\in\mathbb{R},F_x(t)>\alpha\}. \tag{3.1}$$

For fixed x, the two functions differ only on a countable set. Each author proves that the function is a $\mu\times\mathcal{L}$ measurable function on $\Theta\times]0,1[$ and for any positive Carathéodory function ψ

$$\int_\Theta\int_\mathbb{R}\psi(x,\lambda)\mathrm{d}\tau_x(\lambda)\mu(\mathrm{dx})=\int_\Theta\int_0^1\psi(x,\mathbf{u}(x,\alpha))d\alpha\ \mu(\mathrm{~dx})=\int_\Theta\int_0^1\psi(x,\widetilde{\mathbf{u}}(x,\alpha))d\alpha\,\mu(\mathrm{~dx}).$$

A sequence of Young measures $(\tau^n)_n$ is said to converge narrowly towards τ if $\int_{\Theta\times\mathbb{R}}\psi\,\mathrm{d}\tau^n$ converges towards $\int_{\Theta\times\mathbb{R}}\psi\,\mathrm{d}\tau$ for all bounded Carathéodory function ψ.

Consider now $(u_n)_n\subset L^1(\Theta,\mu,\mathbb{R})$ and denote by τ^n the associated Young measures.
If the sequence $(u_n)_n$ is assumed to be bounded in $L^1(\Theta)$, the theorem of Prohorov for Young measures (BALDER [7], SAADOUNE-VALADIER [66] and VALADIER [73]) ensures that a subsequence $(\tau^{n_k})_k$ of $(\tau^n)_n$ and a Young measure τ exist such that τ^{n_k} converges narrowly towards τ.
Moreover
i) for μ-a.e. x in Θ, $supp(\mathrm{~d}\tau_x)\subset\cap_{p=1}^\infty\overline{\cup_{n\geq p}\{u_n(x)\}}$.
ii) for any Carathéodory function ψ such that the sequence of functions $\{\psi(.,u_n(.))\}_n$ is uniformly integrable,

$$\int_\Theta\psi(x,u_n(x))\,\mu(\mathrm{~dx})\to\int_{\Theta\times\mathbb{R}}\psi(x,\lambda)\,\mathrm{d}\tau$$

(if the sequence $(u_n)_n$ is uniformly integrable, the above convergence still holds if one assumes that $|\psi(x,\lambda)|\leq\alpha(x)+k|\lambda|$ where $k\geq0$ and $\alpha\in L^1(\Theta)$).
iii) for any measurable function ψ, l.s.c. with respect to its second variable and such that $\{\psi(.,u_n(.))^-\}_n$ is uniformly integrable,

$$\liminf_{n\to\infty}\int_\Theta\psi(x,u_n(x))\,\mu(\mathrm{~dx})\geq\int_{\Theta\times\mathbb{R}}\psi(x,\lambda)\,\mathrm{d}\tau.$$

As a consequence, if u_n converges weakly to some u in L^1, it converges strongly to u in L^1, if and only if τ^n converges narrowly to τ_u (*i.e.* if $\mathbf{u}$, or resp. $\widetilde{\mathbf{u}}$, is independent of α).

3.2 In $Q \times \Omega$

Consider in the sequel a bounded sequence (u_n) in $\mathcal{N}^2_w(0,T;L^2(\mathbb{R}^d))$.

For any $M > 0$, if one denotes by $Q_M =]0,T[\times B(0,M)$, the sequence is bounded in $L^2(Q_M \times \Omega)$ and it converges, up to a subsequence still denoted (u_n), in the sense of the Young measures in $Q_M \times \Omega$ to a given τ^M.
Setting $K > M$, a new subsequence converges in the sense of the Young measures in $Q_K \times \Omega$ to τ^K. Thus, for any $v \in L^1(Q_M \times \Omega)$ (extended by 0 in $[Q_K \backslash Q_M] \times \Omega$) and any bounded continuous function f, one gets that
$\int_{Q_M \times \Omega} v[\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\tau^{\mathrm{M}}(\lambda) - \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\tau^{\mathrm{K}}(\lambda)]\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{dP} \to 0.$
Thus, for any bounded continuous function f, $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\tau^{\mathrm{M}}(\lambda) = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\tau^{\mathrm{K}}(\lambda)$ in $Q_M \times \Omega$ and $\tau^M = \tau^K$ restricted to $Q_M \times \Omega$.

Therefore, by a diagonal extraction of subsequences, there exists a Young measure τ on $Q \times \Omega \times \mathbb{R}$ such that,
if $\psi : (t,x,\omega,\lambda) \in Q \times \Omega \times \mathbb{R} \mapsto \psi(t,x,\omega,\lambda)$ such that $\psi(.,u_n)$ is uniformly integrable, then

$$E\int_Q \psi(.,u_n)\,\mathrm{dt}\,\mathrm{dx} \to E\int_Q \int_{\mathbb{R}} \psi(.,\lambda)\,\mathrm{d}\tau_{(t,x,\omega)}(\lambda)\,\mathrm{dt}\,\mathrm{dx}.$$

To prove it, recall that a bounded sequence $\psi(.,u_n)$ in $L^1(Q \times \Omega)$ is uniformly integrable when (denote $\mathcal{L}^m$ the measure of Lebesgue in $\mathbb{R}^m$)

1. $\forall \epsilon > 0,\ \exists \delta > 0,\ \mathcal{L}^{d+1} \otimes P(A) < \delta \Longrightarrow \sup_n \int_A |\psi(.,u_n)|\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{dP} < \epsilon$.
2. $\forall \epsilon > 0,\ \exists M_\epsilon > 0,\ \sup_n \int_{[Q \backslash Q_{M_\epsilon}] \times \Omega} |\psi(.,u_n)|\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{dP} < \epsilon$.[‡]

Set $\epsilon > 0$. Thanks to the first item of the uniform integrability, for any positive K, $|\psi(.,u_n)|$ is uniformly integrable in $Q_K \times \Omega$ and

$$\int_{Q_K \times \Omega} |\psi(.,u_n)|\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{dP} \to \int_{Q_K \times \Omega \times \mathbb{R}} |\psi|\,\mathrm{d}\tau.$$

[‡]This is needed since $\mathcal{L}^d$ is not finite on $\mathbb{R}^d$. This condition is useless when one considers the U.I. in a bounded measure space.

In particular, for any $K > M_\epsilon$

$$\int_{[Q_K \backslash Q_{M_\epsilon}]\times\Omega} |\psi(.,u_n)| \,\mathrm{dx\,dt\,dP} \to \int_{[Q_K \backslash Q_{M_\epsilon}]\times\Omega\times\mathbb{R}} |\psi| \,\mathrm{d}\tau$$

and, for any $K > M_\epsilon$

$$\int_{[Q_K \backslash Q_{M_\epsilon}]\times\Omega\times\mathbb{R}} |\psi| \,\mathrm{d}\tau \le \epsilon.$$

Then the theorem of Beppo Levi yields

$$\int_{[Q \backslash Q_{M_\epsilon}]\times\Omega\times\mathbb{R}} |\psi| \,\mathrm{d}\tau \le \epsilon.$$

Using the notation $\mathbf{u}$, one gets that $\psi(.,\mathbf{u}) \in L^1(Q\times\Omega\times]0,1[)$ and

$$\begin{aligned}
&\left|\int_{Q\times\Omega} \psi(.,u_n)\,\mathrm{dx\,dt\,dP} - \int_{Q\times\Omega\times]0,1[} \psi(.,\mathbf{u})\,\mathrm{d}\alpha\,\mathrm{dx\,dt\,dP}\right| \\
\le\ &\left|\int_{Q_{M_\epsilon}\times\Omega} |\psi(.,u_n)|\,\mathrm{dx\,dt\,dP} - \int_{Q_{M_\epsilon}\times\Omega\times]0,1[} |\psi(.,\mathbf{u})|\,\mathrm{d}\alpha\,\mathrm{dx\,dt\,dP}\right| \\
&+ \int_{[Q\backslash Q_{M_\epsilon}]\times\Omega} |\psi(.,u_n)|\,\mathrm{dx\,dt\,dP} + \int_{[Q\backslash Q_{M_\epsilon}]\times\Omega\times]0,1[} |\psi(.,\mathbf{u})|\,\mathrm{d}\alpha\,\mathrm{dx\,dt\,dP}.
\end{aligned}$$

Then

$$\limsup_n \left|\int_{Q\times\Omega} \psi(.,u_n)\,\mathrm{dx\,dt\,dP} - \int_{Q\times\Omega\times]0,1[} \psi(.,\mathbf{u})\,\mathrm{d}\alpha\,\mathrm{dx\,dt\,dP}\right| \le 2\epsilon$$

and the result holds since the above inequality is satisfied for any $\epsilon > 0$.

Assume that $(\psi(t,x,\omega,\lambda))$ is bounded in $L^p(Q\times\Omega)$ for a given $p \in]1,+\infty]$. Then, one gets that $\psi(.,u_n)$ converges weakly (resp. *-weakly if $p = +\infty$) to $\int_0^1 \psi(.,\mathbf{u})\ \mathrm{d}\alpha$ in $L^p(Q\times\Omega)$.
Indeed, up to a subsequence $\psi(.,u_{n_k})$ converges weakly in $L^p(Q\times\Omega)$ (resp. *-weakly if $p = +\infty$) to an element called χ.
But, for any $\varphi \in L^q(Q\times\Omega)$ where q is the conjugate of p, $(\varphi\psi(.,u_n))$ is uniformly integrable[§].
Thus, at the limit, $\int_{Q\times\Omega} \varphi\chi\,\mathrm{dt\,dx\,dP} = \int_{Q\times\Omega\times]0,1[} \psi(.,\mathbf{u})\,\mathrm{d}\alpha\varphi\,\mathrm{dt\,dx\,dP}$. Then the limit is identified and the subsequence is not needed anymore.

In particular, if (u_n) is a bounded sequence in $L^p(Q\times\Omega)$ for a given $p \in]1,+\infty]$, then, $\mathbf{u} \in L^p(Q\times\Omega\times]0,1[)$.

[§]It is based on the fact that for any set A, $\int_A |\varphi\psi(.,u_n)|\,\mathrm{dt\,dx\,dP} \le C(\|\psi(.,u_n)\|_{L^p})[\int_A |\varphi|^q\,\mathrm{dt\,dx\,dP}]^{1/q}$.

3.3 Predictability and Itô integral

Let us revisited the measurability of $\mathbf{u}$ with respect to all variables (t,x,ω,α) as proposed by E. Yu. Panov.

Since for any $f \in C_b(\mathbb{R})$, $f(u_n)$ converges to $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\nu_{(t,x,\omega)}$ in $L^\infty(Q\times\Omega)$ weak-*, one gets that $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\nu_{(t,x,\omega)}$ is a $\mathbb{R}$-valued $\mathcal{L}^{d+1}\otimes P$ measurable function.

Therefore, $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\nu_{(t,x,\omega)}$ is a $\mathbb{R}$-valued $\mathcal{L}^{d+1}\otimes P$ measurable function for any bounded f in the union of Baire classes of continuous functions, thus for any Borel function. In particular, for any real number c, $(t,x,\omega)\mapsto \nu_{(t,x,\omega)}(]-\infty,c[)$ is measurable.

Let us recall that $\mathbf{u}(t,x,\omega,\alpha) = \inf\{c,\ \nu_{(t,x,\omega)}(]-\infty,c[) > \alpha\}$.

Set $\mu\in\mathbb{R}$ and denote by

$E_\mu = \{(t,x,\omega,\lambda),\ \mathbf{u}(t,x,\omega,\lambda) < \mu\}$ and $F_\mu = \{(t,x,\omega,\lambda),\ \nu_{(t,x,\omega)}(]-\infty,\mu[) > \lambda\}$.

Consider $(t,x,\omega,\lambda)\in E_\mu$. Then, $\mathbf{u}(t,x,\omega,\lambda) < \mu$ and by definition of the infimum, there exists $c \in]\mathbf{u}(t,x,\omega,\lambda),\mu[$ such that $\nu_{(t,x,\omega)}(]-\infty,c[) > \lambda$ and thus $\nu_{(t,x,\omega)}(]-\infty,\mu[) > \lambda$.

Consider $(t,x,\omega,\lambda)\in F_\mu$. Then, $\nu_{(t,x,\omega)}(]-\infty,\mu[) > \lambda$ and by left-continuity of the repartition function, there exists $c<\mu$ with $\nu_{(t,x,\omega)}(]-\infty,c[) > \lambda$. Then, by definition of $\mathbf{u}$, $\mathbf{u}(t,x,\omega,\lambda)\le c<\mu$.

Thus, $E_\mu = F_\mu$ and $\mathbf{u}$ is measurable since F_μ is measurable for any μ. Note that $(t,x,\omega)\mapsto \nu_{(t,x,\omega)}(]-\infty,\mu[)$ is also measurable for the σ-field $\mathcal{P}_T\times L(\mathbb{R}^d)$ where we recall that $\mathcal{P}_T$ denotes the predictable σ-field and $L(\mathbb{R}^d)$ the Lebesgue's one. So, $\mathbf{u}$ is measurable for the σ-field $\mathcal{P}_T\times L(\mathbb{R}^d\times]0,1[)$.

In particular, if $\psi(.,u_n)$ is bounded in $\mathcal{N}^2_w(0,T;L^2(\mathbb{R}^d))$, it converges weakly to $\int_0^1 \psi(\mathbf{u}(.,\alpha))\,\mathrm{d}\alpha$ in $\mathcal{N}^2_w(0,T;L^2(\mathbb{R}^d))$.

Since the Itô integration $u\mapsto \int_0^t u dw$ is an isometric transformation from $\mathcal{N}^2_w(0,T;L^2(\mathbb{R}^d))$ to $\mathcal{M}^2_T(L^2(\mathbb{R}^d))$, the space of all $L^2(\mathbb{R}^d)$-valued continuous, square integrable martingales for the norm of $C([0,T],L^2(\Omega,L^2(\mathbb{R}^d)))$, one gets that $\int_0^t \psi(.,u_n)dw$ converges weakly to $\int_0^t\int_0^1 \psi(.,\mathbf{u}(.,\alpha))d\alpha dw$ in $C([0,T],L^2(\Omega,L^2(\mathbb{R}^d)))$.

Appendix J

Numerical examples related with oil-recovery processes

Here we collect simulations we have implemented using data from the paper of Holden-Risebro [49]. The authors presented numerical examples motivated by oil-recovery processes and we get back here those presented in their paper. Following Holden-Risebro:
"We assume that we have a one-dimensional oil reservoir where a slug of gas is located in the middle of the reservoir. This slug is surrounded by oil. For some reason it is desired to extract this gas. This is done by injecting oil at the left end of the reservoir and extracting oil, and later on gas, at the right end. We assume that the oil has some natural tendency to change into gas, the rate of this reaction is proportional to the product of the oil and gas. The reaction rate is, however, uncertain, and we model this by adding a noise term.
We let u denote the gas saturation, *i.e.*, the fraction of the available pore volume occupied by gas. We assume that only gas and oil are present so that the oil saturation is given by $1-u$."
As the flux function f, they use for simplicity

$$f(u) = \frac{u^2}{u^2 + (1-u)^2},$$

this type of flux function has the correct s-shape and is often used for model studies. They set the reaction terms to be

$$h(x,t,u) = k_1 u(1-u), \quad g(u) = k_2 u(1-u)(u - \frac{1}{2}), \quad \text{where} \quad k_1, k_2 \in \mathbb{R}.$$

The initial saturation is given by

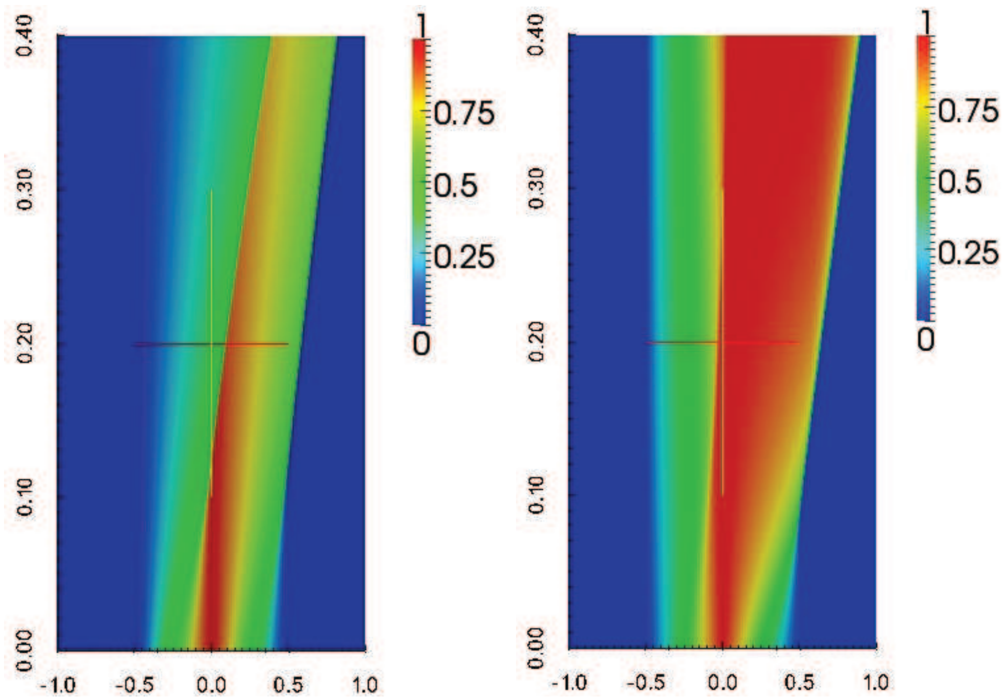
$$u_0(x) = \begin{cases} 0 & \text{for} \quad x \leqslant -\frac{1}{2}, \\ 2x+1 & \text{for} \quad -\frac{1}{2} < x \leqslant 0, \\ -2x+1 & \text{for} \quad 0 < x \leqslant \frac{1}{2}, \\ 0 & \text{for} \quad \frac{1}{2} < x. \end{cases}$$

Remark 0.1 *As explained in the general introduction of this manuscript, adding a stochastic perturbation in the model is a suitable choice to take into account this imperfect knowledge of the studied phenomenon.*

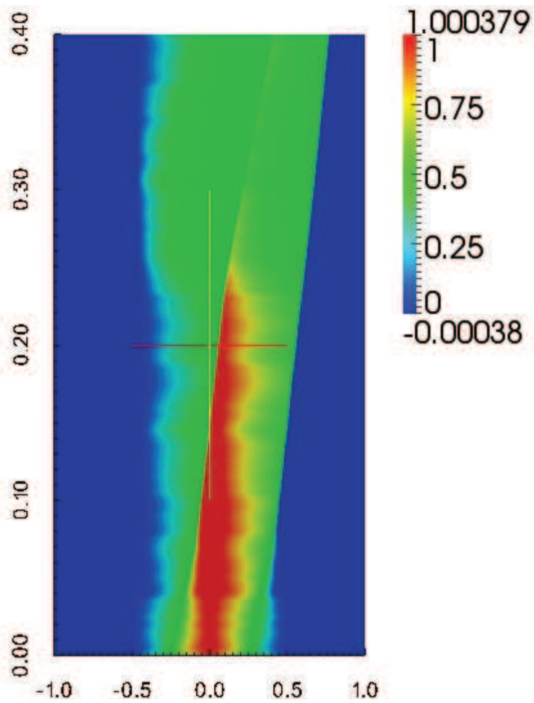
And the authors proposed the following model for finding the gas saturation u:

$$\partial_t(u - \int_0^{\cdot} g(u)dW) + f(u)_x \quad = \quad h(x,t,u).$$

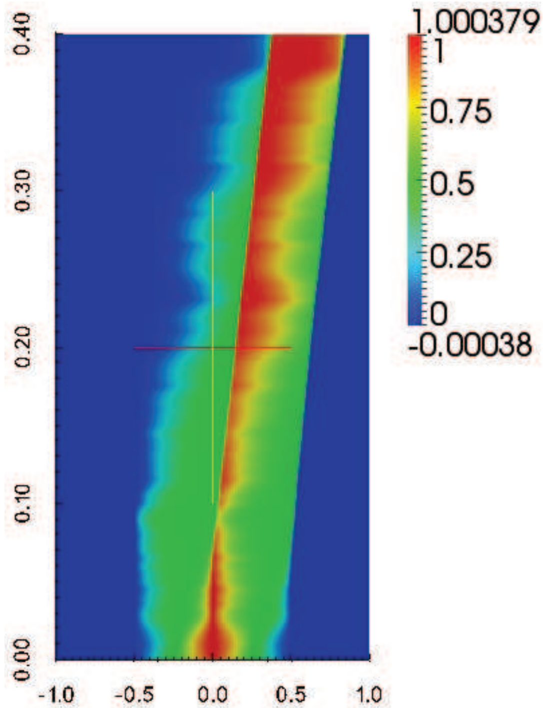
We get back the numerical experiments introduced in [49], for the time step $\Delta_t = 0.005$ and the space step $\Delta_x = 0.01$ in the (x-t)-plane. Note that in the case $k_2 \neq 0$, we propose three sample path simulations (*i.e* three simulations of the Brownian motion). For comments on these simulations, see [49] Section 2.



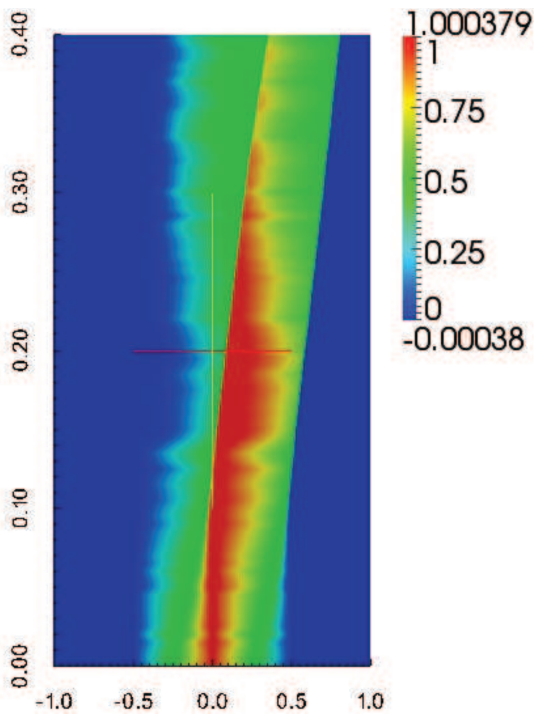
(a) $k_1 = k_2 = 0$ (b) $k_1 = 10,\ k_2 = 0$



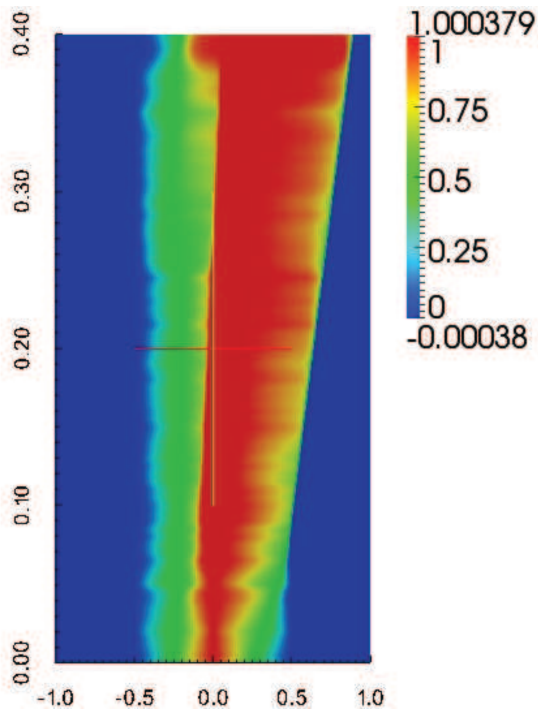
(c) $k_1 = 0,\ k_2 = 10$



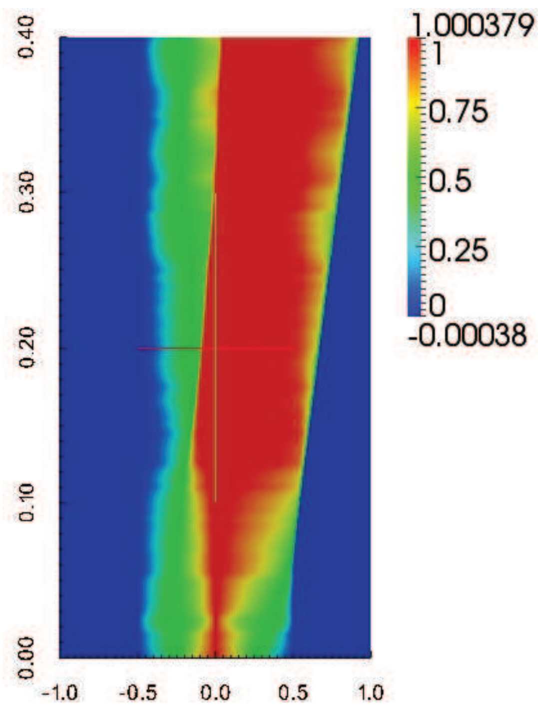
(d) $k_1 = 0,\ k_2 = 10$



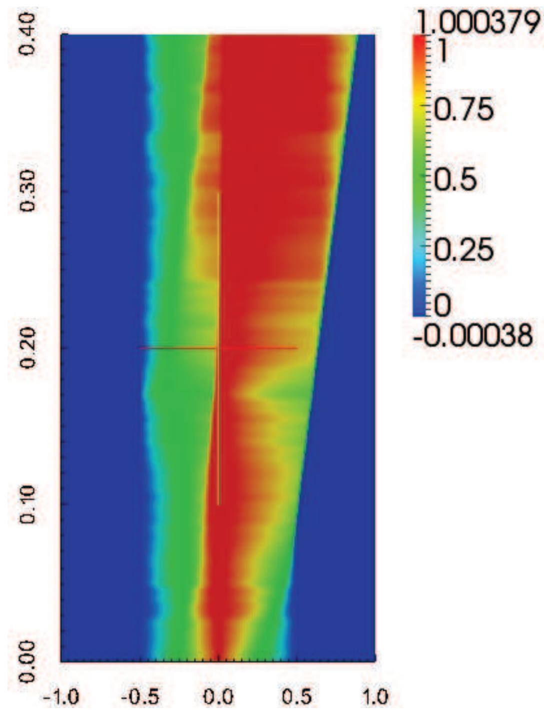
(e) $k_1 = 0,\ k_2 = 10$



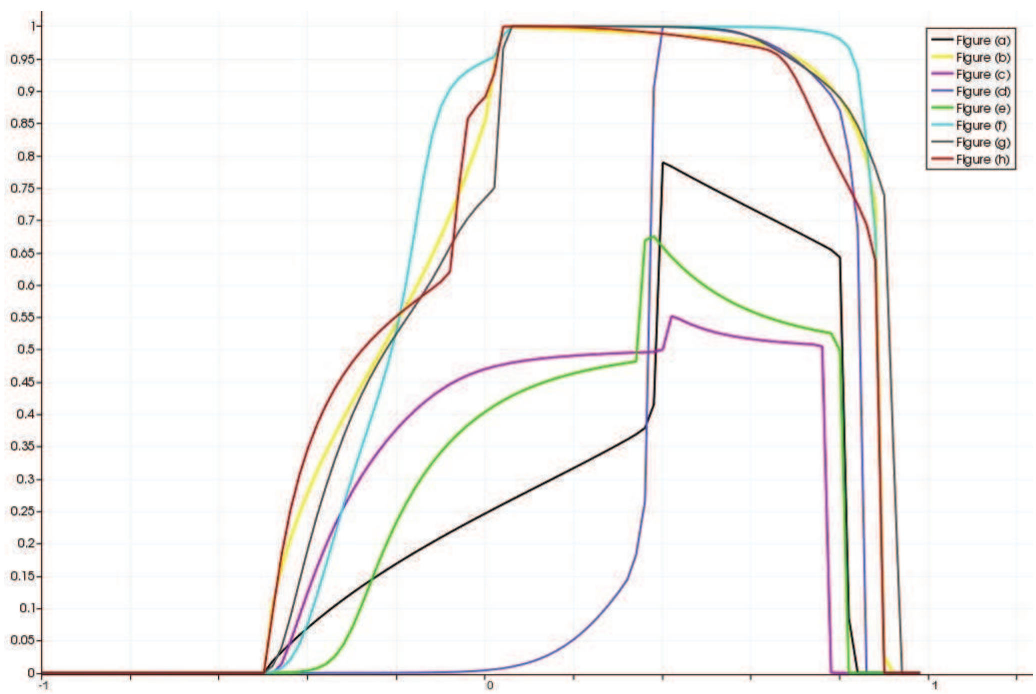
(f) $k_1 = k_2 = 10$



(g) $k_1 = k_2 = 10$



(h) $k_1 = k_2 = 10$



(i) Pathwises for $t = 0.4$

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<u>Résumé :</u> Cette thèse s'inscrit dans le domaine mathématique de l'analyse des **équations aux dérivées partielles** (EDP) non-linéaires **stochastiques**. Nous nous intéressons à des EDP paraboliques et hyperboliques que l'on perturbe stochastiquement au sens d'Itô. Il s'agit d'introduire l'aléatoire *via* l'ajout d'une intégrale stochastique (intégrale d'Itô) qui peut dépendre ou non de la solution, on parle alors de **bruit multiplicatif** ou **additif**. La présence de la variable de probabilité ne nous permet pas d'utiliser tous les outils classiques de l'analyse des EDP. Notre but est d'adapter les techniques connues dans le cadre déterministe aux EDP non linéaires stochastiques en proposant des méthodes alternatives. Les résultats obtenus sont décrits dans les cinq chapitres de cette thèse :

- Dans le **Chapitre I**, nous étudions une perturbation stochastique des **équations de Barenblatt**. En utilisant une semi-discrétisation implicite en temps, nous établissons l'existence et l'unicité d'une solution dans le cas additif, et grâce aux propriétés de la solution nous sommes en mesure d'étendre ce résultat au cas multiplicatif à l'aide d'un théorème de point fixe.
- Dans le **Chapitre II**, nous considérons une classe d'équations de type Barenblatt stochastiques dans un **cadre abstrait**. Il s'agit là d'une généralisation des résultats du Chapitre I.
- Dans le **Chapitre III**, nous travaillons sur l'étude du **problème de Cauchy** pour une loi de conservation stochastique. Nous montrons l'existence d'une solution par une méthode de **viscosité artificielle** en utilisant des arguments de compacité donnés par la théorie des **mesures de Young**. L'unicité repose sur une adaptation de la méthode de **dédoublement des variables de Kruzhkov**.
- Dans le **Chapitre IV**, nous nous intéressons au **problème de Dirichlet** pour la loi de conservation stochastique étudiée au Chapitre III. Le point remarquable de l'étude repose sur l'utilisation des **semi-entropies de Kruzhkov** pour montrer l'unicité.
- Dans le **Chapitre V**, nous introduisons une **méthode de splitting** pour proposer une approche numérique du problème étudié au Chapitre IV, suivie de quelques simulations de l'équation de Burgers stochastique dans le cas monodimensionnel.

<u>Mots clés :</u> EDP stochastique, bruit multiplicatif, bruit additif, processus prévisible, formule d'Itô, équation hyperbolique du premier ordre, lois de conservation, problème de Cauchy, problème de Dirichlet, mesure de Young, entropie de Kruzhkov, opérateur monotone, viscosité artificielle, équation parabolique, discrétisation en temps, méthode de splitting, schéma d'Euler, schéma de Godunov.

AMS-Code : 60H15, 35R60, 35L60, 35L45, 35L65.

<u>Abstract:</u> This thesis deals with the mathematical field of **stochastic nonlinear partial differential equations' analysis**. We are interested in parabolic and hyperbolic PDE stochastically perturbed in the Itô sense. We introduce randomness by adding a stochastic integral (Itô integral), which can depend or not on the solution. We thus talk about a **multiplicative noise** or an **additive** one. The presence of the random variable does not allow us to apply systematically classical tools of PDE analysis. Our aim is to adapt known techniques of the deterministic setting to nonlinear stochastic PDE analysis by proposing alternative methods. Here are the obtained results:

- In **Chapter I**, we investigate on a stochastic perturbation of **Barenblatt equations**. By using an implicit time discretization, we establish the existence and uniqueness of the solution in the additive case. Thanks to the properties of such a solution, we are able to extend this result to the multiplicative noise using a fixed-point theorem.
- In **Chapter II**, we consider a class of stochastic equations of Barenblatt type but in an **abstract frame**. It is about a generalization of results from Chapter I.
- In **Chapter III**, we deal with the study of the **Cauchy problem** for a stochastic conservation law. We show existence of solution *via* an **artificial viscosity** method. The compactness arguments are based on **Young measure** theory. The uniqueness result is proved by an adaptation of the **Kruzhkov doubling variable**s technique.
- In **Chapter IV**, we are interested in the **Dirichlet problem** for the stochastic conservation law studied in Chapter III. The remarkable point is the use of the **Kruzhkov semi-entropies** to show the uniqueness of the solution.
- In **Chapter V**, we introduce a **splitting method** to propose a numerical approach of the problem studied in Chapter IV. Then we finish by some simulations of the stochastic Burgers' equation in the one dimensional case.

<u>Keywords:</u> Stochastic PDE, multiplicative stochastic perturbation, additive noise, predictable process, Itô formula, first order hyperbolic equation, conservation law, Cauchy Problem, Dirichlet problem, Young measure, measure valued-solution, Kruzhkov's entropy, parabolic regularization, parabolic equation, monotone operators, time discretization, splitting method, Euler scheme, Godunov scheme.

AMS-Code : 60H15, 35R60, 35L60, 35L45, 35L65.