UNIVERSITÉ DE BRETAGNE OCCIDENTALE
And
HANOI NATIONAL UNIVERSITY OF EDUCATION

PHAM HOANG HA

On unicity problems of meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ and the ramification of the Gauss maps of complete minimal surfaces

Doctoral Thesis in Mathematics

Supervisors:
Gerd DETHLOFF (PR, Brest) and Do Duc THAI (PR, ENS Hanoi)

Referees:
Tien-Cuong DINH (PR, Paris 6) and Carlo GASBARRI (PR, Strasbourg)

Jury members:
Gerd DETHLOFF (PR, Brest), Carlo GASBARRI (PR, Strasbourg), Pascal THOMAS (PR, Toulouse 3), Huy Khoai HA (PR, Hanoi Research Institute of Mathematics), Huy Vui HA (PR, Hanoi Research Institute of Mathematics), Mau Hai LE (PR, ENS Hanoi).

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Introduction

This thesis consists of two parts.

The first part deals with the uniqueness problems of meromorphic mappings under some conditions on the inverse images of divisors which was started by R. Nevanlinna [43] in 1926. He showed that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values then $f \equiv g$, and that g is a special type of linear fractional transformation of f if they have the same inverse images counted with multiplicities for four distinct values.

In 1975, H. Fujimoto generalized Nevanlinna's results to the case of meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. He showed [18] that for two linearly nondegenerate meromorphic mappings f and g of $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$, if they have the same inverse images counted with multiplicities for $3N+2$ hyperplanes in general position in $\mathbb{P}^N(\mathbb{C})$, then $f \equiv g$ and there exists a projective linear transformation L of $\mathbb{P}^N(\mathbb{C})$ onto itself such that $g = L.f$ if they have the same inverse images counted with multiplicities for $3N+1$ hyperplanes in general position in $\mathbb{P}^N(\mathbb{C})$. After that, this problem has been studied intensively by a number of mathematicans as H. Fujimoto([18],[28],...), W. Stoll([58]), L. Smiley([57]), M. Ru([55]), G. Dethloff - T. V. Tan([12], [13], [14]...), D. D. Thai - S. D. Quang([63], [64]) and so on.

Here we introduce the necessary notations to state the results.

Let $f : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbb{P}^N(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on $\mathbb{C}^n$ and $f(z) = \big(f_0(z) : \cdots : f_N(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of codimension ≥ 2. Let H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$ given by $H = \{a_0\omega_0 + ... + a_N\omega_N = 0\}$, where $A := (a_0, ..., a_N) \neq (0, ..., 0)$. We set $(f, H) = \sum_{i=0}^{N} a_i f_i$. Then we can define the corresponding divisor $\nu_{(f,H)}(z)$ which is rephrased as the intersection multiplicity of the image of f and H at $f(z)$.

For every $z \in \mathbb{C}^n$, we set

$$\nu_{(f,H),\leq k}(z) = \begin{cases} 0 & \text{if } \nu_{(f,H)}(z) > k, \\ \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) \leq k, \end{cases}$$

$$\nu_{(f,H),>k}(z) = \begin{cases} \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) > k, \\ 0 & \text{if } \nu_{(f,H)}(z) \leq k. \end{cases}$$

Take a meromorphic mapping f of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$, a positive integer d, a positive integer k or $k = \infty$ and q hyperplanes $H_1, ..., H_q$ in $\mathbb{P}^N(\mathbb{C})$ located in general position with

$$\dim\{z \in \mathbb{C}^n : \nu_{(f,H_i),\leqslant k}(z) > 0 \text{ and } \nu_{(f,H_j),\leqslant k}(z) > 0\} \leq n-2 \ (1 \leq i < j \leq q),$$

and consider the set $\mathcal{F}(f, \{H_j\}_{j=1}^q, k, d)$ of all meromorphic maps $g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ satisfying the conditions

(a) g is linearly nondegenerate over $\mathbb{C}$,

(b) $\min\ (\nu_{(f,H_j),\leq k}, d) = \min\ (\nu_{(g,H_j),\leq k}, d)$ $(1 \leq j \leq q)$,

(c) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : \nu_{(f,H_j),\leq k}(z) > 0\}$.

When $k = \infty$, for brevity denote $\mathcal{F}(f, \{H_j\}_{j=1}^q, \infty, d)$ by $\mathcal{F}(f, \{H_j\}_{j=1}^q, d)$. Denote by $\sharp\ S$ the cardinality of the set S.

The unicity problem of meromorphic mappings means that one gives an estimate for the cardinality of the set $\mathcal{F}(f, \{H_j\}_{j=1}^q, k, d)$. Some natural questions arise and we state the following.

Question 1. How about the number of hyperplanes (or fixed targets) in $\mathbb{P}^N(\mathbb{C})$ are used?

Question 2. How about the truncated multiplicities (d and k)?

Question 3. Whether the fixed targets (hyperplanes) can be generalized to moving targets (moving hyperplanes) or hypersurfaces?

On the question 1 and 2, we list some known results:
Smiley [57] $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{3N+2}, 1) = 1$, Thai-Quang [64] $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{3N+1}, 1) = 1, N \geq 2$, Dethloff-Tan [15] $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{[2.75N]}, 1) = 1$ for $N \geq N_0$(where the number N_0 can be explicitly calculated) and Chen-Yan [6] $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{2N+3}, 1) = 1$.

When $q < 2N + 3$, there are some results which were given by Tan [62] and Quang [51],[52]. Those results lead us to the question.

What can we say about the unicity theorems with truncated multiplicities in the case where $q \leq 2N + 2$?

The first purpose of this thesis is to study these problems. Firstly, we will give a new aspect for the unicity problem with $q = 2N + 2$, and we also study the unicity theorems with ramification of truncations.

The second purpose of this thesis is to give some answers relative to the question 3. Our results are following the results of Ru [55], Dethloff-Tan [14], Thai-Quang [63].

On the other hand, there are many interesting unicity theorems for meromorphic functions on $\mathbb{C}$ given by certain conditions of derivations. We would like to study the unicity problems of such type in several complex variables for fixed and moving targets.

Parallel to the development of Nevanlinna theory, the value distribution theory of the Gauss map of minimal surfaces immersed in $\mathbb{R}^m$ was studied by many mathematicans, such as R. Osserman [47], S.S. Chern [7], F. Xavier [66], H. Fujimoto [20]-[24], S. J. Kao [38], M. Ru [53]-[54] and others.

Let M now be a non-flat minimal surface in $\mathbb{R}^3$, or more precisely, a connected oriented minimal surface in $\mathbb{R}^3$. By definition, the Gauss map G of M is the map which maps each point $p \in M$ to the unit normal vector $G(p) \in S^2$ of M at p. Instead of G, we study the map $g := \pi \circ G : M \to \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} (= \mathbb{P}^1(\mathbb{C}))$ for the stereographic projection π of S^2 onto $\mathbb{P}^1(\mathbb{C})$. By associating a holomorphic local coordinate $z = u + \sqrt{-1}v$ with each positive isothermal coordinate system (u, v), M is considered as an open Riemann surface with a conformal metric ds^2 and by the assumption of minimality of M, g is a meromorphic function on M. After that, we can generalize to the definition of Gauss map of minimal surfaces in $\mathbb{R}^m$. So there are many analogous results between the Gauss maps and meromorphic mappings. One of them is the small Picard theorem.

In 1965, R. Osserman [47] showed that the complement of the image of the Gauss map of a nonflat complete minimal surface immersed in $\mathbb{R}^3$ is of logarithmic capacity zero in $\mathbb{P}^1(\mathbb{C})$. In 1981, a remarkable improvement was given by F. Xavier [66] that the Gauss map of a nonflat complete minimal surface immersed in $\mathbb{R}^3$ can omit at most six points in $\mathbb{P}^1(\mathbb{C})$. In 1988, H. Fujimoto [20] reduced the number six to four and this bound is sharp: In fact, we can see that the Gauss map of Scherk's surface omits four points in $\mathbb{P}^1(\mathbb{C})$. In 1991, S. J. Kao [38] showed that the Gauss map of an end of a non-flat complete minimal surface in $\mathbb{R}^3$ that is conformally an annulus $\{z|0 < 1/r < |z| < r\}$ must also assume every value, with at most 4 exceptions. In 2007, Jin-Ru [37] generalized Kao's results for the case $m > 3$.

On the other hand, in 1993, M. Ru [54] studied the Gauss map of minimal surface in $\mathbb{R}^m$ with ramification. That are generalizations of the above-mentioned results. A natural question is that how about the Gauss map of minimal surfaces on annular ends with ramification. The last purpose of this thesis answer to this question for the case $m = 3, 4$. We refer to Dethloff-Ha-Thoan [10] for the case $m > 3$. We would like to note that the aspect of results in this thesis are different from their results.

We now sketch the content of each chapter of the present thesis

In chapter 1, we study the unicity theorems with truncated multiplicities of meromorphic mappings in several complex variables for few fixed targets. In particular, we give a new unicity theorem for the above-mentioned first purpose of this thesis. After that we study the unicity theorems with ramification of truncations which is an improvement of Thai-Quang's results in [64]. The last of this chapter we give a unicity theorem of meromorphic mappings with a conditions on derivations.

In chapter 2, we study the unicity theorems with truncated multiplicities of meromorphic mappings in several complex variables sharing few moving targets. In particular, we improve strongly the results of Dethloff- Tan [14]. Beside that, we also give a unicity theorem of meromorphic mappings for moving targets with a conditions on derivations.

In chapter 3, we introduce the Gauss map of minimal surfaces in $\mathbb{R}^m$ and we study the ramification of the Gauss map on annular ends in minimal surfaces in $\mathbb{R}^3, \mathbb{R}^4$. In particular, we improve the results of S. J. Kao [38] by using the ideas of H. Fujimoto [20] and M. Ru [54].

PHAM HOANG HA
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Chapter 1

Unicity theorems with truncated multiplicities of meromorphic mappings in several complex variables for few fixed targets

The unicity theorems with truncated multiplicities of meromorphic mappings of $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^N(\mathbb{C})$ sharing a finite set of fixed hyperplanes in $\mathbb{P}^N(\mathbb{C})$ have been studied intensively by H. Fujimoto, L. Smiley, S. Ji, M. Ru, D.D. Thai, G. Dethloff, T.V. Tan, S.D. Quang, Z. Chen, Q. Yan and others. The unicity problem has grown into a huge theory.

With the notations in §1.1, we report here briefly the unicity problems with multiplicities of meromorphic mappings

Theorem A.(Smiley [57]) *If* $q \geq 3N+2$ *then* $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^q, 1) = 1$.

Theorem B.(Thai-Quang [64]) *If* $N \geq 2$ *then* $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{3N+1}, 1) = 1$.

Theorem C.(Dethloff-Tan [15]) *There exists a positive integer* N_0 *(which can be explicitly calculated) such that* $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^q, 1) = 1$ *for* $N \geq N_0$ *and* $q = [2.75N]$.

Theorem D.(Chen-Yan [6]) *If* $N \geq 1$ *then* $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{2N+3}, 1) = 1$.

Theorem E.(Tan [62]) *For each mapping* $g \in \mathcal{F}(f, \{H_i\}_{i=1}^{2N+2}, N+1)$, *there exist a constant* $\alpha \in \mathbb{C}$ *and a pair* (i, j) *with* $1 \leq i < j \leq q$, *such that*

$$\frac{(H_i, f)}{(H_j, f)} = \alpha \frac{(H_i, g)}{(H_j, g)}.$$

Theorem F. (Quang [51]) *Let f_1 and f_2 be two linearly nondegenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ $(N \geq 2)$ and let $H_1, ..., H_{2N+2}$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ located in general position such that*

$$\dim\{z \in \mathbb{C}^n : \nu_{(f_1,H_i)}(z) > 0 \text{ and } \nu_{(f_1,H_j)}(z) > 0\} \leq n - 2$$

for every $1 \leq i < j \leq 2N + 2$. Assume that the following conditions are satisfied.

(a) $\min\{\nu_{(f_1,H_j),\leq N}, 1\} = \min\{\nu_{(f_2,H_j),\leq N}, 1\}$ $(1 \leq j \leq 2N + 2)$,

(b) $f_1(z) = f_2(z)$ *on* $\bigcup_{j=1}^{2N+2}\{z \in \mathbb{C}^n : \nu_{(f_1,H_j)}(z) > 0\}$,

(c) $\min\{\nu_{(f_1,H_j),\geq N}, 1\} = \min\{\nu_{(f_2,H_j),\geq N}, 1\}$ $(1 \leq j \leq 2N + 2)$,

Then $f_1 \equiv f_2$.

Theorem G. (Quang [52]) *If $N \geq 2$ then $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{2N+2}, 1) \leq 2$.*

In the first part of this chapter, we would like to study the unicity theorems for the case $q \leq 2N + 2$. In particular, we shall prove Theorem 1.2 (Ha-Quang [33]) which gives a new aspect of them in the first part of this chapter.

In [64], the authors showed that

Theorem H. (Thai-Quang [64]) (a) *If $N = 1$, then $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{3N+1}, k, 2) \leq 2$ for $k \geq 15$.*

(b) *If $N \geq 2$, then $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{3N+1}, k, 2) \leq 2$ for $k \geq 3N + 3 + \dfrac{4}{N-1}$.*

(c) *If $N \geq 4$, then $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{3N}, k, 2) \leq 2$ for $k > 3N + 7 + \dfrac{24}{N-3}$.*

(d) *If $N \geq 6$, then $\sharp\ \mathcal{F}(f, \{H_i\}_{i=1}^{3N-1}, k, 2) \leq 2$ for $k > 3N + 11 + \dfrac{60}{N-5}$.*

The second part of this chapter studies the unicity problems of meromorphic mapping with ramification of truncations. We are going to improve Theorem H by Theorem 1.3 (Ha [31]). In particular, we use different truncations k_i for each hyperplanes $H_i (1 \leq i \leq q)$, and we then give its corollaries.

As far as we know, there are many interesting unicity theorems for meromorphic functions on $\mathbb{C}$ given by certain conditions of derivations. We will give a unicity theorem of such type in several complex variables for fixed targets. That is a unicity theorem with truncated multiplicities in the case where $N + 4 \leq q < 2N + 2$. We will prove Theorem 1.4 (Ha-Quang [33]) in the last part of this chapter.

1.1 Basic notions and auxiliary results from Nevanlinna theory

1.1.1. We set $||z|| = \big(|z_1|^2 + \cdots + |z_n|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and define

$$B(r) := \{z \in \mathbb{C}^n : ||z|| < r\}, \quad S(r) := \{z \in \mathbb{C}^n : ||z|| = r\} \ (0 < r < \infty).$$

Define

$$v_{n-1}(z) := \big(dd^c||z||^2\big)^{n-1} \qquad \text{and}$$

$$\sigma_n(z) := d^c\log||z||^2 \wedge \big(dd^c\log||z||^2\big)^{n-1} \text{on} \quad \mathbb{C}^n \setminus \{0\}.$$

1.1.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^n$. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, we set $|\alpha| = \alpha_1 + ... + \alpha_n$ and $\mathcal{D}^\alpha F = \dfrac{\partial^{|\alpha|}F}{\partial^{\alpha_1}z_1...\partial^{\alpha_n}z_n}$. We define the mapping $\nu_F : \Omega \to \mathbf{Z}$ by

$$\nu_F(z) := \max\ \{m : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < m\} \ (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbb{C}^n$ a mapping $\nu : \Omega \to \mathbf{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood U of a $(\subset \Omega)$ such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq n-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq n-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(n-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n-2$, and we define the divisors $\nu_\varphi,\ \nu^\infty_\varphi$ by $\nu_\varphi := \nu_F,\ \nu^\infty_\varphi := \nu_G$, which are independent of the choices of F and G. Hence, they are globally well-defined on Ω.

1.1.3. For a divisor ν on $\mathbb{C}^n$ and for positive integers k, d (or $k, d = \infty$), we define the counting functions of ν as follows. Set

$$\nu^{(d)}(z) = \min\ \{d, \nu(z)\},$$

$$\nu^{(d)}_{\leq k}(z) = \begin{cases} 0 & \text{if } \nu(z) > k, \\ \nu^{(d)}(z) & \text{if } \nu(z) \leq k. \end{cases}$$

$$\nu^{(d)}_{>k}(z) = \begin{cases} \nu^{(d)}(z) & \text{if } \nu(z) > k, \\ 0 & \text{if } \nu(z) \le k. \end{cases}$$

We define $n(t)$ by

$$n(t) = \begin{cases} \int\limits_{|\nu| \cap B(t)} \nu(z) v_{n-1}(z) & \text{if } n \ge 2 \\ \sum\limits_{|z| \le t} \nu(z) & \text{if } n = 1 \end{cases}, \text{ where } v_{n-1}(z) := \big(dd^c ||z||^2\big)^{n-1}.$$

Similarly, we define $n^{(d)}(t),\ n^{(d)}_{\le k}(t),\ n^{(d)}_{>k}(t)$.

Define

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2n-1}} dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{(d)}),\ N(r, \nu^{(d)}_{\le k}),\ N(r, \nu^{(d)}_{>k})$ and denote them by $N^{(d)}(r, \nu),\ N^{(d)}_{\le k}(r, \nu),\ N^{(d)}_{>k}(r, \nu)$ respectively.

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a nonzero meromorphic function. Define $N_\varphi(r) = N(r, \nu_\varphi),\ N^{(d)}_\varphi(r) = N^{(d)}(r, \nu_\varphi),\ N^{(d)}_{\varphi, \le k}(r) = N^{(d)}_{\le k}(r, \nu_\varphi),\ N^{(d)}_{\varphi, >k}(r) = N^{(d)}_{>k}(r, \nu_\varphi)$.

For brevity we will omit the superscript ${}^{(d)}$ if $d = \infty$.

Now, take a meromorphic mapping f of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$ and q hyperplanes $H_1, ..., H_q$ in $\mathbb{P}^N(\mathbb{C})$ located in general position with

$$\dim\{z \in \mathbb{C}^n : \nu_{(f, H_i), \leqslant k}(z) > 0 \text{ and } \nu_{(f, H_j), \leqslant k}(z) > 0\} \le n - 2 \ (1 \le i < j \le q),$$

and consider the set $\mathcal{F}(f, \{H_j\}_{j=1}^q, k, d)$ of all meromorphic maps $g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ satisfying the conditions

(a) g is linearly nondegenerate over $\mathbb{C}$,

(b) $\min\ (\nu_{(f, H_j), \le k}, d) = \min\ (\nu_{(g, H_j), \le k}, d)\ (1 \le j \le q)$,

(c) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : \nu_{(f, H_j), \le k}(z) > 0\}$.

When $k = \infty$, for brevity denote $\mathcal{F}(f, \{H_j\}_{j=1}^q, \infty, d)$ by $\mathcal{F}(f, \{H_j\}_{j=1}^q, d)$. Denote by $\sharp\ S$ the cardinality of the set S.

1.1.4. Let $f : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbb{P}^N(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on $\mathbb{C}^n$ and

$f(z) = \big(f_0(z) : \cdots : f_N(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of codimension ≥ 2.
Set $\|f\| = \big(|f_0|^2 + \cdots + |f_N|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T(r, f) = \int\limits_{S(r)} \log\|f\|\sigma_n - \int\limits_{S(1)} \log\|f\|\sigma_n.$$

Let H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$ given by $H = \{a_0\omega_0 + ... + a_N\omega_N = 0\}$, where $A := (a_0, ..., a_N) \neq (0, ..., 0)$. We set $(f, H) = \sum_{i=0}^N a_i f_i$. Then we can define the corresponding divisor $\nu_{(f,H)}$ which is rephrased as the intersection multiplicity of the image of f and H at $f(z)$. Moreover, we define the proximity function of H by

$$m_{f,H}(r) = \int_{S(r)} \log \frac{||f|| \cdot ||H||}{|(f, H)|}\sigma_n - \int_{S(1)} \log \frac{||f|| \cdot ||H||}{|(f, H)|}\sigma_n,$$

where $||H|| = (\sum_{i=0}^N |a_i|^2)^{\frac{1}{2}}$.
Let φ be a nonzero meromorphic function on $\mathbb{C}^n$, which are occasionally regarded as a meromorphic mapping into $\mathbb{P}^1(\mathbb{C})$. The proximity function of φ is defined by

$$m(r, \varphi) := \int_{S(r)} \log\max\ (|\varphi|, 1)\sigma_n.$$

1.1.5. Consider a vector-valued meromorphic function $F = (f_0, ..., f_N)$. For each $a \in \mathbb{C}^n$, we denote by $\mathcal{M}_a$ the set of all germs of meromorphic functions at a and, for $\kappa = 1, 2, ...$, by $\mathcal{F}^\kappa$ the $\mathcal{M}_a-$submodule of $\mathcal{M}_a^{N+1}$ which is generated by the set $\{\mathcal{D}^\alpha F := (\mathcal{D}^\alpha f_0, ..., \mathcal{D}^\alpha f_N); |\alpha| \leq \kappa\}$. Set $l_F(\kappa) := rank_{\mathcal{M}_a}\mathcal{F}^\kappa$, which does not depend on each $a \in \mathbb{C}^n$. For a meromorphic map $f = (f_0 : f_1 : \cdots : f_N) : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$, we set $l_f(\kappa) := l_{(f_0,\cdots,f_N)}(\kappa)$.

Assume that meromorphic functions $f_0, ..., f_N$ are linearly independent over $\mathbb{C}$. For $N+1$ vectors $\alpha^i := (\alpha_{i1}, ..., \alpha_{in})(0 \leq i \leq N)$ composed of nonnegative integers α_{ij}, we call a set $\alpha = (\alpha^0, \alpha^1, ..., \alpha^N)$ an admissible set for $F = (f_0, ..., f_N)$ if $\{\mathcal{D}^{\alpha^0}F, ..., \mathcal{D}^{\alpha^{l_F(\kappa)}}F\}$ is a basis of $\mathcal{F}^\kappa$ for each $\kappa = 1, 2, ..., \kappa_0 := \min\{\kappa'; l_F(\kappa') = N+1\}$. By definition, for an admissible set $(\alpha^0, \alpha^1, ..., \alpha^N)$ we have

$$\det\big(\mathcal{D}^{\alpha^0}F, ..., \mathcal{D}^{\alpha^N}F\big) \not\equiv 0.$$

1.1.6. As usual, by the notation $''|| \ P''$ we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

The following results play essential roles in Nevanlinna theory (see Noguchi-Ochiai [46], Stoll [58],[59]).

1.1.7. The first main theorem. *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. Then*

$$N_{(f,H)}(r) + m_{f,H}(r) = T(r,f) \ (r > 1).$$

1.1.8. The second main theorem. *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, ..., H_q$ be hyperplanes in general position in $\mathbb{P}^N(\mathbb{C})$. Then*

$$|| \quad (q - N - 1)T(r,f) \leq \sum_{i=1}^{q} N^{(N)}_{(f,H_i)}(r) + o(T(r,f)).$$

1.1.9. Lemma. (Thai-Quang [64]) *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping. Let $H_1, H_2, ..., H_q$ be q hyperplanes in $\mathbb{P}^N(\mathbb{C})$ located in general position. Assume that $k \geq N - 1$. Then*

$$\Big|\Big| \left(q - N - 1 - \frac{Nq}{k+1}\right) T(r,f) \leq \sum_{j=1}^{q} \left(1 - \frac{N}{k+1}\right) N^{(N)}_{(f,H_j),\leq k}(r) + o(T(r,f)) \ .$$

1.1.10. Logarithmic derivative lemma. *Let f be a nonzero meromorphic function on $\mathbb{C}^n$. Then*

$$\Big|\Big| \quad m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O(\log^+ T(r,f)) \ (\alpha \in \mathbf{Z}^n_+).$$

1.1.11. Denote by ${\mathcal{M}^*}_n$ the abelian multiplicative group of all nonzero meromorphic functions on $\mathbb{C}^n$. Then the multiplicative group ${\mathcal{M}^*}_n/\mathbb{C}^*$ is a torsion free abelian group.

Let G be a torsion free abelian group and $A = (a_1, a_2, ..., a_q)$ a $q-$tuple of elements a_i in G. Let $q \geq r > s > 1$. We say that the $q-$tuple A has the property $(P_{r,s})$ if any r elements $a_{l(1)}, ..., a_{l(r)}$ in A satisfy the condition that for any given $i_1, ..., i_s$ $(1 \leq i_1 < ... < i_s \leq r)$, there exist $j_1, ..., j_s$ $(1 \leq j_1 < ... < j_s \leq r)$ with $\{i_1, ..., i_s\} \neq \{j_1, ..., j_s\}$ such that $a_{l(i_1)}...a_{l(i_s)} = a_{l(j_1)}...a_{l(j_s)}$.

1.1.12. Proposition. (Fujimoto [18]) *Let G be a torsion free abelian group and $A = (a_1, ..., a_q)$ a $q-$tuple of elements a_i in G. If A has the property $(P_{r,s})$ for some r, s with $q \geq r > s > 1$, then there exist $i_1, ..., i_{q-r+2}$ with $1 \leq i_1 < ... < i_{q-r+2} \leq q$ such that $a_{i_1} = a_{i_2} = ... = a_{i_{q-r+2}}$.*

Take 3 mappings f^1, f^2, f^3 with reduced representations $f^k := (f^k_0 : ... : f^k_N)$ and set

$T(r) := \sum_{k=1}^{3} T(r, f^k)$. For each $c = (c_0, ..., c_N) \in \mathbb{C}^{N+1} \setminus \{0\}$, we define $(f^k, c) := \sum_{i=0}^{N} c_i f_i^k$ $(0 \le k \le N)$. Denote by $\mathcal{C}$ the set of all $c \in \mathbb{C}^{N+1} \setminus \{0\}$ such that

$$\dim\{z \in \mathbb{C}^n : (f^k, H_j)(z) = (f^k, c)(z) = 0\} \le n - 2$$

1.1.13. Lemma. *Let $H_1, H_2, ..., H_q$ be q hyperplanes in $\mathbb{P}^N(\mathbb{C})$ located in general position. Assume that $\min(\nu_{(f^k,H_i)}, d) = \min(\nu_{(f^1,H_i)}, d)(1 \le k \le 3), 1 \le d \le N$ and $q \ge N + 2$. Then*

$$|| \ T(r, f^k) = O(T(r, f^1)) \text{ for each } (1 \le k \le 3).$$

Proof. By the Second Main Theorem, we have

$$|| \quad (q-N-1)T(r, f^k) \le \sum_{i=1}^{q} N^{(N)}_{(f^k,H_i)}(r) + o(T(r, f^k)) \le \sum_{i=1}^{q} \frac{N}{d} \cdot N^{(d)}_{(f^k,H_i)}(r) + o(T(r, f^k))$$

$$= \sum_{i=1}^{q} \frac{N}{d} \cdot N^{(d)}_{(f^1,H_i)}(r) + o(T(r, f^k)) \le q\frac{N}{d} T(r, f^1) + o(T(r, f^k)).$$

Hence $||$ $T(r, f^k) = O(T(r, f^1))$. Q.E.D.

1.1.14. Lemma. (Ji [35]) *$\mathcal{C}$ is dense in $\mathbb{C}^{N+1}$.*

1.1.15. Lemma. (Fujimoto [28]) *For every $c \in \mathcal{C}$, we put $F_c^{jk} = \dfrac{(f^k, H_j)}{(f^k, c)}$. Then*

$$T(r, F_c^{jk}) \le T(r, f^k) + o(T(r)).$$

1.1.16. Definition. (Fujimoto [28]) Let $F_0, ..., F_M$ be meromorphic functions on $\mathbb{C}^n$, where $M \ge 1$. Take a set $\alpha := (\alpha^0, ..., \alpha^{M-1})$ whose components α^k are composed of n nonnegative integers, and set $|\alpha| = |\alpha^0| + ... + |\alpha^{M-1}|$. We define Cartan's auxiliary function by

$$\Phi^\alpha \equiv \Phi^\alpha(F_0, ..., F_M) := F_0 F_1 \cdots F_M \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \mathcal{D}^{\alpha^0}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^0}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^0}(\frac{1}{F_M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_M}) \end{vmatrix}$$

1.1.17. Proposition. (Fujimoto [19]) *Let $\alpha = (\alpha^0, \cdots, \alpha^N)$ be an admissible set for $F = (f_0, \cdots, f_N)$ and let h be a holomorphic function. Then,*

$$det\Big(\mathcal{D}^{\alpha^0}(hF), \cdots, \mathcal{D}^{\alpha^N}(hF)\Big) = h^{N+1} det\Big(\mathcal{D}^{\alpha^0}(F), \cdots, \mathcal{D}^{\alpha^N}(F)\Big)$$

1.1.18. Lemma. (Fujimoto [28]) *If* $\Phi^\alpha(F, G, H) = 0$ *and* $\Phi^\alpha(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}) = 0$ *for all* α *with* $|\alpha| \leq 1$, *then one of the following assertions holds :*

(*i*) $F = G, G = H$ *or* $H = F$.

(*ii*) $\frac{F}{G}, \frac{G}{H}$ *and* $\frac{H}{F}$ *are all constant.*

1.1.19. Lemma. *Suppose that* $\Phi^\alpha(F_0, ..., F_M) \not\equiv 0$ *with* $|\alpha| \leq \dfrac{M(M-1)}{2}$. *If*

$$\nu^{([d])} := \min\{\nu_{F_0,\leq k_0}, d\} = \min\{\nu_{F_1,\leq k_1}, d\} = \cdots = \min\{\nu_{F_M,\leq k_M}, d\}$$

for some $d \geq |\alpha|$, *then* $\nu_{\Phi^\alpha}(z_0) \geq \min\{\nu^{([d])}(z_0), d-|\alpha|\}$ *for every* $z_0 \in \{z : \nu_{F_0,\leq k_0}(z) > 0\} \setminus A$, *where* A *is an analytic subset of codimension* ≥ 2.

Proof. Set $H_s := \{z : \nu_{F_s,\leq k_s}(z) > 0\}$, then by the assumption we have $H_0 = H_1 = ... = H_M := H$. Denote by A the set of all singularities of H. Then A is an analytic set of dimension at most $n-2$. We assume that $z_0 \in H \setminus A$. We choose a nonzero holomorphic function h on a neighborhood U of z_0 such that dh has no zero and $H \cap U = \{z \in U; h(z) = 0\}$. Set $m_s := \nu_{F_s}(z_0)$ and $\varphi_s := \dfrac{1}{F_s}$ for $0 \leq s \leq M$. We can write $\varphi_s = h^{-m_s}\widetilde{\varphi}_s$ on a neighborhood $V(\subset U)$ of z_0, where $\widetilde{\varphi}_s$ are nowhere vanishing holomorphic functions on V.

We first consider the case $\nu^{([d])}(z_0) = d$. We have

$$\Phi^\alpha = \begin{vmatrix} F_0 & F_1 & \cdots & F_M \\ F_0.\mathcal{D}^{\alpha^0}(\frac{1}{F_0}) & F_1.\mathcal{D}^{\alpha^0}(\frac{1}{F_1}) & \cdots & F_M.\mathcal{D}^{\alpha^0}(\frac{1}{F_M}) \\ \vdots & \vdots & \vdots & \vdots \\ F_0.\mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_0}) & F_1.\mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_1}) & \cdots & F_M.\mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_M}) \end{vmatrix} = \textstyle\sum_{i=0}^M (-1)^i F_i \psi_i,$$

where $\psi_i := det\left(\dfrac{D^{\alpha^l}\varphi_k}{\varphi_k}; k = 0, ..., i-1, i+1, ..., M;\ l = 0, 1, ..., M-1\right)$ are meromorphic functions.

By induction on $|\alpha^l|$, we can write each $\dfrac{\mathcal{D}^{\alpha^l}\varphi_k}{\varphi_k}$ as $\dfrac{\mathcal{D}^{\alpha^l}\varphi_k}{\varphi_k} = \dfrac{\psi_{k,l}}{h^{|\alpha^l|}}$, where $\psi_{k,l}$ is a holomorphic function, and

$$\psi_i = \sum_{l=(l_1,...,l_M)} \epsilon(l) \frac{\mathcal{D}^{\alpha^{l_1}}\varphi_0}{\varphi_0} ... \frac{\mathcal{D}^{\alpha^{l_i}}\varphi_{i-1}}{\varphi_{i-1}} . \frac{\mathcal{D}^{\alpha^{l_{i+1}}}\varphi_{i+1}}{\varphi_{i+1}} ... \frac{\mathcal{D}^{\alpha^{l_M}}\varphi_M}{\varphi_M},$$

where $l = (l_1, ..., l_M)$ runs through all permutations of $\{0, 1, ..., M-1\}$ and $\epsilon(l)$ denotes the signature of a permutation l. This implies that $\nu^\infty_{\psi_i} \leq |\alpha|$. By the assumption $\nu_{F_i,\leq k_i}(z_0) \geq \nu^{([d])}(z_0) = d$, we have $\nu_{\Phi^\alpha}(z_0) \geq d - |\alpha|$.

After that, we consider the case $1 \leq \nu^{([d])}(z_0) < d$. Then, by the assumption, we get

$$m^* := m_0 = m_1 = \cdots = m_M = \nu^{([d])}(z_0).$$

We now write

$$\Phi^\alpha = \frac{1}{\varphi_0\varphi_1\cdots\varphi_M}\det\Big(\mathcal{D}^{\alpha^l}(\varphi_k-\varphi_0); k=1,...,M;\ l=0,1,....,M-1\Big),$$

and $\varphi_k-\varphi_0 = h^{-m^*}(\widetilde{\varphi}_k-\widetilde{\varphi}_0)$, where $\widetilde{\varphi}_k-\widetilde{\varphi}_0$ is a holomorphic function.

By applying Proposition 1.1.17, it implies that

$$\Phi^\alpha = \frac{h^{m^*(M+1)}}{\widetilde{\varphi}_0\widetilde{\varphi}_1...\widetilde{\varphi}_M}.\frac{1}{h^{m^*M}}\det\Big(\mathcal{D}^{\alpha^l}(\widetilde{\varphi}_k-\widetilde{\varphi}_0); k=1,...,M;\ l=0,1,....,M-1\Big),$$

and hence

$$\Phi^\alpha = \frac{h^{m^*}}{\widetilde{\varphi}_0\widetilde{\varphi}_1...\widetilde{\varphi}_M}\det\Big(\mathcal{D}^{\alpha^l}(\widetilde{\varphi}_k-\widetilde{\varphi}_0); k=1,...,M;\ l=0,1,....,M-1\Big).$$

This yields that $\nu_{\Phi^\alpha}(z_0)\geq m^*$. The proof is completed.

1.1.20. Lemma. *Suppose that the assumptions in Lemma* 1.1.19 *are satisfied. If* $F_0=\cdots=F_M\not\equiv 0,\infty$ *on an analytic subset* H*, which is defined in the proof of Lemma 1.1.19, then* $\nu_{\Phi^\alpha}(z_0)\geq M,\ \forall\ z_0\in H$.

Proof. By using the same proof of Lemma 1.1.19, we now must only show that $\nu_{\Phi^\alpha}(z_0)\geq M$ for all regular points z_0 of H with $F_k(z_0)\neq 0,\infty$ $(0\leq k\leq M)$. Taking a holomorphic function h on a neighborhood U of z_0 such that dh has no zero and $H\cap U=\{z\in U|\ h(z)=0\}$, we write $\psi_k := \frac{1}{F_k}-\frac{1}{F_0} = h\widetilde{\psi}_k$ $(1\leq k\leq M)$ with nonzero holomorphic functions $\widetilde{\psi}_k$ on a neighborhood of z_0. We now use Proposition 1.1.17 to have

$$\begin{aligned}\Phi^\alpha &= F_0F_1...F_M\det\Big(\mathcal{D}^{\alpha^l}\widetilde{\psi}_k; k=1,...,M;\ l=0,1,....,M-1\Big)\\ &= F_0F_1...F_Mh^M\det\Big(\mathcal{D}^{\alpha^l}\psi_k; k=1,...,M;\ l=0,1,....,M-1\Big).\end{aligned}$$

Thus, we get $\nu_{\Phi^\alpha}(z_0)\geq M$.

1.1.21. Lemma. *Let* $f:\mathbb{C}^n\to\mathbb{P}^N(\mathbb{C})$ *be a linearly nondegenerate meromorphic mapping. Let* $H_1,H_2,...,H_q$ *be* q *hyperplanes in* $\mathbb{P}^N(\mathbb{C})$ *located in general position. Assume that* $k_j\geq N-1$ $(1\leq j\leq q)$*. Then*

$$\Big|\Big|\Big(q-N-1-\sum_{j=1}^q\frac{N}{k_j+1}\Big)T(r,f)\leq\sum_{j=1}^q\Big(1-\frac{N}{k_j+1}\Big)N^{(N)}_{(f,H_j),\leq k_j}(r)+o(T(r,f))\ .$$

Proof. By the Second Main Theorem, we have

$$\| \quad (q-N-1)T(r,f) \le \sum_{j=1}^{q} N^{(N)}_{(f,H_j)}(r) + o(T(r,f))$$

$$= \sum_{j=1}^{q} N^{(N)}_{(f,H_j),\le k_j}(r) + \sum_{j=1}^{q} N^{(N)}_{(f,H_j),>k_j}(r) + o(T(r,f))$$

$$\le \sum_{j=1}^{q} N^{(N)}_{(f,H_j),\le k_j}(r) + \sum_{j=1}^{q} \frac{N}{k_j+1} N_{(f,H_j),>k_j}(r) + o(T(r,f))$$

$$= \sum_{j=1}^{q} N^{(N)}_{(f,H_j),\le k_j}(r) + \sum_{j=1}^{q} \frac{N}{k_j+1} \Big(N_{(f,H_j)}(r) - N_{(f,H_j),\le k_j}(r) \Big) + o(T(r,f))$$

$$\le \sum_{j=1}^{q} \big(1 - \frac{N}{k_j+1}\big) N^{(N)}_{(f,H_j),\le k_j}(r) + \sum_{j=1}^{q} \frac{N}{k_j+1} T(r,f) + o(T(r,f)).$$

Thus, we have a desired inequality. Q.E.D.

1.1.22. Lemma. *Assume that there exists* $\Phi^\alpha = \Phi^\alpha(F_c^{j_0 0}, ..., F_c^{j_0 M}) \not\equiv 0$ *for some* $c \in \mathcal{C}, |\alpha| \le \dfrac{M(M-1)}{2}, 2 \ge |\alpha|$ *and the assumptions in Lemma* 1.1.19 *are satisfied. Then, for each* $0 \le i \le M$, *the following holds:*

$$\| \; N^{(2-|\alpha|)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) + M \sum_{j \ne j_0} N^{(1)}_{(f^i,H_j),\le k_{ij}}(r) \le N(r,\nu_{\Phi^\alpha}) \le T(r) + \sum_{l=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^l,H_{j_0}),>k_{lj_0}}(r) + o(T(r)).$$

Proof. The first inequality is deduced immediately from Lemmas 1.1.19 and 1.1.20. On the other hand, we also have

$$N(r,\nu_{\Phi^\alpha}) \le T(r,\Phi^\alpha) + O(1) = N(r,\nu^\infty_{\Phi^\alpha}) + m(r,\Phi^\alpha) + O(1). \qquad (1.1.1)$$

We easily see that a pole of Φ^α is a zero or a pole of some $F_c^{j_0 l}$ and Φ^α is holomorphic at all zeros with multiplicities $\le k_{lj_0}$ of $F_c^{j_0 l}$ because of Lemma 1.1.19 ($l \in \{0, ..., M\}$). Assume that z_0 is a zero of $F_c^{j_0 l}$ with multiplicity $> k_{lj_0}$. We also see that if z_0 is a pole of $\dfrac{\mathcal{D}^{\alpha_i}(1/F_c^{j_0 l})}{(1/F_c^{j_0 l})}$, then it has multiplicity $\le |\alpha_i|$. Thus, if z_0 is a pole of Φ^α then it has multiplicity $\le |\alpha| = \sum_{i=0}^{M-1} |\alpha_i| \le \frac{M(M-1)}{2}$. This implies that

$$N(r,\nu^\infty_{\Phi^\alpha}) \le \sum_{i=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^i,H_{j_0}),>k_{ij_0}}(r) + \sum_{i=0}^{M} N(r,\nu^\infty_{F_c^{j_0 i}}) \qquad (1.1.2)$$

and

$$m(r,\Phi^\alpha) \le \sum_{i=0}^{M} m(r,F_c^{j_0 i}) + O\bigg(\sum m\Big(r, \frac{\mathcal{D}^{\alpha_i}(\varphi_c^{j_0 k})}{\varphi_c^{j_0 k}}\Big)\bigg) + O(1)$$

$$\leq \sum_{i=0}^{M} m(r, F_c^{j_0 i}) + o(T(r)) \qquad (1.1.3),$$

where $\varphi_c^{j_0 k} = 1/F_c^{j_0 k}$. By (1.1.1), (1.1.2) and (1.1.3), we get

$$N(r, \nu_{\Phi^\alpha}) \leq \sum_{i=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^i, H_{j_0}), > k_{ij_0}}(r) + \sum_{i=0}^{M} T(r, F_c^{j_0 i}) + o(T(r))$$

$$\leq T(r) + \sum_{i=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^i, H_{j_0}), > k_{ij_0}}(r) + o(T(r)). \quad \text{Q.E.D.}$$

1.2 A unicity theorem with truncated multiplicities of meromorphic mappings in several complex variables sharing $2N+2$ hyperplanes

Theorem 1.2. (Ha-Quang [33]) *Let f^1 and f^2 be two linearly nondegenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ $(N \geq 2)$ and let $H_1, ..., H_{2N+2}$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ located in general position such that*

$$\dim\{z \in \mathbb{C}^n : \nu_{(f^1, H_i)}(z) > 0 \text{ and } \nu_{(f^1, H_j)}(z) > 0\} \leq n - 2$$

for every $1 \leq i < j \leq 2N+2$. Let m be a positive integer such that

$$m > \binom{2N+2}{N+1}\left[\binom{2N+2}{N+1} - 2\right].$$

Assume that the following conditions are satisfied.

(a) $\min\{\nu_{(f^1, H_j)}, 1\} = \min\{\nu_{(f^2, H_j)}, 1\}$ $(1 \leq j \leq 2N+2)$,

(b) $f^1(z) = f^2(z)$ *on* $\bigcup_{j=1}^{2N+2}\{z \in \mathbb{C}^n : \nu_{(f^1, H_j)}(z) > 0\}$,

(c) $\min\{\nu_{(f^1, H_j)}(z), \nu_{(f^2, H_j)}(z)\} > N$ *or* $\nu_{(f^1, H_j)}(z) \equiv \nu_{(f^2, H_j)}(z) \pmod{m}$ *for all* $z \in (f^1, H_j)^{-1}(0)$ $(1 \leq j \leq 2N+2)$.

Then $f^1 \equiv f^2$.

Proof. Suppose that $f^1 \not\equiv f^2$. For each $i \in \{1, ..., q\}$, we define a divisor ν_i as follows

$$\nu_i(z) := \begin{cases} 1 & \text{if } \min\{\nu_{(f^1, H_i)}(z), \nu_{(f^2, H_i)}(z)\} > N, \\ 1 & \text{if } \nu_{(f^1, H_i)}(z) = \nu_{(f^2, H_i)}(z)) < N, \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1.2.1. *Assume that $i, j \in \{1, 2, ..., 2N+2\}$ such that*

$$P_{ij} = \frac{(f^1, H_i)}{(f^1, H_j)} - \frac{(f^2, H_i)}{(f^2, H_j)} \not\equiv 0.$$

Then, we have

$$\sum_{s=1}^{2}\sum_{v=i,j}\big(2N^{(N)}_{(f^s,H_v)}(r)-NN^{(1)}_{(f^s,H_j)}(r)+N(r,\nu_v)\big)+\sum_{s=1}^{2}\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f^s,H_v)}(r)$$

$$\leq 2\sum_{s=1}^{2}T(r,f^s)+O(1) \quad (1.2.1)$$

Proof . For each $z\in(f^1,H_v)^{-1}(0)$, we consider the three following cases.

Case 1: $\min\,(\nu_{(f^1,H_v)}(z),\nu_{(f^2,H_v)}(z))>N$.

We have

$$\begin{aligned}\min\{\nu_{(f^1,H_v)}(z),\nu_{(f^2,H_v)}(z)\}&\geq N+1\\ &=\sum_{s=1}^{2}\min\{\nu_{(f^s,H_v)}(z),N\}-N+\nu_v(z).\end{aligned}$$

Case 2: $\nu_{(f^1,H_v)}(z)=\nu_{(f^2,H_v)}(z)<N$.

We have

$$\begin{aligned}\min\{\nu_{(f^1,H_v)}(z),\nu_{(f^2,H_v)}(z)\}&=\sum_{s=1}^{2}\min\{\nu_{(f^s,H_v)}(z),N\}-\nu_{(f^1,H_v)}(z)\\ &\geq\sum_{s=1}^{2}\min\{\nu_{(f^s,H_v)}(z),N\}-N+\nu_v(z).\end{aligned}$$

Case 3: z is not satisfied Case 1 and Case 2.

Then $\nu_v(z)=0$. We have

$$\begin{aligned}\min\{\nu_{(f^1,H_v)}(z),\nu_{(f^2,H_v)}(z)\}&\geq\sum_{s=1}^{2}\min\{\nu_{(f^s,H_v)}(z),N\}-N\\ &=\sum_{s=1}^{2}\min\{\nu_{(f^s,H_v)}(z),N\}-N+\nu_v(z).\end{aligned}$$

From the above cases, for every $z\in(f^1,H_v)^{-1}(0)$, we have

$$\min\{\nu_{(f^1,H_v)}(z),\nu_{(f^2,H_v)}(z)\}\geq\sum_{s=1}^{2}\min\{\nu_{(f^s,H_v)}(z),N\}-N+\nu_v(z).$$

By this inequality and by the definition of P_{ij}, it is easy to see that

$$\begin{aligned}\nu_{P_{ij}}(z)&\geq\min\{\nu_{(f^1,H_i)}(z),\nu_{(f^2,H_i)}(z)\}+\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}\nu^{(1)}_{(f^1,H_v)}(z)\\ &\geq\sum_{s=1}^{2}\left(\nu^{(N)}_{(f^s,H_i)}(z)-\frac{N}{2}\nu^{(1)}_{(f^s,H_i)}(z)\right)+\nu_i+\frac{1}{2}\sum_{s=1}^{2}\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}\nu^{(1)}_{(f^s,H_v)}(z).\end{aligned}$$

This yields that

$$2N_{P_{ij}}(r) \geq \sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_i)}(r) - NN^{(1)}_{(f^s,H_i)}(r) + N(r,\nu_i)\big)$$
$$+ \sum_{s=1}^{2} \sum_{\substack{v=1\\ v\neq i,j}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r) \quad (1.2.2).$$

On the other hand, we have

$$\begin{aligned} m(r,P_{ij}) &\leqslant m\left(r, \frac{(f^1,H_i)}{(f^1,H_j)}\right) + m\left(r, \frac{(f^2,H_i)}{(f^2,H_j)}\right) + O(1) \\ &\leqslant T\left(r, \frac{(f^1,H_i)}{(f^1,H_j)}\right) - N\left(r, \frac{(f^1,H_j)}{(f^1,H_i)}\right) + T\left(r, \frac{(f^2,H_i)}{(f^2,H_j)}\right) \\ &- N\left(r, \frac{(f^2,H_i)}{(f^2,H_j)}\right) + O(1) \\ &\leqslant T(r,f^1) + T(r,f^2) - N_{\frac{(f^1,H_j)}{(f^1,H_i)}}(r) - N_{\frac{(f^2,H_j)}{(f^2,H_i)}}(r) + O(1) \\ &= T(r,f^1) + T(r,f^2) - N_{(f^1,H_j)}(r) - N_{(f^2,H_j)}(r) + O(1) \end{aligned}$$

and

$$N_{\frac{1}{P_{ij}}}(r) \leq N(r,\mu_j), \quad \text{where } \mu_j(z) = \max\{\nu_{(f^1,H_j)}(z), \nu_{(f^2,H_j)}\}(z).$$

For every $z \in (f^1,H_j)^{-1}(0)$, it is easy to see that

$$\begin{aligned} \nu_{(f^1,H_j)}(z) + \nu_{(f^2,H_j)}(z) - \mu_j(z) &= \min\{\nu_{(f^1,H_j)}(z), \nu_{(f^2,H_j)}(z)\} \\ &\geq \min\{\nu_{(f^1,H_j)}(z), N\} + \min\{\nu_{(f^2,H_j)}(z), N\} - N + \nu_j(z). \end{aligned}$$

Hence

$$\sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_i)}(r) - NN^{(1)}_{(f^s,H_i)}(r) + N(r,\nu_i)\big) + \sum_{s=1}^{2} \sum_{\substack{v=1\\ v\neq i,j}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r)$$
$$\leq 2N_{P_{ij}}(r) \leq 2T(r,P_{ij}) = 2N_{\frac{1}{P_{ij}}}(r) + 2m(r,P_{ij}) + O(1).$$

$$\begin{aligned} &\leq 2\sum_{s=1}^{2} T(r,f^s) + 2(N(r,\mu_j) - N_{(f^1,H_j)}(r) - N_{(f^2,H_j)}(r)) + O(1) \\ &\leq 2\sum_{s=1}^{2} T(r,f^s) - 2(N^{(N)}_{(f^1,H_j)}(r) + N^{(N)}_{(f^2,H_j)}(r) - NN^{(1)}_{(f^1,H_j)}(r) + N(r,\nu_j)) \\ &+ O(1) \\ &\leq 2\sum_{s=1}^{2} T(r,f^s) - \sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_j)}(r) - NN^{(1)}_{(f^s,H_j)}(r) + N(r,\nu_j)\big) + O(1). \end{aligned}$$

This implies that

$$\sum_{s=1}^{2}\sum_{v=i,j}\big(2N^{(N)}_{(f^s,H_v)}(r)-NN^{(1)}_{(f^s,H_j)}(r)+N(r,\nu_v)\big)+\sum_{s=1}^{2}\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f^s,H_v)}(r)$$
$$\leq 2\sum_{s=1}^{2}T(r,f^s)+O(1).$$

This concludes Claim 1.2.1.

Claim 1.2.2. *For every $1\leq i\leq 2N+2$, we have*

$$||\quad N(r,\nu_i)=o(\sum_{s=1}^{2}T(r,f^s)).$$

Proof . By changing indices if necessary, we may assume that

$$\underbrace{\frac{(f^1,H_1)}{(f^2,H_1)}\equiv\frac{(f^1,H_2)}{(f^2,H_2)}\equiv\cdots\equiv\frac{(f^1,H_{k_1})}{(f^2,H_{k_1})}}_{\text{group 1}}\not\equiv\underbrace{\frac{(f^1,H_{k_1+1})}{(f^2,H_{k_1+1})}\equiv\cdots\equiv\frac{(f^1,H_{k_2})}{(f^2,H_{k_2})}}_{\text{group 2}}$$

$$\not\equiv\underbrace{\frac{(f^1,H_{k_2+1})}{(f^2,H_{k_2+1})}\equiv\cdots\equiv\frac{(f^1,H_{k_3})}{(f^2,H_{k_3})}}_{\text{group 3}}\not\equiv\cdots\not\equiv\underbrace{\frac{(f^1,H_{k_{s-1}+1})}{(f^2,H_{k_{s-1}+1})}\equiv\cdots\equiv\frac{(f^1,H_{k_s})}{(f^2,H_{k_s})}}_{\text{group } s},$$

where $k_s=2N+2$.

For each $1\leq i\leq 2N+2$, we set

$$\chi(i)=\begin{cases} i+N & \text{if } i\leq N+2,\\ i-N-2 & \text{if } i>N+2.\end{cases}$$

Since $f^1\not\equiv f^2$, the number of elements of every group is at most N. Hence $\frac{(f^1,H_i)}{(f^2,H_i)}$ and $\frac{(f^1,H_{\chi(i)})}{(f^2,H_{\chi(i)})}$ belong to distinct groups. This means that $\frac{(f^1,H_i)}{(f^2,H_i)}\not\equiv\frac{(f^1,H_{\chi(i)})}{(f^2,H_{\chi(i)})}$ $(1\leq i\leq 2N+2)$. Hence

$$P_{\chi(i)i}=\frac{(f^1,H_{\chi(i)})}{(f^1,H_i)}-\frac{(f^2,H_{\chi(i)})}{(f^2,H_i)}\not\equiv 0\ (1\leq i\leq 2N+2).$$

Summing up both sides of (1.2.1) over all pairs $(i,\chi(i))$, we have

$$\sum_{s=1,2}\sum_{i=1}^{2N+2}\Big(4N^{(N)}_{(f^s,H_i)}(r)+2N(r,\nu_i)\Big)\leq 2(2N+2)\sum_{s=1}^{2}T(r,f^s)+O(1)\quad (1.2.3)$$

Then, by the Second Main Theorem we have

$$\| \ 2(2N+2)\sum_{s=1}^{2} T(r,f^s) \geq \sum_{s=1,2}\sum_{i=1}^{2N+2}\Big(4N^{(N)}_{(f^s,H_i)}(r)+2N(r,\nu_i)\Big)+O(1)$$

$$\geq 4(N+1)\sum_{s=1,2} T(r,f^s)+4\sum_{i=1}^{2N+2} N(r,\nu_i)$$

$$+\, o(\sum_{s=1,2} T(r,f^s)) \quad (1.2.4).$$

This implies that

$$\| \ N(r,\nu_i)=o(\sum_{s=1,2} T(r,f^s)).$$

Claim 1.2.2 is proved.

Claim 1.2.3. *For $i=1,...,2N+2$, the following assertions hold*

$$(i) \ \| \ \sum_{s=1}^{2}\sum_{v=\chi(i),i}\big(2N^{(N)}_{(f^s,H_v)}(r)-NN^{(1)}_{(f^s,H_j)}(r)\big)+\sum_{s=1}^{2}\sum_{\substack{v=1\\ v\neq\chi(i),i}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r)$$

$$=2\sum_{s=1}^{2} T(r,f^s)+o(\sum_{s=1}^{2} T(r,f^s)) \quad (1.2.5)$$

$$(ii) \ \| \ 2N_{P_{\chi(i)i}}(r)=\sum_{s=1}^{2}\big(2N^{(N)}_{(f^s,H_{\chi(i)})}(r)-NN^{(1)}_{(f^s,H_{\chi(i)})}(r)\big)$$

$$+\sum_{s=1}^{2}\sum_{\substack{v=1\\ v\neq\chi(i),i}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r)+o(\sum_{s=1}^{2} T(r,f^s)) \quad (1.2.6)$$

Proof. Since the inequality (1.2.4) becomes an equality, the inequalities (1.2.1) and (1.2.2) must become equalities for all $P_{\chi(i)i}$. Moreover, we have $\| \ N(r,\nu_{\chi(i)})=N(r,\nu_i)=o(\sum_{s=1}^{2} T(r,f^s))$. Then Claim 1.2.3 is proved.

Claim 1.2.4. *For $i, j \in \{1, ..., 2N+2\}$ with $P_{ij} \not\equiv 0$, the following assertions hold*

$$(i) \parallel \sum_{s=1}^{2} \sum_{v=i,j} \big(2N^{(N)}_{(f^s,H_v)}(r) - NN^{(1)}_{(f^s,H_v)}(r)\big) + \sum_{s=1}^{2} \sum_{\substack{v=1 \\ v\neq i,j}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r)$$

$$= 2\sum_{s=1}^{2} T(r, f^s) + o(\sum_{s=1}^{2} T(r, f^s)) \quad (1.2.7)$$

$$(ii) \parallel 2N_{P_{ij}}(r) = \sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_i)}(r) - NN^{(1)}_{(f^s,H_i)}(r)\big)$$

$$+ \sum_{s=1}^{2} \sum_{\substack{v=1 \\ v\neq i,j}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r) + o(\sum_{s=1}^{2} T(r, f^s)) \quad (1.2.8)$$

Proof . Since $P_{ij} \not\equiv 0$, $\frac{(f^1,H_i)}{(f^1,H_j)}$ and $\frac{(f^2,H_i)}{(f^2,H_j)}$ belong to two distinct groups. Therefore, by changing indices again we may assume that $i = \chi(j)$. Then Claim 1.2.4 is deduced from Claim 1.2.3.

Now we return to prove the theorem. We consider two arbitrary indices $i, j \in \{1, ..., 2N+2\}$. Since $f^1 \not\equiv f^2$, there exists an index k such that $P_{ik} \not\equiv 0$ and $P_{jk} \not\equiv 0$. By (1.2.7), we have

$$\parallel \sum_{s=1}^{2} \sum_{v=i,k} \big(2N^{(N)}_{(f^s,H_v)}(r) - NN^{(1)}_{(f^s,H_v)}(r)\big) + \sum_{s=1}^{2} \sum_{\substack{v=1 \\ v\neq i,k}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r)$$

$$= \sum_{s=1}^{2} \sum_{v=j,k} \big(2N^{(N)}_{(f^s,H_v)}(r) - NN^{(1)}_{(f^s,H_v)}(r)\big) + \sum_{s=1}^{2} \sum_{\substack{v=1 \\ v\neq j,k}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r)$$

$$+ o(\sum_{s=1}^{2} T(r, f^s)) = 2\sum_{s=1}^{2} T(r, f^s) + o(\sum_{s=1}^{2} T(r, f^s)).$$

Thus

$$\parallel \sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_i)}(r) - (N+1)N^{(1)}_{(f^s,H_i)}(r)\big) = \sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_j)}(r)$$

$$- (N+1)N^{(1)}_{(f^s,H_j)}(r)\big) + o(\sum_{s=1}^{2} T(r, f^s)) \quad (1.2.9)$$

Combining (1.2.7) and (1.2.9), we get

$$\parallel 2\sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_i)}(r) - (N+1)N^{(1)}_{(f^s,H_i)}(r)\big) + \sum_{s=1}^{2} \sum_{v=1}^{2N+2} N^{(1)}_{(f^s,H_v)}(r)$$

$$= 2\sum_{s=1}^{2} T(r, f^s) + o(\sum_{s=1}^{2} T(r, f^s)) \quad (1.2.10)$$

Assume that $H_i = \{a_{i0}\omega_0 + \cdots + a_{iN}\omega_N = 0\}$. We set $h_i = \dfrac{(f^1, H_i)}{(f^2, H_i)}$ $(1 \le i \le 2N+2)$. Then $\dfrac{h_i}{h_j} = \dfrac{(f^1, H_i)\cdot(f^2, H_j)}{(f^1, H_j)\cdot(f^2, H_i)}$ does not depend on representations of f^1 and f^2 respectively. Since $\sum_{k=0}^{N} a_{ik} f_{1k} - h_i \cdot \sum_{k=0}^{N} a_{ik} f_{2k} = 0$ $(1 \le i \le 2N+2)$, it implies that $\det(a_{i0}, ..., a_{iN}, a_{i0}h_i, ..., a_{iN}h_i; 1 \le i \le 2N+2) = 0$.

For each subset $I \subset \{1, 2, ..., 2N+2\}$, put $h_I = \prod_{i\in I} h_i$. Denote by $\mathcal{I}$ the set of all combinations $I = (i_1, ..., i_{N+1})$ with $1 \le i_1 < ... < i_{N+1} \le 2N+2$.

For each $I = (i_1, ..., i_{N+1}) \in \mathcal{I}$, define

$$A_I = (-1)^{\frac{(N+1)(N+2)}{2} + i_1 + ... + i_{N+1}} \cdot \det(a_{i_r l}; 1 \le r \le N+1, 0 \le l \le N) \cdot$$

$$\det(a_{j_s l}; 1 \le s \le N+1, 0 \le l \le N),$$

where $J = (j_1, ..., j_{N+1}) \in \mathcal{I}$ such that $I \cup J = \{1, 2, ..., 2N+2\}$.

Then $\sum_{I\in\mathcal{I}} A_I h_I = 0$.

Take $I_0 \in \mathcal{I}$. Then

$$A_{I_0} h_{I_0} = -\sum_{I\in\mathcal{I}, I\ne I_0} A_I h_I, \text{ i.e. } h_{I_0} = -\sum_{I\in\mathcal{I}, I\ne I_0} \frac{A_I}{A_{I_0}} h_I.$$

Remark that for each $I \in \mathcal{I}$, then $\frac{A_I}{A_{I_0}} \not\equiv 0$.

Denote by t the minimal number satisfying the following:

There exist t elements $I_1, ..., I_t \in \mathcal{I} \setminus \{I_0\}$ and t nonzero constants $b_i \in \mathbb{C}$ such that $h_{I_0} = \sum_{i=1}^{t} b_i h_{I_i}$.

It is easy to see that $t \le \begin{pmatrix} 2N+2 \\ N+1 \end{pmatrix} - 1$.

Since $h_{I_0} \not\equiv 0$ and by the minimality of t, it follows that the family $\{h_{I_1}, ..., h_{I_t}\}$ is linearly independent over $\mathbb{C}$.

Assume that $t \ge 2$.

Consider the meromorphic mapping $h : \mathbb{C}^n \to \mathbb{P}^{t-1}(\mathbb{C})$ with a reduced representation $h = (dh_{I_1} : ... : dh_{I_t})$, where d is meromorphic on $\mathbb{C}^n$.

If z is a zero (a pole, resp.) of h_i, then $\nu_{(f^1,H_i)}(z) \ne \nu_{(f^2,H_i)}(z)$. Hence $\max\{\nu_{(f^1,H_i)}(z), \nu_{(f^2,H_i)}(z)\} > N$ or $|\nu_{(f^1,H_i)}(z) - \nu_{(f^2,H_i)}(z)| > m$. Therefore, $\nu_i(z) = 1$ or z is either zero or pole of h_i with multiplicity at least m. This easily implies that if z is a zero of dh_I then $\nu_i(z) = 1$ with one of indices $i \in \{1, ..., 2N+2\}$ or z is zero of dh_I with multiplicity at least m. We thus have, for every $z \notin (f^1)^{-1}(H_i) \cap (f^1)^{-1}(H_j)(1 \le$

$i < j \le 2N+2)$.

$$\min\{1, \nu_{dh_I}(z)\} \le \sum_{i=1}^{2N+2} \nu_i(z) + \frac{1}{m}\nu_{dh_I}(z).$$

This implies that

$$|| \ N^{(1)}_{dh_I}(r) \le \sum_{i=1}^{2N+2} N(r,\nu_i)(r) + \frac{1}{m}N_{dh_I}(r) \le \frac{1}{m}T(r,h) + o(\sum_{s=1}^{2} T(r,f^s))$$

for each $I \in \mathcal{I}$.

By the Second Main Theorem, we have

$$\begin{aligned} || \quad T(r,h) &\le \sum_{i=1}^{t} N^{(t-1)}_{dh_{I_i}}(r) + N^{(t-1)}_{dh_{I_0}}(r) + o(T(r,h)) \\ &\le (t-1)\Big(\sum_{i=1}^{t} N^{(1)}_{dh_{I_i}}(r) + N^{(1)}_{dh_{I_0}}(r)\Big) + o(T(r,h)) \\ &\le \frac{(t-1)(t+1)}{m}T(r,h) + o(T(r,h)) + o(\sum_{s=1}^{2} T(r,f^s)). \end{aligned}$$

This yields that $|| \quad T(r,h) = o(\sum_{s=1}^{2} T(r,f^s))$.

Consider the hyperplanes $\tilde{H}_1 = \{w_1 = 0\}$, $\tilde{H}_2 = \{w_2 = 0\}$, $\tilde{H}_3 = \{b_1w_1 + ... + b_tw_t = 0\}$ in $\mathbb{P}^{t-1}(\mathbb{C})$. Then

$$T(r,h) \ge T\Big(r, \frac{(h,\tilde{H}_1)}{(h,\tilde{H}_2)}\Big) + O(1) = T\Big(r, \frac{h_{I_1}}{h_{I_2}}\Big) + O(1) \ge N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1} + O(1),$$

$$T(r,h) \ge T\Big(r, \frac{(h,\tilde{H}_2)}{(h,\tilde{H}_3)}\Big) + O(1) = T\Big(r, \frac{h_{I_2}}{h_{I_0}}\Big) + O(1) \ge N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1} + O(1),$$

$$T(r,h) \ge T\Big(r, \frac{(h,\tilde{H}_3)}{(h,\tilde{H}_1)}\Big) + O(1) = T\Big(r, \frac{h_{I_0}}{h_{I_1}}\Big) + O(1) \ge N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r) + O(1).$$

Hence $3T(r,h) \ge N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1}(r) + N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r) + O(1)$.

Since $\dfrac{h_I}{h_J} = 1$ on the set $\bigcup_{j \in ((I \cup J)\setminus(I \cap J))^c} E_j$,
where $E_j = \{z \in \mathbb{C}^n : \nu_{(f,H_j)}(z) > 0\}$ and $((I_1 \cup I_2) \setminus (I_1 \cap I_2))^c \cup ((I_2 \cup I_0) \setminus (I_2 \cap I_0))^c \cup ((I_0 \cup I_1) \setminus (I_0 \cap I_1))^c = \{1, ..., 2N+2\}$), it implies that

$$N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1}(r) + N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r) \ge \sum_{i=1}^{2N+2} N^{(1)}_{(f^s,H_i)}(r).$$

Hence $|| \ 3T(r,h) \ge \sum_{i=1}^{2N+2} N^{(1)}_{(f^s,H_i)}(r) + O(1) = \dfrac{N+1}{N} \cdot T(r,f^s) + o(T(r,f^s))$ $(s = 1,2)$. Then $|| \ T(r,f^s) = 0$ $(s = 1,2)$. This is a contradiction. Thus, $t = 1$. Then $\dfrac{h_{I_0}}{h_{I_1}} =$

$constant \neq 0$. Hence, for each $I \in \mathcal{I}$, there is $J \in \mathcal{I} \setminus \{I\}$ such that $\dfrac{h_I}{h_J} = constant \neq 0$. Consider the free abelian subgroup generated by the family $\{[h_1], ..., [h_{2N+2}]\}$ of the torsion free abelian group $\mathcal{M}^*{}_n/\mathbb{C}^*$. Then the family $\{[h_1], ..., [h_{2N+2}]\}$ has the property $P_{2N+2,N+1}$. It implies that there exist $2N + 2 - 2N = 2$ elements, without loss of generality we may assume that they are $[h_1], [h_2]$, such that $[h_1] = [h_2]$. Then $\dfrac{h_1}{h_2} = \chi \in \mathbb{C}^*$.

Suppose that $\chi \neq 1$.

Since $\dfrac{h_1(z)}{h_2(z)} = 1$ for each $z \in \bigcup_{i=3}^{2N+2} (f^1)^{-1}(H_i) \setminus ((f^1)^{-1}(H_1) \cup (f^1)^{-1}(H_2)$, it implies that $\bigcup_{i=3}^{2N+2} (f^1)^{-1}(H_i) = \emptyset$. By the Second Main Theorem, we have

$$|| \quad (2N - N - 1)T(r, f^1) \leq \sum_{i=3}^{2N+2} N^{(N)}_{(f^1,H_i)}(r) + o(T(r, f^1)) = o(T(r, f^1)).$$

This is a contradiction. Thus, $\chi = 1$, i.e, $h_1 = h_2$. By changing reduced representations of f^1, f^2 if necessary, we may assume that $(f^1, H_1) = (f^2, H_1)$. This yields that $(f^1, H_2) = (f^2, H_2) \quad (1.2.11)$.

Now we consider

$$\begin{aligned} P_{\chi(N+3)(N+3)} = P_{1(N+3)} &= \frac{(f^1, H_1)}{(f^1, H_{N+3})} - \frac{(f^2, H_1)}{(f^2, H_{N+3})} \\ &= \frac{(f^1, H_1)((f^2, H_{N+3}) - (f^1, H_{N+3}))}{(f^1, H_{N+3})(f^2, H_{N+3})} \not\equiv 0. \end{aligned}$$

Since $(f^1, H_i)(z) = (f^2, H_i)(z)$ on $\bigcup_{j=1}^{2N+2} (f^1)^{-1}(H_j) \setminus ((f^1)^{-1}(H_1) \cap (f^1)^{-1}(H_2))$ for each $1 \leq i \leq 2N + 2$, it implies that

$$\begin{aligned} 2N_{P_{1(N+3)}}(r) &\geq 2N_{(f^1,H_1)}(r) + \sum_{\substack{v=1 \\ v\neq N+3}}^{2N+2} 2N^{(1)}_{(f^1,H_v)}(r) \\ &\geq \sum_{s=1}^{2} (2N_{(f^s,H_1)}(r) - NN^{(1)}_{(f^s,H_1)}(r)) + \sum_{s=1}^{2} \sum_{\substack{v=1 \\ v\neq N+3}}^{2N+2} N^{(1)}_{(f^s,H_v)}(r) \quad (1.2.12) \end{aligned}$$

Combining (1.2.8) and (1.2.12), we get

$$||N^{(1)}_{(f^1,H_1)}(r) = N^{(1)}_{(f^2,H_1)}(r) = o(\sum_{s=1}^{2} T(r, f^s)) \quad (1.2.13)$$

From (1.2.9) and (1.2.13), for each $i \in \{1, ..., 2N + 2\}$ we have

$$|| \quad \sum_{s=1}^{2} \big(2N^{(N)}_{(f^s,H_i)}(r) - (N + 1)N^{(1)}_{(f^s,H_i)}(r)) = o(\sum_{s=1}^{2} T(r, f^s)) \quad (1.2.14)$$

On ther other hand, for every $z \in (f^1, H_i)^{-1}(0)$, if $\nu_i(z) = 0$ then either $\nu_{(f^1,H_i)}(z) = \nu_{(f^2,H_i)}(z) = N$ or $|\nu_{(f^1,H_i)}(z) - \nu_{(f^2,H_i)}(z)| \geq m$, hence

$$\nu_{(f^1,H_i)}(z) + \nu_{(f^2,H_i)}(z) \geq 2N.$$

Thus

$$\begin{aligned} \| \sum_{s=1}^{2} 2N^{(N)}_{(f^s,H_i)}(r) &\geq \sum_{s=1}^{2} 2NN^{(1)}_{(f^s,H_i)}(r) + 2NN(r,\nu_i) \\ &= \sum_{s=1}^{2} 2NN^{(1)}_{(f^s,H_i)}(r) + o(\sum_{s=1}^{2} T(r,f^s)). \end{aligned}$$

This implies that

$$\begin{aligned} \| \sum_{s=1}^{2}\big(2N^{(N)}_{(f^s,H_i)}(r) - (N+1)N^{(1)}_{(f^s,H_i)}(r)\big) &\geq (N-1)\sum_{s=1}^{2} N^{(1)}_{(f^s,H_i)}(r) \\ &\quad + o(\sum_{s=1}^{2} T(r,f^s)). \end{aligned}$$

From this inequality and (1.2.14), it follows that

$$\sum_{s=1}^{2} N^{(1)}_{(f^s,H_i)}(r) = o(\sum_{s=1}^{2} T(r,f^s)) \ (1 \leq i \leq 2N+2).$$

By the Second Main Theorem, we have

$$\| \sum_{s=1}^{2}(N+1)T(r,f^s) \leq \sum_{s=1}^{2}\sum_{v=1}^{2N+2} N^{(N)}_{(f^s,H_v)}(r) + o(\sum_{s=1}^{2} T(r,f^s)) = o(\sum_{s=1}^{2} T(r,f^s)).$$

This is a contradiction. Hence $f^1 \equiv f^2$. Theorem 1.2 is proved. □

1.3 A unicity theorem for meromorphic mapping sharing few fixed targets with ramification of truncations

Theorem 1.3. (Ha [31]) *Let $f^1, f^2, f^3 : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be three meromorphic mappings and let $\{H_i\}_{i=1}^q$ be hyperplanes in general position. Let $d, k, k_{1i}, k_{2i}, k_{3i}$ be integers with $1 \leq k_{1i}, k_{2i}, k_{3i} \leq \infty$ $(1 \leq i \leq q)$. We set $M = max\{k_{ji}\}$, $m = min\{k_{ji}\}$ $(1 \leq j \leq 3, 1 \leq i \leq q)$, $k = max\{\sharp\{i \in \{1, 2 \cdots, q\} \mid k_{ji} = m\} \mid 1 \leq j \leq 3\}$. Define by $d = 0$ if $M = m$ and $d = min\{k_{ji} - m > 0 \mid 1 \leq j \leq 3; 1 \leq i \leq q\}$ if $M \neq m$. Assume that the* *following conditions are satisfied*

(i) dim$\{z \in \mathbb{C}^n : \nu_{(f^j,H_i),\le k_{ji}} > 0$ *and* $\nu_{(f^j,H_l),\le k_{jl}} > 0\} \le n-2$

$(1 \le j \le 3; 1 \le i < l \le q)$

(ii) min$(\nu_{(f^j,H_i),\le k_{ji}}, 2) =$ *min* $(\nu_{(f^t,H_i),\le k_{ti}}, 2) \quad (1 \le j < t \le 3; 1 \le i \le q)$

(iii) $f^1 \equiv f^j$ *on* $\bigcup_{\alpha=1}^q \{z \in \mathbb{C}^n : \nu_{(f^1,H_\alpha),\le k_{1\alpha}}(z) > 0\} \quad (1 \le j \le 3)$.

Then $f^1 \equiv f^2$ *or* $f^2 \equiv f^3$ *or* $f^3 \equiv f^1$ *if one of the following conditions is satisfied*

1) $N \ge 2, 3N-1 \le q \le 3N+1, m > 3N+1+\dfrac{16}{3(N-1)}$ *and*

$(2q-5N-3) > \dfrac{2Nk}{m+1} + \dfrac{2N(q-k)}{m+d+1} - \dfrac{3N^2+N}{M+1}$.

2) $N = 1, q = 4$ *and* $\dfrac{3(2k+1)}{m+1} + \dfrac{6(4-k)}{m+d+1} + \dfrac{6k}{M(m+1)} + \dfrac{24-6k}{M(m+d+1)} < 1 + \dfrac{12}{M}$.

Before proving, we now give some corollaries that are given directly from Theorem 1.3.

*) Theorem 1.3 is deduced immediately from the theorem 1.3 by choosing $M = m$ and $k = q$.

*) When $k = 1, M = m+d$ and $d = 1$ or $d = 2$, by using the case 1 of Theorem 1.3, we have the following

Corollary 1. *Let* $f^1, f^2, f^3 : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ *be three meromorphic mappings and let* $\{H_i\}_{i=1}^{3N+1}$ *be hyperplanes in general position. Let* k_i *be positive integers with* $1 \le i \le 3N+1$ *satisfying the following conditions*

(i) dim$\{z \in \mathbb{C}^n : \nu_{(f^j,H_i),\le k_i} > 0$ *and* $\nu_{(f^j,H_l),\le k_l} > 0\} \le n-2$ *(* $1 \le i < l \le 3N+1$*)*

(ii) min$(\nu_{(f^j,H_i),\le k_i}, 2) =$ *min* $(\nu_{(f^t,H_i),\le k_i}, 2) \quad (1 \le j < t \le 3; 1 \le i \le 3N+1)$

(iii) $f^1 \equiv f^j$ *on* $\bigcup_{\alpha=1}^{3N+1} \{z \in \mathbb{C}^n : \nu_{(f^1,H_\alpha),\le k_\alpha}(z) > 0\} \quad (1 \le j \le 3)$.

Then $f^1 \equiv f^2$ *or* $f^2 \equiv f^3$ *or* $f^3 \equiv f^1$ *if one of the following conditions is satisfied*

a) $N \ge 2, k_j = k_1 + 1$ *for every* $2 \le j \le 3N+1$ *and* $k_1 > 3N+2+\dfrac{14}{3(N-1)}$.

b) $N \ge 2, k_j = k_1 + 2$ *for every* $2 \le j \le 3N+1$ *and* $k_1 > 3N+1+\dfrac{16}{3(N-1)}$.

*) When $k = 1$ and $M = m+d$, by using the proof for the case 2 of Theorem 1.3, we have the following

Corollary 2. *Let* $f^1, f^2, f^3 : \mathbb{C}^n \longrightarrow \mathbb{P}^1(\mathbb{C})$ *be three meromorphic functions and let* $\{H_i\}_{i=1}^4$ *be distinct points. Let* k_i $(1 \le i \le 4)$ *be positive integers satisfying the following conditions*

(i) dim$\{z \in \mathbb{C}^n : \nu_{(f^j,H_i),\le k_i} > 0$ *and* $\nu_{(f^j,H_l),\le k_l} > 0\} \le n-2$

($1 \le j \le 3; 1 \le i < l \le 4$*)*

(ii) $min(\nu_{(f^j,H_i),\le k_i}, 2) = min(\nu_{(f^t,H_i),\le k_i}, 2)$ $(1 \le j < t \le 3; 1 \le i \le 4)$

(iii) $f^1 \equiv f^j$ *on* $\bigcup_{\alpha=1}^{4}\{z \in \mathbb{C}^n : \nu_{(f^1,H_\alpha),\le k_\alpha}(z) > 0\}$ $(1 \le j \le 3)$

Assume that one of the following conditions is satisfied

a) $k_1 = 9, k_2 = k_3 = k_4 = 66.$

b) $k_1 = 10, k_2 = k_3 = k_4 = 36.$

c) $k_1 = 11, k_2 = k_3 = k_4 = 26.$

d) $k_1 = 12, k_2 = k_3 = k_4 = 21.$

e) $k_1 = 13, k_2 = k_3 = k_4 = 18.$

f) $k_1 = 14, k_2 = k_3 = k_4 = 16.$

Then $f^1 \equiv f^2$ *or* $f^2 \equiv f^3$ *or* $f^3 \equiv f^1$*.*

Proof. **Case 1.** $N \ge 2, 3N-1 \le q \le 3N+1, m > 3N+1+\dfrac{16}{3(N-1)}$ and

$$(2q-5N-3) > \frac{2Nk}{m+1} + \frac{2N(q-k)}{m+d+1} - \frac{3N^2+N}{M+1}.$$

Firstly, we need the following.

Claim 1.3.1. Denote by $\mathcal{Q}$ the set of all indices $j_0 \in \{1, 2, ..., q\}$ satisfying the following: There exist $c \in \mathcal{C}$ and $\alpha = (\alpha_0, \alpha_1)$ with $|\alpha| \le 1$ such that $\Phi^\alpha(F_c^{j_0 1}, F_c^{j_0 2}, F_c^{j_0 3}) \not\equiv 0$.

Then $\mathcal{Q}$ is an empty set.

Proof. Assume that $\mathcal{Q}$ is non-empty. For every $1 \le i \le 3$ and $j_0 \in \mathcal{Q}$, by Lemma 1.1.22, we have

$$\Big\|\quad N^{(1)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) + 2\sum_{j\ne j_0} N^{(1)}_{(f^i,H_j),\le k_{ij}}(r) \le T(r) + \sum_{l=1}^{3} N^{(1)}_{(f^l,H_{j_0}),>k_{lj_0}}(r) + o(T(r)),$$

and hence

$$\Big\|\quad N^{(N)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) + 2\sum_{j\ne j_0} N^{(N)}_{(f^i,H_j),\le k_{ij}}(r) \le N\cdot T(r) + N\sum_{l=1}^{3} N^{(1)}_{(f^l,H_{j_0}),>k_{lj_0}}(r) + o(T(r)).$$

This implies that

$$\begin{aligned}\Big\|\quad &\sum_{i=1}^{3}\Big(N^{(N)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) + 2\sum_{j\ne j_0} N^{(N)}_{(f^i,H_j),\le k_{ij}}(r)\Big)\\ &\le 3NT(r) + 3N\sum_{i=1}^{3} N^{(1)}_{(f^i,H_{j_0}),>k_{ij_0}}(r) + o(T(r))\\ &\le 3NT(r) + \sum_{i=1}^{3}\Big(\frac{3N}{k_{ij_0}+1}\Big) N_{(f^i,H_{j_0}),>k_{ij_0}}(r) + o(T(r))\end{aligned}$$

$$\leq 3NT(r) + \sum_{i=1}^{3}\left(\frac{3N}{k_{ij_0}+1}\right)\left(N_{(f^i,H_{j_0})}(r) - N_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)\right) + o(T(r)) \qquad (1.3.1)$$

Hence we see

$$\Big\| \ \sum_{i=1}^{3}\left(2\sum_{j=1}^{q} N^{(N)}_{(f^i,H_j),\leq k_{ij}}(r)\right) \leq 3NT(r) + \sum_{i=1}^{3}\left(\frac{3N}{k_{ij_0}+1}\right)N_{(f^i,H_{j_0})}(r) +$$

$$+ \sum_{i=1}^{3}(1 - \frac{3N}{k_{ij_0}+1})N^{(N)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r) + o(T(r)) \qquad (1.3.2)$$

On the other hand, since $1 - \dfrac{3N}{k_{ij_0}+1} > 0$ and

$$\max\{N^{(N)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r); N_{(f^i,H_{j_0})}(r)\} \leq T(r,f^i) + o(T(r,f^i)) \text{ for every } 1 \leq i \leq 3, \qquad (1.3.3)$$

we have

$$\Big\| \quad 2\sum_{i=1}^{3}\sum_{j=1}^{q} N^{(N)}_{(f^i,H_j),\leq k_{ij}}(r) \leq (3N+1)T(r) + o(T(r)). \qquad (1.3.4)$$

Using Lemma 1.1.21, we have

$$\Big\|\left(q - N - 1 - \sum_{j=1}^{q}\frac{N}{k_{ij}+1}\right)T(r,f^i) \leq \sum_{j=1}^{q}\left(1 - \frac{N}{k_{ij}+1}\right)N^{(N)}_{(f^i,H_j),\leq k_{ij}}(r) + o(T(r,f^i))$$

$$\Big\|\left(q - N - 1 - \frac{Nk}{m+1} - \frac{N(q-k)}{m+d+1}\right)T(r,f^i) \leq \left(1 - \frac{N}{M+1}\right)\sum_{j=1}^{q} N^{(N)}_{(f^i,H_j),\leq k_{ij}}(r) + o(T(r,f^i))$$

$$\Big\|\left(q - N - 1 - \frac{Nk}{m+1} - \frac{N(q-k)}{m+d+1}\right)T(r) \leq \left(1 - \frac{N}{M+1}\right)\sum_{i=1}^{3}\sum_{j=1}^{q} N^{(N)}_{(f^i,H_j),\leq k_{ij}}(r) + o(T(r)). \qquad (1.3.5)$$

From (1.3.4) and (1.3.5), we have

$$\Big\| \ 2\left(q - N - 1 - \frac{Nk}{m+1} - \frac{N(q-k)}{m+d+1}\right)T(r) \leq (3N+1)(1 - \frac{N}{M+1})T(r) + o(T(r)).$$

Letting $r \to +\infty$, we get

$$\Big\| \ 2\left(q - N - 1 - \frac{Nk}{m+1} - \frac{N(q-k)}{m+d+1}\right) \leq (3N+1)(1 - \frac{N}{M+1}),$$

and hence

$$(2q - 5N - 3) \leq \frac{2Nk}{m+1} + \frac{2N(q-k)}{m+d+1} - \frac{3N^2+N}{M+1}(1.3.6).$$

This is a contradiction. So we have $\sharp\mathcal{Q}=0$

Claim 1.3.2. If $\sharp\Big(\{1,2,...,q\}\setminus\mathcal{Q}\Big)\geq 3N-1$ and $N\geq 2$ then $f^1\equiv f^2$, or $f^2\equiv f^3$, or $f^3\equiv f^1$.

Proof

Indeed, assume that $1,...,3N-1\notin\mathcal{Q}$. By the density of $\mathcal{C}$, it implies that

$$\Phi^\alpha(F_j^{i1},F_j^{i2},F_j^{i3})=0\ (1\leq i,j\leq 3N-1,|\alpha|\leq 1).$$

Thus, there exists $\chi_{ij}\neq 0$ such that $F_j^{i1}=\chi_{ij}F_j^{i2}$, or $F_j^{i2}=\chi_{ij}F_j^{i3}$ or $F_j^{i3}=\chi_{ij}F_j^{i1}$. We may assume that $F_j^{i1}=\chi_{ij}F_j^{i2}$.

Suppose $\chi_{ij}\neq 1$. Then we have the following:

If $\nu_{(f^1,H_l),\leq k_{1l}}(z)>0$ $(l\neq i,j)$, then $\nu_{(f^1,H_i)}(z)>0$ or $\nu_{(f^1,H_j)}(z)>0$.

So we get

$\sum_{l\neq i,j}\nu^{(1)}_{(f^1,H_l),\leq k_{1l}}(z)\leq\nu^{(1)}_{(f^1,H_i),>k_{1i}}(z)+\nu^{(1)}_{(f^1,H_j),>k_{1j}}(z)$ outside a finite union of analytic sets of dimension $\leq n-2$. Hence

$$\sum_{l\neq i,j}N^{(1)}_{(f^1,H_l),\leq k_{1l}}(r)\leq N^{(1)}_{(f^1,H_i),>k_{1i}}(r)+N^{(1)}_{(f^1,H_j),>k_{1j}}(r)$$

$$\leq\frac{1}{k_{1i}+1}N_{(f^1,H_i),>k_{1i}}(r)+\frac{1}{k_{1j}+1}N_{(f^1,H_j),>k_{1j}}(r)$$

$$\leq\frac{1}{k_{1i}+1}N_{(f^1,H_i)}(r)+\frac{1}{k_{1j}+1}N_{(f^1,H_j)}(r)\leq\frac{2}{m+1}T(r,f^1).$$

By Lemma 1.1.21 and since $k_{1l}\geq N-1$, we have

$$\Big|\Big|\Big(q-N-3-\sum_{l\neq i,j}\frac{N}{k_{1l}+1}\Big)T(r,f^1)\leq\sum_{l\neq i,j}\Big(1-\frac{N}{k_{1l}+1}\Big)N^{(N)}_{(f^1,H_l),\leq k_{1l}}(r)+o(T(r,f^1)).$$

This yields that

$$\begin{aligned}\Big(q-N-3-\sum_{l\neq i,j}\frac{N}{m+1}\Big)T(r,f^1)&\leq\sum_{l\neq i,j}\Big(1-\frac{N}{M+1}\Big)N^{(N)}_{(f^1,H_l),\leq k_{1l}}(r)+o(T(r,f^1))\\&\leq N\Big(1-\frac{N}{M+1}\Big)\sum_{l\neq i,j}N^{(1)}_{(f^1,H_l),\leq k_{1l}}(r)+o(T(r,f^1))\\&\leq\Big(1-\frac{N}{M+1}\Big)\frac{2N}{m+1}T(r,f^1)+o(T(r,f^1)).\end{aligned}$$

Hence

$$\Big(q-N-3-\frac{N(q-2)}{m+1}\Big)\leq\Big(1-\frac{N}{M+1}\Big)\frac{2N}{m+1}.$$

This means that

$$q - N - 3 - \frac{N(q-2)}{m+1} \le \frac{2N}{m+1} - \frac{2N^2}{(m+1)(M+1)}.$$

Thus

$$q - N - 3 \le \frac{Nq}{m+1} - \frac{2N^2}{(m+1)(M+1)} \quad (1.3.7)$$

Moreover, since $N \ge 2,\ 3N+1 \ge q$ and $m > 3N + 1 + \dfrac{16}{3(N-1)}$, we have

$$\frac{(3N-3)}{2} \ge \frac{Nq}{m+1} \text{ and } \frac{Nk}{m+1} + \frac{N(q-k)}{m+d+1} \ge \frac{Nq}{m+d+1} \ge \frac{Nq}{M+1} \ge \frac{3N^2+N}{2(M+1)}.$$

This implies that

$$\frac{5N+3}{2} + \frac{Nk}{m+1} + \frac{N(q-k)}{m+d+1} - \frac{3N^2+N}{2(M+1)} > N + 3 + \frac{Nq}{m+1} - \frac{2N^2}{(m+1)(M+1)}.$$

Combining the hypothesis and (1.3.7), we get a contradiction. Hence $\chi_{ij} = 1$.

We define the subsets I_1, I_2 and I_3 by

$I_1 = \{i :\ 1 \le i \le 3N-2 \text{ and } F^{i1}_{3N-1} = F^{i2}_{3N-1}\}$,

$I_2 = \{i :\ 1 \le i \le 3N-2 \text{ and } F^{i2}_{3N-1} = F^{i3}_{3N-1}\}$,

$I_3 = \{i :\ 1 \le i \le 3N-2 \text{ and } F^{i3}_{3N-1} = F^{i1}_{3N-1}\}$.

Then one of them contains at least N indices. We may assume that $\sharp I_1 \ge N$. Then $f^1 \equiv f^2$. Thus the Claim is proved.

From Claim 1.3.1 and Claim 1.3.2 and $q \ge 3N-1$, Case 1 is proved.

Case 2. Assume that $N = 1$ and $q = 4$.

For each $j_0 \in \mathcal{Q}$, from (1.3.1), we get

$$\Big\| \sum_{i=1}^{3}\Big(2\sum_{j=1}^{q} N^{(1)}_{(f^i,H_j),\le k_{ij}}(r)\Big) \le 3T(r) +$$

$$+ \sum_{i=1}^{3}\Big(\frac{3}{k_{ij_0}+1}\Big)(N_{(f^i,H_{j_0})}(r) - N^{(1)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r)) + \sum_{i=1}^{3} N^{(1)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) + o(T(r))$$

and $N^{(1)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) \le N_{(f^i,H_{j_0})}(r) \le T(r,f^i) + o(T(r))$ $(1 \le i \le 3)$.

Hence

$$\Big\| 2\sum_{i=1}^{3}\sum_{j=1}^{4} N^{(1)}_{(f^i,H_j),\le k_{ij}}(r) \le 3(1+\frac{1}{m_{j_0}+1})T(r) + \sum_{i=1}^{3}(1-\frac{3}{m_{j_0}+1})N^{(1)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) + o(T(r))$$

$$\le 3(1+\frac{1}{m_{j_0}+1})T(r) + \sum_{i=1}^{3}(1-\frac{3}{m_{j_0}+1})N^{(1)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) + o(T(r)), \quad (1.3.8)$$

where $m_j = min\{k_{ij} \mid 1 \leq i \leq 3\}(1 \leq j \leq 4)$.

On the other hand, from Lemma 1.1.21, we have

$$\Big\|\Big(2-\sum_{j=1}^{4}\frac{1}{k_{ij}+1}\Big)T(r,f^i) \leq \sum_{j=1}^{4}\Big(1-\frac{1}{k_{ij}+1}\Big)N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r)+o(T(r,f^i)).$$

It implies that

$$\Big(2-\frac{k}{m+1}-\frac{4-k}{m+d+1}\Big)T(r,f^i) \leq \sum_{j=1}^{4}\Big(1-\frac{1}{M+1}\Big)N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r)+o(T(r,f^i)).$$

Hence

$$\Big(2-\frac{k}{m+1}-\frac{4-k}{m+d+1}\Big)T(r) \leq \sum_{i=1}^{3}\sum_{j=1}^{4}\Big(1-\frac{1}{M+1}\Big)N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r)+o(T(r)) \quad (1.3.9)$$

From (1.3.8) and (1.3.9), we have

$$\Big\|2\Big(2-\frac{k}{m+1}-\frac{4-k}{m+d+1}\Big)(\frac{M+1}{M})T(r) \leq 3(1+\frac{1}{m_{j_0}+1})T(r)$$

$$+\sum_{i=1}^{3}(1-\frac{3}{m_{j_0}+1})N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)+o(T(r)).$$

This yields that

$$\sum_{i=1}^{3}N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r) \geq \Big(\frac{m_{j_0}+1}{m_{j_0}-2}\Big)\Big(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-$$

$$-3(1+\frac{1}{m_{j_0}+1})\Big)T(r)+o(T(r)).$$

Hence

$$\sum_{i=1}^{3}N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r) \geq \Big(\frac{m_{j_0}+1}{m_{j_0}-2}\Big)\times\times\Big(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})$$

$$-3(1+\frac{1}{m_{j_0}+1})\Big)T(r)+o(T(r)) \quad (1.3.10)$$

Assume that $\sharp\mathcal{Q} \geq 3$, i.e, $\mathcal{Q} \supset \{j_0, j_1, j_2\}$.

By (1.3.10), we get

$$\Big\|\sum_{i=1}^{3}\sum_{s=0}^{2}N^{(1)}_{(f^i,H_{j_s}),\leq k_{ij_s}}(r) \geq \sum_{s=0}^{2}\Big(\frac{m_{j_s}+1}{m_{j_s}-2}\Big)\times\Big(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})$$

$$-3(1+\frac{1}{m_{j_s}+1})\Big)T(r)+o(T(r)).(1.3.11)$$

Since there exists $c \in \mathcal{C}$ such that $F_c^{j_01} - F_c^{j_02} \not\equiv 0$, it implies that

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le N_{F_c^{j_01}-F_c^{j_02}}(r) \le T(r,f^1) + T(r,f^2) + O(1).$$

Similarly, we have

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le T(r,f^2) + T(r,f^3) + O(1)$$

and

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le T(r,f^3) + T(r,f^1) + O(1).$$

Hence

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le \frac{2}{3}\cdot T(r) + O(1) \ (1 \le i \le 3)$$

and

$$\sum_{i=1}^{3}\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le 2.T(r) + O(1) \ (1.3.12)$$

From (1.3.11) and (1.3.12), we have

$$2.T(r) \ge \sum_{s=0}^{2}\left(\frac{m_{j_s}+1}{m_{j_s}-2}\right)\left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{m_{j_s}+1})\right)T(r)+o(T(r)).$$

Letting $r \to +\infty$, we get

$$2 \ge \sum_{s=0}^{2}\left(\frac{m_{j_s}+1}{m_{j_s}-2}\right)\left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{m_{j_s}+1})\right).$$

On the other hand, the following function is increasing for $t > 2$

$$f(t) = \left(\frac{t+1}{t-2}\right)\left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{t+1})\right)$$

So we get

$$2 \ge 3.\left(\frac{m+1}{m-2}\right)\left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{m+1})\right).$$

This means that

$$\frac{2(m-2)}{3(m+1)} \ge \left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{m+1})\right).$$

Thus, we get

$$\frac{3(2k+1)}{m+1} + \frac{6(4-k)}{m+d+1} + \frac{6k}{M(m+1)} + \frac{24-6k}{M(m+d+1)} \ge 1 + \frac{12}{M}.$$

This is a contradiction (remarking that the equality does not happen if $\max_{1\le j\le 4}\{m_j\} > m$). Hence $\sharp\mathcal{Q} \le 2$.

We now use the same argument in [64] to complete Case 2.

Without loss of generality, we may assume that $1, 2 \notin \mathcal{Q}$. By the density of $\mathcal{C}$ in $\mathbb{C}^2$, it implies that $\Phi^\alpha(F_j^{i0}, F_j^{i1}, F_j^{i2}) = 0$ for each $1 \le i \le 2, 1 \le j \le 2$ and for each $\alpha = (\alpha_0, \alpha_1)$ with $|\alpha| \le 1$, where $F_j^{ik} = \dfrac{(f^k, H_i)}{(f^k, H_j)}$.

Applying Lemma 1.1.18 for $i = 1, j = 2$, we have the following two cases.

(i) There exist $0 \le l_1 < l_2 \le 2$ such that $F_2^{1l_1} = F_2^{1l_2}$. Then $f^{l_1} \equiv f^{l_2}$.

(ii) There are two distinct constants $\alpha, \beta \in \mathbb{C} \setminus \{0, 1\}$ such that $F_2^{10} = \alpha F_2^{11} = \beta F_2^{12}$.

We may assume that $H_1 = \{\omega_0 = 0\}$, $H_2 = \{\omega_1 = 0\}$, $H_3 = \{\omega_0 - c\omega_1 = 0\}$ $(c \in \mathbb{C}\backslash\{0\})$. Then

$$\frac{f_0^0}{f_1^0} = \alpha\frac{f_0^1}{f_1^1} = \beta\frac{f_0^2}{f_1^2},$$

$$(f^1, H_3) = 0 \Leftrightarrow f_0^1 - cf_1^1 = 0 \Leftrightarrow (f_0^0 - c\alpha f_1^0)\left(\frac{f_1^1}{\alpha f_1^0}\right) = 0$$

$$(f^2, H_3) = 0 \Leftrightarrow f_0^2 - cf_1^2 = 0 \Leftrightarrow (f_0^0 - c\beta f_1^0)\left(\frac{f_1^2}{\beta f_1^0}\right) = 0.$$

Hence $\{z \in \mathbb{C}^n : \nu_{(f^0,H_3),\le k_{03}}(z) > 0\} \subset \bigcup_{i=0}^2 I(f^i)$. So that $N^{(1)}_{(f^0,H_3),\le k_{03}}(r) = 0$, and

$\nu_{(f^1,H_3)}(z) = \nu_{f_0^0 - c\alpha f_1^0}(z)$ and $\nu_{(f^2,H_3)}(z) = \nu_{f_0^0 - c\beta f_1^0}(z)$ for $z \notin I(f^0) \cup I(f^1) \cup I(f^2)$

Thus, we have $\nu_{(f^1,H_3)}(z) = \nu_{f_0^0 - c\alpha f_1^0}(z)$ $(z \in \mathbb{C}^n)$ and $\nu_{(f^2,H_3)}(z) = \nu_{f_0^0 - c\beta f_1^0}(z)$ $(z \in \mathbb{C}^n)$.

Put $H_3' = \{\omega_0 - c\alpha\omega_1 = 0\}, H_3'' = \{\omega_0 - c\beta\omega_1 = 0\}$. Then we have the following:

- H_3, H_3', H_3'' are in general position.
- $\nu_{(f^0,H_3')} = \nu_{(f^1,H_3)}$. This yields $\nu^{(1)}_{(f^0,H_3'),\le k_{13}} = \nu^{(1)}_{(f^1,H_3),\le k_{13}} = \nu^{(1)}_{(f^0,H_3),\le k_{03}}$
- $\nu_{(f^0,H_3'')} = \nu_{(f^2,H_3)}$. This yields $\nu^{(1)}_{(f^0,H_3''),\le k_{23}} = \nu^{(1)}_{(f^2,H_3),\le k_{23}} = \nu^{(1)}_{(f^0,H_3),\le k_{03}}$

By Lemma 1.1.21, we have

$$\Big\|\left(3-1-1-\sum_{j=0}^{2}\frac{1}{k_{j3}+1}\right)T(r,f^0) \le (1-\frac{1}{1+k_{03}})N^{(1)}_{(f^0,H_3),\le k_{03}}(r)+(1-\frac{1}{1+k_{13}})N^{(1)}_{(f^0,H_3'),\le k_{13}}(r)+$$

$$+(1-\frac{1}{1+k_{23}})N^{(1)}_{(f^0,H_3''),\le k_{23}}(r) + o(T(r,f^0))$$

$$\Rightarrow \left(1-\frac{3}{m+1}\right)T(r,f^0) \le \left(1-\frac{1}{M+1}\right)\left(N^{(1)}_{(f^0,H_3),\le k_{03}}(r)+N^{(1)}_{(f^0,H_3'),\le k_{13}}(r)+N^{(1)}_{(f^0,H_3''),\le k_{23}}(r)\right)$$

$$+o(T(r,f^0))$$

$$\Rightarrow \left(1-\frac{3}{m+1}\right)T(r,f^0) \le \left(1-\frac{1}{M+1}\right)\left(N^{(1)}_{(f^0,H_3),\le k_{03}}(r)+N^{(1)}_{(f^0,H_3),\le k_{03}}(r)+N^{(1)}_{(f^0,H_3),\le k_{03}}(r)\right)$$

$$+o(T(r,f^0)) = 3(1-\frac{1}{M+1})N^{(1)}_{(f^0,H_3),\le k_{03}}(r) + o(T(r,f^0))$$

So we get

$$\left(1-\frac{3}{m+1}\right)T(r,f^1) \le o(T(r,f^0))$$

This is a contradiction. Case 2 of Theorem 1.3 is proved. □

1.4 A unicity theorem for meromorphic mapping sharing few fixed targets with a conditions on derivations

Take a meromorphic mapping f of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$, a positive integer d, a positive integer k or $k=\infty$ and q hyperplanes $H_1, ..., H_q$ in $\mathbb{P}^N(\mathbb{C})$ located in general position with

$$\dim\{z \in \mathbb{C}^n : \nu_{(f,H_i)}(z) > 0 \text{ and } \nu_{(f,H_j)}(z) > 0\} \le n-2 \ (1 \le i < j \le q),$$

and consider the set $\mathcal{G}(f, \{H_j\}_{j=1}^q, k, d)$ of all meromorphic maps $g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ satisfying the conditions

(a) g is linearly nondegenerate over $\mathbb{C}$,

(b) $\min\{\nu_{(f,H_j),\le k}, d\} = \min\{\nu_{(g,H_j),\le k}, d\}$ $(1 \le j \le q)$,

(c) Let $f = (f_0 : \cdots : f_N)$ and $g = (g_0 : \cdots : g_N)$ be reduced representations of f and g, respectively. Then, for each $0 \leqslant j \leqslant N$ and for each $\omega \in \bigcup_{i=1}^q \{z \in \mathbb{C}^n : \nu_{(f,H_i),\leqslant k}(z) > 0\}$, the following two conditions are satisfied:

(i) If $f_j(\omega) = 0$ then $g_j(\omega) = 0$,

(ii) If $f_j(\omega)g_j(\omega) \neq 0$ then $\mathcal{D}^\alpha\left(\frac{f_i}{f_j}\right)(\omega) = \mathcal{D}^\alpha\left(\frac{g_i}{g_j}\right)(\omega)$ for each n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers with $|\alpha| = \alpha_1 + ... + \alpha_n \leqslant d$ and for each $i \neq j$, where $\mathcal{D}^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 ... \partial^{\alpha_n} z_n}$.

Remark that the condition (c) does not depend on the choice of reduced representations.

The last part of this chapter proves the following.

Theorem 1.4. (Ha-Quang [33]) *If $N \geq 4$ and $2 \leqslant d \leqslant N-1$, then* $\sharp\, \mathcal{G}(f, \{H_i\}_{i=1}^{3N+2-2d}, k, d) = 1$ *for each* $k > \dfrac{3dN^2 - 2N^2 + 2Nd - 2Nd^2}{2(d-1)N + d - 2d^2} - 1.$

Proof. Suppose that there exists a mapping $g \in \mathcal{G}(f, \{H_i\}_{i=1}^{3N+2-2d}, k, d)$ with reduced representation $g = (g_0 : \cdots : g_N)$ such that $g \not\equiv f$. Then there exist indices i and j $(0 \leqslant i < j \leqslant N)$ such that $P_{ij} = \dfrac{(f, H_i)}{(f, H_j)} - \dfrac{(g, H_i)}{(g, H_j)} \not\equiv 0$. Define

$$I = I(f) \cup I(g) \cup_{1 \leqslant\ t < s \leqslant 3N+2-2d} \{z \in \mathbb{C}^n \ |\nu_{(f,H_t),\leqslant k}(z)\nu_{(f,H_s),\leqslant k}(z) > 0\}.$$

Then I is an analytic set of codimension 2 or emptyset.

Claim 1.4.1. *The following assertion holds*

$$\sum_{v=1}^{3N+2-2d} N^{(d)}_{(f,H_v),\leqslant k}(r) \leqslant T(r,f) + T(r,g) + o(T(r,f) + T(r,g))$$

Proof . We fix a point $z \notin I$ satisfying $\nu_{(f,H_t),\leqslant k}(z) > 0$ $(t \neq j)$. Suppose that $f_l(z) \cdot g_l(z) = 0$ $(0 \leqslant l \leqslant N)$. Then $g_l(z) = 0$ $(0 \leqslant l \leqslant N)$. This means that $z \in I(g)$. This is impossible. Hence, there exists an index l such that $f_l(z) \cdot g_l(z) \neq 0$. This implies that

$$\begin{aligned} \mathcal{D}^\alpha P_{ij}(z) &= \mathcal{D}^\alpha\left(\frac{(f,H_i)}{(f,H_j)} - \frac{(g,H_i)}{(g,H_j)}\right)(z) \\ &= \mathcal{D}^\alpha\left(\frac{\sum_{v=0}^N \frac{f_v}{f_l} a_{iv}}{\sum_{v=0}^N \frac{f_v}{f_l} a_{jv}} - \frac{\sum_{v=0}^N \frac{g_v}{g_l} a_{iv}}{\sum_{v=0}^N \frac{g_v}{g_l} a_{jv}}\right)(z) = 0,\ \forall |\alpha| \leqslant d. \end{aligned}$$

Hence $\nu_{P_{ij}}(z) \geq d$. We have $\nu_{P_{ij}} \geq \sum_{\substack{t=1 \\ t\neq j}}^{3N+2-2d} d \min\{1, \nu_{(f,H_t),\leqslant k}\}$ outside an analytic set of codimension 2. This yields that

$$N_{P_{ij}}(r) \geq \sum_{\substack{t=1 \\ t\neq j}}^{3N+2-2d} N^{(d)}_{(f,H_t),\leqslant k}(r).$$

Using the argument in the proof of Theorem 1.2, we have

$$m(r, P_{ij}) \leqslant T(r,f) + T(r,g) - N_{\frac{(f,H_j)}{(f,H_i)}}(r) - N_{\frac{(g,H_j)}{(g,H_i)}}(r) + O(1)$$

and

$$N_{\frac{1}{P_{ij}}}(r) \leqslant N(r, \nu_j), \quad \text{where } \nu_j = \max\{\nu_{\frac{(f,H_j)}{(f,H_i)}}, \nu_{\frac{(g,H_j)}{(g,H_i)}}\}.$$

Hence

$$\sum_{\substack{v=1\\ v\neq j}}^{3N+2-2d} N^{(d)}_{(f,H_v),\leqslant k}(r) \leqslant N_{P_{ij}}(r)$$

$$\begin{aligned}
&\leqslant T(r,P_{ij})\\
&= N_{\frac{1}{P_{ij}}}(r) + m(r,P_{ij}) + O(1)\\
&\leqslant T(r,f) + T(r,g) + N(r,\nu_j) - N_{\frac{(f,H_j)}{(f,H_i)}}(r)\\
&- N_{\frac{(g,H_j)}{(g,H_i)}}(r) + o(T(r,f)+T(r,g)).
\end{aligned}$$

This gives

$$\Big(N_{\frac{(f,H_j)}{(f,H_i)}}(r) + N_{\frac{(g,H_j)}{(g,H_i)}}(r) - N(r,\nu_j)\Big) + \sum_{\substack{v=1\\ v\neq j}}^{3N+2-2d} N^{(d)}_{(f,H_v),\leqslant k}(r)$$

$$\leqslant T(r,f) + T(r,g) + o(T(r,f)+T(r,g)).$$

On the other hand, we have

$$\nu_j(z) - \nu_{\frac{(f,H_j)}{(f,H_i)}}(z) - \nu_{\frac{(g,H_j)}{(g,H_i)}}(z) + \nu^{(d)}_{(f,H_j),\leqslant k}(z) =$$
$$\nu^{(d)}_{(f,H_j),\leqslant k}(z) - \min\{\nu_{\frac{(f,H_j)}{(f,H_i)}}(z), \nu_{\frac{(g,H_j)}{(g,H_i)}}(z)\} \leqslant 0$$

ouside an analytic set of codimension 2. Hence

$$N(r,\nu_i) - N_{\frac{(f,H_j)}{(f,H_i)}}(r) - N_{\frac{(g,H_j)}{(g,H_i)}}(r) + N^{(d)}_{(f,H_j),\leqslant k}(r) \leqslant 0.$$

This yields that

$$\sum_{v=1}^{3N+2-2d} N^{(d)}_{(f,H_v),\leqslant k}(r) \leqslant T(r,f) + T(r,g) + o(T(r,f)+T(r,g)).$$

This concludes Claim 1.4.1.

From Claim 1.4.1 we have the following

$$\sum_{v=1}^{3N+2-2d} N^{(N)}_{(f,H_v),\leqslant k}(r) \leqslant \frac{N}{d}(T(r,f)+T(r,g)) + o(T(r,f)+T(r,g)).$$

By using Lemma 1.1.9 , we also have

$$|| \sum_{i=1}^{3N+2-2d} N^{(N)}_{(f,H_i),\leqslant k}(r)) \geq \frac{(2N+1-2d)(k+1) - N(3N+2-2d)}{k+1-N} T(r,f)$$
$$+ o(T(r,f))$$

and

$$\Big\| \sum_{i=1}^{3N+2-2d} N^{(N)}_{(g,H_i),\leqslant k}(r)\Big)\geq \frac{(2N+1-2d)(k+1)-N(3N+2-2d)}{k+1-N}T(r,g) + o(T(r,g)).$$

This implies that

$$\Big\| \frac{2N}{d}((T(r,f)+T(r,g)) \geq \left(\frac{(2N+1-2d)(k+1)-N(3N+2-2d)}{k+1-N}\right)\times (T(r,f)+T(r,g))+o((T(r,f)+T(r,g))).$$

Letting $r \longrightarrow \infty$, we have

$$\frac{2N}{d} \geq \frac{(2N+1-2d)(k+1)-N(3N+2-2d)}{k+1-N},$$

and hence

$$k+1 \leqslant \frac{3dN^2-2N^2+2Nd-2Nd^2}{2(d-1)N+d-2d^2}.$$

This is a contradiction. Thus, we have $\sharp\, \mathcal{G}(f,\{H_i\}_{i=1}^{3N+2-2d},k,d)=1$ and Theorem 1.4 is proved. □

Chapter 2

Unicity theorems with truncated multiplicities of meromorphic mappings in several complex variables sharing small identical sets

The unicity theorems with truncated multiplicities of meromorphic mappings of $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^N(\mathbb{C})$ sharing a finite set of fixed (or moving) hyperplanes in $\mathbb{P}^N(\mathbb{C})$ have received much attention in the last few decades, and they are related to many problems in Nevanlinna theory and hyperbolic complex analysis .

For moving targets and truncated multiplicites, the following results are best and due to Dethloff-Tan [14]. They proved the following (see §2.1 for notations).

Theorem of Dethloff-Tan [14] *Let $f, g : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ $(N \geq 2)$ be two nonconstant meromorphic mappings, and let $\{a_j\}_{j=1}^{3N+1}$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position such that $(f, a_i) \not\equiv 0,\ (g, a_i) \not\equiv 0$ $(1 \leqslant i \leqslant 3N+1)$ and f is linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^{3N+1})$. Set $M = 3N(N+1)\left[\binom{2N+2}{N+1}\right]^2\left[\binom{2N+2}{N+1}-1\right]+N(3N+4)$. Assume that the following conditions are satisfied.*

(i) $\dim\{z \in \mathbb{C}^n : \nu_{(f,a_i),\leqslant M}(z) > 0$ *and* $\nu_{(f,a_j),\leqslant M}(z) > 0\} \leqslant n-2$
$(1 \leqslant i \leqslant N+3,\ 1 \leqslant j \leqslant 3N+1)$.

(ii) $\min\{\nu_{(f,a_i)}, M\} = \min\{\nu_{(g,a_i)}, M\}$ $((1 \leqslant i \leqslant 3N+1)$.

(iii) $f(z) = g(z)$ *on* $\bigcup_{j\in D}\{z \in \mathbb{C}^n : \nu_{(f,a_j),\leqslant M}(z) > 0\}$, *where D is an arbitrary subset of $\{1, \cdots, 3N+1\}$ with $\sharp D = N+4$.*

Then $f \equiv g$.

We would like to emphasize here that the assumption $\sharp D = N+4$ in the above-mentioned theorem is essential in their proofs. It seems to us that some key techniques in their proofs could not be used for $\sharp D < N+4$.

The first main purpose of the present chapter is to give a unicity theorem with truncated multiplicities of meromorphic mappings in several complex variables sharing $N+2$ moving targets. In particular, we prove Theorem 2.2 (Ha-Quang-Thai [34]). It is an improvement of the above-mentioned theorem of Dethloff-Tan.

In this chapter, we also would like to study the unicity problems of meromorphic mappings in several complex variables for moving targets with conditions on derivations. We will prove Theorem 2.3 (Ha-Quang-Thai [34]) in the last part of this chapter.

2.1 Preliminaries

2.1.1. Let $f : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbb{P}^N(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on $\mathbb{C}^n$ and $f(z) = \big(f_0(z) : \cdots : f_N(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of codimension ≥ 2. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_N|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T(r, f) = \int\limits_{S(r)} \log\|f\|\sigma_n - \int\limits_{S(1)} \log\|f\|\sigma_n.$$

Let a be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representation $a = (a_0 : \cdots : a_N)$. The proximity function $m_{f,a}(r)$ is defined by

$$m_{f,a}(r) = \int\limits_{S(r)} \log\frac{||f|| \cdot ||a||}{|(f,a)|}\sigma_n - \int\limits_{S(1)} \log\frac{||f|| \cdot ||a||}{|(f,a)|}\sigma_n,$$

where $\|a\| = \big(|a_0|^2 + \cdots + |a_N|^2\big)^{1/2}$.

If $f, a : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ are meromorphic mappings such that $(f, a) \not\equiv 0$, then **the First Main Theorem for moving targets** in value distribution theory (see Ru-Stoll [56]) states

$$T(r, f) + T(r, a) = m_{f,a}(r) + N_{(f,a)}(r).$$

2.1.2. Let $a_1, \dots, a_q$ $(q \geq N+1)$ be q meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jN})$ $(1 \leqslant j \leqslant q)$. We say that $a_1, \dots, a_q$ are located in general position if $\det(a_{j_k l}) \not\equiv 0$ for any $1 \leqslant j_0 < j_1 < \dots < j_N \leqslant q$. We also say that $a_1, \dots, a_q$ are located in pointwise general position if the hyperplanes $a_1(z), \dots, a_q(z)$ are in general position as a set of fixed hyperplanes at every point $z \in \mathbb{C}^n$.

Let $\mathcal{M}_n$ be the field of all meromorphic functions on $\mathbb{C}^n$. Denote by $\mathcal{R}\Big(\big\{a_j\big\}_{j=1}^q\Big) \subset \mathcal{M}_n$ the smallest subfield which contains $\mathbb{C}$ and all $\dfrac{a_{jk}}{a_{jl}}$ with $a_{jl} \not\equiv 0$. Define $\widetilde{\mathcal{R}}\Big(\big\{a_j\big\}_{j=1}^q\Big) \subset \mathcal{M}_n$ to be the smallest subfield which contains all $h \in \mathcal{M}_n$ with $h^k \in \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^q\Big)$ for some positive integer k.

Let f be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_N)$. We say that f is linearly nondegenerate over $\mathcal{R}\Big(\big\{a_j\big\}_{j=1}^q\Big)$ $\Big(\widetilde{\mathcal{R}}(\big\{a_j\big\}_{j=1}^q)\Big)$ if $f_0, \dots, f_N$ are linearly independent over $\mathcal{R}\Big(\big\{a_j\big\}_{j=1}^q\Big)$ $(\widetilde{\mathcal{R}}\Big(\big\{a_j\big\}_{j=1}^q\Big)$, respectively).

Let f, a be two meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representations $f = (f_0 : \cdots : f_N)$, $a = (a_0 : \cdots : a_N)$ respectively. Put $(f, a) = \sum_{i=0}^N a_i f_i$. We say that a is "small" with respect to f if $T(r, a) = o(T(r, f))$ as $r \to \infty$.

Let f and a be nonconstant meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. For every $z \in \mathbb{C}^n$, we set

$$\nu_{(f,a),\leq k}(z) = \begin{cases} 0 & \text{if } \nu_{(f,a)}(z) > k, \\ \nu_{(f,a)}(z) & \text{if } \nu_{(f,a)}(z) \leq k, \end{cases}$$

$$\nu_{(f,a),>k}(z) = \begin{cases} \nu_{(f,a)}(z) & \text{if } \nu_{(f,a)}(z) > k, \\ 0 & \text{if } \nu_{(f,a)}(z) \leq k. \end{cases}$$

2.1.3. The second main theorem for moving targets. (Thai-Quang [63]) *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. Let $\{a_j\}_{j=1}^q$ $(q \geq N+2)$ be meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position such that f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Then*

$$\| \ \frac{q}{N+2} T(r, f) \leq \sum_{j=1}^q N_{(f,a_j)}^{(N)}(r) + o(T(r, f)) + O(\max_{1 \leq j \leq q} T(r, a_j)).$$

2.2 A unicity theorem with truncated multiplicities of meromorphic mappings in several complex variables sharing few moving targets

In this section, we prove the following.

Theorem 2.2. (Ha-Quang-Thai [34]) *Let k be a positive integer or $k = \infty$ and d be a positive integer or $d = \infty$ such that the following is satisfied*

$$\Big(\frac{3}{d+1}+\frac{6}{k+1}\Big)\binom{2N+2}{N+1}\Big[\binom{2N+2}{N+1}-2\Big]< \Big(\frac{N+2}{N(N+2)(N(N+2)+1)}-\frac{2N+2}{k+1}\Big).$$

Let $f, g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ $(N \geq 2)$ be two nonconstant meromorphic mappings, and let $\{a_j\}_{j=1}^{3N+1}$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position such that $\dim\{z \in \mathbb{C}^n : \nu_{(f,a_i),\leqslant k}(z)\nu_{(f,a_j),\leqslant k}(z) > 0\} \leqslant n-2$ $(1 \leqslant i < j \leqslant 3N+1)$.

Assume that f, g are linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^{3N+1})$ and the following are satisfied.

(i) $\min\,(\nu_{(f,H_j),\leqslant k}, d) = \min\,(\nu_{(g,H_j),\leqslant k}, d)$ $(1 \leqslant j \leqslant 3N+1)$.

(ii) $f(z) = g(z)$ *on* $\bigcup_{j\in D}\{z \in \mathbb{C}^n : \nu_{(f,a_j),\leqslant N(N+2)}(z) > 0\}$, *where D is an arbitrary subset of* $\{1, \cdots, 3N+1\}$ *with* $\sharp D = N+2$.

Then $f \equiv g$.

Proof. Assume that f, g, a_i have reduced representations

$$f = (f_0 : ... : f_N),\ g = (g_0 : ... : g_N),\ a_i = (a_{i0} : ... : a_{iN}).$$

(i) Consider any $2N+2$ meromorphic mappings of $\{a_1, ..., a_{3N+1}\}$, to say, $a_1, ..., a_{2N+2}$. Define $h_i = \dfrac{(f, a_i)}{(g, a_i)}$ $(1 \leqslant i \leqslant 2N+2)$. Then $\dfrac{h_i}{h_j} = \dfrac{(f, a_i)\cdot(g, a_j)}{(g, a_i)\cdot(f, a_j)}$ does not depend on representations of f and g . Since $\sum_{k=0}^{N} a_{ik}f_k - h_i \cdot \sum_{k=0}^{N} a_{ik}g_k = 0$ $(1 \leqslant i \leqslant 2N+2)$, it implies that $\det\,(a_{i0}, ..., a_{iN}, a_{i0}h_i, ..., a_{iN}h_i; 1 \leqslant i \leqslant 2N+2) = 0$.

For each subset $I \subset \{1, 2, ..., 2N+2\}$, put $h_I = \prod_{i\in I} h_i$. Denote by $\mathcal{I}$ the set of all $N+1$-tuples $I = (i_1, ..., i_{N+1})$ with $1 \leqslant i_1 < ... < i_{N+1} \leqslant 2N+2$.

For each $I = (i_1, ..., i_{N+1}) \in \mathcal{I}$, define

$$A_I = (-1)^{\frac{(N+1)(N+2)}{2}+i_1+...+i_{N+1}} \cdot \det(a_{i_rl}; 1 \leqslant r \leqslant N+1, 0 \leqslant l \leqslant N)\cdot$$
$$\det(a_{j_sl}; 1 \leqslant s \leqslant N+1, 0 \leqslant l \leqslant N),$$

where $J = (j_1, ..., j_{N+1}) \in \mathcal{I}$ such that $I \cup J = \{1, 2, ..., 2N+2\}$.

Then $\sum_{I\in\mathcal{I}} A_I h_I = 0$.

(ii) Take $I_0 \in \mathcal{I}$. Then $A_{I_0} h_{I_0} = -\sum_{I\in\mathcal{I}, I\neq I_0} A_I h_I$, and hence $h_{I_0} = -\sum_{I\in\mathcal{I}, I\neq I_0} \frac{A_I}{A_{I_0}} h_I$.

Notice that

$$A_I \not\equiv 0 \ (I \in \mathcal{I}) \ \text{ and } \frac{A_I}{A_{I_0}} \in \mathcal{R}(\{a_i\}_{i=1}^{3N+1}) \ (I \in \mathcal{I}).$$

Denote by t the minimal number satisfying the following:

There exist t elements $I_1, ..., I_t \in \mathcal{I} \setminus \{I_0\}$ and t nonzero meromorphic functions $b_i \in \mathcal{R}(\{a_i\}_{i=1}^{3N+1})$ such that

$$h_{I_0} = \sum_{i=1}^{t} b_i h_{I_i} \quad (2.2.1).$$

Since $h_{I_0} \not\equiv 0$ and by the minimality of t, it follows that the family $\{b_1 h_{I_1}, ..., b_t h_{I_t}\}$ is linearly independent over $\mathbb{C}$.

Assume that $t \geq 2$.

Put $b_0 = -1$. Then $\sum_{i=0}^{t} b_i h_{I_i} = 0$.

Put $I = \bigcap_{i=0}^{t} I_i,\ I'_i = I_i \setminus I \neq \emptyset\ (0 \leqslant i \leqslant t)$ and $\tilde{I} = \bigcup_{i=0}^{t} I'_i,\ I' = \bigcap_{i=1}^{t} I'_i,\ I''_i = I'_i \setminus I' \neq \emptyset\ (1 \leqslant i \leqslant t)$. We have $\dfrac{h_{I'_0}}{h_{I'}} = \sum_{i=1}^{t} b_i h_{I''_i} \quad (2.2.2)$.

Consider the meromorphic mapping $h : \mathbb{C}^n \to \mathbb{P}^{t-1}(\mathbb{C})$ with a reduced representation $h = (\tilde{h} h_{I''_1} : ... : \tilde{h} h_{I''_t})$, where $\tilde{h}$ is meromorphic on $\mathbb{C}^n$ satisfying $\nu_{\tilde{h}} \leqslant \sum_{i\in\cup_{j=1}^{t} I''_j} \nu^{\infty}_{h_i}$.

Consider the meromorphic mapping $b : \mathbb{C}^n \to \mathbb{P}^{t-1}(\mathbb{C})$ with a reduced representation $b = (\psi b_1 : ... : \psi b_t)$, where ψ is meromorphic on $\mathbb{C}^n$. We get

$$T(r, b) = o(T(r, f)) \text{ and } N_{\psi b_i}(r) \leqslant N_{\psi b_1}(r) + N_{\frac{b_i}{b_1}}(r) = o(T(r, f)) \ (0 \leqslant i \leqslant t).$$

If z is a zero (a pole, respectively) of h_i, then $\nu_{(f,a_i)}(z) \neq \nu_{(g,a_i)}(z)$. Hence $\nu_{(f,a_i)}(z) > d$ or $\nu_{(g,a_i)}(z) > d$. Thus, we have $\min\{1, \nu^{\infty}_{h_i}(z)\} + \min\{1, \nu_{h_i}(z)\} \leqslant \min\{1, \nu_{(f,a_i),>d}(z)\}$. This yields that $N^{(1)}_{h_i}(r) + N^{(1)}_{\frac{1}{h_i}}(r) \leqslant N^{(1)}_{(f,a_i),>d}(r) + N^{(1)}_{(g,a_i),>d}(r) \quad (2.2.3)$.

Consider the meromorphic mapping $h' : \mathbb{C}^n \to \mathbb{P}^{t-1}(\mathbb{C})$ with a reduced representation

$$h' = \Big(\frac{1}{\tilde{h}'} \psi b_1 \tilde{h} h_{I''_1} : ... : \frac{1}{\tilde{h}'} \psi b_t \tilde{h} h_{I''_t}\Big).$$

By (2.2.1), the mapping h' is linearly nondegenerate over $\mathbb{C}$. By the Second Main

Theorem for hyperplanes, it follows that

$$\begin{aligned} || \quad T(r,h') &\leqslant \sum_{i=1}^{t} N^{(t-1)}_{\frac{1}{h'}\psi b_i \tilde{h} h_{I''_i}}(r) + N^{(t-1)}_{\frac{1}{h'}\psi \tilde{h}\frac{h_{I'_0}}{h_{I'}}}(r) + o(T(r,h')) \\ &\leqslant (t-1)\cdot \sum_{i=1}^{t} N^{(1)}_{\tilde{h} h_{I''_i}}(r) + (t-1)\cdot N^{(1)}_{\tilde{h}\cdot\frac{h_{I'_0}}{h_{I'}}}(r) + o(T(r,f)) \\ &\quad + o(T(r,h')) \quad (2.2.4). \end{aligned}$$

Since $N^{(1)}_{\tilde{h} h_{I''_i}}(r) \leqslant O(T(r,f))$ and $N^{(1)}_{\tilde{h}\cdot\frac{h_{I'_0}}{h_{I'}}}(r) \leqslant O(T(r,f))$, we have

$$|| \quad T(r,h') \leqslant O(T(r,f)).$$

Define $I'' = \bigcup_{i=1}^{t} I''_i$. Denote by $\mathcal{W}$ the set $\bigcup_{i\in I''}\{z : \nu_{(f,a_i),>k}(z) > 0\}$. Then

$$N^{(1)}_{\tilde{h} h_{I''_i}}(r) = N^{(1)}_{h_{I''_i}}(r) + N^{(1)}_{\frac{1}{h_{I''\setminus I''_i}}}(r) + \sum_{j\in I''}(N^{(1)}_{(f,a_j),>k}(r) + N^{(1)}_{(g,a_j),>k}(r))$$

and

$$N^{(1)}_{\tilde{h}.\frac{h_{I'_0}}{h_{I'}}}(r) = N^{(1)}_{h_{I'_0}}(r) + N^{(1)}_{\frac{1}{h_{(I''\cup I')\setminus I'_0}}}(r) + \sum_{j\in I''}(N^{(1)}_{(f,a_j),>k}(r) + N^{(1)}_{(g,a_j),>k}(r)).$$

For each $J \subset \{1,2,...,2N+2\}$, put $J^c = \{1,2,...,2N+2\}\setminus J$. It is easy to see that

$$I''_i \subset I_i \text{ and } I''\setminus I''_i \subset I^c_i \ (1 \leqslant i \leqslant t),$$

$$I'_0 \subset I_0 \text{ and } (I''\cup I')\setminus I'_0 = \tilde{I}\setminus I'_0 = \tilde{I}\setminus(I_0\setminus I) = (\tilde{I}\cup I)\setminus I_0 \subset I^c_0.$$

Hence

$$\begin{aligned} &N^{(1)}_{\tilde{h} h_{I''_i}}(r) \leqslant N^{(1)}_{h_{I_i}}(r) + N^{(1)}_{\frac{1}{h_{I^c_i}}}(r) + \sum_{j=1}^{2N+2}(N^{(1)}_{(f,a_j),>k}(r) + N^{(1)}_{(g,a_j),>k}(r)) \\ \text{and } &N^{(1)}_{\tilde{h}.\frac{h_{I'_0}}{h_{I'}}}(r) \leqslant N^{(1)}_{h_{I_0}}(r) + N^{(1)}_{\frac{1}{h_{I^c_0}}}(r) + \sum_{j=1}^{2N+2}(N^{(1)}_{(f,a_j),>k}(r) + N^{(1)}_{(g,a_j),>k}(r)). \end{aligned}$$

Combining with (2.2.4), we deduce that

$$\begin{aligned}
|| \ T(r,h') &\leqslant (t-1)\sum_{i=0}^{t}\Bigg(N^{(1)}_{h_{I_i}}(r)+N^{(1)}_{\frac{1}{h_{I_i^c}}}(r)+\sum_{j=1}^{2N+2}\Big(N^{(1)}_{(f,a_j),>k}(r)\\
&\quad + N^{(1)}_{(g,a_j),>k}(r)\Big)\Bigg)+o(T(r,f))\\
&= (t-1)\sum_{i=0}^{t}\Bigg(\sum_{j\in I_i}N^{(1)}_{h_j}(r)+\sum_{j\in I_i^c}N^{(1)}_{\frac{1}{h_j}}(r)+\sum_{j=1}^{2N+2}\Big(N^{(1)}_{(f,a_j),>k}(r)\\
&\quad + N^{(1)}_{(g,a_j),>k}(r)\Big)\Bigg)+o(T(r,f))\\
&\leqslant \left[\binom{2N+2}{N+1}-2\right]\sum_{I\in\mathcal{I}}\Bigg(\sum_{i\in I}\Big(N^{(1)}_{h_i}(r)+N^{(1)}_{\frac{1}{h_i}}(r)\Big)\\
&\quad + \sum_{j=1}^{2N+2}\Big(N^{(1)}_{(f,a_j),>k}(r)+N^{(1)}_{(g,a_j),>k}(r)\Big)\Bigg)+o(T(r,f))\\
&= \frac{1}{2}\binom{2N+2}{N+1}\left[\binom{2N+2}{N+1}-2\right]\Bigg(\sum_{i=1}^{2N+2}\Big(N^{(1)}_{h_i}(r)+N^{(1)}_{\frac{1}{h_i}}(r)\Big)\\
&\quad + 2\sum_{j=1}^{2N+2}\Big(N^{(1)}_{(f,a_j),>k}(r)+N^{(1)}_{(g,a_j),>k}(r)\Big)\Bigg)+o(T(r,f)) \quad (2.2.5).
\end{aligned}$$

From (2.2.3) and (2.2.5) we get

$$\begin{aligned}
|| \ T(r,h') \leqslant &\frac{1}{2}\binom{2N+2}{N+1}\left[\binom{2N+2}{N+1}-2\right]\sum_{i=1}^{2N+2}\Big(N^{(1)}_{(f,a_i),>d}(r)+N^{(1)}_{(g,a_i),>d}(r)\\
&+2N^{(1)}_{(f,a_i),>k}(r)+2N^{(1)}_{(g,a_i),>k}(r)\Big)+o(T(r,f)) \ (2.2.6)
\end{aligned}$$

Consider the hyperplanes $H_1=\{w_1=0\},\ H_2=\{w_2=0\},\ H_3=\{w_1+...+w_t=0\}$ in $\mathbb{P}^{t-1}(\mathbb{C})$. Then

$$\begin{aligned}
|| \quad T(r,h') &\geq T\Big(r,\frac{(h',H_1)}{(h',H_2)}\Big)+O(1)=T\Big(r,\frac{b_1h_{I_1''}}{b_2h_{I_2''}}\Big)+O(1)\\
&= T\Big(r,\frac{b_1h_{I_1}}{b_2h_{I_2}}\Big)+O(1)=T\Big(r,\frac{h_{I_1}}{h_{I_2}}\Big)+o(T(r,f))\\
&\geq N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1}(r)+o(T(r,f)),
\end{aligned}$$

$$\| \quad T(r,h') \ge T\Big(r, \frac{(h',H_2)}{(h',H_3)}\Big) + O(1) = T\Big(r, \frac{b_2 h_{I_2}}{h_{I_0}}\Big) + O(1)$$
$$= T\Big(r, \frac{h_{I_2}}{h_{I_0}}\Big) + o(T(r,f))$$
$$\ge N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1}(r) + o(T(r,f)),$$
$$\| \quad T(r,h') \ge T\Big(r, \frac{(h',H_3)}{(h',H_1)}\Big) + O(1) = T\Big(r, \frac{h_{I_0}}{b_1 h_{I_1}}\Big) + O(1)$$
$$= T\Big(r, \frac{h_{I_0}}{h_{I_1}}\Big) + o(T(r,f))$$
$$\ge N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r) + o(T(r,f)).$$

Hence $\| \quad 3T(r,h') \ge N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1}(r) + N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r) + o(T(r,f)).$

Since $\dfrac{h_I}{h_J} = 1$ on the set $\bigcup_{j \in D \setminus ((I\cup J)\setminus (I\cap J))} E_j \setminus \mathcal{W}$, where $E_j = \{z \in \mathbb{C}^n : \nu_{(f,a_j),\leqslant N(N+2)}(z) > 0\}$ and

$$(D \setminus ((I_1 \cup I_2) \setminus (I_1 \cap I_2))) \cup (D \setminus ((I_2 \cup I_0) \setminus (I_2 \cap I_0))) \cup (D \setminus ((I_0 \cup I_1) \setminus (I_0 \cap I_1))) = D,$$

we have that

$$N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1}(r) + N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r) \ge \sum_{i\in D} N^{(1)}_{(f,a_i),\leqslant N(N+2)}(r) - \sum_{i=1}^{2N+2} (N^{(1)}_{(f,a_i),>k}(r) + N^{(1)}_{(g,a_i),>k}(r)).$$

Hence

$$\| \quad 3T(r,h') \ge \sum_{i\in D} N^{(1)}_{(f,a_i),\leqslant N(N+2)}(r) - \sum_{i=1}^{2N+2} (N^{(1)}_{(f,a_i),>k}(r) + N^{(1)}_{(g,a_i),>k}(r)) + o(T(r,f)) \quad (2.2.7).$$

On the other hand, we have

$$\Big\| \quad \sum_{i\in D} N^{(1)}_{(f,a_i),\leqslant N(N+2)}(r) = \sum_{i\in D} (N^{(1)}_{(f,a_i)}(r) - N^{(1)}_{(f,a_i),>N(N+2)}(r))$$
$$\ge \frac{N+2}{N(N+2)} T(r,f) - \frac{N+2}{N(N+2)+1} T(r,f) + o(T(r,f))$$
$$= \frac{N+2}{N(N+2)(N(N+2)+1)} T(r,f) + o(T(r,f)).$$

By the same way, we have

$$\Big\| \quad \sum_{i\in D} N^{(1)}_{(g,a_i),\leqslant N(N+2)}(r) \ge \frac{N+2}{N(N+2)(N(N+2)+1)} T(r,g) + o(T(r,g))$$

From (2.2.6) and (2.2.7) we get

$$\Big\Vert \quad 3\binom{2N+2}{N+1}\Big[\binom{2N+2}{N+1}-2\Big]\sum_{i=1}^{2N+2}(N^{(1)}_{(f,a_i),>d}(r)+N^{(1)}_{(g,a_i),>d}(r)$$

$$+2N^{(1)}_{(f,a_i),>k}(r)+2N^{(1)}_{(g,a_i),>k}(r))$$

$$\geq \frac{N+2}{N(N+2)(N(N+2)+1)}(T(r,f)+T(r,g))$$

$$-\sum_{i=1}^{2N+2}(N^{(1)}_{(f,a_i),>k}(r)+N^{(1)}_{(g,a_i),>k}(r))+o(T(r,f)+T(r,g))\ (2.2.8).$$

From (2.2.8) we also obtain

$$\Big\Vert \quad \Big(\frac{3}{d+1}+\frac{6}{k+1}\Big)\binom{2N+2}{N+1}\Big[\binom{2N+2}{N+1}-2\Big](T(r,f)+T(r,g))$$

$$\geqslant \Big(\frac{N+2}{N(N+2)(N(N+2)+1)}-\frac{2N+2}{k+1}\Big)(T(r,f)+T(r,g))$$

$$+o(T(r,f)+T(r,g)).$$

Letting $r\to\infty$, we get

$$\Big(\frac{3}{d+1}+\frac{6}{k+1}\Big)\binom{2N+2}{N+1}\Big[\binom{2N+2}{N+1}-2\Big]\geqslant\Big(\frac{N+2}{N(N+2)(N(N+2)+1)}-\frac{2N+2}{k+1}\Big).$$

This is a contradiction. Thus, $t=1$. Then $\dfrac{h_{I_0}}{h_{I_1}}=b_1\in\mathcal{R}(\{a_i\}_{i=1}^{3N+1})$.

Hence, for each $I\in\mathcal{I}$, there is $J\in\mathcal{I}\setminus\{I\}$ such that $\dfrac{h_I}{h_J}\in\mathcal{R}(\{a_i\}_{i=1}^{3N+1})$.

(iii) Denote by $\mathcal{M}^*{}_n$ the abelian multiplicative group of all nonzero meromorphic functions on $\mathbb{C}^n$. Define $\mathcal{J}\subset\mathcal{M}^*{}_n$ to be the smallest subgroup which contains all $h\in\mathcal{M}^*{}_n$ with $h^k\in\mathcal{R}(\{a_i\}_{i=1}^q)$ for some positive integer k. Then the multiplicative group $\mathcal{M}^*{}_n/\mathcal{J}$ is a torsion free abelian group.

Consider the free abelian subgroup generated by the family $\{[h_1],...,[h_{3N+1}]\}$ of the torsion free abelian group $\mathcal{M}^*{}_n/\mathcal{J}$, where $h_i=\dfrac{(f,a_i)}{(g,a_i)}$ $(1\leqslant i\leqslant 3N+1)$. Then the family $\{[h_1],...,[h_{3N+1}]\}$ has the property $P_{2N+2,N+1}$. It implies that there exist $3N+1-2N=N+1$ elements, to say, $[h_1],...,[h_{N+1}]$, such that $[h_1]=...=[h_{N+1}]$. Then $\dfrac{h_i}{h_j}\in\mathcal{J}$ $(1\leqslant i<j\leqslant N+1)$, and hence

$$T(r,\frac{h_i}{h_j})=o(T(r,f))\ (1\leqslant i<j\leqslant N+1)\quad .$$

Consider the following four cases.

Case 1. Suppose that there exist three indices $\{i,j,k\},\ (1\leqslant i<j<k\leqslant N+1)$ such that $h_i\not\equiv h_j\not\equiv h_k\not\equiv h_i$.

We have

$$\begin{aligned} T(r,\frac{h_i}{h_j}) &\geq N_{\frac{h_i}{h_j}-1}(r)+O(1)\\ &\geq \sum_{l\in D\setminus\{i,j\}} N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r)-\sum_{s\in\{i,j\}} N^{(1)}_{(f,a_s),>k}(r)+O(1). \end{aligned}$$

Hence $N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r)\leqslant \sum_{s\in\{i,j\}} N^{(1)}_{(f,a_s),>k}(r)+o(T(r,f)),\ \forall l\in D\setminus\{i,j\}$.

Similarly, we also have $N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r)\leqslant \sum_{s\in\{j,k\}} N^{(1)}_{(f,a_s),>k}(r)+o(T(r,f))$ for each $l\in D\setminus\{j,k\}$ and $N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r)\leqslant \sum_{s\in\{i,k\}} N^{(1)}_{(f,a_s),>k}(r)+o(T(r,f))$ for each $l\in D\setminus\{i,k\}$. So, we have

$$N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r)\leqslant \sum_{s\in\{i,j,k\}} N^{(1)}_{(f,a_s),>k}(r)+o(T(r,f))$$

for each $l\in D$. This implies that

$$\begin{aligned} \|\ T(r,f) &\leqslant \sum_{l\in D} N^{(N)}_{(f,a_l)}(r)+o(T(r,f))\\ &\leqslant \sum_{l\in D} N^{(N)}_{(f,a_l),>N(N+2)}(r)+N(2N+2)\sum_{s\in\{i,j,k\}} N^{(1)}_{(f,a_s),>k}(r)+o(T(r,f))\\ &\leqslant \left(\frac{N(N+2)}{N(N+2)+1}+\frac{3N(2N+2)}{k+1}\right)T(r,f)+o(T(r,f)). \end{aligned}$$

Then $\|\quad T(r,f)=o(T(r,f))$. This is a contradiction.

Case 2. Assume that there exist two subsets I and J of the set $\{1,\cdots,N+1\}$ with $I\cap J=\emptyset,\ I\cup J=\{1,\cdots,N+1\},\ \sharp I\geq 2,\ \sharp J\geq 2$ such that

$$h_i=h_j\ \forall i,j\in I \text{ and } h_i=h_j\ \forall i,j\in J \text{ and } h_k\not\equiv h_l\ \forall k\in I,\ \forall l\in J.$$

Choose elements $i,k\in I$ and $j,t\in J$.We have

$$\begin{aligned} T(r,\frac{h_i}{h_j}) &\geq N_{\frac{h_i}{h_j}-1}(r)+O(1)\\ &\geq \sum_{l\in D\setminus\{i,j\}} N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r)-\sum_{s\in\{i,j\}} N^{(1)}_{(f,a_s),>k}(r)+O(1). \end{aligned}$$

Hence $N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r)\leqslant \sum_{s\in\{i,j\}} N^{(1)}_{(f,a_s),>k}(r)+o(T(r,f)),\ \forall l\in D\setminus\{i,j\}$.

Similarly, we also have $N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r) \leqslant \sum_{s\in\{k,t\}} N^{(1)}_{(f,a_s),>k}(r) + o(T(r,f))$ for each $l \in D \setminus \{k,t\}$. So, we have

$$N^{(1)}_{(f,a_l),\leqslant N(N+2)}(r) \leqslant \sum_{s\in\{i,j,k,t\}} N^{(1)}_{(f,a_s),>k}(r) + o(T(r,f)) \ \forall l \in D.$$

Repeating the argument in Case 1, we have $T(r,f) = o(T(r,f))$. This is a contradition.

Case 3. Assume that $h_1 = \cdots = h_N \not\equiv h_{N+1}$.

By the condition (i) in the hypothesis of Theorem 2, we see that h_i is a holomorphic function for each $1 \leqslant i \leqslant N$.. Without loss of generality, we may assume that $1 = h_1 = \cdots = h_N \not\equiv h_{N+1}$. It is easy to see that there exist meromorphic functions c_{li} $(N+2 \leqslant l \leqslant 3N+1,\ 1 \leqslant i \leqslant N+1)$ such that

$$a_l = \sum_{i=1}^{N+1} c_{li}a_i \ (N+2 \leqslant l \leqslant 3N+1) \text{ and } N_{c_{li}}(r) + N_{\frac{1}{c_{li}}}(r) = o(T(r,f)).$$

Hence

$$\begin{aligned}(f,a_l) &= \sum_{i=1}^{N+1} c_{li}(f,a_i),\\ (g,a_l) &= \sum_{i=1}^{N} c_{li}(f,a_i) + \frac{c_{li}}{h_{N+1}}(f,a_{N+1})\\ &= (f,a_l) + c_{li}(\frac{1}{h_{N+1}} - 1)(f,a_{N+1}) \ (N+2 \leqslant l \leqslant 3N+1).\end{aligned}$$

By the conditions (i) and (ii), it is easy to see that if $\nu^{(1)}_{(f,a_l),\leqslant k}(z) = 1$ and $(f,a_{N+1})(z) \neq 0$ then $(c_{li}(\frac{1}{h_{N+1}} - 1))(z) = 0$. This implies the following

$$\begin{aligned}N^{(1)}_{(f,a_l),\leqslant k}(r) - N^{(1)}_{(f,a_{N+1}),>k}(r) &\leqslant N_{\frac{1}{h_{N+1}}-1}(r) + o(T(r,f))\\ &= o(T(r,f)) \ (N+2 \leqslant l \leqslant 3N+1).\end{aligned}$$

Thus, we have

$$N^{(1)}_{(f,a_l),\leqslant k}(r) \leqslant N^{(1)}_{(f,a_{N+1}),>k}(r) + o(T(r,f)) \leqslant \frac{1}{k+1}T(r,f) + o(T(r,f)).$$

On the other hand, we have

$$\begin{aligned}
|| \quad T(r,f) &\leqslant \frac{2N}{N+2}\sum_{l=N+2}^{3N+1} N^{(N)}_{(f,a_l)}(r) + o(T(r,f)) \\
&\leqslant \frac{2N^2}{N+2}\sum_{l=N+2}^{3N+1} (N^{(1)}_{(f,a_l),\leqslant k}(r) + N^{(1)}_{(f,a_l),>k}(r)) + o(T(r,f)) \\
&\leqslant \frac{8N^3}{(N+2)(k+1)} T(r,f) + o(T(r,f)).
\end{aligned}$$

Then $|| \quad T(r,f) = o(T(r,f))$. This is a contradiction.

Case 4. $h_1 = \cdots = h_{N+1}$.

This yields $f \equiv g$. The Theorem 2.2 is proved. □

2.3 A unicity theorem for meromorphic mapping with a conditions on derivations

In the present section, we will prove the following.

Theorem 2.3. (Ha-Quang-Thai [34]) *Let $f, g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be two meromorphic mappings, and k be a positive integer with $k > 2N^3 + 12N^2 + 6N - 1$. Let $\{a_t\}_{t=1}^{N+2}$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position such that*

$$\dim\{z \in \mathbb{C}^n : \nu_{(f,a_s),\leqslant k}(z)\nu_{(f,a_t),\leqslant k}(z) > 0\} \leqslant n-2 \ (1 \leqslant s < t \leqslant N+2).$$

Assume that f, g are linearly nondegenerate over $\mathcal{R}(\{a_t\}_{t=1}^{N+2})$ and the following are satisfied.

(i) min $(\nu_{(f,a_t),\leqslant k}, 1) =$ min $(\nu_{(g,a_t),\leqslant k}, 1)$ $(1 \leqslant t \leqslant N+2)$.

(ii) Let $f = (f_0 : \cdots : f_N)$ and $g = (g_0 : \cdots : g_N)$ be reduced representations of f and g, respectively. Then, for each $0 \leqslant j \leqslant N$ and for each $\omega \in \bigcup_{t=1}^{N+2}\{z \in \mathbb{C}^n : \nu_{(f,a_t),\leqslant k}(z) > 0\}$, the following two conditions are satisfied:

(a) If $f_j(\omega) = 0$ then $g_j(\omega) = 0$,

(b) If $f_j(\omega)g_j(\omega) \neq 0$ then $\mathcal{D}^\alpha\left(\frac{f_i}{f_j}\right)(\omega) = \mathcal{D}^\alpha\left(\frac{g_i}{g_j}\right)(\omega)$ for each n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers with $|\alpha| = \alpha_1 + ... + \alpha_n \leqslant 2N$ and for each $i \neq j$, where $\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 ... \partial^{\alpha_n} z_n}$.

Then $f \equiv g$.

Remark that the condition (ii) in Theorem 2.3 does not depend on the choice of reduced representations.

Proof. Assume that $f \not\equiv g$ and f, g, a_i have reduced representations

$$f = (f_0 : ... : f_N),\ g = (g_0 : ... : g_N),\ a_i = (a_{i0} : ... : a_{iN}).$$

Lemma 2.3.1. *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping such that f is linearly nondegenerate over $\mathbb{C}$. Let $a_1, a_2, ..., a_{N+2}$ be $N+2$ "small" (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ located in general position. Then, for each $k \geq N-1$, we have*

$$\Big|\Big| \Big(1 - \frac{N(N+2)}{k+1}\Big) T(r,f) \leqslant \sum_{j=1}^{N+2} \Big(1 - \frac{N}{k+1}\Big) N^{(N)}_{(f,a_j),\leqslant k}(r) + o(T(r,f)).$$

Proof . By the Second Main Theorem (see [63])

$$\begin{aligned} ||\quad T(r,f) &\leqslant \sum_{j=1}^{N+2} N^{(N)}_{(f,a_j)}(r) + o(T(r,f)) \\ &\leqslant \sum_{j=1}^{N+2} N^{(N)}_{(f,a_j),\leqslant k}(r) + \sum_{j=1}^{N+2} \frac{N}{k+1} N_{(f,a_j),>k}(r) + o(T(r,f)) \\ &\leqslant \sum_{j=1}^{N+2} (1 - \frac{N}{k+1}) N^{(N)}_{(f,a_j),\leqslant k}(r) + \frac{N(N+2)}{k+1} T(r,f) + o(T(r,f)). \end{aligned}$$

Hence

$$\Big|\Big| \Big(1 - \frac{N(N+2)}{k+1}\Big) T(r,f) \leqslant \sum_{j=1}^{N+2} \Big(1 - \frac{N}{k+1}\Big) N^{(N)}_{(f,a_j),\leqslant k}(r) + o(T(r,f)).$$

Claim 2.3.2 *The following holds*

$$\begin{aligned} (2N+1) \sum_{v=1}^{N+2} N^{(1)}_{(f,a_v),\leqslant k}(r) \leqslant \Big(1 + \frac{4N+2}{k+1}\Big) &(T(r,f) + T(r,g)) \\ &+ o(T(r,f) + T(r,g)) \end{aligned}$$

Proof. Fix an index j ($0 \leqslant j \leqslant N$). Since $g \not\equiv f$, there exists an index i ($0 \leqslant i \leqslant N$) such that $P_{ij} = \frac{(f,a_i)}{(f,a_j)} - \frac{(g,a_i)}{(g,a_j)} \not\equiv 0$.

We set $I = I(f) \cup I(g) \cup \cup_{1 \leqslant\ t<s \leqslant N+2}\{z \in \mathbb{C}^n\ |\nu_{(f,a_t),\leqslant k}(z) \cdot \nu_{(f,a_s),\leqslant k}(z) > 0\}$. Then I is an analytic subset of codimension 2 or an empty set.

Denote by ν_0 the divisor

$$\nu_0 := (\max\{0, (2N+1) - \nu_{(f,a_j)} - \nu_{(g,a_j)}\}) \cdot (\min\{1, \nu_{(f,a_j),\leqslant k}\}).$$

We show that $\nu_{P_{ij}} \geq \sum_{\substack{s=1\\ s\neq j}}^{N+2}(2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}\} + \nu_0 - (2N+1)\nu^{(1)}_{(f,a_j),>k}$ outside an analysis set of codimension 2.

Indeed, we fix a point $z \in \cup_{i=1}^{N+2}\{w : \nu_{(f,a_i),\leqslant k}(w) > 0\} \setminus I$.

If $(f, a_j)(z) \neq 0$, suppose that $f_l(z) \cdot g_l(z) = 0$ $(0 \leqslant l \leqslant N)$. Then $g_l(z) = 0$ $(0 \leqslant l \leqslant N)$. This means that $z \in I(g)$. This is impossible. Hence, there exist an index l such that $f_l(z)g_l(z) \neq 0$. This implies that

$$\begin{aligned}\mathcal{D}^\alpha P_{ij}(z) &= \mathcal{D}^\alpha\Big(\frac{(f,a_i)}{(f,a_j)} - \frac{(g,a_i)}{(g,a_j)}\Big)(z)\\ &= \mathcal{D}^\alpha\Big(\frac{\sum_{v=0}^N \frac{f_v}{f_l}a_{iv}}{\sum_{v=0}^N \frac{f_v}{f_l}a_{jv}} - \frac{\sum_{v=0}^N \frac{g_v}{g_l}a_{iv}}{\sum_{v=0}^N \frac{g_v}{g_l}a_{jv}}\Big)(z) = 0\ (|\alpha| \leqslant 2N).\end{aligned}$$

Hence, in this case $\nu_{P_{ij}}(z) \geq 2N+1$. (2.3.1)

Similarly, if $(f, a_j)(z) = 0$ then

$$\begin{aligned}\mathcal{D}^\alpha\big((f,a_i)(g,a_j) - (g,a_i)(f,a_i)\big)(z) = \mathcal{D}^\alpha\big((f_lg_l)(&\sum_{v=0}^N \frac{f_v}{f_l}a_{iv}\sum_{v=0}^N \frac{g_v}{g_l}a_{jv}\\ &- \sum_{v=0}^N \frac{g_v}{g_l}a_{iv}\sum_{v=0}^N \frac{f_v}{f_l}a_{jv})\big)(z) = 0\ (|\alpha| \leqslant 2N).\end{aligned}$$

So, in this case we have $\nu_{((f,a_i)(g,a_j)-(f,a_j)(g,a_i))}(z) \geq 2N+1$. (2.3.2)

Suppose that $\nu_{(f,a_j),>k}(z) = 0$. We now consider two cases.

Case 1. Assume that $\nu_{(f,a_t),\leqslant k}(z) > 0$ for some t with $t \neq j$.

Then $\nu_{(f,a_s),\leqslant k}(z) = 0$ $(s \neq t)$, especially $\nu_{(f,a_j),\leqslant k}(z) = 0$. Hence $\nu_0(z) = 0$ and $\sum_{\substack{s=1\\ s\neq j}}^{N+2}(2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} = 2N+1$. From (2.3.1), we have

$$\nu_{P_{ij}}(z) \geq \sum_{\substack{t=1\\ t\neq j}}^{N+2}(2N+1)\min\{1, \nu_{(f,a_t),\leqslant k}(z)\} + \nu_0(z) - (2N+1)\nu^1_{(f,a_j),>k}(z)\ (2.3.3)$$

Case 2. Assume that $\nu_{(f,a_j),\leqslant k}(z) > 0$.

This follows that $\nu_{(f,a_t),\leqslant k}(z) = 0$ for each $t \neq j$.

Then $\sum_{\substack{s=1\\ s\neq j}}^{N+2}(2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} = 0$.

On the other hand, since $P_{ij} = \dfrac{(f,a_i)(g,a_j)-(f,a_j)(g,a_i)}{(f,a_j)(g,a_j)}$ and by (2.3.2), we have

$$\begin{aligned}\nu_{P_{ij}}(z) &= \nu_{((f,a_i)(g,a_j)-(f,a_j)(g,a_i))}(z) - \nu_{(f,a_j)}(z) - \nu_{(g,a_j)}(z)\\ &\geq (2N+1) - \nu_{(f,a_j)}(z) - \nu_{(g,a_j)}(z).\end{aligned}$$

Combining with $\nu_{P_{ij}}(z) \geq 0$, we have

$$\begin{aligned}\nu_{P_{ij}}(z) &\geq \max\{0, (2N+1) - \nu_{(f,a_j)}(z) - \nu_{(g,a_j)}(z)\}\\ &\geq (\max\{0, (2N+1) - \nu_{(f,a_j)}(z) - \nu_{(g,a_j)}(z)\}) \cdot (\min\{1, \nu_{(f,a_j),\leqslant k}(z)\})\\ &= \nu_0(z)\\ &= \sum_{\substack{s=1\\ s\neq j}}^{N+2} (2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} + \nu_0(z) - (2N+1)\nu^{(1)}_{(f,a_j),>k}(z) \quad (2.3.4)\end{aligned}$$

If $\nu_{(f,a_j),>k}(z) > 0$ then $\nu_0(z) = 0$ and

$$\sum_{\substack{s=1\\ s\neq j}}^{N+2} (2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} \leqslant 2N+1.$$

It implies that

$$\nu_{P_{ij}}(z) \geqslant 0 \geqslant \sum_{\substack{s=1\\ s\neq j}}^{N+2} (2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} + \nu_0(z) - (2N+1)\nu^{(1)}_{(f,a_j),>k}(z) \quad (2.3.5)$$

Combining (2.3.4) with (2.3.5), we have

$$\nu_{P_{ij}}(z) \geq \sum_{\substack{s=1\\ s\neq j}}^{N+2} (2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} + \nu_0(z) - (2N+1)\nu^{(1)}_{(f,a_j),>k}(z) \quad (2.3.6)$$

for each $z \in \cup_{i=1}^{N+2}\{w : \nu_{(f,a_i),\leqslant k}(w) > 0\} \setminus I$.

We also see that if $z \notin \cup_{i=1}^{N+2}\{w : \nu_{(f,a_i),\leqslant k}(w) > 0\}$, then

$$\sum_{\substack{s=1\\ s\neq j}}^{N+2} (2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} + \nu_0(z) = 0.$$

It implies that

$$\nu_{P_{ij}}(z) \geq \sum_{\substack{s=1\\ s\neq j}}^{N+2} (2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} + \nu_0(z) - (2N+1)\nu^{(1)}_{(f,a_j),>k}(z) \quad (2.3.7)$$

From (2.3.6) and (2.3.7), for each $z \notin I$, we have

$$\nu_{P_{ij}}(z) \geq \sum_{\substack{s=1\\ s\neq j}}^{N+2} (2N+1)\min\{1, \nu_{(f,a_s),\leqslant k}(z)\} + \nu_0(z) - (2N+1)\nu^{(1)}_{(f,a_j),>k}(z).$$

This yields that

$$N_{P_{ij}}(r) \geq (2N+1)\sum_{j\neq t=1}^{N+2} N^{(1)}_{(f,a_t),\leqslant k}(r) + N(r,\nu_0) - (2N+1)N^{(1)}_{(f,a_j),>k}(r).$$

We now show that

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) - \nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) \leqslant &- (2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} + \nu_0(z)\\ &+ (2N+1)\nu^{(1)}_{(f,a_i),>k}(z)\end{aligned}$$

for each $z \notin I$.

Indeed, it is easy to see that

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) - \nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) \leqslant &\max\{\nu_{\frac{(f,a_j)}{(f,a_i)}}(z),\ \nu_{\frac{(g,a_j)}{(g,a_i)}}(z)\}\\ &- \nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) \leqslant 0.\end{aligned}$$

Fix $z \notin I$. We consider two cases.

Case 1. Assume that $(f,a_i)(z) \neq 0$.

If $\nu_{(f,a_j),\leqslant k}(z) > 0$, then

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) =&\max\{0, \nu_{(f,a_j)} + \nu_{(g,a_j)} - \nu_{((f,a_i)(g,a_j)-(f,a_j)(g,a_i))}\}(z)\\ \leqslant&\nu_{(f,a_j)}(z) + \nu_{(g,a_j)}(z) - (2N+1) + \nu_0(z)\\ =&\nu_{\frac{(f,a_j)}{(f,a_i)}}(z) + \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) - (2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} + \nu_0(z)\\ &+ (2N+1)\nu^{(1)}_{(f,a_i),>k}(z).\end{aligned}$$

If $\nu_{(f,a_j),\leqslant k}(z) = 0$, then

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) - &\nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) \leqslant 0\\ &\leqslant - (2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} + \nu_0(z) + (2N+1)\nu^{(1)}_{(f,a_i),>k}(z).\end{aligned}$$

So, we have

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) - \nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) \leqslant &- (2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} + \nu_0(z)\\ &+ (2N+1)\nu^{(1)}_{(f,a_i),>k}(z).\end{aligned}$$

Case 2. Assume that $(f, a_i)(z) = 0$.

If $\nu_{(f,a_i),\leqslant k}(z) > 0$, then $\nu_{(f,a_j),\leqslant k}(z) = 0$. It implies that

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) - \nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) &\leqslant 0\\ &\leqslant -(2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} + \nu_0(z) + (2N+1)\nu^{(1)}_{(f,a_i),>k}(z).\end{aligned}$$

If $\nu_{(f,a_i),>k}(z) > 0$, then

$$(2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} \leqslant (2N+1)\nu^{(1)}_{(f,a_i),>k}(z).$$

It implies that

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) - \nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) &\leqslant 0\\ &\leqslant -(2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} + \nu_0(z) + (2N+1)\nu^{(1)}_{(f,a_i),>k}(z).\end{aligned}$$

From *Case 1* and *Case 2*, we obtain

$$\begin{aligned}\nu_{\frac{1}{P_{ij}}}(z) - \nu_{\frac{(f,a_j)}{(f,a_i)}}(z) - \nu_{\frac{(g,a_j)}{(g,a_i)}}(z) \leqslant &-(2N+1)\min\{1, \nu_{(f,a_j),\leqslant k}(z)\} + \nu_0(z)\\ &+ (2N+1)\nu^{(1)}_{(f,a_i),>k}(z)\end{aligned}$$

for each $z \notin I$.

This yields that

$$\begin{aligned}N_{\frac{1}{P_{ij}}}(r) - N_{\dfrac{(f,a_j)}{(f,a_i)}}(r) - N_{\dfrac{(g,a_j)}{(g,a_i)}}(r) \leqslant &-(2N+1)N^{(1)}_{(f,a_j),\leqslant k}(r) + N(r, \nu_0)\\ &+ (2N+1)N^{(1)}_{(f,a_i),>k}(r).\end{aligned}$$

We now have

$$\begin{aligned}m(r, P_{ij}) &\leqslant m\left(r, \frac{(f,a_i)}{(f,a_j)}\right) + m\left(r, \frac{(g,a_i)}{(g,a_j)}\right) + o(T(r,f) + T(r,g))\\ &\leqslant T(r,f) + T(r,g) - N\left(r, \frac{(f,a_j)}{(f,a_i)}\right) - N\left(r, \frac{(g,a_j)}{(g,a_i)}\right) + o(T(r,f))\\ &\qquad + o(T(r,g))\\ &\leqslant T(r,f) + T(r,g) - N_{\frac{(f,a_j)}{(f,a_i)}}(r) - N_{\frac{(g,a_j)}{(g,a_i)}}(r) + o(T(r,f) + T(r,g)).\end{aligned}$$

Hence

$$\begin{aligned}
&(2N+1)\sum_{\substack{v=1\\ v\neq j}}^{N+2} N^{(1)}_{(f,a_v),\leqslant k}(r) + N(r,\nu_0) - (2N+1)N^{(1)}_{(f,a_j),>k}(r)\\
&\leqslant N_{P_{ij}}(r) \leqslant T(r,P_{ij}) = N_{\frac{1}{P_{ij}}}(r) + m(r,P_{ij}) + O(1)\\
&\leqslant T(r,f) + T(r,g) + N_{\frac{1}{P_{ij}}}(r) - N_{\frac{(f,a_j)}{(f,a_i)}}(r) - N_{\frac{(g,a_j)}{(g,a_i)}}(r) + o(T(r,f)+T(r,g))\\
&\leqslant T(r,f) + T(r,g) - (2N+1)N^{(1)}_{(f,a_j),\leqslant k}(r) + N(r,\nu_0)\\
&\qquad\qquad + (2N+1)N^{(1)}_{(f,a_i),>k}(r) + o(T(r,f)+T(r,g)).
\end{aligned}$$

This implies that

$$(2N+1)\sum_{v=1}^{N+2} N^{(1)}_{(f,a_v),\leqslant k}(r) \leqslant (1+\frac{4N+2}{k+1})(T(r,f)+T(r,g)) + o(T(r,f)+T(r,g)).$$

The Claim 2.3.2 is proved.

From Claim 2.3.2, we have

$$\sum_{v=1}^{N+2} N^{(N)}_{(f,a_v),\leqslant k}(r) \leqslant (\frac{N}{2N+1}+\frac{2N}{k+1})(T(r,f)+T(r,g)) + o(T(r,f)+T(r,g)).$$

Similarly, we also have

$$\begin{aligned}
\sum_{v=1}^{N+2} N^{(N)}_{(g,a_v),\leqslant k}(r) &\leqslant N\sum_{v=1}^{N+2} N^{(1)}_{(g,a_v),\leqslant k}(r) = N\sum_{v=1}^{N+2} N^{(1)}_{(f,a_v),\leqslant k}(r)\\
&\leqslant \Big(\frac{N}{2N+1}+\frac{2N}{k+1}\Big)\Big(T(r,f)+T(r,g)\Big) + o(T(r,f)+T(r,g)).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{v=1}^{N+2}(N^{(N)}_{(f,a_v),\leqslant k}(r) + N^{(N)}_{(g,a_v),\leqslant k}(r))\\
&\leqslant \Big(\frac{2N}{2N+1}+\frac{4N}{k+1}\Big)\Big(T(r,f)+T(r,g)\Big) + o(T(r,f)+T(r,g)).
\end{aligned}$$

On the other hand, by Claim 2.3.1, it implies that

$$||\ \sum_{i=1}^{N+2} N^{(N)}_{(f,a_i),\leqslant k}(r) \geq \frac{(k+1)-N(N+2)}{k+1-N}T(r,f)$$

and

$$||\ \sum_{i=1}^{N+2} N^{(N)}_{(g,a_i),\leqslant k}(r) \geq \frac{(k+1)-N(N+2)}{k+1-N}T(r,g).$$

Hence

$$\begin{aligned} &\| \left(\frac{2N}{2N+1} + \frac{4N}{k+1}\right)((T(r,f) + T(r,g)) \\ &\geq \frac{(k+1) - N(N+2)}{k+1-N}(T(r,f) + T(r,g)) + o((T(r,f) + T(r,g))). \end{aligned}$$

Letting $r \longrightarrow \infty$, we have

$$\frac{2N}{2N+1} + \frac{4N}{k+1} \geq \frac{(k+1) - N(N+2)}{k+1-N}.$$

Then $\dfrac{2N}{2N+1} \geq \dfrac{(k+1) - N(N+6)}{k+1-N}$. Hence $k+1 \leqslant 2N^3 + 12N^2 + 6N$. This is a contradiction. Thus, $f \equiv g$ and Theorem 2.3 is proved. □

Chapter 3

Value distribution of the Gauss map of minimal surfaces on annular ends

Let M be a non-flat minimal surface in $\mathbb{R}^3$, or more precisely, a connected oriented minimal surface in $\mathbb{R}^3$. By definition, the Gauss map G of M is the map which maps each point $p \in M$ to the unit normal vector $G(p) \in S^2$ of M at p. Instead of G, we study the map $g := \pi \circ G : M \to \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} (= \mathbb{P}^1(\mathbb{C}))$ for the stereographic projection π of S^2 onto $\mathbb{P}^1(\mathbb{C})$. By associating a holomorphic local coordinate $z = u + \sqrt{-1}v$ with each positive isothermal coordinate system (u, v), M is considered as an open Riemann surface with a conformal metric ds^2 and by the assumption of minimality of M, g is a meromorphic function on M.

In 1988, H. Fujimoto [20] proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most 4 points, and the bound is sharp. In 1991, S. J. Kao [38] showed that the Gauss map of an end of a non-flat complete minimal surface in $\mathbb{R}^3$ that is comformally an annulus $\{z | 0 < 1/r < |z| < r\}$ must also assume every value, with at most 4 exceptions.

On the other hand, in 1993, M. Ru [54] studied the Gauss map of minimal surface in $\mathbb{R}^m$ with ramification. In this chapter, we shall study the Gauss map of minimal surfaces in $\mathbb{R}^3, \mathbb{R}^4$ on annular ends with ramification. In particular, we prove Theorem 3.4.6, Theorem 3.4.7 (Dethloff-Ha [9]). We would like to refer the case $\mathbb{R}^m (m > 3)$ with another aspect to Dethloff-Ha-Thoan [10].

3.1 Minimal surface in $\mathbb{R}^m$

We recall some basic facts in differential geometry.

Let M be a connected oriented real 2-dimentional differential manifold and $x = (x_1, ..., x_m) : M \longrightarrow \mathbb{R}^m$ an immersion.

For each point $p \in M$, take a system of local coordinates (u_1, u_2) around p which are positively oriented. The tangent plane of M at p is given by

$$T_p(M) := \left\{ \lambda \frac{\partial x}{\partial u_1} + \mu \frac{\partial x}{\partial u_2} | \lambda, \mu \in \mathbb{R} \right\}$$

and the normal space of M at p is given by

$$N_p(M) := \left\{ N \in T_p\mathbb{R}^m | \left(N, \frac{\partial x}{\partial u_1} \right) = \left(N, \frac{\partial x}{\partial u_2} \right) = 0 \right\},$$

where (X, Y) denotes the inner product of vectors X and Y.

The metric ds^2 on M induced from the standard metric on $\mathbb{R}^m$, called *the first fundamental form* on M, is given by

$$\begin{aligned} ds^2 = |dx|^2 := (dx, dx) &= \left(\frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2, \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 \right) \\ &= g_{11} du_1^2 + 2 g_{12} du_1 du_2 + g_{22} du_2^2, \end{aligned}$$

where $g_{ij} := \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right), 1 \leqslant i, j \leqslant 2$.

The second fundamental form of M with respect to a unit normal vector N is given by

$$d\sigma^2 := \sum_{1 \leqslant i,j \leqslant 2} b_{ij}(N) du_i du_j,$$

where $b_{ij}(N) := \left(\frac{\partial^2 x}{\partial u_i \partial u_j}, N \right), (1 \leqslant i, j \leqslant 2)$.

3.1.1 Proposition. (Fujimoto [25]) *For an arbitrary given regular curve in M, $\gamma : (a, b) \longrightarrow M, \gamma(t) = (u_1(t), u_2(t))$. it holds that*

$$k_\gamma(N) := \left(\frac{d^2 x}{ds^2}, N \right) = \frac{d\sigma^2}{ds^2} = \frac{\sum_{ij} b_{ij} u_i' u_j'}{\sum_{ij} g_{ij} u_i' u_j'}$$

$\forall N \in N_{\gamma(t)}(M)$.

Then we may see that $k_\gamma(N)$ depends only on N and the tangent vector of γ at p. Take a nonzero vector $N \in N_p(M)$ and a unit tangent vector $T \in T_p(M)$. Choose a curve $x(s)$ in M with arclength parameter s such that $x(0) = p$ and $(dx/ds)(0) = T$, and define *the normal curvature* of M in the direction T with respect to the normal vector N by

$$k(N,T) := \left(\frac{d^2x}{ds^2}, N\right).$$

We note α the plane which includes the vectors N and T and let γ be the curve which is defined as the intersection of α and M. By elementary calculation, we can show that $k(N,T)$ is the reciprocal of radius of curvature for the curve γ in the plane α. Set

$$k_1(N) := \min\{k(N,T); T \in T_p(M), |T| = 1\},$$
$$k_2(N) := \max\{k(N,T); T \in T_p(M), |T| = 1\},$$

The mean curvature of M for the direction N at p is defined by

$$H_p(N) := \frac{k_1(N) + k_2(N)}{2}$$

We remark that we may prove the following for the calculation of the mean curvature

$$H_p(N) = \frac{g_{11}b_{22}(N) + g_{22}b_{11}(N) - 2g_{12}b_{12}(N)}{2\big(g_{11}g_{22} - g_{12}^2\big)}.$$

(see Fujimoto [25] for example).

3.1.2 Definition. A surface M is called a minimal surface in $\mathbb{R}^m$ if $H_p(N) = 0$ for all $p \in M$ and $N \in N_p(M)$.

Let M be a surface with a metric ds^2. A system of local coordinates (u_1, u_2) on an open set U in M is called a system of *isothermal coordinates* on U if ds^2 can be represented as

$$ds^2 = \lambda^2(du_1^2 + du_2^2),$$

for some positive $\mathbb{C}^\infty$ function λ on U.

3.1.3 Theorem. (S. S. Chern, [7]) *For every surface M, there is a system of isothermal local coordinates whose domains cover the whole M.*

3.1.4 Proposition. *For an oriented surface M with a metric ds^2, if we take two positively oriented isothermal local coordinates (u,v) and (x,y), then $w = u + \sqrt{-1}v$ is a holomorphic function in $z = x + \sqrt{-1}y$ on the common domain.*

Let $x : M \longrightarrow \mathbb{R}^m$ be an oriented surface with a Riemannian metric ds^2. With each positive isothermal local coordinate system (u,v) we associate the complex function $z = u + \sqrt{-1}v$. By Proposition 3.1.4, we may regard M as a Riemann surface. Then the metric ds^2 is given by

$$ds^2 = \lambda_z^2(du^2 + dv^2),$$

where $\lambda_z{}^2 = \left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial u}\right) = \left(\frac{\partial x}{\partial v}, \frac{\partial x}{\partial v}\right)$.

Set complex differentiations

$$\frac{\partial x_i}{\partial z} := \frac{1}{2}\left(\frac{\partial x_i}{\partial u} - \sqrt{-1}\frac{\partial x_i}{\partial v}\right), \frac{\partial x_i}{\partial \overline{z}} := \left(\overline{\frac{\partial x_i}{\partial z}}\right),$$

Then

$$\begin{aligned}
\lambda_z{}^2 &= \sum_{i=1}^{n}\left(\frac{\partial x_i}{\partial u}\right)^2 \\
&= \sum_{i=1}^{n} 2\left(\frac{1}{4}\left(\frac{\partial x_i}{\partial u}\right)^2 + \frac{1}{4}\left(\frac{\partial x_i}{\partial v}\right)^2\right) \\
&= 2\sum_{i=1}^{n}\frac{1}{2}\left(\frac{\partial x_i}{\partial u} - \sqrt{-1}\frac{\partial x_i}{\partial v}\right)\frac{1}{2}\left(\frac{\partial x_i}{\partial u} + \sqrt{-1}\frac{\partial x_i}{\partial v}\right) \\
&= 2\sum_{i=1}^{n}\frac{\partial x_i}{\partial z}.\overline{\frac{\partial x_i}{\partial z}} \\
&= 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_n}{\partial z}\right|^2\right)
\end{aligned}$$

So we can rewrite the metric as

$$ds^2 = 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_n}{\partial z}\right|^2\right)|dz|^2$$

Define the Laplacian $\Delta_z = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ in terms of the complex local coordinate $z = u + \sqrt{-1}v$. If we take another complex local coordinate ξ, then we have $\Delta_\xi = |dz/d\xi|^2\Delta_z$. Since $\lambda_\xi = \lambda_z|dz/d\xi|$, the operator $\Delta = (1/\lambda_z^2)\Delta_z$ does not depend on the choice of complex local coordinate z, which is called the Laplace-Bertrami operator.

3.1.5 Proposition. (Fujimoto [25]) *It holds that*

(i) $\big(\Delta x, X\big) = 0$, *for each* $X \in T_p(M)$,

(ii) $\big(\Delta x, N\big) = 2H(N)$, *for each* $N \in N_p(M)$.

3.1.6 Theorem. (Fujimoto [25]) *Let* $x = (x_1, ..., x_n) : M \longrightarrow \mathbb{R}^m$ *be a surface immersed in* $\mathbb{R}^m$*, which is considered as a Riemann surface. Then* M *is minimal if and only if each* x_i *is a harmonic function on* M*, namely*

$$\Delta_z x_i = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)x_i = 0, \quad (1 \leq i \leq n)$$

for every holomorphic local coordinate $z = u + \sqrt{-1}v$.

3.1.7 Corollary. *There exists no compact minimal surface without boundary in* $\mathbb{R}^m$.

Let $x : M \longrightarrow \mathbb{R}^m$ be a minimal surface immersed in $\mathbb{R}^m$.

3.1.8 Definition. A continuous curve $\gamma(t)$ $(0 \leq t < 1)$ in M is said to be divergent in M if, for each compact set, there is some t_0 such that $\gamma(t) \notin K$ for any $t \geq t_0$.

3.1.9 Definition. We define the distance $d(p)$ $(\leq +\infty)$ from a point $p \in M$ to the boundary of M as the greatest lower bound of the lengths of all continuous curves which are divergent in M.

3.1.10 Definition. A minimal surface M immersed in $\mathbb{R}^m$ is said to be complete if the image in $\mathbb{R}^m$ of every divergent curve on M has infinite length (equivalently, $d(p) = +\infty$ for all $p \in M$).

3.2 The Gauss map of minimal surfaces

Let $x := (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ be a surface immersed in $\mathbb{R}^m$.

We consider the set of all oriented 2 -planes in $\mathbb{R}^m$ which contain the origin and denote it by Π .

To clarify the set Π, we regard it as a subset of the $(m-1)$-dimensional complex projective space $\mathbb{P}^{m-1}(\mathbb{C})$ as follows. To each $P \in \Pi$, taking a positively oriented basis $\{X, Y\}$ of P such that

$$| X |=| Y |, (X, Y) = 0, \tag{3.2.1}$$

we assign the point $\phi(P) = \pi(X - \sqrt{-1}Y)$, where π denotes the canonical projection from $\mathbb{C}^m - \{0\}$ onto $\mathbb{P}^{m-1}(\mathbb{C})$, namely, the map which maps each $p = (w_1, \cdots, w_m) \neq (0, \cdots, 0)$ to the equivalence class

$$(w_1, \cdots, w_m) := \{(cw_1, \cdots, cw_m); c \in \mathbb{C} - \{0\}\}.$$

For another positive basis $\{\tilde{X}, \tilde{Y}\}$ of P satisfying (3.2.1) we can find a real number θ such that

$$\tilde{X} = r(\cos\theta \cdot X + \sin\theta \cdot Y),$$
$$\tilde{Y} = r(-\sin\theta \cdot X + \cos\theta \cdot Y),$$

where $r := \dfrac{|\tilde{X}|}{|X|}$. Therefore, we can write

$$\tilde{X} - \sqrt{-1}\tilde{Y} = re^{\sqrt{-1}\theta}(X - \sqrt{-1}Y).$$

This shows that the value $\phi(P)$ does not depend on the choice of a positive basis of P satisfying 3.2.1 but only on P. On the other hand, $\phi(P)$ is contained in the quadric

$$Q_{m-2}(\mathbb{C}) := \{(w_1, \cdots, w_m); w_1^2 + \cdots + w_m^2 = 0)\} \subset \mathbb{P}^{m-1}(\mathbb{C}).$$

We can show that ϕ is bijective and we identify Π with Q_{m-2}.

We consider a surface $x := (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$. For each point $P \in M$, the oriented plane $T_p(M)$ is canonically identified with an element of Π after the parallel translation which maps p to the origin.

3.2.1 Definition. The (generalized) Gauss map of a surface M is defined as the map of M into $Q_{m-2}(\mathbb{C})$ which maps each point $p \in M$ to $\phi(T_p(M))$.

For a system of positively oriented isothermal local coordinates (u, v) the vectors $X = \frac{\partial x}{\partial u}, Y = \frac{\partial x}{\partial v}$ give a positive basis of $T_p(M)$ satisfying the condition (3.2.1). Therefore, the Gauss map of M is locally given by

$$G = \phi(X - \sqrt{-1}Y) = (\frac{\partial x_1}{\partial z} : \cdots : \frac{\partial x_m}{\partial z})$$

where $z = u + \sqrt{-1}v$. We may write $G = (\omega_1 : \cdots : \omega_m)$ with globally defined holomorphic forms $\omega_i := dx_i \equiv \frac{\partial x_i}{\partial z} dz \quad (1 \leq i \leq m)$.

3.2.2 Proposition. (Fujimoto [25]) *A surface $x : M \to \mathbb{R}^m$ is minimal if and only if the Gauss map $G : M \to \mathbb{P}^{m-1}(\mathbb{C})$ is holomorphic.*

We say that a holomorphic $1-$form ω on a Riemann surface M has no real periods if

$$Re \int_\gamma \omega = 0$$

for every closed cycle in M. If ω has no real period, then the quantity

$$x(z) = Re \int_{\gamma_{z_0}^z} \omega$$

depends only on z and z_0 for a piecewise smooth curve $\gamma_{z_0}^z$ in M joining z_0 and z and so x is a well-defined function of z on M, which we denote by

$$x(z) = Re \int_{z_0}^z \omega$$

from here on. Related to Proposition 3.2.2, we recall here the following construction theorem of minimal surfaces.

3.2.3 Theorem. (Fujimoto [25] for example) *Let M be an open Riemann surface and* *let* $\omega_1, \omega_2, ..., \omega_m$ *be holomorphic forms on* M *such that they have no common zero, no real periods and locally satisfy the identity*

$$f_1^2 + f_2^2 + \cdots + f_n^2 = 0$$

for holomorphic function f_i *with* $\omega_i = f_i dz$. *Set*

$$x_i = 2Re \int_{z_0}^{z} \omega_i,$$

for an arbitrarily fixed point z_0 *of* M. *Then, the surface* $x = (x_1, ..., x_m) : M \longrightarrow \mathbb{R}^m$ *is a minimal surface immersed in* $\mathbb{R}^m$ *such that the Gauss map is the map* $G = (\omega_1 : \cdots : \omega_m) : M \longrightarrow \mathbb{Q}_{m-2}(\mathbb{C})$ *and the induced metric is given by*

$$ds^2 = 2(|\omega_1|^2 + \cdots + |\omega_m|^2). \tag{3.2.2}$$

Now, let M be a Riemann surface with a metric ds^2 which is conformal, namely, represented as

$$ds^2 = \lambda_z^2 |dz|^2$$

with a positive C^∞ function λ_z in term of a holomorphic local coordinate z.

3.2.4 Definition. For each point $p \in M$ we define *the Gaussian curvature* of M at p by

$$K \equiv K_{ds^2} := -\Delta \log \lambda_z \Big(= -\frac{\Delta_z \log \lambda_z}{\lambda_z^2} \Big).$$

For a minimal surface M immersed in $\mathbb{R}^m$, using (3.2.2), we can show that

$$K \equiv K_{ds^2} = -4 \frac{|\tilde{g} \wedge \tilde{g}'|^2}{|\tilde{g}|^6} = -4 \frac{\sum_{j<k} |g_j g_k' - g_k g_j'|^2}{(\sum_{j=1}^{m} |g_j|^2)^3} \tag{3.2.3}$$

where $\tilde{g} = (g_1, ..., g_m), g_j = \frac{\partial x_j}{\partial z}, 1 \le j \le m.$

This implies that the curvature of a minimal surface is always non-positive.

If a minimal surface is flat (i.e., the Gauss curvature vanishes everywhere), then (3.2.3) implies that $g_i/g_{i_0} = const. (1 \le i \le n)$ for some i_0 with $g_{i_0} \not\equiv 0$ and, therefore, that the Gauss map g is a constant map.

3.2.5 Proposition. (Fujimoto [25]) *For a minimal surface* M *immersed in* $\mathbb{R}^m$, M *is flat, or equivalently, the Gauss map of* M *is a constant if and only if it lies in a plane.*

3.3 Meromorphic functions with ramification

Let f be a nonconstant holomorphic map of a disc $\Delta_R := \{z \in \mathbb{C}; |z| < R\}$ into $\mathbb{P}^1(\mathbb{C})$, where $0 < R < \infty$. Take a reduced representation $f = (f_0 : f_1)$ on Δ_R and define

$$||f|| := (|f_0|^2 + |f_1|^2)^{1/2}, W(f_0, f_1) := f_0 f_1' - f_1 f_0'.$$

Let $a^j (1 \leq j \leq q)$ be q distinct points in $\mathbb{P}^1(\mathbb{C})$. We may assume $a^j = (a_0^j : a_1^j)$ with $|a_0^j|^2 + |a_1^j|^2 = 1 (1 \leq j \leq q)$, and set

$$F_j := a_0^j f_1 - a_1^j f_0 \ (1 \leq j \leq q).$$

3.3.1 Definition. One says that the meromorphic function f is ramified over a point $a = (a_0 : a_1) \in \mathbb{P}^1(\mathbb{C})$*with multiplicity at least* e if all the zeros of the function $F := a_0 f_1 - a_1 f_0$ have orders at least e. If the image of f omits a, one will say that f is *ramified over* a *with multiplicity* ∞.

3.3.2 Proposition. (Fujimoto [19, Propostion 2.1]) *For each* $\epsilon > 0$ *there exist positive constants* C_1 *and* μ *depending only on* $a^1, \cdots, a^q$ *and on* ϵ *respectively such that*

$$\Delta \log \left(\frac{||f||^\epsilon}{\Pi_{j=1}^q \log(\mu ||f||^2/|F_j|^2)} \right) \geq \frac{C_1 ||f||^{2q-4} |W(f_0, f_1)|^2}{\Pi_{j=1}^q |F_j|^2 \log^2(\mu ||f||^2/|F_j|^2)}$$

3.3.3 Lemma. *Suppose that* $q - 2 - \sum_{j=1}^q \frac{1}{m_j} > 0$ *and* f *is ramified over* a^j *with multiplicity at least* m_j *for each* $j (1 \leq j \leq q)$. *Then there exist positive constants* C *and* $\mu (> 1)$ *depending only on* a^j *and* $m_j (1 \leq j \leq q)$ *which satisfy that if we set*

$$v := \frac{C ||f||^{q-2-\sum_{j=1}^q \frac{1}{m_j}} |W(f_0, f_1)|}{\Pi_{j=1}^q |F_j|^{1-\frac{1}{m_j}} \log(\mu ||f||^2/|F_j|^2)}$$

on $\Delta_R - \{F_1 ... F_q = 0\}$ *and* $v = 0$ *on* $\Delta_R \cap \{F_1 ... F_q = 0\}$, *then* v *is continuous on* Δ_R *and satisfies the condition* $\Delta \log v \geq v^2$ *in the sense of distributions.*

Proof. First, we prove the continuousness of v.
Obviously, v is continuous on $\Delta_R - \{F_1 ... F_q = 0\}$.
Take a point z_0 with $F_i(z_0) = 0$ for some i. Then $F_j(z_0) \neq 0$ for all $j \neq i$ and $\nu_{F_i}(z_0) \geq m_i$. Changing indices if necessary, we may assume that $f_0(z_0) \neq 0$, then $a_0^i \neq 0$. Hence, we get

$$\nu_W(z_0) = \nu_{\frac{(a_0^i \frac{f_1}{f_0} - a_1^i)'}{a_0^i}}(z_0) = \nu_{\frac{(F_i/f_0)'}{a_0^i}}(z_0) = \nu_{F_i}(z_0) - 1.$$

Thus,

$$\nu_{v\Pi_{j=1}^q \log(\mu||f||^2/|F_j|^2)}(z_0) = \nu_W(z_0) - \sum_{j=1}^{q}(1-\frac{1}{m_j})\nu_{F_j}(z_0)$$
$$= \nu_{F_i}(z_0) - 1 - (1-\frac{1}{m_i})\nu_{F_i}(z_0) = \frac{\nu_{F_i}(z_0)}{m_i} - 1 \geq 0. \ (*)$$

So, $\lim_{z\to z_0} v(z) = 0$. This implies that v is continuous on Δ_R.
Now, we choose constants C and μ such that C^2 and μ satisfy the inequality in Proposition 3.3.2 for the case $\epsilon = q - 2 - \sum_{j=1}^q \frac{1}{m_j}$. Then we have

$$\begin{aligned}\Delta \log v &\geq \Delta \log \frac{||f||^{q-2-\sum_{j=1}^q \frac{1}{m_j}}}{\Pi_{j=1}^q \log(\mu||f||^2/|F_j|^2)}\\ &\geq C^2 \frac{||f||^{2q-4}|W(f_0,f_1)|^2}{\Pi_{j=1}^q|F_j|^2\log^2(\mu||f||^2/|F_j|^2)}\\ &\geq C^2 \frac{||f||^{2q-4-2\sum_{j=1}^q\frac{1}{m_j}}|W(f_0,f_1)|^2}{\Pi_{j=1}^q|F_j|^{2-\frac{2}{m_j}}\log^2(\mu||f||^2/|F_j|^2)}\\ &= v^2 \text{ (by } |F_j| \leq ||f||(1\leq j\leq q)).\end{aligned}$$

Lemma 3.3.3 is proved.

3.3.4 Lemma. (Generalized Schwarz Lemma [1]) *Let v be a nonnegative real-valued continuous subharmonic function on Δ_R. If v satisfies the inequality $\Delta \log v \geq v^2$ in the sense of distributions, then*

$$v(z) \leq \frac{2R}{R^2-|z|^2}.$$

3.3.5 Lemma. *For every δ with $q - 2 - \sum_{j=1}^q \frac{1}{m_j} > q\delta > 0$ and f is ramified over a^j with multiplicity at least m_j for each $j(1\leq j\leq q)$, there exists a positive constant C_0 such that*

$$\frac{||f||^{q-2-\sum_{j=1}^q\frac{1}{m_j}-q\delta}|W(f_0,f_1)|}{\Pi_{j=1}^q|F_j|^{1-\frac{1}{m_j}-\delta}} \leq C_0\frac{2R}{R^2-|z|^2}.$$

Proof. By using an argument as in (*) of the proof of Lemma 3.3.3, the above inequality is correct on $\{F_1...F_q = 0\}$ for every $C_0 > 0$ (the left hand side of the above inequality is zero).
If $z \notin \{F_1...F_q = 0\}$, using Lemma 3.3.3 and Lemma 3.3.4, we get

$$\frac{C||f||^{q-2-\sum_{j=1}^q\frac{1}{m_j}}|W(f_0,f_1)|}{\Pi_{j=1}^q|F_j|^{1-\frac{1}{m_j}}\log(\mu||f||^2/|F_j|^2)} \leq \frac{2R}{R^2-|z|^2},$$

where C and μ are the constants given in Lemma 3.3.3.
On the other hand, for a given $\delta > 0$, it holds that

$$\lim_{x\to 0} x^{\delta}\log(\mu/x^2) < +\infty$$

so we can set

$$\overline{C} := \sup_{0<x\leq 1} x^{\delta}\log(\mu/x^2)(< +\infty).$$

Then we have

$$\begin{aligned}
&\frac{||f||^{q-2-\sum_{j=1}^{q}\frac{1}{m_j}-q\delta}|W(f_0,f_1)|}{\Pi_{j=1}^{q}|F_j|^{1-\frac{1}{m_j}-\delta}}\\
&= \frac{||f||^{q-2-\sum_{j=1}^{q}\frac{1}{m_j}}|W(f_0,f_1)|}{\Pi_{j=1}^{q}|F_j|^{1-\frac{1}{m_j}}}\prod_{j=1}^{q}\left(\frac{|F_j|}{||f||}\right)^{\delta}\\
&= \frac{||f||^{q-2-\sum_{j=1}^{q}\frac{1}{m_j}}|W(f_0,f_1)|}{\Pi_{j=1}^{q}|F_j|^{1-\frac{1}{m_j}}\log(\mu||f||^2/|F_j|^2)}\prod_{j=1}^{q}(\frac{|F_j|}{||f||})^{\delta}\log(\mu||f||^2/|F_j|^2)\\
&\leq \frac{\overline{C}^q||f||^{q-2-\sum_{j=1}^{q}\frac{1}{m_j}}|W(f_0,f_1)|}{\Pi_{j=1}^{q}|F_j|^{1-\frac{1}{m_j}}\log(\mu||f||^2/|F_j|^2)}\\
&\leq \frac{\overline{C}^q}{C}\frac{2R}{R^2-|z|^2}.
\end{aligned}$$

This gives Lemma 3.3.5.

For our purpose, we shall give the following result which is contained in a classical results of Nevanlinna (Nevanlinna [44]). We give here a direct proof of this result by using Lemma 3.3.5.

3.3.6 Proposition. *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be a holomorphic map. For arbitrary distinct points $a^1, ..., a^q \in \mathbb{P}^1(\mathbb{C})$ suppose that f is ramified over a^j with multiplicity at least m_j for each $j, (1 \leq j \leq q)$ satisfying*

$$\sum_{j=1}^{q}(1-\frac{1}{m_j}) > 2.$$

Then f is constant.

Proof. Assume that f is non-constant. Without loss of generality, we may assume $F_j(0) \neq 0 (1 \leq j \leq q)$ and $W(f_0, f_1)(0) \neq 0$. By our assumptions, for every $R > 0$ and δ with

$$\sum_{j=1}^{q}(1-\frac{1}{m_j}) - 2 > q\delta > 0,$$

we apply Lemma 3.3.5 to the map $f|_{\Delta_R} : \Delta_R \to \mathbb{P}^1(\mathbb{C})$ to show that

$$\frac{||f||^{q-2-\sum_{j=1}^{q}\frac{1}{m_j}-q\delta}|W(f_0,f_1)|}{\Pi_{j=1}^{q}|F_j|^{1-\frac{1}{m_j}-\delta}} \le C_0\frac{2R}{R^2-|z|^2}.$$

By substituting $z = 0$ into the above inequality we conclude that R has to be bounded by a constant depending only on a^j, m_j and on the values of $f, F_j, W(f_0, f_1)$ at the origin. This is a contradiction.

3.4 The Gauss map of minimal surfaces with ramification

3.4.1 Definition. One says that a holomorphic map $g : A \to \mathbb{P}^{m-1}(\mathbb{C})$ of an open Riemann surface A into $\mathbb{P}^{m-1}(\mathbb{C})$ is ramified over a hyperplane $H = \{(w_0 : \cdots : w_{m-1}) \in \mathbb{P}^{m-1}(\mathbb{C}) : a_0w_0 + ... + a_{m-1}w_{m-1} = 0\}$ *with multiplicity at least* e if all the zeros of the function $(g, H) := a_0g_0 + ... + a_{m-1}g_{m-1}$ have orders at least e, where $g = (g_0 : ... : g_{m-1})$ is a reduced representation of g. If the image of g omits H, one will say that g is *ramified over* H *with multiplicity* ∞.

3.4.2 Theorem. (Ru [54]) *Let M be a complete minimal surface immersed in $\mathbb{R}^m$ and assume that the Gauss map g of M is $k-$nondegenerate (that is $g(M)$ is contained in a $k-$dimensional linear subspace of $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension), $1 \le k \le m-1$. Let $\{H_j\}_{j=1}^q$ be hyperplanes in general position in $\mathbb{P}^{m-1}(\mathbb{C})$. If g is ramified over H_j with multiplicity at least m_j for each j, then*

$$\sum_{j=1}^{q}(1-\frac{k}{m_j}) \le (k+1)(m-\frac{k}{2}-1)+m.$$

On the other hand, when $m = 3$, then the following holds.

3.4.3 Theorem. (Ru [54]) *Let M be a non-flat complete minimal surface in $\mathbb{R}^3$. If there are q $(q > 4)$ distinct points $a^1, ..., a^q \in \mathbb{P}^1(\mathbb{C})$ such that the Gauss map of M is ramified over a^j with multiplicity at least m_j for each j, then $\sum_{j=1}^q(1-\frac{1}{m_j}) \le 4$.*

3.4.4 Corollary. *The Gauss map g of a non-flat complete minimal surface in $\mathbb{R}^3$ assumes every value on the unit sphere with the possible exception of at most four values.*

3.4.5 Theorem. (Kao [38]) *Let M be a non-flat complete minimal surface in $\mathbb{R}^3$ with the Gauss map g and let A be an annular end of M which is conformal to $\{z| \; 0 < 1/r <$*

$|z| < r\}$, *where* z *is a conformal coordinate. The Gauss map* g *assumes every value on the unit sphere infinitely often, with the possible exception of at most four values on* A.

3.4.6 Theorem. (Dethloff-Ha [9]) *Let* M *be a non-flat complete minimal surface in* $\mathbb{R}^3$ *and let* A *be an annular end of* M *which is conformal to* $\{z|\ 0 < 1/r < |z| < r\}$, *where* z *is a conformal coordinate. If there are* q $(q > 4)$ *distinct points* $a^1, ..., a^q \in \mathbb{P}^1(\mathbb{C})$ *such that the Gauss map of* M *is ramified over* a^j *with multiplicity at least* m_j *for each* j *on* A, *then* $\sum_{j=1}^{q}(1 - \frac{1}{m_j}) \leq 4$.

Proof. For convenience, we recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^3$.

Let $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be a non-flat complete minimal surface and $g : M \to \mathbb{P}^1(\mathbb{C})$ the Gauss map. Let A be an annular end of M, that is, $A = \{z|0 < 1/r \leq |z| < r < \infty\}$, where z is a conformal coordinate. Set $\phi_i := \partial x_i/\partial z$ $(i = 1, 2, 3)$ and $\phi := \phi_1 - \sqrt{-1}\phi_2$. Then, the (classical) Gauss map $g : M \to \mathbb{P}^1(\mathbb{C})$ is given by

$$g = \frac{\phi_3}{\phi_1 - \sqrt{-1}\phi_2},$$

and the metric on M induced from $\mathbb{R}^3$ is given by

$$ds^2 = |\phi|^2(1 + |g|^2)^2|dz|^2 \text{ (see Fujimoto [25]).}$$

Take a reduced representation $g = (g_0 : g_1)$ on M and set $||g|| = (|g_0|^2 + |g_1|^2)^{1/2}$. Then we can rewrite $ds^2 = |h|^2||g||^4|dz|^2$, where $h := \phi/g_0^2$. In fact, h is a holomorphic map without zeros.

Since by assumption M is not flat, g is not constant.

Assume that the theorem does not hold. Without loss of generality we may assume that g is ramified over a^j with multiplicity at least $m_j \geq 2$ for all $1 \leq j \leq q$ on A for given q distinct points $a^1, ..., a^q$ in $\mathbb{P}^1(\mathbb{C})$ and

$$\sum_{j=1}^{q}(1 - \frac{1}{m_j}) > 4.$$

Take δ with

$$\frac{q - 4 - \sum_{j=1}^{q}\frac{1}{m_j}}{q} > \delta > \frac{q - 4 - \sum_{j=1}^{q}\frac{1}{m_j}}{q + 2},$$

and set $p = 2/(q - 2 - \sum_{j=1}^{q}\frac{1}{m_j} - q\delta)$. Then

$$0 < p < 1, \ \frac{p}{1-p} > \frac{\delta p}{1-p} > 1 \ (3.4.1).$$

Consider the open subset

$$A_1 = Int(A) - \{z|W(g_0, g_1)(z).W(g_0, g_1)(1/z) = 0\}$$

of A and we define a new metric

$$d\tau^2 = |h|^{\frac{2}{1-p}} \left(\frac{\Pi_{j=1}^q |G_j|^{1-\frac{1}{m_j}-\delta}}{|W(g_0, g_1)|} \right)^{\frac{2p}{1-p}} |dz|^2 \quad (3.4.2)$$

on A_1, where $G_j := a_0^j g_1 - a_1^j g_0$:

We can show that $d\tau$ is continuous and nowhere vanishing on A_1. Indeed, h is without zeroes on A_1 and for each $z_0 \in A_1$ with $G_j(z_0) \neq 0$ for all $j = 1, ..., q$ then $d\tau$ is continuous at z_0.

Now, suppose there exists a point $z_0 \in A_1$ with $G_j(z_0) = 0$ for some j. Then $G_i(z_0) \neq 0$ for all $i \neq j$ and $\nu_{G_j}(z_0) \geq m_j$. Changing the indices if necessary, we may assume that $g_0(z_0) \neq 0$ then $a_0^j \neq 0$. So, we get

$$\nu_{W(g_0,g_1)}(z_0) = \nu_{\frac{(a_0^j \frac{g_1}{g_0} - a_1^j)'}{a_0^j}}(z_0) = \nu_{\frac{(G_j/g_0)'}{a_0^j}}(z_0) = \nu_{G_j}(z_0) - 1 > 0.$$

This is a contradition with $z_0 \in A_1$.Thus, $d\tau$ is continuous and nowhere vanishing on A_1. Now, it is easy to see that $d\tau$ is flat.

We now prove the following claim.

Claim 1. *$d\tau^2$ is complete on the set $\{z||z| = r\} \cup \{z|W(g_0, g_1)(z) = 0\}$, i.e., the set $\{z||z| = r\} \cup \{z|W(g_0, g_1)(z) = 0\}$ is at infinite distance from any interior point in A_1.*

If $W(g_0, g_1)(z_0) = 0$, then we have two cases.

Case 1. $G_j(z_0) = 0$ for some $j \in \{1, 2, ..., q\}$.

Then we have $G_i(z_0) \neq 0$ for all $i \neq j$ and $\nu_{G_j}(z_0) \geq m_j$. By the same argument as above we can show that

$$\nu_{W(g_0,g_1)}(z_0) = \nu_{G_j}(z_0) - 1.$$

Thus,

$$\begin{aligned} \nu_{d\tau}(z_0) &= \frac{p}{1-p}((1 - \frac{1}{m_j} - \delta)\nu_{G_j}(z_0) - \nu_{W(g_0,g_1)}(z_0)) \\ &= \frac{p}{1-p}(1 - (\frac{1}{m_j} + \delta)\nu_{G_j}(z_0)) \\ &\leq -\frac{2\delta p}{1-p}. \end{aligned}$$

Case 2. $G_j(z_0) \neq 0$ for all $1 \leq j \leq q$.
It is easily to see that $\nu_{d\tau}(z_0) \leq -\dfrac{p}{1-p}$.
So we can find a positive constant C such that

$$|d\tau| \geq \frac{C}{|z - z_0|^{\delta p/(1-p)}}|dz|$$

in a neighborhood of z_0 and combining with (3.4.1) then $d\tau$ is complete on $\{z|W(g_0, g_1)(z) = 0\}$.

Now assume that $d\tau$ is not complete on $\{z||z| = r\}$. There exists $\gamma : [0, 1) \to A_1$, where $\gamma(1) \in \{z||z| = r\}$, so that $|\gamma| < \infty$. Furthermore, we may also assume $dist(\gamma(0); \{z||z| = 1/r\}) > 2|\gamma|$. Consider a small disk Δ with center at $\gamma(0)$. Since $d\tau$ is flat, Δ is isometric to an ordinary disk in the plane. Let $\Phi : \{|w| < \eta\} \to \Delta$ be the isometry. Extend Φ, as a local isometry into A_1, to the largest disk $\{|w| < R\} = \Delta_R$. Then $R \leq |\gamma|$. The reason that Φ cannot be extended to a larger disk is that the image goes to the outside boundary $\{z||z| = r\}$ of A_1. More precisely, there exists a point w_0 with $|w_0| = R$ so that $\Phi(\overline{0, w_0}) = \Gamma_0$ is a divergent curve on A.
The map $\Phi(w)$ is locally biholomorphic, and the metric on Δ_R induced from ds^2 through Φ is given by

$$\Phi^* ds^2 = |h_\circ \Phi|^2 ||g_\circ \Phi||^4 |\frac{dz}{dw}|^2 |dw|^2 \ (3.4.3).$$

On the other hand, Φ is isometric, we have

$$|dw| = |d\tau| = \left(\frac{|h| \Pi_{j=1}^q |G_j|^{(1-\frac{1}{m_j}-\delta)p}}{|W(g_0, g_1)|^p} \right)^{\frac{1}{1-p}} |dz|$$

$$\Rightarrow |\frac{dw}{dz}|^{1-p} = \frac{|h| \Pi_{j=1}^q |G_j|^{(1-\frac{1}{m_j}-\delta)p}}{|W(g_0, g_1)|^p}.$$

Set $f := g(\Phi)$, $f_0 := g_0(\Phi)$, $f_1 := g_1(\Phi)$ and $F_j := G_j(\Phi)$. Since

$$W(f_0, f_1) = (W(g_0, g_1)_o \Phi) \frac{dz}{dw},$$

we obtain

$$|\frac{dz}{dw}| = \frac{|W(f_0, f_1)|^p}{|h(\Phi)| \Pi_{j=1}^q |F_j|^{(1-\frac{1}{m_j}-\delta)p}} \ (3.4.4).$$

By (3.4.3) and (3.4.4), therefore, we get

$$\Phi^* ds^2 = \left(\frac{||f||^2 |W(f_0, f_1)|^p}{\Pi_{j=1}^q |F_j|^{(1-\frac{1}{m_j}-\delta)p}} \right)^2 |dw|^2$$

$$= \left(\frac{||f||^{q-2-\sum_{j=1}^q \frac{1}{m_j} - q\delta} |W(f_0, f_1)|}{\Pi_{j=1}^q |F_j|^{1-\frac{1}{m_j}-\delta}} \right)^{2p} |dw|^2.$$

Using the Lemma 3.3.5, we obtain

$$\Phi^* ds^2 \leqslant C_0^{2p}.(\frac{2R}{R^2 - |w|^2})^{2p}|dw|^2.$$

Since $0 < p < 1$, it then follows that

$$d_{\Gamma_0} \leqslant \int_{\Gamma_0} ds = \int_{\overline{0,w_0}} \Phi^* ds \leqslant C_0^p. \int_0^R (\frac{2R}{R^2 - |w|^2})^p |dw| < +\infty,$$

where d_{Γ_0} denotes the distance of the divergent curve Γ_0 in M, contradicting the assumption of completeness of M. Claim 1 is proved.

We now define

$$\begin{aligned} d\tilde{\tau}^2 &= \left(|h(z)h(\frac{1}{z})|. \frac{\Pi_{j=1}^q |G_j(z)G_j(\frac{1}{z})|^{(1-\frac{1}{m_j}-\delta)p}}{|W(g_0,g_1)(z)W(g_0,g_1)(\frac{1}{z})|^p} \right)^{\frac{2}{1-p}} |dz|^2 \\ &= \lambda^2(z)|dz|^2, \end{aligned}$$

on A_1. Then $d\tilde{\tau}^2$ is complete and flat on A_1 by Claim 1. Let $u(z) = \log \lambda(z)$. Then $u(z)$ is a harmonic function on A_1. Let D be the universal covering surface of A_1. In a neighborhood of any point of D, we may introduce an analytic function $k(z)$ whose real part is $u(z)$, and the mapping

$$w(z) = \int e^{k(z)} dz$$

satisfies

$$|\frac{dw}{dz}| = |e^{k(z)}| = e^{u(z)} = \lambda \ (3.4.5).$$

Thus the length of any curve on D with respect to the metric $d\tilde{\tau}$ is equal to the length of its image in the $w-$plane. By the simple connectivity of D, there exists a global map of D into the $w-$plane which satisfies (3.4.5), and by the completeness of D, this map must be a one-to-one map of D onto the entire $w-$plane. Thus D is conformally equivalent to the plane, which is impossible by Proposition 3.3.6 . This proves Theorem 3.4.6. □

We now recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^4$.

Let $x = (x_1, x_2, x_3, x_4) : M \to \mathbb{R}^4$ be a non-flat complete minimal surface in $\mathbb{R}^4$. As is well-known, the set of all oriented 2-planes in $\mathbb{R}^4$ is canonically identified with the quadric

$$Q_2(\mathbb{C}) := \{(w_1 : ... : w_4) | w_1^2 + ... + w_4^2 = 0\}$$

in $\mathbb{P}^3(\mathbb{C})$. By definition, the Gauss map $g : M \to Q_2(\mathbb{C})$ is the map which maps each point p of M to the point of $Q_2(\mathbb{C})$ corresponding to the oriented tangent plane of M at p. The quadric $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$(e.g. Fujimoto [25]). By suitable identifications we may regard g as a pair of meromorphic functions $g = (g^1, g^2)$ on M. Let A be an annular end of M, that is, $A = \{z|0 < 1/r \leq |z| < r < \infty\}$, where z is a conformal coordinate.
Set $\phi_i := \partial x_i/dz$ for $i = 1, ..., 4$. Then, g^1 and g^2 are given by

$$g^1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2},\ g^2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}$$

and the metric on M induced from $\mathbb{R}^4$ is given by

$$ds^2 = |\phi|^2(1 + |g^1|^2)(1 + |g^2|^2)|dz|^2,$$

where $\phi := \phi_1 - \sqrt{-1}\phi_2$.
Take reduced representations $g^l = (g^l_0 : g^l_1)$ on M and set $||g^l|| = (|g^l_0|^2 + |g^l_1|^2)^{1/2}$ for $l = 1, 2$. Then we can rewrite

$$ds^2 = |h|^2||g^1||^2||g^2||^2|dz|^2 \ (3.4.6),$$

where $h := \phi/(g^1_0 g^2_0)$.
3.4.7 Theorem. (Dethloff-Ha [9]) *Suppose that M is a complete non-flat minimal surface in $\mathbb{R}^4$ and $g = (g^1, g^2)$ is the Gauss map of M. Let A be an annular end of M which is conformal to $\{z|0 < 1/r < |z| < r\}$, where z is the conformal coordinate. Let $a^{11}, ..., a^{1q_1}, a^{21}, ..., a^{2q_2}$ be $q_1 + q_2$ $(q_1, q_2 > 2)$ distinct points in $\mathbb{P}^1(\mathbb{C})$.*

(i) In the case $g^l \not\equiv$ constant $(l = 1, 2)$, if g^l is ramified over a^{lj} with multiplicity at least m_{lj} for each j $(l = 1, 2)$ on A, then
$\gamma_1 = \sum_{j=1}^{q_1}(1 - \frac{1}{m_{1j}}) \leq 2$, *or* $\gamma_2 = \sum_{j=1}^{q_2}(1 - \frac{1}{m_{2j}}) \leq 2$, *or*

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} \geq 1.$$

(ii) In the case where one of g^1 and g^2 is constant, say $g^2 \equiv$ constant, if g^1 is ramified over a^{1j} with multiplicity at least m_{1j} for each j, we have the following

$$\gamma_1 = \sum_{j=1}^{q_1}(1 - \frac{1}{m_{1j}}) \leq 3.$$

Proof. We first study the case $g^l \not\equiv constant$, for $l = 1, 2$. If g^l is ramified over a^{lj} with multiplicity at least m_{lj} for each $j, (l = 1, 2)$ and $\gamma_1 > 2, \gamma_2 > 2$, and

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} < 1.$$

Choose $\delta_0(> 0)$ such that $\gamma_l - 2 - q_l\delta_0 > 0$ for all $l = 1, 2$, and

$$\frac{1}{\gamma_1 - 2 - q_1\delta_0} + \frac{1}{\gamma_2 - 2 - q_2\delta_0} = 1.$$

If we choose a positive constant $\delta(< \delta_0)$ sufficiently near to δ_0 and set

$$p_l := 1/(\gamma_l - 2 - q_l\delta), (l = 1, 2),$$

we have

$$0 < p_1 + p_2 < 1, \ \frac{\delta p_l}{1 - p_1 - p_2} > 1 (l = 1, 2) \ (3.4.7).$$

Consider the open subset

$$A_2 = Int(A) - \{z | \Pi_{l=1,2} W(g_0^l, g_1^l)(z).W(g_0^l, g_1^l)(1/z) = 0\}$$

of A and we now define a new metric

$$d\tau^2 = \left(|h| \frac{\Pi_{j=1}^{q_1} |G_j^1|^{(1-\frac{1}{m_{1j}}-\delta)p_1} \Pi_{j=1}^{q_2} |G_j^2|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W(g_0^1, g_1^1)|^{p_1} |W(g_0^2, g_1^2)|^{p_2}} \right)^{\frac{2}{1-p_1-p_2}} |dz|^2$$

on A_2, where $G_j^l := a_0^{lj} g_1^l - a_1^{lj} g_0^l (l = 1, 2)$.
Using the same arguments as in the proof of Theorem 3.4.6, we may see that $d\tau$ is flat and continuous on A_2. We shall prove the following.

Claim 2. *$d\tau^2$ is complete on the set $\{z||z| = r\} \cup \{z|\Pi_{l=1,2} W(g_0^l, g_1^l)(z) = 0\}$, i.e., the set $\{z||z| = r\} \cup \{z|\Pi_{l=1,2} W(g_0^l, g_1^l)(z) = 0\}$ is at infinite distance from any interior point in A_2.*

By the same method as the proof of Claim 1, we may show that $d\tau$ is complete on $\{z|\Pi_{l=1,2} W(g_0^l, g_1^l)(z) = 0\}$.

In the case, $d\tau$ is complete on $\{z||z| = r\}$, we shall prove by reduction to absurdity. Assume $d\tau$ is not complete on $\{z||z| = r\}$. There exists $\gamma : [0, 1) \to A_2$, where $\gamma(1) \in \{z||z| = r\}$ so that $|\gamma| < \infty$. Furthermore, we may also assume $dist(\gamma(0), \{z||z| = 1/r\}) > 2|\gamma|$. Consider a small disk Δ with center at $\gamma(0)$. Since $d\tau$ is flat, Δ is isometric to an ordinary disk in the plane. Let $\Phi : \{|w| < \eta\} \to \Delta$ be the isometry.

Extend Φ as a local isometry into A_2, to the largest disk $\{|w| < R\} = \Delta_R$. Then $R \leq |\gamma|$. The reason that Φ cannot be extended to a larger disk is that the image goes to the outside boundary $\{z||z| = r\}$ of A_2. More precisely, there exists a point w_0 with $|w_0| = R$ so that $\Phi(\overline{0, w_0}) = \Gamma_0$ is a divergent curve on A.

The map $\Phi(w)$ is locally biholomorphic, and the metric on Δ_R induced from ds^2 through Φ is given by

$$\Phi^* ds^2 = |h_\circ \Phi|^2 ||g^1_\circ \Phi||^2 ||g^2_\circ \Phi||^2 |\frac{dz}{dw}|^2 |dw|^2 \ (3.4.8).$$

On the other hand, Φ is isometric, we have

$$|dw| = |d\tau| = \left(|h| \frac{\Pi_{j=1}^{q_1} |G_j^1|^{(1-\frac{1}{m_{1j}}-\delta)p_1} \Pi_{j=1}^{q_2} |G_j^2|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W(g_0^1, g_1^1)|^{p_1} |W(g_0^2, g_1^2)|^{p_2}}\right)^{\frac{1}{1-p_1-p_2}} |dz|$$

$$\Rightarrow |\frac{dw}{dz}|^{1-p_1-p_2} = |h| \frac{\Pi_{j=1}^{q_1} |G_j^1|^{(1-\frac{1}{m_{1j}}-\delta)p_1} \Pi_{j=1}^{q_2} |G_j^2|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W(g_0^1, g_1^1)|^{p_1} |W(g_0^2, g_1^2)|^{p_2}}.$$

For each $l = 1, 2$, we set $f^l := g^l(\Phi)$, $f_0^l := g_0^l(\Phi)$, $f_1^l := g_1^l(\Phi)$ and $F_j^l := G_j^l(\Phi)$. Since

$$W(f_0^l, f_1^l) = (W(g_0^l, g_1^l)_\circ \Phi) \frac{dz}{dw}, (l = 1, 2),$$

we obtain

$$|\frac{dz}{dw}| = \frac{\Pi_{l=1,2} |W(f_0^l, f_1^l)|^{p_l}}{|h(\Phi)| \Pi_{l=1,2} \Pi_{j=1}^{q_l} |F_j^l|^{(1-\frac{1}{m_{lj}}-\delta)p_l}} \ (3.4.9).$$

By (3.4.8) and (3.4.9), we get

$$\Phi^* ds^2 = \left(\Pi_{l=1,2} \frac{||f^l|| (|W(f_0^l, f_1^l)|)^{p_l}}{\Pi_{j=1}^{q_l} |F_j^l|^{(1-\frac{1}{m_{lj}}-\delta)p_l}}\right)^2 |dw|^2$$

$$= \Pi_{l=1,2} \left(\frac{||f^l||^{q_l - 2 - \sum_{j=1}^{q_l} \frac{1}{m_{lj}} - q_l \delta} |W(f_0^l, f_1^l)|}{\Pi_{j=1}^{q} |F_j^l|^{1-\frac{1}{m_{lj}}-\delta}}\right)^{2p_l} |dw|^2.$$

Using the Lemma 3.3.5, we obtain

$$\Phi^* ds^2 \leqslant C_0^{2(p_1+p_2)} . (\frac{2R}{R^2 - |w|^2})^{2(p_1+p_2)} |dw|^2.$$

Since $0 < p_1 + p_2 < 1$ by (3.4.7), it then follows that

$$d_{\Gamma_0} \leqslant \int_{\Gamma_0} ds = \int_{\overline{0, w_0}} \Phi^* ds \leqslant C_0^{p_1+p_2} . \int_0^R (\frac{2R}{R^2 - |w|^2})^{p_1+p_2} |dw| < +\infty,$$

where d_{Γ_0} denotes the distance of the divergent curve Γ_0 in M, contradicting the assumption of completeness of M. Claim 2 is proved.

Define $d\tilde{\tau}^2 = \lambda^2(z)|dz|^2$ on A_2, where

$$\lambda(z) = \left(|h(z)|\frac{\Pi_{j=1}^{q_1}|G_j^1(z)|^{(1-\frac{1}{m_{1j}}-\delta)p_1}\Pi_{j=1}^{q_2}|G_j^2(z)|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W(g_0^1,g_1^1)(z)|^{p_1}|W(g_0^2,g_1^2)(z)|^{p_2}}\right)^{\frac{1}{1-p_1-p_2}}$$
$$\times\left(|h(1/z)|\frac{\Pi_{j=1}^{q_1}|G_j^1(1/z)|^{(1-\frac{1}{m_{1j}}-\delta)p_1}\Pi_{j=1}^{q_2}|G_j^2(1/z)|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W(g_0^1,g_1^1)(1/z)|^{p_1}|W(g_0^2,g_1^2)(1/z)|^{p_2}}\right)^{\frac{1}{1-p_1-p_2}}.$$

By Claim 2, $d\tilde{\tau}$ is complete and flat on A_2.

We now use the same arguments as the latter part of the proof of Theorem 3.4.6. This implies Theorem 3.4.7(i).

We finally consider the case where $g^2 \equiv constant$ and $g^1 \not\equiv constant$. Suppose that $\gamma_1 > 3$. We can choose δ with

$$\frac{\gamma_1 - 3}{q_1} > \delta > \frac{\gamma_1 - 3}{q_1 + 1},$$

and set $p = 1/(\gamma_1 - 2 - q_1\delta)$. Then

$$0 < p < 1, \ \frac{p}{1-p} > \frac{\delta p}{1-p} > 1.$$

Set

$$d\tau^2 = |h|^{\frac{2}{1-p}}\left(\frac{\Pi_{j=1}^{q_1}|G_j^1|^{1-\frac{1}{m_{1j}}-\delta}}{|W(g_0^1,g_1^1)|}\right)^{\frac{2p}{1-p}}|dz|^2.$$

By exactly the same arguments as in the proof of Theorem 3.4.6, we get Theorem 3.4.7(ii). □

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