Université de Lorraine
UFR M.I.M
École Doctorale IAEM Lorraine
D.F.D. Mathématiques

Thèse

présentée pour l'obtention du titre de

Docteur en Mathématiques Appliquées

de l'Université de Lorraine

par

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Quelques équations d'évolution non-linéaires de type hyperbolique-parabolique : existence et étude qualitative

Soutenue le vendredi 22 juin 2012 devant le jury composé de

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Remerciements

Je tiens tout d'abord à témoigner ma vive reconnaissance au Professeur Ralph CHILL pour avoir dirigé mes travaux de recherche avec beaucoup d'attention et de patience, je salue ses grandes qualités pédagogiques et humaines.

Je voudrais exprimer ma profonde gratitude aux Professeurs Mohamed Ali JENDOUBI et Zakaria BELHACHMI d'avoir accepté d'être rapporteurs de ma thèse et membres du jury. Leurs commentaires et suggestions sur la version préliminaire m'ont été très précieux.

Je remercie vivement les Professeurs Alain HARAUX, Marius TUCSNAK et Fatiha ALABAU-BOUSSOUIRA qui m'ont fait l'honneur de participer à mon jury.

Un grand merci à tous mes collègues doctorants et à tout le personnel administratif et technique de notre laboratoire LMAM.

Enfin, je pense à ma mère Nada, mon père Ali, mes frères, mes soeurs. Leur amour, affection et soutien sont au-dessus de tous les remerciements.

Résumé

L'objectif principal de cette thèse concerne l'étude du comportement asymptotique des solutions globales de quelques équations, et systèmes couplés des équations, d'évolution non linéaires avec différents types d'amortissements et des conditions sur le bord. Sous la condition basique que la non linéarité est analytique, on prouve que les énergies associées vérifient des inégalités de type Łojasiewicz et on obtient des résultats de convergence avec l'estimation de la vitesse de convergence. Pour tous les modèles étudiés dans cette thèse, on s'intéresse aux questions d'existence et d'unicité des solutions bornées à images relativement compactes dans leurs espaces d'énergie naturelles. Cette thèse est constituée de trois parties principales.

Dans la première partie on prouve un résultat de convergence général avec l'estimation du taux de décroissance des solutions bornées d'une équation d'évolution abstraite non autonome avec dissipation linéaire. Le résultat permet de retrouver et généraliser de manière naturelle des résultats connus, mais aussi il s'applique à une classe très générale des équations et des systèmes couplés avec divers types de couplage et avec diverses conditions sur le bord.

La deuxième partie est consacrée à l'étude des équations du second ordre avec dissipation non linéaire et des conditions dynamiques classiques sur le bord. On prouve l'existence et l'unicité des solutions globales bornées à images relativement compactes et on montre la convergence vers l'équilibre.

Finalement, on s'intéresse à des équations d'évolution dégénérée de type hyperbolique-parabolique avec des conditions dynamiques de type mémoire sur le bord. On prouve l'existence et l'unicité des solutions globales bornées à images relativement compactes et on prouve la convergence avec l'estimation de la vitesse de convergence.

Le premier chapitre de cette thèse consiste en une introduction préliminaire développant non seulement l'histoire des recherches reliées à nos modèles et leurs résultats décrits dans la littérature, mais aussi en présentant les énoncés de nos résultats obtenus avec les idées des démonstrations. On y discute la complexité de la problématique et l'on y présente la justification de l'étude.

Abstract

The main goal of this thesis is the study of the asymptotic behavior of global solutions to some nonlinear evolutions equations and coupled systems with different types of dissipation and boundary conditions. Under the assumption that the nonlinear term is real analytic, we construct an appropriate Lyapunov energy and we use the Łojasiewicz-Simon inequality to show the convergence, and the convergence rate, of global weak solutions to single steady states. Far all models studied in this thesis, we are in addition interested in the questions of the existence and uniqueness of global bounded solutions having relatively compact range in the natural energy space. This thesis consists of three main parts.

In the first part, we present a unified approach to study the asymptotic behavior and the decay rate to a steady state of bounded weak solutions for an abstract non autonomous nonlinear equation with linear dissipation. This result allows us to find and to generalize, in a natural way, known results but it applies to a quite general class of equations and coupled systems with different kinds of coupling and various boundary conditions.

The second part is devoted to the study of a nonautonomous semilinear second order equation with nonlinear dissipation and a dynamical boundary condition. We prove the existence and uniqueness of global, bounded, weak solutions having relatively compact range in the natural energy space and we show that every weak solution converges to an equilibrium.

Finally, we consider a nonautonomous, semilinear, hyperbolic-parabolic equation subject to a dynamical boundary condition of memory type. We prove the existence and uniqueness of global bounded solutions having relatively compact range and we show the convergence of global weak solutions to single steady states. We prove also an estimate for the convergence rate.

The first chapter of this thesis consist of a preliminary introduction developing not only the story of researches linked to our models and the results described in the literature, but presenting also our main results as well the ideas of their proofs. There we discuss the complexity of our problems and we present a justification for our studies.

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Chapitre 1

Introduction générale

L'évolution au cours du temps de plusieurs modèles appliqués à la physique, la biologie, la chimie, l'ingénierie peut être reformulée en langage mathématique en utilisant les équations aux dérivées partielles. Dans le traitement de ces équations l'on aborde les questions d'existence, d'unicité et de régularité de leurs solutions. Cependant, une grande part de la compréhension de ces équations vient de leur étude qualitative. L'étude du comportement asymptotique lorsque $t \to \infty$ est un problème essentiellement global dont la résolution, en pratique, est à la fois difficile et majeure.

L'étude du comportement au cours du temps des solutions des équations d'évolution non linéaires dissipatives a suscité l'intérêt de beaucoup de mathématiciens depuis longtemps. La recherche dans ce domaine a été axée principalement sur deux aspects :
L'un porte sur le comportement asymptotique des familles de solutions globales pour des données initiales dans un ensemble borné dans un espace de Sobolev afin de trouver un ensemble compact invariant qui absorbe ces solutions : c'est un attracteur. Nous nous référons aux trois ouvrages : Temam [67], Hale [34], et Babin et Vishik [10] pour une étude approfondie de ce sujet. Pour l'équation des ondes semi-linéaires, nous nous référons aux travaux de Chueshov et Lasiecka [19]-[24] avec dissipation seulement sur le bord.
L'autre aspect est l'étude de la convergence vers un équilibre de solutions globales bornées lorsque le temps passe à l'infini : c'est le sujet principal de cette thèse.
Pour motiver notre travail, nous présentons les modèles originels en nous appuyant sur la revue de la littérature.
Soit Ω un ouvert borné régulier de $\mathbb{R}^N$ $(N \geq 1)$ de frontière Γ, considérons l'équation de la chaleur

$$u_t - \Delta u + f(x,u) = 0, \tag{1.1}$$

et l'équation des ondes avec dissipation linéaire

$$u_{tt} + u_t - \Delta u + f(x,u) = 0, \tag{1.2}$$

avec des données initiales et l'une des conditions classiques suivantes sur le bord :

$$u = 0 \quad \text{sur } \Gamma \text{ (Dirichlet)}, \tag{1.3}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{sur } \Gamma \text{ (Neumann)}, \tag{1.4}$$

$$u + \frac{\partial u}{\partial n} = 0 \quad \text{sur } \Gamma \text{ (Robin)}, \tag{1.5}$$

$$u_t + u + \frac{\partial u}{\partial n} = 0 \quad \text{sur } \Gamma \text{ (dynamique)}, \tag{1.6}$$

et $f : \Omega \times \mathbb{R} \to \mathbb{R}$ une fonction non linéaire.
Une question essentielle dans l'approche qualitative est la suivante :
étant donnée une solution globale bornée de l'équation (1.1) ou (1.2), y a-t-il convergence vers une solution stationnaire ?
Rappelons qu'une solution stationnaire de l'équation (1.1) ou (1.2) est une fonction ϕ, solution de l'équation suivante :

$$\begin{cases} -\Delta\phi + f(x,\phi) = 0 & x \in \Omega, \\ T(\phi) = 0 & x \in \partial\Omega, \end{cases} \tag{1.7}$$

avec

$$\begin{cases} T(\phi) = \phi & \text{si la condition sur le bord est de type (1.3)}, \\ T(\phi) = \frac{\partial \phi}{\partial n} & \text{si la condition est de type (1.4)}, \\ T(\phi) = \phi + \frac{\partial \phi}{\partial n} & \text{si la condition est de type (1.5) ou (1.6)}. \end{cases}$$

Notre question devient alors : peut-on trouver une fonction ϕ, solution de (1.7) pour laquelle :

$$\|u(t) - \phi\|_{H^1(\Omega)} \to 0, \quad \text{quand} \quad t \to \infty. \tag{1.8}$$

Une autre question importante reliée à la première : une fois que la convergence a eu lieu, peut-on estimer la vitesse de convergence ?
Il existe plusieurs facteurs qui interfèrent : la dimension du domaine Ω, la régularité de f, le type de condition sur le bord, la régularité de la solution, et l'ordre (en temps) de l'équation.

Notons que les équations (1.1) et (1.2) peuvent être réécrites sous la forme des équations de type gradient suivantes :

$$u_t + E'(u(t)) = 0, \quad t \geq 0, \tag{1.9}$$

et

$$u_{tt} + u_t + E'(u(t)) = 0, \quad t \geq 0, \tag{1.10}$$

avec $E'(u)$ le gradient de l'énergie fonctionnelle $E : H^1(\Omega) \to \mathbb{R}$ donnée par

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \int_\Omega F(x,u) \, dx, \;\; F(x,u) = \int_0^u f(x,s) \, ds, \tag{1.11}$$

si la condition sur le bord est de type (1.3) ou (1.4), et

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_\Gamma |u|^2 \, dx + \int_\Omega F(x,u) \, dx,$$

si la condition sur le bord est de type (1.5) ou (1.6). Les solutions stationnaires de l'équation (1.1) ou (1.2) sont alors des fonctions $\phi \in H^1(\Omega)$ telles que $E'(\phi) = 0$.

Remarquons qu'une solution à image relativement compacte converge dans le sens de (1.8) si et seulement si son ensemble ω-limite $\omega(u)$ défini par

$$\omega(u) = \{\psi \in H^1(\Omega) : \exists\ t_n \to +\infty \text{ tel que } \lim_{n\to\infty} \|u(t_n) - \psi\|_{H^1(\Omega)} = 0\}$$

est réduit à un point.
Par le principe d'invariance de La Salle, si u a une image relativement compacte dans $H^1(\Omega)$, alors son ensemble ω-limite est non vide, compact, connexe, et il ne contient que des solutions stationnaires $\phi \in H^1(\Omega)$, [41]. Ainsi, la réponse à la première question est positive si l'ensemble des solutions stationnaires est discret. Mais ce n'est pas toujours le cas si l'ensemble des points stationnaires est continu. En effet, dans [63], J. Palis et W. De Melo ont donné un exemple d'une fonction $F : \mathbb{R}^2 \to \mathbb{R}$ de classe C^∞ pour laquelle il existe une trajectoire globale et bornée du système gradient suivant :

$$\begin{cases} \dot{u}(t) + \nabla F(u(t)) = 0, \quad t \geq 0, \\ u(0) = u_0 \in \mathbb{R}^2, \end{cases} \tag{1.12}$$

qui ne converge pas.
D'autre part, un contre-exemple a été donné par P. Polácik et Rybakowski [64] et ensuite par P. Polácik et F. Simondon [65] : ils ont décrit une non linéarité f de classe C^∞ pour laquelle il existe une solution globale, bornée de l'équation de la chaleur, avec la condition de Dirichlet sur le bord, et l'ensemble ω-limite de la solution est difféomorphe au cercle unitaire S^1. (voir aussi [5] et [48], où la convergence n'a pas lieu).

Si $\Omega = (a, b) \subset \mathbb{R}$ est un intervalle, alors la convergence vers un point d'équilibre est obtenue sous des hypothèses générales sur f (f est de classe C^1 ou Lipschitz) ; voir T. J. Zelenyak [73] et H. Matano [61] pour l'équation de la chaleur avec condition de Dirichlet sur le bord.
J. Hale et G. Raugel [43] ont montré la convergence des solutions globales et bornées lorsque f est de classe C^1 et quand la dimension du noyau de l'opérateur linéarisé en un point d'équilibre est inférieure ou égale à 1. Cette hypothèse est vérifiée en dimension 1, mais n'est pas toujours satisfaite en dimension supérieure.

Dans le cas d'une dimension supérieure à 1, quelques hypothèses ont été posées sur f dans le but de prouver la convergence des solutions globales et bornées.
Dans la littérature, on décrit les hypothèses suivantes :

$$f(\cdot, s) \text{ est monotone,} \tag{1.13}$$

$$c_1|s|^\beta \leq c_2 F(\cdot, s) \leq s f(\cdot, s), \;\; s \in \mathbb{R}, \;\; c_i > 0, \;\; \beta \geq 2 \tag{1.14}$$

$$f(\cdot, s) \text{ est analytique.} \tag{1.15}$$

Sous les hypothèses (1.13) et (1.14), la stabilisation de (1.1) et (1.2) a été largement étudiée, avec diverses conditions sur le bord. Les résultats de nombreux auteurs sont basés sur les lemmes (inégalités intégrales) dûs à A. Haraux [35, 36], V. Komornik [50] et P. Martinez [60], la méthode des multiplicateurs donnée par Komornik et Zuazua [51], et la méthode de Nakao (inégalité différentielle) [62].
Notons que si f est strictement monotone, alors E' est strictement monotone, ainsi l'ensemble des points d'équilibre est réduit à un seul point. Alors, la convergence des solutions bornées est obtenue directement par La Salle.
Remarquons aussi que la monotonie de E' est équivalente à la convexité de E.
Un exemple type d'une fonction vérifiant les hypothèses (1.13) et (1.14) est donné par

$$f(s) = |s|^{\alpha}s, \;\; s \in \mathbb{R}, \;\; \alpha \in \mathbb{R}^{+}. \tag{1.16}$$

L'hypothèse d'analyticité a été introduite pour la première fois dans l'étude du comportement asymptotique des solutions des équations d'évolution par Simon [66]. Il a montré la convergence des solutions globales et bornées de l'équation de la chaleur (1.1) avec la condition de Dirichlet et la non linéarité analytique, en remarquant que cette équation est réécrite sous la forme (1.9).
L'outil clé utilisé dans ce résultat de convergence donné par Simon est une inégalité du gradient reliant l'énergie E, donnée par (1.11), à son gradient autour d'un point critique.

Théorème 1.0.1 (Simon [66], Théorème 3, p. 537)**.** *Soit $\phi \in C^{2,p}(\bar{\Omega}),\ p \in (0,1)$ une solution classique du problème stationnaire suivant :*

$$\begin{cases} \Delta\phi = f(x,\phi) & \textit{dans } \Omega, \\ \phi = 0 & \textit{sur } \Gamma. \end{cases}$$

Supposons que f vérifie les hypothèses suivantes :

$$\begin{cases} f \textit{ est analytique par rapport à } s, \textit{uniformément par rapport à } x, \\ f, \frac{\partial f}{\partial s}, \textit{ et } \frac{\partial^2 f}{\partial^2 s} \textit{ sont bornées sur } \;\Omega \times [-r,r], \;\forall r > 0. \end{cases}$$

Alors il existe $\theta \in (0, \frac{1}{2}]$ et $\sigma > 0$ tels que pour tout $\psi \in C^{2,p}(\bar{\Omega})$,

$$\|\psi - \phi\|_{C^{2,p}(\Omega)} < \sigma \Longrightarrow | \; E(\psi) - E(\phi) \; |^{1-\theta} \leq \|\Delta\psi - f(\cdot,\psi)\|_{L^2(\Omega)}. \tag{1.17}$$

En prouvant l'inégalité (1.17) Simon a utilisé le résultat fondamental dû à S. Łojasiewicz [57, 58] (théorème de Łojasiewicz) qui affirme : si $\Gamma : \mathbb{R}^N \to \mathbb{R}$ est une fonction réelle analytique dans un voisinage d'un point critique donné $a \in \mathbb{R}^N$ (i.e., $\nabla\Gamma(a) = 0$), alors il existe des constantes $\theta \in (0, \frac{1}{2}]$, $\sigma > 0$ telles que

$$| \; \Gamma(x) - \Gamma(a) \; |^{1-\theta} \leq \|\nabla\Gamma(x)\|, \quad \forall x \in \mathbb{R}^N, \; |x-a| \leq \sigma. \tag{1.18}$$

Les constantes θ et σ dépendent de la fonction Γ et du point a. Il existe des exemples de fonctions non analytiques qui vérifient cette inégalité, [44] et [15]. Łojasiewicz a utilisé l'inégalité (1.18) pour prouver la convergence des solutions bornées du système gradient du premier ordre (1.12).

Le travail de L. Simon ci-dessus est publié en 1983. Ce travail à été repris et précisé par A. Haraux et M. A. Jendoubi qui (spécialement Jendoubi [46, 47]) ont simplifié la méthode apportant la preuve originale de Simon. En effet, dans [47] Jendoubi a reformulé la procédure de démonstration de Simon pour prouver le même résultat de convergence des solutions bornées de l'équation de la chaleur avec des conditions de Dirichlet sur le bord.
De plus, par construction d'une nouvelle fonction de Lyapunov [46], Jendoubi a généralisé ce résultat pour l'équation des ondes (1.2) avec des conditions de Dirichlet sur le bord.

La démonstration donnée par Jendoubi semble être plus naturelle parce qu'elle est basée sur la construction de fonctions de Lyapunov et d'inégalités différentielles qui sont des outils classiques pour ce genre de problèmes. De plus, la preuve de Jendoubi est applicable pour d'autres problèmes de type gradient, de dimension finie ou infinie.

En prouvant les résultats de convergence, Jendoubi a donné une généralisation de l'inégalité (1.17) applicable dans l'espace d'énergie $H^1(\Omega)\times L^2(\Omega)$. Cette nouvelle inégalité s'appelera l'inégalité de Łojasiewicz-Simon.

Théorème 1.0.2 (Jendoubi [47]). *Soit $\phi \in W^{2,p}(\Omega)$, $p \geq \frac{N}{2}$, $p > 2$ telle que*

$$\begin{cases} \Delta\phi = f(x,\phi) & \text{dans } \Omega, \\ \phi = 0 & \text{sur } \Gamma. \end{cases}$$

Supposons que f vérifie les hypothèses du théorème 1.0.1. Alors il existe $\theta \in (0, \frac{1}{2}]$ et $\sigma > 0$ tels que pour tout $\psi \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$,

$$\|\psi - \phi\|_{W^{2,p}(\Omega)} < \sigma \Longrightarrow | E(\psi) - E(\phi) |^{1-\theta} \leq \|\Delta\psi - f(\cdot,\psi)\|_{L^2(\Omega)}. \tag{1.19}$$

L'originalité de ces résultats provient du fait qu'aucune condition de croissance sur f n'était imposée. Mais ces résultats étaient uniquement valables pour les solutions fortes. Ils sont alors en quelque sorte restrictifs, en pratique difficilement vérifiables.

En ajoutant une condition de croissance sur f et en partant de la preuve du théorème de Simon [66], A. Haraux et M. A. Jendoubi [38] ont donné une généralisation de l'inégalité (1.19) applicable pour prouver la convergence des solutions faibles.

Théorème 1.0.3 (Haraux et Jendoubi [38]). *Soit $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ telle que*

$$\begin{cases} \Delta\phi = f(x,\phi) & \text{dans } \Omega, \\ \phi = 0 & \text{sur } \Gamma. \end{cases}$$

Supposons que f vérifie les hypothèses suivantes :
(F1) : $f(x,s)$ est analytique par rapport à s, uniformément par rapport à x.

(F2) :

$$\begin{cases} \textit{Si } N=1: \ f(x,s) \textit{ et } \dfrac{\partial f}{\partial s}(x,s) \textit{ sont bornées sur } \Omega\times[-r,r], \ \forall r>0, \\ \textit{Si } N\geq 2: f(\cdot,0)\in L^\infty(\Omega) \textit{ et } |\dfrac{\partial f}{\partial s}(x,s)| \leq \rho(1+|s|^\mu), \ (x,s)\in\Omega\times\mathbb{R}, \\ \textit{où } \rho\geq 0 \textit{ et } \mu\geq 0, (N-2)\mu<2. \end{cases}$$

Alors il existe $\theta\in(0,\frac{1}{2}]$ et $\sigma>0$ tels que pour tout $\psi\in H_0^1(\Omega)$,

$$\|\psi-\phi\|_{H^1(\Omega)}<\sigma \Longrightarrow | E(\psi)-E(\phi) |^{1-\theta} \leq \|\Delta\psi - f(\cdot,\psi)\|_{H^{-1}(\Omega)}. \tag{1.20}$$

Se reposant sur cette nouvelle inégalité, Haraux et Jendoubi ont prouvé les mêmes résultats de convergence pour des solutions bornées dans l'espace d'énergie de l'équation des ondes [38] et l'estimation de la vitesse de convergence en fonction de la valeur de θ [39].
Un exemple type d'une énergie fonctionelle E vérifiant cette inégalité dans un espace de Hilbert $V \hookrightarrow L^2(\Omega)$, avec une injection compacte, est donné par :

$$E(u)=\frac{1}{2}a(u,u)+\int_\Omega F(u)\,dx, \ u\in V,$$

où $a: V\times V\to\mathbb{R}$ est une forme bilinéaire, continue, symétrique, coercive et la fonction F est une primitive d'une non-linéarité analytique f vérifiant certaine condition de croissance. En général, il est très difficile de vérifier cette inégalité pour des fonctions non analytiques. Notons aussi que l'hypothèse d'analyticité est suffisante mais non nécessaire. Nous nous référons à [15, 44] pour la compréhension de cette inégalité.

Nous décrivons brièvement la preuve de Jendoubi pour l'équation de la chaleur et l'équation des ondes avec des conditions de Dirichlet sur le bord.

La méthodologie

Soit $u\in C(\mathbb{R}^+; H^1(\Omega))$ une solution bornée de l'équation (1.1) à image relativement compacte, c'est à dire l'ensemble $\mathcal{O}(u)=\{u(t); t\geq 0\}$ est inclus dans un sous ensemble compact de $H^1(\Omega)$. Le but est de prouver la convergence de u dans $H^1(\Omega)$ vers un point stationnaire ϕ, $E'(\phi)=0$. Pour cela, il suffit de prouver que $\|u(t)-\phi\|_{L^2(\Omega)}\to 0$. Cela est justifié par le fait que cette dernière convergence combinée à la compacité est suffisante pour prouver que l'ensemble $\omega(u)$ est réduit à un seul point ϕ.
Une condition impliquant la convergence de u dans $L^2(\Omega)$ est la suivante :

$$I(u)=\int_0^\infty \|u_t\|_{L^2(\Omega)}\,dt<\infty. \tag{1.21}$$

La méthode de Jendoubi sert à prouver (1.21) en suivant les étapes suivantes.

Premièrement, il est clair que l'énergie E donnée par (1.11) est une fonction de Lyapunov pour l'équation (1.1). En multipliant (1.1) par u_t on obtient

$$\frac{d}{dt}E(u(t)) \leq -\|u_t\|^2_{L^2(\Omega)}. \tag{1.22}$$

Comme dans le principe d'invariance de La Salle, on déduit facilement de cette dernière inégalité les résultats intermédiaires suivants :
(i) $u_t \in L^2(\mathbb{R}^+; L^2(\Omega))$,
(ii) E est constante sur $\omega(u)$, et $\omega(u)$ est inclus dans l'ensemble des points stationnaires,
(iii) $\lim\limits_{t\to\infty} E(u(t)) = E(\phi) = E_\infty < \infty$ pour tout $\phi \in \omega(u)$.

D'autre part, par la compacité et la continuité de u, il existe $\phi \in \omega(u)$ et pour tout $\sigma > 0$ il existe un intervalle de temps (t_1, t_2) dans lequel $\|u(t) - \phi\|_{H^1(\Omega)} \leq \sigma$. Donc on peut appliquer l'inégalité (1.20) pour la fonction $W(t) = E(u(t)) - E(\phi)$ et obtenir :

$$|W(t)|^{1-\theta} \leq \|E'(u(t))\|_{H^{-1}(\Omega)} = \|u_t(t)\|_{H^{-1}(\Omega)}, \quad \text{pour tout } t \in (t_1, t_2).$$

Notons que $W(t)$ (par (1.22) et (iii)) est décroissante et $\lim\limits_{t\to\infty} W(t) = 0$. On distingue alors deux cas :
s'il existe $t_0 \geq 0$ telle que $W(t_0) = 0$, alors $W(t) = 0$ pour tout $t \geq t_0$. Ce qui nous donne la convergence en utilisant (1.22).
Dans le deuxième cas, $W(t) > 0$ pour tout $t > 0$, et on peut dériver $W(t)^\theta$:

$$-\frac{d}{dt}W(t)^\theta = -\theta W(t)^{\theta-1}\frac{d}{dt}W(t) \geq \frac{C_\theta \|u_t(t)\|^2_{L^2(\Omega)}}{\|u_t(t)\|_{L^2(\Omega)}} \geq C_\theta \|u_t(t)\|_{L^2(\Omega)}.$$

On déduit alors en intégrant sur (t_1, t_2) que

$$\int_{t_1}^{t_2} \|u_t(t)\|_{L^2(\Omega)}\, dt \leq C W(t_1)^\theta.$$

Cette dernière inégalité permet de contrôler $\sup\limits_{(t_1,t_2)} \|u(t,\cdot) - \phi\|_{H^1(\Omega)}$ et prouver qu'une fois que la solution u rentre dans un petit voisinage de ϕ, elle y reste pour toujours. On peut alors faire tendre t_2 vers l'infini en obtenant (1.21).

Pour l'équation des ondes, il est clair que la l'énergie $K(t) = \frac{1}{2}\|u_t\|^2_{L^2(\Omega)} + E(u)$ (E est donnée par (1.11)) est une fonction de Lyapunov :

$$\frac{d}{dt}K(t) \leq -\|u_t\|^2_{L^2(\Omega)}. \tag{1.23}$$

Ce qui nous donne facilement :
$i)$ $u_t \in L^2(\mathbb{R}^+, L^2(\Omega))$ et $u_t(t) \longrightarrow 0$ dans $L^2(\Omega)$,
$ii)$ E est constante sur $\omega(u)$, et $\omega(u)$ est inclus dans l'ensemble des points stationnaires,

iii) $\lim_{t\to\infty} E(u(t)) = E(\phi) = E_\infty < \infty$ pour tout $\phi \in \omega(u)$.
Mais, l'estimation d'énergie obtenue en (1.23) n'est pas suffisante pour appliquer la même technique pour cette équation. En fait, l'existence du terme $-\|E'(u)\|^2_{H^{-1}(\Omega)}$ dans l'estimation d'énergie (1.23) est essentielle pour faire le calcul, (pour l'équation de la chaleur, l'existence de ce terme est assurée dans l'estimation (1.22) car $\|E'(u)\|_{L^2(\Omega)} = \|u_t\|_{L^2(\Omega)}$). L'exigence de ce terme est reliée à l'existence de $\|E'(u)\|_{H^{-1}(\Omega)}$ donnée par l'inégalité de Łojasiewicz-Simon.
Pour cela Jendoubi a modifié cette dernière énergie K en ajoutant un terme supplémentaire :

$$G(t) = \frac{1}{2}\|u_t\|_{L^2(\Omega)} + E(u) + \varepsilon(-\Delta u + f(x,u), u_t)_{H^{-1}(\Omega)},$$

où $\varepsilon > 0$ est assez petit. On prouve facilement alors

$$\frac{d}{dt}G(t) \leq -C\big(\|u_t\|^2_{L^2(\Omega)} + \|E'(u)\|^2_{H^{-1}(\Omega)}\big). \tag{1.24}$$

La suite de la démonstration est identique à celle de l'équation du premier ordre avec $W(t) = G(t) - E_\infty$. On a

$$|W(t)|^{1-\theta} \leq C\big(\|u_t\|_{L^2(\Omega)} + \|E'(u(t))\|_{H^{-1}(\Omega)}\big), \quad \text{pour tout } t \in (t_1, t_2). \tag{1.25}$$

On distingue les deux cas, puis on combine (1.24) et (1.25) pour obtenir :

$$-\frac{d}{dt}W(t)^\theta \geq C\big(\|u_t\|_{L^2(\Omega)} + \|E'(u)\|_{H^{-1}(\Omega)}\big).$$

Il reste à intégrer cette dernière inégalité dans l'intervalle (t_1, t_2) et faire tendre t_2 vers l'infini pour obtenir la convergence.
En utilisant la même technique, S. Z. Huang et P. Takàc [42] ont généralisé le résultat de convergence de l'équation (1.1) dans le cas non autonome. Puis, R. Chill et M. A. Jendoubi [16] ont prouvé un résultat de convergence général pour les deux systèmes abstraits suivants :

$$\begin{cases} \dot{u}(t) + M(u(t)) = g(t), \ t \geq 0, \\ u(0) = u_0, \ u_0 \in H, \end{cases} \tag{1.26}$$

$$\begin{cases} \ddot{u}(t) + \dot{u}(t) + M(u(t)) = g(t), \ t \geq 0, \\ u(0) = u_0, \ \dot{u}(t) = u_1, \ (u_0, u_1) \in V \times H, \end{cases} \tag{1.27}$$

où V et H sont deux espaces de Hilbert, $\mathcal{V} \hookrightarrow \mathcal{H}$ avec une injection continue. La fonction M est la première dérivée d'une fonction $E \in C^1(V, \mathbb{R})$ qui vérifie l'inégalité de Łojasiewicz-Simon dans un point de $\omega(u)$ et $g \in L^2(\mathbb{R}^+, H)$ est telle qu'il existe $\delta > 0$ tel que

$$\sup_{t\in\mathbb{R}_+}(1+t)^{1+\delta}\int_t^\infty \|g(s)\|^2_{\mathcal{H}}\, ds < \infty. \tag{1.28}$$

Chill et Jendoubi [16] ont appliqué ces résultats abstraits aux équations des ondes et de la chaleur non autonome avec des conditions de Dirichlet sur le bord.

Ce travail de Chill et Jendoubi est complété par I. Ben Hassen [13]. En admettant les résultats de convergence dans [16], Ben Hassen a estimé la vitesse de convergence des solutions des (1.26) et (1.27).

Cette technique basée sur l'inégalité de Łojasiewicz-Simon a été appliquée pour prouver des résultats de convergence des solutions bornées de plusieurs équations, et systèmes couplés des équations, semi-linéaires de type gradient, telles que l'équation de Cahn-Hilliard (Cahn-Hilliard equation) [18, 45, 70], systèmes de champ de phase (phase field systems) [2, 3, 49, 72], équations de diffusion dégénérées (degenerate diffusion equations) [30], équations différentielles ordinaires du second ordre (second order ODEs) [40], équations intégrales associées à des problèmes d'évolution (evolutionary integral equations) [3, 2, 17], équations d'évolution non-autonomes (non-autonomous evolutionary equations) [42, 16, 13], équations d'évolution avec dissipation non linéaire (evolutionary equations with nonlinear dissipation) [25, 26, 11, 12].

Cependant, dans la plupart des travaux cités ci-dessus, les équations d'évolution traitées sont soumises à des conditions de type Dirichlet ou de type Neumann sur le bord (le travail de Chill et Jendoubi peut être appliqué pour des équations avec la condition de Robin). Néanmoins, pour des conditions complexes sur le bord (comme les conditions dynamiques qui sont très importantes du point de vue physique et mathématique) les résultats sont limités. À notre connaissance, les seuls résultats positifs sont donnés par H. Wu et S. Zheng [71, 70] pour l'équation des ondes et l'équation de Cahn-Hilliard et par H. Wu et M. Grasselli [72] pour un système couplé de type parabolique-hyperbolique (sous la condition dynamique classique (1.6)). Notons que pour ces travaux les résultats de convergence ont été prouvés pour des solutions bien régulières et uniquement dans le cas autonome.
Aussi, pour des équations mixtes, parabolique-hyperbolique (1.37), il n'y a pas de résultat abouti. De plus, des résultats manquent pour des systèmes couplés, avec plusieurs sortes de couplage et des conditions sur le bord. Ces questions sont l'une des motivations (voir Remarque 1.0.3 pour des arguments supplémentaires et applications) pour étudier une équation abstraite générale enveloppant la majorité des travaux précédemment cités et répondant à notre problématique.

Chapitre 2

La première partie de ce travail (chapitre 2) porte sur la généralisation des résultats [16] et [13] sur une équation abstraite contenant une classe plus large des équations et des systèmes couplés ; en particulier, des équations de type mixte (1.37), des conditions dynamiques sur le bord, des systèmes couplés ondes-ondes, ondes-chaleur, chaleur-chaleur, avec différents types de couplage et des conditions complexes sur le bord.
Plus précisément, on considère trois espaces de Hilbert $\mathcal{V}$, $\mathcal{W}$ et $\mathcal{H}$ tels que $\mathcal{V} \subset \mathcal{W} \subset \mathcal{H}$, avec des injections denses et continues. On identifie $\mathcal{H}$ avec son espace dual

$\mathcal{H}'$ de façon à ce que :

$$\mathcal{V} \hookrightarrow \mathcal{W} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{W}' \hookrightarrow \mathcal{V}',$$

avec des injections denses et continues.
On étudie le comportement asymptotique des solutions bornées de l'équation d'évolution abstraite suivante :

$$(A\dot{u})^{\cdot} + B\dot{u} + M(u) = g, \quad t \in \mathbb{R}^{+}. \tag{1.29}$$

Ici, $M = E'$ est la dérivée d'une fonction $E \in C^2(\mathcal{V})$, $A : \mathcal{H} \to \mathcal{H}$ est un opérateur auto-adjoint positif, $B : \mathcal{W} \to \mathcal{W}'$ est un opérateur linéaire borné qui vérifie la condition de coercivité suivante :

$$(Bu, u)_{\mathcal{W}',\mathcal{W}} \geq \varrho \|u\|^2_{\mathcal{W}}, \quad u \in \mathcal{W}, \tag{1.30}$$

pour un $\varrho > 0$, et $g \in L^2(\mathbb{R}^+, \mathcal{H})$ vérifiant (1.28).
Selon le choix de l'opérateur A, l'équation (1.29) contient en particulier des équations d'évolution du premier ordre ($A = 0$), du second ordre (par exemple, $A = I_{\mathcal{H}}$), et aussi d'ordre mixte (par exemple, A est une projection). Notre résultat de convergence généralise, unifie et étend des résultats existants dans la littérature.

Définition 1.0.1. *Une fonction $u : \mathbb{R}^+ \to \mathcal{V}$ est une solution (faible) de l'équation* (1.29) *si*

$$u \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}) \cap H^1_{loc}(\mathbb{R}^+, \mathcal{W}),$$
$$A\dot{u} \in H^1_{loc}(\mathbb{R}^+, \mathcal{V}'),$$

et si u vérifie l'équation différentielle (1.29) *dans $\mathcal{V}'$, pour presque tout $t \in \mathbb{R}^+$.*

On montre les résultats suivants :

Théorème 1.0.4. *Soit $u : \mathbb{R}^+ \to \mathcal{V}$ une solution faible de l'équation* (1.29) *et supposons que :*
(H1) $u \in H^1_{loc}(\mathbb{R}^+, \mathcal{V})$ et $A\dot{u} \in H^1_{loc}(\mathbb{R}^+, \mathcal{H})$.
(H2) L'ensemble $\{(u(t), A^{\frac{1}{2}}\dot{u}(t)) : t \geq 1\}$ est relativement compact dans $\mathcal{V} \times \mathcal{H}$.
(H3) Il existe $\phi \in \omega(u)$ tel que E vérifie l'inégalité de Łojasiewicz-Simon en ϕ d'exposant θ.
(H4) Si $K : \mathcal{V}' \to \mathcal{V}$ est l'opérateur de dualité, alors, pour tout $v \in \mathcal{V}$, l'opérateur $K \circ M'(v) \in \mathcal{L}(\mathcal{V})$ se prolonge en un opérateur linéaire borné sur $\mathcal{H}$ et $K \circ M' : \mathcal{V} \to \mathcal{L}(\mathcal{H})$ transforme les ensembles bornés en ensembles bornés.
(H5) g vérifie (1.28) *pour une constante $\delta > 0$.*
Alors :

$$\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}} + \|u(t) - \phi\|_{\mathcal{V}} \underset{t\to\infty}{\longrightarrow} 0,$$

et il existe une constante $C' > 0$ telle que pour tout $t \geq 0$ on a :

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'(1+t)^{-\eta}, \text{ où } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\}.$$

De plus, si $g = 0$ et $\theta = \frac{1}{2}$, alors il existe deux constantes C'', $\xi > 0$ telles que pour tout $t \geq 0$ on a :

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'' e^{-\xi t}.$$

Corollaire 1.0.1 (cas où $A = 0$)**.** *Soit $u : \mathbb{R}^+ \to \mathcal{V}$ une solution de l'équation suivante :*

$$B\dot{u} + M(u) = g, \tag{1.31}$$

où $\mathcal{V}$, $\mathcal{W}$, B, M et g sont définis comme dans le théorème 1.0.4. Supposons que :
(H1) $u \in W^{1,2}_{loc}(\mathbb{R}^+, \mathcal{V})$.
(H2) L'ensemble $\{u(t) : t \geq 1\}$ est relativement compact dans $\mathcal{V}$.
(H3) Il existe $\phi \in \omega(u)$ telle que E vérifie l'inégalité de Łojasiewicz-Simon en ϕ d'exposant θ.
Alors, $u(t) \to \phi$ dans $\mathcal{V}$ et il existe une constante $C' > 0$ telle que pour tout $t \geq 0$ on a :

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'(1+t)^{-\eta}, \text{ où } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\}.$$

De plus, si $g = 0$ et $\theta = \frac{1}{2}$, alors il existe deux constantes C'', $\xi > 0$ telles que pour tout $t \geq 0$ on a :

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'' e^{-\xi t}.$$

Pour prouver le Théorème 1.0.4, on applique la même idée que celle décrite dans la méthodologie (avec quelques changements de la démarche concernant le cas non-autonome) en construisant l'énergie perturbée suivante :

$$G(t) = \frac{1}{2}\|A^{\frac{1}{2}}\dot{u}(t)\|^2_{\mathcal{H}} + E(u(t)) + \varepsilon(M(u(t)), A\dot{u}(t))_{\mathcal{V}'}, \tag{1.32}$$

où $\varepsilon > 0$ est assez petit.
On montre que G est une fonctionnelle de Lyapunov pour l'équation (1.29) par l'inégalité suivante :

$$\frac{d}{dt}G(t) \leq -C_1\big(\|\dot{u}\|^2_{\mathcal{W}}+\|M(u)\|^2_{\mathcal{V}'}\big) + C_2\|g\|^2_{\mathcal{H}}, \tag{1.33}$$

où C_1 et C_2 sont deux constantes positives.
Remarquons l'existence du terme positif dans l'inégalité (1.33). Pour traiter cette difficulté on utilise le lemme suivant prouvé dans [30] et utilisé dans [42].

Lemme 1.0.1 (Feireisl et Simondon [30])**.** *Soit $Z \geq 0$ une fonction mesurable dans $\mathbb{R}^+$ telle que*

$$Z \in L^2(\mathbb{R}^+), \qquad \|Z\|_{L^2(\mathbb{R}^+)} \leq Y.$$

Soit $\mathcal{D} \subseteq \mathbb{R}^+$ un ensemble ouvert, $\alpha \in (1,2)$, et $w > 0$ telles que :

$$\Big(\int_t^\infty Z(s)^2\, ds\Big)^\alpha \leq wZ^2(t) \text{ pour tout } t \in \mathcal{D}.$$

Alors, $Z \in L^1(\mathcal{D})$ et il existe une constante positive $c = c(\alpha, w, Y)$ indépendante de $\mathcal{D}$ telle que :

$$\int_{\mathcal{D}} Z(s)\, ds \le c.$$

On applique ce lemme en utilisant l'inégalité de Łojasievicz-Simon et en choisissant $Z = \|\dot{u}\|_{\mathcal{W}} + \|M(u)\|_{\mathcal{V}'}$, et on obtient que $\|\dot{u}\|_{\mathcal{W}}$ est intégrable dans $\mathbb{R}^+$. Cette intégrabilité aboutit à l'existence de la limite de $u(t)$ dans $\mathcal{W}$ et, par compacité, dans $\mathcal{V}$.
Pour estimer la vitesse de convergence entre la solution et sa limite on utilise l'approche des fonctionnelles de Lyapunov. Une fois la fonctionnelle d'énergie adéquate choisie, on cherche à obtenir des inégalités différentielles. L'idée directrice pour y parvenir consiste à appliquer l'inégalité de Łojasiewicz-Simon.
Le résultat de décroissance polynômiale est basé sur le lemme suivant dû à Ben Hassen [13].

Lemme 1.0.2. *Soit $\zeta \in W^{1,1}_{loc}(\mathbb{R}^+; \mathbb{R}^+)$. On suppose qu'il existe des constantes $K_1 > 0,\ K_2 \ge 0,\ k > 1$ et $\lambda > 0$ telles que pour tout $t \ge 0$ on a :*

$$\zeta'(t) + K_1 \zeta(t)^k \le K_2 (1+t)^{-\lambda}.$$

Alors, il existe une constante positive m telle que :

$$\zeta(t) \le m(1+t)^{-\nu},\ \text{où } \nu = \inf\{\frac{1}{k-1}, \frac{\lambda}{k}\}.$$

L'énergie choisie pour prouver la convergence polynômiale et exponentielle est la suivante :

$$\zeta(t) = G(t) - E(\phi) + C_2 \int_t^{\infty} \|g(s)\|^2_{\mathcal{H}} ds,$$

où ϕ est la limite de la solution $u(t)$ et G est donné par (1.32). Utilisons l'inégalité (1.33), on obtient

$$\frac{d}{dt}\zeta(t) = \frac{d}{dt}G(t) - C_2\|g(t)\|^2_{\mathcal{H}} \le -C_1\big(\|\dot{u}(t)\|^2_{\mathcal{W}} + \|M(u(t))\|^2_{\mathcal{V}'}\big). \tag{1.34}$$

À partir de cette estimation, de l'hypothèse (1.28), et à l'aide de l'inégalité de Łojasiewicz-Simon, on prouve l'inégalité différentielle suivante :

$$C_3 \frac{d}{dt}\zeta(t) + \zeta(t)^{2(1-\theta)} \le C_4 (1+t)^{-2(1-\theta)(1+\delta)},$$

où C_3 et C_4 sont deux constantes positives.
On applique alors le Lemme 1.0.2, et on obtient la convergence polynômiale.
Dans le cas où $g = 0$ et $\theta = \frac{1}{2}$ on dérive $\zeta^{\frac{1}{2}}(t)$, on utilise l'inégalité (1.34) et l'inégalité de Łojasiewicz-Simon, on obtient l'inégalité différentielle suivante :

$$v'(t) \le -Cv(t),\ C > 0,\ t \ge T,$$

avec

$$v(t) = \int_t^{\infty} (\|\dot{u}(t)\|_{\mathcal{W}} + \|M(u(t))\|_{\mathcal{V}'}) \, ds.$$

La résolution de cette inégalité différentielle permet l'obtention de la décroissance exponentielle de v. Puis, on remarque que :

$$\|u(t) - \phi\|_{\mathcal{W}} \leq \lim_{t' \to \infty} \|u(t) - u(t')\|_{\mathcal{W}} \leq v(t), t \geq T.$$

on obtient alors le résultat de décroissance exponentielle de la solution u.

Remarque 1.0.1. *Notons que le Théorème 1.0.4 reste vrai si l'hypothèse* $(H1)$ *est remplacée par l'hypothèse, plus faible, suivante :*
(H1') $u \in C(\mathbb{R}^+, \mathcal{V}) \cap H^1_{loc}(\mathbb{R}^+, \mathcal{H})$, $A\dot{u} \in H^1_{loc}(\mathbb{R}^+, \mathcal{V}')$ *et, pour des constantes* C_1, $C_2 > 0$, *on a l'estimation* (1.33), *où la fonction* G *est donnée par* (1.32). *Cette remarque est importante dans certaines applications où l'inégalité* (1.33) *peut être verifiée pour des solutions faibles* (u *est seulement différentiable à valeurs dans* $\mathcal{H}$) *par approximation et par arguments de densités.*

Applications

On commence par la remarque suivante concernant les questions d'existence et de précompacité des solutions faibles pour les applications données ci-dessus.

Remarque 1.0.2. *Pour toutes les applications données ci-dessous, les méthodes utilisées dans les chapitres 3 et 4 sont applicables pour prouver :*

1. *l'existence et l'unicité des solutions faibles globales,*
2. *les solutions faibles sont des limites des solutions fortes,*
3. *toute solution faible bornée a une image relativement compacte dans l'espace d'énergie naturelle.*

La première application du Théorème 1.0.4 est l'étude du comportement asymptotique de l'équation des ondes semi-linéaires avec des conditions dynamiques sur le bord. Soit $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ un ensemble ouvert borné ayant une frontière régulière Γ. On considère l'équation suivante :

$$\begin{cases} u_{tt} + u_t - \Delta u + f(x,u) = g_1 & \text{dans } \mathbb{R}^+ \times \Omega, \\ b(x)u_t + \partial_\nu u + a(x)u = g_2 & \text{sur } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \;\; u_t(0) = u_1. \end{cases} \tag{1.35}$$

avec des données $a \in W^{1,\infty}(\Gamma)$, $b \in L^\infty(\Gamma)$, $b(x) \geq b_0 > 0$, $g_1 \in L^2(\mathbb{R}^+ \times \Omega)$ et $g_2 \in L^2(\mathbb{R}^+ \times \Gamma)$. La non-linéarité f vérifie les conditions $(F1)$ et $(F2)$ du Théorème 1.0.3.
On prouve que cette équation est un cas particulier de l'équation (1.29). En effet, soient

$$\mathcal{H} = \mathcal{W} = L^2(\Omega) \times L^2(\Gamma),$$

et

$$\mathcal{V} = \{\boldsymbol{u} = (u, v) \in \mathcal{H};\ u \in H^1(\Omega),\ v = \ {}^t u = \ \text{trace } u\}.$$

On munit $\mathcal{H}$ et $\mathcal{V}$ des produits scalaires usuels et on montre que $\mathcal{V}$ s'injecte dans $\mathcal{H}$, avec une injection dense, compacte et on définit les opérateurs A et B dans $\mathcal{H}$ par :

$$A(u, v) = (u, 0) \text{ et } B(u, v) = (u, bv), \quad (u, v) \in \mathcal{H}.$$

On définit la fonction d'énergie $E : \mathcal{V} \to \mathbb{R}$ par

$$E(\boldsymbol{u}) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega F(x, u)\, dx + \frac{1}{2}\int_\Gamma a(x)|{}^t u|^2\, d\sigma.$$

Avec ce choix des espaces fonctionnels, des opérateurs et de l'énergie E on prouve que si $u \in H^1(\Omega)$ est une solution de l'équation (1.35), alors la fonction $\boldsymbol{u} = (u,\ {}^t u) \in \mathcal{V}$ est une solution de l'équation abstraite (1.29). Ensuite, on montre que l'énergie E est de classe C^2 dans $\mathcal{V}$ et qu'elle vérifie l'inégalité de Łojasiewicz-Simon pour tout point stationnaire $\phi \in \mathcal{V}$. Par conséquent, le Théorème 1.0.4 est appliqué pour montrer la convergence avec l'estimation de la vitesse de convergence des solutions bornées de l'équation (1.35).
Un résultat similaire est obtenu pour l'équation de chaleur avec des conditions dynamiques sur le bord :

$$\begin{cases} u_t - \Delta u + f(x, u) = g_1 & \text{dans } \mathbb{R}^+ \times \Omega, \\ bu_t + \partial_\nu u + au = g_2 & \text{sur } \ \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases} \tag{1.36}$$

Ici, $g = (g_1, g_1)$, f, $a(x)$, $b(x)$, B, E, $\mathcal{V}$ et $\mathcal{H} = \mathcal{W}$ sont définies comme dans le premier exemple, et $A = 0$. En utilisant les mêmes arguments que dans la première application, le résultat de convergence découle du Corollaire 1.0.1.
On peut utiliser le Théorème 1.0.4 pour prouver la convergence des solutions bornées de l'équation mixte de type hyperbolique-parabolique suivante :

$$\begin{cases} K_1(x)u_{tt} + K_2(x)u_t - \Delta u + f(x, u) = g & \text{dans } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{sur } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0,\ \sqrt{K_1}u_t(0) = \sqrt{K_1}u_1, \end{cases} \tag{1.37}$$

avec $K_1(x),\ K_2(x) \in L^\infty(\Omega)$, $K_1(x) \geq 0$, $K_2(x) \geq \beta > 0$ et f est définie comme dans les exemples précédents. Pour écrire l'équation (1.37) sous la forme abstraite (1.29), soient $\mathcal{H} = \mathcal{W} = L^2(\Omega)$ et $\mathcal{V} = H_0^1(\Omega)$, on définit les opérateurs A et B dans $\mathcal{H}$ par :

$$(Au)(x) = K_1(x)u(x), \quad (Bu)(x) = K_2(x)u(x),$$

et on définit l'énergie $E : H_0^1(\Omega) \to \mathbb{R}$ par :

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \int_\Omega F(x, v)\, dx, \quad v \in H_0^1(\Omega).$$

L'énergie E vérifie l'inégalité de Łojasiewicz-Simon pour tout point critique $\phi \in \mathcal{V}$ (Théorème 1.0.3). On peut alors appliquer le Théorème 1.0.4 pour obtenir la convergence des solutions bornées de l'équation (1.37).
Dans le même cadre des équations de type mixte on peut appliquer notre théorème pour l'équation suivante :

$$\begin{cases} K_1(x)u_{tt} + c_1 u_t - c_2 \Delta u_t - \Delta u + f(x,u) = g & \text{dans } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{sur } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \ \sqrt{K_1} u_t(0) = \sqrt{K_1} u_1. \end{cases} \tag{1.38}$$

Ici $c_1, \ c_2 \geq 0, \ c_1 + c_2 > 0$ et $g, \ K_1, f$ sont définies comme dans l'exemple précédent.
Soient $\mathcal{V} = H_0^1(\Omega)$, $\mathcal{H} = L^2(\Omega)$ et soit $\mathcal{W} = \mathcal{V}$ si $c_2 > 0$ et $\mathcal{W} = \mathcal{H}$ si $c_2 = 0$. L'équation (1.38) peut être réécrite sous la forme abstraite (1.29) si l'on définit l'opérateur A et l'énergie E comme dans l'exemple précédent et si l'on définit l'opérateur $B = c_1 I_{\mathcal{V}} - c_2 \Delta : \mathcal{W} \to \mathcal{W}'$, où Δ est l'opérateur de Laplace avec conditions limites de Dirichlet.
Comme dernier exemple on étudie le système couplé suivant :
Soit $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ un ensemble ouvert borné ayant une frontière régulière Γ. On suppose que $\Gamma_0, \Gamma_1 \subseteq \Gamma$ est une partition de Γ (le cas $\Gamma_1 = \varnothing$ n'est pas exclu). On considère le système :

$$\begin{cases} \alpha_1 u_{tt} + u_t - \Delta u + \dfrac{\partial f}{\partial u}(x,u,v) = g_1 & \text{dans } \mathbb{R}^+ \times \Omega, \\ \alpha_2 v_{tt} + v_t - \Delta v + \dfrac{\partial f}{\partial v}(x,u,v) = g_2 & \text{dans } \mathbb{R}^+ \times \Omega, \\ b u_t + \frac{\partial u}{\partial n} + a u = g_3 & \text{sur } \mathbb{R}^+ \times \Gamma_0, \\ u = 0 & \text{sur } \mathbb{R}^+ \times \Gamma_1, \\ v = 0 & \text{sur } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{1.39}$$

Ici, $\alpha_i \geq 0$, $i = 1, 2$, $(g_1, g_2, g_3) \in L^2(\mathbb{R}^+ \times \Omega)^2 \times L^2(\mathbb{R}^+ \times \Gamma_0)$. En plus, $f = f(x,u,v) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ est un fonction de classe C^2 vérifiant les hypothèses suivantes :
(F1) f est analytique par rapport à $(u,v) \in \mathbb{R}^2$, uniformément par rapport à $x \in \Omega$ et (u,v) dans un ensemble borné de $\mathbb{R}^2$.
(F2) Il existe des constantes $\rho > 0$, $\mu \geq 0$, $(N-2)\mu < 2$ telles que

$$\begin{cases} \text{Si } N = 1 : \dfrac{\partial f}{\partial u}(x,u,v), \ \dfrac{\partial f}{\partial v}(x,u,v), \ \text{et } \nabla^2_{u,v} f(x,u,v) \text{ sont bornées sur } \Omega \times [-r,r]^2, \\ \forall r > 0. \\ \text{Si } N \geq 2 : (\dfrac{\partial f}{\partial u}(\cdot,0,0), \dfrac{\partial f}{\partial v}(\cdot,0,0)) \in (L^\infty(\Omega))^2 \text{ et pour tout } (x,u,v) \in \Omega \times \mathbb{R}^2, \\ \qquad |\nabla^2_{u,v} f(x,u,v)| \leq \rho(1 + |u|^\mu + |v|^\mu), \end{cases}$$

où $\nabla^2_{u,v} f(x,u,v)$ est la dérivée seconde de f par rapport à u et v.
Pour réécrire le système (1.39) dans le cas abstrait, on choisit :

$$\mathcal{H} = \mathcal{W} = (L^2(\Omega))^2 \times L^2(\Gamma_0)$$

et on définit l'espace d'énergie $\mathcal{V}$ par :

$$\mathcal{V} = \{\boldsymbol{u} = (u_1, u_2, u_3) \in \mathcal{H} \ ; \ u_1 \in H^1_{0,\Gamma_1}(\Omega), u_2 \in H^1_0(\Omega) \text{ et } u_3 = {}^t u_1\},$$

où $H^1_{0,\Gamma_1} = \{u \in H^1(\Omega) \ ; \ {}^t u = 0 \text{ on } \Gamma_1\}$. On prouve que $\mathcal{V} \hookrightarrow \mathcal{H}$, avec une injection dense et compacte.
L'énergie fonctionnelle $E : \mathcal{V} \longrightarrow \mathbb{R}$ est définie par :

$$E(\boldsymbol{u}) = \frac{1}{2}\int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2)\, dx + \int_\Omega f(x, u_1, u_2)\, dx + \frac{1}{2}\int_{\Gamma_0} a|u_1|^2\, d\sigma.$$

Les opérateurs A et B sont définis dans $\mathcal{H}$ par :

$$A(u, v, w) = (\alpha_1 u, \alpha_2 v, 0) \text{ et } B(u, v, w) = (u, v, bw), \quad \text{pour tout } (u, v, w) \in \mathcal{H}.$$

Similairement à la première application, si (u, v) est une solution faible de (1.39) alors $\boldsymbol{u} = (u, v, \ {}^t u)$ est une solution faible (1.29), avec $M(\boldsymbol{u}) = E'(\boldsymbol{u})$ et $g = (g_1, g_2, g_3)$. Sous les hypotèses (**F1**) et (**F2**), on montre que l'énergie E est de classe C^2 dans $\mathcal{V}$ et qu'elle vérifie l'inégalité de Łojasiewicz-Simon pour tout point sationnaire $\phi \in \mathcal{V}$. Par conséquent le Théorème 1.0.4 est appliqué pour montrer la convergence avec l'estimation de la vitesse de convergence des solutions bornées du système (1.39).

Dans le chapitre 2, nous nous sommes limités à l'énoncé et à la démonstration des exemples les plus typiques. Les remarques ci-dessous précisent les généralisations éventuelles et les conditions d'applicabilité du Théorème 1.0.4.

Remarque 1.0.3. *1. Le Théorème 1.0.4 peut être généralisé à d'autres opérateurs elliptiques à coefficients réguliers avec non linéarité analytique.*

2. Concernant les systèmes couplés ; le même résultat de convergence reste vrai si la deuxième équation dans le système (1.39) *est remplacée par :*

$$\alpha_2 v_{tt} + v_t + \alpha u_t - \Delta v + \frac{\partial f}{\partial v}(x, u, v) = g_2 \quad \textit{dans} \ \ \mathbb{R}^+ \times \Omega, \ 0 \le \alpha < 2$$

Dans ce cas, il suffit de choisir l'opérateur $B : \mathcal{H} \to \mathcal{H}$ tel que :

$$B(u, v, w) = (u, v + \alpha u, bw), \ \textit{pour tout} \ (u, v, w) \in \mathcal{H}.$$

De plus, on peut choisir d'autres types de conditions sur le bord pour u et v, en particulier, v peut vérifier des conditions de type dynamique sur le bord. D'autre part, le Théorème 1.0.4 peut être appliqué à des systèmes couplés des équations de type mixte, soit :

$$\begin{cases} K_1(x)u_{tt} + K_2(x)u_t - \Delta u + \dfrac{\partial f}{\partial u}(x, u, v) = g_1 & \textit{dans } \mathbb{R}^+ \times \Omega, \\ K_3(x)v_{tt} + K_4(x)v_t - \Delta v + \dfrac{\partial f}{\partial v}(x, u, v) = g_2 & \textit{dans } \mathbb{R}^+ \times \Omega, \\ b(x)u_t + \frac{\partial u}{\partial n} + a(x)u = g_3 & \textit{sur } \mathbb{R}^+ \times \Gamma_0, \\ u = 0 & \textit{sur } \mathbb{R}^+ \times \Gamma_1, \\ b_1(x)v_t + \frac{\partial v}{\partial n} + a_1(x)v = g_4 & \textit{sur } \mathbb{R}^+ \times \Gamma, \end{cases}$$

où

$$\begin{cases} (K_1, K_2, K_3, K_4) \in (L^\infty(\Omega))^4, \ \ K_1, K_3 \geq 0, \ \ K_2, K_4 > 0, \\ (a, a_1, b, b_1) \in W^{1,\infty}(\Gamma_0) \times W^{1,\infty}(\Gamma) \times L^\infty(\Gamma_0) \times L^\infty(\Gamma), \ \ b, \ \ b_1 > 0. \end{cases}$$

Finalement, on peut appliquer le résultat pour des systèmes couplés de n équations ($n \geq 2$) (voir [49] avec $n = 3$).

3. *Concernant les conditions dynamiques sur le bord, notre résultat de convergence reste vrai pour les équations, ou les systèmes couplés, du second ordre avec le condition dynamique suivant :*

$$C(x)u_{tt} + u_t + \frac{\partial u}{\partial n} + a(x)u = g \quad sur \ \Gamma,$$

où $C \in L^\infty(\Gamma)$, $C(x) \geq 0$.
Dans [53] J.L. Lions a étudié l'équation des ondes linéaires sous cette condition sur le bord, avec $C = 1, g = 0$ (voir aussi [28], où $c = c(u)$ dépend de u et $a = g = 0$). Voir aussi [31, 32] pour les motivations physiques.
Aussi, le même résultat reste vrai pour la condition dynamique non-linéaire sur le bord :

$$u_t + \frac{\partial u}{\partial n} + h(x, u) = g,$$

avec une non linéarité analytique h qui vérifie des conditions de croissance similaires à f.

4. *Dans les exemples précédents on a décrit des équations pour lesquelles $\mathcal{W} = \mathcal{H}$ ou $\mathcal{W} = \mathcal{V}$. Considérons l'équation de plaque avec dissipation intermédiaire :*

$$\begin{cases} u_{tt} - \Delta^2 u - \Delta u_t - \Delta u + f(u) = g & dans \ \ \mathbb{R}^+ \times \Omega, \\ u = \Delta u = 0 & sur \ \ \mathbb{R}^+ \times \Gamma. \end{cases}$$

On arrive à une situation pour laquelle

$$\mathcal{H} = L^2(\Omega), \ \ \mathcal{W} = H_0^1(\Omega) \ \ et \ \mathcal{V} = H^2(\Omega) \cap H_0^1(\Omega).$$

Chapitre 3

Remarquons que pour toutes les équations du premier chapitre, la stabilisation des solutions se fait par des opérateurs de stabilisation linéaires. Dans les cas où l'amortissement est non linéaire, et pour ce type de non linéarité f analytique, il y a peu de résultats dans la littérature. Selon notre connaissance, les seuls résultats positifs dans ce contexte sont donnés par L. Chergui [25, 26], I. Ben Hassen et L. Chergui [11] et I. Ben Hassen et A. Haraux [12].
Dans [25], L. Chergui a prouvé un résultat de convergence avec l'estimation de la vitesse de convergence des solutions bornées du système (en dimension finie) suivant :

$$\ddot{U}(t) + \|\dot{U}(t)\|^\alpha \dot{U}(t) + \nabla F(U(t)) = 0, \ \ t \in \mathbb{R}^+, \tag{1.40}$$

avec des données initiales dans $\mathbb{R}^N$ ($N \geq 1$), $F : \mathbb{R}^N \to \mathbb{R}$ une fonction analytique et $\alpha > 0$ assez petite.
Dans [26], L. Chergui a illustré le même résultat de convergence dans le cadre de dimension infinie ; en fait, il a prouvé un résultat de convergence de l'équation des ondes avec la dissipation non linéaire suivante :

$$u_{tt} + |u_t|^\alpha u_t - \Delta u + f(x, u) = 0, \ (t, x) \in \mathbb{R}^+ \times \Omega, \tag{1.41}$$

avec des données initiales et des conditions de type Dirichlet homogènes sur le bord. Le résultat de convergence de L. Chergui a été généralisé par I. Ben Hassen et L. Chergui dans [11] pour le cas non-autonome.
Notons que pour l'équation (1.41), seul le résultat de convergence a été établi sans donner une estimation de la vitesse de convergence. C'est aussi le cas pour l'équation non-autonome [11]. Ce fût l'une des motivations pour I. Ben Hassen et A. Haraux [12] d'estimer la vitesse de convergence (avec des non linéarités f particulières) des solutions du système abstrait du second ordre suivant :

$$\ddot{u} + g(\dot{u}(t)) + M(u) = 0, \quad t \in \mathbb{R}^+,$$

où M est un opérateur de type gradient associé à une fonctionnelle positive et g est un opérateur non linéaire dissipatif, sous des conditions reliant l'exposant de Łojasiewicz de la fonctionelle et la valeur de la dissipation au voisinage de l'origine.

Le troisième chapitre de cette thèse traite de l'équation des ondes semilinéaire non-autonome avec dissipation non linéaire et des conditions dynamiques sur le bord. Plus précisément, on étudie l'équation suivante :

$$u_{tt} + |u_t|^\alpha u_t - \Delta u + f(x, u) = g, \tag{1.42}$$

soumise aux conditions suivantes sur le bord :

$$\partial_\nu u + u + u_t = 0, \tag{1.43}$$

et avec des données initiales :

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega. \tag{1.44}$$

Ici, f est définie comme dans les exemples précédents et on suppose qu'il existe deux constantes $\eta, \delta > 0$ telles que :

$$\|g(t)\|_2 \leq \frac{\eta}{(1 + t)^{1+\delta+\alpha}}. \tag{1.45}$$

Pour cette équation, on prouve d'abord l'existence des solutions globales et bornées, sous la condition supplémentaire portant sur la non linéarité f :
(*F3*) il existe $\lambda < \lambda_1$ et $C \geq 0$ telles que pour tout $s \in \mathbb{R}$ et tout $x \in \Omega$,

$$F(x, s) \geq -\lambda \frac{s^2}{2} - C,$$

où $\lambda_1 > 0$ est la constante de l'inégalité de type Poincaré suivante :

$$\int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |{}^t u|^2 \, d\sigma \geq \lambda_1 \int_\Omega |u|^2 \, dx, \ u \in H^1(\Omega) \text{ et } {}^t u = \text{trace } u.$$

Définition 1.0.2. *Soit* $J := [0,\tau)$ *où* $\tau \in (0,\infty]$. *Une fonction* $u \in C(J; H^2(\Omega)) \cap C^1(J; H^1(\Omega)) \cap C^2(J; L^2(\Omega))$ *est dite une* solution forte *de* (1.42)-(1.43), *si* u *vérifie les données initiales* $u(0) = u_0$, $u_t(0) = u_1$, *et si les équations* (1.42)-(1.43) *sont satisfaites p.p sur* J. *Une fonction* $u \in C(J; H^1(\Omega)) \cap C^1(J; L^2(\Omega))$ *est dite une* solution faible *de* (1.42)-(1.43), *si* u *vérifie les données initiales* $u(0) = u_0$, $u_t(0) = u_1$ *et s'il existe une suite* $(g^\mu) \subseteq H^1_{loc}(J; L^2(\Omega))$ *et une suite* (u^μ) *de solutions fortes correspondantes telles que* $g^\mu \to g$ *dans* $L^2_{loc}(J; L^2(\Omega))$ *et* $u^\mu \to u$ *dans* $C(J; H^1(\Omega)) \cap C^1(J; L^2(\Omega))$.

En appliquant le théorie du semi-groupe non-linéaire et en utilisant une idée de Chueshov, Eller et Lasiecka [24] on obtient le premier résultat du Chapitre 3.

Théorème 1.0.5. *Soit* $0 \le \alpha \le \frac{2}{N-2}$ *si* $N \ge 3$, *et* $\alpha \in \mathbb{R}^+$ *si* $N \le 2$. *Supposons que* f *vérifie les conditions (F2) et (F3)*

(I) **Solution faible :** *Soient* $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ *et soit* $g \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$. *Alors, le système* (1.42)-(1.44) *admet une unique solution faible globale. De plus, cette solution vérifie :*

(T1) $u_t \in L^{\alpha+2}(\mathbb{R}^+; L^{\alpha+2}(\Omega))$ *et* ${}^t u_t \in L^2(\mathbb{R}^+; L^2(\Gamma))$.

(T2) (u, u_t) *est bornée dans* $H^1(\Omega) \times L^2(\Omega)$.

(T3) (Inégalité d'énergie) pour tout $t, t' \in \mathbb{R}^+$, $t' \le t$ *:*

$$\mathcal{E}_u(t) + \frac{\alpha+1}{\alpha+2} \int_{t'}^{t} \int_\Omega |u_t|^{\alpha+2}\, dxds + \int_{t'}^{t} \int_\Gamma |{}^t u_t|^2\, d\sigma ds \le \mathcal{E}_u(t') + \frac{\alpha+1}{\alpha+2} \int_{t'}^{t} \|g\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}}\, ds. \tag{1.46}$$

où $\mathcal{E}_u$ *est l'énergie de la solution* u *définie par :*

$$\mathcal{E}_u(t) = \frac{1}{2} \int_\Omega |u_t|^2\, dx + \frac{1}{2} \int_\Omega |\nabla u|^2\, dx + \int_\Omega F(x,u)\, dx + \frac{1}{2} \int_\Gamma |{}^t u|^2\, d\sigma. \tag{1.47}$$

(T4) La formule variationnelle suivante est vraie pour toute $\phi \in H^1(\Omega)$ *:*

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u_t \phi\, dx + \int_\Omega \nabla u \nabla \phi\, dx + \int_\Omega |u_t|^\alpha u_t \phi\, dx + \int_\Omega f(x,u) \phi\, dx \\ &\quad + \int_{\partial\Omega} {}^t u_t\, {}^t \phi\, d\sigma + \int_{\partial\Omega} {}^t u\, {}^t \phi\, d\sigma = \int_\Omega g \phi\, dx. \end{aligned} \tag{1.48}$$

(II) **Solution forte :** *Supposons, de plus, que* $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$, $g \in H^1_{loc}(\mathbb{R}^+, L^2(\Omega))$ *et que les conditions de compatibilité suivantes sont vraies :*

$$u_0 + \partial_\nu u_0 + u_1 = 0, \quad \Gamma.$$

Alors la solution faible est forte.

De plus, on prouve que chaque solution bornée a une image relativement compacte dans l'espace d'énergie naturelle.

Théorème 1.0.6. *Soit α défini similairement au théorème 1.0.5. Alors pour toute solution faible u de* (1.42)-(1.44)*, la fonction $U = (u, u_t)$ est uniformément continue de $\mathbb{R}^+$ à valeurs dans $H^1(\Omega) \times L^2(\Omega)$, et $\bigcup_{t\geq 0}\{U(t)\}$ est relativement compacte dans $H^1(\Omega) \times L^2(\Omega)$.*

D'autre part, et comme mentionné dans le paragraphe méthodologie, les résultats intermédiaires suivants doivent être prouvés. Soit E la fonction donnée par :

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_\Gamma |u|^2 \, d\sigma + \int_\Omega F(x,u) \, dx.$$

Ainsi, on prouve :

Lemme 1.0.3. *Soit $u : \mathbb{R}^+ \to H^1(\Omega)$ une solution faible bornée du système* (1.42)-(1.44)*. Supposons que f vérifie (F1), (F2) et que $\alpha \in [0,1)$. Alors :*
(i) $\lim\limits_{t\to\infty} \|u_t(t)\|_{L^2(\Omega)} = 0$.
(ii) La fonction E est constante sur $\omega(u)$, est $E(\phi) = E_\infty$ =constante, pour tout $\phi \in \omega(u)$.
(iii) L'ensemble ω-limite est un sous ensemble de l'ensemble des points stationnaires.

La preuve de ce lemme découle de l'inégalité d'énergie (1.46). De plus, pour prouver le résultat de convergence, on a besoin d'un exposant de Łojasiewicz uniforme pour tous les points critiques. En pratique, l'existence d'un tel exposant de Łojasiewicz uniforme revient à supposer que l'ensemble des points d'équilibre est compact . Une condition suffisante sur f, qui implique la compacité de cet ensemble est donnée par L. Chergui [26]. Dans ce cas (l'ensemble des points d'équilibre est compact), et comme l'ensemble des points d'équilibre attire la trajectoire à l'infini, on obtient la propriété suivante :
Il existe $\theta \in]0, \frac{1}{2}]$, $\beta > 0$ et $T > 0$ tels que pour tout $t \geq T$:

$$|E(u(t)) - E_\infty|^{1-\theta} \leq \beta \|E'(u(t))\|_*. \tag{1.49}$$

(avec $\|\cdot\|_* = \|\cdot\|_{H^{-1}(\Omega)}$). Finalement, à l'aide de la construction d'une nouvelle fonction de Lyapunov on prouve que si l'exposant de l'inégalité de Łojasiewicz est assez grand ou α est assez petite, alors la solution faible converge vers un équilibre.

Théorème 1.0.7. *Soit $u : \mathbb{R}^+ \to H^1(\Omega)$ une solution faible bornée du système* (1.42)-(1.44)*. Supposons que f vérifie (F1), (F2) et que*
- *si $N \leq 2$ alors $\alpha \in [0, \frac{\theta}{1-\theta}[$,*
- *si $N \geq 3$ alors $\alpha \in [0, \frac{\theta}{1-\theta}[\cap [0, \frac{4}{N-2}[$,*

où θ est donnée par (1.49)*. Alors il existe $\phi \in H^1(\Omega)$, solution du problème stationnaire suivant :*

$$\begin{cases} -\Delta\phi + f(x,\phi) = 0 & \textit{dans } \Omega, \\ \partial_\nu \phi + \phi = 0 & \textit{sur } \partial\Omega, \end{cases}$$

telle que

$$\|u_t(t)\|_2 + \| u(t) - \phi \|_{H^1(\Omega)} \longrightarrow 0 \quad \textit{quand } t \to \infty.$$

Pour prouver le théorème 1.0.7, on utilise la fonction de Lyapunov suivante :

$$\begin{aligned} G(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + E(u(t)) - E_\infty + \varepsilon\|u(t)\|_*^\alpha (E'(u(t)), u_t(t))_* + \\ & \int_t^\infty (g(s), u_t(s))_2 \, ds + \varepsilon(\alpha+1) \int_t^\infty \|u(s)\|_*^\alpha \|g(s)\|_*^2 \, ds, \end{aligned}$$

qui vérifie, pour un $\varepsilon > 0$ assez petit et pour $T > 0$ assez grand,

$$\frac{d}{dt}G(t) \leq -C\|u_t\|_*^\alpha \{\|u_t\|_2^2 + \|E'(u)\|_*^2\} - C\|^t u_t\|_{2,\Gamma}^2, \text{ pour tout } t > T. \tag{1.50}$$

Notons que le même résultat de convergence reste vrai pour les solutions bornées de l'équation (1.42) sous les conditions suivantes sur le bord

$$u + \frac{\partial u}{\partial n} + |u_t|^\rho u_t = h(t,x),$$

avec $\rho \geq 0$ et $h \in L^2(\mathbb{R}^+ \times \Gamma)$:

$$\|h(t)\|_{2,\Gamma} \leq \frac{\eta}{(1+t)^q}, \quad q = \frac{\rho+1}{\rho+2}[\alpha + 2 + \delta(\frac{\alpha+2}{\alpha+1})]. \tag{1.51}$$

Chapitre 4

Dans le quatrième chapitre de cette thèse on étudie une équation non autonome semi-linéaire de type mixte avec des conditions dynamiques de type mémoire sur le bord :

$$\begin{cases} K_1(x)u_{tt} + K_2(x)u_t - \Delta u + f(x,u) = g_1 & \text{dans } \mathbb{R}^+ \times \Omega, \\ \partial_\nu u + \mu(x)u + k * u_t = g_2 & \text{sur } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \ \sqrt{K_1}u_t(0) = \sqrt{K_1}u_1. \end{cases} \tag{1.52}$$

Ici, $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ est un ensemble ouvert borné connexe non vide et on suppose que sa frontière Γ est régulière , ν est le vecteur normal sur le bord, K_1, $K_2 \in L^\infty(\Omega)$ sont deux fonctions positives, $K_2(x) \geq k_0 > 0$, $\mu \in W^{1,\infty}(\Gamma)$ est une fonction positive non identiquement nulle sur Γ, $k \in L^1_{loc}(\mathbb{R}^+)$ est un noyau positif singulier vérifiant (*K1*) et (*K2*) (voir ci-dessous), et $k * v$ représente le produit de convolutions $(k * v)(t) = \int_0^t k(t-s)v(s)\, ds$, $t \geq 0$.

Pour cette équation, on prouve d'abord l'existence et l'unicité des solutions globales bornées à images relativement compactes dans l'espace d'énergie naturelle. Ainsi, par construction d'une nouvelle fonction de Lyaponov, et en utilisant l'inégalité de Łojasiewicz-Simon, on obtient la convergence des solutions bornées vers l'équilibre, quand g_1 et g_2 tendent vers 0 assez rapidement quand t tend vers l'infini. Finalement, on estime la vitesse de convergence (exponentielle et polynômiale) de la différence entre les solutions et leurs limites.

Concernant la question d'existence, on mentionne deux points.
Premièrement, on traite une équation dégénérée ($K_1 \geq 0$). Pour cela, on perturbe l'équation (1.52) en ajoutant le terme $\varepsilon u''$, $\varepsilon > 0$. Après cette perturbation, on obtient $K_1(x) + \varepsilon > 0$ comme un coefficient de u''. Alors, en utilisant la méthode de Faedo-Galerkin pour la nouvelle équation, on obtient des estimations a priori pour les solutions indépendantes de $\varepsilon > 0$, dans lesquelles on peut passer à la limite quand ε tend vers zéro, obtenant une fonction u qui est la solution recherchée.
Deuxièmement, pour bien clarifier la procédure d'approximation, on prouve aussi l'existence des solutions fortes. Pour cela, nous avons besoin de dériver l'équation approchée de (1.52) par rapport au temps t. Mais cette procédure nous amène à des difficultés techniques quand on va estimer le terme $K_1 u''(0)$ dans la norme $L^2(\Omega)$. Pour surmonter cette difficulté, on transforme le problème (1.52) en un problème équivalent avec des données initiales nulles.

Concernant les équations non linéaires avec terme mémoire, il existe des difficultés pour démontrer l'existence de solutions bornées à images relativement compactes dans leurs espaces fonctionnels. Cependant, la plus grande difficulté dans un tel problème consiste à trouver la fonction de Lyapunov appropriée pour prouver la convergence des solutions bornées.
Un autre facteur essentiel est la régularité du noyau dans le terme de convolution. Pour ce type de non linéarité f et de noyau singulier, il existe deux techniques pour construire une telle énergie qui s'accumule à l'inégalité de Łojasiewicz-Simon pour obtenir le résultat de convergence. L'une d'elle est basée sur l'idée de Dafermos [29]. Cette technique a été adaptée par R. Chill et E. Fašangová [17], afin d'obtenir un résultat de convergence pour l'équation des ondes, quand la dissipation est à la fois linéaire et de type mémoire. En fait, ils ont prouvé la convergence des solutions bornées à image relativement compacte de l'équation abstraite suivante :

$$\ddot{u} + \dot{u} + k * \dot{u} + M(u) = 0, \quad t \in \mathbb{R}^+. \tag{1.53}$$

Ici, $M = E'$ est la dérivée d'une fonction E de classe C^2 qui vérifie l'inégalité de Łojasiewicz-Simon. Le noyau $k \in L^1_{loc}(\mathbb{R}^+)$ est positif, convexe et vérifie l'hypothèse suivante :

$$dk'(s) + Ck'(s)\, ds \geq 0, \tag{1.54}$$

où dk' est la dérivée de k' au sens de distribution, et la convolution $k * v$ est définie par

$$k * v(t) = \int_0^t k(t-s)v(s)\, ds, \ t \geq 0. \tag{1.55}$$

Un exemple type des noyaux vérifiant l'hypothèse (1.54) est le noyau singulier donné par :

$$k(s) = s^{-\alpha} e^{-\beta s}, s \geq 0, \ \alpha \in [0,1), \ \beta > 0. \tag{1.56}$$

La même idée a été utilisée par S. Aizicovici et E.Feireisl [3] pour prouver un résultat de convergence d'une équation de champ de phase avec terme mémoire (phase-field model with memory) (voir aussi [2]).

L'autre méthode de construction d'une fonction de Lyapunov a été développée par R. Zacher et V. Vergara [68] pour prouver (en utilisant l'inégalité de Łojasiewicz) la convergence vers l'équilibre, en dimension finie, des solutions régulières bornées des problèmes d'ordre (en temps) plus petit que 1

$$\frac{d}{dt}[k * (u - u_0)](t) + \nabla\mathcal{E}(u) = f(t),\ u(0) = u_0,$$

et d'ordre entre 1 et 2

$$\frac{d}{dt}[k * (\dot{u} - u_1)](t) + \nabla\mathcal{E}(u) = f(t),\ u(0) = u_0,\ \dot{u}(0) = u_1,$$

où $\mathcal{E}$ est une fonction de classe C^2 qui vérifie l'inégalité de Łojasiewicz, avec une classe importante de noyaux singuliers. La construction de cette fonction de Lyapunov repose sur le lemme suivant :

Lemme 1.0.4. *Soient $\mathcal{H}$ un espace de Hilbert, $T > 0$, et $b \in L^1_{loc}(\mathbb{R}^+)$ une fonction positive et décroissante telle qu'il existe un noyau positif $k \in L^1_{loc}(\mathbb{R}^+)$ tel que $b * k = 1$ dans $(0, \infty)$. Supposons que $v \in L^2(0, T; \mathcal{H})$, $b * v \in H^1(0, T; \mathcal{H})$ et $b * \|v\|^2_{\mathcal{H}} \in W^{1,1}(0, T; \mathcal{H})$. Alors on a :*

$$(v(t), \frac{d}{dt}(b * v)(t))_{\mathcal{H}} \geq \frac{1}{2}\frac{d}{dt}(b * \|v\|^2_{\mathcal{H}})(t) + \frac{1}{2}b(t)\|v\|^2_{\mathcal{H}},\quad p.p \text{ sur } (0, T). \tag{1.57}$$

Remarque 1.0.4. *a) Sous les mêmes hypothèses sur le noyau b, l'inégalité* (1.57) *reste vraie pour tout $v \in H^1(0, T; \mathcal{H})$, [68, Remarque 2.1].*
b) Pour des noyaux plus réguliers, $b \in W^{1,1}(0, T)$, l'inégalité (1.57) *reste vraie pour tout $v \in L^2(0, T; \mathcal{H})$, [68, Lemme 2.2].*

Une classe importante de noyaux singuliers :
Zacher et Vergara ont considéré une classe de noyaux singuliers dont la forme est décrite dans les hypothèses suivantes :
(*K1*) $k \in L^1_{loc}(\mathbb{R}^+)$ est décroissant, positif tel qu'il existe un noyau positif décroissant $b \in L^1_{loc}(\mathbb{R}^+)$ tel que :

$$(b * k)(t) = 1.$$

(*K2*) Il existe $\gamma > 0$ et $a \in L^1(\mathbb{R}^+)$ décroissant et strictement positif tels que :

$$b(t) = a(t) + \gamma(1 * a)(t),\ t > 0.$$

Un exemple de tels noyaux est donné par :

$$\begin{cases} k(t) = g_{1-s}(t)e^{-wt}, & t > 0,\ s \in (0,1),\ w > 0, \\ b(t) = g_{1-s}(t)e^{-wt} + w[1 * (g_{1-s}e^{-w})(t)], & t > 0,\ s \in (0,1),\ w > 0, \end{cases} \tag{1.58}$$

où g_β désigne le noyau de Riemann-Liouville, i.e.

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \ t > 0, \ \beta > 0.$$

Pour ce type noyaux on remarque :
- ces noyaux explosent en $t = 0$,
- le noyau donné par (1.56) est un cas particulier,
- l'inégalité (1.57) reste vraie pour tout $v \in L^2(0, T; \mathcal{H})$, $b*v \in H^1(0, T; \mathcal{H})$ (sans avoir besoin de condition de régularité $b*\|v\|^2_{\mathcal{H}} \in W^{1,1}(0, T; \mathcal{H})$)[68, Exemple 2.1].

De plus, nos résultats restent vrais pour le cas des noyaux non singuliers. On peut notamment remplacer l'hypothèse (*K1*) par l'hypothèse (*K1'*) : il existe $b_0 \geq 0$ et un noyau positif et décroissant $b \in L^1_{loc}(\mathbb{R}^+)$ tels que $b_0 k(t) + (b * k)(t) = 1$ pour tout $t \geq 0$ (voir [75]). Nos résultats sont alors, en particulier, valides dans le cas non singulier $s = 0$ de l'exemple (1.58), c'est à dire pour le cas de noyaux k de la forme $k(t) = e^{wt}$, $t \geq 0$, $w > 0$.
Dans [75], Zacher a utilisé le Lemme 1.0.4 pour prouver qu'un amortissement de type mémoire est suffisant pour obtenir la convergence des solutions régulières, en dimension finie, de l'équation suivante :

$$\begin{cases} \ddot{u} + a * \dot{u} + \nabla \mathcal{E}(u) = f(t), \quad t \in \mathbb{R}^+, \\ u(0) = u_0, \ \dot{u}(0) = u_1 \end{cases}$$

où $\mathcal{E}(u) \in C^2(\mathbb{R}^N)$ vérifie l'inégalité de Łojasiewicz, le noyau a vérifie (*K1*), (*K2*) et l'hypothèse supplémentaire suivante :
(*K3*) Il existe $T > 0$ telle que $k \in H^1(T; \infty)$, et

$$\int_T^\infty \Big(\int_t^\infty k(s)^2 + k'(s)^2 ds \Big)^{\frac{1}{2}} \, dt < \infty.$$

Notons que le noyau k donné par (1.58) vérifie (*K3*).
Notre premier résultat du Chapitre 4 est le théorème d'existence suivant :

Théorème 1.0.8. *Supposons que f vérifie les hypothèses (F2), (F3) du Chapitre 3 et que le noyau k vérifie les hypothèses (K1), (K2).*

(I) **Solutions fortes :** *Soient $g_1 \in W^{1,2}_{loc}(\mathbb{R}^+, L^2(\Omega))$, et $g_2 \in L^2_{loc}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \cap W^{2,2}_{loc}(\mathbb{R}^+, L^2(\Gamma))$. On suppose que u_0, $u_1 \in H^2(\Omega)^2$ vérifient la condition de compatibilité suivante :*

$$\begin{cases} -\Delta u_0 + f(x, u_0) = g_1(0) - K_2 u_1 & \text{dans } \Omega, \\ \partial_\nu u_0 + \mu(x) u_0 = g_2(0) & \text{sur } \Gamma. \end{cases} \tag{1.59}$$

Alors, l'équation (1.52) *admet une unique solution forte u.*

(II) **Solutions faibles :** *Soit* $(g_1, g_2) \in L^2(\mathbb{R}^+; L^2(\Omega)) \times L^2(\mathbb{R}^+; L^2(\Gamma))$ *et* $(u_0, u_1) \in \bar{D}$*, où*

$$\bar{D} = \{(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega) \text{ telle que la condition } (1.59) \text{ est vraie } \}.$$

Alors, l'équation (1.52) *admet une unique solution faible globale. De plus, cette solution faible satisfait les propriétés suivantes :*
(T1) $(u, K_1^{\frac{1}{2}} u_t)$ *est bornée dans* $H^1(\Omega) \times L^2(\Omega)$.
(T2) $(u_t, v) \in L^2(\mathbb{R}^+; L^2(\Omega)) \times L^2(\mathbb{R}^+; L^2(\Gamma)$*, où* $v = \frac{d}{dt}(k * (u - u_0))$.
(T3) Soit $G : \mathbb{R}^+ \to \mathbb{R}$ *l'énergie de la solution* u *donnée par*

$$G(t) = \frac{1}{2}\|K_1^{\frac{1}{2}} u_t(t)\|_2^2 + E(u(t)) + \frac{1}{2}a * \|v(t)\|_\Gamma^2 - (g_2(t), a * v(t))_\Gamma + \\ \frac{1}{2k_0}\int_t^\infty \|g_1(s)\|_2^2 \, ds + d\int_t^\infty (\|g_2(s)\|_\Gamma^2 + \|g_2'(s)\|_\Gamma^2) \, ds. \quad (1.60)$$

où $d = \|a\|_{L^1(\mathbb{R}^+)} max(\gamma, \gamma^{-1})$*. Alors* G *est décroissante, et*

$$\frac{d}{dt}G(t) \leq -\frac{k_0}{2}\|u_t(t)\|_2^2 - \frac{b_\infty}{2}\|v(t)\|_\Gamma^2 - \frac{\gamma}{4}a * \|v(t)\|_\Gamma^2, \; t > 0. \quad (1.61)$$

(T4) La formule variationnelle suivante est vraie pour toute $\phi \in H^1(\Omega)$

$$\frac{d}{dt}\int_\Omega K_1(x)u_t\phi \, dx + \int_\Omega K_2(x)u_t\phi \, dx + \int_\Omega \nabla u \nabla\phi \, dx + \\ + \int_\Omega f(x, u)\phi \, dx + \frac{d}{dt}\int_\Gamma (k * (u - u_0))\phi \, d\sigma + \int_\Gamma \mu(x)u\phi \, d\sigma \\ = \int_\Omega g_1\phi \, dx + \int_\Gamma g_2\phi \, d\sigma.$$

Remarque 1.0.5. *Quand* $K_1(x) \geq C_0 > 0$*, on remplace* (1.59) *par la condition suivante :*

$$\partial_\nu u + \mu(x)u = g_2(0) \text{ sur } \Gamma.$$

La précompacité des images des solutions faibles de (1.52) est une propriété importante jouant un rôle crucial dans la preuve du théorème de convergence décrit ci-dessous. Ainsi on prouve

Théorème 1.0.9. *Soit* $u : \mathbb{R}^+ \to H^1(\Omega)$ *une solution faible bornée de* (1.52)*. Alors, la fonction* $U = (u, K_1^{\frac{1}{2}} u_t)$ *est uniformément continue de* $\mathbb{R}^+$ *à valeurs dans* $H^1(\Omega) \times L^2(\Omega)$*, et* $\bigcup_{t \geq 0}\{U(t)\}$ *est relativement compacte dans* $H^1(\Omega) \times L^2(\Omega)$.

Le théorème suivant décrit la convergence des solutions faibles bornées de l'équation (1.52) vers l'équilibre.

Théorème 1.0.10. *Soit* $u : \mathbb{R}^+ \to H^1(\Omega)$ *une solution globale faible bornée de l'équation* (1.52)*. Supposons que* f *vérifie (F1), (F2) du Chapitre 3. Alors il existe* $\phi \in H^1(\Omega)$*, solution du problème stationnaire suivant :*

$$\begin{cases} -\Delta\phi + f(x, \phi) = 0 & \text{dans } \Omega, \\ \partial_\nu\phi + \mu\phi = 0 & \text{sur } \Gamma, \end{cases}$$

telle que

$$\|K_1^{\frac{1}{2}} u_t(t)\|_2 + \|u(t)-\phi\|_{H^1(\Omega)} \longrightarrow 0 \quad \textit{quand } t \to \infty.$$

Pour prouver le Théorème 1.0.10, on construit la fonction de Lyapunov $W : \mathbb{R}^+ \to \mathbb{R}$ suivante :

$$\begin{aligned} W(t) &= \frac{1}{2}\|K_1^{\frac{1}{2}} u_t\|_2^2 + E(u) - E_\infty + \frac{1}{2} a * \|v\|_\Gamma^2 - (g_2, a * v)_\Gamma + \varepsilon(E'(u(t)), K_1 u_t)_* \\ &\quad + (\frac{1}{2k_0} - C_\varepsilon) \int_t^\infty \|g_1(s)\|_2^2 \, ds + (d - C_\varepsilon) \int_t^\infty (\|g_2(s)\|_\Gamma^2 + \|g_2'(s)\|_\Gamma^2) \, ds, \end{aligned}$$

où $C_\varepsilon < \inf\{\frac{1}{2k_0}, d\}$. On prouve que cette energie vérifie, pour un $\varepsilon > 0$ assez petit,

$$\frac{d}{dt} W(t) \leq -C\big(\|u_t\|_2^2 + \|E'(u)\|_*^2 + \|v\|_\Gamma^2 + a * \|v\|_\Gamma^2\big), \quad t > 0.$$

Finalement, en utilisant l'énergie W et le Lemme 1.0.2, on prouve aussi que l'exposant de Łojasiewicz θ dans l'inégalité de Łojasiewicz-Simon détermine la vitesse de décroissance des solutions u vers l'équilibre ϕ.

Théorème 1.0.11. *Soit $\theta = \theta_\phi$ l'exposant de Łojasiewicz de E en ϕ, où ϕ est donnée dans le théorème 1.0.10. Alors, les assertions suivantes sont vraies :*
(i) Si $\theta \in (0, \frac{1}{2})$, alors il existe une constante $C > 0$ telle que pour tout $t \geq 0$ on a

$$\|u(t) - \phi\|_2 \leq C(1+t)^{-\xi},$$

où

$$\begin{cases} \xi = \inf\{\frac{\theta}{1-\theta}, \frac{\delta}{2}\}, & \textit{si} \ \ (g_1, g_2) \neq (0,0), \\ \xi = \frac{\theta}{1-2\theta}, & \textit{si} \ (g_1, g_2) = (0,0). \end{cases}$$

(ii) Si $\theta = \dfrac{1}{2}$ et $(g_1, g_2) = (0,0)$. Alors il existe des constantes $C, \kappa > 0$ telles que

$$\|u(t) - \phi\|_2 \leq C e^{-\theta \kappa t}.$$

Caractéristiques et difficultés mathématiques

On décrit brièvement les nouvelles caractéristiques et difficultés mathématiques des problèmes considérés dans cette thèse.

1. On traite des équations de type mixte, hyperbolique-parabolique. Pour ce type d'équations, la question d'existence n'est en général pas triviale et il n'existe que des résultats d'existence dans des cas particuliers (par exemple, la méthode des semi-groupes n'est en général pas applicable).

2. On prouve la compacité et la convergence des solutions faibles. Le manque de régularité nous cause des difficultés de calculs. Pour cela, et pour tous les modèles étudiés dans cette thèse, on montre aussi l'existence de solutions fortes. Les solutions faibles sont définies comme des limites uniformes des solutions fortes, et l'on peut alors dériver les énergies associées par passage à la limite.
3. On traite les conditions dynamiques sur le bord. Les opérateurs elliptiques correspondent à des conditions non homogènes sur le bord. On prouve l'inégalité de Łojasiewicz-Simon pour l'énergie associée qui nous permet d'obtenir un résultat de convergence.
4. Afin d'appliquer l'idée de Simon, pour prouver les résultats de convergence, nous devons construire de nouvelles énergies fonctionnelles (en perturbant l'énergie fonctionnelle originale en ajoutant des termes auxiliaires) qui varient d'un problème à un autre.
5. Par obtention des estimations d'énergie délicates et en construisant des inégalités différentielles, on obtient des estimations de la vitesse de convergence vers l'équilibre.
6. On traite le cas non-autonome. La difficulté technique provenant des termes sources est non triviale. La technique utilisée pour prouver le résultat de convergence dans le chapitre 4 est direct et reprend naturellement le contexte du cas autonome, sans avoir besoin de discussion supplémentaire [16] ou du lemme 1.0.1. Remarquons aussi que cette technique peut être appliquée dans le deuxième chapitre.
7. On traite le cas d'amortissement non-linéaire et d'amortissement de type mémoire, qui entraînent des difficultés dans la construction de la fonction de Lyapunov.
8. On étudie des systèmes couplés, ondes-ondes, ondes-chaleur, chaleur-chaleur, des systèmes couplés des équations de type mixte, avec différents types de couplage et divers types de conditions sur le bord.

Après ces considérations générales décrivant le cadre de cette thèse, les chapitres suivants comporteront une introduction qui devrait permettre d'avoir une idée plus précise des poblèmes traités.

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Chapitre 2

Asymptotic behavior and decay rate estimates for a class of semilinear evolution equations of mixed order

Le résultat de ce chapitre fait l'objet d'un article publié dans
Nonlinear Analysis 74 (2011), 2309–2326.

2.1 Introduction

Consider three Hilbert spaces $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{H}$ such that $\mathcal{V} \subset \mathcal{W} \subset \mathcal{H}$, with dense and continuous injections. We will identify $\mathcal{H}$ with its dual space $\mathcal{H}'$, so that

$$\mathcal{V} \hookrightarrow \mathcal{W} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{W}' \hookrightarrow \mathcal{V}',$$

with dense and continuous injections.
In this article we study the convergence to equilibrium and the rate of decay as time tends to infinity of bounded solutions of the nonlinear, nonautonomous problem

$$(A\dot{u})^{\cdot} + B\dot{u} + M(u) = g, \quad t \in \mathbb{R}^+. \tag{2.1}$$

Here, $M = E'$ for some $E \in C^2(\mathcal{V})$, $A : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint and positive semidefinite operator, $B : \mathcal{W} \to \mathcal{W}'$ is a bounded linear operator satisfying the coercivity condition

$$(Bu, u)_{\mathcal{W}',\mathcal{W}} \geq \varrho \|u\|_{\mathcal{W}}^2, \quad u \in \mathcal{W}, \tag{2.2}$$

for some $\varrho > 0$, and $g \in L^2(\mathbb{R}^+, \mathcal{H})$ is such that there exists $\delta > 0$ such that

$$\sup_{t \in \mathbb{R}_+} (1+t)^{1+\delta} \int_t^\infty \|g(s)\|_{\mathcal{H}}^2 ds < \infty. \tag{2.3}$$

Depending on the choice of A, equation (2.1) contains as particular cases evolution equations of first order, second order but also mixed order so that our convergence

results unify and extend existing results in the literature.
Existence theorems concerning local or global solutions of (2.1) have been given by several authors for appropriate initial data and under some appropriate additional conditions on the operators (see [9, 19, 29] ; they are not the subject of this article). The proof of convergence is deduced from the so-called Łojasiewicz-Simon inequality. This inequality was first proved by S. Łojasiewicz [22]-[24] for analytic functions of several variables, then it was extended by L. Simon [28] to analytic functionals defined on infinite-dimensional Banach spaces. His original approach was then considerably simplified and subsequently adapted by Jendoubi [16] to obtain convergence results for a much larger class of equations including semilinear hyperbolic systems with weak damping (see also [6, 13, 17] for refinements and simplifications). Starting from this inequality and some differential inequalities, we discuss the polynomial and the exponential decay to an equilibrium.
In the case when $A = B = I_{\mathcal{H}}$, recently Chill and Jendoubi [7] have proved convergence to equilibrium of bounded solutions of equation (2.1) under the same assumption on E and g. They applied their abstract results to the following semilinear wave (respectively, heat) equation

$$ku_{tt} + u_t - \Delta u + f(u) = g, \text{ where } k = 1 \text{ (respectively, } k = 0), \tag{2.4}$$

subject to Dirichlet boundary conditions, where f is a real analytic function satisfying some growth assumption. Note that in the case when $k = 0$, Huang and Takàc [15] have proved the same result under the same condition on f and g. Moreover, the autonomous case, i.e. $g = 0$, has been considered by many authors under several assumptions on the nonlinearity f or the domain Ω. In fact, if $\Omega = (a, b) \subseteq \mathbb{R}$ is an interval, then convergence to a single equilibrium holds under very general hypotheses on f and the boundary conditions ; see T. J. Zelenyak [32] and H. Matano [25]. On the other hand, analyticity of the function f helps to overcome the difficulties encountered in higher-dimensional cases ; see L. Simon [28] and [12, 13, 16, 20]. In [2] Alvarez and Attouch proved convergence to equilibrium of the solution of (2.4) under the Neumann boundary condition, where $g = 0$ and $k = 1$. The work of Chill and Jendoubi is completed by [3], where Ben Hassen has proved the polynomial rate of convergence to equilibrium.
Our abstract result can be applied to the Eq.(2.4) subject to the following dynamical boundary condition

$$b(x)u_t + \partial_\nu u + a(x)u = g_2, \quad \mathbb{R}^+ \times \partial\Omega, \tag{2.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary, $a \in W^{1,\infty}(\partial\Omega)$, $b \in L^\infty(\partial\Omega)$, $b(x) \geq b_0 > 0$.
Note that, in the case when $g_1 = g_2 = 0$, $a = b = k = 1$, Wu and Zheng [30] (refer to [8]) have proved existence and convergence of strong solutions of (2.4)-(2.5) to a single stationary state.
We also apply our abstract results to the nonlinear mixed problem

$$K_1(x)u_{tt} + K_2(x)u_t - \Delta u + |u|^p u = g, \quad \mathbb{R}^+ \times \Omega, \tag{2.6}$$

with zero Dirichlet boundary conditions, where K_1 and K_2 are nonnegative functions satisfying some appropriate conditions. Note that K_1 can vanish on a part of Ω. Physical motivations for studying (2.6) come from several problems of continuum mechanics, such as turbulence, combustion, material aging, transonic flows, etc. The existence for this type of equation was studied by Bensoussan-Papanicolau-Lions in [4], Medeiros [26]. Lima [19] analyzed the Eq.(2.6) in a nonlinear abstract framework. In Lar'kin [18], (2.6) was studied with zero initial conditions and more general nonlinearities, the functions K_1 and K_2 depend also on t.
Our abstract result can be applied to study the following semilinear damped wave equation

$$K_1(x)u_{tt} + c_1u_t - c_2\Delta u_t - \Delta u + f(x,u) = g, \quad \mathbb{R}^+ \times \Omega \tag{2.7}$$

subject to Dirichlet boundary conditions. In [27], under some assumptions regarding $f(x,u) = f(u)$, Xu Runzhang and Liu Yacheng have studied asymptotic behavior of (2.7) where $K = 1$ and $g = c_1 = 0$.
As a final example we can treat the following coupled system $(\alpha_i \geq 0)$:

$$\begin{cases} \alpha_1 u_{tt} + u_t - \Delta u + f_1'(x,u,v) = g_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \alpha_2 v_{tt} + v_t - \Delta v + f_2'(x,u,v) = g_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ b(x)u_t + \frac{\partial u}{\partial n} + a(x)u = g_3 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \\ v = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases}$$

Here, $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary Γ and let Γ_0, $\Gamma_1 \subseteq \Gamma$ be two open subsets such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \varnothing$.
$f = f(x,u,v) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, analytic in $(u,v) \in \mathbb{R}^2$, satisfying some growth conditions.
The study of this system is motivated by the study of various related models (see [5],[21], [31], [33] and the references therein).

2.2 Preliminaries

Let $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{H}$ be as in the Introduction. We introduce the following operators which play an auxiliary role in our proof and statement of Theorem 2.3.1. Let $K : \mathcal{V}' \to \mathcal{V}$ be the duality mapping given by the identities

$$(u,v)_{\mathcal{V}'} = (u, Kv)_{\mathcal{V}',\mathcal{V}} = (u, Kv)_{\mathcal{H}} \text{ for every } u \in \mathcal{H}, v \in \mathcal{V}'.$$

A function $u : \mathbb{R}^+ \to \mathcal{V}$ is called a (weak) solution of equation (2.1) if

$$\begin{aligned} &u \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}) \cap H^1_{loc}(\mathbb{R}^+, \mathcal{W}), \\ &A\dot{u} \in H^1_{loc}(\mathbb{R}^+, \mathcal{V}'), \end{aligned}$$

and if u satisfies the differential equation (2.1) in $\mathcal{V}'$, for almost every t $\in \mathbb{R}^+$.
We define the ω-limit set of a function $u : \mathbb{R}^+ \to \mathcal{V}$ by

$$\omega(u) = \{\phi \in \mathcal{V} : \exists\ t_n \to +\infty \text{ such that } \lim_{n\to\infty} \|u(t_n) - \phi\|_{\mathcal{V}} = 0\}.$$

If $u : \mathbb{R}^+ \to \mathcal{V}$ is a continuous function such that the range $\{u(t) : t \geq 1\}$ is relatively compact in $\mathcal{V}$, then it is well-known that the ω-limit set $\omega(u)$ is non-empty, compact and connected [14]. With an additional geometric condition on the function E and its derivative M we show that the ω-limit set of every global and bounded solution with compact range is actually reduced to one point, that is, the solution converges.

Définition 2.2.1. *A function $E \in C^1(\mathcal{V})$ satisfies the Łojasiewicz-Simon inequality near some point $\phi \in \mathcal{V}$, if there exist constants $\theta \in (0, \frac{1}{2}]$, $\beta \geq 0$ and $\sigma > 0$ such that for all $v \in \mathcal{V}$ with $\|v - \phi\|_{\mathcal{V}} \leq \sigma$ one has*

$$| E(v) - E(\phi) |^{1-\theta} \leq \beta \|M(v)\|_{\mathcal{V}'}.$$

To prove the convergence to equilibrium, we use the following lemma proved in [10] and used in [15].

Lemma 2.2.1 (Feireisl and Simondon [10])**.** *Let $Z \geq 0$ be a measurable function on $\mathbb{R}^+$ such that*

$$Z \in L^2(\mathbb{R}^+), \qquad \|Z\|_{L^2(\mathbb{R}^+)} \leq Y.$$

Let $\mathcal{D} \subseteq \mathbb{R}^+$ be open, and let $\alpha \in (1,2)$, $w > 0$ be such that

$$\Big(\int_t^\infty Z(s)^2 ds\Big)^\alpha \leq w Z^2(t) \quad for \ a.e. \ t \in \mathcal{D}.$$

Then, $Z \in L^1(\mathcal{D})$ and there exists a positive constant $c = c(\alpha, w, Y)$ independent of $\mathcal{D}$ such that

$$\int_{\mathcal{D}} Z(s) ds \leq c.$$

The following lemma can be found in [3]. We us it to obtain the polynomial decay rate to equilibrium.

Lemma 2.2.2 (Ben Hassen)**.** *Let $\zeta \in W^{1,1}_{loc}(\mathbb{R}^+, \mathbb{R}^+)$. We suppose that there exist constants $K_1 > 0$, $K_2 \geq 0$, $k > 1$ and $\lambda > 0$ such that for almost every $t \geq 0$ we have*

$$\zeta'(t) + K_1 \zeta(t)^k \leq K_2 (1+t)^{-\lambda}.$$

Then, there exists a positive constant m such that

$$\zeta(t) \leq m(1+t)^{-\nu}, \quad \nu = \inf\{\frac{1}{k-1}, \frac{\lambda}{k}\}.$$

We let C $\geq$0 be such that $\|u\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{W}} \leq C^2 \|u\|_{\mathcal{V}}$, for every $u \in \mathcal{V}$. Other positive constants in the calculations will be denoted by C_i $(i \geq 1)$.

2.3 Main results

Theorem 2.3.1. *Let $u : \mathbb{R}^+ \to \mathcal{V}$ be a solution of equation* (2.1), *and assume that :*
(H1) $u \in H^1_{loc}(\mathbb{R}^+, \mathcal{V})$, $A\dot{u} \in H^1_{loc}(\mathbb{R}^+, \mathcal{H})$.
(H2) The set $\{(u(t), A^{\frac{1}{2}}\dot{u}(t)) : t \geq 1\}$ is relatively compact in $\mathcal{V} \times \mathcal{H}$.
(H3) There exists $\phi \in \omega(u)$ such that E satisfies the Łojasiewicz-Simon inequality with exponent θ near ϕ.
(H4) If $K : \mathcal{V}' \to \mathcal{V}$ denotes the duality map, then the operator $K \circ M'(v) \in \mathcal{L}(\mathcal{V})$ extends to a bounded linear operator on $\mathcal{H}$ for every $v \in \mathcal{V}$, and $K \circ M' : \mathcal{V} \to \mathcal{L}(\mathcal{H})$ maps bounded sets into bounded sets.
(H5) g satisfies (2.3) *for some $\delta > 0$.*
Then

$$\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}} + \|u(t) - \phi\|_{\mathcal{V}} \underset{t\to\infty}{\longrightarrow} 0,$$

and there exists a constant $C' > 0$ such that for all $t \geq 0$ we have

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'(1+t)^{-\eta}, \text{ where } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\}.$$

If, in addition, $g = 0$ and $\theta = \dfrac{1}{2}$, then there exist constants C'', $\xi > 0$ such that for all $t \geq 0$ we have

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'' e^{-\xi t}.$$

Proof. We proceed in three steps :

2.3.1 The convergence result

We let $\mathcal{S} = \{\phi \in \mathcal{V},\ M(\phi) = 0\}$ and ε be a real positive constant which will be fixed in what follows. Let $G : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$G(t) = \frac{1}{2}\|A^{\frac{1}{2}}\dot{u}(t)\|^2_{\mathcal{H}} + E(u(t)) + \varepsilon(M(u(t)), A\dot{u}(t))_{\mathcal{V}'}. \tag{2.8}$$

By assumption $(H1)$, the function G is differentiable and

$$\begin{aligned}
\frac{d}{dt}G(t) &= ((A\dot{u})^{\cdot}, \dot{u})_{\mathcal{V}',\mathcal{V}} + (M(u), \dot{u})_{\mathcal{V}',\mathcal{V}} + \varepsilon(M'(u)\dot{u}, A\dot{u})_{\mathcal{V}'} + \varepsilon(M(u), (A\dot{u})^{\cdot})_{\mathcal{V}'} \\
&= (-B\dot{u} - M(u) + g, \dot{u})_{\mathcal{V}',\mathcal{V}} + (M(u), \dot{u})_{\mathcal{V}',\mathcal{V}} + \varepsilon(K \circ M'(u)\dot{u}, A\dot{u})_{\mathcal{H}} \\
&\quad + \varepsilon(M(u), -B\dot{u} - M(u) + g)_{\mathcal{V}'} \\
&\leq -\varrho\|\dot{u}\|^2_{\mathcal{W}} + \|g\|_{\mathcal{H}}\|\dot{u}\|_{\mathcal{H}} + \varepsilon\|K \circ M'(u)\|_{\mathcal{L}(\mathcal{H})}\|A\|_{\mathcal{L}(\mathcal{H})}\|\dot{u}\|^2_{\mathcal{H}} \\
&\quad -\varepsilon\|M(u)\|^2_{\mathcal{V}'} + \varepsilon(M(u), g - B\dot{u})_{\mathcal{V}'} \\
&\leq -\frac{\varrho}{2}\|\dot{u}\|^2_{\mathcal{W}} + \frac{C^2}{2\varrho}\|g\|^2_{\mathcal{H}} + \varepsilon C^2\|K \circ M'(u)\|_{\mathcal{L}(\mathcal{H})}\|A\|_{\mathcal{L}(\mathcal{H})}\|\dot{u}\|^2_{\mathcal{W}} \\
&\quad -\frac{\varepsilon}{2}\|M(u)\|^2_{\mathcal{V}'} + \varepsilon C_0^2\|B\|^2_{\mathcal{L}(\mathcal{W}',\mathcal{W})}\|\dot{u}\|^2_{\mathcal{W}} + \varepsilon C_0^4\|g\|^2_{\mathcal{H}},
\end{aligned}$$

where C_0 is such that $\|u\|_{\mathcal{V}'} \leq C_0\|u\|_{\mathcal{W}'} \leq C_0^2\|u\|_{\mathcal{H}}$, for every $u \in \mathcal{H}$. By choosing $\varepsilon > 0$ small enough such that

$$\varepsilon\big(C^2\|K \circ M'(u)\|_{\mathcal{L}(\mathcal{H})}\|A\|_{\mathcal{L}(\mathcal{H})} + C_0^2\|B\|^2_{\mathcal{L}(\mathcal{W}',\mathcal{W})}\big) < \frac{\varrho}{2} \text{ for all } t \geq 0,$$

we see that there exist constants C_1, $C_2 > 0$ such that

$$\frac{d}{dt}G(t) \leq -C_1\big(\|\dot{u}\|^2_{\mathcal{W}}+\|M(u)\|^2_{\mathcal{V}'}\big) + C_2\|g\|^2_{\mathcal{H}}. \tag{2.9}$$

From this inequality, we obtain that :
(i) $\dot{u} \in L^2(\mathbb{R}^+, \mathcal{W})$ and $M(u) \in L^2(\mathbb{R}^+, \mathcal{V}')$.
(ii) $\lim_{t\to\infty} \|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}} = 0$.
(iii) E is constant on $\omega(u)$.
(iv) $\omega(u) \subseteq \mathcal{S}$.
In fact, let $0 \leq t \leq t' < \infty$ and integrate (2.9) over $[t, t']$. Then

$$G(t') - G(t) + C_1\int_t^{t'} (\|\dot{u}(s)\|^2_{\mathcal{W}}+\|M(u(s))\|^2_{\mathcal{V}'})ds \leq C_2\int_t^{\infty} \|g(s)\|^2_{\mathcal{H}}ds. \tag{2.10}$$

From this inequality, and since G is bounded on $\mathbb{R}^+$ and $g \in L^2(\mathbb{R}^+, \mathcal{H})$, we obtain (i).
We deduce from (i) and the equation (2.1) that $(A\dot{u})^{\cdot} \in L^2(\mathbb{R}^+, \mathcal{V}')$. Hence $A\dot{u}$ is uniformly continuous with values in $\mathcal{V}'$. In addition, since $A\dot{u} \in L^2(\mathbb{R}^+, \mathcal{V}')$, this implies $\lim_{t\to\infty} \|A\dot{u}(t)\|_{\mathcal{V}'} = 0$. Using this, $(H2)$, and the fact that A is self-adjoint, we obtain (ii).
Let $\phi \in \omega(u)$. Then there exists an unbounded increasing sequence (t_n) in $\mathbb{R}^+$ such that $u(t_n) \to \phi$ in $\mathcal{V}$. Using (ii) and the regularity of E, we obtain

$$G(t_n) \to E(\phi). \tag{2.11}$$

On the other hand, by (H5), we have $\int_t^\infty \|g(s)\|^2_{\mathcal{H}}ds \searrow 0$ when $t \nearrow \infty$, and by (i) we have $(\int_t^\infty (\|\dot{u}(s)\|^2_{\mathcal{W}}+\|M(u(s))\|^2_{\mathcal{V}'})ds) \searrow 0$ when $t \nearrow \infty$. Then, from (2.10) and (2.11) we obtain

$$G(t) \to E(\phi).$$

Finally we have (using (2.8) and (ii))

$$E(\phi) = \lim_{t\to\infty} E(u(t)) = E_\infty \text{ for all } \phi \in \omega(u), \tag{2.12}$$

and the property (iii) is proved.
Moreover, since $\dot{u} \in L^2(\mathbb{R}^+, \mathcal{W})$ we obtain

$$u(t_n + s) = u(t_n) + \int_{t_n}^{t_n+s} \dot{u}(\rho)d\rho \to \phi \text{ in } \mathcal{W}, \text{ for every } s \in [0, 1].$$

This, together with the compactness of the trajectory, implies that $u(t_n+s) \to \phi$ in $\mathcal{V}$ for every $s \in [0,1]$. Hence $M(u(t_n+s)) \to M(\phi)$ in $\mathcal{V}'$ for every $s \in [0,1]$. Finally, using the dominated convergence theorem and assertion (i), we have

$$\begin{aligned} \|M(\phi)\|_{\mathcal{V}'} &= \int_0^1 \|M(\phi)\|_{\mathcal{V}'} ds = \lim_{n\to\infty} \int_0^1 \|M(u(t_n+s))\|_{\mathcal{V}'} ds \\ &\leq \lim_{n\to\infty} \int_0^1 \|M(u(t_n+s))\|^2_{\mathcal{V}'} ds \ = \ 0. \end{aligned}$$

This proves $\phi \in \mathcal{S}$ as desired.
Now, we use $(H5)$ and (2.9) to obtain, for all $t \in \mathbb{R}^+$,

$$-\int_t^\infty \frac{d}{dt} G(s) ds \geq C_1 \int_t^\infty \left(\|\dot{u}(s)\|^2_{\mathcal{W}} + \|M(u(s))\|^2_{\mathcal{V}'}\right) ds - \frac{C_3}{(1+t)^{(1+\delta)}}.$$

Hence, for all $t \in \mathbb{R}^+$ we have

$$G(t) - E_\infty \geq C_1 \int_t^\infty \left(\|\dot{u}(s)\|^2_{\mathcal{W}} + \|M(u(s))\|^2_{\mathcal{V}'}\right) ds - \frac{C_3}{(1+t)^{(1+\delta)}}. \tag{2.13}$$

We simplify our notation by introducing the auxiliary functions Z, $z : \mathbb{R}^+ \to \mathbb{R}^+$

$$Z(t)^2 = \|\dot{u}(t)\|^2_{\mathcal{W}} + \|M(u(t))\|^2_{\mathcal{V}'}$$

and

$$z(t) = \frac{C_3}{(1+t)^{(1+\delta)}}.$$

By virtue of hypothesis $(H3)$ of Theorem 2.3.1, there exists $\phi \in \omega(u)$, $\beta \geq 0$, $\sigma > 0$ and $0 < \theta \leq \frac{1}{2}$ such that

$$\mid E(v) - E(\phi) \mid^{1-\theta} \leq \beta \|M(v)\|_{\mathcal{V}'}, \text{ for every } v \in B_\sigma(\phi) = \{\psi \in \mathcal{V}, \|\psi - \phi\|_{\mathcal{V}} \leq \sigma\}.$$

In addition, by continuity of E, we can choose $\sigma > 0$ small enough, so that

$$\mid E(v) - E(\phi) \mid \leq 1 \ \text{ for all } v \in B_\sigma(\phi). \tag{2.14}$$

Now our aim is to apply Lemma 2.2.1 to show that once the solution u enters a small neighborhood of a stationary state $\phi \in \omega(u)$, then it must remain there for all t large enough and converge.
Let $t_n \nearrow \infty$ such that $\lim_{n\to\infty} \|u(t_n) - \phi\|_{\mathcal{V}} = 0$ and $\|u(t_n) - \phi\|_{\mathcal{V}} \leq \dfrac{\sigma}{2}$ for all n. For every n we define

$$\tau(t_n) = \sup\{t' \geq t_n : \sup_{s\in[t_n,t']} \|u(s) - \phi\|_{\mathcal{V}} \leq \sigma\}.$$

By continuity of u, $\tau(t_n) > t_n$ for every n.
We show that there exists n_0 such that $\tau(t_{n_0}) = \infty$. In fact, assume that this is not true. Then there exists a subsequence of (t_n) (again denoted by (t_n)) $\nearrow \infty$ such

that $t_{n+1} > \tau(t_n)$ for every n.
Let $J_n = (t_n, \tau(t_n))$ and $\mathcal{D} = \bigcup_n J_n$. Then for all $t \in \mathcal{D}$:

$$| E(u(t)) - E_\infty |^{1-\theta} \leq \beta \|M(u(t))\|_{\mathcal{V}'}. \tag{2.15}$$

Let $\theta' \in (0, \theta]$ be such that

$$0 \leq 2\theta' < 1 - \frac{1}{1+\delta} < 1. \tag{2.16}$$

Then, by (2.15) and (2.14), we have for all $t \in \mathcal{D}$:

$$| E(u(t)) - E_\infty |^{1-\theta'} \leq \beta \|M(u(t))\|_{\mathcal{V}'}. \tag{2.17}$$

On the other hand, by the Cauchy-Schwarz inequality, we deduce for all $t \in \mathcal{D}$:

$$\begin{aligned} | G(t) - E_\infty |^{1-\theta'} &= | \frac{1}{2}\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^2 + E(u(t)) - E_\infty + \varepsilon(M(u(t)), A\dot{u}(t))_{\mathcal{V}'} |^{1-\theta'} \\ &\leq C_4\big(\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^{2(1-\theta')} + | E(u(t) - E_\infty |^{1-\theta'} + \|M(u(t))\|_{\mathcal{V}'}^{1-\theta'} \|A\dot{u}(t)\|_{\mathcal{H}}^{1-\theta'}\big). \end{aligned}$$

Thanks to Young's inequality and (2.17), we obtain for every $t \in \mathcal{D}$:

$$| G(t) - E_\infty |^{1-\theta'} \leq C_5\big(\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^{2(1-\theta')} + \|M(u(t))\|_{\mathcal{V}'} + \|A\dot{u}(t)\|_{\mathcal{H}}^{\frac{1-\theta'}{\theta'}}\big).$$

By (ii), and for t large enough, we have

$$\|A\dot{u}(t)\|_{\mathcal{H}} \leq 1 \text{ and } \|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}} \leq 1.$$

As $2(1-\theta') \geq 1$ and $\frac{1-\theta'}{\theta'} \geq 1$, we obtain

$$\begin{aligned} &\|A\dot{u}(t)\|_{\mathcal{H}}^{\frac{1-\theta'}{\theta'}} \leq \|A\dot{u}(t)\|_{\mathcal{H}} \leq \|A\|_{\mathcal{L}(\mathcal{H})} \|\dot{u}(t)\|_{\mathcal{H}} \\ \text{and} \quad &\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^{2(1-\theta')} \leq \|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}} \leq \|A^{\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H})} \|\dot{u}(t)\|_{\mathcal{H}}. \end{aligned}$$

Then, for every $t \in \mathcal{D}$,

$$| G(t) - E_\infty |^{1-\theta'} \leq C_6(\|\dot{u}(t)\|_{\mathcal{H}} + \|M(u(t))\|_{\mathcal{V}'}) \leq \sqrt{2(1+C^2)}C_6 Z(t).$$

It follows that

$$| G(t) - E_\infty | \leq C_7 Z(t)^{\frac{1}{1-\theta'}}, \quad \text{for every } t \in \mathcal{D}.$$

By using this inequality and (2.13), we conclude that

$$\int_t^\infty Z(s)^2 ds - \frac{1}{C_1} z(t) \leq C_8 Z(t)^{\frac{1}{1-\theta'}}, \quad \text{for every } t \in \mathcal{D}. \tag{2.18}$$

On the other hand, by (2.16), we have for all $t \in \mathcal{D}$

$$\begin{aligned}\int_t^{\infty} z(s)^{2(1-\theta')} ds &= \int_t^{\infty} \frac{C_9}{(1+s)^{2(1-\theta')(1+\delta)}} ds \\ &\leq z(t) \int_1^{\infty} \frac{C_{10}}{(1+s)^{(1-2\theta')(1+\delta)}} \leq C_{11} z(t).\end{aligned} \tag{2.19}$$

Moreover, we have

$$\int_t^{\infty} (Z(s) + z(s)^{1-\theta'})^2 ds \leq 2(\int_t^{\infty} Z(s)^2 ds + \int_t^{\infty} z(s)^{2(1-\theta')} ds).$$

We combine this inequality with (2.18) and (2.19) in order to conclude that

$$\begin{aligned}\int_t^{\infty} (Z(s) + z(s)^{1-\theta'})^2 ds &\leq C_{12}(Z(t)^{\frac{1}{1-\theta'}} + z(t)) \\ &\leq C_{13}(Z(t) + z(t)^{1-\theta'})^{\frac{1}{1-\theta'}}, \quad \text{for every } t \in \mathcal{D}.\end{aligned}$$

Thus, we can apply Lemma 2.2.1 to conclude that

$$\int_{\mathcal{D}} (Z(t) + z(t)^{1-\theta'}) dt < +\infty.$$

This implies

$$\int_{\mathcal{D}} \|\dot{u}(t)\|_{\mathcal{W}} dt < +\infty \quad \text{and} \quad \lim_{n\to\infty} \int_{t_n}^{\tau(t_n)} \|\dot{u}(t)\|_{\mathcal{W}} dt = 0. \tag{2.20}$$

Now, by definition of $\tau(t_n)$:

$$\|u(\tau(t_n)) - \phi\|_{\mathcal{V}} = \sigma.$$

Then, by the compactness hypothesis, there exists a subsequence $(\tau(t_{n_k}))$ of $(\tau(t_n))$ and $\psi \in \mathcal{V}$ such that $u(\tau(t_{n_k})) \to \psi$ in $\mathcal{V}$. We obtain

$$\|\psi - \phi\|_{\mathcal{V}} = \sigma.$$

On the other hand, by (2.20), we have

$$0 < \|\psi - \phi\|_{\mathcal{W}} \leq \limsup_{k\to\infty} \Big(\|u(\tau(t_{n_k})) - \psi\|_{\mathcal{W}} + \int_{t_{n_k}}^{\tau(t_{n_k})} \|\dot{u}(s)\|_{\mathcal{W}} ds + \|u(t_{n_k}) - \phi\|_{\mathcal{W}} \Big) = 0,$$

which is a contradiction. Hence $\tau(t_{n_0}) = \infty$ for some n_0 large enough. Thus, by (2.20), the function $\|\dot{u}\|_{\mathcal{W}}$ is absolutely integrable on $[t_{n_0}, \infty)$, which implies that $\lim_{t\to\infty} u(t)$ exists in $\mathcal{W}$. By compactness and since $\phi \in \omega(u)$, $\lim_{t\to\infty} u(t) = \phi$ in $\mathcal{V}$.

2.3.2 The polynomial decay

To estimate the rate of decay of the difference between a solution and its limit, let

$$K(t) = G(t) - E(\phi) + C_2 \int_t^\infty \|g(s)\|_{\mathcal{H}}^2 ds.$$

By assumption (H5), K is well-defined. Moreover, we have

$$\frac{d}{dt}K(t) = \frac{d}{dt}G(t) - C_2\|g(t)\|_{\mathcal{H}}^2.$$

We combine this with inequality (2.9) to obtain

$$\frac{d}{dt}K(t) \leq -C_1\big(\|\dot{u}(t)\|_{\mathcal{W}}^2 + \|M(u(t))\|_{\mathcal{V}'}^2\big). \tag{2.21}$$

Then the function K is nonincreasing and $\lim\limits_{t\to\infty} K(t) = 0$. It follows that $K(t) \geq 0$ for all $t \in \mathbb{R}^+$. By Young's inequality we obtain

$$\begin{aligned} K(t)^{2(1-\theta)} \quad & \leq C_{14} \mid E(u(t)) - E(\phi) \mid^{2(1-\theta)} + C_{15}\Big(\tfrac{1}{2}\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^2 \\ & + \varepsilon \mid \big(M(u(t)), A\dot{u}(t)\big)_{\mathcal{V}'} \mid + C_2 \textstyle\int_t^\infty \|g(s)\|_{\mathcal{H}}^2 ds\Big)^{2(1-\theta)}. \end{aligned} \tag{2.22}$$

On the other hand, since $\lim\limits_{t\to\infty} (\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}} + \|u(t) - \phi\|_{\mathcal{V}}) = 0$, there exists $T > 0$ such that for all $t \geq T$

$$\|A\dot{u}(t)\|_{\mathcal{H}} \leq 1, \ \|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}} \leq 1 \text{ and } \|u(t) - \phi\|_{\mathcal{V}} < \sigma. \tag{2.23}$$

Using this, (2.22), and assumption (H3), we infer

$$\begin{aligned} K(t)^{2(1-\theta)} \leq C_{14}\beta^2\|M(u(t))\|_{\mathcal{V}'}^2 + C_{15}\Big(\frac{1}{2}\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^2 + \varepsilon\big|\big(Mu(t), A\dot{u}(t)\big)_{\mathcal{V}'}\big| \\ + \int_t^\infty \|g(s)\|_{\mathcal{H}}^2 ds\Big)^{2(1-\theta)}. \end{aligned}$$

Now, by using (2.21) together with the Cauchy-Schwarz inequality, we obtain for all $t \in [T, \infty[$

$$\begin{aligned} K(t)^{2(1-\theta)} \leq -C_{16}\frac{d}{dt}K(t) + C_{17}\Big(\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^{4(1-\theta)} + \|M(u(t))\|_{\mathcal{V}'}^{2(1-\theta)}\|A\dot{u}(t)\|_{\mathcal{V}'}^{2(1-\theta)} \\ + \big(\int_t^\infty \|g(s)\|_{\mathcal{H}}^2 ds\big)^{2(1-\theta)}\Big). \end{aligned}$$

By using Young's inequality, we obtain for all $t \geq T$

$$\begin{aligned} C_{16}\frac{d}{dt}K(t) \leq -K(t)^{2(1-\theta)} + C_{18}\Big(\|A^{\frac{1}{2}}\dot{u}(t)\|_{\mathcal{H}}^{4(1-\theta)} + \|M(u(t))\|_{\mathcal{V}'}^2 + \|A\dot{u}(t)\|_{\mathcal{V}'}^{\frac{2(1-\theta)}{\theta}} \\ + \big(\int_t^\infty \|g(s)\|_{\mathcal{H}}^2 ds\big)^{2(1-\theta)}\Big). \end{aligned}$$

Using (2.23) and the fact that $\inf\{4(1-\theta),\ \frac{2(1-\theta)}{\theta}\} \geq 2$, we obtain for all $t \geq T$

$$C_{16}\frac{d}{dt}K(t) \leq -K(t)^{2(1-\theta)}+C_{19}\Big(\|\dot{u}(t)\|^2_{\mathcal{H}}+\|M(u(t))\|^2_{\mathcal{V}'}+(1+t)^{-2(1-\theta)(1+\delta)}\Big). \quad (2.24)$$

Combining this and (2.21) to obtain the following differential inequality for all $t \geq T$

$$C_{20}\frac{d}{dt}K(t) + K(t)^{2(1-\theta)} \leq C_{19}(1+t)^{-2(1-\theta)(1+\delta)}.$$

Then we apply Lemma 2.2.2 to obtain

$$K(t) \leq C_{21}(1+t)^{-\gamma}, \quad (2.25)$$

where $\gamma = \inf\left\{\frac{1}{1-2\theta}, 1+\delta\right\}$.
Due to (2.21) we have

$$\|\dot{u}(t)\|^2_{\mathcal{W}} \leq -\frac{1}{C_1}\frac{d}{dt}K(t).$$

Then, by integrating over $[t, 2t]$ and by using (2.25), we obtain for all t $\geq$ T

$$\int_t^{2t}\|\dot{u}(s)\|^2_{\mathcal{W}}ds \leq C_{22}(1+t)^{-\gamma}.$$

Note that for every $t \in \mathbb{R}^+$,

$$\int_t^{2t}\|\dot{u}(s)\|_{\mathcal{W}}ds \leq \sqrt{t}\Big(\int_t^{2t}\|\dot{u}(s)\|^2_{\mathcal{W}}ds\Big)^{\frac{1}{2}}.$$

It follows that

$$\int_t^{2t}\|\dot{u}(s)\|_{\mathcal{W}}ds \leq C_{23}(1+t)^{\frac{1-\gamma}{2}}.$$

Therefore we obtain for all $t \geq T$

$$\int_t^{\infty}\|\dot{u}(s)\|_{\mathcal{W}}ds \leq \sum_{k=0}^{\infty}\int_{2^k t}^{2^{k+1}t}\|\dot{u}(s)\|_{\mathcal{W}}ds \leq C_{24}\sum_{k=0}^{\infty}(2^k t)^{\frac{1-\gamma}{2}} \leq C_{25}(1+t)^{\frac{1-\gamma}{2}}.$$

Then, for all $t \geq T$

$$\|u(t)-\phi\|_{\mathcal{W}} \leq C_{26}(1+t)^{-\eta}, \quad \text{where } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\}. \quad (2.26)$$

Finally, by continuity of u, we have (2.26) for all $t \geq 0$.

2.3.3 The exponential decay

Suppose that $g = 0$ and $\theta = \dfrac{1}{2}$. Then $K(t) = G(t) - E(\phi)$ and (2.21) remains true. If there exists $T_0 \geq T$ such that $K(T_0) = 0$, then $K(t) = 0$ for all $t \geq T_0$. Thus, by inequality (2.21), the function u is constant for $t \geq T_0$, i.e. $u = \phi$ for $t \geq T_0$. In this case, there remains nothing to prove. We may therefore suppose in the following that $K(t) > 0$ for every $t \geq T$. Hence, we have

$$-\frac{d}{dt}(K(t))^{\theta} = -\theta K'(t)(K(t))^{\theta-1}. \tag{2.27}$$

On the other hand we have for all $t > T$ (using again the same arguments as in the proof of the estimate of $(K(t))^{2(1-\theta)}$)

$$(K(t))^{1-\theta} \leq C_{27}\big(\|\dot{u}(t)\|_{\mathcal{W}}+\|M(u(t))\|_{\mathcal{V}'}\big). \tag{2.28}$$

Combining (2.27), (2.21), and (2.28) to obtain for all $t > T$

$$-\frac{d}{dt}(K(t))^{\theta} \geq C_{28}\big(\|\dot{u}(t)\|_{\mathcal{W}}+\|M(u(t))\|_{\mathcal{V}'}\big). \tag{2.29}$$

Integrating (2.29) on (t, t'), where $t' \geq t > T$ and using the fact that $K(t) \geq 0$ we obtain (letting $t' \to \infty$)

$$\int_t^{\infty} \big(\|\dot{u}(s)\|_{\mathcal{W}}+\|M(u(s))\|_{\mathcal{V}'}\big)ds \leq C_{29}(K(t))^{\theta}, \quad \text{for all } t > T. \tag{2.30}$$

Moreover, from (2.28), we have for all $t > T$

$$(K(t))^{\theta} = ((K(t))^{1-\theta})^{\frac{\theta}{1-\theta}} \leq (C_{27})^{\frac{\theta}{1-\theta}}\big(\|\dot{u}(t)\|_{\mathcal{W}}+\|M(u(t))\|_{\mathcal{V}'}\big)^{\frac{\theta}{1-\theta}}. \tag{2.31}$$

By combining (2.30) and (2.31), we obtain, for all $t > T$,

$$\int_t^{\infty} \big(\|\dot{u}(s)\|_{\mathcal{W}}+\|M(u(s))\|_{\mathcal{V}'}\big)ds \leq C_{30}\big(\|\dot{u}(t)\|_{\mathcal{W}}+\|M(u(t))\|_{\mathcal{V}'}\big)^{\frac{\theta}{1-\theta}}. \tag{2.32}$$

Now, we let for all $t > T$

$$v(t) = \int_t^{\infty} \big(\|\dot{u}(s)\|_{\mathcal{W}}+\|M(u(s))\|_{\mathcal{V}'}\big)ds.$$

We have

$$\|u(t) - \phi\|_{\mathcal{W}} = \lim_{t'\to\infty} \|u(t) - u(t')\|_{\mathcal{W}} \leq v(t), \quad \text{for all } t > T. \tag{2.33}$$

Moreover, the relation (2.32) can be rewritten as follows (when $\theta = \frac{1}{2}$) :

$$v'(t) \leq -C_{31}v(t), \quad \text{for all } t > T. \tag{2.34}$$

Solving this differential inequality and combining with (2.33), we obtain the exponential decay. This completes the proof of the theorem. $\square$

Corollary 2.3.1 (the case $A = 0$)**.** *Let $u : \mathbb{R}^+ \to \mathcal{V}$ be a solution of the following equation*

$$B\dot{u} + M(u) = g, \tag{2.35}$$

where $\mathcal{V}$, $\mathcal{W}$, B, M, and g are as in Theorem 2.3.1. Assume that :
(H1) $u \in W^{1,2}_{loc}(\mathbb{R}^+, \mathcal{V})$.
(H2) The set $\{u(t) : t \geq 1\}$ is relatively compact in $\mathcal{V}$.
(H3)There exists $\phi \in \omega(u)$ such that E satisfies the Łojasiewicz-Simon inequality with exponent θ near ϕ.
Then, $u(t) \to \phi$ in $\mathcal{V}$ and there exists a constant $C' > 0$, such that for all $t \geq 0$ we have

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'(1+t)^{-\eta}, \text{where } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\}.$$

If, in addition, $g = 0$ and $\theta = \dfrac{1}{2}$, then there exist constants C'', $\xi > 0$ such that for all $t \geq 0$ we have

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'' e^{-\xi t}.$$

Proof. This is very similar to the proof of Theorem 2.3.1, since Eq.(2.1) contains as a special case Eq.(2.35) ($A = 0$). Corollary 2.3.1 contains all the necessary assumptions and the estimates in the proof of Theorem 2.3.1 are easily adapted. Here, the Lyapunov function $G(t) = E(u(t))$, for all $t > 0$. Then we have

$$\begin{aligned}\frac{d}{dt}G(t) &= (M(u(t)), \dot{u}(t))_{\mathcal{V}',\mathcal{V}} \\ &= (M(u(t)), \dot{u}(t))_{\mathcal{V}',\mathcal{V}} + \varepsilon(M(u(t)), -B\dot{u}(t) - M(u(t)) + g(t))_{\mathcal{V}'}.\end{aligned}$$

Similarly, inequality (2.9) remains true. As a consequence :
(i) $\dot{u} \in L^2(\mathbb{R}^+, \mathcal{W})$ and $M(u) \in L^2(\mathbb{R}^+, \mathcal{V}')$.
(ii) E is constant on $\omega(u)$.
(iii) $\omega(u) \subseteq \mathcal{S}$.
The remaining estimates in the proof of Theorem 2.3.1 are then easily adapted. □

Remark 2.3.1. *Note that Theorem 2.3.1 remains true if assumption* ($H1$) *is replaced by the weaker assumption*
(H1)' $u \in C(\mathbb{R}^+, \mathcal{V}) \cap H^1_{loc}(\mathbb{R}^+, \mathcal{H})$ and $A\dot{u} \in H^1_{loc}(\mathbb{R}^+, \mathcal{V}')$ and for some C_1, $C_2 > 0$ we have the estimate (2.9)*, where the function G is defined as in* (2.8)*. This remark is important in some applications since inequality* (2.9) *can be verified for less regular solutions* (u *is only differentiable with values in* $\mathcal{H}$) *by approximation and density arguments.*

2.4 Applications

We begin by the following remark concerning the question of existence of solutions.

Remark 2.4.1. *For all applications given below, the methods used in Chapter 3 and Chapter 4 allow one to prove :*

1. *existence and uniqueness of global weak solutions,*
2. *weak solutions are strong limits of strong solutions,*
3. *any bounded weak solution has relatively compact range in the natural energy space.*

Let $\Omega \subseteq \mathbb{R}^N$, $(N \geq 1)$ be a bounded open set with a smooth boundary Γ, ν is the outward normal direction to the boundary and let $f = f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying the following assumptions :
(*F1*) f is analytic in the second variable, uniformly with respect to $x \in \Omega$ and u in bounded subsets of $\mathbb{R}$,
(*F2*) If $N = 1$, then f and $\dfrac{\partial f}{\partial s}$ are bounded in $\Omega \times [-r, r]$, for every $r > 0$.
If $N \geq 2$, then $f(\cdot, 0) \in L^\infty(\Omega)$, and there exist constants $\rho \geq 0$ and $\mu > 0$, $(N-2)\mu < 2$ such that :

$$|\frac{\partial f}{\partial s}(x,s)| \leq \rho(1 + |u|^\mu) \ \text{ for every } s \in \mathbb{R}, \ x \in \Omega.$$

2.4.1 Nonautonomous semilinear wave equation with dynamical boundary condition

Consider the equation

$$\begin{cases} u_{tt} + u_t - \Delta u + f(x,u) = g_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ b(x)u_t + \partial_\nu u + a(x)u = g_2 & \text{on } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \ u_t(0) = u_1. \end{cases} \tag{2.36}$$

Here, $a \in W^{1,\infty}(\Gamma)$, $b \in L^\infty(\Gamma)$, $b(x) \geq b_0 > 0$, and $(g_1, g_2) \in L^2(\mathbb{R}^+; L^2(\Omega) \times L^2(\Gamma))$.

We call a function $u : \mathbb{R}^+ \to H^1(\Omega)$ a weak solution of (2.36) if

$$\begin{aligned} &u \in L^\infty_{loc}(\mathbb{R}^+; H^1(\Omega)) \cap W^{1,\infty}_{loc}(\mathbb{R}^+; L^2(\Omega)), \\ &{}^t u \in H^1_{loc}(\mathbb{R}^+; L^2(\Gamma)), \ \text{where } {}^t u = \ \text{trace } u, \end{aligned}$$

and for every $\phi \in H^1(\Omega)$ we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u_t\phi \, dx + \int_\Omega u_t\phi \, dx + \int_\Omega \nabla u \nabla\phi \, dx + \int_\Omega f(x,u)\phi \, dx \\ &+ \int_\Gamma (b(x)u_t + a(x)u)\phi \, d\sigma = \int_\Omega g_1(t)\phi \, dx + \int_\Gamma g_2(t)\phi \, d\sigma. \end{aligned} \tag{2.37}$$

In order to prove the convergence result, we rewritten Eq.(2.36) as the abstract equation. For this, we proceed in the following steps.

Step 1 (`The corresponding Hilbert spaces`)

Let $\mathcal{H} = \mathcal{W} = L^2(\Omega) \times L^2(\Gamma)$ and let $\mathcal{V} = \{\boldsymbol{u} = (u_1, u_2) \in \mathcal{H} : u_1 \in H^1(\Omega),\ u_2 = {}^t u_1 = \text{trace } u_1\}$. We equip $\mathcal{H}$ with the usual inner product :

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{H}} = (u_1, v_1)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\Gamma)}, \text{ for all } \boldsymbol{u} = (u_1, u_2),\ \boldsymbol{v} = (v_1, v_2) \in \mathcal{H},$$

and we equip $\mathcal{V}$ with the $H^1(\Omega)$ inner product :

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{V}} = (\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)}, \text{ for all } \boldsymbol{u} = (u,\ {}^t u),\ \boldsymbol{v} = (v,\ {}^t v) \in \mathcal{V}.$$

Since the embeddings $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ are compact, the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$ is compact too. Moreover $\mathcal{V}$ is dense in $\mathcal{H}$ by the following argument. Let $\boldsymbol{f} = (f_1, f_2) \in \mathcal{H}$ be arbitrary and let $\varepsilon > 0$. Since the embedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is dense, there exists $u_2 \in H^{\frac{1}{2}}(\Gamma)$ such that $\|u_2 - f_2\|_{L^2(\Gamma)} \leq \frac{\varepsilon}{2}$. By surjectivity of the trace operator $H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$, there exists $\widetilde{u}_1 \in H^1(\Omega)$ such that ${}^t\widetilde{u}_1 = u_2$. In particular, $\widetilde{u} = (\widetilde{u}_1, u_2) \in \mathcal{V}$. Next, since $D(\Omega)$ is dense in $L^2(\Omega)$, there exists $u_1 \in D(\Omega)$ such that $\|u_1 + \widetilde{u}_1 - f_1\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2}$. Note that $\boldsymbol{u} = (u_1 + \widetilde{u}_1, u_2) \in \mathcal{V}$ and $\|\boldsymbol{u} - \boldsymbol{f}\|_{\mathcal{H}} \leq \varepsilon$. As a consequence, $\mathcal{V}$ is dense in $\mathcal{H}$.

Step 2 (*The energy functional*)

We define the energy functional $E : \mathcal{V} \longrightarrow \mathbb{R}$ by,

$$E(\boldsymbol{u}) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega F(x, u) dx + \frac{1}{2}\int_\Gamma a(x)|{}^t u|^2 d\sigma,$$

where $F(x, u) = \int_0^u f(x, s) ds$.

Lemma 2.4.1. *One has $E \in C^2(\mathcal{V})$ and for all $\boldsymbol{u} = (u,\ {}^t u)$, $\Phi = (\phi,\ {}^t\phi), \Psi = (\psi,\ {}^t\psi) \in \mathcal{V}$, we have*

$$(E'(\boldsymbol{u}), \Psi)_{\mathcal{V}', \mathcal{V}} = \int_\Omega \nabla u \nabla \psi dx + \int_\Omega f(x, u)\psi dx + \int_\Gamma a\ {}^t u\ {}^t\psi d\sigma, \tag{2.38}$$

and

$$E''(\boldsymbol{u})(\Phi, \Psi) = \int_\Omega \nabla\phi\nabla\psi dx + \int_\Omega \frac{\partial f}{\partial u}(x, u)\phi\psi dx + \int_\Gamma a\ {}^t\phi\ {}^t\psi d\sigma.$$

Proof. The first and the last term in the definition of E are continuous and quadratic, hence C^∞, and the corresponding formulas for the first and the second derivative hold. Moreover, using (*F1*) and (*F2*), we can show that the function $\mathcal{T} : \mathcal{X} \longrightarrow \mathbb{R}$ given by $\mathcal{T}(u) = \int_\Omega F(x, u) dx$ is C^2 on $\mathcal{X}$, where

$$\begin{cases} \mathcal{X} = L^{\frac{2N}{N-2}} & \text{if } N \geq 3, \\ \mathcal{X} = L^q,\ q > 0 & \text{if } N \leq 2. \end{cases}$$

By the Sobolev embedding theorem, we obtain the results. □

Step 3 (`The Łojasiewicz-Simon inequality`)

We are now in a position to prove the Łojasiewicz-Simon inequality of the energy E. The argument is essentially based on [6, Corollary 3.11]. Choose $p > \frac{N}{2}$, and let

$$\begin{aligned} X &= \{\Phi = (\phi,\ {}^t\phi) \in \mathcal{V}\ ;\ \phi \in W^{2,p}(\Omega)\} \\ \text{and } Y &= L^p(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma). \end{aligned}$$

Lemma 2.4.2. *The derivative E' maps X into Y, and $E' : X \to Y$ is analytic.*

Proof. The assertion is clear for the two linear terms in E' by the trace theorem and by the regularity of a. So we consider only the nonlinear term. By using $(F1)$ and $(F2)$, we obtain that the function $\mathcal{G}(u_1, u_2) = (f(x, u_1), 0)$ is analytic from $L^\infty(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma)$ into itself. Then the claim follows from the the Sobolev embedding theorem which gives $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ (recall that $p > \frac{N}{2}$). □

Lemma 2.4.3. *Let $\mathcal{L} = E''$ be the second derivative of E and let $\boldsymbol{u} \in \mathcal{V}$ be such that $E'(\boldsymbol{u}) = 0$. Then $\mathcal{L}(\boldsymbol{u})$ is a Fredholm operator from $\mathcal{V}$ to $\mathcal{V}'$ and from X to Y, and Ker $\mathcal{L}(\boldsymbol{u})$ is contained in X.*

Proof. First, since $E'(\boldsymbol{u}) = 0$, then u is a weak solution of the stationary problem

$$\begin{cases} -\Delta u + f(x, u) = 0 & \text{in } \Omega, \\ \partial_\nu u + au = 0 & \text{on } \Gamma. \end{cases}$$

By standard regularity theory for elliptic equations (see, for example, the classial paper by Agmon, Douglis and Nirenberg [1, Theorem 15.2]), and by a bootstrap argument, we obtain that $u \in L^\infty(\Omega)$.
Now, we write $\mathcal{L}(\boldsymbol{u}) = \mathcal{A} + \mathcal{B}$, where

$$(\mathcal{A}\Phi, \Psi)_{\mathcal{V}',\mathcal{V}} = \int_\Omega \nabla\phi\nabla\psi dx + \int_\Omega \phi\psi dx + \int_\Gamma {}^t\phi\ {}^t\psi d\sigma,$$

and

$$(\mathcal{B}\Phi, \Psi)_{\mathcal{V}',\mathcal{V}} = \int_\Omega (\frac{\partial f}{\partial u} - 1)\phi\psi dx + \int_\Gamma (a-1){}^t\phi\ {}^t\psi d\sigma.$$

By the Lax-Milgram theorem, the operator $\mathcal{A}$ is an isomorphism from $\mathcal{V}$ to $\mathcal{V}'$, and by standard regularity theory for elliptic equations [1, Theorem 15.2] it is also an isomorphism from X to Y.
On the other hand, $\mathcal{B}$ is a compact operator from $\mathcal{V}$ to $\mathcal{V}'$ (respectively from X to Y). In fact, the embeddings $\mathcal{V} \hookrightarrow L^q(\Omega) \times L^2(\Gamma)$ and $L^{q'}(\Omega) \times L^2(\Gamma) \hookrightarrow \mathcal{V}'$ are compact for every

$$2 \le q < 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N \ge 3, \\ \infty & \text{if } N = 1,\ 2, \end{cases}$$

and $\mathcal{B}$ is well-defined, linear and continuous from $L^q(\Omega) \times L^2(\Gamma)$ into $L^{q'}(\Omega) \times L^2(\Gamma)$ for some $2 \le q < 2^*$ large enough. In addition, the embedding $X \hookrightarrow L^\infty(\Omega) \times$

$W^{1,p}(\Gamma)$ is compact, the embedding $L^p(\Omega) \times W^{1,p}(\Gamma) \hookrightarrow L^p(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma) = Y$ is continuous, and $\mathcal{B}$ is well-defined, linear and continuous from $L^\infty(\Omega) \times W^{1,p}(\Gamma)$ into $L^p(\Omega) \times W^{1,p}(\Gamma)$. It follows that $\mathcal{B}$ is compact. Then, by the Riesz-Schauder theorem, $\mathcal{L}(\boldsymbol{u})$ is a Fredholm operator from $\mathcal{V}$ to $\mathcal{V}'$ and from X to Y.
Let $\Phi = (\phi,\ {}^t\phi) \in \text{Ker } \mathcal{L}(\boldsymbol{u})$. Then, for all $\Psi = (\psi,\ {}^t\psi) \in \mathcal{V}$ we have

$$\int_\Omega \nabla\phi\nabla\psi dx + \int_\Omega \frac{\partial f}{\partial u}(x,u)\phi\psi dx + \int_\Gamma a\ {}^t\phi\ {}^t\psi d\sigma = 0,$$

that is, ϕ is a weak solution to the following nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta\phi + \dfrac{\partial f}{\partial u}(x,u)\phi = 0 & \text{in } \Omega, \\ \partial_\nu\phi + a\phi = 0 & \text{on } \Gamma. \end{cases}$$

Using the fact that $u \in L^\infty(\Omega)$, we obtain $\dfrac{\partial f}{\partial u}(x,u) \in L^\infty(\Omega)$. Moreover, by assumption, $a \in W^{1,\infty}(\Gamma)$. Then, again by the regularity theory for the elliptic problem [1, Theorem 15.2], $\phi \in W^{2,p}(\Omega)$, which implies that $\Phi \in X$. □

Proposition 2.4.1. *The energy function E satisfies the Łojasiewicz-Simon inequality near every equilibrium point $\boldsymbol{u} \in \mathcal{V}$, that is, for every $\boldsymbol{u} \in \mathcal{V}$ with $E'(\boldsymbol{u}) = 0$, there exist $\beta > 0$, $\sigma > 0$ and $0 < \theta \le \frac{1}{2}$ such that*

$$| E(\boldsymbol{u}) - E(\Phi) |^{1-\theta} \le \beta \| E'(\Phi) \|_{\mathcal{V}'}, \textit{ for all } \Phi \in \mathcal{V} \textit{ such that } \| \boldsymbol{u} - \Phi \|_{\mathcal{V}} < \sigma.$$

Proof. Using Lemma 2.4.2 and Lemma 2.4.3, this result is an immediate consequence of [6, Corollary 3.11], applied with $\mathcal{P} \in \mathcal{L}(\mathcal{H})$ the orthogonal projection onto Ker $\mathcal{L}(\boldsymbol{u})$. □

Step 4 (*Reformulation of the problem*)

Lemma 2.4.4. *Let $A,\ B : \mathcal{H} \to \mathcal{H}$ be given by*

$$A(u,v) = (u,0),\ \ B(u,v) = (u,bv).$$

Let u be a solution of equation (2.36)*. Then $\boldsymbol{u} = (u,\ {}^tu)$ is a solution of equation* (2.1)*, where $M(\boldsymbol{u}) = E'(\boldsymbol{u})$ and $g = (g_1, g_2)$.*

Proof. Using the definition of the solution of equation (2.36) and the trace theorem we deduce that $\boldsymbol{u} \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}) \cap H^1_{loc}(\mathbb{R}^+, \mathcal{H})$. Moreover, by (2.37), (2.38) we have

$$(g - E'(\boldsymbol{u}) - B\dot{\boldsymbol{u}}, \Phi)_{\mathcal{V}',\mathcal{V}} = (u_{tt}, \phi)_{H^1(\Omega)',H^1(\Omega)} = \partial_t(u_t, \phi)_{L^2(\Omega)}$$

$$= (A\dot{\boldsymbol{u}}, \Phi)^{\cdot}_{\mathcal{H}} = ((A\dot{\boldsymbol{u}})^{\cdot}, \Phi)_{\mathcal{V}',\mathcal{V}}, \quad \text{for all } \Phi \in \mathcal{V}.$$

Then $A\dot{\boldsymbol{u}} \in H^1_{loc}(\mathbb{R}^+, \mathcal{V}')$ and $\boldsymbol{u}$ is a weak solution of equation (2.1). □

Step 5 (*The convergence result*)

Theorem 2.4.1. *Let $u : \mathbb{R}^+ \to H^1(\Omega)$ be a global bounded weak solution of Eq.*(2.36) *and assume that g satisfies* (2.3). *Then there exists $\phi \in H^1(\Omega)$, solution of*

$$\begin{cases} -\Delta\phi + f(x,\phi) = 0 & \text{in } \Omega, \\ \partial_\nu \phi + a\phi = 0 & \text{on } \Gamma, \end{cases} \tag{2.39}$$

such that

$$\|u_t(t)\|_{L^2} + \| u(t) - \phi \|_{H^1(\Omega)} \longrightarrow 0 \quad \text{as } t \to \infty,$$

and there exists a constant $C' > 0$ such that for all $t \geq 0$ we have

$$\|u(t)-\phi\|_{L^2(\Omega)} + \|u(t)-\phi\|_{L^2(\Gamma)} \leq C'(1+t)^{-\eta}, \quad \text{where } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\},$$

and where θ is the Łojasiewicz exponent (associated with $\Phi = (\phi,\ {}^t\phi)$) given in Proposition 2.4.1.
If, in addition, $g = 0$ and $\theta = \frac{1}{2}$, then there exist constants $C'' > 0$, $\xi > 0$ such that for all $t \geq 0$ we have

$$\|u(t)-\phi\|_{L^2(\Omega)} + \|u(t)-\phi\|_{L^2(\Gamma)} \leq C'' e^{-\xi t}.$$

Proof. First, it is easy to verify that A and B satisfy the hypotheses of the operators used in the abstract case. Moreover, assumption $(H1)$ and $(H2)$ of Theorem 2.3.1 are satisfied by Remark 2.3.1 and Remark 2.4.1. Assumption $(H3)$ of Theorem 2.3.1 is satisfied by Proposition 2.4.1. Then it remains to verify assumption $(H4)$ (note that the assumptions $E \in C^2(\mathcal{V})$ and that g satisfies (2.3) are verified by Lemma 2.4.1 and assumption of Theorem 2.4.1).
For this, let $L : \mathcal{V} \to \mathcal{V}'$ be the linear operator associated with the inner product on the space $\mathcal{V}$ and let $K = L^{-1}$. We equip $\mathcal{V}'$ with the inner product

$$(g_1, g_2)_{\mathcal{V}'} = (Kg_1, Kg_2)_{\mathcal{V}}, \; g_1, g_2 \in \mathcal{V}'.$$

Then for all $\boldsymbol{u} \in \mathcal{H}, \boldsymbol{v} \in \mathcal{V}'$, we have

$$(\boldsymbol{u}, K\boldsymbol{v})_{\mathcal{H}} = (\boldsymbol{u}, K\boldsymbol{v})_{\mathcal{V}',\mathcal{V}} = (LK\boldsymbol{u}, K\boldsymbol{v})_{\mathcal{V}',\mathcal{V}} = (K\boldsymbol{u}, K\boldsymbol{v})_{\mathcal{V}} = (\boldsymbol{u}, \boldsymbol{v})_{\mathcal{V}'}.$$

Moreover, for all $\boldsymbol{u} \in \mathcal{V}, \boldsymbol{v} = (v_1, v_2) \in \mathcal{H}$, we have

$$K \circ M'(\boldsymbol{u})\boldsymbol{v} = \boldsymbol{v} + L^{-1}\big(\frac{\partial f}{\partial u}(.,u)v_1, av_2\big) + L^{-1}\boldsymbol{v}, \text{ in } \mathcal{H}.$$

From this, the growth assumption on f and the Sobolev embedding theorem, it is not difficult to deduce that the condition $(H4)$ of Theorem 2.3.1 is satisfied. Then Theorem 2.3.1 applies. As a consequence, Theorem 2.4.1 is proved. □

2.4.2 Semilinear parabolic equation with dynamical boundary condition

Similar results are obtained for the following semilinear parabolic equation :

$$\begin{cases} u_t - \Delta u + f(x,u) = g_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ bu_t + \partial_\nu u + au = g_2 & \text{on } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \;\; u_t(0) = u_1. \end{cases} \tag{2.40}$$

Here, $g = (g_1, g_1)$, Ω, Γ, ν, f, $a(x)$, and $b(x)$ are as in the first example.

A function $u : \mathbb{R}^+ \to H^1(\Omega)$ is called a weak solution of equation (2.40) if $u \in L^\infty_{loc}(\mathbb{R}^+; H^1(\Omega)) \cap W^{1,\infty}_{loc}(\mathbb{R}^+; L^2(\Omega))$, ${}^t u \in H^1_{loc}(\mathbb{R}^+; L^2(\Gamma))$, and for all $\phi \in H^1(\Omega)$ one has

$$\int_\Omega (u_t + f(x,u))\phi dx + \int_\Omega \nabla u \nabla \phi dx + \int_\Gamma (b\ {}^t u_t + a\ {}^t u)\phi d\sigma =$$
$$= \int_\Omega g_1(t)\ {}^t\phi dx + \int_\Gamma g_2(t)\ {}^t\phi d\sigma.$$

We have the following result.

Theorem 2.4.2. *Let $u : \mathbb{R}^+ \to H^1(\Omega)$ be a global bounded weak solution of equation* (2.40) *and assume that g satisfies assumption* (2.3). *Then, there exists $\phi \in H^1(\Omega)$, solution of* (2.39) *such that $\lim_{t\to\infty} u(t) = \phi$ in $H^1(\Omega)$. Moreover, there exists a constant $C' > 0$ such that for all $t \geq 0$ we have*

$$\|u(t) - \phi\|_{L^2(\Omega)} + \|u(t) - \phi\|_{L^2(\Gamma)} \leq C'(1+t)^{-\eta}, \quad \text{wher } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\},$$

where θ is the Łojasiewicz exponent (associated with $\Phi = (\phi,\ {}^t\phi)$) given by Proposition 2.4.1.
If, in addition, $g = 0$ and $\theta = \frac{1}{2}$, then there exist constants C'', $\xi > 0$ such that for all $t \geq 0$ we have

$$\|u(t) - \phi\|_{L^2(\Omega)} + \|u(t) - \phi\|_{L^2(\Gamma)} \leq C'' e^{-\xi t}.$$

Proof. Similarly as in the proof of Theorem 2.4.1, we rewrite Eq.(2.40) in an abstract setting in the Hilbert space $\mathcal{H}$ as Eq.(2.35), where $\mathcal{V}$, $\mathcal{W}$, B and E are as in the first example. Using Proposition 2.4.1, Remark 2.3.1, and Remark 2.4.1, the claim follows from Corollary 2.3.1. □

2.4.3 Nonlinear hyperbolic-parabolic partial differential equations

Let $K_1(x)$, $K_2(x) \in L^\infty(\Omega)$, $K_1(x) \geq 0$, $K_2(x) \geq \beta > 0$, let $p > 0$ and consider the mixed hyperbolic-parabolic equation

$$\begin{cases} K_1(x)u_{tt} + K_2(x)u_t - \Delta u + |u|^p u = g & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \ \sqrt{K_1}u_t(0) = \sqrt{K_1}u_1. \end{cases} \tag{2.41}$$

A function $u : \mathbb{R}^+ \to H_0^1(\Omega)$ is called a weak solution of equation (2.41) if

$$u \in L^\infty_{loc}(\mathbb{R}^+; H_0^1(\Omega)) \cap H^1_{loc}(\mathbb{R}^+; L^2(\Omega)),$$
$$K_1^{\frac{1}{2}} u_t \in L^\infty_{loc}(\mathbb{R}^+; L^2(\Omega)),$$

and for all $\phi \in H^1(\Omega)$ one has

$$\frac{d}{dt}\int_\Omega K_1 u_t \phi \, dx + \int_\Omega K_2 u_t \phi \, dx + \int_\Omega \nabla u \nabla \phi \, dx + \int_\Omega |u|^p u \phi \, dx = \int_\Omega g\phi \, dx.$$

Let $\mathcal{H} = \mathcal{W} = L^2(\Omega)$ and $\mathcal{V} = H_0^1(\Omega)$ and define the operators A and B in $\mathcal{H}$ by

$$(Au)(x) = K_1(x)u(x), \quad (Bu)(x) = K_2(x)u(x).$$

The above model (2.41) may be rewritten as equation (2.1) on the space $\mathcal{H}$, where the energy functional $E : H_0^1(\Omega) \to \mathbb{R}$ is given by

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx + \frac{1}{p+2}\int_\Omega |v|^{p+2} \, dx, \quad v \in H_0^1(\Omega).$$

We have the following result.

Theorem 2.4.3. *Let $u : \mathbb{R}^+ \to H_0^1(\Omega)$ be a global bounded weak solution of equation (2.41). Suppose that $p \in (0, \frac{2}{N-2})$ and g satisfies assumption (2.3). Then*

$$u(t) \to 0 \quad \text{in } H^1(\Omega),$$

and there exists a constant $C' > 0$ such that for all $t \geq 0$ we have

$$\|u(t)\|_{L^2(\Omega)} \leq C'(1+t)^{-\frac{\delta}{2}}.$$

In addition, if $g = 0$, then there exist constants C'', $\xi > 0$ such that for all $t \geq 0$ we have

$$\|u(t)\|_{L^2(\Omega)} \leq C'' e^{-\xi t}.$$

Proof. It is clear that A and B satisfy the hypotheses of the operators used in the abstract case. In addition, it is easy to prove that $E \in C^2(H_0^1(\Omega))$ is strictly convex, positive and $E(0) = 0$. Then, the set of all stationary points is reduced to the point 0. Since the ω-limit set of every global and bounded solution u consists only of equilibrium points and is non-empty, then $\omega(u) = \{0\}$. Moreover, by ([6], Example 4.9) the functional E satisfies the Łojasiewicz-Simon inequality near 0 with Łojasiewicz exponent $\theta = \frac{1}{2}$.
The duality mapping $K : H^{-1}(\Omega) \to H_0^1(\Omega)$ is given by $Kv = (-\Delta)^{-1}v$, so that $K \circ M'(v) = I + (-\Delta)^{-1}|v|^p$. From this, and the Sobolev embedding theorem (here, we need $p \in (0, \frac{2}{N-2})$), it is not difficult to deduce that condition $(H4)$ of Theorem 2.3.1 is satisfied. The claim follows from Theorem 2.3.1. □

2.4.4 Semilinear evolutionary damped wave equation of mixed order

Consider the following damped, mixed problem :

$$\begin{cases} K_1(x)u_{tt} + c_1 u_t - c_2\Delta u_t - \Delta u + f(x,u) = g & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \ \sqrt{K_1}u_t(0) = \sqrt{K_1}u_1. \end{cases} \tag{2.42}$$

Here c_1, $c_2 \geq 0$, $c_1 + c_2 > 0$, $g \in L^2_{Loc}(\mathbb{R}^+; L^2(\Omega))$, and K_1 are as above.
Let $\mathcal{V} = H_0^1(\Omega)$, $\mathcal{H} = L^2(\Omega)$ and let $\mathcal{W} = \mathcal{V}$ if $c_2 > 0$ and $\mathcal{W} = \mathcal{H}$ if $c_2 = 0$. The above equation can be rewritten as abstract equation (2.1) if one puts A the multiplication operator associated with K_1, $B = c_1 I_{\mathcal{V}} - c_2\Delta : \mathcal{W} \to \mathcal{W}'$, where $-\Delta$ the Dirichlet-Laplace operator, and if the energy $E : H_0^1(\Omega) \to \mathbb{R}$ is given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega F(x,u)dx, \quad \text{where } F(x,u) = \int_0^u f(x,s)ds.$$

It is easy to verify that the operator B satisfies (2.2). Moreover, under the hypotheses on f (similarly as in Proposition 2.4.1), we have that $E \in C^2(\mathcal{V})$. In addition, the energy E satisfies the Łojasiewicz-Simon gradient inequality with $\theta \in (0, \dfrac{1}{2}]$ near every critical point ; see [13]. Hence, by Theorem 2.3.1, we obtain the following result on convergence of bounded solutions.

Theorem 2.4.4. *Let $u : \mathbb{R}^+ \to H_0^1(\Omega)$ be a global bounded weak solution of the Eq.(2.42) and assume that g satisfies assumption (2.3). Then $\lim_{t\to\infty} u(t) = \phi$ exists in $H_0^1(\Omega)$, ϕ is a stationary solution of (2.42), and there exists a constant $C' > 0$ such that for all $t \geq 0$ we have*

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'(1+t)^{-\eta}, \text{where } \eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\}.$$

In addition, if $g = 0$ and $\theta = \dfrac{1}{2}$, then there exist constants C'', $\xi > 0$ such that for all $t \geq 0$ we have

$$\|u(t) - \phi\|_{\mathcal{W}} \leq C'' e^{-\xi t}.$$

2.4.5 System of first and / or second order equations

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with smooth boundary Γ and let Γ_0, $\Gamma_1 \subseteq \Gamma$ be two open subsets such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \varnothing$. We consider the following coupled system :

$$\begin{cases} \alpha_1 u_{tt} + u_t - \Delta u + \dfrac{\partial f}{\partial u}(x,u,v) = g_1 \text{ in } \mathbb{R}^+ \times \Omega, \\ \alpha_2 v_{tt} + v_t - \Delta v + \dfrac{\partial f}{\partial v}(x,u,v) = g_2 \text{ in } \mathbb{R}^+ \times \Omega, \\ b u_t + \frac{\partial u}{\partial n} + a u = g_3 \text{ on } \mathbb{R}^+ \times \Gamma_0, \\ u = 0 \text{ on } \mathbb{R}^+ \times \Gamma_1, \\ v = 0 \text{ on } \mathbb{R}^+ \times \Gamma, \\ (u(0), v(0)) = (u_0, v_0), \\ (\sqrt{\alpha_1} u_t(0), \sqrt{\alpha_2} v_t(0)) = (\sqrt{\alpha_1} u_1, \sqrt{\alpha_2} v_1). \end{cases} \tag{2.43}$$

Here, $\alpha_i \geq 0$, $(i = 1, 2)$, $(g_1, g_2, g_3) \in L^2(\mathbb{R}^+ \times \Omega)^2 \times L^2(\mathbb{R}^+ \times \Gamma_0)$. The function $f = f(x,u,v) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function satisfying the following assumptions :
(F1) f is analytic in $(u,v) \in \mathbb{R}^2$, uniformly with respect to $x \in \Omega$ and (u,v) in bounded subsets of $\mathbb{R}^2$,
(F2) If $N = 1$, then $\dfrac{\partial f}{\partial u}, \dfrac{\partial^2 f}{\partial u^2}, \dfrac{\partial f}{\partial v}, \dfrac{\partial^2 f}{\partial v^2}$ and $\dfrac{\partial^2 f}{\partial u \partial v}$ are bounded in $\Omega \times [-r,r]^2$, for every $r > 0$.
If $N \geq 2$, then $(\dfrac{\partial f}{\partial u}(\cdot,0,0), \dfrac{\partial f}{\partial v}(\cdot,0,0)) \in (L^\infty(\Omega))^2$, and there exist constants $\rho \geq 0$, $\mu > 0$, and $(N-2)\mu < 2$ such that :

$$|\nabla^2_{u,v} f(x,u,v)| \leq c(1 + |u|^\mu + |v|^\mu) \ \text{ for all } (u,v) \in \mathbb{R}^2,\ x \in \Omega,$$

where $\nabla^2_{u,v} f(x,u,v)$ is the second derivative of f with respect to u and v.
A global (weak) solution of equation (2.43) is a function (u,v) satisfying the following properties

- $u \in L^\infty_{loc}(\mathbb{R}^+; H^1(\Omega)) \cap H^1_{loc}(\mathbb{R}^+; L^2(\Omega))$, $\sqrt{\alpha_1} u_t \in L^\infty_{loc}(\mathbb{R}^+; L^2(\Omega))$, ${}^t u|_{\Gamma_1} = 0$, and ${}^t u|_{\Gamma_0} \in H^1_{loc}(\mathbb{R}^+, L^2(\Gamma_0))$.
- $v \in L^\infty_{loc}(\mathbb{R}^+; H^1_0(\Omega)) \cap H^1_{loc}(\mathbb{R}^+; L^2(\Omega))$, $\sqrt{\alpha_2} v_t \in L^\infty_{loc}(\mathbb{R}^+; L^2(\Omega))$.
- $\forall(\phi_1, \phi_2) \in H^1(\Omega) \times H^1_0(\Omega)$ one has

$$\frac{d}{dt}\int_\Omega \alpha_1 u_t \phi_1 \, dx + \frac{d}{dt}\int_\Omega \alpha_2 v_t \phi_2 \, dx + \int_\Omega \nabla u \nabla \phi_1 \, dx + \int_\Omega \nabla v \nabla \phi_2 \, dx$$

$$+ \int_\Omega (u_t + \frac{\partial f}{\partial u}(x,u,v))\phi_1 \, dx + \int_{\Gamma_0} (b\, {}^t u_t + a\, {}^t u)\, {}^t\phi_1 \, d\sigma + \int_\Omega (v_t + \frac{\partial f}{\partial v}(x,u,v))\phi_2 \, dx$$

$$= \int_\Omega (g_1(t)\phi_1 + g_2(t)\phi_2) \, dx + \int_{\Gamma_0} g_3(t)\, {}^t\phi_1 \, d\sigma.$$

In order to obtain a convergence result, we abstractly rewrite Eq.(2.43) on the space

$$\mathcal{H} = \mathcal{W} = (L^2(\Omega))^2 \times L^2(\Gamma_0).$$

Setting $H^1_{0,\Gamma_1} = \{u \in H^1(\Omega)\ ;\ {}^t u = 0 \text{ on } \Gamma_1\}$, we define the energy space as follows :

$$\mathcal{V} = \{\boldsymbol{u} = (u_1, u_2, u_3) \in \mathcal{H}\ ;\ u_1 \in H^1_{0,\Gamma_1}(\Omega), u_2 \in H^1_0(\Omega) \text{ and } u_3 = {}^t u_1\}.$$

We equip $\mathcal{H}$ with the usual inner product :

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{H}} = (u_1, v_1)_{L^2(\Omega)} + (u_2, v_2)_{L^2(\Omega)} + (u_3, v_3)_{L^2(\Gamma_0)},$$

for all $\boldsymbol{u} = (u_1, u_2, u_3), \boldsymbol{v} = (v_1, v_2, v_3) \in \mathcal{H}$.
And we equip $\mathcal{V}$ with the following inner product :

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{V}} = (\nabla u_1, \nabla v_1)_{L^2(\Omega)} + (\nabla u_2, \nabla v_2)_{L^2(\Omega)} + (u_1, v_1)_{L^2(\Omega)},$$

for all $\boldsymbol{u} = (u_1, u_2, {}^t u_1),\ \boldsymbol{v} = (v_1, v_2, {}^t v_1) \in \mathcal{V}$.
Similarly as in the first application, we can show that $\mathcal{V} \hookrightarrow \mathcal{H}$ is dense and compact.

We define the energy functional $E : \mathcal{V} \longrightarrow \mathbb{R}$ by,

$$E(\boldsymbol{u}) = \frac{1}{2}\int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2)\ dx + \int_\Omega f(x, u_1, u_2)\ dx + \frac{1}{2}\int_{\Gamma_0} a|{}^t u_1|^2\ d\sigma.$$

Lemma 2.4.5. *One has $E \in C^2(\mathcal{V})$ and for all $\boldsymbol{u} = (u_1, u_2, {}^t u_1),\ \Phi = (\phi_1, \phi_2, {}^t\phi_1), \Psi = (\psi_1, \psi_2, {}^t\psi_1) \in \mathcal{V}$, we have*

$$\begin{aligned}(E'(\boldsymbol{u}), \Psi)_{\mathcal{V}',\mathcal{V}} &= \int_\Omega (\nabla u_1 \nabla \psi_1 + \nabla u_2 \nabla \psi_2)\ dx + \int_\Omega \frac{\partial f}{\partial u}(x, u_1, u_2)\psi_1\ dx \\ &\quad + \int_\Omega \frac{\partial f}{\partial v}(x, u_1, u_2)\psi_2\ dx + \int_{\Gamma_0} a\ {}^t u_1\ {}^t\psi_1\ d\sigma,\end{aligned}$$

and

$$\begin{aligned}E''(\boldsymbol{u})(\Phi, \Psi) &= \int_\Omega \nabla\phi_1 \nabla\psi_1 + \nabla\phi_2\nabla\psi_2\ dx + \int_\Omega \frac{\partial^2 f}{\partial u^2}(x, u_1, u_2)\phi_1\psi_1\ dx \\ &\quad + \int_{\Gamma_0} a\ {}^t\phi_1\ {}^t\psi_1\ d\sigma + \int_\Omega \frac{\partial^2 f}{\partial v^2}(x, u_1, u_2)\phi_2\psi_2\ dx \\ &\quad + \int_\Omega \frac{\partial^2 f}{\partial u v}(x, u_1, u_2)\phi_1\psi_2\ dx + \int_\Omega \frac{\partial^2 f}{\partial v u}(x, u_1, u_2)\phi_2\psi_1\ dx.\end{aligned}$$

Proof. Using (**F1**) and (**F2**), we can show that the function $\mathcal{T}\ : \mathcal{X}^2 \longrightarrow \mathbb{R}$ given by $\mathcal{T}(u) = \int_\Omega f(x, u_1, u_2)dx$ is C^2, where

$$\mathcal{X} := \begin{cases} (L^{\frac{2N}{N-2}})^2 & \text{if } N \geq 3, \\ (L^q)^2\ (0 < q < \infty) & \text{if } N = 1,\ 2. \end{cases}$$

By the Sobolev embedding theorem, we obtain the results. □

Now, let A and B be the two operators on $\mathcal{H}$ given by

$$A(u,v,w) = (\alpha_1 u, \alpha_2 v, 0) \text{ and } B(u,v,w) = (u,v,bw), \quad \text{for all } (u,v,w) \in \mathcal{H}.$$

Similarly as in the first application, if (u,v) is a weak solution of (2.43) then $\boldsymbol{u} = (u, v, \ {}^t u)$ is a weak solution of (2.1), where $M(\boldsymbol{u}) = E'(\boldsymbol{u})$ and $g = (g_1, g_2, g_3)$.

Lemma 2.4.6. *Choose $p > \frac{N}{2}$. Let :*
$X = \{\Phi = (\phi_1, \phi_2, \ {}^t\phi_1) \in \mathcal{V} \ ; \ \phi_1 \in W^{2,p}(\Omega) \cap H^1_{0,\Gamma_1}(\Omega)$ and $\phi_2 \in W^{2,p}(\Omega)\}$ and $Y = (L^p(\Omega))^2 \times W^{1-\frac{1}{p},p}(\Gamma_0)$. Then E' is analytic from X to Y.

Proof. By using (**F1**) and (**F2**), we obtain that the function :

$$\mathcal{G}(u_1, u_2, u_3) = \big(\frac{\partial f}{\partial u}(x, u_1, u_2), \frac{\partial f}{\partial v}(x, u_1, u_2), 0\big)$$

is analytic from $(L^\infty(\Omega))^2 \times W^{1-\frac{1}{p},p}(\Gamma_0)$ into itself. By the Sobolev embedding theorem, we obtain the result. $\square$

Lemma 2.4.7. *Let $\mathcal{L} = E''$ be the second derivative of E and $\boldsymbol{u} \in \mathcal{V}$ be such that $E'(\boldsymbol{u}) = 0$. Then $\mathcal{L}(\boldsymbol{u})$ is a Fredholm operator from $\mathcal{V}$ to $\mathcal{V}'$ and from X to Y, and Ker $\mathcal{L}(\boldsymbol{u})$ is contained in X.*

Proof. As in the proof of Lemma 2.4.3, one shows that $\mathcal{L}(\boldsymbol{u}) = \mathcal{A} + \mathcal{B}$ is a Fredholm operator, where $\mathcal{A}$, $\mathcal{B}$: $\mathcal{V} \to \mathcal{V}'$ are given by

$$(\mathcal{A}\Phi, \Psi)_{\mathcal{V}',\mathcal{V}} = \int_\Omega (\nabla\phi_1 \nabla\psi_1 + \nabla\phi_2\nabla\psi_2)\, dx + \int_\Omega (\phi_1\psi_1 + \phi_2\psi_2)\, dx + \\ + \int_{\Gamma_0} {}^t\phi_1 \ {}^t\psi_1 \, d\sigma,$$

and

$$(\mathcal{B}\Phi, \Psi)_{\mathcal{V}',\mathcal{V}} = \int_\Omega \big(\frac{\partial^2 f}{\partial u^2}(x, u_1, u_2) - 1\big)\phi_1\psi_1 \, dx + \int_\Omega \big(\frac{\partial^2 f}{\partial v^2}(x, u_1, u_2) - 1\big)\phi_2\psi_2 \, dx \\ + \int_\Omega \frac{\partial^2 f}{\partial uv}(x, u_1, u_2)\phi_1\psi_2 \, dx + \int_\Omega \frac{\partial^2 f}{\partial vu}(x, u_1, u_2)\phi_2\psi_1 \, dx + \\ + \int_{\Gamma_0} (a-1) \ {}^t\phi_1 \ {}^t\psi_1 \, d\sigma.$$

Note that, for some $2 \le q < 2^*$, where

$$2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N \ge 3, \\ \infty & \text{if } N = 1, \, 2, \end{cases}$$

the function $\mathcal{B}$ is well-defined, linear and continuous from $(L^q(\Omega))^2 \times L^2(\Gamma_0)$ into $(L^{q'}(\Omega))^2 \times L^2(\Gamma_0)$. Also, $\mathcal{B}$ is well-defined, linear and continuous from $(L^\infty(\Omega))^2 \times$

$W^{1,p}(\Gamma_0)$ into $(L^p(\Omega))^2 \times W^{1,p}(\Gamma_0)$. Then, using the Sobolev embedding theorem and arguing as in the first application, we obtain that $\mathcal{L}(\boldsymbol{u})$ is a Fredholm operator from $\mathcal{V}$ to $\mathcal{V}'$ and from X to Y
Now, let $\Phi = (\phi_1, \phi_2, {}^t\phi_1) \in \text{Ker } \mathcal{L}(\boldsymbol{u})$. Then for all $\Psi = (\psi_1, \psi_2, {}^t\psi_1) \in \mathcal{V}$ we have $E''(\boldsymbol{u})(\Phi, \Psi) = 0$. Putting $\psi_2 = 0$ (respectively $\psi_1 = 0$) and using the regularity theory for the elliptic problem (2.44) (respectively (2.45)), we obtain $\phi_1 \in W^{2,p}(\Omega)$ (respectively $\phi_2 \in W^{2,p}(\Omega)$), where

$$\begin{cases} -\Delta\phi_1 + \dfrac{\partial^2 f}{\partial u^2}(x, u_1, u_2)\phi_1 + \dfrac{\partial^2 f}{\partial vu}(x, u_1, u_2)\phi_2 = 0 & \text{in } \Omega, \\ \partial_\nu \phi_1 + a\phi_1 = 0 & \text{on } \Gamma_0, \\ \phi_1 = 0 & \text{on } \Gamma_1, \end{cases} \tag{2.44}$$

and

$$\begin{cases} -\Delta\phi_2 + \dfrac{\partial^2 f}{\partial v^2}(x, u_1, u_2)\phi_2 + \dfrac{\partial^2 f}{\partial uv}(x, u_1, u_2)\phi_1 = 0 & \text{in } \Omega, \\ \phi_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.45}$$

□

Starting from Lemmas 2.4.5-2.4.7, we can apply [6, Corollary 3.11] to prove that the energy function E satisfies the Łojasiewicz-Simon inequality near every equilibrium point $\Phi \in \mathcal{V}$. By arguments similar to those used in the proof of Theorem 2.4.1 we obtain :

Theorem 2.4.5. *Let (u, v) be a global bounded weak solution of system (2.43) and assume that g satisfies (2.3).*
Then, there exists $(\phi_1, \phi_2) \in H^1(\Omega) \times H^1_0(\Omega)$ solution of the following nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta\phi_1 + \dfrac{\partial f}{\partial \phi_1}(x, \phi_1, \phi_2) = 0 \ \textit{in } \mathbb{R}^+ \times \Omega, \\ -\Delta\phi_2 + \dfrac{\partial f}{\partial \phi_2}(x, \phi_1, \phi_2) = 0 \ \textit{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial \phi_1}{\partial n} + a\phi_1 = 0 \ \textit{on } \mathbb{R}^+ \times \Gamma_0, \\ \phi_1 = 0 \ \textit{on } \mathbb{R}^+ \times \Gamma_1, \\ \phi_2 = 0 \ \textit{on } \mathbb{R}^+ \times \Gamma, \end{cases}$$

such that

$$\|u_t(t)\|_{L^2(\Omega)} + \|v_t(t)\|_{L^2(\Omega)} + \| u(t) - \phi_1 \|_{H^1(\Omega)} + \| v(t) - \phi_2 \|_{H^1_0(\Omega)} \longrightarrow 0,$$

and there exists a constant $C' > 0$ such that for all $t \geq 0$ we have

$$\|u(t) - \phi_1\|_{L^2(\Omega)} + \|u(t) - \phi_1\|_{L^2(\Gamma_0)} + \|v(t) - \phi_2\|_{L^2(\Omega)} \leq C'(1 + t)^{-\eta},$$

where $\eta = \inf\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\}$ *and* θ *is the Łojasiewicz exponent associated with* $\Phi = (\phi_1, \phi_2, {}^t\phi_1)$.
If, in addition, $g = 0$ *and* $\theta = \dfrac{1}{2}$, *then there exist constants* C'', $\xi > 0$ *such that for all* $t \geq 0$ *we have*

$$\|u(t) - \phi_1\|_{L^2(\Omega)} + \|u(t) - \phi_1\|_{L^2(\Gamma_0)} + \|v(t) - \phi_2\|_{L^2(\Omega)} \leq C'' e^{-\xi t}.$$

Remark 2.4.2. *Similar results of convergence still hold if the second equation of* (2.43) *is replaced by*

$$\alpha_2 v_{tt} + v_t + \alpha_3 u_t - \Delta v + \frac{\partial f}{\partial v}(x, u, v) = g_2, \ 0 \leq \alpha_3 < 2, \tag{2.46}$$

or

$$\alpha_2 v_{tt} + \alpha_4 v_t - \alpha_5 \Delta v_t - \Delta v + \frac{\partial f}{\partial v}(x, u, v) = g_2, \ \alpha_4, \alpha_5 \geq 0, \ \alpha_4 + \alpha_5 > 0. \tag{2.47}$$

In fact, in the first case, it is sufficient to take $\mathcal{W} = \mathcal{H}$ *and let* $B : \mathcal{H} \to \mathcal{H}$ *be given by* $B(u_1, u_2, u_3) = (u_1, u_2 + \alpha_3 u_1, bu_3)$. *In the second case, it is sufficient to take* $B : \mathcal{W} \to \mathcal{W}'$ *given by* $B(u, v, w) = (u, \alpha_4 v - \alpha_5 \Delta v, bw)$, *where* $\mathcal{W} = \mathcal{V}$ *if* $\alpha_5 > 0$ *and* $\mathcal{W} = \mathcal{H}$ *if* $\alpha_5 = 0$.

Remark 2.4.3. *Similar results of convergence still hold for the following equation*

$$\begin{cases} K_1(x)u_{tt} + K_2(x)u_t - \Delta u + f(x, u) = g_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ b(x)u_t + \frac{\partial u}{\partial n} + a(x)u = g_3 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases}$$

and the following system

$$\begin{cases} K_1(x)u_{tt} + K_2(x)u_t - \Delta u + \dfrac{\partial f}{\partial u}(x, u, v) = g_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ K_3(x)v_{tt} + K_4(x)v_t - \Delta v + \dfrac{\partial f}{\partial v}(x, u, v) = g_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ b(x)u_t + \frac{\partial u}{\partial n} + a(x)u = g_3 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \\ b_1(x)v_t + \frac{\partial v}{\partial n} + a_1(x)v = g_4 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases}$$

where

$$\begin{cases} (K_1, K_2, K_3, K_4) \in (L^\infty(\Omega))^4, \ K_1, K_3 \geq 0, \ K_2, K_4 > 0, \\ (a, a_1, b, b_1) \in W^{1,\infty}(\Gamma_0) \times W^{1,\infty}(\Gamma) \times L^\infty(\Gamma_0) \times L^\infty(\Gamma), \ b, \ b_1 > 0. \end{cases}$$

Remark 2.4.4. *In the previous examples we have described situations in which* $\mathcal{W} = \mathcal{H}$ *or* $\mathcal{W} = \mathcal{V}$. *By considering damped plate equations with intermediate damping, that is, fourth order equations with a damping of the type* $-\Delta u_t$ *or similar, one arrives at situations in which one has* $\mathcal{W} \neq \mathcal{H}$ *and* $\mathcal{W} \neq \mathcal{V}$. *We do not go into details.*

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Chapitre 3

Existence and asymptotic behavior of solutions to semilinear wave equations with nonlinear damping and dynamical boundary condition

Le résultat de ce chapitre fait l'objet d'un article accépté à
Journal of Dynamics and Differential Equations.

3.1 Introduction and main results

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary Γ. In this article we consider the following semilinear wave equation with nonlinear degenerate damping

$$u_{tt} + |u_t|^\alpha u_t - \Delta u + f(x,u) = g \text{ in } (0,\infty) \times \Omega, \tag{3.1}$$

subject to the dissipative boundary condition

$$\partial_\nu u + u + u_t = 0 \text{ in } (0,\infty) \times \Gamma \tag{3.2}$$

and the initial condition

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 \text{ in } \Omega. \tag{3.3}$$

Here, $\alpha \geq 0$ is a constant, ν denotes the outer normal vector to the boundary, and the function $g \in L^2_{loc}(\mathbb{R}^+, L^2(\Omega))$ is such that for all $t \in \mathbb{R}^+$ and some $\eta \geq 0$, $\delta > 0$,

$$\|g(t)\|_2 \leq \frac{\eta}{(1+t)^{1+\delta+\alpha}}. \tag{3.4}$$

Moreover, $f = f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying the following assumptions :
(F1) f is analytic in the second variable, uniformly with respect to $x \in \Omega$ and u in bounded subsets of $\mathbb{R}$.

$(F2)$ If $N = 1$, then f and $\dfrac{\partial f}{\partial s}$ are bounded in $\Omega \times [-r, r]$, for every $r > 0$.
If $N \geq 2$, then $f(\cdot, 0) \in L^\infty(\Omega)$, and there exist constants $\rho \geq 0$ and $\mu > 0$, $(N-2)\mu < 2$ such that :

$$|\frac{\partial f}{\partial s}(x,s)| \leq \rho(1 + |u|^\mu) \text{ for every } s \in \mathbb{R},\ x \in \Omega.$$

$(F3)$ There exists $\lambda < \lambda_1$ and $C \geq 0$ such that for every $s \in \mathbb{R}$ and every $x \in \Omega$,

$$F(x,s) \geq -\lambda \frac{s^2}{2} - C,$$

where $F(x,s) := \int_0^s f(x,r)\ dr$, and $\lambda_1 > 0$ is the best Sobolev constant in the following Poincaré type inequality

$$\int_\Omega |\nabla u|^2 + \int_\Gamma |{}^t u|^2 \geq \lambda_1 \int_\Omega |u|^2,\ u \in H^1(\Omega) \text{ and } {}^t u = \text{trace } u.$$

Remark 3.1.1. *(a) The growth condition (F2) implies that the Nemytskii operator associated f is locally Lipschitz continuous from $H^1(\Omega)$ into $L^2(\Omega)$.*
(b) The condition (F3) is used in Theorem 3.1.1 below in order to obtain existence of global and *bounded solutions. If the condition $\lambda < \lambda_1$ is dropped, then one obtains existence of global, but not necessarily bounded solutions.*

We study well-posedness of the system (3.1)-(3.3) in the energy space $H^1(\Omega) \times L^2(\Omega)$, and – as a main goal – the asymptotic behaviour of weak solutions when $t \to \infty$. In particular, for every initial value in the natural energy space we prove existence of a global solution which converges for large times to a stationary solution.
The case $\alpha = 0$ has been studied by the author in [17]. He has established a convergence result and decay rate estimates for bounded weak solutions under the assumption that f verifies $(F1)$, $(F2)$ and g satisfies (3.4). Note that, in the case when $\alpha = 0$ and $g = 0$, Wu and Zheng [16] have proved existence and convergence of a strong solution of (3.1)-(3.3) to a single stationary state under the same condition on f. Both articles [17] and [16] were confined to the three-dimensional case, but the results are extendable to the case of arbitrary space dimensions if f is assumed to be subcritical (condition $(F2)$).
Recently, Chergui [7] and Ben Hassen & Chergui [3] have studied the asymptotic behaviour of solutions of Eq.(3.1) under Dirichlet boundary conditions. They proved a convergence result for bounded weak solutions in the autonomous and nonautonomous case, respectively, where f verifies $(F1)$, $(F2)$ and g satisfies (3.4). Ben Hassen & Haraux [4] have in addition proved a decay estimate if the underlying energy is positive.

We introduce the finite energy space $\mathcal{H} = H^1(\Omega) \times L^2(\Omega)$, where $H^1(\Omega)$ is equipped with the norm

$$\|u\|_{H^1(\Omega)} = \Big(\int_\Omega |\nabla u|^2\ dx + \int_\Gamma {}^t u^2 d\sigma \Big)^{\frac{1}{2}}.$$

The inner product (respectively the norm) in $\mathcal{H}$, $H^1(\Omega)$, $H^1(\Omega)'$, $L^2(\Omega)$ and $L^2(\Gamma)$ is denoted by $(\cdot,\cdot)_{\mathcal{H}}$, $(\cdot,\cdot)_{H^1(\Omega)}$, $(\cdot,\cdot)_*$, $(\cdot,\cdot)_2$, and $(\cdot,\cdot)_{2,\Gamma}$ (respectively, by $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_*$, $\|\cdot\|_2$, and $\|\cdot\|_{2,\Gamma}$). The norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$.

Similarly as in Chueshov, Eller and Lasiecka [5] (see actualy, [1, 2, 15]), we define strong and weak solutions of Eqns. (3.1)-(3.3) as follows.

Définition 3.1.1. *Let $J := [0,\tau)$ with $\tau \in (0,\infty]$. A function $u \in C(J; H^2(\Omega)) \cap C^1(J; H^1(\Omega)) \cap C^2(J; L^2(\Omega))$ is called a* strong solution *of* (3.1)-(3.3)*, if u satisfies the initial conditions $u(0) = u_0$, $u_t(0) = u_1$, and if the equations* (3.1)-(3.2) *are satisfied a.e on J. A function $u \in C(J; H^1(\Omega)) \cap C^1(J; L^2(\Omega))$ is called a* weak solution *of* (3.1)-(3.3)*, if it satisfies the initial conditions $u(0) = u_0$, $u_t(0) = u_1$ and if there exists a sequence $(g^\mu) \subseteq H^1_{loc}(J; L^2(\Omega))$ and a sequence (u^μ) of corresponding strong solutions such that $g^\mu \to g$ in $L^2_{loc}(J; L^2(\Omega))$ and $u^\mu \to u$ in $C(J; H^1(\Omega)) \cap C^1(J; L^2(\Omega))$.*

Appliying nonlinear semigroup theory and using an idea from [5] we obtain our first main result which reads as follows.

Theorem 3.1.1. *Assume that f satisfies the conditions (F2) and (F3). Let $0 \leq \alpha \leq \frac{2}{N-2}$ if $N \geq 3$, and $\alpha \in \mathbb{R}^+$ if $N \leq 2$.*

(I) **Weak solutions.** *Let $(u_0, u_1) \in \mathcal{H}$, and let $g \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$ satisfy* (3.4)*. Then there exists a unique, global weak solution to Eqns.* (3.1)-(3.3)*. In addition, this weak solution satisfies the following properties :*

(T1) $u_t \in L^{\alpha+2}(\mathbb{R}^+; L^{\alpha+2}(\Omega))$ and ${}^t u_t \in L^2(\mathbb{R}^+; L^2(\Gamma))$.

(T2) (u, u_t) is bounded with values in $\mathcal{H}$.

(T3) (Energy inequality). For all t, $t' \in \mathbb{R}^+$, $t' \leq t$:

$$\mathcal{E}_u(t) + \frac{\alpha+1}{\alpha+2}\int_{t'}^{t}\int_\Omega |u_t|^{\alpha+2} + \int_{t'}^{t}\int_\Gamma |{}^t u_t|^2 \leq \mathcal{E}_u(t') + \frac{\alpha+1}{\alpha+2}\int_{t'}^{t} \|g\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}}, \quad (3.5)$$

where $\mathcal{E}_u$ is the energy of the solution u :

$$\mathcal{E}_u(t) = \frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega F(x,u)\,dx + \frac{1}{2}\int_\Gamma |{}^t u|^2\,d\sigma. \quad (3.6)$$

(T4) (Variational equality). For all $\phi \in H^1(\Omega)$ one has

$$\frac{d}{dt}\int_\Omega u_t\phi\,dx + \int_\Omega \nabla u \nabla\phi\,dx + \int_\Omega |u_t|^\alpha u_t \phi\,dx + \int_\Omega f(x,u)\phi\,dx$$
$$+ \int_\Gamma {}^t u_t\,{}^t\phi d\sigma + \int_\Gamma {}^t u\,{}^t\phi d\sigma = \int_\Omega g\phi\,dx. \quad (3.7)$$

(II) **Strong solutions.** *Assume, in addition, that $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$, $g \in H^1_{loc}(\mathbb{R}^+, L^2(\Omega))$, and that the following compatability condition on the boundary holds :*

$$u_0 + \partial_\nu u_0 + u_1 = 0 \text{ on } \Gamma.$$

Then the weak solution is strong.

An important property for the global weak solution of (3.1)-(3.3) is the relative compactness of its range, which plays a crucial role in the proof of the convergence result below.

Theorem 3.1.2. *Let f and α be as in Theorem 3.1.1. Then for every weak solution u of* (3.1)-(3.3)*, the function $U = (u, u_t)$ is uniformly continuous from $\mathbb{R}^+$ into $\mathcal{H}$, and $\bigcup_{t\geq 0}\{U(t)\}$ is relatively compact in $\mathcal{H}$.*

Our basic argument in the proof of the convergence result below is the Łojasiewicz-Simon inequality for the energy functional $E : H^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega F(x,u)\, dx + \frac{1}{2}\int_\Gamma |{}^t u|^2\, d\sigma.$$

By the regularity and growth condition of f, the function E is twice continuously Fréchet differentiable. If $E'(u) \in H^1(\Omega)'$ and $E''(u) \in \mathcal{L}(H^1(\Omega), H^1(\Omega)')$ denote the first and second derivative at a point $u \in H^1(\Omega)$, respectively, then for all ϕ, $\psi \in H^1(\Omega)$

$$(E'(u), \psi)_{H^1(\Omega)', H^1(\Omega)} = \int_\Omega \nabla u \nabla \psi\, dx + \int_\Omega f(x,u)\psi\, dx + \int_\Gamma {}^t u\, {}^t\psi\, d\sigma, \tag{3.8}$$

and

$$(E''(u)\phi, \psi)_{H^1(\Omega)', H^1(\Omega)} = \int_\Omega \nabla\phi\nabla\psi\, dx + \int_\Omega \frac{\partial f}{\partial u}(x,u)\phi\psi\, dx + \int_\Gamma {}^t\phi\, {}^t\psi\, d\sigma.$$

The proof of the following proposition – in the case $N = 3$ – can be found in [17, Proposition 9] ; the proof for general space dimensions can be easily adapted. Recall that the norm in $H^1(\Omega)'$ is denoted by $\|\cdot\|_*$.

Proposition 3.1.1. *Under the assumptions (F1) and (F2) on the function f the energy functional $E \in C^2(H^1(\Omega))$ satisfies the Łojasiewicz-Simon inequality near every equilibrium point $\phi \in H^1(\Omega)$, that is, for every $\phi \in H^1(\Omega)$ with $E'(\phi) = 0$, there exist $\beta_\phi > 0$, $\sigma_\phi > 0$ and $0 < \theta_\phi \leq \frac{1}{2}$ such that*

$$|E(\phi) - E(\psi)|^{1-\theta_\phi} \leq \beta_\phi \parallel E'(\psi) \parallel_*$$

for all $\psi \in H^1(\Omega)$ such that $\parallel \phi - \psi \parallel_{H^1(\Omega)} < \sigma_\phi$. The number θ_ϕ is called the Łojasiewicz exponent of E at ϕ.

Moreover, in order to prove convergence of global solutions, we need a uniform Łojasiewicz exponent θ independent of every equilibrium point ϕ. In practical situations, the existence of this uniform Łojasiewicz exponent amounts to suppose that the set of all equilibrium points is compact. A sufficient condition on f, which is slightly stronger than our condition (*F3*) and which implies compactness of this set, has been given by Chergui [7]. In this case (the set of all equilibrium points is compact), and since the set of all equilibrium points attracts the trajectoty at infinity, we obtain the following property :

There exists a uniform Łojasiewicz exponent $\theta \in]0, \frac{1}{2}]$, $\beta > 0$ and $T > 0$ such that for all $t \geq T$

$$|E(u(t)) - E_\infty|^{1-\theta} \leq \beta \|E'(u(t))\|_*. \tag{3.9}$$

During the last decade the Łojasiewicz-Simon inequality has been used in the study of the asymptotic behaviour of bounded solutions of many different evolution equations, see e.g [12, 8, 14, 13], and the references given therein. For a detailed study of the Łojasiewicz-Simon inequality we refer to [6].
Note that compared with LaSalle's invariance principle, a significant advantage of the approach based on the so-called Łojasiewicz-Simon inequality consists in the fact that this method also works if the set of equilibria is not discrete.
We define the ω-limit set of a function $u : \mathbb{R}^+ \to H^1(\Omega)$ by

$$\omega(u) = \{\phi \in H^1(\Omega) : \exists\, t_n \to +\infty \text{ such that } \lim_{n\to\infty} \|u(t_n) - \phi\|_{H^1(\Omega)} = 0\}.$$

If $u : \mathbb{R}^+ \to H^1(\Omega)$ is a continuous function such that the range $\{u(t) : t \geq 1\}$ is relatively compact in $H^1(\Omega)$, then it is well-known that the ω-limit set $\omega(u)$ is nonempty, compact and connected [11]. We show that the system (3.1)-(3.3) is gradient-like in the sense that the ω-limit set of every global bounded solution is a subset of the set of stationary solutions.

Lemma 3.1.1. *Let $u : \mathbb{R}^+ \to H^1(\Omega)$ be a weak solution of equation* (3.1)-(3.3)*, and assume that f satisfies (F1)-(F3) and that $\alpha \in [0, 1)$. Then :*
(i) $\lim_{t\to\infty} \|u_t(t)\|_2 = 0$.
(ii) The function E is constant on $\omega(u)$, that is $E(\phi) = E_\infty$ =const, for all $\phi \in \omega(u)$.
(iii) The ω-limit set is a subset of the set of stationary solutions.

The main result of this paper is the following stabilization result.

Theorem 3.1.3. *Let $u : \mathbb{R}^+ \to H^1(\Omega)$ be a bounded weak solution of equation* (3.1)-(3.3)*. Assume that f satisfies (F1)-(F3) and that :*
- *if $N \leq 2$ then $\alpha \in [0, \frac{\theta}{1-\theta})$,*
- *if $N \geq 3$ then $\alpha \in [0, \frac{\theta}{1-\theta}) \cap [0, \frac{4}{N-2})$,*

where θ given by (3.9)*. Then there exists $\phi \in H^1(\Omega)$, solution of the stationary problem*

$$\begin{cases} -\Delta\phi + f(x, \phi) = 0 & \text{in } \Omega, \\ \partial_\nu \phi + \phi = 0 & \text{on } \Gamma, \end{cases}$$

such that

$$\|u_t(t)\|_2 + \| u(t) - \phi \|_{H^1(\Omega)} \longrightarrow 0 \quad \text{as } t \to \infty.$$

Throughout the following we denote by C a generic positive constant which may vary from line to line, which may depend on g, f, α and the measure of Ω, but which can be chosen independently of $t \in \mathbb{R}^+$.

The paper is organized as follows : in Section 2 we study the existence and uniqueness of a global bounded solution of (3.1)-(3.3) (proof of Theorem 3.1.1). Section 3 is devoted to the compactness result for solutions (proof of Theorem 3.1.2). The convergence result for solutions is proved in Section 4 (proof of Lemma 3.1.1 and Theorem 3.1.3).

3.2 Existence and uniqueness of a global, bounded solution : Proof of Theorem 3.1.1

Proof of Theorem 3.1.1. In order to apply semigroup theory, we rewrite the system (3.1)-(3.3) as an abstract Cauchy problem. For this and as in [5], we introduce the following spaces and operators. Let $\Delta_R : D(\Delta_R) \subseteq L^2(\Omega) \to L^2(\Omega)$ be the Robin-Laplacian defined by

$$\Delta_R u = \Delta u \text{ with domain } D(\Delta_R) = \{u \in H^2(\Omega) |\ \partial_\nu u + u = 0 \text{ on } \Gamma\}.$$

This densely defined operator is injective and self-adjoint. Moreover, it can be extended to a continuous linear operator $\Delta_R : H^1(\Omega) \to H^1(\Omega)'$ via the duality

$$(-\Delta_R u, v)_{H^1(\Omega)', H^1(\Omega)} = \int_\Omega \nabla u \nabla v + \int_\Gamma {}^t u\ {}^t v\ d\sigma.$$

From this equality one sees that the negative Robin-Laplacian is positive. Hence, we can define its fractional powers. From [9] we have $D((-\Delta_R)^{\frac{1}{2}}) \equiv H^1(\Omega)$.
Let $R : H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega)$ be the Robin map which is defined as follows :

$$Rp = q \Leftrightarrow \begin{cases} \Delta q = 0 & \text{in } \Omega \\ q + \partial_\nu q = p & \text{on } \Gamma. \end{cases}$$

It is well known (see [5]) that R is continuous for every $s \in \mathbb{R}$, and that the following trace result holds true :

$$R^* \Delta_R v = -{}^t v, \text{ for all } v \in H^1(\Omega) = D((-\Delta_R)^{\frac{1}{2}}).$$

Now we introduce a nonlinear operator A on $\mathcal{H}$ with the domain

$$D(A) = \{(u, v) \in H^2(\Omega) \times H^1(\Omega) \mid u + \partial_\nu u + v = 0 \ \text{ on } \ \Gamma\}$$

by setting

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta_R(u + R({}^t v)) + |v|^\alpha v \end{pmatrix}.$$

It is easy to verify that for all $(u, v)^T \in D(A), u + R({}^t v) \in D(\Delta_R)$. Then the problem (3.1)-(3.3) is equivalent to

$$\frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} + A\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Since $|v|^\alpha v$ considered as a mapping from $H^1(\Omega)$ into $L^2(\Omega)$ is bounded, hemi-continuous and monotone, then similarly as in [5] one can check that A is monotone on H and the range of $I+A$ is H. Thus, A is a maximal monotone operator. Moreover, it follows from the Remark 3.1.1 that the operator $C(u,v)^T = (0, f(x,u))^T$ is locally Lipschitz continuous from $\mathcal{H}$ to $\mathcal{H}$. Hence, by nonlinear semigroup theory (see, for example, [1, Theorem 2.2, p.131]), for every $(u_0,u_1) \in D(A)$ and every $g \in H^1_{loc}(\mathbb{R}^+; L^2(\Omega))$ there exists a unique strong, maximal solution $(u,u_t)^T$ to (3.1)-(3.3) on the interval $[0, t_{max})$. Moreover, if $t_{max} < \infty$, we must have $\lim_{t \nearrow t_{max}} \|(u,u_t)\|_{\mathcal{H}} = +\infty$.
For a strong solution u, and for every t, $t' \in (0, t_{max})$, $t' < t$, an integration by parts yields the standard energy inequality (3.5). It follows from condition ($F2$) that

$$\int_\Omega |F(x,u_0)| \leq C(1+\|u_0\|_{H^1}^{\mu+2}),$$

where $C \geq 0$ is a constant depending only on the constants from condition ($F2$) (including the norm $\|f(\cdot,0)\|_{L^\infty}$) and the constant of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{\mu+2}(\Omega)$. It follows from this inequality and the definition of $\mathcal{E}_u$ that there exists a constant $C_1 \geq 0$ which is independent of the initial data such that

$$\mathcal{E}_u(0) \leq C_1\,(1+\|u_1\|_{L^2}^2+\|u_0\|_{H^1}^{\mu+2}). \tag{3.10}$$

On the other hand, by using condition ($F3$), one easily shows that there exists a positive constant C_2 depending on λ and λ_1, and a positive constant C_3 depending on f and the measure of Ω such that

$$\|(u(t),u_t(t))\|_{\mathcal{H}}^2 \leq C_2 \mathcal{E}_u(t) + C_3 \text{ for every } t \in (0, t_{max}). \tag{3.11}$$

We combine (3.5), (3.10) and (3.11) to obtain the a priori estimate

$$\|(u(t),u_t(t))\|_{\mathcal{H}}^2 + \int_0^t \|^t u_t\|_\Gamma^2 ds \leq C_4\,(1+\|u_1\|_{L^2}^2+\|u_0\|_{H^1}^{\mu+2} + \int_0^t \|g\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}}) \tag{3.12}$$

for every $t \in (0, t_{max})$, where $C_4 \geq 0$ depends only on the constants C_1, C_2, C_3 and on α, but is independent of the initial data and of t_{max}. This a priori estimate gives that $t_{max} = \infty$, that is, there exists a global strong solution. In addition, by the decay condition (3.4) on the function g, this global, strong solution is bounded.
We next show the continuous dependence of strong (and then also weak) solutions on the data. Let u^μ ($\mu = 1$, 2) be two strong solutions of (3.1)-(3.3), corresponding to the initial data (u_0^μ, u_1^μ) and the forcing terms g^μ. Setting $w = u^1 - u^2$, $g = g^1 - g^2$, one has

$$\begin{cases} w_{tt} + |u_t^1|^\alpha u_t^1 - |u_t^2|^\alpha u_t^2 - \Delta w + f(x,u^1) - f(x,u^2) = g \text{ in } (0,\infty)\times\Omega, \\ \partial_\nu w + w + w_t = 0 \text{ on } (0,\infty)\times\Gamma, \\ w(0) = u_0^1 - u_0^2,\ w_t(0) = u_1^1 - u_1^2. \end{cases} \tag{3.13}$$

We multiply the equation (3.13) with w_t and integrate over Ω, in order to find that

$$\frac{d}{dt}\frac{1}{2}(\|w_t\|_2^2+\|\nabla w\|_2^2+\|w_t\|_\Gamma^2)+\|w_t\|_\Gamma^2+\int_\Omega(|u_t^1|^\alpha u_t^1-|u_t^2|^\alpha u_t^2)(u_t^1-u_t^2)\,dx$$
$$+\int_\Omega(f(u^1)-f(u^2))(u_t^1-u_t^2)\,dx=\int_\Omega g w_t\,dx.$$

Integrating this equality over $(0,t)$, using the monotonicity of the function $s\mapsto|s|^\alpha s$ and the fact that the Nemytskii operator generated by f is locally Lipschitz continuous from $H^1(\Omega)$ into $L^2(\Omega)$ (note that u^1 and u^2 are bounded in $C(\mathbb{R}^+,H^1(\Omega))$ by (3.12)), we obtain

$$\frac{1}{2}(\|w_t(t)\|_2^2+\|w(t)\|_{H^1}^2)+\int_0^t\|w_t\|_\Gamma^2\,ds\le C\int_0^t\|w(s)\|_{H^1(\Omega)}^2\,ds+$$
$$+C\int_0^t\|w_t(s)\|_2^2\,ds+\frac{1}{2}\int_0^t\|g(s)\|_2^2\,ds+\frac{1}{2}(\|w_t(0)\|_2^2+\|w(0)\|_{H^1}^2).$$

From this inequality and Gronwall's lemma we infer that, for every $t\ge0$,

$$\|w_t(t)\|_2^2+\|w(t)\|_{H^1(\Omega)}^2+\int_0^t\|{}^tu_t\|_\Gamma^2\,ds$$
$$\le\frac{1}{2}e^{Ct}\Big(\int_0^t\|g(s)\|_2^2\,ds+\|w_t(0)\|_2^2+\|w(0)\|_{H^1}^2\Big).\quad(3.14)$$

The uniqueness of strong solutions is an immediate consequence of this inequality. However, it also allows us to prove existence and uniqueness of weak solutions. In order to see this, note first that the domain $D(A)$ and $H^1_{loc}(\mathbb{R}^+;L^2(\Omega))$ are dense in $H^1(\Omega)\times L^2(\Omega)$ and $L^2_{loc}(\mathbb{R}^+;L^2(\Omega))$, respectively. Hence, given $(u_0,u_1)\in H^1(\Omega)\times L^2(\Omega)$ and $g\in L^2_{loc}(\mathbb{R}_+;L^2(\Omega))$, there exist sequences $((u_0^\mu,u_1^\mu))_\mu\subseteq D(A)$, and $(g^\mu)_\mu\subseteq H^1_{loc}(\mathbb{R}^+;L^2(\Omega))$ such that

$$(u_0^\mu,u_1^\mu)\to(u_0,u_1)\ \text{ in }\ H^1(\Omega)\times L^2(\Omega),\quad\text{and}\quad g^\mu\to g\ \text{ in }\ L^2_{loc}(\mathbb{R}^+;L^2(\Omega)).$$

Let, for each $\mu\in\mathbb{N}$, u^μ be the unique strong solution to (3.1)-(3.3). By the estimate (3.12), $(u^\mu,\ {}^tu_t^\mu)$ is uniformly bounded in $C_b(\mathbb{R}^+;H^1(\Omega))\cap C_b^1(\mathbb{R}^+;L^2(\Omega)\times L^2(\mathbb{R}^+;L^2(\Gamma))$. Moreover, by the estimate (3.14), $(u^\mu,\ {}^tu_t^\mu)$ is a Cauchy sequence in $C(\mathbb{R}^+;H^1(\Omega))\cap C^1(\mathbb{R}^+;L^2(\Omega))\times L^2(\mathbb{R}^+;L^2(\Gamma))$. Let u be its limit. Then clearly $u(0)=u_0$ and $u_t(0)=u_1$, so that u is a weak solution to (3.1)-(3.3). We have thus proved existence of a weak solution. However, from the definition of weak solutions as locally uniform limits of strong solutions one easily sees that the energy inequality (3.5), the estimate (3.12), and the a priori estimate (3.14) remain true for any weak solution, respectively any pair of weak solutions ; in particular, we have proved property (*T3*). The uniqueness of weak solutions is again an immediate consequence of the a priori estimate (3.14). From the estimate (3.12) we obtain that every weak solution is bounded with values in $\mathcal{H}$ (property (*T2*)). Moreover, the properties in

($T1$) are immediate consequences of the energy inequality and the boundedness of weak solutions. Finally, in order to prove the variational equality ($T4$) we note first that this equality is satisfied pointwise (in time) for any strong solution. However, by using again that weak solutions are locally uniform limits of strong solutions, one sees that this equality remains valid for all weak solutions. □

3.3 Compact range of global and bounded solutions : Proof of Theorem 3.1.2

In this section we obtain a compactness result which generalizes the previous results in [10] to the case of dynamical boundary condition. The major difference with the result of [10] is the fact that the convergence of g to 0 provides an integrability result on some power of u_t, (property ($T1$) in Theorem 3.1.1).

In order to prove Theorem 3.1.2, let us list two lemmas. Let X be a (real) Banach space equipped with the norm $\|\cdot\|_X$ and let $S^2(\mathbb{R}^+;X)$ be the Stepanov space defined by

$$S^2(\mathbb{R}^+;X) = \{g \in L^2_{loc}(\mathbb{R}^+;X), \sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|g(s)\|_X^2 \, ds < \infty\}.$$

For any $h > 0$, $t \geq 0$ and any $g \in S^2(\mathbb{R}^+;X)$ we denote by $g^h(t)$ the difference $g(t+h) - g(t)$ and we say that g is S^1-uniformly continuous with values in X if

$$\sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|g^h(s)\|_X^2 \, ds \to 0 \text{ as } h \to 0.$$

Lemma 3.3.1 ([3]). *Assume that f satisfies (F2) and that g satisfies* (3.4). *Then the source term $H(t) = g(t) - f(t,u)$ is S^1-uniformly continuous in $L^2(\Omega)$ and $H \in S^2(\mathbb{R}^+, L^2(\Omega))$.*

Lemma 3.3.2 ([10]). *Let X, Y be two Banach spaces endowed respectively with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Assume that X is compactly embedded into Y. Then :*
(a) If $u : \mathbb{R}^+ \to Y$ is uniformly continuous and

$$\sup_{t\geq 0, \delta\in[0,1]} \| \int_t^{t+\delta} u(s) \, ds\|_X < \infty,$$

then $\bigcup_{t\geq 0}\{u(t)\}$ is relatively compact in Y.
(b) If $u \in C^1(\mathbb{R}^+, Y)$ is bounded with values in X, and if u' is uniformly continuous with values in Y, then $\bigcup_{t\geq 0}\{u'(t)\}$ is relatively compact in Y.

Proof of Theorem 3.1.2. We proceed in two steps.
`Step 1`. We show that the function $(u(t), u_t(t))$ is uniformly continuous with values in $H^1(\Omega) \times L^2(\Omega)$. For all $t \geq 0$, $h \geq 0$ we let $u^h(t) = u(t+h) - u(t)$. Since $u_t \in L^{\alpha+2}(\mathbb{R}^+, L^{\alpha+2}(\Omega))$ and $g(t) - f(t,u) \in S^2(\mathbb{R}^+, L^2(\Omega))$ we have

$$\sup_{t\geq 0} \int_t^{t+1} \|u_{tt} - \Delta u\|_{\frac{\alpha+2}{\alpha+1}} \, ds \leq C. \tag{3.15}$$

From this estimate and the fact that ${}^t u_t \in L^2(\mathbb{R}^+; L^2(\Gamma))$ we deduce easily the inequality

$$\int_t^{t+1} \|u^h(s)\|^2_{H^1(\Omega)}\, ds \le \int_t^{t+1} (\ {}^t u^h(s),\ {}^t u^h(s))_{2,\Gamma}\, ds + (\nabla u^h(s), \nabla u^h(s))_2\, ds$$

$$\le \int_t^{t+1} \|u_t^h(s)\|_2^2\, ds + \|u_t^h(t)\|_2 \|u^h(t)\|_2 + \|u_t^h(t+1)\|_2 \|u^h(t+1)\|_2$$

$$+ C_1 \sup_{[t,t+1]} \|{}^t u^h\|_{2,\Gamma} + C_2 \sup_{[t,t+1]} \|u^h\|_{\alpha+2}$$

$$\le \int_t^{t+1} \|u_t^h(s)\|_2^2\, ds + C_1 \sup_{[t,t+1]} \|{}^t u^h\|_{2,\Gamma} + C_3 \sup_{[t,t+1]} \|u^h\|_{\alpha+2},$$

where C_1, C_3 do not depend on time t. Since $(u_t,\ {}^t u_t) \in L^{\alpha+2}(\mathbb{R}^+; L^{\alpha+2}(\Omega)) \times L^2(\mathbb{R}^+; L^2(\Gamma))$, then $(u,\ {}^t u)$ is uniformly continuous from $\mathbb{R}^+$ into $L^{\alpha+2}(\Omega) \times L^2(\Gamma)$. Using this and the last inequality, we obtain

$$\int_t^{t+1} \|u^h(s)\|^2_{H^1(\Omega)}\, ds \le \int_t^{t+1} \|u_t^h(s)\|_2^2\, ds + \phi_1(h), \tag{3.16}$$

where $\phi_1(h) \to 0$ as $h \to 0$.
Moreover, since $u_t \in L^{\alpha+2}(\mathbb{R}^+; L^{\alpha+2}(\Omega))$, then we have

$$\begin{aligned}\int_t^{t+1} \|u_t^h(s)\|_2^2\, ds &\le \Big(\int_t^{t+1} \|u_t^h(s)\|_2^{\alpha+2}\, ds\Big)^{\frac{2}{\alpha+2}} \\ &\le C\Big(\int_0^{\infty} \|u_t^h(s)\|_{\alpha+2}^{\alpha+2}\, ds\Big)^{\frac{2}{\alpha+2}} \to 0 \text{ as } h \to 0.\end{aligned} \tag{3.17}$$

Actually, the fact that the right-hand side of this inequality tends to 0 as $h \to 0$ is nothing else than the fact that the left-translation semigroup on the space $L^{\alpha+2}(\mathbb{R}^+; L^{\alpha+2}(\Omega))$ is strongly continuous. This property follows easily for compactly supported, continuous functions u from the bounded convergence theorem. The strong continuity for arbitrary functions in $L^{\alpha+2}(\mathbb{R}^+; L^{\alpha+2}(\Omega))$ then follows from a density argument. By using the preceding inequality and (3.16), we obtain

$$\int_t^{t+1} \|u^h(s)\|^2_{H^1(\Omega)}\, ds \le \phi_2(h) \tag{3.18}$$

where $\phi_2(h) \to 0$ as $h \to 0$.
Now we introduce $K_h(t) = \|u_t^h(t)\|_2^2 + \|u^h(t)\|^2_{H^1(\Omega)}$. Using Lemma 3.3.1, it is easy to deduce that for any $t \ge 0$ and $\theta \in [t, t+1]$

$$\begin{aligned}K_h(t+1) - K_h(\theta) &\le C \int_t^{t+1} \|(g - f(x,u))^h\|_2^2\, dxds \\ &\le \phi_3(h), \ \text{where } \phi_3(h) \to 0 \ \text{ as } \ h \to 0.\end{aligned} \tag{3.19}$$

In the other hand, combining (3.17) and (3.18), we get

$$\int_t^{t+1} K_h(\theta)\,d\theta \leq \phi_4(h), \text{ where } \phi_4(h) \to 0 \text{ as } h \to 0.$$

Then, by integrating (3.19) over $[t, t+1]$ with respect to θ, we obtain

$$K_h(t+1) \leq \phi_3(h) + \int_t^{t+1} K_h(\theta)\,d\theta \leq \phi_5(h),$$

which tends to 0 as $h \to 0$. This concludes the proof of `Step 1`.
`Step 2`. We show that (u, u_t) has relatively compact range in $H^1(\Omega) \times L^2(\Omega)$. By applying Lemma 3.3.2(b) with $Y = L^2(\Omega)$ and $X = H^1(\Omega)$, we obtain immediately that $\bigcup_{t\geq 0}\{u_t(t)\}$ is relatively compact in $L^2(\Omega)$. To prove that $\bigcup_{t\geq 0}\{u(t)\}$ is relatively compact in $H^1(\Omega)$, we remark that

$$u_t(t+h) - u_t(t) - \int_t^{t+h} \Delta u(s)ds + \int_t^{t+h} |u_t|^\alpha u_t\, ds = \int_t^{t+h} (g(s) - f(x, u(s)))ds$$

By using $(F2)$, Lemma 3.3.1, the property that $u_t \in L^{\alpha+2}(\mathbb{R}^+; L^{\alpha+2}(\Omega))$ and the fact that (u, u_t) is bounded in $H^1(\Omega) \times L^2(\Omega)$, we obtain

$$\sup_{t\geq 0, \delta\in[0,1]} \| \int_t^{t+\delta} \Delta u(s)\, ds\|_{\frac{\alpha+2}{\alpha+1}} < \infty.$$

By applying Lemma 3.3.2(a) with $Y = H^1(\Omega)$ and $X = \{\phi \in H^1(\Omega); \Delta\phi \in L^{\frac{\alpha+2}{\alpha+1}}\}$, we obtain the claim. □

3.4 Convergence of global solutions : Proof of Lemma 3.1.1 and Theorem 3.1.3

In the following proof we identify the dual of $H^1(\Omega)$ through the embedding $j : H^1(\Omega) \to L^2(\Omega) \times L^2(\Gamma)$, $u \mapsto (u,\ {}^t u)$ and the identification of $L^2(\Omega) \times L^2(\Gamma)$ with its dual. In particular, pairs $(g, h) \in L^2(\Omega) \times L^2(\Gamma)$ act as linear functionals on $H^1(\Omega)$ via integration on Ω and on the boundary Γ :

$$u \mapsto \int_\Omega ug + \int_\Gamma {}^t uh.$$

For simplicity of notation, we identify single elements $g \in L^2(\Omega)$ resp. $h \in L^2(\Gamma)$ with functionals on $H^1(\Omega)$ by identifying them with the elements $(g, 0)$ and $(0, h)$, respectively.

Proof of Lemma 3.1.1. From (T2) we have that u_t is bounded in $L^2(\Omega)$. Then $|u_t|^\alpha u_t$ is bounded in $L^{\frac{2}{\alpha+1}}(\Omega)$. Note that if $s \in \mathbb{R}^+$ is such that $\frac{N}{2}\alpha \leq s$, then the space $H^s(\Omega)$ is continuously embedded into $L^{\frac{2}{1-\alpha}}(\Omega)$, which implies that $L^{\frac{2}{\alpha+1}}(\Omega)$ is continuously embedded in $H^s(\Omega)'$. It follows that $|u_t|^\alpha u_t$ is bounded in $(H^s(\Omega))'$. Then,

by this, the growth assumption ($F2$), and the equation (3.1), we obtain that u_{tt} is the sum of a function which is bounded with values in $H^s(\Omega)'$, and a function which is square integrable with values in $H^s(\Omega)'$. Hence, u_t is uniformly continuous with values in $(H^s(\Omega))'$. By using this and the fact that $u_t \in L^{\alpha+2}(\mathbb{R}^+, (H^s(\Omega))')$ we deduce that $\lim\limits_{t\to\infty} \|u_t\|_{(H^s(\Omega))'} = 0$. Using the relative compactness of the range of the function u_t with values in $L^2(\Omega)$, we obtain (i).
Now we rewrite (3.5) for all t, $t' \in \mathbb{R}^+$, $t \leq t'$:

$$\mathcal{E}_u(t') - \mathcal{E}_u(t) + \frac{\alpha+1}{\alpha+2}\int_t^{t'}\int_\Omega |u_t|^{\alpha+2}\, dxds + \int_t^{t'}\int_\Gamma |{}^t u_t|^2\, d\sigma ds \leq$$
$$\leq (\frac{\alpha+1}{\alpha+2})\int_t^{t'} \|g\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}}\, ds. \quad (3.20)$$

Let $\phi \in \omega(u)$. Then there exists an unbounded increasing sequence (t_n) in $\mathbb{R}^+$ such that $u(t_n) \to \phi$ in $H^1(\Omega)$. Using (i) and the regularity of E, we obtain

$$\mathcal{E}_u(t_n) \to E(\phi), \text{ as } n \text{ tends to infinity.} \quad (3.21)$$

On the other hand, by (3.4), we have $\lim\limits_{t\to\infty}\int_t^\infty \|g(s)\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}}\, ds = 0$, and by (T1) we have $\lim\limits_{t\to\infty}\int_t^\infty (\|u_t(s)\|_{\alpha+2}^{\alpha+2} + \|{}^t u_t\|_{2,\Gamma}^2)\, ds = 0$. Then, from (3.20) and (3.21) we obtain

$$\mathcal{E}_u(t) \to E(\phi), \text{ as } t \text{ tends to infinity.}$$

Finally we have (using the definition of $\mathcal{E}_u(t)$ and property (i))

$$E(\phi) = \lim_{t\to\infty} E(u(t)) = E_\infty \text{ for all } \phi \in \omega(u),$$

and the property (ii) is proved.
Moreover, since $u_t \in L^{\alpha+2}(\mathbb{R}^+, L^{\alpha+2}(\Omega))$ we obtain

$$u(t_n+s) = u(t_n) + \int_{t_n}^{t_n+s} u_t(\rho)\, d\rho \to \phi \text{ in } L^{\alpha+2}(\Omega), \text{ for every } s \in [0,1].$$

This, together with the relative compactness of the trajectory in $H^1(\Omega)$, implies that $u(t_n+s) \to \phi$ in $H^1(\Omega)$ for every $s \in [0,1]$. Hence $E'(u(t_n+s)) \to E'(\phi)$ in $H^1(\Omega)'$ for every $s \in [0,1]$. Finally, using the dominated convergence theorem, (T4) (that is, the variational equality (3.7)), (T1), (i), and (3.4), we have for all $\psi \in H^1(\Omega)$

$$(E'(\phi),\psi)_{H^1(\Omega)',H^1(\Omega)} = \int_0^1 (E'(\phi),\psi)_{H^1(\Omega)',H^1(\Omega)}\, ds$$
$$= \lim_{n\to\infty}\int_0^1 (E'(u(t_n+s)),\psi)_{H^1(\Omega)',H^1(\Omega)}\, ds$$
$$= \lim_{n\to\infty}\int_0^1 \Big(\int_\Omega \nabla u(t_n+s)\nabla\psi\, dx + \int_\Omega f(x,u(t_n+s))\psi\, dx + \int_\Gamma {}^t u(t_n+s)\ {}^t\psi\, d\sigma\Big)ds$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\int_0^1 \Big(-\frac{d}{dt}\int_\Omega u_t(t_n+s)\psi dx - \int_\Omega (|u_t|^\alpha u_t - g)(t_n+s)\psi\, dx \\
&\quad - \int_\Gamma {}^t u_t(t_n+s)\ {}^t\psi\, d\sigma\Big) ds \\
&= \lim_{n\to\infty}\Big[\int_0^1 \Big(\int_\Omega (-|u_t|^\alpha u_t + g)(t_n+s)\psi\, dx - \int_\Gamma {}^t u_t(t_n+s)\ {}^t\psi\, d\sigma\Big) ds \\
&\quad + \int_\Omega (u_t(t_n) - u_t(t_n+1))\psi\, dx\Big] \\
&= 0.
\end{aligned}
$$

This proves (iii) as desired.

□

Proof of Theorem 3.1.3. Let ε be a real positive constant which will be fixed in the sequel and let $G : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$
\begin{aligned}
G(t) = \frac{1}{2}\|u_t(t)\|_2^2 + E(u(t)) - E_\infty + \varepsilon\|u(t)\|_*^\alpha (E'(u(t)), u_t(t))_* + \\
+ \int_t^\infty (g(s), u_t(s))_2\, ds + \varepsilon(\alpha+1)\int_t^\infty \|u_t(s)\|_*^\alpha \|g(s)\|_*^2\, ds.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{d}{dt}G(t) &= -\|u_t\|_{\alpha+2}^{\alpha+2} - \|{}^t u_t\|_{2,\Gamma}^2 + \varepsilon\|u_t\|_*^\alpha (E''(u)u_t, u_t)_* + \varepsilon\|u_t\|_*^\alpha (E'(u), u_{tt})_* \\
&\quad + \alpha\varepsilon\|u_t\|_*^{\alpha-2}(E'(u), u_t)_*(u_t, u_{tt})_* - \varepsilon(\alpha+1)\|u_t\|_*^\alpha\|g\|_*^2 \\
&= -\|u_t\|_{\alpha+2}^{\alpha+2} - \|{}^t u_t\|_{2,\Gamma}^2 + \varepsilon\|u_t\|_*^\alpha (E''(u)u_t, u_t)_* - \varepsilon\|u_t\|_*^\alpha \|E'(u)\|_*^2 \\
&\quad - \varepsilon\|u_t\|_*^\alpha (E'(u), |u_t|^\alpha u_t)_* + \varepsilon\|u_t\|_*^\alpha (E'(u), g)_* - \varepsilon\|u_t\|_*^\alpha (E'(u), {}^t u_t)_* \\
&= -\alpha\varepsilon\|u_t\|_*^{\alpha-2}(E'(u), u_t)_*(u_t, |u_t|^\alpha u_t)_* - \alpha\varepsilon\|u_t\|_*^{\alpha-2}(E'(u), u_t)_*^2 \\
&\quad + \alpha\varepsilon\|u_t\|_*^{\alpha-2}(E'(u), u_t)_*(u_t, g)_* - \alpha\varepsilon\|u_t\|_*^{\alpha-2}(E'(u), u_t)_*(u_t, {}^t u_t)_* \\
&\quad - \varepsilon(\alpha+1)^2\|u_t\|_*^\alpha\|g\|_*^2 \\
&\leq -\|u_t\|_{\alpha+2}^{\alpha+2} - \|u_t\|_{2,\Gamma}^2 - \varepsilon\|u_t\|_*^\alpha \|E'(u)\|_*^2 + \varepsilon(1+\alpha)\|u_t\|_*^\alpha\|E'(u)\|_*\||u_t|^\alpha u_t\|_* \\
&\quad + \varepsilon(1+\alpha)\|u_t\|_*^\alpha\|E'(u)\|_*\|g\|_* + \frac{\alpha\varepsilon}{4}\|u_t\|_*^\alpha\|{}^t u_t\|_*^2 \\
&\quad + \varepsilon\|u_t\|_*^\alpha\|E'(u)\|_*\|{}^t u_t\|_* + \varepsilon\|u_t\|_*^\alpha (E''(u)u_t, u_t)_* - \varepsilon(\alpha+1)\|u_t\|_*^\alpha\|g\|_*^2.
\end{aligned}
$$

The subsequent simple lemma will be used in the estimates of $\dfrac{d}{dt}G(t)$.

Lemma 3.4.1. *There exists a constant $C > 0$ such that, for every $t \geq 0$,*

$$
\|E'(u)\|_*\||u_t|^\alpha u_t\|_* \leq \frac{1}{4(\alpha+1)}\|E'(u)\|_*^2 + C\|u_t\|_{\alpha+2}^{\alpha+2},
$$

and

$$\|u_t\|_*^\alpha (E''(u)u_t, u_t)_* \le C\|u_t\|_{\alpha+2}^{\alpha+2}.$$

Proof of Lemma 3.4.1. Since u is bounded in $H^1(\Omega)$ then $E'(u)$ is bounded in $H^1(\Omega)'$. Let $C_E = 1 + \sup_{t\in\mathbb{R}^+} \|E'(u(t))\|_*$. Using Young's inequality, we obtain

$$\begin{aligned}\|E'(u)\|_* \||u_t|^\alpha u_t\|_* &\le \frac{1}{4(\alpha+1)C_E^\alpha}\|E'(u)\|_*^{\alpha+2} + C\||u_t|^\alpha u_t\|_*^{\frac{\alpha+2}{\alpha+1}} \\ &\le \frac{1}{4(\alpha+1)}\|E'(u)\|_*^2 + C\|u_t\|_{\alpha+2}^{\alpha+2},\end{aligned}$$

where we have used the fact that

$$\||u_t|^\alpha u_t\|_*^{\frac{\alpha+2}{\alpha+1}} \le C\|u_t\|_{\alpha+2}^{\alpha+2}, \tag{3.22}$$

for all $\alpha \in [0, \frac{4}{N-2}[$. This can be proved as follows :

– If $N \le 2$, by the Sobolev embedding $L^{\frac{\alpha+2}{\alpha+1}} \hookrightarrow H^1(\Omega)'$ we get

$$\||u_t|^\alpha u_t\|_*^{\frac{\alpha+2}{\alpha+1}} \le C(\||u_t|^\alpha u_t\|_{\frac{\alpha+2}{\alpha+1}})^{\frac{\alpha+2}{\alpha+1}} \le C\|u_t\|_{\alpha+2}^{\alpha+2}.$$

– If $N \ge 3$, once again by the Sobolev embedding $L^{\frac{2N}{N+2}} \hookrightarrow H^1(\Omega)'$ we get

$$\||u_t|^\alpha u_t\|_*^{\frac{\alpha+2}{\alpha+1}} \le C\|u_t\|_{\frac{2N(\alpha+1)}{N+2}}^{\alpha+2}. \tag{3.23}$$

Since $\alpha \in [0, \frac{4}{N-2}[(N \ge 3)$, it follows that $\frac{2N(\alpha+1)}{N+2} \le \alpha + 2$. Consequently $L^{\alpha+2}(\Omega)$ is continuously embedded in $L^{\frac{2N(\alpha+1)}{N+2}}$. This together with (3.23) implies (3.22).

In order to prove the second estimate, let $L : H^1(\Omega) \to H^1(\Omega)'$ be the linear operator associated with the inner product on the space $H^1(\Omega)$ and let $K = L^{-1}$. We equip $H^1(\Omega)'$ with the inner product

$$(g_1, g_2)_* = (Kg_1, Kg_2)_{H^1(\Omega)},\ g_1,\ g_2 \in H^1(\Omega)'.$$

Then for all $u \in L^2(\Omega)$, $v \in H^1(\Omega)'$, we have

$$(u, Kv)_2 = (u, Kv)_{H^1(\Omega)', H^1(\Omega)} = (LKu, Kv)_{H^1(\Omega)', H^1(\Omega)} = (Ku, Kv)_{H^1(\Omega)} = (u, v)_*.$$

Moreover, for all $u \in H^1(\Omega), v \in L^2(\Omega)$, we have

$$K \circ E''(u)v = v + L^{-1}\big(\frac{\partial f}{\partial u}(x, u)v\big) \text{ in } L^2(\Omega).$$

From this, the growth assumption on f and the Sobolev embedding theorem, it is not difficult to deduce that the operator $K \circ E''(v) \in \mathcal{L}(H^1(\Omega))$ extends to a bounded linear operator on $L^2(\Omega)$ for every $v \in H^1(\Omega)$, and $K \circ E'' : H^1(\Omega) \to \mathcal{L}(L^2(\Omega))$ maps bounded sets into bounded sets. Then

$$(E''(u)u_t, u_t)_* = (K \circ E''(u)u_t, u_t)_2 \leq \|K \circ E''(u)\|_{\mathcal{L}(L^2(\Omega))}\|u_t\|_2^2 \leq C\|u_t\|_2^2$$

and

$$\|u_t\|_*^\alpha (E''(u)u_t, u_t)_* \leq C\|u_t\|_*^\alpha \|u_t\|_2^2 \leq C\|u_t\|_{\alpha+2}^{\alpha+2}.$$

□

By using this lemma, the estimates

$$\|E'(u)\|_*\|g\|_* \leq \frac{1}{4(\alpha+1)}\|E'(u)\|_*^2 + (\alpha+1)\|g\|_*^2 \text{ and}$$
$$\|E'(u)\|_*\|{}^t u_t\|_* \leq \frac{1}{4}\|E'(u)\|_*^2 + \|{}^t u_t\|_*^2 \leq \frac{1}{4}\|E'(u)\|_*^2 + C\|{}^t u_t\|_{2,\Gamma}^2,$$

and the fact that $\|u_t\|_2 < 1$ for $t > T$, choosing $\varepsilon > 0$ small enough, then

$$\frac{d}{dt}G(t) \leq -C\|u_t\|_*^\alpha\{\|u_t\|_2^2 + \|E'(u)\|_*^2\} - C\|{}^t u_t\|_{2,\Gamma}^2. \tag{3.24}$$

Then the function G is nonincreasing and $\lim_{t\to\infty} G(t) = 0$. It follows that $G(t) \geq 0$ for all $t \in \mathbb{R}^+$. If there exists $T_0 \geq T$ such that $G(T_0) = 0$, then $G(t) = 0$ for all $t \geq T_0$. By the inequality (3.24), the function u is then constant for $t \geq T_0$, i.e., $u = \phi$ for $t \geq T_0$. In this case, there remains nothing to prove. We may therefore suppose in the following that $G(t) > 0$ for every $t \geq T$.

Let $\gamma = \alpha + 1 + \delta(\dfrac{\alpha+2}{\alpha+1})$. Then $\gamma(1 - \frac{\alpha}{\alpha+1}) > 1$ and there exists $\zeta > 0$ such that $\gamma(1-\theta') > 1$ for all $\theta' \in [\frac{\alpha}{\alpha+1}, \frac{\alpha}{\alpha+1} + \zeta[$. In particular there exists θ_0 such that $\frac{\alpha}{\alpha+1} < \theta_0 < \inf\{\theta, \frac{\alpha}{\alpha+1} + \zeta\}$ that is $\theta_0 > (1-\theta_0)\alpha$ and $\gamma(1-\theta_0) > 1$. Note that (3.9) is satisfied with θ replaced by θ_0.

Now, let $\beta = \theta_0 - \alpha(1-\theta_0)$. Then $\beta > 0$ and

$$-\frac{1}{\beta}\frac{d}{dt}(G(t)^\beta) = \frac{-G'(t)}{\{G(t)^{1-\theta_0}\}^{1+\alpha}}. \tag{3.25}$$

By applying the Cauchy-Schwarz inequality we obtain

$$G(t)^{1-\theta_0} \leq C\Big\{\|u_t\|_2^{2(1-\theta_0)} + |E(u) - E_\infty|^{(1-\theta_0)} + \|u_t\|_*^{(\alpha+1)(1-\theta_0)}\|E'(u)\|_*^{(1-\theta_0)}$$
$$+ \Big(\int_t^\infty |(g, u_t)_2|\, ds\Big)^{(1-\theta_0)} + \Big(\int_t^\infty \|g\|_2^2\, ds\Big)^{(1-\theta_0)}\Big\}. \tag{3.26}$$

By Hölder's inequality,

$$2\Big(\int_t^\infty |(g, u_t)_2|\, ds\Big) \leq C\int_t^\infty \|g\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}}\, ds + \int_t^\infty \|u_t\|_{\alpha+2}^{\alpha+2}\, ds$$
$$\leq C(1+t)^{-\gamma} + \int_t^\infty \|u_t\|_{\alpha+2}^{\alpha+2}\, ds. \tag{3.27}$$

On the other hand, let

$$Z(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} + E(u(t)) - E_\infty + \int_t^\infty (g, u_t)_2\, ds. \tag{3.28}$$

The function Z is positive, $\lim\limits_{t\to\infty} Z(t) = 0$ and

$$Z'(t) = -\|{}^t u_t\|^2_{2,\Gamma} - \|u_t\|^{\alpha+2}_{\alpha+2}.$$

Then

$$\int_t^\infty \|{}^t u_t\|^2_{2,\Gamma}\, ds + \int_t^\infty \|u_t\|^{\alpha+2}_{\alpha+2}\, ds = -\int_t^\infty Z'(s)\, ds = Z(t). \tag{3.29}$$

By combining (3.27), (3.28) and (3.29), we obtain

$$\int_t^\infty |(g, u_t)_2|\, ds \le C(1+t)^{-\gamma} + C\Big(\frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} + E(u(t)) - E_\infty\Big). \tag{3.30}$$

Moreover, by (3.4) we have

$$\int_t^\infty \|g\|_2^2\, ds \le C(1+t)^{-(1+2\delta+2\alpha)} \le C(1+t)^{-\gamma}. \tag{3.31}$$

Then, by combining (3.26), (3.30) and (3.31), we obtain

$$G(t)^{1-\theta_0} \le C\Big\{\|u_t\|_2^{2(1-\theta_0)} + |E(u) - E_\infty|^{(1-\theta_0)} + (1+t)^{-\gamma(1-\theta_0)}$$
$$+\|u_t\|_*^{(\alpha+1)(1-\theta_0)}\|E'(u)\|_*^{(1-\theta_0)}\Big\}.$$

By Young's inequality, we have

$$\|u_t\|_*^{(\alpha+1)(1-\theta_0)}\|E'(u)\|_*^{(1-\theta_0)} \le \|u_t\|_*^{(\alpha+1)\frac{1-\theta_0}{\theta_0}} + \|E'(u)\|_*.$$

Moreover, by Lemma 3.1.1 (i), we can assume that $\|u_t\|_2 \le 1$ for all $t \ge T$. Note that $\frac{1-\theta_0}{\theta_0} \ge 1$ and $2(1-\theta_0) \ge 1$. It follows that

$$G(t)^{1-\theta_0} \le C\Big\{\|u_t\|_2 + |E(u) - E_\infty|^{(1-\theta_0)} + \|E'(u)\|_* + (1+t)^{-\gamma(1-\theta_0)}\Big\}.$$

Using (3.9), we obtain for every $t \ge T$

$$G(t)^{1-\theta_0} \le C\Big\{\|u_t\|_2 + \|E'(u)\|_* + (1+t)^{-\gamma(1-\theta_0)}\Big\}. \tag{3.32}$$

Then, by combining (3.24), (3.25) and (3.32), we obtain for all $t \ge T$

$$-C\frac{d}{dt}(G(t)^\beta) + \frac{1}{(1+t)^{\gamma(1-\theta_0)}} \ge$$
$$\ge \frac{\|u_t\|_*^\alpha\{\|u_t\|_2^2 + \|E'(u)\|_*^2\} + \|{}^t u_t\|^2_{2,\Gamma}}{\{\|u_t\|_2 + \|E'(u)\|_* + (1+t)^{\gamma(\theta_0-1)}\}^{1+\alpha}} + \frac{1}{(1+t)^{\gamma(1-\theta_0)}}$$
$$\ge \frac{\frac{1}{C^\alpha}\|u_t\|_*^\alpha\{\|u_t\|_2^2 + \|E'(u)\|_*^2\} + \|{}^t u_t\|^2_{2,\Gamma}}{\{\|u_t\|_2 + \|E'(u)\|_* + (1+t)^{\gamma(\theta_0-1)}\}^{1+\alpha}} + \frac{1}{(1+t)^{\gamma(1-\theta_0)}},$$

where $C^\alpha > 1$ is such that $\|u_t\|_*^\alpha \le \frac{C^\alpha}{2}\|u_t\|_2^\alpha$. Now we use the following lemma.

Lemma 3.4.2 ([3]). *Let A, a, b, $c \in \mathbb{R}^+$ be such that $a+b+c>0$ and $2A^\alpha \le a^\alpha$. Then we have*

$$\frac{A^\alpha (a+b)^2}{(a+b+c)^{\alpha+1}} + c \ge A^\alpha a^{1-\alpha}.$$

We use this lemma with $A = \frac{1}{C}\|u_t\|_*$, $a = \|u_t\|_2$, $b = \|E'(u)\|_*$ and $c = \dfrac{1}{(1+t)^{\gamma(1-\theta_0)}}$. Then we obtain

$$-C\frac{d}{dt}(G(t)^\beta) + \frac{1}{(1+t)^{\gamma(1-\theta_0)}} \ge C\|u_t\|_*^\alpha \|u_t\|_2^{1-\alpha} \ge C\|u_t\|_*. \tag{3.33}$$

Hence, by integrating (3.33), we obtain for every $t \ge T$

$$\int_T^t \|u_t\|_*\, ds \le CG(T)^\beta + \frac{C}{(1+T)^{\gamma(1-\theta_0)-1}}.$$

Thus $\|u_t\|_*$ is integrable on $[T,+\infty)$, which implies that $\lim_{t\to\infty} u(t,.)$ exists in $(H^1(\Omega))'$. By compactness (Theorem 3.1.2), $\lim_{t\to\infty} u(t,.)$ exists in $H^1(\Omega)$. This is the claim. □

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Chapitre 4

Well-posedness and asymptotic behaviour of a nonautonomous, semilinear hyperbolic-parabolic equation with dynamical boundary condition of memory type

4.1 Introduction

The main purpose of this work is to study the existence and the asymptotic behaviour of global weak solutions to the semilinear degenerate wave equation with boundary conditions of memory type given by

$$\begin{cases} K_1(x)u_{tt} + K_2(x)u_t - \Delta u + f(x,u) = g_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu u + \mu(x)u + k * u_t = g_2 & \text{on } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0, \ \sqrt{K_1}u_t(0) = \sqrt{K_1}u_1. & \end{cases} \tag{4.1}$$

Here, $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ is a bounded open connected set with smooth boundary Γ, ν denotes the outer normal vector to the boundary. The coefficients K_1, $K_2 \in L^\infty(\Omega)$, $\mu \in W^{1,\infty}(\Gamma)$ and $k \in L^1_{loc}(\mathbb{R}^+)$ are nonnegative functions, $K_2(x) \geq k_0 > 0$, μ is not identically zero on Γ, and $k * v$ stands for the convolution on the positive half-line, that is, $(k * v)(t) = \int_0^t k(t-s)v(s)\,ds$ $(t \geq 0)$.
The boundary condition arises in mathematical models for the motion of viscoelastic materials. For such materials, the feedback operator is a convolution operator in time. We consider also the case in which the kernel is singular ; a typical example for the kernel k we have in mind is given by

$$k(t) = \frac{1}{\Gamma(1-\beta)}\, t^{-\beta}\, e^{-wt} \quad (\beta \in (0,1),\ w > 0), \tag{4.2}$$

where Γ is the Gamma function.
The nonlinearity $f = f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be a C^2 function satisfying

the following assumptions :
(*F1*) The function f is analytic in the second variable, uniformly with respect to $x \in \Omega$ and u in bounded subsets of $\mathbb{R}$,
(*F2*) One has $f(\cdot, 0) \in L^\infty(\Omega)$, and there exist constants $\rho \geq 0$ and $\alpha \in (0,1)$, $(N-2)\alpha < 2$ such that :

$$|\frac{\partial f}{\partial u}(x,u)| \leq \rho(1 + |u|^\alpha) \text{ for every } u \in \mathbb{R},\ x \in \Omega.$$

(*F3*) There exists $\lambda < \lambda_1$ and $C \geq 0$ such that for every $u \in \mathbb{R}$ and every $x \in \Omega$,

$$F(x,u) \geq -\lambda \frac{u^2}{2} - C,$$

where $F(x,u) = \int_0^u f(x,s)\ ds$ ($x \in \Omega$, $u \in \mathbb{R}$), and $\lambda_1 > 0$ is the best Sobolev constant in the following Poincaré type inequality

$$\int_\Omega |\nabla u|^2 + \int_\Gamma \mu(x)|u|^2 \geq \lambda_1 \int_\Omega |u|^2 \quad (u \in H^1(\Omega)).$$

We study well-posedness of the equation (4.1) in the energy space $\mathcal{H} = H^1(\Omega) \times L^2(\Omega)$ and the asymptotic behaviour of weak solutions when $t \to \infty$. In particular, for every initial values in the natural energy space we prove the existence and uniqueness of a global, bounded solution of (4.1). In addition, we prove that every global, bounded solution has relatively compact range in $\mathcal{H}$. Then, by using a new Lyapunov functional and the Łojasiewicz-Simon inequality, we show that if g_1 and g_2 tend to 0 sufficiently fast at infinity, then the solution of (4.1) converges to a single steady state. Finally, we show that the decay rate to equilibrium is either exponential or polynomial.

Concerning existence of solutions, we carefully note that the function K_1 may vanish on Ω or on a subset of Ω. Equation (4.1) thus includes the semilinear diffusion equation ($K_1 = 0$), the semilinear wave equation ($K_1 = 1$), and mixed hyperbolic-parabolic problems ($K_1 \geq 0$). In our existence proof below, we shall first replace K_1 by $K_1 + \varepsilon$ and prove existence of solutions for this perturbed, purely hyperbolic problem by means of a Faedo-Galerkin method. We shall further obtain a priori estimates for the solutions which are independent of $\varepsilon > 0$, in such way that we can pass to the limit when ε tends to zero, obtaining thus a function u which is the solution of the problem (4.1). By derivating the equation with respect to time, we shall also prove the existence of strong solutions if the data are regular enough.

We recall that the basic argument in the proof of the convergence results is the Łojasiewicz inequality which was generalized first by L. Simon [19], then by A. Haraux and M. A. Jendoubi [14, 16, 17] (see below for the definition of the Łojasiewicz-Simon inequality).
Concerning the convergence to steady state for nonlinear equations with memory there is a technical difficulty consisting in proving that the solution of such problems

are bounded and have relatively compact range in the natural energy space. However, the more complicated problem is to find an appropriate Lyapunov functional in order to investigate the asymptotic behaviour of global, bounded solutions. For the type of kernel k and nonlinearity f as above, we note that there are up to now two techniques to construct an appropriate Lyapunov functional which allows one to apply the Łojasiewicz-Simon inequality in order to obtain a convergence result. The first technique goes back to C. Dafermos [10], and this technique was recently adaptated by S. Aizicovici and E. Feireisl [1] in order to obtain a convergence result for a phase-field model with memory (see also S. Aizicovici and H. Petzeltová [2]), and then by R. Chill and E. Fašangová [8] in order to obtain a convergence results for the wave equation, where the dissipation is both frictional and with memory :

$$u_{tt} + u_t + k * u_t - \Delta u + f(x, u) = 0 \text{ in } \mathbb{R}^+ \times \Omega.$$

Recently, R. Zacher and V. Vergara [20] have developed a second technique to find Lyapunov functions for ordinary differential equations, in finite-dimensional spaces, of order less than 1, and of order between 1 and 2 in time, which combined with the Łojasiewicz inequality leads to a proof of convergence of global, bounded solutions to a single steady state.
In [22], Zacher has proved that, still in the finite dimensional case, the dissipation given through the memory term is strong enough to guarantee convergence of global, bounded and regular solutions of the following second order equation

$$\ddot{u} + k * \dot{u} + \nabla E(u) = g,$$

when the nonlinear potential E satisfies the Łojasiewicz inequality. In his proof, Zacher used the Łojasiewicz inequality together with the method of higher order energies. In this direction it is important to mention to work of F. Alabau-Boussouira, J. Prüss, and R. Zacher [3], too, where the autonomuous, linear case ($f = K_2 = g_1 = g_2 = 0$, $K_1 = 1$) was studied under the same boundary condition.

Concerning the nonautonomous, nonlinear case, the source terms introduce nonstandard difficulties. The convergence proof given here is direct and naturally generalizes the autonomous case, without using the additional discussion from R. Chill and M.-A. Jendoubi [9] or the additional integral lemma from S. Z. Huang and P. Takác [15] (see also E. Feireisl and F. Simondon [11] and the author article [21]).

Remark 4.1.1 (Related boundary conditions)**.** *For the well-posedness of the Robin-type problem, we assume that the coefficient μ on the boundary is not identically zero almost everywhere on Γ (with respect to the surface measure). However, the following variants of (dynamical) boundary conditions may also be studied. Assume, for example, that $\Gamma = \Gamma_0 \cup \Gamma_1$ for two closed, disjoint subsets Γ_0, $\Gamma_1 \subseteq \Gamma$. Then the results of this paper (existence and uniqueness of global, bounded solutions, relative compactness of their range in the energy space, convergence to equilibrium and decay rate estimates) still hold for the following boundary condition*

$$\begin{cases} u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu u + \mu(x)u + k * u_t = g_2 & \text{on } \mathbb{R}^+ \times \Gamma_1, \end{cases} \tag{4.3}$$

where $\mu \in W^{1,\infty}(\Gamma_1)$ *is such that*

$$\begin{cases} \text{if } \Gamma_0 \neq \varnothing, \text{ then } \mu \geq 0, \\ \text{if } \Gamma_0 = \varnothing, \text{ then } \mu \text{ is not identically zero almost everywhere on } \Gamma_1. \end{cases} \tag{4.4}$$

Also, the results of this paper still hold when the boundary feedback is both frictional and of memory type

$$\begin{cases} u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu u + \mu(x)u + b(x)u_t + k * u_t = g_2 & \text{on } \mathbb{R}^+ \times \Gamma_1, \end{cases} \tag{4.5}$$

where b *is a nonnegative function on* Γ_1 *and* $\mu \in W^{1,\infty}(\Gamma_1)$ *satisfying* (4.4). *This boundary condition has been studied in [3], when* $g_2 = 0$*; see also [21], when the feedback is only frictional (that is,* $k = 0$*).*
An other boundary condition, with more regular kernels, has been studied by several authors (see, for example, M. L. Santos [18], M. M. Cavalcanti et al. [6, 7] and the references therein), namely the boundary condition

$$\begin{cases} u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ u + h * \partial_\nu u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1. \end{cases} \tag{4.6}$$

Here, the relaxation function h *belongs to* $W^{1,\infty}(0,\infty)$ *and is assumed to be positive and non-increasing. By differentiating the equation* (4.6) *and by applying the inverse Volterra operator, we obtain*

$$\partial_\nu u = -\rho\big(u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\big) \text{ on } \mathbb{R}^+ \times \Gamma_1,$$

where $\rho = \frac{1}{h(0)}$, *and* k_1 *is the resolvent kernel satisfying*

$$k_1 + \rho h' * k_1 = -\rho h'.$$

Observe that

$$k_1' * u = \frac{d}{dt}(k_1 * u) - k_1(0)u = k_1 * u_t + k_1(t)u_0 - k_1(0)u.$$

Then, the boundary condition (4.6) *can be rewritten in the following form*

$$\begin{cases} u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu u + \rho u_t + \rho k_1 * u_t = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \end{cases} \tag{4.7}$$

which is a particular case of the boundary condition (4.5).

Throughout the following :
- The inner product (respectively the norm) in the spaces $H^1(\Omega)$, $H^1(\Omega)'$, $L^2(\Omega)$ and $L^2(\Gamma)$ is denoted by $(\cdot,\cdot)_{H^1(\Omega)}$, $(\cdot,\cdot)_*$, $(\cdot,\cdot)_2$, and $(\cdot,\cdot)_\Gamma$ (respectively, by $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_*$, $\|\cdot\|_2$, and $\|\cdot\|_\Gamma$). The norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$.

- We denote by C (sometime C_i) a generic positive constant which may vary from line to line, which may depend on g_1, g_2, f, and the measure of Ω, but which can be chosen independently of $t \in \mathbb{R}^+$.

The remaining part of this paper is organized as follows. In Section 2, we state the assumptions on the kernel and the source terms, and we state the main results. The existence and uniqueness of solutions to problem (4.1) is proved in Section 3. Section 4 is devoted to the compactness results. In the final Section 5, we prove the convergence of global bounded solutions and we obtain an estimate on the convergence rate.

4.2 Assumptions and main results

Before stating our main results, we present several assumptions about the initial data, the source terms, and the memory kernel.

Assumptions on the source terms and the kernel

For the global existence and uniqueness for weak solutions, we assume that the functions g_1, g_2 satisfy the regularity condition

$$g_1 \in L^2(\mathbb{R}^+; L^2(\Omega)) \text{ and } g_2 \in L^2(\mathbb{R}^+; L^2(\Gamma)), \tag{G1}$$

and for our convergence result we assume in addition the a decay condition, namely that there exist constants $\eta_0 \geq 0$ and $\delta > 0$ such that for all $t \in \mathbb{R}^+$

$$\|g_2(t)\|_\Gamma + \int_t^\infty (\|g_1(s)\|_2^2 + \|g_2'(s)\|_\Gamma^2)\, ds \leq \frac{\eta_0}{(1+t)^{1+\delta}}. \tag{G2}$$

Condition (G2) implies in turn that $g_2 \in L^1(\mathbb{R}^+; L^2(\Gamma))$, $\|g_2(t)\|_\Gamma \searrow 0$ and there exists $\eta \geq 0$ such that

$$\int_t^\infty (\|g_1(s)\|_2^2 + \|g_2(s)\|_\Gamma^2 + \|g_2'(s)\|_\Gamma^2)\, ds \leq \frac{\eta}{(1+t)^{1+\delta}}. \tag{G2$'$}$$

Concerning the kernel k we suppose that

$$\begin{aligned}&\text{there exists a nonnegative and nonincreasing kernel } b \in L^1_{loc}(\mathbb{R}^+)\\ &\text{such that } b * k = 1, \text{ and}\end{aligned} \tag{K1}$$

$$\begin{aligned}&\text{there are } \gamma > 0 \text{ and } a \in L^1(\mathbb{R}^+) \text{ strictly positive and nonincreasing,}\\ &\text{such that } b = a + \gamma\,(1 * a).\end{aligned} \tag{K2}$$

Remark 4.2.1. *(a) Condition* (K1) *implies that the kernel k is nonnegative.*
(b) The conditions (K1) *and* (K2) *together imply that $b(t) \geq b_\infty = \lim_{s\to\infty} b(s) = \gamma \|a\|_{L^1(\mathbb{R}^+)} > 0$ for every $t > 0$.*
(c) It follows further from conditions (K1) *and* (K2) *that $k \in L^1(\mathbb{R}^+)$. Indeed, since*

k is nonnegative (see (a)), the condition (K1) *implies* $(b * k)(t) \leq 1$ *for every* $t \geq 0$. *Using the lower bound for b from (ii) and positivity of k, we see that* $\|k\|_{L^1(\mathbb{R}^+)} \leq \dfrac{1}{b_\infty}$.
(d) For each $\gamma > 0$ the unique solution of the equation in (K2) *is given by*

$$a = b - \gamma\,(e^{-\gamma\cdot} * b).$$

(e) Typical examples for the kernels b and k which satisfy the conditions (K1) *and* (K2) *are given by*

$$b(t) = g_{1-s}(t)e^{-wt} + w[1 * (g_{1-s}e^{-w})(t)] \quad (s \in (0,1),\, w > 0),$$

k and g are given by (4.2).

In fact, our method can be adapted to the more general case when the kernel k is completely positive, that is, the condition (*K1*) can be weakened to the condition that

(K1′) there exist $b_0 > 0$ and a nonnegative and nonincreasing kernel $b \in L^1_{loc}(\mathbb{R}^+)$ such that $b_0 k(t) + (b * k)(t) = 1$ for all $t \geq 0$ (see [22]).

This condition allows one to include the nonsingular case $\beta = 0$ in the example (4.2). In particular, our results are still valid for $k(t) = e^{-wt}$ $(t \geq 0,\, w > 0)$.

Existence and uniqueness of global, bounded solutions

Throughout the following, a function $u : \mathbb{R}^+ \to H^2(\Omega)$ is called a global *strong solution* of (4.1), if

$$\begin{cases} u \in L^\infty_{loc}(\mathbb{R}^+; H^2(\Omega)) \cap W^{1,\infty}_{loc}(\mathbb{R}^+; H^1(\Omega)) \cap W^{2,2}_{loc}(\mathbb{R}^+; L^2(\Omega)), \\ K_1^{\frac{1}{2}} u_t \in W^{1,\infty}_{loc}(\mathbb{R}^+; L^2(\Omega)), \end{cases}$$

if it satisfies the initial conditions $u(0) = u_0$ and $(K_1)^{\frac{1}{2}} u_t(0) = (K_1)^{\frac{1}{2}} u_1$, and if it satisfies the differential equation (4.1) almost everywhere on $\mathbb{R}^+$. A function $u \in C(\mathbb{R}^+; H^1(\Omega)) \cap W^{1,2}_{loc}(\mathbb{R}^+; L^2(\Omega))$ is called a global *weak solution* of (4.1), if it satisfies the initial conditions $u(0) = u_0$ and $(K_1)^{\frac{1}{2}} u_t(0) = (K_1)^{\frac{1}{2}} u_1$, and if there exists a sequence (u^μ) of strong solutions such that

$$\begin{aligned} u^\mu &\to u && \text{in } C(\mathbb{R}^+; H^1(\Omega)) \cap W^{1,2}_{loc}(\mathbb{R}^+; L^2(\Omega)), \\ K_1^{\frac{1}{2}} u_t^\mu &\to K_1^{\frac{1}{2}} u_t && \text{in } C(\mathbb{R}^+; L^2(\Omega)). \end{aligned}$$

Our first main result, which establishes the global well-posedness of the equation (4.1), reads as follows.

Theorem 4.2.1 (Existence and uniqueness of global, bounded solutions)**.** *Assume that the function f satisfies the conditions (F2) and (F3), and that the kernel k satisfies the conditions* (K1) *and* (K2).

(I) **Strong solutions :** *Let*

$$g_1 \in W^{1,2}_{loc}(\mathbb{R}^+; L^2(\Omega)) \text{ and } g_2 \in L^1_{loc}(\mathbb{R}^+; H^{\frac{1}{2}}(\Gamma)) \cap W^{2,2}_{loc}(\mathbb{R}^+; L^2(\Gamma)), \quad (4.8)$$

and let the initial values $(u_0, u_1) \in H^2(\Omega) \times H^2(\Omega)$ *satisfy the compatibility conditions*

$$\begin{cases} -\Delta u_0 + f(x, u_0) = g_1(0) - K_2 u_1 & \text{in } \Omega, \\ \partial_\nu u_0 + \mu(x) u_0 = g_2(0) & \text{on } \Gamma. \end{cases} \quad (4.9)$$

Then the problem (4.1) *possesses a unique, global, strong solution.*

(II) **Weak solutions :** *Let* g_1 *and* g_2 *satisfy the regularity condition* (G1) *and let* $(u_0, u_1) \in \bar{D}$*, where*

$$D = \{(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega);\ (4.9) \text{ is holds } \}. \quad (4.10)$$

Then the problem (4.1) *possesses a unique global weak solution* u*. In addition, this weak solution satisfies the following properties :*
(T1) $(u, K_1^{\frac{1}{2}} u_t)$ *is bounded in* $H^1(\Omega) \times L^2(\Omega)$*.*
(T2) $(u_t, v) \in L^2(\mathbb{R}^+; L^2(\Omega)) \times L^2(\mathbb{R}^+; L^2(\Gamma))$*, where* $v = \frac{d}{dt}(k * (u - u_0))$*.*
(T3) Let $G : \mathbb{R}^+ \to \mathbb{R}$ *be the energy of the solution* u *given by*

$$\begin{aligned} G(t) =& \frac{1}{2}\|K_1^{\frac{1}{2}} u_t\|_2^2 + E(u) + \frac{1}{2} a * \|v\|_\Gamma^2 - (g_2, a * v)_\Gamma + \\ &+ \frac{1}{2k_0} \int_t^\infty \|g_1(s)\|_2^2 \, ds + d \int_t^\infty (\|g_2(s)\|_\Gamma^2 + \|g_2'(s)\|_\Gamma^2) \, ds. \end{aligned}$$

where $d = \|a\|_{L^1(\mathbb{R}^+)} \max(\gamma, \gamma^{-1})$*. Then* G *is nonincreasing and*

$$\frac{d}{dt} G(t) \le -\frac{k_0}{2}\|u_t\|_2^2 - \frac{b_\infty}{2}\|v\|_\Gamma^2 - \frac{\gamma}{4} a * \|v\|_\Gamma^2, \ t > 0. \quad (4.11)$$

(T4) The following variational equality holds for all $\phi \in H^1(\Omega)$

$$\begin{aligned} &\frac{d}{dt}\int_\Omega K_1(x) u_t \phi \, dx + \int_\Omega K_2(x) u_t \phi \, dx + \int_\Omega \nabla u \nabla \phi \, dx + \\ &+ \int_\Omega f(x, u)\phi \, dx + \frac{d}{dt}\int_\Gamma (k * (u - u_0))\phi \, d\sigma + \int_\Gamma \mu(x) u \phi \, d\sigma \\ = &\int_\Omega g_1 \phi \, dx + \int_\Gamma g_2 \phi \, d\sigma. \end{aligned}$$

Remark 4.2.2. *(a) When* $K_1 \ge C > 0$*, we replace the compatibility conditions* (4.9) *by the compatibility conditions*

$$\partial_\nu u_0 + \mu(x) u_0 = g_2(0) \quad \text{on } \Gamma.$$

(b) Note that for every $u_1 \in H^2(\Omega) \subseteq L^2(\Omega)$ *the problem* (4.9) *admits at most one solution* $u_0 \in H^2(\Omega)$*.*

Compactness property of solutions

In the following theorem, we state an additional property of global weak solutions of (4.1) which is of crucial importance for the study of their asymptotic behaviour, namely the relative compactness of their range.

Theorem 4.2.2. *Let $u : \mathbb{R}^+ \to H^1(\Omega)$ be a global bounded weak solution of (4.1). Then the function $U = (u, K_1^{\frac{1}{2}} u_t)$ is uniformly continuous from $\mathbb{R}^+$ with values in $H^1(\Omega) \times L^2(\Omega)$, and $\bigcup_{t\geq 0}\{U(t)\}$ is relatively compact in $H^1(\Omega) \times L^2(\Omega)$.*

The Łojasiewicz-Simon inequality for the underlying energy

Our basic argument in the proof of the convergence result below is the Łojasiewicz-Simon inequality for the energy functional $E : H^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \int_\Omega F(x,u) \, dx + \frac{1}{2}\int_\Gamma \mu(x)|u|^2 \, d\sigma.$$

By the regularity and growth condition on f, the function E is twice continuously Fréchet differentiable [21]. If $E'(u) \in H^1(\Omega)'$ and $E''(u) \in \mathcal{L}(H^1(\Omega), H^1(\Omega)')$ denote the first and second derivative at a point $u \in H^1(\Omega)$, respectively, then for all ϕ, $\psi \in H^1(\Omega)$

$$(E'(u), \psi)_{H^1(\Omega)', H^1(\Omega)} = \int_\Omega \nabla u \nabla \psi \, dx + \int_\Omega f(x,u)\psi \, dx + \int_\Gamma \mu(x) u \psi \, d\sigma,$$

and

$$(E''(u)\phi, \psi)_{H^1(\Omega)', H^1(\Omega)} = \int_\Omega \nabla \phi \nabla \psi \, dx + \int_\Omega \frac{\partial f}{\partial u}(x,u)\phi\psi \, dx + \int_\Gamma \mu(x)\phi\psi \, d\sigma.$$

The proof of the following proposition – in the case $N = 3$ – can be found in [21, Proposition 9] ; the proof for general space dimensions can be easily adapted. Recall that the norm in $H^1(\Omega)'$ is denoted by $\| \cdot \|_*$.

Proposition 4.2.1. *Under the assumptions (F1) and (F2) on the function f the energy functional $E \in C^2(H^1(\Omega))$ satisfies the Łojasiewicz-Simon inequality near every equilibrium point $\phi \in H^1(\Omega)$, that is, for every $\phi \in H^1(\Omega)$ with $E'(\phi) = 0$, there exist $\beta_\phi > 0$, $\sigma_\phi > 0$ and $0 < \theta_\phi \leq \frac{1}{2}$ such that*

$$|E(\phi) - E(\psi)|^{1-\theta_\phi} \leq \beta_\phi \, \| \, E'(\phi) \, \|_*$$

for all $\psi \in H^1(\Omega)$ such that $\|\phi - \psi\|_{H^1(\Omega)} < \sigma_\phi$. The number θ_ϕ is called the Łojasiewicz exponent of E at ϕ.

Convergence to equilibrium and decay rate

The following theorem describes the asymptotic behaviour of global weak solutions to the problem (4.1).

Theorem 4.2.3. *Let $u : \mathbb{R}^+ \to H^1(\Omega)$ be a global, bounded, weak solution of equation* (4.1)*. Suppose that f satisfies (F1), (F2), and that (g_1, g_2) satisfies the growth condition (G2). Then, there exists $\phi \in H^1(\Omega)$, solution of the stationary problem*

$$\begin{cases} -\Delta\phi + f(x,\phi) = 0 & \text{in } \Omega, \\ \partial_\nu \phi + \mu\phi = 0 & \text{on } \Gamma, \end{cases}$$

such that

$$\|K_1^{\frac{1}{2}} u_t(t)\|_2 + \|u(t) - \phi\|_{H^1(\Omega)} \longrightarrow 0 \text{ as } t \to \infty.$$

From the proof of Theorem 4.2.3 and the differential inequality given below (Lemma 4.5.2), we deduce in addition that the Łojasiewicz exponent θ in the Łojasiewicz-Simon inequality determines the decay rate of the solution u to the steady state ϕ.

Theorem 4.2.4. *Let $\theta = \theta_\phi$ be the Łojasiewicz exponent of E at ϕ, where ϕ is given by Theorem 4.2.3. Then, the following assertions hold :*

(i) If $\theta \in (0, \frac{1}{2})$, then there exists a constant $C > 0$ such that

$$\|u(t) - \phi\|_2 \le C\,(1+t)^{-\xi} \text{ for every } t \ge 0,$$

where

$$\xi = \begin{cases} \inf\{\frac{\theta}{1-\theta}, \frac{\delta}{2}\} & \text{if } (g_1, g_2) \ne (0,0), \\ \frac{\theta}{1-2\theta} & \text{if } (g_1, g_2) = (0,0). \end{cases}$$

(ii) If $\theta = \dfrac{1}{2}$ and $(g_1, g_2) = (0,0)$, then there exist constants C, $\kappa > 0$ such that

$$\|u(t) - \phi\|_2 \le Ce^{-\theta\kappa t}.$$

4.3 Existence and uniqueness of a global, bounded solution : Proof of Theorem 4.2.1

In this section, we prove the existence and uniqueness of strong/weak solutions of the problem (4.1), that is, we prove Theorem 4.2.1. First, we prove the existence and the uniqueness of strong solutions satisfying the properties (T1)-(T4), when the initial data and the source terms are sufficiently smooth. Then we extend the same results to weak solutions by using an approximation argument.

For the convenience of the reader, we recall here explicitly some auxiliary lemmas which will be used in the proof below. We begin with the subsequent simple lemma [20, Lemma 2.1].

Lemma 4.3.1. *Let $\mathcal{H}$ be a Hilbert space and $T > 0$. Suppose that $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative. Then for any $v \in L^2([0,T];\mathcal{H})$ there holds*

$$\|(k * v)(t)\|^2_{\mathcal{H}} \leq (k * \|v\|^2_{\mathcal{H}})(1 * k)(t)) \text{ for a.e. } t \in (0,T).$$

The second lemma is due to Vergara and Zacher [20]. It is one key to find a proper Lyapunov function for the problem (4.1).

Lemma 4.3.2. *Let $\mathcal{H}$ be a Hilbert space, $T > 0$, and $b \in L^1_{loc}(\mathbb{R}^+)$ be nonnegative and nonincreasing such that $b * k = 1$ in $(0,\infty)$ for some nonnegative kernel $k \in L^1_{loc}(\mathbb{R}^+)$. Suppose that $v \in L^2(0,T;\mathcal{H})$ is such that $b * v \in H^1(0,T;\mathcal{H})$ as well as $b * \|v\|^2_{\mathcal{H}} \in W^{1,1}(0,T)$. Then*

$$(v(t), \frac{d}{dt}(b * v)(t))_{\mathcal{H}} \geq \frac{1}{2}\frac{d}{dt}(b * \|v\|^2_{\mathcal{H}})(t) + \frac{1}{2}b(t)\|v\|^2_{\mathcal{H}} \text{ for a.e. } t \in (0,T). \tag{4.12}$$

Remark 4.3.1. *(a) Under the same assumptions on the kernel b, the inequality* (4.12) *in Lemma 4.3.2 is also satisfied for any function $v \in H^1(0,T;\mathcal{H})$, [20, Remark 2.1].*
(b) For the kernels k and b given as in the Remark 4.2.1 (e), the inequality (4.12) *in Lemma 4.3.2 is also satisfied for any function $v \in L^2(0,T;\mathcal{H})$ such that $b * v \in H^1(0,T;\mathcal{H})$, [20, Example 2.1].*

Proof of Theorem 4.2.1. **Existence of strong solution.** We transform the problem (4.1) into an equivalent problem with null initial data. In fact, let us consider the change of variables

$$v(x,t) = u(x,t) - \phi(x,t),$$

where

$$\phi(x,t) = u_0(x) + tu_1(x).$$

Due to this change of variables and the regularity of the initial data we get the following equivalent problem for the variable v :

$$\begin{cases} K_1 v_{tt} + K_2 v_t - \Delta v + f(x, v + \phi) = \mathcal{F} & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu v + \mu(x) v + k * v_t = \mathcal{G} & \text{on } \mathbb{R}^+ \times \Gamma, \\ v(0) = 0, \quad (K_1)^{\frac{1}{2}} v_t(0) = 0. \end{cases} \tag{4.13}$$

Here,

$$\mathcal{F} = -K_2 u_1 + \Delta\phi + g_1, \text{ and}$$
$$\mathcal{G} = g_2 - (\partial_\nu \phi + \mu(x)\phi + k * u_1).$$

We note that if v is a solution of the modified problem (4.13) in $[0,T]$, then $u = v + \phi$ is a solution of (4.1) on the same interval.

Since $K_1 \geq 0$, we first perturb the problem (4.13) by the term εv_{tt} ($\varepsilon > 0$) and we apply a Faedo–Galerkin method in order to solve the perturbed problem. Then we shall pass to the limit with $\varepsilon \to 0$ in the perturbed problem and obtain the solution for the problem (4.13).

Let $K_{1\varepsilon} := K_1 + \varepsilon$, and consider the perturbed problem :

$$\begin{cases} K_{1\varepsilon}(x)v_{\varepsilon tt} + K_2 v_{\varepsilon t} - \Delta v_\varepsilon + f(x, v_\varepsilon + \phi) = \mathcal{F} & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu v_\varepsilon + \mu(x)v_\varepsilon + k * v_{\varepsilon t} = \mathcal{G} & \text{on } \mathbb{R}^+ \times \Gamma, \\ v_\varepsilon(0) = 0, \quad (K_{1\varepsilon})^{\frac{1}{2}} v_{\varepsilon t}(0) = 0. \end{cases} \tag{4.14}$$

Let $(w_i)_{i\in\mathbb{N}}$ be a total family in $H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$, and let V_m be the subspace of $H^2(\Omega)$ which is spanned by the first m vectors $w_1, \ldots, w_m$. Consider the following weak formulation of an approximated problem, namely to find a solution

$$v_{\varepsilon m}(t) := \sum g_{im}(t) w_i,$$

of the ordinary differential equation

$$\begin{aligned} &(K_{1\varepsilon} v''_{\varepsilon m}(t), w)_2 + (K_2 v'_{\varepsilon m}(t), w)_2 + (\nabla v_{\varepsilon m}(t), \nabla w)_2 + (f(v_{\varepsilon m}(t) + \phi), w)_2 \\ &+ (\mu(x) v_{\varepsilon m}(t), w)_\Gamma + \int_0^t k(t-s)(v'_{\varepsilon m}(s), w)_\Gamma = (\mathcal{F}, w)_2 + (\mathcal{G}, w)_\Gamma \qquad (4.15) \\ &\qquad\qquad\qquad \text{for every } w \in V_m, \\ &v_{\varepsilon m}(0) = 0, \quad v'_{\varepsilon m}(0) = 0. \end{aligned}$$

By standard arguments from the theory of ordinary differential equations, one proves the existence and uniqueness of a maximal solution of (4.15) on some interval $[0, t_{\varepsilon m}]$. We show that this solution can be extended to the whole interval $[0, T]$ by using the first estimate as follows.

First estimate. Taking $w = v'_{\varepsilon m}$ in (4.15), we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{1}{2}\|K_{1\varepsilon}^{\frac{1}{2}} v'_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\nabla v_{\varepsilon m}\|_2^2 + \int_\Omega F(x, v_{\varepsilon m} + \phi)\, dx + \frac{1}{2}\|\mu^{\frac{1}{2}} v_{\varepsilon m}\|_\Gamma^2\Big) + \\ &+ \|K_2^{\frac{1}{2}} v'_{\varepsilon m}\|_2^2 + (k * v'_{\varepsilon m}, v'_{\varepsilon m})_\Gamma = (\mathcal{F}, v'_{\varepsilon m})_2 + (\mathcal{G}, v'_{\varepsilon m})_\Gamma + \int_\Omega f(v_{\varepsilon m} + \phi) u_1\, dx. \end{aligned} \tag{4.16}$$

Let $w_{\varepsilon m} = k * v'_{\varepsilon m}$. We use property (K1) in order to write

$$v'_{\varepsilon m} = \frac{d}{dt}([b * k] * v'_{\varepsilon m}) = \frac{d}{dt}(b * w_{\varepsilon m}),$$

which yields

$$(k * v'_{\varepsilon m}, v'_{\varepsilon m})_\Gamma = (w_{\varepsilon m}, \frac{d}{dt}(b * w_{\varepsilon m}))_\Gamma.$$

Then, by Lemma 4.3.2,

$$(k * v'_m, v'_{\varepsilon m})_\Gamma \geq \frac{1}{2}\frac{d}{dt}(b * \|w_{\varepsilon m}\|_\Gamma^2)(t) + \frac{1}{2} b(t)\|w_{\varepsilon m}\|_\Gamma^2. \tag{4.17}$$

Using this inequality and the decomposition $b = a + \gamma(1 * a)$, we find

$$(k * v'_m, v'_{\varepsilon m})_\Gamma \geq \frac{1}{2}\frac{d}{dt}(a*\|w_{\varepsilon m}\|_\Gamma^2)(t) + \frac{\gamma}{2}(a*\|w_{\varepsilon m}\|_\Gamma^2)(t) + \frac{b_\infty}{2}\|w_{\varepsilon m}\|_\Gamma^2. \tag{4.18}$$

where $b_\infty = \lim_{t\to\infty} b(t) = \gamma\|a\|_{L^1(\mathbb{R}^+)}$. Using the last inequality, (4.16), and the fact that $K_2 > k_0 > 0$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|K_{1\varepsilon}^{\frac{1}{2}}v'_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\nabla v_{\varepsilon m}\|_2^2 + \int_\Omega F(x, v_{\varepsilon m} + \phi)dx + \frac{1}{2}\|\mu^{\frac{1}{2}}v_{\varepsilon m}\|_\Gamma^2 \\
&\quad + \frac{1}{2}a*\|w_{\varepsilon m}\|_\Gamma^2\Big) + k_0\|v'_{\varepsilon m}\|_2^2 + \frac{b_\infty}{2}\|w_{\varepsilon m}\|_\Gamma^2 + \frac{\gamma}{2}a*\|w_{\varepsilon m}\|_\Gamma^2 \\
&\leq (\mathcal{F}, v'_{\varepsilon m})_2 + (\mathcal{G}, v'_{\varepsilon m})_\Gamma + \int_\Omega f(v_{\varepsilon m} + \phi)u_1 dx.
\end{aligned} \tag{4.19}$$

Integrating (4.19) over the interval $(0, t)$, observing that $v_{\varepsilon m}(0) = v'_{\varepsilon m}(0) = 0$, it follows that

$$\begin{aligned}
&\frac{1}{2}\|K_{1\varepsilon}^{\frac{1}{2}}v'_{\varepsilon m}\|_2^2 + k_0\int_0^t \|v'_{\varepsilon m}\|_2^2\, ds + \frac{1}{2}\|\nabla v_{\varepsilon m}\|_2^2 + \int_\Omega F(x, v_{\varepsilon m} + \phi)\, dx + \\
&+ \frac{1}{2}\|\mu^{\frac{1}{2}}v_{\varepsilon m}\|_\Gamma^2 + \frac{1}{2}a*\|w_{\varepsilon m}\|_\Gamma^2 + \frac{b_\infty}{2}\int_0^t \|w_{\varepsilon m}\|_\Gamma^2\, ds + \frac{\gamma}{2}\int_0^t a*\|w_{\varepsilon m}\|_\Gamma^2\, ds \\
&\leq \int_0^t ((\mathcal{F}, v'_{\varepsilon m})_2 + (\mathcal{G}, v'_{\varepsilon m})_\Gamma)\, ds + \int_0^t\int_\Omega f(v_m + \phi)u_1\, dxds + \int_\Omega F(x, u_0)\, dx.
\end{aligned} \tag{4.20}$$

Next, we shall estimate some terms in (4.20). In fact, by (*F3*) we have

$$\int_\Omega F(x, v_{\varepsilon m} + \phi)\, dx \geq -\frac{\lambda}{2}\int_\Omega |v_{\varepsilon m} + \phi|^2\, dx - C \geq -C\|v_{\varepsilon m}\|_2^2 - C. \tag{4.21}$$

By the Cauchy-Schwarz inequality and since $\mathcal{F} \in L^2(0, T; L^2(\Omega))$

$$\int_0^t (\mathcal{F}, v'_{\varepsilon m})_2\, ds \leq \frac{1}{k_0}\int_0^t \|\mathcal{F}\|_2^2\, ds + \frac{k_0}{4}\int_0^t \|v'_{\varepsilon m}\|_2^2\, ds \leq \frac{k_0}{4}\int_0^t \|v'_{\varepsilon m}\|_2^2\, ds + C. \tag{4.22}$$

Moreover, by Lemma 4.3.1, (*K1*), and Young's inequality we have

$$\begin{aligned}
(\mathcal{G}, v'_{\varepsilon m})_\Gamma &= (\mathcal{G}, \frac{d}{dt}b * w_{\varepsilon m})_\Gamma = (\mathcal{G}, \frac{d}{dt}a * w_{\varepsilon m})_\Gamma + \gamma(\mathcal{G}, a * w_{\varepsilon m})_\Gamma \\
&= \frac{d}{dt}(\mathcal{G}, a * w_{\varepsilon m})_\Gamma - (\mathcal{G}', a * w_{\varepsilon m})_\Gamma + \gamma(\mathcal{G}, a * w_{\varepsilon m})_\Gamma \\
&\leq \frac{d}{dt}(\mathcal{G}, a * w_{\varepsilon m})_\Gamma + \|a\|_{L^1(\mathbb{R}^+)}(\gamma\|\mathcal{G}\|_\Gamma^2 + \gamma^{-1}\|\mathcal{G}'\|_\Gamma^2) \\
&\quad + \frac{\gamma}{4\|a\|_{L^1(\mathbb{R}^+)}}\|a * w_{\varepsilon m}\|_\Gamma^2 \\
&\leq \frac{d}{dt}(\mathcal{G}, a * w_{\varepsilon m})_\Gamma + d(\|\mathcal{G}\|_\Gamma^2 + \|\mathcal{G}'\|_\Gamma^2) + \frac{\gamma}{4}a*\|w_{\varepsilon m}\|_\Gamma^2,
\end{aligned} \tag{4.23}$$

where $d = \|a\|_{L^1(\mathbb{R}^+)} \max(\gamma, \gamma^{-1})$. Then

$$\int_0^t (\mathcal{G}, v'_{\varepsilon m})_\Gamma \, ds \le (\mathcal{G}, a * w_{\varepsilon m})_\Gamma + d \int_0^t (\|\mathcal{G}\|_\Gamma^2 + \|\mathcal{G}'\|_\Gamma^2) \, ds + \frac{\gamma}{4} \int_0^t a * \|w_{\varepsilon m}\|_\Gamma^2 \, ds$$

$$\le \frac{1}{4} a * \|w_{\varepsilon m}\|_\Gamma^2 + C\|\mathcal{G}\|_\Gamma^2 + d \int_0^t (\|\mathcal{G}\|_\Gamma^2 + \|\mathcal{G}'\|_\Gamma^2) \, ds + \frac{\gamma}{4} \int_0^t a * \|w_{\varepsilon m}\|_\Gamma^2 \, ds$$

$$\le \frac{1}{4} a * \|w_{\varepsilon m}\|_\Gamma^2 + \frac{\gamma}{4} \int_0^t a * \|w_{\varepsilon m}\|_\Gamma^2 + C. \tag{4.24}$$

Also, by the growth condition $(F2)$, Caushy-Schwarz inequality, Young's inequality, we have

$$\int_0^t \int_\Omega f(v_{\varepsilon m} + \phi) u_1 \, dxds \le C \int_0^t \int_\Omega (1 + |v_{\varepsilon m} + \phi|^{1+\alpha}) u_1 \, dxds \le$$

$$\le C + C \int_0^t \int_\Omega |v_{\varepsilon m}|^{1+\alpha} u_1 \, dxds \le C + C \int_0^t (\| |v_{\varepsilon m}|^{1+\alpha} \|_2 \|u_1\|_2) \, ds$$

$$\le C + C \int_0^t \|v_{\varepsilon m}\|_{2(\alpha+1)}^{1+\alpha} \, ds \le C + C \int_0^t \|v_{\varepsilon m}\|_{H^1(\Omega)}^{1+\alpha} \, ds$$

$$\le C + C \int_0^t \|v_{\varepsilon m}\|_{H^1(\Omega)}^2 \, ds. \tag{4.25}$$

Combining (4.20)–(4.25), we obtain

$$\frac{1}{2}\|K_{1\varepsilon}^{\frac{1}{2}} v'_{\varepsilon m}\|_2^2 + \frac{3k_0}{4} \int_0^t \|v'_{\varepsilon m}\|_2^2 \, ds + \frac{1}{2}\|\nabla v_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\mu^{\frac{1}{2}} v_{\varepsilon m}\|_\Gamma^2 +$$

$$+ \frac{1}{4} a * \|w_{\varepsilon m}\|_\Gamma^2 + \frac{b_\infty}{2} \int_0^t \|w_{\varepsilon m}\|_\Gamma^2 \, ds + \frac{\gamma}{4} \int_0^t a * \|w_{\varepsilon m}\|_\Gamma^2 \, ds \le$$

$$\le C \Big(\int_0^t \|v_{\varepsilon m}\|_2^2 \, ds + \int_0^t \|v_{\varepsilon m}\|_{H^1(\Omega)}^2 \, ds + \|v_{\varepsilon m}\|_2^2 + 1 \Big). \tag{4.26}$$

Observe that

$$C\|v_{\varepsilon m}\|_2^2 = C \int_0^t \frac{d}{dt} \|v_{\varepsilon m}(s)\|_2^2 \, ds \le \frac{C^2}{k_0} \int_0^t \|v_{\varepsilon m}(s)\|_2^2 \, ds + \frac{k_0}{4} \int_0^t \|v'_{\varepsilon m}(s)\|_2^2 \, ds.$$

Using this inequality and (4.26), we obtain

$$\frac{1}{2}\|K_{1\varepsilon}^{\frac{1}{2}} v'_{\varepsilon m}\|_2^2 + \frac{k_0}{2} \int_0^t \|v'_{\varepsilon m}\|_2^2 \, ds + \frac{1}{2}\|\nabla v_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\mu^{\frac{1}{2}} v_{\varepsilon m}\|_\Gamma^2 +$$

$$+ \frac{1}{4} a * \|w_{\varepsilon m}\|_\Gamma^2 + \frac{b_\infty}{2} \int_0^t \|w_{\varepsilon m}\|_\Gamma^2 \, ds + \frac{\gamma}{4} \int_0^t a * \|w_{\varepsilon m}\|_\Gamma^2 \, ds$$

$$\le C \int_0^t \|v_{\varepsilon m}\|_{H^1(\Omega)}^2 \, ds + C. \tag{4.27}$$

By using this inequality and Gronwall's inequality, we obtain that

$$\|K_{1\varepsilon}^{\frac{1}{2}} v'_{\varepsilon m}\|_2^2 + \int_0^T \|v'_{\varepsilon m}\|_2^2 \, ds + \|v_{\varepsilon m}\|_{H^1(\Omega)}^2 + a * \|w_{\varepsilon m}\|_\Gamma^2 + \int_0^T \|w_{\varepsilon m}\|_\Gamma^2 \, ds \leq C_T, \quad (4.28)$$

where C_T is a positive constant independent of m, ε, and t.

Second estimate. Next, we estimate $v''_{\varepsilon m}(0)$. Indeed, taking $\psi = v''_{\varepsilon m}(0)$ in (4.15) and noting that $v_{\varepsilon m}(0) = v'_{\varepsilon m}(0) = 0$, we obtain

$$\|K_{1\varepsilon}^{\frac{1}{2}} v''_{\varepsilon m}(0)\|_2^2 + (f(u_0) - \mathcal{F}(0), v''_m(0))_2 + (\mathcal{G}(0), v''_{\varepsilon m}(0))_\Gamma = 0.$$

Using the assumptions on the initial data, we obtain

$$\|K_{1\varepsilon}^{\frac{1}{2}} v''_{\varepsilon m}(0)\|_2 = 0. \quad (4.29)$$

Also, taking the derivative of (4.15) with respect to time t, taking $w = v''_{\varepsilon m}(t)$, and arguing as in the first estimate, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{1}{2}\|K_{1\varepsilon}^{\frac{1}{2}} v''_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\nabla v'_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\mu^{\frac{1}{2}} v'_{\varepsilon m}\|_\Gamma^2 + \frac{1}{2} a * \|z_{\varepsilon m}\|_\Gamma^2\Big) \quad (4.30)\\ &+ k_0 \|v''_{\varepsilon m}\|_2^2 + \frac{b_\infty}{2}\|z_{\varepsilon m}\|_\Gamma^2 + \frac{\gamma}{2} a * \|z_{\varepsilon m}\|_\Gamma^2 + \int_\Omega f'(v_{\varepsilon m} + \phi)(v'_{\varepsilon m} + u_1) v''_{\varepsilon m} \, dx \\ &\leq (\mathcal{F}', v''_{\varepsilon m})_2 + (\mathcal{G}', v''_{\varepsilon m})_\Gamma, \text{ where } z_{\varepsilon m} = k * v''_{\varepsilon m}. \end{aligned}$$

Integrating this inequality over the interval $(0,t)$ and noticing $v_{\varepsilon m}(0) = v'_{\varepsilon m}(0) = \|K_{1\varepsilon}^{\frac{1}{2}} v''_{\varepsilon m}(0)\|_2 = 0$, it follows that

$$\begin{aligned} &\frac{1}{2}\|K_{1\varepsilon}^{\frac{1}{2}} v''_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\nabla v'_{\varepsilon m}\|_2^2 + \frac{1}{2}\|\mu^{\frac{1}{2}} v'_{\varepsilon m}\|_\Gamma^2 + \frac{1}{2} a * \|z_{\varepsilon m}\|_\Gamma^2 + \\ &\qquad + k_0 \int_0^t \|v''_{\varepsilon m}\|_2^2 \, ds + \frac{b_\infty}{2}\int_0^t \|z_{\varepsilon m}\|_\Gamma^2 \, ds + \frac{\gamma}{2}\int_0^t a * \|z_{\varepsilon m}\|_\Gamma^2 \, ds \leq \\ &\leq -\int_0^t \int_\Omega f'(v_{\varepsilon m} + \phi)(v'_{\varepsilon m} + u_1) v''_{\varepsilon m} \, dx ds + \int_0^t ((\mathcal{F}', v''_{\varepsilon m})_2 + (\mathcal{G}', v''_{\varepsilon m})_\Gamma) \, ds. \quad (4.31) \end{aligned}$$

Next, we shall estimate the nonlinear terms of (4.31). For this, by using Hölder's inequality and the first estimate, we obtain

$$\begin{aligned}
&\int_0^t \int_\Omega f'(v_{\varepsilon m} + \phi)(v'_{\varepsilon m} + u_1) v''_{\varepsilon m} \, dxds \leq \\
\leq & C \int_0^t \int_\Omega (1 + |v_{\varepsilon m} + \phi|^\alpha)(v'_{\varepsilon m} + u_1) v''_{\varepsilon m} \, dxds \\
\leq & C \int_0^t \left(\|1 + |v_{\varepsilon m} + \phi|^\alpha\|_N \|v'_{\varepsilon m} + u_1\|_{\frac{2N}{N-2}} \|v''_{\varepsilon m}\|_2 \right) ds \\
\leq & C \int_0^t \left((1 + \|v_{\varepsilon m} + \phi\|^\alpha_{N\alpha}) \|v'_{\varepsilon m} + u_1\|_{\frac{2N}{N-2}} \|v''_{\varepsilon m}\|_2 \right) ds \\
\leq & \frac{k_0}{4} \int_0^t \|v''_{\varepsilon m}\|_2^2 \, ds + C \int_0^t \|v'_{\varepsilon m} + u_1\|^2_{\frac{2N}{N-2}} \, ds \\
\leq & \frac{k_0}{4} \int_0^t \|v''_{\varepsilon m}\|_2^2 \, ds + C \int_0^t \|v'_{\varepsilon m}\|^2_{H^1(\Omega)} \, ds + C. \qquad (4.32)
\end{aligned}$$

Again, by the Cauchy-Schwarz inequality and since $\mathcal{F}' \in L^2([0,T]; L^2(\Omega))$

$$\int_0^t (\mathcal{F}', v''_{\varepsilon m})_2 \, ds \leq \frac{k_0}{4} \int_0^t \|v''_{\varepsilon m}\|_2^2 \, ds + C. \qquad (4.33)$$

Moreover, similarly as in (4.24),

$$\int_0^t (\mathcal{G}', v''_{\varepsilon m})_\Gamma \, ds \leq \frac{1}{4} a * \|z_{\varepsilon m}\|_\Gamma^2 + \frac{\gamma}{4} \int_0^t a * \|z_{\varepsilon m}\|_\Gamma^2 \, ds + C. \qquad (4.34)$$

Combining (4.31)–(4.34) and applying Gronwall's inequality, we obtain

$$\|K_{1\varepsilon}^{\frac{1}{2}} v''_{\varepsilon m}\|_2^2 + \int_0^T \|v''_{\varepsilon m}\|_2^2 \, ds + \|v'_{\varepsilon m}\|^2_{H^1(\Omega)} + a * \|z_{\varepsilon m}\|_\Gamma^2 + \int_0^T \|z_{\varepsilon m}\|_\Gamma^2 \, ds \leq C_T, \quad (4.35)$$

where C_T is a positive constant independent of m, ε, and t.

Passing to the limit. Using the estimates (4.28) and (4.35), and by passing to the limit (first $m \to \infty$, and then $\varepsilon \to 0$), we see that there exists a strong solution $u \in W^{1,\infty}_{loc}(\mathbb{R}^+; H^1(\Omega)) \cap W^{2,2}_{loc}(\mathbb{R}^+; L^2(\Omega))$, $K_1^{\frac{1}{2}} u_t \in W^{1,\infty}_{loc}(\mathbb{R}^+; L^2(\Omega))$. In addition, u satisfies, for every $t \geq 0$, the inhomogeneous Neumann problem :

$$\begin{cases} -\Delta u = -K_1 u_{tt} - K_2 u_t - f(x,u) + g_1 & \text{in } L^2(\Omega), \\ \partial_\nu u = -\mu u - k * u_t + g_2 & \text{in } H^{\frac{1}{2}}(\Gamma). \end{cases} \qquad (4.36)$$

The theory of elliptic problems gives us $u \in L^\infty_{loc}(\mathbb{R}^+; H^2(\Omega))$.

Boundedness and energy estimate for strong solutions : Now, let u be a global strong solution of (4.1), and let $v = k * u_t$. We take the inner product of the equation (4.1) with u_t in order to find that

$$\frac{d}{dt}\Big(\frac{1}{2}\|K_1^{\frac{1}{2}}u_t\|_2^2 + E(u)\Big) + (K_2(x)u_t, u_t)_2 + (v, u_t)_\Gamma = (g_1, u_t)_2 + (g_2, u_t)_\Gamma.$$

Using that K_2 is strictly positive and the Cauchy-Schwarz inequality, we find

$$\frac{d}{dt}\Big(\frac{1}{2}\|K_1^{\frac{1}{2}}u_t\|_2^2 + E(u) + \frac{1}{2k_0}\int_t^\infty \|g_1(s)\|_2^2\, ds\Big) + (k * u_t, u_t)_\Gamma \leq -\frac{k_0}{2}\|u_t\|_2^2 + (g_2, u_t)_\Gamma. \quad (4.37)$$

Using the regularity of the strong solution, Remark 4.3.1, and arguing as in (4.18), we obtain

$$(k * u_t, u_t)_\Gamma \geq \frac{1}{2}\frac{d}{dt}a*\|v\|_\Gamma^2 + \frac{b_\infty}{2}\|v\|_\Gamma^2 + \frac{\gamma}{2}a*\|v\|_\Gamma^2, \quad (4.38)$$

where $b_\infty = \lim\limits_{t\to\infty} b(t) = \gamma\|a\|_{L^1(\mathbb{R}^+)}$. Moreover, by Lemma 4.3.1 and by Young's inequality, we have (as in (4.23))

$$(g_2, u_t)_\Gamma \leq \frac{d}{dt}(g_2, a * v)_\Gamma + d(\|g_2\|_\Gamma^2 + \|g_2'\|_\Gamma^2) + \frac{\gamma}{4}a*\|v\|_\Gamma^2 \quad (t > 0). \quad (4.39)$$

Combining (4.37), (4.38) and (4.39), one obtains (4.11) for every strong solution. In addition, from the condition (*F2*) we have

$$\int_\Omega |F(x, u_0)| \leq C(1 + \|u_0\|_{H^1}^{\alpha+2}),$$

where $C \geq 0$ is a constant depending only on the constants from condition (*F2*) (including the norm $\|f(\cdot, 0)\|_{L^\infty}$) and the constant of the embedding $H^1(\Omega) \hookrightarrow L^{\alpha+2}(\Omega)$. It follows from this inequality and the definition of G that there exists a constant $C_1 \geq 0$ which is independent of the initial data such that

$$G(0) \leq C_1\,(1 + \|K_1^{\frac{1}{2}}u_1\|_{L^2}^2 + \|u_0\|_{H^1}^{\alpha+2}). \quad (4.40)$$

On the other hand, by using condition (*F3*), the definition of G, the boundedness of g_2 with values in $L^2(\Gamma)$, and the following estimates given by Lemma 4.3.1, that is,

$$(g_2, a * v)_\Gamma \leq \|a\|_{L^1(\mathbb{R}^+)}\|g_2\|_\Gamma^2 + \frac{1}{4}a*\|v\|_\Gamma^2, \quad (4.41)$$

one easily shows that there exists a positive constant C_2 depending on λ and λ_1, and a positive constant C_3 depending on f, g_2 and the measure of Ω such that, for every $t \geq 0$,

$$\|u(t)\|_{H^1(\Omega)}^2 + \|K_1^{\frac{1}{2}}u_t(t)\|_2^2 \leq C_2 G(t) + C_3. \quad (4.42)$$

We combine (4.11), (4.40) and (4.42) to obtain the a priori estimate

$$\begin{aligned}&\|u(t)\|^2_{H^1(\Omega)}+\|K_1^{\frac{1}{2}}u_t(t)\|_2^2+\int_0^t\|u_t(s)\|_2^2\,ds+\int_0^t\|v(s)\|_\Gamma^2\,ds\\ \leq &C_4\left(1+\|K_1^{\frac{1}{2}}u_1\|^2_{L^2}+\|u_0\|^{\mu+2}_{H^1}\right)\quad(t\geq 0),\end{aligned}\tag{4.43}$$

where $C_4\geq 0$ depends only on the constants C_1, C_2, C_3 and on g_1, but is independent of the initial data. This a priori estimate gives the boundedness of strong solutions.

Uniqueness and continuous dependence. Next we show the continuous dependence of strong solutions on the initial data. Let u^μ $(\mu=1,\,2)$ be two strong solutions of (4.1), corresponding to the initial data (u_0^μ,u_1^μ) and the forcing terms (g_1^μ,g_2^μ) $(\mu=1,\,2)$. Setting $w=u^1-u^2$, $g_1=g_1^1-g_1^2$, and $g_2=g_2^1-g_2^2$, one has

$$\begin{cases}K_1w_{tt}+K_2w_t-\Delta w+f(x,u^1)-f(x,u^2)=g_1 & \text{in }\mathbb{R}^+\times\Omega,\\ \partial_\nu w+\mu w+k*w_t=g_2 & \text{on }\mathbb{R}^+\times\Gamma,\\ w(0)=u_0^1-u_0^2,\ \sqrt{K_1}w_t(0)=\sqrt{K_1}u_1^1-\sqrt{K_1}u_1^2. & \end{cases}\tag{4.44}$$

Let $h=k*w_t$. We multiply the equation (4.44) with w_t and integrate over Ω, in order to find that

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\|K_1^{\frac{1}{2}}w_t\|_2^2+\frac{1}{2}\|\nabla w\|_2^2+\frac{1}{2}\|\mu^{\frac{1}{2}}w\|_\Gamma^2+\frac{1}{2}a*\|h\|_\Gamma^2-(g_2,a*h)_\Gamma+\\ &\qquad+\frac{1}{2k_0}\int_t^\infty\|g_1(s)\|_2^2\,ds+d\int_t^\infty(\|g_2(s)\|_\Gamma^2+\|g_2'(s)\|_\Gamma^2)\,ds\Big)+\\ &\qquad+\int_\Omega(f(u^1)-f(u^2))(u_t^1-u_t^2)\,dx+\frac{k_0}{2}\|w_t\|_2^2+\frac{b_\infty}{2}\|h\|_\Gamma^2+\frac{\gamma}{4}a*\|h\|_\Gamma^2\leq 0,\end{aligned}$$

where we have used (4.38) and (4.39), when (u,v) are remplaced by (w,h). Integrating this inequality over $(0,t)$, using (4.41), and the fact that the Nemytskii operator generated by f is locally Lipschitz continuous from $H^1(\Omega)$ into $L^2(\Omega)$ (note that u^1 and u^2 are bounded in $C(\mathbb{R}^+,H^1(\Omega))$ by (4.43)), we obtain

$$\begin{aligned}&\frac{1}{2}\|K_1^{\frac{1}{2}}w_t\|_2^2+\frac{1}{2}\|\nabla w\|_2^2+\frac{1}{2}\|\mu^{\frac{1}{2}}w\|_\Gamma^2+\frac{1}{4}a*\|h\|_\Gamma^2+\frac{k_0}{4}\int_0^t\|w_t\|_2^2\,ds+\frac{b_\infty}{2}\int_0^t\|h\|_\Gamma^2\,ds\\ &\qquad+\frac{\gamma}{4}\int_0^t a*\|h\|_\Gamma^2\,ds+\frac{1}{2k_0}\int_t^\infty\|g_1(s)\|_2^2\,ds+d\int_t^\infty(\|g_2(s)\|_\Gamma^2+\|g_2'(s)\|_\Gamma^2)\,ds\\ &\leq C\int_0^t\|w(s)\|^2_{H^1(\Omega)}\,ds+\frac{1}{2k_0}\int_0^\infty\|g_1(s)\|_2^2\,ds+d\int_0^\infty(\|g_2(s)\|_\Gamma^2+\|g_2'(s)\|_\Gamma^2)\,ds\\ &\qquad+C(\|K_1^{\frac{1}{2}}w_t(0)\|_2^2+\|w(0)\|^2_{H^1}).\end{aligned}\tag{4.45}$$

From this inequality and Gronwall's lemma we infer that, for every $t \geq 0$,

$$\begin{aligned}\|K_1^{\frac{1}{2}} w_t(t)\|_2^2 + \|w(t)\|_{H^1(\Omega)}^2 + \int_0^t \|w_t(s)\|_2^2 \, ds + \int_0^t \|h\|_\Gamma^2 \, ds \\ \leq Ce^{Ct} \Big(\int_0^t \|g_1(s)\|_2^2 \, ds + \int_t^\infty (\|g_2(s)\|_\Gamma^2 + \|g_2'(s)\|_\Gamma^2) \, ds \\ + \|K_1^{\frac{1}{2}} w_t(0)\|_2^2 + \|w(0)\|_{H^1}^2 \Big). \quad (4.46)\end{aligned}$$

The continuous dependence of strong solutions on initial data, and the uniqueness of strong solutions are both an immediate consequence of this inequality.

Existence and uniqueness of weak solutions. Let $(u_0, u_1) \in \bar{D}$ and $(g_1, g_2) \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega) \times L^2_{loc}(\mathbb{R}^+; L^2(\Gamma))$. Then there exists a sequence $((u_0^\mu, u_1^\mu))_\mu \subseteq H^2(\Omega) \times H^2(\Omega)$ satisfying the compatibility condition (4.9), and a sequence $((g_1^\mu, g_2^\mu))_\mu \subseteq H^1_{loc}(\mathbb{R}^+; L^2(\Omega))$ such that

$$\begin{aligned} &(u_0^\mu, u_1^\mu) \to (u_0, u_1) \quad \text{in } H^1(\Omega) \times L^2(\Omega), \quad \text{and} \\ &(g_1^\mu, g_2^\mu) \to (g_1, g_2) \quad \text{in } L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \times L^2_{loc}(\mathbb{R}^+; L^2(\Gamma)). \end{aligned}$$

Then, for each $\mu \in \mathbb{N}$, there exists a unique strong solution u^μ to the problem (4.1). By the estimate (4.43) we have

$$\begin{aligned} &u^\mu && \text{is uniformly bounded in } C_b(\mathbb{R}^+; H^1(\Omega)), \\ &u_t^\mu && \text{is uniformly bounded in } L^2(\mathbb{R}^+; L^2(\Omega)), \\ &K_1^{\frac{1}{2}} u_t^\mu && \text{is uniformly bounded in } C_b(\mathbb{R}^+; L^2(\Omega)), \\ &k * u_t^\mu && \text{is uniformly bounded in } L^2(\mathbb{R}^+; L^2(\Gamma)). \end{aligned} \quad (4.47)$$

Moreover, by the estimate (4.46) we have

$$\begin{aligned} &u^\mu && \text{is a Cauchy sequence in } C(\mathbb{R}^+; H^1(\Omega)), \\ &u_t^\mu && \text{is a Cauchy sequence in } L^2(\mathbb{R}^+; L^2(\Omega)), \\ &K_1^{\frac{1}{2}} u_t^\mu && \text{is a Cauchy sequence in } C(\mathbb{R}^+; L^2(\Omega)), \\ &k * u_t^\mu && \text{is a Cauchy sequence in } L^2(\mathbb{R}^+; L^2(\Gamma)). \end{aligned} \quad (4.48)$$

The convergences given by (4.47) and (4.48) are sufficient to obtain a weak solution u to problem (4.1) as the strong limit of the above sequence of strong solutions, that is

$$\begin{aligned} &u^\mu \to u && \text{in } C(\mathbb{R}^+; H^1(\Omega)), \\ &u_t^\mu \to u_t && \text{in } L^2(\mathbb{R}^+; L^2(\Omega)), \\ &K_1^{\frac{1}{2}} u_t^\mu \to K_1^{\frac{1}{2}} u_t && \text{in } C(\mathbb{R}^+; L^2(\Omega)), \\ &k * u_t^\mu \to v = \tfrac{d}{dt}(k * (u - u_0)) && \text{in } L^2(\mathbb{R}^+; L^2(\Gamma)). \end{aligned} \quad (4.49)$$

However, from (4.49), one easily sees that the energy inequality (4.11), the estimate (4.43), and the a priori estimate (4.46) remain true for any weak solution, respectively any pair of weak solutions. The uniqueness of weak solutions is again an immediate consequence of the a priori estimate (4.46). From the estimate (4.43) we obtain that every weak solution is bounded (property (*T1*)). Moreover, by (4.11), the boundedness of u in $H^1(\Omega)$, the continuity of E, and (4.41), the energy function G is decreasing and bounded from below, and therefore

$$\lim_{t\to\infty} G(t) = \inf_{t\geq 0} G(t) = G_\infty \text{ exists.} \tag{4.50}$$

From this and the energy inequality (4.11) we obtain (*T2*). Finally, in order to prove the variational equality (*T4*) we note first that this equality is satisfied pointwise (in time) for any strong solution. However, by using again that weak solutions are locally uniform limits of strong solutions, one sees that this equality remains valid for all weak solutions. □

4.4 Compact range of global and bounded solutions : Proof of Theorem 4.2.2

In this section we obtain a compactness result which generalizes the previous results in [13] to the case of dynamical boundary conditions. In order to prove Theorem 4.2.2, let us list two lemmas for which we need the following notation. Let X be a (real) Banach space equipped with the norm $\|\cdot\|_X$ and let $S^2(\mathbb{R}^+; X)$ be the Stepanov space defined by

$$S^2(\mathbb{R}^+; X) = \big\{g \in L^2_{loc}(\mathbb{R}^+; X), \sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|g(s)\|_X^2 \, ds < \infty\big\}.$$

For any $h > 0$, $t \geq 0$ and any $g \in S^2(\mathbb{R}^+; X)$ we denote by $g^h(t)$ the difference $g(t+h) - g(t)$ and we say that g is S^1-uniformly continuous with values in X if

$$\sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|g^h(s)\|_X^2 \, ds \to 0 \text{ as } h \to 0.$$

Lemma 4.4.1 ([5]). *Assume that f satisfies (F2) and that g_1 satisfies (G2). Then the source term $H(t) = g_1(t) - f(t,u)$ is S^1-uniformly continuous in $L^2(\Omega)$ and $H \in S^2(\mathbb{R}^+, L^2(\Omega))$.*

Lemma 4.4.2 ([13]). *Let X and Y be two Banach spaces endowed respectively with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Assume that X is compactly embedded into Y. Then :*

(a) If $u : \mathbb{R}^+ \to Y$ is uniformly continuous and

$$\sup_{t\geq 0, \delta\in[0,1]} \Big\| \int_t^{t+\delta} u(s) \, ds \Big\|_X < \infty,$$

then $\bigcup_{t\geq 0}\{u(t)\}$ is precompact in Y.

(b) If $u \in C^1(\mathbb{R}^+, Y)$ is bounded with values in X, and if u' is uniformly continuous with values in Y, then $\bigcup_{t\geq 0}\{u'(t)\}$ is precompact in Y.

Proof of Theorem 4.2.2. We proceed in two steps.
`Step 1`. We first show that the function $(u(t), K_1^{\frac{1}{2}}u_t(t))$ is uniformly continuous with values in $H^1(\Omega)\times L^2(\Omega)$. For all $t \geq 0$, $h \geq 0$, we let $u^h(t) = u(t+h) - u(t)$. Since $u_t \in L^2(\mathbb{R}^+; L^2(\Omega))$ and $g_1 - f(\cdot, u) \in S^2(\mathbb{R}^+, L^2(\Omega))$, we have

$$\sup_{t\geq 0}\int_t^{t+1} \|K_1 u_{tt} - \Delta u\|_2 \, ds \leq C.$$

From this estimate and (4.1), we deduce easily the inequality

$$\begin{aligned}
&\int_t^{t+1} \|u^h(s)\|^2_{H^1(\Omega)} \, ds \leq \\
&\leq C\Big\{\int_t^{t+1} (\mu u^h(s), \mu u^h(s))_\Gamma + (\nabla u^h(s), \nabla u^h(s))_2 \, ds\Big\}\\
&\leq C\Big\{\int_t^{t+1} (\mu u^h(s) + \partial_\nu u^h(s), \mu u^h(s))_\Gamma + (-\Delta u^h(s), u^h(s))_2 \, ds\Big\}\\
&\leq C\Big\{\int_t^{t+1} \|K_1^{\frac{1}{2}} u_t^h(s)\|_2^2 \, ds + \|K_1 u_t^h(t)\|_2 \|u^h(t)\|_2\\
&\quad + \|K_1 u_t^h(t+1)\|_2\|u^h(t+1)\|_2 + \int_t^{t+1} \|g_2^h - v^h\|_\Gamma^2 \, ds + \sup_{[t,t+1]}\|u^h\|_2\Big\}\\
&\leq C\Big\{\int_t^{t+1} \|K_1^{\frac{1}{2}} u_t^h(s)\|_2^2 \, ds + \int_t^{t+1} \|g_2^h - v^h\|_\Gamma^2 \, ds + C_3 \sup_{[t,t+1]}\|u^h\|_2\Big\}.
\end{aligned}$$

Since $u_t \in L^2(\mathbb{R}^+; L^2(\Omega))$, u is uniformly continuous from $\mathbb{R}^+$ into $L^2(\Omega)$. Using this and the last inequality, we obtain

$$\int_t^{t+1} \|u^h(s)\|^2_{H^1(\Omega)} \, ds \leq C\Big\{\int_t^{t+1} \|K_1^{\frac{1}{2}} u_t^h(s)\|_2^2 \, ds + \int_t^{t+1} \|g_2^h - v^h\|_\Gamma^2 \, ds\Big\} + \phi_1(h), \tag{4.51}$$

where $\phi_1(h) \to 0$ as $h \to 0$. Moreover, since $u_t \in L^2(\mathbb{R}^+; L^2(\Omega))$ and since the left-shift semigroup on the space $L^2(\mathbb{R}^+; L^2(\Omega))$ is strongly continuous, then we have

$$\int_t^{t+1} \|K_1^{\frac{1}{2}} u_t^h(s)\|_2^2 \, ds \to 0 \text{ as } h \to 0. \tag{4.52}$$

Similarly, since g_2, $v \in L^2(\mathbb{R}^+; L^2(\Gamma))$,

$$\int_t^{t+1} \|g_2^h - v^h\|_\Gamma^2 \, ds \to 0 \text{ as } h \to 0. \tag{4.53}$$

By using the last two limits and the inequality (4.51), we obtain

$$\int_t^{t+1} \|u^h(s)\|^2_{H^1(\Omega)}\, ds \le \phi_2(h), \tag{4.54}$$

where $\phi_2(h) \to 0$ as $h \to 0$. Now we introduce

$$V_h(t) = \frac{1}{2}\big(\|K_1^{\frac{1}{2}} u_t^h(t)\|_2^2 + \|\nabla u^h(t)\|_2^2 + \|\mu^{\frac{1}{2}} u^h(t)\|_\Gamma^2 + a{*}\|v^h\|_\Gamma^2(t)\big).$$

Since $a{*}\|v\|_\Gamma^2 \in L^1(\mathbb{R}^+)$,

$$\int_t^{t+1} a{*}\|v^h\|_\Gamma^2(s)\, ds \to 0 \text{ as } h \to 0. \tag{4.55}$$

Combining (4.52), (4.54), and (4.55), we obtain

$$\int_t^{t+1} V_h(\theta)\, d\theta \le \phi_3(h), \text{ where } \phi_3(h) \to 0 \text{ as } h \to 0. \tag{4.56}$$

On the other hand, for a strong solution u, by taking the derivative of $V_h(t)$ with respect to t, and by using (4.1) and (4.38), we obtain

$$\begin{aligned}\frac{d}{dt}V_h(t) &\le (g_1^h - f^h(x,u), u_t^h)_2 + (g_2^h, u_t^h)_\Gamma - \Big((K_2 u_t^h, u_t^h) \\ &\quad + \frac{\gamma}{2} a{*}\|v^h\|_\Gamma^2 + \frac{1}{2} a{*}\|v^h\|_\Gamma^2 + \frac{b_\infty}{2}\|v^h\|_\Gamma^2\Big) \\ &\le (g_1^h - f^h(x,u), u_t^h)_2 + (g_2^h, u_t^h)_\Gamma. \end{aligned} \tag{4.57}$$

Integrating (4.57) over $[\theta, t+1]$ with $\theta \in [t, t+1]$, using Lemma 4.4.1, the fact that $u(t)$ is bounded in $H^1(\Omega)$, $u_t \in L^2(\mathbb{R}^+; L^2(\Omega))$, and $g_2' \in L^2(\mathbb{R}^+; L^2(\Gamma))$, we deduce that for any $t \ge 0$:

$$\begin{aligned} &V_h(t+1) - V_h(\theta) \le \\ &\le C\int_t^{t+1} (\|(g_1 - f(x,u))^h(s)\|_2^2 + \|u_t^h(s)\|_2^2)\, ds + \int_\theta^{t+1} (g_2^h, u_t^h)_\Gamma(s)\, ds \\ &\le C\int_t^{t+1} (\|(g_1 - f(x,u))^h(s)\|_2^2 + \|u_t^h(s)\|_2^2)\, ds - \int_\theta^{t+1} ((g_2')^h, u^h)_\Gamma(s)\, ds \\ &\quad + C\sup_{[t,t+1]} \|g_2^h(s)\|_\Gamma \\ &\le C\int_t^{t+1} (\|(g_1 - f(x,u))^h(s)\|_2^2 + \|u_t^h(s)\|_2^2)\, ds + C\int_t^{t+1} \|(g_2')^h(s)\|_\Gamma^2\, ds \\ &\quad + C\sup_{[t,t+1]} \|g_2^h(s)\|_\Gamma \\ &\le \phi_4(h), \text{ where } \phi_4(h) \to 0 \text{ as } h \to 0. \end{aligned} \tag{4.58}$$

By an approximation argument, the inequality (4.58) still holds for all weak solutions. Then, by integrating (4.58) over $[t, t+1]$ with respect to θ and by using (4.56), we obtain

$$V_h(t+1) \leq \phi_4(h) + \int_t^{t+1} V_h(\theta)\, d\theta \leq \phi_5(h),$$

which tends to 0 as $h \to 0$. This concludes the proof of `Step 1`.

`Step 2`. We show that $(u(t), K_1^{\frac{1}{2}} u_t(t))$ is relatively compact in $H^1(\Omega) \times L^2(\Omega)$. By applying Lemma 4.4.2 (b) with $Y = L^2(\Omega)$ and $X = H^1(\Omega)$, we obtain immediately that $\bigcup_{t \geq 0}\{K_1^{\frac{1}{2}} u_t(t)\}$ is relatively compact in $L^2(\Omega)$. To prove that $\bigcup_{t\geq 0}\{u(t)\}$ is relatively compact in $H^1(\Omega)$, we remark that

$$K_1 u_t(t+h) - K_1 u_t(t) - \int_t^{t+h} \Delta u(s)\, ds + \int_t^{t+h} K_2 u_t(s)\, ds = \int_t^{t+h} (g_1(s) - f(x, u(s)))\, ds$$

By using $(F2)$, Lemma 4.4.1, $u_t \in L^2(\mathbb{R}^+; L^2(\Omega))$ and the fact that $(u, K_1^{\frac{1}{2}} u_t)$ is bounded with values in $H^1(\Omega) \times L^2(\Omega)$, we obtain

$$\sup_{t \geq 0, \delta \in [0,1]} \| \int_t^{t+\delta} \Delta u(s)\, ds \|_2 < \infty.$$

By applying Lemma 4.4.2 (a) with $Y = H^1(\Omega)$ and $X = \{\phi \in H^1(\Omega); \Delta\phi \in L^2(\Omega)\},$, we obtain the claim. $\square$

4.5 Convergence and decay rate of global weak solutions : Proof of Theorem 4.2.3 and Theorem 4.2.4

In this section we study the long-time stabilization of global bounded solutions of (4.1), that is, we prove Theorems 4.2.3 and 4.2.4. Let us recall that the ω-limit set of a continuous function $u : \mathbb{R}^+ \to H^1(\Omega)$ is defined by

$$\omega(u) = \{\phi \in H^1(\Omega) : \exists\, t_n \to +\infty \text{ such that } \lim_{n\to\infty} \|u(t_n) - \phi\|_{H^1(\Omega)} = 0\}.$$

From well-known results on dynamical systems [12], if u is a continuous function having in addition relatively compact range, then the ω-limit set of u is a non-empty, compact, and connected subset of $H^1(\Omega)$. Moreover, since our system has a continuous Lyapunov functional G, we prove the following lemma which is fundamental for the proof of Theorem 4.2.3.

Lemma 4.5.1. *Let u be a global bounded weak solution of equation* (4.1)*, and* $v = \frac{d}{dt}(k * (u - u_0))$*. Then :*

(i) The function E is constant on $\omega(u)$, and

$$E(\phi) = \lim_{t\to\infty} E(u(t)) = E_\infty < \infty, \text{ for all } \phi \in \omega(u).$$

(ii) $\lim_{t\to\infty} \|K_1^{\frac{1}{2}} u_t\|_2 = \lim_{t\to\infty} a * \|v\|_\Gamma^2 = 0.$

(iii) $E'(\phi) = 0$, *for all* $\phi \in \omega(u)$.

(iv) There exists a uniform Łojasiewicz exponent $\theta \in]0, \frac{1}{2}]$, $\beta > 0$ *and* $T > 0$ *such that for all* $t \geq T$

$$|E(u(t)) - E_\infty|^{1-\theta} \leq \beta \|E'(u(t))\|_*. \tag{4.59}$$

Proof. Let $\phi \in \omega(u)$. Then there exists an unbounded increasing sequence (t_n) in $\mathbb{R}^+$ such that $u(t_n) \to \phi$ in $H^1(\Omega)$. Since $u_t \in L^2(\mathbb{R}^+, L^2(\Omega))$, we have

$$u(t_n + s) = u(t_n) + \int_{t_n}^{t_n+s} u_t(\rho)\, d\rho \to \phi \text{ in } L^2(\Omega) \text{ for every } s \in [0,1].$$

This, together with the relative compactness of the trajectory in $H^1(\Omega)$, implies that $u(t_n+s) \to \phi$ in $H^1(\Omega)$ for every $s \in [0,1]$. Then, by continuity of E, $E(u(t_n+s)) \to E(\phi)$ in $H^1(\Omega)'$ for every $s \in [0,1]$. Using the dominated convergence theorem,

$$E(\phi) = \lim_{n\to\infty} \int_0^1 E(u(t_n + s))\, ds.$$

Therefore, by integrating $G(t_n + .)$ in $[0,1]$, we obtain

$$E(\phi) = \lim_{n\to\infty} \int_0^1 G(t_n + s)\, ds = G_\infty,$$

where we have used (T2), (G1), (4.50), and following estimate :

$$\Big|\int_{t_n}^{t_n+1} (g_2(s), a * v(s))_\Gamma\, ds\Big|^2 \leq \int_{t_n}^{t_n+1} \|g_2(s)\|_\Gamma^2\, ds + \|a\|_{L^1(\mathbb{R}^+)} \int_{t_n}^{t_n+1} a * \|v(s)\|_\Gamma^2\, ds.$$

Since ϕ was chosen arbitrarily in $\omega(u)$, this implies that E is constant on $\omega(u)$. Moreover, by the relative compactness of u with values in $H^1(\Omega)$, we obtain $\lim_{t\to\infty} E(u(t)) = G_\infty = E_\infty$. Then assertion (i) is proved. From this, the definition of G and since $g_2(t)$ and the integral terms in G tend to 0 as $t \to \infty$, we obtain assertion (ii).
In order to prove (iii), let $\phi \in \omega(u)$ and choose $t_n \to \infty$ such that $u(t_n) \to \phi$ in $H^1(\Omega)$. We have already seen that this implies $u(t_n + s) \to \phi$ in $H^1(\Omega)$ for every $s \in [0,1]$. Hence $E'(u(t_n + s)) \to E'(\phi)$ in $H^1(\Omega)'$ for every $s \in [0,1]$. Finally, using the dominated convergence theorem, (T2), (T4), (ii), and (G1), we have for all $\psi \in H^1(\Omega)$

$$(E'(\phi), \psi)_{H^1(\Omega)', H^1(\Omega)} = \int_0^1 (E'(\phi), \psi)_{H^1(\Omega)', H^1(\Omega)}\, ds$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int_0^1 (E'(u(t_n+s)), \psi)_{H^1(\Omega)', H^1(\Omega)}\, ds \\
&= \lim_{n\to\infty} \int_0^1 \Big(\int_\Omega \nabla u(t_n+s) \nabla\psi \, dx + \int_\Omega f(x, u(t_n+s))\psi \, dx + \int_\Gamma \mu u(t_n+s)\ \psi \, d\sigma \Big) ds \\
&= \lim_{n\to\infty} \int_0^1 \Big(-\frac{d}{dt}\int_\Omega K_1 u_t(t_n+s)\psi\, dx - \int_\Omega (K_2 u_t - g_1)(t_n+s)\psi\, dx \\
&\quad - \int_\Gamma (v - g_2)(t_n+s)\ \psi\, d\sigma \Big) ds \\
&= \lim_{n\to\infty} \Big[\int_0^1 \Big(\int_\Omega (-K_2 u_t + g_1)(t_n+s)\psi\, dx - \int_\Gamma (v-g_2)(t_n+s)\ \psi\, d\sigma\Big) ds \\
&\quad + \int_\Omega (K_1 u_t(t_n) - K_1 u_t(t_n+1))\psi\, dx \Big] \\
&= 0.
\end{aligned}
$$

This proves (iii). □

After the previous preparation, we are ready to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. Let $W_0(t) : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$W_0(t) = G(t) - E_\infty + \varepsilon (E'(u(t)), K_1 u_t)_* \quad (t \geq 0).$$

Then, by (T3) and (T4), we have

$$
\begin{aligned}
\frac{d}{dt} W_0(t) &= \frac{d}{dt} G(t) + \varepsilon (E''(u) u_t, K_1 u_t)_* + \varepsilon (E'(u), K_1 u_{tt})_* \\
&\leq -\frac{k_0}{2}\|u_t\|_2^2 - \frac{b_\infty}{2}\|v\|_\Gamma^2 - \frac{\gamma}{4} a{*}\|v\|_\Gamma^2 + \varepsilon (E''(u)u_t, K_1 u_t)_* \qquad (4.60) \\
&\quad + \varepsilon (E'(u), -E'(u) - K_2 u_t - v + g_1(t) + g_2(t))_*.
\end{aligned}
$$

Arguing as in the Chapter 3, we have

$$(E''(u)u_t, K_1 u_t)_* \leq C\, \|u_t\|_2^2$$

and

$$
\begin{aligned}
(E'(u), -E'(u) - K_2 u_t - v + g_1(t) + g_2(t))_* \leq -\frac{1}{2}\|E'(u)\|_*^2 + C\big(\|u_t\|_2^2 \\
+\|v\|_\Gamma^2 + \|g_1(t)\|_2^2 + \|g_2(t)\|_\Gamma^2\big).
\end{aligned}
$$

Combining (4.60) and the last two inequalities, and choosing $\varepsilon > 0$ small enough, we obtain

$$\frac{d}{dt} W(t) \leq -C\big(\|u_t\|_2^2 + \|E'(u)\|_*^2 + \|v\|_\Gamma^2 + a{*}\|v\|_\Gamma^2\big) \quad (t > 0), \qquad (4.61)$$

where $W : \mathbb{R}^+ \to \mathbb{R}$ is the energy given by

$$W(t) = \frac{1}{2}\|K_1^{\frac{1}{2}}u_t\|_2^2 + E(u) - E_\infty + \frac{1}{2}a{*}\|v\|_\Gamma^2 - (g_2, a*v)_\Gamma + \varepsilon(E'(u(t)), K_1u_t)_* + (\frac{1}{2k_0} - C_\varepsilon)\int_t^\infty \|g_1(s)\|_2^2\,ds + (d - C_\varepsilon)\int_t^\infty (\|g_2(s)\|_\Gamma^2 + \|g_2'(s)\|_\Gamma^2)\,ds, \tag{4.62}$$

where $C_\varepsilon < \inf\{\frac{1}{2k_0}, d\}$. Thus, the function W is nonincreasing and $\lim\limits_{t\to\infty} W(t) = 0$. It follows that $W(t) \geq 0$ for all $t \in \mathbb{R}^+$. If there exists $T_0 \geq 0$ such that $W(T_0) = 0$. Then $W(t) = 0$ for all $t \geq T_0$. Therefore, by the inequality (4.61), $u_t = 0$ for all $t \geq T_0$, and the function u is constant for $t \geq T_0$, that is, $u(t) = \phi$ for $t \geq T_0$. In this case, there remains nothing to prove. We may therefore suppose in the following that $W(t)$ is strictly positive on $\mathbb{R}^+$.

Now, Let θ be as in Lemma 4.5.1 (iv). and let $\theta_0 \in (0, \theta]$ be such that

$$(1+\delta)(1-\theta_0) > 1, \tag{4.63}$$

that is, $\theta_0 < \dfrac{\delta}{1+\delta}$. Note that (4.59) is satisfied with θ replaced by θ_0. Using Youngâs inequality, we deduce from the definitions of W and Lemma 4.3.1 that for every $t \geq 0$

$$W(t)^{1-\theta_0} \leq C\Big\{\|K_1^{\frac{1}{2}}u_t\|_2^{2(1-\theta_0)} + (a{*}\|v(t)\|_\Gamma^2)_2^{\frac{2(1-\theta_0)}{2}} + |E(u) - E_\infty|^{(1-\theta_0)} + \|g_2(t)\|_\Gamma + (a{*}\|v(t)\|^2)_\Gamma^{\frac{1-\theta_0}{2\theta_0}} + \Big(\int_t^\infty (\|g_1(s)\|_2^2 + \|g_2(s)\|_\Gamma^2 + \|g_2'(s)\|_\Gamma^2)\,ds\Big)^{1-\theta_0} + \|K_1u_t\|_2^{\frac{1-\theta_0}{\theta_0}} + \|E'(u)\|_*\Big\}.$$

On the other hand, by assertions (ii) and (iv) from Lemma 4.5.1, there exists $T > 0$ such that for all $t \geq T$ we have

$$\Big\{\|K_1^{\frac{1}{2}}u_t\|_2 + \|K_1u_t\|_2 + (a{*}\|v(t)\|^2)_2^{\frac{1}{2}}\Big\} < 1$$

and

$$|E(u(t)) - E_\infty|^{1-\theta_0} \leq \beta\|E'(u(t))\|_*.$$

Using this, (G2′) and the fact that $2(1-\theta_0) \geq 1$ and $\frac{1-\theta_0}{\theta_0} \geq 1$, we obtain for all $t \geq T$

$$W(t)^{1-\theta_0} \leq C\Big\{\|u_t\|_2 + (a{*}\|v(t)\|_\Gamma^2)^{\frac{1}{2}} + \|E'(u)\|_* + \|g_2(t)\|_\Gamma + (1+t)^{-(1+\delta)(1-\theta_0)}\Big\}. \tag{4.64}$$

Combining the last inequality and (4.61), we obtain

$$\begin{aligned}
-\frac{d}{dt}W(t)^{\theta_0} &= -\theta_0 W(t)^{\theta_0-1}\frac{d}{dt}W(t)\\
&\geq \frac{C\big(\|u_t\|_2^2+\|E'(u)\|_*^2+\|v\|_\Gamma^2+a*\|v\|_\Gamma^2\big)}{\|u_t\|_2+(a*\|v(t)\|_\Gamma^2)^{\frac{1}{2}}+\|E'(u)\|_*+\|g_2(t)\|_2+(1+t)^{-(1+\delta)(1-\theta_0)}}\\
&\geq C\big(\|u_t\|_2+\|v(t)\|_\Gamma+(a*\|v(t)\|_\Gamma^2)^{\frac{1}{2}}+\|E'(u)\|_*\big)- \qquad (4.65)\\
&\quad - C\big(\|g_2(t)\|_2+(1+t)^{-(1+\delta)(1-\theta_0)}\big).
\end{aligned}$$

From this and the fact that the term $-\frac{d}{dt}W(t)^{\theta_0}+C\big(\|g_2(t)\|_2+(1+t)^{-(1+\delta)(1-\theta_0)}\big)$ is integrable on $[T,+\infty)$, we obtain that $\|u_t\|_2$ is integrable on $[T,+\infty)$, which implies that $\lim_{t\to\infty} u(t,\cdot)$ exists in $L^2(\Omega)$. By the relative compactness of the range of u in $H^1(\Omega)$, $\lim_{t\to\infty} u(t,\cdot)$ exists in $H^1(\Omega)$. This is the claim. $\square$

The following lemma is used in the proof of Theorem 4.2.4, that is, the proof of the convergence rate to equilibrium. Its proof can be found in [4].

Lemma 4.5.2. *Let $\zeta\in W^{1,1}_{loc}(\mathbb{R}^+,\mathbb{R}^+)$. We suppose that there exist constants $K_1>0$, $K_2\geq 0$, $k>1$ and $\lambda>0$ such that for almost every $t\geq 0$ we have*

$$\zeta'(t)+K_1\zeta(t)^k\leq K_2(1+t)^{-\lambda}.$$

Then there exists a positive constant m such that

$$\zeta(t)\leq m(1+t)^{-\nu},\ \text{ where } \nu=\inf\{\frac{1}{k-1},\frac{\lambda}{k}\}.$$

Proof of Theorem 4.2.4. We proceed in two steps.
`Step 1 (Polynomial decay).` First, we note that the inequalities (4.64) and (4.65) are satisfied when θ_0 is replaced by the initial exponent θ given by Lemma 4.5.1 (iv). By using (4.64) together with Youngâs inequality, we obtain for every $t\in[T,\infty[$

$$\begin{aligned}
W(t)^{2(1-\theta)}\leq C\Big\{&\|u_t\|_2^2+(a*\|v(t)\|_\Gamma^2)+\|E'(u)\|_*^2\\
&+\|g_2(t)\|_\Gamma^2+(1+t)^{-2(1+\delta)(1-\theta)}\Big\}. \qquad (4.66)
\end{aligned}$$

Using this, (G2), and (4.61), we obtain the following differential inequality for every $t\geq T$

$$C\frac{d}{dt}W(t)+W(t)^{2(1-\theta)}\leq C(1+t)^{-2(1+\delta)(1-\theta)}. \qquad (4.67)$$

Then we may apply Lemma 4.5.2 in order to obtain

$$W(t)\leq C(1+t)^{-\gamma}, \qquad (4.68)$$

where $\gamma = \inf\{\frac{1}{1-2\theta}, 1+\delta\}$. By using again (4.61), we have

$$-\frac{d}{dt}W(s) \geq C\|u_t(t)\|_2^2.$$

Integrating this inequality over $[t, 2t]$ $(t \geq T)$ and using (4.68), we obtain

$$\int_t^{2t} \|u_t(s)\|_2^2 \, ds \leq C(1+t)^{-\gamma}.$$

Note that for every $t \in \mathbb{R}^+$,

$$\int_t^{2t} \|u_t(s)\|_2 \, ds \leq t^{\frac{1}{2}}\Big(\int_t^{2t} \|u_t(s)\|_2^2 \, ds\Big)^{\frac{1}{2}}.$$

It follows that

$$\int_t^{2t} \|u_t(s)\|_2 \, ds \leq C(1+t)^{\frac{1-\gamma}{2}} \text{ for every } t \geq T.$$

Therefore we obtain for every $t \geq T$

$$\int_t^{\infty} \|u_t(s)\|_2 \, ds \leq \sum_{k=0}^{\infty} \int_{2^k t}^{2^{k+1}t} \|u_t(s)\|_2 \, ds \leq C\sum_{k=0}^{\infty} (2^k t)^{\frac{1-\gamma}{2}} \leq C(1+t)^{\frac{1-\gamma}{2}}.$$

Then, for all $t \geq T$

$$\|u(t) - \phi\|_2 \leq \int_t^{\infty} \|u_t(s)\|_2 \, ds \leq C(1+t)^{-\xi}, \quad \text{where } \xi = \inf\{\frac{\theta}{1-\theta}, \frac{\delta}{2}\}.$$

`Step 2 (Exponential decay).` Suppose that $g_1 = 0$ and $g_2 = 0$. Then (4.67) becomes

$$-\frac{d}{dt}W(t) \geq CW(t)^{2(1-\theta)}.$$

Since $W(t) > 0$, for sufficiently large times t, we obtain from this inequality that

$$\begin{cases} (\frac{-1}{1-2\theta})\frac{d}{dt}W(t)^{-(1-2\theta)} \leq -C & \text{if } \theta \in (0, \frac{1}{2}), \\ \frac{d}{dt}(\ln W(t)) \leq -C & \text{if } \theta = \frac{1}{2}. \end{cases}$$

Hence, integrating these differential inequalities, we obtain that there exists a constant $C > 0$ such that, for every large $t > 0$,

$$\begin{cases} W(t) \leq C(1+t)^{-\frac{1}{1-2\theta}} & \text{if } \theta \in (0, \frac{1}{2}), \\ W(t) \leq Ce^{-Ct} & \text{if } \theta = \frac{1}{2}. \end{cases} \tag{4.69}$$

Note that the inequality (4.65) (when $g_1 = g_2 = 0$) implies for every $s \geq T$

$$-\frac{d}{dt}W(s)^{\theta} \geq C\|u_t(t)\|_2.$$

Integrating this inequality on the interval $[t, \infty)$ $(t \geq T)$, we obtain

$$\|u(t) - \phi\|_2 \leq \int_t^{\infty} \|u_t(s)\|_2 \, ds \leq CW(t)^{\theta}.$$

This inequality together with the inequality (4.69) implies the claim. □

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