Théorie des invariants des Equations différentielles, Equations d'Abel et de Riccati

Thèse présentée devant

l'Université Bordeaux1

pour obtenir le grade de

Docteur en Mathématiques Pures

par

Oumar Wone

soutenue devant le jury composé de

Guy Casale	Alain Hénaut
Isao Nakai (Rapporteur)	Emmanuel Paul (Rapporteur)
Ahmed Sebbar (Directeur)	Jacques Arthur Weil
Alain Yger	

Mois-Année: Février 2012 Ecole Doctorale Math-Info

Numéro d'Ordre: section: 25

Remerciements

"Remerciements" un bien grand mot. Peut-on réellement exprimer toute notre reconnaissance aux êtres qui nous sont chers, à ceux qui nous ont forgés notre identité, je ne le pense pas. Mais comme il faut respecter la tradition, je vais m'y atteler tant bien que mal.

Tout d'abord, toutes les Louanges à la Cause Suprême!

Mes remerciements sont adressés ensuite aux deux causes relatives qui me sont très chères: mes parents; pour leur soutien sans faille et leur amour. Je remercie aussi du fond du coeur toute ma famille.

La vie est faite de rencontres plus ou moins bonnes, qui forgent notre identité. Le sort m'a fait un cadeau en me faisant rencontrer Ahmed Sebbar. Par son humanisme, ses conseils et sa haute valeur mathématique, il a su guider mes recherches et m'a fait découvrir l'immense oeuvre d'Élie Cartan en particulier ses connexions. Que ma profonde gratitude lui soit exprimée.

L'autre cadeau que le sort m'a fait, fut de me faire rencontrer Alain Hénaut. Un être assez ferme dans son approche mathématique. Je lui exprime ici toute ma reconnaissance et mon respect.

Merci aussi à tous les autres membres du jury, Guy Casale, Isao Nakai, Emmanuel Paul, Jacques Arthur Weil, Alain Yger pour leurs questions et remarques constructives.

Une mention spéciale est pour l'ami Diomba. Merci de m'avoir dit la vérité quand il le fallait et merci pour tes très belles figures.

Enfin j'exprime toute ma gratitude à tous les membres de l'école doctorale, aux autres collègues, à tous les autres travailleurs de l'IMB et ainsi qu'à toute l'Université Bordeaux 1 pour son soutien.

Introduction Générale

La multitude de transformations qui changent une équation différentielle en une autre, de même ordre ou de même forme, a incité les mathématiciens à chercher une classification des équations différentielles sous l'action des groupes de transformations.

L'approche classique amenée à son apogée par Halphen dans [22], consistait à étudier les équations différentielles sous l'action d'un certain pseudo-groupe de transformations locales (ie des transformations localement inversibles) qui conservent à la fois la forme et l'ordre de l'équation, en un mot on se restreignait à certaines substitutions dites de 'symétrie', qui envoyait une solution de l'équation de départ, sur une autre solution de la même équation. On obtient ainsi une classification en fonction de certaines quantités qui ne dépendent que de l'équation elle-même.

Prenons l'exemple de l'équation linéaire homogène

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n = 0,$$

étudiée sous l'action des transformations de la forme

$$y = \eta u, \quad x = \phi(\epsilon).$$

Cette équation considérée par Halphen admet une classification complète en fonction de certaines quantités qui ne dépendent que des p_i.

Une autre façon d'aborder le problème de la classification des équations des équations différentielles est d'adopter des hypothèses moins restrictives. Plus précisément on ne requiert plus que la conservation de l'ordre de l'équation lorsque l'on la soumet à un certain type de pseudo-groupe de transformations.

C'est Sophus Lie qui est l'un des précurseurs d'une telle approche. Lie réussit à classifier les équations du premier ordre

$$\frac{dv}{dt} = f(v, t);$$

sous l'action des transformations ponctuelles les plus générales, c'est à dire des substitutions de la forme

$$x = \xi(v,t), \quad y = \zeta(v,t).$$

Il montre que toutes les équations du premier ordre sont équivalentes à l'équation triviale

$$\frac{dv}{dt} = 0.$$

L'approche de Lie fut ensuite utilisée par son élève A. Tresse qui, dans [58], réalise la classification complète des équations du second ordre

$$\frac{d^2y}{dx^2} = \omega(x,y,y').$$

Mais l'avancée cruciale et définitive dans cette approche de Lie, est due à Elie Cartan.

Cartan contrairement à Lie qui utilisait l'analyse, introduit une vision et une méthode géométrique pour attaquer l'étude des équations différentielles. Il associe à chaque équation différentielle résolue en la plus haute dérivée, de manière univoque, un système pfaffien avec condition d'indépendance. Les variétés intégrales de ce dernier système sont en bijection avec les solutions de l'équation considérée.

Il applique sa méthode à l'étude de l'équation du second ordre sous l'action des transformations ponctuelles, étude que Tresse avait entreprise auparavant, et retrouve dans [10] les résultats de ce dernier. Il étudie de même l'équivalence des équations du troisième ordre résolue en leur plus haute dérivée sous l'action des transformations ponctuelles dans [11]. Cette méthode est désormais connu sous le nom de méthode du repère mobile.

Par la suite l'algorithme de Cartan fut expérimenté par certains de ses élèves, notamment Moshen Hachtroudi et S.S. Chern. Ce dernier parvint à classifier dans [14], les équations du troisième ordre sous l'action des transformations de contact, qui sont une catégorie de changement de variables que l'on peut effectuer sur une équation différentielle. Ces substitutions modifient aussi les dérivées et constituent les transformations les plus générales que l'on peut faire subir à une équation différentielle.

L'approche de Cartan tomba au cours des années qui suivirent sa disparition en désuétude. Cela étant, à notre humble avis, dû à la profondeur de ses idées plutôt qu'à leur manque d'intérêt. Pour corroborer ce fait citons cette remarque qu'a faite Hermann Weyl sur un livre de Cartan (voir [21,

page.vii]):
'Je dois admettre que j'ai trouvé ce livre, tout comme la plupart des papiers de Cartan, difficile d'accès...'

Les années 1960 correspondent au renouveau des idées de Cartan, sous l'impulsion de Sternberg, Guillemin, Spencer et Kuranishi; ceux-là permirent de jeter les bases théoriques modernes de la méthode de Cartan. Voir à ce propos [54].

La réelle présentation pédagogique des idées de Cartan quant à elle, est l'oeuvre de R. B. Gardner [21]. C'est à partir de sa contribution que fut vraiment entamée la résolution de divers Problèmes d'Equivalence autres que ceux déjà considérés par Cartan et ses étudiants. On peut citer à titre d'exemple [8, 18, 37].

Une autre façon d'aborder l'étude qualitative des équations différentielles est d'étudier l'algébricité de leurs solutions sur un certain corps. Cette question dans le cas de l'équation linéaire du second ordre

$$y'' + p_1 y' + p_2 y = 0$$

fut étudiée tout d'abord par Joseph Liouville [30, 31]; puis Auguste Boulanger [7] étudia le cas des équations linéaires homogènes du troisième ordre. La classification complète de l'algébricité des solutions des équations différentielles linéaires homogènes est l'oeuvre d'Emile Picard et d'Ernest Vessiot [48, 59], qui introduisirent un formalisme similaire à la théorie de Galois algébrique. C'est la naissance de la théorie Galois différentielle avec comme objet caractéristique, le groupe de Galois différentiel.

L'étude de l'algébricité des solutions d'une équation non linéaire est quant à elle plus ardue. La cause est la difficulté de construire un groupe de Galois différentiel analogue à celui que l'on obtient dans le cas des équations différentielles linéaires homogènes. Tout au plus l'on tombe sur un groupoïde de Galois (notion introduite par Bernard Malgrange dans les années 2000 dans [34]); un groupoïde étant un ensemble G_R qui est muni d'une loi $\star$ qui vérifie tous les axiomes d'un groupe, mis à part le fait que la loi $\star$, n'est pas une loi de composition interne car elle n'est définie que pour certains couples de G_R.

Enfin une autre domaine de l'étude des équations différentielles concerne l'étude des singularités de leurs solutions. Deux types de singularités peuvent apparaître pour les solutions des équations différentielles: les singularités dites mobiles c'est à dire dépendantes des constantes d'intégration et les singularités fixes.

Dans le cas le plus simple c'est à dire pour les équations différentielles linéaires, toutes les singularités sont fixes: elles se localisent toutes aux singularités de leurs fonctions coefficients.

Par contre dans le cas non linéaire il en est tout autrement. Donc les mathématiciens se sont intéressés aux équations non linéaires possédant une bonne propriété dite propriété de Painlevé: celle que toutes les singularités mobiles de leurs solutions sont des pôles (on peut aussi avoir des singularités essentielles ou des points de ramification).

Pour le cas des équations différentielles du premier ordre, leur classification suivant cette propriété de Painlevé donne l'équation de Riccati

$$\frac{dy}{dx} = c_0(x) + c_1(x)y + c_2(x)y^2$$

et toutes équations provenant de cette dernière par un changement de variable holomorphe de la variable indépendante et une transformation de Möbius de la variable dépendante:

$$x = \xi(v, t), \quad y = \frac{av + b}{cv + d}.$$

Voir [24].

Les classifications des équations d'ordre 2 et 3 possédant la propriété de Painlevé furent quant à elles réalisées par Painlevé et Gambier pour le cas des équations du second ordre et par Chazy pour celui du troisième ordre. Et c'est d'une telle étude que proviennent les six transcendantes de Painlevé. On pourra consulter [45, 46] et [19, 13].

Dans cette thèse nous étudierons l'équation d'Abel du premier ordre

$$\frac{dy}{dx} = c_0(x) + 3c_1(x)y + 3c_2(x)y^2 + c_3(x)y^3;$$

considérée pour la premier fois par Niels Abel dans [1], lors d'une étude sur les fonctions elliptiques.

Notre étude sera d'abord différentielle. Elle utilisera la méthode de Cartan, pour ensuite adopter une méthode plus classique. Cela se fera dans la seconde partie.

Ensuite nous aborderons dans la partie troisième partie, l'étude de certaines propriétés des solutions des équations de Riccati; cette dernière équation provient de l' équation d'Abel en faisant $c_3 \equiv 0$. Lorsque l'on suppose qu'il existe n solutions distinctes d'une équation de Riccati qui satisfont à la même équation algébrique irréductible et séparable:

$$h_n(x) = x^n + a_1(t)x^{n-1} + \cdots + a_n = 0,$$

sur le corps $\mathbb{F}$ des séries de Laurent sur $\mathbb{C}$ convergentes, le bi-rapport ou rapport anharmonique de 4 racines quelconques de cette équation algébrique est constant; eu égard à cette propriété, nous appelons les équations algébriques de ce type, des équations anharmoniques. Les anharmoniques jouissent de propriétés remarquables dont la plus importante est que leur groupe de Galois sont des sous-groupes finis de $PGL(2, \mathbb{C})$. Nous n'aborderons que les petits degrés 4 et 5 et espérons revenir sur les autres cas dans un proche avenir.

Nous examinerons ensuite l'étude du problème d'équivalence des équations différentielles ordinaires du second ordre

$$y'' = f(x, y, y')$$

sous l'action des transformations préservant les aires.

$$\begin{cases} X = \chi(x, y) \\ Y = \phi(x, y) \quad \text{avec } J := \chi_x \phi_y - \chi_y \phi_x \equiv 1. \end{cases}$$

Table des Matières

1 Notions Fondamentales

Nous présentons dans ce chapitre divers concepts pour la plupart dus à Elie Cartan, qui peuvent nous être utiles au cours de cette thèse. Nous commençons tout d'abord par dire ce que nous entendons par l'objet géométrique 'équation différentielle'. On pourra consulter à titre d'exemple, [41, 40, 17].

1.1 Définition géométrique des équations différentielles

Définissons en premier lieu les espaces de jets.

Soit une variété différentielle M lisse ou C^∞ de dimension $n+m$. Puis deux sous-variétés L et $\overline{L}$ de dimension n de M. Soit $p \in L \cap \overline{L}$. Nous dirons que L et $\overline{L}$ ont un contact d'ordre $k \in \mathbb{N}$ si, relativement à un système de coordonnées locales

$$(x_1, \cdots, x_n, y_1, \cdots, y_m)$$

d'origine p, L et $\overline{L}$ sont respectivement représentées comme graphes des fonctions suivantes:

$$f : \mathbb{R}^n \to \mathbb{R}^m, \quad (x_1, \cdots, x_n) \mapsto (f_1(x), \cdots, f_m(x))$$

et

$$\bar{f} : \mathbb{R}^n \to \mathbb{R}^m, \quad (x_1, \cdots, x_n) \mapsto (\bar{f}_1(x), \cdots, \bar{f}_m(x));$$

vérifiant

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(0) = \frac{\partial^{|\alpha|} \bar{f}_i}{\partial x^\alpha}(0)$$

pour tout $i = 1, \cdots, m$ et tout n-uplet $\alpha = (\alpha_1, \cdots, \alpha_n)$ avec $|\alpha| \leqq k$. Cette définition ne dépend pas du système de coordonnées locales choisi. De plus la relation d'existence d'un contact d'ordre k entre L et $\overline{L}$ comme ci-dessus, est une relation d'équivalence sur l'ensemble des sous-variétés de dimension n et passant par p. Nous noterons la classe de L pour une telle relation:

$$[L]_p^k.$$

L'ensemble de toutes les classes d'équivalence sera quant à lui désigné par

$$J_p^k(M,n).$$

L'union $\bigcup_{p\in M} J_p^k(M,n)$, que nous notons $J^k(M,n)$, est appelée l'espace des k-jets de sous-variétés de dimension n dans M.

Soit

$$(x_1,\cdots,x_n,y_1,\cdots,y_m)$$

un système de coordonnées locales dans un domaine ouvert $U\subset M$. Appelons U^k le sous-ensemble de $J^k(M,n)$ formé des points $[L]_p^k$ tels que $p\in U$ et $L=\Gamma_f$ est le graphe d'une application C^∞

$$f:\mathbb{R}^n\to\mathbb{R}^m,\quad (x_1,\cdots,x_n)\to f_1(x),\cdots,f_m(x)).$$

Alors les fonctions

$$x_i,\quad i=1,\cdots,n,\quad y_j^\alpha=\frac{\partial^{|\alpha|}f_j}{\partial x^\alpha},\quad j=1,\cdots,m,|\alpha|\le k$$

définissent un système de coordonnées locales sur U^k. La collection de tels systèmes de coordonnées locales munit $J^k(M,n)$, d'une structure de variété différentielle lisse de dimension

$$n+m\binom{n+k}{n}.$$

Cela est dû au fait qu'une fonction

$$f_j:\mathbb{R}^n\to\mathbb{R},$$

a autant de dérivées d'ordre N qu'il y a de monômes indépendants de degré N dans l'espace des fonctions polynomiales en les n variables $x_1,\cdots,x_n$. Ce nombre est

$$\binom{n+N-1}{n-1}=\binom{n+N-1}{N}.$$

La dimension totale de $J^k(\mathbb{R}^n,\mathbb{R})$ est donc

$$n+1+\sum_1^k\binom{n+i-1}{i}=n+\sum_0^k\binom{n+i-1}{i}=n+\binom{n+k}{k}.$$

Maintenant vu que chaque fonction coordonnée dans l'espace but $\mathbb{R}^m$ nous donne le même ensemble de dérivées indépendantes d'un certain ordre N, nous avons avec [9, chap.1, page.20]

$$\dim J^k(M,n) = n + m\binom{n+k}{n}.$$

Considérons à présent deux entiers positifs $k > l$; la projection naturelle de $J^k(M,n) \to J^l(M,n)$ sera notée $\pi_{k,l}$ et définie comme suit:

$$\pi_{k,l} : J^k(M,n) \to J^l(M,n) : [L]_p^k \to [L]_p^l.$$

Aussi nous identifierons $J^0(M,n)$ avec M (en posant $[L]_p^0$=p pour tout $[L]_p^0 \in J^0(M,n)$). La projection naturelle de $J^k(M,n) \to M$ sera désignée par $\pi_{k,0}$.

Passons à la définition d'une équation différentielle

1.1.1 Définition des équations différentielles

Etant donnée une sous-variété de dimension n, $L \subset M$, nous définissons sa relevée d'ordre k comme étant la sous-variété de $J^k(M,n)$ de la forme

$$L^{(k)} := \{[L]_p^k, p \in L\}.$$

Si dans un système de coordonnées locales $(x_1, \cdots, x_n, y_1, \cdots, y_m)$, la sous-variété L est donnée de manière analogue à ci-dessus, comme le graphe d'une fonction f, alors relativement au système de coordonnées locales correspondant dans $J^k(M,n)$ sa relevée d'ordre k, $L^{(k)}$, a la forme

$$L^{(k)} = \left\{ \left(x_1, \cdots, x_n, y_j^\alpha = \frac{\partial f_j^{|\alpha|}}{\partial x^\alpha} : x_1, \cdots, x_n \in \mathbb{R}; j = 1, \cdots, m \right) \right\}.$$

Nous avons alors la définition suivante

Definition 1.1.1 *Une équation différentielle d'ordre k* (avec n variables indépendantes et m variables dépendantes) est par définition, une sous-variété $\mathcal{E}$ de $J^k(M,n)$, où $\dim M = n + m$. Une solution de l'équation différentielle $\mathcal{E}$ est quant à elle une sous-variété $L \subset M$ de dimension n telle que

$$L^{(k)} \subset \mathcal{E}.$$

Le tableau ci-après montre la correspondance entre l'interprétation géométrique et celle classique des équations différentielles

Classique	Géométrique
Variables $x_1, \cdots, x_n$ et $y_1, \cdots, y_m$	Coordonnées locales sur M
Dérivations possibles jusqu'à l'ordre k	Jets $J^k(M, n)$
$f : \mathbb{R}^n \to \mathbb{R}^m$	sous-variété de dim n, $L \subset M$
Dérivées de f jusqu'à l'ordre k	$L^{(k)} \subset J^k(M, n)$
système d'équations dif. d'ordre k	$\mathcal{E} \subset J^k(M, n)$

1.1.2 Transformations des équations différentielles

Nous déterminons maintenant l'analogue du changement de variables dans le modèle géométrique des équations différentielles que nous venons de donner. Soit

$$\phi : M_1 \to M_2$$

un difféomorphisme arbitraire de variétés différentielles lisses de dimension $n + m$. Il induit un difféomorphisme de k-jets

$$\phi^{(k)} : J^k(M_1, n) \to J^k(M_2, n), \quad [L]_p^k \to [\phi(L)]_{\phi(p)}^k,$$

appelé la prolongation à l'ordre k de ϕ. Ci-dessous nous en donnons un certain nombre de propriétés.

- Si $\phi_1 : M_1 \to M_2$ et $\phi_2 : M_2 \to M_3$ sont deux difféomorphismes de variétés, alors
$$(\phi_1 \circ \phi_2)^{(k)} = \phi_1^{(k)} \circ \phi_2^{(k)}.$$
En particulier pour toute variété différentielle M nous obtenons le morphisme injectif
$$Diff(M_1 \to M_2) \to Diff(J^k(M_1, n) \to J^k(M_2, n))$$
$$\phi \mapsto \phi^{(k)}.$$

- De même pour tout couple $(k, l) \in \mathbb{N}^2$: $k > l$, nous avons
$$\pi_{k,l} \circ \phi^{(k)} = \phi^{(l)} \circ \pi_{k,l},$$
avec $\pi_{k,l} : J^k \to J^l$ la projection canonique associée.

- Aussi pour toute sous-variété $L \subset M_1$ de dimension n, nous avons $\phi^{(k)}\left(L^{(k)}\right) = (\phi(L))^{(k)}$.

- Enfin pour toute équation différentielle $\mathcal{E} \subset J^k(M_1, n)$, le difféomorphisme ϕ envoie les solutions de cette dernière équation sur les solutions de $\phi^{(k)}(\mathcal{E})$.

Usuellement les transformations $\phi^{(k)}$ des équations différentielles, où ϕ est un difféomorphisme arbitraire de la variété différentielle sous-jacente sont appelées les transformations ponctuelles.

1.2 La solution du Problème d'Equivalence de Cartan

1.2.1 Formulation du Problème d'Equivalence

Dans cette section, nous utilisons principalement comme références [21, 26] et supposons que tous les objets que nous manipulons sont lisses (C^∞) hormis dans le cas du théorème d'équivalence général de Cartan (système différentiel en involution) qui requiert l'hypothèse d'analycité réelle (C^ω). Nous utiliserons aussi pour simplifier, la notation de sommation d'Einstein: on somme les indices lorsqu'ils se trouvent en haut et en bas dans une formule.

Le Problème d'Equivalence de Cartan peut être formulé de la manière suivante. Etant donnés deux corepères (bases respectives de l'espace cotangent au-dessus U et V) $(\omega_U^i)_{1\leq i\leq n}$ et $(\overline{\omega}_V^j)_{1\leq j\leq n}$ sur des ouverts U et V de $\mathbb{R}^n$ et un sous-groupe de Lie G de dimension r (un sous-groupe au sens usuel qui est en même temps une sous-variété) de $GL(n, \mathbb{R})$, déterminer tous les difféomorphismes

$$f : U \to V$$

satisfaisant à la condition

$$f^\star(\overline{\omega}_V^i) = \gamma_j^i \omega_U^j \tag{1.2.1}$$

où (γ_j^i) est une fonction sur U et à valeurs dans G. Autrement dit on prescrit au jacobien du difféomorphisme f la condition d'être à valeurs dans G.

Cette formulation du Problème d'Equivalence n'est pas la plus générale que Cartan a eu à étudier; Cartan autorisait aussi que le groupe G varie de manière lisse d'un point à un autre. Nous ne décrirons dans ce qui suit que la solution du Problème d'Equivalence (1.2.1).

Si $U = V$ et $\overline{\omega}_V^i|_q = \omega_U^i|_q$ pour tout $q \in V$, alors les solutions du Problème d'Equivalence de Cartan donné par l'équation (1.2.1) sont appelées des automorphismes et on dit que l'on est en présence du Problème d'auto-équivalence. Nous verrons dans ce qui suit que les solutions d'un tel problème ($U = V$) forment toujours un pseudo-groupe de Lie fini (ses éléments dépendent de constantes arbitraires) ou infini (ses éléments dépendent de fonctions arbitraires).

Si $(\overline{\omega}_V^i)_{1\leq i\leq n}$ a g_V pour pseudo-groupe non trivial de solutions, du problème d'auto-équivalence qui lui est relatif, alors la solution générale du Problème d'Equivalence est donnée par la composée d'une solution particulière de l'équation (1.2.1) et d'une solution quelconque de Problème d'auto équivalence. Par conséquent nous pouvons nous attendre à ce que l'ensemble des solutions du Problème d'Equivalence de Cartan, donnée par l'équation (1.2.1), dépende d'un critère de solvabilité local pour assurer l'existence d'une solution particulière de l'équation (1.2.1) et de la structure de g_V.

1.2.2 La solution du Problème d'Equivalence de Cartan

Le Problème d'Equivalence de Cartan défini par l'équation (1.2.1) peut être regardé à la fois comme un problème d'Analyse et un problème de Géométrie différentielle. En effet, si nous introduisons des coordonnées locales

$$(x^a)_{1\leq a\leq n} \quad \text{sur } U$$

et

$$(\overline{x}^a)_{1\leq a\leq n} \quad \text{sur } V,$$

alors l'équation (1.2.1) se réduit à un système d'équations aux dérivées partielles quasi-linéaires pour les n fonctions inconnues f^a définies par

$$\overline{x}^a \circ f = f^a(x_1, \cdots, x_n).$$

D'un autre coté, l'équation (1.2.1) exprime le fait que la G-structure B_G^1 au-dessus de U dont la forme canonique est donnée par le relevé du corepère $(\omega_U^i)_{1\leq i\leq n}$ est localement isomorphe à la G-structure B_G^2 au-dessus de V dont la forme canonique est donnée par le relevé du corepère $(\overline{\omega}_V^j)_{1\leq j\leq n}$. Nous reviendrons dans ce qui suit sur la notion de G-structure.

Nous verrons dans la suite que divers outils provenant à la fois de l'Analyse et de la Géométrie sont nécessaires pour résoudre le Problème d'Equivalence de Cartan.

Les outils analytiques sont les théorèmes de Cartan d'existence des variétés intégrales des systèmes différentiels extérieurs en involution et de Frobenius pour l'existence de variétés intégrales des systèmes pfaffiens complètement intégrables. L'outil géométrique sera une construction qui produira un ensemble d'invariants locaux pour B^1_G et B^2_G. Et c'est la combinaison de l'information apportée par ces invariants avec les théorèmes d'Analyse sus-nommés qui résoudra le Problème d'Equivalence de Cartan. Entamons à présent la description de la solution du Problème d'Equivalence.

Le groupe G agit de façon naturelle sur $U \times G$ et $V \times G$ par translation le long des fibres au-dessus de U et V. Plus explicitement nous avons une application

$$L_U : G \times (U \times G) \to U \times G : (S, (p, T)) \to S(p, T) := (p, ST),$$

et une application analogue L_V pour $V \times G$. La première étape dans la solution du Problème d'Equivalence consiste à relever les données à ces espaces produits pour obtenir une formulation équivalente du problème originel, mais qui présente une plus grande symétrie entre U et V. Nous posons

$$\omega^i|_{(p,S)} := S^i_j \pi^\star_U(\omega^i_U|_p), \quad \overline{\omega}^i|_{(q,T)} := T^i_j \pi^\star_V(\overline{\omega}^j_V|_q) \tag{1.2.2}$$

où $\pi_U : U \times G \to U$ et $\pi_V : V \times G \to V$ sont les projections canoniques, et nous avons

Proposition 1.2.1 *Il existe un difféomorphisme $f : U \to V$ tel que*

$$f^\star \overline{\omega}^i_V = \gamma^i_j \omega^j_U \tag{1.2.3}$$

pour une fonction (γ^i_j) à valeurs dans G, si et seulement si il existe un difféomorphisme $\tilde{f} : U \times G \to V \times G$ qui satisfait pour tout $p \in U$ et pour tout $S, T \in G$:

$$\tilde{f}^\star \overline{\omega}^i = \omega^i \tag{1.2.4}$$

$$S\tilde{f}(p, T) = \tilde{f}(p, ST). \tag{1.2.5}$$

Preuve. Si $\tilde{f}$ est une application de $U \times G$ dans $V \times G$, alors il existe des applications $H : U \times G \to V$ et $\Lambda : U \times G \to G$ telles que

$$\tilde{f}(p, S) = (H(p, S), \Lambda(p, S))$$

pour tout $p \in U$ et pour tout $S \in G$. Maintenant nous avons $S\tilde{f}(p,T) = (H(p,T), S\Lambda(p,T))$ et $\tilde{f}(p,ST) = (H(p,ST), \Lambda(p,ST))$; donc si $\tilde{f}$ satisfait à l'équation (1.2.5), alors nous avons l'égalité

$$H(p,T) = H(p,ST) \quad \text{et} \quad S\Lambda(p,T) = \Lambda(p,ST),$$

pour tous $p \in U$ et $S,T \in G$. Par conséquent il existe des applications $h : U \to V$ et $\lambda : U \to G$ telles que $h(p) = H(p,S)$ and $S\lambda(p) = \Lambda(p,S)$, pour tout $p \in U$ et tout $S \in G$. Nous pouvons alors écrire

$$\tilde{f}(p,S) = (h(p), S\lambda(p)).$$

Si $\tilde{f}$ satisfait à l'équation (1.2.4), alors nous avons les relations suivantes

$$\begin{aligned} S^i_j \pi^\star_U(\omega^j_U|_p) = \omega^i|_{(p,S)} &= \tilde{f}^\star(\overline{\omega}^i|_{(q,T)}) \\ \tilde{f}^\star(T^i_j \pi^\star_V(\overline{\omega}^j_V|_q)) &= S^i_k(\lambda(p))^k_j \pi^\star(h^\star \overline{\omega}^j_V)|_p. \end{aligned} \tag{1.2.6}$$

De l'équation précédente (1.2.6) nous déduisons que l'équation (1.2.3) est satisfaite avec $f = h$ et $\gamma^i_j = (\lambda^{-1})^i_j$. Inversement, étant donnés un difféomorphisme f et une fonction à valeurs dans G, γ^i_j vérifiant l'équation (1.2.3), nous obtenons un difféomorphisme $\tilde{f} : U \times G \to V \times G$ qui satisfait aux équations (1.2.4) et (1.2.5) en posant

$$\tilde{f}(p,S) = (f(p), S\gamma(p)^{-1}). \tag{1.2.7}$$

■

A partir de maintenant nous supprimons les applications π_U et π_V (l'opération d'image réciproque sur les formes différentielles commute avec la différentiation extérieure des formes différentielles). Les formes relevées ω^i et $\overline{\omega}^i$ satisfont un ensemble d'équations qui sont appelées les premières équations de structure d'Elie Cartan. Sur $U \times G$, nous avons

$$\begin{aligned} d\omega^i &= d(S^i_j \omega^j_U) \\ &= dS^i_j \wedge \omega^j_U + S^i_j d\omega^j_U \\ &= (dS \cdot S^{-1})^i_j \wedge S^i_j \omega^j_U + S^i_j d\omega^j_U. \end{aligned} \tag{1.2.8}$$

La matrice $dS \cdot S^{-1}$ est l'image réciproque vers $U \times G$, sous l'action de la projection $\pi_G : U \times G \to G$, d'une 1-forme sur G invariante à droite (ou encore une forme de Maurer-Cartan) et qui prend ses valeurs dans l'algèbre de Lie $\mathfrak{g}$ de G.

En effet, rappelons que si G est un groupe de Lie et $\{\pi^\rho|_e 1 \leq \rho \leq r := \dim G\}$ est une base de $T_e^\star G$, que nous identifions avec le dual $\mathfrak{g}^\star$ de $\mathfrak{g}$, nous obtenons des 1-formes définies globalement sur G en posant

$$\pi^\rho|_S = R^\star_{S^{-1}}\pi^\rho|_e$$

où la translation à droite $R_S : G \to G$ est définie par

$$R_S : A \to AS^{-1}.$$

Ces 1-formes sont invariantes à droite, vu que

$$R^\star_T\pi^\rho|_{ST} = R^\star_T R^\star_{(ST)^{-1}}\pi^\rho|_e = R^\star_{TT^{-1}}R^\star_{S^{-1}}\pi^\rho|_e = \pi|_S. \tag{1.2.9}$$

L'ensemble $\{\pi^\rho|1 \leq \rho \leq r := \dim G\}$ est de ce fait une base de l'espace des 1-formes invariantes à droite sur G. Dans le cas d'un groupe linéaire, c'est à dire le cas d'un sous-groupe de Lie de $GL(n, \mathbb{R})$, nous avons l'identité

$$R^\star_T d(ST) = dS \cdot T, \tag{1.2.10}$$

ce qui entraine que

$$R^\star_T d(ST) \cdot (ST)^{-1} = dS.S^{-1}, \tag{1.2.11}$$

ce qui veut encore dire que $d(S) \cdot S^{-1}$ est une 1-forme invariante à droite donc prenant ses valeurs dans l'algèbre de Lie de G.

Nous exprimons à présent la matrice $dS \cdot S^{-1}$ dans la base π^ρ; nous pouvons écrire

$$(dS \cdot S^{-1})^i_j = a^i_{j\rho}\pi^\rho \tag{1.2.12}$$

où les $a^i_{j\rho}, 1 \leq i, j \leq n, 1 \leq \rho \leq r$ sont des constantes. Maintenant les 1-formes $\omega^i_U, 1 \leq i \leq n$, forment un corepère sur U, il existe donc $n^2\frac{(n-1)}{2}$ fonctions $T^i_{jk} : U \to \mathbb{R}$, où $1 \leq i, j, k \leq n$ et $T^i_{jk} = -T^i_{kj}$ telles que

$$d\omega^i_U = \frac{1}{2}T^i_{jk}\omega^j_U \wedge \omega^k_U. \tag{1.2.13}$$

En exprimant les 1-formes $\omega^i_U, 1 \leq i \leq n$, en fonction des 1-formes ω^i , $1 \leq i \leq n$, dans le membre de droite de l'équation (1.2.13), nous obtenons

$$d\omega^i_U|_p = \frac{1}{2}T^i_{jk}(p)(S^{-1})^j_m\omega^m|_{(p,S)} \wedge (S^{-1})^k_m\omega^m|_{(p,S)}. \tag{1.2.14}$$

Remplaçons maintenant cette expression (1.2.14) et l'expression donnée par l'équation (1.2.12) dans l'identité donnée par l'équation (1.2.8), nous obtenons les premières équations de structures d'Elie Cartan

$$d\omega^i = a^i_{j\rho}\pi^\rho \wedge \omega^j + \frac{1}{2}\gamma^i_{jk}\omega^j \wedge \omega^k \tag{1.2.15}$$

où les $n^2\frac{(n-1)}{2}$ fonctions $\gamma^i_{jk} : U \times G \to \mathbb{R}$ sont définies par

$$\gamma^i_{jk}(p,S) = S^i_l T^l_{mn}(p)(S^{-1})^m_j (S^{-1})^n_k. \tag{1.2.16}$$

Remarque 1.2.2 *Par un abus de langage qui prend son origine dans les cas où l'on peut associer une connexion invariante à un Problème d'Equivalence de Cartan, les termes* $\frac{1}{2}\gamma^i_{jk}\omega^j \wedge \omega^k$ *dans les premières équations de structure* (1.2.15) sont appelés les termes de torsion.

L'approche que nous avons suivie jusqu'à présent pour déterminer les premières équations de structure est due à Cartan. Nous suivons à présent une autre approche due à Singer et Sternberg [54], qui est complètement équivalente à celle de Cartan et qui est peut-être plus parlante géométriquement. Cela nous permettra aussi de justifier notre affirmation selon laquelle le Problème d'Equivalence de Cartan est le problème d'Equivalence locale des G-structures.

Soit un groupe de Lie G. On dit que G agit de manière différentiable à gauche sur P (variété différentielle), s'il existe un morphisme

$$\begin{aligned} \varphi : G \times P &\to P \\ (a,p) &\to \varphi(a,p), \end{aligned}$$

vérifiant

$$\varphi(a, \varphi(b,p)) = \varphi(ab,p);$$

et tel que l'application

$$\begin{aligned} L_a : P &\to P \\ p &\to \varphi(a,p) \end{aligned}$$

est un difféomorphisme de P sur P. De plus on demande que $L_e = \mathrm{Id}_P$.

On peut par conséquent observer que

$$L_a \circ L_b = L_{ab}.$$

Par la suite nous noterons $L_a(p) = a.p$. Nous avons la notion classique

Définition 1.2.3 *Soient P et M deux variétés différentielles et $\pi : P \to M$ une submersion surjective. Soit G un groupe de Lie agissant à gauche sur P. Alors (P, M, π, G) est un fibré principal au-dessus de M avec groupe structural G si*

- *G agit de manière libre sur P: $\exists\ a : a.p = p \iff a = e$.*

- $\pi(p_1) = \pi(p_2) \iff p_1 = a.p_2$ (p_1 et p_2 sont dans la même orbite si et seulement si ils ont la même projection sur M); on identifie ainsi M avec l'ensemble quotient pour la relation précédente.

- P est localement trivial ie tout $x \in M$ a un voisinage U, tel qu'il existe un difféomorphisme $\psi : \pi^{-1}(U) \to U \times G$: $\psi(p) = (\pi(p), \eta(p))$ avec $\eta(p) \in G$ et $\psi(a.p) = (\pi(p), a.\eta(p))$.

Remarque 1.2.4 *Si $M = \{x\}$ alors P ressemble à G* (l'action de G sur P est transitive car on a une seule orbite (par le point 2 de la définition)). Pour tout $y \in M$, la fibre de la submersion π: $\pi^{-1}(y)$ est une sous-variété. P ressemble à une collection de copies de G indexée de manière différentiable par M.

Soit M une variété différentielle de dimension n et $\mathbb{F}(M)$ le fibré des repères linéaires de M. Nous rappelons que $\mathbb{F}(M)$ est un $GL(n, \mathbb{R})$-fibré principal au-dessus de M avec l'action à gauche de $GL(n, \mathbb{R})$ donnée par

$$(S, u) \to S \cdot u.$$

Un point $u \in \mathbb{F}(M)$ est une base de l'espace tangent T_pM, où $p = \pi(u)$ et $\pi : \mathbb{F}(M) \to M$ représente la projection canonique sur la base. Fixons un espace vectoriel de dimension n sur $\mathbb{R}$ et une de ses bases. Un point u de $\mathbb{F}(M)$ définit de ce fait un isomorphisme de T_pM sur V (que nous noterons encore u par abus de notation); donc nous pouvons identifier $GL(n, \mathbb{R})$ à $GL(V)$ au moyen u. Nous avons la définition suivante

Définition 1.2.5 *Une G-structure B_G au-dessus de M est une réduction de $\mathbb{F}(M)$ au groupe G, c'est à dire une sous-variété de $\mathbb{F}(M)$ telle que pour tout $u \in B_G$ et pour tout $S \in GL(V)$, le point $S \cdot u \in B_G$ si et seulement si $S \in G$.*

Par un abus de notation, nous noterons encore π, la restriction à B_G de la projection canonique $\pi : \mathbb{F}(M) \to M$.

La 1-forme canonique ω sur $\mathbb{F}(M)$ est une forme différentielle à valeurs dans V; ou encore une section du fibré $T^\star\mathbb{F}(M) \bigotimes V$, définie de la manière suivante.

Soit $u \in \mathbb{F}(M)$ et $X \in T_u\mathbb{F}(M)$. Alors $\pi_\star X \in T_pM$, où $p = \pi(u)$, et nous définissons ω par

$$\omega(X) = u(\pi_\star X) \tag{1.2.17}$$

où u est l'isomorphisme de T_pM sur V.

Maintenant comme la G-structure B_G est une sous-variété de $\mathbb{F}(M)$, la restriction de ω à B_G est encore une 1-forme à valeurs dans V, nous la notons encore ω.

Pour obtenir les premières équations de structure (1.2.15), nous avons besoin de savoir comment ω se transforme sous l'action de G. Pour tout $X \in T_u\mathbb{F}(M)$ et $S \in G$, nous avons

$$\pi_\star(X - L_{S^\star}X) = 0 \tag{1.2.18}$$

car par définition d'un $GL(n, \mathbb{R})$-fibré principal, le morphisme π vérifie

$$\pi \circ L_S = \pi;$$

d'où le résultat en passant aux différentielles. En utilisant les équations (1.2.17) et (1.2.18) nous obtenons (nous évaluons ω au vecteur tangent $L_{S^\star}X \in T_{Su}\mathbb{F}(M)$)

$$\begin{aligned}\omega(L_{S^\star}X) &= Su(\pi_\star(L_{S^\star})X) = Su(\pi_\star X)\\ &= S\omega(X)\end{aligned}$$

donc par dualité,

$$L_S^\star\omega = S\omega \tag{1.2.19}$$

pour tout $S \in G$. Le terme $S\omega$ est simplement le résultat de l'action de l'isomorphisme linéaire S sur la forme différentielle ω qui est à valeurs dans V.

Maintenant, nous savons qu'à tout $A \in \mathfrak{g} \subset \mathfrak{gl}(V)$ correspond un champ de vecteurs invariant à droite sur G: $\tilde{A}$, obtenu par translation à droite. Le fait que B_G soit un $GL(n, \mathbb{R})$-fibré principal entraine alors l'existence d'un champ de vecteurs $\tilde{A}$ induit sur B_G et qui est partout tangent aux fibres de π, c'est à dire

$$\pi_\star\tilde{A} = 0;$$

nous obtenons alors de l'équation (1.2.19) (en posant que $S = \exp(tA) \in G$)

$$L^\star_{\exp(tA)}\omega = \exp(tA)\omega. \tag{1.2.20}$$

Nous allons dériver cette équation par rapport à t en $t=0$. Rappelons que $\exp(tA)$ est le flot du champ de vecteurs $\tilde{A}$ qui vaut A en l'identité e. De plus le groupe à un paramètre $\exp(tA)$ induit une action sur B_g via $L_{\exp(tA)}$. La dérivée de Lie de ω relativement à ce dernier groupe à un paramètre est fournie par la formule

$$\mathcal{L}_{\tilde{A}} = \frac{d}{dt}\Big|_{t=0} L^{\star}_{\exp(tA)}\omega = A\omega.$$

Alors en utilisant la formule de A. Weil [53, page 37]

$$\mathcal{L}_{\tilde{A}} = di_{\tilde{A}} + i_{\tilde{A}}d$$

et l'équation (1.2.17) ($\pi_\star\tilde{A} = 0$), nous obtenons

$$i_{\tilde{A}}d\omega = A\omega. \tag{1.2.21}$$

Les équations de structure (1.2.15) sont alors obtenues en choisissant pour trivialisation locale de $\pi^{-1}(B_G)$, $U \times G$ (voir à cet effet [54, page 328]).

Enfin, pour reconnaitre que le Problème d'Equivalence n'est autre que le problème d'Equivalence locale entre G-structures, nous rappelons que deux G-structures B^1_G and B^2_G au-dessus de variétés différentielles M_1 et M_2 sont dites isomorphes s'il existe un difféomorphisme

$$f : M_1 \to M_2$$

tel que

$$f_\star B^1_G = B^2_G,$$

et localement isomorphes en $(p,q) \in M_1 \times M_2$ s'il existe des voisinages ouverts U et V de p et q tels que $B^1_G|_U$ et $B^2_G|_V$ soient isomorphes. En utilisant la définition de la forme canonique sur une G-structure et la proposition 1.2.1, nous voyons que le problème d'Equivalence locale de G-structures et le Problème d'Equivalence de Cartan sont les mêmes.

Nous utilisons à présent les premières équations de structure à notre disposition pour obtenir des conditions nécessaires pour l'existence d'une équivalence.

Sur $U \times G$ nous écrivons

$$d\omega^i = a_{j\rho}\pi^\rho \wedge \omega^j + \frac{1}{2}\gamma^i_{jk}\omega^j \wedge \omega^k \tag{1.2.22}$$

et sur $V \times G$

$$d\overline{\omega}^i = a_{j\rho}\overline{\pi}^\rho \wedge \overline{\omega}^j + \frac{1}{2}\overline{\gamma}^i_{jk}\overline{\omega}^j \wedge \overline{\omega}^k \tag{1.2.23}$$

S'il existe une application $f : U \to V$ qui résout le Problème d'Equivalence (1.2.1), on déduit alors de la proposition 1.2.1 et des équations (1.2.22) et (1.2.23) que sa relevée $\tilde{f} : U \times G \to V \times G$ satisfait

$$a^i_{k\rho}(\tilde{f}^\star \overline{\pi}^\rho - \pi^\rho) \wedge \omega^k + \frac{1}{2}(\overline{\gamma}^i_{jk} \circ \tilde{f} - \gamma^i_{jk})\omega^j \wedge \omega^k = 0 \qquad (1.2.24)$$

Rappelons maintenant un lemme important de Cartan en Algèbre extérieure.

Lemme 1.2.6 *Si α_a, β_b, $1 \le a, b \le n$, sont des éléments d'un espace vectoriel W tels que α_a, $1 \le a \le N$ sont linéairement indépendants, alors l'égalité dans l'algèbre $\Lambda^2 W$ (engendré par les $u \wedge v$ pour u, $v \in W$)*

$$\sum_{a=1}^{N} \alpha_a \wedge \beta_a = 0$$

a lieu si et seulement si

$$\beta_a = \sum_{b=1}^{N} c_{ab}\alpha_b, \quad \text{avec} \quad c_{ab} = c_{ba}.$$

Si l'on applique ce lemme à l'équation (1.2.24) nous obtenons l'identité

$$a^i_{k\rho}(\tilde{f}^\star \overline{\pi}^\rho - \pi^\rho) + \frac{1}{2}(\overline{\gamma}^i_{jk} \circ \tilde{f} - \gamma^i_{jk})\omega^j = b^i_{kj}\omega^j \qquad (1.2.25)$$

avec $b^i_{kj} = b^i_{jk}$. Comme $\dim G = r = \dim \mathfrak{g}$ et que $dS \cdot S^{-1}$ à valeurs dans $\mathfrak{g}$, nous pouvons choisir r 1-formes linéairement indépendantes $a^{i_s}_{k_s\rho}\pi^\rho$, $s = 1, \cdots, r$, parmi les $a_{j\rho}\pi^\rho$ et pour ces valeurs de i et k, l'équation (1.2.25) devient

$$a^{i_s}_{k_s\rho}(\tilde{f}^\star \overline{\pi}^\rho - \pi^\rho) + \frac{1}{2}(\overline{\gamma}^{i_s}_{jk_s} \circ \tilde{f} - \gamma^{i_s}_{jk_s})\omega^j = b^{i_s}_{k_sj}\omega^j. \qquad (1.2.26)$$

En multipliant à gauche cette dernière équation (1.2.26) par l'inverse de la matrice $A^s_\rho := (a^{i_s}_{k_s\rho})$ nous obtenons l'équation

$$\tilde{f}^\star \overline{\pi}^\rho = \pi^\rho + v^\rho_j \omega^j. \qquad (1.2.27)$$

L'insertion de cette identité dans l'équation (1.2.25) nous donne

$$a^i_{k\rho}(v^\rho_j \omega^j) + \frac{1}{2}(\overline{\gamma}^i_{jk} \circ \tilde{f} - \gamma^i_{jk})\omega^j = b^i_{kj}\omega^j.$$

Permutons ensuite les indices k, j dans l'égalité qui précède, il résulte

$$a^i_{j\rho}(v^\rho_k \omega^k) + \frac{1}{2}(\overline{\gamma}^i_{kj} \circ \tilde{f} - \gamma^i_{kj})\omega^k = b^i_{jk}\omega^k.$$

Faisons la différence de ces deux dernières égalités; nous obtenons alors (en utilisant le fait que les ω^j sont linéairement indépendantes)

$$\overline{\gamma}^i_{jk} \circ \tilde{f} = \gamma^i_{jk} + a^i_{j\rho}v^\rho_k - a^i_{k\rho}v^\rho_j. \tag{1.2.28}$$

Nous pouvons maintenant obtenir des conditions nécessaires pour l'existence d'une solution f du Problème d'Equivalence en utilisant les deux précédentes équations (1.2.27) et (1.2.28).

Soit V un espace vectoriel réel de dimension n, dont une base est $\{e_i, 1 \leq i \leq n\}$ et $V^\star$ son dual, muni de la base duale $\{e^{\star j}, 1 \leq j \leq n\}$. Considérons aussi $\{\delta_\sigma|_e, 1 \leq \sigma \leq r\}$, une base de $T_eG \simeq \mathfrak{g}$ et qui est duale à la base $\{\pi^\rho|_e, 1 \leq \rho \leq r\}$ de $T^\star_eG \simeq \mathfrak{g}^\star$. Définissons l'application

$$L : \mathfrak{g} \otimes V^\star \to V \otimes \Lambda^2 V^* : v^\rho_i \delta_\rho|_e \otimes e^{*i} \to (a^i_{j\rho}v^\rho_k - a^i_{k\rho}v^\rho_j)e_i \otimes e^{*j} \wedge e^{*k}.$$

Soit la suite exacte d'espace vectoriels

$$0 \to \mathfrak{g}^{(1)} \xrightarrow{i} \mathfrak{g} \otimes V^\star \xrightarrow{L} V \otimes \Lambda^2 V^* \xrightarrow{pr} \Pi_\mathfrak{g} \to 0 \tag{1.2.29}$$

où

$$\mathfrak{g}^{(1)} = \ker L,\ \Pi_\mathfrak{g} = V \otimes \Lambda^2 V^* / \operatorname{Im} L \tag{1.2.30}$$

Le noyau $\mathfrak{g}^{(1)}$ est appelé la première prolongation de l'algèbre de Lie $\mathfrak{g}$. Considérons

$$\gamma : U \times G \to V \otimes \Lambda^2 V^* : (p, S) \to \frac{1}{2}\gamma^i_{jk}(p, S)e_i \otimes e^{*j} \wedge e^{*k}$$

et définissons le premier tenseur de structure comme l'application

$$\tau_U : U \times G \to \Pi_\mathfrak{g} : (p, S) \to (pr \circ \gamma)(p, S).$$

Nous avons alors la condition nécessaire suivante

Proposition 1.2.7 *S'il existe une application $f : U \to V$ telle que l'égalité* (1.2.1) soit vraie, c'est à dire

$$f^\star(\overline{\varpi}^i_V) = \gamma^i_j \omega^j_U$$

alors l'image réciproque du tenseur de structure sur $V \times G$ par la relevée $\tilde{f} : U \times G \to V \times G$ est le tenseur de structure sur $U \times G$, ce qui veut encore dire que

$$\tau_U = \tau_V \circ \tilde{f} \tag{1.2.31}$$

Preuve. Nous réinterprétons l'équation (1.2.28) sous la forme

$$\tilde{\gamma} \circ \tilde{f} = \gamma \mod \operatorname{Im} L; \tag{1.2.32}$$

nous projetons ensuite sur le quotient $V \otimes \Lambda^2 V^* / \operatorname{Im} L$, via pr et obtenons ainsi

$$\tau_u = \tau_V \circ \tilde{f}.$$

■

Les relations (1.2.27) et (1.2.28) peuvent s'interpréter de façon très originale grâce au fibré principal B_G introduit ci-dessus. Nous dirons qu'un sous-espace de $T_u B_G$ est vertical en u s' il est contenu dans le noyau de π_*, et tout supplémentaire H de $\ker \pi_*$ dans $T_u B_G$, est appelé un sous-espace horizontal au point u. Pour une base fixée $\{\pi^1, \cdots, \pi^r\}$ de 1-formes invariantes à droite sur G, les sous-espaces horizontaux H en u sont paramétrés par les v_i^ρ qui apparaissent dans la formule (1.2.27). En ce sens que pour tout sous-espace horizontal H au point u, il existe des nombres $v_i^\rho(H)$ tels que

$$H = \{X \in T_u B_G | \pi^\rho(X) = v_i^\rho(H)\omega^i(X)\},$$

et inversement.

L'interprétation de l'équation (1.2.28), est qu'elle exprime le fait que la torsion dépend du choix du sous-espace horizontal.

Nous examinons maintenant le problème d'équivalence de e-structures. Les conditions nécessaires données par la proposition 1.2.7 en fonction des invariants locaux τ_U et τ_V, pour l'existence d'une application $f : U \to V$ qui résout le Problème d'Equivalence donnée par l'équation (1.2.1), induisent des conditions nécessaires et suffisantes dans le cas où $G = \{e\}$, c'est à dire dans le cas du problème d'isomorphisme local entre $\{e\}$-structures. Le Problème d'Equivalence se réduit dans ce cas à la détermination de tous les difféomorphismes $f : U \to V$ satisfaisant à

$$f^\star \overline{\omega}_V^i = \omega_U^i, 1 \leq i \leq n \tag{1.2.33}$$

Nous avons

$$d\omega_U^i = \frac{1}{2} C_{jk}^i \omega_U^j \wedge \omega_U^k, \quad d\overline{\omega}_V = \frac{1}{2} \overline{C}_{jk}^i \overline{\omega}_V^j \wedge \overline{\omega}_V^k \tag{1.2.34}$$

et la proposition (1.2.7) donne les conditions nécessaires

$$\overline{C}_{jk}^i \circ f = C_{jk}^i, 1 \leq i, j, k \leq n. \tag{1.2.35}$$

De plus si nous définissons les dérivées covariantes $F_{|i}$ d'une fonction $F: U \to \mathbb{R}$ comme les dérivées de F relatives aux champs de vecteurs duaux des ω_U^i, c'est à dire

$$dF = F_{|i}\omega_U^i, \tag{1.2.36}$$

nous obtenons alors d'autres conditions nécessaires

$$\overline{C}^i_{jk|i_1\cdots i_p} \circ f = C^i_{jk|i_1\cdots i_p}, 1 \leq i, j, k, i_1, \cdots, i_p \leq n, \quad p \geq 1. \tag{1.2.37}$$

Nous pouvons donc définir la famille de fonctions

$$\mathbb{F}_s := \{C^i_{jk}, C^i_{jk|i_1}, C^i_{jk|i_1 i_2}, \cdots, C^i_{jk|i_1\cdots i_s} | 1 \leq i, j, k, i_1, \cdots, i_s \leq n\} \tag{1.2.38}$$

Nous voyons que $\mathbb{F}_s \subset \mathbb{F}_{s+1}$ et posons pour tout $p \in U$

$$k_s(p); = \dim\langle d\mathbb{F}_s\rangle_p \tag{1.2.39}$$

où

$$\begin{aligned}\langle d\mathbb{F}_s\rangle_p := \Big\{ &\mathbb{R}dC^i_{jk} + \mathbb{R}dC^i_{jk|i_1} + \cdots + \mathbb{R}dC_{jk|i_1\cdots i_s} | \\ &1 \leq i, j, k, i_1, \cdots, i_s \leq n \Big\} \subset T_p^\star U\end{aligned} \tag{1.2.40}$$

L'entier $k_s(p)$ fournit donc le nombre maximal d'éléments fonctionnellement indépendants de $\mathbb{F}_s$ au point p. Nous supposons que nous nous trouvons dans une situation non-singulière, c'est à dire une situation où $p \to k_s(p)$ est localement constante: tout point $p \in U$ admet un voisinage U_p tel que

$$k_s(p') = k_s(p), \quad \text{pour tout } p' \in U_p.$$

Faisons à présent l'hypothèse que

$$k_s(p) = k_{s+1}(p) := R,$$

alors nous voyons par définition de l'entier k_{s+1} que les éléments de $\mathbb{F}_{s+1}$ sont fonctions des éléments de $\mathbb{F}_s$. Appliquant la règle de différentiation des fonctions composées et utilisant la définition de la dérivée covariante, on obtient que $k_{s+2} = k_s$. Nous obtenons alors le résultat suivant par récurrence:

$$k_s(p) = k_{s+l}(p), l \geq 1.$$

Par conséquent sous les hypothèses qui précèdent, il existe un ensemble maximal de fonctions indépendantes $\{f^1, \cdots, f^R\}$ telles que tout élément de $g \in \mathbb{F}_s$ s'écrit sous la forme

$$g = F(f^1, \cdots, f^R).$$

Le résultat suivant dû à Elie Cartan [12] va montrer comment les conditions nécessaires (1.2.35) et (1.2.37) peuvent être utilisées pour obtenir des conditions nécessaires et suffisantes pour l'existence d'une Equivalence de $\{e\}$-structures.

Proposition 1.2.8 *Si s désigne le plus petit entier vérifiant $k_s = k_{s+1}$, alors des conditions nécessaires et suffisantes d'existence d'un difféomorphisme $f : U \to V$ tel que*

$$f^{\star}\overline{\omega}^i_V = \omega^i_U, f(p) = q \tag{1.2.41}$$

sont

- $\overline{s} = s$ *et* $k_s = \overline{k}_s := R$
- *Si $\{f^1, \cdots, f^R\}$ est un ensemble maximal de fonctions indépendantes telles que toute fonction de $g \in \mathbb{F}_s$ s'exprime au moyen de $f^1, \cdots, f^R$; et $\{\overline{f}^1, \cdots, \overline{f}^R\}$ est l'ensemble correspondant pour $\overline{\mathbb{F}}_s$, alors la condition de solvabilité locale*

  $$f^{\alpha}(p) = \overline{f}^{\alpha}(q), \alpha = 1, \cdots, R \tag{1.2.42}$$

  est satisfaite
- *Si g est un élément de $\mathbb{F}_{s+1}$ et $\overline{g}$ est un élément de même indice dans $\overline{F}_{s+1}$, alors*

  $$g = F(f^1, \cdots, f^R) \Rightarrow \overline{g} = F(\overline{f}^1, \cdots, \overline{f}^R) \tag{1.2.43}$$

La proposition précédente montre de ce fait qu'une classe d'équivalence de $\{e\}$-structure est caractérisée localement par les relations fonctionnelles entre un système complet d'invariants locaux et leurs dérivées covariantes.

Aussi il est à remarquer que les conditions données dans l'énoncé de la proposition 1.2.8, sont bien sûr toujours vérifiées dans le cas du Problème d'auto-équivalence, car l'application identité en est toujours une solution. En fait, l'ensemble des solutions du Problème d'auto-équivalence de $\{e\}$-structures, est un groupe de Lie pour la composition des applications, dont la dimension est reliée au cardinal d'un ensemble maximal d'invariants fonctionnellement indépendants; c'est ce que montre le corollaire suivant:

Corollaire 1.2.9 *Si $G = \{e\}$, alors l'ensemble des automorphismes forme un groupe de Lie de transformations de dimension $n - R$.*

La proposition 1.2.8 nous conduit naturellement à rechercher une construction grâce à laquelle le Problème d'Equivalence originel (1.2.1) est réduit à un nouveau Problème d'Equivalence, pour lequel G est maintenant remplacé par un sous-groupe de Lie $G_{(1)}$ de G, de telle sorte que les deux Problèmes d'Equivalence aient même ensemble de solutions. Le résultat d'une telle construction est appelée réduction de groupe.

Nous rappelons qu'étant donnés un espace vectoriel W et un groupe G, W est une G-représentation s'il existe une représentation, c'est à dire un homomorphisme de groupes entre G et $GL(W)$. Avec les notations précédentes nous avons alors la proposition suivante

Proposition 1.2.10 *La suite exacte d'espaces vectoriels suivante*

$$0 \to \mathfrak{g}^{(1)} \xrightarrow{i} \mathfrak{g} \otimes V^\star \xrightarrow{L} V \otimes \Lambda^2 V^* \xrightarrow{pr} \Pi_\mathfrak{g} \to 0 \tag{1.2.44}$$

est une suite exacte de G-représentations. C'est à dire qu'il existe une représentation de G vers chacun de ces espaces vectoriels et que ces représentations sont compatibles avec les applications i, L et pr.

Supposons tout d'abord par souci de simplicité que

$$\tau_U(U \times G)$$

est une seule G-orbite dans $\Pi_\mathfrak{g}$ ou encore que l'action de G sur $\tau_U(U \times G)$ est transitive, c'est à dire que $\tau_U(p, S)$ est indépendante de p et ce pour tout p. Cette hypothèse est satisfaite dans les exemples que nous traiterons. Fixons un vecteur

$$\tau_{(1)} = \tau_U(p, S_{(1)})$$

dans $\tau_U(U \times G)$ et désignons par $\tilde{\rho}$ la représentation de G sur $\Pi_\mathfrak{g}$. Soit alors

$$G_{(1)} = \{S \in G | \tilde{\rho}(S)\tau_{(1)} = \tau_{(1)}\}$$

le stabilisateur en $\tau_{(1)}$ sous cette action. Comme G agit transitivement sur $\tau_U(U \times G)$, un choix différent de $\tau_{(1)}$, donnerait un groupe stabilisateur conjugué à $G_{(1)}$. Par conséquent $\tau_U(U \times G)$ est localement difféomorphe au quotient $G/G_{(1)}$. Nous choisissons à présent une fonction quelconque à valeurs dans G, $\overset{(1)}{\delta}_U : U \to G$ dont le graphe $\Gamma(\overset{(1)}{\delta}_U)$ dans $U \times G$ est contenu dans $\tau_U^{-1}(\tau_{(1)})$ et définissons le corepère

$$(\overset{(1)}{\omega}{}^i{}_U = \overset{(1)}{\delta}{}^i_{Uj} \omega^j_U)_{1 \leq i \leq n};$$

réalisant une construction similaire sur $V \times G$, nous obtenons de ce fait le résultat suivant [20]

Proposition 1.2.11 *Les deux Problèmes d'Equivalence de Cartan*

$$f^*\overline{\omega}_V^i = \gamma_j^i \omega_U^j \tag{1.2.45}$$

où $(\gamma_{\mathrm{j}}^{\mathrm{i}})$ *est une fonction à valeurs dans* G*, et*

$$\overset{(1)}{f^*}\overset{(1)}{\overline{\omega}_V^i} = \overset{(1)}{\gamma_j^i}\overset{(1)}{\omega_U^j} \tag{1.2.46}$$

où $\overset{(1)}{\gamma_j^i}$ *est une fonction à valeurs dans* $G_{(1)}$ *ont même ensemble de solutions.*

Les propositions 1.2.8 et 1.2.11 nous suggèrent d'itérer la procédure de réduction de groupe jusqu'à l'obtention d'une famille décroissante de sous-groupes de G

$$G_{(k)} \subset G_{(k-1)} \subset \cdots \subset G_{(1)} \subset G \tag{1.2.47}$$

telle que $G_{(k)} = \{e\}$, ou $G_{(k)} \neq \{e\}$ et l'action de $G_{(k)}$ sur $\Pi_{\mathfrak{g}_{(k)}}$ est triviale.

Si $G_{(k)} = \{e\}$, nous avons réduit le problème originel à un problème équivalent d'équivalence local de $\{e\}$-structures que nous pouvons résoudre grâce à la proposition 1.2.3. Il résulte du corollaire de cette proposition que la solution générale du problème d'auto-équivalence dépend dans ce cas de $n - R$ constantes arbitraires.

Si au contraire $G_{(k)} \neq \{e\}$ et l'action de $G_{(k)}$ est triviale sur $\Pi_{\mathfrak{g}_{(k)}}$, nous avons recours à des résultats plus forts pour résoudre le Problème d'Equivalence. En effet après un certain nombre de réductions, il peut arriver que des paramètres de groupe subsistent, même s'il est impossible de poursuivre la réduction.

Plaçons nous dans un cadre analytique réel et formulons à présent le test d'involution de Cartan [21]. Les équations de structures sont de la forme

$$d\overset{(k)}{\omega^i} = \overset{(k)}{a^i_{j\rho}}\overset{(k)}{\pi^\rho} \wedge \overset{(k)}{\omega^j} + \frac{1}{2}\overset{(k)}{\gamma^i_{jl}}\overset{(k)}{\omega^j} \wedge \overset{(k)}{\omega^l}.$$

Posons

$$(\beta)^i_j := (\overset{(k)}{a^i_{j\rho}}\overset{(k)}{\pi^\rho});$$

cette matrice est appelée une matrice tableau; et la matrice

$$(\beta)^i_j \mod ((\overset{(k)}{\omega^j})) = \overset{(k)}{a^i_{j\rho}}\overset{(k)}{\pi^\rho} \mod (\overset{(k)}{\omega^j}),$$

est appelée la matrice tableau réduite. Dans l'expression précédente

$$\mathrm{mod}\ (\overset{(k)}{\omega}{}^j),$$

veut dire que l'on considère la matrice tableau modulo l'idéal algébrique pour le produit extérieur engendré par les $\overset{(k)}{\omega}{}^j$. Nous définissons alors par récurrence, les caractères de Cartan réduits de la manière suivante. On commence la récurrence en posant $\Sigma_0 = \{0\}$. Choisissons maintenant autant d'entrées indépendantes que possible $\mod \Sigma_0, \Sigma_1, \cdots, \Sigma_{s-1}$ dans la matrice tableau réduit en ne prenant au plus qu'une entrée par ligne. Le nombre de lignes apportant une contribution non nulle est le caractère de Cartan réduit σ_k et la collection de formes indépendantes est notée Σ_s.

Exemple 1.2.12 *La matrice tableau réduite suivante dont toutes les entrées sont indépendantes:*

$$\begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & \alpha_5 \end{pmatrix}$$

a pour caractères de Cartan

$$\sigma_1 = 3, \quad \sigma_2 = 1, \quad \sigma_3 = 1;$$

et

$$\Sigma_1 = \{\alpha_1, \alpha_2, \alpha_3\}, \quad \Sigma_2 = \{\alpha_4\}, \quad \Sigma_3 = \{\alpha_5\}.$$

Le choix des Σ_i ne se fait pas de façon unique.

Les équations de structures

$$d\overset{(k)}{\omega}{}^i = \overset{(k)}{a}{}^i_{j\rho}\overset{(k)}{\pi}{}^\rho \wedge \overset{(k)}{\omega}{}^j + \frac{1}{2}\overset{(k)}{\gamma}{}^i_{jl}\overset{(k)}{\omega}{}^j \wedge \overset{(k)}{\omega}{}^l$$

ne sont pas forcément définies de manière univoque, il est à priori possible de modifier les 1-formes $\overset{(k)}{\pi}{}^\rho$ par des combinaisons linéaires bien choisies des 1-formes $\overset{(k)}{\omega}{}^j$ sans modifier les équations de structures.

Exemple 1.2.13 *Les équations de structures*

$$d\theta_1 = \pi_1 \wedge \theta_1$$
$$d\theta_2 = \pi_2 \wedge \theta_2,$$

où θ_1, θ_2 sont une base de formes d'une variété de dimension 2 et π_1, π_2 sont indépendantes $\mod (\theta_1, \theta_2)$*, peuvent être modifiées en remplaçant respectivement θ_1, θ_2 par*

$$\pi_1 + t_1\theta_1, \quad \textit{resp} \quad \pi_2 + t_2\theta_2,$$

sans changer les équations de structures de θ_1 et θ_2. Le nombre maximal de paramètres arbitraires indépendant que l'on peut introduire grâce à ce procédé est appelé degré d'indétermination. Ici en l'occurrence sa valeur est 2.

Donc dans le cas général, la modification des équations de structures comme expliquée ci-dessus peut introduire des paramètres arbitraires dans les équations de structure; et le nombre de ses paramètres qui sont indépendants est appelé degré d'indétermination. Le test d'involution de Cartan nous dit alors que le système différentiel extérieur donné par les équations de structure

$$d\overset{(k)}{\omega}{}^i = \overset{(k)}{a}{}^i_{j\rho}\overset{(k)}{\pi}{}^\rho \wedge \overset{(k)}{\omega}{}^j + \frac{1}{2}\overset{(k)}{\gamma}{}^i_{jl}\overset{(k)}{\omega}{}^j \wedge \overset{(k)}{\omega}{}^l$$

est en involution si et seulement si

$$\dim(\overset{(1)}{\mathfrak{g}}_{(k)}) = \sum_{1}^{p} j\sigma_j \tag{1.2.48}$$

où p est le nombre des caractères réduits.

La proposition suivante décrit la structure de l'ensemble des solutions dans le cas d'un système en involution [26, page 46]

Proposition 1.2.14 *Si le test d'involution donnée par la condition* (1.2.48) est satisfait et le tenseur de structure obtenu des équations de structure

$$d\overset{(k)}{\omega}{}^i = \overset{(k)}{a}{}^i_{j\rho}\overset{(k)}{\pi}{}^\rho \wedge \overset{(k)}{\omega}{}^j + \frac{1}{2}\overset{(k)}{\gamma}{}^i_{jl}\overset{(k)}{\omega}{}^j \wedge \overset{(k)}{\omega}{}^l$$

a des coordonnées constantes lorsqu'on l'écrit dans une base, alors l'ensemble des solutions du problème d'équivalence forme un pseudo-groupe de Lie pour la composition. Il est transitif. De plus il est infini (dépend de fonctions arbitraires) s'il existe $\overset{(k)}{a}{}^i_{j\rho} \neq 0$ et fini (dépend de constantes arbitraires) dans le cas contraire.

Remarque 1.2.15 *Cette proposition requiert l'hypothèse d'analycité réelle pour les données considérées car elle est basée sur le théorème de Cartan-Kähler. Voir à ce propos* [9, 26].

Exemple 1.2.16 *Les équations de structure*

$$d\theta_1 = \pi_1 \wedge \theta_1$$
$$d\theta_2 = \pi_2 \wedge \theta_2$$

où θ_1, θ_2 *sont une base de formes d'une variété de dimension* 2 *et* π_1, π_2 *sont indépendantes* mod (θ_1, θ_2), *ont pour caractères de Cartan réduits* $\sigma_1 = 2$ *et pour degré d'indétermination* 2. *Le test de Cartan nous dit alors que ce système est en involution. Comme il n'y a pas de termes de torsion, alors la proposition qui précède nous apprend que l'ensemble des solutions d'un tel problème d'équivalence est un pseudo-groupe de Lie transitif et infini.*

Si le test d'involution de Cartan n'est pas satisfait, alors nous devons prolonger le Problème d'Equivalence. Cette procédure de prolongation consiste à prendre en compte l'indétermination dans la définition des équations de structures comme dans l'exemple 1.2.13 et de travailler justement dans un espace où ces formes sont bien définies, c'est à dire en relevant comme précédemment les formes à un espace plus grand [26, page 47]. Nous commençons par écrire les nouvelles équations de structures (en utilisant le lemme de Cartan)

$$\begin{aligned} d\overset{(k)}{\omega^i} &= \overset{(k)}{a^i_{j\rho}}\overset{(k)}{\pi^\rho} \wedge \overset{(k)}{\omega^j} + \frac{1}{2}\overset{(k)}{\gamma^i_{jl}}\overset{(k)}{\omega^j} \wedge \overset{(k)}{\omega^l} \\ d\overset{(k)}{\pi^\rho} &= \overset{(k)}{A^\rho_{si}}\overset{(k)}{\theta^s} \wedge \overset{(k)}{\omega^i} + \overset{(k)}{B^\rho_{\sigma j}}\overset{(k)}{\pi^\sigma} \wedge \overset{(k)}{\omega^j} + \frac{1}{2}\overset{(k)}{C^\rho_{lm}}\overset{(k)}{\omega^l} \wedge \overset{(k)}{\omega^m} \end{aligned} \tag{1.2.49}$$

Les formes $\theta^s, 1 \leq s \leq \dim \mathfrak{g}^{(1)}_{(k)}$, sont égales à l'image réciproque sur $U \times G_{(k)} \times G^{(1)}_{(k)}$ par la projection canonique

$$U \times G_{(k)} \times G^{(1)}_{(k)} \to G^{(1)}_{(k)}$$

de la forme de Maurer-Cartan sur la première prolongation $G^{(1)}_{(k)}$ de $G^{(1)}_{(k)}$ (modulo l'ideal généré par les 1-formes

$$\overset{(k)}{\omega^i}, \quad \overset{(k)}{\pi^\rho},$$

pour $1 \leq i \leq n$ et $1 \leq \rho \leq \dim G^{(1)}_{(k)}$).

Le groupe de Lie $G^{(1)}_{(k)}$ est par définition le groupe de Lie $GL(V^\star + g^\star_{(k)})$ dont les éléments t vérifient

$$t(\omega^i) = \omega^i, \quad 1 \leq i \leq n$$

et

$$t(\overset{(k)}{\pi^\rho}) = \overset{(k)}{\pi^\rho} + v_i^\rho \overset{(k)}{\omega^i}, 1 \leq \rho \leq \dim G_{(k)},$$

où

$$v_i^\rho \overset{(k)}{\delta_\rho} \otimes e^{\star i} \in \mathfrak{g}_{(k)}^{(1)}.$$

(Voir l'équation (1.2.29)).

Nous avons donc un nouveau problème d'équivalence sur $U \times G_{(k)}$ avec corepère associé

$$\overset{(k)}{\omega}{}^A_{U \times G_{(k)}} = (\overset{(k)}{\omega}{}^1_U, \cdots, \overset{(k)}{\omega}{}^n_U, \quad \overset{(k)}{\pi}{}^1 := \overset{(k)}{\omega}{}^{n+1}_{U \times G_{(k)}}, \cdots, \overset{(k)}{\pi}{}^{\dim G_{(k)}} := \overset{(k)}{\omega}{}^{n+\dim G_{(k)}}_{U \times G_{(k)}})$$

et groupe structural $G^1_{(k)}$. Le résultat suivant dit que le problème d'équivalence originel donné par l'équation (1.2.1) et celui donné par la proposition ci-après ont même ensemble de solutions [26].

Proposition 1.2.17 *Il existe une bijection entre les difféomorphismes $f : U \to V$ tels que*

$$f^\star \overline{\omega}^j_V = \gamma^i_j \omega^j_U$$

pour une fonction (γ^i_j) à valeurs dans G, et les difféomorphismes $\hat{f} : U \times G_{(k)} \to V \times G_{(k)}$ tels que

$$\hat{f}^\star \overset{(k)}{\overline{\omega}}{}^A_{V \times G_{(k)}} = \hat{\gamma}^A_B \overset{(k)}{\omega}{}^B_{U \times G_{(k)}} \tag{1.2.50}$$

pour une fonction $(\hat{\gamma}^A_B)$ à valeurs dans $G^{(1)}_{(k)}$.

Enfin nous savons qu'après un certain nombre de prolongations, réductions, nous aboutirons forcément à une $\{e\}$-structure (que l'on pourra résoudre grâce à la proposition(1.2.8)) ou à un système en involution (que l'on pourra résoudre à l'aide de la proposition précédente grâce à la proposition (1.2.14)).

Nous avons terminé notre description de la solution du Problème de Cartan, à part pour deux points. Le premier point concerne le choix du tenseur de structure τ_U et le second traite de la détermination de l'action de G sur $\Pi_{\mathfrak{g}}$.

Au lieu de calculer directement τ_U, Cartan choisissait toujours un complémentaire C à $\operatorname{Im} L$ dans $V \otimes \Lambda^2 V^\star$:

$$V \otimes \Lambda^2 V^\star = \operatorname{Im} L + C. \tag{1.2.51}$$

Une façon de choisir ce complémentaire C est de faire ce que R.B. Gardner appelle absorption de Torsion; ce qui consiste à résoudre autant d'équations de la forme

$$\gamma^i_{jk} = -(a^i_{j\rho} v^\rho_k - a^i_{k\rho} v^\rho_j) \tag{1.2.52}$$

que possible. La détermination de l'action de G sur $\Pi_{\mathfrak{g}}$ se fait quant à elle de manière infinitésimale en utilisant le fait que pour toute 1-forme ω, $d^2\omega = 0$ (voir [21, lecture.4, page 39]).

2 The Abel equation

2.1 Introduction

Our aim in the present part is a study of the Abel Equation

$$\frac{dy}{dx} = c_0(x) + 3c_1(x)y + 3c_2(x)y^2 + c_3(x)y^3. \tag{2.1.1}$$

This latter equation was first studied by Abel in [1], in his study of the theory of elliptic functions. In fact Abel studied an equation of the form

$$\frac{dy}{dx} = \frac{c_0(x) + 3c_1(x)y + 3c_2y^2 + c_3(x)y^3}{b_0(x) + b_1(x)y},$$

which is called the second kind Abel differential equation and which is can be brought to the first kind Abel equation, whenever $b_1 \neq 0$, by the transformation $z = \dfrac{1}{b_0 + b_1y}$.

Our study will be first of all by the method of equivalence of Cartan and then by methods from differential algebra to obtain its invariants under a class of local transformations.

To be more precise, we study the equivalence of the two general first order ODEs

$$\frac{dy}{dx} = f(x, y), \tag{2.1.2}$$

and

$$\frac{dY}{dX} = F(X, Y) \tag{2.1.3}$$

with f and F, C^∞ functions, under the transformations

$$X = \chi(x, y); \quad Y = \varphi(x, y) \text{ with } \begin{cases} \dfrac{\partial \chi}{\partial X} & = \dfrac{\partial \varphi}{\partial Y} \\ \dfrac{\partial \chi}{\partial Y} & = 0, \end{cases} \tag{2.1.4}$$

using the Cartan equivalence method. This enables us to show that equation (2.1.2) admits the fundamental local invariant

$$I = f_{yy},$$

under transformations (2.1.4). Its vanishing is the condition for equation (2.1.2) to be linearizable to the linear first order differential equation under the transformations (2.1.4).

For case of equation (2.1.1), with $c_2, c_3 \neq 0$, $I = 0$ when $y = -\dfrac{c_2}{c_3}$. Injecting this value of y into equation (2.1.1), one gets the identity (an hypersurface in the first jet space of the applications c_i):

$$c_3\frac{dc_2}{dx} - c_2\frac{dc_3}{dx} - 3c_1c_2c_3 + 2c_2^3 + c_0c_3^2 = 0. \tag{2.1.5}$$

Conversely when such a relation is satisfied, $y_0 = -\dfrac{c_2}{c_3}$, verifies equation (2.1.1). The left hand side s_3 of equation (2.1.5) is very important. It was first formed by Roger Liouville in [32, 33]. We now restrict our study and consider some transformations of equation (2.1.1) which preserve its form (the polynomial character of the second member), namely

$$y = \eta(\xi)u(x) + \nu(x); \quad \frac{d\xi}{dx} = \mu(x); \tag{2.1.6}$$

one sees that s_3 is an invariant under these last substitutions; moreover the condition $s_3 = 0$ is preserved under the action of transformations (2.1.6). It is in the sense of classical invariant theory or representation theory, that we mean the invariance: for instance given a binary quadratic form $f_0(x, y) = a_0x^2 + 2a_1xy + a_2y^2$, the linear group $GL(2, \mathbb{C})$ acts on it via the transformation

$$\begin{cases} x &= aX + bY \\ y &= cX + dY, \end{cases}$$

with $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$. This action induces another one on the coordinate ring $\mathbb{C}[a_0, a_1, a_2]$; and an invariant $I(a_0, a_1, a_2)$ of f_0 is a function multiplied by a power of the determinant of g_0 when one replaces the a_i by their new values. It is well known that the binary quadratic form has a unique invariant (up to multiplication by a fixed constant) namely its discriminant $D_0 = a_1^2 - a_0a_2$. It is of degree two in the coefficients of the quadratic form. In the same vein transformations (2.1.6) induce an action on the coefficients

c_i of equation (2.1.1) and on their derivatives. And we look for expressions of the latter which are multiplied by a certain factor only depending on u, μ, ν, when we change the c_is and their derivatives by their new values after transformation (2.1.6) is made.

Starting from s_3, one builds an hierarchy of expressions

$$s_{2n+1} = c_3 \frac{d}{dx} s_{2n-1} - (2n-1) s_{2n-1} \left[\frac{dc_3}{dx} + 3(c_1 c_3 - c_2^2) \right], \quad n \geqslant 2$$

which are also invariants in the previous sense.

We next examine the operator

$$\frac{d}{dx} - \frac{(2n-1)\left(\frac{dc_3}{dx} + 3\left(c_1 c_3 - c_2^2\right)\right)}{c_3}$$

and show that it plays a role of a covariant derivation. Using the expressions $s_{2i+1}, i \geq 2$, we give a necessary and sufficient condition for two equations of the form (2.1.1) to be in the same class. Two such equations of the form (2.1.1) are in same class if and only if they can be transformed one into each other by a substitution of the type (2.1.6).

As it is classically known from the following theorem of Lie:

Theorem 2.1.1 (Lie superposition theorem 1) *The only ordinary differential equations*

$$\frac{dy}{dx} = f(x, y)$$

allowing a set of arbitrary solutions are the Riccati equation

$$\frac{dy}{dx} = c_0(x) + 2c_1(x)y + c_2(x)y^2$$

and any set of equation obtained from it by a change of dependent and independent variables of the form $\psi = \psi(y)$ and $\tau = \tau(x)$.

By "admitting" a set of arbitrary solutions we mean that its general solution can be expressed in terms of arbitrary particular solutions and some constants. More precisely, whenever three distinct solutions of the Riccati equation are known, one knows its general solution, because the cross-ratio of any four solutions of it is constant. Given four distinct of its solutions u_1, u_2, u_3, u, we have explicitly

$$u = \frac{u_3(u_2 - u_1) + k(u_1 - u_3)u_2}{u_2 - u_1 + k(u_1 - u_3)}, \text{ for some } k \in \mathbb{P}^1(\mathbb{C}).$$

The cross-ratio admits a generalization defined for any $2k$, $k \geq 2$ distinct elements called hyper-cross-ratios. We examine for the case of equation (2.1.1), when $2k$, $z_1, \cdots, z_{2k}$ distinct solutions have a constant hyper-cross-ratio. The relation is given by a quadratic relation between the z_is.

Our work is organized as follows: we apply the Cartan procedure to the study of equation (2.1.2) in section 2.2; after that we study in the succeeding section the invariant theory of equation (2.1.1): we determine its canonical form, study its invariants and use the latter to characterize classes of equation (2.1.1). Finally in the last section we examine the degenerate case ($c_3 \equiv 0$) of equation (2.1.1): we mimic the work done in the section 2.3. Our principal results are in sections 2.2, 2.3 and 2.4.

2.2 The Abel equation

In the following we study the first order ODE (2.1.1) using Cartan's method under the pseudo-group of transformations (2.1.4):

$$x = \chi(X, Y), \quad y = \varphi(X, Y);$$

with

$$\begin{cases} \dfrac{\partial \chi}{\partial X} & = \dfrac{\partial \varphi}{\partial Y} \\ \dfrac{\partial \chi}{\partial Y} & = 0. \end{cases}$$

One has the following definition:

Definition 2.2.1 *A pseudo-group on a manifold S is a collection Γ of local diffeomorphisms between open subsets of S satisfying the following properties:*

- *For every open set U in S, the identity map on U, Id_U is in Γ.*
- *If $f \in \Gamma$ then so does f^{-1}.*
- *If $f \in \Gamma$, then the restriction of f to an arbitrary open subset of its domain of definition is in Γ.*
- *If an open set U is the union of open sets $\{U_i\}$ and f is a diffeomorphism from U to some open subset of S such that the restriction of f to every U_i is in Γ, then $f \in \Gamma$.*

- *If $f : U \to V$ and $f' : U' \to V'$ are in Γ and the intersection $V \cap U'$ is not empty, then the following restricted composition*

$$f' \circ f : f^{-1}(V \cap U') \to f'(V \cap U'),$$

  is in Γ.

In order to apply Cartan's method one needs a more particular type of pseudo-group, namely a Lie pseudo-group

Definition 2.2.2 *A Lie pseudo-group is a pseudo-group whose transformations have components which are the solutions of a system of partial differential equations, named equations of definition. A Lie pseudo-group will be said to be of finite dimension if the general solutions of the equations of definition depend on a finite number of arbitrary parameters, infinite if on the contrary these components are functions of arbitrary functions or an infinite number of parameters. As an example we remark that contact transformations form also a Lie pseudo-group. See* [26].

The differential equation (2.1.1) is a natural generalization of the Riccati equation

$$\frac{dy}{dx} = a_0(x) + a_1(x)y + a_2(x)y^2.$$

It was studied by Appell in [2], by Roger Liouville in [32, 33] and Painlevé in [44] to give a few, prior to the moving coframe Cartan Method.

2.2.1 Cartan's method

We recall that two first order ODEs

$$\frac{dy}{dx} = f(x,y)$$

and

$$\frac{dY}{dX} = F(X,Y)$$

are equivalent under the transformations (2.1.4) if and only if there exists such a transformation which sends solutions of the source equation to solutions of the target equation. In order to put the equivalence problem of the two first order ODEs into an equivalence problem of Cartan, one has to put it into a pfaffian system.

Lemma 2.2.3 *The solutions of* (2.1.2) are into one to one correspondence with the integral manifolds of the pfaffian system $(I = \langle\omega_U^1\rangle, dx)$ with independence condition on dx; $\omega_U^1 = dy - f(x,y)dx$; x, y, are the standard coordinates on $J^0(\mathbb{R},\mathbb{R}) := J^0$.

Proof. An integral manifold of $(I = \langle\omega_U^1\rangle, dx)$ is a curve $s : \mathbb{R} \to J^0(\mathbb{R},\mathbb{R})$:

$$s^*\omega_U^1 = 0 \quad \text{and} \quad s^*(\omega_2 = dx) \neq 0.$$

So there exists $c : \mathbb{R} \to \mathbb{R}$ such that $s = (x, c(x))$, ie $\dfrac{dc}{dx} = f(x, c(x))$. ■

One will from now on eliminate the dependance on the open set to unburden the notation (a differential form ω_U^i on U will be simply written ω^i).

Let $\mathfrak{D} := \partial_x + f(x,y)\partial_y$ be the total derivative. A local diffeomorphism Φ of the form (2.1.4) which realizes the equivalence between equations (2.1.2) and (2.1.3) must preserve the integral curves; its jacobian must also satisfy some other conditions. Consider the two coframes

$$\begin{aligned} \omega^1 &= dy - f dx \\ \omega^2 &= dx; \end{aligned} \tag{2.2.1}$$

and

$$\begin{aligned} \omega^2 &= dx \\ \omega^3 &= dy. \end{aligned} \tag{2.2.2}$$

on some open subset U of $J^0(\mathbb{R},\mathbb{R})$. Let ϖ^1, ϖ^2 and ϖ^3 be their respective pullbacks under Φ; one has

$$\begin{pmatrix} \varpi^1 \\ \varpi^2 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

with the last condition expressing the fact that Φ preserves the integral curves; and

$$\begin{pmatrix} \varpi^2 \\ \varpi^3 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix}$$

which means that Φ is a transformation of the prescribed type given by (2.1.4). One has therefore an overdetermined equivalence problem, meaning that the conditions are given on a set of linearly dependent one-forms. In order to put it into a determined one, we must normalize. Let $\Xi := \begin{pmatrix} \overline{\omega}^1 \\ \overline{\omega}^2 \end{pmatrix}$ and $\Gamma := \begin{pmatrix} \overline{\omega}^2 \\ \overline{\omega}^3 \end{pmatrix}$; then

$$\Xi = \frac{1}{a^2} \begin{pmatrix} -ufa - ub & -ua \\ \\ aw & 0 \end{pmatrix} \Gamma.$$

We normalize by setting $u = a$, $w = a$ and $b = -uf$. This induces

$$\tilde{\Xi} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} a\omega^1 \\ a\omega^2 \end{pmatrix}.$$

A fixed coframe of the set $\left\{ \begin{pmatrix} a\omega^1 \\ a\omega^2 \end{pmatrix}, a \neq 0 \right\}$, is given by

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

Now the elements

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} \right\}$$

whose action on $\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$ give $\tilde{\Xi}$ satisfy, $U = a$ and $W = a$; therefore we end up with the following condition on the coframe (ω_1, ω_2):

$$\Xi = \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

One can see [21, lect.5] for a complete description of the normalization process.

A class of equivalence of first order ODEs under fiber preserving transformations is thus a local G-structure $U \times G_f$ (we work on local trivialization of the principal bundle) with (ω^1, ω^2) one forms on U. The structural group is given by

$$G_f = \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix}. \tag{2.2.3}$$

The latter condition of equivalence translates on the one forms associated to the G_f-structures as the existence of a diffeomorphism ϕ such that (with ϕ^*

the pull-back), $\phi^*(\overline{\theta}) = \theta$; $\overline{\theta}$ is associated with the target equation and θ to the source one. We consider the components of the canonical one-form on the principal bundle over J^0 with structure group G_f: (θ_1, θ_2). They read at the point (x, g)

$$\begin{aligned} \theta_1 &= u_1 \omega^1 \\ \theta_2 &= u_1 \omega^2. \end{aligned} \tag{2.2.4}$$

The idea of the Cartan method is the following. One starts with $U \times G_f$ and builds when possible a new principal bundle $\mathcal{P} = U \times H$ (with new structural group H) endowed with a coframe (θ_i, Π_μ) built in a geometric way such that it contains all the local information about $U \times G_f$, through its structure equations.

The structures equations for (θ_1, θ_2) are

$$\begin{aligned} d\theta_1 &= \frac{du_1}{u_1} \wedge \theta_1 + u_1 d(dy - f dx) \\ &= \pi_1 \wedge \theta_1 - \frac{f_y}{u_1} \theta_1 \wedge \theta_2 \\ &= \varpi \wedge \theta_1 \\ d\theta_2 &= d(u_1 \omega_2) = \frac{du_1}{u_1} \wedge \theta_2 \\ &= \varpi \wedge \theta_2; \end{aligned} \tag{2.2.5}$$

with

$$\varpi = \frac{du_1}{u_1} + \frac{f_y}{u_1} \theta_2.$$

We see that ϖ is defined uniquely; one can not modify it with substitutions of the form $\varpi \to \varpi + t_1 \theta_1 + t_2 \theta_2$ because otherwise the structure equations would change; moreover the reduced Cartan character σ_1 equals to 1, the others vanish. So our system is not in involution and we must prolong it. We place ourselves on the local principal bundle $U \times G_f$ with group structure

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and local coordinates} \quad (x, y, u_1)$$

endowed with the coframe $(\theta_1, \theta_2, \varpi_1)$; the new structure equations are

$$\begin{aligned} d\theta_1 &= \varpi \wedge \theta_1 \\ d\theta_2 &= \varpi \wedge \theta_2 \\ d\varpi &= d(f_y dx) \\ &= \frac{f_{yy}}{u_1^2}\theta_1 \wedge \theta_2. \end{aligned} \tag{2.2.6}$$

One has

Theorem 2.2.4 $I = \dfrac{f_{yy}}{u_1^2}$ *is the fundamental relative differential invariant of the equivalence problem. If it vanishes f is a first degree polynomial in y ie (2.1.2) is linear. Let X_1, X_2, X_3 the frame dual to the invariant coframe $(\theta_1, \theta_2, \varpi)$, then the successive derivative of I with respect to X_i are also relative differential invariants, meaning that for any equivalence $\widetilde{\chi} : P \to \overline{P}$, with P associated to the source equation and $\overline{P}$ to the target equation, one has if $\overline{I}$ denotes the expression corresponding to I formed from the target equation,*

$$\overline{I} \circ \widetilde{\chi} = I.$$

Also the dual frame is given by

$$X^1 = \frac{1}{u_1}\partial_y, \quad X^2 = \frac{1}{u_1}(\partial_x + f\partial_y) - f_y\partial_{u_1}, \quad X^3 = u_1\partial_{u_1}.$$

We now consider equation (2.1.1); with $c_2, c_3 \neq 0$. $I = 0$ for the latter equation when $y_0 = -\dfrac{c_2}{c_3}$. If we inject this value in equation (2.1.1), then

$$\frac{1}{c_3^2}\left(-c_3\frac{dc_2}{dx} + c_2\frac{dc_3}{dx}\right) = \frac{c_0c_3^2 - 3c_1c_2c_3 + 2c_2^3}{c_3^2} \Longleftrightarrow s_3 = 0.$$

2.3 Invariants

When submitted to transformations (2.1.6), the differential equation (2.1.1) becomes

$$\frac{d\eta}{d\xi} = \gamma_0 + 3\gamma_1\eta + 3\gamma_2\eta^2 + \gamma_3\eta^3 \tag{2.3.1}$$

where if we denote by u', ν' and ξ', the derivatives of the functions u, ν, ξ, with respect to variable x, the $\gamma_i s$ are respectively given by:

$$\begin{aligned}
\gamma_0 &= \frac{c_0 + 3c_1\nu^2 + 3c_2\nu^2 + c_3\nu^3 - \nu'}{\mu u} \\
\gamma_1 &= \frac{c_1 + 2c_2\nu + c_3\nu^2}{\mu} - \frac{u'}{3\mu u} \\
\gamma_2 &= \frac{u(c_2 + c_3\nu)}{\mu} \\
\gamma_3 &= \frac{u^2 c_3}{\mu}.
\end{aligned} \tag{2.3.2}$$

2.3.1 Weight and Degree

Weight

Consider the particular transformations of the form (2.1.6)

$$\begin{aligned}
y &= \lambda\eta \\
x &= \lambda\xi,
\end{aligned}$$

where λ is a constant. They change equation (2.1.1) into the following

$$\frac{d\eta}{d\xi} = \gamma_0 + 3\gamma_1\eta + 3\gamma_2\eta^2 + \gamma_3\eta^3$$

where

$$\gamma_0 = c_0, \quad \gamma_1 = \lambda c_1, \quad \gamma_2 = \lambda^2 c_2, \quad \gamma_3 = \lambda^3 c_3, \quad \frac{d^k\gamma_i}{d\xi^k} = \lambda^{i+k}\frac{d^k c_i}{dx^k}.$$

We attribute to $\frac{d^k c_i}{dx^k}$ the weight $i + k$.

Degree

As the substitutions

$$\begin{aligned}
y &= \eta \\
x &= \mu\xi,
\end{aligned}$$

where μ is a constant, change equation (2.1.1) into

$$\frac{d\eta}{d\xi} = \gamma_0 + 3\gamma_1\eta + 3\gamma_2\eta^2 + \gamma_3\eta^3$$

with
$$\frac{d^k\gamma_i}{d\xi^k} = \mu^{1+k}\frac{d^k c_i}{dx^k};$$
we attribute to $\dfrac{d^k c_i}{dx^k}$ the degree $1+k$.

Remark 2.3.1 *More generally under transformations of the type*
$$(x, y) = (\lambda^s\xi, \lambda^r\eta),$$
one gets
$$\frac{d^k\gamma_i}{d\xi^k} = \lambda^{(k+1)s+r(i-1)}\frac{d^k c_i}{dx^k}.$$

2.3.2 Canonical form

The substitutions (2.1.6) as shown in identities (2.3.2), when acting on the differential equation (2.1.1), induce a transformation on the coefficients c_i. We have

Definition 2.3.2 *An invariant Z of the differential equation* (2.1.1) is a function on the infinite jet space of its coefficients b_δ, which we remind has locally for coordinates the derivatives of the b_δ up to any order, such that if we denote by t a substitution of type (2.1.6), there exists a function ν depending only on t:
$$t.Z = \nu(t)Z;$$
ν is called a multiplier. When $\nu \equiv 1$, the invariant of the differential equation is called absolute.

It is possible in equation (2.1.1) to eliminate the coefficients of y and y^2. We do so by setting
$$y = Y\mathcal{U} + V, \quad \frac{dX}{dx} = M(x),$$
with $\mathcal{U}$, V, M chosen in order to make the coefficients of Y and Y^2 become zero and the coefficient of Y^3 equal to 1. One finds
$$V = -\frac{c_2}{c_3}, \quad \mathcal{U} = c_3 e^{3\int\frac{c_1c_3-c_2^2}{c_3}dx}, \quad M = c_3\mathcal{U}^2;$$
and equation (2.1.1) becomes
$$\frac{dY}{dX} = Y^3 + J \tag{2.3.3}$$

with
$$J = \frac{s_3}{c_3^3 \mathcal{U}^3}, \quad s_3 = c_0 c_3^2 - 3c_1 c_2 c_3 + 2c_2^3 + c_3 \frac{dc_2}{dx} - c_2 \frac{dc_3}{dx}.$$
We will call equation (2.3.3), the canonical form of equation (2.1.1). As an expression of the $c_i s$ and their derivatives, s_3 is of degree 3 and weight 6.

Given now an expression E of the c_is and their derivatives with respect to x, one will index with a 1 from now on, the functions E_1 of γ_is and their derivatives with respect to ξ, obtained from E by replacing the c_is and their derivatives with the γ_is and their derivatives with respect to ξ.

We see that c_3 is a relative invariant of the differential equation (2.1.1); because from the formulas (2.3.2), one has :
$$\gamma_3 = \frac{u^2}{\mu} c_3.$$
Also
$$\mathcal{U}_1 := \frac{\mathcal{U}}{u}$$
and
$$(s_3)_1 = \frac{u^3}{\mu^3} s_3.$$
Moreover J is an absolute invariant; finally the $\frac{d^k J}{d^k X}$ are all absolute invariants $\left(\frac{dX_1}{d\xi} := \gamma_3 \mathcal{U}_1^2 = \frac{c_3}{\mu} \mathcal{U}^2 \Rightarrow X_1 = X \right)$. Every function of these quantities is an absolute invariant and every absolute invariant is a function of them, as seen immediately by computing that invariant on the canonical form (2.3.3).

Remark 2.3.3 *The relative invariant s_3 was first formed by Roger Liouville in* [33] by pure computations. Moreover to any equation (2.1.1) studied under transformation (2.1.6), one can associate in a unique way the formulas of transformations (2.3.2) and conversely; we do so by identifying the equation with the quadruplet $(c_0, \cdots, c_3)$. We have seen that any absolute invariant of the differential equation (2.1.1) is a function of the expressions $\frac{d^k J}{dX^k}$, which therefore form a functional basis for the absolute invariants of the differential equation.

Under the identification we have just done, one can say that these $\frac{d^k J}{dX^k}$ constitute a basis of differential invariants for the action of the pseudo-group (2.1.6) on the variables $(c_0, \cdots, c_3)$ defined by formulae (2.3.2). The latter differential invariants under the pseudo-group (2.1.6), are expressions of the

c_is which are left fixed under the action of the infinite prolongation of the pseudo-group action (2.3.2) (we look at the transformation induced on the infinite jet with local coordinates $c_i, \frac{dc_i}{dx}, \frac{d^2c_i}{dx^2} \cdots$). This is just another way to define an absolute invariant of the differential equation (2.1.1). The same remark is true for relative invariants of the differential equation (2.1.1); they are all relative differential invariants for the action on the variables c_i of the prolongation of pseudo-group (2.1.6) (see definition 2.2.1).

Let's now seek the most general transformation of the form (2.1.6) which leaves the canonical form invariant. It is a symmetry group of the canonical form, ie a Lie group of transformation which sends solutions of the canonical form to solutions of the canonical form. Using formulas (2.3.3) one must have $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 1$; with $c_1 = c_2 = 0$ and $c_3 = 1$. This leads to $\nu = 0$, $u' = 0$ ie $u = K$ and $\mu = \frac{1}{K^2}$. So we obtain the transformations

$$\begin{aligned} X_2 &= \frac{1}{K^2}(X+h) \\ Y_2 &= KY. \end{aligned} \tag{2.3.4}$$

with K, h constants.

Let $\tau_h^X : (X,Y) \to X+h$ and $H_K^Y : (X,Y) \to KY$, then one can represent transformations (2.3.4) as $(H_{K^{-1}}^X)^2 \circ \tau_h^X$.

We now apply transformations (2.3.4) on equation (2.3.3) to determine how the J changes; one gets

$$J_2 = K^3 J;$$

and in general if $\frac{d^kJ_2}{dX_2^k}$ denotes the new value of $\frac{d^kJ}{dX^k}$ after transformation (2.3.4) is applied:

$$\begin{aligned} J_2 &= K^3 J \\ \frac{dJ_2}{dX_2} &= K^5 \frac{dJ}{dX} \\ \frac{d^iJ_2}{dX_2^i} &= K^{2i+3}\frac{d^iJ}{dX^i}, \quad i \geq 2. \end{aligned} \tag{2.3.5}$$

For instance $\frac{dJ_2}{dX_2} = \frac{dJ_2}{dX}\frac{dX}{dX_2}$.

Proposition 2.3.4 *The group of transformations*

$$X_2 = \frac{1}{K^2}(X+h)$$
$$Y_2 = KY$$
$$\frac{d^i J_2}{dX_2^i} = K^{2i+3}\frac{d^i J}{dX^i}, \quad i \geq 1$$

admits the following infinitesimal generators:

$$\mathcal{V} = Y\partial_Y - 2X\partial_X + \sum_{i\geqslant 0}(2i+3)J_i\partial_{J_i}, \quad J_i := \frac{d^i J}{dX^i},$$

and

$$W = \partial_X.$$

An invariant I in its canonical form must therefore satisfy:

$$W(I) \equiv 0 \quad \textit{and} \quad \sum_{i\geqslant 0}(2i+3)J_i\partial_{J_i}(I) \equiv 0.$$

Proof. Derive with respect to K and h then evaluate at $(K,h)=(1,0)$ and use the fact that an invariant by very definition is independent of y. ■

The $\dfrac{d^k J}{dX^k}$ are absolute invariants which can be expressed with the c_i and their derivatives. To do so one starts with

$$J = \frac{s_3}{c_3^3\mathcal{U}^3}, \quad \frac{dX}{dx} = c_3\mathcal{U}^2, \quad \mathcal{U} = \exp^{3\int\frac{c_1c_3-c_2^2}{c_3}dx}.$$

We recall the expression of $s_3 = c_0c_3^2 - 3c_1c_2c_3 + 2c_2^3 + c_3\dfrac{dc_2}{dx} - c_2\dfrac{dc_3}{dx}$ as given in equation (2.1.5). Let's set with Roger Liouville [33]

$$s_{2n+1} = c_3\frac{d}{dx}s_{2n-1} - (2n-1)s_{2n-1}\left[\frac{dc_3}{dx} + 3(c_1c_3 - c_2^2)\right], \quad n \geqslant 2.$$

We have

Proposition 2.3.5

$$\frac{d^k J}{dX^k} = \frac{s_{2k+3}}{c_3^{2k+3}\mathcal{U}^{2k+3}}, \quad k \geqslant 0,$$

and

$$(s_{2k+1})_1 = \frac{u^{2k+1}}{\mu^{2k+1}} s_{2k+1}, \quad k \geqslant 1.$$

Therefore

$$(s_{2k+1})_1 = 0 \Leftrightarrow s_{2k+1} = 0, \quad k \geqslant 1.$$

As $\dfrac{u}{\mu} = \dfrac{\partial(x,y)}{\partial(\xi,\eta)}$*: the jacobian of transformation* (2.1.6), one can say s_{2k+1} defines a $2k+1$ tensor for any k. Moreover the s_{2k+1} are polynomials in the c_i and their derivatives; they are also of weight $4k+2$ and degree $2k+1$ for $k \geq 1$.

Proof. Let's suppose as recurrence hypothesis that the first formula holds for rank $k \geqslant 0$, one has

$$\begin{aligned}\frac{d^{k+1}J}{dX^{k+1}} &:= \frac{1}{c_3\mathcal{U}^2}\frac{d}{dx}\left(\frac{d^kJ}{dX^k}\right) \\ &= \frac{c_3^{2k+3}\mathcal{U}^{2k+3}\dfrac{ds_{2n+3}}{dx} - (2n+3)\left[\dfrac{dc_3}{dx} + 3(c_1c_3 - c_2^2)\right]c_3^{2n+2}\mathcal{U}^{2n+3}}{c_3^{4k+7}\mathcal{U}^{4k+8}}\end{aligned}$$

which gives the desired results after simplification and using the fact that the $\dfrac{d^kJ}{dX^k}$s are absolute invariants.

We saw that s_3 is of degree 3 and weight 6. Using a straightforward recurrence argument we see that s_{2k+1} is of weight $4k+2$ and degree $2k+1$ for $k \geq 1$ (use the argument that derivation with respect to x increases the weight and the degree by one and the fact that the c_is are of weight i and degree 1). ■

Remark 2.3.6

$$\mathcal{U}s_{2n+1} = (c_3\mathcal{U})\frac{ds_{2n-1}}{dx} - (2n-1)s_{2n-1}\frac{d}{dx}(c_3\mathcal{U}) := [c_3\mathcal{U}, s_{2n-1}]^{(1)}.$$

We call $[,]^{(1)}$*, the Jacobian of Laguerre-Forsyth. Moreover if we define*

$$\nabla_n := c_3\frac{d}{dx} - (2n-1)\left(\frac{dc_3}{dx} + 3\left(c_1c_3 - c_2^2\right)\right),$$

we observe that it sends the relative invariants s_{2n-1} *of degree* $2n-1$ *to the relative invariants* s_{2n+1} *of degree* $2n+1$*. Here one can see an operator analog to*

$$\frac{1}{2i\pi}\frac{d}{d\tau} - \frac{k}{12}E_2$$

with E_2 denoting the Eisenstein series which in the context of modular forms, sends the space of modular forms of weight k to the space of modular forms of weight $k+2$.

Take the expression

$$\frac{dc_3}{dx} + 3\left(c_1c_3 - c_2^2\right);$$

we compute, with the notations of (2.3.2)

$$\frac{d\gamma_3}{d\xi} + 3\left(\gamma_1\gamma_3 - \gamma_2^2\right),$$

$$\begin{aligned}\frac{d\gamma_3}{d\xi} + 3\left(\gamma_1\gamma_3 - \gamma_2^2\right) &= \frac{1}{\mu}\frac{d}{dx}\left(\frac{u^2}{\mu}c_3\right) + 3\frac{u^2}{\mu}c_3\left(\frac{c_1 + 2c_2\nu + c_3\nu^2}{\mu} - \frac{u'}{3\mu u}\right) \\ &\quad - 3\frac{u^2}{\mu^2}\left(c_2^2 + 2c_2c_3\nu + c_3^2\nu^2\right) \\ &= \frac{u^2}{\mu^2}\left(\frac{dc_3}{dx} + 3\left(c_1c_3 - c_2^2\right)\right) + \frac{c_3}{\mu}\frac{d}{dx}\left(\frac{u^2}{\mu}\right) - \frac{u'u}{\mu^2}c_3.\end{aligned}$$

Therefore $\dfrac{\dfrac{d\gamma_3}{d\xi} + 3\left(\gamma_1\gamma_3 - \gamma_2^2\right)}{\gamma_3}$ satisfies

$$\frac{\dfrac{d\gamma_3}{d\xi} + 3\left(\gamma_1\gamma_3 - \gamma_2^2\right)}{\gamma_3} = \frac{1}{\mu}\frac{\dfrac{dc_3}{dx} + 3\left(c_1c_3 - c_2^2\right)}{c_3} + \frac{1}{u^2}\frac{d}{dx}\left(\frac{u^2}{\mu}\right) - \frac{u'}{u\mu}.$$

If we consider the particular transformation of the form (2.1.6)

$$\frac{d\xi}{dx} = \mu(x), \quad y = \eta + \nu, \tag{2.3.6}$$

one remarks that in this case

$$\frac{1}{u^2}\frac{d}{dx}\left(\frac{u^2}{\mu}\right) - \frac{u'}{u\mu} = -\frac{\mu'}{\mu^2} = -\frac{1}{\mu}\frac{\left(\dfrac{d\xi}{dx}\right)'}{\dfrac{d\xi}{dx}}.$$

Definition 2.3.7 *We remind that given a Riemann surface with coordinates $\tilde{z}$, $z, \cdots$, an affine connection on M is an object which is represented by local differentials $r(z)dz$, $\tilde{r}(\tilde{z})d\tilde{z}, \cdots$* (one for each coordinate variable) glued together according to the rule

$$\tilde{r}(\tilde{z})d\tilde{z} = r(z)dz - \{\tilde{z}, z\}_1 dz$$

with $\{\tilde{z}, z\}_1 = \frac{d}{dz} log \frac{d\tilde{z}}{dz}$. *In the presence of an affine connection, it is possible to define, for every* $k \in \mathbb{Z}$, *a covariant derivative* D_k *from* k*-th order differentials to* $k+1$ *ones, by* $\phi(dz)^k \to (D_k\phi)(dz)^{k+1}$, *where*

$$D_k\phi = \frac{\partial \phi}{\partial z} - kr\phi.$$

With this in mind, and considering only substitutions of the form (2.3.6), one has

$$\frac{\frac{d\gamma_3}{d\xi} + 3\left(\gamma_1\gamma_3 - \gamma_2^2\right)}{\gamma_3} d\xi = \frac{\frac{dc_3}{dx} + 3\left(c_1c_3 - c_2^2\right)}{c_3} dx - \{\xi, x\}_1 dx.$$

One has hence the following theorem

Proposition 2.3.8

$$\frac{d}{dx} - \frac{(2n-1)\left(\frac{dc_3}{dx} + 3\left(c_1c_3 - c_2^2\right)\right)}{c_3}$$

transforms almost 'covariantly' when one substitutes the c_i *by the* γ_i.

2.3.3 Classes of Canonical forms

Suppose now that we want to study a characteristic property of a class of equation (2.1.1), class meaning here that they can be brought into each other by means of transformations (2.1.6). The canonical equation depends on two arbitrary constants K and h. And this property if it does exist translates into a differential relation between J and X; the general integral of the latter containing the two arbitrary constants K and h. In fact as

$$\frac{dy}{dx} = y^3 + j,$$

all the canonical equations which are in the same class as the latter are of the form

$$\frac{dY}{dX} = Y^3 + K^3 j(K^2X + h), \quad \text{for a determined j.}$$

Hence a property of a class of equation translates into

$$J = K^3 j(K^2X + h).$$

Elimination of K and h gives a differential equation

$$f(J, J', J'') = 0.$$

This equation must stay unchanged, when one replaces Y by $\frac{Y}{K}$ and X by $(K^2X + h)$ (as it is a property verified by any canonical form in the class considered and transformations (2.3.4) permit us to stay in the same class). Remarking that $\frac{J'^3}{J^5}$ and $\frac{J'J''}{J^4}$ do not vary under such a substitution we now set

$$\begin{aligned} \frac{J'^3}{J^5} &= \kappa \\ \frac{J'J''}{J^4} &= v. \end{aligned} \tag{2.3.7}$$

We can then express J' and J'' in terms of κ, v and J. If we suppose next that $\frac{\partial f}{\partial v} \neq 0$ for some point, then $v = F(\kappa, J)$. This last equation does not vary when J is replaced by K^3J, so F is independent of J; which leads to an equation

$$v = F(\kappa)$$

or again

$$H\left(\frac{s_5^3}{s_3^5}, \frac{s_5 s_7}{s_3^4}\right) = 0.$$

This leads to the proposition

Proposition 2.3.9 *An equivalence class of canonical forms under point transformations* (2.1.6) satisfies a syzygy relation of the form

$$H\left(\frac{s_5^3}{s_3^5}, \frac{s_5 s_7}{s_3^4}\right) = 0.$$

and conversely such a relation gives the absolute invariant J which corresponds to a canonical form (2.3.3).

2.3.4 Lie superposition Principle, Vessiot-Guldberg Algebra and Possible Point configurations

It is well known that to an Abel equation (2.1.1):

$$\frac{dy}{dx} = c_0(x) + 3c_1(x)y + 3c_2(x)y^2 + c_3(x)y^3 := \mathfrak{S}(x, y),$$

is uniquely associated an x-dependent vector field (a family indexed by x of vector fields)

$$\xi(x, y) = \mathfrak{S}(x, y)\partial_y,$$

in fact we have seen in lemma 2.2.3 that the integral manifolds of the pfaffian system on J^0, associated to the Abel equation namely $dy - \mathfrak{S}(x, y)dx = 0$, are in one to one correspondence with the solutions of the Abel differential equation. The vector field $\partial_x + \mathfrak{S}(x, y)\partial_y$ on J^0 which is annihilated by $dy - \mathfrak{S}(x, y)dx = 0$ ie belongs to $\langle dy - \mathfrak{S}(x, y)dx\rangle^\perp$ is therefore associated uniquely to an Abel differential equation. The component $V_A(x, y) = \mathfrak{S}(x, y)\partial_y$ when we project to the y coordinate is called the x-dependent vector field associated to the Abel differential equation.

Given a general first order differential equation

$$\frac{dy}{dx} = f(x, y),$$

one says that it admits a global superposition rule if there exists an x-independent map

$$\Pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

of the form

$$y = \Pi(y_1, \cdots, y_n, k)$$

such that its general solution can be written in the form

$$y(t) = \Pi(y_1(t), y_2(t), \cdots, y_n(t), k),$$

with $y_1(t), \cdots, y_n(t)$, n generic or arbitrary solutions.

The Lie theorem cited in the Introduction of this chapter translates in the context of vector fields as

Theorem 2.3.10 (Lie superposition theorem 2) *A necessary and sufficient condition for a differential equation* $\frac{dy}{dx} = f(x, y)$*, with corresponding* x*-dependent vector field* $V = f(x, y)\partial_y$ *to admit a general superposition rule, is that* $V(x, y)$ *admits a decomposition*

$$V(x, y) = \sum_1^r b_\alpha(x) X_\alpha$$

with the ordinary vector fields X_α*, generating an* r*-dimensional Lie algebra.*

The Lie algebra generated by the X_α is called an r-dimensional Vessiot-Guldberg algebra.

With this in mind we give the following example cited in [25]. Consider the x-dependent vector field

$$V(x,y) = (y^2 + h(x)y^3)\partial_y$$

related to the abel differential equation

$$\frac{dy}{dx} = y^2 + h(x)y^3, \text{with } h(x) \text{ non-constant.}$$

The corresponding vector fields are $X_1 = y^2\partial_y$ and $X_2 = y^3\partial_y$; define for $j \geq 3$, $Z_j := [X_1, X_{j-1}]$; then we see that $Z_j = y^{j+1}\partial_y$ and $[X_1, Z_j] = (n-1)y^{n+2}\partial_y$. This spans an infinite family of independent vector fields, therefore from the above theorem an Abel differential equation does not admit in general a general superposition principle.

Nevertheless one can look for certain relations satisfied by some group of solutions of a given Abel differential equation. To do so we exploit the canonical form (2.3.3). We first of all need to define the notion of hyper-cross-ratio

Definition 2.3.11 *Let $2n$, $n \geq 2$ distinct quantities y_1, y_2, y_3, $\cdots$, y_{2n} of $\mathbb{P}^1(\mathbb{C})$ and consider all the products*

$$(y_{k_1} - y_{\mu_1})(y_{k_2} - y_{\mu_2}) \cdots (y_{k_n} - y_{\mu_n}),$$

where $\{k_1, k_2, \cdots, k_n\}$ and $\{\mu_1, \mu_2, \cdots, \mu_n\}$ form a partition of $\{1, \cdots, 2n\}$ such that

$$\mu_1 < k_1, \mu_2 < k_2, \cdots, \mu_n < k_n.$$

An hyper-cross-ratio will be the quotient of two such distinct products. They are invariant under the action of Möbius transformations on the y_is. When $n = 2$ an hyper-cross-ratio is simply a cross-ratio of four distinct points.

Consider the projective group $(PGL(2,\mathbb{C}))$ of the projective line $\mathbb{P}^1(\mathbb{C})$ and let $[n;\mathbb{C}]$ denote for $n \in \mathbb{N}^\star$, the set of all ordered n-tuples $(A_1, \cdots, A_n)$; where the A_is are distinct points of $\mathbb{P}^1(\mathbb{C})$. Let I be an arbitrary non empty set; given a map

$$\mu : [n;\mathbb{C}] \to I,$$

we call the pair (I,μ) a n-point invariant of $PGL(2,\mathbb{C})$ if and only if for $\gamma \in PGL(2,\mathbb{C})$ and $(A_1,\cdots,A_n) \in [n;\mathbb{C}]$, one has

$$\mu(\gamma(A_1),\cdots,\gamma(A_n)) = \mu(A_1,\cdots,A_n).$$

There is an equivalence relation between n-point invariants given by

$$(I,\mu) \sim (I',\mu') \Longleftrightarrow \exists \alpha : \mu([n;\mathbb{C}]) \to \mu'([n;\mathbb{C}]) : \alpha \circ \mu = \mu'.$$

We would like to characterize all n-point invariants for $n \geq 4$. Let $m = n-3$ and consider the set E_m of all $m-$tuples $(a_1,\cdots,a_m)$, with $a_\nu \in \mathbb{C} - \{0,1\}$. Now for $(P_1,\cdots,P_4) \in [4;\mathbb{C}]$, we set

$$\begin{pmatrix} P_1 & P_2 \\ P_4 & P_3 \end{pmatrix} := (P_2 - P_3)^{-1}(P_1 - P_3)(P_1 - P_4)^{-1}(P_2 - P_4);$$

and by

$$\begin{bmatrix} A_1 & A_2 \\ A_4,\cdots,A_n & A_3 \end{bmatrix},$$

we understand the element

$$\left(\begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix},\cdots,\begin{pmatrix} A_1 & A_2 \\ A_n & A_3 \end{pmatrix}\right)$$

of E_m for $(A_1,\cdots,A_n) \in [n;\mathbb{C}]$ and $n \geq 4$. We have following [6]

Theorem 2.3.12 ([6]) *Let $n \geq 4$ be an integer and*

$$\Omega : E_m \to I$$

an arbitrary map. Define

$$\mu(A_1,\cdots,A_n) := \Omega\left(\begin{bmatrix} A_1 & A_2 \\ A_4,\cdots,A_n & A_3 \end{bmatrix}\right)$$

for $(A_1,\cdots,A_n) \in [n;\mathbb{C}]$. Then (I,μ) is a $n-$point invariant and there are no others.

As a consequence the hyper-cross-ratios of $2n$ things are $2n$-point invariants (they are invariant under all Möbius transformations); they are expressible in a certain sense in terms of $2n-3$ ordinary cross-ratios.

Let $S_0 = \dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$ be a particular hyper-cross-ratio for $n = 2$ of four particular solutions of the Abel differential equation (2.3.3); deriving it with respect to x we get

$$S_0' = (z_1 z_2 + z_3 z_4 - z_1 z_3 - z_2 z_4) S_0.$$

Let

$$S = \frac{(z_1 - z_2)(z_3 - z_4)(z_5 - z_6)}{(z_2 - z_3)(z_4 - z_5)(z_1 - z_6)},$$

be a particular hyper-cross-ratio for $n = 3$. Deriving we find (from the canonical form $z_i' - z_j' = z_i^3 - z_j^3$),

$$S' = (z_1 z_2 + z_3 z_4 + z_5 z_6 - z_2 z_3 - z_4 z_5 - z_1 z_6) S.$$

We remark that $S = k$, with k constant if and only if

$$z_1 z_2 + z_3 z_4 + z_5 z_6 - z_2 z_3 - z_4 z_5 - z_1 z_6 = 0.$$

The result generalizes itself for any hyper-cross-ratios.

2.4 The degenerate case $c_3 \equiv 0$

We would like to mimic the invariant theory of equation (2.1.1), that we have done previously. After all as previously noticed a Riccati equation is an equation of the form (2.1.1) where c_3 is zero. One remarks that the function V used in the building of the canonical form in case of equation (2.1.1) diverges to infinity or is not defined when $c_3 \equiv 0$. What we will show is that one can get an invariant theory for the Riccati equation by starting directly from the general Riccati equation itself. Then one will be able to interpret the invariants which appear in the Riccati setting as some kind of 'semi-invariants' of the Appell setting.

Consider a general Riccati equation

$$y' = c_0(x) + 2c_1(x)y + c_3 y^2 \qquad (2.4.1)$$

One could be tempted to analyze it under the same set of transformations as previously, ie under transformations

$$y = \eta(\xi)u(x) + \nu(x); \quad \frac{d\xi}{dx} = \mu(x).$$

Computing, one gets

$$\frac{d\eta}{d\xi} = \frac{c_0 + 2c_1\nu + c_2\nu^2 - \nu'}{\mu u} + 2\left(\frac{c_1}{\mu} + \frac{c_2\nu}{u} - \frac{u'}{2\mu u}\right)\eta + c_2\frac{u}{\mu}\eta^2. \tag{2.4.2}$$

We next try to determine the canonical form using the same method as for the case of Abel. We first of all set

$$y = YU(x) + V; \quad \frac{dX}{dx} = M(x),$$

and require that the coefficient of Y vanishes and that the coefficient of Y^2 is equal to 1. But we have three unknowns and only two equations which gives us therefore an overdetermined system of equations. The values we get for the unknowns are expressed in terms of V and we have a family $(U(V), M(V), V)$. We from now on set $\nu \equiv 0$; getting therefore the transformations

$$y = \eta(\xi)u(x); \quad \frac{d\xi}{dx} = \mu(x). \tag{2.4.3}$$

This choice is not unnatural. As we know (see for instance [62, chap.1], $y = \eta(\xi)u(x); \quad \frac{d\xi}{dx} = \mu(x)$, is the most general point transformation preserving the general linear differential equation of order n); and the Riccati equation is indeed deeply connected with the second order linear differential.

Our expression for equation (2.4.2) greatly simplifies becoming,

$$\frac{d\eta}{d\xi} = \frac{c_0}{\mu u} + 2\left(\frac{c_1}{\mu} - \frac{u'}{2\mu u}\right)\eta + c_2\frac{u}{\mu}\eta^2. \tag{2.4.4}$$

If we denote by δ_i the natural coefficients of (2.4.4), one sees that

$$\begin{aligned} \delta_0 &= \frac{c_0}{\mu u} \\ \delta_1 &= \frac{c_1}{\mu} - \frac{u'}{2\mu u} \\ \delta_2 &= \frac{uc_2}{\mu}. \end{aligned} \tag{2.4.5}$$

With the same requirements as previously (the vanishing of the coefficient of y and the fact the coefficient of y^2 equals to 1), one immediately gets the canonical form

$$\frac{dY}{dX} = Y^2 + Q; \tag{2.4.6}$$

with $U = e^{2\int c_1 dx}$, $M = c_2 U$ and $Q = \dfrac{c_0}{c_2 U^2}$.

Remark 2.4.1 *A look at formulas* (2.3.2) where we set $c_3 \equiv 0$ and $\nu \equiv 0$, gives us

$$\begin{aligned} \gamma_0 &= \frac{c_0}{\mu u} \\ \gamma_1 &= \frac{c_1}{\mu} - \frac{u'}{3\mu u} \\ \gamma_2 &= \frac{uc_2}{\mu}. \end{aligned} \tag{2.4.7}$$

which amounts to the expressions in equation (2.4.5) modulo a rescaling

$$y' = c_0 + 3c_1 y + 3c_2 y^2.$$

We now consider $J = \dfrac{c_0}{c_2 U^2}$, $U = e^{2\int c_1 dx}$ and X such that $\dfrac{dX}{dx} = c_2 U$. One has with the conventions in the last section:

$$U_1 := e^{2\int \gamma_1 d\xi} = e^{2\int \frac{c_1}{\mu}\mu dx} e^{-\frac{u'}{2u\mu}\mu dx} = \frac{U}{u}.$$

As $\gamma_0 = \dfrac{c_0}{u\mu}$ and $\gamma_2 = \dfrac{c_2 u}{\mu}$, a computation gives that $J_1 = J$. One also has that

$$\frac{dX_1}{d\xi} = \gamma_2 U_1 \Rightarrow X_1 = X.$$

Therefore all the $\dfrac{d^k Q}{dX^k}$ are absolute invariants.

One has that

$$Q = \frac{c_0}{c_2 U^2} := \frac{I_1}{c_2 U^2}, \quad \text{with} \quad U = e^{2\int c_1 dx}.$$

Now if we compute $\dfrac{dQ}{dX}$, we find

$$\frac{dQ}{dX} = \frac{\left[c_2 \dfrac{dc_0}{dx} - c_0 \left(\dfrac{dc_2}{dx} + 4c_1 c_2\right)\right]}{c_2^3 U^3} := \frac{I_3}{c_2^3 U^3}.$$

We now define I_{2k+1}, for $k \geqslant 1$ by the following identity:

$$I_{2k+1} := c_2 \frac{dI_{2k-1}}{dx} - \left((2k-1)\frac{dc_2}{dx} + (2k+2)c_1 c_2\right) I_{2k-1},$$

One then has that

Proposition 2.4.2
$$\frac{d^kQ}{dX^k} = \frac{I_{2k+1}}{c_2^{2k+1}U^{k+2}}.$$
and
$$(I_{2k+1})_1 = \frac{u^{k-1}}{\mu^{2k+1}} I_{2k+1}.$$

Proof. Indeed let's take as recurrence hypothesis that the previous formula is true for all $k \geqslant 1$ then

$$\frac{d^{k+1}Q}{dX^{k+1}} := \frac{(c_2^{2k+1}U^{k+2})\dfrac{dI_{2k+1}}{dX}}{c_2^{4k+3}U^{2k+5}} - \frac{I_{2k+1}\left((2k+1)\dfrac{dc_2}{dx}c_2^{2k}U^{k+2} + 2(k+2)c_1c_2^{2k+1}U^{k+2}\right)}{c_2^{4k+3}U^{2k+5}};$$

simplifying one obtains

$$\frac{d^{k+1}J}{dX^{k+1}} = \frac{\left[c_2\dfrac{dI_{2k+1}}{dX} - I_{2k+1}\left((2k+1)\dfrac{dc_2}{dx} + 2(k+2)c_1c_2\right)\right]}{c_2^{2k+3}U^{k+3}};$$

which is the same as $\dfrac{I_{2k+3}}{c_2^{2k+3}U^{k+3}}$. The I_{2k+1} are of degree $2k+1$ and weight $3k$, for all $k \geqslant 0$. Moreover

$$\frac{(I_{2k+1})_1}{\gamma_2^{2k+1}U_1^{k+2}} = \frac{(I_{2k+1})_1}{\left(\dfrac{u}{\mu}\right)^{2k+1} c_2^{2k+1}\dfrac{U^{k+2}}{u^{k+2}}} = \frac{I_{2k+1}}{c_2^{2k+1}U^{k+2}}.$$

■

Remark 2.4.3 *A look at the definition of*

$$I_{2k+1} := c_2\frac{dI_{2k-1}}{dx} - \left((2k-1)\frac{dc_2}{dx} + (2k+2)c_1c_2\right) I_{2k-1}$$

shows that one has again an operator namely

$$c_2\frac{d}{dx} - ((2k-1)\frac{dc_2}{dx} + (2k+2)c_1c_2),$$

which sends relative invariants of degree $2k-1$ *to relative invariants of degree* $2k+1$*, for all* $k \geqslant 1$*. Using formulas* (2.4.5), one finds with a method analog to a previously used one that

$$\frac{d}{d\xi} - \frac{(2k-1)\dfrac{d\gamma_2}{d\xi} + 2(k+1)\gamma_1\gamma_2}{\gamma_2} = \frac{1}{\mu}\left(\frac{d}{dx} - \frac{(2k-1)\dfrac{dc_2}{dx} + 2(k+1)c_1c_2}{c_2}\right) - \frac{2k-1}{\mu}\frac{d}{dx}\left(\frac{u}{\mu}\right) + (k+1)\frac{u'}{u\mu}. \tag{2.4.8}$$

Remark 2.4.4 *We know that a Riccati equation* (2.4.1) is intimately related to the second order linear ordinary differential equation in the following manner. Set in

$$\frac{dY}{dX} = Y^2 + Q,$$
$$Y = -\frac{S'}{S}.$$

Then it becomes

$$S'' = -QS.$$

If we define as before the relative and absolute invariants of a second linear ordinary differential equation as functions of its coefficients and their derivatives which satisfy a certain multiplier property when submitted to transformations (2.4.3), then all invariants of the Riccati equation are also invariants of the associated second order ODE and conversely. One also sees that Q is only the Laguerre invariant.

As in section (2.3) we look for the most general substitution (2.4.3) which preserve the canonical equation (2.4.6) and find

$$\begin{aligned} X_3 &= \frac{1}{K}(X+h) \\ Y_3 &= KY, \end{aligned} \tag{2.4.9}$$

with K, h constants. The last transformations are generated as a group of substitutions on the variables (X,Y) by $\tau_h^X : (X,Y) \to X+h$ and $H_K^Y : (X,Y) \to KY$. Also the expressions: $\dfrac{d^iQ}{dX^i}$, $i \geq 0$, transform as

$$\frac{d^iQ_3}{dX_3^i} = K^{i+2}\frac{d^iQ}{dX^i}. \tag{2.4.10}$$

We have the following proposition

Proposition 2.4.5 *The substitutions* (2.4.9) and (2.4.10), admit the following infinitesimal generators

$$T = \partial_X$$

and

$$Z = -X\partial_X + Y\partial_Y + \sum_{i\geqslant 0}(i+2)Q_i\partial_{Q_i}, \quad \textit{with} \quad Q_i := \frac{d^iQ}{dX^i}$$

And therefore an invariant I in its canonical form must satisfy

$$T(I) \equiv 0, \quad \sum_{i\geqslant 0}(i+2)Q_i\partial_{Q_i}(I) \equiv 0.$$

As for the classes of canonical forms

$$\frac{dJ}{dX} = Y^2 + Q,$$

One finds with an analog reasoning to an already used one (section (2.3)), that they fulfill an identity of the form

$$g(Q, Q', Q'') \equiv 0.$$

Remark 2.4.6 *All invariants of the Riccati equation under transformations* (2.4.3) are "semi-invariants" for equation (2.1.1); which is a normal fact. They only preserve the invariant property for the particular substitution (2.4.3) of the form (2.1.6).

In the next chapter we do a more algebraic study of the Riccati equation.

3 Certain algebraic equations of degree four and five

3.1 Introduction

We proceed in this chapter to the study of some particular properties of the Riccati equation which is a differential equation of the form mentionned above. We consider an algebraic irreducible and separable equation $h_n(x)$ of degree four or five,

$$h_n(x) = x^n + a_1(t)x^{n-1} + \cdots + a_n = 0 \tag{3.1.1}$$

where the a_i reside in the function field $\mathbb{F} = \mathbb{C}(t)$. This field is naturally endowed with the derivation $\frac{d}{dt}$, which acts trivially on $\mathbb{C}$; ie $\mathbb{C}$ is the field of constants for this derivation.

Suppose all the roots of h_n are solutions of a same Riccati equation

$$u'(t) = \mathfrak{B}_0(t) + \mathfrak{B}_1(t)u + \mathfrak{B}_2(t)u^2, \tag{3.1.2}$$

where $\mathfrak{B}_0(t)$, $\mathfrak{B}_1(t)$, $\mathfrak{B}_2(t)$ are in $\mathbb{F}$.

The cross-ratio of any four solutions x_i ([47]) will be a constant and we will call such equations, anharmonic equations. This fundamental property leads to profound implications; it is indeed by studying the different values taken by the cross-ratio of four roots that one will be able to effectively parameterize the anharmonics. We recover the main result of Autonne [4]

Theorem 3.1.1 • *The Galois group of $h_n(x)$ is isomorphic to a finite subgroup S of $PGL(2,\mathbb{C})$, described in theorem* (3.2.10).

- *Every $h_n(x)$ is of the form $F(x,T)=0$, where the polynomial in two arguments F has numerical coefficients which depend only on S and T is in $\mathbb{F}$.*

The fact that we are dealing with an anharmonic equation intervenes a posteriori. The second set of results gives a more thorough description of the anharmonics:

- *The fourth degree equation*
  - *The first type of h_4 is*

$$X^4+6TX^2+4TX-3T^2=0.$$

    The cross-ratio K of four roots is equianharmonic, ie belongs to $\Omega_1=\{-\rho,-\rho^2\}$ with ρ a primitive third root of unity. G is the group $\mathfrak{U}_4$ or A_4, the alternating group between four letters or again the tetrahedral subgroup of $PGL(2,\mathbb{C})$.
  - *The second type of h_4 is*

$$X^4+6TX^2+4TX+T(T+1)=0.$$

    The cross-ratio is harmonic ie belongs to $\Omega_2=\left\{-1,2,\frac{1}{2}\right\}$. G is isomorphic to $\mathfrak{D}_4$, the dihedral subgroup of $PGL(2,\mathbb{C})$. The latter is generated by the cyclic substitution $\theta_4=|Z,\theta.Z|$ of order four and by the inversion substitution $\epsilon_0=|Z,\frac{1}{Z}|$. θ satisfies $\theta^4=1$.
  - *Finally the third type of h_4 is*

$$(X^2+q)^2-p(2X-1)^2=0,$$

    where we have set

$$\frac{1}{p}=(1-T)^2-(1+T)^2\left(\frac{K-1}{K+1}\right)^2$$

    and

$$\frac{1}{q}=\frac{1}{pT}.$$

    G is isomorphic to $\mathfrak{K}$, the Klein group.

- *The fifth degree equation*

– *In this case there is only one type of h_5; it is obtained from the elimination of ξ between the two equations*

$$X = \frac{\xi - 2}{\xi(\xi - 3) + 1}, \quad \xi(\xi^2 - 5\xi + 5)^2 = T;$$

and its Galois group G is isomorphic to the dihedral group $\mathfrak{D}_5$.

The interest in the Riccati equation has always been huge; it is the only first order differential equation (solved in the first derivative) up to fractional linear transformations of the dependent variable and holomorphic transformation of the independent variable, possessing the Painlevé property. To be more precise, one knows from the Cauchy theorem of existence and unicity of a local solution of a differential equation, that the general solution of a first order differential equation is of the form

$$f(z, C), \quad \text{with } C \text{ an arbitrary constant of } \mathbb{C}.$$

But in general the solution $y(z, C)$ exhibits singular points, meaning points where $y(z, C)$ is not analytical. These singularities are of different types: poles, or possibly less friendly ones called branch points and essential singularities.

In most cases the location of these singularities depends on the constant C. The Riccati equation is the only first order equation whose only movable singularities are poles. One may consult [24] for the latter subject.

Another aspect of the theory of Riccati equations is the linearization property. In fact the substitution

$$u = -\frac{z'}{\mathfrak{B}_2 z},$$

changes the Riccati equation into a second order linear one. Therefore the particular Riccati equation

$$\frac{du}{dt} = u^2 + Q,$$

is related in that way to the first dimensional Schrödinger equation.

$$\frac{d^2 z}{dt^2} = -Qz.$$

Nowadays the Riccati equation is most notably used in the field of linear differential Galois theory of the second order linear differential equation [51, 52, 61] and in models for physical problems. One can have a look at the paper [35], where the Riccati equation is related to Newton laws and to a model for the expansion of the Universe (more precisely the authors relate their model to the Friedmann-Robertson-Lemaître cosmology under which the expansion of the Universe under uniform pressure (baratropic) obeys a Riccati equation).

This chapter is structured as follows: in section 3.2 we give the theoretical foundation necessary in order to read this chapter smoothly, then in section 3.3 we first reprove in its great lines the theorem of Picard of the constance of the cross-ratio of any quadruple of distinct solutions of a Riccati equation as it is central in the analysis; we then prove the main theorem in section 3.3 and finally in section 3.4, we use the Klein invariants to build the Riccati equations of the question. The main results are in section 3.3 and 3.4.

3.2 Some introductory concepts

3.2.1 Projective space of dimension n

In this introductory section 3.2, we use mainly material taken from [3, chap.27]. For further investigation into the beautiful field of projective geometry, one may check [42].

Let $\mathbb{K}$ be a commutative field of characteristic zero and V be a vector space over $\mathbb{K}$ of dimension $n+1$, with $n \geqslant 1$. The set of lines of V is called the n-dimensional Projective space associated to V. It will be denoted $\boldsymbol{Proj}(V)$. Given an $(n+1)$-dimensional vector space as before and the projective space associated to it, one has a canonical surjection

$$V \setminus \{0_V\} \xrightarrow{\varpi_V} \boldsymbol{Proj}(V)$$
$$x \mapsto \mathbb{K}x.$$

This map enables us to identify $\boldsymbol{Proj}(V)$ with the quotient space of $V \setminus \{0_V\}$ under the equivalence relation, saying that x and y are equivalent if and only if:

$$\exists \lambda \in \mathbb{K}^{\star} = \mathbb{K} \setminus \{O\} : x = \lambda y.$$

This is because an element of $\boldsymbol{Proj}(V)$ which is a line of V, is represented by a non zero vector x and two vectors define the same line of V when they are $\mathbb{K}^{*}$ proportional. We will write $\dim(\boldsymbol{Proj}(V)) = n$.

Let us denote by $GL_{\mathbb{K}}(V)$, the group of linear automorphisms from V to V.

$GL_{\mathbb{K}}(V)$ acts in a natural way on $\boldsymbol{Proj}(V)$. Its action consists in assigning to $(g, M = \varpi_V(x)) \in GL_{\mathbb{K}}(V) \times \boldsymbol{Proj}(V)$ the element $g \cdot M = \varpi_V(g(x))$. The kernel of the morphism from $GL_{\mathbb{K}}(V)$ to the group of permutations or bijections of $\boldsymbol{Proj}(V)$ associated to it, is the group $\mathbb{K}^* Id_V$ of the scaling transformations.

We note by $PGL_{\mathbb{K}}(V)$ the quotient of $GL_{\mathbb{K}}(V)$ by its subgroup $H := \mathbb{K}^* Id_V$. Passing to the quotient level, one gets a faithful (its kernel is the identity) action of $PGL_{\mathbb{K}}(V)$ on $\boldsymbol{Proj}(V)$. This action is 2-transitive or also doubly-transitive. This means that for any two pairs $(M_1 \neq M_2)$ and $(M_1' \neq M_2')$ of $(\boldsymbol{Proj}(V))^2$, there exists $g \in PGL_{\mathbb{K}}(V)$ such that $g \cdot M_1 = M_1'$ and $g \cdot M_2 = M_2'$.

The permutations or bijections of $\boldsymbol{Proj}(V)$ defined by the elements of $PGL_{\mathbb{K}}(V)$ will be called the homographies of $\boldsymbol{Proj}(V)$. As the action by homography is a faithful one, one can therefore identify by means of this action, the group $PGL_{\mathbb{K}}(V)$ with the group of homographies of $\boldsymbol{Proj}(V)$. We will make this identification in what follows.

For every automorphism $f \in GL_{\mathbb{K}}(V)$ the corresponding homography in $PGL_{\mathbb{K}}(V)$ will be denoted $f_\star$. From what we have just seen the application

$$\begin{gathered} GL_{\mathbb{K}}(V) \to PGL_{\mathbb{K}}(V) \\ f \mapsto f_\star . \end{gathered}$$

is a surjective group homomorphism of kernel $\mathbb{K}^\star \, \mathrm{Id}_V$.

Given a sequence $(M_1, \cdots, M_{n+2})$ of elements of $\boldsymbol{Proj}(V)$, we will say that it is projectively free when: the M_is are free as elements of V. For instance when $p = 2$, then (M_1, M_2) is projectively free if $M_1 \neq M_2$ (the two lines are not collinear). One will call a projective frame a sequence $(M_1, \cdots, M_{n+2})$, such that any subsequence of $n+1$ terms of it is projectively free.

Theorem 3.2.1 *Given two projective frames $(M_1, \cdots, M_{n+2})$ and $\left(M_1', \cdots, M_{n+2}'\right)$ of $\boldsymbol{Proj}(V)$, there exists a unique $h \in PGL_{\mathbb{K}}(V)$ such that: $h(M_i) = M_i', \forall i \in [\![1, n+2]\!]$.*

Proof. For all $i \in [\![1, n+2]\!]$, let us fix $e_i \in M_i \setminus \{0_V\}$ and $e_i' \in M_i' \setminus \{0_V\}$. By hypothesis the sequences $\beta = (e_1, \cdots, e_{n+1})$ and $\beta' = (e_1', \cdots, e_{n+1}')$ are basis of V. Let $(\lambda_i)_{1\leqslant i\leqslant n}$ (respectively $(\lambda'_i)_{1\leqslant i\leqslant n}$) be the sequence of coordinates of e_{n+2} in β (respectively e'_{n+2} in β'). Using the hypothesis one sees that λ_i as well as $\lambda'_i \neq 0$ for all $i \in [\![1, n+1]\!]$.

This is because if we assume for instance that some λ_i vanishes, we get, as $e_{n+2} = \sum_{j\geqslant 1} \lambda_j e_j$ and the M_is form a projective frame, that $e_{n+2} = 0$, a consequence of the fact that one has a dependence relation between $n+1$ non-vanishing independent vectors; this is a contradiction.

Now for any $(n+1)$-tuple $\alpha = (\alpha_1, \cdots, \alpha_{n+1}) \in (\mathbb{K}^\star)^{n+1}$, we consider the unique element $f_\alpha \in GL_{\mathbb{K}}(V)$:

$$f_\alpha(e_i) = \alpha_i e'_i, \quad \forall i \in [\![1, n+1]\!].$$

One sees that the homography $h \in PGL_{\mathbb{K}}(V)$ will verify $h(M_i) = M'_i$ for all $i \in [\![1, n+1]\!]$ if and only if it is one of the $(f_\alpha)_\star$. Now for every fixed α one has:

$$(f_\alpha)_\star(M_{n+2}) = M'_{n+2}, \text{if and only if there exists } \rho \in \mathbb{K}^\star :$$

$$(f_\alpha)_\star(e_{n+2}) = \rho e'_{n+2}.$$

This means also that $\alpha_i\lambda_i = \rho\lambda'_i, \quad \forall i \in [\![1, n+1]\!]$. The set of $\alpha \in (\mathbb{K}^\star)^{n+1}$ which verify this condition is therefore a line of $(\mathbb{K})^{n+1}$ of direction $\lambda = \left(\frac{\lambda'_i}{\lambda_i}\right)_{1\leqslant i\leqslant n+1}$. Hence the required f_αs form a line and this gives us a unique homography $h = (f_\alpha)_\star$, answering to the question. ∎

A sequence of p elements $(M_1, \cdots, M_p)$ of $\boldsymbol{Proj}(V)$ is said to be p-injective, if the M_is are pairwise distinct. One has the following corollary.

Corollary 3.2.2 *Assume that* $n = 1$ (ie $\dim V = 2$). Then $\boldsymbol{Proj}(V)$ for any two 3-injective sequences (M_1, M_2, M_3) and (M'_1, M'_2, M'_3) of $\boldsymbol{Proj}(V)$, there exists a unique $h \in PGL_{\mathbb{K}}(V) : h(M_i) = M'_i \quad \forall i \in [\![1, 3]\!]$.

3.2.2 Homogeneous Coordinates

Let $\beta = (e_1, \cdots, e_{n+1})$ be a basis of V and $M \in \boldsymbol{Proj}(V)$. The set $C_H(M)$ of the $(n+1)$-tuples $\lambda = (\lambda_i)_{1\leqslant i\leqslant n} \in (\mathbb{K}^\star)^{n+1} \setminus \{0\}$ such that $\varpi_V(\sum_{1\leqslant i\leqslant n+1} \lambda_i e_i) = M$, is a line of $(\mathbb{K})^{n+1}$ without the origin 0. We will call any such a $\lambda = (\lambda_1, \cdots, \lambda_{n+1})$ of $C_H(M)$, a system of homogeneous coordinates of M relatively to the basis β.

Consider now $h \in PGL_{\mathbb{K}}(V)$, an homography ie the f in $GL_{\mathbb{K}}(V)$ such that $f_* = h$ form a line of $Hom_{\mathbb{K}}(V)$ without the origin. The corresponding set of the matrices of these automorphisms is therefore a line of $\mathfrak{m}_{n+1}$ (the $(n+1)$-dimensional matrices). Any matrix of this type is called a matrix of f in β.

If H is any such a matrix and M, $M^{'}$ are two points of $\boldsymbol{Proj}(V)$ of respective homogeneous coordinates $\chi = (x_1, \cdots, x_n)$, $\chi' = (x'_1, \cdots, x'_n)$, one has then the following equivalence:

$$M' = h(M) \iff (\chi') = \rho \left(H \chi^t \right), \tag{3.2.1}$$

where $\rho \in \mathbb{K}^\star$, depends on H, χ and χ'.

3.2.3 Linear Projective Subvarieties

Definition 3.2.3 *With the preceding notations and hypothesises, let* $d \in [\![0, n]\!]$. *One calls a sub-variety of dimension* d *of* $\boldsymbol{Proj}(V)$ *any set of the form* $\mathcal{W} = \varpi_V(W \setminus \{0_V\})$, *where* W *is a sub-vector space of dimension* $d+1$ *of* V. *The linear projective sub-variety is called a projective line when* $d = 1$*; a projective plane when* $d = 2$*; and a projective hyperplane when* $d = n - 1$.

Given a linear projective sub-variety $\mathcal{W}$, there exists a unique sub-vector space of V verifying the definition above; it is precisely $W = \{0_V\} \bigcup \varpi_V^{-1}(\mathcal{W})$. We call it the sub-vector space associated to $\mathcal{W}$.

The linear projective sub-varieties of 0 dimension are the singletons and the unique sub-variety of dimension n is $\boldsymbol{Proj}(V)$. The empty set will be considered to be a particular linear projective sub-variety. Its associated sub-vector space will be $\{0_V\}$ and its dimension -1.

3.2.4 Isomorphisms between Projective Spaces

Let V_1 and V_2 be two vector $\mathbb{K}$ vector spaces of same dimension $n+1$, $n \geqslant 1$. Any isomorphism of $\mathbb{K}$ vector spaces $f : V_1 \to V_2$, induces a bijection $f_\star : \boldsymbol{Proj}(V_1) \to \boldsymbol{Proj}(V_2)$. Every isomorphism of this form $f_\star$ is called a projective isomorphism from $\boldsymbol{Proj}(V_1)$ to $\boldsymbol{Proj}(V_2)$. One also says that there is an homographic correspondence between the two projective spaces, instead of projective isomorphism. The identity map of a projective space is a projective isomorphism.

The product of two projective isomorphisms $u : \boldsymbol{Proj}(V_1) \to \boldsymbol{Proj}(V_2)$ and $v : \boldsymbol{Proj}(V_2) \to \boldsymbol{Proj}(V_3)$, $v \circ u$ is a projective isomorphism. Also the inverse map of a projective isomorphism remains one. When $V_1 = V_2$ one sees that the set of projective isomorphisms of $\boldsymbol{Proj}(V)$ on itself is the group $PGL_{\mathbb{K}}(V)$ of homographies of $\boldsymbol{Proj}(V)$.

Theorem 3.2.1 or a construction similar to it shows immediately that given two projective frames $(M_i)_{1 \leqslant i \leqslant n+2}$ and $(N_i)_{1 \leqslant i \leqslant n+2}$ of $\boldsymbol{Proj}(V_1)$ respectively $\boldsymbol{Proj}(V_2)$, there exists a unique projective isomorphism

$$u : \boldsymbol{Proj}(V_1) \to \boldsymbol{Proj}(V_2) : \quad u(M_i) = N_i, \quad \forall \in [\![1, n+2]\!].$$

Take now a finite dimensional $\mathbb{K}$-vector space V of dimension $n+1$, with $n \geqslant 1$ and basis $\beta = (e_1, \cdots, e_{n+1})$. The linear bijection

$$\begin{aligned} f_\beta : \mathbb{K}^{n+1} &\to V \\ (x_1, \cdots, x_{n+1}) &\mapsto x_1 e_1 + \cdots + x_{n+1} e_{n+1}, \end{aligned}$$

induces a projective isomorphism

$$\begin{aligned} (f_\beta)_\star : \boldsymbol{Proj}(\mathbb{K}^{n+1}) &\to \boldsymbol{Proj}(V) \\ (x_1, \cdots, x_{n+1}) &\mapsto x_1 e_1 + \cdots + x_{n+1} e_{n+1}. \end{aligned}$$

When we apply this isomorphism, we say that we identify $\boldsymbol{Proj}(V)$ to $\boldsymbol{Proj}(\mathbb{K}^{n+1})$ by means of β. This identification extends itself to the corresponding projective groups. In fact the application

$$\begin{aligned} PGL_{\mathbb{K}}(\mathbb{K}^{n+1}) &\to PGL_{\mathbb{K}}(V) \\ h &\mapsto (f_\beta)_\star \circ h \circ (f_\beta)_\star^{-1}, \end{aligned}$$

establishes a group isomorphism between the two sets.

3.2.5 Some Additional Results about the Projective Line

The Cross-Ratio

$\mathbb{K}$ remains a commutative field. We fix a $\mathbb{K}$-plane V. Corollary (3.2.2) has shown that there existed a unique $PGL_{\mathbb{K}}(V)$-orbit of injective 3-sequences in $\boldsymbol{Proj}(V)$. Let $(M_1, \cdots, M_4)$ be a 4-sequence of $\boldsymbol{Proj}(V)$ such that M_1, M_2,

M_3 are pairwise distinct. From what we have seen in the preceding section, there exists a unique projective isomorphism

$$\Psi : \boldsymbol{Proj}(V) \to \widetilde{\mathbb{K}} := Proj(K^2) \simeq \mathbb{K} \cup \{\infty_{\mathbb{K}}\}$$

such that $\Psi(M_1) = 0_{\mathbb{K}}$; $\Psi(M_2) = \infty_K$ and $\Psi(M_3) = 1_{\mathbb{K}}$.

Remark 3.2.4 *One can identify $\boldsymbol{Proj}(\mathbb{K}^2)$ with $\mathbb{K} \cup \{\infty_{\mathbb{K}}\}$ by means of the following map*

$$\begin{aligned} \boldsymbol{Proj}(\mathbb{K}^2) &\to \mathbb{K} \cup \{\infty_{\mathbb{K}}\} \\ M = \varpi_{\mathbb{K}^2}(X, Y) &\mapsto \frac{X}{Y}. \end{aligned}$$

The last isomorphism Ψ between $\boldsymbol{Proj}(V)$ and $\widetilde{\mathbb{K}}$ sends M_4 on an element $\lambda \in \mathbb{K} \setminus \{0_{\mathbb{K}}, 1_{\mathbb{K}}\}$. One sets:

Definition 3.2.5 *Under the above conditions the element $\Psi(M_4)$ of $\mathbb{K}$, is called the cross-ratio of the sequence (M_1, M_2, M_3, M_4) and is denoted $Brp(M_1, M_2, M_3, M_4)$.*

In the following to unburden the notation we will note the cross-ratio of any four above points as $Brp(M_i)_{1\leq i\leq 4}$.

Consider the natural left action of $PGL_{\mathbb{K}}(V)$ on $\mathcal{B}_4(\boldsymbol{Proj}(V))$: the set of the 4-sequences (M_1, M_2, M_3, M_4) of $\boldsymbol{Proj}(V)$ such that M_1, M_2, M_3 are pairwise distinct. This latter action is regular meaning for any two $M = (M_1, M_2, M_3, M_4)$ and $N = (N_1, N_2, N_3, N_4) \in \mathcal{B}_4(\boldsymbol{Proj}(V))$, there exists a unique $h \in PGL_{\mathbb{K}}(V) : h(M_i) = N_i, \forall i \in [\![1, 4]\!]$. The cross-ratio enables us to characterize the orbits of this action.

Proposition 3.2.6

- *Let be given in $\boldsymbol{Proj}(V)$ two elements $(M_i)_{1\leqslant i\leqslant 4}$ and $(N_j)_{1\leqslant j\leqslant 4}$ of $\mathcal{B}_4(\boldsymbol{Proj}(V))$. The condition: "there exists $h \in PGL_{\mathbb{K}}(V)$ such that $h(M_i) = N_i, \forall i \in [\![1, 4]\!]$ " is equivalent to:*

$$Brp(M_i)_{1\leq i\leq 4} = Brp(N_i)_{1\leq i\leq 4}.$$

- *Let $\lambda \in \mathbb{K} \cup \{\infty_{\mathbb{K}}\} \simeq \widetilde{\mathbb{K}}$; then there exists at least one 4-sequence $(M_i)_{1\leq i\leq 4}$ belonging to $\mathcal{B}_4(\boldsymbol{Proj}(V))$ such that: $Brp(M_i)_{1\leq i\leq 4} = \lambda$* ($Brp$ as birapport).

Proof.

- Let Ψ, respectively θ, be the projective isomorphisms from $\boldsymbol{Proj}(V)$ to $\widetilde{\mathbb{K}}$ which sends (M_1, M_2, M_3) respectively (N_1, N_2, N_3) to $(0_{\mathbb{K}}, \infty_{\mathbb{K}}, 1_{\mathbb{K}})$. One has $\eta = \theta^{-1} \circ \Psi \in PGL_{\mathbb{K}}(V)$. Moreover if $\Psi(M_4) = \theta(N_4)$, ie the cross-ratio associated to the sequence $(M_i)_{1 \leq i \leq 4}$ is equal to the one given by the sequence $(N_i)_{1 \leq i \leq 4}$, one sees that η transforms the sequence $(M_i)_{1 \leqslant i \leqslant 4}$ into the sequence $(N_i)_{1 \leqslant i \leqslant 4}$.

  Conversely if there exists $h \in PGL_{\mathbb{K}}(V)$ which sends $(M_i)_{1 \leq i \leq 4}$ into $(N_i)_{1 \leq i \leq 4}$ then $\theta \circ h = \Psi$, because as corollary 3.2.2 shows it, there exists a unique projective isomorphism sending (M_1, M_2, M_3) to $(0_{\mathbb{K}}, \infty_{\mathbb{K}}, 1_{\mathbb{K}})$. So $h = \eta$, hence $\Psi(M_4) = \theta \circ h(M_4) = \theta(N_4)$. We have therefore proved the part the first part of the proposition.

- Let's prove now the second assertion. We first of all choose a basis (e_1, e_2) of V. Next we set $M_1 = \varpi_V(e_2)$, $M_2 = \varpi_V(e_1)$, $M_3 = \varpi_V(e_1 + e_2)$ and $M_4 = \varpi_V(\lambda e_1 + e_2)$. One verifies that $Brp(M_i)_{1 \leq i \leq 4}$ equals to λ, by using the remark previously given (one identifies $\boldsymbol{Proj}(V)$ with $Proj(\mathbb{K}^2)$) and the corollary 3.2.2.

$\blacksquare$

Let $\beta = (e_1, e_2)$ be a basis of V and $(M_i)_{1 \leq i \leq 4}$ an element of $\mathcal{B}_4(\boldsymbol{Proj}(V))$. For all $i \in [\![1, 4]\!]$ let (X_i, Y_i) be an homogeneous coordinate system of M_i in β. We are going to give the explicit value of $\lambda = Brp(M_i)_{1 \leq i \leq 4}$. For all $(i, j) \in [\![1, 4]\!]^2$ we denote by $\Delta_{i,j}$ the determinant $X_i Y_j - X_j Y_i$. By assumption $\Delta_{2,3}$, $\Delta_{3,1}$ and $\Delta_{1,2}$ are non zero (because the lines M_1, M_2 and M_3 are supposed to be non-collinear for any element of $\mathcal{B}_4(\boldsymbol{Proj}(V))$).

Let $\theta_1 = \varpi_V(e_2)$, $\theta_2 = \varpi_V(e_1)$ and $\theta_3 = \varpi_V(e_1 + e_2)$. The definitions show that the homography h which sends (M_1, M_2, M_3) on $(\theta_1, \theta_2, \theta_3)$ sends also M_4 on $\varpi_{\mathbb{K}}(\lambda e_1 + e_2)$ if $\lambda \neq \infty_{\mathbb{K}}$ and on $\varpi_V(e_1)$ if $\lambda = \infty_{\mathbb{K}}$. Let Ψ_1 denote the homography associated with the element $(\theta_i)_{1 \leq i \leq 3}$ sending θ_1 on $0_{\mathbb{K}}$, θ_2 on $\infty_{\mathbb{K}}$ and θ_3 on $1_{\mathbb{K}}$.

If we take the composition of h and Ψ_1: $\Psi_1 \circ h$, we see that it sends M_1 on $0_{\mathbb{K}}$, M_2 on $\infty_{\mathbb{K}}$ and M_3 on $1_{\mathbb{K}}$. Hence by definition and corollary (3.2.2) $\Psi_1 \circ h$ sends M_4 on λ. Taking the inverse one sees that h sends M_4 on $\varpi_V(\lambda e_1 + e_2)$ if $\lambda \neq \infty_{\mathbb{K}}$ and on $\varpi_V(e_1)$ otherwise.

A matrix H of h as above ie one matrix of an element $f \in GL_{\mathbb{K}}(V) : f_* = h$ is

$$H = \begin{pmatrix} -Y_1\Delta_{2,3} & X_1\Delta_{2,3} \\ -Y_2\Delta_{1,3} & X_2\Delta_{1,3} \end{pmatrix}. \tag{3.2.2}$$

One can verify this by computing

$$H\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \quad \forall i \in \{1,2,3\}.$$

Hence a representative system of homogeneous coordinates of $h(M_4)$ in β is given by $(\Delta_{1,4}\Delta_{2,3}, \Delta_{1,3}\Delta_{2,4})$. Therefore the cross-ratio $Brp(M_i)_{1\leq i\leq 4}$ of four points is

$$\frac{\Delta_{1,4}\Delta_{2,3}}{\Delta_{1,3}\Delta_{2,4}} = \begin{cases} \dfrac{\begin{vmatrix} X_1 & X_4 \\ Y_1 & Y_4 \end{vmatrix} \cdot \begin{vmatrix} X_2 & X_3 \\ Y_2 & Y_3 \end{vmatrix}}{\begin{vmatrix} X_1 & X_3 \\ Y_1 & Y_3 \end{vmatrix} \cdot \begin{vmatrix} X_2 & X_4 \\ Y_2 & Y_4 \end{vmatrix}} & \text{if } M_2 \neq M_4 \\ \infty_{\mathbb{K}} & \text{otherwise.} \end{cases} \tag{3.2.3}$$

Assume now that the $(M_i)_{1\leq i\leq 4}$ are all distinct from θ_2 ie $Y_i \neq 0_{\mathbb{K}}$ for all i and set

$$x_i = \frac{X_i}{Y_i}, \quad \forall i.$$

Formula (3.2.3) then becomes:

$$Brp(M_1, M_2, M_3, M_4) = \frac{x_4 - x_1}{x_4 - x_2} \div \frac{x_3 - x_1}{x_3 - x_2}. \tag{3.2.4}$$

This justifies the name cross-ratio.

Let be two $\mathbb{K}$-planes V_1 and V_2. One says that a bijection

$$f : \boldsymbol{Proj}(V_1) \to \boldsymbol{Proj}(V_2)$$

preserves the cross-ratio if and only if one has:

$$Brp(f((M_i))_{1\leq i\leq 4}) = Brp(M_i)_{1\leq i\leq 4},$$

with $(M_i)_{1\leqslant i\leqslant 4} \in \mathcal{B}_4(\boldsymbol{Proj}(V))$. The following proposition is true.

Proposition 3.2.7 *Let be two $\mathbb{K}$-planes V_1 and V_2. For a bijection*

$$f : \boldsymbol{Proj}(V_1) \to \boldsymbol{Proj}(V_2)$$

the condition: h is a projective isomorphism is equivalent to h preserves the cross-ratio.

Finite Homography Subgroups of a Projective Line

Let G be a finite subgroup of $PGL_{\mathbb{K}}(V)$, of order or cardinal $n \geqslant 2$. We will call a singular G-orbit, any G-orbit in $\boldsymbol{Proj}(V)$ of cardinal $< n$. The non-singular G-orbits will be called ordinary.

We remind that a finite group is called tetrahedral when it is isomorphic to $\mathfrak{U}_4$, octahedral when it is isomorphic to $\mathfrak{S}_4$, and icosahedral when it is isomorphic to $\mathfrak{U}_5$.

Let $h = f_\star$ be an homography of $\boldsymbol{Proj}(V)$, where $f \in GL_{\mathbb{K}}(V)$. Let $M = \varpi_V(x) \in \boldsymbol{Proj}(V)$, where $x \in V \setminus \{0_V\}$. One has the following:

$$h(M) = M \iff \varpi_V(f(x)) = M \quad \text{ie} \quad \exists \lambda \in \mathbb{K}^* : f(x) = \lambda x;$$

meaning that the set of fix points of h are the union of linear projective sub-varieties whose associated sub-vector spaces are the eigenspaces of f.

For instance when $n = 1$ and $h \neq \mathrm{Id}_{\boldsymbol{Proj}(V)}$ ie $f \notin \mathbb{K}^* Id_V$, then f admits exactly one line (one dimensional space) as eigenspace if its characteristic polynomial $P_f(X)$ is not separable; two lines (two independent one dimensional spaces) as eigenspaces when $P_f(X)$ is separable with simple roots in $\mathbb{K}$ and finally no eigenvectors if $P_f(X)$ is separable and irreducible.

We now give a description of cyclic subgroups of $PGL_{\mathbb{K}}(V)$. One has the following theorem

Theorem 3.2.8

- *Given two distinct points M_1 and M_2 of $\boldsymbol{Proj}(V)$ and $n \in \mathbb{N}^*$. There exists a unique subgroup of cardinal n in the group of homographies $PGL_{\mathbb{K}}(V)$ which fixes M_1 and M_2 and this group is n-cyclic. The singular orbits of this group are $\{M_1\}$ and $\{M_2\}$.*
- *Let G be a finite subgroup of $PGL_{\mathbb{K}}(V)$ which fixes a point M_2 of the projective space $\boldsymbol{Proj}(V)$. Then G fixes a point $M_1 \in \boldsymbol{Proj}(V) \setminus \{M_2\}$ and G is cyclic.*
- *Every cyclic group of $PGL_{\mathbb{K}}(V)$ has two fixed points.*

The finite dihedral or pyramidal subgroups on their side, are characterized by the following theorem

Theorem 3.2.9 *Let G be a finite n-cyclic subgroup of $PGL_{\mathbb{K}}(V)$, with $n \geqslant 3$. Let ω be a G-orbit of cardinal n in $\boldsymbol{Proj}(V)$. There exists a dihedral subgroup of order $2n$ of $PGL_{\mathbb{K}}(V)$ and a unique one containing G as a subgroup* *and such that ω is one of its orbits. This group has two orbits of cardinal n, one of cardinal 2, the remaining ones being ordinary ones.*

Making ω vary, one obtains in this manner all the sub-groups of cardinal $2n$ of $PGL_{\mathbb{K}}(V)$ of which G is a subgroup and such that the elements not in G are involutions (their second power is the identity).

These groups are non-abelian. There are also abelian dihedral subgroups in $PGL_{\mathbb{K}}(V)$. They are all of cardinal 4. Recapitulating one gets

Theorem 3.2.10 *The finite subgroups of $PGL_{\mathbb{K}}(V)$ are the cyclic, the dihedral, the tetrahedral, the octahedral and the icosahedral ones.*

3.3 Fourth and Fifth Degree Equation whose roots satisfy a same Riccati Equation

3.3.1 The Riccati equation, the constance of its cross-ratio

Proposition 3.3.1 *The Riccati equation is the only first order differential order ordinary differential having the cross-ratio of any four arbitrary of its solutions constant.*

Proof. Let us take four arbitrary roots u_1, u_2, u_3, u_4 of the Riccati equation

$$u'(t) = \mathfrak{B}_0(t) + \mathfrak{B}_1(t)u + \mathfrak{B}_2(t)u^2.$$

one therefore gets the system

$$\begin{cases} u_1'(t) = \mathfrak{B}_0(t) + \mathfrak{B}_1(t)u_1 + \mathfrak{B}_2(t)u_1^2 \\ u_2'(t) = \mathfrak{B}_0(t) + \mathfrak{B}_1(t)u_2 + \mathfrak{B}_2(t)u_2^2 \\ u_3'(t) = \mathfrak{B}_0(t) + \mathfrak{B}_1(t)u_3 + \mathfrak{B}_2(t)u_3^2 \\ u_4'(t) = \mathfrak{B}_0(t) + \mathfrak{B}_1(t)u_4 + \mathfrak{B}_2(t)u_4^2. \end{cases} \tag{3.3.1}$$

Thus as

$$(\mathfrak{B}_0(t), \mathfrak{B}_1(t), \mathfrak{B}_2(t)) \neq (0, 0, 0),$$

the following determinant vanishes

$$\begin{vmatrix} \dfrac{du_1}{dt} & u_1^2 & u_1 & 1 \\ \\ \dfrac{du_2}{dt} & u_2^2 & u_2 & 1 \\ \\ \dfrac{du_3}{dt} & u_3^2 & u_3 & 1 \\ \\ \dfrac{du_4}{dt} & u_4^2 & u_4 & 1 \end{vmatrix} = 0.$$

Computation of the latter determinant gives [47]

$$\begin{aligned} &\frac{du_1}{dt}(u_2-u_3)(u_2-u_4)(u_3-u_4)+\frac{du_2}{dt}(u_3-u_4)(u_3-u_1)(u_4-u_1) \\ &+\frac{du_3}{dt}(u_4-u_1)(u_4-u_2)(u_1-u_2)+\frac{du_4}{dt}(u_1-u_2)(u_1-u_3)(u_2-u_3) \\ &=0. \end{aligned}$$

and this is equivalent to the fact that

$$\begin{aligned} &(u_1-u_2)(u_3-u_4)\frac{d}{dt}((u_1-u_4)(u_3-u_2)) \\ &-(u_1-u_4)(u_3-u_2)\frac{d}{dt}((u_1-u_2)(u_3-u_4))=0 \end{aligned}$$

hence the constance of the cross-ratio $\dfrac{(u_1-u_2)(u_3-u_4)}{(u_1-u_4)(u_3-u_2)}$.

The converse is easy computation. One replaces u_4 by u, takes the cross-ratio of the four quantities u_1, u_2, u_3, u, which is assumed to have a constant value v_0, derives and sees that u satisfies a Riccati equation. ■

Another consequence this implies is that the general solution y of the Riccati can be written in terms of three particular solutions

$$y=\frac{y_3(y_2-y_1)+k(y_1-y_3)y_2}{y_2-y_1+k(y_1-y_3)}, \quad \text{for some } k\in\mathbb{P}^1(\mathbb{C}).$$

Therefore the Riccati admits a general superposition principle (its general solution is expressed with three generic solutions and a constant).

In the following we will not distinguish between the h_n (3.1.1) which differ only from a substitution

$$\mathfrak{M}=\left|Z,\frac{ZA(t)+B(t)}{ZC(t)+D(t)}\right|,$$

with $AC - BD \neq 0$ and $A,\ B,\ C,\ D \in \mathbb{F}_1 := \mathbb{C}(a_i),\ i \in \{1, \cdots, 4\}$ or $i \in \{1, \cdots, 5\}$.

This identification makes sense because such a transformation changes a solution of Riccati equation to another solution of a Riccati differential equation; preserves the constance of the cross-ratio of any four elements in some field extension of $\mathbb{F}_1$.

Moreover it preserves the Galois group of the equation h_n. We will take advantage of similar substitutions in order to assume that the sum a_1 of the roots of f is zero; or to multiply x by an appropriate element of $\mathbb{F}_1$. The goal being the simplification of the expressions.

3.3.2 Fourth Degree Equation

Study of the Symmetric Group $\mathfrak{S}_4$

The symmetric group $\mathfrak{S}_4$ between four elements $E = \{1, \cdots, 4\}$ is the group of bijections between these elements. It has 24 elements called permutations. It is generated as a group by the following permutations: $\alpha = (1,2)(3,4)$, $\beta = (1,3)(2,4)$, $\delta = (4)(1,2,3)$, $\epsilon = (1)(2)(3,4)$.

If ϵ lacks, the resulting subgroup is generated by $\alpha,\ \beta,\ \delta$: $\langle \alpha, \beta, \delta \rangle$. Its order is 12 and It is called the alternating subgroup of 4 variables. Moreover $\alpha\beta = \beta\alpha = (1,4)(2,3)$.

If δ lacks, the subgroup of order 8, is generated by $\alpha,\ \beta,\ \epsilon$. It is dihedral.

Finally if δ and ϵ lack, the subgroup is generated by $\alpha,\ \beta$. It becomes $\mathfrak{K}$ called the Klein group.

Let us have a closer look at the cross-ratio of the four distinct roots x_1, x_2, x_3, x_4 in an algebraic closure $\bar{\mathbb{F}}$.

Set

$$\{i,j\} := x_i - x_j, \quad i,j = 1,2,3,4;$$

take the cross-ratio

$$K = K_0 = \frac{\{3,1\}\{4,2\}}{\{3,2\}\{4,1\}} \tag{3.3.2}$$

of the x_i, and consider the way it changes under the action of $\mathfrak{S}_4$ (we remind that a permutation σ of $\mathfrak{S}_4$ acts on x_i by the action $(\sigma, x_i) \to x_{\sigma(i)}$). Assume that

$$\tau_1 = \{2,3\}\{1,4\}, \quad \tau_2 = \{3,1\}\{2,4\}, \quad \tau_3 = \{1,2\}\{3,4\};$$

then

$$\tau_1 + \tau_2 + \tau_3 = 0.$$

This is how the elements α, β, γ, δ transform the τ_i; α and β leave them unchanged; while δ induces the 3-cycle (τ_1, τ_2, τ_3), ie

$$\delta = (\tau_1, \tau_2, \tau_3).$$

The action of ϵ is the following: it sends τ_1 to $-\tau_2$, τ_2 to $-\tau_1$ and τ_3 to $-\tau_3$. Therefore one can write this action schematically as

$$\epsilon = \begin{vmatrix} \tau_1 & -\tau_2 \\ \tau_2 & -\tau_1 \\ \tau_3 & -\tau_3 \end{vmatrix}.$$

We next examine the action on K.

The action of the Klein group $\mathfrak{K}$ on it is trivial. Its elements leave K invariant.

The effect of δ on K is given by

$$\delta(K_0) = \delta\left(-\frac{\tau_2}{\tau_1}\right) = -\frac{\tau_3}{\tau_2} = \frac{\tau_1 + \tau_2}{\tau_2} = \frac{K_0 - 1}{K_0} := K_1,$$

$$\delta(K_1) = -\frac{\tau_1}{\tau_3} = \frac{\tau_1}{\tau_1 + \tau_2} = \frac{1}{1 - K_0} := K_2,$$

and

$$\delta(K_2) = K_0.$$

ϵ on his side acts on K in the following manner

$$\epsilon(K_0) = -\frac{\tau_1}{\tau_2} = \frac{1}{K_0} := K_3,$$

$$\epsilon(K_1) = -\frac{\tau_3}{\tau_1} = \frac{1}{K_2} := K_4,$$

and

$$\epsilon(K_2) = -\frac{\tau_2}{\tau_3} = \frac{1}{K_1} := K_5.$$

When the six elements K_i are distinct, they form an orbit of cardinal six under the action of the symmetric group $\mathfrak{S}_4$ on them. The latter set when parameterized by K_0, equals the following set

$$\left\{K_0, \frac{K_0 - 1}{K_0}, \frac{1}{1 - K_0}, \frac{1}{K_0}, 1 - K_0, \frac{K_0}{K_0 - 1}\right\}. \tag{3.3.3}$$

The kernel of the previous action is therefore of cardinal four. It is the Klein group $\mathfrak{K}$, which is thus a normal subgroup of $\mathfrak{S}_4$.

If we denote by $\mathfrak{R}$ the image of this action; one sees that

$$\mathfrak{R} \simeq \frac{\mathfrak{S}_4}{\mathfrak{K}}.$$

In this group $\mathfrak{R}$, the identity corresponds to $\mathfrak{K}$; δ can be identified with the product of 3-cycles

$$\vartheta = (K_0 K_1 K_2)(K_3 K_5 K_4);$$

while ϵ takes the form

$$\mathfrak{s} = (K_0 K_3)(K_1 K_5)(K_2 K_4).$$

When the orbit set of the previous action of $\mathfrak{S}_4$ on the K_i is less than six, ie when some of the K_is are identical; one sees that only one of two cases can take place.

First of all

$$K_0 = K_1 = K_2; \quad K_3 = K_4 = K_5$$

and

$$K_0^2 - K_0 + 1 = 0;$$

this gives

$$K_0 = K_1 = K_2 = -\rho, \quad K_3 = K_4 = K_4 = -\rho^2;$$

with ρ satisfying $\rho^2 + \rho + 1 = 0$ or in other words ρ is a primitive third root of unity.

The orbit in this case is therefore formed of only the two elements $-\rho$, $-\rho^2$. One often calls it the equianharmonic orbit.

The last case which can happen is a compound of three other particular cases (none of the considered cross-ratio can takes the value 1 because this would lead to the vanishing of one the ratios or again to the fact that one of them takes the value ∞).

$$K_0 = K_3 = -1; \quad K_1 = K_4 = 2; \quad K_2 = K_5 = \frac{1}{2};$$

or

$$K_0 = K_5 = 2; \quad K_1 = K_3 = \frac{1}{2}; \quad K_2 = K_4 = -1;$$

and

$$K_0 = K_5 = \frac{1}{2}; \quad K_1 = K_4 = -1, \quad K_2 = K_5 = 2.$$

The resulting orbit is hence
$$\left\{-1, \frac{1}{2}, 2\right\}$$
and it is called the harmonic orbit.

This finishes our description of $\mathfrak{S}_4$ and of its action on the cross-ratio.

Construction of the anharmonics of degree four

Let us take a general equation of the fourth degree
$$h_4(x) = f(x) = x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$
in $\mathbb{F}[x]$; we suppose it separable and irreducible. As we have seen linear fractional transformations or homographies preserve the anharmonicity and the Galois group. So we may choose one of them and annihilate a_1. From now on, we suppose
$$f(x) = x^4 + 6a_2x^2 + 4a_3x + a_4.$$
We recall from classical invariant theory that given a field $\mathbb{K}$ of characteristic zero. A binary form is by definition an expression of the form
$$f(x, y) = f_0x^n + \binom{n}{1} f_1x^{n-1}y + \binom{n}{2} f_2x^{n-2}y^2 + \cdots + f_ny^n$$
with coefficients in $\mathbb{K}$ and n the degree of the form f. Another used notation is
$$f(x, y) = (f_0, f_1, \cdots , f_n)(x, y)^n.$$
The set V_n of binary forms of degree n is a $\mathbb{K}$-vector space of dimension $n + 1$ and can be identified with the space of polynomials $\mathbb{K}[f_0, f_1, \cdots , f_n]$ on which the group $SL(2, \mathbb{K})$ acts in the following way:
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{K}), f \in V_n, \quad (gf)(x, y) = f(ax + by, cx + dy).$$
When $\mathbb{K} = \mathbb{C}$, the generating elements of the tangent Lie algebra of $\mathfrak{sl}(2, \mathbb{C})$ are realized by
$$t_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$
they act on V_n by the derivations $y\dfrac{\partial}{\partial x}$, $x\dfrac{\partial}{\partial y}$ and $x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

Indeed if we compute [40] the one parameter subgroups of transformations associated with the t_i, we get respectively

$$\exp\left(t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ty \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y+tx \end{pmatrix}$$

and

$$\begin{pmatrix} \exp(t)x \\ \exp(-t)y \end{pmatrix}.$$

Derivation with respect to t and evaluation at $t=0$ gives the three infinitesimal transformations

$$y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y} \quad \text{and} \quad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y};$$

but as

$$\left[y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right] = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

it suffices to know the action only the action of

$$y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}$$

on V_n.

Their action on the ring $\mathbb{K}\left[f_0, f_1, \cdots, f_n\right]$, is by the two fundamental differential operators

$$D := f_0\frac{\partial}{\partial f_1} + 2f_1\frac{\partial}{\partial f_2} + \cdots + nf_{n-1}\frac{\partial}{\partial f_n}$$

$$\Delta := nf_1\frac{\partial}{\partial f_0} + (n-1)f_2\frac{\partial}{\partial f_1} + \cdots + f_n\frac{\partial}{\partial f_{n-1}}.$$

An invariant denotes an homogeneous polynomial expression of the f_i which is annihilated by both D and Δ. One may consult [23], for further study of classical invariant theory.

In the case $n=4$ with $\mathbb{K}=\mathbb{F}$ and for

$$f_4(x,y) = f_0x^4 + 4f_1x^3y + 6f_2x^2y^2 + 6f_3xy^3 + f_4y^4;$$

it is well-known that the algebra of invariants for $SL(2,\mathbb{F})$:

$$\mathfrak{A} = \mathbb{F}\left[a_0, a_1, \cdots, a_n\right]^{SL(2,\mathbb{F})}$$

is freely generated (by generators and relations) by the two invariants

$$g_2 = \begin{vmatrix} f_0 & f_2 \\ f_2 & f_4 \end{vmatrix} - 4 \begin{vmatrix} f_1 & f_2 \\ f_2 & f_3 \end{vmatrix}, \quad \text{and} \quad g_3 = \begin{vmatrix} f_0 & f_1 & f_2 \\ f_1 & f_2 & f_3 \\ f_2 & f_3 & f_4 \end{vmatrix}.$$

The invariants g_2 and g_3 are related to the cross ratio K (3.3.17) by a remarkable formula; in fact one remarks that by setting (see [3, chap.27, page.400] or [15, pages.284,285,297])

$$I = 2g_2, \quad \text{and} \quad J = 6g_3,$$

$$\Omega = \frac{I^3}{J^2} = 24\frac{(K^2-K+1)^3}{(K+1)^2(k-2)^2(2K-1)^2}. \tag{3.3.4}$$

Moreover if we set

$$\Omega_0 = \frac{g_2^3}{d_{f_4}}$$

with $d_{f_4} = g_2^3 - 27g_3^2$ the discriminant of the quartic then,

$$\Omega_0 = \frac{4}{27}\frac{(K^2-K+1)^3}{K^2(K-1)^2}.$$

We get therefore a similar relation relating the absolute rational invariant j_a and the cross-ratio. This invariant j_a characterizes the isomorphism between two elliptic curves as Riemann surfaces. One can check [16, chap.9] for further details about that subject.

For the particular case of

$$h_4(x) = f(x) = x^4 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

one therefore gets (after homogenization) the two invariants

$$I = 2(a_4 + 3a_2^2), \quad J = 6(a_2a_4 - a_3^2 - a_2^3).$$

Our intention is now to find the Galois group G for the h_4 and build them explicitly. We remind first that G is a subgroup of automorphisms of $\mathfrak{S}_4$ and that it fixes all the elements of $\mathbb{F}$, by the fundamental theorem of the Galois theory. The construction will result from a series of lemmas.

Lemma 3.3.2 *If the Galois group G of h_4 contains $\delta = (4)(123)$, then*

$$\Omega = I = 0;$$

if it contains $\epsilon = (1)(2)(3,4)$ then

$$\Omega = \infty, \quad J = 0.$$

Proof. As δ is in the Galois group G, it must fix $K = K_0$; but we have seen in subsection (3.3.2), that this happens only when

$$K_0 = K_1 = K_2, \quad K_3 = K_4 = K_5; \quad K_0^2 - K_0 + 1 = 0.$$

therefore

$$\Omega = I = 0;$$

and we are in the equianharmonic case.

As for the case when $\epsilon \in G$, we see again from subsection (3.3.2), that its occurrence leads to the case implying the vanishing of J and therefore to the fact the $\Omega = \infty$. We are in the harmonic case. ■

Remark 3.3.3 *As the equianharmonic and harmonic cases exclude mutually, one sees that that Galois group can not be the full group $\mathfrak{S}_4$. Therefore it is contained in the alternate group $\mathfrak{U}_4$, generated by α, β and δ, with $I = 0$* (the cardinal of the Galois group is divisible by four because it is transitive; the reason for the latter argument being the assumption of irreducibility and separability of h_4).

Or in the eight order group generated by α, β and ϵ with $J = 0$, which is dihedral.

Or finally in the normal subgroup $\mathfrak{K}$ (Klein group) generated by α and β.

Normally one could also have had the occurrence of the 4-cyclic group C_4, but that would mean that its $h_4(x) = x^4 - T$, for some $T \in \mathbb{F} \setminus \mathbb{C}$ and h_4 would be already of the required form. Moreover the lemma which follows shows that it leads to a degenerate Riccati equation.

We now deal with the construction of h_4 in the harmonic and equianharmonic cases.

Lemma 3.3.4 *The coefficient a_3 can not be zero.*

Proof. Otherwise h_4 is of the form $x^4+bx^2+c = 0 = \left(\left(x^2 + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}\right)$; and in a splitting field its roots are of the u, v, $-u$, $-v$. Expressing that these roots are solutions of the Riccati equation (3.1.2), we get:

$$u' = \mathfrak{B}_0 + \mathfrak{B}_1 u + \mathfrak{B}_2 u^2, \quad -u' = \mathfrak{B}_0 - \mathfrak{B}_1 u + \mathfrak{B}_2 u^2;$$

and

$$v' = \mathfrak{B}_0 + \mathfrak{B}_1 v + \mathfrak{B}_2 v^2, \quad -v' = \mathfrak{B}_0 - \mathfrak{B}_1 v + \mathfrak{B}_2 v^2.$$

Hence

$$0 = \mathfrak{B}_0 + \mathfrak{B}_2 u^2 = \mathfrak{B}_0 + \mathfrak{B}_2 v^2.$$

If one has $0 = \mathfrak{B}_0 = \mathfrak{B}_2$, then the Riccati equation is degenerate (not the most general Riccati equation). Otherwise

$$u^2 - v^2 = 0;$$

a fact which $u = v$ or $u = -v$, a contradiction. ■

In the following we always suppose that we are dealing with the most general Riccati equation.

Lemma 3.3.5 *If $a_2 = 0$, then h_4 is neither harmonic nor equianharmonic.*

Proof. In fact, then

$$I = 2a_4, \quad J = -6a_3^2.$$

For $I = 0$, $a_4 = 0$, and zero is a root of h_4 (which is not possible as it would lead to the vanishing of $\mathfrak{B}_0(t)$, a case we have already excluded). For $J = 0$, $a_3 = 0$, a contradiction from the last lemma. ■

We will therefore suppose that $a_2 a_3 \neq 0$ in the construction of the h_4 equianharmonic and harmonic.

We multiply (which is licit) x by an element r^{-1} of $\mathbb{F}_1$; h_4 becomes

$$x^4 + 6a_2 r^2 x^2 + 4a_3 r^3 x + r^4 a_4 = 0.$$

Set

$$T = a_2 r^2 = a_3 r^3,$$

then as

$$a_2 a_3 \neq 0, \quad r = a_2 a_3^{-1}, \quad T = a_2^3 a_3^{-2}$$

and

$$f(x) = x^4 + 6Tx^2 + 4Tx + \mathfrak{C} = 0, \quad \mathfrak{C} = a_4 a_2^4 a_3^{-4}.$$

Then by very definition (replace the initial a_2, a_3 and a_4 by their respective new values $a_2 r^2$, $a_3 r^3$ and $\mathfrak{C}$.)

$$I = 2(\mathfrak{C} + 3T^2), \quad J = 6(T\mathfrak{C} - T^3 - T^2).$$

When

$$I = 0, \quad \mathfrak{C} = -3T^2$$

$$f(x) = x^4 + 6Tx^2 + 4Tx - 3T^2. \tag{3.3.5}$$

When

$$J = 0, \quad \mathfrak{C} = T^2 + T$$

$$f(x) = x^4 + 6Tx^2 + 4Tx + T(T+1). \tag{3.3.6}$$

In both cases, $f(x)$ is a polynomial in x and T with numerical coefficients. This is the first two set of results announced in the first part of theorem 3.1.1.

Let us now deal with the construction of the h_4, with $IJ \neq 0$. Its Galois group must coincide with $\mathfrak{K}$ formed of the four substitutions 1, $\alpha = (1,2)(3,4)$, $\alpha = (1,3)(2,4)$ and $\alpha\beta = (1,4)(2,3)$.

First of all consider

$$x_1 + x_2 + x_3 + x_4 = 0,$$

then

$$x_1 + x_2 - (x_3 + x_4) \neq 0,$$

because otherwise

$$x_1 + x_2 = x_3 + x_4 = 0$$

ie

$$x_1 = -x_2; \quad x_3 = -x_4$$

and h_4 would be a biquadratic polynomial, a_3 would vanish, a fact which would contradict the lemma 3.3.4.

One remarks that the expression

$$(x_1 + x_2 - (x_3 + x_4))^2 = 2^4 p, \tag{3.3.7}$$

is invariable under the action of α and β. The same is true of

$$\begin{cases} x_1x_2 + x_3x_4 = 2q \\ \dfrac{x_1x_2 - x_3x_4}{x_1 + x_2 - (x_3 + x_4)} = \frac{1}{2}r \end{cases} \tag{3.3.8}$$

We will use these invariants (p, q, r), to parameterize the solutions x_1, x_2, x_3 and x_4. Using formula (3.3.7) one can choose the x_i such that

$$\begin{cases} x_1 + x_2 = 2\sqrt{p} \\ x_3 + x_4 = -2\sqrt{p}. \end{cases}$$

This induces from formulas (3.3.8)

$$\begin{cases} x_1x_2 = q + r\sqrt{p} \\ x_3x_4 = q - r\sqrt{p}. \end{cases}$$

and

$$\begin{cases} x_1 = \sqrt{p} + \sqrt{p-q-r\sqrt{p}} = g + h \\ x_2 = \sqrt{p} - \sqrt{p-q-r\sqrt{p}} = g - h \\ x_3 = -\sqrt{p} + \sqrt{p-q+r\sqrt{p}} = -g + k \\ x_4 = -\sqrt{p} - \sqrt{p-q+r\sqrt{p}} = -g + k. \end{cases} \tag{3.3.9}$$

with

$$g^2 = p, \quad h^2 = p - q - r\sqrt{p}$$

and

$$k^2 = p - q + r\sqrt{p}.$$

In the above expressions of equation (3.3.9), one can set $r = 1$. First of all we remark that we may assume $r \neq 0$, because otherwise, one would have $h = k$, $x_1 = -x_4$, $x_2 = -x_3$, and we would have a biquadratic h_4. As we have already seen, this not possible. We scale, as previously done, the x_i by r. We can therefore divide the two sides of the second equation (3.3.8) by r, and this is the same as requiring $r = 1$.

Let

$$K = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} = \frac{p + q + \Delta}{p + q - \Delta}$$

because from the formulas 3.3.9

$$\begin{aligned} (x_3 - x_1)(x_4 - x_2) &= (2g + (h - k))(2g - (h - k)) \\ &= \left(4g^2 - \left(h^2 - 2hk + k^2\right)\right) \\ &= \left(4p - \left(2(p - q) - 2\sqrt{(p-q)^2 - p}\right)\right) \\ &= 2(p + q + \Delta). \end{aligned} \tag{3.3.10}$$

In the same way, we find

$$(x_3 - x_2)(x_4 - x_1) = 2(p + q - \Delta).$$

Hence

$$(p - q)^2 - p = (p + q)^2 \left[\frac{K - 1}{K + 1}\right]^2 \tag{3.3.11}$$

As $p \neq 0$, we can set $q = pT$; equation (3.3.11) then becomes

$$\begin{cases} p = \dfrac{1}{(1-T)^2 - \left(\dfrac{K-1}{K+1}\right)^2 (1+T)^2} \\ q = \dfrac{T}{(1-T)^2 - \left(\dfrac{K-1}{K+1}\right)^2 (1+T)^2} \end{cases} \tag{3.3.12}$$

We build immediately the corresponding h_4 with formulas of equation (3.3.9) with $r = 1$;

$$x_1 - g = h, \quad g^2 = p,$$

$$h^2 = p - q - g, \quad k^2 = p - q + g,$$

So we have in succession

$$x_1^2 + p - 2x_1 g = h^2 = p - q - g,$$

$$x_1^2 + q = g(2x_1 - 1),$$

$$(x_1^2 + q)^2 = p(2x_1 - 1)^2.$$

Every other root would give the same result, hence h_4 is the equation

$$f(x) = (x^2 + q)^2 - p(2x - 1)^2 = 0. \tag{3.3.13}$$

But by the formulas (3.4), p and q are rational in T; therefore $f(x)$ remains a polynomial in x and T, with numerical coefficients. This finishes the proof of theorem 3.1.1 for the anharmonic $h_4(x)$.

Remark 3.3.6 *In the previous, while choosing a square root, we have always taken the positive one.*

Remark 3.3.7 *All the anharmonics h_4 we have obtained are resoluble by radicals, meaning that the Galois groups are solvable.*

In the equianharmonic case, ie when

$$\Omega = I = 0,$$

one has the sequence

$$\{e\} \subset \mathfrak{K} \subset \mathfrak{U}_4.$$

In the harmonic case, ie when

$$\Omega = \infty, \quad \textit{and} \quad J = 0,$$

the sequence is

$$\{e\} \subset \mathfrak{K} \subset \mathfrak{D}_4.$$

The remaining case corresponds to the sequence

$$\{e\} \subset \mathfrak{K}.$$

3.3.3 Fifth Degree Equation

Let $f(x) = 0 \in \mathbb{F}[x]$ be an irreducible and separable h_5, of distinct roots x_0, $x_1 \cdots$, x_4; $\mathbb{F}_f$ its roots field. We designate by G its Galois group. Let G_0 be the subgroup of G which fixes x_0. The latter must fix the inverse cross-ratio

$$K^{-1} = \frac{(x_3 - x_2)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)}.$$

We suppose (we shall prove it later) that K is neither harmonic nor equianharmonic. Then G_0 which we can identify with a subgroup of $\mathfrak{S}_4$ which fixes K, will contain neither $\delta = (0)(1, 2, 3)(4)$ nor $\epsilon = (0)(1)(2)(3, 4)$ (see subsection 3.3.2). It can therefore be identified with a subgroup of $\mathfrak{K}$ and its index $(G_0 : 1)$ is either 1, 2 or 4.

Its possible presentations are

$$\{e\}$$

or

$$\{e, (0)(1, 2)(3, 4)\},$$

or

$$\{e, (0)(1, 3)(2, 4)\},$$

and finally

$$\{e, (0)(1, 2)(3, 4), (0)(1, 3)(2, 4), (0)(1, 4)(2, 3)\}.$$

We suppose in the following that the index $(G_0 : 1)$ is not 1. Because if it happened, as G is transitive, one has from the index formula that $\sharp(G) = 5$. So G is cyclic and $h_5(x)$ has the form

$$h_5(x) = x^5 - T$$

for some $T \in \mathbb{F} \setminus \mathbb{C}$ and is already of the required form. We will not examine this case in the following.

As no non trivial element of the Klein group $\mathfrak{K}$ fixes an element of the set $\{x_1, x_2, x_3, x_4\}$, one sees that G will contain no substitution distinct from the identity e leaving fixed more than a root of $f(x)$.

So what is the structure of G? Firstly one sees that G certainly does contain the circular permutation

$$\sigma = (0, 1, 2, 3, 4) = |i, i + 1|(\text{mod} 5).$$

This is because G is transitive and therefore of order divisible by 5 which is a prime; so from the Cauchy theorem, there is a subgroup of order 5 in G, hence a circular permutation.

Now let $\sigma = (0, 1, 2, 3, 4)$; one has

$$\sigma^2(0) = 2, \quad \sigma^2(1) = 3, \quad \sigma^2(2) = 4, \quad \sigma^2(3) = 0, \quad \sigma^2(4) = 1.$$

Using this one sees that the Galois group G does not contain

$$(0)(1, 3)(2, 4) \simeq \beta;$$

such a fact would indeed lead to the fact that $\sigma^2\beta$ is an element of G. But the latter permutation has the two fixed points 3 and 4 as

$$\sigma^2\beta(3) = \sigma^2(1) = 3, \quad \text{and} \quad \sigma^2\beta(4) = \sigma^2(2) = 4.$$

We show similarly that $(0)(1, 2)(3, 4) \simeq \alpha$ is not an element of G, using the fact that $(0, 1, 2, 3, 4) \circ (0)(1, 2)(3, 4)$ have the fix points 2 and 4.

So $G_0 = \{e, (0)(1, 4)(2, 3) \simeq \alpha\beta = \tau\}$ and $(G_0 : 1) = 2$.

As a consequence the fixed field of G_0 is $\mathbb{F}[x_0]$. Now using the fundamental theorem of the Galois theory, one can say that $[\mathbb{F}_f : \mathbb{F}[x_0]] = (G_0 : 1) = 2$. Thus $\mathbb{F}_f = \mathbb{F}[x_0, \zeta]$, with ζ algebraic of degree 2 over $\mathbb{F}[x_0]$.

Let us now see what G really looks like. We remark that $\tau^{-1}\sigma\tau = \sigma^{-1}$; such an identity leads to

$$\tau\sigma^k = \sigma^{-k}\tau^{-1} = \sigma^{-k}\tau;$$

therefore G is formed by the following ten substitutions

$$\sigma^k \quad \text{et} \quad \tau\sigma^k, \quad k = 0, 1, 2, 3, 4.$$

And it is isomorphic to the dihedral group $\mathfrak{D}_5$.

We recapitulate the results in the following proposition

Proposition 3.3.8 *The Galois group of an irreducible separable $h_5(x)$ having all its roots solution of a same Riccati equation is the dihedral group $\mathfrak{D}_5$. It is formed by the ten substitutions*

$$\sigma^k \quad et \quad \tau\sigma^k, \quad k = 0, 1, 2, 3, 4;$$

with $\sigma^5 = e$ and $\tau^2 = e$.

It remains to be shown that the cross-ratio is neither harmonic nor equianharmonic and to determine explicitly the form of $h_5(x)$. We first show the result concerning the cross-ratio.

We set

$$\mu_0 = x_1 - x_4, \quad \mu_1 = x_2 - x_0, \quad \mu_2 = x_3 - x_1, \quad \mu_3 = x_4 - x_2, \quad \mu_4 = x_0 - x_3.$$

We will use the notation $\{i, j\} = x_i - x_j$.

The permutation σ which is necessarily in G permutes the μ_i circularly

$$\sigma = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4).$$

We have

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0,$$

and also

$$\mu_0 + \mu_1 + \mu_2 + \mu_3 + \mu_4 = 0.$$

Finally by an easy computation, one sees

$$\begin{cases} 5x_0 = 2(\mu_4 - \mu_1) + \mu_2 - \mu_3, \\ 5x_1 = 2(\mu_0 - \mu_2) + \mu_3 - \mu_4, \\ 5x_2 = 2(\mu_1 - \mu_3) + \mu_4 - \mu_0, \\ 5x_3 = 2(\mu_2 - \mu_4) + \mu_0 - \mu_1, \\ 5x_0 = 2(\mu_3 - \mu_0) + \mu_1 - \mu_2. \end{cases} \tag{3.3.14}$$

This results from the fact that

$$2(\mu_4 - \mu_1) + \mu_2 - \mu_3 = 4x_0 - (x_1 + x_2 + x_3 + x_4) = 5x_0;$$

to obtain the remaining identities, just apply the cyclic permutation σ successively.

Define now the six quantities a, b, c, d, e, μ as

$$a = \mu_2\mu_3, \quad b = \mu_3\mu_4, \quad c = \mu_4\mu_0, \quad d = \mu_0\mu_1, \quad e = \mu_1\mu_2,$$

$$\mu = \mu_0\mu_1\mu_2\mu_3\mu_4.$$

Then

$$\mu^2 = abcde,$$

and

$$\mu\mu_1 = bde, \quad \mu\mu_2 = cea, \quad \mu\mu_2 = dab, \quad \mu\mu_3 = ebc, \quad \mu\mu_0 = acd.$$

Next we multiply the formulas (3.3.14) by μ and replace the $\mu\mu_i$s by their just given values, we get the relations

$$\begin{cases} 5\mu x_0 = 2be(c-d) + a(ec-bd), \\ 5\mu x_1 = 2ca(d-e) + b(ad-ce), \\ 5\mu x_2 = 2db(e-a) + c(be-da), \\ 5\mu x_3 = 2ec(a-b) + d(ca-eb), \\ 5\mu x_4 = 2ad(b-c) + e(db-ac) \end{cases} \tag{3.3.15}$$

We remark that if

$$a = b = c = d = e, \quad \mu x_0 = \mu x_1 = \mu x_2 = \mu x_3 = \mu x_4 = 0;$$

but

$$\mu^2 = abcde \neq 0;$$

because otherwise for instance $a = 0 \Longrightarrow \mu_2 = 0$ or $\mu_3 = 0 \Longrightarrow x_3 = x_1$ or $x_2 = x_4$. So

$$x_0 = \cdots = x_4 = 0,$$

a contradiction. Therefore none of the the five letters $a, \cdots, e$ is zero and they are not all equal.

We now take advantage of the μ_is and of a, b, c, d, e to parameterize the cross-ratio K. One has

$$K^{-1} = \frac{\{3,1\}\{4,2\}}{\{3,2\}\{4,1\}} = \frac{\mu_2\mu_3}{\mu_0(\mu_1+\mu_4)} = \frac{a}{d+c}$$

ie

$$-Ka + c + d = 0.$$

The substitution $\sigma \in G$ permutes circularly a, b, c, d, e leaving at the same time K invariable. Therefore one has the following system of linear homogeneous equations

$$\begin{cases} -Ka+c+d=0, \\ -Kb+d+e=0, \\ -Kc+e+a=0, \\ -Kd+a+b=0, \\ -Ke+b+c=0. \end{cases} \tag{3.3.16}$$

We have seen that a, b, c, d, e can not take the value zero; so the determinant

$$\begin{vmatrix} -K & 0 & 1 & 1 & 0 \\ 0 & -K & 0 & 1 & 1 \\ 1 & 0 & -K & 0 & 1 \\ 1 & 1 & 0 & -K & 0 \\ 0 & 1 & 1 & 0 & -K \end{vmatrix} = -K^5+5K^3-5K+2$$

$$= -(K-2)(K^2+K-1)^2 = 0.$$

If $K = 2$, then $a = b = c = d = e$; this results from the fact that the identities (3.3.16) lead to

$$\begin{cases} a &= \dfrac{b+c+d+e}{4} \\ b &= \dfrac{a+b+c+e}{4} \\ c &= \dfrac{a+b+d+e}{4} \\ d &= \dfrac{a+b+c+e}{4} \\ e &= \dfrac{a+b+c+d}{4}; \end{cases}$$

and this gives

$$a = b = c = d = e;$$

a contradiction as previously seen. Therefore

$$K^2+K-1=0.$$

Let θ be a primitive fifth root of 1 and $\rho_1 = \theta+\theta^4$, $\rho_1^2 = 2+\theta^2+\theta^3$; we want to parameterize the values of K using θ.

As

$$\theta^4+\cdots+\theta+1=0,$$

then

$$\rho_1^2 + \rho_1 - 1 = 0.$$

The two values of K are now seen to be

$$K = \theta + \theta^4 \quad \text{and} \quad K = \theta^2 + \theta^3. \tag{3.3.17}$$

We are therefore neither in the equianharmonic case, nor in the harmonic one. As these two latter cases correspond respectively to the fact that K belongs to

$$\{-\rho, -\rho^2\}, \quad \text{with } \rho \text{ a primitive root of unity;}$$

respectively to

$$\left\{-1, \frac{1}{2}, 2\right\}.$$

We regroup these facts in the following proposition

Proposition 3.3.9 *The cross-ratio of any four roots of an $h_5(x)$ is neither equianharmonic nor harmonic.*

Parameterization of the anharmonic $h_5(x)$

Let $K^2+K-1 = 0$ for example by setting $K = \theta+\theta^4$; we want to parameterize a, b, c, d, e with two independent variables u and v such that all the former are expressed linearly with u and v with coefficients in $\mathbb{C}$.

Assuming that fact, one sees that the substitutions σ and τ of the group G translate on u and v as two linear homogeneous substitutions $\mathcal{S}$ and $\mathcal{T}$. We impose moreover that $\mathcal{S}$ is put into canonical form:

$$\mathcal{S} = (u \to \theta^\alpha u \quad v \to \theta^\beta v).$$

As $\sigma^5 = e$, one also has that $\mathcal{S}^5 = e$. Also because $\tau^2 = e$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$, it results that $\mathcal{T}^2 = e$, $\mathcal{T}\mathcal{S}\mathcal{T} = \mathcal{S}^{-1}$; therefore by an easy computation,

$$\mathcal{T} = (u \to v, v \to u)$$

and

$$\alpha + \beta = (\text{mod} 5).$$

$\mathcal{T}$ or its counterpart τ must permute a, b, c, d, e in the following way

$$\mathcal{T} = (a)(b, e)(c, d) = \tau$$

and $\mathcal{S}$ or again σ must act as

$$\mathcal{S} = \sigma = (a, b, c, d, e).$$

Therefore

$$\begin{gathered} a = u + v, \quad c = \theta^{2\alpha}u + \theta^{2\beta}v, \\ b = \theta^{\alpha}u + \theta^{\beta}v, \quad d = \theta^{3\alpha}u + \theta^{3\beta}v, \\ e = \theta^{4\alpha} + \theta^{4\beta}, \end{gathered}$$

We insert now our results in the first of the equations (3.3.16) and get

$$\begin{gathered} Ka = c + d, \\ (\theta + \theta^4)(u + v) = (\theta^{2\alpha} + \theta^{3\alpha})u + (\theta^{2\beta} + \theta^{3\beta})v, \end{gathered}$$

hence

$$\alpha = 2(\mathrm{mod}5), \quad \beta = 3(\mathrm{mod}5)$$

or

$$\alpha = 3(\mathrm{mod}5), \quad \beta = 2(\mathrm{mod}5).$$

We will assume that $\alpha = 2(\bmod 5)$. One will consequently have the following parameterization

$$\begin{cases} a = u + v, \\ b = \theta^2 u + \theta^3 v, \\ c = \theta^4 u + \theta v, \\ d = \theta u + \theta^4 v, \\ e = \theta^3 u + \theta^2 v. \end{cases} \tag{3.3.18}$$

Remark 3.3.10 *The construction above induces a faithful linear representation of $\mathfrak{D}_5$ into $\mathbb{C}^2$.*

The fifth degree equation which admits a, b, c, d, e for roots is

$$\Upsilon(x) = \lambda^5 - 5uv\lambda^3 + 5u^2v^2\lambda - (u^5 + v^5) = 0. \tag{3.3.19}$$

Therefore to recapitulate, the two substitutions $\sigma = (0, 1, 2, 3, 4)$ and $\tau = (0)(1, 4)(2, 3)$ of G act respectively on the μ_i as the two permutations

$$(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$$

and

$$\left|\begin{array}{ccccc} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ -\mu_0 & -\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 \end{array}\right|.$$

They act on the $a, \cdots, e$ as (a, b, c, d, e) and $(a)(b, e)(c, d)$; and finally they act on u and v as the two substitutions

$$\mathcal{S} = (u \to \theta^\alpha u \quad v \to \theta^\beta v), \quad \mathcal{T} = (u \to v, v \to u).$$

At the same time we remark that

$$\mu = \mu_0\mu_1\mu_2\mu_3\mu_4 \quad \text{such that} \quad \mu^2 = abcde$$

is invariable under the action of σ and that it is changed to its opposite when τ acts on it.

Lastly, the two expressions $u^5 + v^5 = p$ and $q = uv$ remain invariable under the action of $\mathcal{T}$ and $\mathcal{S}$, ie under the action of G. Thus they lie all in $\mathbb{F}$ by the fundamental theorem of Galois theory.

In what precedes, we have worked with the root x_0 of h_5. We have considered the subgroup G_0 of the elements of G which left it fixed. We saw G_0 was $\{e, (0)(1, 4)(2, 3)\}$; we have also seen that it fixed a (we are presently reasoning by isomorphism). Had we worked with x_1 and b, we would have similar results by circular permutation.

Therefore by construction, we see that anyone of the couples

$$\{(a, x_0); (b, x_1); (c, x_2); (d, x_3); (e, x_4)\};$$

is fixed by the same substitutions of G (as a and x_0 are fixed by the same substitutions of G, one is a polynomial of the other one). Hence

$$x_0 = \psi(a), \quad \psi(X) \in \mathbb{F}(X).$$

We now let σ act on

$$x_0 - \psi(a) = 0,$$

we get similar relations

$$x_1 - \psi(b) = 0,$$
$$x_2 - \psi(c) = 0,$$
$$x_3 - \psi(d) = 0,$$
$$x_4 - \psi(e) = 0.$$

Construction of the $h_5(x)$

Let us build the previous ψ. We will then as before give a parameterization of $h_5(x)$.

We put the values of a, b, c, d, e in equation (3.3.18) in the formulas of equation (3.3.15). We obtain the next lemma whose proof results by simple computation. We therefore omit it.

Lemma 3.3.11

$$5\mu x_0 = R(u-v)(u^2+v^2)$$

with

$$R = 2(\theta^2 - \theta^3) + \theta - \theta^4.$$

We remark that the expression

$$\frac{(u^5-v^5)}{\mu} = r$$

is invariant under the action of $\mathcal{S}$ and $\mathcal{T}$. This is due on the one hand, to the fact $\mathcal{S}(u) = \theta^{\alpha} u, \quad \mathcal{S}(v) = \theta^{\beta} v$ and to the fact that $\mathcal{S}$ leaves μ unchanged; the remaining side results from the action of $\mathcal{T}$ on μ and $(u^5 - v^5)$. The latter substitution changes as is easily seen, both the signs of μ and $(u^5 - v^5)$.

Therefore r lies in $\mathbb{F}$.

Consider the expression

$$\begin{aligned}\frac{u^5-v^5}{u-v} &= u^4 + u^3v + u^2v^2 + uv^3 + v^4 \\ &= u^4 + v^4 + uv(u^2+v^2) + u^2v^2.\end{aligned}$$

As (3.3.18)

$$uv = q, \quad u + v = a;$$

one obtains

$$(u^4+v^4) = (u^2+v^2)^2 - 2u^2v^2 = ((u+v)^2 - 2uv)^2 - 2u^2v^2 = a^4 - 4qa^2 + 2q^2;$$

in the same vein one gets

$$uv(u^2+v^2) = q(a^2 - 2q).$$

Hence

$$\frac{u^5-v^5}{u-v} = a^2(a^2 - 3q) + q^2.$$

We now use the previous identities to determine the value of x_0 in terms a. One has from lemma 3.3.11

$$\begin{aligned} x_0 &= \frac{R(u-v)(u^2+v^2)}{5\mu} \\ &= \frac{Rr(u-v)(u^2+v^2)}{5(u^5-v^5)} \\ &= \frac{rR}{5} \cdot \frac{a^2-2q}{a^2(a^2-3q)+q^2} \\ &:= \psi(a); \end{aligned} \tag{3.3.20}$$

so

$$\psi(X) = \frac{rR}{5} \cdot \frac{X^2-2q}{X^2(X^2-3q)+q^2}.$$

We set $a^2 = q\xi$, and deduce from it that

$$x_0 = \frac{Rr}{5q} \cdot \frac{\xi-2}{\xi(\xi-3)+1}.$$

As a is a root of equation (3.3.19), we have

$$\Upsilon(a) = a(a^4 - 5qa^2 + 5q^2) - p = 0,$$

where we remind that we have set

$$uv = q, \quad u^5 + v^5 = p.$$

Thus

$$q^2 a(\xi^2 - 5\xi + 5) = p.$$

Taking the square of the last expression and dividing appropriately, one gets

$$\xi(\xi^2 - 5\xi + 5)^2 = \frac{p^2}{q^5} := T.$$

In the last results we have privileged the couple (x_0, a), if rather than working with it, we handled the couple x_1 and b, the result would have been identical. Therefore our construction is independent of the root chosen at the start of the process.

If we summarize, we see that h_5 is obtained by eliminating ξ between the two equations

$$x = \frac{Rr}{5q} \cdot \frac{\xi-2}{\xi(\xi-3)+1}, \quad \xi(\xi^2 - 5\xi + 5)^2 = T.$$

As previously seen, it is licit to multiply all the x_i by the element of $\mathbb{F}_1$

$$\frac{Rr}{5q};$$

such an operation is equivalent to set $\dfrac{Rr}{q} = 1$ (the Galois group and the anharmonicity are preserved).

Therefore one has the following proposition

Proposition 3.3.12 *To get the anharmonic $h_5(x)$, one has to eliminate ξ between*

$$\xi(\xi^2 - 5\xi + 5)^2 = T \quad \textit{and} \quad x = \frac{\xi - 2}{\xi(\xi - 3) + 1}.$$

The resultant is a polynomial of the fifth degree between x and T with numerical coefficients.The algebraic relation between x and T is of genus zero (they are rationally expressible with the same parameter ξ).

A question remains to be examined, namely: is the fifth degree equation that we have just built an anharmonic? In other words, are all the cross-ratios formed from any set of roots in $\mathbb{C}$? The answer is happily yes. To see this let us name by $\mathfrak{R}_i$ the system of the six distinct cross-ratios formed by the four roots different from x_i. Equation (3.3.3) shows that the $\mathfrak{R}_i$ are rational functions of anyone of them.

We have shown (3.3.17) that K admitted the parameterization

$$K = \theta + \theta^4, \quad \text{or} \quad K = \theta^2 + \theta^3,$$

with θ a primitive fifth root of unity. Thus K which belongs to $\mathfrak{R}_0$ is constant. So are all the remaining elements of $\mathfrak{R}_0$.

The substitution $\sigma \in G$ which permutes circularly the five systems $\mathfrak{R}_i, i = 0, \cdots, 4$ leaves K constant. Therefore every one of the systems $\mathfrak{R}_i$ has a constant term, a fact which leads to the constance of its all terms.

This ends the proof of theorem 3.1.1.

Remark 3.3.13 *The h_5 we have just built is again solvable by radicals; as it admits the following sequence*

$$\{e\} \subset C_5 = \langle (0, 1, 2, 3, 4) \rangle \subset \mathfrak{D}_5.$$

Moreover one remarks that $\mathfrak{D}_5$ is part of a larger family of groups, namely the Frobenius groups which are defined as follows

Definition 3.3.14 *See* [29]. Let $p \geq 5$ be a prime number. $\mathfrak{S}_p$ the symmetric group; a subgroup H of S_p is a called a Frobenius group, when it is transitive, when all its substitutions have at most a fix point and such that there exists at least a substitution of H having a fixed point.

3.4 The Construction of the Riccati equations

To begin, we determine the stabilizer of the roots of the anharmonics $h_4(x)$ and $h_5(x)$. In order to do so, we use first of all the following lemma.

Lemma 3.4.1 *Let x_i be a root of h_n, $n \in \{4, 5\}$ and x_j another root of it with $i \neq j$; then the stabilizers G_{x_i} and G_{x_j} of x_i and x_j under the action of the Galois group G, are conjugate.*

Proof. We know that the Galois group G acts transitively on the roots; therefore there exits some $\sigma_0 \in G$ such that

$$\sigma_0(x_i) = x_j.$$

Now consider the stabilizers of x_i respectively x_j: G_{x_i} respectively G_{x_j}. One has

$$\sigma_0^{-1} G_{x_j} \sigma_0 \subset G_{x_i}.$$

Indeed let $g \in G_{x_j}$ then $\sigma_0^{-1} g \sigma_0(x_i) = \sigma_0^{-1} g(x_j) = \sigma_0^{-1}(x_j) = x_i$. We show in the same vein that

$$\sigma_0 G_{x_i} \sigma_0^{-1} \subset G_{x_j},$$

and this gives the complete proof of the lemma. ∎

We can therefore reason on any of the stabilizers for the respective cases we will consider. We remind that the anharmonics h_4 fell into three categories:

The first h_4 corresponded to the equation

$$f(x) = x^4 + 6Tx^2 + 4Tx - 3T^2 = 0;$$

we were in the equianharmonic case, ie $I = 0$ or $\Omega_1 = \{-\rho, \rho^2\}$, with ρ a primitive third root of unity. The Galois group of the equation was $\mathfrak{U}_4 = \langle \alpha, \beta, \delta \rangle$ ie the tetrahedral group.

The second h_4 was

$$f(x) = f(x) = x^4 + 6Tx^2 + 4Tx + T(T+1) = 0;$$

its Galois group was the dihedral group $\mathfrak{D}_4 = \langle \alpha, \beta, \epsilon \rangle$ (contains the 4-cycle (1423)). We were in the harmonic case ie $\Omega_2 = \left\{-1, 2, \frac{1}{2}\right\}$ or $J = 0$.

Finally when we were neither in the harmonic case nor in the equianharmonic one, h_4 took the form

$$f(x) = (x_1^2 + q)^2 = p(2x_1 - 1)^2,$$

with

$$\begin{cases} p = \dfrac{1}{(1-T)^2 - \left(\dfrac{K-1}{K+1}\right)^2 (1+T)^2} \\ q = \dfrac{T}{(1-T)^2 - \left(\dfrac{K-1}{K+1}\right)^2 (1+T)^2}. \end{cases}$$

And the Galois group was the Klein group $\mathfrak{K}$ generated by α and β.

For the case of h_5, only one case happened to exist. It resulted from the elimination of ξ between the following two equations

$$\xi(\xi^2 - 5\xi + 5)^2 = T \quad \text{and} \quad x = \frac{\xi - 2}{\xi(\xi - 3) + 1}.$$

The Galois group of the latter equation was the dihedral group $\mathfrak{D}_5$ generated by $\sigma = (01234)$ and $\tau = (0)(14)(23)$.

Thus in the tetrahedral case (first h_4), the stabilizer of x_4 is $\langle \delta \rangle$ (by inspection). It has cardinal three. From the previous lemma the stabilizers of the the other roots are also of cardinal three. Therefore they are cyclic subgroups of order three.

For the second h_4, the Galois group was $\mathfrak{D}_4$. One sees that the stabilizer of x_1 is generated by ϵ. Thus all the stabilizers of any of its roots are cyclic subgroups of order 2.

For the case of the Klein group $\mathfrak{K}$, the stabilizer of any root of the corresponding h_4 is the identity. Lastly we have seen at the beginning of section (3.3), that the stabilizer of the root x_0 of h_5 was of cardinal 2 and therefore is a cyclic group of order 2.

To summarize one has the following proposition

Proposition 3.4.2 *The stabilizer of any root of an anharmonic h_n, $n \in \{4, 5\}$ is a cyclic subgroup of the Galois group of the equation. Moreover all the stabilizers of the roots of any h_n are conjugate.*

It is well-known (see [27]) that the tetrahedral group $\mathfrak{U}_4$ is isomorphic to the group of linear fractional transformation generated by the elements

$$\Theta_2 = |Z, -Z|; \quad \epsilon_0 = \left|Z, \frac{1}{Z}\right|; \quad \left|Z, \frac{1-i}{1+i}\frac{Z-1}{Z+1}\right|;$$

for the case of the dihedral groups $\mathfrak{D}_4$ one knows that it has an isomorphism with the subgroup of linear fractional transformations generated by

$$\Theta_4 = |Z, iZ|; \quad \epsilon_0 = \left|Z, \frac{1}{Z}\right|.$$

The Klein group $\mathfrak{K}$ is on his side isomorphic to the group generated by

$$\Theta_2 = |Z, -Z|; \quad \epsilon_0 = \left|Z, \frac{1}{Z}\right|.$$

Finally the dihedral group $\mathfrak{D}_5$ is isomorphic to the group of fractional linear transformations generated by

$$\Theta_5 = \left|Z, e^{\frac{2i\pi}{5}} Z\right|; \quad \epsilon_0 = \left|Z, \frac{1}{Z}\right|.$$

Klein in [27] also associates to each of the previous groups of fractional linear transformations, an absolute invariant.

For the tetrahedral case, this invariant is given by the formula

$$\Psi_{tetr} = \frac{\psi(Z)}{\phi(z)} = \left(\frac{Z^4 + 1 + 2i\sqrt{3}Z^2}{Z^4 + 1 - 2i\sqrt{3}Z^2}\right)^3.$$

For the dihedral cases $\mathfrak{D}_4$, respectively $\mathfrak{D}_5$, respectively$\mathfrak{K}$. The absolute invariant is given by

$$\Psi_{dih} = \frac{\psi(Z)_1}{\phi(Z)_1} = \frac{1}{Z^m} + Z^m, \quad \text{with } m \in \{4, 5, 2\} \text{ in previous adopted order.}$$

Let now S be the finite group of linear fractional transformation isomorphic to G. The properties of the absolute invariants Ψ are the following:

- They are left invariant under the transformation of the concerned groups, with the action given by

$$(h_f, \Psi) \to h_f.\Psi : Z \to \Psi(h_f.Z).$$

- Every other absolute invariant for the group is a rational function with coefficients in $\mathbb{C}$ of Ψ.

We shall use these previous invariants to give a new parameterization of the roots of the anharmonics. Then we will use the constance of the cross-ratio to determine a form for the general solution of the Riccati equation and finally build the Riccati equation into consideration explicitly.

First of all we describe a procedure to parameterize the roots. We take any root x_i of an anharmonic h_n; and consider its stabilizer G_{x_i}. We have seen in proposition 3.4.2 that it is a cyclic subgroup of the Galois group G of h_n. We call its order p and note N the order of G. Thus by transitivity

$$N = np.$$

Call S_i the group corresponding to G_{x_i}. S_i is a cyclic of cardinal p. Let σ^i be its generator. One knows from theorem 3.2.8, that it admits two fixed points. Consider η_i as one of them.

We take a generic variable $U \in \mathbb{F}_1 \setminus \mathbb{C}$ and denote by Ψ the absolute invariant in all the cases.

The equation

$$\Psi(\Delta) = \frac{\psi(\Delta)}{\phi(\Delta)} = U, \tag{3.4.1}$$

has got

$$N = np$$

roots in an algebraic closure. We suppose that their list is given by Δj, where $0 \leq j \leq N-1$. Let be the equation

$$\frac{\psi(\eta)}{\phi(\eta)} = \frac{\psi(\eta_0)}{\phi(\eta_0)} = \frac{\psi(\eta_1)}{\phi(\eta_1)} = \cdots = \frac{\psi(\eta_{n-1})}{\phi(\eta_{n-1})}. \tag{3.4.2}$$

If η_j is one root it, then for any σ^j in its stabilizer S_j, η_j is a solution of the equation

$$\frac{\psi(\sigma.\eta)}{\phi(\sigma.\eta)} = \frac{\psi(\eta_j)}{\phi(\eta_j)}; \tag{3.4.3}$$

therefore η_j is root of equation (3.4.2) of multiplicity p. Hence we see that all the roots of equation (3.4.2) have multiplicity p and that equation (3.4.2) is the p-th power of a polynomial of degree n which we will call H. Let us try to take advantage of the polynomial H and of Ψ to parameterize a root x_i of an anharmonic h_n.

Proposition 3.4.3 *Consider one of the Δ as defined by equation* (3.4.1) and η_j the roots of H. We have seen the sum of the roots of an anharmonic can be taken to be zero. Impose the condition

$$x_i := P(\Delta)\left[\frac{1}{\Delta - \eta_i} + Q(\Delta)\right]$$

and that

$$P(s.\Delta) = \frac{ad - bc}{(c\Delta + d)^2} P(\Delta)$$

with

$$s(Z) = \frac{aZ + b}{cZ + d}, \quad ad - cb \neq 0.$$

Then x_i admits the parameterization

$$x_i = \frac{\left[\dfrac{1}{\Delta - \eta_i} - \dfrac{H'(\Delta)}{nH(\Delta)}\right]}{\Psi'(\Delta)}.$$

Proof. As

$$\sum_i x_i = 0,$$

one has

$$\sum_i x_i = P\left(nQ + \sum_i \frac{1}{\Delta - \eta_i}\right) = 0.$$

But

$$\sum_i \frac{1}{\Delta - \eta_i} = \frac{H'(\Delta)}{H(\Delta)};$$

hence

$$Q(\Delta) = -\frac{H'(\Delta)}{nH(\Delta)}.$$

Now if P and P_1 are two polynomials satisfying the conditions of the proposition, then

$$\frac{P_1}{P},$$

is left invariant by all linear fractional transformations, therefore it is a rational function with numerical coefficients of $\Psi(\Delta)$. As the latter is supposed to be equal to $U \in \mathbb{F}$, $\dfrac{P_1}{P} \in \mathbb{F}$. P_1 differs from P only by a multiplicative factor belonging to $\mathbb{F}$. Suppressing this factor is equivalent by the below representation of x_i:

$$x_i := P(\Delta)\left[\frac{1}{\Delta - \eta_i} + Q(\Delta)\right];$$

to multiply x_i by an element of $\mathbb{F}$. This does not change any of the fundamental properties of the anharmonics (their Galois group and the constance of the cross-ratio). Therefore it suffices to get an arbitrary $P(\Delta)$, in order to solve the problem. Let Ψ' be the derivative of the absolute invariant. As

$$\Psi\left(\frac{a\Delta+b}{c\Delta+d}\right)=\Psi(\Delta),$$

then

$$\Psi'\left(\frac{a\Delta+b}{c\Delta+d}\right)=\Psi'(\Delta)\frac{\partial\Delta}{\partial\left(\dfrac{a\Delta+b}{c\Delta+d}\right)}=\Psi'(\Delta)\frac{(c\Delta+d)^2}{ad-bc}.$$

Now we just take

$$P(\Delta):=\frac{1}{\Psi'(\Delta)}.$$

This completes the proof. ■

One has therefore for expression of the x_i

$$x_i=\frac{\left[\dfrac{1}{\Delta-\eta_i}-\dfrac{H'(\Delta)}{nH(\Delta)}\right]}{\Psi'(\Delta)}:=\mathfrak{f}(\Delta,\eta_i) \tag{3.4.4}$$

with

$$\Psi(\Delta)=U \quad \text{and} \quad H(\eta_i)=0.$$

We now show that the general solution of the Riccati equation admits a representation in the form $u=\mathfrak{f}(\Delta,C)$, C an arbitrary constant.

Lemma 3.4.4 *The general solution u of the given Riccati equation* (3.1.2) *can be written in the form*

$$u=\mathfrak{f}(\Delta,C).$$

Proof. Take three solutions x_1, x_2, x_3 with respective representations

$$x_1=\mathfrak{f}(\Delta,\eta_1),\quad x_2=\mathfrak{f}(\Delta,\eta_2),\quad x_3=\mathfrak{f}(\Delta,\eta_3).$$

Define ζ:

$$u=\mathfrak{f}(\Delta,\zeta).$$

One knows that the cross-ratio of u, x_1, x_2, x_3, is a constant K_c . So

$$\frac{(u-x_1)(x_3-x_1)}{(x_3-x_2)(u-x_2)}=K_c.$$

But by very construction

$$\frac{(u-x_1)(x_3-x_2)}{(x_3-x_1)(u-x_2)}=\frac{(\zeta-\eta_1)(\eta_3-\eta_2)}{(\eta_3-\eta_1)(\zeta-\eta_2)}=K_c.$$

So ζ is a constant whose value depends on K_c. It is the arbitrary constant C. ■

Now we have that

$$\Psi(\Delta) = U$$

and

$$u = \mathfrak{f}(\Delta, C).$$

This leads to

$$C = \Delta - \left[u\Psi'(\Delta) + \frac{H'(\Delta)}{nH(\Delta)} \right]^{-1}.$$

Taking the derivative $\dfrac{d}{d\Delta}$, we get

$$\Psi' \frac{du}{d\Delta} + u\Psi'' + \frac{d}{d\Delta}\left(\frac{H'}{nH}\right) + \left(u\Psi' + \frac{H'}{nH}\right)^2 = 0. \tag{3.4.5}$$

This gives the form of the Riccati equation (3.1.1) associated to the algebraic equation (3.1.2) all of whose roots are solutions of the Riccati equation (3.1.1).

4 Equivalence problem of certain second order ODEs

4.1 Introduction

Our aim in the present chapter is the study of second order ODEs $y'' = f(x, y, y')$ under the pseudo-group of area preserving transformations

$$\begin{cases} X = \chi(x, y) \\ Y = \phi(x, y) \quad \text{with } J := \chi_x \phi_y - \chi_y \varphi_x \equiv 1; \end{cases} \tag{4.1.1}$$

using the Cartan equivalence method. The equivalence of second order ODEs (ordinary differential equations) was studied by various authors; most notably Cartan himself in [10], and before him by Tresse in [57, 58]. Cartan associated to every second order ODE a normal projective connection in the sense that it is uniquely characterized by the function f and its derivatives. In [38, 39] second order ODEs under general point transformations were studied under a modern perspective. Moreover in [43, lemma.2.2], Ozawa and Sato studied the linearization under area preserving transformations of the class of second order ODEs with

$$f(x, y, y') = A(x, y)y'^3 + 3B(x, y)y'^2 + 3C(x, y)y' + D(x, y).$$

The main point here is the analysis of the area preserving hypothesis in full generality. More precisely we associate to each second order ODE studied under area preserving transformations, a uniquely defined normal affine connection over the first jet space analogous to the normal projective connection built by Cartan in [10]. Moreover we rediscover in more geometric terms the conditions for the linearization of [43, lemma2.2].

The remark is that general point transformations give rise to a projective geometry associated to second order ODEs

$$y'' = f(x, y, y')$$

while the area preserving transformations associate to second order ODEs an affine geometry.

4.2 Equivalence of second order ODEs under area preserving transformations

Given two ODEs

$$y'' = f(x, y, y') \tag{4.2.1}$$

and

$$Y'' = F(X, Y, Y'), \tag{4.2.2}$$

they are considered point equivalent if there exists a local diffeomorphism $\Phi : U \to V$ which maps solutions of equation (4.2.1) to those of equation (4.2.2), with U and V open subsets of $J^1(\mathbb{R}, \mathbb{R})$ respectively associated to the source and the target equation. The problem we consider is therefore to determine local conditions in terms of invariants for the existence of such transformations.

In all this chapter, $\mathfrak{D}$ will denote the total derivative

$$\mathfrak{D} := \partial_x + y'\partial_y + f\partial_{y'}.$$

4.2.1 Cartan Method

In order to apply the Cartan equivalence method we discussed in the first chapter, one has to express the differential equation (4.2.1) into a pfaffian system; this achieved by the following result.

Lemma 4.2.1 *The solutions of* (4.2.1) *are in one-to-one correspondence with the integral manifolds of the Pfaffian system with independence condition on* ω^3_U: $(I_f, \omega^3_U := dx)$ *on* $J^1(\mathbb{R}, \mathbb{R})$ *where* I_f *is generated as an algebraic ideal (relatively to the wedge product) by the 1-forms*

$$\omega^1_U := dy - y'dx, \quad \omega^2_U := dy' - f dx - \frac{1}{3} f_{y'}(dy - y'dx) \tag{4.2.3}$$

and (x, y, y') *are standard coordinates on* $J^1(\mathbb{R}, \mathbb{R})$ *in which the contact form is given by* ω^1_U.

Proof. An integral manifold of (I_f, ω_U^3) with independence condition on ω_U^3 is by very definition a curve $s : \mathbb{R} \to J^1(\mathbb{R}, \mathbb{R})$ such that $s^\star\omega_U^1 = s^\star\omega_U^2 = 0$ and $s^\star\omega_U^3 \neq 0$. The conditions $s^\star\omega_U^1 = 0$ and $s^\star\omega_U^3 \neq 0$ will be satisfied if and only there exists a map $c : \mathbb{R} \to \mathbb{R} : x \to c(x)$ such that $s = j^1 c$ where

$$j^1 c : \mathbb{R} \to J^1(\mathbb{R}, \mathbb{R}) : x \to \left(x, c(x), \frac{dc}{dx} \right).$$

The condition $s^\star\omega_U^2 = (j^1 c)^\star\omega_U^2 = 0$ is then equivalent to

$$\frac{d^2 c}{dx^2} = f \left(x, c, \frac{dc}{dx} \right).$$

■

We consider the coframe basis

$$\begin{aligned} \omega_U^1 &= dy - y' dx \\ \omega_U^2 &= dy' - f dx - \frac{1}{3} f_{y'} (dy - y' dx) \\ \omega_U^3 &= dx. \end{aligned} \tag{4.2.4}$$

From now on we eliminate dependence on open subset U. Under point transformations (4.1.1), the first derivative $\dfrac{dy}{dx}$ changes into $\dfrac{dY}{dX} = \dfrac{\mathfrak{D}(\phi)}{\mathfrak{D}(\chi)}$. Therefore ω^1 becomes

$$\bar{\omega}^1 := d\phi - \frac{\mathfrak{D}(\phi)}{\mathfrak{D}(\chi)} d(\chi) = \frac{J}{\chi_x + y'\chi_y} \omega^1 := u_1 \omega^1. \tag{4.2.5}$$

Equally ω^3 is changed into

$$\bar{\omega}^3 = \chi_y \omega^1 + (\chi_x + y'\chi_y)\omega^3. \tag{4.2.6}$$

Let's now look at the law of transformation of ω^2; one has the following lemma.

Lemma 4.2.2 *Under point transformations* (4.1.1), ω^2 is transformed in $\bar{\omega}^2$ with

$$\bar{\omega}^2 = \frac{J}{(\chi_x + y'\chi_y)^2} \omega^2 + \frac{1}{3} \frac{J_x + y' J_y}{(\chi_x + y'\chi_y)^2} \omega^1. \tag{4.2.7}$$

Proof. One has by very definition that

$$dY' = d \left(\frac{\mathfrak{D}(\phi)}{\mathfrak{D}(\chi)} \right). \tag{4.2.8}$$

Computing we get

$$\begin{aligned} dY' &= \frac{1}{(\chi_x + y'\chi_y)^2}\left(Jdy' + \left[C_1 + B_1 y' + A_1 y'^2\right] dx\right) \\ &+ \frac{1}{(\chi_x + y'\chi_y)^2}\left[C_2 + B_2 y' + A_2 y'^2\right] dy \end{aligned} \tag{4.2.9}$$

with

$$\begin{aligned} C_1 &= \chi_x\phi_{xx} - \phi_x\chi_{xx} \\ B_1 &= (\chi_x\phi_{xy} - \phi_x\chi_{xy}) + (\chi_y\phi_{xx} - \phi_y\chi_{xx}) \\ A_1 &= \chi_y\phi_{xy} - \phi_y\chi_x \\ C_2 &= \chi_x\phi_{xy} - \phi_x\chi_{xy} \\ B_2 &= (\chi_x\phi_{yy} - \phi_x\chi_{yy}) + (\chi_y\phi_{xy} - \phi_y\chi_{xy}) \\ A_2 &= \chi_y\phi_{yy} - \phi_y\chi_{yy}. \end{aligned} \tag{4.2.10}$$

Now we remark that on the set of solutions of equation (4.2.1), we have $dy = y'dx$, $dy' = fdx$ (with similar identities for the corresponding capital variables) and $dX = (\chi_x + y'\chi_y)dx$. Therefore

$$(\chi_x + y'\chi_y)^3 F = Jf + D + Cy' + By'^2 + Ay'^3 \tag{4.2.11}$$

with

$$D = C_1, \quad C = C_2 + B_1, \quad B = B_2 + A_1 \quad \text{and} \quad A = A_2.$$

One also sees that $B_2 = 2A_1 + \frac{\partial J}{\partial y}$, $B_1 = 2C_2 - \frac{\partial J}{\partial x}$ and finally $\frac{\partial Y'}{\partial y'} = \frac{J}{(\chi_x + y'\chi_y)^2}$. Consider now equation (4.2.11). Derivation of it with respect to y' yields

$$3\chi_y(\chi_x + y'\chi_y)^2 F + J(\chi_x + y'\chi_y)F_{Y'} = Jf_{y'} + 3Ay'^2 + 2By' + C \tag{4.2.12}$$

This gives

$$\begin{aligned} -\frac{1}{3}F_{Y'}(dY - Y'dX) &= \frac{\chi_y Jf(dy - y'dx)}{(\chi_x + y'\chi_y)^3} - \frac{1}{3}f_{y'}\frac{(dy - y'dx)}{(\chi_x + y'\chi_y)^2} \\ &+ \frac{\chi_y(Ay'^3 + By'^2 + Cy' + D)(dy - y'dx)}{(\chi_x + y'\chi_y)^3} \\ &- \frac{1}{3}\frac{(3Ay'^2 + 2By' + C)(dy - y'dx)}{(\chi_x + y'\chi_y)^2} \end{aligned} \tag{4.2.13}$$

and

$$
\begin{aligned}
dY' - FdX = &\frac{J}{(\chi_x + y'\chi_y)^2}dy' - \frac{Jf\chi_x}{(\chi_x + y'\chi_y)^3}dx - \frac{Jf\chi_y}{(\chi_x + y'\chi_y)^3}dy \\
&+ \frac{((C_1 + B_1y' + A_1y'^2)dx + (C_2 + B_2y' + A_2y'^2)dy)}{(\chi_x + y'\chi_y)^2} \\
&- \frac{(Ay'^3 + By'^2 + Cy' + D)(\chi_x dx + \chi_y dy)}{(\chi_x + y'\chi_y)^3}
\end{aligned}
\tag{4.2.14}
$$

Hence

$$
\begin{aligned}
\bar{\omega}^2 = &\frac{J}{(\chi_x + y'\chi_y)^2}\omega^2 \\
&+ \frac{[3(C_1 + B_1y' + A_1y'^2) + y'(3Ay'^2 + 2By' + C)]\,dx}{3(\chi_x + y'\chi_y)^2} \\
&- \frac{3\,[Ay'^3 + By'^2 + Cy' + D]}{3(\chi_x + y'\chi_y)^2}dx + \frac{[3(C_2 + B_2y' + A_2y'^2)]}{3(\chi_x + y'\chi_y)^2}dy \\
&- \frac{(3Ay'^2 + 2By' + C)}{3(\chi_x + y'\chi_y)^2}dy
\end{aligned}
\tag{4.2.15}
$$

Therefore

$$
\begin{aligned}
\bar{\omega}^2 = &\frac{J}{(\chi_x + y'\chi_y)^2}\omega^2 + \frac{[3B_1 - 2C_1 + (3A_1 - B)y']}{3(\chi_x + y'\chi_y)^2}y'dx \\
&+ \frac{[3C_2 - C + (3B_2 - 2B)y']}{3(\chi_x + y'\chi_y)^2}dy.
\end{aligned}
\tag{4.2.16}
$$

As

$$3C_2 - C = J_x, \quad 3A_1 - B = -J_y, \quad 3B_2 - 2B = J_y \quad \text{and} \quad 3B_1 - 2C = -J_x,$$

the result follows. ■

Consequently one sees that when $J \equiv 1$ (area preserving condition (4.1.1)), then

$$
\begin{aligned}
\overline{\omega}^1 &= \frac{1}{\chi_x + y'\chi_y}\omega^1, \\
\overline{\omega}^2 &= \frac{1}{(\chi_x + y'\chi_y)^2}\omega^2, \\
\overline{\omega}^3 &= \chi_y\omega^1 + (\chi_x + y'\chi_y)\omega^3.
\end{aligned}
$$

Therefore a class of equivalence of second order ODEs under point transformations (4.1.1) is given a local G-structure $U \times G_b$, ie a local sub-principal bundle of the bundle of frames over U. It is defined by the property that $(\omega^1, \omega^2, \omega^3)$ is defined on it and with the structural group parameterized by

$$G_b := \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_1^2 & 0 \\ u_2 & 0 & \dfrac{1}{u_1} \end{pmatrix}. \tag{4.2.17}$$

As we can set

$$u_1 := \frac{1}{\chi_x + y'\chi_y} \quad \text{and} \quad u_2 := \chi_y.$$

Two second order ODEs are locally equivalent under transformations (4.1.1) when their corresponding G_b-structures are.

On the local principal bundle $U \times G_b$ there exists three well-defined canonical one-forms $(\theta_1, \theta_2, \theta_3)$: the components of the canonical $\mathbb{R}^3$-valued form θ existing on the frame bundle of $U \subset J^1(\mathbb{R}, \mathbb{R})$. Let (x) denote x, y, y' and (g) be coordinates in G_b given by (4.2.17) such that (x, g) is a coordinate system on $U \times G_b$ compatible with the local trivialization. Then the θ^i at the point (x, g) are given by

$$\begin{aligned} \theta_1 &= u_1\omega^1 \\ \theta_2 &= u_1^2\omega^2 \\ \theta_3 &= u_2\omega^1 + \frac{1}{u_1}\omega^3 \end{aligned} \tag{4.2.18}$$

The equivalence condition of two second order ODEs translates on the one-forms associated to the G_b-structures as the existence of a diffeomorphism ϕ: $\phi^*(\bar{\theta}) = \theta$; $\bar{\theta}$ is associated with target equation and θ to the source one; see [40].

4.3 Connections on Principal Bundles

Before we discuss the equivalence problem (4.2.18), we recall with some details, the important notion of connection.

According to the current "standard model" of elementary particle physics, every fundamental force (boson) is associated with a kind of curvature. But the curvatures involved are not only the curvatures of space-time, but curvatures associated with the notion of a connection on a principal bundle. We assume that we are dealing with a right principal G-bundle. We remind that it is a quadruple (P, M, G, π). P is the total space, M the base manifold, G is a Lie group acting on the right on P and π the projection is a submersion (see definition 1.2.3). We will note it (the principal bundle) $\pi : P \to M$ or $G \to P \to M$.

Let $q \in P$, $x = \pi(q) \in M$, $p \in \pi^{-1}(x)$. Let the group right action be denoted by Φ:

$$\begin{aligned} &\Phi : G \times P \to P \\ &\Phi(g, p) = \Phi_g(p) := p.g \end{aligned} \tag{4.3.1}$$

We can think of the group as an action which pushes points in the bundle along the fibres.

4.3.1 Horizontal spaces, vertical spaces, and connections: the two view points

Taking p as previously. We define the vertical subspace $V_p \subset T_pP$ as

$$V_p = \ker(\pi_\star)_p, \text{with } \pi_\star \text{ the differential of } \pi.$$

A vector field X on P is vertical if $X_p \in V_p$ for any p. The Lie bracket of two vertical vector fields is again vertical. Indeed consider two vector fields X and Y belonging to V and $f \in C^\infty(M)$. We have

$$\begin{aligned} \pi_\star \left[X, Y\right]_p (f) &:= \left[X, Y\right]_p (f \circ \pi_\star) \\ &= X_p(Y(f \circ \pi)) - Y_p(X(f \circ \pi)) \\ &= X_p((\pi_\star \circ Y)(f)) - Y_p((\pi_\star \circ X)(f)) \\ &= 0; \end{aligned}$$

where we have used the fact that

$$(\pi_\star \circ X)(f) = X(f \circ \pi) \quad \text{and} \quad (\pi_\star \circ Y)(f) = Y(f \circ \pi).$$

Therefore the vertical subspaces define a G-invariant distribution $V \subset TP$; because $\pi \circ \Phi_g(p) = \pi$ implies that we have

$$(\Phi_g)_\star V_p = V_{p.g}.$$

In the absence of any extra structure, there is no natural way to choose a complement to V_p in T_pP. This is in a sense what a connection provides.

Definition 4.3.1 *A connection on P is a smooth choice of horizontal subspaces $H_p \subset T_pP$ complementary to V_p:*

$$T_pP = V_p \oplus H_p$$

and such that $(\Phi_g)_\star H_p = H_{p.g}$. This also means that a connection is a G-invariant distribution $H \subset TP$ complementary to V.

We visualize below the local representation of a connection (inspired by [60] and authorized by its author) and horizontal subspaces

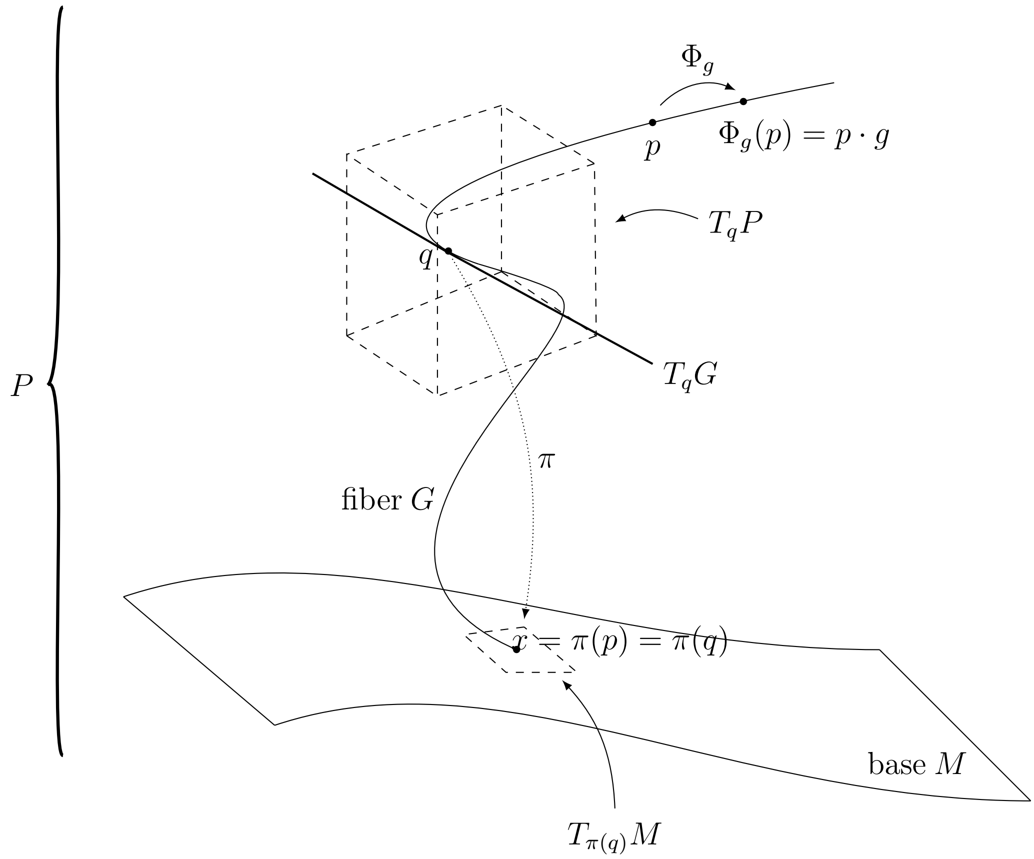

Figure 4.1: Visualizing a local representation of a principal bundle.

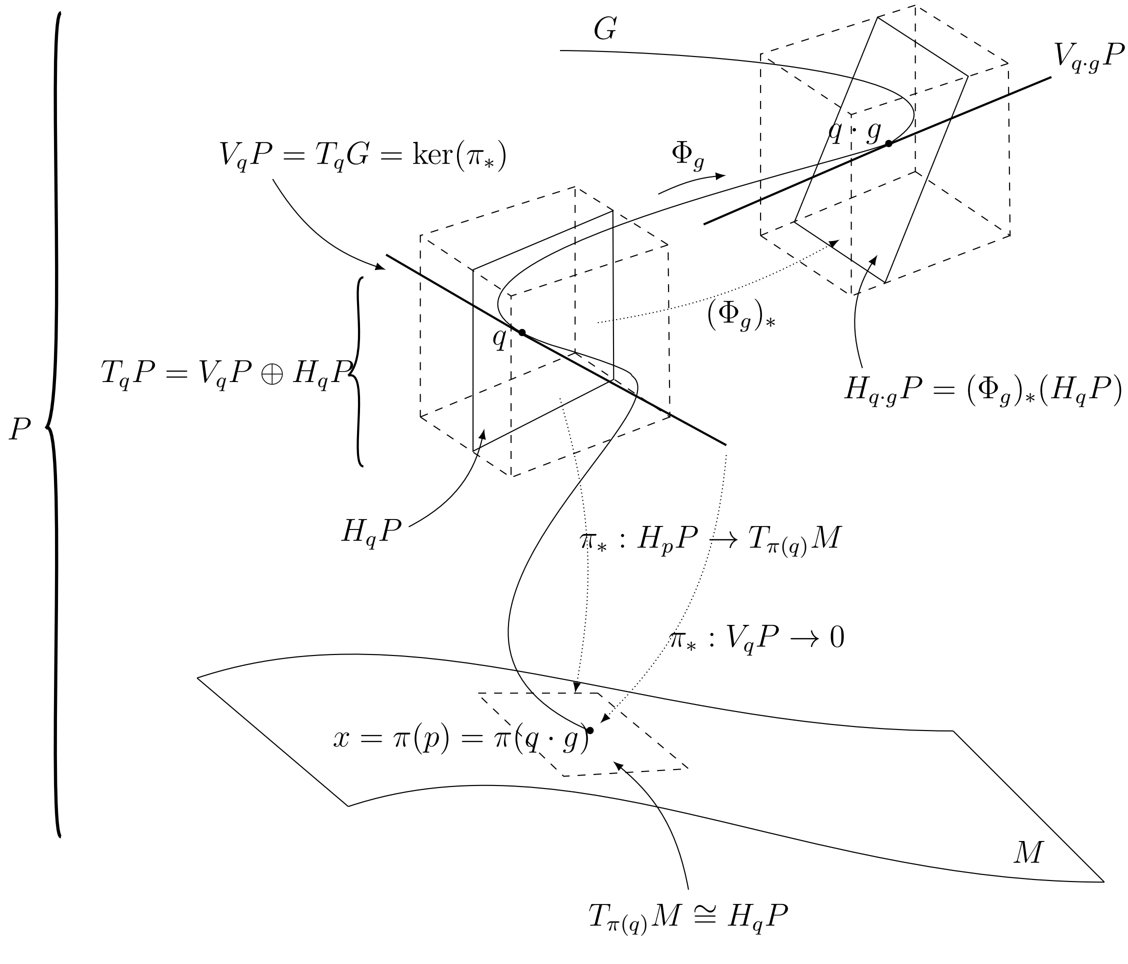

Figure 4.2: Horizontal spaces.

The action of G on P defines a map $\varrho : \mathfrak{g} \to \mathcal{H}(P)$, with $\mathcal{H}(P)$ the set of vector fields on P. It assigns to every $v \in \mathfrak{g}$, the vector field $\varrho(v)$ whose value at p is given by

$$\varrho_p(v) = \frac{d}{dt}(p.e^{tv})|_{t=0}.$$

Notice that

$$\pi_\star \varrho_p(v) = \frac{d}{dt}\pi(p.e^{tv})|_{t=0} = \frac{d}{dt}\pi(p)|_{t=0} = 0,$$

whence $\varrho(v)$ is a vertical vector field. In fact, since G acts freely on P, the map $v \to \varrho_p(v)$ is an isomorphism between $\mathfrak{g}$ and V_p for every p.

Lemma 4.3.2

$$(\Phi_g)_\star \varrho(v) = \varrho(ad_{g^{-1}}v).$$

Proof. By definition, at $p \in P$ we have

$$\begin{aligned}
(\Phi_g)_\star \varrho_p(v) &= \frac{d}{dt}\Phi_g(p.e^{tv})|_{t=0} \\
&= \frac{d}{dt}(p.e^{tv}g)|_{t=0} \\
&= \frac{d}{dt}(p.gg^{-1}e^{tv}g)|_{t=0} \\
&= \frac{d}{dt}(p.ge^{tag_{g^{-1}}v})|_{t=0} \\
&= \varrho_{p.g}(ad_{g^{-1}}v).
\end{aligned} \tag{4.3.2}$$

■

The horizontal subspace $H_p \subset T_pP$ being a linear subspace, is annihilated by $k = \dim G$ linear equations $T_pP \to \mathbb{R}$ (its orthogonal). In other words, H_p is the kernel of k one-forms at p, the components of a one form ω at p with values in a k-dimensional vector space. There is a natural such vector space, namely the Lie algebra of G, and since ω annihilates horizontal vectors (vectors belonging to H_p for any p), it is defined by what it does on vertical vectors, and we do have a natural map $V_p \to \mathfrak{g}$ given by the inverse of ϱ_p. This prompts the following definition

Definition 4.3.3 *The connection one-form of a connection $H \subset TP$ is the $\mathfrak{g}$-valued one form ω defined by*

$$\omega(X) = \begin{cases} v & \text{if } X = \varrho(v)\ , \\ 0 & \text{if } X \text{ is horizontal;} \end{cases} \tag{4.3.3}$$

and obeys to the identity

$$(\Phi_g)_\star \omega = ad_{g^{-1}} \omega.$$

Now a form on P is said horizontal if it it annihilates the vertical vectors. The following picture shows how the previous definition works

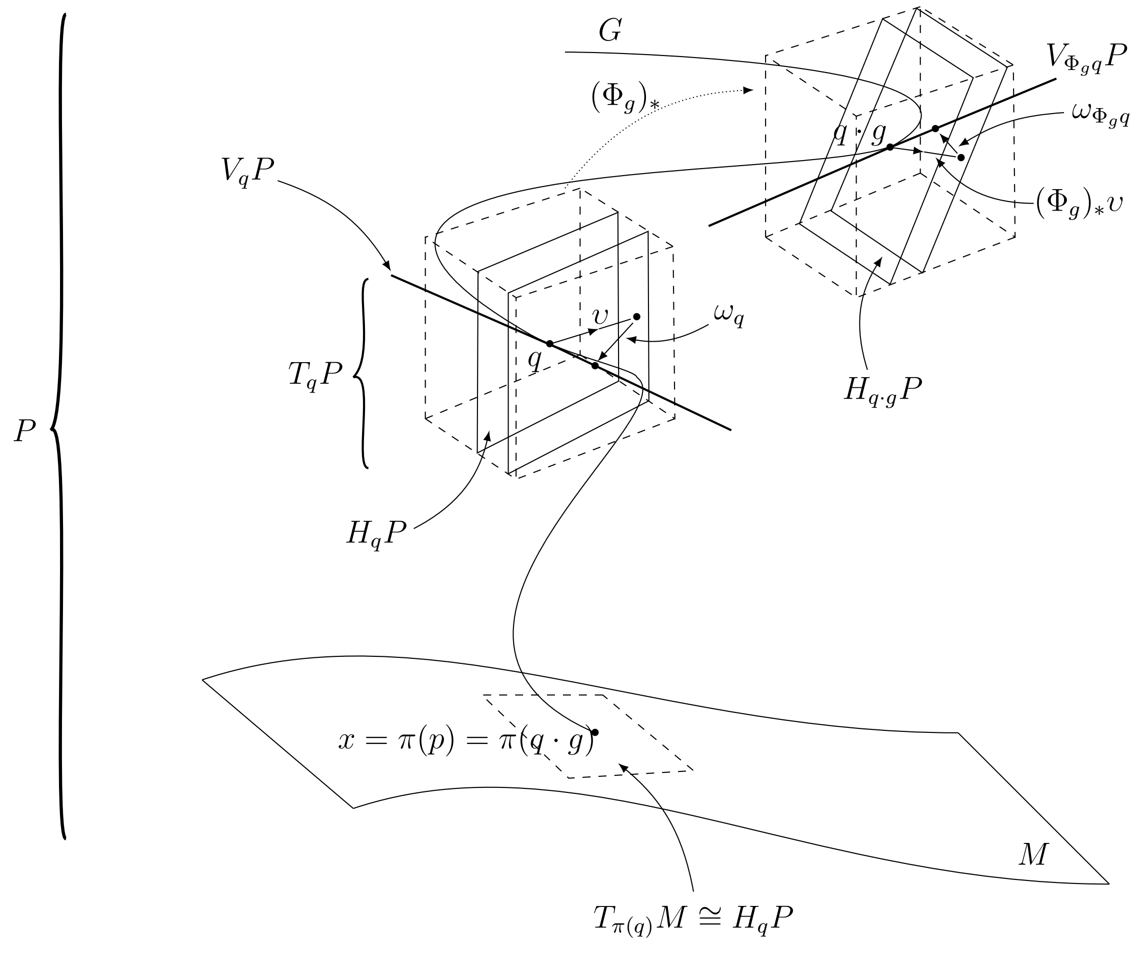

Figure 4.3: G-equivariance of the connection form.

Remark 4.3.4 *A choice of connection (form) is in fact a choice of projection of T_pP on V_p with the complement H_p varying in an equivariant manner* $(\Phi_{g\star}H_p = H_{p.g})$.

Let see how one can project T_pP on the vertical space V_p.

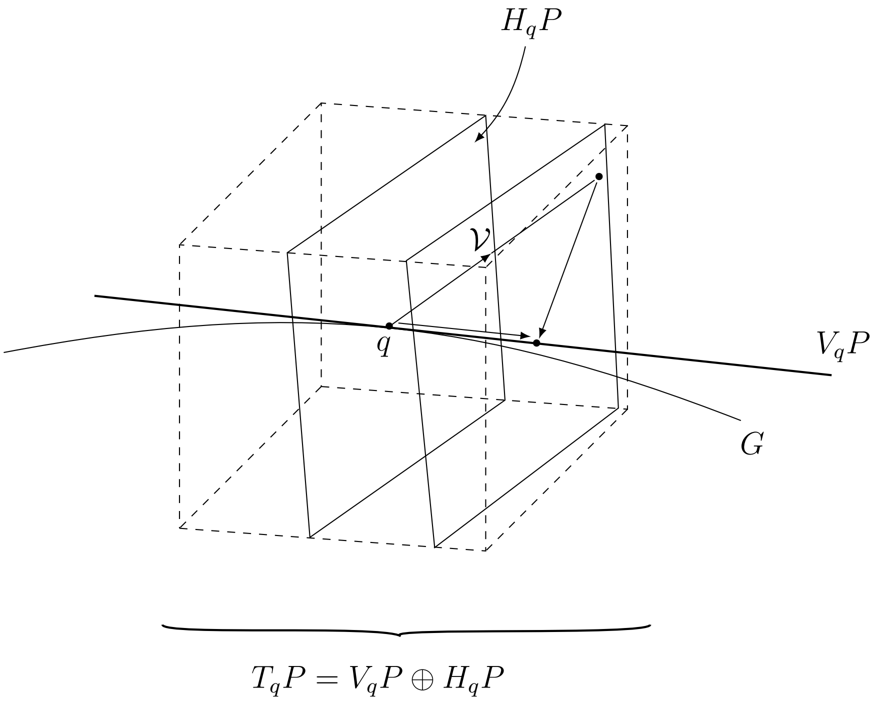

Figure 4.4: Using H_qP to project $\mathcal{V} \in T_qP$ on to V_qP.

Let ω_1 a 1-form with values in $\mathfrak{g}$ with and ω_2 another 1-form with values in $\mathfrak{g}$. We define the 2-form

$$[\omega_1, \omega_2] : (X, Y) \to [\omega_1(X), \omega_2(Y)] - [\omega_1(Y), \omega_2(X)]$$

Consider now a connection with form ω; its curvature is defined by the formula

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{4.3.4}$$

We can thus rewrite the previous equation in the form

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)].$$

When ω is given in matrix form, ie there exists a representation of $\mathfrak{g}$ in some $M_n(\mathbb{R})$, the matrix $[\omega(X), \omega(Y)]$ (the commutator of $\omega(X)$ and $\omega(Y)$) is just $\omega(X) \wedge \omega(Y)$:

$$[\omega(X), \omega(Y)] = \omega(X) \wedge \omega(Y);$$

because if $\omega = \omega_{ij}$ then

$$\omega \wedge \omega(X, Y) = \sum_k \omega_{ik} \wedge \omega_{kj}(X, Y) = (\omega_{ik}(X))(\omega_{kj}(Y)) - (\omega_{ik}(Y))(\omega_{kj}(X)),$$

with $(\omega_{ij}(X))$ denoting the matrix with components $\omega_{ij}(X)$.

The curvature hence takes the form

$$\Omega = d\omega + \omega \wedge \omega$$

Next we define with [50] the notions of Cartan geometry and connection

Definition 4.3.5 *A Cartan geometry* $\xi = (P, \omega, G, H)$ *on a manifold* M *modeled on* $(\mathfrak{g}, \mathfrak{h})$ *with group* H *consists of the following data:*

- *a smooth manifold* M
- *a principal right* H *bundle* P *over* M*.*
- *a* $\mathfrak{g}$*-valued* 1*-form* ω *on* P (called Cartan connection) which satisfies the following conditions:
  1. *for each point* $p \in P$*, the linear map* $\omega_p : T_pP \to \mathfrak{g}$ *is a linear isomorphism;*
  2. $(\Phi_h)^\star \omega = \mathrm{ad}_{\mathrm{h}^{-1}}\omega$ *for all* $h \in H$*;*
  3. $\omega(\varrho(v)) = v$ *for all* $v \in \mathfrak{h}$ *with* $\varrho(v)$ *the associated vector field.*

The curvature of a Cartan connection ω is defined analogously and verifies the relation

$$\Omega = d\omega + \omega \wedge \omega,$$

when ω is a matrix of one-forms. If it vanishes the model geometry is called flat.

Remark 4.3.6 *A Cartan connection in* P *is not a connection in the usual sense since it is not* $\mathfrak{h}$*-valued. It can however be considered as a connection in a certain larger bundle. One may check* [28] for further precisions.

4.4 Solution of the Equivalence Problem

We reconsider the Cartan equivalence problem with structural group (4.2.18). The first structure equations are

$$\begin{aligned}
d\theta_1 &= \pi_1 \wedge \theta_1 + \theta_3 \wedge \theta_2 + \frac{u_2}{u_1}\theta_2 \wedge \theta_1 + \frac{1}{3}u_1 f_{y'}\theta_3 \wedge \theta_1 \\
d\theta_2 &= 2\pi_1 \wedge \theta_2 + \frac{2}{3}u_1 f_{y'}\theta_3 \wedge \theta_2 + \frac{1}{3u_1}\left(f_{y'y'} - 2u_1u_2 f_{y'}\right)\theta_1 \wedge \theta_2 \\
&\quad + u_1^2\left(f_y + \frac{2}{9}f_{y'}^2 - \frac{1}{3}\mathfrak{D}(f_{y'})\right)\theta_3 \wedge \theta_1 \qquad (4.4.1)\\
d\theta_3 &= \frac{u_2}{u_1}\left(\pi_2 + \pi_1\right) \wedge \theta_1 - \pi_1 \wedge \theta_3 + \frac{u_2}{u_1}\theta_3 \wedge \theta_2 + \left(\frac{u_2}{u_1}\right)^2 \theta_2 \wedge \theta_1 \\
&\quad + \frac{1}{3}u_2 f_{y'}\theta_3 \wedge \theta_1;
\end{aligned}$$

with

$$\pi_1 = \frac{du_1}{u_1}, \quad \pi_2 = \frac{du_2}{u_2}.$$

Our next task is to do what Gardner [21] calls Lie Algebra compatible absorption of torsion. Such a process uses the linearity property of wedge product, in order to modify the one forms π_1 and π_2 (they are not well-defined for the moment). We now set

$$\begin{aligned}
\alpha_1 &= \pi_1 + \frac{u_2}{u_1}\theta_2 + \frac{1}{3}u_1 f_{y'}\theta_3 + \frac{1}{6u_1}\left(f_{y'y'} - 2u_1u_2 f_{y'}\right)\theta_1 \\
\alpha_2 &= \frac{u_2}{u_1}\left(\pi_1 + \pi_2\right) + \left(\frac{u_2}{u_1}\right)^2 \theta_2 + \frac{1}{3}u_2 f_{y'}\theta_3 - \frac{1}{6u_1}\left(f_{y'y'} - 2u_1u_2 f_{y'}\right)\theta_3.
\end{aligned} \qquad (4.4.2)$$

At the end of this first process of absorption, one therefore has the following new structure equations

$$\begin{aligned}
d\theta_1 &= \alpha_1 \wedge \theta_1 + \theta_3 \wedge \theta_2 \\
d\theta_2 &= 2\alpha_1 \wedge \theta_2 + u_1^2\left(f_y + \frac{2}{9}f_{y'}^2 - \frac{1}{3}\mathfrak{D}(f_{y'})\right)\theta_3 \wedge \theta_1 \\
d\theta_3 &= \alpha_2 \wedge \theta_1 - \alpha_1 \wedge \theta_3.
\end{aligned} \qquad (4.4.3)$$

At this stage we do not have any way for further reduction process because as we see in equation (4.4.3), $f_y + \frac{2}{9}f_{y'}^2 - \frac{1}{3}\mathfrak{D}(f_{y'})$ is an invariant for second order ODEs submitted to area preserving transformations and we want to treat the problem in a unified manner.

The reduced Cartan character is $\sigma_1 = 2$ (the others vanish) and the structure equations admit the freedom

$$\begin{cases} \alpha_1 \to \alpha_1 \\ \beta_1 \to \beta_1 + t_1\theta_1. \end{cases} \tag{4.4.4}$$

Thus the indetermination degree is one; therefore we have to prolong the equivalence problem. The idea of the process of prolongation is the following: one takes into account the freedom of the coframes given by the identities (4.4.4) and lift the equivalence problem on the bundle $U \times G_b \times G^{prol}$ with

$$G^{prol} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

where the block matrix t, is

$$\begin{pmatrix} 0 & 0 \\ t_1 & 0 \end{pmatrix}.$$

On the bundle $U \times G_b \times G^{prol}$, there are five fixed one forms (θ_i, Ω_μ), $1 \le i \le 3$, $1 \le \mu \le 2$ given by the following formula

$$\begin{pmatrix} \theta_i \\ \Omega_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} \theta_i \\ \alpha_1 \\ \beta_1 \end{pmatrix} \tag{4.4.5}$$

Therefore the new coframe on $U \times G_b \times G^{prol}$ read

$$\begin{aligned}
\theta_1 &= u_1\omega^1 \\
\theta_2 &= u_1^2\omega^2 \\
\theta_3 &= u_2\omega^1 + \frac{1}{u_1}\omega^3 \\
\Omega_1 &= \pi_1 + \frac{u_2}{u_1}\theta_2 + \frac{1}{3}u_1 f_{y'}\theta_3 + \frac{1}{6u_1}\left(f_{y'y'} - 2u_1u_2f_{y'}\right)\theta_1 \\
\Omega_2 &= \frac{u_2}{u_1}\left(\pi_1 + \pi_2\right) + \left(\frac{u_2}{u_1}\right)^2\theta_2 + \frac{1}{3}u_2 f_{y'}\theta_3 \\
&\quad - \frac{1}{6u_1}\left(f_{y'y'} - 2u_1u_2f_{y'}\right)\theta_3 + t_1\theta_1.
\end{aligned} \tag{4.4.6}$$

We compute $d\Omega_1$:

$$
\begin{aligned}
d\Omega_1 = {} & \Omega_2 \wedge \theta_2 \\
& + \left(t_1 + \frac{1}{6u_1^3} f_{y'y'y'} + \frac{2}{3}\frac{u_2^2}{u_1} f_{y'} - \frac{1}{2}\frac{u_2}{u_1^2} f_{y'y'} \right) \theta_2 \wedge \theta_1 \\
& + \left(-\frac{1}{3} f_{y'y} - \frac{1}{18} f_{y'} f_{y'y'} + \frac{1}{6}\mathfrak{D}(f_{y'y'}) \right) \theta_3 \wedge \theta_1 \\
& + u_1 u_2 \left(f_y - \frac{1}{3}\mathfrak{D}(f_{y'}) + \frac{2}{9} f_{y'}^2 \right) \theta_3 \wedge \theta_1 .
\end{aligned}
\tag{4.4.7}
$$

The computation of $d\Omega_2$ gives

$$
\begin{aligned}
d\Omega_2 = {} & dt_1 \wedge \theta_1 + 3t_1 \Omega_1 \wedge \theta_1 + 2\Omega_2 \wedge \Omega_1 \\
& + \left(-\frac{1}{2u_1} f_{y'y'} + \frac{4}{3} u_2 f_{y'} \right) \Omega_2 \wedge \theta_1 \\
& + \left(t_1 + \frac{2}{3}\frac{u_2^2}{u_1} f_{y'} + \frac{1}{6u_1^3} f_{y'y'y'} - \frac{1}{2} f_{y'y'} \frac{u_2}{u_1^2} \right) \theta_3 \wedge \theta_2 \\
& - \frac{u_2^2}{3u_1^3} \left(f_{y'y'} - 2u_1 u_2 f_{y'} \right) \theta_2 \wedge \theta_1 \\
& + \frac{1}{u_1^2} \left(\frac{1}{6} \left(f_{y'y'y} + \frac{1}{3} f_{y'} f_{y'y'y'} \right) - \frac{1}{36} f_{y'y'}^2 \right) \theta_3 \wedge \theta_1 \\
& + \frac{u_2}{u_1} \left(-\frac{1}{18} f_{y'} f_{y'y'} - \frac{1}{6}\mathfrak{D}(f_{y'y'}) - \frac{2}{3} f_{y'y} \right) \theta_3 \wedge \theta_1 \\
& + u_2^2 \left(f_y + \frac{1}{3}\mathfrak{D}(f_{y'}) \right) \theta_3 \wedge \theta_1 .
\end{aligned}
\tag{4.4.8}
$$

We recollect the torsion and do reduction by setting

$$
t_1 = -\frac{2}{3}\frac{u_2^2}{u_1} f_{y'} - \frac{1}{6u_1^3} f_{y'y'y'} + \frac{1}{2}\frac{u_2}{u_1^2} f_{y'y'} ;
$$

we obtain an $\{e\}$-structure on a sub-bundle $\mathfrak{P} = U \times G_b$.

$$
\begin{aligned}
d\theta_1 &= \Omega_1 \wedge \theta_1 + \theta_3 \wedge \theta_2 \\
d\theta_2 &= 2\Omega_1 \wedge \theta_2 + u_1^2 \left(f_y + \frac{2}{9} f_{y'}^2 - \frac{1}{3} \mathfrak{D}(f_{y'}) \right) \theta_3 \wedge \theta_1 \\
d\theta_3 &= \Omega_2 \wedge \theta_1 - \Omega_1 \wedge \theta_3 \\
d\Omega_1 &= \Omega_2 \wedge \theta_2 + \left(-\frac{1}{3} f_{y'y} - \frac{1}{18} f_{y'} f_{y'y'} + \frac{1}{6} \mathfrak{D}(f_{y'y'}) \right) \theta_3 \wedge \theta_1 \\
&\quad + u_1 u_2 \left(f_y - \frac{1}{3} \mathfrak{D}(f_{y'}) + \frac{2}{9} f_{y'}^2 \right) \theta_3 \wedge \theta_1 \\
d\Omega_2 &= 2\Omega_2 \wedge \Omega_1 - \frac{1}{6u_1^5} f_{y'y'y'y'} \theta_2 \wedge \theta_1 \\
&\quad + \frac{1}{u_1^2} \left(\frac{1}{6} f_{y'y'y} - \frac{1}{6} \mathfrak{D}(f_{y'y'y'}) - \frac{1}{9} f_{y'} f_{y'y'y'} + \frac{1}{18} f_{y'y'}^2 \right) \theta_3 \wedge \theta_1 \\
&\quad + 2\frac{u_2}{u_1} \left(-\frac{1}{18} f_{y'} f_{y'y'} + \frac{1}{6} \mathfrak{D}(f_{y'y'}) - \frac{1}{3} f_{y'y} \right) \theta_3 \wedge \theta_1 \\
&\quad + u_2^2 \left(f_y - \frac{1}{3} \mathfrak{D}(f_{y'}) + \frac{2}{9} f_{y'}^2 \right) \theta_3 \wedge \theta_1
\end{aligned} \tag{4.4.9}
$$

In the previous structure equations all the non-zero coefficients are invariants of the equivalence problem. Moreover we can use them in order to give necessary and sufficient conditions for equivalence under area preserving transformations to a given equation. We first have the following proposition

Proposition 4.4.1 *The invariants of the equation* (4.4.9) vanish if and only if

1. $f_{y'y'y'y'} = 0$, ie

$$f(x, y, y') = A(x, y)y'^3 + 3B(x, y)y'^2 + 3C(x, y)y' + D(x, y)$$

2. and

$$
\begin{aligned}
D_y - C_x &= 2(BD - C^2) \\
C_y - B_x &= (AD - BC) \\
B_y - A_x &= 2(AC - B^2).
\end{aligned} \tag{4.4.10}
$$

Proof. From the structure equations (4.4.9), one sees that the invariants vanish if and only if

$$
\begin{aligned}
&\frac{1}{6} f_{y'y'y'y'} = 0 \\
&-\frac{1}{6}\left(2f_{y'y} + \frac{1}{3} f_{y'} f_{y'y'} - \mathfrak{D}(f_{y'y'})\right) = 0 \\
&f_y + \frac{2}{9} f_{y'}^2 - \frac{1}{3}\mathfrak{D}(f_{y'}) = 0 \\
&-\frac{1}{18}\left(f_{y'y'}^2 - 2f_{y'} f_{y'y'y'}\right) - \frac{1}{6}\left(f_{y'y'y} - \mathfrak{D}\left(f_{y'y'y'}\right)\right) = 0.
\end{aligned}
\tag{4.4.11}
$$

The first condition gives us the form

$$
f = A(x,y)y'^3 + 3B(x,y)y'^2 + 3C(x,y)y' + D(x,y)
$$

for the function f. Using this we get three other conditions (by exploiting the just given form of f)

$$
\begin{aligned}
&(D_y - C_x - 2(BD - C^2)) + 2(C_y - B_x - (AD - BC))y' \\
&+ (B_y - A_x - 2(AC - B^2))y'^2 = 0 \\
&6(C_y - B_x - (AD - BC)) + 6(B_y - A_x - 2(AC - B^2))y' = 0 \\
&A_x - B_y + 2(AC - B^2) = 0
\end{aligned}
\tag{4.4.12}
$$

with the last relations resulting from the last three relations of equation (4.4.11) respectively. Therefore if the specified conditions hold, all the invariants vanish and conversely as a polynomial in one variable vanishes if and only if its coefficients vanish do. ■

In the most symmetric case when we deal with the equivalence problem of

$$
y'' = 0
$$

Then the structure equations become the Maurer-Cartan equations for the special affine Lie algebra

$$
\mathfrak{asl}(2,\mathbb{R}) = \mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R});
$$

and $\mathfrak{P}$ is locally the special affine Lie group

$$
ASL(2,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix}, x \in \mathbb{R}^2, A \in SL(2,\mathbb{R}) \right\}.
$$

One thus has the following corollary

Corollary 4.4.2 *A second order ordinary differential equation $y'' = f(x,y,y')$ is equivalent under an area preserving transformation to $y'' = 0$ if and only if*

$$\begin{aligned} &f(x,y,y') = A(x,y)y'^3 + 3B(x,y)y'^2 + 3C(x,y)y' + D(x,y) \\ &D_y - C_x = 2(BD - C^2) \\ &C_y - B_x = (AD - BC) \\ &B_y - A_x = 2(AC - B^2). \end{aligned} \tag{4.4.13}$$

Moreover, there is a unique equivalence class of second order ordinary differential equations admitting the maximal Lie group of symmetries $ASL(2,\mathbb{R})$, namely the equivalence class of $y'' = 0$.

Proof. The conditions given by (4.4.13) are verified by $y'' = 0$. Since they are equivalent (from the previous proposition) to the vanishing of the invariants of the structure equations (4.4.9), every equation equivalent to $y'' = 0$ under area preserving transformations must satisfy them. The second part is proved by observing that if we assume the invariants of the structure equations are constant, then by the Jacobi identity they all vanish and we end up with the Maurer-Cartan equations of $ASL(2,\mathbb{R})$. ■

This solves in particular the linearization problem and rediscovers the condition given in [43, lemma2.2]

4.4.1 Further reduction

When $f_{y'y'y'y'} \neq 0$ one can continue the reduction procedure and get an invariant coframe on U. In effect reconsider equation (4.2.11); deriving it twice with respect to y' and using the fact that $\dfrac{\partial Y'}{\partial y'} = \dfrac{J}{(\chi_x + y'\chi_y)^2}$, one gets

$$\frac{J^2}{(\chi_x + y'\chi_y)} F_{Y'Y'} + 4J\chi_y F_{Y'} + 6\chi_y(\chi_x + y'\chi_y)F = Jf_{y'y'} + 6Ay' + B. \tag{4.4.14}$$

A new derivation yields

$$\frac{J^3}{(\chi_x + y'\chi_y)^3} F_{Y'Y'Y'} + \frac{3J^2\chi_y F_{Y'Y'}}{(\chi_x + y'\chi_y)^2} + \frac{6J\chi_y^2 F_{Y'}}{(\chi_x + y'\chi_y)} = Jf_{y'y'y'} + 6A \tag{4.4.15}$$

and a final derivation gives the relative differential invariant $I_0 := f_{y'y'y'y'}$ satisfying

$$\frac{J^3}{(\chi_x + y'\chi_y)^5} F_{Y'Y'Y'Y'} = f_{y'y'y'y'}. \tag{4.4.16}$$

Assume $f_{y'y'y'y'}$ does not vanish identically and set

$$f_{y'y'y'y'} = e^{5g} \quad \text{and} \quad F_{Y'Y'Y'Y'} = e^{5G}.$$

One has

$$\begin{cases} dX = (\chi_x + y'\chi_y)dx + \chi_y(dy - y'dx) \\ dY = (\phi_x + y'\phi_y)dx + \phi_y(dy - y'dx). \end{cases} \tag{4.4.17}$$

The relation (4.4.16) written logarithmically yields

$$G - \log(\chi_x + y'\chi_y) = g - \frac{3}{5}\log(J).$$

Derivation of the previous identity once with respect to y' gives

$$G_{Y'}\frac{J}{(\chi_x + y'\chi_y)^2} - \frac{\chi_y}{(\chi_x + y'\chi_y)} = g_{y'}. \tag{4.4.18}$$

From the previous identity one finds:

$$G_{Y'} = \frac{\chi_y(\chi_x + y'\chi_y)}{J} + g_{y'}\frac{(\chi_x + y'\chi_y)^2}{J}. \tag{4.4.19}$$

Therefore

$$G_{Y'}(dY - Y'dX) = \chi_y(dy - y'dx) + g_{y'}(dy - y'dx)(\chi_x + y'\chi_y), \tag{4.4.20}$$

using equation (4.2.5). Subtracting the first identity of (4.4.17), one gets

$$dX - G_{Y'}(dY - Y'dX) = (\chi_x + y'\chi_y)\left(dx - g_{y'}(dy - y'dx)\right)$$

So we have the following three semi-invariant one forms

$$\begin{aligned} \varsigma_1 &= e^{-G}\left(dX - G_{Y'}(dY - Y'dX)\right) \\ &= J^{\frac{3}{5}}e^{-g}\left(dx - g_{y'}(dy - y'dx)\right) \\ \varsigma_2 &= e^{G}\left(dY - Y'dX\right) = J^{\frac{2}{5}}e^{g}\left(dy - y'dx\right) \\ \varsigma_3 &= e^{2G}\left(dY' - FdX - \frac{1}{3}F_{Y'}(dY - Y'dX)\right) \\ &= J^{-\frac{1}{5}}e^{2g}\left(dy' - fdx - \frac{1}{3}f_{y'}(dy - y'dx)\right) \\ &+ \frac{1}{3}J^{-\frac{6}{5}}(J_x + y'J_y)e^{2g}\left(dy - y'dx\right). \end{aligned} \tag{4.4.21}$$

We denote the analog coframe to ς_i, without the capital letters, again by ς_i. Taking the exterior derivatives of these latter one-forms one obtains

$$\begin{aligned} d\varsigma_1 &= A_0\varsigma_3 \wedge \varsigma_2 + B_0\varsigma_1 \wedge \varsigma_2 \\ d\varsigma_2 &= -\varsigma_3 \wedge \varsigma_1 + C_0\varsigma_1 \wedge \varsigma_2 \\ d\varsigma_3 &= D_0\varsigma_3 \wedge \varsigma_1 + E_0\varsigma_3 \wedge \varsigma_2 + I_0\varsigma_1 \wedge \varsigma_2. \end{aligned} \tag{4.4.22}$$

With

$$\begin{aligned}
A_0 &= e^{-4g}\left(g_{y'}^2 - g_{y'y'}\right)\\
B_0 &= e^{-g}\left(g_y + g_{y'}\mathfrak{D}(g) - \mathfrak{D}(g_{y'})\right)\\
C_0 &= e^{g}\left(\frac{1}{3}f_{y'} + \mathfrak{D}(g)\right)\\
D_0 &= -2C_0\\
E_0 &= -e^{-g}\left(\left(\frac{4}{3}f_{y'}g_{y'} + \frac{1}{3}f_{y'y'}\right) + 2\left(g_y + g_{y'}\mathfrak{D}(g)\right)\right)\\
I_0 &= e^{2g}\left(f_y + \frac{2}{9}f_{y'}^2 - \frac{1}{3}\mathfrak{D}(f_{y'})\right)
\end{aligned} \tag{4.4.23}$$

Setting $J \equiv 1$, the σ_i become invariant one-forms and the coefficients A_0, B_0, C_0, D_0, E_0, I_0 become invariants of the differential equation. All the other invariants are expressed in terms of them by taking covariant derivatives with respect to the coframe dual to $\varsigma_1, \varsigma_2, \varsigma_3$.

4.5 Normal Cartan Affine Connection

Reexamining the case of the equation

$$y'' = 0$$

one can write equation (4.4.9) in matrix form as:

$$d\omega_0 + \omega_0 \wedge \omega_0 = 0,$$

with

$$\omega_0 = \begin{pmatrix} 0 & 0 & 0 \\ \theta_3 & \Omega_1 & -\Omega_2 \\ \theta_1 & \theta_2 & -\Omega_1 \end{pmatrix}.$$

Consider the two dimensional sub-algebra $\mathfrak{h}$ of $\mathfrak{asl}(2,\mathbb{R})$ annihilated by the ideal

$$< \theta_1, \theta_2, \theta_3 >;$$

generated by the one forms θ_1, θ_2, θ_3. As the latter are are closed under exterior differentiation meaning that

$$d\theta_i \wedge \theta_1 \wedge \theta_2 \wedge \theta_3 = 0, \quad i = 1, 2, 3;$$

one sees therefore from Frobenius theorem [54] that $\mathfrak{h}$ is completely integrable and $U \times G_b$ is foliated by two dimensional leaves tangent to $\mathfrak{h}$.

Thus if H_2 be the connected simply-connected Lie subgroup of $ASL(2,\mathbb{R})$ with Lie algebra $\mathfrak{h}$, parameterized in the following manner:

$$H_2 := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_1 & -u_2 \\ 0 & 0 & \dfrac{1}{u_1} \end{pmatrix}, \quad \text{with } u_1,\, u_2 \in \mathbb{R} \right\}$$

one sees that we have a principal bundle

$$H_2 \to \mathfrak{P} \xrightarrow{\pi} U;$$

it is a homogeneous space and

$$(ASL(2,\mathbb{R}), H_2)$$

is a flat Cartan connection over $U \times G_b$ because its curvature vanishes.

We want to associate to a general second order ordinary differential equation submitted to area preserving transformations, a $\mathfrak{asl}(2,\mathbb{R})$-valued Cartan connection on an appropriate local principal bundle over some open set U of $J^1(\mathbb{R},\mathbb{R})$, with structure group H_2.

In order to do so, we start with a connection matrix ω with values in the special affine algebra $\mathfrak{asl}(2,\mathbb{R})$ and work with local coordinates (x, y, y', u_1, u_2) compatible with the local trivialization $U \times G_b$. We fix the cross-section $\sigma : U \to U \times G_b$ characterized by

$$(u_1 = 1, u_2 = 0);$$

let ϑ the value of the connection ω on the section σ. We impose some natural conditions on ϑ, and then lift it to a connection matrix on $U \times G_b$ via the formula (with $h \in H_2$ a generic element)

$$\omega = h^{-1}\vartheta h + h^{-1}dh. \tag{4.5.1}$$

$$h^{-1}dh$$

is the left invariant Maurer-Cartan form on H_2. Moreover the curvature is in this case given by the formula

$$d\omega + \omega \wedge \omega = h^{-1}(d\vartheta + \vartheta \wedge \vartheta)h \tag{4.5.2}$$

as one has the Maurer-Cartan equation for the Lie group H_2

$$d(h^{-1}dh) + (h^{-1}dh) \wedge (h^{-1}dh) = 0.$$

The given one form ϑ can be written in component form as

$$\vartheta = \begin{pmatrix} 0 & 0 & 0 \\ \theta^1 & \theta^1_1 & \theta^1_2 \\ \theta^2 & \theta^2_1 & \theta^2_2 \end{pmatrix}. \tag{4.5.3}$$

Now since ϑ must be with values in $\mathfrak{asl}(2, \mathbb{R})$, then

$$\theta^1_1 + \theta^2_2 = 0;$$

the structure equations of this connection, derived from the structure equations of $\mathfrak{asl}(2, \mathbb{R})$ are thus given by

$$\begin{aligned} d\theta^1 &= -\theta^1_1 \wedge \theta^1 - \theta^1_2 \wedge \theta^2 + \Theta^1 \\ d\theta^2 &= -\theta^2_1 \wedge \theta^1 - \theta^2_2 \wedge \theta^2 + \Theta^2 \\ d\theta^1_1 &= -\theta^1_2 \wedge \theta^2_1 + \Theta^1_1 \\ d\theta^2_1 &= -2\theta^2_1 \wedge \theta^1_1 + \Theta^2_1 \\ d\theta^1_2 &= 2\theta^1_2 \wedge \theta^1_1 + \Theta^1_2. \end{aligned} \tag{4.5.4}$$

We want the connection matrix to define a connection over the open set U of $J^1(\mathbb{R}, \mathbb{R})$; so we choose a basis of one forms θ^1, θ^2, Π of U. We also want this basis to define the general solutions of equation (4.2.1) (modulo pull-back). Finally we look for a uniquely defined connection in term of the derivatives of f.

With the previous specifications, Θ^i and Θ^j_i must be horizontal with respect to the fibration $\pi : \mathfrak{P} \to U$ ie must be in the algebraic ideal with respect to wedge product, generated by $< \theta^1, \theta^2, \Pi >$. See [28, prop.2, page.220].

The structure equations (4.5.4) hence are more explicitly written in the form

$$\begin{aligned} d\theta^1 &= -\theta^1_1 \wedge \theta^1 - \theta^1_2 \wedge \theta^2 + \Theta^1 \\ d\theta^2 &= -\theta^2_1 \wedge \theta^1 - \theta^2_2 \wedge \theta^2 + \Theta^2 \\ d\theta^1_1 &= -\theta^1_2 \wedge \theta^2_1 + R^1_{113}\theta^1 \wedge \Pi + R^1_{123}\theta^2 \wedge \Pi + R^1_{112}\theta^1 \wedge \theta^2 \\ d\theta^2_1 &= -2\theta^2_1 \wedge \theta^1_1 + R^2_{113}\theta^1 \wedge \Pi + R^2_{123}\theta^2 \wedge \Pi + R^2_{112}\theta^1 \wedge \theta^2 \\ d\theta^1_2 &= 2\theta^1_2 \wedge \theta^1_1 + R^1_{213}\theta^1 \wedge \Pi + R^1_{223}\theta^2 \wedge \Pi + R^1_{212}\theta^1 \wedge \theta^2. \end{aligned} \tag{4.5.5}$$

We choose $\theta^1 = dx$, $\theta^2 = dy - y'dx$ and $\Pi = dy' - f dx + \mu(dy - y'dx)$. This implies $\Pi = \theta^2_1$.

We set now the following "normalization" condition in order to fix the connection:

$$\begin{aligned} \Theta^1 = \Theta^2 &= 0 \\ R^1_{113} = R^1_{123} = R^2_{113} &= R^2_{123} = 0. \end{aligned} \tag{4.5.6}$$

If Θ^1 and Θ^2 vanishes, then

$$\theta_1^1 \wedge dx + \theta_2^1 \wedge (dy - y'dx) = 0$$

and

$$-dy' \wedge dx = -(dy' - fdx + \mu(dy - y'dx)) \wedge dx - \theta_2^2 \wedge (dy - y'dx)$$

This gives

$$\begin{aligned}
\theta_1^1 &= -\mu dx + \delta(dy - y'dx) \\
\theta_2^1 &= \delta dx + \nu(dy - y'dx) \\
\theta_2^2 &= \mu dx - \delta(dy - y'dx)
\end{aligned} \tag{4.5.7}$$

for functions μ, δ, ν on $J^1(\mathbb{R}, \mathbb{R})$. Using the other "normalization" conditions, we get after computation

$$\begin{aligned}
R^1_{113} &= \mu_{y'} + 2\delta \\
R^1_{123} &= \nu - \delta_{y'} \\
R^2_{113} &= f_{y'} + 3\mu \\
R^2_{123} &= -(\mu_{y'} + 2\delta)
\end{aligned} \tag{4.5.8}$$

and this gives

$$\mu = -\frac{1}{3} f_{y'}, \quad \delta = \frac{1}{6} f_{y'y'}, \quad \nu = \frac{1}{6} f_{y'y'y'} \tag{4.5.9}$$

To summarize we have the following result

$$\begin{aligned}
\theta^1 &= dx \\
\theta^2 &= dy - y'dx \\
\theta_1^1 &= \frac{1}{3} f_{y'} dx + \frac{1}{6} f_{y'y'} (dy - y'dx) \\
\theta_2^1 &= \frac{1}{6} f_{y'y'} dx + \frac{1}{6} f_{y'y'y'} (dy - y'dx) \\
\theta_1^2 &= dy' - fdx - \frac{1}{3} f_{y'} (dy - y'dx).
\end{aligned} \tag{4.5.10}$$

Consider now the remaining components of the curvature of ϑ

$$\begin{aligned}
\Theta_1^2 &= R^2_{112} \theta^1 \wedge \theta^2 \\
\Theta_1^1 &= R^1_{112} \theta^1 \wedge \theta^2 \\
\Theta_2^1 &= R^1_{223} \theta^2 \wedge \theta_1^2 + R^1_{212} \theta^1 \wedge \theta^2.
\end{aligned} \tag{4.5.11}$$

We have

$$
\begin{aligned}
R^1_{223} &= -\frac{1}{6} f_{y'y'y'y'} \\
R^1_{112} &= -\frac{1}{6}\left(2f_{y'y} + \frac{1}{3} f_{y'} f_{y'y'} - \mathfrak{D}(f_{y'y'})\right) \\
R^2_{112} &= f_y + \frac{2}{9} f^2_{y'} - \frac{1}{3}\mathfrak{D}(f_{y'}) \\
R^1_{212} &= -\frac{1}{18}\left(f^2_{y'y'} - 2f_{y'} f_{y'y'y'}\right) - \frac{1}{6}\left(f_{y'y'y} - \mathfrak{D}\left(f_{y'y'y'}\right)\right)
\end{aligned}
\tag{4.5.12}
$$

For the previous "normalization" condition (4.5.6) to have geometrical meaning, the curvature must be invariant under the adjoint action *ad*. Now:

$$
\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{u_1} & u_2 \\ 0 & 0 & u_1 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 \\ 0 & R^1_{112}\theta^1 \wedge \theta^2 & R^1_{223}\theta^2 \wedge \theta^2_1 + R^1_{212}\theta^1 \wedge \theta^2 \\ 0 & R^2_{112}\theta^1 \wedge \theta_2 & -R^1_{112}\theta^1 \wedge \theta^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \tau \end{pmatrix}
\end{aligned}
\tag{4.5.13}
$$

with

$$
\begin{aligned}
\alpha &= \frac{1}{u_1} R^1_{112}\theta^1 \wedge \theta^2 + u_2 R^2_{112}\theta^1 \wedge \theta^2 \\
\beta &= \frac{1}{u_1}\left(R^1_{223}\theta^2 \wedge \theta^2_1 + R^1_{212}\theta^1 \wedge \theta^2\right) - u_2 R^1_{112}\theta^1 \wedge \theta^2 \\
\gamma &= u_1 R^2_{112}\theta^1 \wedge \theta_2 \\
\tau &= -u_1 R^1_{112}\theta^1 \wedge \theta^2
\end{aligned}
\tag{4.5.14}
$$

Multiplying the previous result of equation (4.5.14) with a generic element of H_2:

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & u_1 & -u_2 \\ 0 & 0 & \frac{1}{u_1} \end{pmatrix}
$$

one obtains the matrix of two forms

$$
\begin{pmatrix} 0 & 0 & 0 \\ 0 & u_1\alpha & \beta_1 \\ 0 & u_1\gamma & -u_1\alpha \end{pmatrix}
\tag{4.5.15}
$$

with

$$
\beta_1 = -u_2\alpha + \frac{1}{u_1}\beta.
$$

Therefore the "normalization" that we imposed previously is *ad* invariant. Finally we remark that computation of ω via the formula of equation (4.5.1) gives the matrix

$$\omega = \omega_0 = \begin{pmatrix} 0 & 0 & 0 \\ \theta_3 & \Omega_1 & -\Omega_2 \\ \theta_1 & \theta_2 & -\Omega_1 \end{pmatrix} \tag{4.5.16}$$

which is canonically associated to the coframe $(\theta_1, \theta_2, \theta_3, \Omega_1, \Omega_2)$ (it results from the remark that in our notation $\theta^1 = \omega^3$, $\theta^2 = \omega^1$, $\theta^2_1 = \omega^2$, from the use of formulae (4.2.18) and computation). One thus may formulate the following theorem

Theorem 4.5.1 *To every second order ordinary differential equation $y'' = f(x, y, y')$ studied under area preserving* (4.1.1) transformations, there is associated a canonical coframe $(\theta_1, \theta_2, \theta_3, \Omega_1, \Omega_2)$ on a local principal H_2 bundle $U \times G_b$. This coframe solves the local equivalence problem of second order differential equation under the substitutions (4.1.1). Moreover it enables us to endow $U \times G_b$ with a normal Cartan affine connection given in matrix form by

$$\omega_0 = \begin{pmatrix} 0 & 0 & 0 \\ \theta_3 & \Omega_1 & -\Omega_2 \\ \theta_1 & \theta_2 & -\Omega_1 \end{pmatrix}$$

whose curvature contains all the points relative invariants of the differential equation $y'' = f(x, y, y')$ submitted to area preserving transformations.

So this gives a more geometric interpretation of the area preserving relative invariants for second order ordinary differential

$$y'' = f(x, y, y')$$

and ends our study of it.

Conclusion

Cette thèse fut l'occasion pour nous d'aborder divers concepts de la théorie des équations différentielles. Tout d'abord nous nous sommes intéressés à l'équation d'Abel du premier type et avons réalisé son étude grâce à la méthode de Cartan qui nous a permis d'éclairer l'approche classique de cette étude. Nous avons par la suite examiné certaines équations de Riccati type algébrique. Enfin notre intérêt s'est aussi porté sur le problème d'équivalence de Cartan pour les équations du second ordre sous l'action des transformations ponctuelles à jacobien unitaire (préservant les aires). Nous pouvons à ce propos remarquer que nos calculs ((4.2.18) montrent que l'on peut sans aucun changement, les appliquer pour associer à toutes les équations du second ordre étudiée sous l'action du pseudo-groupe des transformations au jacobien constant, une connexion de Cartan affine normale.

Une direction vers laquelle on pourrait poursuivre ce travail de recherche serait, celle concernant l'étude de l'existence ou non de solutions algébriques pour l'équation d'Abel. La difficulté réside dans l'absence d'analogue du birapport pour l'équation d'Abel.

Enfin une deuxième piste de réflexion concerne l'étude des connexions Cartan normales pour les algèbres de Lie semi-simples développée par N. Tanaka [55, 56] et pour les variétés filtrées introduite par T. Morimoto [36].

Bibliography

[1] Niels Abel. Précis d'une théorie des fonctions elliptiques. *J. Reine. Angew. Math*, 4:309–348, 1829.

[2] Paul Appell. Sur les invariants de quelques équations différentielles. *Journal de Mathématiques pures et appliquées*, 5:361–423, 1889.

[3] Jean Marie Arnaudiès and José Bertin. *Surfaces de* Riemann*, Groupes Polyédraux et Equation de* Halphen. Ellipses Paris, 2000.

[4] Léon Autonne. Sur certaines équations des quatrièmes et cinquièmes degrés. *Bull. Soc. Math. France*, 28:90–107, 1900.

[5] Léon Autonne. Sur les équations algébriques dont toutes les racines sont solutions d'une même équation de Riccati. *Journal de Mathématiques pures et appliquées*, 6:157–214, 1900.

[6] Walter Benz. The n-point-invariants of the projective line and cross-ratio of $n-$tuples. *Annales Polonici Mathematici*, 26:53–60, 1972.

[7] Auguste Boulanger. *Contribution à l'étude des équations différentielles linéaires et homogènes intégrables algébriquement.* PhD thesis, Faculté des Sciences de Paris, France, 1897.

[8] R. Bryant. Two exotic holonomies in dimension four, path geometries and twistor theory. *Proc. Symp. Pure Math*, 53:33–88, 1991.

[9] R. Bryant, S.S. Chern, R. Gardner, H. Goldshmidt, and P. Griffiths. *Exterior differential systems.* Springer-Verlag, N.Y., 1991.

[10] E. Cartan. Sur les variétés à connexion projective. *Bull. Soc. Math. France*, 52:205–241, 1924.

[11] E. Cartan. La geometria de las equaciones differentiales de tercer orden. *Revista Mat. Hisp.-Amer*, 1:3–33, 1941.

[12] Elie Cartan. Les sous-groupes des groupes continus de transformations. *Annales de L'école normale supérieure*, 25:57–194, 1908.

[13] Jean Chazy. Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. *Acta Math.*, 33:317–385, 1910.

[14] S. S. Chern. The geometry of the third order differential equations. *Sci.Rep.Nat. Tsing Hua Univ*, 4:97–111, 1950.

[15] Alfred Clebsch. *Leçons de géométrie, trad.* A. B*enois*, volume 1. Gauthier-Villars, 1879.

[16] Igor Dolgachev. Lectures on modular forms, 1998. Available at www.math.lsa.umich.edu/ idolga/.

[17] B. Doubrov, B. Komrakov, and T. Morimoto. Equivalence of holonomic differential equations. *Lobachevskii Journal of Mathematics*, 3:39–71, 1999.

[18] Raouf Dridi. *Utilisation de la méthode d'équivalence de* C*artan dans la construction d'un solveur d'équations différentielles.* PhD thesis, Université de Lille1, France, 2007.

[19] B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes. *Acta Math.*, 33:1–55, 1910.

[20] Robert B. Gardner. *Differential geometric methods interfacing control theory.* Birkhaüser, Boston, 1983.

[21] Robert B. Gardner. *The Method of Equivalence and Its Applications.* Number 58 in National conference series in Applied Mathematics. Society For Industrial and Applied Mathematics, 1989.

[22] G. H. Halphen. Mémoire sur la réduction des équations différentielles aux formes intégrables. In *Oeuvres Complètes de* G. H H*alphen*, volume 3. Gauthier-Villars, 1950.

[23] David Hilbert. *Theory of Algebraic Invariants.* Cambridge University Press, 1993.

[24] Einar Hille. *Ordinary differential equation in the complex domain.* John Wiley and Sons, 1976.

[25] Javier de Lucas José F. Carinena and Manuel F. Ranada. A geometric approach to integrability of Abel differential equations. *Int J Theo Phys*, 50:2114–2124, 2011.

[26] Niky Kamran. *On the equivalence Problem of* E*lie* C*artan*, volume 65 of *2*. Académie Royale de Belgique, 1989.

[27] Félix Klein. Uber lineare differential gleichungen. *Mathematische Annalen*, 12:167–179, 1877.

[28] S. Kobayashi and T. Nagano. On projective connections. *Journal of Mathematics and Mechanics*, 13:215–235, 1964.

[29] Audille Lecacheux. Constructions de polynômes génériques à groupe de Galois résoluble. *Acta Arithmetica*, 86:207–216, 1998.

[30] Joseph Liouville. Sur la détermination des intégrales dont la valeur est algébrique. *Journal de l'école polytechnique*, 22:1–19, 1833.

[31] Joseph Liouville. Mémoire sur l'intégration d' une classe d'équation différentielle en quantités finies explicites. *Journal de Mathématiques pures et appliquées*, 4:423–456, 1839.

[32] R. Liouville. Sur certaines équations du premier ordre. *Comptes Rendus Acad. Sci.*, 103:476–479, 1886.

[33] R. Liouville. Sur une classe d'équations différentielles du premier ordre et les formations invariantes qui s'y rapportent. *Comptes Rendus Acad. Sci.*, 105:460–463, 1887.

[34] Bernard Malgrange. *Le groupoïde de Galois d'un feuilletage*, volume 38. Cassini, 2001.

[35] Nowakowski Marek and Rosu Haret. Newton's laws of motion in the form of a Riccati equation. *Physical review.* E*, statistical, non linear and soft matter physics*, 65:47602.1–47602.4, 2002.

[36] T. Morimoto. Geometric structures on filtered manifolds. *Hokkaido Math. J.*, 22:263–347, 1993.

[37] Sylvain Neut. *Implantation et nouvelles applications de la méthode d'équivalence* Cartan. PhD thesis, Université de Lille1, France, 2003.

[38] Ezra T. Newmann and Pawel Nurowski. Projective connections associated with second order odes. *Classical and quantum gravity*, 20:2325–2335, 2003.

[39] Pawel Nurowski and George A. J. Sparling. Three dimensional Cauchy-Rieman structures and second order odes. *Classical and quantum gravity*, 20:4995–5016, 2003.

[40] Peter Olver. *Equivalence, Invariants and Symmetry.* Springer Verlag, 1995.

[41] Peter J. Olver. *Application of* Lie *Groups to Differential Equations.* GTM 107. Springer Verlag, 1986.

[42] Valentin Ovsienko and Sergei Tabachnikov. Projective geometry old and new: from Schwarzian derivative to cohomology of diffeomorphism groups, 2005. Available at math.univ-lyon1.fr/ ovsienko/.

[43] T. Ozawa and H. Sato. Linearization of ordinary differential equations by area preserving transformations. *Nagoya Math. J.*, 156:109–122, 1999.

[44] Paul Painlevé. Sur les équations différentielles du premier ordre. *Comptes rendus Acad. Sci.*, 107:724–726, 1888.

[45] Paul Painlevé. Mémoire sur les équations différentielles dont l'intégrale générale est uniforme. *Bull. Soc. Math. France*, 28:201–261, 1900.

[46] Paul Painlevé. Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme. *Acta Math.*, 25:1–85, 1902.

[47] Emile Picard. Applications de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches. *Ann. Sci. Ec. Norm. Sup.*, 2(6):342–343, 1877.

[48] Emile Picard. Sur les équations différentielles linéaires et les groupes algébriques de transformations. *Annales de la Faculté de Sciences Toulouse*, 1:1–15, 1877.

[49] Austin Roche. *A new method for functional decomposition of rational invariants, and the solutions of Abel's differential equation via the equivalence method*. PhD thesis, Simon Fraser University, Canada, 2010.

[50] R. Sharpe. *Differential Geometry: Cartan's generalization of Klein's Erlangen program*, volume 104. Springer, 2000.

[51] Michael F. Singer and Félix Ulmer. Galois groups of second order and third order linear differential equations. *Journal of Symbolic computation*, 16:9–36, 1993.

[52] Michael F. Singer and Félix Ulmer. Liouvillian and algebraic solutions of second and third order linear differential equations. *Journal of Symbolic computation*, 16:37–73, 1993.

[53] Shlomo Sternberg. Semi-Riemann Geometry and General Relativity. Available at www.math.harvard.edu/ shlomo/.

[54] Shlomo Sternberg. *Lectures on Differential geometry*. Englewood Cliffs, Prince-Hall, 1964.

[55] N. Tanaka. On differential systems, graded Lie algebras and pseudo-groups. *J. Math. Kyoto Univ.*, 10:1–82, 1970.

[56] N. Tanaka. On the equivalence problem associated with simple graded Lie algebras. *Hokkaido Math. J*, 8:23–84, 1979.

[57] A. Tresse. Sur les invariants différentiels des groupes continus de transformations. *Acta Math.*, 18:1–88, 1894.

[58] A. Tresse. Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre $y'' = \omega(x, y, y')$. *Hirsel, Leipzig*, pages 1–87, 1896.

[59] Ernest Vessiot. Sur l'intégration des équations différentielles linéaires. *Ann. Sci. Ec. Norm. Sup.*, 9:197–280, 1892.

[60] Rupert Way. *Dynamics in the Hopf bundle, geometric phase and implications for dynamical systems.* PhD thesis, University of Surrey, U.K, 2008.

[61] J. A. Weil. *Constante et Polynômes de Darboux en algèbre différentielle: applications aux systèmes différentielles linéaires.* PhD thesis, Ecole polytechnique, 1995.

[62] E. Wilczynski. *Projective Differential Geometry of Curves and Ruled Surfaces.* B. G. Teubner, 1906.