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À Tristan

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Résumé

Ce mémoire est composé de deux parties indépendantes.

Dans la première partie, des ondes de surface, solutions de problèmes aux limites hyperboliques non linéaires, sont étudiées : on construit une solution BKW sous forme de développement infini en puissance de ε. On le justifie rigoureusement, en construisant une solution exacte, qui admet ce développement asymptotique. On montre que la solution n'est pas nécessairement localisée à l'ordre $O(\varepsilon^\infty)$ sur la frontière, même lorsque le terme source l'est ; l'exemple d'un cas particulier de l'élasticité est traité.

La deuxième partie est dédiée à l'étude de la réflexion d'ondes non linéaires discontinues, pour des problèmes aux limites hyperboliques, faiblement bien posés, ni fortement stables, ni fortement instables. On étudie comment les singularités d'une solution striée sont réfléchies lorsque la solution atteint la frontière. On prouve des estimations striées et en normes infinies. On montre qu'une discontinuité du gradient de la solution à travers un hyperplan peut être réfléchie en une discontinuité de la solution elle-même.

mots clés : Problèmes aux limites non linéaires, Ondes de surface, Développement asymptotique rigoureux, Développement BKW, Élasticité, Ondes de Rayleigh non linéaires, Rectification, Réflexion de discontinuités, Problèmes aux limites hyperboliques non linéaires faiblement stables, Condition faible de Lopatinski, Condition (WR), Perte d'une dérivée, Solutions striées

Abstract

Two distinctive topics are investigated in this dissertation.

The first part deals with surface waves, solutions of hyperbolic nonlinear boundary value problems. We construct BKW solutions in the weakly nonlinear regime with infinite expansion in powers of ε. We rigorously justify this expansion,constructing exact solutions, which admit the asymptotic expansions. We also show that the solution is not necessarily localized at the order $O(\varepsilon^\infty)$ in the interior, even if the data are; a particular case of elasticity is studied: we prove that fast oscillatory elastic surface waves can produce non trivial internal non oscillatory displacements.

The second part is dedicated to the reflection of non linear discontinuous waves, for weakly well-posed hyperbolic boundary value problems, satisfying the (WR) condition, which has been introduced in [1, 12], that is in a case where the IBVP is neither strongly stable, nor strongly unstable. We study how the singularities of a striated solution are reflected when the solution hits the boundary. We prove striated estimates and L^∞ estimates and observe the loss of one derivative: we show that a discontinuity of the gradient of the solution across an hyperplane can be reflected in a discontinuity across an hyperplane of the solution itself.

keywords: Nonlinear boundary value problems, Surface waves, Rigorous asymptotic expansion, WKB expansion, Elasticity, Non linear Rayleigh waves, Rectification, Reflection of discontinuities, Nonlinear weakly stable hyperbolic IBVP, Weak Lopatinski condition, (WR) condition, Loss of one derivative, Striated solutions

Table des matières

Introduction générale

Cette thèse concerne trois problèmes différents dans le cadre de la propagation d'ondes non linéaires pour des problèmes aux limites hyperboliques faiblement bien posés : la propagation d'ondes de surface, l'exemple des ondes d'élasticité et la réflexion de sauts à travers des hypersurfaces.

0.1 Solutions oscillantes et problèmes aux limites, présentation du Chapitre 1

L'étude des solutions oscillantes de systèmes hyperboliques linéaires généraux commence avec l'article de Lax [57](1957), qui construit des solutions de la forme

$$u^\varepsilon(x) = \mathrm{Re}(a^0(x)e^{i\phi/\varepsilon} + \varepsilon a^1(x)e^{i\phi/\varepsilon} + \varepsilon^2 a^2(x)e^{i\phi/\varepsilon} + \cdots).$$

En 1969, Choquet-Bruhat ([20]) a montré que la généralisation formelle naturelle des développements de Lax au cadre non linéaire consiste à tenir compte de la non linéarité en remplaçant les termes $a^k(x)e^{i\phi/\varepsilon}$ par une fonction périodique générale $U^k(x, \phi/\varepsilon)$ de ϕ/ε.

L'étude rigoureuse de la propagation et de l'interaction des oscillations pour des opérateurs ou des systèmes non linéaires hyperboliques a commencé avec les travaux de Joly et Rauch ([47, 48, 49, 51]). Dans ce type de problème, on s'intéresse aux solutions d'une équation semi-linéaire d'ordre N (ou d'un système $N \times N$ d'ordre 1), munie des variables $x = (x_0, \dots, x_d)$, strictement hyperbolique par rapport à la variable $t = x_0$, avec second membre $\mathcal{C}^\infty$ et données de Cauchy oscillantes, c'est-à-dire de la forme $v\big(x, \frac{\phi_1^0}{\varepsilon}, \dots, \frac{\phi_q^0}{\varepsilon}\big)$, où $\phi_1^0, \dots, \phi_q^0$ sont les phases considérées et $v(x, \theta_1, \dots, \theta_q)$ est $\mathcal{C}^\infty$, à support compact en x et périodique par rapport à chaque θ_j. Il s'agit alors d'étudier si, grâce au caractère oscillant des données, il peut y avoir existence sur un intervalle de temps uniforme en ε et si, sur cet intervalle de temps, la solution admet un développement asymptotique à l'aide de fonctions oscillantes de la même forme que les données. Lorsque $d > 1$, où d est la dimension d'espace, l'équation eikonale est non linéaire (lorsque $d = 1$, elle peut être factorisée en facteurs linéaires), d'où la focalisation associée à des points critiques des phases $\phi_1^0, \ldots, \phi_q^0$ (ou des points critiques de combinaisons linéaires des ϕ_j) peut amener le temps d'existence des solutions à tendre vers 0 lorsque $\varepsilon \to 0$, des exemples sont construits dans [42, 44].

Dans le cas des systèmes semi-linéaires multidimensionnels monophasés, Joly et Rauch ([51]) ont montré qu'il existe une solution sur un intervalle de temps indépendant de ε, admettant un développement asymptotique donné par l'optique géométrique non linéaire de la forme

$$u^\varepsilon(x) = U^0(x, \phi(x)/\varepsilon) + \varepsilon U^1(x, \phi(x)/\varepsilon) + \cdots + \varepsilon^k U^k(x, \phi(x)/\varepsilon) + o(\varepsilon^k), \tag{0.1.1}$$

où les profils $U^j(x, \theta)$ sont 2π-périodiques, et $k \geq 1$ est un entier arbitraire. Pour cela, ils ont introduit, en particulier, des techniques de moyennisation des systèmes hyperboliques permettant de construire les profils U^j.

Pour des développements à plusieurs phases, où l'on doit tenir compte des interactions ("résonances") d'oscillations, les premiers résultats concernent le cas mono-dimensionnel : Joly et Rauch ([47, 48, 49]) ont prouvé l'existence d'une solution sur un intervalle de temps uniforme et ont obtenu une représentation asymptotique de celle-ci. Ce résultat a par la suite été étendu au cas quasi-linéaire mono-dimensionnel par Joly, Métivier et Rauch ([41]).

Dans tous ces problèmes multidimensionnels, le phénomène d'interaction des oscillations est évité : si les données d'un problème semi-linéaire oscillent sur des phases caractéristiques $\phi_1, \ldots, \phi_q$, la présence des non-linéarités dans le second membre de l'équation fait s'attendre en général à ce que la solution oscille sur toutes les phases caractéristiques ϕ s'écrivant sous la forme $\phi = \sum n_j \phi_j$ avec $n_j \in \mathbb{Z}$ pour tout j ; lorsque ϕ est une nouvelle phase caractéristique, la nouvelle oscillation se propage et on dit qu'il y a résonance. Dans le cas où $q = 1$ ou bien où $q = 2$ mais l'opérateur est d'ordre 2, de telles interactions ne font pas apparaître de nouvelles phases caractéristiques (à cause de la convexité du cône d'onde dans le second cas).

La situation est radicalement différente lorsque l'on considère des opérateurs d'ordre strictement supérieur à 2 ou des données oscillant sur au moins trois phases. Joly et Rauch ont en effet construit une exemple ([50]) dans lequel l'ensemble des directions des différentielles des phases caractéristiques, obtenues par interactions à partir d'une famille de trois phases sur lesquelles oscillent les données du problème, est dense - ce qui ôte tout espoir d'obtenir une description raisonnable de la solution. Ils ont alors conjecturé que le seul cas où l'on peut espérer des résultats avec des données oscillant sur au moins trois phases est celui d'un opérateur d'ordre 2 avec non-linéarité quadratique. Delort ([25]) a étudié ce problème en dimension 2 ou 3 d'espace sous l'hypothèse supplémentaire que la non-linéarité est compatible à l'opérateur au sens de Hanouzet-Joly ([35]). Il prouve que si $(\phi_k)_{k\in K}$ est une famille finie de phases caractéristiques dont les différentielles sont deux à deux indépendantes en tout point et si le second membre de l'équation est somme de fonctions de la forme $f_k(x, \frac{\phi_k}{\varepsilon})$ où pour tout $k \in K,\ f_k(x,\theta)$ est 2π-périodique en $\theta \in \mathbb{R}$, il existe une solution sur un intervalle de temps uniforme en ε, qui, en outre, admet un développement asymptotique à l'ordre 2 en ε.

Dans le cas des systèmes hyperboliques quasi-linéaires, Choquet-Bruhat a montré dans [20] que les développements naturels à considérer étaient ceux de la forme (0.1.1) dont le premier terme est une solution donnée indépendante de ϕ/ε :

$$u^\varepsilon(x) = u^0(x) + \varepsilon U^1(x, \phi(x)/\varepsilon) + \cdots + \varepsilon^k U^k(x, \phi(x)/\varepsilon) + o(\varepsilon^k), \quad (0.1.2)$$

de sorte que le problème se présente comme un problème de perturbation oscillante de petite amplitude et de haute fréquence d'une solution donnée. De nombreux travaux, dont notamment ceux de Hunter, Keller, Majda et Rosales ([40, 60, 63, 39]), s'intéressent aux propriétés de telles solutions formelles ainsi qu'à des développements plus généraux à plusieurs phases.

En ce qui concerne la justification dans un cadre quasi-linéaire général des développements (0.1.2), Guès ([31]) a étendu les résultats semi-linéaires mutli-dimensionnels de Joly et Rauch ([51]) ; il a prouvé l'existence d'une solution sur un intervalle de temps uniforme lorsque la donnée oscille sur une seule phase et admet un développement asymptotique à un ordre assez grand devant la demi-dimension de l'espace-temps. Sa méthode consiste à montrer d'abord que l'équation étudiée admet une solution asymptotique (par résolution des équations données par l'optique géométrique non linéaire) puis à établir que la différence entre la solution cherchée et cette solution asymptotique vérifie une équation dans laquelle le terme non-linéaire est multiplié par une puissance de ε assez grande pour pouvoir utiliser les théorèmes d'injection de Sobolev. Cela permet alors de résoudre ce dernier problème par une méthode classique d'approximations successives. Dans [30], Guès considère des développements à peu de termes mono-phases, et tout particulièrement de la forme

$$u^\varepsilon(x) = u^0(x) + \varepsilon\mathcal{U}(x, \phi(x)/\varepsilon) + o(\varepsilon), \quad (0.1.3)$$

où $\mathcal{U}(x,\theta)$ est une fonction presque-périodique en la variable θ. Il complète [31] en levant l'obstruction de l'ordre du développement asymptotique assez grand devant la demi-dimension.

Joly, Métivier et Rauch ont donné une justification rigoureuse d'optiques géométriques non linéaires pour des solutions continues de problèmes de Cauchy semi-linéaires et quasi-linéaires dans un espace libre ([44, 43, 45]). Des domaines avec frontière ont été considérés dans le cas non résonant par Chikhi ([19]) et dans le cas d'oscillations résonantes, dans des situations particulières où les modes glancing (phases pour lesquelles les champs de vecteurs associés sont tangents à la frontière) ne sont pas présents, par Williams ([80]). Dans [2], Majda et Artola construisent dans le cas non glancing des développements formels solutions de problèmes aux limites pour des lois de conservation hyperboliques, des phases linéaires et des états constants dans le passé. Cheverry ([18]) a étudié la propagation d'oscillations près d'un point diffractif pour des problèmes scalaires du second ordre (non résonants) comportant une seule phase.

Williams ([81]) donne une justification rigoureuse d'optiques géométriques pour une classe de problèmes aux limites semi-linéaires bien posés au sens de Kreiss, avec des données oscillantes dépendant de nombreuses phases, où à la fois les interactions résonnantes et les modes glancing sont présents. Les erreurs tendent vers 0 dans L^2 lorsque la longueur d'onde ε tend vers 0. Afin d'atteindre des erreurs tendant vers 0 dans L^∞ lorsque ε tend vers 0, Williams incorpore dans [82] les profils des couches limites glancing, elliptiques et hyperboliques dans les développements. L'existence des couches limites glancing et elliptiques, qui sont petites dans L^2 mais pas dans L^∞, était déjà manifeste dans [81], mais elles étaient absorbées dans les termes d'erreur. L'analyse de la couche limite glancing amène à introduire une troisième échelle $1/\sqrt{\varepsilon}$ en plus des échelles oscillantes ($1/\varepsilon$) et spatiales (1) habituelles (voir [26, 46, 32, 33] pour différents problèmes à trois échelles). Williams construit aussi des exemples montrant qu'en présence de modes glancing d'ordre au moins 3, le temps maximal T_ε d'existence de la solution exacte tend vers 0 lorsque $\varepsilon \to 0$. Le mécanisme d'explosion est différent des types de focalisations qui se manifestent en l'absence de frontières.

Dans le premier chapitre, on considère la propagation d'ondes de surface haute fréquence non linéaires, qui surviennent par exemple en géophysique ou magnétohydrodynamique. Il s'agit d'ondes, telles que des ondes sismiques, qui se propagent le long de la frontière d'un domaine et décroissent exponentiellement dans l'intérieur. Le régime dit faiblement non linéaire correspond aux amplitudes telles que les effets non linéaires affectent la propagation de la composante oscillante principale. Le cadre mathématique est l'analyse de solutions haute fréquence de problèmes aux limites. Ici, on considérera, par souci de simplicité, des lois de conservation du premier ordre, dans un espace de dimension deux, sur un domaine qui est un demi-plan :

$$\begin{cases} \partial_t \vec{u} + \partial_x(f(\vec{u})) + \partial_y(g(\vec{u})) = \vec{h} & y > 0 \\ C\vec{u}_{|y=0} = 0. \end{cases} \tag{0.1.4}$$

Notre analyse s'étend à des dimensions plus élevées, à des domaines réguliers plus généraux et à des systèmes d'ordre plus élevé tels qu'en élasticité, mais nos résultats sont présentés dans le cas plus simple (0.1.4). Nous choisissons aussi le cadre des problèmes aux limites dissipatifs symétriques au sens de [58], avec une frontière non caractéristique $y = 0$.

Les conditions mathématiques pour l'existence d'ondes de surfaces sont bien comprises : la condition de Lopatinski faible est satisfaite mais la condition plus forte de Lopatinski uniforme est violée pour des fréquences dans la région non hyperbolique (voir e.g. [78, 11, 36]). Cela signifie que pour tout état $\vec{u_0}$ fixé et toutes fréquences tangentielles au temps $(\tau, \xi) \neq (0, 0)$ avec $\mathrm{Im}\tau \leq 0$,

1) lorsque $\mathrm{Im}\tau < 0$, l'ODE de dimension un

$$\partial_Y v + ig'(\vec{u_0})^{-1}(\tau \mathrm{Id} + \xi f'(\vec{u_0}))v = 0, \qquad Cv_{|Y=0} = 0 \tag{0.1.5}$$

admet seulement la solution triviale $v = 0$ dans $L^2([0, \infty[)$; ceci est en effet impliqué par l'hypothèse de dissipativité.

2) il existe des fréquences réelles (τ, ξ) telles que l'équation (0.1.5) a une solution non triviale exponentiellement décroissante.

Des énoncés plus précis sont donnés ci-dessous. On note $(-\omega, k)$ une des fréquences de la condition 2), on cherche des ondes de surface oscillantes, c'est-à-dire des solutions du problème aux limites localisées près de la frontière et telles que leur trace sur la frontière ont des oscillations rapides de phase $(kx - \omega t)/\varepsilon$. Plus précisément, on cherche des solutions admettant un développement asymptotique de la forme

$$\vec{u}(t, x, y) \sim \vec{u}_0 + \sum_{n \geq 1} \varepsilon^n \vec{u}_n \Big(t, x, y, \frac{kx - \omega t}{\varepsilon}, \frac{y}{\varepsilon}\Big) \tag{0.1.6}$$

avec des profiles $\vec{u}_n(t, x, y, \theta, Y)$ périodiques par rapport à la variable θ et exponentiellement convergeant vers une limite $\underline{\vec{u}_n}(t, x, y)$ lorsque la variable Y tend vers $+\infty$.

Ce type de développements a été considéré par de nombreux auteurs, voir par exemple [36, 37, 54, 55, 38, 79, 52] et leurs références. La trace sur la frontière du terme principal est polarisée selon une direction donnée :

$$\vec{u}_1(t, x, 0, \theta, 0) = \mathrm{u}_+(t, x, \theta)\vec{R}_+ + \mathrm{u}_-(t, x, \theta)\vec{R}_- \tag{0.1.7}$$

où $\vec{R}_-$ est la trace sur $Y = 0$ de la solution spéciale de 2) et $\vec{R}_+$ est son complexe conjugué (qui satisfait la même propriété pour la fréquence $(\omega, -k)$); de plus, dans (0.1.7), u représente une fonction scalaire, périodique en θ, de séries de Fourier $\sum_{n\neq 0} u^n e^{in\theta}$ et $u_+ = \sum_{n>0} u^n e^{in\theta}$ [resp. $u_- = \sum_{n<0} u^n e^{in\theta}$].

L'amplitude scalaire u satisfait une équation de propagation de la forme

$$\partial_t u + v\partial_x u + \partial_\theta a(u, u) = 0 \tag{0.1.8}$$

où a est une forme bilinéaire *non locale* agissant sur les fonctions périodiques en θ (voir [36] pour une obtention formelle).

Hunter ([36]) a initié l'étude de ces équations d'amplitude, associées à des ondes de surface dans le régime faiblement non linéaire (voir aussi [9] pour l'obtention de l'équation d'onde associée à des ondes de surface le long de chocs neutralement stables). Il s'agit de généralisations non locales de l'équation de Burger. Benzoni-Gavage ([7]) a prouvé que ces équations sont bien posées localement en temps dans des espaces de Sobolev, sous une condition de stabilité explicite, originellement exhibée par Hunter dans [36]. Benzoni-Gavage, Coulombel et Tzvetkov ([8]) ont prouvé que cette condition est non seulement suffisante mais aussi nécessaire pour que les équations soient bien posées dans des espaces de Sobolev.

Les résultats principaux de ce chapitre sont doubles :

1) *obtenir les équations de propagation dans un cadre général et construire des développements asymptotiques à tout ordre n. Il semble que, dans la littérature, seulement l'équation vérifiée par le terme principal a été obtenue (voir les références ci-desssus) et étudiée (voir [7, 37]).* Mais l'analyse de termes plus élevés révèle un phénomène nouveau et peut-être inattendu : en général, les correcteurs $\vec{u}_n$ pour $n \geq 2$ ne sont pas purement localisés près de la frontière ; par exemple, en général $\vec{u}_2$ dépend de la variable lente y et $\vec{u}_2$ ne décroît pas vers 0 lorsque Y tend vers $+\infty$. Cette analyse se conclut par la construction de solutions approchées à tout ordre ε^n.

2) *construire des solutions exactes du problème original qui admettent l'expansion asymptotique obtenue à la première étape*, en adaptant l'analyse

de O. Guès ([31, 30]). Cela équivaut à prouver la stabilité des solutions approchées.

0.2 Applications, cas particulier de l'élasticité, présentation du Chapitre 2

Les ondes de Rayleigh, qui sont des ondes acoustiques sur des demi-espaces élastiques, constituent un exemple important d'ondes de surface. Elles apparaissent en séismologie et sont utilisées dans des dispositifs ultrasoniques. Zabolotskaya, Kalyanasundaram et al., Lardner et Parker ont obtenu des équations du type (0.1.8) pour des ondes de Rayleigh : dans le cas isotropique, Zabolotskaya ([79]) a étudié les ondes de Rayleigh planes et circulaires, Kalyanasundaram et al. ([52]) ont étudiés des formes d'ondes périodiques ; en utilisant des techniques multi-échelles, Lardner ([54]) a étendu cette méthode au cas d'ondes ayant une forme d'onde initiale générale et Parker ([68]) traite le cas anisotropique, en utilisant un système de coordonnées se déplaçant à la vitesse de Rayleigh, il évite l'emploi de techniques multi-échelles. Parker et Talbot ([69]) ont construit des ondes régulières solutions de ces équations. Taylor ([78]) a étudié la propagation de singularités, à la vitesse de Rayleigh, le long d'une frontière courbe, dans le cas de l'élasticité linéaire.

Les avancées dans le domaine des ondes de surface élastiques non linéaires ont été motivées dans une large mesure par le développement des dispositifs SAW (surface acoustic wave), qui exécutent des opérations non linéaires de traitement d'image et emploient des cristaux piézoélectriques. Hamilton et al. ([34]) ont généralisé le cas isotropique étudié dans [79] afin d'inclure l'anisotropie du cristal, sans restriction à une symétrie du matériau, une orientation de la surface libre par rapport aux axes du cristal ou une direction de propagation dans le plan de la surface libre.

Des ondes de surface peuvent aussi se propager sur des discontinuités, comme des discontinuités de contact ou des ondes de choc, dans des solutions de systèmes de lois de conservation. Alì et Hunter étudient dans [38] un problème modèle pour la propagation d'ondes de surface, non linéaires et exponentiellement décroissantes, sur une discontinuité, à savoir la propagation d'ondes de surface sur une discontinuité tangentielle en magnétohydrodynamique. Des ondes de surface sur des discontinuités ont été étudiées par Artola et Majda ([1, 3]) dans le cas de feuilles de vortex compressibles et par Majda et Rosales ([61, 62]) dans le cas d'ondes de détonation. Les ondes de surface dans ces problèmes émettent des "bulk waves" dans l'intérieur du fluide et donc diffèrent qualitativement des véritables ondes de surface, telles que les ondes Rayleigh, dont l'énergie est localisée sur la frontière et décroît exponentiellement dans l'intérieur.

Dans le deuxième chapitre, on prouve que des ondes de surface élastiques rapidement oscillantes peuvent produire un déplacement interne non oscillant non trivial. Ce phénomène a été observé et expliqué dans le chapitre précédent pour des systèmes du premier ordre généraux.

En reprenant le cadre de [54], on considère un milieu élastique qui occupe au repos le demi-plan $y > 0$ et qui est déformé en tension plane dans les directions x et y. Les composantes du déplacement sont notées $u(t, x, y)$ et $v(t, x, y)$ dans les directions x et y. On suppose que le milieu est isotropique et hyperélastique. Les équations de mouvement et les conditions sur le bord sans tension $y = 0$ prennent la forme :

$$\partial_{tt}u^\varepsilon - r\partial_{xx}u^\varepsilon - (r-1)\partial_{xy}v^\varepsilon - \partial_{yy}u^\varepsilon = \partial_x F_1 + \partial_y F_2 \tag{0.2.1a}$$
$$\partial_{tt}v^\varepsilon - \partial_{xx}v^\varepsilon - (r-1)\partial_{xy}u^\varepsilon - r\partial_{yy}v^\varepsilon = \partial_x G_1 + \partial_y G_2 \tag{0.2.1b}$$

sur $y > 0$ et

$$\partial_y u^\varepsilon + \partial_x v^\varepsilon = -F_2 + f^\varepsilon \tag{0.2.2a}$$
$$(r-2)\partial_x u^\varepsilon + r\partial_y v^\varepsilon = -G_2 + g^\varepsilon \tag{0.2.2b}$$

sur $y = 0$.

On étudie un cas particulier des équations données dans [54] (voir aussi [68]), où les termes non linéaires quadratiques sont donnés par :

$$F_1 = \partial_y u^\varepsilon \partial_x v^\varepsilon \tag{0.2.3}$$
$$F_2 = \partial_x v^\varepsilon(\partial_x u^\varepsilon + \partial_y v^\varepsilon) \tag{0.2.4}$$
$$G_1 = \partial_y u^\varepsilon(\partial_x u^\varepsilon + \partial_y v^\varepsilon) \tag{0.2.5}$$
$$G_2 = \partial_y u^\varepsilon \partial_x v^\varepsilon = F_1. \tag{0.2.6}$$

On considère des ondes de surface de la forme :

$$U^\varepsilon(t,x,y) = \begin{pmatrix} u^\varepsilon(t,x,y) \\ v^\varepsilon(t,x,y) \end{pmatrix} \sim \sum_{k=2}^{\infty} \varepsilon^k U_k\left(t, x, y, \frac{x-ct}{\varepsilon}, \frac{y}{\varepsilon}\right),$$

sur $y > 0$, avec des profils

$$U_k(t,x,y,Y,\theta) = \underline{U}_k(t,x,y) + U_k^*(t,x,\theta,Y),$$

où U_k^* est périodique par rapport à la variable θ et exponentiellement décroissant vers 0 en la variable Y.

En notant $(-c, 1)$ une des fréquences telles qu'il existe des ondes de surface associées à la phase $\varphi(t,x) = -ct + x$ (à un changement de variables homothétique près, il n'y a pas de restriction à supposer que le nombre d'onde spatial est $k = 1$), on cherche des ondes de surface oscillantes, c'est-à-dire des solutions du problème aux limites localisées près de la frontière et telles que la trace sur la frontière a des oscillations rapides de phase $\theta = \frac{x-ct}{\varepsilon}$.

Le premier terme U_2 est déterminé par exemple dans [54] et [68]. En particulier, il est purement localisé près de la frontière, c'est-à-dire $\underline{U}_2 = 0$, et U_2^* est déterminé par une inconnue scalaire $\alpha(t,x,\theta)$, qui satisfait une équation de propagation (2.2.2). Notre objectif principal est de prouver que, en général, le correcteur U_3 n'est pas purement localisé près de la frontière, c'est-à-dire $\underline{U}_3$ n'est pas nul, même si c'est le cas pour le terme source. $\underline{U}_3$ est une solution des équations linéarisées de l'élasticité, avec des termes de bord déterminés par α qui ne s'annulent pas en général. U_3 dépend de la variable lente y et ne décroît pas vers 0 lorsque Y tend vers $+\infty$, même si le terme source est exponentiellement décroissant vers 0.

Les autres termes du développement peuvent être déterminés en suivant l'analyse du premier chapitre. Dans ce chapitre, on a prouvé, dans le cas de systèmes du premier ordre généraux, l'existence d'une solution exacte admettant le développement asymptotique considéré. Cette analyse devrait pouvoir s'étendre à l'élasticité.

0.3 Ondes conormales, ondes striées, ondes discontinues, présentation du Chapitre 3

Dans le dernier chapitre, on considère la réflexion d'ondes non linéaires discontinues, pour des problèmes aux limites *faiblement* bien posés.

L'étude de la propagation de singularités pour des solutions suffisamment régulières d'équations hyperboliques non linéaires a commencé avec les travaux de Bony ([14]), Rauch ([70]) et Lascar ([56]). En général, d'autres singularités non présentes dans la théorie linéaire apparaissent, comme le montrent Rauch et Reed ([72]) et Beals ([4]). Ces singularités non linéaires additionnelles sont en toutes dimensions plus faibles que celles du problème linéaire correspondant. Dans un espace de dimension 1, leur localisation est très limitée ([71]), même pour des données initiales très générales. Mais, pour des dimensions plus grandes, les résultats de [4] impliquent que des hypothèses supplémentaires de régularité des données ou de la solution dans le passé sont nécessaires afin de limiter la dissémination des singularités. Une condition appropriée est celle de "conormalité" par rapport à des hypersurfaces. Il a été prouvé par Bony ([13, 15, 16]) et Melrose et Ritter ([64]) que la régularité est préservée pour certaines interactions d'ondes conormales, voir aussi les travaux de Rauch et Reed ([73, 74]) et Beals ([5]).

Le cadre des *solutions striées* est bien adapté à la description de la réflexion de discontinuités. Ce sont des distributions conormales par rapport à un feuilletage de codimension deux de l'espace-temps. Les solutions striées apparaissent naturellement dans de nombreuses situations physiques où les solutions sont régulières par rapport à toutes les variables, exceptées deux. Rauch et Reed ([73, 75]) ont exhibé cette classe de solutions striées pour des systèmes hyperboliques du premier ordre à deux vitesses, c'est-à-dire dont le polynôme caractéristique n'admet que deux racines distinctes et de multiplicité constante dans toutes les directions. Pour les systèmes à deux vitesses, le problème semi-linéaire possède la propriété que singularités ou oscillations se croisent sans interagir. Comme pour le problème linéaire, il n'y a pas création de singularités ou d'oscillations. Rauch et Reed ont montré que la propriété d'être strié par rapport à l'intersection transverse de deux feuilletages caractéristiques, d'un système à deux vitesses, est propagée par ce système. Ils se sont intéressés à l'exemple important de distributions striées fourni par les oscillations à hautes fréquences sur deux phases. Dans un espace strié dont les dérivations annulent les deux phases, ces oscillations restent uniformément bornées par rapport à la fréquence. Rauch et Reed ont montré que ce type d'oscillations sur deux phases, avec des développements d'ordre quelconque, est propagé par les systèmes à deux vitesses : ils ont résolu le problème de Cauchy pour un problème hyperbolique semi-linéaire à deux vitesses.

Des problèmes aux limites hyperboliques non linéaires ont aussi été considérés. En dimension 1, de nouvelles singularités apparaissent de façon contrôlée (voir Berning et Reed [12] et Oberguggenberger [67]). En toutes dimensions, les singularités non linéaires seront plus faibles que dans le cas linéaire (voir Sablé-Tougeron [76]). Beals et Métivier ([6]) considèrent la propagation de régularité lorsque la solution du problème mixte est conormale par rapport à une seule hypersurface caractéristique dans le passé, ils montrent que si l'hypersurface coupe la frontière du domaine transversalement, et si seulement une hypersurface caractéristique réfléchie est issue de l'intersection, alors la solution sera conormale par rapport à l'union de ces surfaces. Chikhi [19]) a adapté la notion d'onde striée pour le problème mixte, dans le cadre d'un problème aux limites hyperbolique à deux vitesses, non caractéristique, de type maximal dissipatif. Il a démontré que cette notion se propage et l'a appliquée à la réflexion des oscillations.

Depuis les travaux de Kreiss ([53]), il est établi qu'une condition nécessaire pour qu'un problème aux limites hyperbolique linéaire soit fortement bien posé est la condition de Lopatinski forte (voir section 3.4). Dans le cas de multiplicités constantes, Métivier ([66]) a montré que la condition de Lopatinski forte est aussi suffisante pour que le problème soit fortement bien posé. Le cas strictement hyperbolique a été traité par Kreiss ([53]). Benzoni, Rousset, Serre et Zumbrun ([10]) ont montré qu'il existait, outre la classe des problèmes aux limites hyperboliques instables au sens de Hadamard (condition de Lopatinski faible violée), sans aucune estimations, et celle des problèmes fortement stables (condition de Lopatinski forte), avec des estimations sans perte de dérivée, une troisième classe "ouverte", c'est-à-dire stable par petites perturbations des coefficients dans les équations et les conditions aux limites. Cette classe est nommée "weakly stable of real type" ou (WR), car elle a une propriété de faible stabilité (estimées avec perte d'une dérivée) et car elle est caractérisée par un ensemble caractéristique "réel" pour le déterminant de Lopatinski. Les transitions d'une classe ouverte à une autre sont caractérisées dans [10]. En résumé, la condition (WR) est satisfaite lorsque la condition de Lopatinski faible est vérifiée et reste vérifiée pour des perturbations. Sous la condition (WR), le problème aux limites n'est ni fortement stable, ni fortement instable (voir section 3.2.1). En particulier, cela implique la perte d'une dérivée dans l'estimation a priori principale, par rapport au cas de problèmes fortement stables.

On étudies des problèmes aux limites pour des équations d'onde sur un demi-plan, de la forme :

$$q_0(\partial_t, \partial_x, \partial_y)u + l(\partial_t, \partial_x, \partial_y)u = f + F(\,.\,, u) \quad x > 0 \tag{0.3.1a}$$

$$(\partial_x + \beta\partial_t + v \cdot \partial_y)\, u + cu = g + G(\,.\,, u) \quad x = 0, \tag{0.3.1b}$$

où $q_0(\tau, \xi, \eta)$ est un polynôme du second ordre strictement hyperbolique et l est du premier ordre (voir l'hypothèse 3.2.1).

On suppose que la condition (WR), introduite dans [10, 77], est satisfaite. Cette condition est plus faible que la condition de Lopatinski uniforme qui caractérise les problèmes aux limites fortement bien posés. La réflexion d'ondes non linéaires discontinues pour des problèmes satisfaisant la condition forte de Lopatinski est étudiée dans [65, 19].

Dans ce chapitre, les solutions striées sont régulières par rapport à un ensemble de dérivées Y_j, où les champs Y_j pour $j = 1, \ldots, d-1$ sont tangents à la striation $\{\varphi_+ = a_+, \varphi_- = a_-\}$, correspondant aux phases :

$$\varphi_\pm = \xi_\pm x + \omega_0 t + k_0 \cdot y, \tag{0.3.2}$$

avec des fréquences *réelles* $(\xi_\pm, \omega_0, k_0)$. Les phases $\varphi_\pm$ qui portent les éventuelles singularités des solutions sont caractéristiques, i.e. satisfont

$$q_0(\omega_0, \xi_\pm, k_0) = 0. \tag{0.3.3}$$

On suppose que $\xi_+ \neq \xi_-$, ce qui, dans la classification des fréquences au bord pour des problèmes aux limites hyperboliques, signifie que (ω_0, k_0) est un point hyperbolique.

Une des propriétés du problème est qu'il s'écrit (à un facteur multiplicatif strictement positif près) :

$$-X_+ X_- u - Q(Y)u = f + F(\,.\,, u) \quad x > 0 \tag{0.3.4}$$

où $X_\pm = \widetilde{X}_\pm + c_\pm$ sont des opérateurs du premier ordre, avec des termes principaux

$$\widetilde{X}_\pm = \partial_x + \sigma_\pm \partial_{\tilde{t}} + \nu_\pm \cdot \partial_{\tilde{y}}, \tag{0.3.5}$$

tels que $\tilde{X}_+$ [resp. $\tilde{X}_-$] est tangent à la foliation $\{\varphi_+ = const\}$ [resp. $\{\varphi_- = const\}$]. Cela signifie que

$$\tilde{X}_+(\varphi_+) = 0 \quad \text{et} \quad \tilde{X}_-(\varphi_-) = 0. \tag{0.3.6}$$

Sous l'hypothèse 3.2.1, on peut fixer les plus et moins de telle façon que $\sigma_- < 0$ et $\sigma_+ > 0$. Le champ $\tilde{X}_-$ est *sortant*, ce qui signifie qu'il propage le signal vers la gauche par rapport à x pour des temps croissants, tandis que le champ $\tilde{X}_+$ est *entrant* ce qui signifie qu'il le propage vers la droite par rapport à x.

Dans ce cadre, la question principale sur laquelle on se penche dans ce chapitre est la suivante : *si une solution u est régulière par rapport aux Y_j et X_-, et singulière en X_+ dans le passé, comment est réfléchie la singularité lorsque la solution atteint la frontière ? Typiquement, si u (ou ∇u) a une discontinuité à travers $\Sigma_- = \{\varphi_- = 0\}$ dans le passé, comment est-elle réfléchie ?*

Cette question est étudiée dans [19, 65] lorsque le problème aux limites est fortement stable, le résultat principal peut être résumé de la façon suivante : les discontinuités de u (resp. ∇u) à travers $\Sigma_- = \{\varphi_- = 0\}$ sont réfléchies en des discontinuités de u (resp. ∇u) à travers $\Sigma_+ = \{\varphi_+ = 0\}$.

Ici, on étudie le cas où le problème aux limites est seulement faiblement stable et satisfait la condition (WR), et où *la fréquence tangentielle* (ω_0, k_0) *est précisément un point où la condition de Lopatinski est violée.* Dans ce cas, on obtient que les conditions aux bords s'écrivent sous la forme :

$$X_+ u + \check{v} \cdot Y u + \check{c} u = g + G(\,.\,, u), \quad x = 0. \tag{0.3.7}$$

De plus, le coefficient de ∂_t dans le champ de vecteurs tangent $\check{v} \cdot Y$ ne s'annule pas.

Le résultat principal de ce chapitre est que, dans ce cas, *les discontinuités de* ∇u *à travers* $\Sigma_- = \{\varphi_- = 0\}$ *sont réfléchies en des discontinuités de* u *à travers* $\Sigma_+ = \{\varphi_+ = 0\}$, se référer au Théorème 3.2.27 pour un énoncé précis. L'esprit de ce résultat est en accord complet avec la perte d'une dérivée sus-mentionnée pour les problèmes satisfaisant la condition (WR).

Des estimations L^2 sont prouvées dans [77] pour l'équation des ondes, en utilisant un problème auxiliaire fortement bien posé pour Pu où P est un opérateur bien choisi ; la perte d'une dérivée est observée lors du passage d'estimations pour Pu à des estimations pour u. Soulignons le fait que, contrairement au cas fortement stable, les termes du premier ordre $l(\partial_t, \partial_x, \partial_y)u$ dans (3.1.7a) et le terme constant cu dans (3.1.7b) ne peuvent pas être considérés comme de simples perturbations, à cause de la perte d'une dérivée dans les estimations. Cependant, la condition (WR) ne fait intervenir que les termes principaux $q_0(\partial_t, \partial_x, \partial_y)$ à l'intérieur et $\partial_x + \beta\partial_t + v \cdot \partial_y$ à la frontière. Dans ce chapitre, on donne une preuve directe des estimations L^2 qui prend en compte de façon détaillée l'absorption des termes lu et cu dans les estimations. Il existe dans la littérature d'autres travaux où des estimations d'énergie pour des problèmes hyperboliques faiblement stables sont données (voir [21, 22, 24] et leurs références). Ici, nous soulignons le fait que pour des problèmes (WR) (3.1.7) en dimension $d \geq 3$, la condition de Lopatinski est nécessairement violée en des points glancing, pour un large ensemble de paramètres (voir Remarque Remark 3.4.9). Il se semble pas qu'il existe un traitement général de cette situation dans la littérature, notre analyse semble donc être nouvelle aussi à cet égard.

Chapter 1

Rigorous Weakly Nonlinear Geometric Optics for Surface Waves

This chapter is concerned with surface waves, solutions of hyperbolic nonlinear boundary value problems. We construct BKW solutions in the weakly nonlinear regime with infinite expansion in powers of ε. We rigorously justify this expansion, constructing exact solutions, which admit the asymptotic expansions. We also show that the solution is not necessarily localized at the order $\mathcal{O}(\varepsilon^\infty)$ in the interior, even if the data are.

1.1 Introduction

In this paper we consider the propagation of nonlinear high frequency surface waves, which occur for instance in geophysics and magneto-hydrodynamics. They are waves, such as seismic waves, which propagate along the boundary of the domain. The so-called weakly nonlinear regime corresponds to those amplitudes such that the nonlinear effects affect the propagation of the main oscillatory component. The mathematical setting is the analysis of high frequency solutions of boundary value problems. Here we consider, for simplicity, first order conservation laws in space dimension two, on a domain which is a half plane:

$$\begin{cases} \partial_t \vec{u} + \partial_x(f(\vec{u})) + \partial_y(g(\vec{u})) = \vec{h} & y > 0 \\ C\vec{u}_{|y=0} = 0. \end{cases} \tag{1.1.1}$$

Our analysis extends to higher dimensions, general smooth domains and to higher order systems such as elasticity but we present our results in the simpler case (1.1.1). We also choose the framework of symmetric dissipative boundary value problems in the sense of [58], with non characteristic boundary $y = 0$.

The mathematical conditions for the existence of surface waves are well understood: the weak Lopatinski condition is satisfied but the stronger uniform condition fails at some frequencies in the nonhyperbolic region (see e.g. [78, 11, 36]). This means that for all fixed state $\vec{u_0}$ and time-tangential frequencies $(\tau, \xi) \neq (0, 0)$ with $\mathrm{Im}\tau \leq 0$,

1) when $\mathrm{Im}\tau < 0$, the one dimensional ode

$$\partial_Y v + ig'(\vec{u_0})^{-1}(\tau \mathrm{Id} + \xi f'(\vec{u_0}))v = 0, \qquad Cv_{|Y=0} = 0 \tag{1.1.2}$$

has only the trivial solution $v = 0$ in $L^2([0, \infty[)$; this is indeed implied by the dissipativity assumption.

2) there are real frequencies (τ, ξ) such that the equation (1.1.2) has a nontrivial exponentially decaying solution.

More precise statements are given below. Denoting by $(-\omega, k)$ one of the special frequencies of condition 2), one looks for oscillatory surface waves, that is for solutions of the boundary value problem localized near the boundary and such that the trace on the boundary has rapid oscillations with the phase $(kx - \omega t)/\varepsilon$. More precisely, we seek solutions which admit asymptotic expansions of the form

$$\vec{u}(t, x, y) \sim \vec{u}_0 + \sum_{n \geq 1} \varepsilon^n \vec{u}_n\Big(t, x, y, \frac{kx - \omega t}{\varepsilon}, \frac{y}{\varepsilon}\Big) \tag{1.1.3}$$

with profiles $\vec{u}_n(t, x, y, \theta, Y)$ which are periodic with respect to the variable θ and converging to a limit $\underline{\vec{u}_n}(t, x, y)$ with an exponential rate as the fast variable Y tends to $+\infty$.

This kind of expansions has been considered by many authors, see for instance [36, 37, 54, 38, 79, 52] and their references. For instance, the trace of the main term on the boundary is polarized along a given direction:

$$\vec{u}_1(t, x, 0, \theta, 0) = \mathrm{u}_+(t, x, \theta)\vec{R}_+ + \mathrm{u}_-(t, x, \theta)\vec{R}_- \tag{1.1.4}$$

where $\vec{R}_-$ is the trace at $Y = 0$ of the special solution in 2) and $\vec{R}_+$ is its complex conjugate (which satisfies the same property for the frequency

$(\omega, -k)$); moreover, in (1.1.4), u denotes a scalar function, periodic in θ with Fourier series $\sum_{n\neq 0} \mathrm{u}^n e^{in\theta}$ and $\mathrm{u}_+ = \sum_{n>0} \mathrm{u}^n e^{in\theta}$ [resp. $\mathrm{u}_- = \sum_{n<0} \mathrm{u}^n e^{in\theta}$].

The scalar amplitude u satisfies a propagation equation of the form

$$\partial_t \mathrm{u} + v\partial_x \mathrm{u} + \partial_\theta a(\mathrm{u}, \mathrm{u}) = 0 \tag{1.1.5}$$

where a is a *nonlocal* bilinear form acting on periodic functions in θ (see [36] for formal derivation and e.g. [54, 38] for a couple of applications among many).

The main objective of the present paper is twofold:

1) derive the propagation equations in a general setting and construct asymptotic expansions at any order n. It seems that, in the literature, only the equation for the main order has been obtained (see references above) and studied (see [7]). But the analysis of higher terms reveals a new and maybe unexpected phenomenon: an important remark is that, in general, the correctors $\vec{u}_n$ for $n \geq 2$ are not purely localized near the boundary; for instance, in general, $\vec{u}_2$ does depend on the slow variable y and $\vec{u}_2$ does not decay to 0 as Y tends to $+\infty$. This analysis ends with the construction of approximate solutions at any order ε^n.

2) construct exact solutions of the original equation which admit the asymptotic expansions found in the first step, adapting the analysis of O. Guès ([31, 30]). This amounts to prove the stability of the approximate solutions.

1.2 Statement of the main results

1.2.1 Structural assumptions

We consider a system (1.1.1) with C^∞ flux functions f and g. It is assumed to be symmetric hyperbolic in the sense of Friedrichs, with maximal dissipative boundary conditions on the non characteristic boundary $\{y = 0\}$, that is:

Assumption 1.2.1. *For $\vec{u}$ in a neighbourhood of $\vec{u}_0$:*

i) there exists a positive definite symmetric matrix $S(\vec{u})$ such that $S(\vec{u})f'(\vec{u})$ and $S(\vec{u})g'(\vec{u})$ are symmetric.

ii) $g'(\vec{u})$ is invertible

iii) when $\vec{u}$ satisfies the boundary condition $C\vec{u} = 0$, then the matrix $S(\vec{u})g'(\vec{u})$ is non positive on $Ker\, C$ and the rank of C is equal to the number of positive eigenvalues of $Sg'(\vec{u})|_{y=0}$.

We consider surface waves associated to the phase

$$\varphi(t,x) = -\omega t + x, \qquad \omega \in \mathbb{R} \tag{1.2.1}$$

(there is no restriction in assuming that the spatial wave number $k = 1$). We assume that $(-\omega, 1)$ is an *elliptic* frequency:

Assumption 1.2.2. *For τ near ω, all the eigenvalues of the real matrix*

$$A(\tau) = -(g'(\vec{u}_0))^{-1}(-\tau Id + f'(\vec{u}_0))$$

are simple and none is real.

In particular, the eigenvalues are pairwise complex conjugate, and the dimension of the system (1.1.1) must be even. We denote it by $2M$.

Moreover, in this context, the maximal dissipation property requires that the number of boundary conditions must be equal to M. Indeed, from the maximal dissipation property, the number of boundary conditions is equal to the number of positive eigenvalues of $S(\vec{u}_0)g'(\vec{u}_0)$, which is equal to the number of positive eigenvalues of $g'(\vec{u}_0)$, since, on the one hand, $g'(\vec{u}_0)$ and the similar matrix $S(\vec{u}_0)^{\frac{1}{2}}g'(\vec{u}_0)S(\vec{u}_0)^{-\frac{1}{2}}$ have the same eigenvalues, and, on the other hand, $S(\vec{u}_0)^{\frac{1}{2}}g'(\vec{u}_0)S(\vec{u}_0)^{-\frac{1}{2}} = S(\vec{u}_0)^{-\frac{1}{2}}[S(\vec{u}_0)g'(\vec{u}_0)]S(\vec{u}_0)^{-\frac{1}{2}}$ and $S(\vec{u}_0)g'(\vec{u}_0)$ are congruent real symmetric matrices and thus have the same number of positive eigenvalues. It remains to show that the number of positive eigenvalues of $g'(\vec{u}_0)$ is equal to the number of eigenvalues of $A(\tau) = -(g'(\vec{u}_0))^{-1}(-\tau Id + f'(\vec{u}_0))$ with positive imaginary part.

For ξ, η real numbers, the eigenvalues of $\xi f'(\vec{u}_0)+\eta g'(\vec{u}_0)$ are real, because they are equal to the eigenvalues of $S(\vec{u}_0)^{\frac{1}{2}}[\xi f'(\vec{u}_0) + \eta g'(\vec{u}_0)]S(\vec{u}_0)^{-\frac{1}{2}} = S(\vec{u}_0)^{-\frac{1}{2}}[S(\vec{u}_0)(\xi f'(\vec{u}_0)+\eta g'(\vec{u}_0))]S(\vec{u}_0)^{-\frac{1}{2}}$, which is a real symmetric matrix. Thus, by contradiction, the roots η of $\det[(\alpha - i\gamma)\mathrm{Id} + \xi f'(\vec{u}_0) + \eta g'(\vec{u}_0)]$ for $(\alpha, \xi, \gamma) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^*$ have a non-zero imaginary part. Therefore the integer values function

$$\begin{aligned} \phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* &\to \mathbb{N} \\ (\alpha,\xi,\gamma) &\mapsto \mathrm{Card}\,\{\eta / \det[(\alpha - i\,\gamma)\mathrm{Id} + \xi f'(\vec{u}_0) + \eta g'(\vec{u}_0)] = 0 \\ &\qquad\qquad \text{and } \mathrm{Im}\,\eta > 0\} \end{aligned}$$

is continuous, therefore it is constant.

With $\xi = 0$ and $(\alpha, \gamma) \in \mathbb{R}\times\mathbb{R}_+^*$, we obtain that this number is equal to the number of positive eigenvalues of $g'(\vec{u}_0)$, indeed:

$$\det[(\alpha - i\gamma)\mathrm{Id} + \eta g'(\vec{u}_0)] = 0 \Longleftrightarrow \exists\lambda \in \mathrm{Sp}(g'(\vec{u}_0)), \eta = \frac{-\alpha + i\gamma}{\lambda}.$$

From the assumption 1.2.2, that the eigenvalues of $A(\tau)$ are not real, we get that ϕ is continuous in $(\alpha, \xi, \gamma) = (-\tau, 1, 0)$. Thus, with $\xi = 1$ and $\alpha = -\tau$ and by passing to the limit as $\gamma \to 0^+$, we obtain that the number of positive eigenvalues of $g'(\vec{u}_0)$ is equal to the number of eigenvalues of $A(\tau) = -(g'(\vec{u}_0))^{-1}(-\tau Id + f'(\vec{u}_0))$ with positive imaginary part, that is M.

For our analysis, it is more convenient to introduce the eigenvalues of $iA(\tau)$: using the assumption, we can denote them by

$$\lambda_1, \dots, \lambda_M, \quad -\lambda_1^*, \dots, -\lambda_M^* \qquad \text{with } \operatorname{Re}(\lambda_j) > 0 \tag{1.2.2}$$

with z^* denoting the complex conjugate of z. We introduce next right eigenvectors $\vec{R}_j$ and left eigenvectors $\vec{L}_j$

$$iA\vec{R}_j = \lambda_j \vec{R}_j, \qquad i\vec{L}_j\, A = \lambda_j \vec{L}_j,$$

normalized by the condition

$$\vec{L}_j \vec{R}_k = \delta_{jk}.$$

Note that $\vec{R}_j^*$ and $\vec{L}_j^*$ are right and left eigenvectors respectively, associated to the eigenvalue λ_j^*. Note also that these eigenvalues are smooth functions of τ in a neighbourhood of ω. We choose smooth eigenvectors, as we may.

Recall that the Lopatinski condition concerns the invertibility of the matrix C on the positive space of iA, that is the invertibility of the matrix

$$CR(\tau) := \big[C\vec{R}_1(\tau), \dots, C\vec{R}_M(\tau)\big]. \tag{1.2.3}$$

The key condition for the existence of surface waves associated to the phase (1.2.1) is that the uniform Lopatinski condition fails at $(-\omega, 1)$. We assume that it fails in a generic way:

Assumption 1.2.3. *The matrix C has dimension $M \times 2M$, and $\phi(\tau) := \det CR(\tau)$ vanishes at first order at $\tau = \omega$ that is*

$$\phi(\omega) = 0, \qquad \partial_\tau \phi(\omega) \neq 0. \tag{1.2.4}$$

Moreover, we assume that zero is a simple eigenvalue of the matrix $CR(\omega)$ that is $\dim\ker CR(\omega) = 1$.

We denote by $\vec{\rho}, \vec{\sigma}$ left and right eigenvectors corresponding to the zero eigenvalue of the matrix $CR(\omega)$:

$$\vec{\rho} = (\rho_1, \dots, \rho_M)^t, \qquad \vec{\sigma} = (\sigma_1, \dots, \sigma_M) \tag{1.2.5}$$

$$CR(\omega)\vec{\rho} = 0, \qquad \vec{\sigma} CR(\omega) = 0. \tag{1.2.6}$$

1.2.2 Statement of the problem

We seek solutions $\vec{u}^\varepsilon$ of

$$\partial_t \vec{u}^\varepsilon + \partial_x(f(\vec{u}^\varepsilon)) + \partial_y(g(\vec{u}^\varepsilon)) = \vec{h}^\varepsilon \qquad \text{in } y > 0 \tag{1.2.7a}$$

satisfying the boundary condition:

$$C\vec{u}^\varepsilon{}_{|y=0} = 0, \tag{1.2.7b}$$

the condition in the past:

$$\forall t \leq 0,\ \vec{u}^\varepsilon(t) = \vec{u}_0 \tag{1.2.7c}$$

and admitting asymptotic expansions

$$\vec{u}(t,x,y) \sim \vec{u}_0 + \sum_{n\geq 1} \varepsilon^n \vec{u}_n\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) \tag{1.2.7d}$$

with profiles $\vec{u}_i(t,x,y,\theta,Y)$ in the space S defined below:

Definition 1.2.4. *i) $\underline{S}$ denotes the space of functions $\underline{u}(t,x,y)$ in H^∞ on $[0,T]\times\mathbb{R}\times\mathbb{R}_+$, $H^\infty = \bigcap H^s$ denoting the intersection of usual Sobolev spaces.*

ii) S^* denotes the space of functions $u^*(t,x,\theta,Y)$ in H^∞ on $[0,T]\times\mathbb{R}\times\mathbb{T}\times\mathbb{R}_+$, which are periodic in θ and decay exponentially in the variable Y as well as their derivatives, meaning that there is $\delta > 0$ such that for all α, there is C_α such that

$$\big\|\partial^\alpha_{t,x,\theta,Y} u^*(t,\cdot,\theta,Y)\big\|_{L^2(\mathbb{R})} \leq C_\alpha e^{-\delta Y} \tag{1.2.8}$$

iii) $S = \underline{S} \oplus S^*$ is the space of functions $u(t,x,y,\theta,Y) = \underline{u}(t,x,y) + u^*(t,x,\theta,Y)$.

These spaces depend on the chosen time interval $[0,T]$. When necessary, we make this dependence explicit in the notations, writing $S([0,T])$ etc.

The unperturbed constant state $\vec{u}_0$ is supposed to be a solution of the homogeneous problem. This reduces to the condition

$$C\vec{u}_0 = 0. \tag{1.2.9}$$

We look for solutions such that the main non constant term $\vec{u}_1$ is purely localized on the boundary, that is such that $\underline{\vec{u}_1} = 0$ and

$$\vec{u}^\varepsilon(t,x,y) = \vec{u}_0 + \varepsilon \vec{u}_1^*\Big(t,x,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) + O(\varepsilon^2). \tag{1.2.10}$$

Remark 1.2.5. *The assumption $\underline{\vec{u}_1} = 0$ is made in order to simplify some expressions, but it is in fact satisfied by any solution, see remark 1.5.1 below.*

The equations (1.1.1) must be supplemented by Cauchy data on $\{t = 0\}$. To avoid the technicalities induced by compatibility conditions at the corner $\{t = y = 0\}$, we consider data $\vec{h}^\varepsilon$ which vanish identically in the past, and solutions equal to the constant solution $\vec{u}_0$ in the past:

$$\vec{u}^\varepsilon{}_{|\{t\leq 0\}} = \vec{u}_0, \qquad \vec{h}^\varepsilon{}_{|\{t\leq 0\}} = 0. \tag{1.2.11}$$

The surface wave is ignited by the source term $\vec{h}^\varepsilon$, which we assume to be small and localized near the boundary

$$\vec{h}^\varepsilon(t,x,y) = \varepsilon \vec{h}_1\Big(t, x, \frac{x-\omega t}{\varepsilon}, \frac{y}{\varepsilon}\Big). \tag{1.2.12}$$

More generally, we can take

$$\vec{h}^\varepsilon(t,x,y) \sim \sum_{k\geq 1} \varepsilon^k \vec{h}_k\Big(t, x, \frac{x-\omega t}{\varepsilon}, \frac{y}{\varepsilon}\Big) \tag{1.2.13}$$

with profiles $\vec{h}_k \in S^*(]-\infty, T_0])$ such that $\forall t \leq 0$, $\vec{h}_k(t) = 0$.

1.2.3 Main results

A) *Formal asymptotics; the fast equation.*

Substituting the expansion (1.2.7d) in (1.2.7a), and ordering in powers of ε, we obtain the following equations:

$$\begin{cases} \mathcal{L}_0 \vec{u}_1 = 0 \\ \mathcal{L}_0 \vec{u}_{k+1} + \vec{G}_k(\vec{u}_0, \ldots, \vec{u}_k, \vec{h}_k) = 0, \qquad \text{for } k \geq 1, \end{cases} \tag{1.2.14}$$

with

$$\mathcal{L}_0 = (-\omega Id + f'(\vec{u}_0))\partial_\theta + g'(\vec{u}_0)\partial_Y. \tag{1.2.15}$$

The expression of $\vec{G}_k(\vec{u}_0, \ldots, \vec{u}_k, \vec{h}_k)$ is given in section 1.3. The boundary conditions read

$$C\vec{u}_{k|y=Y=0} = 0, \qquad \text{for } k \geq 1. \tag{1.2.16}$$

This cascade of equations leads to the following differential equation in the fast variables (θ, Y):

$$\partial_Y \vec{u} = A(\omega)\partial_\theta \vec{u} - (g'(\vec{u}_0))^{-1}\vec{G}, \tag{1.2.17}$$

$$C\vec{u}|_{y=Y=0} = 0. \tag{1.2.18}$$

This equation is analyzed in details in section 1.4, using Fourier expansions in the variable θ. For instance, we use the notations

$$\vec{G}(t,x,y,\theta,Y) = \sum_{n\in\mathbb{Z}} \vec{G}^n(t,x,y,Y)e^{in\theta}.$$

Note that when $\vec{G} \in S$, then the Fourier coefficients $\vec{G}^n$ are exponentially decaying in Y and do not depend on the slow variable y when $n \neq 0$. On the other hand $\vec{G}^0$ is the sum of a function independent of Y, $\underline{\vec{G}}^0(t,x,y)$, and of an exponentially decaying function in Y independent of y, $\vec{G}^{0,*}(t,x,Y)$.

Proposition 1.2.6. *When $\vec{G} = 0$, the solutions in S of the homogeneous equations* (1.2.17) (1.2.18) *are functions with Fourier coefficients of the form*

$$\vec{u}^n(t,x,Y) = h^n(t,x)e^{inYA(\omega)}\vec{R}_\pm, \quad \textit{when} \ \pm n > 0, \tag{1.2.19}$$

with $\vec{R}_- = \sum \rho_j \vec{R}_j$ and $\vec{R}_+ = \sum \rho_j^* \vec{R}_j^*$, and

$$\vec{u}^0(t,x,y,Y) = \underline{\vec{u}}^0(t,x,y), \qquad \textit{with} \quad C\underline{\vec{u}}^0(t,x,0) = 0 \tag{1.2.20}$$

when $n = 0$.

Note that $\vec{R}_+$ [resp. $\vec{R}_-$] belongs to the negative [resp. positive] space of $iA(\omega)$, so that (1.2.19) defines exponentially decaying functions of Y.

Proposition 1.2.7. *If G belongs to $\underline{S}\oplus S^*$, then the problem* (1.2.17) (1.2.18) *has a solution $\vec{u} \in S$, if and only if*

$$\underline{G}^0(t,x,y) = 0, \tag{1.2.21}$$

$$\forall n \neq 0, \qquad \int_0^\infty \vec{L}(n,Y).\vec{G}^n(t,x,Y)\,dY = 0, \tag{1.2.22}$$

where the functions $\vec{L}(n,Y)$ are defined in Section 1.4.

Moreover, for $\vec{G}$ satisfying these conditions, there is a continuous partial inverse $\vec{G} \mapsto \vec{u} = \mathcal{R}\vec{G}$, also defined in Section 1.4. The general solution is the sum of $\mathcal{R}\vec{G}$ and a solution of the homogeneous problem described in Proposition 1.2.6.

B) *Formal asymptotics; the leading term.*

A corollary of Proposition 1.2.6 is that $\vec{u}_1 = \vec{u}_1^*$ is determined from its boundary data $\vec{u}^*_{1|Y=0}$ which have the form

$$\vec{u}_1^*(t,x,\theta,0) = \sum_{n>0} K_1(n,x,t)e^{in\theta}\vec{R}_+ + \sum_{n<0} K_1(n,x,t)e^{in\theta}\vec{R}_-. \tag{1.2.23}$$

It is convenient to set $K_1(0, x, t) = 0$ and introduce

$$K_1(\theta, x, t) = \sum_{n \in \mathbb{Z}} K_1(n, x, t) e^{in\theta}.$$

Thus $\vec{u}_1$ is determined by the scalar function with zero mean $K_1(\theta, x, t)$.

The compatibility conditions of Proposition 1.2.7 for the existence of $\vec{u}_2 \in S$ satisfying $\mathcal{L}_0 \vec{u}_2 + \vec{G}_1(\vec{u}_0, \vec{u}_1, \vec{h}_1) = 0$ and (1.2.16), are shown to be equivalent to an equation for K_1 of the form:

$$\partial_t K_1 + v \partial_x K_1 + \partial_\theta a(K_1, K_1) = H_1 \tag{1.2.24}$$

where H_1 is related to $\vec{h}_1^*$. The nonlocal bilinear form a on Fourier expansions reads

$$a(K_1, K_1) = \sum_{n \in \mathbb{Z}} \Big(\sum_{l \in \mathbb{Z}} \Lambda(n-l, l) K_1(l, x, t) K_1(n-l, x, t) \Big) e^{in\theta} \tag{1.2.25}$$

with coefficients $\Lambda(n-l, l)$ given in section 1.5.2.

The coefficients $\Lambda(n, l)$ are shown to satisfy the properties:

$$\forall n \in \mathbb{Z}, \ \ \forall l \in \mathbb{Z} \ \ \forall \alpha > 0 \qquad \begin{aligned} &\Lambda(n, l) = 0 \ \text{ if } nl(n+l) = 0 \\ &\Lambda(-n, -l) = \Lambda^*(n, l) \\ &\Lambda(n, l) = \Lambda(l, n) \\ &\Lambda(\alpha n, \alpha l) = \Lambda(n, l). \end{aligned}$$

In order to prove a local-in-time existence of K_1, we assume that $\Lambda(n, l)$ satisfies the following supplementary condition:

Assumption 1.2.8. *There exists a constant C such that*

$$\forall |m| < |n| \quad |\Lambda(m, n) - \Lambda(m, -m-n)| \leq C \left| \frac{m}{n} \right|. \tag{1.2.26}$$

This assumption holds under the stability condition $\Lambda(1, 0^-) = \Lambda^*(1, 0^+)$ and regularity conditions on Λ (for example if Λ is $\mathcal{C}^1$ outsides the lines $k = 0$, $l = 0$, and $k + l = 0$ and has $\mathcal{C}^1$ continuations to the sectors delimited by these lines), see Sylvie Benzoni-Gavage [7]. The stability condition $\Lambda(1, 0^-) = \Lambda^*(1, 0^+)$ has been pointed out by Hunter [36] as formally ensuring the linearized stability of constant states. It has been showed in [8] that the latter condition, as conjectured by Hunter, is not only sufficient for well-posedness in Sobolev spaces but also necessary.

Extending [7] to the case of Assumption 1.2.8 and to the case of solutions depending on x, the equation for K_1 is solved in section 1.7:

Theorem 1.2.9. *There is* $T > 0$*, such that the equation* (1.2.24) *has a unique solution* $K_1 \in H^\infty(]-\infty, T] \times \mathbb{R} \times \mathbb{T})$ *vanishing when* $t \le 0$.

At this stage, the profile $\vec{u}_1$ is known.

C) *Formal asymptotics; correctors.*

In section 1.8, iterating this method, we obtain the other terms $\vec{u}_k$ such that the equations in (1.2.14) are satisfied at all order k.

For instance, knowing $\vec{u}_1$, the equations

$$\mathcal{L}_0 \vec{u}_2 + \vec{G}_1(\vec{u}_0, \vec{u}_1, \vec{h}_1) = 0, \qquad C\vec{u}_2|_{y=Y=0} = 0,$$

can be solved in S, since $\vec{G}_1(\vec{u}_0, \vec{u}_1, \vec{h}_1)$ satisfies the compatibility conditions of Proposition 1.2.7, and

$$\vec{u}_2 = \mathcal{R}\vec{G}_1(\vec{u}_0, \vec{u}_1, \vec{h}_1) + \vec{v}_2$$

where $\vec{v}_2$ is a solution of the homogeneous problem, thus determined by a scalar function $K_2(t, x, \theta)$ and a function $\underline{\vec{v}}_2^0(t, x, y)$. The compatibility conditions (1.2.22) for $\vec{G}_2(\vec{u}_0, \vec{u}_1, \vec{u}_2, \vec{h}_1, \vec{h}_2)$ give a linear equation for K_2, of the form,

$$\partial_t K_2 + v\partial_x K_2 + \partial_\theta\big(a(K_1, K_2) + a(K_2, K_1)\big) = H_2, \tag{1.2.27}$$

determining K_2. Similarly, the compatibility condition (1.2.21) for $\vec{G}_2$ gives an equation for $\underline{\vec{v}}_2^0$, of the form

$$\partial_t \underline{\vec{v}}_2^0 + f'(\vec{u}_0)\partial_x \underline{\vec{v}}_2^0 + g'(\vec{u}_0)\partial_y \underline{\vec{v}}_2^0 = \underline{\vec{H}_2^0}, \qquad C\underline{\vec{v}}^0_{2|y=0} = 0, \tag{1.2.28}$$

determining $\underline{\vec{v}}_2^0$, where $\underline{\vec{H}_2^0}$ is related to $\underline{\vec{h}_2^0}$ and $\underline{\mathcal{R}\vec{G}_1}(\vec{u}_0, \vec{u}_1, \vec{h}_1)$.

This analysis, repeated at all orders, yields the following result:

Theorem 1.2.10. *There is a unique sequence of profiles* $\vec{u}_k \in S(]-\infty, T])$ *for* $k \ge 1$*, which vanish when* $t \le 0$ *and such that the equations (1.2.14) and the boundary conditions (1.2.16) are satisfied.*

Remark 1.2.11. It is important to point out that, *in general*, the component $\underline{\vec{v}}_2$ and thus $\underline{\vec{u}}_2$ does not vanish. More precisely, the equation for $\underline{\vec{u}}_2$ reads

$$\begin{cases} \partial_t \underline{\vec{u}}_2 + f'(\vec{u}_0)\partial_x \underline{\vec{u}}_2 + g'(\vec{u}_0)\partial_y \underline{\vec{u}}_2 = \underline{\vec{h}_2^0}, \\ C\underline{\vec{u}}_{2|y=0} = -C\displaystyle\int_0^\infty (g'(\vec{u}_0))^{-1}\vec{G}_1^{0,*}(t, x, Y)dY. \end{cases} \tag{1.2.29}$$

with $\vec{G}_1^{0,*}$ denoting the exponentially decaying part of $\vec{G}_1^0$, the 0-Fourier coefficient of $\vec{G}_1 = \vec{G}_1(\vec{u}_0, \vec{u}_1, \vec{h}_1)$. In general, this term does not vanish, and therefore, even if data h are localized on the boundary, that is $\underline{\vec{h}_k^0} = 0$ at all order k, we get $\underline{\vec{u}}_2 \neq 0$. This observation, that the corrector $\vec{u}_2$ is not purely localized on the boundary seems new.

D) *Approximate solutions.*

Because of the substitution

$$u^\varepsilon(t,x,y) = \vec{u}\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big)$$

we measure the smoothness of functions u^ε using the derivatives $\varepsilon\partial_{t,x,y}$.

Definition 1.2.12. *We say that a family f^ε is $\mathcal{O}(\varepsilon^k)$ in $H^s_\varepsilon([0,T]\times\mathbb{R}\times\mathbb{R}_+)$ if there is a constant C such that for all $|\alpha|\le s$:*

$$\varepsilon^{|\alpha|}\sup_{t\le T}\|\partial^\alpha_{t,x,y}f^\varepsilon(t)\|_{L^2(\mathbb{R}\times\mathbb{R}_+)} \le C\varepsilon^k\,. \tag{1.2.30}$$

A corollary of Theorem 1.2.10 is the following:

Theorem 1.2.13. *Suppose that $\{\vec{u}_k\}_{k\ge1}$ is given by Theorem 1.2.10 and let*

$$\vec{u}^\varepsilon_{app}(t,x,y) = \vec{u}_0 + \sum_{k=1}^{M}\varepsilon^k\vec{u}_k\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big). \tag{1.2.31}$$

Suppose that

$$\vec{h}^\varepsilon(t,x,y) - \sum_{k=1}^{M-1}\varepsilon^k\vec{h}_k\Big(t,x,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) = \mathcal{O}(\varepsilon^M) \tag{1.2.32}$$

in H^s_ε. Then, $\vec{u}^\varepsilon_{app}$ is an approximate solution of (1.2.7a) (1.2.7b) (1.2.7c) *in the sense that*

$$\begin{cases} \partial_t\vec{u}^\varepsilon_{app} + \partial_x(f(\vec{u}^\varepsilon_{app})) + \partial_y(g(\vec{u}^\varepsilon_{app})) - \vec{h}^\varepsilon = \mathcal{O}(\varepsilon^M) \quad in\ H^s_\varepsilon,\\ C\vec{u}^\varepsilon_{app|y=0} = 0,\\ \forall t\le 0,\ \vec{u}^\varepsilon_{app}(t) = \vec{u}_0. \end{cases} \tag{1.2.33}$$

E) *Exact solutions.*

Having constructed approximate solutions, the problem is to construct exact solutions of (1.2.7a) (1.2.7b) (1.2.7c) close to $\vec{u}^\varepsilon_{app}$. The equation for the residual $\vec{u}^\varepsilon - \vec{u}^\varepsilon_{app}$ is given and solved in section 1.9, yielding the following result.

Theorem 1.2.14. *Suppose that $\vec{u}^\varepsilon_{app}$ is a family of approximate solutions* (1.2.31) *and $\vec{h}^\varepsilon$ satisfies* (1.2.32) *with $M \geq 3$ and $s \geq 3$. Then, there is $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$, the problem* (1.2.7a) (1.2.7b) *has a unique solution $\vec{u}^\varepsilon$ equal to $\vec{u}_0$ in the past and*

$$\vec{u}^\varepsilon - \vec{u}^\varepsilon_{app} = \mathcal{O}(\varepsilon^M) \qquad \text{in } H^s_\varepsilon. \tag{1.2.34}$$

The proof of this result relies on an extension to dissipative boundary value problems of a theorem proved by O.Guès in [31, 30], see also [27]. It concerns equations with rapidly oscillating coefficients and we state it here for its own interest. The proof is given in Section 1.10.

Theorem 1.2.15. *Consider a boundary value problem on $\mathbb{R}^d_+ := \{x_d \geq 0\}$:*

$$A_0(a,u)\partial_t u + \sum_{j=1}^{d} A_j(a,u)\partial_{x_j} u = F(b,u) \tag{1.2.35}$$

$$Cu|_{x_d=0} = 0 \tag{1.2.36}$$

$$\forall t \leq 0 \;\; u(t) = 0. \tag{1.2.37}$$

We assume that the problem is hyperbolic symmetric and the boundary not characteristic. The coefficients a and b are assumed to have rapid oscillations, more precisely, we assume that they belong to the Sobolev space $W^{s,\infty}(]-\infty, T] \times \mathbb{R}^d_+)$ and that their derivatives satisfy:

$$\forall \;\; 0 \leq |\alpha| \leq s \quad \varepsilon^{(|\alpha|-1)_+} \, \|\partial^\alpha_{t,x} a\|_{L^\infty} \leq C_1\,, \tag{1.2.38}$$

$$\forall \;\; 0 \leq |\alpha| \leq s \quad \varepsilon^{|\alpha|} \, \|\partial^\alpha_{t,x} b\|_{L^\infty} \leq C_1'\,. \tag{1.2.39}$$

for some integer $s > \frac{d}{2} + 1$. In (1.2.38)*, m_+ denotes* $\max\{m, 0\}$.

We suppose that 0 *is an exact solution in the past $\{t \leq 0\}$ and that for $t \geq 0$ it is an approximate solution, meaning that $f = F(b, 0)$ vanishes in the past and satisfies for $|\alpha| \leq s$ and $t \leq T$:*

$$\varepsilon^{|\alpha|} \, \|\partial^\alpha_{t,x} f^\varepsilon(t)\|_{L^2(\mathbb{R}^d_+)} \leq \varepsilon^M C_2\,. \tag{1.2.40}$$

with $M > 1 + d/2$.

Then, there exist $\varepsilon_0 > 0$ and C_3, depending only on the constants C_1, C_2 and coefficients A_j and F, such that for $\varepsilon \in]0, \varepsilon_0]$, the boundary value problem (1.2.35)-(1.2.36)-(1.2.37) has a unique solution $u \in \bigcap_{0 \leq k \leq s} \mathcal{C}^k([0,T]; H^{s-k})$ which satisfies for $0 \leq |\alpha| \leq s$ and $t \leq T$:

$$\varepsilon^{|\alpha|} \, \|\partial^\alpha_{t,x} u(t)\|_{L^2(\mathbb{R}^d_+)} \leq \varepsilon^M C_3\,. \tag{1.2.41}$$

1.3 The cascade of equations for the $\vec{u}_i$

In this section we derive the system (1.2.14) for the $\vec{u}_i$. The main point is that the space of asymptotic expansions with profiles in S is stable by nonlinear mappings.

Lemma 1.3.1. *Let ϕ be a function in $\mathcal{C}^\infty(\mathbb{R}^{2M})$ and $\vec{u} = \vec{u}_0 + \sum_{j=1}^n \varepsilon^j \vec{u}_j$ with $\vec{u}_j \in S$. Let*

$$u^\varepsilon(t,x,y) = \vec{u}\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big).$$

Then there are profiles $\varphi_j \in S$, for $j = 1,\dots,n$, with φ_j depending only on $(\vec{u}_0,\dots,\vec{u}_j)$ such that

$$\phi(\vec{u}^\varepsilon) = \phi(\vec{u}_0) + \sum_{j=1}^n \varepsilon^j \varphi_j\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) + \varepsilon^{n+1} R_n^\varepsilon\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) \tag{1.3.1}$$

where $R_n^\varepsilon(t,x,y,\theta,Y)$ is a bounded family of functions which are H^∞ in each variables and converge with an exponential rate when $Y \to \infty$.

Furthermore, if $\underline{\vec{u}_1} = 0$, that is if $\vec{u}_1 \in S^*$, then

$$\varphi_1(\vec{u}_0,\vec{u}_1) = \phi'(\vec{u}_0)\vec{u}_1, \qquad \varphi_2(\vec{u}_0,\vec{u}_1,\vec{u}_2) = \phi'(\vec{u}_0)\vec{u}_2 + \frac{1}{2}\phi''(\vec{u}_0)\vec{u}_1^*\vec{u}_1^*$$

and for $j \geq 3$:

$$\begin{aligned}\varphi_j(\vec{u}_0,\dots,\vec{u}_j) = \phi'(\vec{u}_0)\vec{u}_j + \phi''(\vec{u}_0)\underline{\vec{u}}_{j-1|y=0}\vec{u}_1^* + \phi''(\vec{u}_0)\vec{u}_{j-1}^*\vec{u}_1^* \\ + \psi_j(\vec{u}_0,\dots,\vec{u}_{j-2})\end{aligned}$$

Proof. Perform a Taylor expansion

$$\phi(\vec{u}) = \phi(\vec{u}_0) + \sum_{k=1}^n \frac{1}{k!}\phi^{(k)}(\vec{u}_0)(\vec{u}-\vec{u}_0)^k + O(\varepsilon^{n+1}). \tag{1.3.2}$$

For simplicity, we use notations as if the functions were scalar. Next,

$$(\vec{u}-\vec{u}_0)^k = \sum_{j=1}^n \sum_{p_1+\cdots+p_k=j\,,\,p_i>0} \varepsilon^j \vec{u}_{p_1}\dots\vec{u}_{p_k} + O(\varepsilon^{n+1}).$$

The functions $\vec{u}_j$ belong to S, therefore they can be written in the form $\vec{u}_j = \underline{\vec{u}_j} + \vec{u}_j^*$ with $\underline{\vec{u}_j}$ in $\underline{S}$ and $\vec{u}_j^*$ in S^*, then:

$$\vec{u}_{p_1}\dots\vec{u}_{p_k} = \underline{\vec{u}}_{p_1}\dots\underline{\vec{u}}_{p_k} + \cdots + \underline{\vec{u}}_{l_1}\dots\underline{\vec{u}}_{l_r}\vec{u}_{l_{r+1}}^*\dots\vec{u}_{l_k}^* + \cdots.$$

The first term $\underline{\vec{u}}_{p_1} \dots \underline{\vec{u}}_{p_k}$ clearly belongs to $\underline{S}$, it remains to make sure that the other terms $\underline{\vec{u}}_{l_1} \dots \underline{\vec{u}}_{l_r} \vec{u}^*_{l_{r+1}} \dots \vec{u}^*_{l_k}$ are equal to expressions of the form $\sum_{j=1}^n \varepsilon^j \vec{V}_j + O(\varepsilon^{n+1})$ with $\vec{V}_j$ in S^*, after the substitution $\theta = (x - \omega t)/\varepsilon$ and $Y = y/\varepsilon$. This is equivalent to showing that if $\underline{a}$ belongs to $\underline{S}$ and b^* belongs to S^*, then

$$\underline{a} b^* \Big(t, x, y, \frac{x - \omega t}{\varepsilon}, \frac{y}{\varepsilon}\Big) = \sum_{j=1}^n \varepsilon^j \vec{V}_j \Big(t, x, y, \frac{x - \omega t}{\varepsilon}, \frac{y}{\varepsilon}\Big) + O(\varepsilon^{n+1})$$

with $\vec{V}_j$ belonging to S^*. Taylor expanding $\underline{a}$ in y, we see that, after the substitution, $\underline{a} b^*$ is equivalent to

$$\underline{a}(t, x, 0) b^*(t, x, \theta, Y) + \sum_{j=1}^n \varepsilon^j \partial_y^j \underline{a}(t, x, 0) Y^j b^*(t, x, \theta, Y) + O(\varepsilon^{n+1}).$$

Observing that an exponentially decreasing function in the variable Y multiplied by a power of Y is still exponentially decreasing, we obtain that $\underline{a} b^*$ is in the required form.

Observing that the only term depending on $\vec{u}_j$ comes from the term $k = 1$ of equation (1.3.2) and the terms depending on $\vec{u}_{j-1}$ come from the term $k = 2$, we obtain the expression of φ_j. $\square$

With the notations of Lemma 1.3.1:

$$\begin{aligned}
\partial_x(f(\vec{u})) &= \sum_j \varepsilon^{j-1} \partial_\theta f_j + \varepsilon^j \partial_x f_j \\
&= \partial_\theta f_1 + \sum_{j=1}^n \varepsilon^j [\partial_\theta f_{j+1} + \partial_x f_j] + o(\varepsilon^n) \\
&= f'(\vec{u}_0) \partial_\theta \vec{u}_1 + \varepsilon [f'(\vec{u}_0)(\partial_\theta \vec{u}_2 + \partial_x \vec{u}_1) + f''(\vec{u}_0) \partial_\theta \vec{u}_1^* \vec{u}_1^*] \\
&+ \sum_{j=2}^n \varepsilon^j [f'(\vec{u}_0)(\partial_\theta \vec{u}_{j+1} + \partial_x \vec{u}_j) + f''(\vec{u}_0) \underline{\vec{u}_j}(y = 0) \partial_\theta \vec{u}_1^* \\
&+ f''(\vec{u}_0) \partial_\theta \vec{u}_j^* \vec{u}_1^* + f''(\vec{u}_0) \vec{u}_j^* \partial_\theta \vec{u}_1^* + \psi_j(\vec{u}_0, \dots, \vec{u}_{j-1})] + o(\varepsilon^n)
\end{aligned}$$

$$\partial_t \vec{u} = -\omega \partial_\theta \vec{u}_1 + \varepsilon(\partial_t \vec{u}_1 - \omega \partial_\theta \vec{u}_2) + \textstyle\sum_{j=2}^n \varepsilon^j (\partial_t \vec{u}_j - \omega \partial_\theta \vec{u}_{j+1}) + o(\varepsilon^n)$$

$$
\begin{aligned}
\partial_y(g(\vec{u})) &= \sum_j \varepsilon^{j-1}\partial_Y g_j + \varepsilon^j \partial_y g_j \\
&= \partial_Y g_1 + \sum_{j=1}^{n} \varepsilon^j [\partial_Y g_{j+1} + \partial_y g_j] + o(\varepsilon^n) \\
&= g'(\vec{u}_0)\partial_Y \vec{u}_1 + \varepsilon[g'(\vec{u}_0)(\partial_Y \vec{u}_2 + \partial_y \vec{u}_1) + g''(\vec{u}_0)\partial_Y \vec{u}_1^* \vec{u}_1^*] \\
&+ \sum_{j=2}^{n} \varepsilon^j [g'(\vec{u}_0)(\partial_Y \vec{u}_{j+1} + \partial_y \vec{u}_j) + g''(\vec{u}_0)\underline{\vec{u}_j}(y=0)\partial_Y \vec{u}_1^* \\
&+ g''(\vec{u}_0)\partial_Y \vec{u}_j^* \vec{u}_1^* + g''(\vec{u}_0)\vec{u}_j^* \partial_Y \vec{u}_1^* + \psi_j(\vec{u}_0, \ldots, \vec{u}_{j-1})] + o(\varepsilon^n).
\end{aligned}
$$

Equating the coefficient of ε^0 in equation (1.2.7a) to zero yields:

$$f'(\vec{u}_0)\partial_\theta \vec{u}_1 - \omega \partial_\theta \vec{u}_1 + g'(\vec{u}_0)\partial_Y \vec{u}_1 = 0$$

that is

$$\mathcal{L}_0 \vec{u}_1 = 0$$

with

$$\mathcal{L}_0 = (-\omega Id + f'(\vec{u}_0))\partial_\theta + g'(\vec{u}_0)\partial_Y.$$

From the coefficient of ε^1, we get:

$$\mathcal{L}_0 \vec{u}_2 + \vec{G}_1(\vec{u}_0, \vec{u}_1, \vec{h}_1) = 0$$

with

$$
\begin{aligned}
\vec{G}_1(\vec{u}_0, \vec{u}_1, \vec{h}_1) = \partial_t \vec{u}_1 + f'(\vec{u}_0)\partial_x \vec{u}_1 + f''(\vec{u}_0)\partial_\theta \vec{u}_1^* \vec{u}_1^* \\
+ g'(\vec{u}_0)\partial_y \vec{u}_1 + g''(\vec{u}_0)\partial_Y \vec{u}_1^* \vec{u}_1^* - \vec{h}_1.
\end{aligned}
$$

From the coefficient of ε^k, we get:

$$\mathcal{L}_0 \vec{u}_{k+1} + \vec{G}_k(\vec{u}_0, ..., \vec{u}_k, \vec{h}_k) = 0$$

with

$$
\begin{aligned}
&\vec{G}_k(\vec{u}_0, ..., \vec{u}_k, \vec{h}_k) = \partial_t \vec{u}_k + f'(\vec{u}_0)\partial_x \vec{u}_k + f''(\vec{u}_0)\underline{\vec{u}_k}(y=0)\partial_\theta \vec{u}_1^* + f''(\vec{u}_0)\partial_\theta \vec{u}_k^* \vec{u}_1^* \\
&+ f''(\vec{u}_0)\vec{u}_k^* \partial_\theta \vec{u}_1^* + g'(\vec{u}_0)\partial_y \vec{u}_k + g''(\vec{u}_0)\underline{\vec{u}_k}(y=0)\partial_Y \vec{u}_1^* + g''(\vec{u}_0)\partial_Y \vec{u}_k^* \vec{u}_1^* \\
&+ g''(\vec{u}_0)\vec{u}_k^* \partial_Y \vec{u}_1^* + \psi_j(\vec{u}_0, \ldots, \vec{u}_{k-1}) - \vec{h}_k.
\end{aligned}
$$

Then, finally:

$$\begin{cases} \mathcal{L}_0\vec{u}_1 = 0 \\ \vdots \\ \mathcal{L}_0\vec{u}_{k+1} + \vec{G}_k(\vec{u}_0, \dots, \vec{u}_k, \vec{h}_k) = 0 \end{cases}$$

with

$$\mathcal{L}_0 = (-\omega Id + f'(\vec{u}_0))\partial_\theta + g'(\vec{u}_0)\partial_Y$$

and $\vec{G}_k$ belonging to $\underline{S} \oplus S^*$.

By the assumptions $\vec{u}_1, \dots, \vec{u}_k, \vec{h}_k$, and therefore also $\vec{G}_k$, are periodic in θ, we can write their decomposition in Fourier series:

$$\vec{u}_k = \sum_{n\in\mathbb{Z}} e^{in\theta}\vec{u}_k^{\,n}(Y),\ \vec{G}_k = \sum_{n\in\mathbb{Z}} e^{in\theta}\vec{G}_k^{\,n}(Y) \text{ and } \vec{h}_k = \sum_{n\in\mathbb{Z}} e^{in\theta}\vec{h}_k^n(Y).$$

We obtain then the following equations:

$$\partial_Y\vec{u}_1^{\,n} = inA\vec{u}_1^{\,n}$$

$$\forall k \geq 1 \quad \partial_Y\vec{u}_{k+1}^{\,n} = inA\vec{u}_{k+1}^{\,n} - (g'(\vec{u}_0))^{-1}\vec{G}_k^n$$

with

$$A(\omega) = -(g'(\vec{u}_0))^{-1}(-\omega Id + f'(\vec{u}_0)).$$

1.4 Equation of limit layer

We seek $\vec{u}$ belonging to S solution of a problem:

$$\partial_Y\vec{u} = A\partial_\theta\vec{u} - (g'(\vec{u}_0))^{-1}\vec{G} \tag{1.4.1}$$

$$C\vec{u}|_{y=Y=0} = 0 \tag{1.4.2}$$

with $\vec{G}$, and therefore $(g'(\vec{u}_0))^{-1}\vec{G}$, belonging to S.

Thus, writing the Fourier coefficients:

$$\partial_Y\vec{u}^n = inA\vec{u}^n - (g'(\vec{u}_0))^{-1}\vec{G}^n \tag{1.4.3}$$

$$C\vec{u}^n|_{y=Y=0} = 0. \tag{1.4.4}$$

Let us prove the following proposition:

Proposition 1.4.1. *If $\vec{G}$ belongs to $\underline{S} \oplus S^*$, then the problem*

$$\partial_Y \vec{u} = A\partial_\theta \vec{u} - (g'(\vec{u}_0))^{-1}\vec{G} \tag{1.4.5}$$

$$C\vec{u}|_{y=Y=0} = 0 \tag{1.4.6}$$

has a solution $\vec{u} \in \underline{S} \oplus S^*$ if and only if

$$\underline{\vec{G}^0}(t, x, y) = 0$$

and

$$\forall n \neq 0 \quad \int_0^\infty \vec{L}(n, s).\vec{G}^n(t, x, s)\, ds = 0$$

where

$$\vec{L}(n, Y) = \sum_{j=1}^{M} \tau_j \exp(-n\lambda_j Y)(g'(\vec{u}_0)^{-1})^t \vec{L}_j^t \quad \textit{if } n > 0$$

and

$$\vec{L}(-n, Y) = \vec{L}^*(n, Y) \quad \forall n \in \mathbb{Z}.$$

The solutions are functions with Fourier coefficients of the form:

$$\vec{u}^0 = \underline{\vec{u}^0}(t, x, y) + \int_Y^\infty (g'(\vec{u}_0))^{-1}\vec{G}^0(t, x, s)\, ds$$

$$\vec{u}^n = \sum_{j=1}^{M} V_j(n, Y)\vec{R}_j + W_j(n, Y)\vec{R}_j^{\ *}$$

with

if $n > 0$:

$$V_j = \int_Y^\infty \exp(n\lambda_j(Y - s))F_j(n, s)\, ds$$

$$W_j = [g_{known,j} + K(n, x, t)\rho_j^{\ *}] \exp(-n\lambda_j^* Y) - \int_0^Y \exp(-n\lambda_j^*(Y - s))H_j(n, s)\, ds$$

if $n < 0$:

$$V_j = [f_{known,j} + K(n, x, t)\rho_j] \exp(n\lambda_j Y) - \int_0^Y \exp(n\lambda_j(Y - s))F_j(n, s)\, ds$$

$$W_j = \int_Y^\infty \exp(-n\lambda_j^*(Y - s))H_j(n, s)\, ds$$

where $F_j = \vec{L}_j\ g'(\vec{u}_0)^{-1}\vec{G}^n$, $H_j = \vec{L}_j^{\ *}\ g'(\vec{u}_0)^{-1}\vec{G}^n$
and $f_{known} = (f_{known,1}, \ldots, f_{known,M})^t$ and $g_{known} = (g_{known,1}, \ldots, g_{known,M})^t$

are known functions, depending on $\vec{G}$, whose expressions are given by (1.4.11) and (1.4.12).

$\underline{\vec{u}^0}$ is an unknown function lying in $\underline{S}$ which has to satisfy the boundary condition

$$C\underline{\vec{u}^0}(t,x,y=0) = -C\int_0^\infty (g'(\vec{u}_0))^{-1}\vec{G}^0(t,x,s)\,ds$$

and $K(\theta,x,t) = \sum_{n\in\mathbb{Z}} K(n,x,t)e^{in\theta}$ is an unknown scalar-valued function with zero mean.

We begin to prove a lemma, which will be useful for the convergence of the series.

Lemma 1.4.2. *If F belongs to S^*,*

then $IF = \sum_{n\neq 0} I_n F^n e^{in\theta}$ belongs to S^*,

where $I_n F^n$ is in the form

$\int_Y^\infty \exp(n\lambda(Y-s))F^n(s)\,ds$ *or* $\int_0^Y \exp(-n\lambda^*(Y-s))F^n(s)\,ds$

if $n>0$ and

$\int_Y^\infty \exp(-n\lambda^*(Y-s))F^n(s)\,ds$ *or* $\int_0^Y \exp(n\lambda(Y-s))F^n(s)\,ds$

if $n<0$,

where λ is a complex number with non-negative real parts.

Proof. F belongs to S^*: there exist $\delta>0$ and $\gamma_n>0$ where $(\gamma_n)_n$ is a rapidly decreasing sequence such that $\forall t,x,Y$ $\|F^n(t,x,Y)\|_{L^2(\mathbb{R})} \leq \gamma_n e^{-\delta Y}$ then for all λ with non-negative real parts,
for all $n>0$:

$$\begin{aligned}\left\|\int_Y^\infty \exp(n\lambda(Y-s))F^n(s)\,ds\right\|_{L^2(\mathbb{R})} &\leq \int_Y^\infty \exp(n\lambda(Y-s))\|F^n(s)\|_{L^2(\mathbb{R})}\,ds\\ &\leq \int_Y^\infty \exp(n\lambda(Y-s))\gamma_n e^{-\delta s}\,ds\\ &= \frac{-1}{n\lambda+\delta}\left[\exp(n\lambda(Y-s))\gamma_n e^{-\delta s}\right]_Y^\infty\\ &\leq \frac{\gamma_n}{n\lambda+\delta}e^{-\delta Y}\end{aligned}$$

As above, we prove: for all $n>0$, if $0<\delta<\frac{1}{2}\Re(\lambda)$

$$\left\|\int_0^Y \exp(-n\lambda^*(Y-s))F^n(s)\,ds\right\|_{L^2(\mathbb{R})} \leq \frac{\gamma_n}{n\lambda^*-\delta}e^{-\delta Y} \tag{1.4.7}$$

for all $n > 0$:

$$\left\| \int_Y^\infty \exp(-n\lambda^*(Y-s))F^n(s)\,ds \right\|_{L^2(\mathbb{R})} \leq \frac{\gamma_n}{-n\lambda^* + \delta} e^{-\delta Y} \tag{1.4.8}$$

and if $0 < \delta < \frac{1}{2}\Re(\lambda)$

$$\left\| \int_0^Y \exp(n\lambda(Y-s))F^n(s)\,ds \right\|_{L^2(\mathbb{R})} \leq \frac{\gamma_n}{-n\lambda - \delta} e^{-\delta Y}. \tag{1.4.9}$$

Applying the assumption F belonging to S^* and differentiating under the integration sign, we obtain that the derivatives of these expressions are bounded by similar terms. We deduce then that IF belongs to S^*, since the sequences $(\gamma_n)_n$ are rapidly decreasing, and decreasing δ, if necessary, so that $0 < \delta < \frac{1}{2}\Re(\lambda)$.

$\square$

For $n = 0$, the equation is $\partial_Y \vec{u}^0 = -(g'(\vec{u}_0))^{-1}\vec{G}^0$.

$\vec{u}^0$ is then in the form

$$\vec{u}^0 = \underline{\vec{u}^0}(t,x,y) + \int_Y^\infty (g'(\vec{u}_0))^{-1}\vec{G}^0(t,x,s)\,ds$$

with $\underline{\vec{u}^0}$ an independent of Y function, that is belonging to $\underline{S}$. In order to obtain $\vec{u}^0$ belonging to S, necessarily $\int_Y^\infty (g'(\vec{u}_0))^{-1}\vec{G}^0\,ds$ has to belong to S^*, that is has to decay exponentially in Y, therefore $\underline{\vec{G}^0}$, the independent of Y part of $\vec{G}^0$, has to be equal to zero. We then obtain the following condition:

$$\underline{\vec{G}^0} = 0.$$

This condition will allow us to write an equation satisfied by the part belonging to $\underline{S}$ of the 0 order Fourier coefficient of the foregoing term in the power series in ε.

$\vec{u}^0$ satisfies the boundary condition if and only if

$$C\underline{\vec{u}^0}(t,x,y=0) = -C\int_0^\infty (g'(\vec{u}_0))^{-1}\vec{G}^0(t,x,s)\,ds.$$

For $n \neq 0$, we write the decompositions in the basis $\vec{R}_1, \ldots, \vec{R}_M, \vec{R}_1{}^*, \ldots, \vec{R}_M{}^*$ of $\mathbb{C}^{2M}$:

$$\vec{u}^n = \sum_{j=1}^M V_j(n,Y)\vec{R}_j + W_j(n,Y)\vec{R}_j{}^*$$

$$\vec{G}^n = \sum_{j=1}^{M} F_j(n,Y) g'(\vec{u}_0)\vec{R}_j + H_j(n,Y) g'(\vec{u}_0)\vec{R}_j{}^*$$

where $F_j = \vec{L}_j\ g'(\vec{u}_0)^{-1}\vec{G}^n$ and $H_j = \vec{L}_j{}^*\ g'(\vec{u}_0)^{-1}\vec{G}^n$,
therefore substituting these expressions into equation (1.4.3):

$$\partial_Y V_j = n\lambda_j V_j - F_j(n,Y)$$

$$\partial_Y W_j = -n\lambda_j^* W_j - H_j(n,Y).$$

Resolving these equations, we obtain:

$$V_j = V_j^0(n)\, \exp\left(n\lambda_j Y\right) - \int_0^Y \exp(n\lambda_j(Y-s)) F_j(n,s)\, ds$$

$$W_j = W_j^0(n)\, \exp\left(-n\lambda_j^* Y\right) - \int_0^Y \exp(-n\lambda_j^*(Y-s)) H_j(n,s)\, ds.$$

To be bounded in the Y variable implies:

$$V_j^0(n) = \int_0^\infty \exp(-n\lambda_j s) F_j(n,s)\, ds \quad \text{if } n > 0$$

$$W_j^0(n) = \int_0^\infty \exp(n\lambda_j^* s) H_j(n,s)\, ds \quad \text{if } n < 0.$$

We still have to determine $V_j^0(n)$ for $n < 0$ and $W_j^0(n)$ for $n > 0$. The boundary condition is equivalent to:

$$\begin{aligned}\text{if } n<0 \quad & [C\vec{R}_1, \ldots, C\vec{R}_M](V_1^0(n), \ldots, V_M^0(n))^t = \\ & -[C\vec{R}_1{}^*, \ldots, C\vec{R}_M{}^*]\Big(\int_0^\infty \exp(n\lambda_1^* s) H_1(n,s)\, ds, \ldots, \\ & \int_0^\infty \exp(n\lambda_M^* s) H_M(n,s)\, ds\Big)^t\end{aligned}$$

and

$$\begin{aligned}\text{if } n>0 \quad & [C\vec{R}_1{}^*, \ldots, C\vec{R}_M{}^*](W_1^0(n), \ldots, W_M^0(n))^t = \\ & -[C\vec{R}_1, \ldots, C\vec{R}_M]\Big(\int_0^\infty \exp(-n\lambda_1 s) F_1(n,s)\, ds, \ldots, \\ & \int_0^\infty \exp(-n\lambda_M s) F_M(n,s)\, ds\Big)^t.\end{aligned}$$

Thus the following conditions are necessary:

$$\text{if } n<0 \quad [C\vec{R}_1{}^*,\ldots,C\vec{R}_M{}^*]\Big(\int_0^\infty \exp(n\lambda_1^* s)H_1(n,s)\,ds,\ldots,$$

$$\int_0^\infty \exp(n\lambda_M^* s)H_M(n,s)\,ds\Big)^t \in \mathrm{Im}[C\vec{R}_1,\ldots,C\vec{R}_M] \qquad (1.4.10a)$$

$$\text{if } n>0 \quad [C\vec{R}_1,\ldots,C\vec{R}_M]\Big(\int_0^\infty \exp(-n\lambda_1 s)F_1(n,s)\,ds,\ldots,$$

$$\int_0^\infty \exp(-n\lambda_M s)F_M(n,s)\,ds\Big)^t \in \mathrm{Im}[C\vec{R}_1{}^*,\ldots,C\vec{R}_M{}^*]. \qquad (1.4.10b)$$

Under these conditions, since 0 is a simple eigenvalue of the matrix $[C\vec{R}_1,\ldots,C\vec{R}_M]$ and $\vec{\rho}$ is a corresponding right eigenvector, the boundary condition will be satisfied if and only if:

$$V_j^0(n) = f_{known,j} + K(n,x,t)\rho_j \quad \text{if } n<0$$

and

$$W_j^0(n) = g_{known,j} + K(n,x,t)\rho_j{}^* \quad \text{if } n>0$$

where f_{known}=$(f_{known,1},\ldots,f_{known,M})^t$ and g_{known}=$(g_{known,1},\ldots,g_{known,M})^t$ are defined by:

$$f_{known} = -A^{-1}[C\vec{R}_1{}^*,\ldots,C\vec{R}_M{}^*]\Big(\int_0^\infty \exp(n\lambda_1^* s)H_1(n,s)\,ds,\ldots,$$

$$\int_0^\infty \exp(n\lambda_M^* s)H_M(n,s)\,ds\Big)^t \qquad (1.4.11)$$

where A is the matrix corresponding to the matrix $[C\vec{R}_1,\ldots,C\vec{R}_M]$, restricted to a supplement of its kernel and corestricted to its image.

$$g_{known} = -B^{-1}[C\vec{R}_1,\ldots,C\vec{R}_M]\Big(\int_0^\infty \exp(-n\lambda_1 s)F_1(n,s)\,ds,\ldots,$$

$$\int_0^\infty \exp(-n\lambda_M s)F_M(n,s)\,ds\Big)^t \qquad (1.4.12)$$

where B is the matrix corresponding to the matrix $[C\vec{R}_1{}^*,\ldots,C\vec{R}_M{}^*]$, restricted to a supplement of its kernel and corestricted to its image.

f_{known} and g_{known} are known functions depending on F_j and H_j, that is depending on G, they decay exponentially in Y since G decays exponentially. Thus finally, the expression of $\vec{u}$ is determined, except an unknown scalar-valued function

$$K(\theta, x, t) = \sum_{n \in \mathbb{Z}^*} K(n, x, t) e^{in\theta}$$

and an unknown function $\underline{\vec{u}^0}$ that lies in $\underline{S}$.

Remark 1.4.3. *The $K(n, x, t)$ expressions appear in the Fourier coefficients $\vec{u}^n$ of order $n \neq 0$. It is convenient to set $K(0, x, t) = 0$ and write $K(\theta, x, t) = \sum_{n \in \mathbb{Z}} K(n, x, t) e^{in\theta}$.*

It remains to study the resolubility conditions (1.4.10). Since $\vec{\sigma}^*$ is a left eigenvector corresponding to the simple eigenvalue zero of $[C\vec{R}_1{}^*, \ldots, C\vec{R}_M{}^*]$, we obtain $\mathrm{Im}[C\vec{R}_1{}^*, \ldots, C\vec{R}_M{}^*] = \left(\mathrm{span}\ \vec{\sigma}^{*\,t}\right)^{\perp}$. Therefore, the condition (1.4.10b) can be written under the form

$$\forall n > 0 \ \ \vec{\sigma}^*[C\vec{R}_1, \ldots, C\vec{R}_M]\Big(\int_0^\infty \exp(-n\lambda_1 s) F_1(n, s)\, ds, \ldots, \int_0^\infty \exp(-n\lambda_M s) F_M(n, s)\, ds\Big)^t = 0$$

Using the notations $\vec{\tau} = [\tau_1, \ldots, \tau_M] = \vec{\sigma}^*[C\vec{R}_1, \ldots, C\vec{R}_M]$:

$$\forall n > 0 \ \ \sum_{j=1}^{M} \tau_j \int_0^\infty \exp(-n\lambda_j s) F_j(n, s)\, ds = 0.$$

Since $F_j = \vec{L}_j \ g'(\vec{u}_0)^{-1} \vec{G}^n = (g'(\vec{u}_0)^{-1})^t \vec{L}_j^t \,.\, \vec{G}^n$:

$$\int_0^\infty \sum_{j=1}^{M} \tau_j \exp(-n\lambda_j s)(g'(\vec{u}_0)^{-1})^t \vec{L}_j^t \,.\, \vec{G}^n\, ds = 0.$$

Similarly for $n < 0$ we obtain:

$$\int_0^\infty \sum_{j=1}^{M} \tau_j^* \exp(n\lambda_j^* s)(g'(\vec{u}_0)^{-1})^t \vec{L}_j^{*t} \,.\, \vec{G}^n\, ds = 0$$

Thus finally this gives the following resolubility condition:

$$\int_0^\infty \vec{L}(n, s).\vec{G}^n\, ds = 0 \tag{1.4.13}$$

where

$$\vec{L}(n,Y) = \sum_{j=1}^{M} \tau_j \exp(-n\lambda_j Y)(g'(\vec{u}_0)^{-1})^t \vec{L}_j^t \quad \text{if } n > 0$$

$$\vec{L}(n,Y) = \sum_{j=1}^{M} \tau_j^* \exp(n\lambda_j^* Y)(g'(\vec{u}_0)^{-1})^t \vec{L}_j^{*t} \quad \text{if } n < 0,$$

the following properties $\vec{L}(n,Y)$ result from these expressions:

$$\begin{aligned} \vec{L}(-n,Y) &= \vec{L}^*(n,Y) \quad \forall n \in \mathbb{Z} \\ \vec{L}(\alpha^{-1}n, \alpha Y) &= \vec{L}(n,Y) \quad \forall \alpha > 0. \end{aligned}$$

This resolubility condition allows us to write an equation satisfied by the unknown scalar-valued function K of the foregoing term in the power series in ε.

It remains to establish the convergence of the Fourier series. The convergence of $K(n)$ is self-evident, since they are Fourier coefficients of a regular function K. The terms of the form $K(n,x,t)\rho_j{}^* \exp\left(-n\lambda_j^* Y\right)$ for $n > 0$ and $K(n,x,t)\rho_j \exp\left(n\lambda_j Y\right)$ for $n < 0$ are bounded by a constant multiplied by $K(n)$, their convergence follows. The other terms are in one of the forms appearing in Lemma 1.4.2, with $F \in S^*$. The lemma gives us the convergence and the fact that the sums belong to S^*. We proved the convergence of the Fourier series and the belonging of the sum to S. This finishes the proof of Proposition 1.4.1.

1.5 Determination of $\vec{u}_1$

1.5.1 Expression of $\vec{u}_1$

The Fourier coefficients of $\vec{u}_1$ satisfy the equation:

$$\partial_Y \vec{u}_1^{\,n} = inA\vec{u}_1^{\,n}. \tag{1.5.1}$$

It is an equation of limit layer with $\vec{G} = 0$. The resolubility conditions are immediately satisfied, therefore, applying the previous proposition, we obtain:

for $n = 0$, necessarily $\vec{u}_1^{\,0} = 0$, since $\underline{\vec{u}_1} = 0$,

for $n \neq 0$,

$$\vec{u}_1^{\,n} = \sum_{j=1}^{M} K_1(n,x,t)\rho_j{}^* \exp(-n\lambda_j^* Y)\vec{R}_j{}^* \quad \text{if } n > 0$$

$$\vec{u}_1^{\,n} = \sum_{j=1}^{M} K_1(n,x,t)\rho_j \exp(n\lambda_j Y)\vec{R}_j \ \text{ if } n<0$$

where $K_1(n,x,t)$ is a scalar-valued function.
Thus

$$\vec{u}_1^{\,n} = K_1(n,x,t)\vec{R}(n,Y)$$

with

$$\begin{aligned} \vec{R}(n,Y) &= \sum_{j=1}^{M} \rho_j^{\,*} \exp(-n\lambda_j^* Y)\vec{R}_j^{\,*} \quad \text{if } n>0 \\ \vec{R}(n,Y) &= \sum_{j=1}^{M} \rho_j \exp(n\lambda_j Y)\vec{R}_j \quad \text{if } n<0. \end{aligned} \tag{1.5.2}$$

We deduce the following properties of $\vec{R}(n,Y)$:

$$\vec{R}(-n,Y) = \vec{R}^*(n,Y) \quad \forall n\in\mathbb{Z}$$

$$\vec{R}(\alpha^{-1}n,\alpha Y) = \vec{R}(n,Y) \quad \forall \alpha>0.$$

Thus finally $\vec{u}_1$ solution of the equation (1.5.1) and satisfying the boundary condition is in the form

$$\vec{u}_1 = \sum_{n\in\mathbb{Z}} e^{in\theta}\, K_1(n,x,t)\, \vec{R}(n,Y)$$

with $K_1(\theta,x,t) = \sum_{n\in\mathbb{Z}} K_1(n,x,t)e^{in\theta}$ an unknown scalar-valued function with zero mean.

Remark 1.5.1. *The computation above can be repeated without the assumption (1.2.10) $\underline{\vec{u}_1} = 0$, we get the boundary condition $C\underline{\vec{u}_1}|_{y=0} = 0$, the condition in the past $\forall t \leq 0, \underline{\vec{u}_1}(t) = 0$. From the equation satisfied by $\vec{u}_2$, we get the resolubility condition $\underline{\vec{G}_1^0}(t,x,y) = 0$, that is, since $\underline{\vec{h}_1} = 0$, $\partial_t\underline{\vec{u}_1} + f'(\vec{u}_0)\partial_x\underline{\vec{u}_1} + g'(\vec{u}_0)\partial_y\underline{\vec{u}_1} = 0$. Therefore $\underline{\vec{u}_1} = 0$ necessarily.*

1.5.2 Necessary condition for the existence of a solution $\vec{u}_2$

The Fourier coefficients $\vec{u}_2^{\,n}$ of $\vec{u}_2$ satisfy the equation:

$$\partial_Y \vec{u}_2^{\,n} = inA\vec{u}_2^{\,n} - (g'(\vec{u}_0))^{-1}\vec{G}_1^n. \tag{1.5.3}$$

It is an equation of limit layer with $\vec{G} = \vec{G}_1$.

It follows from the resolubility condition described in section 1.4:

$$\int_0^\infty \vec{L}(n,s).\vec{G}_1^{\,n}(t,x,s)\,ds = 0 \tag{1.5.4}$$

where the expression of $\vec{L}(n,Y)$ is given in section 1.4.

This necessary condition for the existence of $\vec{u}_2$ allows us to write an equation verified by the scalar-valued function $K_1(\theta,x,t) = \sum_{n\in\mathbb{Z}} K_1(n,x,t)e^{in\theta}$.

$\vec{u}_1$ does not depend on y since $\underline{\vec{u}_1} = 0$. $\vec{G}_1^{\,n}$ is in the form:

$$\vec{G}_1 = \partial_t\vec{u}_1 + f'(\vec{u}_0)\partial_x\vec{u}_1 + \frac{1}{2}\partial_\theta f''(\vec{u}_0)\vec{u}_1^*\vec{u}_1^* + g''(\vec{u}_0)\partial_Y\vec{u}_1^*\vec{u}_1^* - \vec{h}_1$$

thus

$$\begin{aligned}\vec{G}_1^{\,n} = \partial_t\vec{u}_1^{\,n} + f'(\vec{u}_0)\partial_x\vec{u}_1^{\,n} + \sum_{l\in\mathbb{Z}}\frac{1}{2}inf''(\vec{u}_0)\vec{u}_1^{\,l}\vec{u}_1^{\,n-l} \\ + \sum_{l\in\mathbb{Z}}\frac{1}{2}g''(\vec{u}_0)\vec{u}_1^{\,l}\partial_Y\vec{u}_1^{\,n-l} + \frac{1}{2}g''(\vec{u}_0)\vec{u}_1^{\,n-l}\partial_Y\vec{u}_1^{\,l} - \vec{h}_1^n\end{aligned}$$

We have

$$\vec{u}_1^{\,n} = K_1(n,x,t)\,\vec{R}(n,Y),$$

we introduce $\vec{R}'(n,Y)$ the function defined by:

$$\partial_Y\vec{R}(n,Y) = in\vec{R}'(n,Y).$$

From the expression of $\vec{R}(n,Y)$ (1.5.2), we get the following expression and properties of $\vec{R}'(n,Y)$:

$$\begin{aligned}\vec{R}'(n,Y) &= i\sum_{j=1}^{M}\lambda_j{}^*\rho_j{}^*\exp(-n\lambda_j^*Y)\vec{R}_j{}^* \quad \text{if } n>0 \quad (1.5.5)\\ \vec{R}'(-n,Y) &= \vec{R}'^*(n,Y) \quad \forall n\in\mathbb{Z}\\ \vec{R}'(\alpha^{-1}n,\alpha Y) &= \vec{R}'(n,Y) \quad \forall\alpha>0.\end{aligned}$$

We obtain the following expression of $\vec{G}_1^{\,n}$:

$$\begin{aligned}\vec{G}_1^{\,n} =& \partial_t K_1(n,x,t)\vec{R}(n,Y) + \partial_x K_1(n,x,t)f'(\vec{u}_0)\vec{R}(n,Y) + \\ &\sum_{l\in\mathbb{Z}}\Big[\frac{1}{2}inf''(\vec{u}_0)\vec{R}(l,Y)\vec{R}(n-l,Y) + \frac{1}{2}i(n-l)g''(\vec{u}_0)\vec{R}(l,Y)\vec{R}'(n-l,Y)\\ &+\frac{1}{2}ilg''(\vec{u}_0)\vec{R}(n-l,Y)\vec{R}'(l,Y)\Big]\,K_1(l,x,t)K_1(n-l,x,t) - \vec{h}_1^n.\end{aligned}$$

It follows from the condition (1.5.4) $\int_0^\infty \vec{L}(n,s).\vec{G}\,_1^n\,ds = 0$:

$$\partial_t K_1(n,x,t) + v\partial_x K_1(n,x,t) \\ + in\sum_{l\in\mathbb{Z}} \Lambda(n-l,l)K_1(l,x,t)K_1(n-l,x,t) = H_1^n \tag{1.5.6}$$

with, if $nl(n+l)\neq 0$

$$\Lambda(n,l)=\left[\int_0^\infty \vec{L}(n+l,s).\vec{R}(n+l,s)ds\right]^{-1}\int_0^\infty \vec{L}(n+l,s).\left(\frac{1}{2}f''(\vec{u}_0)\vec{R}(l,s)\vec{R}(n,s)\right. \\ \left.+\frac{1}{2}\frac{n}{n+l}g''(\vec{u}_0)\vec{R}(l,s)\vec{R}'(n,s)+\frac{1}{2}\frac{l}{n+l}g''(\vec{u}_0)\vec{R}(n,s)\vec{R}'(l,s)\right)ds$$

and $\Lambda(n,l) = 0$ if $nl(n+l) = 0$.

Remark 1.5.2. Since K_1 is with zero mean (that is $K_1(n = 0,x,t) = 0$), we can set $\Lambda(n,l) = 0$ if $n = 0$ or $l = 0$ and $H_1^n = 0$ if $n = 0$. Furthermore, the coefficient in in equation (1.5.6) allows us to set $\Lambda(n,l) = 0$ if $n+l = 0$.

Since $\vec{R}(n,Y)$, $\vec{R}'(n,Y)$ and $\vec{L}(n,Y)$ satisfy the properties:

$$\begin{aligned} \vec{R}(-n,Y) &= \vec{R}^*(n,Y) \quad \forall n\in\mathbb{Z} \\ \vec{R}(\alpha^{-1}n,\alpha Y) &= \vec{R}(n,Y) \quad \forall\alpha>0 \end{aligned}$$

we deduce the following properties of $\Lambda(n,l)$:

$$\forall n\in\mathbb{Z},\ \forall l\in\mathbb{Z}\ \forall\alpha>0 \qquad \Lambda(n,l) = 0 \text{ if } nl(n+l)=0 \tag{1.5.7a}$$
$$\Lambda(-n,-l) = \Lambda^*(n,l) \tag{1.5.7b}$$
$$\Lambda(n,l) = \Lambda(l,n) \tag{1.5.7c}$$
$$\Lambda(\alpha n,\alpha l) = \Lambda(n,l). \tag{1.5.7d}$$

Therefore it is sufficient to know the expression of $\Lambda(n,l)$ for $n+l>0,\ l>0,\ n>0$ and $n+l>0,\ l>0,\ n<0$, since the expression of $\Lambda(n,l)$ for other (n,l) values can be deduced from the previous properties. We set $n+l>0$, then

$$\begin{aligned} \int_0^\infty \vec{L}(n+l,s).\vec{R}(n+l,s)ds &= \sum_{p,q}\int_0^\infty e^{(-(n+l)\lambda_p s)} \\ &\quad e^{(-(n+l)\lambda q^* s)}\,ds\ \tau_p\rho_q^*(g'(\vec{u}_0)^{-1})^t\vec{L}_p.\vec{R}_q^* \\ &= \sum_{p,q}\frac{\tau_p\rho_q^*}{(n+l)(\lambda_p+\lambda_q^*)}(g'(\vec{u}_0)^{-1})^t\vec{L}_p.\vec{R}_q^* \\ &= \frac{N}{n+l} \end{aligned}$$

where

$$N = \sum_{p,q} \frac{\tau_p \rho_q^*}{(\lambda_p + \lambda_q^*)} (g'(\vec{u}_0)^{-1})^t \vec{L}_p . \vec{R}_q^* . \tag{1.5.8}$$

If $n+l>0$, $l>0$, $n>0$, then

$$\begin{aligned}
\Lambda(n,l) =& \frac{n+l}{N} \int_0^\infty \sum_{j,p,q} \tau_j \exp(-(n+l)\lambda_j s)(g'(\vec{u}_0)^{-1})^t \vec{L}_j . \\
& [\frac{1}{2} f''(\vec{u}_0) \rho_p^* \exp(-l\lambda_p^* s) \vec{R}_p^* \rho_q^* \exp(-n\lambda_q^* s) \vec{R}_q^* \\
& + i\frac{1}{2} \frac{n\lambda_q^* + l\lambda_p^*}{n+l} g''(\vec{u}_0) \rho_p^* \exp(-l\lambda_p^* s) \vec{R}_p^* \rho_q^* \exp(-n\lambda_q^* s) \vec{R}_q^*] \; ds \\
=& \frac{n+l}{N} \sum_{j,p,q} \frac{\tau_j \rho_p^* \rho_q^*}{(n+l)\lambda_j + l\lambda_p^* + n\lambda_q^*} (g'(\vec{u}_0)^{-1})^t \vec{L}_j . \\
& [\frac{1}{2} f''(\vec{u}_0) \vec{R}_p^* \vec{R}_q^* + i\frac{1}{2} \frac{n\lambda_q^* + l\lambda_p^*}{n+l} g''(\vec{u}_0) \vec{R}_p^* \vec{R}_q^*].
\end{aligned}$$

If $n+l>0$, $l>0$, $n<0$, then

$$\begin{aligned}
\Lambda(n,l) =& \frac{n+l}{N} \int_0^\infty \sum_{j,p,q} \tau_j \exp(-(n+l)\lambda_j s)(g'(\vec{u}_0)^{-1})^t \vec{L}_j . \\
& [\frac{1}{2} f''(\vec{u}_0) \rho_p^* \exp(-l\lambda_p^* s) \vec{R}_p^* \rho_q \exp(n\lambda_q s) \vec{R}_q \\
& + i\frac{1}{2} \frac{-n\lambda_q + l\lambda_p^*}{n+l} g''(\vec{u}_0) \rho_p^* \exp(-l\lambda_p^* s) \vec{R}_p^* \rho_q \exp(n\lambda_q s) \vec{R}_q] \; ds \\
=& \frac{n+l}{N} \sum_{j,p,q} \frac{\tau_j \rho_p^* \rho_q}{(n+l)\lambda_j + l\lambda_p^* - n\lambda_q} (g'(\vec{u}_0)^{-1})^t \vec{L}_j . \\
& [\frac{1}{2} f''(\vec{u}_0) \vec{R}_p^* \vec{R}_q + i\frac{1}{2} \frac{-n\lambda_q + l\lambda_p^*}{n+l} g''(\vec{u}_0) \vec{R}_p^* \vec{R}_q]
\end{aligned}$$

thus finally:

If $n+l>0$, $l>0$ and $n>0$:

$$\begin{aligned}
\Lambda(n,l) =& \frac{n+l}{N} \sum_{j,p,q} \frac{\tau_j \rho_p^* \rho_q^*}{(n+l)\lambda_j + l\lambda_p^* + n\lambda_q^*} (g'(\vec{u}_0)^{-1})^t \vec{L}_j . \\
& [\frac{1}{2} f''(\vec{u}_0) \vec{R}_p^* \vec{R}_q^* + i\frac{1}{2} \frac{n\lambda_q^* + l\lambda_p^*}{n+l} g''(\vec{u}_0) \vec{R}_p^* \vec{R}_q^*].
\end{aligned}$$

If $n+l>0$, $l>0$ and $n<0$:

$$\Lambda(n,l) = \frac{n+l}{N}\sum_{j,p,q}\frac{\tau_j\rho_p^*\rho_q}{(n+l)\lambda_j+l\lambda_p^*-n\lambda_q}(g'(\vec{u}_0)^{-1})^t\vec{L}_j.$$
$$[\frac{1}{2}f''(\vec{u}_0)\vec{R}_p^*\vec{R}_q+i\frac{1}{2}\frac{-n\lambda_q+l\lambda_p^*}{n+l}g''(\vec{u}_0)\vec{R}_p^*\vec{R}_q].$$

$$H_1^n = \left[\int_0^\infty \vec{L}(n,s).\vec{R}(n,s)\,ds\right]^{-1}\int_0^\infty \vec{h}_1^n.\vec{L}(n,s)\,ds \text{ if } n\neq 0 \qquad (1.5.9)$$

and $H_1^0=0$ (see remark 1.5.2).

v depends on n:

$$v(n) = \left[\int_0^\infty \vec{L}(n,s).\vec{R}(n,s)\,ds\right]^{-1}\int_0^\infty f'(\vec{u}_0)\vec{R}(n,s).\vec{L}(n,s)\,ds$$

From the expression of v and the properties of $\vec{L}$ and $\vec{R}$, it follows:

$$\forall n,\ v(-n)=v(n)^*.$$

Furthermore

$$\forall n>0,\ v(n) = \frac{\sum_{p,q}\frac{\rho_p^*\tau_q}{n(\lambda_p^*+\lambda_q)}f'(\vec{u}_0)\vec{R}_p^*.(g'(\vec{u}_0)^{-1})^t\vec{L}_p}{\frac{N}{n}}$$
$$= \frac{\sum_{p,q}\frac{\rho_p^*\tau_q}{\lambda_p^*+\lambda_q}f'(\vec{u}_0)\vec{R}_p^*.(g'(\vec{u}_0)^{-1})^t\vec{L}_p}{\sum_{p,q}\frac{\tau_p\rho_q^*}{(\lambda_p+\lambda_q^*)}(g'(\vec{u}_0)^{-1})^t\vec{L}_p.\vec{R}_q^*}.$$

thus $v(n)$ depends only on the sign of n.

We considered $\theta=\frac{x-\tau t}{\varepsilon}$, with τ near ω, under the hypothesis $\phi(\tau) = det[C\vec{R}_1,\ldots,C\vec{R}_M]$ vanishing at the order 1 in $\tau=\omega$. If we consider $\theta=\frac{\xi x+\tau t}{\varepsilon}$ and $\phi(\tau,\xi) = det[C\vec{R}_1,\ldots,C\vec{R}_M]$, the previous hypothesis is equivalent to $\phi(-\omega,1)=0$ and $\partial_\tau\phi(-\omega,1)\neq 0$. Therefore, by continuity, for all τ near $-\omega$ and for all ξ near 1, $\partial_\tau\phi(\tau,\xi)\neq 0$ and we can write, by implicit function theorem and Taylor expansion:

in the neighbourhood of $(-\omega,1)$,

$$\phi(\tau,\xi)=e(\tau,\xi)(\tau+\mu(\xi))$$

where $e(\tau,\xi)\neq 0$, $-\omega+\mu(1)=0$ and μ is real for ξ real.

v depends only on $f'(\vec{u}_0)$ and on $g'(\vec{u}_0)$, that is the linearized terms of the equation. In the following section, we establish the expression of the speed of surface waves for a linear problem, the result applies to v, since v corresponds to the speed of the linearized problem. Thus, the results of the following section give: $v=\partial_\xi\mu(1)$, that is

$$v=\frac{\partial_\xi\phi}{\partial_\tau\phi}(-\omega,1).$$

In particular, v is real, but v depends only on the sign of n and $\forall n,\ v(-n)=v(n)^*$, therefore:

v is independent of n.

Finally, we proved that the scalar-valued function with zero mean $K_1(\theta,x,t)$, whose Fourier coefficients with respect to θ are the $K_1(n,x,t)$, satisfies the equation:

$$\partial_t K_1(\theta,x,t)+v\partial_x K_1(\theta,x,t)+\partial_\theta a(K_1(\theta,x,t),K_1(\theta,x,t))=H_1 \quad (1.5.10)$$

where H_1 is the scalar-valued function whose Fourier coefficients are H_1^n, given by (1.5.9), and a is a bilinear form defined by its Fourier coefficients with respect to the variable θ:

$$a(u,v)_n(x,t)=\sum_{l\in\mathbb{Z}}\Lambda(n-l,l)u_{n-l}(x,t)v_l(x,t). \quad (1.5.11)$$

1.6 Speed of surface waves

1.6.1 Result

We consider the linear part of system

$$\partial_t u+A\partial_x u+B\partial_y u=0.$$

We introduce

$$G(\tau,\xi)=-iB^{-1}\big(\tau\mathrm{Id}+\xi A\big).$$

We consider a neighbourhood of $(\tau_0,\xi_0)\in\mathbb{R}^d\backslash\{0\}$ and we denote $G(\tau_0,\xi_0)$ by G_0.

Assumption 1.6.1. Spectrum$(G_0)\cap i\mathbb{R}=\emptyset$.

Thus Spectrum$(G(\tau,\xi))\cap i\mathbb{R}=\emptyset$ in a neighbourhood of (τ_0,ξ_0).

Notations 1.6.2. We denote by $\mathbb{E}^-(\tau,\xi) \subset \mathbb{C}^{2N}$ [resp. $\mathbb{E}^+(\tau,\xi)$] the space of dimension N generated by the complex eigenvectors corresponding to the eigenvalues with real part non-positive [resp. non-negative]. Then

$$\mathbb{C}^{2N} = \mathbb{E}^-(\tau,\xi) \oplus \mathbb{E}^+(\tau,\xi).$$

We denote by $\Pi^\pm(\tau,\xi)$ the projectors corresponding to these spaces.

We can also choose basis $(r_1^\pm,\dots,r_N^\pm)$ of $\mathbb{E}^\pm$ and denote by $R^\pm(\tau,\xi)$ the column matrix $[r_1^\pm,\dots,r_N^\pm]$, thus

$$\mathbb{E}^\pm(\tau,\xi) = R^\pm(\tau,\xi)\mathbb{C}^N.$$

We denote $R^\pm(\tau_0,\xi_0)$ by $R_0^\pm$.

Asymptotic solutions

We seek asymptotic solution in the form

$$u^\varepsilon(t,x,y) = e^{i(\tau_0 t+\xi_0 x)/\varepsilon}\big(u_0(t,x,\frac{y}{\varepsilon}) + \varepsilon u_1 \dots\big)$$

the corresponding equations are

$$\begin{aligned}
&\partial_Y u_0 - G_0 u_0 = 0,\\
&\partial_Y u_1 - G_0 u_1 = -B^{-1}\big(\partial_t u_0 + A\partial_x u_0\big) := F_0.
\end{aligned}$$

The solution bounded in Y is

$$u_0(t,x,Y) = e^{YG_0}R_0^- b_0(t,x).$$

Then $F_0(t,x,Y)$ is bounded (and decays exponentially). Writing $u_1 = u_1^- + u_1^+$ with $u_1^\pm = \Pi_0^\pm u_1$, we get that $u_1^-(0) = R_0^- b_1$ is arbitrary in $\mathbb{E}_0^-$ and that

$$u_1^+(t,x,Y) = -\int_Y^\infty e^{(Y-Y')G_0}\Pi_0^+ F_0(t,x,Y')dY'.$$

Thus

$$\begin{aligned}
u_1^+(t,x,0) &= -\int_0^\infty e^{-YG_0}\Pi_0^+ F_0(t,x,Y)dY\\
&= K_t\partial_t b_0 + K_x\partial_x b_0
\end{aligned}$$

with

$$\begin{aligned}
K_t &= \int_0^\infty e^{-YG_0}\Pi_0^+ B^{-1} e^{YG_0} R_0^- dY,\\
K_x &= \int_0^\infty e^{-YG_0}\Pi_0^+ B^{-1} A e^{YG_0} R_0^- dY.
\end{aligned}$$

Lemma 1.6.3. *We have*

$$K_t = \Pi_0^+ \frac{d}{ds}_{|s=0} \Pi^-(\tau_0 + s, \xi_0) R_0^-, \tag{1.6.1}$$

$$K_x = \Pi_0^+ \frac{d}{ds}_{|s=0} \Pi^-(\tau_0, \xi_0 + s) R_0^-. \tag{1.6.2}$$

Boundary condition

We consider boundary condition

$$Cu_{|y=0} = 0$$

where C is a matrix $N \times 2N$.

Assumption 1.6.4. *i) The rank of CR_0^- is $N-1$.*

ii) In a neighbourhood of (τ_0, ξ_0), the determinant of CR^- is in the form

$$\det CR^-(\tau, \xi) = e(\tau, \xi)(\tau + \mu(\xi))$$

where $e(\tau_0, \xi_0) \neq 0$, $\tau_0 + \mu(\xi_0) = 0$ and μ is real for ξ real.

We denote by β_0 a vector lying in $\ker CR_0^-$ and σ_0 a vector lying in cokerCR_0^-:

$$CR_0^- \beta_0 = 0, \qquad \sigma_0 CR_0^- = 0.$$

In particular, 0 is a simple eigenvalue of CR_0^-. Thus, there exists $\beta(\tau, \xi)$ of class C^∞ in a neighbourhood of (τ_0, ξ_0) and eigenvector of CR^- for an eigenvalue near 0. From (ii), it follows

$$CR^- \beta = \tilde{e}(\tau + \mu(\xi))\beta \tag{1.6.3}$$

such that $\beta(\tau_0, \xi_0) = \beta_0$ and $\tilde{e}(\tau_0, \xi_0) \neq 0$.

Surface waves transport

The boundary conditions for the asymptotic solutions are in the form:

$$Cu_0(t, x, 0) = 0,$$
$$Cu_1(t, x, 0) = 0.$$

The first condition is equivalent to

$$b_0(t, x) = a(t, x)\beta_0$$

with a scalar. The resolubility condition in $\Pi_0^- u_1$ of the second becomes

$$\sigma_0 C(K_t \partial_t b_0 + K_x \partial_x b_0) = 0$$

that is the transport equation for the amplitude a:

$$(\sigma_0 C K_t \beta_0)\partial_t a + (\sigma_0 C K_x \beta_0)\partial_x a = 0. \tag{1.6.4}$$

Lemma 1.6.5. *We have* $\sigma_0 C K_x \beta_0 = \partial_\xi \mu(\xi_0)\sigma_0 C K_t \beta_0$. *In particular the transport* (1.6.4) *is in the form*

$$e_0\Big(\partial_t + \partial_\xi \mu(\xi_0)\partial_x\Big),$$

with $e_0 \neq 0$.

Proof. Lemma 1.6.3 gives

$$\sigma_0 C K_t \beta_0 = \sigma_0 C \Pi_0^+ (\partial_\tau \Pi^-)(\tau_0, \xi_0) R_0^- \beta_0.$$

We have

$$\Pi^-(\tau, \xi) R^-(\tau, \xi) = R^-(\tau, \xi).$$

Differentiating and left multiplying by Π_0^+, we obtain

$$\Pi_0^+ (\partial_\tau \Pi^-)(\tau_0, \xi_0) R_0^- = \Pi_0^+ (\partial_\tau R^-)(\tau_0, \xi_0).$$

We used the fact that the third term is equal to zero, since $\Pi_0^+ \Pi_0^- = 0$. From $\sigma_0 C \Pi_0^- = 0$ it follows

$$\sigma_0 C K_t \beta_0 = \sigma_0 C (\partial_\tau R^-)(\tau_0, \xi_0)\beta_0. \tag{1.6.5}$$

Similarly

$$\sigma_0 C K_x \beta_0 = \sigma_0 C (\partial_\xi R^-)(\tau_0, \xi_0)\beta_0. \tag{1.6.6}$$

We differentiate 1.6.3 with respect to τ and evaluate at (τ_0, ξ_0):

$$C \partial_\tau R^- \beta_0 + C R_0^- \partial_\tau \beta = \tilde{e}_0 \beta_0$$

(the other terms are equal to zero at (τ_0, ξ_0)). Left multiplying by σ_0,

$$\sigma_0 C \partial_\tau R^-(\tau_0, \xi_0)\beta_0 = \tilde{e}_0 \sigma_0 \beta_0 \neq 0.$$

Similarly:

$$\sigma_0 C \partial_\xi R^-(\tau_0, \xi_0)\beta_0 = \tilde{e}_0 \sigma_0 \beta_0 \partial_\xi \mu(\xi_0).$$

Comparing with (1.6.5) (1.6.6), this finishes the proof of the lemma. $\square$

1.6.2 Proof of Lemma 1.6.3

Let $G = G_0 + tG_1$ be a matrix whose eigenvalues do not lie in $i\mathbb{R}$. The projector $\Pi^-(G)$ is

$$\Pi^-(t) = \frac{1}{2i\pi} \int_{\Gamma^-} (G - z)^{-1} dz$$

where Γ^- is a simple curve oriented in the direct direction, lying in the half plane $\{\mathrm{Re} z < 0\}$ and surrounding the eigenvalues of G with non-positive real parts. We have

$$\frac{d}{dt}_{|t=0} \Pi^- = \frac{-1}{2i\pi} \int_{\Gamma^-} (G_0 - z)^{-1} G_1 (G_0 - z)^{-1} dz. \tag{1.6.7}$$

We solve

$$\begin{aligned} \partial_Y u_0 - G_0 u_0 &= 0, \\ \partial_Y u_1 - G_0 u_1 &= G_1 u_0. \end{aligned}$$

We have

$$u_0(Y) = e^{YG_0} b_0, \quad b_0 \in \mathbb{E}_0^-$$

and

$$\Pi^+ u_1(0) = K b_0$$

with

$$K = -\int_0^\infty e^{-YG_0} \Pi_0^+ G_1 e^{YG_0} \Pi_0^- dY.$$

We have

$$e^{YG_0} \Pi_0^- = \frac{1}{2i\pi} \int_{\Gamma^-} e^{Yz} (G_0 - z)^{-1} dz.$$

Thus

$$K = -\frac{1}{2i\pi} \int_{\Gamma^-} \int_0^\infty e^{(z - G_0)Y} \Pi_0^+ dY G_1 (G_0 - z)^{-1} dz.$$

Since the eigenvalues of $G_0 \Pi_0^+$ lie in $\{\mathrm{Re}\, z' \geq 0\}$ and $z \in \Gamma^- \subset \{\mathrm{Re}\, z' < 0\}$, we obtain

$$\int_0^\infty e^{(z - G_0)Y} \Pi_0^+ dY = \Pi_0^+ (G_0 - z)^{-1}.$$

Thus

$$K = -\frac{1}{2i\pi} \int_{\Gamma^-} \Pi_0^+ (G_0 - z)^{-1} G_1 (G_0 - z)^{-1} \Pi_0^- dz.$$

With (1.6.7) it yields

$$K = \Pi_0^+ \frac{d\Pi^-}{dt}_{|t=0} \Pi_0^-. \tag{1.6.8}$$

1.7 Analysis of the propagation equation

1.7.1 Notations

In this section we consider functions

$$\begin{aligned} \varphi :\ & \mathbb{T} \times \mathbb{R} \times I \longrightarrow \mathbb{R} \\ & (\theta, x, t) \longmapsto \varphi(\theta, x, t) \end{aligned}$$

where $\mathbb{T}$ is a circle of length 2π and I a time interval, periodic in θ.

We introduce φ_n the Fourier coefficients of φ with respect to the variable θ:

$$\varphi_n(x,t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\theta, x, t) e^{-in\theta}\, d\theta$$

and $\hat{\varphi}_n$ the Fourier transform of these Fourier coefficients with respect to the variable x:

$$\hat{\varphi}_n(\xi, t) = \int_{\mathbb{R}} \varphi_n(x,t) e^{-ix\xi}\, dx.$$

The Fourier transform of the function φ with respect to the variable (θ, x) is then:

$$\begin{aligned} \hat{\varphi} :\ & \mathbb{Z} \times \mathbb{R} \times I \longrightarrow \mathbb{R} \\ & (n, \xi, t) \longmapsto \hat{\varphi}_n(\xi, t). \end{aligned}$$

We denote by $H^s(\mathbb{T} \times \mathbb{R})$ the Hilbert space

$$H^s(\mathbb{T} \times \mathbb{R}) = \left\{ \varphi : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R} \Big/ \int_{\mathbb{Z} \times \mathbb{R}} (1 + n^2 + \xi^2)^s |\, \hat{\varphi}_n(\xi)|^2\, d(n, \xi) < \infty \right\}$$

The scalar product and the norm are defined by:

$$\langle \varphi, \psi \rangle_s = \int_{(n,\xi) \in \mathbb{Z} \times \mathbb{R}} (1 + n^2 + \xi^2)^s \hat{\varphi}_n(\xi) \hat{\psi}_{-n}(-\xi)\, d(n, \xi)$$

$$\|\varphi\|_s = \left(\int_{(n,\xi) \in \mathbb{Z} \times \mathbb{R}} (1 + n^2 + \xi^2)^s |\, \hat{\varphi}_n(\xi)|^2\, d(n, \xi) \right)^{\frac{1}{2}},$$

where $\int_{n \in \mathbb{Z}} dn = \sum_{n \in \mathbb{Z}}$.

1.7.2 Evolution equation

Consider the following Cauchy problem:

$$\partial_t u + v\partial_x u + \partial_\theta a(u,u) = \phi \tag{1.7.1a}$$

$$u(\theta, x, 0) = 0 \tag{1.7.1b}$$

where $u : \mathbb{T} \times \mathbb{R} \times I \longrightarrow \mathbb{R}$. The bilinear mapping a is defined by its Fourier coefficients with respect to the variable θ:

$$a(u,v)_k(x,t) = \sum_{n\in\mathbb{Z}} S(-k, k-n, n) u_{k-n}(x,t) v_n(x,t), \tag{1.7.2}$$

and $\phi : \mathbb{T} \times \mathbb{R} \times I \longrightarrow \mathbb{R}$ is a function with zero mean with respect to the variable $\theta \in \mathbb{T}$ (that is $\phi_0 = 0$).

Assumption 1.7.1. *We assume that the kernel $S : \mathbb{Z}^3 \longrightarrow \mathbb{C}$ satisfies the following conditions:*

$$S(k,m,n) = 0 \text{ if } kmn = 0 \tag{1.7.3a}$$

$$S(k,m,n) = S^*(-k,-m,-n) \tag{1.7.3b}$$

$$S(k,m,n) = S(k,n,m) \tag{1.7.3c}$$

$$S(\eta k, \eta m, \eta n) = S(k,m,n) \quad \forall \eta > 0. \tag{1.7.3d}$$

In order to prove the local existence of a solution, we suppose that S satisfies the following additional conditions:

Assumption 1.7.2. *There exists a constant C such that*

$$\forall k,m,n \in \mathbb{Z}, \quad |S(k,m,n)| \leq C \tag{1.7.4}$$

Assumption 1.7.3. *There exists a constant C such that*

$$\forall |m| < |n| \quad |S(-m-n, m, n) - S(n, m, -m-n)| \leq C \left|\frac{m}{n}\right| \tag{1.7.5}$$

From the equation (1.5.10), established in section 1.5.2, the scalar-valued function $K_1(\theta, x, t)$ is solution of the problem (1.7.1), if we use the notations $\Phi = H_1$ and $\Lambda(n,l) = S(-n-l, n, l)$.

$H_1 = \sum_{n\in\mathbb{Z}} H_1^n e^{in\theta}$ is with zero mean, since $H_1^0 = 0$ (see remark 1.5.2).

The conditions (1.7.3a), (1.7.3b), (1.7.3c) and (1.7.3d) are satisfied: they follow from the properties (1.5.7a), (1.5.7b), (1.5.7c) and (1.5.7d) of $\Lambda(n,l)$.

The assumption (1.7.4) is satisfied; using the fact that the real parts of the eigenvalues λ_j are positive, we can bound all the expressions depending on (l, n) that appear in the expression of $\Lambda(n, l)$:

$$\forall\, n+l>0,\, l>0,\, n>0 \quad \left|\frac{n+l}{(n+l)\lambda_j + l\lambda_p^* + n\lambda_q^*}\right| \leq \frac{1}{\mathrm{Re}\,(\lambda_j)}$$

$$\left|\frac{n\lambda_q^* + l\lambda_p^*}{n+l}\right| \leq \max_{1\leq i\leq M}|\lambda_i|$$

$$\forall\, n+l>0,\, l>0,\, n<0 \quad \left|\frac{n+l}{(n+l)\lambda_j + l\lambda_p^* - n\lambda_q}\right| \leq \frac{1}{\mathrm{Re}\,(\lambda_j)}$$

$$\left|\frac{-n\lambda_q + l\lambda_p^*}{(n+l)\lambda_j + l\lambda_p^* - n\lambda_q}\right| \leq \frac{(-n+l)\max_{1\leq i\leq M}|\lambda_i|}{2l\underbrace{\min_{1\leq i\leq M}\mathrm{Re}\,\lambda_i}_{>0}}$$

$$\leq \frac{\max_{1\leq i\leq M}|\lambda_i|}{\min_{1\leq i\leq M}\mathrm{Re}\,\lambda_i} \;\; (\text{since} - n < l).$$

The assumption (1.7.5) is equivalent to the assumption (1.2.26)

$$\forall |m| < |n| \quad |\Lambda(m,n) - \Lambda(m, -m-n)| \leq C\left|\frac{m}{n}\right|.$$

In order to simplify the equation, we perform a change of variables. We introduce $\tilde{u}:\ \mathbb{T}\times\mathbb{R}\times I \longrightarrow \mathbb{R}$ such that:

$$\tilde{u}(\theta, \tilde{x}, t) = u(\theta, \tilde{x} + vt, t).$$

We then get

$$\begin{aligned}(\partial_t \tilde{u})(\theta, \tilde{x}, t) &= (\partial_t u + v\partial_x u)(\theta, \tilde{x}+vt, t)\\ &= -\partial_\theta a(u,u)(\theta, \tilde{x}+vt, t)\\ &= -\partial_\theta a(\tilde{u},\tilde{u})(\theta, \tilde{x}, t)\end{aligned}$$

The problem is reduced to the Cauchy problem:

$$\partial_t u + \partial_\theta a(u,u) = \phi \tag{1.7.6a}$$

$$u(\theta, x, 0) = 0. \tag{1.7.6b}$$

1.7.3 Properties of the bilinear mapping

Let us establish properties of the bilinear form defined by (1.7.2).

Proposition 1.7.4. *Let $a(u,v)$ be the bilinear mapping defined by (1.7.2), where $S:\ \mathbb{Z}^3 \longrightarrow \mathbb{C}$ satisfies (1.7.3a)-(1.7.5), then*
(a) for all $s>1$, $a:\ \ H^s \times H^s \longrightarrow H^s$ is a bounded symmetric bilinear mapping and for u,v real-valued functions, $a(u,v)$ is a real-valued function
(b) for all $s>2$, for all $u,v\ \in H^s$,

$$\partial_\theta a(u,v) = a(\partial_\theta u, v) + a(u, \partial_\theta v), \tag{1.7.7}$$

(c) for all $s>2$, there exists a constant $C_s>0$ such that, for all real-valued functions $u,w \in H^s$,

$$\left| \int_{\mathbb{T}\times\mathbb{R}} \partial_\theta a(u,w) w \, d(\theta,x) \right| \leq C_s \|u\|_s \|w\|_0^2. \tag{1.7.8}$$

Proof. *(a)* a is obviously a symmetric bilinear mapping.

If u and v are real-valued functions, $u^*_{-m} = u_m$ and $v^*_{-m} = v_m$, therefore, from the assumption (1.7.3b) $S(k,m,n) = S^*(-k,-m,-n)$, we obtain:

$$\begin{aligned}(a(u,v)_{-k})^* =& \sum_{n\in\mathbb{Z}} S^*(k, n-k, -n) u^*_{n-k} v^*_{-n} \\ =& \sum_{n\in\mathbb{Z}} S(-k, k-n, n) u_{k-n} v_n \\ =& a(u,v)_k\end{aligned}$$

thus $a(u,v)$ is a real-valued function.

Let us show that a is bounded in H^s.

$$\|a(u,v)\|_s^2 = \int_{\mathbb{Z}\times\mathbb{R}} (1+k^2+\xi^2)^s \left| \int_{\mathbb{Z}\times\mathbb{R}} S(-k, k-n, n) \hat{u}_{k-n}(\xi-\eta) \hat{v}_n(\eta) \, d(n,\eta) \right|^2 d(k,\xi)$$

from the assumption (1.7.4), S is bounded, thus

$$\|a(u,v)\|_s^2 \leq C \int_{\mathbb{Z}\times\mathbb{R}} (1+k^2+\xi^2)^s \left(\int_{\mathbb{Z}\times\mathbb{R}} |\, \hat{u}_{k-n}(\xi-\eta)| |\, \hat{v}_n(\eta)| \, d(n,\eta) \right)^2 d(k,\xi).$$

It is sufficient to show that w defined by

$$\hat{w}_k(\xi) = \int_{(n,\eta)\in\mathbb{Z}\times\mathbb{R}} |\, \hat{u}_{k-n}(\xi-\eta)| |\, \hat{v}_n(\eta)| \, d(n,\eta)$$

belongs to H^s. We write $\hat{w}_k(\xi)$ in the form

$$\hat{w}_k(\xi) = \hat{w}_k^I(\xi) + \hat{w}_k^{II}(\xi)$$

with

$$\hat{w}_k^I(\xi) = \int_{|(n,\eta)|\geq|(k,\xi)|/2} |\,\hat{u}_{k-n}(\xi-\eta)||\,\hat{v}_n(\eta)|\,d(n,\eta)$$

and

$$\hat{w}_k^{II}(\xi) = \int_{|(n,\eta)|<|(k,\xi)|/2} |\,\hat{u}_{k-n}(\xi-\eta)||\,\hat{v}_n(\eta)|\,d(n,\eta).$$

$$(1+|(k,\xi)|^2)^{s/2}\hat{w}_k^I(\xi) \leq \int_{\mathbb{Z}\times\mathbb{R}} |\,\hat{u}_{k-n}(\xi-\eta)|(1+4|(n,\eta)|^2)^{s/2}|\,\hat{v}_n(\eta)|\,d(n,\eta).$$

For $s>1$, $(1+|(k,\xi)|^2)^{s/2}\hat{u} \in L^2(\mathbb{Z}\times\mathbb{R}) \Rightarrow \hat{u} \in L^1(\mathbb{Z}\times\mathbb{R})$ since $\hat{u} = \frac{(1+|(k,\xi)|^2)^{s/2}}{(1+|(k,\xi)|^2)^{s/2}}\hat{u}$ and $\frac{1}{(1+|(k,\xi)|^2)^{s/2}} \in L^2(\mathbb{Z}\times\mathbb{R})$ for $s>1$.

Therefore $(1+|(k,\xi)|^2)^{s/2}\hat{w}_k^I(\xi)$ is less than a convolution integral of a function in L^1 and a function in L^2 and thus belongs to L^2 from the Young's inequality. Similarly, we show that $(1+|(k,\xi)|^2)^{s/2}\hat{w}_k^{II}(\xi)$ belongs to L^2. Thus finally $w \in H^s$ and a is bounded.

(b) In order to prove (1.7.7), we note that

$$\begin{aligned}
(\partial_\theta a(u,v))_k =& ik(a(u,v))_k \\
=& ik\sum_n S(-k,k-n,n)u_{k-n}v_n \\
=& \sum_n i(k-n)S(-k,k-n,n)u_{k-n}v_n \\
&+ \sum_n inS(-k,k-n,n)u_{k-n}v_n \\
=& (a(\partial_\theta u, v))_k + (a(u,\partial_\theta v))_k.
\end{aligned}$$

(c) By Fourier analysis, we have to estimate:

$$\mathrm{Re}\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\hat{w}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{w}_{-k}(-\xi)d\xi d\eta,$$

which, by the assumption (1.7.3b), is equal to the half of:

$$\begin{aligned}
&\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\hat{w}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{w}_{-k}(-\xi)d\xi d\eta \\
&\quad -\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(k,n-k,-n)\hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta.
\end{aligned}$$

We make the change of variable $((k,\xi),(n,\eta)) \mapsto ((k-n,\xi-\eta),(-n,-\eta))$ in the first integral here above and obtain:

$$\begin{aligned}
&2\mathrm{Re}\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\hat{w}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{w}_{-k}(-\xi)d\xi d\eta \\
&= \sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} i((k-n)S(n-k,k,-n)-kS(k,n-k,-n)) \\
&\qquad\qquad \hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta \\
&= I_1 + I_2,
\end{aligned}$$

with

$$I_1 = \sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} -inS(n-k,k,-n)\hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta$$

$$\begin{aligned}
I_2 = &\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ik(S(n-k,k,-n)-S(k,n-k,-n)) \\
&\qquad\qquad \hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta.
\end{aligned}$$

From the assumption that S is bounded, we get

$$|I_1| \leq C\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} |\hat{w}_{n-k}(\eta-\xi)|\,|n\hat{u}_{-n}(-\eta)|\,|\hat{w}_k(\xi)|d\xi d\eta,$$

thus, by Cauchy-Schwarz inequality and L_1-L_2 convolution estimates:

$$|I_1| \leq C\|\widehat{\partial_\theta u}\|_{L^1}\|\widehat{w}\|_{L^2}^2.$$

To estimate the integral I_2, we split it into

$$\begin{aligned}
I_2 = &\sum_{|k|\leq|n|}\int_{\mathbb{R}\times\mathbb{R}} ik(S(n-k,k,-n)-S(k,n-k,-n)) \\
&\qquad\qquad \hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta \\
&+ \sum_{|k|>|n|}\int_{\mathbb{R}\times\mathbb{R}} ik(S(n-k,k,-n)-S(k,n-k,-n)) \\
&\qquad\qquad \hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta.
\end{aligned}$$

The first term satisfies

$$\left|\sum_{|k|\le|n|}\int_{\mathbb{R}\times\mathbb{R}} ik(S(n-k,k,-n)-S(k,n-k,-n))\hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta\right|$$
$$\le 2C\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}}|\hat{w}_{n-k}(\eta-\xi)|\,|n\hat{u}_{-n}(-\eta)|\,|\hat{w}_k(\xi)|d\xi d\eta$$

thus, as for I_1, this term is bounded by $C\|\widehat{\partial_\theta u}\|_{L^1}\|\widehat{w}\|_{L^2}^2$. It remains to estimate the second term

$$\sum_{|k|>|n|}\int_{\mathbb{R}\times\mathbb{R}} ik(S(n-k,k,-n)-S(k,n-k,-n))\hat{w}_{n-k}(\eta-\xi)\hat{u}_{-n}(-\eta)\hat{w}_k(\xi)d\xi d\eta.$$

From the assumptions (1.7.3c) and (1.7.5),

$$\begin{aligned}|S(n-k,k,-n)-S(k,n-k,-n)| &= |S(n-k,-n,k)-S(k,-n,n-k)|\\ &\le C\left|\frac{n}{k}\right| \quad \forall |n|<|k|\end{aligned}$$

thus, the modulus of the second term is bounded by

$$C\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}}|\hat{w}_{n-k}(\eta-\xi)|\,|n\hat{u}_{-n}(-\eta)|\,|\hat{w}_k(\xi)|d\xi d\eta$$

and therefore is bounded by $C\|\widehat{\partial_\theta u}\|_{L^1}\|\widehat{w}\|_{L^2}^2$.

Finally, we have

$$|I_2|\le C\|\widehat{\partial_\theta u}\|_{L^1}\|\widehat{w}\|_{L^2}^2.$$

Adding the estimates of $|I_1|$ and $|I_2|$, we obtain

$$\left|\mathrm{Re}\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\hat{w}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{w}_{-k}(-\xi)d\xi d\eta\right|\le C\|\widehat{\partial_\theta u}\|_{L^1}\|\widehat{w}\|_{L^2}^2. \tag{1.7.9}$$

Since, for $s>1+d/2=2$ (here $d=\dim(\mathbb{T}\times\mathbb{R})=2$), $\|\widehat{\partial_\theta u}\|_{L^1}\le C_s\|u\|_s$, we get

$$\left|\mathrm{Re}\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\hat{w}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{w}_{-k}(-\xi)d\xi d\eta\right|\le C_s\|u\|_s\|w\|_0^2.$$

This finishes the proof of (1.7.8). □

1.7.4 Local existence

We are now ready to proceed with the proof of the existence and uniqueness theorem:

Theorem 1.7.5. *We assume that $S : \mathbb{Z}^3 \longrightarrow \mathbb{C}$ satisfies the assumptions (1.7.3a)-(1.7.5), a is defined by (1.7.2) and $s \geq 3$. Then for all real-valued function ϕ with zero mean with respect to the variable $\theta \in \mathbb{T}$, such that $\phi \in C^0([0, T_1]\,; H^3\,(\mathbb{T} \times \mathbb{R}))$ with $T_1 > 0$, the Cauchy problem (1.7.6) has a unique real-valued local solution*

$$u \in \mathcal{C}^0([0, t^*[\,; H^3\,(\mathbb{T} \times \mathbb{R})) \cap \mathcal{C}^1([0, t^*[\,; H^2\,(\mathbb{T} \times \mathbb{R})) \tag{1.7.10}$$

where

$$t^* = min(T_1, \frac{\pi}{2\sqrt{CK_3}}) \tag{1.7.11}$$

with $C = \sup_{0 \leq t \leq T_1} \|\phi(t)\|_3$ and K_3 the constant obtained in the following proposition. Furthermore, the solution is with zero mean (that is $u_0 = 0$).

Moreover if $\phi \in C^0([0, t^*[; H^s\,(\mathbb{T} \times \mathbb{R}))$, then the solution u satisfies:

$$u \in \mathcal{C}^0([0, t^*[\,; H^s\,(\mathbb{T} \times \mathbb{R})) \cap \mathcal{C}^1([0, t^*[\,; H^{s-1}\,(\mathbb{T} \times \mathbb{R})). \tag{1.7.12}$$

Proof. Let $P^N : H^s(\mathbb{T} \times \mathbb{R}) \longrightarrow H^s(\mathbb{T} \times \mathbb{R})$ be the orthogonal projection

$$P^N \left(\sum_{k \in \mathbb{Z}} \varphi_k(x) e^{ik\theta} \right) = \sum_{k=-N}^{N} \varphi_k(x) e^{ik\theta}. \tag{1.7.13}$$

We introduce the approximation

$$u^N(\theta, x, t) = \sum_{k=-N}^{N} u_k^N(x, t) e^{ik\theta}$$

that satisfies the ODE system

$$\partial_t u^N + P^N \partial_\theta a(u^N, u^N) = P^N \phi \tag{1.7.14}$$

$$u^N(\theta, x, 0) = 0.$$

All functions will be taken with values in $\mathbb{R}$ and we shall use repeatedly the corresponding property $(\hat{u}_{-k}(-\xi) = \hat{u}_k(\xi)^*)$ in the Fourier variables.

The main estimate which let us prove the theorem is given by the following proposition.

Proposition 1.7.6. *If $s \geq 3$, there exists a constant K_s such that for all N, the solution $u^N(\theta, x, t)$ of (1.7.14) satisfies:*

$$\left| \frac{d}{dt} \left\| u^N \right\|_s^2 \right| \leq 2K_s \left\| u^N \right\|_3 \left\| u^N \right\|_s^2 + 2\|\phi\|_s \left\| u^N \right\|_s . \tag{1.7.15}$$

Proof. Taking the scalar product of (1.7.14) and u^N yields:

$$\frac{d}{dt}\langle u^N, u^N\rangle_s + 2\mathrm{Re}\langle P^N \partial_\theta a(u^N, u^N), u^N\rangle_s = 2\mathrm{Re}\langle P^N \phi, u^N\rangle_s.$$

Thus

$$\left| \frac{d}{dt} \left\| u^N \right\|_s^2 \right| \leq 2 \left| \mathrm{Re}\langle \partial_\theta a(u^N, u^N), u^N\rangle_s \right| + 2|\mathrm{Re}\langle P^N \phi, u^N\rangle_s|.$$

We drop the N-superscripts to simplify the notation. It is sufficient to establish the following lemma to finish the proof of the proposition.

Lemma 1.7.7. *For $s \geq 3$, there exists a constant K_s such that*

$$\left|\mathrm{Re}\langle \partial_\theta a(u, u), u\rangle_{H^s}\right| \leq K_s \|u\|_{H^3} \|u\|_{H^s}^2. \tag{1.7.16}$$

Proof. 1. From the property (1.7.7), we can substitute $\partial_\theta a(u, u)$ by $a(u, \partial_\theta u)$.

2. By Fourier analysis, we have to estimate

$$\mathrm{Re}\left(\sum_{k,n} \int_{\mathbb{R}\times\mathbb{R}} S(-k, k-n, n) i\ n\ \delta(k, \xi)^{2s} \hat{u}_{k-n}(\xi - \eta) \hat{u}_n(\eta) \hat{u}_{-k}(-\xi) d\xi d\eta \right)$$

with $\delta(k, \xi) = \sqrt{1 + k^2 + |\xi|^2}$.

3. We write

$$\delta(k, \xi)^s = \delta(n, \eta)^s + \Delta_s,$$

with $\Delta_s = \delta(k, \xi)^s - \delta(n, \eta)^s$ and obtain:

$$\sum_{k,n} \int_{\mathbb{R}\times\mathbb{R}} S(-k, k-n, n) i\ n\ \delta(k, \xi)^{2s} \hat{u}_{k-n}(\xi - \eta) \hat{u}_n(\eta) \hat{u}_{-k}(-\xi) d\xi d\eta = I_1 + I_2, \tag{1.7.17}$$

with

$$I_1 = \sum_{k,n} \int_{\mathbb{R}\times\mathbb{R}} S(-k, k-n, n) i\ n\ \delta(k, \xi)^s \Delta_s \hat{u}_{k-n}(\xi - \eta) \hat{u}_n(\eta) \hat{u}_{-k}(-\xi) d\xi d\eta, \tag{1.7.18}$$

$$I_2 = \sum_{k,n} \int_{\mathbb{R}\times\mathbb{R}} S(-k, k-n, n) i\ n\ \delta(k,\xi)^s \delta(n,\eta)^s \hat{u}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{u}_{-k}(-\xi) d\xi d\eta. \tag{1.7.19}$$

4. Since $\delta(k,\xi) = d((1,0,0),(0,k,\xi))$ for d the distance defined by the norm $\|\,.\,\|_2$ of $\mathbb{R}^3$, we get from the triangular inequality:
$\forall k, n \in \mathbb{Z}\ \forall \xi, \eta \in \mathbb{R}\ \ |\delta(k,\xi) - \delta(n,\eta)| \leq \delta(k-n, \xi-\eta)$, thus

$$\forall k, n \in \mathbb{Z}\ \forall \xi, \eta \in \mathbb{R}\ \ \delta(k,\xi)^{s-1} \leq C_s \left(\delta(k-n,\xi-\eta)^{s-1} + \delta(n,\eta)^{s-1}\right),$$

with C_s a constant (depending on s). In order to simplify the notation, we will denote by C_s any constant, possibly depending on s, which may vary from one line to another. Using that

$$\begin{aligned} |\Delta_s| &= |\delta(k,\xi)^s - \delta(n,\eta)^s| \\ &\leq C_s |\delta(k,\xi) - \delta(n,\eta)| (\delta(k,\xi)^{s-1} + \delta(n,\eta)^{s-1}), \end{aligned}$$

the previous inequalities imply:

$$|\Delta_s| \leq C_s \delta(k-n,\xi-\eta)(\delta(k-n,\xi-\eta)^{s-1} + \delta(n,\eta)^{s-1}). \tag{1.7.20}$$

5. Applying the assumption S is bounded and the estimate (1.7.20), we get that the modulus of the first integral I_1 (1.7.18) is bounded by:

$$\begin{aligned} |I_1| \leq C_s \sum_{k,n} \int_{\mathbb{R}\times\mathbb{R}} |n| \delta(k,\xi)^s \delta(k-n,\xi-\eta)(\delta(k-n,\xi-\eta)^{s-1} + \delta(n,\eta)^{s-1}) \\ |\hat{u}_{k-n}(\xi-\eta)||\hat{u}_n(\eta)||\hat{u}_{-k}(-\xi)| d\xi d\eta. \end{aligned}$$

By Cauchy-Schwarz inequality and $L^1 - L^2$ convolution estimates, we obtain

$$\begin{aligned} \sum_{k,n} \int_{\mathbb{R}\times\mathbb{R}} |n||\hat{u}_n(\eta)|\ \delta(k-n,\xi-\eta)^s |\hat{u}_{k-n}(\xi-\eta)|\ \delta(k,\xi)^s |\hat{u}_{-k}(-\xi)|\, d\xi d\eta \\ \leq C_s \|\widehat{\partial_\theta u}\|_{L^1(\mathbb{Z}\times\mathbb{R})}\ \|u\|^2_{H^s(\mathbb{T}\times\mathbb{R})}. \end{aligned}$$

Noticing that $|n| \leq \delta(n,\eta)$ and $\delta(k-n,\xi-\eta) \leq 1 + |k-n| + |\xi-\eta|$, and proceeding as before, we have

$$\begin{aligned} \sum_{k,n} \int_{\mathbb{R}\times\mathbb{R}} \delta(k,\xi)^s |\hat{u}_{-k}(-\xi)\ |\delta(k-n,\xi-\eta)|\hat{u}_{k-n}(\xi-\eta)|\ |n|\delta(n,\eta)^{s-1}|\hat{u}_n(\eta)|\, d\xi d\eta \\ \leq C_s (\|\widehat{u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_\theta u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_x u}\|_{L^1(\mathbb{Z}\times\mathbb{R})})\ \|u\|^2_{H^s(\mathbb{T}\times\mathbb{R})}. \end{aligned}$$

Adding the two estimates, we obtain:

$$|I_1| \le C_s(\|\widehat{u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_\theta u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_x u}\|_{L^1(\mathbb{Z}\times\mathbb{R})})\ \|u\|^2_{H^s(\mathbb{T}\times\mathbb{R})}. \quad (1.7.21)$$

6. We now concentrate on the real part of the second integral I_2 (1.7.19). I_2 can be decomposed as the sum

$$I_2 = \sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} S(-k,k-n,n)i(n-k)\delta(k,\xi)^s\delta(n,\eta)^s\hat{u}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{u}_{-k}(-\xi)d\xi d\eta$$
$$+\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} S(-k,k-n,n)ik\ \delta(k,\xi)^s\delta(n,\eta)^s\hat{u}_{k-n}(\xi-\eta)\hat{u}_n(\eta)\hat{u}_{-k}(-\xi)d\xi d\eta$$

Similarly as in 5., we get that the modulus of the first integral here above admits the bound

$$C_s\|\widehat{\partial_\theta u}\|_{L^1(\mathbb{Z}\times\mathbb{R})}\ \|u\|^2_{H^s(\mathbb{T}\times\mathbb{R})}.$$

It remains to estimate the real part of the second integral

$$\mathrm{Re}\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\ \hat{u}_{k-n}(\xi-\eta)\ \delta(n,\eta)^s\hat{u}_n(\eta)\ \delta(k,\xi)^s\hat{u}_{-k}(-\xi)d\xi d\eta,$$

introducing $w \in L^2$ the function defined by

$$\forall k,\xi\ \ \hat{w}_k(\xi) = \delta(k,\xi)^s\hat{u}_k(\xi),$$

the real part of the second integral is equal to:

$$\mathrm{Re}\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\ \hat{u}_{k-n}(\xi-\eta)\ \hat{w}_n(\eta)\ \hat{w}_{-k}(-\xi)d\xi d\eta.$$

By the change of variable $((n,\eta),(k,\xi)) \mapsto ((k-n,\xi-\eta),(k,\xi))$ and from the assumption on S (1.7.3c), this is equal to:

$$\mathrm{Re}\sum_{k,n}\int_{\mathbb{R}\times\mathbb{R}} ikS(-k,k-n,n)\ \hat{w}_{k-n}(\xi-\eta)\ \hat{u}_n(\eta)\ \hat{w}_{-k}(-\xi)d\xi d\eta,$$

from the proof of the part (c) of Proposition 1.7.4 (see inequality (1.7.9)), the modulus of this real part is bounded by $C_s\|\widehat{\partial_\theta u}\|_{L^1}\|\widehat{w}\|^2_{L^2}$, it follows from the definition of w $\|\widehat{w}\|_{L^2} = \|\widehat{u}\|_{L^s}$, therefore, we obtain the estimate of the real part of I_2:

$$|\mathrm{Re}(I_2)| \le C_s\|\widehat{\partial_\theta u}\|_{L^1(\mathbb{Z}\times\mathbb{R})}\ \|u\|^2_{H^s(\mathbb{T}\times\mathbb{R})}. \quad (1.7.22)$$

7. From the estimates of $|I_1|$ (1.7.21) and of $|\mathrm{Re}(I_2)|$ (1.7.22), the equality (1.7.17) permits us to bound the left on side of (1.7.16) by

$$C_s\ (\|\widehat{u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_\theta u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_x u}\|_{L^1(\mathbb{Z}\times\mathbb{R})})\ \|u\|^2_{H^s(\mathbb{T}\times\mathbb{R})}.$$

8. From the Sobolev embedding, it follows that for $s' > 1 + \frac{d}{2}$ (here $d = \dim(\mathbb{T}\times\mathbb{R}) = 2$) and thus for $s' = 3$, we have

$$\|\widehat{u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_\theta u}\|_{L^1(\mathbb{Z}\times\mathbb{R})} + \|\widehat{\partial_x u}\|_{L^1(\mathbb{Z}\times\mathbb{R})}\ \le K_{s'}\ \|u\|_{H^{s'}(\mathbb{T}\times\mathbb{R})}.$$

This finishes the proof of Lemma 1.7.7. $\square$

From Lemma 1.7.7, we get for $0 \le t \le T_1$:

$$\begin{aligned}\left|\frac{d}{dt}\,\|u^N\|_s^2\right| &\le 2\left|\mathrm{Re}\langle\partial_\theta a(u^N,u^N),u^N\rangle_s\right| + 2|\langle P^N\phi,u^N\rangle_s| \\ &\le 2K_s\|u^N\|_3\|u^N\|_s^2 + 2\|\phi\|_s\|u^N\|_s.\end{aligned}$$

This finishes the proof of Proposition 1.7.6.

$\square$

The assumption $\phi \in C^0([0,T_1]\,;H^3\,(\mathbb{T}\times\mathbb{R}))$ gives the existence of $C = \sup_{0\le t\le T_1}\|\phi(t)\|_3$, it follows from Proposition 1.7.6 applied to $s = 3$:

$$\forall N\ \forall 0 \le t \le T_1\quad \left|\frac{d}{dt}\,\left\|u^N\right\|_3^2\right| \le 2K_3\|u^N\|_3^3 + 2C\left\|u^N\right\|_3. \tag{1.7.23}$$

The solution of the differential equation $y' = 2K_3y^{3/2} + 2Cy^{1/2}$ which is equal to zero at $t = 0$ is $t \mapsto \frac{C}{K_3}\tan^2(\sqrt{CK_3}t)$, thus by comparison of differential equations, we deduce from the estimate (1.7.23) that:

$$\forall N\,\forall 0 \le t < min(T_1, \frac{\pi}{2\sqrt{CK_3}})\ \ \|u^N\|_3^2(t) \le \frac{C}{K_3}\tan^2(\sqrt{CK_3}t),$$

we obtain finally

$$\forall N\,\forall 0 \le t < t_*\ \ \|u^N\|_3(t) \le \left|\sqrt{\frac{C}{K_3}}\tan(\sqrt{CK_3}t)\right|, \tag{1.7.24}$$

with $t_* = min(T_1, \frac{\pi}{2\sqrt{CK_3}})$. Thus the approximations u^N exist for $0 \le t < t_*$.
Classical arguments finish the proof of the existence of a solution u in $\mathcal{C}^0([0,t^*[\,;H^3\,(\mathbb{T}\times\mathbb{R}))\cap\mathcal{C}^1([0,t^*[\,;H^2\,(\mathbb{T}\times\mathbb{R}))$.

For $0 \leq T < t_*$, from the inequality (1.7.24), $(u^N)_N$ is a bounded family of $\mathcal{C}^0([0,T], H^3(\mathbb{T}\times\mathbb{R}))$, therefore, from the equation (1.7.14) satisfied by u^N, $(\partial_t u^N)_N$ is a bounded family of $\mathcal{C}^0([0,T], H^2(\mathbb{T}\times\mathbb{R}))$. From Ascoli's theorem, there is a subsequence of $(u^N)_N$, which converges in the continuous functions space $\mathcal{C}^0([0,T], H^2_w(\mathbb{T}\times\mathbb{R}))$, where $H^2_w(\mathbb{T}\times\mathbb{R})$ is the Sobolev space $H^2(\mathbb{T}\times\mathbb{R})$ equipped with the weak topology. We obtain the existence of a solution $u \in \mathcal{C}^0([0,T], H^2_w(\mathbb{T}\times\mathbb{R}))$. Since $(u^N)_N$ is a bounded family of $\mathcal{C}^0([0,T], H^3(\mathbb{T}\times\mathbb{R}))$, $u \in L^\infty([0,T], H^3(\mathbb{T}\times\mathbb{R}))$ and we get $u \in \mathcal{C}^0([0,T], H^3_w(\mathbb{T}\times\mathbb{R}))$. From the equation (1.7.6a) satisfied by u, it follows:

$$u \in \mathcal{C}^0([0,T], H^3_w(\mathbb{T}\times\mathbb{R})) \cap \mathcal{C}^1([0,T], H^2_w(\mathbb{T}\times\mathbb{R})),$$

and finally

$$u \in \mathcal{C}^0([0,t_*[, H^3_w(\mathbb{T}\times\mathbb{R})) \cap \mathcal{C}^1([0,t_*[, H^2_w(\mathbb{T}\times\mathbb{R})).$$

The continuity of u from $[0,t_*[$ to $H^3(\mathbb{T}\times\mathbb{R})$ in the strong topology can be proved as in [7], showing that $t \mapsto \|u(t)\|_{H^3(\mathbb{T}\times\mathbb{R})}$ is continuous in time, using the integrated form of (1.7.23) for u. Thus

$$u \in \mathcal{C}^0([0,t_*[, H^3(\mathbb{T}\times\mathbb{R})) \cap \mathcal{C}^1([0,t_*[, H^2(\mathbb{T}\times\mathbb{R})).$$

Now, we assume that $\phi \in C^0([0,t^*[\,; H^s(\mathbb{T}\times\mathbb{R}))$ with $s \geq 3$. From the previous case $s = 3$, $u \in C^0([0,t^*[\,; H^3(\mathbb{T}\times\mathbb{R}))$, therefore, for $0 \leq T < t^*$, the constants $C_s = \sup_{0\leq t\leq T} \|\phi(t)\|_s$ and $M = \sup_{0\leq t\leq T} \|u(t)\|_3$ are well defined. Since $\forall N\ \forall t\ \ \|u_N(t)\|_3 \leq \|u(t)\|_3$, Proposition 1.7.6 gives

$$\begin{aligned}\forall N\ \forall 0 \leq t \leq T \quad \left|\frac{d}{dt}\left\|u^N\right\|_s^2\right| &\leq 2K_s\left\|u^N\right\|_3\left\|u^N\right\|_s^2 + 2\|\phi(t)\|_s\left\|u^N\right\|_s \\ &\leq 2K_s M\left\|u^N\right\|_s^2 + 2C_s\left\|u^N\right\|_s.\end{aligned}$$

By comparison of differential equations:

$$\forall N\ \forall\, 0 \leq t \leq T \quad \left\|u^N(t)\right\|_s^2 \leq \frac{C_s^2}{K_s^2 M^2}(e^{2K_s M t} - 1)^2 \leq \frac{C_s^2}{K_s^2 M^2}(e^{2K_s M T} - 1)^2$$

Since the upper-bound doesn't depend on N, we obtain, for all $0 \leq T < t^*$, by the same arguments as in the case $s = 3$:

$$u \in \mathcal{C}^0([0,T]\,; H^s_w(\mathbb{T}\times\mathbb{R})) \cap \mathcal{C}^1([0,T]\,; H^{s-1}_w(\mathbb{T}\times\mathbb{R})),$$

thus

$$u \in \mathcal{C}^0([0,t^*[\,; H^s_w(\mathbb{T}\times\mathbb{R})) \cap \mathcal{C}^1([0,t^*[\,; H^{s-1}_w(\mathbb{T}\times\mathbb{R})).$$

As before, one shows that $t \mapsto \|u(t)\|_{H^s(\mathbb{T}\times\mathbb{R})}$ is continuous and so that

$$u \in \mathcal{C}^0([0,t^*[\,; H^s(\mathbb{T}\times\mathbb{R})) \cap \mathcal{C}^1([0,t^*[\,; H^{s-1}(\mathbb{T}\times\mathbb{R})).$$

In order to prove the uniqueness, we assume that u, v are two solutions of the problem (1.7.6) with the regularity (1.7.10). By integration, we show that

$$\begin{aligned}\frac{d}{dt}\|u-v\|^2_{L^2(\mathbb{T}\times\mathbb{R})} &= \frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}}(u-v)^2\\ &= 2\int_{\mathbb{T}\times\mathbb{R}}(u-v)\partial_t(u-v)\\ &= -2\int_{\mathbb{T}\times\mathbb{R}}(u-v)[\partial_\theta a(u,u)-\partial_\theta a(v,v)]\end{aligned}$$

Since a is a symmetric bilinear mapping, we obtain

$$\frac{d}{dt}\|u-v\|^2_{L^2(\mathbb{T}\times\mathbb{R})} == -2\int_{\mathbb{T}\times\mathbb{R}}(u-v)\partial_\theta a(u+v,u-v),$$

thus for $s>2$ and applying the property (1.7.8)

$$\left|\frac{d}{dt}\|u-v\|^2_{L^2(\mathbb{T}\times\mathbb{R})}\right| \le C_s\|u+v\|_s\|u-v\|^2_{L^2(\mathbb{T}\times\mathbb{R})}.$$

By applying the Gronwall inequality, we obtain $u=v$, the uniqueness is proved.

□

1.8 Existence and uniqueness of the profiles $\{\vec{u}_k\}_{k\geq 1}$: proof of theorem (1.2.10)

During the first step we studied the equation

$$\mathcal{L}_0\vec{u}_1 = 0$$

and obtained an expression of $\vec{u}_1$ with an unknown scalar-valued function.

During the second step, from the study of the equation

$$\mathcal{L}_0\vec{u}_2 + \vec{G}_1(\vec{u}_0,\vec{u}_1,\vec{h}_1) = 0,$$

we established an equation verified by the unknown scalar-valued function that determines $\vec{u}_1$. Furthermore, applying Proposition (1.4.1), we obtain:

$$\vec{u}\,_2^0 = \underline{\vec{u}\,_2^0}(t,x,y) + \int_Y^\infty \left(g'(\vec{u}_0)\right)^{-1} \vec{G}\,_1^0(t,x,s)\,ds$$

and for $n \neq 0$

$$\vec{u}\,_2^n = \sum_{j=1}^{M} V_{2,j}(n,Y)\vec{R}_j + W_{2,j}(n,Y)\vec{R}_j\,^*$$

with

if $n > 0$:

$$V_{2,j} = \int_Y^\infty \exp(n\lambda_j(Y-s))F_j^1(n,s)\,ds$$

$$W_{2,j} = [g_{2,known} + K_2(n,x,t)\rho_j^*]\exp\left(-n\lambda_j^* Y\right) - \int_0^Y \exp(-n\lambda_j^*(Y-s))H_j^1(n,s)ds$$

if $n < 0$:

$$V_{2,j} = [f_{2,known} + K_2(n,x,t)\rho_j]\ \exp\left(n\lambda_j Y\right) - \int_0^Y \exp(n\lambda_j(Y-s))F_j^1(n,s)\,ds$$

$$W_{2,j} = \int_Y^\infty \exp(-n\lambda_j^*(Y-s))H_j^1(n,s)\,ds$$

where $f_{2,known}$ and $g_{2,known}$ are known functions depending on $\vec{G}_1$ and $\underline{\vec{u}_2^0}$ is an unknown function lying in $\underline{S}$ which has to satisfy the boundary condition

$$C\underline{\vec{u}_2^0}(t,x,y=0) = -C\int_0^\infty \left(g'(\vec{u}_0)\right)^{-1} \vec{G}\,_1^0(t,x,s)\,ds$$

and K_2 an unknown scalar-valued function.

At this stage, $\vec{u}_1$, verifying the equation $\mathcal{L}_0\vec{u}_1 = 0$, the boundary condition $C\vec{u}_1(Y=0) = 0$ and the condition in the past $\forall t \leq 0, \vec{u}_1(t) = 0$, is then completely determined, $\vec{u}_2$ is known, except a scalar-valued function $K_2(n,x,t)$ and an independent of Y function $\underline{\vec{u}\,_2^0}(t,x,y)$, which has to verify the boundary condition

$$C\underline{\vec{u}\,_2^0}(t,x,y=0) = -C\int_0^\infty \left(g'(\vec{u}_0)\right)^{-1} \vec{G}\,_1^0(t,x,s)\,ds.$$

During the k-th step, $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{k-1}$, satisfying the equation $\mathcal{L}_0\vec{u}_i + \vec{G}_{i-1}(\vec{u}_0,\ldots,\vec{u}_{i-1},\vec{h}_{i-1}) = 0$, the boundary condition $C\vec{u}_i(Y=0) = 0$

and the condition in the past $\forall t \leq 0,\ \vec{u}_i(t) = 0$, are completely determined. Furthermore, we establish the following expression of the Fourier coefficients of $\vec{u}_k^{\,n}$: for $n = 0$,

$$\vec{u}_k^{\,0} = \underline{\vec{u}_k^{\,0}}(t,x,y) + \int_Y^\infty (g'(\vec{u}_0))^{-1} \vec{G}_{k-1}^{\,0}(t,x,s)\,ds$$

and for $n \neq 0$

$$\vec{u}_k^{\,n} = \sum_{j=1}^{M} V_{k,j}(n,Y)\vec{R}_j + W_{k,j}(n,Y)\vec{R}_j^{\,*}$$

with

if $n > 0$:

$$V_{k,j} = \int_Y^\infty \exp(n\lambda_j(Y-s))F_j^{k-1}(n,s)\,ds$$

$$W_{k,j} = [g_{k,known} + K_k(n,x,t)\rho_j^*]\exp(-n\lambda_j^* Y) - \int_0^Y \exp(-n\lambda_j^*(Y-s))H_j^{k-1}(n,s)ds$$

if $n < 0$:

$$V_{k,j} = [f_{k,known} + K_k(n,x,t)\rho_j]\exp(n\lambda_j Y) - \int_0^Y \exp(n\lambda_j(Y-s))F_j^{k-1}(n,s)\,ds$$

$$W_{k,j} = \int_Y^\infty \exp(-n\lambda_j^*(Y-s))H_j^{k-1}(n,s)\,ds$$

where $f_{k,known}$ and $g_{k,known}$ are known functions depending on $\vec{G}_{k-1}$, that decay exponentially in the variable Y.

We still have to determine the scalar-valued function $K_k(n,x,t)$ and the independent of Y function $\underline{\vec{u}_k^{\,0}}(t,x,y)$, which has to satisfy the boundary condition

$$C\underline{\vec{u}_k^{\,0}}(t,x,y=0) = -C\int_0^\infty (g'(\vec{u}_0))^{-1} \vec{G}_{k-1}^{\,0}((t,x,s)\,ds.$$

In order to write an equation satisfied by these unknown functions, we study the equation satisfied by $\vec{u}_{k+1}$:

$$\partial_Y \vec{u}_{k+1}^{\,n} = inA\vec{u}_{k+1}^{\,n} - (g'(\vec{u}_0))^{-1}\vec{G}_k^n.$$

We get from the first resolubility condition of section 1.4: $\underline{\vec{G}_k^{\,0}} = 0$, where $\vec{G}_k$ is given by

$\vec{G}_k(\vec{u}_0, ..., \vec{u}_k, \vec{h}_k) = \partial_t \vec{u}_k + f'(\vec{u}_0)\partial_x \vec{u}_k + f''(\vec{u}_0)\underline{\vec{u}_k}(y=0)\partial_\theta \vec{u}_1^* + f''(\vec{u}_0)\partial_\theta \vec{u}_k^* \vec{u}_1^* +$
$f''(\vec{u}_0)\vec{u}_k^*\partial_\theta \vec{u}_1^* + g'(\vec{u}_0)\partial_y \vec{u}_k + g''(\vec{u}_0)\underline{\vec{u}_k}(y=0)\partial_Y \vec{u}_1^*$
$+ g''(\vec{u}_0)\partial_Y \vec{u}_k^* \vec{u}_1^* + g''(\vec{u}_0)\vec{u}_k^*\partial_Y \vec{u}_1^* + \psi_j(\vec{u}_0, \ldots, \vec{u}_{k-1}) - \vec{h}_k,$

we then obtain

$$\partial_t \underline{\vec{u}_k^0} + f'(\vec{u}_0)\partial_x \underline{\vec{u}_k^0} + g'(\vec{u}_0)\partial_y \underline{\vec{u}_k^0} + \underline{F}(\vec{u}_0, \ldots, \vec{u}_{k-1}) = 0.$$

We established an equation satisfied by $\underline{\vec{u}_k^0}$.

We get from the second resolubility condition:

$$\int_0^\infty \vec{L}(n,s).\vec{G}_k^n(t,x,s)\, ds = 0 \tag{1.8.1}$$

the expression of $L(n,s)$ is given in section 1.4.

$\vec{G}_k^n$, for $n \neq 0$, is given by

$$\begin{aligned}&\vec{G}_k^n(\vec{u}_0, ..., \vec{u}_k, \vec{h}_k) = \partial_t \vec{u}_k^n + f'(\vec{u}_0)\partial_x \vec{u}_k^n + f''(\vec{u}_0)\underline{\vec{u}_k}(y=0)\partial_\theta \vec{u}_1^{*,n} \\ &+ in \sum_{l\in\mathbb{Z}} f''(\vec{u}_0)\vec{u}_k^{*,n-l}\vec{u}_1^{*,l} + g''(\vec{u}_0)\underline{\vec{u}_k}(y=0)\partial_Y \vec{u}_1^{*,n} + \\ &\sum_{l\in\mathbb{Z}} g''(\vec{u}_0)\partial_Y \vec{u}_k^{*,n-l}\vec{u}_1^{*,l} + \sum_{l\in\mathbb{Z}} g''(\vec{u}_0)\vec{u}_k^{*,l}\partial_Y \vec{u}_1^{*,n-l} + \psi_j^n(\vec{u}_0, \ldots, \vec{u}_{k-1}) - \vec{h}_k^n.\end{aligned}$$

It follows from the condition (1.8.1):

$$\begin{aligned}\partial_t K_k(n,x,t) + v\partial_x K_k(n,x,t) + in \sum_{l\in\mathbb{Z}} \Lambda(n-l,l) K_k(n-l,x,t) K_1(l,x,t) \\ + in \sum_{l\in\mathbb{Z}} \Lambda(n-l,l) K_1(n-l,x,t) K_k(l,x,t) = f_{k-1}^n(\vec{u}_0, \ldots, \vec{u}_{k-1}, \vec{h}_k, \underline{\vec{u}_k})\end{aligned}$$

with v and $\Lambda(n-l,l)$ defined in section 1.5.2 and $f_{k-1}^n(\vec{u}_0, \cdots, \vec{u}_{k-1}, \vec{h}_k, \underline{\vec{u}_k})$ a function depending on $\vec{u}_0, \cdots, \vec{u}_{k-1}, \vec{h}_k$ and on the known terms in the expression of $\vec{u}_k$, in particular $\underline{\vec{u}_k}$.

Consequently the scalar-valued function $K_k(\theta, x, t)$, whose Fourier coefficients with respect to θ are $K_k(n,x,t)$, satisfies the equation

$$\partial_t K_k + v\partial_x K_k + \partial_\theta(a(K_k, K_1) + a(K_1, K_k)) = f_{k-1} \tag{1.8.2}$$

where the bilinear form a is defined in section 1.5.2 by (1.5.11) and f_{k-1} is the scalar-valued function whose Fourier coefficients are f_{k-1}^n.

We obtained an equation satisfied by $K_k(n,x,t)$: $\vec{u}_k$ is completely determined. This finishes the proof of theorem 1.2.10.

1.9 Approximate solution and problem satisfied by the residual: proof of Theorems 1.2.13 and 1.2.14

Let us deduce Theorem 1.2.13 from Theorem 1.2.10. Suppose that $\{\vec{u}_k\}_{k\geq 1}$ is given by Theorem 1.2.10 and let

$$\vec{u}^\varepsilon_{app}(t,x,y) = \vec{u}_0 + \sum_{k=1}^{M} \varepsilon^k \vec{u}_k\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big). \tag{1.9.1}$$

Suppose that

$$\vec{h}^\varepsilon(t,x,y) - \sum_{k=1}^{M-1} \varepsilon^k \vec{h}_k\Big(t,x,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) = \mathcal{O}(\varepsilon^M) \tag{1.9.2}$$

in H^s_ε, where $\mathcal{O}(\varepsilon^M)$ in H^s_ε is defined by (1.2.30).

We introduce

$$\phi(\vec{u}^\varepsilon) = \partial_t \vec{u}^\varepsilon + \partial_x(f(\vec{u}^\varepsilon)) + \partial_y(g(\vec{u}^\varepsilon)).$$

From Lemma (1.3.1):

$$\begin{aligned}\phi(\vec{u}^\varepsilon_{app}) &= \phi^\varepsilon(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon})\\ &= \sum_{k=0}^{M-1} \varepsilon^k \varphi_k\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) + \varepsilon^M R^\varepsilon_M\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big),\end{aligned}$$

where $R^\varepsilon_M = \mathcal{O}(1)$ in H^s_ε.

From the assumption (1.9.2), there is $\tilde{R}^\varepsilon_M = \mathcal{O}(1)$ in H^s_ε, such that:

$$\begin{aligned}\phi(\vec{u}^\varepsilon_{app}) - \vec{h}^\varepsilon = \sum_{k=0}^{M-1} \varepsilon^k &\left(\varphi_k\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) - \vec{h}_k\Big(t,x,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big)\right)\\ &+\varepsilon^M \tilde{R}^\varepsilon_M\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big).\end{aligned}$$

The equations (1.2.14) have been obtained by equating the coefficients of ε^k to zero (see section 1.3), therefore, since the profiles $\vec{u}_k$ satisfy the equations (1.2.14), for $0 \leq k \leq M-1$:

$$\forall y > 0 \quad \varphi_k\Big(t,x,y,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) - \vec{h}_k\Big(t,x,\frac{x-\omega t}{\varepsilon},\frac{y}{\varepsilon}\Big) = 0.$$

Thus

$$\phi(\vec{u}^{\varepsilon}_{app}) - \vec{h}^{\varepsilon} = \mathcal{O}(\varepsilon^M) \quad \text{in } H^s_\varepsilon. \tag{1.9.3}$$

Furthermore, the profiles $\{\vec{u}_k\}_{k\geq 1}$ vanish when $t \leq 0$, and satisfy the boundary conditions (1.2.16) $C\vec{u}_{k|y=Y=0} = 0$, for $k \geq 1$ and the constant state $\vec{u}_0$ satisfies the condition (1.2.9) $C\vec{u}_0 = 0$. Therefore, the boundary condition $C\vec{u}^{\varepsilon}_{app|y=0} = 0$, and the condition in the past $\forall t \leq 0,\ \vec{u}^{\varepsilon}_{app}(t) = \vec{u}_0$ are satisfied.

Finally we get that $\vec{u}^{\varepsilon}_{app}$ is an approximate solution of (1.2.7a) (1.2.7b) (1.2.7c) in the sense that

$$\begin{cases} \partial_t \vec{u}^{\varepsilon}_{app} + \partial_x(f(\vec{u}^{\varepsilon}_{app})) + \partial_y(g(\vec{u}^{\varepsilon}_{app})) - \vec{h}^{\varepsilon} = \mathcal{O}(\varepsilon^M) \quad in\ H^s_\varepsilon, \\ C\vec{u}^{\varepsilon}_{app|y=0} = 0, \\ \forall t \leq 0,\ \vec{u}^{\varepsilon}_{app}(t) = \vec{u}_0. \end{cases} \tag{1.9.4}$$

This finishes the proof of Theorem 1.2.13.

Let us prove Theorem 1.2.14:
we seek an exact solution $\vec{u}^{\varepsilon}_{ex}$ of the problem (1.2.7) in the form

$$\vec{u}^{\varepsilon}_{ex}(t,x,\theta,Y) = \vec{u}^{\varepsilon}_{app}(t,x,\theta,Y) + \vec{r}(t,x,\theta,Y).$$

$\vec{u}^{\varepsilon}_{ex}$ satisfies (1.2.7a) $\phi(\vec{u}^{\varepsilon}_{ex}) - \vec{h}^{\varepsilon} = 0$, therefore,
from (1.9.3) $\phi(\vec{u}^{\varepsilon}_{app}) - \vec{h}^{\varepsilon} = \mathcal{O}(\varepsilon^M)$ in H^s_ε, we get

$$\phi(\vec{u}^{\varepsilon}_{ex}) - \phi(\vec{u}^{\varepsilon}_{app}) = \mathcal{O}(\varepsilon^M) \quad \text{in } H^s_\varepsilon,$$

this means that there is $G = \mathcal{O}(1)$ in H^s_ε such that

$$\begin{aligned} \partial_t \vec{u}^{\varepsilon}_{ex} - \partial_t \vec{u}^{\varepsilon}_{app} + f'(\vec{u}^{\varepsilon}_{ex})\partial_x \vec{u}^{\varepsilon}_{ex} - f'(\vec{u}^{\varepsilon}_{app})\partial_x \vec{u}^{\varepsilon}_{app} \\ + g'(\vec{u}^{\varepsilon}_{ex})\partial_y \vec{u}^{\varepsilon}_{ex} - g'(\vec{u}^{\varepsilon}_{app})\partial_y \vec{u}^{\varepsilon}_{app} = \varepsilon^M G, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \partial_t \vec{u}^{\varepsilon}_{ex} - \partial_t \vec{u}^{\varepsilon}_{app} + f'(\vec{u}^{\varepsilon}_{ex})(\partial_x \vec{u}^{\varepsilon}_{ex} - \partial_x \vec{u}^{\varepsilon}_{app}) + (f'(\vec{u}^{\varepsilon}_{ex}) - f'(\vec{u}^{\varepsilon}_{app}))\partial_x \vec{u}^{\varepsilon}_{app} \\ + g'(\vec{u}^{\varepsilon}_{ex})(\partial_y \vec{u}^{\varepsilon}_{ex} - \partial_y \vec{u}^{\varepsilon}_{app}) + (g'(\vec{u}^{\varepsilon}_{ex}) - g'(\vec{u}^{\varepsilon}_{app}))\partial_y \vec{u}^{\varepsilon}_{app} = \varepsilon^M G. \end{aligned}$$

Let us establish an equation where appear only $\vec{u}^{\varepsilon}_{app}$ and $\vec{r}$:

$$\begin{aligned} f'(\vec{u}^{\varepsilon}_{ex}) - f'(\vec{u}^{\varepsilon}_{app}) &= f'(\vec{u}^{\varepsilon}_{app} + \vec{r}) - f'(\vec{u}^{\varepsilon}_{app}) \\ &= A_f(\vec{u}^{\varepsilon}_{app}, \vec{r})\vec{r} \end{aligned}$$

with $A_f(\vec{u}^{\varepsilon}_{app}, \vec{r}) = \int_0^1 f''(\vec{u}^{\varepsilon}_{app} + t\vec{r})dt$.
Similarly

$$g'(\vec{u}^{\varepsilon}_{ex}) - g'(\vec{u}^{\varepsilon}_{app}) = A_g(\vec{u}^{\varepsilon}_{app}, \vec{r})\vec{r}$$

with $A_g(\vec{u}^{\varepsilon}_{app}, \vec{r}) = \int_0^1 g''(\vec{u}^{\varepsilon}_{app} + t\vec{r})dt$.
Thus finally

$$\begin{aligned}\partial_t \vec{r} + f'(\vec{u}^{\varepsilon}_{app} + \vec{r})\partial_x \vec{r} + A_f(\vec{u}^{\varepsilon}_{app}, \vec{r})\partial_x \vec{u}^{\varepsilon}_{app}\, \vec{r} + g'(\vec{u}^{\varepsilon}_{app} + \vec{r})\partial_y \vec{r} \\ + A_g(\vec{u}^{\varepsilon}_{app}, \vec{r})\partial_y \vec{u}^{\varepsilon}_{app}\, \vec{r} = \varepsilon^M G\end{aligned}$$

therefore, using the notations

$$B_f(\vec{u}^{\varepsilon}_{app}, \partial_x \vec{u}^{\varepsilon}_{app}, \vec{r}) = A_f(\vec{u}^{\varepsilon}_{app}, \vec{r})\partial_x \vec{u}^{\varepsilon}_{app} = \int_0^1 f''(\vec{u}^{\varepsilon}_{app} + t\vec{r})dt\; \partial_x \vec{u}^{\varepsilon}_{app} \tag{1.9.5a}$$

$$B_g(\vec{u}^{\varepsilon}_{app}, \partial_y \vec{u}^{\varepsilon}_{app}, \vec{r}) = A_g(\vec{u}^{\varepsilon}_{app}, \vec{r})\partial_y \vec{u}^{\varepsilon}_{app} = \int_0^1 g''(\vec{u}^{\varepsilon}_{app} + t\vec{r})dt\; \partial_y \vec{u}^{\varepsilon}_{app}, \tag{1.9.5b}$$

$$\begin{aligned}\partial_t \vec{r} + f'(\vec{u}^{\varepsilon}_{app} + \vec{r})\partial_x \vec{r} + g'(\vec{u}^{\varepsilon}_{app} + \vec{r})\partial_y \vec{r} = \\ \epsilon^M G - B_f(\vec{u}^{\varepsilon}_{app}, \partial_x \vec{u}^{\varepsilon}_{app}, \vec{r})\vec{r} - B_g(\vec{u}^{\varepsilon}_{app}, \partial_y \vec{u}^{\varepsilon}_{app}, \vec{r})\vec{r}.\end{aligned}$$

Furthermore $\vec{r}$ satisfies the boundary condition $C\vec{r}|_{y=0} = 0$, since $\vec{u}^{\varepsilon}_{ex}$ and $\vec{u}^{\varepsilon}_{app}$ satisfy it, and the condition in the past $\forall t \leq 0$, $\vec{r}(t) = 0$, since the profiles $\vec{u}_i$ satisfy it. We proved the proposition:

Proposition 1.9.1. *Suppose that $\vec{u}^{\varepsilon}_{app}$ is an approximate solution* (1.2.31) *of the problem (1.2.7) given by Theorem 1.2.13, then $\vec{u}^{\varepsilon}_{ex}$ in the form*

$$\vec{u}^{\varepsilon}_{ex}(t, x, \theta, Y) = \vec{u}^{\varepsilon}_{app}(t, x, \theta, Y) + \vec{r}(t, x, \theta, Y)$$

is solution of the problem (1.2.7) if and only if $\vec{r}$ is solution of:

$$\begin{aligned}\partial_t \vec{r} + f'(\vec{u}^{\varepsilon}_{app} + \vec{r})\partial_x \vec{r} + g'(\vec{u}^{\varepsilon}_{app} + \vec{r})\partial_y \vec{r} = \\ \epsilon^M G - B_f(\vec{u}^{\varepsilon}_{app}, \partial_x \vec{u}^{\varepsilon}_{app}, \vec{r})\vec{r} - B_g(\vec{u}^{\varepsilon}_{app}, \partial_y \vec{u}^{\varepsilon}_{app}, \vec{r})\vec{r} \quad \text{in } y > 0,\end{aligned} \tag{1.9.6a}$$

where $G = \mathcal{O}(1)$ in H^s_ε and B_f and B_g are defined by (1.9.5),
satisfies the boundary condition:

$$C\vec{r}|_{y=0} = 0 \tag{1.9.6b}$$

and the condition in the past

$$\forall t \leq 0,\ \vec{r}(t) = 0. \tag{1.9.6c}$$

Consequently the problem satisfied by $\vec{r}$ is in the form:

$$\begin{aligned}
&A_0(\vec{a},\vec{u})\partial_t\vec{u} + \sum_{j=1}^{d} A_j(\vec{a},\vec{u})\partial_{x_j}\vec{u} = F(\vec{b},\vec{u}) \quad x_d > 0 \\
&C\vec{u}|_{x_d=0} = 0 \\
&\forall t \leq 0 \;\; \vec{u}(t) = 0.
\end{aligned}$$

with $d = 2$, $\vec{a} = \vec{u}^{\varepsilon}_{app}$, $\vec{b} = (\vec{u}^{\varepsilon}_{app}, \partial_x \vec{u}^{\varepsilon}_{app}, \partial_y \vec{u}^{\varepsilon}_{app})$,
$A_0(\vec{a},\vec{u}) = \mathrm{Id}$, $A_1(\vec{a},\vec{u}) = f'(\vec{u}^{\varepsilon}_{app} + \vec{u})$, $A_2(\vec{a},\vec{u}) = g'(\vec{u}^{\varepsilon}_{app} + \vec{u})$
and $F(\vec{b},\vec{u}) = \epsilon^M G - B_f(\vec{u}^{\varepsilon}_{app}, \partial_x \vec{u}^{\varepsilon}_{app}, \vec{u})\vec{u} - B_g(\vec{u}^{\varepsilon}_{app}, \partial_y \vec{u}^{\varepsilon}_{app}, \vec{u})\vec{u}$.

We will consider in the following section boundary value problems of this type, show that the assumptions of Theorem 1.2.15 are satisfied by the problem (1.9.6) correspondant to the residual $\vec{r}$ and prove Theorem 1.2.15. Therefore, for $M \geq 3 > 1+d/2 = 2$ and $s \geq 3 > 1+d/2 = 2$, Theorem 1.2.15 gives the existence of ε_0 such that for $\varepsilon \in]0,\varepsilon_0]$, the boundary value problem (1.9.6) has a unique solution $\vec{r}$ in $\bigcap_{0\leq k\leq s} \mathcal{C}^k([0,T];H^{s-k})$ and $\vec{r} = \mathcal{O}(\varepsilon^M)$ in H^s_ε. We then obtain from Proposition 1.9.1 that, for $\varepsilon \in]0,\varepsilon_0]$, the problem (1.2.7) has a unique solution $\vec{u}^\varepsilon$ and $\vec{u}^\varepsilon - \vec{u}^{\varepsilon}_{app} = \mathcal{O}(\varepsilon^M)$ in H^s_ε. This finishes the proof of Theorem 1.2.14.

1.10 Equations with rapidly varying coefficients: proof of theorem 1.2.15

We consider the boundary value problem:

$$A_0(a,u)\partial_t u + \sum_{j=1}^{d} A_j(a,u)\partial_{x_j} u = F(b,u) \quad x_d > 0, \tag{1.10.1a}$$

$$Cu|_{x_d=0} = 0 \qquad \text{with } C \text{ of constant rank}, \tag{1.10.1b}$$

$$\forall t \leq 0 \;\; u(t) = 0. \tag{1.10.1c}$$

Under the hypothesis C of constant rank, there is no restriction in assuming that C is constant, using a change of variable $v = Mu$.

We assume that the problem is symmetric hyperbolic, that is that the matrices A_j are self adjoint with A_0 positive definite.

We assume that the boundary is not characteristic, that is that

$$A_d(a,u)|_{x_d=0} \text{ is non singular} \tag{1.10.2}$$

and that the boundary conditions are maximal dissipative, thas is that

$$\forall a\ \forall u\in \ker C\ \forall z\in \ker C\ \ (A_d(a,u)|_{x_d=0}z.z) \le 0, \tag{1.10.3a}$$

$$\operatorname{rank}(C) = \operatorname{Card}\{\lambda \in \operatorname{Sp}(A_d(a,u)|_{x_d=0})\ /\lambda > 0\}\,. \tag{1.10.3b}$$

These assumptions hold for the problem satisfied by the residual: they are direct consequences of the hypothesis (1.2.1).

a is assumed to have rapid oscillations, more precisely, we assume that a belongs to the Sobolev space $W^{s,\infty}(]-\infty,T]\times\mathbb{R}^d_+)$, where we denote $\mathbb{R}^{d-1}\times\mathbb{R}_+$ by $\mathbb{R}^d_+$, and that its derivatives satisfy:

$$\forall\ \ 0 \le |\alpha| \le s \quad \varepsilon^{(|\alpha|-1)_+}\,\|\partial^\alpha_{t,x} a\|_{L^\infty([0,T]\times\mathbb{R}^d_+)} \le C_1\,, \tag{1.10.4}$$

for some integer $s > \frac{d}{2}+1$.

Note that these assumptions hold for families of data a, which are the sum of a constant state with amplitude $\mathcal{O}(1)$ and a state with amplitude $\mathcal{O}(\varepsilon)$ at frequencies $|\xi| \approx \varepsilon^{-1}$, and therefore hold for $a = \vec{u}^\varepsilon_{app}$ and $T = t^*$ given by Theorem 1.7.5.

We assume that $b \in W^{s,\infty}(]-\infty,T]\times\mathbb{R}^d_+)$ and that its derivative satisfy

$$\forall\ \ 0 \le |\alpha| \le s \quad \varepsilon^{|\alpha|}\,\|\partial^\alpha_{t,x} b\|_{L^\infty([0,T]\times\mathbb{R}^d_+)} \le C_1'\,. \tag{1.10.5}$$

These assumptions allow for families of data b, with amplitude $\mathcal{O}(1)$ at frequencies $|\xi| \approx \varepsilon^{-1}$, and therefore allow for $b = (\vec{u}^\varepsilon_{app}, \partial_x\vec{u}^\varepsilon_{app}, \partial_y\vec{u}^\varepsilon_{app})$.

We suppose that $u = 0$ is almost a solution, that is that $f = F(b,0)$ vanishes in the past and satisfies for $|\alpha| \le s$ and $t \le T$:

$$\varepsilon^{|\alpha|}\,\|\partial^\alpha_{t,x} f^\varepsilon(t)\|_{L^2(\mathbb{R}^d_+)} \le \varepsilon^M C_2\,. \tag{1.10.6}$$

This assumption holds for the problem correspondant to the residual since, from Proposition 1.9.1, $f = F(b,0) = \varepsilon^M G$ with $G = \mathcal{O}(1)$ in H^s_ε.

Our aim in this section is to prove the following theorem:

Theorem 1.10.1. *Under the assumptions above, if $M > 1+d/2$, there exist $\varepsilon_0 > 0$ and C_3, depending only on the constants C_1, C_1', C_2 and coefficients A_j and F, such that:*

for $\varepsilon \in]0,\varepsilon_0]$, the boundary value problem (1.10.1a)-(1.10.1b)-(1.10.1c) has a unique solution $u \in \cap_{0\le k\le s}\mathcal{C}^k([0,T];H^{s-k}_\varepsilon)$ which satisfies for $0 \le |\alpha| \le s$ and $0 \le t \le T$:

$$\varepsilon^{|\alpha|}\,\|\partial^\alpha_{t,x} u(t)\|_{L^2(\mathbb{R}^d_+)} \le \varepsilon^M C_3\,, \tag{1.10.7}$$

When u is a function of $[0,T] \times \mathbb{R}^d_+$, for fixed t, $u(t)$ is the function of $\mathbb{R}^d_+$ obtained by freezing t.

We introduce the following norms:
for u a function of $\mathbb{R}^d_+$

$$\|u\|_{H^s_\varepsilon} = \sup_{|\alpha|\le s} \varepsilon^{|\alpha|} \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^d_+)} , \tag{1.10.8}$$

for u a function of $[0,T] \times \mathbb{R}^d_+$

$$\begin{aligned} \|u(t)\|_{CH^s_\varepsilon} &= \sup_{0\le k\le s} \varepsilon^k \|\partial_t^k u(t)\|_{H^{s-k}_\varepsilon} \\ &= \sup_{|\alpha|\le s} \varepsilon^{|\alpha|} \|\partial_{t,x}^\alpha u(t)\|_{L^2(\mathbb{R}^d_+)} . \end{aligned} \tag{1.10.9}$$

The main new ingredients are the weighted Sobolev inequalities and a multiplication lemma:

Proposition 1.10.2.

$$\|u(t)\|_{L^\infty(\mathbb{R}^d_+)} \le K\,\varepsilon^{-\frac{d}{2}} \|u(t)\|_{CH^{s-1}_\varepsilon}, \tag{1.10.10}$$

$$\|\partial_{t,x} u(t)\|_{L^\infty(\mathbb{R}^d_+)} \le K\,\varepsilon^{-\frac{d}{2}-1} \|u(t)\|_{CH^s_\varepsilon}. \tag{1.10.11}$$

Lemma 1.10.3. *For $\sigma > \frac{d}{2}$ and $l \ge 0$, $m \ge 0$ with $l+m \le \sigma$, we have*

$$\big\|u(t)v(t)\big\|_{CH^{\sigma-l-m}_\varepsilon} \le C\varepsilon^{-\frac{d}{2}} \big\|u(t)\big\|_{CH^{\sigma-l}_\varepsilon} \big\|v(t)\big\|_{CH^{\sigma-m}_\varepsilon}. \tag{1.10.12}$$

Proof. These inequalities can be proved with the help of the following remark: denoting by $\| \,.\, \|_{CH^s}$ the norm $\| \,.\, \|_{CH^s_1}$, we have
$\|u(t)\|_{CH^s_\varepsilon} = \varepsilon^{d/2} \|\tilde u(\frac{t}{\varepsilon})\|_{CH^s}$ where $u(t,x) = \tilde u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) = \tilde u(\tilde t, \tilde x)$.

Then we have:

$$\begin{aligned} \|u(t)\|_{L^\infty(\mathbb{R}^d_+)} = \|\tilde u(\frac{t}{\varepsilon})\|_{L^\infty(\mathbb{R}^d_+)} &\le C_s \|\tilde u(\frac{t}{\varepsilon})\|_{H^{s-1}(\mathbb{R}^d_+)} \quad \text{since } s-1 > \tfrac{d}{2} \\ &\le C_s \|\tilde u(\frac{t}{\varepsilon})\|_{CH^{s-1}} \end{aligned}$$

therefore, applying the remark, $\|u(t)\|_{L^\infty(\mathbb{R}^d_+)} \le C_s \varepsilon^{-\frac{d}{2}} \|u(t)\|_{CH^{s-1}_\varepsilon}$. The second inequality can be proved similarly: from $\partial_x u(t,x) = \varepsilon^{-1} \partial_{\tilde x} \tilde u(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ it follows $\|\partial_x u(t)\|_{L^\infty(\mathbb{R}^d_+)} = \varepsilon^{-1} \|\partial_{\tilde x} \tilde u(\frac{t}{\varepsilon})\|_{L^\infty(\mathbb{R}^d_+)}$, we apply the Sobolev inequality for $s-1 > \frac{d}{2}$, and observe that $\|\partial_{\tilde x} \tilde u(\frac{t}{\varepsilon})\|_{H^{s-1}(\mathbb{R}^d_+)} \le \|\tilde u(\frac{t}{\varepsilon})\|_{H^s(\mathbb{R}^d_+)}$.

The result for the derivative with respect to t is obtained by the same way, from $\partial_t u(t,x) = \varepsilon^{-1}\partial_{\tilde t}\tilde u(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$ and $\|\partial_{\tilde t}\tilde u(\frac{t}{\varepsilon})\|_{H^{s-1}} \leq \|\tilde u(\frac{t}{\varepsilon})\|_{CH^s}$.

The lemma is a consequence of the similar result for the norm $\| \,.\, \|_{H^s(\mathbb{R}^d_+)}$: for $\sigma > \frac{d}{2}$ and $l \geq 0$, $m \geq 0$ with $l+m \leq \sigma$, we have

$$\big\|u(t)v(t)\big\|_{H^{\sigma-l-m}(\mathbb{R}^d_+)} \leq C\big\|u(t)\big\|_{H^{\sigma-l}(\mathbb{R}^d_+)}\big\|v(t)\big\|_{H^{\sigma-m}(\mathbb{R}^d_+)}.$$

Therefore for $k \leq \sigma - l - m$

$$\begin{aligned}\big\|\partial_t^k\,(u(t)v(t))\,\big\|_{H^{\sigma-l-m-k}(\mathbb{R}^d_+)} &\leq C \sup_{k_1+k_2=k} \big\|\partial_t^{k_1}u(t)\partial_t^{k_2}v(t)\big\|_{H^{\sigma-l-k_1-m-k_2}(\mathbb{R}^d_+)}\\ &\leq C \sup_{k_1+k_2=k} \big\|\partial_t^{k_1}u(t)\big\|_{H^{\sigma-l-k_1}(\mathbb{R}^d_+)}\\ &\qquad\qquad \big\|\partial_t^{k_2}v(t)\big\|_{H^{\sigma-m-k_2}(\mathbb{R}^d_+)}\\ &\leq C\big\|u(t)\big\|_{CH^{\sigma-l}(\mathbb{R}^d_+)}\big\|v(t)\big\|_{CH^{\sigma-m}(\mathbb{R}^d_+)}.\end{aligned}$$

Passing to the sup we get

$$\big\|u(t)v(t)\big\|_{CH^{\sigma-l-m}(\mathbb{R}^d_+)} \quad\leq\quad C\big\|u(t)\big\|_{CH^{\sigma-l}(\mathbb{R}^d_+)}\big\|v(t)\big\|_{CH^{\sigma-m}(\mathbb{R}^d_+)}.$$

Applying the remark to $u(t), v(t)$ and $uv(t)$ we obtain the required result. $\square$

We are now ready to proceed with the proof of the theorem.

Proof. We write $u = \varepsilon^M v$, $f = \varepsilon^M g$ and $F(b,u) = f + \varepsilon^M G(b,u)v$. The problem for v reads:

$$L(a,u,\partial)v := A_0(a,u)\partial_t v + \sum_{j=1}^{d} A_j(a,u)\partial_{x_j} v - G(b,u)v = g \quad (1.10.13a)$$

$$Cv|_{x_d=0} = 0 \quad \text{with } C \text{ of constant rank} \quad (1.10.13b)$$

$$\forall t \leq 0 \;\; v(t) = 0. \quad (1.10.13c)$$

The theorem can be deduced from the following proposition:

Proposition 1.10.4. *There exist $\varepsilon_0 > 0$ and C_3, depending only on the constants C_1, C_1', C_2 and on the coefficients A_j and F, such that, for $\varepsilon \in$*

$]0,\varepsilon_0]$ *and for* $T' \in [0,T]$*, if* v*, solution on* $[0,T']$ *of the problem (1.10.13a)-(1.10.13b)-(1.10.13c), satisfies*

$$\|v(t)\|_{CH^s_\varepsilon} \leq C_3 \;\; \forall t \in [0,T'] \tag{1.10.14}$$

then v *satisfies*

$$\|v(t)\|_{CH^s_\varepsilon} \leq \frac{C_3}{2} \;\; \forall t \in [0,T']. \tag{1.10.15}$$

Assuming the proposition, we discuss the proof of the theorem. The problem is symmetric hyperbolic, under the assumption (1.10.13c), v satisfies the condition in the past $\forall t \in]-\infty,0]$, $\|v(t)\|_{CH^s_\varepsilon} = 0 \leq \frac{C_3}{2}$, then there exist $T' > 0$, depending on ε and on C_3, and a unique solution v such that

$$\forall 0 \leq k \leq s,\; \partial_t^k v \in C^0([0,T']; H^{s-k}(\mathbb{R}^d_+)),$$

that is $v \in CH^s_\varepsilon([0,T'])$, where we use the notation

$$CH^s_\varepsilon([0,\tilde{T}]) = \cap_{0\leq k\leq s} \mathcal{C}^k([0,\tilde{T}]; H^{s-k}_\varepsilon),$$

with v verifying

$$\forall t \in [0,T'],\;\; \|v(t)\|_{CH^s_\varepsilon} \leq C_3. \tag{1.10.16}$$

We define $T_0 = \sup\left\{\tilde{T} / \forall t \in [0,\tilde{T}],\; \|v(t)\|_{CH^s_\varepsilon} \leq C_3\right\}$, (1.10.16) implies $T_0 \geq T'$. Let us prove $T_0 = T$. By contradiction, we assume that $T_0 < T$. For $0 \leq t < T_0$, by definition of the least upper bound, $\exists \tilde{T},\; t \leq \tilde{T} \leq T_0,\; \forall s \in [0,\tilde{T}],\; \|v(s)\|_{CH^s_\varepsilon} \leq C_3$ hence $\|v(t)\|_{CH^s_\varepsilon} \leq C_3$, this gives $\forall t \in [0,T_0[,\; \|v(t)\|_{CH^s_\varepsilon} \leq C_3$. Since $0 < T' \leq T_0$, we have $0 < T_0 - \frac{T'}{2} < T_0$, thus $\forall t \in [0, T_0 - \frac{T'}{2}],\; \|v(t)\|_{CH^s_\varepsilon} \leq C_3$. The proposition allows us to write $\forall t \in [0, T_0 - \frac{T'}{2}],\; \|v(t)\|_{CH^s_\varepsilon} \leq \frac{C_3}{2}$. By definition of T' and since the data in the past $]-\infty, T_0 - \frac{T'}{2}]$ are upper bounded, we get $\forall t \in [0, T_0 + \frac{T'}{2}],\; \|v(t)\|_{CH^s_\varepsilon} \leq C_3$, which is contrary to the definition of T_0, since $T_0 + \frac{T'}{2} > T_0$.

We proved that there exist $\varepsilon_0 > 0$ and C_3, depending only on the constants C_1, C_1', C_2 and on the coefficients A_j and F, such that

for $\varepsilon \in]0,\varepsilon_0]$, the boundary value problem (1.10.1a)-(1.10.1b)-(1.10.1c) has a solution $u \in \cap_{0\leq k\leq s} \mathcal{C}^k([0,T]; H^{s-k}_\varepsilon)$ satisfying for $0 \leq |\alpha| \leq s$ and $0 \leq t \leq T$:

$$\varepsilon^{|\alpha|}\, \|\partial^\alpha_{t,x} u(t)\|_{L^2(\mathbb{R}^d_+)} \leq \varepsilon^M C_3\,. \tag{1.10.17}$$

$\square$

In order to demonstrate the proposition, let us prove some lemma.

We begin by this L^2 estimate:

Lemma 1.10.5. *There exist constants C and K, which depend only on C_1, C_1', C_2 and $\varepsilon_0 > 0$ which depends in addition on C_3, such that for $\varepsilon \leq \varepsilon_0$ and u verifying (1.10.1a)-(1.10.1b)-(1.10.1c) on $[0, T']$, we have for $t \in [0, T']$:*

$$\left\|v(t)\right\|_{L^2(\mathbb{R}^d_+)} \leq C \int_0^t e^{K(t-t')} \left\|L(a,u,\partial)v(t')\right\|_{L^2(\mathbb{R}^d_+)} dt'. \tag{1.10.18}$$

Proof. It follows from the weighted Sobolev inequalities (1.10.10)-(1.10.11):

$$\begin{aligned} \|\varepsilon^M v(t)\|_{L^\infty(\mathbb{R}^d_+)} &\leq K\, \varepsilon^{M-\frac{d}{2}} \|v(t)\|_{CH^{s-1}_\varepsilon}, \\ \|\varepsilon^M \partial_x v(t)\|_{L^\infty(\mathbb{R}^d_+)} &\leq K\, \varepsilon^{M-\frac{d}{2}-1} \|v(t)\|_{CH^s_\varepsilon}. \end{aligned}$$

Since $M > \frac{d}{2} + 1$, these inequalities, combined with the inequality (1.10.14), allow us to bound the Lipschitz norm of the coefficients $A_j(a, \varepsilon^M v)$ and the norm L^∞ of $G(b, \varepsilon^M v)$ by a constant independent of C_3 if ε is sufficiently small.

We denote $\langle A_0(a,u)v, v\rangle_{L^2(\mathbb{R}^d_+)}$ by $\mathcal{E}(t)$.

Since A_0 is continuous and since the $\|\,.\,\|_\infty$ norm of a and u is bounded, we have: $\exists \tilde{C}\ \forall a\, \forall u\ \ \tilde{C}\,\mathrm{Id} \leq A_0(a,u) \leq \tilde{C}^{-1}\mathrm{Id}$, we then obtain the following result:

$$\tilde{C}\|v(t)\|^2_{L^2} \leq \mathcal{E}(t) \leq \tilde{C}^{-1}\|v(t)\|^2_{L^2} \tag{1.10.19}$$

We write

$$\begin{aligned} \partial_t(A_0(a,u)v.v) + \sum_{j=1}^d \partial_{x_j}(A_j(a,u)v.v) = 2[A_0(a,u)\partial_t v + \sum_{j=1}^d A_j(a,u)\partial_{x_j} v].v \\ +\mathrm{div}(A_0, \ldots, A_d)v.v \end{aligned}$$

thus, integrating on $\mathbb{R}^d_+$

$$\begin{aligned} \partial_t \mathcal{E}(t) = \int_{\mathbb{R}^d_+} (\mathrm{div}(A_0, \ldots, A_d)v.v + 2L(a,u,\partial)v.v + 2G(b,u)v.v)\ dx \\ + \int_{\mathbb{R}^{d-1}} (A_d(a,u)v.v)\,|_{x_d=0}\, dx'. \end{aligned}$$

Since $\forall t \, \forall x' \;\; u(t,x',0) \in \ker C$, $v(t,x',0) \in \ker C$, we obtain from the assumption (1.10.3a) and the Cauchy-Schwarz inequality:

$$\partial_t \mathcal{E}(t) \leq \|\mathrm{div}(A_0,\ldots,A_d)\|_{L^\infty} \|v\|_{L^2}^2 + 2\|L(a,u,\partial)v\|_{L^2}\|v\|_{L^2} \\ +2\|G(b,u)\|_{L^\infty}\|v\|_{L^2}^2.$$

Applying the left inequality of (1.10.19) and simplifying by $\sqrt{\mathcal{E}(t)}$, we get:

$$\partial_t \sqrt{\mathcal{E}(t)} \leq \frac{1}{2}\tilde{C}^{-1}\|\mathrm{div}(A_0,\ldots,A_d)\|_{L^\infty}\sqrt{\mathcal{E}(t)} + \tilde{C}^{-1/2}\|L(a,u,\partial)v\|_{L^2} \\ +\tilde{C}^{-1}\|G(b,u)\|_{L^\infty}\sqrt{\mathcal{E}(t)}.$$

therefore by comparison of differential equations and since $v(0)=0$:

$$\sqrt{\mathcal{E}(t)} \leq \int_0^t \exp\left(\int_{t'}^t \tilde{C}^{-1}\left(\frac{1}{2}\|\mathrm{div}(A_0,\ldots,A_d)\|_{L^\infty} + \|G(b,u)\|_{L^\infty}\right) ds\right) \\ \tilde{C}^{-1/2}\|L(a,u,\partial)v(t')\|_{L^2}\, dt'.$$

Combining this with the fact that the Lipschitz norm of the coefficients $A_j(a,\varepsilon^M v)$ and the norm L^∞ of $G(b,\varepsilon^M)$ are bounded and applying the left inequality of (1.10.19), we obtain the inequality (1.10.18). $\square$

As a result of this L^2 estimate, we derive a CH^s_ε norm estimate.

Lemma 1.10.6. *There exist constants C and K, depending only on C_1, C_1', C_2 and $\varepsilon_0 > 0$ which depends in addition on C_3, such that for $\varepsilon \leq \varepsilon_0$ and u verifying (1.10.1a)-(1.10.1b)-(1.10.1c) on $[0,T']$, we have for $t \in [0,T']$:*

$$\forall t < T' \; \|v(t)\|_{CH^s_\varepsilon} \leq C\int_0^t e^{K(t-t')}\left(\|v(t')\|_{CH^s_\varepsilon} + \|g(t')\|_{CH^s_\varepsilon}\right) dt'. \tag{1.10.20}$$

Proof. The first step is to localize the problem in order to commute with A_d^{-1}.

Let $\{\phi_i(x)\}$ be a smooth partition of unity for $\mathbb{R}^d_+$, i.e. $\sum \phi_i = 1$ on $\mathbb{R}^d_+$, we introduce $v_i := \phi_i v$, then v_i satisfies $L(a,u,\partial)v_i = g_i$, $Cv_i|_{x_d=0} = 0$ and $\forall t \leq 0 \;\; v_i(t) = 0$ with $g_i = \phi_i g + \sum_{j=1}^d A^j(a,u)\partial_{x_j}\phi_i\, v$. From now on we suppose that v vanishes outside of a set with a small diameter.

1) We begin by the case where this set contains a portion of the boundary $\{x_d = 0\}$: by choosing the set sufficiently small, we obtain that A_d is non singular on this set from the assumption (1.10.2) $A_d(a,u)|_{x_d=0}$ invertible. We

commute the equation with the weighted tangential derivative $\varepsilon^{|\alpha|}\partial^{\alpha}_{t,x'}$ with $x' = (x_1, \dots, x_{d-1})$: we will obtain an estimate of $\sup_{|\alpha|\le s} \|\varepsilon^{|\alpha|}\partial^{\alpha}_{t,x'}v(t)\|_{L^2(\mathbb{R}^d_+)}$. Left multiplying the equation by A_d^{-1}, we will deduce an estimate of $\|v(t)\|_{CH^s_\varepsilon}$.

Let us prove an estimate of the commutators:

Lemma 1.10.7. *For $|\alpha| \le s$,*

$$\forall t \in [0, T'] \ \ \|[\varepsilon^{|\alpha|}\partial^{\alpha}_{t,x'}, L(a, u, \partial)]v(t)\|_{L^2(\mathbb{R}^d_+)} \le K\|v(t)\|_{CH^s_\varepsilon} \tag{1.10.21}$$

with K depending only on C_1, C'_1 and C_2 and for $\varepsilon \le \varepsilon_0$ where $\varepsilon_0 > 0$ is dependent of C_1, C'_1, C_2 and C_3.

Proof. The commutator $[\varepsilon^{|\alpha|}\partial^{\alpha}_{t,x'}, A_j\partial_j]v$ is linear combination of terms

$$\varepsilon^{|\alpha|}B(a, u)\partial^{\beta^1}_{t,x'}a \dots \partial^{\beta^q}_{t,x'}a \ \partial^{\alpha^1}_{t,x'}u \dots \partial^{\alpha^p}_{t,x'}u \ \partial^{\gamma}_{t,x}v$$

with $0 < |\beta^j| \le |\alpha|$, $0 < |\alpha^k| \le |\alpha|$, $0 < |\gamma| \le |\alpha|$ and $\sum|\beta^j| + \sum|\alpha^k| + |\gamma| \le |\alpha| + 1 \le s + 1$.

Let us bound the L^2 norm of these terms.

The L^∞ norm of $B(a, u)$ is bounded by a constant which is independent of C_3 if ε is sufficiently small, applying the weighted Sobolev inequality (1.10.10)-(1.10.11).

Let us examine the case $p \ge 1$.

Applying p times the multiplication lemma (1.10.12):

$$\|\partial^{\alpha^1}_{t,x'}u \dots \partial^{\alpha^p}_{t,x'}u \, \partial^{\gamma}_{t,x}v\|_{CH^\delta_\varepsilon} \le C\varepsilon^{-pd/2}\|\partial^{\alpha^1}_{t,x'}u\|_{CH^{s-|\alpha^1|}_\varepsilon} \dots \|\partial^{\alpha^p}_{t,x'}u\|_{CH^{s-|\alpha^p|}_\varepsilon} \|\partial^{\gamma}_{t,x}v\|_{CH^{s-|\gamma|}_\varepsilon}$$

with $\delta = s - 1 - (|\alpha^1| - 1) - \dots - (|\alpha^p| - 1) - (|\gamma| - 1) = s - 1 - \underbrace{(\sum|\alpha^j| + |\gamma|)}_{\le s+1} + \underbrace{p + 1}_{\ge 2} \ge 0$

We note that

$$\|\partial^{\alpha}_{t,x}u\|_{CH^{s-|\alpha|}_\varepsilon} = \sup_{|\delta|\le s-|\alpha|} \varepsilon^{|\delta|}\|\partial^{\alpha+\delta}_{t,x}u\|_{L^2} \le \varepsilon^{-|\alpha|}\|u\|_{CH^s_\varepsilon},$$

thus, the inequality (1.10.14) gives:

$$\|\partial^{\alpha^1}_{t,x'}u \dots \partial^{\alpha^p}_{t,x'}u \, \partial^{\gamma}_{t,x}v\|_{L^2} \le C\varepsilon^{-pd/2-(\sum|\alpha^k|+|\gamma|)+pM}C_3^p\|v\|_{CH^s_\varepsilon}.$$

From the assumption (1.10.4) on a and the fact that the L^∞ norm of $B(a,u)$ is bounded, it follows:

$$\|\varepsilon^{|\alpha|}B(a,u)\partial_{t,x'}^{\beta^1}a\dots\partial_{t,x'}^{\beta^q}a\ \partial_{t,x'}^{\alpha^1}u\dots\partial_{t,x'}^{\alpha^p}u\ \partial_{t,x}^{\gamma}v\|_{L^2}\le\varepsilon^{\mu}CC_1^qC_3^p\|v\|_{CH^s_\varepsilon}$$

with $\mu=|\alpha|+pM-(\sum(|\beta^i|-1)_++\sum|\alpha^k|+|\gamma|)-pd/2$.

Since $\sum(|\beta^i|-1)_+\le\sum|\beta^i|$, we have

$$\mu\ge|\alpha|+p(M-d/2)-\underbrace{(\sum|\beta^i|+\sum|\alpha^k|+|\gamma|)}_{\le|\alpha|+1}\ge p(M-d/2)-1>p-1\ge 0$$

since $M>d/2+1$ and $p\ge 1$.

We finally get $\mu>0$, consequently, for ε sufficiently small, $\varepsilon^\mu CC_1^qC_3^p$ is bounded by a constant which is independent of C_3.

We still have to examine the case $p=0$, in this case there is no dependence on C_3: we apply the hypothesis (1.10.4) on a, the estimate of the L^∞ norm of $B(a,u)$ and the inequality $\|\partial_{t,x}^\gamma v\|_{L^2}\le\varepsilon^{-\gamma}\|v\|_{CH^s_\varepsilon}$,

$$\|\varepsilon^{|\alpha|}B(a,u)\partial_{t,x'}^{\beta^1}a\dots\partial_{t,x'}^{\beta^q}a\partial_{t,x}^{\gamma}v\|_{L^2}\le\varepsilon^{\mu}CC_1^q\|v\|_{CH^s_\varepsilon}$$

with $\mu=|\alpha|-(\sum(|\beta^i|-1)_++|\gamma|)$. If $q>0$ then $\sum(|\beta^i|-1)_+\le\sum|\beta^i|-1$ therefore $\mu\ge|\alpha|-(\sum|\beta^i|+|\gamma|)+1\ge 0$. If $q=0$, since $|\gamma|\le\alpha$, we also obtain $\mu\ge 0$.

Finally we always have

$$\|\varepsilon^{|\alpha|}B(a,u)\partial_{t,x'}^{\beta^1}a\dots\partial_{t,x'}^{\beta^q}a\ \partial_{t,x'}^{\alpha^1}u\dots\partial_{t,x'}^{\alpha^p}u\ \partial_{t,x}^{\gamma}v\|_{L^2}\le K\|v\|_{CH^s_\varepsilon}$$

with K depending only on C_1 and C_2 and for $\varepsilon\le\varepsilon_0$, where ε_0 depends in addition on C_3.

We proceed in a similar way with the commutator $[\varepsilon^{|\alpha|}\partial_{t,x'}^{\alpha},G(b,u)]v$, the only difference is that the assumption (1.10.5) on b is weaker than the assumption (1.10.4) on a, but this is compensated by the fact that the number of derivatives is smaller by one:

the commutator is linear combination of terms

$$\varepsilon^{|\alpha|}B(a,b,u)\partial_{t,x'}^{\beta^1}a\dots\partial_{t,x'}^{\beta^q}a\,\partial_{t,x'}^{\delta^1}b\dots\partial_{t,x'}^{\delta^r}b\,\partial_{t,x'}^{\alpha^1}u\dots\partial_{t,x'}^{\alpha^p}u\ \partial_{t,x'}^{\gamma}v$$

with $\sum|\beta^j|+\sum|\delta^i|+\sum|\alpha^k|+|\gamma|\le|\alpha|$.

Similarly we obtain:

$$\begin{aligned}\|\varepsilon^{|\alpha|}B(a,b,u)\partial_{t,x'}^{\beta^1}a\dots\partial_{t,x'}^{\beta^q}a\ \partial_{t,x'}^{\delta^1}b\dots\partial_{t,x'}^{\delta^r}b\ \partial_{t,x'}^{\alpha^1}u\dots\partial_{t,x'}^{\alpha^p}u\ \partial_{t,x'}^{\gamma}v\|_{L^2}&\\ \le\varepsilon^{\mu}CC_1^qC_1'^rC_3^p\|v\|_{CH^s_\varepsilon}&\end{aligned}$$

with $\mu = |\alpha| + p(M - d/2) - \underbrace{(\sum(|\beta^j| - 1)_+ + \sum|\delta^i| + \sum|\alpha^k| + |\gamma|)}_{\leq|\alpha|}$, thus $\mu \geq p(M - d/2)$ we get $\mu > 0$ for $p \geq 1$, it is therefore bounded by a constant which is independent of C_3 for ε sufficiently small and in the case $p = 0$, where there is no dependence on C_3, we have $\mu \geq 0$.

Finally, we obtain the inequality (1.10.21): this finishes the proof of Lemma 1.10.7. □

Applying the L^2 estimate (1.10.18) to $\varepsilon^{|\alpha|}\partial^\alpha_{t,x'}v$:

$$\|\varepsilon^{|\alpha|}\partial^\alpha_{t,x'}v(t)\|_{L^2(\mathbb{R}^d_+)} \leq C\int_0^t e^{K(t-t')}\Big(\|[\varepsilon^{|\alpha|}\partial^\alpha_{t,x'}, L(a,u,\partial)]v(t')\|_{L^2(\mathbb{R}^d_+)} + \|\varepsilon^{|\alpha|}\partial^\alpha_{t,x'}g(t')\|_{L^2(\mathbb{R}^d_+)}\Big)\,dt'$$

Applying Lemma 1.10.7, we then obtain, for $t \in [0, T']$:

$$\forall|\alpha| \leq s \quad \|\varepsilon^{|\alpha|}\partial^\alpha_{t,x'}v(t)\|_{L^2(\mathbb{R}^d_+)} \leq C\int_0^t e^{K(t-t')}\big(\|v(t')\|_{CH^s_\varepsilon} + \|\varepsilon^{|\alpha|}\partial^\alpha_{t,x'}g(t')\|_{L^2(\mathbb{R}^d_+)}\big)\,dt'$$

thus by passing to the sup, for $t \in [0, T']$:

$$\sup_{|\alpha|\leq s}\|\varepsilon^{|\alpha|}\partial^\alpha_{t,x'}v(t)\|_{L^2(\mathbb{R}^d_+)} \leq C\int_0^t e^{K(t-t')}\big(\|v(t')\|_{CH^s_\varepsilon} + \|g(t')\|_{CH^s_\varepsilon}\big)\,dt' \tag{1.10.22}$$

In order to get a bound for the CH^s_ε norm of v, it is necessary to bound the derivatives of v which include derivatives with respect to x_d: for this purpose, we will use the equation verified by v.

We have: $A_0(a,u)\partial_t v + \sum_{j=1}^d A_j(a,u)\partial_{x_j}v - G(b,u)v = g$, thus

$$\partial_{x_d}v = A_d^{-1}[g + G(b,u)v - A_0(a,u)\partial_t v - \sum_{j=1}^{d-1}A_j(a,u)\partial_{x_j}v].$$

We then obtain:

$$\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d}v = \varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d-1}[A_d^{-1}(g+G(b,u)v - A_0(a,u)\partial_t v - \sum_{j=1}^{d-1} A_j(a,u)\partial_{x_j}v)].$$

$\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d-1}A_d^{-1}(a,u)g$ is linear combination of terms

$$\varepsilon^{|\alpha|}B(a,u)\partial_{t,x}^{\beta^1}a\ldots\partial_{t,x}^{\beta^q}a\ \partial_{t,x}^{\alpha^1}u\ldots\partial_{t,x}^{\alpha^p}u\ \partial_{t,x}^{\gamma}g$$

with $\sum|\beta^j|+\sum|\alpha^k|+|\gamma|\leq|\alpha|-1\leq s-1$
whose L^2 norm is bounded by $\varepsilon^{\mu}C_1^qC_3^p\|g\|_{CH_\varepsilon^{s-1}}$,
where $\mu = |\alpha| + p\underbrace{(M-d/2)}_{>0} - (\underbrace{\sum(|\beta^i|-1)_+ + \sum|\alpha^j|+|\gamma|}_{\leq|\alpha|-1}) \geq 1$
if $p=0$, there is no dependence on C_3, if $p>0$, we obtain $\mu>1$, therefore $\|\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d-1}A_d^{-1}(a,u)g\|_{L^2}\leq\varepsilon C\|g\|_{CH_\varepsilon^{s-1}}$ with C a constant which is independent of C_3 for ε sufficiently small.

$\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d-1}\left(A_d^{-1}(a,u)G(b,u)v\right)$ is linear combination of terms:

$$\varepsilon^{|\alpha|}B(a,b,u)\partial_{t,x}^{\beta^1}a\ldots\partial_{t,x}^{\beta^q}a\ \partial_{t,x}^{\delta^1}b\ldots\partial_{t,x}^{\delta^q}b\ \partial_{t,x}^{\alpha^1}u\ldots\partial_{t,x}^{\alpha^p}u\ \partial_{t,x}^{\gamma}v$$

with $\sum|\beta^j|+\sum|\delta^i|+\sum|\alpha^k|+|\gamma|\leq|\alpha|-1\leq s-1$ and $\gamma=(\gamma_0,\gamma',\gamma_d)$ such that $\gamma_d\leq\alpha_d-1$, similarly we obtain:
$\|\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d-1}\left(A_d^{-1}(a,u)G(b,u)v\right)\|_{L^2}\leq\varepsilon C\sup_{|\beta|\leq s,\,\beta_d\leq\alpha_d-1}\varepsilon^{|\beta|}\|\partial_{t,x}^{\beta}v\|_{L^2}$
with C an independent of C_3 constant for ε sufficiently small.

$\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d-1}\left(A_d^{-1}(a,u)A_j(a,u)\partial_{x_j}v\right)$ is linear combination of terms:

$$\varepsilon^{|\alpha|}B(a,u)\partial_{t,x}^{\beta^1}a\ldots\partial_{t,x}^{\beta^q}a\ \partial_{t,x}^{\alpha^1}u\ldots\partial_{t,x}^{\alpha^p}u\ \partial_{t,x}^{\gamma}v$$

with $\sum|\beta^j|+\sum|\alpha^k|+|\gamma|\leq|\alpha|\leq s$ and $\gamma=(\gamma_0,\gamma',\gamma_d)$, $\gamma_d\leq\alpha_d-1$, whose L^2 norm is bounded by $\varepsilon^{\mu}C_1^qC_3^p\sup_{|\beta|\leq s,\,\beta_d\leq\alpha_d-1}\varepsilon^{|\beta|}\|\partial_{t,x}^{\beta}v\|_{L^2}$
where $\mu = |\alpha| + p\underbrace{(M-d/2)}_{\geq 0} - (\underbrace{\sum(|\beta^i|-1)_+ + \sum|\alpha^j|+|\gamma|}_{\leq|\alpha|}) > 0$ for $p>0$.

Thus:

$$\|\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d-1}\left(A_d^{-1}(a,u)A_j(a,u)\partial_{x_j}v\right)\|_{L^2}\leq C\sup_{|\beta|\leq s,\,\beta_d\leq\alpha_d-1}\varepsilon^{|\beta|}\|\partial_{t,x}^{\beta}v\|_{L^2}$$

with C an independant of C_3 constant for ε sufficiently small.

We then obtain:

$$\|\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d}v\|_{L^2} \leq \varepsilon C\|g\|_{CH_\varepsilon^{s-1}} + C \sup_{|\beta|\leq s,\, \beta_d\leq\alpha_d-1} \varepsilon^{|\beta|}\|\partial_{t,x}^{\beta}v\|_{L^2}$$

with C depending only on C_1, C_1' and C_2 and for $\varepsilon \leq \varepsilon_0$ for $\varepsilon_0 > 0$ depending on C_1, C_1', C_2 and C_3.

Therefore by induction:

$$\begin{aligned}&\forall\alpha = (\alpha_0, \alpha', \alpha_d), |\alpha| \leq s,\\ &\quad \|\varepsilon^{|\alpha|}\partial_t^{\alpha_0}\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d}v\|_{L^2} \leq \varepsilon\, C\|g\|_{H_\varepsilon^{s-1}} + C \sup_{|\beta|\leq s} \varepsilon^{|\beta|}\|\partial_{t,x'}^{\beta}v\|_{L^2}\end{aligned}$$

By passing to the sup, we finally obtain:

$$\|v\|_{CH_\varepsilon^s} \leq \varepsilon\, C\|g\|_{CH_\varepsilon^{s-1}} + C \sup_{|\beta|\leq s} \varepsilon^{|\beta|}\|\partial_{t,x'}^{\beta}v\|_{L^2}.$$

Let us bound the first term: since f vanishes in the past, $\forall x \in \mathbb{R}_+^d\ g(0,x) = 0$ thus $\varepsilon g(t,x) = \int_0^t \varepsilon\partial_t g(t',x)\,dt'$ then $\varepsilon\|g(t)\|_{CH_\varepsilon^{s-1}} \leq \int_0^t \|\varepsilon\partial_t g(t')\|_{CH_\varepsilon^{s-1}}dt' \leq \int_0^t \|g(t')\|_{CH_\varepsilon^s}\,dt'$.
We then get:

$$\|v(t)\|_{CH_\varepsilon^s} \leq C\int_0^t \|g(t')\|_{CH_\varepsilon^s}\,dt' + C\sup_{|\beta|\leq s}\varepsilon^{|\beta|}\|\partial_{t,x'}^{\beta}v\|_{L^2}.$$

Therefore, it follows from (1.10.22):

$$\|v(t)\|_{CH_\varepsilon^s} \leq C\int_0^t \|g(t')\|_{CH_\varepsilon^s}\,dt' + C\int_0^t e^{K(t-t')}\left(\|v(t')\|_{CH_\varepsilon^s} + \|g(t')\|_{CH_\varepsilon^s}\right)\,dt'.$$

Since $\forall\, 0 \leq t' \leq t,\ 1 \leq e^{K(t-t')}$, we finally obtain the inequality (1.10.20):

$$\|v(t)\|_{CH_\varepsilon^s} \leq C\int_0^t e^{K(t-t')}\left(\|v(t')\|_{CH_\varepsilon^s} + \|g(t')\|_{CH_\varepsilon^s}\right)\,dt'. \tag{1.10.23}$$

2) In order to complete the proof of Lemma 1.10.6, it remains to examine the case where the set does not intersect the boundary. The problem is reduced to a Cauchy problem, with data equal to zero in the past, we can then commute the equation with the weighted derivative $\varepsilon^{|\alpha|}\partial_{t,x}^{\alpha}$ where $x = (x_1, \ldots, x_d)$. As before, we prove that for $|\alpha| \leq s$,

$$\left\|[\varepsilon^{|\alpha|}\partial_{t,x}^{\alpha}\,,\, L(a,u,\partial)]v(t)\right\|_{L^2} \leq K\left\|v(t)\right\|_{CH_\varepsilon^s}.$$

We then obtain directly the inequality (1.10.20) of Lemma 1.10.6. □

The CH^s_ε norm estimate (1.10.20) allows us to prove the proposition.

Proof. We will absorb the term in v of the right member of the inequality (1.10.20). We introduce $\phi(t) = \int_0^t e^{K(t-t')}\|v(t')\|_{CH^s_\varepsilon} dt'$, $\phi'(t) = \|v(t)\|_{CH^s_\varepsilon} + K\phi(t)$ thus, from (1.10.20):

$$\phi'(t) \le C \int_0^t e^{K(t-t')}\|g(t')\|_{CH^s_\varepsilon}\, dt' + (K+C)\phi(t)$$

then $\frac{d}{dt}\left(e^{-(K+C)t}\phi(t)\right) \le C\int_0^t e^{-Ct}e^{-Kt'}\|g(t')\|_{CH^s_\varepsilon}\, dt'$. Since $\phi(0) = 0$, we obtain by integration:

$$e^{-(K+C)t}\phi(t) \le C \int_{s=0}^t \int_{t'=0}^s e^{-Cs}e^{-Kt'}\|g(t')\|_{CH^s_\varepsilon}\, dt' ds$$

thus by Fubini theorem:

$$e^{-(K+C)t}\phi(t) \le C \int_{t'=0}^t \left(\int_{s=t'}^t e^{-Cs} ds\right) e^{-Kt'}\|g(t')\|_{CH^s_\varepsilon}\, dt'.$$

We write $\int_{s=t'}^t e^{-Cs}ds = \frac{1}{C}(-e^{-Ct} + e^{-Ct'}) \le \frac{1}{C}e^{-Ct'}$ and we obtain:

$$\phi(t) \le C \int_{t'=0}^t e^{(K+C)(t-t')}\|g(t')\|_{CH^s_\varepsilon}\, dt'.$$

thus from (1.10.20), it follows:

there exist C and K, depending only on C_1, C_1' and C_2, and ε_0, depending in addition on C_3, such that for $\varepsilon \le \varepsilon_0$ and v satisfying (1.10.16) $\forall t \in [0, T']$ $\|v(t)\|_{CH^s_\varepsilon} \le C_3$, we have, for $t \in [0, T']$

$$\|v(t)\|_{CH^s_\varepsilon} \le C \int_0^t e^{(K+C)(t-t')}\|g(t')\|_{CH^s_\varepsilon}\, dt'. \tag{1.10.24}$$

We choose C_3 such that $C_3 \ge 2\frac{CC_2}{K+C}e^{(K+C)T}$ and ε_0 corresponding to this choice.

Thus (1.10.24) shows that if v satisfies $\|v(t)\|_{CH^s_\varepsilon} \le C_3$ on $[0, T']$ then v satisfies $\|v(t)\|_{CH^s_\varepsilon} \le \frac{C_3}{2}$ on $[0, T']$.

Indeed, the assumption (1.10.6) gives $\forall 0 \le t \le T$, $\|g(t)\|_{CH^s_\varepsilon} \le C_2$, therefore (1.10.24) allows us to write:

$$\forall t \le T' \;\|v(t)\|_{CH^s_\varepsilon} \le CC_2 \int_0^t e^{(K+C)(t-t')}\, dt' \le \frac{CC_2}{K+C}e^{(K+C)T} \le \frac{C_3}{2}.$$

$\square$

This finishes the proof of Theorem (1.2.15).

Chapter 2

Internal rectification for elastic surface waves

We prove that fast oscillatory elastic surface waves can produce non trivial internal non oscillatory displacements, illustrating on the example of elasticity a phenomena observed in the first chapter for general first order systems.

2.1 Introduction

We prove that fast oscillatory elastic surface waves can produce non trivial internal non oscillatory displacements. This phenomenon was observed and explained in the first chapter for general first order systems. The goal of this chapter is to prove that it does occur in the case of elastic waves.

For simplicity, we consider surface waves in space dimension two, on a domain which is a half-plane. We seek surface waves which admit asymptotic expansions of the form

$$U^\varepsilon(t,x,y) = \begin{pmatrix} u^\varepsilon(t,x,y) \\ v^\varepsilon(t,x,y) \end{pmatrix} \sim \sum_{k=2}^{\infty} \varepsilon^k U_k \left(t, x, y, \frac{x-ct}{\varepsilon}, \frac{y}{\varepsilon} \right),$$

on $y > 0$, with profiles $U_k(t,x,y,Y,\theta) = \underline{U}_k(t,x,y) + U_k^*(t,x,\theta,Y)$, where U_k^* is periodic with respect to the variable θ and exponentially decaying to 0 in the variable Y. U_k^* contributes to a term localized near the boundary, rapidly oscillating on the boundary, while $\underline{U}_k$ is a slowly varying mean field, which is not localized on the boundary.

The first term U_2 is determined for example in [54], [68] or [69] (see also [36, 7] and the first chapter for the analysis of the equation). In particular, it is purely localized near the boundary, that is $\underline{U}_2 = 0$, and U_2^* is determined by a scalar unknown $\alpha_2(t, x, \theta)$, which satisfies a propagation equation (2.2.2). Our main aim in what follows is to prove that, in general, the corrector U_3 is not purely localized near the boundary, that is $\underline{U}_3$ does not vanish, even if the source term is purely localized near the boundary. $\underline{U}_3$ is a solution of the linearized equations of elasticity, with boundary terms which are determined by α_2 and in general do not vanish. U_3 does depend on the slow variable y and does not decay to 0 as Y tends to $+\infty$, even if the source term is exponentially decaying to 0.

The other terms of the expansion can be determined as in the first chapter. In the first chapter, we have proved in the case of general first order systems the existence of an exact solution admitting the given asymptotic expansion. This analysis, which is not done here, is expected to be extendable to elasticity.

Following [54], we consider an elastic medium which in the rest state occupies the half-space $y > 0$ and which is deformed in plane strain in the x and y directions. The displacement components are denoted by $u(t, x, y)$ and $v(t, x, y)$ in the x- and y-directions. Assuming the medium is isotropic and hyperelastic, the equations of motion and boundary conditions on the stress-free boundary $y = 0$ take the form

$$\partial_{tt} u^\varepsilon - r\partial_{xx} u^\varepsilon - (r-1)\partial_{xy} v^\varepsilon - \partial_{yy} u^\varepsilon = \partial_x F_1 + \partial_y F_2 \tag{2.1.1a}$$

$$\partial_{tt} v^\varepsilon - \partial_{xx} v^\varepsilon - (r-1)\partial_{xy} u^\varepsilon - r\partial_{yy} v^\varepsilon = \partial_x G_1 + \partial_y G_2 \tag{2.1.1b}$$

in $y > 0$ and

$$\partial_y u^\varepsilon + \partial_x v^\varepsilon = -F_2 + f^\varepsilon \tag{2.1.2a}$$

$$(r-2)\partial_x u^\varepsilon + r\partial_y v^\varepsilon = -G_2 + g^\varepsilon \tag{2.1.2b}$$

on $y = 0$,
where f^ε and g^ε are given source terms and F_1, F_2, G_1 and G_2 are quadratic terms.

We consider a particular case of the equations given in [54] (see also [68]), where the quadratic nonlinear terms are given by:

$$F_1 = \partial_y u^\varepsilon \partial_x v^\varepsilon \tag{2.1.3}$$

$$F_2 = \partial_x v^\varepsilon (\partial_x u^\varepsilon + \partial_y v^\varepsilon) \tag{2.1.4}$$

$$G_1 = \partial_y u^\varepsilon (\partial_x u^\varepsilon + \partial_y v^\varepsilon) \tag{2.1.5}$$

$$G_2 = \partial_y u^\varepsilon \partial_x v^\varepsilon = F_1. \tag{2.1.6}$$

Denoting by $(-c, 1)$ one frequency such that there exist surface waves associated to the phase $\varphi(t, x) = -ct + x$ (up to a homothetic change of variable, there is no restriction in assuming that the spatial wave number $k = 1$), one looks for oscillatory surface waves, that is for solutions of the boundary value problem localized near the boundary and such that the trace on the boundary has rapid oscillations with the phase $\theta = \frac{x-ct}{\varepsilon}$.

2.2 Statement of the main results

2.2.1 Statement of the problem

Surface waves are real solutions

$$U^\varepsilon(t, x, y) = \begin{pmatrix} u^\varepsilon(t, x, y) \\ v^\varepsilon(t, x, y) \end{pmatrix}$$

satisfying the equations (2.1.1) in $y > 0$ and the boundary conditions (2.1.2) on $y = 0$, that admit asymptotic expansions

$$U(t, x, y) \sim \sum_{k=2}^{\infty} \varepsilon^k U_k(t, x, y, \frac{x - ct}{\varepsilon}, \frac{y}{\varepsilon}), \tag{2.2.1}$$

with profiles $U_k(t, x, y, \theta, Y) = \begin{pmatrix} u_k(t, x, y, \theta, Y) \\ v_k(t, x, y, \theta, Y) \end{pmatrix}$ such that $u_k(t, x, y, \theta, Y) = \underline{u}_k(t, x, y) + u_k^*(t, x, \theta, Y)$ and $v_k(t, x, y, \theta, Y) = \underline{v}_k(t, x, y) + v_k^*(t, x, \theta, Y)$ belong to the space $S = \underline{S} \oplus S^*$ defined in the first chapter (see Definition 1.2.4), that is u_k^* and v_k^* are periodic in θ and exponentially decaying in Y.

We denote by U_k^n the Fourier coefficient, with respect to θ, of order n of the profile U_k. From the definition of $S = \underline{S} \oplus S^*$, for $n \neq 0$, U_k^n is of the form $U_k^n = U_k^{n,*}(t, x, Y)$, with $U_k^{n,*} \in S^*$ and U_k^0 is of the form $U_k^0 = \underline{U}_k(t, x, y) + U_k^{0,*}(t, x, Y)$, with $\underline{U}_k \in \underline{S}$ and $U_k^{0,*} \in S^*$.

For the sake of definitiveness, we suppose that the solution vanishes identically in the past:

$$\forall t \leq 0 \quad U^\varepsilon(t) = 0,$$

and, to fix the ideas, that it is ignited by source terms f^ε and g^ε on the boundary, which we assume to be small and localized near the boundary:

$$f(t, x) \sim \sum_{k=2}^{\infty} \varepsilon^k f_k(t, x, \frac{x - ct}{\varepsilon}),$$

$$g(t,x) \sim \sum_{k=2}^{\infty} \varepsilon^k g_k(t,x,\frac{x-ct}{\varepsilon}),$$

with profiles f_k and g_k belonging to S^*, that is exponentially decaying to 0 as Y tends to $+\infty$, with zero mean and vanishing identically in the past:

$$\forall t \leq 0 \quad f^\varepsilon(t) = g^\varepsilon(t) = 0.$$

Remark 2.2.1. *The order of magnitude $U = O(\varepsilon^2)$ and $f, g = O(\varepsilon^2)$ as $\varepsilon \to 0$ corresponds to the regime of weakly nonlinear geometric optics where the nonlinear effects are present in the propagation of the leading term U_2.*

2.2.2 Main results

Theorem 2.2.2. *The profiles of main order u_2 and v_2 belong to S^*, that is are purely localized on the boundary: $\underline{u}_2$ and $\underline{v}_2$ vanish.*

u_2^* and v_2^* are determined by a scalar unknown

$$\alpha_2(t,x,\theta) = \sum_{n\in\mathbb{Z}^*} \alpha_2(n,t,x)e^{in\theta}$$

(see section 2.4 for an explicit form of u_2^* and v_2^*), which solves an equation of the form

$$\partial_t \alpha_2 + c\partial_x \alpha_2 + a(\alpha_2, \alpha_2) = l(f_2, g_2) \qquad (2.2.2)$$

where $l(f_2, g_2)$, up to a multiplicative constant, has Fourier coefficients of order n equal to $\frac{p_1}{|n|} f_2^n + \frac{q}{in} g_2^n$ (p_1 and q given by (2.4.4) and (2.4.6)) and a is a nonlocal bilinear form such that the Fourier coefficients of order n of $a(\alpha_2, \alpha_2)$ are equal to

$$\sum_{k'=1}^{k} \Lambda_1(k,k')k'(k-k')\alpha_2(k')\alpha_2(k-k') + \sum_{k'=k}^{\infty} \Lambda_2(k,k')(k-k')\alpha_2(k')\alpha_2(k-k'),$$

where the expressions of the kernels Λ_1 and Λ_2 are given in [54].

This theorem is proved in section 2.8 and 2.9: we show that $\underline{u}_2$ and $\underline{v}_2$ solve the linearized equations of elasticity with homogeneous boundary conditions, so that they must vanish. The relation between u_2^*, v_2^* and α_2 as well as the equation for α_2 are obtained in [54].

The main results of the chapter are the following theorem and corollary.

Theorem 2.2.3. *The profiles of higher order u_3 and v_3 are of the form*

$$u_3 = \underline{u}_3 + u_3^* \quad v_3 = \underline{v}_3 + v_3^*,$$

with $u_3^*, v_3^* \in S^*$ and $\underline{u}_3, \underline{v}_3 \in \underline{S}$ satisfying the equations

$$\partial_{tt}\underline{u}_3 - r\partial_{xx}\underline{u}_3 - (r-1)\partial_{xy}\underline{v}_3 - \partial_{yy}\underline{u}_3 = 0 \tag{2.2.3a}$$
$$\partial_{tt}\underline{v}_3 - \partial_{xx}\underline{v}_3 - (r-1)\partial_{xy}\underline{u}_3 - r\partial_{yy}\underline{v}_3 = 0, \tag{2.2.3b}$$

on $\{y > 0\}$ and the boundary conditions

$$\partial_y\underline{u}_3 + \partial_x\underline{v}_3 = \underbrace{-\frac{2}{r}\frac{q}{p_1+p_2}C_r}_{\neq 0}\sum_{n\in Z^*}|n|\partial_x(|\alpha_2(n,t,x)|)^2 \tag{2.2.4}$$

$$(r-2)\partial_x\underline{u}_3 + r\partial_y\underline{v}_3 = 0, \tag{2.2.5}$$

on $y = 0$.

The equation satisfied by $\underline{u}_3$ and $\underline{v}_3$ is obtained in section 2.10 and the boundary condition is obtained in section 2.9.

Remark 2.2.4. *u_3^* and v_3^* are determined by u_2, v_2 and a scalar unknown $\alpha_3(t,x,\theta)$, which solves the linearized equation of* (2.2.2)*.*

Corollary 2.2.5. *If f_2 and g_2 satisfy $\partial_x l(f_2, g_2)|_{t=0} \neq 0$, then $\begin{pmatrix} \underline{u}_3 \\ \underline{v}_3 \end{pmatrix}$ does not vanish.*

Proof. Since u_2 and v_2, and thus α_2, vanish in the past, we have

$$\partial_t\alpha_2|_{t=0} = l(f_2, g_2)|_{t=0},$$

and therefore, for t small

$$\alpha_2 \sim tl(f_2, g_2),$$

we then obtain that $\partial_x l(f_2, g_2)|_{t=0} \neq 0$ yields $\sum_{n\in Z^*}|n|\partial_x(|\alpha_2(n,t,x)|)^2 \neq 0$.

The right-hand side term of the first boundary condition satisfied by $\underline{u}_3$ and $\underline{v}_3$ does not vanish, we then obtain that $\begin{pmatrix} \underline{u}_3 \\ \underline{v}_3 \end{pmatrix}$ does not vanish. $\square$

Remark 2.2.6. *For example, we can take f_1 and f_2 such that their Fourier coefficients of order* 1 *satisfy $\partial_x\big(p_1 f_2^1 - iqg_2^1\big)|_{t=0} \neq 0$ in order to have $\partial_x l(f_2, g_2)|_{t=0} \neq 0$ and thus $\begin{pmatrix} \underline{u}_3 \\ \underline{v}_3 \end{pmatrix} \neq 0$.*

2.3 The cascade of equations and boundary conditions

We introduce the linear terms:

$$L_{ff}(U) := \begin{pmatrix} (c^2 - r)\partial_{\theta\theta}u - (r-1)\partial_{\theta Y}v - \partial_{YY}u \\ (c^2 - 1)\partial_{\theta\theta}v - (r-1)\partial_{\theta Y}u - r\partial_{YY}v \end{pmatrix}$$
$$L_{fs}(U) := \begin{pmatrix} -2c\partial_{\theta t}u - 2r\partial_{\theta x}u - (r-1)\partial_{\theta y}v - (r-1)\partial_{Yx}v - 2\partial_{Yy}u \\ -2c\partial_{\theta t}v - 2\partial_{\theta x}v - (r-1)\partial_{\theta y}u - (r-1)\partial_{Yx}u - 2r\partial_{Yy}v \end{pmatrix}$$
$$L_{ss}(U) := \begin{pmatrix} \partial_{tt}u - r\partial_{xx}u - (r-1)\partial_{xy}v - \partial_{yy}u \\ \partial_{tt}v - \partial_{xx}v - (r-1)\partial_{xy}u - r\partial_{yy}v \end{pmatrix}$$

$$l_f(U) := \begin{pmatrix} \partial_Y u + \partial_\theta v \\ (r-2)\partial_\theta u + r\partial_Y v \end{pmatrix}$$
$$l_s(U) := \begin{pmatrix} \partial_y u + \partial_x v \\ (r-2)\partial_x u + r\partial_y v \end{pmatrix}$$

and the quadratic terms

$$\begin{aligned} A(\partial_{X_{1,2}}; \partial_{Z_{1,2}}; \partial_{W_{1,2}})(U_m, U_l) := {}& \partial_{X_1} C(\partial_{Z_{1,2}}; \partial_{W_{1,2}})(U_m, U_l) \\ & + \partial_{X_2} D(\partial_{Z_{1,2}}, \partial_{W_{1,2}})(U_m, U_l), \end{aligned}$$

with

$$C(\partial_{Z_{1,2}}; \partial_{W_{1,2}})(U_m, U_l) := \begin{pmatrix} \partial_{Z_2}u_m \partial_{W_1} v_l \\ \partial_{Z_2}u_m(\partial_{W_1}u_l + \partial_{W_2}v_l) \end{pmatrix},$$
$$D(\partial_{Z_{1,2}}; \partial_{W_{1,2}})(U_m, U_l) := \begin{pmatrix} \partial_{Z_1}v_m(\partial_{W_1}u_l + \partial_{W_2}v_l) \\ \partial_{Z_2}u_m \partial_{W_1} v_l \end{pmatrix},$$

and

$$\begin{aligned} A_{fff} :={}& A(\partial_{\theta,Y}; \partial_{\theta,Y}; \partial_{\theta,Y}) \\ A_{ffs} :={}& A(\partial_{\theta,Y}; \partial_{\theta,Y}; \partial_{x,y}) + A(\partial_{\theta,Y}; \partial_{x,y}; \partial_{\theta,Y}) + A(\partial_{x,y}; \partial_{\theta,Y}; \partial_{\theta,Y}) \\ A_{fss} :={}& A(\partial_{\theta,Y}; \partial_{x,y}; \partial_{x,y}) + A(\partial_{x,y}; \partial_{\theta,Y}; \partial_{x,y}) + A(\partial_{x,y}; \partial_{x,y}; \partial_{\theta,Y}) \\ A_{sss} :={}& A(\partial_{x,y}; \partial_{x,y}; \partial_{x,y}) \end{aligned}$$

$$\begin{aligned} D_{ff} :={}& D(\partial_{\theta,Y}; \partial_{\theta,Y}) \\ D_{fs} :={}& D(\partial_{\theta,Y}; \partial_{x,y}) + D(\partial_{x,y}; \partial_{\theta,Y}) \\ D_{ss} :={}& D(\partial_{x,y}; \partial_{x,y}). \end{aligned}$$

Plugging the expression (2.2.1) of U^ε into the equations and boundary conditions and collecting the powers of ε yield, for all $k \geq 2$,

$$L_{ff}(U_k) + L_{fs}(U_{k-1}) + L_{ss}(U_{k-2}) = \sum_{m+l=k+1} A_{fff}(U_m, U_l)$$
$$+ \sum_{m+l=k} A_{ffs}(U_m, U_l) + \sum_{m+l=k-1} A_{fss}(U_m, U_l) + \sum_{m+l=k-2} A_{sss}(U_m, U_l),$$

on$\{Y > 0, y > 0\}$ and

$$l_f(U_k) + l_s(U_{k-1}) = - \sum_{m+l=k+1} D_{ff}(U_m, U_l) - \sum_{m+l=k} D_{fs}(U_m, U_l)$$
$$- \sum_{m+l=k-1} D_{ss}(U_m, U_l) + \begin{pmatrix} f_{k-1} \\ g_{k-1} \end{pmatrix},$$

on $Y = y = 0$.

We thus obtain:

$$R(u_k, v_k) = H_{k-1}(u_{k-1}, v_{k-1}, \ldots, u_2, v_2) \tag{2.3.1a}$$
$$T(u_k, v_k) = K_{k-1}(u_{k-1}, v_{k-1}, \ldots, u_2, v_2) \tag{2.3.1b}$$

on $\{Y > 0, y > 0\}$,
where

$$R(u, v) = (c^2 - r)\partial_{\theta\theta} u - (r - 1)\partial_{\theta Y} v - \partial_{YY} u \tag{2.3.2}$$
$$T(u, v) = (c^2 - 1)\partial_{\theta\theta} v - (r - 1)\partial_{\theta Y} u - r\partial_{YY} v \tag{2.3.3}$$

and

$$r(u_k, v_k) = h_{k-1}(u_2, v_2, \ldots, u_{k-1}, v_{k-1}) \tag{2.3.4a}$$
$$t(u_k, v_k) = k_{k-1}(u_2, v_2, \ldots, u_{k-1}, v_{k-1}) \tag{2.3.4b}$$

on $Y = y = 0$,
where

$$r(u, v) = \partial_Y u + \partial_\theta v \tag{2.3.5}$$
$$t(u, v) = (r - 2)\partial_\theta u + r\partial_Y v. \tag{2.3.6}$$

The expressions of the right-hand sides are given by:

$$\begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix} = -L_{fs}(U_{k-1}) - L_{ss}(U_{k-2}) + \sum_{m+l=k+1} A_{fff}(U_m, U_l)$$
$$+ \sum_{m+l=k} A_{ffs}(U_m, U_l) + \sum_{m+l=k-1} A_{fss}(U_m, U_l) + \sum_{m+l=k-2} A_{sss}(U_m, U_l)$$

$$\begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix} = -l_s(U_{k-1}) - \sum_{m+l=k+1} D_{ff}(U_m, U_l) - \sum_{m+l=k} D_{fs}(U_m, U_l)$$
$$- \sum_{m+l=k-1} D_{ss}(U_m, U_l) + \begin{pmatrix} f_{k-1} \\ g_{k-1} \end{pmatrix},$$

In particular, for $k = 1, 2, 3$ and 4, the expressions of H_k, K_k, h_k and k_k are given by:

$$H_1 = K_1 = 0 \tag{2.3.7}$$
$$h_1 = k_1 = 0 \tag{2.3.8}$$

$$\begin{aligned} H_2 =& 2c\partial_{\theta t} u_2 + 2r\partial_{\theta x} u_2 + (r-1)\partial_{\theta y} v_2 + (r-1)\partial_{Yx} v_2 + 2\partial_{Yy} u_2 \\ &+ \partial_\theta \big[\partial_Y u_2 \partial_\theta v_2\big] + \partial_Y \big[\partial_\theta v_2 (\partial_\theta u_2 + \partial_Y v_2)\big] \end{aligned} \tag{2.3.9}$$

$$\begin{aligned} K_2 =& 2c\partial_{\theta t} v_2 + 2\partial_{\theta x} v_2 + (r-1)\partial_{\theta y} u_2 + (r-1)\partial_{Yx} u_2 + 2r\partial_{Yy} v_2 \\ &+ \partial_\theta \big[\partial_Y u_2 (\partial_\theta u_2 + \partial_Y v_2)\big] + \partial_Y \big[\partial_Y u_2 \partial_\theta v_2\big] \end{aligned} \tag{2.3.10}$$

$$h_2 = -\partial_y u_2 - \partial_x v_2 - \partial_\theta v_2 (\partial_\theta u_2 + \partial_Y v_2) + f_2 \tag{2.3.11}$$
$$k_2 = -(r-2)\partial_x u_2 - r\partial_y v_2 - \partial_Y u_2 \partial_\theta v_2 + g_2 \tag{2.3.12}$$

$$\begin{aligned} H_3 =& 2c\partial_{\theta t} u_3 + 2r\partial_{\theta x} u_3 + (r-1)\partial_{\theta y} v_3 + (r-1)\partial_{Yx} v_3 + 2\partial_{Yy} u_3 \\ &- \partial_{tt} u_2 + r\partial_{xx} u_2 + (r-1)\partial_{xy} v_2 + \partial_{yy} u_2 \\ &+ \partial_x \big[\partial_Y u_2 \partial_\theta v_2\big] + \partial_y \big[\partial_\theta v_2 (\partial_\theta u_2 + \partial_Y v_2)\big] \\ &+ \partial_\theta \big[\partial_Y u_2 (\partial_\theta v_3 + \partial_x v_2) + \partial_\theta v_2 (\partial_Y u_3 + \partial_y u_2)\big] \\ &+ \partial_Y \big[\partial_\theta v_2 (\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) + (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2)\big] \end{aligned} \tag{2.3.13}$$

$$
\begin{aligned}
K_3 =& 2c\partial_{\theta t}v_3 + 2\partial_{\theta x}v_3 + (r-1)\partial_{\theta y}u_3 + (r-1)\partial_{Yx}u_3 + 2r\partial_{Yy}v_3 \\
&- \partial_{tt}v_2 + \partial_{xx}v_2 + (r-1)\partial_{xy}u_2 + r\partial_{yy}v_2 \\
&+ \partial_x\big[\partial_Y u_2(\partial_\theta u_2 + \partial_Y v_2)\big] + \partial_y\big[\partial_Y u_2\partial_\theta v_2\big] \\
&+ \partial_\theta\big[\partial_Y u_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) + (\partial_Y u_3 + \partial_y u_2)(\partial_\theta u_2 + \partial_Y v_2)\big] \\
&+ \partial_Y\big[\partial_\theta v_2(\partial_Y u_3 + \partial_y u_2) + \partial_Y u_2(\partial_\theta v_3 + \partial_x v_2)\big]
\end{aligned}
\tag{2.3.14}
$$

$$
\begin{aligned}
h_3 =& -\partial_y u_3 - \partial_x v_3 \\
&- \partial_\theta v_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) - (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2) + f_3
\end{aligned}
\tag{2.3.15}
$$

$$
\begin{aligned}
k_3 =& -(r-2)\partial_x u_3 - r\partial_y v_3 \\
&- \partial_\theta v_2(\partial_Y u_3 + \partial_y u_2) - \partial_Y u_2(\partial_\theta v_3 + \partial_x v_2) + g_3.
\end{aligned}
\tag{2.3.16}
$$

$$
\begin{aligned}
H_4 =& 2c\partial_{\theta t}u_4 + 2r\partial_{\theta x}u_4 + (r-1)\partial_{\theta y}v_4 + (r-1)\partial_{Yx}v_4 + 2\partial_{Yy}u_4 \\
&- \partial_{tt}u_3 + r\partial_{xx}u_3 + (r-1)\partial_{xy}v_3 + \partial_{yy}u_3 \\
&+ \partial_x\big[(\partial_Y u_3 + \partial_y u_2)\partial_\theta v_2 + \partial_Y u_2(\partial_\theta v_3 + \partial_x v_2)\big] \\
&+ \partial_y\big[(\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2) + \partial_\theta v_2(\partial_\theta u_3 + \partial_x u_2 + \partial_Y v_3 + \partial_y v_2)\big] \\
&+ \partial_\theta\big[(\partial_Y u_3 + \partial_y u_2)(\partial_\theta v_3 + \partial_x v_2) + \partial_Y u_2(\partial_\theta v_4 + \partial_x v_3) \\
&\qquad + \partial_\theta v_2(\partial_Y u_4 + \partial_y u_3)\big] \\
&+ \partial_Y\big[(\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_3 + \partial_x u_2 + \partial_Y v_3 + \partial_y v_2) \\
&\qquad + \partial_\theta v_2(\partial_\theta u_4 + \partial_Y v_4 + \partial_x u_3 + \partial_y v_3) + (\partial_\theta v_4 + \partial_x v_3)(\partial_\theta u_2 + \partial_Y v_2)\big]
\end{aligned}
\tag{2.3.17}
$$

$$
\begin{aligned}
K_4 =& 2c\partial_{\theta t}v_4 + 2\partial_{\theta x}v_4 + (r-1)\partial_{\theta y}u_4 + (r-1)\partial_{Yx}u_4 + 2r\partial_{Yy}v_4 \\
&- \partial_{tt}v_3 + \partial_{xx}v_3 + (r-1)\partial_{xy}u_3 + r\partial_{yy}v_3 \\
&+ \partial_x\big[(\partial_Y u_3 + \partial_y u_2)(\partial_\theta u_2 + \partial_Y v_2) + \partial_Y u_2(\partial_\theta u_3 + \partial_x u_2 + \partial_Y v_3 + \partial_y v_2)\big] \\
&+ \partial_y\big[(\partial_Y u_3 + \partial_y u_2)\partial_\theta v_2 + \partial_Y u_2(\partial_\theta v_3 + \partial_x v_2)\big] \\
&+ \partial_\theta\big[(\partial_Y u_3 + \partial_y u_2)(\partial_\theta u_3 + \partial_x u_2 + \partial_Y v_3 + \partial_y v_2) \\
&\qquad + \partial_Y u_2(\partial_\theta u_4 + \partial_Y v_4 + \partial_x u_3 + \partial_y v_3) + (\partial_Y u_4 + \partial_y u_3)(\partial_\theta u_2 + \partial_Y v_2)\big] \\
&+ \partial_Y\big[(\partial_\theta v_3 + \partial_x v_2)(\partial_Y u_3 + \partial_y u_2) + \partial_\theta v_2(\partial_Y u_4 + \partial_y u_3) \\
&\qquad + \partial_Y u_2(\partial_\theta v_4 + \partial_x v_3)\big]
\end{aligned}
\tag{2.3.18}
$$

$$
\begin{aligned}
h_4 = &-\partial_y u_4 - \partial_x v_4 \\
&- (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_3 + \partial_x u_2 + \partial_Y v_3 + \partial_y v_2) \\
&- \partial_\theta v_2(\partial_\theta u_4 + \partial_Y v_4 + \partial_x u_3 + \partial_y v_3) - (\partial_\theta v_4 + \partial_x v_3)(\partial_\theta u_2 + \partial_Y v_2) + f_4
\end{aligned} \tag{2.3.19}
$$

$$
\begin{aligned}
k_4 = &-(r-2)\partial_x u_4 - r\partial_y v_4 \\
&- (\partial_\theta v_3 + \partial_x v_2)(\partial_Y u_3 + \partial_y u_2) \\
&- \partial_\theta v_2(\partial_Y u_4 + \partial_y u_3) - \partial_Y u_2(\partial_\theta v_4 + \partial_x v_3) + g_4.
\end{aligned} \tag{2.3.20}
$$

2.4 Form of the oscillatory parts of the profile U_2, that is of the Fourier coefficients U_2^n, $n \neq 0$

For $n \neq 0$, u_2^n and v_2^n satisfy the equations

$$-\partial_{YY} u_2^n - i(r-1)n\partial_Y v_2^n - (c^2 - r)n^2 u_2^n = 0 \tag{2.4.1a}$$

$$-r\partial_{YY} v_2^n - i(r-1)n\partial_Y u_2^n - (c^2 - 1)n^2 v_2^n = 0 \tag{2.4.1b}$$

on $Y > 0$ and the boundary conditions

$$\partial_{\tilde{Y}} u_2^n - \tilde{v}_2^n = 0 \tag{2.4.2a}$$

$$(r-2)u_2^n + r\partial_{\tilde{Y}} \tilde{v}_2^n = 0 \tag{2.4.2b}$$

on $Y = 0$.

2.4.1 Equations for Fourier coefficients u_2^n, v_2^n, $n \neq 0$

For $n \neq 0$, we introduce

$$\tilde{Y} = |n|Y, \quad \tilde{v}_2^n = -i\mathrm{sign}(n)v_2^n,$$

we obtain

$$-\partial_{\tilde{Y}\tilde{Y}} u_2^n + (r-1)\partial_{\tilde{Y}} \tilde{v}_2^n - (c^2 - r)u_2^n = 0 \tag{2.4.3a}$$

$$-r\partial_{\tilde{Y}\tilde{Y}} \tilde{v}_2^n - (r-1)\partial_{\tilde{Y}} u_2^n - (c^2 - 1)\tilde{v}_2^n = 0. \tag{2.4.3b}$$

We have considered $\theta = \frac{x-ct}{\varepsilon}$. For the general phase $\theta = \frac{\xi x + \tau t}{\varepsilon}$, we obtain, for $n \neq 0$

$$
\begin{aligned}
-\partial_{\tilde{Y}\tilde{Y}} u_2^n + (r-1)\xi\partial_{\tilde{Y}} \tilde{v}_2^n - (\tau^2 - r\xi^2)u_2^n = 0 \\
-r\partial_{\tilde{Y}\tilde{Y}} \tilde{v}_2^n - (r-1)\xi\partial_{\tilde{Y}} u_2^n - (\tau^2 - \xi^2)\tilde{v}_2^n = 0.
\end{aligned}
$$

Thus, introducing $\tilde{U}_2^n = \begin{pmatrix} u_2^n \\ \tilde{v}_2^n \end{pmatrix}$, we have

$$\partial_{\tilde{Y}\tilde{Y}} \tilde{U}_2^n - B \partial_{\tilde{Y}} \tilde{U}_2^n - D \tilde{U}_2^n = 0,$$

with $D = \begin{pmatrix} -(\tau^2 - r\xi^2) & 0 \\ 0 & -\frac{\tau^2-\xi^2}{r} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & (r-1)\xi \\ (\frac{1}{r}-1)\xi & 0 \end{pmatrix}$.

We obtain the first order equation satisfied by $\check{U}_2^n = \begin{pmatrix} \tilde{U}_2^n \\ \partial_{\tilde{Y}} \tilde{U}_2^n \end{pmatrix}$:

$$\partial_{\tilde{Y}} \check{U}_2^n - A \check{U}_2^n = 0,$$

with $A = \begin{pmatrix} O & I_2 \\ D & B \end{pmatrix}$. We denote $A(-c, 1)$ by A_0.

The negative eigenvalues of A are defined by

$$\left\{ \begin{array}{l} \lambda_1 < 0 \\ \lambda_1^2 = \xi^2 - \tau^2 \end{array} \right. \qquad \left\{ \begin{array}{l} \lambda_2 < 0 \\ \lambda_2^2 = \xi^2 - \frac{\tau^2}{r} \end{array} \right.$$

We denote by p_1 (resp. p_2) $\lambda_1(-c,1)$ (resp. $\lambda_2(-c,1)$), that is we define p_1 and p_2 by

$$p_1^2 = 1 - c^2, \quad p_1 < 0 \tag{2.4.4}$$

$$p_2^2 = 1 - \frac{c^2}{r}, \quad p_2 < 0. \tag{2.4.5}$$

We also introduce q defined by

$$q^2 = p_1 p_2, \quad q > 0. \tag{2.4.6}$$

The corresponding right eigenvectors are

$$R_1 = \begin{pmatrix} \lambda_1 \\ -\xi \\ \lambda_1^2 \\ -\lambda_1 \xi \end{pmatrix} \qquad R_2 = \begin{pmatrix} -\xi \\ \lambda_2 \\ -\lambda_2 \xi \\ \lambda_2^2 \end{pmatrix}.$$

We denote by R_1^0 (resp. R_2^0) $R_1(-c,1)$ (resp. $R_2(-c,1)$).

From the expressions of R_1^0 and R_2^0, we obtain the following form of u_2^n and v_2^n:

$$u_2^n(t,x,Y) = q(q\alpha_2(n,t,x)e^{p_1 \tilde{Y}} - \beta_2(n,t,x)e^{p_2 \tilde{Y}}) \tag{2.4.7}$$

$$\tilde{v}_2^n(t,x,Y) = p_2(-\alpha_2(n,t,x)e^{p_1 \tilde{Y}} + q\beta_2(n,t,x)e^{p_2 \tilde{Y}}). \tag{2.4.8}$$

Remark 2.4.1. *u_2 and v_2 being real, α_2 and β_2 satisfy*

$$\forall n \in \mathbb{Z},\ \alpha_2(-n,t,x) = \overline{\alpha}_2(n,t,x), \quad \beta_2(-n,t,x) = \overline{\beta}_2(n,t,x),$$

where $\overline{z}$ denotes the complex conjugate of z.

2.4.2 The boundary conditions

The boundary conditions read

$$\begin{aligned} \partial_{\tilde{Y}} u_2^n - \xi \tilde{v}_2^n &= 0 \\ \xi(r-2)u_2^n + r\partial_{\tilde{Y}}\tilde{v}_2^n &= 0 \end{aligned}$$

that is

$$C\check{U}_2^n = 0,$$

with $C = \begin{pmatrix} D & I_2 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & -\xi \\ \xi(1-\frac{2}{r}) & 0 \end{pmatrix}$. We denote $C(-c,1)$ by C_0.

We introduce

$$\begin{aligned} \phi(\tau,\xi) &= \det[CR_1, CR_2] \\ &= \frac{1}{r}[(2\xi^2-\tau^2)^2 - 4\lambda_1\lambda_2\xi^2] \\ &= \frac{1}{r}[(2\xi^2-\tau^2)^2 - 4\sqrt{\xi^2-\tau^2}\sqrt{\xi^2-\frac{\tau^2}{r}}\xi^2]. \end{aligned}$$

There exists a surface wave related to the phase $\varphi = x - ct$ if and only if there is a non trivial oscillating solution of the equations (2.4.1) and the boundary conditions (2.4.2), that is if and only if

$$\phi(-c,1) = 0.$$

The following assumption has thus to be satisfied in order to have the existence of surface waves:

Assumption 2.4.2. *We assume that*

$$(2-c^2)^2 = 4q^2. \tag{2.4.9}$$

Since q and $2-c^2$ are both positive, we have $2-c^2 = 2q$.

ϕ is homogeneous in (τ,ξ) of degree 4. $\phi(-c,1) = 0$ yields $\phi(\tau,\xi) = e(\tau,\xi)(\tau + c\xi)$, with e such that $e(-c,1) \neq 0$. From the first chapter (see section 1.6), we thus obtain that the speed of propagation is equal to c: c is the Rayleigh speed.

The matrix $M_0 = [C_0 R_1^0, C_0 R_2^0] = \begin{pmatrix} 2-c^2 & -2p_2 \\ -\frac{2}{r}p_2 & \frac{1}{r}(2-c^2) \end{pmatrix}$ has a kernel and a cokernel equal to

$$\mathrm{Ker} M_0 = \mathrm{Span}\begin{pmatrix} 2p_2 \\ 2-c^2 \end{pmatrix} \tag{2.4.10}$$

$$\mathrm{Coker} M_0 = \mathrm{Span}\begin{pmatrix} p_1 & rq \end{pmatrix}. \tag{2.4.11}$$

The boundary conditions satisfied by u_2^n and v_2^n thus yield

$$\forall t, x, y \ \ \alpha_2(n,t,x) = \beta_2(n,t,x). \tag{2.4.12}$$

Therefore

$$u_2^n(t,x,Y) = q\alpha_2(n,t,x)(qe^{p_1\tilde{Y}} - e^{p_2\tilde{Y}}) \tag{2.4.13}$$

$$v_2^n(t,x,Y) = i\mathrm{sign}(n)p_2\alpha_2(n,t,x)(-e^{p_1\tilde{Y}} + qe^{p_2\tilde{Y}}). \tag{2.4.14}$$

2.5 Form of U_3^n, $n \neq 0$, and equation for α_2

2.5.1 The equations

2.5.2 Form of the profile U_3 and resolubility condition

a) Form of the profile U_3

We seek a solution $\check{U}_3^n$ of the equation

$$\partial_{\tilde{Y}}\check{U}_3^n - A_0\check{U}_3^n = \check{F}_2^n,$$

in $Y > 0$,
under the form $\check{U}_3^n = \check{U}_3^{n,hom} + \check{U}_3^{n,part}$, with $\check{U}_3^{n,hom}$ a homogeneous solution of the equation, that is satisfying $\partial_{\tilde{Y}}\check{U}_3^{n,hom} - A_0\check{U}_3^{n,hom} = 0$ and $\check{U}_3^{n,part}$ a particular solution, that is satisfying $\partial_{\tilde{Y}}\check{U}_3^{n,part} - A_0\check{U}_3^{n,part} = \check{F}_2^n$.

$\check{U}_3^{n,hom}$ is of the form

$$\check{U}_3^{n,hom} = p_2\alpha_3(n,t,x)e^{p_1\tilde{Y}}R_1^0 + q\beta_3(n,t,x)e^{p_2\tilde{Y}}R_2^0.$$

Let us determine a particular solution $\check{U}_3^{n,part}$. We seek $u_3^{n,part}$ and $\tilde{v}_3^{n,part}$ satisfying the equations

$$\begin{aligned} -\partial_{\tilde{Y}\tilde{Y}} u_3^{n,part} + (r-1)\partial_{\tilde{Y}} \tilde{v}_3^{n,part} - (c^2 - r) u_3^{n,part} &= \frac{H_2^n}{|n|^2} \\ -r\partial_{\tilde{Y}\tilde{Y}} \tilde{v}_3^{n,part} - (r-1)\partial_{\tilde{Y}} u_3^{n,part} - (c^2 - 1)\tilde{v}_3^{n,part} &= \frac{K_2^n}{in|n|}. \end{aligned}$$

Grouping the terms proportional to $e^{p_1\tilde{Y}}$ and to $e^{p_2\tilde{Y}}$, we write $\frac{H_2^n}{|n|^2}$ and $\frac{K_2^n}{in|n|}$ under the form

$$\begin{aligned} \frac{H_2^n}{|n|^2} &= \frac{i}{n}\Big[L_1 e^{p_1\tilde{Y}} + L_2 e^{p_2\tilde{Y}} + \sum_{i,n'} R_i(n,n')e^{T_i(n,n')\tilde{Y}}\Big] \\ \frac{K_2^n}{in|n|} &= \frac{i}{n}\Big[M_1 e^{p_1\tilde{Y}} + M_2 e^{p_2\tilde{Y}} + \sum_{i,n'} S_i(n,n')e^{T_i(n,n')\tilde{Y}}\Big]. \end{aligned}$$

The final terms represent symbolically the terms which are not proportional to $e^{p_1\tilde{Y}}$ or to $e^{p_2\tilde{Y}}$. The coefficients and exponents in these terms are given in [54].

We introduce $I_\alpha(n)$ and $J_\alpha(n)$ defined by

$$I_\alpha(n) = \sum_{n'=1}^{n} n'(n-n')\alpha(n')\alpha(n-n')$$

$$J_\alpha(n) = \sum_{n'=n}^{\infty} n'(n-n')\alpha(n')\alpha(n-n').$$

The coefficients of $e^{p_1\tilde{Y}}$ and of $e^{p_2\tilde{Y}}$ are given by:

$$\begin{aligned} L_1 &= 2cq^2\partial_t\alpha_2 + (r+1)q^2\partial_x\alpha_2 + q^4 I_{\alpha_2}(n) \\ L_2 &= -2cq\partial_t\alpha_2 - (2r + p_2^2 - rp_2^2)q\partial_x\alpha_2 + q^2p_2^2(2-p_2^2)I_{\alpha_2}(n) \\ M_1 &= -2cp_2\partial_t\alpha_2 - p_2(2 + rp_1^2 - p_1^2)\partial_x\alpha_2 - q^4 p_1 I_{\alpha_2}(n) \\ M_2 &= 2cp_2q\partial_t\alpha_2 + (r+1)qp_2\partial_x\alpha_2 + q^2p_2(1-2p_2^2)I_{\alpha_2}(n). \end{aligned}$$

Writing $u_3^{n,part}$ and $\tilde{v}_3^{n,part}$ under the form

$$\begin{aligned} u_3^{n,part} &= (A_1\tilde{Y} + U_1)e^{p_1\tilde{Y}} + (A_2\tilde{Y} + U_2)e^{p_2\tilde{Y}} + \sum_{i,n'} W_i(n,n')e^{T_i(n,n')\tilde{Y}} \\ \tilde{v}_3^{n,part} &= (B_1\tilde{Y} + V_1)e^{p_1\tilde{Y}} + (B_2\tilde{Y} + V_2)e^{p_2\tilde{Y}} + \sum_{i,n'} X_i(n,n')e^{T_i(n,n')\tilde{Y}}, \end{aligned}$$

and plugging these expressions into the equations, we obtain the equations

$$(r-1)[U_1+p_1V_1]-2p_1A_1+(r-1)B_1=\frac{i}{n}L_1$$
$$-(r-1)p_1[U_1+p_1V_1]-2p_1rB_1-(r-1)A_1=\frac{i}{n}M_1$$

$$(r-1)p_2[V_2+p_2U_2]-2p_2A_2+(r-1)B_2=\frac{i}{n}L_2$$
$$-(r-1)[V_2+p_2U_2]-2p_2rB_2-(r-1)A_2=\frac{i}{n}M_2$$

$$A_1+p_1B_1=0$$
$$B_2+p_2A_2=0$$

$$-T_i^2W_i+(r-1)T_iX_i-(c^2-r)W_i=\frac{i}{n}R_i$$
$$-rT_i^2X_i-(r-1)T_iW_i-(c^2-1)X_i=\frac{i}{n}S_i$$

We eliminate A_1 and B_2 and obtain

$$(r-1)[U_1+p_1V_1]+(1-2c^2+r)B_1=\frac{i}{n}L_1$$
$$-(r-1)p_1[U_1+p_1V_1]-(r+1)p_1B_1=\frac{i}{n}M_1$$

$$(r-1)p_2[V_2+p_2U_2]-(1+r)p_2A_2=\frac{i}{n}L_2$$
$$-(r-1)[V_2+p_2U_2]+(r-2c^2+1)A_2=\frac{i}{n}M_2.$$

Therefore

$$\frac{i}{n}(M_1+p_1L_1)=-2c^2p_1B_1$$
$$\frac{i}{n}(L_2+p_2M_2)=-2c^2p_2A_2.$$

We thus obtain

$$
\begin{aligned}
B_1 &= -\frac{i}{n}\frac{M_1 + p_1 L_1}{2p_1 c^2}\\
A_2 &= -\frac{i}{n}\frac{L_2 + p_2 M_2}{2p_2 c^2}\\
A_1 &= -p_1 B_1 = \frac{i}{n}\frac{M_1 + p_1 L_1}{2c^2}\\
B_2 &= -p_2 A_2 = \frac{i}{n}\frac{L_2 + p_2 M_2}{2c^2}
\end{aligned}
$$

Substituting these expressions into the equations satisfied by U_1, V_1, U_2 and V_2, we obtain

$$
\begin{aligned}
U_1 + p_1 V_1 &= \frac{i}{n}\frac{1}{r-1}\Big[\frac{1+r}{2c^2}L_1 + \frac{1}{p_1}\Big(\frac{1+r}{2c^2} - 1\Big)M_1\Big]\\
V_2 + p_2 U_2 &= -\frac{i}{n}\frac{1}{r-1}\Big[\frac{1}{p_2}\Big(\frac{1+r}{2c^2} - 1\Big)L_2 + \frac{1+r}{2c^2}M_2\Big].
\end{aligned}
$$

We take

$$V_1 = U_2 = 0$$

and

$$
\begin{aligned}
U_1 &= \frac{i}{n}\frac{1}{r-1}\Big[\frac{1+r}{2c^2}L_1 + \frac{1}{p_1}\Big(\frac{1+r}{2c^2} - 1\Big)M_1\Big]\\
V_2 &= -\frac{i}{n}\frac{1}{r-1}\Big[\frac{1}{p_2}\Big(\frac{1+r}{2c^2} - 1\Big)L_2 + \frac{1+r}{2c^2}M_2\Big].
\end{aligned}
$$

W_i and X_i are solution of a Cramer system, they are given by:

$$
\begin{aligned}
W_i &= \frac{i}{n}\frac{(1 - c^2 - rT_i)R_i - (r-1)T_i S_i}{r(T_i^2 - p_1^2)(T_i^2 - p_2^2)}\\
X_i &= \frac{i}{n}\frac{(r - c^2 - T_i^2)S_i + (r-1)T_i R_i}{r(T_i^2 - p_1^2)(T_i^2 - p_2^2)}.
\end{aligned}
$$

b) Resolubility condition

$\check{U}_3^n$ satisfies the boundary condition

$$C_0 \check{U}_3^n = G_2^n,$$

on $Y = 0$.

The boundary condition reads

$$\begin{aligned} C_0\check{U}_3^n = G_2^n \Leftrightarrow C_0\check{U}_3^{n,hom} &= G_2^n - C_0\check{U}_3^{n,part} \\ &\Leftrightarrow p_2\alpha_3(n,t,x)C_0R_1^0 + q\beta_3(n,t,x)C_0R_2^0 = G_2^n - C_0\check{U}_3^{n,part} \\ &\Leftrightarrow [C_0R_1^0, C_0R_2^0]\begin{pmatrix} p_2\alpha_3(n,t,x) \\ q\beta_3(n,t,x) \end{pmatrix} = G_2^n - C_0\check{U}_3^{n,part}. \end{aligned}$$

There exist α_3 and β_3 such that the boundary condition is satisfied if and only if

$$G_2^n - C_0\check{U}_3^{n,part} \in \mathrm{Range}[C_0R_1^0, C_0R_2^0]$$

that is, from the expression (2.4.11) of the Cokernel of $[C_0R_1^0, C_0R_2^0]$, if and only if

$$\begin{pmatrix} p_1 & rq \end{pmatrix}(G_2^n - C_0\check{U}_3^{n,part}) = 0.$$

We then obtain the resolubility condition

$$-p_1\tilde{v}_3^{n,part} + p_1\partial_{\tilde{Y}}u_3^{n,part} + q(r-2)u_3^{n,part} + rq\partial_{\tilde{Y}}\tilde{v}_3^{n,part} = p_1\frac{h_2^n}{|n|} + q\frac{k_2^n}{in}. \quad (2.5.1)$$

Consider the case $n > 0$.

From the expressions (2.3.11) and (2.3.12) of h_2 and k_2, we get:

$$\frac{h_2^n}{|n|} = \frac{i}{n}\left[p_2(1-q)\partial_x\alpha_2 + qp_2(q-1)(-1+p_2^2)I_{\alpha_2}(n)\right] + \frac{f_2^n}{|n|}$$

$$\frac{k_2^n}{in} = \frac{i}{n}\left[(r-2)q(q-1)\partial_x\alpha_2 + qp_2(qp_1-p_2)(1-q)[-2J_{\alpha_2}(n) + I_{\alpha_2}(n)\right] + \frac{g_2^n}{in}.$$

For $n < 0$, there are similar expressions, that can be deduced from the relation $\alpha_2(-n) = \bar{\alpha}_2(n)$.

Plugging into the equation (2.5.1) these expressions of $\frac{h_2^n}{|n|}$ and $\frac{k_2^n}{in}$, and the expression of $u_3^{n,part}$ and $\tilde{v}_3^{n,part}$ obtained in the section 2.5.2, we obtain an equation of the form:

$$\partial_t\alpha_2 + c\partial_x\alpha_2 + a(\alpha_2,\alpha_2) = l(f_2, g_2), \quad (2.5.2)$$

where $l(f_2, g_2)$, up to a multiplicative constant, has Fourier coefficients of order n equal to $\frac{p_1}{|n|}f_2^n + \frac{q}{in}g_2^n$. The contributions to the quadratic term $a(\alpha_2, \alpha_2)$ are the quadratic terms R_i and S_i and the factors of $I_{\alpha_2}(n)$ and $J_{\alpha_2}(n)$ in L_1, L_2, M_1 and M_2. The bilinear form $a(\alpha_2,\alpha_2)$ has Fourier coefficients of order n equal to

$$\sum_{k'=1}^{k}\Lambda_1(k,k')k'(k-k')\alpha_2(k')\alpha_2(k-k') + \sum_{k'=k}^{\infty}\Lambda_2(k,k')(k-k')\alpha_2(k')\alpha_2(k-k'),$$

where the expressions of the kernels Λ_1 and Λ_2 are given in [54].

2.6 Equation and boundary condition for Fourier coefficients U_k^0

2.6.1 Equations

u_k^0 and v_k^0 satisfy

$$-\partial_{YY} u_k^0 = H_{k-1}^0 \tag{2.6.1a}$$
$$-r\partial_{YY} v_k^0 = K_{k-1}^0. \tag{2.6.1b}$$

Since $u_k^0, v_k^0 \in S$, we obtain the resolubility conditions

$$\underline{H}_{k-1}^0 = \underline{K}_{k-1}^0 = 0. \tag{2.6.2}$$

Remark 2.6.1. *The resolubility conditions* (2.6.2) *yield equations satisfied by $\underline{u}_{k-3}^0$ and $\underline{v}_{k-3}^0$ on $\{Y > 0, y > 0\}$.*

The equations 2.6.1 yield

$$u_k^0 = \underline{u}_k^0(t, x, y) + u_k^{0,*}(t, x, y, Y) \tag{2.6.3}$$
$$v_k^0 = \underline{v}_k^0(t, x, y) + v_k^{0,*}(t, x, y, Y), \tag{2.6.4}$$

with $\underline{u}_k^0, \underline{v}_k^0 \in \underline{S}$ unknown functions that have to be determined and $u_k^{0,*}, v_k^{0,*} \in S^*$ known functions given by the following expressions:

$$u_k^{0,*} = -\int_Y^\infty \Big(\int_s^\infty H_{k-1}^0(t, x, y, s')\, ds' \Big)\, ds \tag{2.6.5}$$
$$v_k^{0,*} = -\frac{1}{r} \int_Y^\infty \Big(\int_s^\infty K_{k-1}^0(t, x, y, s')\, ds' \Big)\, ds. \tag{2.6.6}$$

2.6.2 Boundary conditions for Fourier coefficients u_k^0 and v_k^0

The boundary conditions for u_k^0 and v_k^0 read

$$\partial_Y u_k^0 = h_{k-1}^0 \tag{2.6.7a}$$
$$r\partial_Y v_k^0 = k_{k-1}^0 \tag{2.6.7b}$$

on $Y = y = 0$. Plugging the expressions (2.6.3) and (2.6.4) in (2.6.7), we obtain

$$\int_0^\infty H_{k-1}^0(t, x, y, s)\, ds = h_{k-1}^0(t, x, y, Y) \tag{2.6.8a}$$
$$\int_0^\infty K_{k-1}^0(t, x, y, s)\, ds = k_{k-1}^0(t, x, y, Y), \quad \text{on } Y = y = 0. \tag{2.6.8b}$$

Remark 2.6.2. *The boundary conditions* (2.6.7) *yield boundary conditions satisfied by $\underline{u}_{k-1}^0$ and $\underline{v}_{k-1}^0$ on $y = 0$.*

2.6.3 Determination of u_l^0 and v_l^0

The Fourier coefficients u_l^0 and v_l^0 are determined in 3 steps. First, from the equations (2.6.1) with $k = l$, satisfied by u_l^0 and v_l^0, we obtain expressions of u_l^0 and v_l^0, where $\underline{u}_l^0 \in \underline{S}$ and $\underline{v}_l^0 \in \underline{S}$ have to be determined. Afterwards, the boundary conditions (2.6.7) with $k = l + 1$ satisfied by u_{l+1}^0 and v_{l+1}^0 yield boundary conditions satisfied by $\underline{u}_l^0$ and $\underline{v}_l^0$ on $y = 0$. Finally, it follows from the equations (2.6.1) with $k = l + 2$, satisfied by u_{l+2}^0 and v_{l+2}^0, the resolubility conditions (2.6.2) $H_{l+1}, K_{l+1} \in S^*$, leading to equations satisfied by $\underline{u}_l^0$ and $\underline{v}_l^0$ on $y > 0$.

2.7 Form of U_2^0

The Fourier coefficients u_2^0 and v_2^0 satisfy

$$\partial_{YY} u_2^0 = 0 \tag{2.7.1a}$$

$$\partial_{YY} v_2^0 = 0 \tag{2.7.1b}$$

on $\{Y > 0, y > 0\}$. Therefore, since $u_2^0 \in S$, $v_2^0 \in S$,

$$u_2^0(t, x, y, Y) = \underline{u}_2(t, x, y) \tag{2.7.2}$$

$$v_2^0(t, x, y, Y) = \underline{v}_2(t, x, y), \tag{2.7.3}$$

with $\underline{u}_2 \in \underline{S}$ and $\underline{v}_2 \in \underline{S}$. The boundary conditions read

$$\partial_Y u_2^0 = 0 \tag{2.7.4a}$$

$$r\partial_Y v_2^0 = 0, \tag{2.7.4b}$$

on $Y = y = 0$.

It follows from the expressions (2.7.2) and (2.7.3) of u_2^0 and v_2^0 that the boundary conditions (2.7.4) are automatically satisfied.

2.8 Form of U_3^0 and boundary condition for $\underline{u}_2$

The equations (2.6.1) for $k = 3$ satisfied by u_3^0 and v_3^0 yield the resolubility conditions

$$\underline{H}_2^0 = \underline{K}_2^0 = 0. \tag{2.8.1}$$

These conditions are automatically satisfied, since all the terms in H_2 and K_2 are derived with respect to θ or Y and thus belong to S^* (for $f = \underline{f} + f^* \in$

$S = \underline{S} + S^*$, $\partial_\theta f = \partial_\theta f^* \in S^*$ and $\partial_Y f = \partial_Y f^* \in S^*$, since the functions in $\underline{S}$ are independent of θ and Y).

From the equations (2.6.1) for $k = 3$, we obtain the expressions (2.6.3) and (2.6.4) of u_3^0 and v_3^0 where $\underline{u}_3, \underline{v}_3 \in \underline{S}$ have to be determined.

The boundary conditions (2.6.7) for $k = 3$ yield

$$\int_0^\infty H_2^0(t, x, 0, s)\, ds = h_2^0(t, x, 0, 0) \tag{2.8.2a}$$

$$\int_0^\infty K_2^0(t, x, 0, s)\, ds = k_2^0(t, x, 0, 0). \tag{2.8.2b}$$

From the expression (2.3.9) of H_2, we obtain

$$H_2^0 = \partial_Y \Big[(r-1)\partial_x v_2^0 + 2\partial_y u_2^0\Big] + \partial_Y \big[[\partial_\theta v_2(\partial_\theta u_2 + \partial_Y v_2)]^0\big],$$

where, for $f \in S$, $[f]^0$ denotes the Fourier coefficient of order 0 of f.

Remark 2.8.1. *For $f = \underline{f} + f^* \in S = \underline{S} + S^*$*

$$\begin{aligned}\int_Y^\infty \partial_Y f(t, x, y, \theta, s)\, ds &= \lim_{Y \to +\infty} f(t, x, y, \theta, Y) - f(t, x, y, \theta, Y) \\ &= \underline{f}(t, x, y) - f(t, x, y, \theta, Y) \\ &= -f^*(t, x, \theta, Y).\end{aligned}$$

From this remark, and since $u_2^0 = \underline{u}_2$ and $v_2^0 = \underline{v}_2$ belong to $\underline{S}$ and $\partial_\theta v_2(\partial_\theta u_2 + \partial_Y v_2)$ belongs to S^*, we obtain

$$\int_Y^\infty H_2^0(t, x, y, s)\, ds = -[\partial_\theta v_2(\partial_\theta u_2 + \partial_Y v_2)]^0. \tag{2.8.3}$$

Since $h_2^0 = -\partial_y u_2^0 - \partial_x v_2^0 - [\partial_\theta v_2(\partial_\theta u_2 + \partial_Y v_2)]^0$, the equation (2.8.2a) yields $\partial_y u_2^0 + \partial_x v_2^0 = 0$, that is, since $u_2^0 = \underline{u}_2$ and $v_2^0 = \underline{v}_2$:

$$\partial_y \underline{u}_2 + \partial_x \underline{v}_2 = 0,$$

on $y = 0$. Similarly, the equation (2.8.2a) yields:

$$(r-2)\partial_x \underline{u}_2 + r\partial_y \underline{v}_2 = 0,$$

on $y = 0$.

We thus obtained that the boundary conditions (2.6.7) for $k = 3$ are equivalent to the boundary conditions on $\underline{u}_2$ and $\underline{v}_2$:

$$\partial_y \underline{u}_2 + \partial_x \underline{v}_2 = 0 \tag{2.8.4a}$$

$$(r-2)\partial_x \underline{u}_2 + r\partial_y \underline{v}_2 = 0 \tag{2.8.4b}$$

on $y = 0$.

2.9 Equation for $\underline{u}_2$ and boundary condition for $\underline{u}_3$

The equations (2.6.1) for $k=4$ satisfied by u_4^0 and v_4^0 yields the resolubility conditions

$$\underline{H}_3^0 = \underline{K}_3^0 = 0. \tag{2.9.1}$$

From the expressions (2.3.13) and (2.3.14) of H_3 and K_3, we then obtain from the resolubility condition (2.9.1) the following equations satisfied by $\underline{u}_2$ and $\underline{v}_2$:

$$\partial_{tt}\underline{u}_2 - r\partial_{xx}\underline{u}_2 - (r-1)\partial_{xy}\underline{v}_2 - \partial_{yy}\underline{u}_2 = 0 \tag{2.9.2a}$$
$$\partial_{tt}\underline{v}_2 - \partial_{xx}\underline{v}_2 - (r-1)\partial_{xy}\underline{u}_2 - r\partial_{yy}\underline{v}_2 = 0, \tag{2.9.2b}$$

on $\{y>0\}$.

From the equations (2.9.2) and the boundary conditions (2.8.4), we then obtain $\underline{u}_2 = 0$ and $\underline{v}_2 = 0$, that is:

$$u_2^0 = v_2^0 = 0. \tag{2.9.3}$$

The boundary conditions (2.6.7) for $k=4$ yield

$$\int_0^\infty H_3^0(t,x,0,s)\,ds = h_3^0(t,x,0,0) \tag{2.9.4a}$$
$$\int_0^\infty K_3^0(t,x,0,s)\,ds = k_3^0(t,x,0,0). \tag{2.9.4b}$$

From the expression (2.3.13) of H_3, and since $u_2^0 = v_2^0 = 0$ and $u_2, v_2 \in S^*$ and thus u_2, v_2 independent of y, we obtain

$$\begin{aligned} H_3^0 = {} & \partial_Y\Big[(r-1)\partial_x v_3^0 + 2\partial_y u_3^0\Big] + \partial_x\big[\partial_Y u_2 \partial_\theta v_2\big]^0 \\ & + \partial_Y\big[\partial_\theta v_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) + (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2)\big]^0, \end{aligned}$$

thus

$$\begin{aligned} \int_Y^\infty H_3^0(t,x,y,s)\,ds = {} & \int_Y^\infty \partial_x\big[\partial_Y u_2 \partial_\theta v_2\big]^0 ds - (r-1)\partial_x v_3^{0,*} - 2\partial_y u_3^{0,*} \\ & - \big[\partial_\theta v_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) + (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2)\big]^0. \end{aligned}$$

From the expression (2.3.15) of h_3, we then obtain that the equation (2.9.4a) is equivalent to

$$\begin{aligned} \partial_y \underline{u}_3 + \partial_x \underline{v}_3 = {} & (r-2)\partial_x v_3^{0,*} \\ & - \int_0^\infty \partial_x\big[\partial_Y u_2 \partial_\theta v_2\big]^0 + \partial_y\big[\partial_\theta v_2(\partial_\theta u_2 + \partial_Y v_2)\big]^0 ds, \end{aligned} \tag{2.9.5}$$

on $Y = y = 0$.

From the definition of $v_3^{0,*}$ (2.6.6):

$$
\begin{aligned}
v_3^{0,*} &= -\frac{1}{r}\int_Y^\infty \Big(\int_s^\infty K_2^0(t,x,y,s')\,ds'\Big)\,ds \\
&= \frac{1}{r}\int_Y^\infty [\partial_Y u_2 \partial_\theta v_2]^0\,ds.
\end{aligned}
$$

Therefore, equation (2.9.5) yields the boundary condition satisfied by $\underline{u}_3$ and $\underline{v}_3$:

$$
\partial_y \underline{u}_3 + \partial_x \underline{v}_3 = -\frac{2}{r}\int_0^\infty \partial_x[\partial_Y u_2 \partial_\theta v_2]^0\,ds, \tag{2.9.6}
$$

on $y = 0$.

Similarly, we obtain from the boundary condition (2.9.4b),

$$
(r-2)\partial_x \underline{u}_3 + r\partial_y \underline{v}_3 = \int_0^\infty \partial_x[(\partial_\theta v_2 - \partial_Y u_2)(\partial_\theta u_2 + \partial_Y v_2)]^0\,ds, \tag{2.9.7}
$$

on $y = 0$.

Plugging the expressions (2.4.7) and (2.4.8) of u_2^n and v_2^n, $n \neq 0$ and since $u_2^0 = v_2^0 = 0$, we obtain:

$$
\begin{aligned}
[\partial_Y u_2 \partial_\theta v_2]^0 &= \sum_{n\in\mathbb{Z}^*} \partial_Y u_2^n(-in)v_2^{-n} \\
&= \sum_{n\in\mathbb{Z}^*} -|n|^2 q p_2 |\alpha_2(n,t,x)|^2 \Big[-qp_1 e^{2|n|p_1 Y} - qp_2 e^{2|n|p_2 Y} \\
&\quad + \big(q^2 p_1 + p_2\big) e^{|n|(p_1+p_2)Y} \Big].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_0^\infty [\partial_Y u_2 \partial_\theta v_2]^0\,ds &= \sum_{n\in\mathbb{Z}^*} -|n| q p_2 |\alpha_2(n,t,x)|^2 \Big[q - \frac{q^2 p_1 + p_2}{p_1 + p_2}\Big] \\
&= \sum_{n\in\mathbb{Z}^*} -|n| \frac{q}{p_1+p_2} |\alpha_2(n,t,x)|^2 \Big[q(p_1p_2 + p_2^2) - q^2 p_1 p_2 - p_2^2\Big] \\
&= \sum_{n\in\mathbb{Z}^*} |n| \frac{q}{p_1+p_2} C_r |\alpha_2(n,t,x)|^2
\end{aligned}
$$

with

$$
C_r = -q(p_1p_2 + p_2^2) + q^2 p_1 p_2 + p_2^2,
$$

that is, since $q^2 = p_1 p_2$, $p_2^2 = 1 - \frac{c^2}{r}$ and, from the equation (2.4.9) satisfied by c, $q = 1 - \frac{c^2}{2}$,

$$\begin{aligned} C_r &= -q^3 - qp_2^2 + q^4 + p_2^2 \\ &= q^3(q-1) + p_2^2(1-q) \\ &= (1-q)(p_2^2 - q^3) \\ &= \frac{c^2}{2}\big[1 - \frac{c^2}{r} - \big(1 - \frac{c^2}{2}\big)^3\big]. \end{aligned}$$

c satisfies the equation (2.4.9), therefore

$$\big(1 - \frac{c^2}{2}\big)^4 = q^4 = p_1^2 p_2^2 = \big(1 - c^2\big)\big(1 - \frac{c^2}{r}\big),$$

thus

$$\begin{aligned} C_r &= \frac{c^2}{2}\Big[\frac{\big(1 - \frac{c^2}{2}\big)^4}{1 - c^2} - \big(1 - \frac{c^2}{2}\big)^3\Big] \\ &= \frac{c^4}{4}\frac{\big(1 - \frac{c^2}{2}\big)^3}{1 - c^2} \end{aligned}$$

Since $1 - c^2 = p_1^2 > 0$, we have $1 - \frac{c^2}{2} > 0$ and therefore

$$C_r = \frac{c^4}{4}\frac{\big(1 - \frac{c^2}{2}\big)^3}{1 - c^2} > 0. \tag{2.9.8}$$

The boundary condition (2.9.6) is thus equivalent to

$$\partial_y \underline{u}_3 + \partial_x \underline{v}_3 = \underbrace{-\frac{2}{r}\frac{q}{p_1 + p_2} C_r}_{\neq 0} \sum_{n \in Z^*} |n| \partial_x(|\alpha_2(n,t,x)|)^2) \tag{2.9.9}$$

on $y = 0$.

We compute the right-hand side term of the boundary condition (2.9.6):

$$\begin{aligned} [(\partial_\theta v_2 - \partial_Y u_2)(\partial_\theta u_2 + \partial_Y v_2)]^0 = \sum_{n \in \mathbb{Z}^*} in|n||\alpha_2(n,t,x)|^2 \\ \Big[p_2\big(-e^{|n|p_1 Y} + qe^{|n|p_2 Y}\big) + q\big(qp_1 e^{|n|p_1 Y} - p_2 e^{|n|p_2 Y}\big)\Big] \\ \Big[q\big(qe^{|n|p_1 Y} - e^{|n|p_2 Y}\big) + p_2\big(-p_1 e^{|n|p_1 Y} + qp_2 e^{|n|p_2 Y}\big)\Big]. \end{aligned}$$

Thus

$$
\begin{aligned}
&[(\partial_\theta v_2 - \partial_Y u_2)(\partial_\theta u_2 + \partial_Y v_2)]^0|_{y=0} \\
&\quad = \sum_{n\in\mathbb{Z}^*} qp_2(p_1^2-1)(p_2^2-1)\underbrace{in|n||\alpha_2(n,t,x)|^2 e^{|n|(p_1+p_2)Y}}_{odd\,with\,respect\,to\,n} \\
&\quad = 0
\end{aligned}
$$

The boundary condition (2.9.7) is thus equivalent to

$$
(r-2)\partial_x \underline{u}_3 + r\partial_y \underline{v}_3 = 0, \tag{2.9.10}
$$

on $y = 0$.

Finally, we obtained the boundary conditions for $\underline{u}_3$ and $\underline{v}_3$:

$$
\partial_y \underline{u}_3 + \partial_x \underline{v}_3 = \underbrace{-\frac{2}{r}\frac{q}{p_1+p_2}C_r}_{\neq 0} \sum_{n\in Z^*} |n|\partial_x(|\alpha_2(n,t,x)|)^2 \tag{2.9.11}
$$

$$
(r-2)\partial_x \underline{u}_3 + r\partial_y \underline{v}_3 = 0, \tag{2.9.12}
$$

on $y = 0$.

The right-hand side term of the first equation does not vanish if and only if $\exists n \in Z^* \partial_x(|\alpha_2(n,t,x)^2) \neq 0$, that is if and only if $\partial_x\big(\frac{1}{2\pi}\int_0^{2\pi} |\alpha_2(t,x,\theta)|^2\, d\theta\big) \neq 0$, where $\alpha_2(t,x,\theta)$ is the function with Fourier coefficients of order $n \neq 0$ equal to $\alpha_2(n,t,x)$ and of order 0 equal to 0.

In this case, we obtain that $\underline{u}_3$ and $\underline{v}_3$ do not vanish.

2.10 Equation for $\underline{u}_3$

The equations (2.6.1) for $k = 5$ satisfied by u_5^0 and v_5^0 yields the resolubility conditions

$$
\underline{H}_4^0 = \underline{K}_4^0 = 0. \tag{2.10.1}
$$

From the expressions (2.3.17) and (2.3.18) of H_4 and K_4, we then obtain from the resolubility condition (2.10.1) the following equations satisfied by $\underline{u}_3$ and $\underline{v}_3$:

$$
\partial_{tt}\underline{u}_3 - r\partial_{xx}\underline{u}_3 - (r-1)\partial_{xy}\underline{v}_3 - \partial_{yy}\underline{u}_3 = 0 \tag{2.10.2a}
$$

$$
\partial_{tt}\underline{v}_3 - \partial_{xx}\underline{v}_3 - (r-1)\partial_{xy} - r\partial_{yy}\underline{v}_3 = 0, \tag{2.10.2b}
$$

on $\{y > 0\}$.

Chapter 3

Reflection of discontinuities for nonlinear weakly stable boundary value problems

This chapter is concerned with the reflection of non linear discontinuous waves, for *weakly* well-posed hyperbolic boundary value problems, satisfying the (WR) condition, which has been introduced in [10, 77], that is in a case where the IBVP is neither strongly stable, nor strongly unstable. We study how the singularities of a striated solution are reflected when the solution hits the boundary. We prove striated estimates and L^∞ estimates and observe the loss of one derivative: we show that a discontinuity of the gradient of the solution across a hyperplane can be reflected in a discontinuity across a hyperplane of the solution itself.

3.1 Introduction

We consider the reflection of non linear discontinuous waves, for *weakly* well-posed boundary value problems. In this paper, we study boundary value problems for wave equations on a half space $\mathbb{R}_t \times \mathbb{R}_x^+ \times \mathbb{R}_y^{d-1}$, with a given integer $d \geq 2$, of the form:

$$q_0(\partial_t, \partial_x, \partial_y)u + l(\partial_t, \partial_x, \partial_y)u = f + F(\,.\,, u) \quad x > 0 \tag{3.1.1a}$$

$$(\partial_x + \beta\partial_t + v \cdot \partial_y)\, u + cu = g + G(\,.\,, u) \quad x = 0, \tag{3.1.1b}$$

where $q_0(\tau, \xi, \eta)$ is a second order strictly hyperbolic homogeneous polynomial and l is a first order homogeneous polynomial (see Assumption 3.2.1). We assume that the (WR) condition (for "weakly stable of real type") which has been introduced in [10, 77], is satisfied. This condition is weaker than the uniform Lopatinski condition which characterizes the strongly well-posed IBVPs. In short, the (WR) condition holds when the weak Lopatinski condition is satisfied and remains satisfied for perturbations. The (WR) condition implies that the IBVP is neither strongly stable, nor strongly unstable (see section 3.2.1). In particular, it implies that there is a loss of one derivative in the main a-priori estimate, compared to strongly stable problems.

The reflections of nonlinear discontinuous waves for problems satisfying the uniform Lopatinski condition is studied in [65, 19]. The framework of *striated solutions* (see [19, 73]) is well adapted to the description of reflection of discontinuities. Striated solutions are regular with respect to a special set of derivatives Y_j, where the independent vector fields Y_j for $j = 1, \ldots, d-1$ are tangent to the striation $\{\varphi_+ = a_+, \varphi_- = a_-\}$, corresponding to the phases:

$$\varphi_\pm = \xi_\pm x + \omega_0 t + k_0 \cdot y, \tag{3.1.2}$$

with *real* frequencies $(\xi_\pm, \omega_0, k_0)$. Striated solutions arise naturally in a variety of physical situations where the solutions are smooth in all but two variables. The phases $\varphi_\pm$ which carry the possible singularities of the solutions are characteristic, i.e. satisfy

$$q_0(\omega_0, \xi_\pm, k_0) = 0. \tag{3.1.3}$$

We assume that $\xi_+ \neq \xi_-$, which, in the classification of boundary frequencies for hyperbolic boundary value problems, means that (ω_0, k_0) is a hyperbolic point.

One of the properties of the problem is that it reads (up to the multiplication by a positive factor):

$$-X_+ X_- u - Q(Y) u = f + F(\,.\,, u) \quad x > 0 \tag{3.1.4}$$

where $Q(Y)$ is a polynomial in $Y = (Y_1, \ldots, Y_{d-1})$ of degree 2 and $X_\pm = \widetilde{X}_\pm + c_\pm$ are first order operators, with principal parts $\widetilde{X}_\pm$ such that $\widetilde{X}_+$ [resp. $\widetilde{X}_-$] is tangent to the foliation $\{\varphi_+ = const\}$ [resp. $\{\varphi_- = const\}$]. This means that

$$\tilde{X}_+(\varphi_+) = 0 \quad \text{and} \quad \tilde{X}_-(\varphi_-) = 0. \tag{3.1.5}$$

Under Assumption 3.2.1, we can label the pluses and minuses so that the field $\widetilde{X}_-$ is *outgoing*, meaning that it propagates signal towards the left in x for increasing times, while the field $\widetilde{X}_+$ is *incoming* meaning that it propagates to the right in x.

In this framework, the main question addressed in this paper is the following : *if a solution u is regular with respect to the Y_j and X_-, and singular in X_+ in the past, how is this singularity reflected when it hits the boundary? Typically, if u (or ∇u) has a discontinuity across $\Sigma_- = \{\varphi_- = 0\}$ in the past, how is it reflected?*

This question is studied in [19, 65] when the IBVP is strongly stable, and the main result can be summarized in the following manner: discontinuities of u (resp. ∇u) across $\Sigma_- = \{\varphi_- = 0\}$ are reflected in discontinuities of u (resp. ∇u) across $\Sigma_+ = \{\varphi_+ = 0\}$.

In this paper, we study the case where the IBVP is only weakly stable, satisfying the (WR) condition, and where *the tangential frequency (ω_0, k_0) is precisely a point where the Lopatinski condition fails.* In this case, it turns out that the boundary condition reads:

$$X_+ u + \check{v} \cdot Y u + \check{c} u = g + G(\,.\,, u), \quad x = 0. \tag{3.1.6}$$

Moreover, the coefficient of ∂_t in the tangent vector field $\check{v} \cdot Y$ does not vanish.

The main outcome of the present paper is that, in this case, *discontinuities of ∇u across $\Sigma_- = \{\varphi_- = 0\}$ are reflected in discontinuities of u across $\Sigma_+ = \{\varphi_+ = 0\}$*, see Theorem 3.2.27 for a precise statement. The spirit of this result is in complete accordance with the above mentioned loss of one derivative for the problems satisfying the (WR) condition.

The main step in the analysis is to solve the problem in spaces of striated functions. Denote by $H^{0,m}(\mathbb{R}^d_{t,y})$ space of functions in $L^2(\mathbb{R}^d)$, such that all the derivatives, of order less than or equal to m, with respect to the set Y_j, are in $L^2(\mathbb{R}^d)$. Then we study and solve locally in time the problem in spaces of functions $u \in L^\infty \cap L^2_x(H^{0,m}(\mathbb{R}^d_{t,y}))$, such that $X_+ u \in L^\infty \cap L^2_x(H^{0,m-2}(\mathbb{R}^d_{t,y}))$, and $u|_{x=0} \in H^{0,m}(\mathbb{R}^d_{t,y})$, provided that m is large enough (see Theorem 3.2.26 for a precise statement). From this, an analysis of piecewise Lipschitz solutions is given in section 3.8.

The nonlinear problem is solved by Picard's iterations, using linear problems of the form

$$q_0(\partial_t, \partial_x, \partial_y) u + l(\partial_t, \partial_x, \partial_y) u = f, \quad x > 0, \tag{3.1.7a}$$

$$(\partial_x + \beta \partial_t + v \cdot \partial_y)\, u + cu = g, \quad x = 0. \tag{3.1.7b}$$

The analysis of such linear problem is twofold : first one proves striated estimates in spaces $L^2_x(H^{0,m}(\mathbb{R}^d_{t,y}))$. Next, knowing these estimates, one proves L^∞ estimates.

Commuting the equations with the vector fields Y_j which are tangent to the boundary, the proof of the striated estimates reduces immediately to the case $m = 0$. L^2 energy estimates are proved in [77] for the wave equation, using an auxiliary strongly well-posed problem for Pu where P is a well chosen operator; the loss of one derivative is observed when passing from estimates for Pu to estimates for u. We stress here that, in contrast with the strongly stable case, the first order terms $l(\partial_t, \partial_x, \partial_y)u$ in (3.1.7a) and the constant term cu in (3.1.7b) cannot be considered as simple perturbations, because of the loss of one derivative in the estimates. However, the (WR) condition only involves the principal terms $q_0(\partial_t, \partial_x, \partial_y)$ in the interior and $\partial_x + \beta\partial_t + v \cdot \partial_y$ on the boundary. In the present paper, we give a direct proof of the L^2 estimates which gives a detailed account of how the terms lu and cu are absorbed in the estimates. There are other places in the literature where energy estimates for weakly stable hyperbolic problems are given (see [22, 24] and references therein). We point out here that for (WR) problems (3.1.7) in dimension $d \geq 3$, for a large set of parameters (see Remark 3.4.9), the Lopatinski condition necessarily fails at glancing points, that is at points (τ, η) such that the equation $q_0(\tau, \xi, \eta)$ has a double real root in ξ. It does not seem that there is a general treatment of this situation in the literature, so our analysis seems also to be new in this respect.

The new point in the derivation of the L^∞ estimate is the following. The $L^2_x(H^{0,m}(\mathbb{R}^d_{t,y}))$ estimates provide $L^2_x(H^{0,m-2}(\mathbb{R}^d_{t,y}))$ estimates for the term $Q(Y)u$ and the equation 3.1.7a reads

$$-X_- X_+ u = f + Q(Y)u. \tag{3.1.8}$$

Integrating along the characteristics of the outgoing field X_-, one obtains bounds for X_+u and the trace of $(X_+u)_{|x=0}$. If the Lopatinski condition were satisfied at (ω_0, k_0), then the boundary condition and the Y_j would provide an elliptic system for the trace of u on the boundary. In our case, the boundary condition yields a transport equation for $u_{|x=0}$:

$$(\check{v} \cdot Y + \check{c})u_{|x=0} = g - (X_+u)_{|x=0}. \tag{3.1.9}$$

Integrating along the characteristics of $\check{v} \cdot Y$, one gets estimates for $u_{|x=0}$. The loss of one derivative is clear at that point, since this method shows that $u_{|x=0}$ has the smoothness of $(X_+u)_{|x=0}$. The L^∞ estimates are obtained using this reduction of the equations, together with a partial Sobolev embedding in the directions of the striation and integrations along the characteristics of X_-, X_+ and $\check{v} \cdot Y$. The details are given in section 3.6.

3.2 Statement of the main results

3.2.1 Structural assumptions and the (WR) condition

Let us consider the equation and the boundary condition:

$$q(\partial_t, \partial_x, \partial_y)u = q_0(\partial_t, \partial_x, \partial_y)u + l(\partial_t, \partial_x, \partial_y)u = f + F(\,.\,, u) \quad x > 0, \tag{3.2.1a}$$

$$m(\partial_t, \partial_x, \partial_y)u = (\partial_x + \beta\partial_t + v \cdot \partial_y)\, u + cu = g + G(\,.\,, u) \quad x = 0. \tag{3.2.1b}$$

Assumption 3.2.1. *i) q_0 is a quadratic form of signature $(+1, -1, \ldots, -1)$, strictly hyperbolic in the time direction and such that the boundary $\{x = 0\}$ is not time like:*

$$q_0(1, 0, 0, \ldots, 0) > 0, \tag{3.2.2}$$

$$q_0(0, 1, 0, \ldots, 0) < 0. \tag{3.2.3}$$

ii) l is a real linear form and c is a real constant.

For simplicity, we assume, as we can (up to the multiplication by a positive factor), that $q_0(0, 1, 0, \ldots, 0) = -1$.

Remark 3.2.2. *The problem being hyperbolic in the time direction, if ξ, η are real, then the roots ζ of $q_0(\zeta, \xi, \eta)$ are real.*

Assumption 3.2.3. *The functions $F : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{d-1} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy*

$$F(\,.\,; 0) \equiv 0, \;\; G(\,.\,; 0) \equiv 0,$$

F (resp. G) and their derivatives $\partial^\alpha F$ (resp. $\partial^\alpha G$) are continuous on $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{d-1} \times \mathbb{R}$ (resp. $\mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$) and satisfy:

$$\begin{aligned} &\forall \alpha \in \mathbb{N}^{d+2}, \forall R > 0, \exists C > 0, \\ &\forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{d-1}, \forall |u| \leq R, \quad |\partial^\alpha F(t, x, y, u)| \leq C, \\ &\forall \alpha \in \mathbb{N}^{d+1}, \forall R > 0, \exists C > 0, \\ &\forall (t, y) \in \mathbb{R} \times \mathbb{R}^{d-1}, \forall |u| \leq R, \quad |\partial^\alpha G(t, y, u)| \leq C. \end{aligned}$$

Remark 3.2.4. *All the results of this paper can be extended to complex fields l and complex valued solutions u, assuming that F and G are $\mathcal{C}^\infty$ functions of* $\mathrm{Re}(u)$ *and* $\mathrm{Im}(u)$*.*

We begin by the study of the linear problem:

$$q(\partial_t,\partial_x,\partial_y)u = q_0(\partial_t,\partial_x,\partial_y)u + l(\partial_t,\partial_x,\partial_y)u = f \quad x>0, \tag{3.2.4a}$$
$$m(\partial_t,\partial_x,\partial_y)u = (\partial_x + \beta\partial_t + v\cdot\partial_y)\,u + cu = g \quad x=0. \tag{3.2.4b}$$

We consider the weak and uniform Lopatinski conditions on the problem without the first order terms $l(\partial_t,\partial_x,\partial_y)$ in the equation and the constant term c in the boundary condition:

$$q_0(\partial_t,\partial_x,\partial_y)u = f \quad x>0, \tag{3.2.5a}$$
$$m_0(\partial_t,\partial_x,\partial_y)u = (\partial_x + \beta\partial_t + v\cdot\partial_y)\,u = g \quad x=0. \tag{3.2.5b}$$

Performing a Laplace transform in the time variable t and a Fourier transform in the tangential space variables y, we obtain from the problem (3.2.5) an ODE and a boundary condition of the form:

$$q_0(i\zeta,\partial_x,i\eta)w = \phi \quad x>0, \tag{3.2.6a}$$
$$m_0(i\zeta,\partial_x,i\eta)w = (\partial_x + i\beta\zeta + iv\cdot\eta)w = \psi, \quad x=0, \tag{3.2.6b}$$

with $\zeta = \tau - i\gamma$, τ,γ real numbers and $\eta \in \mathbb{R}^{d-1}$.

The Lopatinski condition is satisfied at $(\zeta = \tau - i\gamma, \eta)$, $\gamma > 0$, when the problem (3.2.6) with $\phi = 0$ and $\psi = 0$ has only $w = 0$ as a solution in $L^2(\mathbb{R}_+)$.

The weak Lopatinski condition is satisfied when the Lopatinski condition is satisfied at each $(\zeta = \tau - i\gamma, \eta)$, $\gamma > 0$.

For all $\zeta = \tau - i\gamma$, $\gamma > 0$ and η, let $\lambda_{0,+}$ be defined by:

$$\begin{aligned}&q_0(\tau - i\gamma, \lambda_{0,+}, \eta) = 0\\ &\mathrm{Im}\lambda_{0,+} > 0.\end{aligned}$$

Introducing the Lopatinski determinant at $(\zeta = \tau - i\gamma, \eta)$, $\gamma > 0$:

$$D(\tau - i\gamma, \eta) = \lambda_{0,+} + \beta(\tau - i\gamma) + v\cdot\eta,$$

we obtain that the Lopatinski condition is satisfied at $(\zeta = \tau - i\gamma, \eta)$, $\gamma > 0$, if and only if the Lopatinski determinant does not vanish:

$$D(\tau - i\gamma, \eta) = \lambda_{0,+} + \beta(\tau - i\gamma) + v\cdot\eta \neq 0.$$

Notations 3.2.5. *Denote by* $q_0^\flat(\cdot,\cdot)$ *the symmetric bilinear form on* $\mathbb{E}^* := \mathbb{R}^{1+d}$*associated to* q_0 *and by* q_0^* *the dual quadratic form of* q_0 *on* $\mathbb{E}$*, that is defined by*

$$q_0^*(X) = q_0(Q^{-1}X) = Q^{-1}X \cdot X,$$

where $\Xi \cdot X$ *denotes the canonical duality pairing on* $\mathbb{E}^* \times \mathbb{E}$ *and* Q *is the linear map from* $\mathbb{E}^*$ *to* $\mathbb{E}$ *such that*

$$q_0(\Xi) = \Xi \cdot Q\Xi.$$

Let $\mathbb{F}$ *be defined by* $\mathbb{F} := \{e_0, e_1\}^{\perp(q_0)}$*, where* $e_0 = (1, 0, \ldots)$ *is timelike and* $e_1 = (0, 1, 0 \ldots)$ *is conormal to the boundary. Let* $m_0^\flat \in \mathbb{E}$ *denote the linear form on* $\mathbb{E}^*$ *equal to the boundary condition* m_0 *on* $\mathbb{F}$ *and vanishing on* $\{e_0, e_1\}$*.*

We introduce

$$\tilde{\beta} = \Big(q_0(e_0) + \big(q_0^b(e_0, e_1)\big)^2\Big)^{-1/2}\big(\beta + q_0^b(e_0, e_1)\big).$$

Remark 3.2.6. *Since* $\mathbb{F}$ *and* $\operatorname{Span}(e_0, e_1)$ *are orthogonal for* q_0 *and* q_0 *is definite negative on* $\mathbb{F}$*, we have*

$$q_0^*(m_0^\flat) \leq 0.$$

Remark 3.2.7. *For the wave equation:*

$$(c^{-2}\partial_t^2 - \triangle_{x,y})u = f \quad x > 0, \tag{3.2.7a}$$
$$(\partial_x + \beta\partial_t + v \cdot \partial_y)\, u = g \quad x = 0, \tag{3.2.7b}$$

$\tilde{\beta}$ *and* $-q_0^*(m_0^\flat)$ *are given by:*

$$\tilde{\beta} = c\beta, \quad -q_0^*(m_0^\flat) = |v|^2.$$

We then prove (see section 3.4):

Proposition 3.2.8. *The weak Lopatinski condition is satisfied if and only if*
$\tilde{\beta} \leq \sqrt{1 - q_0^*(m_0^\flat)}$ *and* $\tilde{\beta} \neq 1$ *in dimension* $d = 2$*,*
$\tilde{\beta} < 1$ *in dimension* $d \geq 3$*.*

Remark 3.2.9. *In the case of the wave equation* (3.2.7)*, we obtain that the weak Lopatinski condition is satisfied if and only if*
$c\beta \leq \sqrt{1 + |v|^2}$ *and* $c\beta \neq 1$ *in dimension* $d = 2$*,*
$c\beta < 1$ *in dimension* $d \geq 3$*.*

Under the weak Lopatinski condition, for all $|\tau| + |\gamma| + |\eta| = 1$, $\gamma > 0$, there is a constant $C_{\tau,\gamma,\eta}$ such that the L^2 solution w of the problem (3.2.6) with $\phi = 0$ satisfies:

$$|w(0)| \leq C_{\tau,\gamma,\eta}|\psi|. \tag{3.2.8}$$

The Lopatinski condition is satisfied at $(\underline{\tau}, \underline{\eta})$ when, for all $(\tau - i\gamma, \eta)$, $\gamma > 0$, in a neighbourhood $\mathcal{V}_{(\underline{\tau},\underline{\eta})}$ of $(\underline{\tau}, \underline{\eta})$, the Lopatinski condition is satisfied at $(\tau - i\gamma, \eta)$ and there is a constant C, independent of (τ, γ, η), such that, for all $(\tau - i\gamma, \eta)$, $\gamma > 0$ in $\mathcal{V}_{(\underline{\tau},\underline{\eta})}$, the L^2 solution w of the problem (3.2.6) with $\phi = 0$ satisfies:

$$|w(0)| \leq C|\psi|.$$

At a hyperbolic point (τ, η), the square root $\lambda_{0,+}$, and thus the Lopatinski determinant D, can be extended by continuity: we can define the value $D(\tau, \eta)$ for all (τ, η) hyperbolic points. We then obtain that the Lopatinski condition is satisfied at a hyperbolic point (τ, η) if and only if

$$D(\tau, \eta) \neq 0.$$

The uniform Lopatinski condition is satisfied when the Lopatinski condition is satisfied at each point $(\tau - i\gamma, \eta)$, $\gamma \geq 0$, $(\tau - i\gamma, \eta) \neq (0, 0)$. Under the uniform Lopatinski condition, the constant C in the estimate (3.2.8) is uniform for $|\tau| + |\gamma| + |\eta| = 1$, $\gamma > 0$.

Proposition 3.2.10. *The uniform Lopatinski condition is satisfied if and only if* $\tilde{\beta} < -\sqrt{-q_0^*(m_0^\flat)}$.

Remark 3.2.11. *In the case of the wave equation* (3.2.7)*, we obtain that the uniform Lopatinski condition is satisfied if and only if* $c\beta < -|v|$.

In the case $-\sqrt{-q_0^*(m_0^\flat)} \leq \tilde{\beta} \leq \sqrt{1 - q_0^*(m_0^\flat)}$ and $\tilde{\beta} \neq 1$ in dimension $d = 2$ and $-\sqrt{-q_0^*(m_0^\flat)} \leq \tilde{\beta} < 1$ in dimension $d \geq 3$, the weak Lopatinski condition is satisfied whereas the uniform Lopatinski condition is violated. This weak stability without a strong stability has been studied in [10, 77], where is introduced the (WR) condition:

Definition 3.2.12. *The (WR) condition is satisfied when:*

i) the weak Lopatinski condition is satisfied

ii) the Lopatinski condition is not satisfied at, at least, one point $(\underline{\tau}, \underline{\eta})$*, which is hyperbolic and such that the Lopatinski determinant satisfies*

$$\partial_\tau D(\underline{\tau}, \underline{\eta}) \neq 0.$$

Proposition 3.2.13. *The (WR) condition is satisfied if and only if*

$-\sqrt{-q_0^*(m_0^\flat)} < \tilde\beta < \sqrt{1 - q_0^*(m_0^\flat)}$ *and* $\tilde\beta \neq 1$ *in dimension* $d = 2$*,*

$-\sqrt{-q_0^*(m_0^\flat)} < \tilde\beta < 1$ *in dimension* $d \geq 3$*.*

Remark 3.2.14. *In the case of the wave equation* (3.2.7)*, we obtain that the (WR) condition is satisfied if and only if*

$-|v| < c\beta < \sqrt{1+|v|^2}$ *and* $c\beta \neq 1$ *in dimension* $d = 2$*,*

$-|v| < c\beta < 1$ *in dimension* $d \geq 3$*.*

This result, and the previous characterizations of the weak and uniform Lopatinski conditions, are consistent with [77].

3.2.2 The geometry

The frame

Assumption 3.2.15. *We assume that the (WR) condition is satisfied. Let* $(\omega_0, k_0) \in \mathbb{R} \times \mathbb{R}^{d-1}$ *be a hyperbolic point, where the Lopatinski condition is violated.*

Remark 3.2.16. *Lemma 3.4.4 implies that the Lopatinski condition is violated at a hyperbolic point* $(\omega_0, k_0 = 0)$ *if and only if* $\tilde\beta = 1$*. Since* $\tilde\beta \neq 1$ *under the (WR) condition (the weak Lopatinski condition is not satisfied for* $\tilde\beta = 1$*), we obtain that, necessarily,* $k_0 \neq 0$*.*

Let

$$\varphi_0 = \omega_0 t + k_0 \cdot y$$

be the "phase" on the boundary.

We denote by $\xi_\pm$ the continuous limit of $\lambda_{0,\pm}$ when $\tau \to \omega_0$, $\gamma \xrightarrow{>} 0$ and $\eta \to k_0$, where $\lambda_{0,\pm}$ is defined by

$$\begin{aligned} &q_0(\tau - i\gamma, \lambda_{0,\pm}, \eta) = 0 \\ &\pm \mathrm{Im}\lambda_{0,\pm} > 0. \end{aligned}$$

Since the Lopatinski condition is violated at (ω_0, k_0), ξ_+ satisfies:

$$\xi_+ = -(\beta\omega_0 + v \cdot k_0). \tag{3.2.9}$$

The characteristic phases $\varphi_\pm$, which are equal to φ_0 on the boundary $\{x = 0\}$, are

$$\varphi_\pm = \xi_\pm x + \omega_0 t + k_0 \cdot y.$$

The striation

The "striation" is the family of spaces of dimension $d-1$ in $\mathbb{R}^{1+d}$ given by the equations $\{\varphi_+ = c_+, \varphi_- = c_-\}$ parametrized by constants c_+ and c_-. These spaces are also given by $\{x = c, \varphi_0 = c_0\}$ with other constants c and c_0. The fields tangent to the striation are the fields Y such that $Y\varphi_+ = Y\varphi_- = 0$, that is $Yx = Y\varphi_0 = 0$. They generate a space of dimension $d-1$, endowed with a basis $\{Y_1, \ldots, Y_{d-1}\}$. A convenient choice of the basis will be made later on at (3.3.6).

Proposition 3.2.17. *The equation 3.2.4a reads:*

$$-X_+X_-u - Q(Y)u = f \quad x > 0,$$

where $Q(Y)$ is a polynomial in Y of degree 2
and $X_\pm$ are first order operators such that:

$$X_\pm = \widetilde{X}_\pm + c_\pm,$$

with $c_\pm$ constants and $\widetilde{X}_\pm$ satisfying

$$\widetilde{X}_\pm(\varphi_\pm) = 0,\ \widetilde{X}_\pm(\varphi_\mp) \neq 0.$$

Furthermore, the coefficient of ∂_x in X_+ and X_- is equal to 1 and the coefficient of ∂_t in X_+ (resp. X_-) is positive (resp. negative).

The boundary condition is of the form:

$$X_+u + \check{v}\cdot Yu + \check{c}u = g \quad x = 0, \tag{3.2.10}$$

with $\check{c}$ a constant and $\check{v}$ a constant vector such that the coefficient of ∂_t in $\check{v}\cdot Y$ does not vanish.

3.2.3 Main results

We introduce the notations:

$$\Omega^+ := \mathbb{R}_t \times \mathbb{R}_x^+ \times \mathbb{R}_y^{d-1}\ ,\ \ \Omega^0 := \mathbb{R}^d_{t,y}.$$

On the space $H^s(\Omega^0)$, we introduce the weighted norm:

Definition 3.2.18.

$$\forall u \in H^s(\Omega^0), \quad \left|u\right|_{\gamma,s} := \left\|\Lambda^s\,\mathcal{F}\big(e^{-\gamma t}u\big)\right\|_{L^2(\Omega^0)}, \tag{3.2.11}$$

where $\mathcal{F}$ is the Fourier transform with respect to (t,y) and $\Lambda = \big(\tau^2 + \gamma^2 + |\eta|^2\big)^{\frac{1}{2}}$.

For the functions depending on $x \geq 0$, we define on $L^2_x(H^s(\Omega^0))$ the following norm:

Definition 3.2.19.

$$\forall u \in L^2_x(H^s(\Omega^0)), \quad \left\|u\right\|_{\gamma,s} := \Big(\int_0^\infty \left|u(\,\cdot\,,x)\right|^2_{\gamma,s} dx \Big)^{\frac{1}{2}}. \tag{3.2.12}$$

We introduce the striated space $H^{s,m}(\Omega^0)$ with respect to the set of derivatives Y_j:

Definition 3.2.20. $H^{s,m}(\Omega^0)$ *is the space of functions in* $H^s(\Omega^0)$ *such that for all* $\alpha \in \mathbb{N}^{d-1}$ *of length* $|\alpha| \leq m$, $Y^\alpha u = Y_1^{\alpha_1} \dots Y_{d-1}^{\alpha_{d-1}} u \in H^s(\Omega^0)$.

$H^{s,m}(\Omega^0)$ and $L^2_x(H^{s,m}(\Omega^0))$ are equipped with the following norms:

Definition 3.2.21.

$$\forall u \in H^{s,m}(\Omega^0), \quad \left|u\right|_{m,\gamma,s} := \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \left|Y^\alpha u\right|_{\gamma,s}, \tag{3.2.13}$$

$$\forall u \in L^2_x(H^{s,m}(\Omega^0)), \quad \left\|u\right\|_{m,\gamma,s} := \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \left\|Y^\alpha u\right\|_{\gamma,s}. \tag{3.2.14}$$

For $T > 0$, we introduce the notations:

$$\Omega^+_T := \Omega^+ \cap \{t \leq T\} =]-\infty, T]_t \times \mathbb{R}^+_x \times \mathbb{R}^{d-1}_y,$$
$$\Omega^0_T := \Omega^0 \cap \{t \leq T\} =]-\infty, T]_t \times \mathbb{R}^{d-1}_y.$$

We introduce the space $H^s(\Omega^0_T) = \{u|_{\Omega^0_T} / \, u \in H^s(\Omega^0)\}$ and the striated space $H^{s,m}(\Omega^0_T) = \{u|_{\Omega^0_T} / \, u \in H^{s,m}(\Omega^0)\}$. For general s, the norms $\left|\,.\,\right|_{\gamma,s,T}$, $\left|\,.\,\right|_{m,\gamma,s,T}$, $\left\|\,.\,\right\|_{\gamma,s,T}$ and $\left\|\,.\,\right\|_{m,\gamma,s,T}$ are defined in the usual way. For $s = 0$, we use the equivalent norms (for the construction of extensions, see Lemma 3.7.2):

Definition 3.2.22.

$$\forall u \in H^0(\Omega^0_T), \ \ \left|u\right|_{\gamma,0,T} := \left\|e^{-\gamma t} u\right\|_{L^2(\Omega^0_T)},$$

$$\forall u \in L^2_x(H^0(\Omega^0_T)), \ \ \left\|u\right\|_{\gamma,0,T} := \Big(\int_0^\infty \left|u(\,\cdot\,,x)\right|^2_{\gamma,0,T} dx \Big)^{\frac{1}{2}},$$

$$\forall u \in H^{0,m}(\Omega^0_T), \ \ \left|u\right|_{m,\gamma,0,T} := \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \left|Y^\alpha u\right|_{\gamma,0,T},$$

$$\forall u \in L^2_x(H^{0,m}(\Omega^0_T)), \ \ \left\|u\right\|_{m,\gamma,0,T} := \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \left\|Y^\alpha u\right\|_{\gamma,0,T}.$$

For $T > 0$ and $m \in \mathbb{N}$, we introduce the space:

$$\mathbb{H}^{m,T} := \big\{ u \in L^2_x(H^{0,m}(\Omega^0_T)) / \partial_x u \in L^2_x(H^{-1,m}(\Omega^0_T)), \partial^2_x u \in L^2_x(H^{-2,m}(\Omega^0_T)),$$
$$u(0) \in H^{0,m}(\Omega^0_T), \partial_x u(0) \in H^{-1,m}(\Omega^0_T) \big\}.$$

In section 3.5.4, we obtain the striated estimate:

Theorem 3.2.23. *Under the (WR) condition, there exists $\gamma_0 > 0$ such that, for all $T > 0$, $m \in \mathbb{N}$ and for all f in $L^2_x(H^{0,m}(\Omega^0_T))$ and g in $H^{0,m}(\Omega^0_T)$ functions vanishing in the past, the linear problem* (3.2.4) *has a solution u vanishing in the past which belongs to the space $\mathbb{H}^{m,T}$ and satisfies the following estimate:*

$$\forall \gamma \geq \gamma_0 \quad \gamma^2 \Big(\big\| \partial_x u \big\|_{m,\gamma,-1,T} + \big\| u \big\|_{m,\gamma,0,T} \Big)$$
$$+ \gamma^{\frac{3}{2}} \Big(\big| \partial_x u(0) \big|_{m,\gamma,-1,T} + \big| u(0) \big|_{m,\gamma,0,T} \Big) \leq C \left(\big\| f \big\|_{m,\gamma,0,T} + \gamma^{\frac{1}{2}} \big| g \big|_{m,\gamma,0,T} \right), \tag{3.2.15}$$

with C a constant independent of T and γ.

We prove this result for $T = \infty$ and, by classical arguments, we deduce the theorem for all $T > 0$.

Remark 3.2.24. *Under the (WR) condition, the L^2 estimate (that is the estimate for $m = 0$) gives an estimate of $\gamma^2 \big\| u \big\|_{L^2(\Omega^+_T)}$, whereas, in the case of the uniform Lopatinski condition, an estimate of $\gamma \big\| u \big\|_{H^1(\Omega^+_T)}$ is obtained. We observe a loss of regularity.*

We introduce the space:

$$\mathbb{L}^{\infty,T} := \big\{ u \in L^\infty(\Omega^+_T) / X_+ u \in L^\infty(\Omega^+_T) \big\}.$$

In section 3.6, we obtain the L^∞ estimate:

Theorem 3.2.25. *Under the (WR) condition, there exists $\gamma_0 > 0$ such that, for all $m > 4 + \frac{d-1}{2}$, $\gamma \geq \gamma_0$ and $T > 0$ fixed numbers that satisfy $\gamma T \approx 1$ and for all f in $L^\infty(\Omega^+_T) \cap L^2_x(H^{0,m}(\Omega^0_T))$ and g in $L^\infty(\Omega^0_T) \cap H^{0,m}(\Omega^0_T)$ functions vanishing in the past, the linear problem* (3.2.4) *has a solution u vanishing in the past which belongs to $\mathbb{L}^{\infty,T} \cap \mathbb{H}^{m,T}$ and satisfies the following estimate:*

$$\big\| u \big\|_{L^\infty(\Omega^+_T)} + \big\| X_+ u \big\|_{L^\infty(\Omega^+_T)}$$
$$\leq C\, T \Big(\big\| f \big\|_{m,\gamma,0,T} + \big\| f \big\|_{L^\infty(\Omega^+_T)} + \big\| g \big\|_{L^\infty(\Omega^0_T)} + \big| g \big|_{m,\gamma,0,T} \Big), \tag{3.2.16}$$

with C a constant independent of T and γ.

Here $\gamma T \approx 1$ means that, for given $c > 0$, $C > 0$, γ and T satisfy $c \le \gamma T \le C$.

In section 3.7, we generalize these results to the semi-linear problem and obtain:

Theorem 3.2.26. *Under the (WR) condition, there exist $\gamma_0 > 0$ and $T_0 > 0$ such that, for all $m > 4 + \frac{d-1}{2}$ and $\gamma \ge \gamma_0$ satisfying $\gamma T_0 \approx 1$ and for all f in $L^\infty(\Omega^+_{T_0}) \cap L^2_x(H^{0,m}(\Omega^0_{T_0}))$ and g in $L^\infty(\Omega^+_{T_0}) \cap H^{0,m}(\Omega^0_{T_0})$ functions vanishing in the past , the semi-linear problem* (3.2.1) *has a solution u vanishing in the past, which belongs to $\mathbb{L}^{\infty,T_0} \cap \mathbb{H}^{m,T_0}$ and satisfies the following estimate:*

$$\begin{aligned} &\big\|u\big\|_{m,\gamma,0,T_0} + \big\|u\big\|_{L^\infty(\Omega^+_{T_0})} + \big|u(0)\big|_{m,\gamma,0,T_0} \\ &+ \big\|X_+ u\big\|_{L^\infty(\Omega^+_{T_0})} + \big\|\partial_x u\big\|_{m,\gamma,-1,T_0} + \big|\partial_x u(0)\big|_{m,\gamma,-1,T_0} \\ &\qquad \le C T_0 \left(\big\|f\big\|_{m,\gamma,0,T_0} + \big\|f\big\|_{L^\infty(\Omega^+_{T_0})} + \big\|g\big\|_{L^\infty(\Omega^0_{T_0})} + \big|g\big|_{m,\gamma,0,T_0} \right), \end{aligned} \tag{3.2.17}$$

with C a constant independent of T_0 and γ.

Finally, in section 3.8, we study, for piecewise Lipschitz solutions, how a singularity located on $\Sigma_- = \{\varphi_- = 0\}$ is reflected on $\Sigma_+ = \{\varphi_+ = 0\}$. We obtain:

Theorem 3.2.27. *Under the (WR) condition, suppose that $Y^\alpha f \in L^\infty(\Omega^+_T) \cap L^2_x(H^{0,m}(\Omega^0_T))$ and $Y^\alpha g \in L^\infty(\Omega^0_T) \cap H^{0,m}(\Omega^0_T)$ for all $|\alpha| \le 2$; suppose in addition that they vanish for $t \le 0$ where $m > 4 + \frac{d-1}{2}$ and are piecewise Lipschitz, with*

$$[f]_{\Sigma_-} \ne 0, \qquad [g]_{\Sigma_0} = 0, \tag{3.2.18}$$

where $\Sigma_0 = \Sigma_+ \cap \Sigma_-$.

Suppose that $u \in L^\infty(\Omega^+_T) \cap L^2_x(H^{0,m}(\Omega^0_T))$ satisfies (3.2.1)*. Then,*

i) u and $X_+ u$ are piecewise Lipschitz on Ω^+_T, and the trace $u_{|x=0}$ is piecewise Lipschitz on Ω^0_T.

ii) u is continuous across Σ_-, while the jump of $X_+ u$ across Σ_- satisfies the propagation equation

$$-X_-([X_+ u]_{\Sigma_-}) = [f]_{\Sigma_-}. \tag{3.2.19}$$

Therefore, in general, $\jmath_1 := ([X_+ u]_{\Sigma_-})_{|x=0}$, the trace of $[X_+ u]_{\Sigma_-}$ on the boundary, does not vanish identically on Σ_0.

iii) If $\jmath_1 \ne 0$, then $\jmath_0 = [u_{|x=0}]_{\Sigma_0}$, the jump of the trace $u_{|x=0}$ across Σ_0, does not vanish identically.

iv) The jump of u across Σ_+ satisfies the propagation equation

$$X_+([u]_{\Sigma_+}) = 0, \qquad ([u]_{\Sigma_+})_{|x=0} = \jmath_0 \tag{3.2.20}$$

and therefore, does not vanish identically if $\jmath_1 \neq 0$.

3.3 Preliminary computations

3.3.1 Factorization of the differential equation (3.2.5a)

Lemma 3.3.1. *There is a change of variables*

$$(\zeta, \xi, \eta) \mapsto (\tilde{\zeta}, \xi, \tilde{\eta}) = (a\zeta + b \cdot \eta, \xi, P\eta),$$

where $a > 0$ and P is a non singular matrix of size $d-1$, such that q_0 can be written under the form:

$$\begin{aligned} \forall \zeta, \xi, \eta \;\; q_0(i\zeta, i\xi, i\eta) &= \tilde{q}_0(i\tilde{\zeta}, i\xi, i\tilde{\eta}) \\ &= (\xi + s_0)^2 - \tilde{\zeta}^2 + |\tilde{\eta}|^2, \end{aligned}$$

where $s_0 = \alpha\tilde{\zeta} + \nu \cdot \tilde{\eta}$, $|\alpha| < 1$, $\nu \in \mathbb{R}^{d-1}$ a suitable vector and $|\tilde{\eta}|^2 = \sum_{j=2}^{d} \tilde{\eta}_j^2$.

For all $\tilde{\zeta} = \tilde{\tau} - i\tilde{\gamma}$, $\tilde{\gamma} > 0$, introducing λ_0 such that

$$\lambda_0^2 = \tilde{\zeta}^2 - |\tilde{\eta}|^2 \text{ and } \mathrm{Im}\lambda_0 > 0,$$

we have

$$\forall \xi, \tilde{\eta}, \forall \tilde{\zeta} = \tilde{\tau} - i\tilde{\gamma}, \tilde{\gamma} > 0, \;\; \tilde{q}_0(i\tilde{\zeta}, i\xi, i\tilde{\eta}) = (\xi - \lambda_{0,+})(\xi - \lambda_{0,-})$$

with $\lambda_{0,\pm} = -s_0 \pm \lambda_0$, satisfying $\pm\mathrm{Im}\lambda_{0,\pm} > 0$.

Remark 3.3.2. *The coefficients a and α are given by:*

$$\alpha = \frac{-q_0^b(e_0, e_1)}{\left[q_0(e_0) + \left(q_0^b(e_0, e_1)\right)^2\right]^{1/2}}, \qquad a = \left[q_0(e_0) + \left(q_0^b(e_0, e_1)\right)^2\right]^{1/2}.$$

Proof. The quadratic form q_0 on $\mathbb{E}^* = \mathbb{R}^{1+d}$ satisfies (see Notations 3.2.5):

$$q_0(e_0) > 0, \qquad q_0(e_1) = -1.$$

Set $\tilde{e}_1 = e_1$ and let $\tilde{e}_0 = a^{-1}e_0 - \alpha e_1 \in \mathrm{Span}(e_0, e_1)$ be such that

$$q_0(\tilde{e}_0) = 1, \qquad q_0^\flat(\tilde{e}_0, \tilde{e}_1) = 0, \qquad a > 0.$$

Then, q_0 being of signature $(+,-,-,\dots)$, $-q_0$ is definite positive on $\mathbb{F} = \{e_0, e_1\}^{\perp(q_0)}$. Denote by $\{\tilde{e}_2, \dots, \tilde{e}_d\}$ an orthonormal basis of $-q_0$ on $\mathbb{F}$. We obtain that $\{\tilde{e}_0, \dots, \tilde{e}_d\}$ is a basis of $\mathbb{E}^*$ in which q_0 is diagonal :

$$q_0\Big(\tilde{\zeta}\tilde{e}_0 + \tilde{\xi}e_1 + \sum_{j=2}^{d} \tilde{\eta}_j \tilde{e}_j\Big) = \tilde{\zeta}^2 - \tilde{\xi}^2 - \sum_{j=2}^{d} \tilde{\eta}_j^2.$$

$\tilde{e}_0 = a^{-1}e_0 - \alpha e_1$ yields that $\tilde{\zeta}, \tilde{\xi}, \tilde{\eta}$ satisfying $\zeta e_0 + \xi e_1 + \sum_{j=2}^{d} \eta_j e_j = \tilde{\zeta}\tilde{e}_0 + \tilde{\xi}e_1 + \sum_{j=2}^{d} \tilde{\eta}_j \tilde{e}_j$, where $\{e_0, e_1, e_2, \dots, e_d\}$ are the vectors of the canonical basis of $\mathbb{E}^* = \mathbb{R}^{1+d}$, are of the form $\tilde{\zeta} = a\zeta + b\cdot\eta$, $\tilde{\xi} = \xi + s_0 = \xi + \alpha\tilde{\zeta} + \nu\cdot\tilde{\eta}$ and $\tilde{\eta} = P\eta$, with $b \in \mathbb{R}^{d-1}$ and $\nu \in \mathbb{R}^{d-1}$ suitable vectors, and P a suitable non singular matrix.

The coefficients a and α satisfy

$$a^{-1}q_0^b(e_0, e_1) + \alpha = 0, \qquad a^{-2}q_0(e_0) - 2\alpha a^{-1}q_0^b(e_0, e_1) - \alpha^2 = 1$$

thus

$$\alpha = \frac{-q_0^b(e_0, e_1)}{\big[q_0(e_0) + \big(q_0^b(e_0, e_1)\big)^2\big]^{1/2}}, \qquad a = \big[q_0(e_0) + \big(q_0^b(e_0, e_1)\big)^2\big]^{1/2}.$$

It follows from $q_0(e_0) > 0$ that $|\alpha| < 1$.

For all $\zeta = \tau - i\gamma$, $\gamma > 0$ and $\eta \in \mathbb{R}^{d-1}$, the roots ξ of $q_0(\tau - i\gamma, \xi, \eta)$ cannot be real: thus the number of roots ξ with positive imaginary part does not depend on η. For $\eta = 0$, we can write

$$\begin{aligned} q_0(\zeta, \xi, 0) &= a_{0,0}\zeta^2 + a_{0,1}\zeta\xi - \xi^2 \qquad a_{0,0} = q_0(e_0) > 0, \\ &= -(\xi - \alpha\zeta)(\xi - \beta\zeta), \end{aligned}$$

with α, β such that $\alpha\beta = -a_{0,0} < 0$, thus there is, for $\eta = 0$, one root of q_0 with a positive imaginary part and one root with a negative imaginary part. We have therefore, for all η, one root $\lambda_{0,+} = -s_0 + \lambda_0$ with a positive imaginary part and one root $\lambda_{0,-} = -s_0 - \lambda_0$ with a negative imaginary part. $\square$

3.3.2 Intrinsic expression of the Lopatinski determinant

Lemma 3.3.3. *The Lopatinski determinant $D(\tau - i\gamma, \eta)$ can be written under the form:*

$$\begin{aligned} \forall \tau, \eta, \forall \gamma > 0, \ \ D(\tau - i\gamma, \eta) &= \tilde{D}(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta}) \\ &= \lambda_0 + \tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v}\cdot\tilde{\eta}, \end{aligned} \tag{3.3.1}$$

where

$$\tilde\beta = \Big(q_0(e_0) + \big(q_0^b(e_0,e_1)\big)^2\Big)^{-1/2}\big(\beta + q_0^b(e_0,e_1)\big)$$

and $\tilde v$ is the restriction to $\mathbb{F} = \{e_0, e_1\}^{\perp(q_0)}$ of the linear form m_0.

Introducing

$$|\tilde v|^2 = \sum_{j=2}^{d} |\tilde v_j|^2$$

if $\tilde v = \sum_{j=2}^{d} \tilde v_j Z_j$ in the basis $\{Z_2,\dots,Z_d\}$ of $\mathbb{F}^*$ which is dual to $\{\tilde e_2,\dots,\tilde e_d\}$ orthonormal basis of $-q_0$ on $\mathbb{F}$, we have

$$|\tilde v|^2 = -q_0^*(m_0^\flat),$$

where q_0^* and $m_0^\flat$ have been introduced in Notations 3.2.5.

Proof. With the notations introduced in Lemma 3.3.1 and its proof, the boundary condition reads, since $m_0 \cdot e_1 = 1$,

$$m_0 \cdot \Big(\tilde\zeta \tilde e_0 + \tilde\xi e_1 + \sum_{j=2}^{d} \tilde\eta_j \tilde e_j\Big) = \tilde\xi + \tilde\beta\tilde\zeta + \tilde v \cdot \tilde\eta, \tag{3.3.2}$$

where

$$\begin{aligned}\tilde\beta &= m_0 \cdot \tilde e_0 = a^{-1} m_0 \cdot e_0 - \alpha m_0 \cdot e_1 = a^{-1}\beta - \alpha \\ &= \Big(q_0(e_0) + \big(q_0^b(e_0,e_1)\big)^2\Big)^{-1/2}\big(\beta + q_0^b(e_0,e_1)\big).\end{aligned}$$

Considering $q_{0|\mathbb{F}}$ the restriction of q_0 to $\mathbb{F}$, the intrinsic definition of $|\tilde v|^2$ is

$$|\tilde v|^2 = -q_{0|\mathbb{F}}^*(\tilde v)$$

where $q_{0|\mathbb{F}}^*$ is the dual quadratic form of $q_{0|\mathbb{F}}$ on the dual space $\mathbb{F}^*$.

Since $\mathbb{F}$ and $\mathrm{Span}(e_0, e_1)$ are orthogonal for q_0, we see that

$$-q_{0|\mathbb{F}}^*(\tilde v) = -q_0^*(m_0^\flat) = |\tilde v|^2.$$

The boundary condition (3.3.2) yields the form (3.3.1) of the Lopatinski determinant. $\square$

3.3.3 In the new variables

We introduce the dual variables $\tilde{\zeta} = \tilde{\tau} - i\tilde{\gamma}$ and $\tilde{\eta}$ given by Lemma 3.3.1. The Lopatinski condition is satisfied at $(\tau - i\gamma, \eta)$, $\gamma > 0$ (for simplicity, we may also say at $(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta})$), if and only if the Lopatinski determinant does not vanish:

$$D(\tau - i\gamma, \eta) = \tilde{D}(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta}) = \lambda_0 + \tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v} \cdot \tilde{\eta} \neq 0.$$

A point $(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ is hyperbolic when $\tilde{\tau}^2 > |\tilde{\eta}|^2$ (for simplicity, we may say that $(\tilde{\tau}, \tilde{\eta})$ is hyperbolic). For $(\tilde{\tau}, \tilde{\eta})$ a hyperbolic point,

$$\tilde{D}(\tilde{\tau}, \tilde{\eta}) = -\tilde{\tau}\sqrt{1 - |\tilde{\eta}|^2/\tilde{\tau}^2} + \tilde{\beta}\tilde{\tau} + \tilde{v} \cdot \tilde{\eta}.$$

Remark 3.3.4. *The condition (ii) in the definition 3.2.12 of the (WR) condition can be written under the form:*

ii) the Lopatinski condition is not satisfied at, at least, one point $(\underline{\tilde{\tau}}, \underline{\tilde{\eta}})$, which is hyperbolic and such that the Lopatinski determinant

$$\tilde{D}(\tilde{\tau}, \tilde{\eta}) = -\tilde{\tau}\sqrt{1 - |\tilde{\eta}|^2/\tilde{\tau}^2} + \tilde{\beta}\tilde{\tau} + \tilde{v} \cdot \tilde{\eta}$$

vanishes at $(\underline{\tilde{\tau}}, \underline{\tilde{\eta}})$ and satisfies $\partial_{\tilde{\tau}}\tilde{D}(\underline{\tilde{\tau}}, \underline{\tilde{\eta}}) \neq 0$.

Introduce

$$\omega = a\omega_0 + b \cdot k_0, \quad k = Pk_0.$$

From Assumption 3.2.15, (ω_0, k_0) is a hyperbolic point, that is such that $\omega^2 > |k|^2$, where the Lopatinski condition is violated. Without loss of generality, we assume that $\omega > 0$ and then:

$$-\sqrt{\omega^2 - |k|^2} = -(\tilde{\beta}\omega + \tilde{v} \cdot k). \tag{3.3.3}$$

Introduce the variables $\tilde{t}, \tilde{y}$ corresponding to the dual variables $\tilde{\zeta}$, $\tilde{\eta}$:

$$\tilde{t} = a^{-1}t, \quad \tilde{y} = \left(P^t\right)^{-1}(y - a^{-1}bt).$$

The phase on the boundary $\varphi_0 = \omega_0 t + k_0 \cdot y$ and the characteristic phases $\varphi_\pm = \xi_\pm x + \omega_0 t + k_0 \cdot y$ can be written under the form

$$\varphi_0 = \omega\tilde{t} + k \cdot \tilde{y},$$
$$\varphi_\pm = \xi_\pm x + \omega\tilde{t} + k \cdot \tilde{y},$$

with

$$\xi_\pm = -\alpha\omega - \nu \cdot k \mp \sqrt{\omega^2 - |k|^2}. \tag{3.3.4}$$

Note that

$$\xi_+ < \xi_-. \tag{3.3.5}$$

For $j = 2, \ldots, d-1$, we define the fields tangent to the striation $\{x = c, \varphi_0 = c_0\}$:

$$Y_j = \delta^{-1}\big(\omega\partial_{\tilde{y}_j} - k_j\partial_{\tilde{t}}\big), \tag{3.3.6}$$

with $\delta = \sqrt{\omega^2 - |k|^2}$.

3.4 The Lopatinski condition

3.4.1 The weak Lopatinski condition: proof of Proposition 3.2.8

The Lopatinski condition is not satisfied at $(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta}), \tilde{\gamma} > 0$ if and only if the Lopatinski determinant vanishes at $(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta})$, that is:

$$\tilde{D}(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta}) = \lambda_0 + \tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v}\cdot\tilde{\eta} = 0. \tag{3.4.1}$$

If $\tilde{\beta} \leq 0$, then, for all $(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta}), \tilde{\gamma} > 0$,

$$\mathrm{Im}(\tilde{D}(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta})) = \mathrm{Im}\lambda_0 - \tilde{\beta}\tilde{\gamma} \geq \mathrm{Im}\lambda_0 > 0,$$

therefore the weak Lopatinski condition is satisfied when $\tilde{\beta} \leq 0$.

We assume now that $\tilde{\beta} > 0$. Since $\mathrm{Im}\lambda_0 > 0$ and $\mathrm{Im}(\tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v}\cdot\tilde{\eta}) = -\tilde{\beta}\tilde{\gamma} < 0$, we get that

$$\begin{aligned}\tilde{D}(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta}) = 0 &\Leftrightarrow \lambda_0^2 - (\tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v}\cdot\tilde{\eta})^2 = 0\\ &\Leftrightarrow (\tilde{\tau} - i\tilde{\gamma})^2 - |\tilde{\eta}|^2 - (\tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v}\cdot\tilde{\eta})^2 = 0.\end{aligned}$$

Then, the weak Lopatinski condition is satisfied if, for all $\tilde{\eta}$, the roots $\tilde{\zeta}$ of

$$P_0 := \tilde{\zeta}^2 - |\tilde{\eta}|^2 - (\tilde{\beta}\tilde{\zeta} + \tilde{v}\cdot\tilde{\eta})^2 \tag{3.4.2}$$

are all in $\mathrm{Im}\tilde{\zeta} \geq 0$.

The equation can be written under the form

$$(1 - \tilde{\beta}^2)\tilde{\zeta}^2 - 2\tilde{\beta}(\tilde{v}\cdot\tilde{\eta})\tilde{\zeta} - \big(|\tilde{\eta}|^2 + (\tilde{v}\cdot\tilde{\eta})^2\big) = 0.$$

In the degenerate case $\tilde{\beta}^2 = 1$, the equation is $2\tilde{\beta}\tilde{v}\cdot\tilde{\eta}\,\tilde{\zeta} + |\tilde{\eta}|^2 + (\tilde{v}\cdot\tilde{\eta})^2 = 0$. The Lopatinski condition is not satisfied at $(\tilde{\tau} - i\tilde{\gamma}, \tilde{\eta} = 0)$, $\tilde{\gamma} > 0$: the weak Lopatinski condition is therefore not satisfied.

In the non degenerate case $\tilde\beta^2 \neq 1$, since the equation has real coefficients, the roots are all in $\mathrm{Im}\tilde\zeta \geq 0$ if and only if the roots are real. The discriminant is

$$\begin{aligned}\Delta &= \tilde\beta^2(\tilde v\cdot\tilde\eta)^2 + (1-\tilde\beta^2)\big(|\tilde\eta|^2 + (\tilde v\cdot\tilde\eta)^2\big)\\ &= (1-\tilde\beta^2)|\tilde\eta|^2 + (\tilde v\cdot\tilde\eta)^2.\end{aligned} \tag{3.4.3}$$

The roots are real when $\Delta \geq 0$.

For $d = 2$, we get

$$\Delta = (1-\tilde\beta^2+\tilde v^2)\tilde\eta^2.$$

Thus

$$\forall\tilde\eta,\ \Delta\geq 0 \Leftrightarrow \tilde\beta^2 \leq 1+\tilde v^2.$$

For $d \geq 3$, choosing the coordinates such that
$\tilde v = (|\tilde v|, 0, \ldots, 0)$ and $\tilde\eta = (\tilde\eta_1, \tilde\eta')$, we get

$$\Delta = (1-\tilde\beta^2)(\tilde\eta_1^2 + |\tilde\eta'|^2) + \tilde\eta_1^2|\tilde v|^2,$$

thus $\Delta \geq 0$ for all $\tilde\eta$ if and only if $\tilde\beta^2 \leq 1$.

We proved that, if $\tilde\beta \leq 0$, the weak Lopatinski condition is satisfied and if $\tilde\beta > 0$, the weak Lopatinski condition is satisfied if and only if

$\tilde\beta^2 \leq 1+\tilde v^2$ and $\tilde\beta^2 \neq 1$ if $d = 2$,

$\tilde\beta^2 < 1$ if $d \geq 3$.

Finally, we obtain that the weak Lopatinski condition is satisfied if and only if

$\tilde\beta \leq \sqrt{1+\tilde v^2}$ and $\tilde\beta \neq 1$ if $d = 2$,

$\tilde\beta < 1$ if $d \geq 3$.

3.4.2 The uniform Lopatinski condition: proof of Proposition 3.2.10

When the square root λ_0, which has been defined for $(\tilde\tau - i\tilde\gamma, \tilde\eta), \tilde\gamma > 0$, can be extended at $(\tilde\tau, \tilde\eta)$ by continuity, the Lopatinski condition is satisfied at $(\tilde\tau, \tilde\eta)$ if the value of λ_0 at $(\tilde\tau, \tilde\eta)$ does not satisfy the equation (3.4.1).

The elliptic case: $\tilde\tau^2 - |\tilde\eta|^2 < 0$.

In this case, the limit of λ_0 is

$$\lambda_0 = i\sqrt{|\tilde\eta|^2 - \tilde\tau^2}.$$

Since $\tilde\beta$ and $\tilde v$ are real, the equation (3.4.1) is never satisfied.

Lemma 3.4.1. *The Lopatinski condition is always satisfied in the elliptic zone.*

The glancing case: $\tilde{\tau}^2 - |\tilde{\eta}|^2 = 0$, $(\tilde{\tau}, \tilde{\eta}) \neq (0,0)$**.**

In this case, the limit of λ_0 is $\lambda_0 = 0$.

Lemma 3.4.2. *The Lopatinski condition is not satisfied at a glancing point* $(\tilde{\tau}, \tilde{\eta})$ *if and only if*

$$\big(\tilde{\beta}\tilde{\tau} + \tilde{v} \cdot \tilde{\eta}\big)^2 = \tilde{\tau}^2 - |\tilde{\eta}|^2 = 0. \tag{3.4.4}$$

In particular, we have

$$\tilde{\beta}\tilde{\tau} + \tilde{v} \cdot \tilde{\eta} = 0, \qquad (\tilde{v} \cdot \tilde{\eta})^2 = \tilde{\beta}^2 |\tilde{\eta}|^2. \tag{3.4.5}$$

This lemma implies immediately the following proposition.

Proposition 3.4.3. *The Lopatinski condition is satisfied at all the glancing points if and only if*
i) $\tilde{\beta}^2 \neq |\tilde{v}|^2$ *in dimension* $d = 2$,
ii) $\tilde{\beta}^2 > |\tilde{v}|^2$ *in dimension* $d \geq 3$.

The hyperbolic case: $\tilde{\tau}^2 - |\tilde{\eta}|^2 > 0$.

The sign being deduced by continuity from the case $\tilde{\eta} = 0$, where the limit of λ_0 is $\lambda_0 = -\tilde{\tau}$, we obtain that the limit of λ_0 is

$$\lambda_0 = -\varepsilon\sqrt{\tilde{\tau}^2 - |\tilde{\eta}|^2}, \qquad \varepsilon = \operatorname{sign}\tilde{\tau},$$

that is

$$\lambda_0 = -\tilde{\tau}\sqrt{1 - \frac{|\tilde{\eta}|^2}{\tilde{\tau}^2}}.$$

Lemma 3.4.4. *The Lopatinski condition is violated at a hyperbolic point if and only if*

$$\big(\tilde{\beta}\tilde{\tau} + \tilde{v} \cdot \tilde{\eta}\big)^2 = \tilde{\tau}^2 - |\tilde{\eta}|^2 > 0, \tag{3.4.6}$$

$$\tilde{\tau}(\tilde{\beta}\tilde{\tau} + \tilde{v} \cdot \tilde{\eta}) > 0. \tag{3.4.7}$$

By homogeneity and symmetry, we can fix $\tilde{\tau} = 1$, in which case the conditions are:

$$\big(\tilde{\beta} + \tilde{v} \cdot \tilde{\eta}\big)^2 = 1 - |\tilde{\eta}|^2 > 0, \tag{3.4.8}$$

$$(\tilde{\beta} + \tilde{v} \cdot \tilde{\eta}) > 0. \tag{3.4.9}$$

In particular, (3.4.8) means that $\tilde{\eta}$ belongs to the ellipsoid

$$\Gamma := \big\{\tilde{\eta} \;/ |\tilde{\eta}|^2 + (\tilde{v} \cdot \tilde{\eta} + \tilde{\beta})^2 = 1 \;\big\}. \tag{3.4.10}$$

Lemma 3.4.5. *The Lopatinski condition is satisfied at each hyperbolic point if and only if the set* Γ *is contained in the half space* $\{\tilde{v} \cdot \tilde{\eta} + \tilde{\beta} \leq 0\}$.

To simplify the computations, we choose the coordinates such that $\tilde{v} = (|\tilde{v}|, 0, \ldots, 0)$ and $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}')$; then we obtain that the equation of Γ is

$$\begin{aligned}
&|\tilde{\eta}'|^2 + (1 + |\tilde{v}|^2)\tilde{\eta}_1^2 + 2\tilde{\beta}|\tilde{v}|\tilde{\eta}_1 + \tilde{\beta}^2 - 1 \\
&= |\tilde{\eta}'|^2 + (1 + |\tilde{v}|^2)\Big(\tilde{\eta}_1 + \frac{\tilde{\beta}|\tilde{v}|}{1 + |\tilde{v}|^2}\Big)^2 - \frac{1 - \tilde{\beta}^2 + |\tilde{v}|^2}{1 + |\tilde{v}|^2} \\
&= 0.
\end{aligned}$$

In particular, the condition

$$1 - \tilde{\beta}^2 + |\tilde{v}|^2 \geq 0 \tag{3.4.11}$$

is necessary in order to have Γ not empty.

Under this condition, we introduce

$$R^2 := \frac{1 - \tilde{\beta}^2 + |\tilde{v}|^2}{1 + |\tilde{v}|^2} \leq 1.$$

The set Γ is contained in $\{|\tilde{\eta}'| \leq R\}$. The roots in $\tilde{\eta}_1$ are

$$\tilde{\eta}_1^{\pm} = -\frac{\tilde{\beta}|\tilde{v}|}{1 + |\tilde{v}|^2} \pm \sqrt{\frac{R^2 - |\tilde{\eta}'|^2}{1 + |\tilde{v}|^2}}$$

and

$$\tilde{v} \cdot \tilde{\eta} + \tilde{\beta} = |\tilde{v}|\tilde{\eta}_1 + \tilde{\beta} = \frac{\tilde{\beta}}{1 + |\tilde{v}|^2} \pm |\tilde{v}|\sqrt{\frac{R^2 - |\tilde{\eta}'|^2}{1 + |\tilde{v}|^2}}.$$

Thus Γ is contained in $\{\tilde{v} \cdot \tilde{\eta} + \tilde{\beta} \leq 0\}$ if and only if

$$\tilde{\beta} + |\tilde{v}|\sqrt{R^2(1 + |\tilde{v}|^2)} = \tilde{\beta} + |\tilde{v}|\sqrt{1 - \tilde{\beta}^2 + |\tilde{v}|^2} \leq 0.$$

This condition is never satisfied for $\tilde{\beta} > 0$. For $\tilde{\beta} \leq 0$, the condition is equivalent to

$$\tilde{\beta}^2 \geq |\tilde{v}|^2(1 + |\tilde{v}|^2 - \tilde{\beta}^2) \geq 0,$$

that is

$$(1 + |\tilde{v}|^2)(\tilde{\beta}^2 - |\tilde{v}|^2) \geq 0.$$

We obtain that, when the condition (3.4.11) is satisfied, the Lopatinski condition is satisfied at each hyperbolic point if and only if $\tilde{\beta} \leq 0$ and

$\tilde{\beta}^2 \geq |\tilde{v}|^2$. Furthermore, when the condition (3.4.11) is not satisfied (that is when Γ is empty), the Lopatinski condition is satisfied at each hyperbolic point. Finally the Lopatinki condition is satisfied at each hyperbolic point if and only if $\tilde{\beta} > \sqrt{1+|\tilde{v}|^2}$ or $\tilde{\beta} \leq -|\tilde{v}|$. Since $\tilde{\beta} \leq \sqrt{1+|\tilde{v}|^2}$ under the weak Lopatinski condition, we get the following proposition.

Proposition 3.4.6. *Under the weak Lopatinski condition, the Lopatinski condition is satisfied at each hyperbolic point if and only if*

$$\tilde{\beta} \leq -|\tilde{v}|. \tag{3.4.12}$$

3.4.3 The (WR) condition: proof of Proposition 3.2.13

Lemma 3.4.7. *Under the weak Lopatinski condition,*

in dimension $d = 2$, the Lopatinski condition is violated at a hyperbolic point where the Lopatinski determinant $\tilde{D}$ and its derivative $\partial_{\tilde{\tau}}\tilde{D}$ vanish if and only if $\tilde{\beta} = \sqrt{1+\tilde{v}^2}$;

in dimension $d \geq 3$, if the Lopatinski condition is violated at a hyperbolic point, then the Lopatinski determinant $\tilde{D}$ vanishes at this point, whereas its derivative $\partial_{\tau}\tilde{D}$ does not vanish at this point.

Remark 3.4.8. *It follows from this lemma that, under the weak Lopatinski condition, if the Lopatinski determinant vanishes at a hyperbolic point, where its derivative $\partial_{\tilde{\tau}}\tilde{D}$ does not vanish, then the Lopatinski determinant vanishes and its derivative $\partial_{\tilde{\tau}}\tilde{D}$ does not vanish at each hyperbolic point where the Lopatinski condition is violated.*

Proof. We assume that the weak Lopatinski condition is satisfied.

If the Lopatinski determinant

$$\tilde{D}(\tilde{\tau},\tilde{\eta}) = \lambda_0 + \tilde{\beta}\tilde{\tau} + \tilde{v}\cdot\tilde{\eta}$$

vanishes at a hyperbolic point $(\underline{\tilde{\tau}},\underline{\tilde{\eta}})$ and satisfies $\partial_{\tilde{\tau}}\tilde{D}(\underline{\tilde{\tau}},\underline{\tilde{\eta}}) = 0$, then $\underline{\tilde{\tau}}$ is a root of order 2 of the polynomial in $\tilde{\zeta}$:

$$P_0 = \lambda_0^2 - \left(\tilde{\beta}\tilde{\zeta} + \tilde{v}\cdot\underline{\tilde{\eta}}\right)^2 = \tilde{\zeta}^2 - |\underline{\tilde{\eta}}|^2 - \left(\tilde{\beta}\tilde{\zeta} + \tilde{v}\cdot\underline{\tilde{\eta}}\right)^2.$$

Conversely, let $(\underline{\tilde{\tau}},\underline{\tilde{\eta}})$ be a hyperbolic point such that $\underline{\tilde{\tau}}$ is a root of order 2 of the the polynomial P_0.

From Lemma 3.4.4, the Lopatinski condition is violated at the hyperbolic point $(\underline{\tilde{\tau}},\underline{\tilde{\eta}})$ if and only if

$$\underline{\tilde{\tau}}(\tilde{\beta}\underline{\tilde{\tau}} + \tilde{v}\cdot\underline{\tilde{\eta}}) > 0.$$

Since $\underline{\tilde{\tau}}$ is a root of order 2 of P_0, we obtain by derivation that $\underline{\tilde{\tau}} - \tilde{\beta}(\tilde{\beta}\underline{\tilde{\tau}} + \tilde{v} \cdot \underline{\tilde{\eta}}) = 0$. Thus

$$\tilde{\beta}\underline{\tilde{\tau}}(\tilde{\beta}\underline{\tilde{\tau}} + \tilde{v} \cdot \underline{\tilde{\eta}}) = \underline{\tilde{\tau}}^2 > 0.$$

Thus, we get that the Lopatinski condition is violated at $(\underline{\tilde{\tau}}, \underline{\tilde{\eta}})$ if and only if

$$\tilde{\beta} > 0. \tag{3.4.13}$$

Under the weak Lopatinski condition, $\tilde{\beta} \neq 1$ (cf. Proposition 3.2.8); thus, if $\tilde{\beta} > 0$, then $\underline{\tilde{\eta}} \neq 0$ (otherwise, from $\underline{\tilde{\tau}}$ root of P_0 and $1 - \tilde{\beta}^2 \neq 0$, it follows $\underline{\tilde{\tau}} = 0$, which is absurd).

The discriminant $\Delta = (1 - \tilde{\beta}^2)|\underline{\tilde{\eta}}|^2 + (\tilde{v} \cdot \underline{\tilde{\eta}})^2$ of P_0 vanishes, since P_0 has a double root.

In dimension $d = 2$, the discriminant vanishes if and only if $\tilde{\beta} = \pm\sqrt{1 + \tilde{v}^2}$ or $\underline{\tilde{\eta}} = 0$. Therefore, from (3.4.13), the Lopatinski condition is violated at the hyperbolic point $(\underline{\tilde{\tau}}, \underline{\tilde{\eta}})$ if and only if

$$\tilde{\beta} = \sqrt{1 + \tilde{v}^2}.$$

Furthermore, in this case, the Lopatinski determinant $\tilde{D}$ and its derivative $\partial_{\tilde{\tau}}\tilde{D}$ vanish at the hyperbolic point $(\underline{\tilde{\tau}}, \underline{\tilde{\eta}})$, since the determinant vanishes (that is (3.4.1) is satisfied) and $\underline{\tilde{\tau}}$ is a root of order 2 of P_0. Finally, the Lopatinski condition is violated at a hyperbolic point where the Lopatinski determinant $\tilde{D}$ and its derivative $\partial_{\tilde{\tau}}\tilde{D}$ vanish if and only if $\tilde{\beta} = \sqrt{1 + \tilde{v}^2}$.

In dimension $d \geq 3$, the Lopatinski condition cannot be violated at $(\underline{\tilde{\tau}}, \underline{\tilde{\eta}})$. Indeed, if $\tilde{\beta} > 0$, then it follows from the weak Lopatinski condition (that is $\tilde{\beta} < 1$), that $1 - \tilde{\beta}^2 > 0$ and thus, since $\underline{\tilde{\eta}} \neq 0$, we get that the discriminant Δ does not vanish, which is absurd. Finally, if the Lopatinski condition is violated at a hyperbolic point, then the Lopatinski determinant $\tilde{D}$ vanishes at this point, whereas its derivative $\partial_{\tilde{\tau}}\tilde{D}$ does not vanish. □

The Lopatinski condition is not satisfied at, at least, one hyperbolic point, whereas the weak Lopatinski condition is satisfied, when:

$-|\tilde{v}| < \tilde{\beta} \leq \sqrt{1 + \tilde{v}^2}$ and $\tilde{\beta} \neq 1$ in dimension $d = 2$

$-|\tilde{v}| < \tilde{\beta} < 1$ in dimension $d \geq 3$.

We deduce then Proposition 3.2.13 from Lemma 3.4.7.

Remark 3.4.9. *From Propostion 3.4.3, in dimension $d \geq 3$, the Lopatinski condition fails at a glancing point if and only if $\tilde{\beta}^2 \leq |\tilde{v}|^2$. Therefore, for parameters such that $|\tilde{v}| \geq 1$, we obtain that, in dimension $d \geq 3$, if the (WR) condition is satisfied, then the Lopatinski condition fails at a glancing point.*

3.5 Energy estimates

Let us consider the equation and the boundary condition:

$$q(\partial_t, \partial_x, \partial_y)u = q_0(\partial_t, \partial_x, \partial_y)u + l(\partial_t, \partial_x, \partial_y)u = f \quad x > 0, \tag{3.5.1a}$$

$$(\partial_x + \beta\partial_t + v \cdot \partial_y + c)\, u = g \quad x = 0. \tag{3.5.1b}$$

3.5.1 Factorization of the differential equation (3.5.1a)

Lemma 3.5.1. *i) q can be written under the form*

$$\begin{aligned} q(i\zeta, i\xi, i\eta) &= \tilde{q}(i\tilde{\zeta}, i\xi, i\tilde{\eta}) \\ &= (\xi + s_0 + is_1)^2 - \tilde{\zeta}^2 + |\tilde{\eta}|^2 + p(i\tilde{\zeta}, i\tilde{\eta}), \end{aligned}$$

with $\tilde{\zeta}$, $\tilde{\eta}$ and s_0 defined in Lemma 3.3.1,
s_1 a real constant and
p a polynomial of degree 1 which does not depend on ξ.

ii) λ_0 defined in Lemma 3.3.1 satisfies

$$\mathrm{Im}\lambda_0 \geq \tilde{\gamma}, \qquad |\lambda_0|\mathrm{Im}\lambda_0 \gtrsim \gamma\Lambda. \tag{3.5.2}$$

iii) There is $\gamma_0 > 0$ such that, for all $\zeta = \tau - i\gamma, \gamma \geq \gamma_0$: there is a unique λ satisfying

$$\lambda^2 = \tilde{\zeta}^2 - |\tilde{\eta}|^2 - p(i\tilde{\zeta}, i\tilde{\eta}),$$

$$\mathrm{Im}\lambda > 0;$$

furthermore, λ is of the form $\lambda = \lambda_0 + \lambda'$, with $\lambda' = O(\frac{\Lambda}{|\lambda_0|})$, where Λ is defined in Definition 3.2.18, and satisfy

$$\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} - |s_1| \gtrsim \tilde{\gamma} \qquad |\lambda|(\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} - |s_1|) \gtrsim \gamma\Lambda; \tag{3.5.3}$$

introducing

$$\lambda_\pm := -s_0 - is_1 \pm \lambda,$$

we have

$$\tilde{q}(i\tilde{\zeta}, i\xi, i\tilde{\eta}) = (\xi - \lambda_+)(\xi - \lambda_-),$$

and

$$\pm\mathrm{Im}\lambda_\pm > 0.$$

Remark 3.5.2. *The notation $a \gtrsim b$ denotes $\exists C > 0,\ a \geq Cb$, with C a constant independent of the parameters under consideration.*

Proof. i) Lemma 3.3.1 yields, with $\tilde{l}$ the real linear form such that $l(i\zeta, i\xi, i\eta) = \tilde{l}(i\tilde{\zeta}, i\xi, i\tilde{\eta})$:

$$\begin{aligned} q(i\zeta, i\xi, i\eta) &= \tilde{q}(i\tilde{\zeta}, i\xi, i\tilde{\eta}) = \tilde{q}_0(i\tilde{\zeta}, i\xi, i\tilde{\eta}) + \tilde{l}(i\tilde{\zeta}, i\xi, i\tilde{\eta}) \\ &= (\xi + s_0)^2 - \tilde{\zeta}^2 + |\tilde{\eta}|^2 + \tilde{l}(i\tilde{\zeta}, i\xi, i\tilde{\eta}). \end{aligned}$$

We choose s_1 a constant, such that all the terms depending on ξ are in the square $(\xi + s_0 + is_1)^2$, more precisely, we choose $s_1 = \frac{l(0,1,0,\ldots,0)}{2} \in \mathbb{R}$; we thus get:

$$q(i\zeta, i\xi, i\eta) = \tilde{q}(i\tilde{\zeta}, i\xi, i\tilde{\eta}) = (\xi + s_0 + is_1)^2 - \tilde{\zeta}^2 + |\tilde{\eta}|^2 + p(i\tilde{\zeta}, i\tilde{\eta}),$$

with p a polynomial of degree 1 which does not depend on ξ.

For all $\tilde{\zeta} = \tilde{\tau} - i\tilde{\gamma}$, $\tilde{\gamma} > 0$,

$$\tilde{\zeta}^2 - |\tilde{\eta}|^2 - p(i\tilde{\zeta}, i\tilde{\eta}) = \lambda_0^2 - p(i\tilde{\zeta}, i\tilde{\eta}) = \lambda_0^2 \Big(1 - \frac{p(i\tilde{\zeta}, i\tilde{\eta})}{\lambda_0^2}\Big).$$

ii) We prove now (3.5.2).

From $\lambda_0^2 = \tilde{\zeta}^2 - |\tilde{\eta}|^2 = \tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2 - 2i\tilde{\tau}\tilde{\gamma}$, we get

$$\begin{aligned} |\lambda_0|^4 &= \big(\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2\big)^2 + 4\tilde{\tau}^2\tilde{\gamma}^2 \\ &= (\tilde{\tau}^2 - |\tilde{\eta}|^2)^2 + \tilde{\gamma}^2\big(2\tilde{\tau}^2 + 2|\tilde{\eta}|^2 + \tilde{\gamma}^2\big). \end{aligned}$$

Thus

$$|\lambda_0|^4 \geq \tilde{\gamma}^2\big(\tilde{\tau}^2 + |\tilde{\eta}|^2 + \tilde{\gamma}^2\big) \approx \tilde{\gamma}^2\Lambda^2,$$

that is

$$|\lambda_0|^2 \geq \tilde{\gamma}\Lambda. \tag{3.5.4}$$

Writing $\lambda_0^2 = |\lambda_0|^2 e^{i\theta}$ with $\theta \in]0, 2\pi[$ yields $\mathrm{Im}\lambda_0 = |\lambda_0| \sin(\frac{1}{2}\theta)$, since $\mathrm{Im}\lambda_0 > 0$.

1) Let us prove $|\lambda_0|\mathrm{Im}\lambda_0 \gtrsim \gamma\Lambda$.

a) If $\theta \in [\pi/4, 7\pi/4]$ then $\sin(\frac{1}{2}\theta) \geq \sin(\pi/8)$ and

$$\mathrm{Im}\lambda_0 \gtrsim |\lambda_0|.$$

From (3.5.4), we get $|\lambda_0|\mathrm{Im}\lambda_0 \gtrsim \tilde{\gamma}\Lambda \gtrsim \gamma\Lambda$.

b) If $\theta \in]0, \pi/4] \cup [7\pi/4, 2\pi[$, then $\cos\theta > 0$, thus $\mathrm{Re}(\lambda_0^2) = |\lambda_0|^2 \cos\theta \geq 0$. Since $\mathrm{Re}(\lambda_0^2) = \tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2$, we get $|\tilde{\eta}|^2 + \tilde{\gamma}^2 \leq \tilde{\tau}^2$ and thus

$$\Lambda^2 = \tau^2 + \gamma^2 + |\eta|^2 \lesssim \tilde{\tau}^2 + \tilde{\gamma}^2 + |\tilde{\eta}|^2 \lesssim \tilde{\tau}^2.$$

Moreover $|\sin\theta| = 2\underbrace{\sin(\frac{1}{2}\theta)}_{>0}|\cos(\frac{1}{2}\theta)| \leq 2\sin(\frac{1}{2}\theta)$, therefore

$$|\lambda_0|\mathrm{Im}\lambda_0 = |\lambda_0|^2\sin(\frac{1}{2}\theta) \geq \frac{1}{2}|\lambda_0|^2|\sin\theta| = \frac{1}{2}|\mathrm{Im}(\lambda_0^2)| = |\tilde{\tau}|\tilde{\gamma} \gtrsim \Lambda\gamma.$$

2) Let us show that $\mathrm{Im}\lambda_0 \geq \tilde{\gamma}$.

From $\lambda_0^2 = \tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2 - 2i\tilde{\tau}\tilde{\gamma} = (\mathrm{Re}\lambda_0 + i\mathrm{Im}\lambda_0)^2$, we get

$$(\mathrm{Re}\lambda_0)^2 - (\mathrm{Im}\lambda_0)^2 = \tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2 =: S,$$

$$2\mathrm{Re}\lambda_0\mathrm{Im}\lambda_0 = -2\tilde{\tau}\tilde{\gamma},$$

that is

$$(\mathrm{Re}\lambda_0)^2(-(\mathrm{Im}\lambda_0)^2) = -\tilde{\tau}^2\tilde{\gamma}^2 =: P.$$

Therefore $(\mathrm{Re}\lambda_0)^2$ and $-(\mathrm{Im}\lambda_0)^2$ are the roots of $X^2 - SX + P$ and thus are equal to

$$\frac{\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2}{2} \pm \frac{\sqrt{(\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2)^2 + 4\tilde{\tau}^2\tilde{\gamma}^2}}{2}.$$

Since $-(\mathrm{Im}\lambda_0)^2 \leq 0 \leq (\mathrm{Re}\lambda_0)^2$, we get
$2(\mathrm{Im}\lambda_0)^2 = \sqrt{(\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2)^2 + 4\tilde{\tau}^2\tilde{\gamma}^2} - \left(\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2\right).$

Writing

$$\begin{aligned}\left(\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2\right)^2 + 4\tilde{\tau}^2\tilde{\gamma}^2 &= \left(\tilde{\tau}^2 + \tilde{\gamma}^2 - |\tilde{\eta}|^2\right)^2 + 4|\tilde{\eta}|^2\tilde{\gamma}^2 \\ &\geq \left(\tilde{\tau}^2 + \tilde{\gamma}^2 - |\tilde{\eta}|^2\right)^2\end{aligned}$$

yields

$$\begin{aligned}2(\mathrm{Im}\lambda_0)^2 &\geq |\tilde{\tau}^2 + \tilde{\gamma}^2 - |\tilde{\eta}|^2| - \left(\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2\right) \\ &\geq \tilde{\tau}^2 + \tilde{\gamma}^2 - |\tilde{\eta}|^2 - \left(\tilde{\tau}^2 - \tilde{\gamma}^2 - |\tilde{\eta}|^2\right) \\ &= 2\tilde{\gamma}^2,\end{aligned}$$

and finally $\mathrm{Im}\lambda_0 \geq \tilde{\gamma}$.

This finishes the proof of (3.5.2).

iii) For γ large, $p(i\tilde{\zeta}, i\tilde{\eta}) = O(\Lambda)$; therefore, since $|\lambda_0|^2 \gtrsim \gamma\Lambda$ from (3.5.2), we get $\mathrm{Re}(1 - \frac{p(i\tilde{\zeta},i\tilde{\eta})}{\lambda_0^2}) > 0$ for γ large. We can define $\lambda = \lambda_0\sqrt{1 - \frac{p(i\tilde{\zeta},i\tilde{\eta})}{\lambda_0^2}}$, where, for $a = \rho e^{i\theta}$, $\rho \in \mathbb{R}_+$, $\theta \in]-\pi, \pi[$, $\sqrt{a}$ is defined by $\sqrt{a} = \sqrt{\rho}e^{i\theta/2}$. λ is of the form $\lambda = \lambda_0 + \lambda'$, with $\lambda' = O(\frac{p(i\tilde{\zeta}, i\tilde{\eta})}{|\lambda_0|}) = O(\frac{\Lambda}{|\lambda_0|})$. Since, from (3.5.2), $\frac{\Lambda}{|\lambda_0|} \lesssim \frac{\mathrm{Im}\lambda_0}{\gamma}$, we get that, for γ large, $\mathrm{Im}\lambda > 0$.

We prove now (3.5.3).

From (3.5.2), we have the estimates

$$|\mathrm{Im}\lambda'| \leq |\lambda'| \lesssim \frac{\Lambda}{|\lambda_0|} \lesssim \frac{\mathrm{Im}\lambda_0}{\gamma} \lesssim \frac{|\lambda_0|}{\gamma}.$$

1)$|\lambda|\mathrm{Im}\lambda \gtrsim \gamma\Lambda$.

It follows from the estimates $|\lambda'| \lesssim \frac{|\lambda_0|}{\gamma}$ and $|\mathrm{Im}\lambda'| \lesssim \frac{\mathrm{Im}\lambda_0}{\gamma}$ that, for γ large,

$$|\lambda| \gtrsim |\lambda_0| \qquad \mathrm{Im}\lambda \gtrsim \mathrm{Im}\lambda_0,$$

therefore, from (3.5.2):

$$|\lambda|\mathrm{Im}\lambda \gtrsim |\lambda_0|\mathrm{Im}\lambda_0 \gtrsim \gamma\Lambda.$$

2) $\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} \gtrsim \tilde{\gamma}$.

Since $|\alpha| < 1$, there exists $\epsilon > 0$ such that $|\alpha| + \epsilon < 1$.
$\mathrm{Im}\lambda \geq \mathrm{Im}\lambda_0 - |\mathrm{Im}\lambda'|$ and $|\mathrm{Im}\lambda'| \lesssim \frac{\mathrm{Im}\lambda_0}{\gamma}$, thus there is $\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$,

$$\mathrm{Im}\lambda \geq (1-\epsilon)\mathrm{Im}\lambda_0.$$

Therefore, $\mathrm{Im}\lambda_0 \geq \tilde{\gamma}$ yields $\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} \geq \underbrace{(1-\epsilon-|\alpha|)}_{>0}\tilde{\gamma}$.

3) $(\mathrm{Im}\lambda - |\alpha|\tilde{\gamma})|\lambda| \gtrsim \mathrm{Im}\lambda|\lambda|$.

$\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} \geq c\tilde{\gamma}$, with $c = 1 - \epsilon - |\alpha| > 0$, that is $\mathrm{Im}\lambda - (|\alpha| + c)\tilde{\gamma} \geq 0$: to obtain $\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} \geq c'\,\mathrm{Im}\lambda$, that is $(1-c')\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} \geq 0$, it is sufficient to have $\frac{|\alpha|}{1-c'} = |\alpha| + c$ and $c' \leq 1$. Thus, with $c' = \frac{c}{|\alpha|+c} > 0$, we get $\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} \geq c'\,\mathrm{Im}\lambda$ and finally $(\mathrm{Im}\lambda - |\alpha|\tilde{\gamma})|\lambda| \gtrsim \mathrm{Im}\lambda|\lambda|$.

4) $(\mathrm{Im}\lambda - |\alpha|\tilde{\gamma})|\lambda| \gtrsim \gamma\Lambda$.

Applying 3) and 1), we get:

$$(\mathrm{Im}\lambda - |\alpha|\tilde{\gamma})|\lambda| \gtrsim \mathrm{Im}\lambda|\lambda| \gtrsim \gamma\Lambda.$$

Finally, s_1 being a constant, we get from 4) and 2) the estimates (3.5.3) for γ large.

For all $\tilde{\zeta} = \tilde{\tau} - i\tilde{\gamma}$, γ large, the roots in ξ of $\tilde{q}(i\tilde{\zeta}, i\xi, i\tilde{\eta})$ are equal to

$$\lambda_\pm = -s_0 - is_1 \pm \lambda.$$

From (3.5.3), there is $\gamma_0 > 0$ such that, for all $\gamma \geq \gamma_0$,
$\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} \gtrsim \tilde{\gamma}$; since $\mathrm{Im}\lambda_\pm = \alpha\tilde{\gamma} \pm \mathrm{Im}\lambda - s_1$, we get that, for γ large, $\pm\mathrm{Im}\lambda_\pm > 0$. $\square$

Recall that $\tilde{\gamma} = a\gamma$. Proving an estimate with γ is equivalent to proving the same estimate with $\tilde{\gamma}$ (only the constants change).

3.5.2 Optimal L^2 estimates

It follows from Lemma 3.5.1 that taking the tangential Fourier Laplace transform of the problem (3.5.1) yields the following ODE and boundary condition:

$$(\partial_x - i\lambda_+)(\partial_x - i\lambda_-)\hat{u} = -\hat{f} \tag{3.5.5}$$
$$\partial_x \hat{u}(0) + (i\mu + c)\hat{u}(0) = \hat{g}, \tag{3.5.6}$$

where $\hat{u}, \hat{f}$ and $\hat{g}$ are the Fourier Laplace transforms of u, f and g, and $\mu = \beta\zeta + v \cdot \eta = \beta(\tau - i\gamma) + v \cdot \eta$.

Remark 3.5.3. *The form* (3.3.2) *of the boundary condition yields*

$$\lambda_{0,+} + \mu = \lambda_{0,+} + \beta(\tau - i\gamma) + v \cdot \eta = \lambda_0 + \tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v} \cdot \tilde{\eta},$$

that is, since $\lambda_{0,+} = \lambda_0 - s_0 = \lambda_0 - (\alpha(\tilde{\tau} - i\tilde{\gamma}) + \nu \cdot \tilde{\eta})$*,*

$$\mu = (\tilde{\beta} + \alpha)\tilde{\zeta} + (\tilde{v} + \nu) \cdot \tilde{\eta} = (\tilde{\beta} + \alpha)(\tilde{\tau} - i\tilde{\gamma}) + (\tilde{v} + \nu) \cdot \tilde{\eta}.$$

Introducing $\hat{u}_\pm = \partial_x \hat{u} - i\lambda_\mp \hat{u}$, we obtain the equations:

$$\partial_x \hat{u}_+ - i\lambda_+ \hat{u}_+ = -\hat{f} \quad x > 0, \tag{3.5.7}$$
$$\partial_x \hat{u}_- - i\lambda_- \hat{u}_- = -\hat{f} \quad x > 0. \tag{3.5.8}$$

Writing $\hat{u} = i\frac{\hat{u}_- - \hat{u}_+}{2\lambda}$ and $\partial_x \hat{u} = \frac{\lambda_+ \hat{u}_+ - \lambda_- \hat{u}_-}{2\lambda}$ yields the boundary condition

$$\frac{\lambda_+ \hat{u}_+(0) - \lambda_- \hat{u}_-(0)}{2\lambda} + \frac{\mu - ic}{2\lambda}(\hat{u}_+(0) - \hat{u}_-(0)) = \hat{g}. \tag{3.5.9}$$

Lemma 3.5.4. *For γ large,*

$$-\mathrm{Im}\lambda_- \left\|\hat{u}_-\right\|^2_{L^2(\mathbb{R}^+)} + \left|\hat{u}_-(0)\right|^2 \leq \frac{1}{-\mathrm{Im}\lambda_-}\left\|\hat{f}\right\|^2_{L^2(\mathbb{R}^+)},$$

that is

$$(\mathrm{Im}\lambda - \alpha\tilde{\gamma} + s_1)\left\|\hat{u}_-\right\|^2_{L^2(\mathbb{R}^+)} + \left|\hat{u}_-(0)\right|^2 \leq \frac{1}{\mathrm{Im}\lambda - \alpha\tilde{\gamma} + s_1}\left\|\hat{f}\right\|^2_{L^2(\mathbb{R}^+)}.$$

Proof. For γ large, λ_- is well defined. Furthermore $\mathrm{Im}\lambda_- < 0$: it is possible to divide by $-\mathrm{Im}\lambda_-$.

Multiplying (3.5.8) by $-2\overline{\hat{u}_-}$, taking the real part and integrating on $[0,+\infty[$ yields

$$\begin{aligned}\left|\hat{u}_-(0)\right|^2 + 2(-\mathrm{Im}\lambda_-)\left\|\hat{u}_-\right\|^2_{L^2(\mathbb{R}^+)} &= 2\int_0^\infty \mathrm{Re}(\hat{f}\,\overline{\hat{u}_-}) \\ &\leq 2\int_0^\infty |\hat{f}\,\overline{\hat{u}_-}| \\ &\leq 2\left\|\hat{f}\right\|_{L^2(\mathbb{R}^+)}\left\|\hat{u}_-\right\|_{L^2(\mathbb{R}^+)} \\ &\leq \frac{\left\|\hat{f}\right\|^2_{L^2(\mathbb{R}^+)}}{-\mathrm{Im}\lambda_-} + (-\mathrm{Im}\lambda_-)\left\|\hat{u}_-\right\|^2_{L^2(\mathbb{R}^+)}.\end{aligned}$$

□

Lemma 3.5.5. *For γ large,*

$$\mathrm{Im}\lambda_+\left\|\hat{u}_+\right\|^2_{L^2(\mathbb{R}^+)} \leq \frac{1}{\mathrm{Im}\lambda_+}\left\|\hat{f}\right\|^2_{L^2(\mathbb{R}^+)} + \left|\hat{u}_+(0)\right|^2,$$

that is

$$(\mathrm{Im}\lambda + \alpha\tilde{\gamma} - s_1)\left\|\hat{u}_+\right\|^2_{L^2(\mathbb{R}^+)} \leq \frac{1}{\mathrm{Im}\lambda + \alpha\tilde{\gamma} - s_1}\left\|\hat{f}\right\|^2_{L^2(\mathbb{R}^+)} + \left|\hat{u}_+(0)\right|^2.$$

The proof of this lemma is similar to the proof of Lemma 3.5.4.

The boundary condition yields

$$(\lambda_+ + \mu - ic)\hat{u}_+(0) = (\lambda_- + \mu - ic)\ \hat{u}_-(0) + 2\lambda\hat{g}. \tag{3.5.10}$$

Introducing

$$\rho = \frac{\lambda_+ + \mu - ic}{\Lambda}, \tag{3.5.11}$$

we get from (3.5.10):

$$\left|\rho\hat{u}_+(0)\right| \lesssim \left|\hat{u}_-(0)\right| + \frac{|\lambda|}{\Lambda}\left|\hat{g}\right|, \tag{3.5.12}$$

where Λ has been defined in Definition 3.2.18.

Then, we deduce from Lemmas 3.5.4 and 3.5.5:

Proposition 3.5.6.

$$(\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} - |s_1|)\Big(\big\|\hat{u}_-\big\|^2_{L^2(\mathbb{R}^+)} + \big\|\rho\hat{u}_+\big\|^2_{L^2(\mathbb{R}^+)}\Big) + \big|\hat{u}_-(0)\big|^2 + \big|\rho\hat{u}_+(0)\big|^2 \lesssim \frac{1}{\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} - |s_1|}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)} + \big|\frac{\lambda}{\Lambda}\hat{g}\big|^2. \tag{3.5.13}$$

From Lemma 3.5.1, we have

$$\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} - |s_1| \gtrsim \tilde{\gamma} \qquad |\lambda|(\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} - |s_1|) \gtrsim \gamma\Lambda. \tag{3.5.14}$$

We are now looking for lower bounds of ρ.

3.5.3 Lower bound of ρ under the (WR) condition

Lower bound of $|\lambda_{0,+} + \mu|$: the homogeneous case

Proposition 3.5.7. *If the condition (WR) is satisfied, then*

$$|\lambda_{0,+} + \mu| \gtrsim \gamma. \tag{3.5.15}$$

Proof. When $\tilde{\beta} \leq 0$, from (3.5.2), $\mathrm{Im}(\lambda_{0,+} + \mu) = \mathrm{Im}\lambda_0 - \tilde{\beta}\tilde{\gamma} \geq \mathrm{Im}\lambda_0 \gtrsim \gamma$.

When $\tilde{\beta} > 0$, we factorize the polynomial P_0 defined by (3.4.2) and derive a lower bound for each of its two factors.

The condition (WR) implies that

$$\Delta \approx |\tilde{\eta}|^2 \approx |\eta|^2, \tag{3.5.16}$$

where Δ is the discriminant of the polynomial P_0.

Indeed, when $d = 2$, $\Delta = \underbrace{(1 + \tilde{v}^2 - \tilde{\beta}^2)}_{>0}|\tilde{\eta}|^2$,

when $d \geq 3$, $\underbrace{(1 - \tilde{\beta}^2)}_{>0}|\tilde{\eta}|^2 \leq \Delta \leq (1 - \tilde{\beta}^2 + |\tilde{v}|^2)|\tilde{\eta}|^2$.

Let us denote by $\zeta_\pm$ the roots $\frac{\tilde{\beta}(\tilde{v}\cdot\tilde{\eta})}{1-\tilde{\beta}^2} \pm \frac{\sqrt{\Delta}}{1-\tilde{\beta}^2}$ of

$$\begin{aligned} P_0 =& \lambda_0^2 - (\tilde{\beta}(\tilde{\tau} - i\tilde{\gamma}) + \tilde{v}\cdot\tilde{\eta})^2 \\ =& -(\lambda_{0,+} + \mu)(\lambda_{0,-} + \mu) \\ =& (1 - \tilde{\beta}^2)(\tilde{\tau} - i\tilde{\gamma} - \zeta_+)(\tilde{\tau} - i\tilde{\gamma} - \zeta_-). \end{aligned}$$

When $\tilde{\eta} = 0$, we get $\Delta = 0$ and $\zeta_\pm = 0$.

When $\tilde{\eta} \neq 0$, we get $\Delta \neq 0$ and thus the roots $\zeta_\pm$ are distinct.

Therefore, the functions $f_\pm : (\tilde\tau, \tilde\gamma, \tilde\eta) \mapsto |\tilde\tau - i\tilde\gamma - \zeta_\pm(\tilde\eta)|$ do not vanish simultaneously on the half sphere $\mathcal{S}_+ : \begin{cases} \tilde\tau^2 + \tilde\gamma^2 + |\tilde\eta|^2 = 1 \\ \tilde\gamma \geq 0 \end{cases}$. Indeed, when $\tilde\eta = 0$, their product is equal to $|\tilde\tau - i\tilde\gamma|^2 \neq 0$ and when $\tilde\eta \neq 0$, the roots $\zeta_\pm$ are distinct. We can construct for each point x of $\mathcal{S}_+$ an open neighbourhood $\mathcal{V}_x$ in $\mathcal{S}_+$ such that either f_+ or f_- does not vanish on the closure $\overline{\mathcal{V}_x}$. Suppose that this is f_+, the other case being similar. The compactness of $\overline{\mathcal{V}_x}$ yields the existence of a constant $C_x > 0$ such that

$$\forall(\tilde\tau, \tilde\gamma, \tilde\eta) \in \mathcal{V}_x \;\; f_+(\tilde\tau, \tilde\gamma, \tilde\eta) \geq C_x.$$

Since $\forall(\tilde\tau, \tilde\gamma, \tilde\eta) \in \mathbb{R}^3 \;\; f_-(\tilde\tau, \tilde\gamma, \tilde\eta) = |\tilde\tau - i\tilde\gamma - \zeta_-(\tilde\eta)| \geq \tilde\gamma$, we get

$$\forall(\tilde\tau, \tilde\gamma, \tilde\eta) \in \mathcal{V}_x \;\; f_+ f_-(\tilde\tau, \tilde\gamma, \tilde\eta) \geq C_x \tilde\gamma.$$

The compactness of $\mathcal{S}_+$ yields the existence of a finite cover $\{\mathcal{V}_{x_i}\}_{1\leq i\leq s}$ of $\mathcal{S}_+$. With $C = \min_{1\leq i\leq s} C_{x_i}$, we get

$$\forall(\tilde\tau, \tilde\gamma, \tilde\eta) \in \mathcal{S}_+ \;\; f_+ f_-(\tilde\tau, \tilde\gamma, \tilde\eta) = |(\tilde\tau - i\tilde\gamma - \zeta_+)(\tilde\tau - i\tilde\gamma - \zeta_-)| \geq C\tilde\gamma.$$

Homogeneity yields

$$\forall(\tilde\tau, \tilde\gamma, \tilde\eta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \;\; |(\tilde\tau - i\tilde\gamma - \zeta_+)(\tilde\tau - i\tilde\gamma - \zeta_-)| \geq C\sqrt{\tilde\tau^2 + \tilde\gamma^2 + |\tilde\eta|^2}\,\tilde\gamma,$$

thus

$$\begin{aligned} |(\lambda_{0,+} + \mu)(\lambda_{0,-} + \mu)| &= |1 - \tilde\beta^2||(\tilde\tau - i\tilde\gamma - \zeta_+)(\tilde\tau - i\tilde\gamma - \zeta_-)| \\ &\gtrsim \tilde\gamma\sqrt{\tilde\tau^2 + \tilde\gamma^2 + |\tilde\eta|^2} \\ &\gtrsim \gamma\sqrt{\tau^2 + \gamma^2 + |\eta|^2}. \end{aligned}$$

Since $|\lambda_{0,-} + \mu| \lesssim \sqrt{\tau^2 + \gamma^2 + |\eta|^2}$, we finally proved

$$|\lambda_{0,+} + \mu| \gtrsim \gamma.$$

□

Lower bound of $|\lambda_+ + \mu - ic|$: the general case

Theorem 3.5.8. *If the condition (WR) is satisfied, then there is $\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$:*

$$|\lambda_+ + \mu - ic| \gtrsim \gamma, \tag{3.5.17}$$

that is

$$|\rho| \gtrsim \frac{\gamma}{\Lambda}.$$

Proof. a) When $\tilde{\beta} \leq 0$, the estimate is trivial since

$$\text{Im}(\lambda_+ + \mu - ic) = \text{Im}\lambda - \tilde{\beta}\tilde{\gamma} - s_1 - c \geq \text{Im}\lambda - s_1 - c \gtrsim \gamma - s_1 - c \gtrsim \gamma,$$

for γ large.

b) Suppose that $\tilde{\beta} > 0$. Consider

$$\begin{aligned} P &= -(\lambda_+ + \mu - ic)(\lambda_- + \mu - ic) \\ &= \lambda^2 - \big(\mu - ic - (s_0 + is_1)\big)^2 \\ &= \lambda^2 - \big(\tilde{\beta}\tilde{\zeta} + \tilde{v} \cdot \tilde{\eta} - is_1 - ic\big)^2 \\ &= P_0 + P', \end{aligned}$$

with $P_0 = \lambda_0^2 - \big(\tilde{\beta}\tilde{\zeta} + \tilde{v} \cdot \tilde{\eta}\big)^2$.

$$\begin{aligned} P' = \lambda^2 - \lambda_0^2 - \big[&\big(\tilde{\beta}\tilde{\zeta} + \tilde{v} \cdot \tilde{\eta} - is_1 - ic\big)^2 \\ &\qquad - \big(\tilde{\beta}\tilde{\zeta} + \tilde{v} \cdot \tilde{\eta}\big)^2\big] \\ = 2\lambda_0\lambda' &+ \lambda'^2 + (s_1 + c)^2 + 2i(s_1 + c)\big(\tilde{\beta}\tilde{\zeta} + \tilde{v} \cdot \tilde{\eta}\big). \end{aligned}$$

Therefore, since $\lambda' = O(\Lambda/|\lambda_0|)$ and, from (3.5.2), $\frac{\Lambda}{|\lambda_0|^2} = O(\frac{1}{\gamma})$, we get, for γ large: $P' = O(\Lambda)$.

We have shown, in the proof of Proposition 3.5.7, that

$$|P_0| \gtrsim \gamma\Lambda.$$

We see then that, for γ large:

$$|P| \gtrsim \gamma\Lambda.$$

Since $|\lambda_- + \mu - ic| \lesssim \Lambda$, this implies that $|\lambda_+ + \mu - ic| \gtrsim \gamma$. □

3.5.4 Energy estimate under the (WR) condition

L^2 estimate

We deduce from Proposition 3.5.6 and the estimates (3.5.14):

Theorem 3.5.9. *There is $\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$:*

$$\begin{aligned} \gamma|\rho|^2 &\Big(\big\|\partial_x \hat{u}\big\|^2_{L^2(\mathbb{R}^+)} + \Lambda^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}\Big) \\ &+ |\rho|^2\Big(\big|\partial_x \hat{u}(0)\big|^2 + \Lambda^2\big|\hat{u}(0)\big|^2\Big) \leq C\left(\frac{1}{\gamma}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)} + \big|\hat{g}\big|^2\right), \end{aligned} \tag{3.5.18}$$

with C a constant independent of γ.

Proof. 1) Estimate of $\gamma|\rho|^2\Big(\big\|\partial_x\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}+|\Lambda|^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}\Big)$

$$\begin{aligned}\partial_x\hat{u} &= \frac{\lambda_+\hat{u}_+-\lambda_-\hat{u}_-}{2\lambda}\\ &= \frac{\hat{u}_++\hat{u}_-}{2}+s\frac{\hat{u}_--\hat{u}_+}{2\lambda} \qquad \text{with } s=s_0+is_1=O(\Lambda) \text{ for } \gamma \text{ large}\\ &= \frac{\hat{u}_++\hat{u}_-}{2}-is\hat{u}.\end{aligned}$$

Thus, for γ large, $\big\|\partial_x\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}\lesssim\big\|\hat{u}_+\big\|^2_{L^2(\mathbb{R}^+)}+\big\|\hat{u}_-\big\|^2_{L^2(\mathbb{R}^+)}+\Lambda^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}$.
From Proposition 3.5.6, the estimates (3.5.14) and since $|\rho|\lesssim 1$, we obtain:

$$\begin{aligned}\gamma\big\|\rho\partial_x\hat{u}\big\|^2_{L^2(\mathbb{R}^+)} &\lesssim \gamma\Lambda^2|\rho|^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}\\ &\qquad +(\mathrm{Im}\lambda-|\alpha|\tilde{\gamma}-|s_1|)\big(\big\|\hat{u}_-\big\|^2_{L^2(\mathbb{R}^+)}+\big\|\rho\hat{u}_+\big\|^2_{L^2(\mathbb{R}^+)}\big)\\ &\lesssim \gamma\Lambda^2|\rho|^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}+\frac{1}{\mathrm{Im}\lambda-|\alpha|\tilde{\gamma}-|s_1|}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)}+\big|\frac{\lambda}{\Lambda}\hat{g}\big|^2\\ &\lesssim \gamma\Lambda^2|\rho|^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}+\frac{1}{\gamma}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)}+\big|\hat{g}\big|^2.\end{aligned}$$

From $\hat{u}=i\frac{\hat{u}_--\hat{u}_+}{2\lambda}$ and Proposition 3.5.6, we get

$$(\mathrm{Im}\lambda-|\alpha|\tilde{\gamma}-|s_1|)\big\|\rho\lambda\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}\lesssim\frac{1}{\mathrm{Im}\lambda-|\alpha|\tilde{\gamma}-|s_1|}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)}+\big|\frac{\lambda}{\Lambda}\hat{g}\big|^2;$$

thus, from the estimates (3.5.14),

$$\begin{aligned}\gamma|\rho|^2\Lambda^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)} &\lesssim \frac{\gamma\Lambda^2\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)}}{|\lambda|^2(\mathrm{Im}\lambda-|\alpha|\tilde{\gamma}-|s_1|)^2}+\frac{\gamma\big|\hat{g}\big|^2}{\mathrm{Im}\lambda-|\alpha|\tilde{\gamma}-|s_1|}\\ &\lesssim \frac{1}{\gamma}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)}+\big|\hat{g}\big|^2;\end{aligned}$$

thus, finally

$$\gamma|\rho|^2\Big(\big\|\partial_x\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}+\Lambda^2\big\|\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}\Big)\lesssim\frac{1}{\gamma}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)}+\big|\hat{g}\big|^2.$$

2) Estimate of $|\rho|^2\Big(\big|\partial_x\hat{u}(0)\big|^2+\Lambda^2\big|\hat{u}(0)\big|^2\Big)$

Similarly as for $\big\|\partial_x\hat{u}\big\|^2_{L^2(\mathbb{R}^+)}$, we get

$$|\rho|^2\big|\partial_x\hat{u}(0)\big|^2\lesssim|\rho|^2\Lambda^2\big|\hat{u}(0)\big|^2+\frac{1}{\gamma}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)}+\big|\hat{g}\big|^2.$$

$\hat{u}(0) = i\frac{\hat{u}_-(0)-\hat{u}_+(0)}{2\lambda}$ and the boundary condition $\frac{\lambda_++\mu-ic}{2\lambda}\hat{u}_+(0) = \frac{\lambda_-+\mu-ic}{2\lambda}\hat{u}_-(0) + \hat{g}$ yield

$$i(\lambda_+ + \mu - ic)\hat{u}(0) = \hat{g} - \hat{u}_-(0),$$

thus $|\rho\Lambda\hat{u}(0)| \leq |\hat{u}_-(0)| + |\hat{g}|$ and Proposition 3.5.6 yields

$$|\rho\Lambda\hat{u}(0)|^2 \lesssim \frac{1}{\mathrm{Im}\lambda - |\alpha|\tilde{\gamma} - |s_1|}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)} + \big|\hat{g}\big|^2 \lesssim \frac{1}{\gamma}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)} + \big|\hat{g}\big|^2.$$

Thus, finally

$$|\rho|^2\Big(\big|\partial_x\hat{u}(0)\big|^2 + \Lambda^2\big|\hat{u}(0)\big|^2\Big) \lesssim \frac{1}{\gamma}\big\|\hat{f}\big\|^2_{L^2(\mathbb{R}^+)} + \big|\hat{g}\big|^2.$$

This finishes the proof of the estimate (3.5.18)

□

Using the inverse Fourier transform with respect to (t, y), we deduce from Theorem 3.5.8, Theorem 3.5.9 and Plancherel theorem the following corollary (see section 3.2.3 for the definition of the norms):

Corollary 3.5.10. *Under the (WR) condition, there is $\gamma_0 > 0$ such that, for all $\gamma \geq \gamma_0$, the following estimate holds:*

$$\begin{aligned}\gamma^2&\Big(\big\|\partial_x u\big\|_{\gamma,-1} + \big\|u\big\|_{\gamma,0}\Big)\\ &+ \gamma^{\frac{3}{2}}\Big(\big|\partial_x u(0)\big|_{\gamma,-1} + \big|u(0)\big|_{\gamma,0}\Big) \leq C\left(\big\|f\big\|_{\gamma,0} + \gamma^{\frac{1}{2}}\big|g\big|_{\gamma,0}\right),\end{aligned} \tag{3.5.19}$$

with C a constant independent of γ.

Striated estimates: Proof of Theorem 3.2.23

The fields Y_j, defined in section 3.2.2, commute exactly with the equation and the boundary condition, thus the corollary 3.5.10 yields:

Theorem 3.5.11. *Under the (WR) condition, there is $\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$, the following estimate holds:*

$$\begin{aligned}\gamma^2&\Big(\big\|\partial_x u\big\|_{m,\gamma,-1} + \big\|u\big\|_{m,\gamma,0}\Big)\\ &+ \gamma^{\frac{3}{2}}\Big(\big|\partial_x u(0)\big|_{m,\gamma,-1} + \big|u(0)\big|_{m,\gamma,0}\Big) \leq C\left(\big\|f\big\|_{m,\gamma,0} + \gamma^{\frac{1}{2}}\big|g\big|_{m,\gamma,0}\right),\end{aligned} \tag{3.5.20}$$

with C a constant independent of γ.

We now show how Theorem 3.2.23 follows from Theorem 3.5.11. We first solve the boundary value problem for a fixed $\gamma > 0$. From the general analysis by Coulombel ([23]) about the weak well-posedness of hyperbolic initial boundary value problems, we can pass from a priori energy estimates with a loss of one derivative to weak well-posedness. We thus obtain the well-posedness from the estimate (3.5.20). By passing to the limit as $\gamma \to +\infty$, we get, by classical arguments, the causality principle (see [17]), that is that, if, for $T_0 \in \mathbb{R}$, f and g vanish on $\{t \le T_0\}$, then the striated solution also vanishes on $\{t \le T_0\}$. For f in $L^2_x(H^{0,m}(\Omega^0_T))$ and g in $H^{0,m}(\Omega^0_T)$, let $\overline{f}$ and $\overline{g}$ be functions in $L^2_x(H^{0,m}(\Omega^0))$ and $H^{0,m}(\Omega^0)$ equal to f and g on $\{t \le T\}$ and such that $\|\overline{f}\|_{m,\gamma,0} \le C\|f\|_{m,\gamma,0,T}$ and $|\overline{g}|_{m,\gamma,0} \le C|g|_{m,\gamma,0,T}$ with C a constant not depending on T (for the construction of such extensions see e.g. the proof of Lemma 3.7.2). We have the existence and uniqueness of a solution u in $L^2_x(H^{0,m}(\Omega^0))$ of the problem with $\overline{f}$ and $\overline{g}$ source terms, which, furthermore, satisfies the estimate (3.5.20). Since, from the causality principle, the restriction of u on $\{t \le T\}$ does not depend on the choice of the extensions $\overline{f}$ and $\overline{g}$, we get the existence and uniqueness of a solution in $L^2_x(H^{0,m}(\Omega^0_T))$ of the problem (3.2.4). Moreover, this solution belongs to $\mathbb{H}^{m,T}$ and satisfies the estimate (3.2.15). This finishes the proof of Theorem 3.2.23.

Remark 3.5.12. *For all $\gamma \ge \gamma_0$ and $T > 0$ such that $\gamma T \approx 1$, the estimate* (3.2.15) *of Theorem 3.2.23 yields*

$$\begin{aligned}\|\partial_x u\|_{m,\gamma,-1,T} + \|u\|_{m,\gamma,0,T} + |\partial_x u(0)|_{m,\gamma,-1,T} + |u(0)|_{m,\gamma,0,T} \\ \le C\,T\left(\|f\|_{m,\gamma,0,T} + |g|_{m,\gamma,0,T}\right), \qquad (3.5.21)\end{aligned}$$

with C a constant not depending on T and γ.

3.6 L^∞ estimates: Proof of Theorem 3.2.25

3.6.1 Reduction of the equation

Proposition 3.6.1. *The equation* (3.2.4a) *reads:*

$$-X_+X_-u - Q(Y)u = f \quad x > 0,$$

where $Q(Y)$ is a polynomial in Y of degree 2
and $X_\pm$ are first order operators such that:

$$X_\pm = \widetilde{X}_\pm + c_\pm,$$

with $c_\pm$ constants and $\widetilde{X}_\pm$ satisfying $\widetilde{X}_\pm(\varphi_\pm) = 0$, $\widetilde{X}_\pm(\varphi_\mp) \neq 0$.
More precisely, $\widetilde{X}_\pm$ are given by:

$$\widetilde{X}_\pm = \partial_x + \sigma_\pm \partial_{\tilde{t}} + \nu_\pm \cdot \partial_{\tilde{y}},$$

with $\sigma_\pm = \pm\frac{\omega}{\delta} + \alpha \quad (\sigma_+ > 0, \sigma_- < 0)$ *and* $\nu_\pm = \nu \mp \frac{k}{\delta}$.

Proof. Let us introduce $S = \mathrm{Span}(e_1^*, \varphi_0) \subset E^* := \mathbb{R}^{1+d}$, with $e_1^* = x$. Let Ψ be the orthogonal of S for q_0; $q_0|_\Psi$ is definite negative.

Let $(\Psi_1, \ldots, \Psi_{d-1})$ be an orthonormal basis of Ψ for $-q_0$ and let $(X_0, X_1, \tilde{Y}_1, \ldots, \tilde{Y}_{d-1})$ be the dual basis of $(e_1^*, \varphi_0, \Psi_1, \ldots, \Psi_{d-1})$. Thus $\forall i$ $\langle \tilde{Y}_i, e_1^* \rangle = \langle \tilde{Y}_i, \varphi_0 \rangle = 0$: the fields $\tilde{Y}_i$ are tangent to the striation, that is belong to the space generated by $(Y_1, \ldots, Y_{d-1})$.

Writing $\xi = \xi_S + \xi_\Psi$, with $\xi_S \in S$ and $\xi_\Psi \in \Psi$, we obtain, since $q_0|_S$ is of signature $(1, -1)$, the following form of $q_0(\xi)$:

$$\begin{aligned} q_0(\xi) &= -\widetilde{X}_+\widetilde{X}_- - \sum \tilde{Y}_j^2 \\ &= -\widetilde{X}_+\widetilde{X}_- - \tilde{Q}(Y), \end{aligned}$$

with $\tilde{Q}(Y)$ a polynomial in Y of degree 2 and $\widetilde{X}_\pm$ in $\mathrm{Span}(X_0, X_1)$.

Since $q_0(0, 1, 0, \ldots, 0) = -1$, $\widetilde{X}_\pm$ can be written under the form

$$\widetilde{X}_\pm = \partial_x + \sigma_\pm \partial_{\tilde{t}} + \nu_\pm \cdot \partial_{\tilde{y}}.$$

Furthermore, $\widetilde{X}_\pm$ are proportional to $dq_0(d\varphi_\pm)$.

Indeed, $\tilde{Y}_j$ being tangent to the striation, $\tilde{Y}_j(d\varphi_\pm) = 0$ and, $\varphi_\pm$ being characteristic phases, $q_0(d\varphi_\pm) = 0$. Thus $\widetilde{X}_+(d\varphi_\pm)\widetilde{X}_-(d\varphi_\pm) = 0$. Since $\widetilde{X}_\pm \in \mathrm{Span}(X_0, X_1)$, $\widetilde{X}_\pm(d\varphi_+) = \widetilde{X}_\pm(d\varphi_-) = 0$ implies $\widetilde{X}_\pm = 0$, which is absurd. Therefore, we obtain that either $\widetilde{X}_+(d\varphi_+) = \widetilde{X}_-(d\varphi_-) = 0$ or $\widetilde{X}_+(d\varphi_-) = \widetilde{X}_-(d\varphi_+) = 0$. We label the plus and minus so that $\widetilde{X}_+(d\varphi_+) = \widetilde{X}_-(d\varphi_-) = 0$ (and thus $\widetilde{X}_+(d\varphi_-) \neq 0$, $\widetilde{X}_-(d\varphi_+) \neq 0$).

With $\phi_{q_0}(\xi, \xi') = -\frac{1}{2}\widetilde{X}_+(\xi)\widetilde{X}_-(\xi') - \frac{1}{2}\widetilde{X}_+(\xi')\widetilde{X}_-(\xi) - \sum \tilde{Y}_j(\xi)\tilde{Y}_j(\xi')$ the bilinear form associated to q_0, we get $\phi_{q_0}(\xi, d\varphi_+) = -\frac{1}{2}\widetilde{X}_+(\xi)\widetilde{X}_-(d\varphi_+)$. Since $\widetilde{X}_-(d\varphi_+) \neq 0$, we obtain that $\widetilde{X}_+$ is proportional to $dq_0(d\varphi_+)$ and, similarly, that $\widetilde{X}_-$ is proportional to $dq_0(d\varphi_-)$.

$$\begin{aligned} \frac{1}{2} dq_0(d\varphi_\pm) &= -\left[(\xi_\pm + \alpha\omega + \nu \cdot k)(\partial_x + \alpha\partial_{\tilde{t}} + \nu \cdot \partial_{\tilde{y}}) - \omega\partial_{\tilde{t}} + k \cdot \partial_{\tilde{y}}\right] \\ &= -\left[\mp\sqrt{\omega^2 - k^2}(\partial_x + \alpha\partial_{\tilde{t}} + \nu \cdot \partial_{\tilde{y}}) - \omega\partial_{\tilde{t}} + k \cdot \partial_{\tilde{y}}\right] \\ &= -\left[\mp\delta\partial_x + (\mp\delta\alpha - \omega)\partial_{\tilde{t}} + (\mp\delta\nu + k) \cdot \partial_{\tilde{y}}\right]. \end{aligned}$$

It follows from $\widetilde{X}_\pm$ proportional to $dq_0(d\varphi_\pm)$ that $\sigma_\pm = \pm\frac{\omega}{\delta} + \alpha$ and $\nu_\pm = \nu \mp \frac{k}{\delta}$. Furthermore, $|\alpha| < 1$ yields $\sigma_+ > 0$ and $\sigma_- < 0$.

Finally, we introduce two constants c_+, c_- such that, with

$$X_\pm = \widetilde{X}_\pm + c_\pm,$$

we get

$$q = q_0 + l = -X_+X_- - Q(Y),$$

with $Q(Y)$ a polynomial in Y of degree 2 (Q is of the form $Q = \tilde{Q} + \check{Q}$ with $\check{Q}$ a polynomial of degree 1). $\square$

3.6.2 The propagation field

Proposition 3.6.2. *The boundary condition* (3.2.4b) *reads:*

$$X_+u + \check{v} \cdot Yu + \check{c}u = g \quad x = 0 \tag{3.6.1}$$

with

$$\check{v} = \omega^{-1}(k + \delta\tilde{v}), \tag{3.6.2}$$

and

$$\check{c} = c - c_+. \tag{3.6.3}$$

Proof. The boundary condition reads (see Remark 3.5.3):

$$\partial_x u + (\tilde{\beta} + \alpha)\partial_{\tilde{t}} u + (\tilde{v} + \nu) \cdot \partial_{\tilde{y}} u + cu = g \quad x = 0.$$

Thus, writing

$$\partial_{\tilde{t}} = \frac{\omega T + (k - \delta\nu) \cdot Y}{\omega\alpha + \delta + k \cdot \nu} = \frac{\omega T + (k - \delta\nu) \cdot Y}{(\tilde{\beta} + \alpha)\omega + (\tilde{v} + \nu) \cdot k},$$

$$\omega\partial_{\tilde{y}_j} = \delta Y_j + k_j \frac{\omega T + (k - \delta\nu) \cdot Y}{\omega\alpha + \delta + k \cdot \nu} = \delta Y_j + k_j \frac{\omega T + (k - \delta\nu) \cdot Y}{(\tilde{\beta} + \alpha)\omega + (\tilde{v} + \nu) \cdot k},$$

where $T = \sigma_+\partial_{\tilde{t}} + \nu_+ \cdot \partial_{\tilde{y}}$, we get the expression (3.6.1) of the boundary condition. $\square$

In (3.6.1) appears the field tangent to the striation

$$H = \check{v} \cdot Y = w_1 \cdot \partial_{\tilde{y}} - w_2\partial_{\tilde{t}}, \tag{3.6.4}$$

with $w_1 = \tilde{v} + \delta^{-1}k$, $w_2 = \delta^{-1}\omega - \tilde{\beta}$.

The Lopatinski determinant in the neighbourhood of (ω, k) reads

$$\tilde{D}(\tilde{\tau}, \tilde{\eta}) = -\sqrt{\tilde{\tau}^2 - |\tilde{\eta}|^2} + \tilde{\beta}\tilde{\tau} + \tilde{v} \cdot \tilde{\eta}.$$

The propagation field associated to the equation $\tilde{D} = 0$ is

$$H_{Lop} := \partial_{\tilde{\tau}} \tilde{D} \partial_{\tilde{t}} + (\partial_{\tilde{\eta}} \tilde{D}) \cdot \partial_{\tilde{y}},$$

the derivatives being calculated at (ω, k).

From (3.6.4), we then obtain:

Proposition 3.6.3. *$H = H_{Lop}$ and H_{Lop} is tangent to the striation.*

We then deduce from the (WR) condition:

Proposition 3.6.4. *The coefficient of ∂_t in H does not vanish.*

Proof. From the (WR) condition, the derivative $\partial_{\tilde{\tau}} \tilde{D}$ of the Lopatinski determinant $\tilde{D}$ does not vanish at (ω, k), that is $w_2 \neq 0$. Since $\partial_{\tilde{y}} = P\partial_y$ and $\partial_{\tilde{t}} = a\partial_t + b \cdot \partial_y$, $a > 0$, we get $H = (P^t w_1 - w_2 b) \cdot \partial_y - a w_2 \partial_t$. The coefficient of ∂_t in H is equal to $-a w_2$ and thus does not vanish. □

Proposition 3.2.17 follows from Proposition 3.6.1, Proposition 3.6.2 and Proposition 3.6.4.

3.6.3 Additional L^2 regularity

We have the equation:

$$X_-(X_+ u) = -f - Q(Y)u. \tag{3.6.5}$$

The field X_- being outgoing ($\sigma_- < 0$), the problem $X_- v = h$ is well-posed without boundary condition, thus:

Proposition 3.6.5. *For all f in $L^2_x(H^{0,m}(\Omega^0_T))$ and g in $H^{0,m}(\Omega^0_T)$ functions vanishing in the past, the solution u of the problem (3.5.1) vanishing in the past satisfies*

$$\exists \gamma_0 > 0 \; \forall \gamma \geq \gamma_0 \;\; \forall T > 0 \;\; \gamma \|X_+ u\|_{m-2,\gamma,0,T} \leq C\big(\|f\|_{m-2,\gamma,0,T} + \|u\|_{m,\gamma,0,T}\big),$$

with C a constant independent of T, γ, u, f and g.

Remark 3.6.6. *From now on, C will be a constant, independent of T, γ, u, f and g, which may vary from an expression to another.*

Proof. Let us introduce $v = X_+ u$ and $h = -f - Q(Y)u$. The equation 3.6.5 reads $X_- v = h$, with

$$\begin{aligned} X_- &= \widetilde{X}_- + c_- \\ &= \partial_x + \sigma_- \partial_{\tilde{t}} + \nu_- \cdot \partial_{\tilde{y}} + c_- \\ &= \partial_x + a\sigma_- \partial_t + (b\sigma_- + P^t \nu_-) \cdot \partial_y + c_-, \end{aligned}$$

where $\sigma_- < 0$, $a > 0$.

We introduce $v_\gamma = e^{-\gamma t} v$ and $h_\gamma = e^{-\gamma t} h$.

$$e^{-2\gamma t}(X_- v)\, v = (\widetilde{X}_- v_\gamma)\, v_\gamma - \gamma a |\sigma_-| v_\gamma^2 + c_- v_\gamma^2,$$

$$2(\widetilde{X}_- v_\gamma)\, v_\gamma = \partial_x(v_\gamma^2) - a|\sigma_-|\partial_t(v_\gamma^2) + (b\sigma_- + P^t \nu_-) \cdot \partial_y(v_\gamma^2);$$

thus, integrating on $\Omega_T^+ = \mathbb{R}^{1+d} \cap \{t \le T\} \cap \{x > 0\}$, we get

$$2 \int_{\Omega_T^+} (\widetilde{X}_- v_\gamma)\, v_\gamma = - \int_{\Omega_T^+ \cap \{x=0\}} v_\gamma^2 - a|\sigma_-| \int_{\Omega_T^+ \cap \{t=T\}} v_\gamma^2 \le 0.$$

Thus, for γ large

$$\begin{aligned} \frac{\gamma a |\sigma_-|}{2} \int_{\Omega_T^+} v_\gamma^2 &\le \gamma a |\sigma_-| \int_{\Omega_T^+} v_\gamma^2 - c_- \int_{\Omega_T^+} v_\gamma^2 \\ &\le - \int_{\Omega_T^+} e^{-2\gamma t}(X_- v)\, v = - \int_{\Omega_T^+} h_\gamma\, v_\gamma, \end{aligned}$$

that is

$$\exists \gamma_0 \forall \gamma \ge \gamma_0 \ \ \forall T > 0 \ \ \gamma \|v\|_{0,\gamma,0,T} \le C \|h\|_{0,\gamma,0,T} \le C\big(\|f\|_{0,\gamma,0,T} + \|u\|_{2,\gamma,0,T}\big).$$

Applying Y^α to (3.6.5), for $|\alpha| \le m-2$, we obtain, since Y^α exactly commutes with the equation:

$$\gamma \|X_+ u\|_{m-2,\gamma,0,T} \le C\big(\|f\|_{m-2,\gamma,0,T} + \|u\|_{m,\gamma,0,T}\big).$$

□

Notations 3.6.7. *From now on, we consider $\gamma \ge \gamma_0$, where γ_0 is given by Theorem 3.2.23, and T such that $\gamma T \approx 1$. We work on Ω_T^+ and we assume that f in $L^2_x(H^{0,m}(\Omega_T^0))$ and g in $H^{0,m}(\Omega_T^0)$ vanish on $\{t \le 0\}$. By the causality principle, it implies that u, solution of the problem* (3.5.1)*, also vanishes on $\{t \le 0\}$.*

Remark 3.6.8. *For all $\gamma \geq \gamma_0$, $T > 0$ such that $\gamma T \approx 1$, we have, for all u vanishing in the past,*

$$\forall (t,x,y) \in \Omega_T^+,\ c|u(t,x,y)| \leq e^{-\gamma T}|u(t,x,y)| \leq e^{-\gamma t}|u(t,x,y)| \leq |u(t,x,y)|.$$

Therefore $\|u\|_{\gamma,0,T} \approx \|u\|_{L^2(\Omega_T^+)} := \|u\|_{0,T}$. *Thus, since $\gamma \geq \gamma_0$, we obtain* $\|u\|_{m,0,T} := \sum_{|\alpha|\leq m} \|Y^\alpha u\|_{0,T} \leq C \sum_{|\alpha|\leq m} \gamma^{m-|\alpha|} \|Y^\alpha u\|_{\gamma,0,T} = C\|u\|_{m,\gamma,0,T}$.

3.6.4 Sobolev embeddings related to the striation

We consider the variables $(\check{t}, \check{y})$ with $\check{t} = \delta^{-1}(\omega\tilde{t} + k\cdot\tilde{y})$ and $\check{y}_j = \delta^{-1}(\omega\tilde{y}_j + k_j\tilde{t})$ (recall $\delta = \sqrt{\omega^2 - |k|^2}$) such that the striation is given by the spaces family ($\check{t} = constant, x = constant$) of dimension $d-1$. The fields Y_j generate and are generated by the fields $\partial_{\check{y}_j}$.

From the expression of $\tilde{t}$, $\tilde{y}$, ω_0 and k_0 defined in section 3.2.2, we get $\check{t} = \delta^{-1}[a^{-1}(\omega - P^{-1}k\cdot b)t + P^{-1}k\cdot y] = \delta^{-1}[\omega_0 t + k_0\cdot y]$ and $\check{y} = \delta^{-1}[a^{-1}(k - \omega(P^t)^{-1}b)t + \omega(P^t)^{-1}y]$.

Lemma 3.6.9. *For all $m > \frac{d-1}{2}$, and $u \in H^{0,m}(\Omega_T^+)$ there exists a function $u^* \in L^2(\mathbb{R}\times\mathbb{R}^{+*})$, such that:*

$$\forall (t,x,y) \in \Omega_T^+\ \ \big|u(t,x,y)\big| \leq u^*(\check{t},x), \quad \textit{with} \quad \|u^*\|_{L^2(\mathbb{R}\times\mathbb{R}^{+*})} \leq C\|u\|_{m,\gamma,0,T}.$$

Proof. When (t,x,y) is in $\Omega_T^+ = \mathbb{R}^{1+d} \cap \{x > 0\} \cap \{t \leq T\}$, $(\check{t},x,\check{y})$ is in $\mathbb{R}^{1+d} \cap \{x > 0\} \cap \{\omega\check{t} \leq k\cdot\check{y} + \delta a^{-1}T\}$. For each leaf $F = \{\check{t} = \underline{t}, x = \underline{x}\}$, $\underline{x} > 0$, $G_{\underline{t},\underline{x}} = F \cap \mathbb{R}^{1+d} \cap \{\omega\check{t} \leq k\cdot\check{y} + \delta a^{-1}T\}$ is a half-space of dimension $d-1$ in which we use the Sobolev embedding $H^m(G_{\underline{t},\underline{x}}) \subset L^\infty(G_{\underline{t},\underline{x}})$:

$$\forall t,x,y \in \Omega_T^+,\ \ |u(t,x,y)| = |\check{u}(\check{t},x,\check{y})| \leq u^*(\check{t},x),$$

with $u^*(\check{t},x) = \|\check{u}\|_{L^\infty(G_{\check{t},x})} \leq C\|\check{u}\|_{H^m(G_{\check{t},x})}$, thus $\|u^*\|_{L^2(\mathbb{R}\times\mathbb{R}^{+*})} \leq C\|u\|_{m,0,T}$ (the norm $\|\cdot\|_{m,0,T}$ is defined in Remark 3.6.8).

From $\|u\|_{m,0,T} \leq C\|u\|_{m,\gamma,0,T}$, we get $\|u^*\|_{L^2(\mathbb{R}\times\mathbb{R}^{+*})} \leq C\|u\|_{m,\gamma,0,T}$. □

3.6.5 Partial L^∞ estimate of u

Proposition 3.6.5 yields an estimate of X_+u in $L^2_x(H^{0,m-2}(\Omega_T^0))$ and Theorem 3.2.23 yields an estimate of u on the boundary. Integrating on the characteristics of X_+ and applying Lemma 3.6.9, we obtain

Proposition 3.6.10. *For all $|\alpha| < m - 2 - \frac{d-1}{2}$, there exists a function $u^{**} \in L^2(\mathbb{R})$, such that:*

$$\forall (t,x,y) \in \Omega_T^+ \ \ \left|Y^\alpha u(t,x,y)\right| \le u^{**}(\check{t} - cx), \tag{3.6.6}$$

with c a constant depending on ω, k, b, P, ν and σ_+, and

$$\left\|u^{**}\right\|_{L^2(\mathbb{R})} \le CT\left\|f\right\|_{m-2,\gamma,0,T} + C\big(\left\|u\right\|_{m,\gamma,0,T} + \left|u_{|x=0}\right|_{m-2,\gamma,0,T}\big). \tag{3.6.7}$$

Proof. We introduce $h = Y^\alpha X_+ u$.
From Proposition 3.6.5, we have $h \in L^2_x(H^{0,m-2-|\alpha|}(\Omega_T^0))$ and $\|h\|_{m-2-|\alpha|,\gamma,0,T} \le C\|X_+u\|_{m-2,\gamma,0,T} \le CT\big(\|f\|_{m-2,\gamma,0,T} + \|u\|_{m,\gamma,0,T}\big)$.

The characteristic of $X_+ = \partial_x + a\sigma_+\partial_t + (b\sigma_+ + P^t\nu_+)\cdot\partial_y + c_+$ containing (t_0, x_0, y_0) is $(t,x,y) = (a\sigma_+(x-x_0)+t_0, x, (b\sigma_+ + P^t\nu_+)(x-x_0)+y_0)$. Thus, for $0 \le t \le T$ and $x \ge \frac{T}{a\sigma_+}$, the characteristic containing (t,x,y) intersects $\{t=0\}$ before $\{x=0\}$.

Let us begin by the case where the characteristic containing (t,x,y) intersects the boundary $\{x=0\}$ before $\{t=0\}$, thus, necessarily, $x \le \frac{T}{a\sigma_+}$.

$$\begin{aligned} &Y^\alpha u(t,x,y) = \\ &\quad \int_0^x e^{-c_+(x-s)} h(a\sigma_+(s-x)+t, s, (b\sigma_+ + P^t\nu_+)(s-x)+y)\, ds \\ &\quad + Y^\alpha u(t - a\sigma_+ x, 0, y - (b\sigma_+ + P^t\nu_+)x) e^{-c_+ x}. \end{aligned}$$

For (t,x,y), we have $\check{t} = \delta^{-1}[\omega_0 t + k_0 \cdot y]$, thus, for $(t_1, x_1, y_1) = (t - a\sigma_+ x, 0, y - (b\sigma_+ + P^t\nu_+)x)$, we have $\check{t}_1 = \check{t} - cx$, with c a constant depending on ω, k, b, P, ν and σ_+. Thus, from Lemma 3.6.9 (adapted to the functions defined on the boundary $\{x=0\}$), we have:

$$\begin{aligned} |Y^\alpha u(t - a\sigma_+ x, 0, y - (b\sigma_+ + P^t\nu_+)x) e^{-c_+ x}| &\le e^{|c_+|\frac{T}{a\sigma_+}} u_1^*(\check{t} - cx) \\ &\le C u_1^*(\check{t} - cx), \end{aligned}$$

with $\|u_1^*\|_{L^2(\mathbb{R})} \le C|Y^\alpha u|_{x=0}|_{m-2-|\alpha|,\gamma,0,T} \le C|u|_{x=0}|_{m-2,\gamma,0,T}$.

Similarly, defining h^* from h as u^* was defined from u in the proof of Lemma 3.6.9, we have

$\|h^*\|_{L^2(\mathbb{R}\times\mathbb{R}^{+*})} \leq C\|h\|_{m-2-|\alpha|,\gamma,0,T} \leq CT\big(\|f\|_{m-2,\gamma,0,T} + \|u\|_{m,\gamma,0,T}\big)$ and

$$\begin{aligned}\int_0^x & e^{-c_+(x-s)}|h(a\sigma_+(s-x)+t, s, (b\sigma_+ + P^t\nu_+)(s-x)+y)|\,ds \\ &\leq e^{|c_+|\frac{T}{a\sigma_+}} \int_0^x h^*(\check{t}+c(s-x), s)\,ds \\ &\leq e^{|c_+|\frac{T}{a\sigma_+}} \sqrt{\frac{T}{a\sigma_+}} \sqrt{\int_{\mathbb{R}} \big(h^*(\check{t}+c(s-x), s)\big)^2\,ds} \\ &\leq Ch^{**}(\check{t}-cx),\end{aligned}$$

with $h^{**}(t) := \sqrt{\int_{\mathbb{R}} \big(h^*(t+cs, s)\big)^2\,ds}$ and therefore $\|h^{**}\|_{L^2(\mathbb{R})} \leq C\|h^*\|_{L^2(\mathbb{R}\times\mathbb{R}^{+*})} \leq CT\big(\|f\|_{m-2,\gamma,0,T} + \|u\|_{m,\gamma,0,T}\big)$, we thus get (3.6.6) with $u^{**} = Cu_1^* + Ch^{**}$.

It remains to study the case where the characteristic containing (t, x, y) intersects $\{t = 0\}$ before $\{x = 0\}$.

Since u vanishes in the past, we get

$$Y^\alpha u(t,x,y) = \int_{x-\frac{t}{a\sigma_+}}^x e^{-c_+(x-s)} h(a\sigma_+(s-x)+t, s, (b\sigma_+ + P^t\nu_+)(s-x)+y)\,ds,$$

thus, from Lemma 3.6.9,

$$\begin{aligned}|Y^\alpha u(t,x,y)| &\leq \int_{x-\frac{t}{a\sigma_+}}^x e^{-c_+(x-s)} h^*(\check{t}+c(s-x), s)\,ds \\ &\leq e^{|c_+|\frac{T}{a\sigma_+}} \sqrt{\frac{T}{a\sigma_+}} \left(\int_{\mathbb{R}} \big(h^*(\check{t}+c(s-x), s)\big)^2\,ds\right)^{\frac{1}{2}} \\ &\leq Ch^{**}(\check{t}-cx).\end{aligned}$$

□

Remark 3.6.11. *The scalar nature of the equation* (3.2.4a) *is crucial in order to integrate along characteristics.*

3.6.6 L^∞ estimate of X_+u

Proposition 3.6.12. *For all $m > 4 + \frac{d-1}{2}$, we have*

$$\begin{aligned}\big\|X_+u\big\|_{L^\infty(\Omega_T^+)} &\leq CT\big(\big\|f\big\|_{m-2,\gamma,0,T} + \big\|f\big\|_{L^\infty(\Omega_T^+)}\big) \\ &\quad + C\big(\big\|u\big\|_{m,\gamma,0,T} + \big|u_{|x=0}\big|_{m-2,\gamma,0,T}\big).\end{aligned}$$

Proof. $X_-(X_+u) = -f - Q(Y)u$ with $X_- = \partial_x + a\sigma_-\partial_t + (b\sigma_- + P^t\nu_-)\cdot\partial_y + c_-$. We introduce $h := -f - Q(Y)u$. Q is a polynomial of degree 2, thus, applying Proposition 3.6.10 to α such that $|\alpha| \leq 2$, we obtain

$$\forall m > 4 + \frac{d-1}{2} \geq |\alpha| + 2 + \frac{d-1}{2},$$
$$\forall t, x, y \in \Omega_T^+ \;\; |h(t,x,y)| \leq C\big(\|f\|_{L^\infty(\Omega_T^+)} + u^{**}(\check{t} - cx)\big),$$

with $\|u^{**}\|_{L^2(\mathbb{R})} \leq CT\|f\|_{m-2,\gamma,0,T} + C\big(\|u\|_{m,\gamma,0,T} + |u|_{x=0}|_{m-2,\gamma,0,T}\big)$.

The characteristic of X_- containing (t_0, x_0, y_0) is $(t,x,y) = (-a|\sigma_-|(x-x_0) + t_0, x, (b\sigma_- + P^t\nu_-)(x-x_0) + y_0)$; thus, since u vanishes in the past:

$$\forall (t,x,y) \in \Omega_T^+, \;\; X_+u(t,x,y) =$$
$$\int_{x+\frac{t}{a|\sigma_-|}}^{x} e^{-c_-(x-s)} h(-a|\sigma_-|(s-x) + t, s, (b\sigma_- + P^t\nu_-)(s-x) + y)\, ds.$$

For $(t_2, x_2, y_2) = (-a|\sigma_-|(s-x) + t, s, (b\sigma_- + P^t\nu_-)(s-x) + y)$, $\check{t}_2 = \check{t} + c'(x-s)$, with c' a constant depending on ω, k, b, P, ν and σ_-, thus

$$\forall (t,x,y) \in \Omega_T^+,$$
$$\begin{aligned}
|X_+u(t,x,y)| &\leq Ce^{|c_-|\frac{T}{a|\sigma_-|}} \int_x^{x+\frac{t}{a|\sigma_-|}} (\|f\|_{L^\infty(\Omega_T^+)} + u^{**}(\check{t} + c'x - (c+c')s))\, ds \\
&\leq Ce^{|c_-|\frac{T}{a|\sigma_-|}} \Big[\frac{T}{a|\sigma_-|} \|f\|_{L^\infty(\Omega_T^+)} + \\
&\qquad \sqrt{\frac{T}{a|\sigma_-|}} \left(\int_x^{x+\frac{T}{a|\sigma_-|}} (u^{**}(\check{t} + c'x - (c+c')s))^2\, ds \right)^{\frac{1}{2}} \Big] \\
&\leq CT\|f\|_{L^\infty(\Omega_T^+)} + C\|u^{**}\|_{L^2(\mathbb{R})} \\
&\leq CT\big(\|f\|_{L^\infty(\Omega_T^+)} + \|f\|_{m-2,\gamma,0,T}\big) \\
&\qquad + C\big(\|u\|_{m,\gamma,0,T} + |u|_{x=0}|_{m-2,\gamma,0,T}\big).
\end{aligned}$$

□

3.6.7 L^∞ estimate of the trace of u

We use the boundary condition

$$Hu + \check{c}u = g - X_+u \quad x = 0,$$

where H is the transport field (3.6.4). Since the coefficient of ∂_t in H does not vanish, we can propagate L^∞ regularity along the integral curves of H.

Lemma 3.6.13.

$$\big\|u_{|x=0}\big\|_{L^\infty(\Omega_T^+\cap\{x=0\})} \leq CT\big(\big\|g\big\|_{L^\infty(\Omega_T^0)} + \big\|X_+u\big\|_{L^\infty(\Omega_T^+)}\big). \tag{3.6.8}$$

Proof. $H = w_1 \cdot \partial_{\tilde{y}} - w_2\partial_{\tilde{t}} = \check{w}_1\partial_y - \check{w}_2\partial_t$ with $\check{w}_2 = aw_2 \neq 0$. The characteristic of H containing (t_0, x_0, y_0) is $(t, x, y) = (-\check{w}_2 s, x_0, (s+\frac{t_0}{\check{w}_2})\check{w}_1 + y_0)$; from u vanishing in the past, we get:

$$\begin{aligned}
&\forall (t, y) \in \Omega_T^0,\ \ |u(t, 0, y)| = \\
&\qquad \Big|\int_0^{-\frac{t}{\check{w}_2}} e^{-\check{c}(-\frac{t}{\check{w}_2}-s)}(g - X_+u)(-\check{w}_2 s, 0, (s + \frac{t}{\check{w}_2})\check{w}_1 + y)\, ds\Big| \\
&\qquad\qquad \leq e^{|\check{c}|\frac{T}{\check{w}_2}}\frac{T}{\check{w}_2}\Big(\big\|g\big\|_{L^\infty(\Omega_T^0)} + \|X_+u|_{x=0}\|_{L^\infty(\Omega_T^+\cap\{x=0\})}\Big)\,;
\end{aligned}$$

thus

$$\|u|_{x=0}\|_{L^\infty(\Omega_T^+\cap\{x=0\})} \leq CT\big(\big\|g\big\|_{L^\infty(\Omega_T^0)} + \|X_+u\|_{L^\infty(\Omega_T^+)}\big).$$

□

3.6.8 L^∞ estimate of u

Lemma 3.6.14.

$$\big\|u\big\|_{L^\infty(\Omega_T^+)} \leq CT\big\|X_+u\big\|_{L^\infty(\Omega_T^+)} + C\big\|u_{|x=0}\big\|_{L^\infty(\Omega_T^+\cap\{x=0\})}. \tag{3.6.9}$$

Proof. When the characteristic $(t, x, y) = (a\sigma_+(x - x_0) + t_0, x,$ $(b\sigma_+ + P^t\nu_+)(x - x_0) + y_0)$ intersects the boundary $\{x = 0\}$ before $\{t = 0\}$, we have, necessarily, $x \leq \frac{T}{a\sigma_+}$, thus:

$$\begin{aligned}
&\forall (t, x, y) \in \Omega_T^+,\ |u(t, x, y)| = \\
&\Big|\int_0^x e^{-c_+(x-s)}X_+u(a\sigma_+(s - x) + t, s, (b\sigma_+ + P^t\nu_+)(s - x) + y)\, ds \\
&\qquad\qquad + u(t - a\sigma_+x, 0, y - (b\sigma_+ + P^t\nu_+)x)e^{-c_+x}\Big| \\
&\qquad\quad \leq e^{|c_+|\frac{T}{a\sigma_+}}\frac{T}{a\sigma_+}\|X_+u\|_{L^\infty(\Omega_T^+)} + e^{|c_+|\frac{T}{a\sigma_+}}\|u|_{x=0}\|_{L^\infty(\Omega_T^+\cap\{x=0\})}.
\end{aligned}$$

When the characteristic intersects $\{t = 0\}$ before $\{x = 0\}$, we have, from u

vanishing in the past:

$$\forall (t,x,y) \in \Omega_T^+,\ |u(t,x,y)| =$$
$$\Big| \int_{x-\frac{t}{a\sigma_+}}^{x} e^{-c_+(x-s)} X_+ u(a\sigma_+(s-x)+t, s, (b\sigma_+ + P^t\nu_+)(s-x)+y)\, ds \Big|$$
$$\leq e^{|c_+|\frac{T}{a\sigma_+}} \frac{T}{a\sigma_+} \|X_+ u\|_{L^\infty(\Omega_T^+)}.$$

□

Finally, Theorem 3.2.25 follows from Lemma 3.6.14, Lemma 3.6.13, Proposition 3.6.12, Theorem 3.2.23 and Remark 3.5.12.

3.7 The semi-linear problem

We consider the semi-linear problem (3.2.1):

$$\begin{aligned} q(\partial_t, \partial_x, \partial_y)u &= f + F(\,.\,, u) \quad x > 0, \\ m(\partial_t, \partial_x, \partial_y)u &= g + G(\,.\,, u) \quad x = 0, \end{aligned}$$

with $m(\partial_t, \partial_x, \partial_y) = \partial_x + \beta\partial_t + v \cdot \partial_y + c$,
f in $L^\infty(\Omega_T^+) \cap L_x^2(H^{0,m}(\Omega_T^0))$,
and g in $L^\infty(\Omega_T^0) \cap H^{0,m}(\Omega_T^0)$,
where $m > 4 + \frac{d-1}{2}$.

We assume that the (WR) condition is satisfied.

We introduce the following norms:

Definition 3.7.1.

$$\begin{aligned} &\forall u \in L^\infty(\Omega^+) \cap L_x^2(H^{0,m}(\Omega^0)), \\ &\qquad \|u\|_{\infty,m,\gamma,0} = \|u\|_{L^\infty(\Omega^+)} + \|u\|_{m,\gamma,0}, \\ &\forall u \in L^\infty(\Omega_T^+) \cap L_x^2(H^{0,m}(\Omega_T^0)), \\ &\qquad \|u\|_{\infty,m,\gamma,0,T} = \|u\|_{L^\infty(\Omega_T^+)} + \|u\|_{m,\gamma,0,T}, \\ &\forall v \in L^\infty(\Omega^0) \cap H^{0,m}(\Omega^0), \\ &\qquad |v|_{\infty,m,\gamma,0} = \|v\|_{L^\infty(\Omega^0)} + |v|_{m,\gamma,0}, \\ &\forall v \in L^\infty(\Omega_T^0) \cap H^{0,m}(\Omega_T^0), \\ &\qquad |v|_{\infty,m,\gamma,0,T} = \|v\|_{L^\infty(\Omega_T^0)} + |v|_{m,\gamma,0,T}. \end{aligned}$$

The notations $\Omega^+, \Omega^0, \Omega_T^+$ and Ω_T^0 have been introduced in section 3.2.3.

3.7.1 Preliminary results

Lemma 3.7.2 (Extension lemma)**.** *For all $T > 0$, $m \in \mathbb{N}$ and u in $L^\infty(\Omega_T^+) \cap L_x^2(H^{0,m}(\Omega_T^0))$, there exists $\overline{u}$ in $L^\infty(\Omega^+) \cap L_x^2(H^{0,m}(\Omega^0))$, equal to u on Ω_T^+ and such that*

$$\left\|\overline{u}\right\|_{\infty,m,\gamma,0} \leq C\left\|u\right\|_{\infty,m,\gamma,0,T}$$

with C a constant not depending on T.

Proof. The vector fields Y_j tangent to the striation are given by:

$$\begin{aligned}Y_j =& \delta^{-1}(\omega\partial_{\tilde{y}_j} - k_j\partial_{\tilde{t}})\\ =& -\delta^{-1}ak_j\partial_t + \delta^{-1}\sum_{1\leq k\leq d-1}(\omega P_{j,k} - k_jb_k)\partial_{y_k}.\end{aligned}$$

Since $k \neq 0$ (see Remark 3.2.16) and $a > 0$, the coefficient of ∂_t does not vanish for at least one of the Y_j. We can thus obtain Z_j such that the Z_j generate the Y_j and are generated by the Y_j and

$$\begin{aligned}Z_1 &= \partial_t + d_1\cdot\partial_y\\ Z_j &= d_j\cdot\partial_y \ \ \forall\, 2\leq j\leq d-1.\end{aligned}$$

We perform a change of variable $u(t,x,y) = v(t,x,z) = v(t,x,y-td_1)$ in order to have

$$\begin{aligned}Z_1u =& \partial_t v,\\ \partial_y u =& \partial_z v.\end{aligned}$$

For (t,x,y) in Ω_T^+, (t,x,z) is also in Ω_T^+.

We use a Babitch extension (see [59]). More precisely, we define $\overline{v}$ such that

$$\begin{aligned}&\forall (t,x,z)\in]-\infty,T]\times\mathbb{R}_x^+\times\mathbb{R}_y^{d-1}, && \overline{v}(t,x,z) = v(t,x,z)\\ &\forall (t,x,z)\in]T,+\infty[\times\mathbb{R}_x^+\times\mathbb{R}_y^{d-1}, && \overline{v}(t,x,z) = Rv(t,x,z)\\ & && \qquad = \sum_{k=1}^m a_k v(T-k(t-T),x,z),\end{aligned}$$

with a_k real numbers which satisfy the condition

$$\forall j = 0,...,m-1,\ \sum_{k=1}^m a_k(-k)^j = 1. \tag{3.7.1}$$

$\overline{v}$ is well defined since, for $t > T$, we get $T - k(t-T) < T$ for $k = 1, ..., m$.

∂_z commutes with R:

$$\partial_z(Rv) = R(\partial_z v)$$

and ∂_t satisfies:

$$\forall j = 0, ..., m-1 \;\; \partial_t^j(Rv) = \sum_{k=1}^{m} a_k (-k)^j \partial_t^j v(T - k(t-T), x, z).$$

From the condition (3.7.1), the traces are equal on $\{t = T\}$.

With $\overline{u}$ defined by

$$\forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{d-1}, \;\; \overline{u}(t, x, y) = \overline{v}(t, x, y - td_1),$$

we get $\overline{u}$ in $L^\infty(\Omega^+) \cap L^2_x(H^{0,m}(\Omega^0))$ and

$$\|\overline{u}\|_{L^\infty(\Omega^+)} + \|\overline{u}\|_{m,\gamma,0} \leq C \left(\|u\|_{L^\infty(\Omega^+_T)} + \|u\|_{m,\gamma,0,T} \right),$$

since $Z_1\overline{u} = \partial_t \overline{v}$ and $\forall j = 2, ..., d-1, \;\; Z_j \overline{u} = d_j \cdot \partial_y \overline{u} = d_j \cdot \partial_z \overline{v}$.

$\square$

Lemma 3.7.3 (Gagliardo-Nirenberg inequality)**.** *For all $n \in \mathbb{N}$ and $\delta \in \mathbb{N}^{d-1}$, $|\delta| \leq n$, there exists a constant $C > 0$, depending on n, δ, d, such that, for all $u \in L^\infty(\Omega^+) \cap L^2_x(H^{0,m}(\Omega^0))$,*

$$\|Y^\delta u\|_{L^{\frac{2n}{|\delta|}}(\Omega^+)} \leq C \|u\|^{1-\frac{|\delta|}{n}}_{L^\infty(\Omega^+)} \|u\|^{\frac{|\delta|}{n}}_{n,\gamma,0}.$$

Proof. This inequality is a particular case of the Gagliardo-Nirenberg inequalities on Ω^+ (see [29], Appendice 2). $\square$

3.7.2 Properties of F and G

Proposition 3.7.4. *For all $T > 0$ and $m \in \mathbb{N}$, if u is in $L^\infty(\Omega^+_T) \cap L^2_x(H^{0,m}(\Omega^0_T))$, then*

$F(\,.\,; u)$ is in $L^\infty(\Omega^+_T) \cap L^2_x(H^{0,m}(\Omega^0_T))$ and satisfies

$$\|F(\,.\,; u)\|_{\infty,m,\gamma,0,T} \leq C(\|u\|_{L^\infty(\Omega^+_T)})\,(1 + \|u\|_{m,\gamma,0,T}),$$

$G(\,.\,; u(0))$ is in $L^\infty(\Omega^0_T) \cap H^{0,m}(\Omega^0_T)$ and satisfies

$$|G(\,.\,; u(0))|_{\infty,m,\gamma,0,T} \leq C(\|u(0)\|_{L^\infty(\Omega^0_T)})\,(1 + |u(0)|_{m,\gamma,0,T}).$$

Proof. It follows from the Assumption 3.2.3 on F that $F(\,.\,;u)$ and $\partial^\alpha F(\,.\,;u)$ are in $L^\infty(\Omega_T^+)$ and satisfy

$$\forall \alpha \in \mathbb{N}^{d+2}, \big\|\partial^\alpha F(\,.\,;u)\big\|_{L^\infty(\Omega_T^+)} \leq C(\big\|u\big\|_{L^\infty(\Omega_T^+)}).$$

For all $|\alpha| \leq m$, $\gamma^{m-|\alpha|}\, Y^\alpha(F(\,.\,;u))$ is a linear combination of terms:

$$\gamma^{m-|\alpha|}\, \partial^\delta F(\,.\,;u) \prod_{i=1}^{k} Y^{\beta_i} u, \quad \text{ with } \sum_{i=1}^{k} |\beta_i| \leq |\alpha|.$$

From Lemma 3.7.2 and the Gagliardo-Nirenberg inequality on Ω_T^+ 3.7.3, we get, for $n = \sum_{i=1}^{k} |\beta_i| \leq |\alpha| \leq m$

$$\big\|Y^{\beta_i} u\big\|_{L^{\frac{2n}{|\beta_i|}}(\Omega_T^+)} \lesssim \big\|u\big\|_{L^\infty(\Omega_T^+)}^{1-\frac{|\beta_i|}{n}} \big\|u\big\|_{n,\gamma,0,T}^{\frac{|\beta_i|}{n}}.$$

The product $\gamma^{m-|\alpha|} \prod_{i=1}^{k} Y^{\beta_i} u$ is thus in $L^2(\Omega_T^+)$ and satisfies

$$\begin{aligned}
\gamma^{m-|\alpha|}\, \|\prod_{i=1}^{k} Y^{\beta_i} u\big\|_{L^2(\Omega_T^+)} &\lesssim \gamma^{m-|\alpha|} \prod_{i=1}^{k} \big\|Y^{\beta_i} u\big\|_{L^{\frac{2n}{|\beta_i|}}(\Omega_T^+)} \\
&\lesssim \gamma^{m-|\alpha|}\, \big\|u\big\|_{L^\infty(\Omega_T^+)}^{k-1} \big\|u\big\|_{n,\gamma,0,T} \\
&\lesssim \gamma^{n-|\alpha|} \big\|u\big\|_{L^\infty(\Omega_T^+)}^{k-1} \big\|u\big\|_{m,\gamma,0,T} \\
&\lesssim \big\|u\big\|_{L^\infty(\Omega_T^+)}^{k-1} \big\|u\big\|_{m,\gamma,0,T}. \qquad (3.7.2)
\end{aligned}$$

Thus, for all $|\alpha| \leq m$, $Y^\alpha(F(\,.\,;u))$ is in $L^2(\Omega_T^+)$, that is $F(\,.\,;u)$ is in $L^2_x(H^{0,m}(\Omega_T^0))$, and

$$\big\|F(\,.\,;u)\big\|_{m,\gamma,0,T} \leq C(\big\|u\big\|_{L^\infty(\Omega_T^+)}) \big\|u\big\|_{m,\gamma,0,T}.$$

The result on G is proved similarly. $\square$

Proposition 3.7.5. *For all $T > 0$ and $m \in \mathbb{N}$,*

$$\begin{aligned}
&\forall u, v \in L^\infty(\Omega_T^+) \cap L^2_x(H^{0,m}(\Omega_T^0)), \\
&\qquad \big\|F(\,.\,;u) - F(\,.\,;v)\big\|_{\infty,m,\gamma,0,T} \leq C\big\|u - v\big\|_{\infty,m,\gamma,0,T}, \\
&\qquad \big|G(\,.\,;u(0)) - G(\,.\,;v(0))\big|_{\infty,m,\gamma,0,T} \leq C'\big|u(0) - v(0)\big|_{\infty,m,\gamma,0,T},
\end{aligned}$$

with $C = C(\big\|u\big\|_{L^\infty(\Omega_T^+)}, \big\|v\big\|_{L^\infty(\Omega_T^+)})\big[1 + \big\|u\big\|_{m,\gamma,0,T} + \big\|v\big\|_{m,\gamma,0,T}\big]$,
$C' = C(\big\|u(0)\big\|_{L^\infty(\Omega_T^0)}, \big\|v(0)\big\|_{L^\infty(\Omega_T^0)})\big[1 + \big|u(0)\big|_{m,\gamma,0,T} + \big|v(0)\big|_{m,\gamma,0,T}\big]$.

Proof. The Assumption 3.2.3 on F yields, for all $\alpha \in \mathbb{N}^{d+2}$,

$$\left\|\partial^\alpha F(\,.\,,u) - \partial^\alpha F(\,.\,,v)\right\|_{L^\infty(\Omega_T^+)} \leq C\left\|u-v\right\|_{L^\infty(\Omega_T^+)}, \tag{3.7.3}$$

with $C = C(\|u\|_{L^\infty(\Omega_T^+)}, \|v\|_{L^\infty(\Omega_T^+)})$.

For all $\alpha \in \mathbb{N}^{d-1}$,$|\alpha| \leq m$, $\gamma^{m-|\alpha|}\left(Y^\alpha(F(\,.\,;u)) - Y^\alpha(F(\,.\,;v))\right)$ is a linear combination of terms:

$$\gamma^{m-|\alpha|}\left(\partial^\delta F(\,.\,;u) - \partial^\delta F(\,.\,;v)\right)\prod_{i=1}^{k} Y^{\beta_i}u,$$

and

$$\gamma^{m-|\alpha|}\,\partial^\delta F(\,.\,;v)\sum_{l=1}^{k}\prod_{i<l} Y^{\beta_i}u\,Y^{\beta_l}(u-v)\prod_{j>l}Y^{\beta_j}v,$$

with $\sum_{i=1}^k |\beta_i| \leq |\alpha|$.

From (3.7.2) and (3.7.3), we get that

$$\begin{aligned}
&\gamma^{m-|\alpha|}\left\|\left(\partial^\delta F(\,.\,;u) - \partial^\delta F(\,.\,;v)\right)\prod_{i=1}^{k} Y^{\beta_i}u\right\|_{L^2(\Omega_T^+)} \\
&\qquad \leq C(\|u\|_{L^\infty(\Omega_T^+)}, \|v\|_{L^\infty(\Omega_T^+)})\,\|u\|_{L^\infty(\Omega_T^+)}^{k-1}\,\|u\|_{m,\gamma,0,T}\,\|u-v\|_{L^\infty(\Omega_T^+)}.
\end{aligned}$$

Similarly as in the proof of Proposition 3.7.4, we get that, for $\gamma_1 = \sum_{i<l}\beta_i$, $\gamma_2 = \sum_{j>l}\beta_j$ and $n = \sum_{i=1}^k |\beta_i| \leq |\alpha| \leq m$

$$\begin{aligned}
&\gamma^{m-|\alpha|}\left\|\partial^\delta F(\,.\,;v)\prod_{i<l} Y^{\beta_i}u\,Y^{\beta_l}(u-v)\prod_{j>l}Y^{\beta_j}v\right\|_{L^2(\Omega_T^+)} \\
&\qquad \leq C\,\|u\|_{m,\gamma,0,T}^{\frac{|\gamma_1|}{n}}\|u-v\|_{m,\gamma,0,T}^{\frac{|\beta_l|}{n}}\|v\|_{m,\gamma,0,T}^{\frac{|\gamma_2|}{n}}\|u-v\|_{L^\infty(\Omega_T^+)}^{1-\frac{|\beta_l|}{n}} \\
&\qquad \leq C\,\|u-v\|_{m,\gamma,0,T}^{\frac{|\beta_l|}{n}}\left[(\|u\|_{m,\gamma,0,T} + \|v\|_{m,\gamma,0,T})\|u-v\|_{L^\infty(\Omega_T^+)}\right]^{1-\frac{|\beta_l|}{n}} \\
&\qquad \leq C\,\left[\|u-v\|_{m,\gamma,0,T} + \left(\|u\|_{m,\gamma,0,T} + \|v\|_{m,\gamma,0,T}\right)\|u-v\|_{L^\infty(\Omega_T^+)}\right],
\end{aligned}$$

with $C = C(\|u\|_{L^\infty(\Omega_T^+)}, \|v\|_{L^\infty(\Omega_T^+)})$.

We finally get that:

$$\begin{aligned}
\|F(\,.\,;u) - F(\,.\,;v)\|_{\infty,m,\gamma,0,T} \leq C(\|u\|_{L^\infty(\Omega_T^+)}, \|v\|_{L^\infty(\Omega_T^+)})\Big[1 + \|u\|_{m,\gamma,0,T} \\
+ \|v\|_{m,\gamma,0,T}\Big]\|u-v\|_{\infty,m,\gamma,0,T}.
\end{aligned}$$

The result on G is proved similarly. □

3.7.3 Proof of Theorem 3.2.26

We solve the semi-linear problem by Picard's iterations. The Picard's iterations are adapted to the problem, despite the loss of one derivative, because the nonlinearity is weak enough. Let $(u_\nu)_{\nu\geq 0}$ be a sequence defined by iteration by:

$u_0 = 0$,

for all $\nu \geq 1$, u_ν is the solution vanishing in the past of the linear problem:

$$q(\partial_t, \partial_x, \partial_y) u_\nu = f + F(\,.\,, u_{\nu-1}) \quad x > 0, \tag{3.7.4a}$$

$$m(\partial_t, \partial_x, \partial_y) u_\nu = g + G(\,.\,, u_{\nu-1}(0)) \quad x = 0. \tag{3.7.4b}$$

By induction, we obtain from Proposition 3.7.4 and from the previous study of the linear problem:

Lemma 3.7.6. *The sequence* $(u_\nu)_{\nu\geq 0}$ *is well-defined in* $\mathbb{L}^{\infty,T} \cap \mathbb{H}^{m,T}$.

Proof. u_0 is in $\mathbb{L}^{\infty,T} \cap \mathbb{H}^{m,T}$. For all $\nu \geq 1$, if $u_{\nu-1}$ is in $\mathbb{L}^{\infty,T} \cap \mathbb{H}^{m,T}$, then, from Proposition 3.7.4, $f + F(\,.\,, u_{\nu-1})$ (resp. $g + G(\,.\,, u_{\nu-1}(0))$) is in $L^\infty(\Omega_T^+) \cap L_x^2(H^{0,m}(\Omega_T^0))$ (resp. in $L^\infty(\Omega_T^0) \cap H^{0,m}(\Omega_T^0)$). Then, from Theorem 3.2.23 and Theorem 3.2.25, the linear problem (3.7.4) has a unique solution vanishing in the past u_ν, which belongs to $\mathbb{L}^{\infty,T} \cap \mathbb{H}^{m,T}$. $\square$

It follows from Theorem 3.2.23, Theorem 3.2.25 and Remark 3.5.12, that u solution vanishing in the past of the linear problem (3.2.4) satisfies the following estimate:

$$\begin{aligned}
&\forall m > 4 + \frac{d-1}{2}, \exists C > 0, \forall \gamma \geq \gamma_0, \forall T > 0 \,/\, \gamma T \approx 1,\\
&\quad \|u\|_{\infty,m,\gamma,0,T} + |u(0)|_{\infty,m,\gamma,0,T}\\
&\qquad + \|X_+ u\|_{L^\infty(\Omega_T^+)} + \|\partial_x u\|_{m,\gamma,-1,T} + |\partial_x u(0)|_{m,\gamma,-1,T}\\
&\qquad\quad \leq C\, T(\|f\|_{\infty,m,\gamma,0,T} + |g|_{\infty,m,\gamma,0,T}).
\end{aligned} \tag{3.7.5}$$

We deduce from this estimate:

Lemma 3.7.7. *There exists* $0 < T_1 \leq T$ *such that the sequence* $(\|u_\nu\|_{\infty,m,\gamma,0,T_1} + |u_\nu(0)|_{\infty,m,\gamma,0,T_1})_{\nu\geq 0}$ *is bounded.*

Proof. For all $\nu \geq 1$, u_ν being the solution vanishing in the past of the linear problem (3.7.4), we get from the estimate (3.7.5):

$$\begin{aligned}
&\|u_\nu\|_{\infty,m,\gamma,0,T} + |u_\nu(0)|_{\infty,m,\gamma,0,T} \leq C\, T\big(\|f\|_{\infty,m,\gamma,0,T}\\
&\quad + \|F(\,.\,, u_{\nu-1})\|_{\infty,m,\gamma,0,T} + |g|_{\infty,m,\gamma,0,T} + |G(\,.\,, u_{\nu-1}(0))|_{\infty,m,\gamma,0,T}\big).
\end{aligned}$$

Thus, from Proposition 3.7.4

$$\begin{aligned}&\big\|u_\nu\big\|_{\infty,m,\gamma,0,T}+\big|u_\nu(0)\big|_{\infty,m,\gamma,0,T}\le C\,T\big(\big\|f\big\|_{\infty,m,\gamma,0,T}+\big|g\big|_{\infty,m,\gamma,0,T}\\&+C'(\big\|u_{\nu-1}\big\|_{L^\infty(\Omega_T^+)})\,(1+\big\|u_{\nu-1}\big\|_{m,\gamma,0,T}+\big|u_{\nu-1}(0)\big|_{m,\gamma,0,T})\big).\end{aligned}$$

We introduce $R=\big\|u_1\big\|_{\infty,m,\gamma,0,T}+\big|u_1(0)\big|_{\infty,m,\gamma,0,T}$
and $C_0=\big\|f\big\|_{\infty,m,\gamma,0,T}+\big|g\big|_{\infty,m,\gamma,0,T}$
and we choose T_1 such that

$$C\,T_1\big(C_0+C'(R)(1+R)\big)\le R.$$

We then obtain by induction that, for all $\nu\ge 0$,

$$\big\|u_\nu\big\|_{\infty,m,\gamma,0,T_1}+\big|u_\nu(0)\big|_{\infty,m,\gamma,0,T_1}\le R.$$

□

Lemma 3.7.8. *There exists T_* in $]0,T_1]$ such that $(u_\nu)_{\nu\ge0}$ is a Cauchy sequence in $L^\infty(\Omega^+_{T_*})\cap L^2_x(H^{0,m}(\Omega^0_{T_*}))$, for the norm $\big\|\,.\,\big\|_{\infty,m,\gamma,0,T_*}+\big|\,.\,(0)\big|_{\infty,m,\gamma,0,T_*}$.*

Proof. For all $\nu\ge 2$, $(u_\nu-u_{\nu-1})$ is the solution vanishing in the past of the linear problem:

$$\begin{aligned}q(\partial_t,\partial_x,\partial_y)(u_\nu-u_{\nu-1})&=F(\,.\,,u_{\nu-1})-F(\,.\,,u_{\nu-2})\quad x>0\\ m(\partial_t,\partial_x,\partial_y)(u_\nu-u_{\nu-1})&=G(\,.\,,u_{\nu-1}(0))-G(\,.\,,u_{\nu-2}(0))\quad x=0.\end{aligned}$$

We have, for all $0<T'\le T_1$, the estimate:

$$\begin{aligned}&\big\|u_\nu-u_{\nu-1}\big\|_{\infty,m,\gamma,0,T'}+\big|(u_\nu-u_{\nu-1})(0)\big|_{\infty,m,\gamma,0,T'}\\&\le C\,T'\big(\big\|F(\,.\,,u_{\nu-1})-F(\,.\,,u_{\nu-2})\big\|_{\infty,m,\gamma,0,T'}\\&\qquad+\big|G(\,.\,,u_{\nu-1}(0))-G(\,.\,,u_{\nu-2}(0))\big|_{\infty,m,\gamma,0,T'}\big).\end{aligned}$$

Since $(\big\|u_\nu\big\|_{\infty,m,\gamma,0,T_1}+\big|u_\nu(0)\big|_{\infty,m,\gamma,0,T_1})_{\nu\ge0}$ is bounded, we get from Proposition 3.7.5:

$$\begin{aligned}&\big\|u_\nu-u_{\nu-1}\big\|_{\infty,m,\gamma,0,T'}+\big|(u_\nu-u_{\nu-1})(0)\big|_{\infty,m,\gamma,0,T'}\le C\,T'\tilde C\\&\qquad\big(\big\|u_{\nu-1}-u_{\nu-2}\big\|_{\infty,m,\gamma,0,T'}+\big|(u_{\nu-1}-u_{\nu-2})(0)\big|_{\infty,m,\gamma,0,T'}\big),\end{aligned}$$

with $\tilde C$ a constant depending on the bound of $(\big\|u_\nu\big\|_{\infty,m,\gamma,0,T_1}+\big|u_\nu(0)\big|_{\infty,m,\gamma,0,T_1})_{\nu\ge0}$.

We finally choose $0<T_*\le T_1$ such that $C\,T_*\tilde C<1$. $(u_\nu)_{\nu\ge0}$ is then a Cauchy sequence in $L^\infty(\Omega^+_{T_*})\cap L^2_x(H^{0,m}(\Omega^0_{T_*}))$, for the norm $\big\|\,.\,\big\|_{\infty,m,\gamma,0,T_*}+\big|\,.\,(0)\big|_{\infty,m,\gamma,0,T_*}$. □

Lemma 3.7.9. *$(u_\nu)_{\nu\geq 0}$ is a Cauchy sequence in $\mathbb{L}^{\infty,T_*}\cap\mathbb{H}^{m,T_*}$, for the norm $\big\|\,\cdot\,\big\|_{\infty,m,\gamma,0,T_*}+\big|\,\cdot\,(0)\big|_{\infty,m,\gamma,0,T_*}+\big\|X_+\,\cdot\,\big\|_{L^\infty(\Omega^+_{T_*})}+\big\|\partial_x\,\cdot\,\big\|_{m,\gamma,-1,T_*}$ $+\big|\partial_x\,\cdot\,(0)\big|_{m,\gamma,-1,T_*}$.*

Proof. The estimate (3.7.5) leads

$$\begin{aligned}
&\big\|u_\nu-u_{\nu'}\big\|_{\infty,m,\gamma,0,T_*}+\big|(u_\nu-u_{\nu'})(0)\big|_{\infty,m,\gamma,0,T_*}+\big\|X_+(u_\nu-u_{\nu'})\big\|_{L^\infty(\Omega^+_{T_*})}\\
&\quad+\big\|\partial_x(u_\nu-u_{\nu'})\big\|_{m,\gamma,-1,T_*}+\big|\partial_x(u_\nu-u_{\nu'})(0)\big|_{m,\gamma,-1,T_*}\\
&\qquad\leq C\,T_*\Big(\big\|F(\,.\,,u_{\nu-1})-F(\,.\,,u_{\nu'-1})\big\|_{\infty,m,\gamma,0,T_*}\\
&\qquad\qquad+\big|G(\,.\,,u_{\nu-1}(0))-G(\,.\,,u_{\nu'-1}(0))\big|_{\infty,m,\gamma,0,T_*}\Big).
\end{aligned}$$

Furthermore, from Lemma 3.7.8 and Proposition 3.7.5, $(F(\,.\,,u_\nu))_{\nu\geq 0}$ (resp. $(G(\,.\,,u_\nu(0)))_{\nu\geq 0}$) is a Cauchy sequence in $L^\infty(\Omega^+_{T_*})\cap L^2_x(H^{0,m}(\Omega^0_{T_*}))$ (resp. in $L^\infty(\Omega^0_{T_*})\cap H^{0,m}(\Omega^0_{T_*})$), for the norm $\big\|\,.\,\big\|_{\infty,m,\gamma,0,T_*}$ (resp. $\big|\,.\,(0)\big|_{\infty,m,\gamma,0,T_*}$). This finishes the proof of Lemma 3.7.9. □

We obtain from Lemma 3.7.9 that $(u_\nu)_{\nu\geq 0}$ converges to a limit u in $\mathbb{L}^{\infty,T_*}\cap\mathbb{H}^{m,T_*}$, for the norm $\big\|\,.\,\big\|_{\infty,m,\gamma,0,T_*}+\big|\,.\,(0)\big|_{\infty,m,\gamma,0,T_*}+\big\|X_+\,.\,\big\|_{L^\infty(\Omega^+_{T_*})}+\big\|\partial_x\,.\,\big\|_{m,\gamma,-1,T_*}+\big|\partial_x\,.\,(0)\big|_{m,\gamma,-1,T_*}$.

Passing to the limit as $\nu\to+\infty$ in the problem (3.7.4), we obtain, since F and G are continuous, that the limit u of the sequence $(u_\nu)_{\nu\geq 0}$ is the solution vanishing in the past of the semi-linear problem (3.2.1).

Furthermore, from the estimate (3.7.5), we have, for $m>4+\frac{d-1}{2}$ fixed integer, the estimate:

$$\begin{aligned}
&\exists C>0,\forall\gamma\geq\gamma_0,\,\forall 0<T\leq T_*\ /\ T\,\gamma\approx 1,\\
&\big\|u_\nu\big\|_{\infty,m,\gamma,0,T}+\big|u_\nu(0)\big|_{\infty,m,\gamma,0,T}\\
&\quad+\big\|X_+u_\nu\big\|_{L^\infty(\Omega^+_T)}+\big\|\partial_x u_\nu\big\|_{m,\gamma,-1,T}+\big|\partial_x u_\nu(0)\big|_{m,\gamma,-1,T}\\
&\qquad\leq C\,T\big(\big\|f\big\|_{\infty,m,\gamma,0,T}+\big\|F(\,.\,,u_{\nu-1})\big\|_{\infty,m,\gamma,0,T}\\
&\qquad\qquad+\big|g\big|_{\infty,m,\gamma,0,T}+\big|G(\,.\,,u_{\nu-1}(0))\big|_{\infty,m,\gamma,0,T}\big).
\end{aligned}$$

Passing to the limit as $\nu \to \infty$, we obtain

$$\begin{aligned}\|u\|_{\infty,m,\gamma,0,T} &+ |u(0)|_{\infty,m,\gamma,0,T}\\ &+ \|X_+u\|_{L^\infty(\Omega_T^+)} + \|\partial_x u\|_{m,\gamma,-1,T} + |\partial_x u(0)|_{m,\gamma,-1,T}\\ &\leq C\,T\big(\|f\|_{\infty,m,\gamma,0,T} + \|F(\,.\,,u)\|_{\infty,m,\gamma,0,T}\\ &\qquad + |g|_{\infty,m,\gamma,0,T} + |G(\,.\,,u(0))|_{\infty,m,\gamma,0,T}\big).\end{aligned}$$

From Proposition 3.7.4 on F and G, and taking $0 < T_0 \leq T_*$ sufficiently small, we obtain:

$$\begin{aligned}\|u\|_{\infty,m,\gamma,0,T_0} &+ |u(0)|_{\infty,m,\gamma,0,T_0}\\ &+ \|X_+u\|_{L^\infty(\Omega_{T_0}^+)} + \|\partial_x u\|_{m,\gamma,-1,T_0} + |\partial_x u(0)|_{m,\gamma,-1,T_0}\\ &\lesssim \|f\|_{\infty,m,\gamma,0,T_0} + |g|_{\infty,m,\gamma,0,T_0}.\end{aligned}$$

This finishes the proof of Theorem 3.2.26.

3.8 Reflection of discontinuities: Proof of Theorem 3.2.27

Recall that we are given the phases

$$\varphi_\pm = \xi_\pm x + \omega_0 t + k_0 \cdot y, \quad \text{such that} \quad \varphi_{\pm|x=0} = \varphi_0 = \omega_0 t + k_0 \cdot y. \tag{3.8.1}$$

Introduce $\Sigma_\pm = \{\varphi_\pm = 0\}$ and $\Sigma_0 = \Sigma_+ \cap \Sigma_- \subset \{x = 0\}$. Recall that the fields Y_j are tangent to both Σ_+ and Σ_- and thus to Σ_0. Moreover, $\widetilde{X}_+$ is tangent to Σ_+ and $\widetilde{X}_-$ is tangent to Σ_-.

Our goal is to study how a singularity located on Σ_- is reflected on Σ_+.

For some $T > 0$, consider the domain

$$\Omega_T^+ = \{t \leq T, x \geq 0\} =]-\infty, T]_t \times \mathbb{R}_x^+ \times \mathbb{R}_y^{d-1}. \tag{3.8.2}$$

Noticing that $\xi_+ < \xi_-$, by (3.3.5), introduce the subdomains

$$\Omega_1 = \Omega_T^+ \cap \{\varphi_- \leq 0\}, \tag{3.8.3}$$

$$\Omega_2 = \Omega_T^+ \cap \{\varphi_+ \leq 0 \leq \varphi_-\}, \tag{3.8.4}$$

$$\Omega_3 = \Omega_T^+ \cap \{\varphi_+ \geq 0\}. \tag{3.8.5}$$

On the boundary $\{x = 0\}$, introduce the subdomains

$$\Omega_1^0 = \Omega_T^0 \cap \{\varphi_0 \leq 0\}, \tag{3.8.6}$$

$$\Omega_3^0 = \Omega_T^0 \cap \{\varphi_0 \geq 0\}, \tag{3.8.7}$$

where

$$\Omega_T^0 = \{t \leq T\} =]-\infty, T]_t \times \mathbb{R}_y^{d-1}. \tag{3.8.8}$$

We say that a function u is piecewise Lipschitz on Ω_T^+, [resp. Ω_T^0] if its restrictions to the Ω_j [resp. Ω_j^0] are uniformly Lipschitz continuous, that is belong to the Sobolev space $W^{1,\infty}(\Omega_j)$ [resp. $W^{1,\infty}(\Omega_j^0)$]. Such functions have continuous traces from above and from below on Σ_+ and Σ_- [resp. Σ_0] and we denote by $[u]_{\Sigma_+}$ and $[u]_{\Sigma_-}$ the jumps of u across Σ_+ and Σ_- [resp. $[u]_{\Sigma_0}$ the jump of u across Σ_0].

We now prove Theorem 3.2.27.

a) We know from the analysis of Section 3.6 that $u \in L^\infty$ and $X_+ u \in L^\infty$. Differentiating twice the equation in the directions Y_j, we conclude that also $Y^\alpha u \in L^\infty$ and $Y^\alpha X_+ u \in L^\infty$ when $|\alpha| \leq 2$. Therefore, $v = X_+ u$ satisfies

$$-X_- v = f + F(\,.\,, u) + Q(Y)u \in L^\infty(\Omega_T^+). \tag{3.8.9}$$

Next, because f is piecewise Lipschitz and X_+ is tangent to Σ_+, we see that $X_+ f \in L^\infty$ on both sides of Σ_-. Therefore, one can differentiate the equation for v outside of Σ_- and

$$-X_-(X_+ v) = X_+ f + (X_+ F)(\,.\,, u) + (\partial_u F)(\,.\,, u)v + Q(Y)v \in L^\infty(\Omega_T^+ \backslash \Sigma_-). \tag{3.8.10}$$

Integrating along the characteristics of the outgoing field X_-, yields

$$X_+ v \in L^\infty(\Omega_T^+ \backslash \Sigma_-). \tag{3.8.11}$$

Therefore, together with (3.8.9) and the known L^∞ estimates for v and the $Y_j v$, we see that v is Lipschitz continuous on $\Omega_T^+ \backslash \Sigma_-$. In particular, v is piecewise Lipschitz, $[v]_{\Sigma_+} = 0$ and $v_0 := v_{|x=0}$ is piecewise Lipschitz.

b) The boundary equation for $u_0 := u_{|x=0}$ reads

$$\check{v} \cdot Y u_0 + \check{c} u_0 - G(\,.\,, u_0) = g - v_0. \tag{3.8.12}$$

The characteristics of $\check{v} \cdot Y$ reach $\{t \leq 0\}$ in finite time since the coefficient of ∂_t does not vanish. Moreover, they are parallel to Σ_0. Therefore, since the right-hand side is piecewise Lipschitz and, from the causality principle, the condition $u_0 = 0$ for $t \leq 0$ is satisfied, the equation (3.8.12) implies that u_0 is piecewise Lipschitz.

c) Next we use the propagation equation for u:

$$X_+ u = v, \qquad u_{|x=0} = u_0. \tag{3.8.13}$$

Because v and u_0 are piecewise Lipschitz and X_+ is parallel to Σ_+, this implies that u itself is piecewise Lipschitz.

d) We now investigate the equations for the jumps. Because $X_+ u = v \in L^\infty(\Omega_T^+)$ and X_+ is transverse to Σ_-, we see that

$$[u]_{\Sigma_-} = 0. \tag{3.8.14}$$

From the regularity of F, we thus get $[F(\,.\,,u)]_{\Sigma_-} = 0$. Similarly, since $X_+ Q(Y)u \in L^\infty$, the traces of $Q(Y)u$ on Σ_- from above and from below in the direction of the characteristics of X_+ are well defined, and coincide. Therefore, taking traces on Σ_- in (3.8.9), yields

$$-X_-([v]_{\Sigma_-}) = [f]_{\Sigma_-}. \tag{3.8.15}$$

Moreover, since v is Lipschitz continuous on $\Omega_T^+ \backslash \Sigma_-$,

$$\jmath_1 := ([v]_{\Sigma_-})_{|x=0} = [v_{|x=0}]_{\Sigma_0} = [v_0]_{\Sigma_0}. \tag{3.8.16}$$

With (3.8.12) this implies that

$$(\check{v} \cdot Y + \check{c})[u_0]_{\Sigma_0} - [G(\,.\,,u_0)]_{\Sigma_0} = [g]_{\Sigma_0} - \jmath_1. \tag{3.8.17}$$

In particular, if $[g]_{\Sigma_0} = 0$ and $\jmath_1 \neq 0$, then the jump $[u_0]_{\Sigma_0}$ cannot vanish identically.

Furthermore, from u continuous across Σ_-, we get $[u_0]_{\Sigma_0} = ([u]_{\Sigma_+})_{|x=0}$

Finally we take traces on Σ_+ in the equation (3.8.13). Since v is bounded and Lipschitz continuous on $\Omega_T^+ \backslash \Sigma_-$, we see that $[u]_{\Sigma_+}$ satisfies (3.2.20).

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