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On the distribution of the values of arithmetical functions

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- 2010 -

This thesis is dedicated to

my patient father, my dear mother and my sweetheart wife.

To be honest, it is impossible for me to explain by words their sincerely and endless kindness and supports.

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Mehdi Hassani
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RÉSUMÉ

Cette thèse est consacrée à l'étude de plusieurs aspects de la répartition des fonctions multiplicatives à valeurs dans l'intervalle $[0,1]$. L'archétype de ces fonctions est la fonction $\nu(n) = \varphi(n)/n$, où $\varphi(n)$ est la fonction d'Euler qui donne le cardinal de $(\mathbb{Z}/n\mathbb{Z})^*$. Cette fonction présente l'avantage technique d'être fortement multiplicative, c'est-à-dire que $\nu(p^\alpha)$ ne dépend pas de α; les résultats que nous présentons pour la fonction ν s'étendent aux fonctions multiplicatives f à valeurs dans l'intervalle $[0,1]$ pour lesquelles il existe un réel $c > 0$ tel que $f(p^\alpha) = 1 - c/p + O(1/p^\alpha)$ pour tout premier p et tout $\alpha \geq 1$; un autre exemple d'importance historique est la fonction $n/\sigma(n)$, où $\sigma(n)$ désigne la somme des diviseurs de l'entier n.

Nous décrivons ici les deux résultats principaux de notre travail, qui donne chacun lieu à une publication.

On sait, depuis les travaux de Imre Kátai, que la suite $(\nu(p-1))_p$ indexée par les nombres premiers p, admet une fonction de répartition F_ν, c'est-à-dire que pour tout réel y la limite

$$F_\nu(y) \;=\; \lim_{x\to\infty} \frac{1}{x} \operatorname{Card}\{n \leq x \,|\, \nu(n) \leq y\}$$

existe. On sait en outre que cette fonction est croissante au sens large sur $\mathbb{R}$, vaut 0 sur $]-\infty, 0]$, vaut 1 sur $[1/2, +\infty[$ et est continue et strictement croissante sur $[0, 1/2]$; en outre, elle est purement singulière, c'est-à-dire qu'en presque tout point, au sens de la mesure de Lebesgue, la fonction F_ν est dérivable à dérivée nulle. Notre premier résultat est d'établir qu'en tout point $x_m = \nu(2m)$, la fonction F_ν n'est pas dérivable à gauche du point x_m. Ce résultat est obtenu par une méthode de moments ; pour l'illustrer, considérons le point $x_1 = 1/2$. On calcule de deux façons l'intégrale $I_k = \int_0^{1/2} (2t)^k dF_\nu(t)$, où k désigne un entier positif.
D'une part, on a

$$I_k \;=\; \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} 2^k \nu^k(n)$$

et on montre, par des méthodes classiques de la théorie des nombres, la minoration $I_k \gg 1/\log(k)$ quand k tend vers l'infini.
D'autre part, si on suppose que F_ν est dérivable à gauche en x_1, on montre par une méthode classique de l'analyse réelle, la majoration $I_k = O(1/k)$ quand k tend vers l'infini.
Il en résulte une contradiction qui prouve que la fonction F_ν n'est pas dérivable à gauche en x_1.

En un point x_m général, on considère les nombres premiers p congrus à 1 modulo $2m$ et on raisonne de façon similaire.

Le second résultat principal concerne une généralisation d'un problème étudié récemment par Jean-Marc Deshouillers, Henryk Iwaniec et Florian Luca, à

savoir la répartition modulo 1 de la suite à croissance linéaire

$$u_n = \sum_{1\leq k\leq n} \nu(k).$$

On étudie ici la moyenne prise, non plus sur tous les entiers, mais sur une suite polynômiale ; ici encore, nous regardons la situation "archétypale" où le polynôme considéré est le polynôme non linéaire le plus simple, à savoir $P(x) = x^2 + 1$. On pose

$$v_n = \sum_{1\leq k\leq n} \nu(k^2 + 1).$$

Nous montrons que la suite $(v_n)_n$ est dense modulo 1. L'argument repose sur une construction combinatoire: pour chaque entier M d'une famille infinie, on construit des entiers $k_1, k_2, \dots k_M$, chacun somme de deux carrés, premiers entre eux deux à deux, tels que les valeurs $\nu(k_j)$ soient petites (disons plus petites que $4/M$ pour fixer les idées), mais telles que leur somme $\sum_{1\leq j\leq M} \nu(k_j)$ soit supérieure à 2. Cette construction, assez technique, ne met en jeu que des moyens élémentaires : théorème des restes chinois, divergence de la somme des inverses des nombres premiers dans une progression arithmétique. Alors, l'utilisation de la théorie du crible, permet de montrer l'existence d'entiers n tels que pour tout j, le nombre $n + j$ soit divisible par k_j, mais $(n + j)/k_j$ n'a que peu de facteurs premiers qui en outre sont tous très grands. Cela implique que $\nu(n+j)$ est proche de $\nu(k_j)$; bien évidemment, $\nu(n+j)$ est petit (toujours inférieur à $4/M$), mais surtout, la somme $\sum_{1\leq j\leq M} \nu(n + j)$ est supérieure à 1. Cela implique la densité modulo 1 de la suite $(v_n)_n$.

Ces différentes études théoriques sont en outre illustrées par plusieurs expérimentations numériques.

ABSTRACT

As its title's indicates, this thesis is about the distribution of arithmetic functions. The word "distribution" refers to various concepts and facts, including density or uniform distribution, and distribution functions of certain sequences. We consider distribution functions of some sequences related to the Euler φ function and divisor function σ. This is a classical matter and many known results like Erdös - Wintner theorem for the existence and continuity of those distribution functions for the sequences defined on positive integers and also on the shifted primes $p+1$, are at hand.

In chapter 1 first we review some known results mentioned the above. Then we study differentiability of the distribution function $F(x)$ defined by

$$\lim_{N\to\infty}\frac{1}{\pi(N)}\mathrm{Card}\Big\{p\le N\big|\frac{\varphi(p-1)}{p-1}\le x\Big\}=F(x).$$

We show that at each point $x_m=\varphi(m)/m$, where m is an even integer, F is not differentiable from the left. For this purpose we use the method of moments.

Chapter 2 contains results on the density modulo 1 of some well known means of the Euler function and divisor function. This is based on the work of J-M. Deshouillers and F. Luca (2008). We elaborate on their work and give a refinement and a generalization.

In chapter 3, we study density modulo 1 of some sequences connected with the mean values of the ratio $\varphi(n^2+1)/(n^2+1)$. Among various sequences, we prove that the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by

$$b_n=\sum_{m\le n}\frac{\varphi(m^2+1)}{m^2+1}$$

is dense modulo 1. Our proof is based on some sieve results which allow us to control the size of prime factors of numbers of the form n^2+1.

After studying density results, we focus on the uniform distribution modulo 1 of some sequences of arithmetic functions. Uniform distribution modulo 1 of sequences containing additive arithmetical functions has been studied in general by H. Delange. Best known examples of such functions are the Omega functions $\omega(n)$ and $\Omega(n)$. Delange gave also a class of Omega functions containing these classical functions and some generalizations of them, and then he gave an analytic method to study uniform distribution modulo 1 of them. In chapter 4, first we introduce the work of Delange and the empirical results of F. Dekking and M. Mendès France, and then, we study three dimensional distributions of related sequences. The new empirical results that we obtain in this chapter shed some light on the work of J-M. Deshouillers and H. Iwaniec [17] and other classical results.

We end our work by adding an appendix, including required and frequently used formulas.

NOTATIONS

$\mathbb{P}$ ..... set of prime numbers $\{2, 3, 5, \dots\}$
$\mathbb{P}_{a,q}$ ..... set of primes p with $p \equiv a \pmod q$
$\mathbb{N}$ ..... set of natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{N}_{a,q}$ ..... set of positive integers n with $n \equiv a \pmod q$
$\mathbb{Z}$ ..... set of integer numbers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
$\mathbb{R}$ ..... set of real numbers
$]a, b[$ and (a, b) ..... open interval with endpoints a and b
$[a, b[$ and $]a, b]$ ..... half-open intervals with endpoints a and b
$\mathbb{C}$ ..... set of complex numbers
$\Re(z)$ ..... real part of $z \in \mathbb{C}$
$\#A$ and $\mathrm{Card}(A)$ ..... both denote cardinal number of the set A
$\gcd(m, n)$ ..... greatest common divisor of the integers m and n
$\varphi(n)$ ..... Euler function, defined by $\#\{m \in \mathbb{N} : m \le n, \gcd(m, n) = 1\}$
$d|n$ ..... d is a positive divisor of n
$p^\alpha \| n$ ..... $p^\alpha | n$ and $p^{\alpha+1} \nmid n$
$\sigma(n)$ ..... divisor function, defined by $\sum_{d|n, d>0} d$
$\mu(n)$ ..... Möbius function of n
$\omega(n), \Omega(n)$ .... omega functions, defined by $\sum_{p|n} 1$ and $\sum_{p^\alpha \| n} \alpha$, respectively
$\nu_N(x)$ ..... distribution function, defined by $\frac{1}{N} \# \{n \le N | f(n) \le x\}$
$\zeta(s)$ ..... the Riemann zeta function, defined by $\sum_{n=1}^{\infty} n^{-s}$
$\sum_p$ ..... a sum over primes p
$\sum_{p \equiv a \ [m]}$ ..... a sum over primes p with $p \equiv a \pmod m$
$\pi(x)$ ..... primes counting function, defined by $\sum_{p \le x} 1$
$\pi(x; q, a)$ ..... number of primes $p \le x$ with $p \equiv a \pmod q$
$f \ll g$ .... Vinogradov's notation, which means $|f(x)| \le cg(x)$ for $x \ge x_0$ and some absolute constant $c \in \mathbb{R}$
$f = O(g)$ ..... Landau's big-oh notation, which means $f \ll g$
$f \ll_m g$ and $f = O_m(g)$ ..... both are same as $f \ll g$ with c is an absolute constant depending on m
$f \gg_m g$ ..... means $g \ll_m f$
$f = o(g)$ ... Landau's small-oh notation, which means $\lim_{x \to \infty} f(x)/g(x) = 0$
$f \asymp g$ ..... means $f \ll g$ and $g \ll f$, simultaneously
γ ..... Euler's constant, defined by $\lim_{N \to \infty} (\sum_{1 \le n \le N} 1/n - \log N)$
$\mathrm{fix}(f)$ .... fixed divisor of the polynomial f with integer coefficients, which is largest integer that divides $f(n)$ for all $n \in \mathbb{Z}$
$\lfloor x \rfloor$ ..... floor of x
$\{x\}$ ..... fractional part of x, that is $x - \lfloor x \rfloor$
$\|\!| x |\!\|$ ..... distance from x to its nearest integer, i.e., $\min(\{x\}, 1 - \{x\})$
$e(x)$ ..... means $e^{2\pi i x}$
$\Gamma(x)$ ..... gamma function at x

KEY WORDS

MATHEMATICS SUBJECT CLASSIFICATION

• **Key Words**

Number theory, Euler φ function, divisor σ function, arithmetical function, multiplicative function, additive function, distribution function, density modulo 1, uniform distribution modulo 1, exponential sums, Weyl sums, Weyl criterion, primes, density, sieve method.

• **Mathematics Subject Classification (MSC) 2010**

11-XX, 11A25, 11A41, 11K06, 11K31, 11K65, 11L03, 11L15, 11N36, 11N37, 11N56, 11N60, 11N64, 11J71.

• **Code-list of Mathematics Subject Classification 2010**

11A25 Arithmetic functions; related numbers; inversion formulas
11A41 Primes
11K06 General theory of distribution modulo 1 [See also 11J71]
11K31 Special sequences
11K65 Arithmetic functions [See also 11Nxx]
11L03 Trigonometric and exponential sums, general
11L15 Weyl sums
11N36 Applications of sieve methods
11N37 Asymptotic results on arithmetic functions
11N56 Rate of growth of arithmetic functions
11N60 Distribution functions associated with additive and positive multiplicative functions
11N64 Other results on the distribution of values or the characterization of arithmetic functions
11J71 Distribution modulo one

CONTENTS

LIST OF FIGURES

1. DISTRIBUTION FUNCTION OF SOME SEQUENCES OF $\varphi(N)$ AND $\sigma(N)$

In this chapter, we study distribution function of some sequences related by the values of the functions $\varphi(n)$ and $\sigma(n)$. We follow history of the subject, and then, we focus on the differentiability of such distribution functions, mainly defined on the shifted primes.

1.1 Introduction

The arithmetical functions f and g are additive and multiplicative, respectively, if for $\gcd(m,n)=1$, they satisfy

$$f(mn)=f(m)+f(n), \qquad \text{and} \qquad g(mn)=g(m)g(n).$$

We assume that a multiplicative function takes only strictly positive values, so that we can consider its logarithm which is an additive function. For an arithmetic function f with real values, we consider the quantity

$$\nu_N(x)=\frac{1}{N}\#\left\{n\leq N | f(n)\leq x\right\}$$

defined for $N\geq 1$ and $x\in\mathbb{R}$. The function ν_N is the distribution function of a probability measure on $\mathbb{R}$. We say a function f admits a distribution function ν if the sequence $(\nu_N(x))_N$ converges to a function $\nu(x)$ at each point of continuity of ν, in other words, where the sequence of the probability measures associated to ν_N converges weakly toward the probability measure whose distribution function is ν. Natural questions are to find conditions under which ν exists, ν is continuous and ν is differentiable.

As it is known, the first published work regarding these questions, is due to I. J. Schoenberg [45] in 1928. He proved that $\varphi(n)/n$ has a continuous distribution function. After Schoenberg, Behrend [2], Chowla [4] and Davenport [7] proved same result for $\sigma(n)/n$. Indeed, letting

$$A(x)=\lim_{N\to\infty}\frac{1}{N}\#\left\{n\leq N : \frac{\sigma(n)}{n}\geq x\right\},$$

they proved that $A(x)$ exists and is continuous for all real x, and consequently, they proved that the density of abundant numbers, $A(2)$, exists.

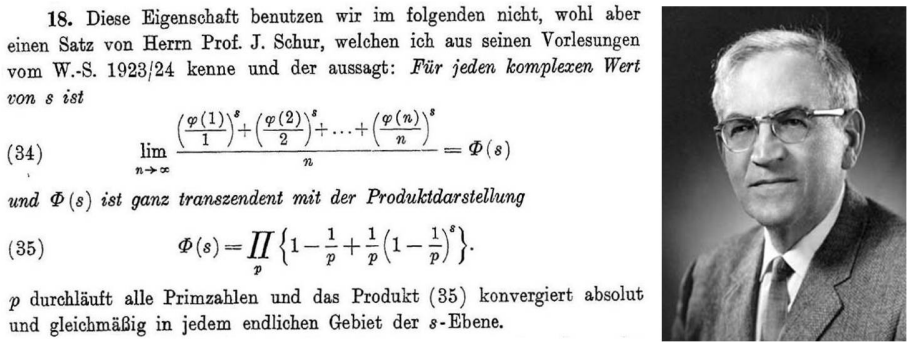
Moreover, Behrend got some lower bound and upper bound for the value of $A(2)$, a question on which many other authors focuced. The table below summarizes bounds for $A(2)$:

Bound	Author	Year
$0.241 < A(2) < 0.314$	Behrend [2]	1933
$0.2441 < A(2) < 0.2909$	Wall, Crews, Johnson [52]	1972
$0.2474 < A(2) < 0.2480$	Deléglise [13]	1998

The result of Deléglise gives

$$A(2) = 0.247\cdots,$$

and answers the following question asked by Henri Cohen: Is the proportion of abundant numbers more or less than a quarter?



18. Diese Eigenschaft benutzen wir im folgenden nicht, wohl aber einen Satz von Herrn Prof. J. Schur, welchen ich aus seinen Vorlesungen vom W.-S. 1923/24 kenne und der aussagt: *Für jeden komplexen Wert von s ist*

$$\lim_{n\to\infty} \frac{\left(\frac{\varphi(1)}{1}\right)^s + \left(\frac{\varphi(2)}{2}\right)^s + \ldots + \left(\frac{\varphi(n)}{n}\right)^s}{n} = \Phi(s) \tag{34}$$

und $\Phi(s)$ ist ganz transzendent mit der Produktdarstellung

$$\Phi(s) = \prod_p \left\{1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right\}. \tag{35}$$

p durchläuft alle Primzahlen und das Produkt (35) konvergiert absolut und gleichmäßig in jedem endlichen Gebiet der s-Ebene.

Fig. 1.1: Isaac Jacob Schoenberg (21 April 1903 - 21 Feb 1990, Romanian mathematician), and a part of his 1926 (published in 1928) paper.

An analytic method (but with weaker numerical results) for the computation of $A(x)$ is given by J. Martinet, J-M. Deshouillers and H. Cohen in [38], where they make use of the classical result

$$g(s) = \int_0^\infty x^s A(x)dx = \frac{1}{s+1} \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{-s} \sum_{k=0}^{\infty} \frac{1}{p^k} \left(1 - \frac{1}{p^{k+1}}\right)^{s+1},$$

for $\Re(s) > 0$. By the inversion of the Mellin transform $g(s)$ of $A(x)$, we have

$$A(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s} g(s)ds.$$

Furthermore

$$\int_0^\infty A(x)dx = \zeta(2) \qquad \text{and} \qquad \int_0^\infty xA(x)dx = \frac{5}{4}\zeta(3).$$

These authors suggest that $A(x)$ should be differentiable everywhere, except at points x which are a value of the function $\sigma(n)/n$. Almost at the same time P. Erdös [23] proved several results in this connection.

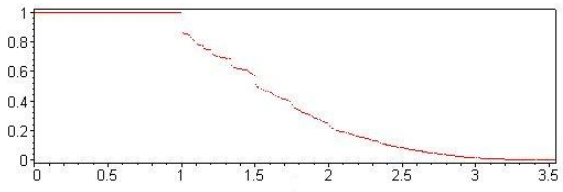


Fig. 1.2: Distribution function $A_N(x) = \frac{1}{N}\#\{n \le N : \sigma(n)/n \ge x\}$ with $N = 2000$.

Studying the distribution function for additive and multiplicative functions in general case was initiated by I. J. Schoenberg. Since the logarithm of a multiplicative function is additive, it will be sufficient to consider additive functions only. Schoenberg [44] proved the following result.

Theorem 1.1.1 (Schoenberg, 1934)**.** *Suppose that* $\sum_p \frac{\min(1,|f(p)|)}{p}$ *converges. Then, the distribution function of* $f(m)$ *exists. If* $f(m)$ *satisfies the supplementary condition that there exists an infinite subset of primes* $\mathcal{P} = \{p_1, p_2, \cdots\}$ *with* $f(p_i) \neq f(p_j)$ *for* $i \neq j$ *and such that* $\sum_{p\in\mathcal{P}} \frac{1}{p}$ *diverges, then the distribution function is continuous. On the other hand, if* $\sum_{f(p)\neq 0} \frac{1}{p}$ *converges, then the distribution function is purely discontinuous.*

In the third part of his investigations on the distribution of additive functions, P. Erdös [22] found a sufficient condition for the existence of the asymptotic distribution function $\nu(x)$ for the distribution functions

$$\nu_N(x) = \frac{1}{N} \sum_{\substack{n\le N \\ f(n)\le x}} 1.$$

This sufficient condition for the existence of the asymptotic distribution function turns out to be a necessary condition as well. This fact due to P. Erdös and A. Wintner [26], solves the problem of the existence of an asymptotic distribution function of an arbitrary (real) additive arithmetical function.

Theorem 1.1.2 (Erdös - Wintner, 1939)**.** *Suppose that* f *is a real additive function. A necessary and sufficient condition for* f *to have a limiting distribution is that for a positive real number* R*, the following series are convergent:*

$$\sum_{|f(p)|\le R} \frac{f(p)}{p}, \qquad \sum_{|f(p)|\le R} \frac{f(p)^2}{p}, \qquad \sum_{|f(p)|>R} \frac{1}{p}.$$

Moreover, the distribution function is continuous if and only if the series $\sum_{f(p)\neq 0} \frac{1}{p}$ *diverges.*

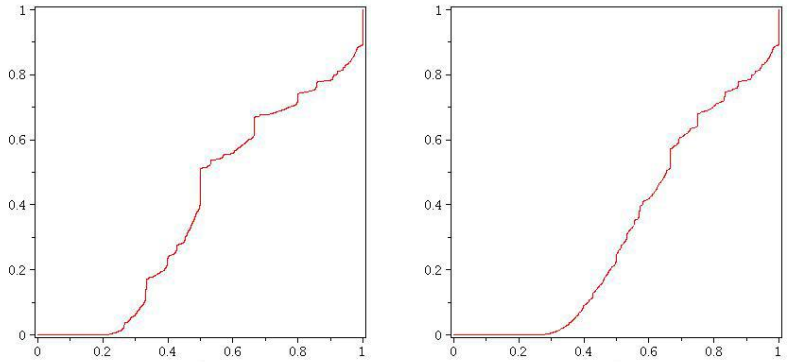


Fig. 1.3: Distribution function $\nu_N(x) = \frac{1}{N}\#\{n \leq N : f(n) \leq x\}$ with $N = 20000$ for the functions $f(n) = \varphi(n)/n$ (left) and $f(n) = n/\sigma(n)$ (right).

Studying the distribution function of arithmetic functions on the shifted primes is also an interesting problem. In 1968, I. Kátai [34] proved the following result:

Theorem 1.1.3 (Kátai, 1968)**.** *Suppose that f is a real additive function such that the series*

$$\sum_p \frac{f^*(p)}{p}, \qquad \sum_p \frac{(f^*(p))^2}{p}, \qquad \sum_{|f(p)|>1} \frac{1}{p},$$

converge. Here

$$f^*(n) = \begin{cases} f(n), & \text{for } |f(n)| \leq 1, \\ 1, & \text{for } |f(n)| > 1. \end{cases}$$

Let

$$\nu_N(x) = \frac{1}{\pi(N)} \sum_{\substack{p \leq N \\ f(p+1) \leq x}} 1.$$

Then, there exists a distribution function ν such that $\nu_N(x)$ tends to $\nu(x)$ at each point of continuity of ν. Moreover, $\nu(x)$ is a continuous function if and only if the series $\sum_{f(p)\neq 0} \frac{1}{p}$ diverges.

The proof of this theorem, is based on the approximation of the moments of some multiplicative functions related to f at shifted primes. Indeed, Kátai obtains the above theorem from the following one:

Theorem 1.1.4 (Kátai, 1968)**.** *Suppose that g is a complex valued multiplicative function with $|g(n)| \leq 1$ for $n = 1, 2, \cdots$, and such that the series*

$$\sum_p \frac{g(p) - 1}{p}$$

converges. Let

$$M(g) = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha) - g(p^{\alpha-1})}{p^{\alpha-1}(p-1)}\right).$$

Then, we have

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\le N} g(p+1) = M(g).$$

Kátai applies his Theorem 1.1.3 to the function $\log g$, where g is a positive multiplicative function, to get a limiting distribution function for $\nu_N(x) = \frac{1}{\pi(N)} \sum_{g(p+1)\le x} 1$. More precisely, the functions $\frac{\varphi(p+1)}{p+1}$ and $\frac{\sigma(p+1)}{p+1}$ have limiting distribution functions.

In 1972, Elliott [19] showed that in the result of Kátai, the mentioned conditions for sufficiency, are also necessary for positive and strongly additive functions. We call the arithmetical function f strongly additive or strongly multiplicative, if it is additive or multiplicative and $f(p^\alpha) = f(p)$ for all primes p and all positive integers α.

In 1975, J-M. Deshouillers [15] considered a class of polynomials F and multiplicative functions g, with some certain conditions, and he showed the following results.

Theorem 1.1.5 (Deshouillers, 1975)**.** *Distribution functions*

$$\nu_N(x) = \frac{1}{\pi(N)} \sum_{\substack{p\le N \\ g(F(p))\le x}} 1,$$

tend to a limiting distribution function $\nu(x)$, where ν is continuous. Also, there exists a real number θ such that ν becomes strictly increasing in $(0,\theta)$ and $\nu(\theta) = 1$. Moreover, the condition $\nu(x) < \nu(y)$ is equivalent by existence of a prime $p_0 > \max\{F(0), \deg(F)\}$ such that $x < g(F(p_0)) < y$.

Theorem 1.1.6 (Deshouillers, 1975)**.** *Suppose that $\lambda(d)$ is the number of integers n satisfying $F(n) \equiv 0 \pmod d$ with $0 \le n < d$ and $\gcd(n,d) = 1$. Let*

$$V(x) = \sum_{\substack{p\le x \\ p\equiv b \pmod a}} g(F(p)).$$

Then

$$V(x) = (1+o(1))\frac{\pi(x)}{\varphi(a)} g(F(p_0)) \prod_{p>z} \left(1 + \sum_{\alpha=1}^{\infty} \frac{\lambda(p^\alpha)(g(p^\alpha) - g(p^{\alpha-1}))}{p^{\alpha-1}(p-1)}\right),$$

where $z > \max\{p_0, q_1, \cdots, q_n\}$, and $F(p_0) = q_1^{\alpha_1} \cdots q_n^{\alpha_n}$.

In 1989, A. Hildebrand [32] introduced an extensive study of the distribution of additive and multiplicative functions on the set of shifted primes. More

precisely, he obtained a complete analogues of the Erdös - Wintner Theorem on shifted primes.

In this chapter, we study differentiability of the distribution of some multiplicative functions on shifted primes. The method introduced in this chapter is applicable to the studying of differentiability of the distribution function of some other arithmetical functions on positive integers.

1.2 On the differentiability of the distribution functions on shifted primes

It has been shown by I. Kátai [34] that the numbers $\varphi(p-1)/(p-1)$, where p runs over the set of primes numbers, has a continuous limiting distribution F, that is to say that for any real x we have

$$\lim_{N\to\infty}\frac{1}{\pi(N)}\mathrm{Card}\Big\{p\le N\big|\frac{\varphi(p-1)}{p-1}\le x\Big\}=F(x).$$

The aim of this chapter is to show that at each point $x_m=\varphi(m)/m$, where m is an even integer, F is not differentiable from the left. More precisely, we shall prove the following.

Theorem 1.2.1. *Let m be any positive even integer and let $x_m=\varphi(m)/m$. We have*

$$\forall A>0,\forall\delta>0,\exists y\in[x_m-\delta,x_m)\ s.t.\ F(x_m)-F(y)\ge A(x_m-y). \quad (1.2.1)$$

To prove Theorem 1.2.1, we shall restrict ourselves to primes congruent to 1 modulo m and obtain the validity of (1.2.1) by contradiction, using the method of moments.

The method used by Kátai [34] immediately leads to the following:

Proposition 1.2.2. *Let g be a positive valued multiplicative number-theoretic function such that the three series*

$$\sum_{|\log g(p)|\le 1}\frac{\log g(p)}{p},\qquad \sum_{|\log g(p)|\le 1}\frac{\log^2 g(p)}{p}\quad \textit{and}\quad \sum_{|\log g(p)|>1}\frac{1}{p} \quad (1.2.2)$$

converge. Then for every $m\ge 1$ there exists a distribution function G_m such that at all points y of continuity of G_m one has:

$$\frac{1}{\pi(N;m,1)}\sum_{\substack{p\le N,\ p\equiv 1\ [m]\\ g(p-1)\le y}}1\longrightarrow G_m(y)\qquad (as\ N\to\infty). \quad (1.2.3)$$

Moreover, G_m is continuous if and only if the series

$$\sum_{\substack{p\equiv 1\ [m]\\ g(p)\neq 1}}\frac{1}{p}$$

diverges.

We apply Kátai's extended proposition to the function g defined by $g(n) = \varphi(n)/n$. It is easily seen that g satisfies all the conditions of Proposition 1.2.2 and thus, for any positive integer m, which we further assume to be even, we have

$$\forall y \in \mathbb{R} : \frac{1}{\pi(N;m,1)} \sum_{\substack{p\le N,\ p\equiv 1\ [m]\\ g(p-1)\le y}} 1 \longrightarrow G_m(y) \qquad (\text{as } N\to\infty), \tag{1.2.4}$$

where G_m is a continuous distribution function.

Relation (1.2.4) indeed means that the sequence (in N) of the empirical measures

$$\nu_{N,m} = \frac{1}{\pi(N;m,1)} \sum_{p\le N,\ p\equiv 1\ [m]} \delta_{(g(p-1))}, \tag{1.2.5}$$

where $\delta_{(a)}$ denotes the Dirac measure at the positive a, weakly converges to Lebesgue-Stieljes measure dG_m. For $p \equiv 1\ [m]$, we have

$$0 \le g(p-1) = \prod_{q\mid p-1} \Big(1-\frac{1}{q}\Big) = \frac{\varphi(m)}{m} \prod_{\substack{q\mid p-1\\ q\nmid m}} \Big(1-\frac{1}{q}\Big) \le \frac{\varphi(m)}{m}.$$

Thus, the support of the measure dG_m is indeed in $[0, x_m]$ with $x_m = \varphi(m)/m$.

Let us consider the continuous function $t \mapsto t^k$, with support in the compact $[0, x_m]$. We have

$$\forall k \ge 1 : \int_0^{x_m} t^k d\nu_{N,m} \longrightarrow \int_0^{x_m} t^k dG_m(t) \qquad (\text{as } N\to\infty). \tag{1.2.6}$$

1.2.1 Computing moments

In this section, we compute the left hand side of (1.2.6) by number-theoretic methods and obtain a lower bound for the right hand side; more precisely, we show that the following is valid for all positive even integers m, and $k \ge 2$:

$$c_{m,k} := \int_0^{x_m} t^k dG_m(t) \gg_m \frac{x_m^k}{\log k}. \tag{1.2.7}$$

By the definition (1.2.5) of the measure $\nu_{N,m}$, we have

$$\int_0^{x_m} t^k d\nu_{N,m} = \frac{1}{\pi(N;m,1)} \sum_{\substack{p\le N\\ p\equiv 1\ [m]}} \Big(\frac{\varphi(p-1)}{p-1}\Big)^k. \tag{1.2.8}$$

In order to compute the sum, we introduce the multiplicative function $h_{m,k}$ defined for ℓ prime and $\alpha \ge 1$:

$$h_{m,k}(\ell^\alpha) = \begin{cases} 1-(1-1/\ell)^k & \text{if } \ell\nmid m \text{ and } \alpha = 1,\\ 0 & \text{if } \ell\mid m \text{ or } \alpha \ge 2. \end{cases} \tag{1.2.9}$$

By the Möbius inversion formula, the function $f_{m,k}$ defined by

$$f_{m,k}(n) = \sum_{d|n} \mu(d) h_{m,k}(d)$$

is multiplicative. We obviously have for a prime ℓ and any $\alpha \geq 1$:

$$f_{m,k}(\ell^\alpha) = 1 - h_{m,k}(\ell) = \begin{cases} (\varphi(\ell)/\ell)^k & \text{if } \ell \nmid m, \\ 1 & \text{if } \ell | m. \end{cases}$$

We thus have

$$\begin{aligned} \sum_{\substack{p \leq N \\ p \equiv 1 \ [m]}} \Big(\frac{\varphi(p-1)}{p-1}\Big)^k &= \Big(\frac{\varphi(m)}{m}\Big)^k \sum_{\substack{p \leq N \\ p \equiv 1 \ [m]}} f_{m,k}(p-1) \\ &= x_m^k \sum_{\substack{p \leq N \\ p \equiv 1 \ [m]}} \sum_{\substack{d | p-1 \\ (d,m)=1}} \mu(d) h_{m,k}(d) \\ &= x_m^k \sum_{\substack{d \leq N-1 \\ (d,m)=1}} \mu(d) h_{m,k}(d) \sum_{\substack{p \leq N \\ p \equiv 1 \ [d] \\ p \equiv 1 \ [m]}} 1 \\ &= x_m^k \sum_{\substack{d \leq N-1 \\ (d,m)=1}} \mu(d) h_{m,k}(d) \pi(N; md, 1). \end{aligned}$$

When d is large, say $d \geq D = \lfloor N^{1/3} \rfloor$, we use the trivial upper bound $N/(md)$ for $\pi(N; md, 1)$, as well as the upper bound k/ℓ for $h_{m,k}(\ell)$. We get

$$\sum_{\substack{d \geq D \\ (d,m)=1}} |\mu(d) h_{m,k}(d)| \frac{N}{md} \leq N \sum_{d \geq D} \frac{k^{\omega(d)}}{d^2} \ll_{m,k} N^{5/6}.$$

We now consider small d's, that is to say $d < D = \lfloor N^{1/3} \rfloor$. We write

$$\pi(N; md, 1) = \frac{\pi(N)}{\varphi(m)\varphi(d)} + E(N; md, 1)$$

and use the Bombieri-Vinogradov theorem which implies

$$\sum_{\substack{d \leq D \\ (d,m)=1}} |E(N; md, 1)| = O_m\Big(\frac{\pi(N)}{\log N}\Big).$$

This relation, combined with the trivial upper bound $|h_{m,k}(d)| \leq 1$, leads to

$$\sum_{\substack{d \leq D \\ (d,m)=1}} |\mu(d) h_{m,k}(d) E(N; md, 1)| = O_m\Big(\frac{\pi(N)}{\log N}\Big).$$

We are left with the main contribution

$$\frac{\pi(N)}{\varphi(m)} \sum_{\substack{d\le D\\(d,m)=1}} \frac{\mu(d)h_{m,k}(d)}{\varphi(d)} = \frac{\pi(N)}{\varphi(m)} \sum_{\substack{d=1\\(d,m)=1}}^{\infty} \frac{\mu(d)h_{m,k}(d)}{\varphi(d)} + o_m(\pi(N)),$$

since, as above, the upper bound k/ℓ for $|h_{m,k}(\ell)|$ implies the absolute convergence of the series. By the definition of $h_{m,k}$, we have

$$\sum_{\substack{d=1\\(d,m)=1}}^{\infty} \frac{\mu(d)h_{m,k}(d)}{\varphi(d)} = \prod_{\ell\nmid m} \Big(1 - \frac{1-(1-1/\ell)^k}{\ell-1}\Big) \ge \prod_{\ell\ge 3} \Big(1 - \frac{1-(1-1/\ell)^k}{\ell-1}\Big).$$

For $3 \le \ell \le k^2$, we use the lower bound

$$\Big(1 - \frac{1-(1-1/\ell)^k}{\ell-1}\Big) \ge 1 - \frac{1}{\ell-1} = \Big(1-\frac{1}{\ell}\Big)\Big(1-\frac{1}{(\ell-1)^2}\Big),$$

and for $\ell > k^2$, we use

$$\Big(1 - \frac{1-(1-1/\ell)^k}{\ell-1}\Big) \ge 1 - \frac{k}{\ell(\ell-1)} \ge 1 - \frac{1}{\ell^{1/2}(\ell-1)}.$$

We thus have

$$\begin{aligned}\prod_{\ell\ge 3} \Big(1 - \frac{1-(1-1/\ell)^k}{\ell-1}\Big) &\ge \prod_{3\le\ell<k^2} \Big(1-\frac{1}{\ell}\Big) \prod_{\ell>3} \Big(1-\frac{1}{(\ell-1)^2}\Big) \prod_{\ell>3} \Big(1-\frac{1}{\ell^{1/2}(\ell-1)}\Big)\\ &\gg \frac{1}{\log k},\end{aligned}$$

where the last inequality comes from Mertens theorem and the absolute convergence of the two infinite products. This proves (1.2.7).

1.2.2 Proof of nondifferentiability

In this section, we assume that Theorem 1.2.1 does not hold and we deduce an upper bound for $\int_0^{x_m} t^k dG_m(t)$ that contradicts (1.2.7), thus proving Theorem 1.2.1.

The negation of Theorem 1.2.1 is

$$\exists A>0, \exists \delta>0, \forall y \in [x_m-\delta, x_m) \text{ s.t. } F(x_m)-F(y) < A(x_m-y). \quad (1.2.10)$$

By the definition of F, we thus have $\forall y \in [x_m-\delta, x_m)$:

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \mathrm{Card}\Big\{p\le N \Big| \frac{\varphi(p-1)}{p-1} \in [y, x_m)\Big\} < A(x_m-y).$$

This implies that, on the same range for y, one has

$$\overline{\lim_{N\to\infty}} \frac{1}{\pi(N)} \mathrm{Card}\Big\{p\le N \Big| p\equiv 1\ [m], \frac{\varphi(p-1)}{p-1} \in [y, x_m)\Big\} < A(x_m-y).$$

By the definition of G_m, the left hand side of the previous inequality is $(G_m(x_m)-G_m(y))\varphi(m)$. We thus have $\forall y \in [x_m-\delta, x_m]$:

$$G_m(x_m) - G_m(y) \leq A_m(x_m - y),$$

where $A_m = A/\varphi(m)$.

Integrating by parts the integral expression of $c_{m,k}$ we obtain

$$\begin{aligned}
c_{m,k} &= \int_0^{x_m} t^k dG_m(t) \\
&= \Big[t^k(G_m(t)-G_m(x_m))\Big]_0^{x_m} - \int_0^{x_m} kt^{k-1}(G_m(t)-G_m(x_m))dt \\
&\leq \int_0^{x_m-\delta} kt^{k-1}dt + A_m\int_{x_m-\delta}^{x_m} kt^{k-1}(x_m-t)dt \\
&= (x_m-\delta)^k + A_m\Big[t^k(x_m-t)\Big]_{x_m-\delta}^{x_m} + A_m\int_{x_m-\delta}^{x_m} t^k dt \\
&= A_m\frac{x_m^k}{k+1} + o_m\Big(\frac{x_m^k}{k}\Big) = O_m\Big(\frac{x_m^k}{k}\Big),
\end{aligned}$$

which contradicts the inequality (1.2.7).

Thus, Theorem 1.2.1 is proved, as well as the nondifferentiability of F from the left at any point $x_m = \varphi(m)/m$, where m is an even integer.

2. DENSITY MODULO 1 OF SOME SEQUENCES CONNECTED WITH THE MEAN VALUES OF $\varphi(N)$ AND $\sigma(N)$

In 2008, J-M. Deshouillers and F. Luca [18] introduced a method for studying density modulo 1 of means the Euler function. In this chapter, we study the work of J-M. Deshouillers and F. Luca, and also we give some remarks and generalizations for their work. Also, we consider density modulo 1 of some sequences of the sum of the divisor function.

2.1 Introduction

We say that the real sequence $\{t_n\}_{n\in\mathbb{N}}$ is dense modulo 1, if the sequence of the fractional parts of the values of t_n is dense in the unit interval $[0,1]$. The aim of a recent work of J-M. Deshouillers and F. Luca is studying the density modulo 1 of some well known mean value functions of the Euler function. For the sequence $\{h_n\}_{n\in\mathbb{N}}$ of harmonic mean of the Euler function, defined by

$$h_n = \frac{n}{\sum\limits_{m\leq n}\frac{1}{\varphi(m)}}, \tag{2.1.1}$$

density modulo 1 is an easy consequence of well known results. Indeed, let

$$R(n) = \sum_{m\leq n}\frac{1}{\varphi(m)}.$$

Then by [36, 39]

$$R(n) = A\log n + B + O\left(\frac{\log n}{n}\right),$$

where

$$A = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \approx 1.94 \qquad \text{and} \qquad B = A\Big(\gamma - \sum_{p}\frac{\log p}{p^2-p+1}\Big) \approx -0.06.$$

This approximation gives, on one hand

$$h_n \asymp \frac{n}{\log n},$$

and on the other hand

$$h_{n+1} - h_n = \left(R(n) - \frac{n}{\varphi(n+1)}\right) \frac{1}{R(n)R(n+1)}.$$

For $n \geq 3$ we have[1]

$$\frac{\log 2}{2} \frac{n}{\log n} < \varphi(n) < n, \tag{2.1.2}$$

which gives $R(n) - \frac{n}{\varphi(n+1)} \asymp \log n$. So, as $n \to \infty$, we obtain

$$h_{n+1} - h_n = O\left(\frac{1}{\log n}\right) = o(1).$$

These estimates trivially yields that $\left\{\{h_n\}\right\}_{n\in\mathbb{N}}$ is dense in $[0,1]$.

Remark 2.1.1. Similarly, we can show that the sequence $\{h_n\}_{n\in\mathbb{N}}$ defined by

$$h_n = \frac{n}{\sum\limits_{m\leq n} \frac{1}{\sigma(m)}},$$

is dense module 1. Indeed, putting $R(n) = \sum\limits_{m\leq n} \frac{1}{\sigma(m)}$, for some real β we have $m \leq \sigma(m) \leq \beta m \log\log m \leq \beta m \log m$, which gives $\log\log n \ll R(n) \ll \log n$ and so

$$h_{n+1} - h_n = \left(R(n) - \frac{n}{\sigma(n+1)}\right) \frac{1}{R(n)R(n+1)} = o(n) \qquad (\text{as } n\to\infty).$$

Also, we have $\lim_{n\to\infty} h_n = \infty$.

Deshouillers and Luca [18] proved the following deeper result:

Theorem 2.1.2. *The sequence with general term defined by (2.1.1), and the sequences of the following general terms all are dense modulo 1;*

$$s_n = \sqrt{\sum_{m\leq n} \varphi(m)}, \quad a_n = \frac{1}{n} \sum_{m\leq n} \varphi(m), \quad g_n = \left(\prod_{m\leq n} \varphi(m)\right)^{\frac{1}{n}}.$$

2.2 Sequences related by square root mean

The method of Deshouillers-Luca can be used to prove the density modulo 1 of some other sequences related to the Euler function. In this section we study this method by reviewing the proof of the above theorem for the sequence $\{s_n\}_{n\in\mathbb{N}}$. We give a generalization of this sequence, and as a corollary we obtain the density modulo 1 of the sequence with general term $H_n = \sum_{m\leq n} \frac{\varphi(m)}{m}$. In next sections, we will apply this method to get some other density results.

[1] More stronger lower bound $\varphi(n) > n/\varrho(n)$ with $\varrho(n) = e^{\gamma}\log\log n + \frac{2.50637}{\log\log n}$ for $n \geq 3$ is obtained in [43].

2.2.1 Square root mean

Let $\varepsilon \in (0, \frac{1}{10})$ and M be a positive integer depending on ε, which we will fix it later. For $j \in \{1, 2, \cdots, M\}$ we put $c_j = \frac{\varphi(j)}{j}$ and

$$\alpha = \frac{3}{\pi^2}.$$

Choose finite disjoint sets of primes $\mathcal{P}_j$ consisting of primes $p > M$ such that

$$\prod_{p \in \mathcal{P}_j} \left(1 - \frac{1}{p}\right) \in \left[\frac{\sqrt{\alpha}\varepsilon}{c_j}, \frac{2\sqrt{\alpha}\varepsilon}{c_j}\right]. \tag{2.2.1}$$

This is possible for sufficiently small values of ε, because $\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)$ tends to zero (Mertens approximation), and the inequality

$$2\sqrt{\alpha}\varepsilon < c_j$$

holds for $j = 1, 2, \cdots, M$, where we choose $M = \lfloor \frac{5}{\varepsilon} \rfloor + 1$. Indeed, considering the minimal order of the Euler function in the interval $[1, M]$, we know that if $M \geq 3$, then

$$\min_{1 \leq m \leq M} \frac{\varphi(m)}{m} \geq \frac{\beta}{\log\log M}$$

where β is a positive constant. Thus, for $j = 1, 2, \cdots, M$, we have

$$c_j \geq \min_{1 \leq m \leq M} \frac{\varphi(m)}{m} \geq \frac{\beta}{\log\log M} = \frac{\beta}{\log\log(\lfloor \frac{5}{\varepsilon} \rfloor + 1)} > 2\sqrt{\alpha}\varepsilon,$$

where the last inequality holds for sufficiently small values of ε. Set $P_j = \prod_{p \in \mathcal{P}_j} p$, and let $x > 0$ be a real number with

$$\log x > \max\left\{ y : y \in \bigcup_{j=1}^{M} \mathcal{P}_j \right\}. \tag{2.2.2}$$

Also, let

$$\mathcal{Q} = \left\{ q \in \mathbb{P} : M < q \leq \log x \right\} - \bigcup_{j=1}^{M} \mathcal{P}_j, \tag{2.2.3}$$

and put $Q = \prod_{q \in \mathcal{Q}} q$. Since $\gcd(P_i, P_j) = 1$ for $i \neq j$, and $\gcd(P_j, M!Q) = 1$ for $j = 1, 2, \cdots, M$, by the Chinese Remainder Lemma, the system

$$\begin{cases} n \equiv 0 \pmod{M!Q}, \\ n \equiv -j \pmod{P_j} \qquad \text{for } j = 1, 2, \cdots, M, \end{cases} \tag{2.2.4}$$

is solvable, and all its positive integer solutions n form an arithmetic progression $n_0 \pmod{N}$ with $N = M!Q \prod_{j=1}^{M} P_j$, where n_0 is taken the least positive integer

in above progression. Considering the Prime Number Theorem, keeping M fixed, we have

$$N = M! \prod_{M<p\le \log x} p = M! \exp(\theta(\log x) - \theta(M)) = M! \exp((1+o(1))\log x),$$

as $x\to\infty$. Therefore, we can choose x to be sufficiently large such that $N < x^2$. Take $n \equiv n_0 \pmod N$ with $n \in [x^2, 2x^2)$, and write $n = n_0 + Nl$ for some $l \ge 1$. Considering (2.2.4), we observe that for $j = 1, 2, \cdots, M$, we have $P_j | n + j$, and also $j|M!|QM!|n$ or $j|n+j$. The choice of the sets $\mathcal{P}_j$ asserts that $\gcd(j, P_j) = 1$. Now, let $p \le \log x$ be a prime factor of $n + j$. If $p \le M$ or $p \in \mathcal{Q}$, then since $M!Q|n$, we get $p|n$ and consequently $p|(n + j) - n = j$. If $p > M$ and $p \notin \mathcal{P}_j$, then $p \in \mathcal{P}_i$ for some $i \ne j$. System (2.2.4) leads to $p|n+i$, and therefore we get $p|(n + i) - (n + j) = i - j$, which is not possible because $0 < |i - j| < M < p$. So, if $p \le \log x$ and $p|n + j$, then $p|jP_j$. On the other hand, we note that since $n + j \le 4x^2$, we have

$$\omega(n+j) \ll \frac{\log(n+j)}{\log\log(n+j)} \ll \frac{\log x}{\log\log x}. \tag{2.2.5}$$

Since $p|jP_j$ gives $p|n + j$, we write

$$\begin{aligned}
\frac{\varphi(n+j)}{n+j} &= \prod_{p|n+j}\left(1-\frac{1}{p}\right) = \prod_{p|jP_j}\left(1-\frac{1}{p}\right)\prod_{\substack{p|n+j\\ p\nmid jP_j}}\left(1-\frac{1}{p}\right)\\
&= \frac{\varphi(jP_j)}{jP_j}\left(1-\frac{1}{\log x}\right)^{O(\frac{\log x}{\log\log x})}\\
&= c_j\frac{\varphi(P_j)}{P_j}\left(1+O\Big(\frac{1}{\log\log x}\Big)\right) \in \left[\frac{\sqrt{\alpha}\varepsilon}{2}, 3\sqrt{\alpha}\varepsilon\right],
\end{aligned} \tag{2.2.6}$$

where the last containment holds by (2.2.1) provided that x is sufficiently large with respect to ε and M. Now, we use the approximations (A.1.2) and (A.1.3), and the relation

$$S(n) := \sum_{m\le n}\varphi(m) = \alpha n^2 + E(n), \tag{2.2.7}$$

with $E(n) \ll n\log n$, to write for each fixed $j = 0, 1, \cdots, M-1$,

$$\begin{aligned}
\sqrt{S(n+j+1)} &= \sqrt{S(n+j)+\varphi(n+j+1)}\\
&= \sqrt{S(n+j)}\sqrt{1+\frac{\varphi(n+j+1)}{S(n+j)}}\\
&= \sqrt{S(n+j)}+\frac{\varphi(n+j+1)}{2\sqrt{S(n+j)}}+O\left(\frac{\varphi(n+j+1)^2}{S(n+j)^{3/2}}\right)\\
&= \sqrt{S(n+j)}+\frac{\varphi(n+j+1)}{2\sqrt{S(n+j)}}+O\left(\frac{1}{x^2}\right),
\end{aligned}$$

and

$$\begin{aligned}\frac{n+j+1}{2\sqrt{S(n+j)}} &= \frac{1}{2\sqrt{\alpha}}\left(1+\frac{1}{n+j}\right)\frac{1}{\sqrt{1+O(\frac{\log(n+j)}{n+j})}}\\ &= \frac{1}{2\sqrt{\alpha}}\left(1+\frac{1}{n+j}\right)\left(1+O\Big(\frac{\log(n+j)}{n+j}\Big)\right)\\ &= \frac{1}{2\sqrt{\alpha}}\left(1+O\Big(\frac{\log x}{x^2}\Big)\right).\end{aligned}$$

Thus, for large x we obtain

$$\begin{aligned}s_{n+j+1}-s_{n+j} &= \sqrt{S(n+j+1)}-\sqrt{S(n+j)}\\ &= \frac{\varphi(n+j+1)}{2\sqrt{\alpha}(n+j+1)}\left(1+O\Big(\frac{\log x}{x^2}\Big)\right)+O\left(\frac{1}{x^2}\right)\\ &= \frac{\varphi(n+j+1)}{2\sqrt{\alpha}(n+j+1)}+O\left(\frac{\log x}{x^2}\right).\end{aligned}$$

These approximations beside the containment (2.2.6), lead us

$$s_{n+j+1}-s_{n+j}\in\left[\frac{\varepsilon}{5},3\varepsilon\right],$$

holding for all $j=0,1,\cdots,M-1$, provided x is sufficiently large. Therefor we get

$$s_{n+M}-s_n=\sum_{j=0}^{M-1}s_{n+j+1}-s_{n+j}\geq M\frac{\varepsilon}{5}=\left(\Big\lfloor\frac{5}{\varepsilon}\Big\rfloor+1\right)\frac{\varepsilon}{5}\geq 1+\frac{\varepsilon}{5}>1.$$

This inequality implies that for each subinterval I of $[0,1]$ with length $>3\varepsilon$ there exist $j\in\{1,2,\cdots,M\}$ such that $s_{n+j}\in I$. Since $\varepsilon\in(0,\frac{1}{10})$ was arbitrary, this completes the proof of density modulo 1 of the sequence $\{s_n\}_{n\in\mathbb{N}}$.

2.2.2 Generalization and the sequence $H_n=\sum_{m\leq n}\frac{\varphi(m)}{m}$

We fix the real $\eta>-2$, and we consider the sequence $\{w_n\}_{n\in\mathbb{N}}$ defined by

$$w_n=\left(\sum_{m\leq n}m^{\eta}\varphi(m)\right)^{\frac{1}{\eta+2}}.$$

We let

$$S_\eta(n)=\sum_{m\leq n}m^{\eta}\varphi(m).$$

Since $S_0(n)=\frac{3}{\pi^2}n^2+O(n\log n)$, using partial summation formula, we have

$$S_\eta(n)=\alpha_\eta n^{\eta+2}+O(n^{\eta+1}\log n),\tag{2.2.8}$$

where

$$\alpha_\eta = \frac{6}{(\eta+2)\pi^2}.$$

We use the approximation (A.1.1) to write

$$\begin{aligned} w_{n+j+1} &= S_\eta(n+j+1)^{\frac{1}{\eta+2}} = \Big(S_\eta(n+j) + (n+j+1)^\eta \varphi(n+j+1)\Big)^{\frac{1}{\eta+2}} \\ &= S_\eta(n+j)^{\frac{1}{\eta+2}} \left(1 + \frac{(n+j+1)^\eta \varphi(n+j+1)}{S_\eta(n+j)}\right)^{\frac{1}{\eta+2}} \\ &= S_\eta(n+j)^{\frac{1}{\eta+2}} + \frac{(n+j+1)^\eta \varphi(n+j+1)}{(\eta+2) S_\eta(n+j)^{1-\frac{1}{\eta+2}}} + O\left(\frac{\varphi(n+j+1)^2}{(n+j)^3}\right). \end{aligned}$$

Thus, we obtain

$$w_{n+j+1} - w_{n+j} = \frac{(n+j+1)^{\eta+1}}{S_\eta(n+j)^{1-\frac{1}{\eta+2}}} \frac{\varphi(n+j+1)}{(\eta+2)(n+j+1)} + O(x^{-2}).$$

But, we have

$$S_\eta(n+j)^{1-\frac{1}{\eta+2}} = \alpha_\eta^{\frac{\eta+1}{\eta+2}} (n+j)^{\eta+1} \left(1 + O\left(\frac{\log x}{x^2}\right)\right),$$

and consequently

$$\frac{(n+j+1)^{\eta+1}}{S_\eta(n+j)^{1-\frac{1}{\eta+2}}} = \alpha_\eta^{-\frac{\eta+1}{\eta+2}} \left(1 + O\left(\frac{\log x}{x^2}\right)\right).$$

Therefore, we obtain

$$w_{n+j+1} - w_{n+j} = \frac{\alpha_\eta^{-\frac{\eta+1}{\eta+2}}}{(\eta+2)} \frac{\varphi(n+j+1)}{n+j+1} + O\left(\frac{\log x}{x^2}\right).$$

We use this approximate identity to prove that $\{w_n\}_{n\in\mathbb{N}}$ is dense modulo 1. To do this, we follow the same argument as in the previous section, but instead containment (2.2.1), we consider the following one

$$\frac{\varphi(P_j)}{P_j} = \prod_{p\in\mathcal{P}_j} \left(1 - \frac{1}{p}\right) \in \left[\frac{(\eta+2)\alpha_\eta^{\frac{\eta+1}{\eta+2}}\varepsilon}{2c_j}, \frac{(\eta+2)\alpha_\eta^{\frac{\eta+1}{\eta+2}}\varepsilon}{c_j}\right].$$

Considering (2.2.6), this containment gives

$$\frac{\varphi(n+j)}{n+j} = c_j \frac{\varphi(P_j)}{P_j} \left(1 + O\Big(\frac{1}{\log\log x}\Big)\right) \in \left[\frac{(\eta+2)\alpha_\eta^{\frac{\eta+1}{\eta+2}}\varepsilon}{4}, \frac{3(\eta+2)\alpha_\eta^{\frac{\eta+1}{\eta+2}}\varepsilon}{2}\right],$$

provided x is sufficiently large. Thus, we get

$$w_{n+j+1} - w_{n+j} \in \left[\frac{\varepsilon}{5}, 3\varepsilon\right],$$

for all $j = 0, 1, \cdots, M-1$, with x sufficiently large. Therefor $\sum_{j=0}^{M-1} w_{n+j+1} - w_{n+j} > 1$, and for each $I \subset [0,1]$ with length $> 3\varepsilon$ there exist $j \in \{1, 2, \cdots, M\}$ such that $w_{n+j} \in I$. Since ε was arbitrary, this completes the proof. So, we have proved the following result.

Theorem 2.2.1. *For every real $\eta > -2$, the sequence defined with general term*

$$w_n = \left(\sum_{m \le n} m^{\eta} \varphi(m) \right)^{\frac{1}{\eta+2}},$$

is dense modulo 1.

This theorem for $\eta = 0$, recovers the result of previous section, and for $\eta = -1$ it gives the following corollary:

Corollary 2.2.2. *The sequence $\{H_n\}_{n\in\mathbb{N}}$ defined by*

$$H_n = \sum_{m \le n} \frac{\varphi(m)}{m},$$

is dense modulo 1.

Remark 2.2.3. Suppose that f be a differentiable function on $(a, +\infty)$ for some fixed $a > 0$, and

$$\lim_{n\to\infty} f(n) \log n = +\infty, \quad \lim_{n\to\infty} (\log n) \Big(\sup_{n<x<n+1} f'(x) \Big) = 0, \quad \lim_{n\to\infty} \frac{f(n)}{n} = 0.$$

Then the sequence $\{K_n\}_{n\in\mathbb{N}}$ defined by

$$K_n = f(n) \sum_{m \le n} \frac{\varphi(m)}{m^2},$$

is dense modulo 1. Indeed, if we let

$$R(n) = \sum_{m \le n} \frac{\varphi(m)}{m^2},$$

then, considering the relation (2.2.7) and the partial summation formula, we get

$$R(n) = 2\alpha \log n + \alpha + O\left(\frac{\log n}{n}\right) \qquad (\text{as } n\to\infty).$$

Also, we have

$$K_{n+1} - K_n = f'(c_n) R(n) + \frac{f(n+1)\varphi(n+1)}{(n+1)^2}.$$

The first condition assumed above for f gives $K_n \to \infty$ and the second and third ones yield $K_{n+1} - K_n \to 0$, as $n\to\infty$. This proves our assertion. We note that, for example, the function

$$f(n) = \frac{n}{\log n \log\log n},$$

satisfies the above mentioned conditions.

2.2.3 The sequence $H_n = \sum_{m\leq n} \frac{m}{\sigma(m)}$

We can modify the proof of density modulo 1 of the studied sequences to get same result for the sequence $\{H_n\}_{n\geq 1}$ defined by

$$H_n = \sum_{m\leq n} \frac{m}{\sigma(m)}$$

To do this, we let $\varepsilon \in (0, \frac{1}{10})$ be small, and we let M be a positive integer which depends on ε. For $j \in \{1, 2, \cdots, M\}$ we put

$$c_j = \frac{j}{\sigma(j)},$$

and we choose disjoint finite sets of primes $\mathcal{P}_j$ consisting of primes $> M$ such that

$$\frac{P_j}{\sigma(P_j)} \in \left(\frac{5\varepsilon}{4c_j}, \frac{7\varepsilon}{4c_j}\right), \tag{2.2.9}$$

where $P_j = \prod_{p\in\mathcal{P}_j}$. This is possible, because if we let

$$f_\alpha(p) = \Big(1 + \frac{1-p^{-\alpha}}{p-1}\Big)^{-1},$$

we have

$$\frac{P_j}{\sigma(P_j)} = \prod_{p\in\mathcal{P}_j} f_\alpha(p) = \prod_{p\in\mathcal{P}_j} \left(1 - \frac{1}{p} + O\Big(\frac{1}{p^2}\Big)\right) \asymp \prod_{p\in\mathcal{P}_j} \left(1 - \frac{1}{p}\right).$$

Mertens formula about $\prod_{p\leq x}(1-\frac{1}{p})$ asserts that this product can be arbitrary small. Also, the inequality $\frac{7\varepsilon}{4c_j} < c$ holds for any fixed positive real c and for sufficiently small values of ε. Indeed, considering the bound $\sigma(n) < \beta n \log\log n$, which holds for some absolute constant β, we have

$$4cc_j > \frac{4c}{\beta \log\log j} \geq \frac{4c}{\beta \log\log M} > 7\varepsilon,$$

where the last inequality holds by taking $M = \lfloor \frac{1}{\varepsilon} \rfloor + 1$, when $\varepsilon > 0$ is sufficiently small. With the notations and structures of the relations (2.2.2) - (2.2.5), we have as $n\to\infty$

$$\begin{aligned}\frac{n+j}{\sigma(n+j)} &= \prod_{p^\alpha \| n+j} f_\alpha(p) = \prod_{p^\alpha \| jP_j} f_\alpha(p) \prod_{\substack{p^\alpha \| n+j \\ p\nmid jP_j}} f_\alpha(p) \\ &= \frac{jP_j}{\sigma(jP_j)} \prod_{\substack{p\mid n+j \\ p\nmid jP_j}} \left(1 - \frac{1}{p} + O\Big(\frac{1}{p^2}\Big)\right) = c_j \frac{P_j}{\sigma(P_j)}(1+o(1)).\end{aligned}$$

Thus, the containment (2.2.9) leads to

$$\frac{n+j}{\sigma(n+j)} \in [\varepsilon, 2\varepsilon],$$

for all $j = 0, 1, \cdots, M-1$, provided x is sufficiently large. Therefore we get

$$H_{n+M-1} - H_{n-1} = \sum_{j=0}^{M-1} \frac{n+j}{\sigma(n+j)} \geq M\varepsilon = \left(\left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1\right)\varepsilon > 1.$$

This inequality implies that for each subinterval I of $[0,1]$ of length $> 2\varepsilon$ there exist $j \in \{0, 1, \cdots, M-1\}$ such that $H_{n+j+1} \in I$. Since ε was arbitrary, this completes the proof of density modulo 1 of the sequence $\{s_n\}_{n\in\mathbb{N}}$.

2.3 The sequence of arithmetic mean

As defined in the Theorem 2.1.2, $a_n = \frac{1}{n}S(n)$, where

$$S(n) = \sum_{m\leq n} \varphi(m) = \alpha n^2 + O(n \log n),$$

and

$$\alpha = \frac{3}{\pi^2}.$$

Using partial summation formula (consider (2.2.8) with $\eta = -1$), we obtain

$$H_n = \sum_{m\leq n} \frac{\varphi(m)}{m} = 2\alpha n + O(\log n). \tag{2.3.1}$$

Therefore for each $a \in \mathbb{N}$ there is $b \in \mathbb{N}$ such that

$$\sum_{a<j\leq b} \frac{\varphi(j)}{j} > \frac{4(b-a)}{\pi^2}. \tag{2.3.2}$$

Indeed, if this were not so, then for some $a \in \mathbb{N}$ and for all $b > a$, we would have

$$\frac{1}{b-a} \sum_{a<j\leq b} \frac{\varphi(j)}{j} \leq \frac{4}{\pi^2}.$$

But, we have

$$\lim_{b\to\infty} \frac{1}{b-a} \sum_{a<j\leq b} \frac{\varphi(j)}{j} = \lim_{b\to\infty} \left(\frac{H_b}{b} - \frac{H_a}{a}\right) \frac{b}{b-a} = \frac{6}{\pi^2} > \frac{4}{\pi^2}.$$

If we take $b = a + E$, then (2.3.2) is equivalent by $\frac{2}{\pi^2}E + O(\log a) > 0$, which yields that the minimal b satisfying this relation is $b = a + O(\log a)$. We define the sequence $\{m_i\}_{i\geq 0}$ with $m_0 = 0$ and

$$m_{i+1} = \min\left\{ b : \sum_{m_i<j\leq b} \frac{\varphi(j)}{j} > \frac{4(b-m_i)}{\pi^2} \right\},$$

which satisfy $m_{i+1} = m_i + O(\log m_i)$ for all $i \geq 2$. Put

$$T_i = \frac{H_{m_{i+1}} - H_{m_i}}{\alpha(m_{i+1} - m_i)} > \frac{4}{3}.$$

Now, let $\varepsilon \in (0, 1/10)$ be small, and let L be the minimal positive integer such that $M = m_L > 7/\varepsilon$. Let $\varepsilon_1 = 7/M < \varepsilon$. For $j = 1, 2, \cdots, M$, let $\mathcal{P}_j$ be disjoint finite sets of primes $p > M$ with the following property: For $i \in \{0, 1, \cdots, L-1\}$ such that $j \in [m_i + 1, m_{i+1}]$ one has

$$\prod_{p \in \mathcal{P}_j} \left(1 - \frac{1}{p}\right) - \frac{1}{T_i} \in (\varepsilon_1, 2\varepsilon_1).$$

Note that the sequence $\{m_i\}_{i \geq 0}$ is increasing, and we have $\frac{1}{T_i} + 2\varepsilon_1 < 1$. We put P_j, x, $\mathcal{Q}$ and Q as in the previous section (the sequence of square roots of $S(n)$'s), and we set n to be a positive integer satisfying the system (2.2.4). As the previous section, we can find such n in $[x^2, 2x^2)$. Further, every prime factor of $n + j$ either divides jP_j, or exceeds $\log x$. Let again i be such that $j \in [m_i + 1, m_{i+1}]$. Thus, for some $\theta_j \in (1, 2)$ we have

$$\begin{aligned}
\frac{\varphi(n+j)}{n+j} &= \prod_{p \mid n+j} \left(1 - \frac{1}{p}\right) = \prod_{p \mid jP_j} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \mid n+j \\ p \nmid jP_j}} \left(1 - \frac{1}{p}\right) \\
&= \frac{\varphi(jP_j)}{jP_j} \left(1 - \frac{1}{\log x}\right)^{O(\frac{\log x}{\log\log x})} = c_j \frac{\varphi(P_j)}{P_j} \left(1 + O\Big(\frac{1}{\log\log x}\Big)\right) \\
&= c_j \left(\frac{1}{T_i} + \theta_j \varepsilon_1\right) \left(1 + O\Big(\frac{1}{\log\log x}\Big)\right) = \frac{c_j}{T_i} + \lambda_j \varepsilon_1, \qquad (2.3.3)
\end{aligned}$$

where $\lambda_j \in (c_j/2, 3c_j)$, provided x is sufficiently large with respect to ε (and consequently with respect to ε_1). We have

$$\begin{aligned}
a_{n+m_{i+1}} - a_{n+m_i} &= \frac{S(n+m_{i+1})}{n+m_{i+1}} - \frac{S(n+m_i)}{n+m_i} \\
&= \frac{S(n+m_{i+1}) - S(n+m_i)}{n+m_{i+1}} - \frac{(m_{i+1} - m_i) S(n+m_i)}{(n+m_i)(n+m_{i+1})}.
\end{aligned}$$

When $m_i < j \leq m_{i+1}$, we can write

$$\frac{1}{n+j+k} = \frac{1}{n+j}\Big(1 - \frac{k}{n+j+k}\Big) = \frac{1}{n+j}\Big(1 + O\Big(\frac{1}{n}\Big)\Big),$$

where $0 \le k < m_{i+1} - m_i$. So, considering (2.3.3) and the definition of T_i, we obtain

$$\begin{aligned}\frac{S(n+m_{i+1}) - S(n+m_i)}{n+m_{i+1}} &= \sum_{m_i < j \le m_{i+1}} \frac{\varphi(n+j)}{n+j}\Big(1 + O\Big(\frac{1}{n}\Big)\Big) \\ &= \sum_{m_i < j \le m_{i+1}} \Big(\frac{c_j}{T_i} + \lambda_j \varepsilon_1\Big)\Big(1 + O\Big(\frac{1}{n}\Big)\Big) \\ &= \alpha(m_{i+1} - m_i) + \Big(\sum_{m_i < j \le m_{i+1}} \lambda_j\Big)\varepsilon_1 + O\Big(\frac{\log n}{n}\Big).\end{aligned}$$

Also, we have

$$\begin{aligned}\frac{m_{i+1} - m_i}{(n+m_i)(n+m_{i+1})} &= \frac{m_{i+1} - m_i}{(n+m_i)^2}\Big(1 - \frac{m_{i+1} - m_i}{n+m_{i+1}}\Big) \\ &= \frac{m_{i+1} - m_i}{(n+m_i)^2}\Big(1 + O\Big(\frac{1}{n}\Big)\Big).\end{aligned}$$

Thus, using (2.2.7), we obtain

$$\begin{aligned}\frac{(m_{i+1} - m_i)S(n+m_i)}{(n+m_i)(n+m_{i+1})} &= \frac{m_{i+1} - m_i}{(n+m_i)^2}\Big(\alpha(n+m)^2 + O(n\log n)\Big)\Big(1 + O\Big(\frac{1}{n}\Big)\Big) \\ &= \alpha(m_{i+1} - m_i) + \Big(1 + O\Big(\frac{\log n}{n}\Big)\Big).\end{aligned}$$

Putting above calculations together and noting that $n < 2x^2$, we obtain

$$a_{n+m_{i+1}} - a_{n+m_i} = \Big(\sum_{m_i < j \le m_{i+1}} \lambda_j\Big)\varepsilon_1 + O\Big(\frac{\log x}{x^2}\Big).$$

For $i = 0, 1, \cdots, L-1$, we put

$$d_i = \sum_{m_i < j \le m_{i+1}} c_j.$$

Since $\varepsilon_1 = 7/M$ and $\lambda_j > c_j/2$, estimating c_j's by the minimal order of the Euler function, we obtain

$$d_i\varepsilon_1 \ge \frac{7\beta}{2M\log\log M} \gg \frac{\varepsilon}{\log\log(7/\varepsilon)}.$$

On the other hand, considering the growth condition on the sequence $\{m_i\}_{i\ge 0}$, we have

$$d_i\varepsilon_1 \le \frac{7(m_{i+1} - m_i)}{M} \ll \frac{\log M}{M} \ll \varepsilon\log(7/\varepsilon).$$

Hence, for sufficiently small values of ε, we obtain

$$d_i\varepsilon_1 \in \Big[\varepsilon^2, \sqrt{\varepsilon}\Big] \qquad (\text{for } i = 0, 1, \cdots, L-1),$$

and then, for sufficiently large values of x with respect to ε, we get

$$a_{n+m_{i+1}} - a_{n+m_i} \in \left[\frac{d_i\varepsilon_1}{2}, 2d_i\varepsilon_1\right] \qquad (\text{for } i = 0, 1, \cdots, L-1).$$

So, considering (2.3.1) we have

$$\begin{aligned} a_{n+M} - a_n &= \sum_{i=0}^{L-1} a_{n+m_{i+1}} - a_{n+m_i} \geq \frac{\varepsilon_1}{2}\sum_{i=0}^{L-1} d_i \\ &= \frac{\varepsilon_1}{2}\sum_{i=0}^{L-1}\sum_{m_i<j\leq m_{i+1}} c_j = \frac{7H_M}{2M} \\ &= 7\alpha + O\left(\frac{\log M}{M}\right) = \frac{21}{\pi^2} + O\left(\varepsilon\log\left(\frac{7}{\varepsilon}\right)\right) > 1, \end{aligned}$$

where the last inequality holds when ε is sufficiently small. Therefore, for such values of ε and for each interval $I \subset [0,1]$ of length $\sqrt{\varepsilon}$, there exists $i \in \{0, 1, \cdots, L-1\}$ such that $\{a_{n+m_i}\} \in I$ and this completes the proof.

2.4 Sequences related by geometric means

2.4.1 The sequence of geometric means

As denoted in Theorem 2.1.2, we define the sequence $\{g_n\}_{n\in\mathbb{N}}$ by

$$g_n = \left(\prod_{m\leq n}\varphi(m)\right)^{\frac{1}{n}}.$$

To achieve density modulo 1 of this sequence, first we need an asymptotic evaluation of it. To get this, we write

$$\log g_n = \frac{1}{n}\sum_{m\leq n}\log\varphi(m) = \frac{1}{n}\sum_{m\leq n}\left(\log m + \sum_{p|m}\log\left(1-\frac{1}{p}\right)\right) = \frac{\log n!}{n} + S,$$

where

$$S = \frac{1}{n}\sum_{m\leq n}\sum_{p|m}\log\left(1-\frac{1}{p}\right).$$

We use Stirling formula in the form given by (A.1.7) to obtain

$$\frac{\log n!}{n} = \log n - 1 + O\left(\frac{\log n}{n}\right).$$

To evaluate S, we change the order of related summations to get

$$S = \frac{1}{n}\sum_{p\le n}\log\left(1-\frac{1}{p}\right)\sum_{\substack{m\le n\\ m\equiv 0\ [p]}} 1 = \frac{1}{n}\sum_{p\le n}\log\left(1-\frac{1}{p}\right)\left\lfloor\frac{n}{p}\right\rfloor$$
$$= \frac{1}{n}\sum_{p\le n}\log\left(1-\frac{1}{p}\right)\left(\frac{n}{p}+O(1)\right) = \sum_{p}\frac{1}{p}\log\left(1-\frac{1}{p}\right)+E,$$

where in the last sum p runs over all primes, and

$$E = -\sum_{p>n}\frac{1}{p}\log\left(1-\frac{1}{p}\right)+O\Big(\frac{1}{n}\sum_{p\le n}\frac{1}{p}\Big) \ll \frac{\log\log n}{n}.$$

Thus, we get

$$\log g_n = \log n - 1 + \sum_{p}\frac{1}{p}\log\left(1-\frac{1}{p}\right)+O\left(\frac{\log n}{n}\right) = \log(\alpha n)+O\left(\frac{\log n}{n}\right),$$

where

$$\alpha = \frac{1}{e}\prod_{p}\left(1-\frac{1}{p}\right)^{\frac{1}{p}}.$$

Using the relation (A.1.4), we obtain

$$g_n = \alpha n + O(\log n). \tag{2.4.1}$$

Also, we need to approximately evaluate the consecutive differences $g_{n+1}-g_n$. To do so, we note that $g_{n+1}^{n+1} = \varphi(n+1)g_n^n$, and then we write

$$g_{n+1}-g_n = g_n^{\frac{n}{n+1}}\left(\varphi(n+1)^{\frac{1}{n+1}} - g_n^{\frac{1}{n+1}}\right).$$

Considering the estimates (A.1.5), (2.4.1) and (A.1.4) we have

$$g_n^{\frac{n}{n+1}} = g_n g_n^{-\frac{1}{n+1}} = \alpha n + O(\log n).$$

Also, we have

$$\varphi(n+1)^{\frac{1}{n+1}} = e^{\frac{\log\varphi(n+1)}{n+1}} = 1+\frac{\log(\varphi(n+1))}{n+1}+O\left(\frac{\log^2 n}{n^2}\right),$$

and similarly, we have

$$g_n^{\frac{1}{n+1}} = e^{\frac{\log g_n}{n+1}} = \exp\left(\frac{1}{n+1}\log\left(\alpha n\Big(1+O\Big(\frac{\log n}{n}\Big)\Big)\right)\right)$$
$$= 1+\frac{\log(\alpha(n+1))}{n+1}+O\left(\frac{\log^2 n}{n^2}\right).$$

Putting these estimates together, we obtain

$$g_{n+1} - g_n = \alpha \log\left(\frac{\varphi(n+1)}{\alpha(n+1)}\right) + O\left(\frac{\log^2 n}{n}\right). \tag{2.4.2}$$

Now, to prove the density modulo 1 of the sequence $\{g_n\}_{n\in\mathbb{N}}$, it is enough to show that for every $\delta > 0$, we can find $M = \lfloor \frac{5}{\delta} \rfloor + 1$ consecutive integers $n+1, n+2, \cdots, n+M$, such that for every $j \in [1, M]$, we have

$$\frac{\delta}{5} \leq \{g_{n+j} - g_{n+j-1}\} \leq \delta. \tag{2.4.3}$$

We need to build a family of integers $n_1, n_2, \cdots, n_M$, such that for every $j \leq M$ we have $j | n_j$ and

$$\frac{\delta}{4} \leq \left\{\alpha \log\left(\frac{\varphi(n_j)}{\alpha n_j}\right)\right\} \leq \frac{\delta}{2}. \tag{2.4.4}$$

Then, using the Chinese Remainder Theorem, as in square root mean, we prove that there exists n such that for any j, the integer $n + j \leq 2x^2$ is the product of n_j by a number of at most $O(\log x / \log\log x)$ prime factors each exceeding $\log x$. So that (2.4.4) implies (2.4.3). To construct the required family of integers $n_1, n_2, \cdots, n_M$, we follow the proof of square root mean (in [18]).

2.4.2 Some generalizations

It is possible to consider various generalizations of g_n, with approximate evaluation of the form $An+o(n)$. In this section, we study some of such generalizations. We keep the notations of computations of g_n.

Generalization 1

We define the sequence with general term

$$G_n = n^{-\eta}\left(\prod_{m\leq n} m^{\eta}\varphi(m)\right)^{\frac{1}{n}}.$$

Approximate Evaluation of the Sequence. We have

$$G_n = n^{-\eta} n!^{\frac{\eta}{n}} g_n.$$

Considering Stirling formula in form introduced by (A.1.8) and applying the approximation (A.1.1), we obtain

$$\left(\frac{en!^{\frac{1}{n}}}{n}\right)^{\eta} = \left(1 + \frac{\log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^2 n}{n^2}\Big)\right)^{\eta}$$
$$= 1 + \frac{\eta \log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^2 n}{n^2}\Big). \tag{2.4.5}$$

This relation with (2.4.1), gives

$$G_n = \alpha e^{-\eta} n + e^{-\eta}\left(\frac{\alpha\eta}{2}+1\right)\log n + \alpha\eta e^{-\eta}\log\sqrt{2\pi} + O\Big(\frac{\log^2 n}{n}\Big),$$

provided $\eta \neq 0$. Indeed, in case $\eta = 0$ we have $G_n = g_n = \alpha n + O(\log n)$. Simply, for every real η we obtain

$$G_n = \alpha e^{-\eta} n + O(\log n).$$

Approximation of the Consecutive Difference. We set

$$e_n = \left(1+\frac{1}{n}\right)^n.$$

We note that

$$G_{n+1}^{n+1} = e_n^{-\eta}\varphi(n+1)G_n^n.$$

So, we may write

$$G_{n+1} - G_n = G_n^{\frac{n}{n+1}}\left(\left(e_n^{-\eta}\varphi(n+1)\right)^{\frac{1}{n+1}} - G_n^{\frac{1}{n+1}}\right).$$

Considering approximate formula for G_n, we have

$$\begin{aligned} G_n^{\frac{n}{n+1}} &= G_n G_n^{-\frac{1}{n+1}} = G_n e^{-\frac{\log G_n}{n+1}} \\ &= G_n\left(1+O\Big(\frac{\log n}{n}\Big)\right) = G_n + O(\log n) = \alpha e^{-\eta} n + O(\log n). \end{aligned}$$

Also, we have

$$\begin{aligned} \left(e_n^{-\eta}\varphi(n+1)\right)^{\frac{1}{n+1}} &= \exp\left(\frac{1}{n+1}\log\left(e_n^{-\eta}\varphi(n+1)\right)\right) \\ &= 1 + \frac{\log\left(e_n^{-\eta}\varphi(n+1)\right)}{n+1} + O\left(\frac{\log^2 n}{n^2}\right). \end{aligned}$$

Similarly, since $G_{n+1} = G_n + O(\log n)$, we have

$$\begin{aligned} G_n^{\frac{1}{n+1}} = e^{\frac{\log G_n}{n+1}} &= 1 + \frac{\log G_{n+1}}{n+1} + O\left(\frac{\log^2 n}{n^2}\right) \\ &= 1 + \frac{\log\left(\alpha e^{-\eta}(n+1)\right)}{n+1} + O\left(\frac{\log^2 n}{n^2}\right). \end{aligned}$$

Putting above estimates together, we obtain

$$G_{n+1} - G_n = \alpha e^{-\eta}\log\left(\frac{e_n^{-\eta}\varphi(n+1)}{\alpha e^{-\eta}(n+1)}\right) + O\left(\frac{\log^2 n}{n}\right).$$

Taylor expansion $(1+\frac{1}{x})^x = e(1+\frac{1}{2x}+O(\frac{1}{x^2}))$, holds as $x\to\infty$. So, we have

$$e_n = e + O\left(\frac{1}{n}\right),$$

and consequently

$$\frac{e_n^{-\eta}}{e^{-\eta}} = e + O\left(\frac{1}{n}\right).$$

Therefore, we obtain

$$G_{n+1} - G_n = \alpha e^{-\eta}\log\left(\frac{\varphi(n+1)}{\alpha(n+1)}\right) + O\left(\frac{\log^2 n}{n}\right). \tag{2.4.6}$$

Generalization 2

We consider for $\eta \neq -1$ the sequence defined by general term

$$G_n = \left(\prod_{m\leq n} m^\eta \varphi(m)\right)^{\frac{1}{n(\eta+1)}}.$$

Approximate Evaluation of the Sequence. Since $G_n^{\eta+1} = n!^{\frac{\eta}{n}} g_n$, from (2.4.5) and (2.4.1) we obtain

$$\begin{aligned} G_n^{\eta+1} &= e^{-\eta} n^\eta \left(1 + \frac{\eta\log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^2 n}{n^2}\Big)\right)(\alpha n + O(\log n)) \\ &= \alpha e^{-\eta} n^{\eta+1}\left(1 + O\Big(\frac{\log n}{n}\Big)\right). \end{aligned}$$

Thus, using (A.1.1), we obtain

$$G_n = (\alpha e^{-\eta})^{\frac{1}{\eta+1}} n + O(\log n).$$

Approximation of the Consecutive Difference. We have

$$G_{n+1}^{n+1} = \Big((n+1)^\eta \varphi(n+1)\Big)^{\frac{1}{\eta+1}} G_n^n,$$

and so, we can write

$$G_{n+1} - G_n = G_n^{\frac{n}{n+1}}\left(\Big((n+1)^\eta\varphi(n+1)\Big)^{\frac{1}{(\eta+1)(n+1)}} - G_n^{\frac{1}{n+1}}\right).$$

From the approximate formula for G_n, we have

$$\begin{aligned} G_n^{\frac{n}{n+1}} &= G_n G_n^{-\frac{1}{n+1}} = G_n e^{-\frac{\log G_n}{n+1}} \\ &= G_n\left(1 + O\Big(\frac{\log n}{n}\Big)\right) = G_n + O(\log n) = (\alpha e^{-\eta})^{\frac{1}{\eta+1}} n + O(\log n). \end{aligned}$$

Also, we have

$$\Big((n+1)^\eta\varphi(n+1)\Big)^{\frac{1}{(\eta+1)(n+1)}} = \exp\left(\frac{1}{(\eta+1)(n+1)}\log\Big((n+1)^\eta\varphi(n+1)\Big)\right)$$
$$= 1+\frac{\log\Big((n+1)^\eta\varphi(n+1)\Big)}{(\eta+1)(n+1)}+O\left(\frac{\log^2 n}{n^2}\right).$$

Since $G_{n+1}=G_n+O(\log n)$, we get

$$G_n^{\frac{1}{n+1}} = e^{\frac{\log G_n}{n+1}} = 1+\frac{\log G_{n+1}}{n+1}+O\left(\frac{\log^2 n}{n^2}\right)$$
$$= 1+\frac{\log\Big((\alpha e^{-\eta})^{\frac{1}{\eta+1}}(n+1)\Big)}{n+1}+O\left(\frac{\log^2 n}{n^2}\right).$$

Putting above estimates together, we get

$$G_{n+1}-G_n = \frac{(\alpha e^{-\eta})^{\frac{1}{\eta+1}}}{\eta+1}\log\left(\frac{\varphi(n+1)}{\alpha e^{-\eta}(n+1)}\right)+O\left(\frac{\log^2 n}{n}\right). \tag{2.4.7}$$

Generalization 3

We consider the sequence with general term

$$G_n = n^{1-\eta}\left(\prod_{m\le n}\varphi(m)\right)^{\frac{\eta}{n}}.$$

Approximate Evaluation of the Sequence. We have

$$G_n = n^{1-\eta}g_n^\eta = n^{1-\eta}\left(\alpha n+O(\log n)\right)^\eta = \alpha^\eta n\left(1+O\Big(\frac{\log n}{n}\Big)\right)^\eta.$$

Thus, we obtain

$$G_n = \alpha^\eta n+O(\log n).$$

Approximation of the Consecutive Difference. Considering

$$G_{n+1}^{n+1} = \big(e_n(n+1)\big)^{1-\eta}\varphi(n+1)^\eta G_n^n,$$

we write

$$G_{n+1}-G_n = G_n^{\frac{n}{n+1}}\left(\Big(\big(e_n(n+1)\big)^{1-\eta}\varphi(n+1)^\eta\Big)^{\frac{1}{n+1}}-G_n^{\frac{1}{n+1}}\right).$$

From the approximate formula for G_n, we obtain

$$G_n^{\frac{n}{n+1}} = G_nG_n^{-\frac{1}{n+1}} = G_ne^{-\frac{\log G_n}{n+1}}$$
$$= G_n\left(1+O\Big(\frac{\log n}{n}\Big)\right) = G_n+O(\log n) = \alpha^\eta n+O(\log n).$$

Also, we have

$$\Big(\big(e_n(n+1)\big)^{1-\eta}\varphi(n+1)^{\eta}\Big)^{\frac{1}{n+1}} = \exp\left(\frac{1}{n+1}\log\Big(\big(e_n(n+1)\big)^{1-\eta}\varphi(n+1)^{\eta}\Big)\right)$$
$$= 1 + \frac{\log\Big(\big(e_n(n+1)\big)^{1-\eta}\varphi(n+1)^{\eta}\Big)}{n+1} + O\left(\frac{\log^2 n}{n^2}\right).$$

Since $G_{n+1} = G_n + O(\log n)$, we obtain

$$G_n^{\frac{1}{n+1}} = e^{\frac{\log G_n}{n+1}} = 1 + \frac{\log G_{n+1}}{n+1} + O\left(\frac{\log^2 n}{n^2}\right)$$
$$= 1 + \frac{\log\big(\alpha^{\eta}(n+1)\big)}{n+1} + O\left(\frac{\log^2 n}{n^2}\right).$$

Putting above estimates together, and considering $e_n = e(1+O(n^{-1}))$, we thus get

$$G_{n+1} - G_n = \alpha^{\eta}\log\left(\frac{e^{1-\eta}}{\alpha^{\eta}}\Big(\frac{\varphi(n+1)}{n+1}\Big)^{\eta}\right) + O\left(\frac{\log^2 n}{n}\right). \tag{2.4.8}$$

3. DENSITY MODULO 1 OF SOME SEQUENCES CONNECTED WITH THE MEAN VALUES OF $\frac{\varphi(N^2+1)}{N^2+1}$ AND $\frac{N^2+1}{\sigma(N^2+1)}$

Based on sieve methods, we study density modulo 1 of some sequences connected with the mean values of the ratio $\varphi(n^2+1)/(n^2+1)$. Our study covers some sequences related to the square root of mean values and various generalizations, as well as sequences involving the arithmetic and harmonic. Finally, we consider density modulo 1 of the sequence of sums of $(n^2+1)/\sigma(n^2+1)$.

3.1 Introduction

The aim of this chapter is to study the density modulo 1 of some sequences of the arithmetic and harmonic means of the sequence $\{r_n\}_{n\in\mathbb{N}}$ defined as

$$r_n = \frac{\varphi(n^2+1)}{n^2+1}.$$

First we show that none of sequences of the arithmetic and harmonic values of r_n are dense modulo 1, and instead we consider some other sequences that are suitable means of the above sequences. Indeed, we focus on the sequences having linear expression in their main terms. In our proofs, we follow the method of Deshouillers-Luca and also some sieve results.

3.2 Sequences related by arithmetic and harmonic means

3.2.1 The sequence with general term $b_n = \sum_{m\leq n} \frac{\varphi(m^2+1)}{m^2+1}$

First, we consider the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by

$$b_n = \sum_{m\leq n} r_m,$$

and we prove that it is dense modulo 1.

To do this, we use sieve methods by applying the following result. We give its proof in the last section.

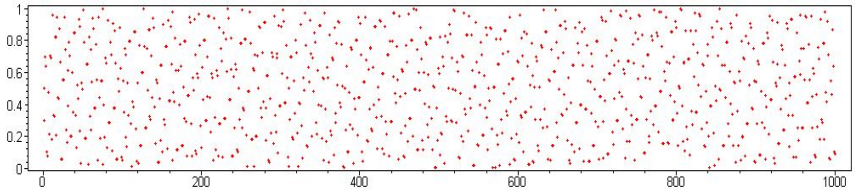


Fig. 3.1: Values of the sequence $b_n = \sum_{m\leq n} \varphi(m^2+1)/(m^2+1) \pmod 1$ for $1 \leq n \leq 1000$. This sequence is dense modulo 1.

Proposition 3.2.1. *Suppose that M be a given sufficiently large integer, let $\{\mathcal{P}_m\}_{1\leq m\leq M}$ be a family of finite and disjoint sets of primes p with $p \equiv 1 \pmod 4$ and $p > M^2+1$, and put $P_m = \prod_{p\in\mathcal{P}_m} p$. Then there exist infinitely many integers n such that for every integer $m = 1, 2, \cdots, M$, the integer $(n+m)^2+1$ is divisible by $(m^2+1)P_m$, and its prime factors that do not divide $(m^2+1)P_m$ are larger than $n^{1/6M}$.*

We take M sufficiently large as in this proposition, and for $m = 1, 2, \cdots, M$, we set $c_m = r_m$. We take $\delta > 0$ to be small, and we choose the family $\{\mathcal{P}_m\}_{1\leq m\leq M}$ as in above proposition with the following additional containment property holding for all $m = 1, 2, \cdots, M$:

$$\frac{\varphi(P_m)}{P_m} \in \left(\frac{5\delta}{4c_m}, \frac{7\delta}{4c_m}\right). \tag{3.2.1}$$

This is possible, because

$$\frac{\varphi(P_m)}{P_m} = \prod_{p\in\mathcal{P}_m}\left(1-\frac{1}{p}\right),$$

and considering Mertens formula for $\prod_{p\leq x}(1-\frac{1}{p})$, this product can be made arbitrarily small by taking x sufficiently large. Also, the inequality $\frac{7\delta}{4c_m} < 1$ holds for sufficiently small values of δ. Indeed, considering the minimal order of the Euler function, we have

$$\min_{1\leq m\leq M} c_m \geq \frac{\beta}{\log\log(M^2+1)},$$

where β is some positive absolute constant. We take

$$M = \left\lfloor \frac{1}{\delta} \right\rfloor + 1.$$

So, we obtain

$$4c_m \geq 4 \min_{1\leq m\leq M} c_m \geq \frac{4\beta}{\log\log(M^2+1)} > 7\delta,$$

where the last inequality holds when $\delta > 0$ is sufficiently small.

Now, using the hypothesis of Proposition 3.2.1, and for $m = 1, 2, \cdots, M$, we write

$$\begin{aligned} r_{n+m} &= \frac{\varphi((n+m)^2+1)}{(n+m)^2+1} = \prod_{p|(n+m)^2+1}\left(1-\frac{1}{p}\right) \\ &= \prod_{p|(m^2+1)P_m}\left(1-\frac{1}{p}\right)\prod_{\substack{p|(n+m)^2+1\\ p\nmid(m^2+1)P_m}}\left(1-\frac{1}{p}\right) \\ &= \frac{\varphi(m^2+1)}{m^2+1}\frac{\varphi(P_m)}{P_m}\prod_{\substack{p|(n+m)^2+1\\ p\nmid(m^2+1)P_m}}\left(1-\frac{1}{p}\right) \\ &= c_m\frac{\varphi(P_m)}{P_m}(1+o(1)) \qquad (\text{as } n\to\infty). \end{aligned} \tag{3.2.2}$$

Note that to get the last asymptotic relation, we use Proposition 3.2.1 to get

$$\prod_{\substack{p|(n+m)^2+1\\ p\nmid(m^2+1)P_m}}\left(1-\frac{1}{p}\right) = \left(1+O(n^{-\frac{1}{6M}})\right)^{\omega((n+m)^2+1)-\omega((m^2+1)P_m)},$$

where $\omega(N) = \sum_{p^a\|N} 1$, and we have

$$\omega((n+m)^2+1) \ll \frac{\log(n+m)}{\log\log(n+m)} \le \frac{\log(n+M)}{\log\log n} \qquad (\text{as } n\to\infty).$$

Also, we have

$$\omega((m^2+1)P_m) = \omega(m^2+1) + |\mathcal{P}_m| \ll \frac{\log m}{\log\log m} + |\mathcal{P}_m| \ll_m 1 \qquad (\text{as } n\to\infty).$$

Thus, as $n\to\infty$ we obtain

$$\prod_{\substack{p|(n+m)^2+1\\ p\nmid(m^2+1)P_m}}\left(1-\frac{1}{p}\right) = 1+O\left(\frac{\log(n+M)}{n^{\frac{1}{6M}}\log\log n}\right) = 1+o(1). \tag{3.2.3}$$

The containment (3.2.1) implis that for every sufficiently small $\delta > 0$ there exist infinitely many (large enough) n such that for all $m = 1, 2, \cdots, M$, with $M = \lfloor \frac{1}{\delta} \rfloor + 1$ we have

$$r_{n+m} \in (\delta, 2\delta).$$

Thus, we have

$$b_{n+M} - b_n = \sum_{m=1}^{M}(b_{n+m} - b_{n+m-1}) = \sum_{m=1}^{M} r_{n+m} > M\delta > 1.$$

This inequality implies that for each subinterval I of $[0,1]$ of length $> 2\delta$ there exist $i \in \{1, 2, \cdots, M\}$ such that $b_{n+i} \in I$, and since $\delta > 0$ was arbitrary small, we get the following result:

Theorem 3.2.2. *The sequence* $\{b_n\}_{n\in\mathbb{N}}$ *defined by*

$$b_n = \sum_{m\leq n} \frac{\varphi(m^2+1)}{m^2+1}$$

is dense modulo 1.

3.2.2 The sequences of arithmetic and harmonic means

About the arithmetic and harmonic means of r_n, we use some known results to get the following result:

Theorem 3.2.3. *The sequences* $\{a_n\}_{n\in\mathbb{N}}$ *and* $\{h_n\}_{n\in\mathbb{N}}$ *defined by*

$$a_n = \frac{1}{n}\sum_{m\leq n} r_m, \quad h_n = \frac{n}{\sum_{m\leq n}\frac{1}{r_m}},$$

are not dense modulo 1.

To prove the assertion for the arithmetic mean we consider the following proposition due to H. Shapiro [48].

Proposition 3.2.4. *Suppose that* $f(x)\in\mathbb{Z}[x]$ *be a polynomial with degree* $h>0$ *and leading coefficient* $a_h>0$*, such that* f *has no repeated root and* $f(n)>0$ *for integer values of* $n\geq 1$*. Let*

$$a = \sum_{n=1}^{\infty} \frac{\mu(n)\rho_f(n)}{n^2},$$

where $\rho_f(n)$ *is the number of the solutions of* $f(x)\equiv 0 \pmod n$*. Then, as* $x\to\infty$ *we have*

$$\sum_{n\leq x}\varphi(f(n)) = \frac{a_h a}{h+1}x^{h+1} + O\left((x\log x)^h\right), \tag{3.2.4}$$

and

$$\sum_{n\leq x}\frac{\varphi(f(n))}{f(n)} = ax + O\left(\log^h x\right). \tag{3.2.5}$$

As a corollary, if we let $S(n)=\sum_{m\leq n} r_m$, then (3.2.5) yields

$$S(n) = \alpha n + O\left(\log^2 n\right), \tag{3.2.6}$$

where

$$\alpha = \sum_{n=1}^{\infty}\frac{\mu(n)\rho_f(n)}{n^2}, \tag{3.2.7}$$

and $\rho_f(n)$ is the number of the solutions of $x^2\equiv -1 \pmod n$. Consequently, we observe that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is not dense modulo 1, because $a_n=\alpha+o(1)$ as $n\to\infty$.

To prove our claim about $\{h_n\}_{n\in\mathbb{N}}$ we note that $0 < h_n < 1$. Also, we let

$$R(n) = \sum_{m\leq n} \frac{1}{r_m},$$

and we consider the bounds

$$\frac{\log 2}{2}\frac{n}{\log n} < \varphi(n) < n, \tag{3.2.8}$$

to get $n \ll R(n) \ll n\log n$. We have

$$h_{n+1} - h_n = \left(R(n) - \frac{n(n^2+1)}{\varphi(n^2+1)}\right)\frac{1}{R(n)R(n+1)}.$$

Above mentioned approximations give us $R(n) - \frac{n(n^2+1)}{\varphi(n^2+1)} \ll \frac{n}{\log n}$, and thus $h_{n+1} - h_n = o(1)$. Therefore, the sequence $\{h_n\}_{n\in\mathbb{N}}$ can not be dense modulo 1.

Remark 3.2.5. About the constant α defined by (3.2.7), we have

$$\alpha = -\frac{1}{4} + \sum_{\substack{n\geq 5\\ n=p_1p_2\cdots p_k\\ p_i\neq p_j\ (i\neq j)\\ p_i\equiv 1\ [4]}} \frac{(-2)^k}{n^2}.$$

Indeed, to get this representation we use the following facts about polynomial congruences:
(i) For primes $p \equiv 1 \pmod 4$ the equation $x^2 \equiv -1 \pmod p$ has two solutions in $\{1, 2, \cdots, p-1\}$, for primes $p \equiv 3 \pmod 4$ it has no solution, and for $p = 2$ it has only one solution.
(ii) Suppose that $f(x) \in \mathbb{Z}[x]$, and (as above) let $\rho_f(n)$ be the number of the solutions of $f(x) \equiv 0 \pmod n$. Then, $\rho_f(n)$ as function of n is multiplicative, i.e., if $\gcd(m, n) = 1$, then $\rho_f(mn) = \rho_f(m)\rho_f(n)$.

3.3 Sequences containing square root

We consider the sequence $\{w_n\}_{n\in\mathbb{N}}$, defined by

$$w_n = \sqrt{\sum_{m\leq n} mr_m}. \tag{3.3.1}$$

Considering the relation (3.2.6) and the partial summation formula, we get

$$w_n^2 = \sum_{m\leq n} mr_m = \frac{\alpha}{2}n^2 + O(n\log^2 n). \tag{3.3.2}$$

Also, considering the approximation (A.1.2), we write

$$\begin{aligned} w_{n+m} &= w_{n+m-1}\sqrt{1+\frac{(n+m)r_{n+m}}{w_{n+m-1}^2}} \\ &= w_{n+m-1}\left(1+\frac{(n+m)r_{n+m}}{2w_{n+m-1}^2}+O\left(\frac{(n+m)^2}{w_{n+m-1}^4}\right)\right). \end{aligned}$$

Thus, considering the order of w_n, we obtain

$$w_{n+m}-w_{n+m-1}=\frac{(n+m)r_{n+m}}{2w_{n+m-1}}+O\left(\frac{1}{n+m}\right).$$

On the other hand, considering (3.3.2), we have

$$2w_{n+m-1}=\sqrt{2\alpha}(n+m)\left(1+O\left(\frac{\log^2(n+m)}{n+m}\right)\right),$$

and this gives

$$\frac{(n+m)r_{n+m}}{2w_{n+m-1}}=\frac{r_{n+m}}{\sqrt{2\alpha}}+O\left(\frac{\log^2(n+m)}{n+m}\right).$$

Therefore, we obtain

$$w_{n+m}-w_{n+m-1}=\frac{r_{n+m}}{\sqrt{2\alpha}}+O\left(\frac{\log^2(n+m)}{n+m}\right). \tag{3.3.3}$$

Now, let M be sufficiently large as in the Proposition 3.2.1. We take $\delta>0$ to be small. For $m=1,2,\cdots,M$, we set $c_m=r_m$, and we choose the family $\{\mathcal{P}_m\}_{1\leq m\leq M}$ of primes as in above mentioned proposition with the following containment property:

$$\frac{\varphi(P_m)}{P_m}\in\left(\frac{5\delta\sqrt{2\alpha}}{4c_m},\frac{7\delta\sqrt{2\alpha}}{4c_m}\right),$$

holding for all $m=1,2,\cdots,M$. Take $M=\left\lfloor\frac{1}{\delta}\right\rfloor+1$, and this is possible for the same reasons as mentioned in the proof of Theorem 3.2.2. This containment together with the relations (3.2.2) and (3.3.3) gives

$$w_{n+m}-w_{n+m-1}\in(\delta,2\delta),$$

provided n is sufficiently large. Proposition 3.2.1 asserts that this containment holds true for all $m=1,2,\cdots,M$, and for infinitely many integers n. So, we can write

$$w_{n+M}-w_n=\sum_{m=1}^{M}(w_{n+m}-w_{n+m-1})>M\delta>1,$$

which implies that for every $I\subset[0,1]$ with length $>2\delta$ there exist $i\in\{1,2,\cdots,M\}$ such that $w_{n+i}\in I$, and since $\delta>0$ is taken arbitrary small, we get the result, summerized below.

Theorem 3.3.1. *The sequence* $\{w_n\}_{n\in\mathbb{N}}$ *defined by*

$$w_n = \sqrt{\sum_{m\leq n} m\frac{\varphi(m^2+1)}{m^2+1}},$$

is dense modulo 1.

An observation on the structure of the sequence w_n leads us to formulate the following result.

Theorem 3.3.2. *The sequence* $\{z_n\}_{n\in\mathbb{N}}$ *defined by*

$$z_n = \sqrt{\sum_{m\leq n} \frac{\varphi(m^2+1)}{am+b}},$$

is dense modulo 1, where $a>0$ *is real number and* $b=b(m)\ll 1$ *is a function of* m*, such that* $am+b>0$ *for all* $m\in\mathbb{N}$*.*

Proof. To prove this result, we show that

$$z_{n+m} - z_{n+m-1} = \frac{r_{n+m}}{\sqrt{2a\alpha}} + O\left(\frac{\log^2(n+m)}{n+m}\right). \tag{3.3.4}$$

Then, using Proposition 3.2.1, the proof follows the same lines as the proof of Theorem 3.3.1. We have

$$\begin{aligned} z_{n+m}^2 &= z_{n+m-1}^2 + \frac{(n+m)^2+1}{a(n+m)+b} r_{n+m} \\ &= z_{n+m-1}^2\left(1 + \frac{(n+m)^2+1}{(a(n+m)+b)z_{n+m-1}^2} r_{n+m}\right). \end{aligned}$$

Using the approximation (A.1.2), we obtain

$$\begin{aligned} z_{n+m} &= z_{n+m-1}\left(1 + \frac{(n+m)^2+1}{2(a(n+m)+b)z_{n+m-1}^2} r_{n+m} + O\Big(\frac{(n+m)^2}{z_{n+m-1}^4}\Big)\right) \\ &= z_{n+m-1} + \frac{(n+m)^2+1}{2(a(n+m)+b)z_{n+m-1}} r_{n+m} + O\Big(\frac{1}{n+m}\Big). \end{aligned}$$

Here, we need to simplify the coefficient of r_{n+m}. To do this, we note that

$$z_n^2 = \sum_{m\leq n}\frac{\varphi(m^2+1)}{am+b} = \sum_{m\leq n}\frac{m^2+1}{am+b} r_m = \sum_{m\leq n}\Big(\frac{m}{a}+O(1)\Big) r_m.$$

Thus, considering (3.3.2) we obtain

$$z_n^2 = \frac{\alpha}{2a} n^2 + O(n\log^2 n),$$

and this yields that

$$z_{n+m-1} = \sqrt{\frac{\alpha}{2a}}(n+m)\left(1+O\Big(\frac{\log^2(n+m)}{n+m}\Big)\right).$$

So, we have

$$\frac{(n+m)^2+1}{2(a(n+m)+b)z_{n+m-1}} = \frac{1}{\sqrt{2a\alpha}} + O\Big(\frac{\log^2(n+m)}{n+m}\Big),$$

from which, we obtain (3.3.4). □

Remark 3.3.3. Similar to Theorem 3.3.1, we can prove that for each fixed real $\eta > 0$, the sequence with general term

$$w_n = \left(\sum_{m\leq n} m^{\eta}\frac{\varphi(m^2+1)}{m^2+1}\right)^{\frac{1}{\eta+1}},$$

is dense modulo 1.

3.4 Completing sieve tools

To complete our proofs of density modulo 1, we should prove the Proposition 3.2.1. The proof of this proposition relies on a sieve result and the following preparatory Lemma 3.4.4. First we need to recall some concepts and related results from the elementary theory of polynomials.

Suppose that $f(x) \in \mathbb{Z}[x]$ is a polynomial with degree h. So that $f(x) = \sum_{k=0}^{h} a_k x^k$. We say that f is primitive, if $\gcd(a_h, \cdots, a_0) = 1$.

Lemma 3.4.1. *The product of two primitive polynomials, is primitive.*

Proof. Suppose that $A(x) = \sum_{k=0}^{n} a_k x^k$ and $B(x) = \sum_{k=0}^{m} b_k x^k$ are two primitive polynomials, and

$$C(x) = A(x)B(x) = \sum_{k=0}^{n+m} c_k x^k.$$

If we suppose that $C(x)$ is not primitive, then there exists prime p such that

$$p|\gcd(c_{n+m}, \cdots, c_0).$$

Since $A(x)$ is primitive, it is not possible for p to divides all its coefficients, and so we may set i to be the largest index such that $p \nmid a_i$. So, $p|a_k$ for $i < k \leq n$. Similarly, we set j to be the largest index such that $p \nmid b_j$, from which $p|b_k$ for $j < k \leq m$. Thus, considering

$$c_{i+j} = \sum_{i<k\leq n} a_k b_{i+j-k} + a_i b_j + \sum_{j<k\leq m} a_{i+j-k} b_k,$$

we obtain $p|a_i b_j$, and so $p|a_i$ or $p|b_j$, which is possible, because of minimal choice for i and j. □

As before let $f(x) \in \mathbb{Z}[x]$. The fixed divisor of f is the largest integer that divides $f(n)$ for all $n \in \mathbb{Z}$. We denote the fixed divisor of f by $\mathrm{fix}(f)$. The following result is well known [3, 41].

Lemma 3.4.2. *Suppose that f is a primitive polynomial with degree h. Then, we have* $\mathrm{fix}(f)|h!$.

Note that in above lemma, equality may occur. Indeed, letting $f(x) = \prod_{k=1}^{h}(x+h-k)$, we have $\mathrm{fix}(f) = h!$. We use above lemmas to get the following required result:

Corollary 3.4.3. *Suppose that $g_1(x), \cdots, g_M(x)$ are primitive polynomials with degrees $h_1, \cdots, h_M$, respectively, and let $G(x) = \prod_{m=1}^{M} g_m(x)$. Then, if $p|\mathrm{fix}(G)$ is a fixed prime divisor of G, we have $p \leq S$, where $S = \sum_{k=1}^{M} h_k$. Moreover, if $p^{\alpha}\|\mathrm{fix}(G)$, then $\alpha \leq \frac{S-1}{p-1}$.*

Proof. $G(x)$ is primitive and its degree is S. So, $\mathrm{fix}(G)|S!$, and we have $p|S!$. Since p is prime, we get $p \leq S$. Also, we have $\alpha \leq v_p(S!)$, where $v_p(S!)$ is the power of p in factorization of $S!$ into primes, and we have $v_p(S!) \leq \frac{S-1}{p-1}$. This completes the proof. □

Lemma 3.4.4. *Let $M \geq 2$ be an integer and let $D = \prod_{m=1}^{M}(m^2+1)$. For integers $m = 1, 2, \cdots, M$, we consider a family $\{\mathcal{P}_m\}_{1\leq m\leq M}$ of finite and distinct sets of primes p with $p \equiv 1 \pmod 4$ and $p > M^2+1$, and with related products $P_m = \prod_{p\in\mathcal{P}_m} p$ and $P = \prod_{m=1}^{M} P_m$. We let*

$$K = \prod_{\substack{p<2M \\ p\nmid D}} p,$$

and $Q = D^2K$. Then, there exists an integer r such that we have

$$\forall m, \forall n \in \mathbb{Z} : \gcd\left(f_m(n), QP\right) = (m^2+1)P_m, \tag{3.4.1}$$

where $f_m(x) = (Q(Px+r)+m)^2+1$. Also, letting $F(x) = \prod_{m=1}^{M} f_m(x)$ we have

$$\mathrm{fix}(F) = DP. \tag{3.4.2}$$

Proof. We first observe that the numbers D, K and P are pairwise coprime. Thus, relation (3.4.1) is equivalent to the pair of relations

$$\forall m, \forall n \in \mathbb{Z} : \gcd\left(f_m(n), Q\right) = (m^2+1), \tag{3.4.3}$$

and

$$\forall m, \forall n \in \mathbb{Z} : \gcd\left(f_m(n), P\right) = P_m. \tag{3.4.4}$$

Expanding $f_m(n)$ easily leads to the first relation. It also leads to

$$\forall m, \forall n \in \mathbb{Z} : f_m(n) \equiv (Qr+m)^2+1 \pmod P. \tag{3.4.5}$$

Now, let $p|P$. Since p is congruent to 1 modulo 4, there exists x_p such that $x_p^2+1 \equiv 0 \pmod{p}$. Since p divides P, there is a unique m such that $p \in \mathcal{P}_m$. Consider the family of congruences

$$Qr + m \equiv x_p \pmod{p}; \tag{3.4.6}$$

Since Q is coprime with P, and the p's are pairwise distinct, we can find, by the Chinese remainder theorem, an integer r such that (3.4.6) holds for any p dividing P.

In order to prove (3.4.5) we have to prove that for p in $\mathcal{P}_m$, then $f_m(n)$ is divisible by p - which is clear by construction - and that for p dividing P but not in $\mathcal{P}_m$, then $f_m(n)$ is not divisible by p. Indeed, if p is not in $\mathcal{P}_m$, then p is in $\mathcal{P}_h$ for some $h \neq m$. We thus have $(Qr+h)^2+1 \equiv 0 \pmod{p}$; then the relation $(Qr+m)^2+1 \equiv 0 \pmod{p}$ cannot hold, otherwise we would have $(2Qr+h+m)(h-m) \equiv 0 \pmod{p}$, whence, since $p > M^2+1 > M \geq |h-m|$, the relation $2Qr+h+m \equiv 0 \pmod{p}$ would hold true. But this last relation and $(Qr+h)^2+1 \equiv 0 \pmod{p}$ would imply $0 \equiv (2Qr+2h)^2+4 \equiv (h-m)^2+4 \pmod{p}$, which is impossible, since we have $p > M^2+1 \geq (M-1)^2+4 \geq |h-m|^2+4$. This proves that our construction leads to our first requirement (3.4.1).

Let us turn our attention to the fixed divisor of F. Since for any m and any n, the number $(m^2+1)P_m$ divides $f_m(n)$, the product $DP = \prod_m (m^2+1)P_m$ divides $F(n)$, thus $DP|\mathrm{fix}(F)$.

The main point in the proof of the converse implication is to show that with our construction, we can write

$$f_m(x) = (m^2+1)P_m g_m(x), \text{ where } g_m \text{ is primitive,}$$

which means that the gcd of the coefficients of g_m is 1, or that the *content* of g_m is 1, with the definition of [37], p. 126. Let us write $g_m(x) = Ax^2+Bx+C$, so that

$$A = \frac{Q^2P^2}{(m^2+1)P_m}, \quad B = \frac{2QP(Qr+m)}{(m^2+1)P_m}, \quad C = \frac{(Qr+m)^2+1}{(m^2+1)P_m},$$

and assume, by contradiction, that there exists a prime p such that $p|\gcd(A,B,C)$. Since $p|A$, one of the following must hold:

1. $p|P$ and $p \nmid P_m$
2. $p|P$ and $p|P_m$
3. $p|Q$ and $p \nmid m^2+1$
4. $p|Q$ and $p|m^2+1$

But, case (1) is excluded by (3.4.4). Assume now that $p|P_m$; by construction, we have $p > 2$ so that $p \nmid 2$, $p \nmid Q$; since $p|C$, we have $p|(Qr+m)^2+1$ and so $p \nmid (Qr+m)$; finally $p \nmid P/P_m$, and so $p \nmid B$, a contradiction which shows

that case (2) cannot hold. Case (3) is simply impossible by the definition of Q. Let us finally assume case (4) and write $p^\alpha \| (m^2+1)$ with $\alpha \geq 1$; then $(Qr+m)^2+1 = Q(Qr^2+2mr)+m^2+1$, where $p^{2\alpha}|Q$ and $p^\alpha \| (m^2+1)$, so that $p^\alpha \| (Qr+m)^2+1$ and thus p does not divide C; a final contradiction which shows that g_m is primitive.

We thus have $F(x) = \prod_m f_m(x) = DPG(x)$, where $G(x) = \prod_m g_m(x)$. We have to show that the fixed divisor of G is 1. We first notice that G is primitive (this comes from Lemma 3.4.1 which is indeed Gauss lemma of [37], p. 127); thus, if a prime p is a fixed divisor of G, a polynomial of degree $2M$, we must have $p \leq 2M$; by our construction, this implies that $p|Q$; since it is a fixed divisor of G, we have $p|G(0)$ and so $p|C$; since $p|Q$ and $p|C$, p must divide some (m^2+1), and this is impossible, as observed in Case (4). Thus $\mathrm{fix}(G) = 1$ and so $\mathrm{fix}(F) = DP$. □

We can now apply standard results from sieve theory. Once we know that G has no prime fixed divisor and that the polynomials g_m are irreducible over $\mathbb{Q}$ (they are obviously irreducible over $\mathbb{R}$), Proposition 3.2.1 follows in a straightforward way from [31] (cf. the proof of Theorem 7.4, p. 219, where the inequality symbol in (6.2) should be reversed). Notice that the constant $1/(6M)$ comes from the fact that G is of degree $2M$ and $\nu_\kappa \leq 3\kappa$ (cf. (4.7) p. 212).

3.5 Sequence with general term $b_n = \sum_{m \leq n} \frac{m^2+1}{\sigma(m^2+1)}$

In this section, we study the sequence with general term

$$s_m = \frac{m^2+1}{\sigma(m^2+1)},$$

where $\sigma(n) = \sum_{d|n} d$.

Theorem 3.5.1. *The sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = \sum_{m\leq n} s_m$ is dense modulo 1.*

Proof. Let M to be sufficiently large as in Proposition 3.2.1. We take $\delta > 0$ to be small, and we choose the family $\{\mathcal{P}_m\}_{1\leq m\leq M}$ as in above mentioned proposition with the following additional containment property holding for all $m = 1, 2, \cdots, M$:

$$\frac{P_m}{\sigma(P_m)} \in \left(\frac{5\delta}{4s_m}, \frac{7\delta}{4s_m}\right). \tag{3.5.1}$$

Indeed, if we let

$$f_\alpha(p) = \left(1 + \frac{1-p^{-\alpha}}{p-1}\right)^{-1},$$

then we have

$$\begin{aligned}\frac{P_m}{\sigma(P_m)} &= \prod_{p\in\mathcal{P}_m} f_\alpha(p) = \prod_{p\in\mathcal{P}_m}\Big(1-\frac{1}{p}+O\Big(\frac{1}{p^2}\Big)\Big)\\ &= \prod_{p\in\mathcal{P}_m}\Big(1-\frac{1}{p}\Big)\prod_{p\in\mathcal{P}_m}\Big(1+O\Big(\frac{1}{p^2}\Big)\Big) \asymp \prod_{p\in\mathcal{P}_m}\Big(1-\frac{1}{p}\Big).\end{aligned}$$

Mertens formula for $\prod_{p\le x}(1-\frac{1}{p})$ asserts that this product can be arbitrary small. Also, the inequality $\frac{7\delta}{4s_m} < c$ holds for any fixed positive real c, and for sufficiently small values of δ. Indeed, considering the bound $\sigma(n) < \beta n\log\log n$, which holds for some absolute constant β, we have

$$4cs_m > \frac{4c}{\beta\log\log(m^2+1)} \ge \frac{4c}{\beta\log\log(M^2+1)} > 7\delta,$$

where the last inequality holds by taking $M = \lfloor\frac{1}{\delta}\rfloor+1$, when $\delta>0$ is sufficiently small. Now, we consider the assumptions of Proposition 3.2.1 and the relation (3.2.3), and for $m = 1, 2, \cdots, M$, we write

$$\begin{aligned}s_{n+m} &= \prod_{p^\alpha\|(n+m)^2+1} f_\alpha(p)\\ &= \prod_{p^\alpha\|(m^2+1)P_m} f_\alpha(p) \prod_{\substack{p^\alpha\|(n+m)^2+1\\ p\nmid(m^2+1)P_m}} f_\alpha(p)\\ &= \frac{m^2+1}{\sigma(m^2+1)}\frac{P_m}{\sigma(P_m)} \prod_{\substack{p\mid(n+m)^2+1\\ p\nmid(m^2+1)P_m}}\Big(1-\frac{1}{p}+O\Big(\frac{1}{p^2}\Big)\Big)\\ &= s_m\frac{P_m}{\sigma(P_m)}(1+o(1)) \qquad (\text{as } n\to\infty). \end{aligned} \tag{3.5.2}$$

The containment (3.5.1) yields that for every sufficiently small $\delta>0$ there exist infinitely many (large enough) n such that for all $m = 1, 2, \cdots, M$, with $M = \lfloor\frac{1}{\delta}\rfloor+1$ we have

$$s_{n+m} \in (\delta, 2\delta).$$

Thus, we have

$$b_{n+M} - b_n = \sum_{m=1}^{M}(b_{n+m}-b_{n+m-1}) = \sum_{m=1}^{M} s_{n+m} > M\delta > 1.$$

This inequality implies that for each subinterval I of $[0,1]$ with length $> 2\delta$ there exist $i\in\{1,2,\cdots,M\}$ such that $b_{n+i}\in I$, and since $\delta>0$ was arbitrary small, we get the result. □

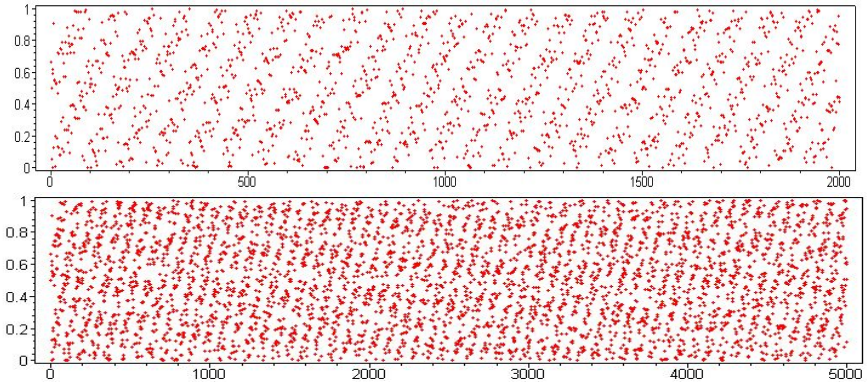


Fig. 3.2: Values of the sequence $b_n = \sum_{m\leq n}(m^2+1)/\sigma(m^2+1) \pmod 1$ for $1 \leq n \leq 2000$ (top) and for $1 \leq n \leq 5000$ (down). This sequence is dense modulo 1.

Remark 3.5.2. As above and below figures show, it seems that there are some parallel patterns in the distribution of the values of the sequence

$$b_n = \sum_{m\leq n}(m^2+1)/\sigma(m^2+1).$$

A natural question is finding the mathematical meaning of this pattern? Below we consider the sequence with general term $y_n = \lambda\{b_n/\lambda\}$ for various values of λ, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

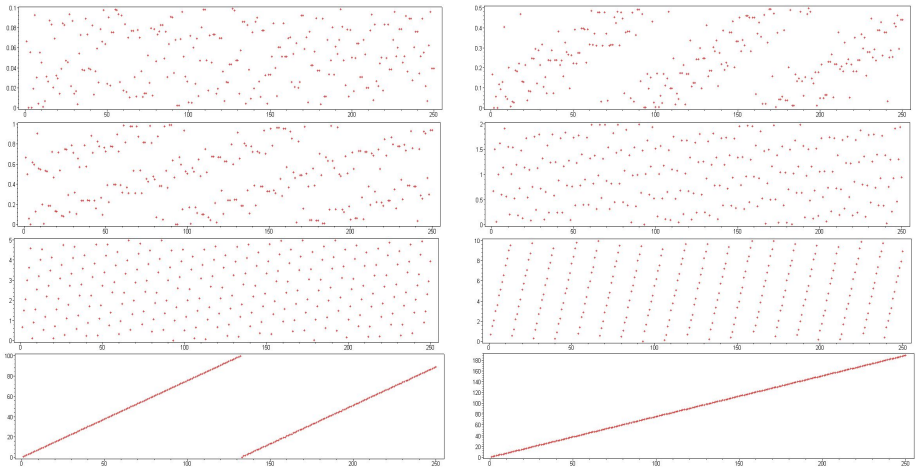


Fig. 3.3: Values of the sequence $\{y_n\}_n \in \mathbb{N}$ defined by $y_n = \lambda\{b_n/\lambda\}$ for $1 \leq n \leq 250$, with $\lambda = 0.1, 0.5, 1, 2, 5, 10, 100, \infty$, from top-left to right-down.

4. UNIFORM DISTRIBUTION MODULO 1 OF SOME SEQUENCES OF ADDITIVE FUNCTIONS

Uniform distribution modulo 1 of sequences containing additive arithmetical functions has been studied in general by H. Delange. Best known examples of such functions are the Omega functions $\omega(n)$ and $\Omega(n)$. Delange gave also a class of Omega functions containing these classical functions and some generalizations of them, and then he gave an analytic method to study uniform distribution modulo 1 of them. After introducing the work of Delange and the empirical results of F. Dekking and M. Mendès France, we study three dimensional distributions of related sequences. The new empirical results that we obtain in this chapter shed some light on the work of J-M. Deshouillers and H. Iwaniec [17] and other classical results.

4.1 Introduction - Weyl criterion

In the years 1909-1910, P. Bohl, W. Sierpinski and H. Weyl showed independently that the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \{\alpha n\} = \alpha n - \lfloor \alpha n \rfloor$, the fractional part of αn, for an irrational value of α, has a property stronger than being dense in the interval $I = [0, 1]$. Indeed, the measure (in Lebesgue sense) of the images of sequence in every sub-interval of I, is equal to the measure of that sub-interval. This is uniform distribution modulo 1 of a_n. Formally, an arbitrary real sequence $\{a_n\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 (u.d. mod 1) if for all real numbers a, b with $0 \leq a < b \leq 1$ we have

$$\lim_{N\to\infty} \frac{1}{N} \#\Big\{ n \leq N : \{a_n\} \in [a, b] \Big\} = b - a.$$

Historically, this formal definition is due to H. Weyl for the year 1914 (see [35] and references therein).

Various extensions of this definition are possible [27, 33, 35], but the more interesting case for our work is above one. There are some analytic criterions for a sequence in order to be u.d. mod 1. For example, it is known that [35] the sequence $\{a_n\}_{n=1}^{\infty}$ is u.d. mod 1 if and only if for every real-valued continuous function f defined on I, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{a_n\}) = \int_0^1 f(t)dt.$$

This implies that $\{a_n\}_{n=1}^{\infty}$ is u.d. mod 1 if and only if for every complex-valued continuous function f on $\mathbb{R}$ with period 1, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(a_n) = \int_0^1 f(t)dt. \tag{4.1.1}$$

As a consequence we obtain another criterion, which asserts that the sequence $\{a_n\}_{n=1}^{\infty}$ is u.d. mod 1 if and only if for every positive integer h we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\{a_n\}^h = \frac{1}{h+1}.$$

The relation (4.1.1) also leads to the following extraordinary, and widely applicable, criterion of Weyl [53]. For whole text, we set

$$e(x) = e^{2\pi i x}.$$

Theorem 4.1.1 (Weyl criterion - 1914)**.** *The sequence $\{a_n\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 if and only if, for every positive integer h we have*

$$\sum_{n\le N} e(ha_n) = o(N),$$

as N tends to infinity.

Satz 1. *Gilt für jede ganze Zahl $m \neq 0$ die Limesgleichung*

$$\sum_{h=1}^{n} e(m\alpha_h) = o(n),$$

so genügen die Zahlen α_n mod. 1 *dem Gesetz der überall gleichmäßig dichten Verteilung.*

Fig. 4.1: Hermann Klaus Hugo Weyl (9 November 1885 - 8 December 1955, German mathematician), and a part of his 1914 (published at 1916) paper indicating his criterion.

Originally, H. Weyl introduced his criterion to study the distribution modulo one of the sequence $a_n = P(n)$ for a given polynomial P. Many problems of uniform distribution have been considered [27, 33, 35]. An explicit version of Weyl's criterion was given by Erdös and Turán [25] in 1948: For any sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, and any $0 \le a < b \le 1$ we have

$$\left|\frac{1}{N}\#\Big\{n\le N : \{a_n\}\in[a,b]\Big\} - (b-a)\right| \le \frac{6}{m+1} + \frac{4}{\pi}\sum_{k=1}^{m}\frac{1}{k}\left|\frac{1}{N}\sum_{n\le N} e(ka_n)\right|.$$

The sequence with general term $a_n = \log n$ is not uniformly distributed modulo 1 [35]. A number theory sequence which behaves very similar to $\log n$ is the sequence with general term $a_n = \frac{1}{n}\sum_{m\leq n}\tau(m)$. We know that

$$a_n = \log n + (2\gamma - 1) + O\Big(\frac{1}{\sqrt{n}}\Big).$$

Considering the Corollary A.2.8, and the criterion of Weyl, this sequence is not uniformly distributed modulo 1 either.

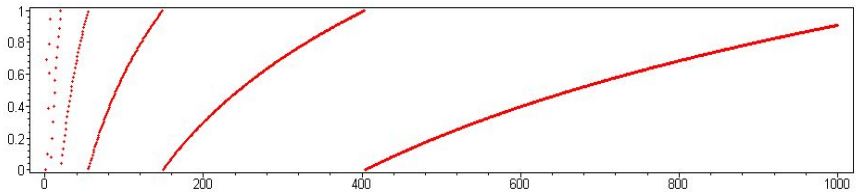


Fig. 4.2: Values of the sequence $a_n = \log n \pmod 1$ for $1 \leq n \leq 1000$. This sequence is not uniformly distributed modulo 1.

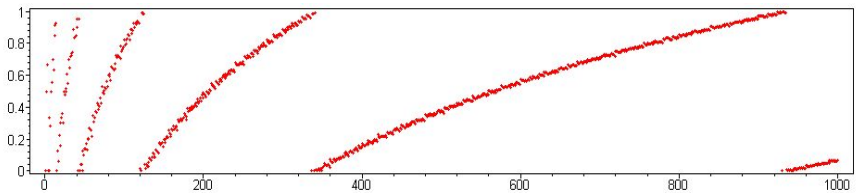


Fig. 4.3: Values of the sequence $a_n = \frac{1}{n}\sum_{m\leq n}\tau(m) \pmod 1$ for $1 \leq n \leq 1000$. This sequence is not uniformly distributed modulo 1.

For arithmetic functions, in case of additive ones, we study the works of H. Delange [9, 10, 11] and [12], which provide a general theory for studying the uniform distribution of such functions. There is no known general theory of studying uniform distribution modulo 1 of multiplicative functions. Of course, there are some remarkable results in special cases. For example one can see [17].

4.2 Additive arithmetical functions - general theory

We recall that the arithmetical function f is additive if for each coprime pair of positive integers m and n, we have $f(mn) = f(m) + f(n)$. If the multiplicative function f is positive, then the function defined by $g(n) = \log f(n)$ is additive.

Most famous examples of additive functions are the functions $\lambda\omega(n)$ and $\lambda\Omega(n)$ defined by

$$\omega(n) = \sum_{p|n} 1, \qquad \Omega(n) = \sum_{p^v \| n} v.$$

The general theory of the uniform distribution modulo 1 of additive functions are formulated by H. Delange [11]. One may find this formulation in Elliott's book [20], too. In this section, we study Delange's work.

Suppose that f is an arithmetical additive function. For a given complex-valued arithmetical function F, we define the *mean-value* of F to be

$$M(F) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(n),$$

provided this limit exists. For each positive integer h, we define

$$F_h(n) = e(hf(n)). \tag{4.2.1}$$

The Weyl criterion asserts that f is u.d. mod 1, if for any positive integer h, $M(F_h)$ exists and we have

$$M(F_h) = 0.$$

We denote by $\mathfrak{M}_0$ the set of all complex-valued multiplicative functions, whose values have modulus not greater than 1. Clearly $F_h \in \mathfrak{M}_0$. We use the following result due to G. Halász [30]:

Theorem 4.2.1. *Let $F \in \mathfrak{M}_0$. In order for $M(F)$ to exist and be zero, it is necessary and sufficient that one of the following conditions be satisfied:*
(C 4.2.1.A) For every real u, we have

$$\sum_p \frac{1}{p}\Big(1 - \Re\big(F(p)p^{-iu}\big)\Big) = +\infty.$$

(C 4.2.1.B) There exists a real u_0 such that

$$\sum_p \frac{1}{p}\Big(1 - \Re\big(F(p)p^{-iu_0}\big)\Big) < +\infty,$$

and $2^{-riu_0}F(2^r) = -1$ for all $r \in \mathbb{N}$.

Remark 4.2.2. If both $M(F)$ and $M(F^2)$ exist and are zero, the condition (C 4.2.1.A) must be satisfied.

Proof. Since $F \in \mathfrak{M}_0$ (assumption of theorem), obviously we have $F^2 \in \mathfrak{M}_0$. We show that if the condition (C 4.2.1.B) is satisfied for F, then neither (C 4.2.1.A) nor (C 4.2.1.B) can be satisfied for F^2. But, our assumption $M(F) = M(F^2) = 0$, and Theorem 4.2.1 imply that for each of F and F^2 one of these two conditions are fulfilled which implies our implication.

Now, we note that if the condition (C 4.2.1.B) is satisfied for F, then for some $u_0 \in \mathbb{R}$ we have

$$\sum_p \frac{1}{p}\Big(1 - \Re\big(F(p)p^{-iu_0}\big)\Big) < +\infty.$$

Since for any complex number z we have $1 - \Re(z^2) \leq 4(1 - \Re(z))$, then we get

$$\sum_p \frac{1}{p}\Big(1 - \Re\big(F(p)^2 p^{-2iu_0}\big)\Big) \leq 4\sum_p \frac{1}{p}\Big(1 - \Re\big(F(p)p^{-iu_0}\big)\Big) < +\infty.$$

This shows that (C 4.2.1.A) doesn't hold for F^2, and on the other hand it shows that if the condition (C 4.2.1.B) is satisfied simultaneously for F and F^2, then it will hold for both of them with same u_0. Thus, if (C 4.2.1.B) holds for F^2, then we should have $2^{-2riu_0}F(2^r)^2 = -1$ for all $r \in \mathbb{N}$. But, $2^{-riu_0}F(2^r) = -1$ implies $2^{-2riu_0}F(2^r)^2 = 1$. □

Theorem 4.2.3. *A necessary and sufficient condition for the real-valued additive function f to be uniformly distributed modulo 1 is that for all $h \in \mathbb{N}$ and all $t \in \mathbb{R}$ we have*

$$\sum_p \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big) = +\infty.$$

Proof. Recall that the function F_h is defined by (4.2.1). We have $F_h \in \mathfrak{M}_0$. A necessary and sufficient condition for the real-valued additive function f to be uniformly distributed modulo 1 is that $M(F_h)$ exists (for all positive integers h) and to be zero. But, if $M(F_h) = 0$ for all positive integers h, then we get $M(F_{2h}) = 0$, too. Since, $F_h^2 = F_{2h}$, above theorem of Halász and its remark, imply that $M(F_h)$ exists for all positive integers h and being zero is equivalent to the condition (C 4.2.1.A), i.e.,

$$\sum_p \frac{1}{p}\Big(1 - \Re\big(F_h(p)p^{-iu}\big)\Big) = +\infty,$$

for all $h \in \mathbb{N}$ and all $u \in \mathbb{R}$. But, we have

$$F_h(p)p^{-iu} = \exp\Big(2\pi i h\Big(f(p) - \frac{u}{2\pi h}\log p\Big)\Big),$$

and so

$$1 - \Re\left(F_h(p)p^{-iu}\right) = 2\sin^2\Big(h\pi\Big(f(p) - \frac{u}{2\pi h}\log p\Big)\Big).$$

If we change u to $2\pi ht$ (now $t \in \mathbb{R}$ instead u), then we get required equivalent condition of the theorem. This completes the proof. □

As the first application, Delange uses above theorem to reprove uniform distribution modulo 1 of the functions $\lambda\omega(n)$ and $\lambda\Omega(n)$, and for a wider class of additive functions.

Theorem 4.2.4. *Let f be additive, and suppose that for all $p \in \mathbb{P}$ we have $f(p) = \alpha$ where α is an irrational number. Then f is u.d. mod 1.*

Proof. If $t = 0$, then we have

$$\sum_p \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big) = \sin^2(\alpha h\pi) \sum_p \frac{1}{p} = +\infty.$$

For $t \neq 0$ we define λ_k for each integer $k \geq 2h|\alpha|$ by

$$\lambda_k = \exp\left(\frac{2k+1}{4h|t|} + \frac{\alpha}{t}\right).$$

We note that, for each integer $r \geq h|\alpha|$, the inequalities $\lambda_{2r} < p \leq \lambda_{2r+1}$ imply

$$r\pi + \frac{\pi}{4} \leq \Big|h\pi\big(f(p) - t\log p\big)\Big| \leq r\pi + \frac{3\pi}{4}.$$

Thus, we get

$$\sum_{\lambda_{2r} < p \leq \lambda_{2r+1}} \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big) \geq \frac{1}{2} \sum_{\lambda_{2r} < p \leq \lambda_{2r+1}} \frac{1}{p}.$$

But, from the approximation[1]

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + C + o\left(\frac{1}{\log x}\right) \qquad (\text{as } x \to \infty),$$

we obtain

$$\sum_{\lambda_{2r} < p \leq \lambda_{2r+1}} \frac{1}{p} = \log \frac{\log \lambda_{2r+1}}{\log \lambda_{2r}} + o\left(\frac{1}{\log \lambda_{2r}}\right) \sim \frac{1}{2r},$$

as $r \to \infty$. This yields that

$$\sum_p \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big) = +\infty,$$

and the proof is complete. □

Corollary 4.2.5. *For any irrational λ, the functions $\lambda\omega(n)$ and $\lambda\Omega(n)$ are uniformly distributed modulo 1.*

[1] More precisely, we have

$$\Big|\sum_{p \leq x} \frac{1}{p} - \log\log x - C\Big| < \frac{1}{2\log^2 x},$$

where $x \geq 286$, and $C = 0.2614972\cdots$.

Corollary 4.2.6. *Let f_1 and f_2 be real-valued additive functions, and suppose that there exists a real constant a such that $f_2(p) - f_1(p) = a \log p$ for all $p \in \mathbb{P}$. Then either both f_1 and f_2 are u.d. mod 1 or none of them is.*

Also, Delange uses the validity of the Theorem 4.2.3 to get the following known result.

Theorem 4.2.7. *Let f be additive, and assume that*

$$\lim_{p\to\infty} f(p) = 0.$$

Then f is u.d. mod 1 if and only if

$$\sum_p \frac{f(p)^2}{p} = +\infty.$$

Proof. Suppose that $\sum_p \frac{f(p)^2}{p} = +\infty$. If $t = 0$, then considering $\sin(h\pi f(p)) = (1 + o(1))h\pi f(p)$ we have

$$\sum_p \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big) = \sum_p \frac{1}{p} \sin^2\big(h\pi f(p)\big) \sim (h\pi)^2 \sum_p \frac{f(p)^2}{p} = +\infty.$$

For $t \neq 0$ we define λ_k by

$$\lambda_k = \exp\left(\frac{2k+1}{4h|t|}\right).$$

We note that when r is sufficiently large, we can get $|f(p)| \leq \frac{1}{12h}$ for $p \geq \lambda_{2r}$. Also, the inequalities $\lambda_{2r} < p \leq \lambda_{2r+1}$ imply

$$r\pi + \frac{\pi}{6} \leq \Big|h\pi\big(f(p) - t\log p\big)\Big| \leq r\pi + \frac{5\pi}{6}.$$

Thus, we get

$$\sum_{\lambda_{2r}<p\leq\lambda_{2r+1}} \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big) \geq \frac{1}{4} \sum_{\lambda_{2r}<p\leq\lambda_{2r+1}} \frac{1}{p},$$

and this gives (similar to the proof of the previous theorem)

$$\sum_p \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big) = +\infty.$$

For the converse, we observe[2] that

$$\sum_p \frac{1}{p} \sin^2\big(h\pi f(p)\big) \leq 2\sum_p \frac{1}{p} \sin^2\big(h\pi t\log p\big) + 2\sum_p \frac{1}{p} \sin^2\Big(h\pi\big(f(p) - t\log p\big)\Big),$$

from which the divergence the series $\sum \frac{1}{p} \sin^2(h\pi(f(p) - t\log p))$ gives the divergence of the series $\sum \frac{1}{p} \sin^2(h\pi f(p))$, and consequently the divergence of $\sum \frac{f(p)^2}{p}$. This completes the proof. □

[2] We use the trivial inequality $\sin^2 a \leq 2\sin^2 b + 2\sin^2(a - b)$.

Finally, we note that if we let $\{\!\{x\}\!\}$ to be the distance from x to its nearest integer, i.e., $\{\!\{x\}\!\} = \min(\{x\}, 1-\{x\})$, we have

$$2\{\!\{x\}\!\} \le |\sin \pi x| \le \pi \{\!\{x\}\!\}.$$

Considering this we can get the following formulation of Theorems 4.2.3 and 4.2.7, as they appear in Elliott's book [20].

Theorem 4.2.8. *In order that the real-valued additive function $f(n)$ should be uniformly distributed modulo 1, it is both necessary and sufficient that for each integer h the series*

$$\sum_p p^{-1} \{\!\{hf(p) - \tau \log p\}\!\}^2$$

diverge for every real number τ.

Theorem 4.2.9. *Let $\{\!\{f(p)\}\!\} \to 0$ as $p \to \infty$. Then the additive function $f(n)$ is uniformly distributed modulo 1 if and only if the series*

$$\sum_p p^{-1} \{\!\{f(p)\}\!\}^2,$$

diverges.

4.2.1 Some results of Selberg and Halász

In this section using some results of A. Selberg and G. Halász, we reprove that for any irrational number λ, the function $\lambda\omega(n)$, and also similarly the function $\lambda\Omega(n)$, are uniformly distributed modulo 1.

Corollary of a result of Selberg

A. Selberg [46] proved that as $x \to \infty$, then

$$\sum_{n \le x} z^{\omega(n)} = F(z) x (\log x)^{z-1} + O\left(x (\log x)^{\Re(z)-2}\right), \tag{4.2.2}$$

uniformly for $|z| \le R$, where R is any positive number and

$$F(z) = \frac{1}{\Gamma(z)} \prod_{p \in \mathbb{P}} \left(1 + \frac{z}{p-1}\right) \left(1 - \frac{1}{p}\right)^z.$$

If we take $z = e(h\lambda)$, where h is a positive integer, we get

$$\Big| \sum_{n \le x} e(h\lambda\omega(n)) \Big| = \Big| \sum_{n \le x} z^{\omega(n)} \Big| \le |F(z)||x| (\log x)^{\Re(z)-1} + O\Big(\frac{x}{\log x}\Big).$$

Now, since λ is irrational we note that $\Re(z) - 1 = \cos(2\pi h\lambda) - 1 < 0$. So, we obtain

$$\Big| \sum_{n \le x} e(h\lambda\omega(n)) \Big| = o(x) \qquad (\text{as } x \to \infty).$$

Weyl's criterion gives u.d. mod 1 of $\lambda\omega(n)$.

Corollary of a result of Halász

Suppose that $E \subseteq \mathbb{P}$, and let

$$E(x) = \sum_{\substack{p \le x \\ p \in E}} \frac{1}{p}.$$

Also, let

$$\Omega(n; E) = \sum_{\substack{p^v \| n \\ p \in E}} v.$$

In 1971, G. Halász [29] showed that if $\delta > 0$, and $\delta \le |z| \le 2 - \delta$, then one has

$$\sum_{n \le x} z^{\Omega(n;E)} \ll x \exp\Big(\Big(|z| - 1 - c_1\big(|z| - \Re(z)\big)\Big)E(x)\Big), \tag{4.2.3}$$

where $c_1 > 0$ is some constant which depends only on δ. Also, if $|z-1| \le \frac{1}{2}$, then for some absolute constants $c_2 > 0$ and $c_3 > 0$ we have

$$\Big|\frac{1}{x}\sum_{n \le x} z^{\Omega(n;E)} - e^{(z-1)E(x)}\Big| \ll |z-1| e^{(\Re(z)-1)E(x)} + e^{(|z|-1)E(x)}\Big(e^{-\frac{c_2}{|z-1|}} + (\log x)^{-c_3}\Big). \tag{4.2.4}$$

If we take in (4.2.3), $z = e(h\lambda)$ where h is any positive integer and λ is an irrational number, and we set $E = \mathbb{P}$, then $\Omega(n; E) = \Omega(n)$ and we get

$$\begin{aligned} \sum_{n \le x} e(h\lambda\Omega(n)) &= \sum_{n \le x} z^{\Omega(n)} \\ &\ll x \exp\Big(\Big(-c_1\big(1 - \cos(2\pi h\lambda)\big)\Big)\log\log x + O(1)\Big) \\ &= x \exp(-c_{h,\lambda} \log\log x) = o(x) \qquad (\text{as } x \to \infty), \end{aligned}$$

where $c_{h,\lambda} > 0$ is absolute constant depending on h and λ. This gives the uniform distribution modulo 1 of the numbers $\lambda\Omega(n)$.

Moreover, let $q \ge 1$ be integer and let a be such that $\gcd(a, q) = 1$. Let $\mathbb{P}_{a,q}$ be the set of prime numbers p such that $p \equiv a \pmod q$. Then, we have

$$\sum_{\substack{p \le x \\ p \in \mathbb{P}_{a,q}}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + O(1).$$

Thus, if we set $E = \mathbb{P}_{a,q}$, $z = e(h\lambda)$ where h is any positive integer and λ is an

irrational number, then we obtain

$$\begin{aligned}\sum_{n\leq x} e(h\lambda\Omega(n;E)) &= \sum_{n\leq x} z^{\Omega(n;E)}\\ &\ll x\exp\Big(\frac{1}{\varphi(q)}\Big(-c_1\big(1-\cos(2\pi h\lambda)\big)\Big)\log\log x+O(1)\Big)\\ &= x\exp(-c_{q,h,\lambda}\log\log x)=o(x) \qquad (\text{as } x\to\infty),\end{aligned}$$

where $c_{q,h,\lambda}>0$ is absolute constant depending on q, h and λ. This shows that the numbers $\lambda\Omega(n;E)$ are uniformly distributed modulo 1. More generally, if $a_1,\cdots,a_k$ are distinct positive integers with $k\leq\varphi(q)$ and are coprime to q, and we set

$$E=\bigcup_{1\leq i\leq k}\mathbb{P}_{a_i,q},$$

then

$$\sum_{\substack{p\leq x\\ p\in E}}\frac{1}{p}=\sum_{i=1}^{k}\sum_{\substack{p\leq x\\ p\in\mathbb{P}_{a_i,q}}}\frac{1}{p}=\frac{k}{\varphi(q)}\log\log x+O(1).$$

Thus, for irrational numbers λ, similarly we obtain the uniform distribution modulo 1 of the numbers $\lambda\Omega(n;E)$. Similar general result holds for the numbers $\lambda\Omega(n;E)$ with λ an irrational number and E be a set of primes such that

$$\lim_{x\to\infty}E(x)=\infty.$$

Above mentioned these corollaries recover some results obtained by Delange, which we will study them in next sections.

4.2.2 Method of Delange

In this section we study the analytic method of Hubert Delange [10] in the theory of uniform distribution modulo 1 of additive functions. Delange uses some Tauberian theorems and also non-vanishing of the Riemann zeta function $\zeta(s)$ in the region $\Re(s)\geq 1$. The following alternative form of the classical Tauberian theorem of Ikehara is required:

Theorem 4.2.10. *Let $\alpha(t)$ be a nondecreasing real function defined for $t\geq 0$ with $\alpha(0)\geq 0$. Suppose that the integral*

$$\int_0^\infty e^{-st}\alpha(t)dt,$$

is convergent for $\Re(s)>a>0$ and equals to $f(s)$. Also, suppose that for each real $y\neq 0$, the function $f(s)$ tends to a finite limit as $s\to a+iy$ in the half plane $\Re(s)>a$, and that as $s\to a$ in this half plane, we have

$$f(s)-\frac{A}{s-a}\ll|s-a|^{-w},$$

where $A > 0$ and $0 < w < 1$. Then, as $t \to +\infty$, we get

$$\alpha(t) \sim Ae^{at}.$$

The proof of this theorem can be obtained from the proof of Ikehara's theorem in [54]. It helps us to get the following result.

Theorem 4.2.11. *Consider the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where a_n's are real or complex numbers satisfying $|a_n| \leq 1$ (so the series absolutely convergent for $\Re(s) > 1$). Suppose that for $\Re(s) > 1$ we have*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = (s-1)^{-\beta - i\gamma} g(s) + h(s), \tag{4.2.5}$$

where the functions g and h are regular for $\Re(s) \geq 1$, β and γ are real numbers, $\beta < 1$, and $(s-1)^{-\beta-i\gamma}$ has its principal value. Then, as $x \to +\infty$, we have

$$\sum_{n \leq x} a_n = o(x).$$

Proof. We set

$$a_n = u_n + iv_n,$$

where u_n and v_n are real. We let

$$A(t) = \sum_{1 \leq n \leq e^t} (1 + u_n), \qquad B(t) = \sum_{1 \leq n \leq e^t} (1 + v_n).$$

Since $|a_n| \leq 1$, so we have $|u_n| \leq 1$ and $|v_n| \leq 1$. Thus, $A(0) = 1 + u_1 \geq 0$, and also $B(0) \geq 0$. Note that if $t_2 \geq t_1 \geq 0$, then

$$A(t_2) - A(t_1) = \sum_{e^{t_1} < n \leq e^{t_2}} (1 + u_n) \geq 0.$$

Thus, both functions A and B are nondecreasing. Also, using partial summation formula (see the Example A.2.5), we have

$$\sum_{n=1}^{\infty} \frac{1 + u_n}{n^s} = s \int_0^{\infty} e^{-st} A(t) dt, \tag{4.2.6}$$

and

$$\sum_{n=1}^{\infty} \frac{1 + v_n}{n^s} = s \int_0^{\infty} e^{-st} B(t) dt. \tag{4.2.7}$$

If D is a domain which is symmetric with respect to the real axis and contains the closed half plane $\Re(s) \geq 1$, and in which g and h are regular, we may write in this domain

$$g(s) = g_1(s) + ig_2(s), \qquad h(s) = h_1(s) + ih_2(s),$$

g_1, g_2, h_1 and h_2 are regular in D and real for s real in D. Indeed, we have

$$g_1(s) = \frac{1}{2}\left(g(s) + \overline{g(\overline{s})}\right), \qquad g_2(s) = \frac{1}{2i}\left(g(s) - \overline{g(\overline{s})}\right),$$
$$h_1(s) = \frac{1}{2}\left(h(s) + \overline{h(\overline{s})}\right), \qquad h_2(s) = \frac{1}{2i}\left(h(s) - \overline{h(\overline{s})}\right),$$

where $\overline{s}$ denotes the conjugate of s. Then, for real $s > 1$ (and hence by analytic continuation for $\Re(s) > 1$) using the assumption (4.2.5) to write

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = (s-1)^{-\beta} e^{-i\gamma \log(s-1)} \Big(g_1(s) + ig_2(s)\Big) + h_1(s) + ih_2(s).$$

In this relation, we utilize the following identity

$$e^{-i\gamma \log(s-1)} = \cos\left(\gamma \log \frac{1}{s-1}\right) + i \sin\left(\gamma \log \frac{1}{s-1}\right),$$

and multiply factors, and then separate real and imaginary parts. Thus, considering $a_n = u_n + iv_n$, and the relations (4.2.6) and (4.2.7), we obtain

$$\int_0^{\infty} e^{-st} A(t)dt = f_A(s), \qquad \int_0^{\infty} e^{-st} B(t)dt = f_B(s),$$

where

$$f_A(s) = \frac{\zeta(s)}{s} + \frac{h_1(s)}{s} + \frac{(s-1)^{-\beta}}{s}\left(g_1(s)\cos\left(\gamma \log \frac{1}{s-1}\right) - g_2(s)\sin\left(\gamma \log \frac{1}{s-1}\right)\right),$$

and

$$f_B(s) = \frac{\zeta(s)}{s} + \frac{h_2(s)}{s} + \frac{(s-1)^{-\beta}}{s}\left(g_1(s)\sin\left(\gamma \log \frac{1}{s-1}\right) + g_2(s)\cos\left(\gamma \log \frac{1}{s-1}\right)\right).$$

The functions $f_A(s)$ and $f_B(s)$ fulfill the conditions for the function $f(s)$ in the Theorem 4.2.10. Thus, as $t \to \infty$ we get

$$A(t) \sim e^t, \qquad B(t) \sim e^t.$$

But, we have $A(t) = \lfloor e^t \rfloor + \sum_{n \le e^t} u_n$ and $B(t) = \lfloor e^t \rfloor + \sum_{n \le e^t} v_n$. So, we obtain

$$\sum_{n \le e^t} u_n = o(e^t), \qquad \sum_{n \le e^t} v_n = o(e^t),$$

and then

$$\sum_{n \le x} u_n = o(x), \qquad \sum_{n \le x} v_n = o(x).$$

These complete the proof. □

Then, Delange uses the following result, the detailed proof of which appeared in his 1956 paper [12].

Theorem 4.2.12. *There exist two functions $\mathcal{G}_1(s,z)$ and $\mathcal{G}_2(s,z)$ with the following properties:*

1. *They are regular in s and z for $|z| < \sqrt{2}$, and s belonging to a certain domain Δ, which contains the closed half plane $\Re(s) \geq 1$.*
2. *For $\Re(s) > 1$ and $|z| \leq 1$ we have*

$$\sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s} = \mathcal{G}_1(s,z)(s-1)^{-z}, \tag{4.2.8}$$

and

$$\sum_{n=1}^{\infty} \frac{z^{\Omega(n)}}{n^s} = \mathcal{G}_2(s,z)(s-1)^{-z}, \tag{4.2.9}$$

where $(s-1)^{-z}$ has its principal value.

For any positive integer h, taking $z = e(h\lambda)$ in (4.2.8) and (4.2.9), we get

$$\sum_{n=1}^{\infty} \frac{e(h\lambda\omega(n))}{n^s} = \mathcal{G}_1(s, e(h\lambda))(s-1)^{-e(h\lambda)},$$

and

$$\sum_{n=1}^{\infty} \frac{e(h\lambda\Omega(n))}{n^s} = \mathcal{G}_2(s, e(h\lambda))(s-1)^{-e(h\lambda)}.$$

Since λ is irrational, we have $\Re(e(h\lambda)) < 1$, and so we can use the Theorem 4.2.11 to get

$$\sum_{n\leq x} e(h\lambda\omega(n)) = o(x), \qquad \sum_{n\leq x} e(h\lambda\Omega(n)) = o(x),$$

as $x \to +\infty$. So, Weyl's criterion gives u.d. mod 1 of the functions $\lambda\omega(n)$ and $\lambda\Omega(n)$.

4.2.3 Delange class of Omega functions

As we see, Delange doesn't use the additivity of the Omega functions, and the key point in his proof is the existence of relations like (4.2.8) and (4.2.9). So, he was able to extend the results obtained for the Omega functions, to other arithmetic (not necessarily additive) functions.

Theorem 4.2.13. *Let $f(n)$ be an integral valued arithmetic function, and suppose that for $|z| \leq 1$ and $\Re(s) > 1$ we have*

$$\sum_{n=1}^{\infty} \frac{z^{f(n)}}{n^s} = \mathcal{G}(s,z)(s-1)^{\alpha-1-\alpha z} + \mathcal{H}(s,z), \tag{4.2.10}$$

where α is a real positive number and, for $|z| \leq 1$, the functions $\mathcal{G}(s,z)$ and $\mathcal{H}(s,z)$ are regular in s for s belonging to a certain domain Δ which contains the closed half plane $\Re(s) \geq 1$. Then, for any irrational number λ, the sequence $\lambda f(n)$ is uniformly distributed modulo 1.

Proof. Suppose that h is any positive integer. We take $z = e(h\lambda)$ in (4.2.10). For $\Re(s) > 1$ we have

$$\sum_{n=1}^{\infty} \frac{e(h\lambda f(n))}{n^s} = \mathcal{G}(s, e(h\lambda))(s-1)^{\alpha-1-\alpha e(h\lambda)} + \mathcal{H}(s, e(h\lambda)).$$

The irrationality of λ gives $\Re(-(\alpha - 1 - \alpha e(h\lambda))) < 1$, and then Theorem 4.2.11 enables us to conclude that

$$\sum_{n \leq x} e(h\lambda f(n)) = o(x) \qquad (\text{as } x \to +\infty).$$

This completes the proof. □

As we see again, the key point is the relation (4.2.10), which holds for the Delange class of arithmetical functions. He introduce this class in his paper [9].

Delange class $\mathcal{D}$

For $E \subset \mathbb{P}$, we define two functions $\omega_E(n)$ and $\Omega_E(n)$ as follows:

$$\omega_E(n) = \sum_{\substack{p|n \\ p \in E}} 1, \qquad \Omega_E(n) = \sum_{\substack{p^v \| n \\ p \in E}} v.$$

We put $\omega_E(1) = \Omega_E(1) = 0$. Then the class of Delange, which we denote by $\mathcal{D}$, consists of all the functions $\omega_E(n)$ and $\Omega_E(n)$ corresponding to sets E which have the following property:
There exist a real positive number $\alpha \leq 1$ and a function $\delta(s)$ regular for $\Re(s) \geq 1$, such that for $\Re(s) > 1$ we have

$$\sum_{p \in E} \frac{1}{p^s} = \alpha \log \frac{1}{s-1} + \delta(s),$$

where $\log \frac{1}{s-1}$ means its principal value. In this case, we say [47] that E has *Dirichlet density* equals to α and we write

$$\mathbf{D}(E) = \alpha.$$

This holds for the set of all primes. Indeed, we have

$$\sum_{p \in \mathbb{P}} \frac{1}{p^s} \sim \log \frac{1}{s-1} \qquad (\text{as } s \to 1),$$

which means $\mathbf{D}(\mathbb{P}) = 1$. In brief, the Delange class of Omega functions is

$$\mathcal{D} = \Big\{\omega_E(n), \Omega_E(n) : \mathbf{D}(E) > 0\Big\}.$$

As a nontrivial example, we consider the theorem on arithmetic progressions, which can be refined in the following way:

Theorem 4.2.14. *Let $q \geq 1$ be integer and let a be such that $\gcd(a, q) = 1$. Let $\mathbb{P}_{a,q}$ be the set of prime numbers p such that $p \equiv a \pmod{q}$. Then, we have*

$$\mathbf{D}(\mathbb{P}_{a,q}) = \frac{1}{\varphi(q)}.$$

Corollary 4.2.15. *Suppose that $\gcd(a, q) = 1$, and let E be the union of one or more distinct arithmetic progressions of primes with same difference q. Then, we have*

$$\omega_E(n) \in \mathcal{D}, \qquad \Omega_E(n) \in \mathcal{D}.$$

Proof. Let $a_1 \neq a_2$ and $\gcd(a_1, q) = \gcd(a_2, q) = 1$. Then, clearly $\mathbb{P}_{a_1,q} \cup \mathbb{P}_{a_2,q} = \varnothing$, and we have

$$\sum_{p \in \mathbb{P}_{a_1,q} \cup \mathbb{P}_{a_2,q}} \frac{1}{p^s} = \sum_{p \in \mathbb{P}_{a_1,q}} \frac{1}{p^s} + \sum_{p \in \mathbb{P}_{a_2,q}} \frac{1}{p^s} \sim \frac{2}{\varphi(q)} \log \frac{1}{s-1} \qquad (\text{as } s \to 1),$$

which means that

$$\mathbf{D}(\mathbb{P}_{a_1,q} \cup \mathbb{P}_{a_2,q}) = \frac{2}{\varphi(q)}.$$

For $a_1 \neq \cdots \neq a_k$ with $k \leq \varphi(q)$, similarly we obtain

$$\mathbf{D}(\cup_{1 \leq i \leq k} \mathbb{P}_{a_i,q}) = \frac{k}{\varphi(q)} > 0.$$

This complete the proof. □

More generally, above discussion leads to the following result of Delange [9, 10].

Theorem 4.2.16. *For any irrational number λ and any $f \in \mathcal{D}$, the sequence defined by $\lambda f(n)$ is uniformly distributed modulo 1.*

Further generalizations

Delange, then considers uniform distribution modulo 1 of the numbers $\lambda f(n)$ for when n runs through a certain infinite set of positive integers, say A, other than set of all positive integers. To do this, he redefines the concept of uniform distribution modulo 1, and also gives a refinement of the Weyl's criterion.

Definition. The numbers $\{u_n\}_{n \in A}$ are uniformly distributed modulo 1, if for $0 \leq t \leq 1$, we have

$$\#\Big\{n \leq x : \{u_n\} \leq t\Big\} = t\nu(x) + o(\nu(x)) \qquad (\text{as } x \to \infty),$$

where

$$\nu(x) = \#\big\{n \le x : n \in A\big\}.$$

Criterion. In order that the numbers $\{u_n\}_{n\in A}$ be uniformly distributed modulo 1, it is necessary and sufficient that for every positive integer h we have

$$\sum_{\substack{n\le x\\ n\in A}} e(hu_n) = o(\nu(x)) \qquad (\text{as } x\to\infty).$$

If A has a positive density, then $o(\nu(x))$ may obviously be replaced by $o(x)$. By density, we mean

$$\mathbf{d}(A) = \lim_{x\to\infty} \frac{1}{x}\#\big\{n \le x : n \in A\big\},$$

provided this limit exists. This is called *natural density* or *asymptotic density*. For example, we know that if $\mathcal{S}$ denotes the set of square-free positive integers, then we have

$$\mathbf{d}(\mathcal{S}) = \frac{6}{\pi^2}.$$

As another example, let $q \ge 1$ be integer and let a be such that $\gcd(a, q) = 1$. Let $\mathbb{N}_{a,q}$ be the set of positive integers n such that $n \equiv a \pmod q$. Then, we have

$$\mathbf{d}(\mathbb{N}_{a,q}) = \frac{1}{q}.$$

We note that we have various kinds of densities [49]. The above definition and criterion and Theorem 4.2.11 imply the following result:

Theorem 4.2.17. *Suppose that $f(n)$ is an integral valued arithmetical function, and $A \subset \mathbb{N}$ with $\mathbf{d}(A) > 0$. Also, suppose that for $\Re(s) > 1$ and $|z| \le 1$ we have*

$$\sum_{n\in A} \frac{z^{f(n)}}{n^s} = \mathcal{G}(s, z)(s-1)^{\alpha-1-\alpha z} + \mathcal{H}(s, z), \tag{4.2.11}$$

where α is a real positive number and, for $|z| \le 1$, the functions $\mathcal{G}(s, z)$ and $\mathcal{H}(s, z)$ are regular in s for s belonging to a certain domain Δ which contains the closed half plane $\Re(s) \ge 1$. Then, for any irrational number λ, the numbers $\{\lambda f(n)\}_{n\in A}$ are uniformly distributed modulo 1.

The paper [12] contains the proof of the validity of the relation (4.2.11) in some cases. Indeed, based on the results of [12], we have the following results:

Theorem 4.2.18. *If $f \in \mathcal{D}$, then for any irrational number λ, the numbers $\{\lambda f(n)\}_{n\in\mathcal{S}}$ are uniformly distributed modulo 1.*

Theorem 4.2.19. *If $f(n) = \omega(n)$ or $\Omega(n)$, then for any irrational number λ, the numbers $\{\lambda f(n)\}_{n\in\mathbb{N}_{a,q}}$ and the numbers $\{\lambda f(n)\}_{n\in\mathcal{S}\cap\mathbb{N}_{a,q}}$ are uniformly distributed modulo 1, provided* $\gcd(a, q) = 1$.

Remark 4.2.20. Using arguments similar to those of paper [12] sections 3.10, 3.10.1, 3.10.3, 3.10.4 and 3.10.5, we can see that the truth of this theorem still holds when $\gcd(a, q) \in \mathcal{S}$.

4.3 Geometry of Weyl sums

In the year 1981, F. Dekking and M. Mendès France [8] introduced an idea of making visible the Weyl sums $\sum_{n\leq N} e(ha_n)$ for a given real sequence a_n and given positive integer h. Indeed, for given $h, N \in \mathbb{N}$ they draw in $\mathbb{R}^2$ a plane curve generated by successively connected lines segment, which joint the point V_n to V_{n+1} with

$$V_n = \left(\sum_{k=1}^{n} \cos(2\pi h a_k), \sum_{k=1}^{n} \sin(2\pi h a_k)\right),$$

for $1 \leq n \leq N$.

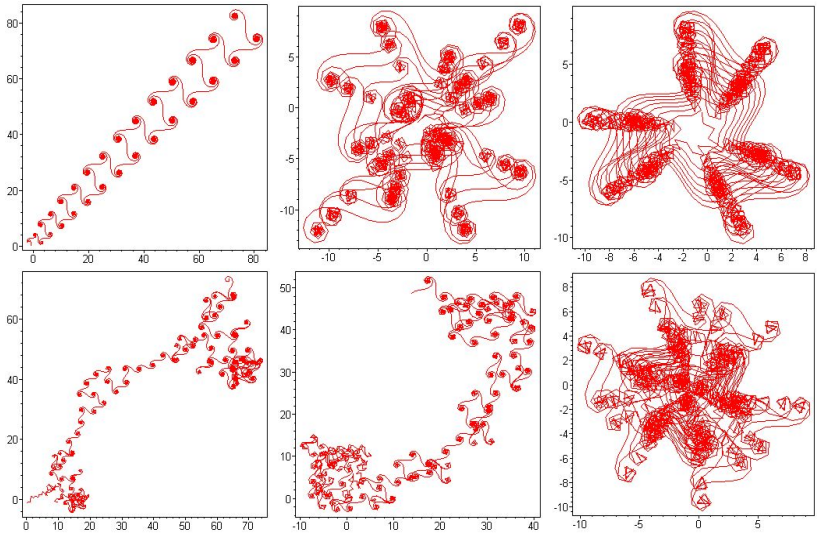


Fig. 4.4: Graph of the Weyl sums $\sum_{n\leq N} e(h(\frac{4}{9}n^{\frac{3}{2}}))$ with $N = 2000$, from left to right respectively for $h = 1, 2, 3$ (top row) and $h = 4, 5, 6$ (down row).

As above figure shows, for various values of h we have various graphs. Usually, we take $h = 1$. See [8], [14] for more classical graphs.

Here, we suggest a three dimensional version of geometric view of Weyl sums, which generalizes the two dimensional case. For given $h, N \in \mathbb{N}$ we consider in $\mathbb{R}^3$ the space curve generated by successively connected lines segment, which joint the point V_n to V_{n+1} with

$$V_n = \left(\sum_{k=1}^{n} \cos(2\pi h a_k), \sum_{k=1}^{n} \sin(2\pi h a_k), n\right),$$

for $1 \leq n \leq N$. Below, we reproduce some classical and new examples with a 3-d view of their graphs. Our 3-d graphs and their level-colors detect more details of the behavior of Weyl sums.

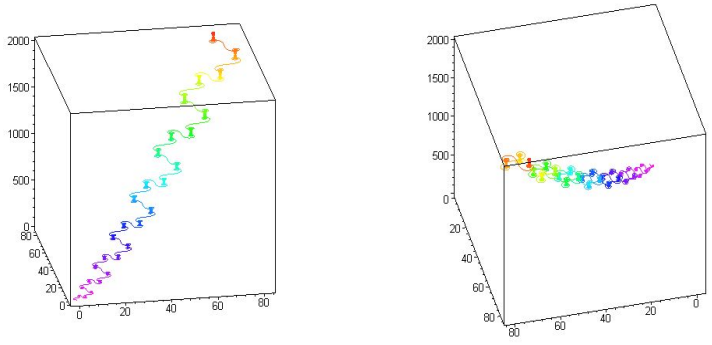


Fig. 4.5: 3-d graphs of the Weyl sum $\sum_{n\leq N} e(\frac{4}{9}n^{\frac{3}{2}})$ with $N = 2000$.

Example 4.3.1. We consider the sequence

$$a_n = \alpha n.$$

Corresponding points $V_n \in \mathbb{R}^2$ lie on a circle with radius $1/(2|\sin(\pi\alpha)|)$ and center $\big(-1/2, (\cot(\pi\alpha))/2\big)$. For irrational values of α, the graph of Weyl sum of a_n is dense in an annulus with raduses $|\cot(\pi\alpha)|/2$ and $1/(2|\sin(\pi\alpha)|)$.

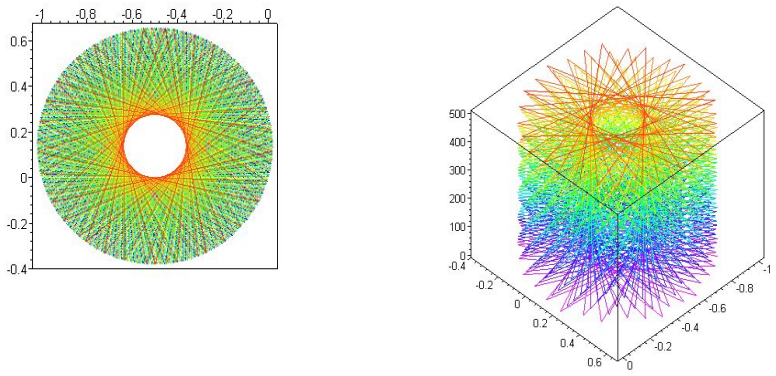


Fig. 4.6: 2-d and 3-d graphs of the Weyl sum $\sum_{n\leq N} e(\sqrt{2}n)$ with $N = 500$.

Example 4.3.2. Figures below show the graph of Weyl sum $\sum_{n\leq N} e(n\log n)$. To interpret the spiral appearance, we note that because of the weak growth of $\log n$, the curve behaves locally like the curve associated with the linear sequence $a_n = c_H n$ where c_H is a local constant with $c_H \approx \log H \pmod 1$ for $n \approx H$. Thus, the curve of $a_n = n\log n$ appears as a succession of annuli, joined by almost straight lines, corresponding to the values of H such that $c_H \approx 0 \pmod 1$. This will happen for the values of $H \approx e^m$ for some $m \in \mathbb{N}$ and $1 \leq m \leq \log N$. In the case of present example, this will happen at heights $H \approx e^m$ with $m = 1, 2, \cdots, 8$, and more visibly for $H \approx 403, 1100, 2980$, as the side view in the following figure shows.

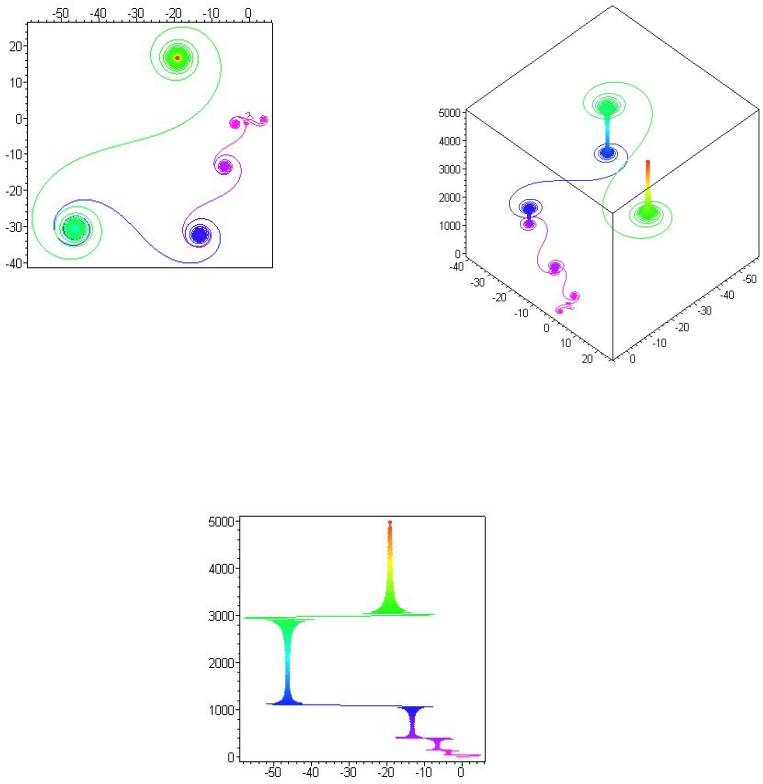


Fig. 4.7: 2-d and 3-d graphs of the Weyl sum $\sum_{n\leq N} e(n\log n)$ with $N = 5000$.

Example 4.3.3. The sequence of the form

$$a_n = \alpha n^2$$

is uniformly distributed modulo 1 if and only if α is irrational. Below we see the graph of the Weyl sums of this sequence for $\alpha = \pi$ and $\alpha = \frac{100}{10001}$.

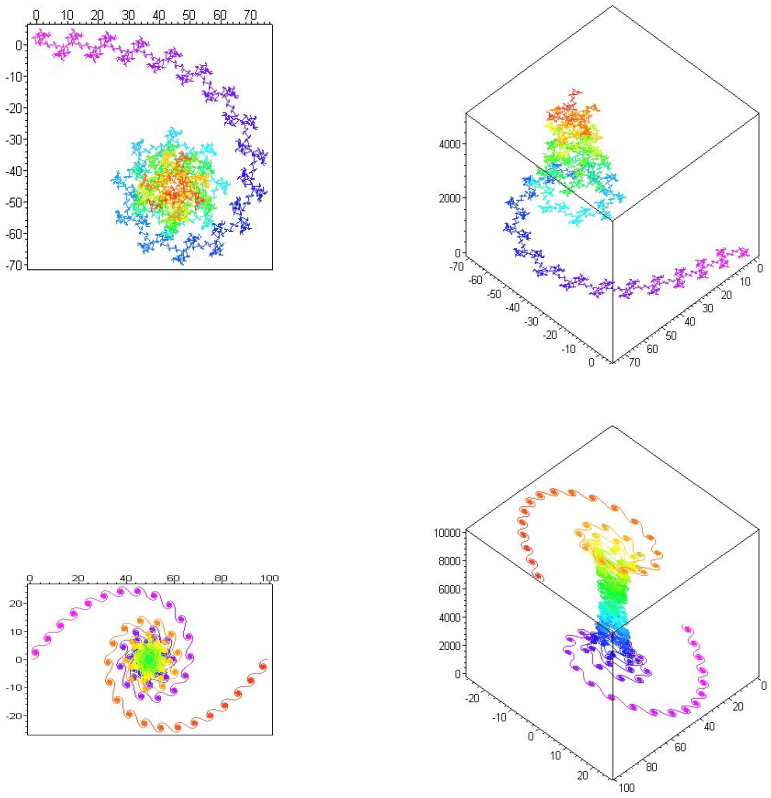


Fig. 4.8: 2-d and 3-d graphs of the Weyl sum $\sum_{n \leq N} e(a_n)$ with $a_n = \pi n^2, N = 5000$ (top) and $a_n = \frac{100}{10001} n^2, N = 10000$ (down).

Example 4.3.4. J-M. Deshouillers and H. Iwaniec [17] proved that the sequence

$$a_n = \frac{1}{n} \sum_{m \leq n} \varphi(m),$$

where φ is Euler function is uniformly distributed modulo 1. Their method can be used to obtain the same result for the sequence

$$a_n = \frac{1}{n} \sum_{m \leq n} \sigma(m).$$

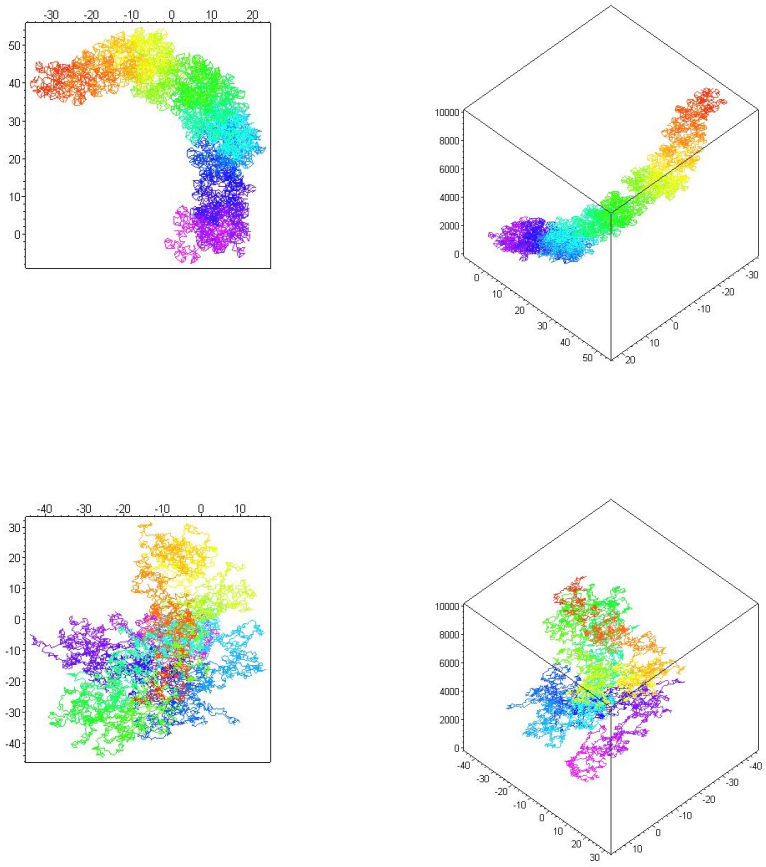


Fig. 4.9: 2-d and 3-d graphs of the Weyl sum $\sum_{n \leq N} e(a_n)$ with $a_n = \frac{1}{n} \sum_{m \leq n} \varphi(m), N = 10000$ (top) and $a_n = \frac{1}{n} \sum_{m \leq n} \sigma(m), N = 10000$ (down).

Example 4.3.5. As we proved in previous chapter, sequences with general terms

$$a_n = \sum_{m \le n} \frac{\varphi(m^2+1)}{m^2+1} \qquad \text{and} \qquad a_n = \sum_{m \le n} \frac{m^2+1}{\sigma(m^2+1)}$$

are dense modulo 1. Figures below show that it is likely that both sequences are uniformly distributed modulo 1.

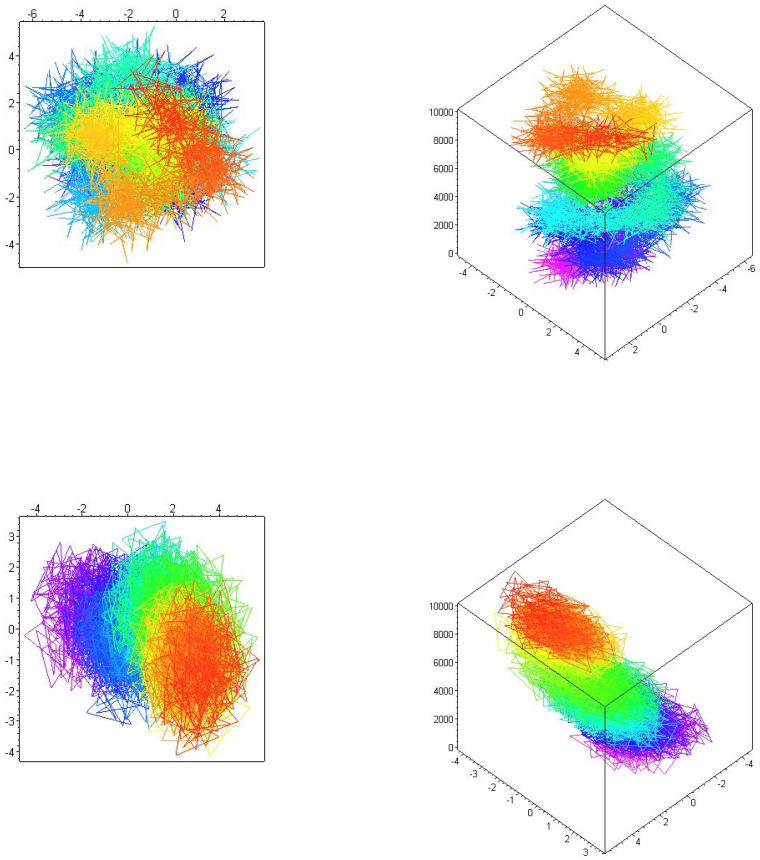


Fig. 4.10: 2-d and 3-d graphs of the Weyl sum $\sum_{n \le N} e(a_n)$ with $a_n = \sum_{m \le n} \frac{\varphi(m^2+1)}{m^2+1}$, $N = 10000$ (top) and $a_n = \sum_{m \le n} \frac{m^2+1}{\sigma(m^2+1)}$, $N = 10000$ (down).

APPENDIX

A. FREQUENTLY USED FORMULAS

A.1 Approximation formulas

We use the following *Taylor expansions*, frequently. All of them hold as $x \to 0$. For every $a \in \mathbb{R}$, we have

$$(1+x)^a = 1 + ax + O(x^2). \tag{A.1.1}$$

More precisely, when $a = \frac{1}{2}$, we have

$$\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2). \tag{A.1.2}$$

Considering the geometric expansion, we obtain

$$\frac{1}{1 \pm x} = 1 + O(x). \tag{A.1.3}$$

Also, we have

$$e^x = 1 + x + O(x^2), \tag{A.1.4}$$

and

$$\log(1 \pm x) = \pm x + O(x^2). \tag{A.1.5}$$

We use *Stirling formula* frequently, which asserts that

$$n! = \Big(\frac{n}{e}\Big)^n \sqrt{2\pi n}\Big(1 + O(\frac{1}{n})\Big), \tag{A.1.6}$$

where here and below $O(\ )$ refers to $n \to \infty$. Taking logarithm and simplifying, we get

$$\log(n!) = n \log n - n + O(\log n). \tag{A.1.7}$$

Also, considering (A.1.4) and (A.1.5), we obtain

$$\begin{aligned} en!^{\frac{1}{n}} &= n\Big(\sqrt{2\pi n}\big(1 + O(\frac{1}{n})\big)\Big)^{\frac{1}{n}} \\ &= n \exp\Big(\frac{1}{n} \log\Big(\sqrt{2\pi n}\big(1 + O(\frac{1}{n})\big)\Big)\Big) \\ &= n\left(1 + \frac{1}{n} \log\Big(\sqrt{2\pi n}\big(1 + O(\frac{1}{n})\big)\Big) + O\Big(\frac{\log^2 n}{n^2}\Big)\right) \\ &= n + \log\sqrt{2\pi n} + O\Big(\frac{\log^2 n}{n}\Big). \end{aligned}$$

So, we have

$$n!^{\frac{1}{n}} = \frac{n}{e} + \frac{1}{e}\log\sqrt{2\pi n} + O\Big(\frac{\log^2 n}{n}\Big). \tag{A.1.8}$$

A.2 Summation formulas

Reducing a Riemann-Stieljes integral to a finite sum

Theorem A.2.1. *Let α be a step function defined on $[a,b]$ with jump α_k at x_k, where $a \le x_1 < x_2 < \cdots < x_n \le b$. Let f be defined on $[a,b]$ in such a way that not both f and α are discontinuous from the right or from the left t each x_k. Then $\int_a^b f d\alpha$ exists and we have*

$$\int_a^b f(x)d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha_k.$$

Corollary A.2.2. *Using integrating by parts, we have*

$$\sum_{k=1}^{n} f(x_k)\alpha_k = \int_a^b f(x)d\alpha(x) = \int_a^b \alpha(x)\frac{d}{dx}(-f(x))dx + f(b)\alpha(b) - f(a)\alpha(a).$$

Corollary A.2.3. *For the sequence a_k, let $f(x) = a_k$ if $k-1 < x \le k$ with $f(0) = 0$. Then*

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x)d\lfloor x\rfloor.$$

Partial summation formula

Partial summation formula is very useful in our computations. Let $\{a_n\}$ be a sequence of complex numbers, and for $t > 0$ set

$$A(t) = \sum_{n\le t} a_n.$$

Also, let $b(n)$ be a continuously differentiable function on the interval $[1,x]$. Then

$$\sum_{1\le n\le x} a_n b(n) = A(x)b(x) - \int_1^x A(t)b'(t)dt.$$

Corollary A.2.4. *With above assumptions, let $b(n)$ be a continuously differentiable function on $[1,\infty)$. Then, using of above formula twice we obtain*

$$\sum_{n>y} a_n b(n) = \lim_{x\to\infty} A(x)b(x) - A(y)b(y) - \int_y^\infty A(t)b'(t)dt.$$

Example A.2.5. Suppose that u_n is a real sequence and let

$$U(x) = \sum_{1\le n\le x} (1+u_n),$$

and $A(t) = U(e^t)$. Then, setting $a_n = 1+u_n$ and $b(n) = n^{-s}$, we have

$$\sum_{1\le n\le x} \frac{1+u_n}{n^s} = \frac{U(x)}{x^s} + s\int_1^x \frac{U(z)}{z^{s+1}}dz.$$

Applying the change of variable $t = \log z$, we get

$$\sum_{1\le n\le x} \frac{1+u_n}{n^s} = \frac{U(x)}{x^s} + s\int_0^{\log x} e^{-st}A(t)dt.$$

Euler-Maclaurin summation formula

Euler-Maclaurin summation formula is another useful tool in our approximations. Suppose that $f(x)$ has $2m$ continuous derivatives in $[a,b]$, for some positive integer m and integers a and b. Let $B_n(x)$ denotes the Bernoulli polynomials defined by

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!},$$

and let B_n are the Bernoulli numbers, defined by $B_n = B_n(0)$, so that $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \cdots$. Then

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\Big(f^{(2r-1)}(b) - f^{(2r-1)}(a)\Big) + \frac{1}{2}\Big(f(a)+f(b)\Big) + R_m,$$

where

$$R_m = -\int_a^b g^{(2m)}(x)\frac{B_{2m}(x-\lfloor x\rfloor)}{(2m)!}dx.$$

Note that

$$|R_m| \le \int_a^b |g^{(2m)}(x)|\frac{|B_{2m}(x-\lfloor x\rfloor)|}{(2m)!}dx,$$

and since $|B_{2m}(x-\lfloor x\rfloor)| \le |B_{2m}|$, we have the following explicit bound

$$|R_m| \le \frac{|B_{2m}|}{(2m)!}\int_a^b |g^{(2m)}(x)|dx.$$

Example A.2.6. For every $D \in \mathbb{N}$ we have

$$\sum_{d>D} \frac{1}{d^2} = \frac{1}{D} - \frac{1}{2D^2} + \frac{c_D}{3D^3},$$

for some real c_D in $[0,1]$.

Splitting the sum $\sum_{n\le X} e(a_n+b_n)$

To apply Weyl criterion some splitting formulas for the sum $\sum_{n\le X} e(a_n+b_n)$ are required. Here we study two kinds of them.

1. Assume that a_n and b_n are two real sequences. Then we have

$$\begin{aligned}\Big|\sum e(a_n+b_n)-\sum e(a_n)\Big| &= \Big|\sum e(a_n)(e(b_n)-1)\Big| \\ &\le \sum \big|e(a_n)(e(b_n)-1)\big| = \sum \big|e(b_n)-1\big|.\end{aligned}$$

But

$$\big|e(b_n)-1\big| = \big|\cos(2\pi b_n)-1+i\sin(2\pi b_n)\big| = 2\big|\sin(\pi b_n)\big|.$$

Since, $|\sin(x)|\le|x|$ holds for every $x\in\mathbb{R}$, we obtain

$$\Big|\sum e(a_n+b_n)-\sum e(a_n)\Big| \le \sum 2\big|\sin(\pi b_n)\big| \le 2\pi\sum\big|b_n\big|.$$

Corollary A.2.7. *For all real sequences a_n and b_n we have*

$$\sum_{n\le X} e(a_n+b_n) = \sum_{n\le X} e(a_n) + O\Big(\sum_{n\le X}|b_n|\Big),$$

as X tends to infinity.

2. We use partial summation formula to obtain the second splitting formula of summations concerning $e(a_n)$. As above, assume that a_n and b_n are two real sequences. Let $A(X):=\sum_{n\le X} e(a_n)$ and $b(n)=b_n$. Then, we have

$$\begin{aligned}\sum_{n\le X} e(a_n+b_n)-\sum_{n\le X} e(a_n) &= \sum_{n\le X} e(a_n)(e(b_n)-1) \\ &= A(X)\big(e\big(b(X)\big)-1\big) - 2\pi i\int_1^X A(t)b'(t)e(b(t))dt.\end{aligned}$$

Thus, we obtain

$$\sum_{n\le X} e(a_n+b_n) = A(X)e(b(X)) - 2\pi i\int_1^X A(t)b'(t)e(b(t))dt,$$

and consequently

$$\Big|\sum_{n\le X} e(a_n+b_n)\Big| \le |A(X)| + 2\pi\int_1^X |A(t)b'(t)|dt.$$

Corollary A.2.8. *Keep above notations, and suppose that a_n is u.d. mod 1, and the sequence b_n satisfies $A(t)b'(t)=o(1)$. Then the sequence a_n+b_n is u.d. mod 1, too.*

BIBLIOGRAPHY

[1] Tom M. Apostol, *Mathematical Analysis*, Addison-Wesley Publishing Company, Inc. 1957.

[2] F. Behrend, Über numeri abundantes. I, II, *Preuss. Akad. Wiss. Sitrzungsber*, (1932), 322-328 and (1933), 280-293.

[3] Manjul Bhargava, Generalized Factorials and Fixed Divisors over Subsets of a Dedekind Domain, *J. Number Theory*, **72** (1998) 67-75.

[4] S. Chowla, On abundant numbers, *J. Indian Math. Soc.*, n. Ser. **1**, 41-44 (1934).

[5] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, On the Lambert W function, *Adv. Comput. Math.* 5 (1996), no. 4, 329-359.

[6] Harold Davenport, *Multiplicative number theory* (Third edition: Revised and with a preface by Hugh L. Montgomery), Graduate Texts in Mathematics, Vol. 74, Springer-Verlag, New York, 2000.

[7] Harold Davenport, Über numeri abundantes, *Preuss. Akad. Wiss. Sitrzungsber*, (1934), 830-834.

[8] F. Dekking and M. Mendès France, Uniform distribution modulo one: a geometrical viewpoint, *Journal für die reine und angewandte Mathematik*, **329** (1981), 143-153.

[9] Hubert Delange, Colloque sur la théorie des nombres, Bruxelles, 1955, Thone, Liége, 1956, pp. 147-161.

[10] Hubert Delange, On some arithmetical functions, *Illinois J. Math.*, **2**, 1958, 81-87.

[11] Hubert Delange, On the distribution modulo 1 of additive functions, *Journal of the Indian Math. Soc.*, **34**, 1970, 215-235.

[12] Hubert Delange, Sur la distribution des entiers ayant certaines propriétés, *Ann. Sci. École Norm. Sup.*, (3) **73** (1956), 15-74.

[13] M. Deléglise, Bounds for the density of abundant integers, *Experiment. Math.*, **7** (1998), no. 2, 137-143.

[14] Jean-Marc Deshouillers, Geometric aspect of Weyl sums, *Elementary and Analytic Theory of Numbers, Banach Center Publications*, Vol. 17, (Warsaw 1985), 75-82.

[15] Jean-Marc Deshouillers, Sur la croissance certaines fonctions de répartition, *Compositio Mathematica*, **31**, No. 3 (1975), 259-284.

[16] Jean-Marc Deshouillers and Mehdi Hassani, Distribution modulo 1 of a linear sequence associated to a multiplicative function evaluated at polynomial arguments, *SCIENCE CHINA Mathematics*, **53**, No. 9 (2010), 2203-2206.

[17] Jean-Marc Deshouillers and Henryk Iwaniec, On the distribution modulo one of the mean values of some arithmetical functions, *Uniform Distribution Theory*, **3** (2008), No.1, 111-125.

[18] Jean-Marc Deshouillers and Florian Luca, On the distribution of some means concerning the Euler function, *Functiones et Approximatio*, *XXXIX*, (2008), 11-20.

[19] P. D. T. A. Elliott, On the limiting distribution of $f(p+1)$ for non-negative additive functions, *Acta Arithmetica*, **25** (1974), 259-264.[1]

[20] Peter D.T.A. Elliott, *Probabilistic Number Theory I: Mean-Value Theorems*, Grundlehren der mathematischen Wissenschaften 239 (A Series of Comprehensive Studies in Mathematics), Springer - Verlag, 1979.

[21] Peter D.T.A. Elliott, *Probabilistic Number Theory II: Centeral Limit Theorems*, Grundlehren der mathematischen Wissenschaften 240 (A Series of Comprehensive Studies in Mathematics), Springer - Verlag, 1980.

[22] P. Erdös, On the density of some sequences of numbers. III, *J. London Math. Soc.*, **13** (1938), 119-127.

[23] P. Erdös, On the distribution of the numbers of the form $\sigma(n)/n$ and on some related questions, *Pasific J. Math.*, Vol. 52, No. 1, 1974, 59-65.

[24] Paul Erdös and Harold N. Shapiro, On the changes of sign of a certain error function, *Canad. J. Math.* **3** (1951), 375-384.

[25] Paul Erdös and Paul Turán, On a problem in the theory of uniform distribution. II, *Nederl. Akad. Wetensch., Proc.* **51**, (1948) 1262-1269, *Indagationes Math.* **10**, 406-413 (1948).

[26] P. Erdös and A. Wintner, Additive arithmetical functions and statistical independence, *Amer. J. Math.*, **61** (1939), 713-721.

[27] A. Granville and Z. Rudnick (Editors), *Equidistribution in Number Theory, An Introduction*, (Proceedings of the NATO Advanced Study Institute on Equidistribution in Number Theory Montreal, Canada, 11-22 July 2005), Springer, 2007.

[1] This paper was received on 16-6-1972.

[28] T.H. Gronwall, Some asymptotic expressions in the theory of numbers, *Trans. Amer. Math. Soc.*, **14** (1913), 113-122.

[29] G. Halász, On the distribution of additive and the mean values of multiplicative arithmetic functions, *Studia Scient. Math. Hung.*, **6**, 211-233.

[30] G. Halász, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hungar.*, **19** (3-4), 1968, 365-403.

[31] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, 1974.

[32] Adolf Hildebrand, Additive and multiplicative functions on shifted primes, *Proc. London Math. Soc.*, (3) **59** (1989), No. 2, 209-232.

[33] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS, 2004.

[34] I. Kátai, On distribution of arithmetical functions on the set prime plus one, *Compositio Mathematica*, **19**, No. 4 (1968), 278-289.

[35] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Dover Publication Inc., Mineola, New York, 2006.

[36] E. Landau, Über die Zahlentheoretische Function $\varphi(n)$ und ihre Beziehung zum Goldbachschen Satz., *Nachr. Königlichen Ges. Wiss. Göttingen, Math.-Phys. Klasse*, Göttingen, 1900, 177-186.

[37] Serge Lang, *Algebra*, Addison-Wesley, 1965.

[38] J. Martinet, J-M. Deshouillers, H. Cohen, La fonction somme des diviseurs, *Séminaire de Théorie des Nombres, 1972-1973 (Univ. Bordeaux I, Talence)*, Exp. No. 11, 10 pp. Lab. Théorie des Nombres, Centre Nat. Recherche Sci., Talence, 1973.

[39] H.L. Montgomery, Primes in arithmetic progressions, *Mich. Math. J.*, **17** (1970), 33-39.

[40] K.K. Norton, Upper bounds for sums of powers of divisor functions, *J. Number Theory*, **40**, 60-85 (1992).

[41] G. Pólya, Über ganzwertige ganze Funktionen, *Rend. Circ. Mat. Palermo*, **40** (1915) 1-16.

[42] Tanguy Rivoal, La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs (There are infinitely many irrational values of the Riemann zeta function at odd integers), *C. R. Acad. Sci. Paris Sér. I Math.* **331** (2000), no. 4, 267-270.

[43] J. Barkley Rosser and Lowell Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.*, **6** (1962), 64-94.

[44] I. J. Schoenberg, On Asymptotic Distributions of Arithmetical Functions, *Transactions of the American Mathematical Society*, Vol. 39, No. 2 (1936) 315-330.[2]

[45] I. Schoenberg, Über die asymptotische verteilung reeller Zahlen mod 1, *Mathematische Zeitschrift*, **28** (1928), 171-200.[3]

[46] Atle Selberg, Note on a paper by L. G. Sathe, *J. Indian Math. Soc.*, Vol. 18 (1954), pp. 83-87.

[47] Jean-Pierre Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics, Vol 7, Springer-Verlag, 1973.

[48] Harold N. Shapiro, *Introduction to the Theory of Numbers*, Dover Publication Inc., Mineola, New York, 2008.

[49] Gérald Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics **46**, Cambridge University Press, Cambridge, 1995.

[50] E. C. Titchmarsh, *The Theory of Functions*, Oxford Science Publications, 1997.

[51] Arnold Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte, XV. VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[52] Charles R. Wall, Phillip L. Crews and Donald B. Johnson, Density Bounds for the Sum of Divisors Function, *Math. Comp.*, **26** (1972), 773-777.

[53] Hermann Weyl, Über die Gleichverteilung von Zahlen mod. Eins, *Mathematische Annalen*, **77** (1916), 313-352.[4]

[54] David Vernon Widder, *The Laplace Transform*, Princeton University Press, 1946.

[2] This paper was presented to the Society on 31 March 1934.
[3] This paper was received on 7-6-1926.
[4] This paper was received on 13-6-1914.

Résumé

Cette thèse est consacrée à l'étude de plusieurs aspects de la répartition des fonctions multiplicatives à valeurs dans l'intervalle $[0,1]$. L'archétype de ces fonctions est la fonction $\nu(n) = \varphi(n)/n$, où $\varphi(n)$ est la fonction d'Euler qui donne le cardinal de $(\mathbb{Z}/n\mathbb{Z})^*$.

On sait, depuis les travaux de Imre Kátai, que la suite $(\nu(p-1))_p$ indexée par les nombres premiers p, admet une fonction de répartition F_ν, c'est-à-dire que pour tout réel y la limite

$$F_\nu(y) \;=\; \lim_{x\to\infty} \frac{1}{x}\,\mathrm{Card}\,\{n \le x \,|\, \nu(n) \le y\}$$

existe. On sait en outre que cette fonction est croissante au sens large sur $\mathbb{R}$, vaut 0 sur $]-\infty,0]$, vaut 1 sur $[1/2,+\infty[$ et est continue et strictement croissante sur $[0,1/2]$; en outre, elle est purement singulière, c'est-à-dire qu'en presque tout point, au sens de la mesure de Lebesgue, la fonction F_ν est dérivable à dérivée nulle. Notre premier résultat est d'établir qu'en tout point $x_m = \nu(2m)$, la fonction F_ν n'est pas dérivable à gauche du point x_m. Ce résultat est obtenu par une méthode de moments.

Le second résultat principal concerne une généralisation d'un problème étudié récemment par Jean-Marc Deshouillers, Henryk Iwaniec et Florian Luca, à savoir la répartition modulo 1 de la suite à croissance linéaire $u_n = \sum_{1\le k\le n} \nu(k)$. On étudie ici la moyenne prise, non plus sur tous les entiers, mais sur une suite polynômiale; ici encore, nous regardons la situation "archétypale" où le polynôme considéré est le polynôme non linéaire le plus simple, à savoir $P(x) = x^2+1$. On pose $v_n = \sum_{1\le k\le n} \nu(k^2+1)$ et on montre que la suite $(v_n)_n$ est dense modulo 1.

Mots-clés. Fonction d'Euler, fonction somme des diviseurs, fonctions arithmétiques multiplicatives, fonctions arithmétiques additives, fonctions de répartition, répartition modulo 1, critère de Weyl, nombres premiers, densités, méthodes de cribles.

Abstract

In this thesis, we study some topics of the distribution of the values of arithmetical functions. We obtain two kinds of results. Our first result is about differentiability of the distribution function $F(x)$ defined by

$$F(x) = \lim_{N\to\infty} \frac{1}{\pi(N)}\mathrm{Card}\Big\{p \le N \Big| \frac{\varphi(p-1)}{p-1} \le x\Big\}.$$

We prove that at each point $x_m = \varphi(m)/m$, where m is an even integer, F is not differentiable from the left. For this purpose we use the method of moments.

The second result is about density modulo 1 of some sequences connected with the mean values of the ratio $r_n = \varphi(n^2+1)/(n^2+1)$. Among various results, we prove that the sequence $b_n = \sum_{m\le n} r_m$, as well as the sequence $a_n = \sum_{m\le n}(m^2+1)/\sigma(m^2+1)$, are dense modulo 1. Our proof is based on some sieve results which allow us to control the size of prime factors of numbers of the form n^2+1.

Key Words. Euler φ function, divisor σ function, multiplicative function, additive function, distribution function, density modulo 1, uniform distribution modulo 1, Weyl criterion, primes, density, sieve method.