COLLEGE DOCTORAL PAYS DE LA LOIRE — ED MATHEMATIQUES STIC

Nantes Université

THÈSE DE DOCTORAT DE

NANTES UNIVERSITÉ

ÉCOLE DOCTORALE N° 641
Mathématiques et Sciences et Technologies de l'Information et de la Communication
Spécialité : *Mathématiques*

Par

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Sur la koszulité de certaines propérades

Thèse présentée et soutenue à Nantes, le 04 septembre 2024
Unité de recherche : LMJL (UMR 6629)

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En mémoire de Zannie.

Remerciements

Durant ces quatre dernières années, de nombreuses personnes ont contribué, de près ou de loin, à l'avancement de cette thèse. Je souhaite donc naturellement commencer par les remercier.

Tout d'abord, un immense merci à mon directeur de thèse Friedrich Wagemann, pour ta sympathie et ta compréhension. Tu t'es toujours rendu disponible au maximum et tu as su me redonner confiance en moi dans des moments difficiles. À mon encadrant Salim Rivière pour ton temps et ta motivation à toute épreuve. Tu as réussi à dégager des créneaux régulièrement pour nos rendez-vous malgré ton emploi du temps de ministre, et tu as su me donner des coups de pression quand il y avait besoin. Également merci à mon troisième encadrant officieux, Johan Leray, pour ta présence qui m'a été précieuse pendant ces quatre années. Tu m'as appris les règles de LaTeX dans les petits détails, et surtout tu avais une compréhension des propérades qui nous a fait gagné beaucoup de temps. Les moments passés tous les trois/quatre à parler de maths étaient vraiment très productifs et très agréables.

Un immense merci également à Vladimir Dotsenko et Martin Markl pour avoir accepté de relire ma thèse et pour m'avoir donné beaucoup de conseils et corrections. Tout particulièrement Vladimir pour les longues discussions en présence ou par mail. Merci aussi à Éric Hoffbeck, Geoffrey Powell et Muriel Livernet d'avoir accepté de faire partie de mon jury de thèse, ainsi que pour les discussions mathématiques ou autres que nous avons pu avoir ensemble. Enfin, merci à Grégory Ginot et Geoffrey Powell d'avoir constitué mon CSI, et à qui j'ai pu présenter trois fois mes travaux de thèse, suite à quoi j'ai reçu de précieux conseils.

Après quatre années passées au LMJL, j'ai également pu faire de bonnes rencontres parmi les personnes du laboratoire. Tout d'abord je tiens à remercier le secrétariat du laboratoire : Anaïs, Béatrice, Caroline et Stéphanie, qui ont toujours été là pour mes questions, qui ont donné de leur temps pour m'expliquer les fonctionnements du laboratoire et des logiciels utilisés. Également merci au personnel informatique : Éric et Saïd, qui sont toujours là pour nous aider sur nos soucis. Merci aussi à Anh, Claude et Naïma pour la gestion de la bibliothèque. J'ai eu la chance de faire partie du comité parité du

LMJL, et donc merci aux personnes que j'ai rencontrées pendant ces réunions : Anne, Fred, Joe, Marianne et Samuel. Enfin, merci aux chercheurs et chercheuses du laboratoire : Hossein A. pour tes mails toujours plus originaux les uns que les autres. Erwan B., pour tes conseils et ton humour. Baptiste C., pour m'avoir fait découvrir Athénor. Nicolas D., pour ton humour et la gestion des services d'enseignement. Damien G., pour la gestion du module de maths et le travail acharné que tu y a mis, et pour ta bonne humeur. Marco G., en fait je ne sais pas trop si je dois te remercier ou te dénoncer pour les nombreuses menaces du regard que tu m'adressais en passant devant mon bureau. En tout cas tes problèmes de maths sont intéressants, même s'il est original de s'en servir pour s'excuser... François L., pour les discussions enrichissantes. Christoph S., pour tes conseils pour l'utilisation de Sagemath, Git, VSCode et Julia. Aymeric S., pour l'organisation de *Maths en Jeans* ainsi que du congrès. Merci également à Thomas Guyard, qui m'a donné énormément de conseils à mon arrivée, et qui m'a aidé à préparer mon atelier Athénor.

Lors de mon arrivée, plusieurs doctorants ont participé à mon intégration, je voulais donc les remercier : Adrian, pour les discussions super intéressantes, que ce soit sur des maths ou autre, et évidemment que tu étais le bienvenu dans mon bureau lorsque tu étais épuisé par l'écriture de ta thèse et que tu avais besoin de parler d'autre chose ! Antoine, pour l'accueil chez toi et pour ton humour. Anh, Maha et Zeinab pour votre accueil dans ce bureau, et pour vos nombreuses explications sur le fonctionnement du laboratoire et du monde de la recherche. Fabien, pour ta joie de vivre et pour m'avoir faire entrer dans le comité parité. Anthony, pour les discussions intéressantes et enrichissantes, et pour m'avoir conseillé une catsitteuse pour Rima. Flavien et Valentin, pour les parties de civ et les moments très agréables passés ensemble.

Bien évidemment, les doctorants que j'ai croisés m'ont aussi marqué et aidé pendant cette thèse, je les remercie donc également. Adrien, qui est toujours partant pour parler de maths, que ce soit de la recherche ou un problème (souvent posé par Marco). Alexandre et Clément, et plus tard Augustin, pour me supporter comme seul cobureau non Nouy-related, particulièrement Alexandre qui doit me supporter quand j'ai besoin de parler (parfois avec Clément, en plus...). Clovis, pour avoir partagé des discussions autour de nos sujets, ta curiosité mathématique m'impressionnera toujours. Damien P., je sais que tu oublies à chaque fois mais je te dois encore un service ! Elric, pour les suggestions et découvertes de jeux de société. Florent et Enzo, pour l'organisation de la murder, c'était incroyable ! Destin, pour ton humour à toute épreuve. Gaspard, pour partager

ma passion pour les jeux de société et m'en avoir fait découvrir plusieurs. Gurvan, pour avoir partagé avec moi de nombreuses galères administratives, à commencer par les heures d'enseignement et en finissant par les documents pour la thèse. Jean, pour le fait que tu ne sois jamais stressé (en tout cas j'ai l'impression), cela me permettait parfois de relativiser. Klervi, pour m'avoir accueilli chez toi plusieurs fois, et pour les longues discussions sur nos vies qui s'en suivaient (Et merci surtout à Méline, Poncho et Toupi, tu ne pensais tout de même pas que j'allais oublier !). Lucas M., pour les discussions intéressantes, et pour m'avoir presque motivé à retourner nager. Thomas et Ziyad, pour votre sens de l'humour et votre enthousiasme pour faire des jeux tous les midi, j'ai l'impression de vous avoir converti au hanabisme, et avoir découvert que ce culte peu parfois créer des tensions... Je voudrais également remercier un doctorant (presque) d'un autre laboratoire, Sinan, on passe toujours de bons moments en conférence ou lorsque tu viens à Nantes. Enfin, même si les interactions avec vous étaient plus rares, elles n'en étaient pas moins agréables : Aline, Bastien, Charbella, Khaled, Marie et Paul.

Je tiens aussi à remercier mes ami.e.s, qui m'ont accompagné lors de ces quatre années : Tout d'abord Éric, même si on ne se voit pas souvent, c'est toujours des moments superbes, merci d'être là. Killian, pour me rappeler à chaque fois à quel point je ne suis vraiment pas très bon à la nage. Mina, avec qui on peut se partager nos vies, nos soucis, mais aussi et surtout beaucoup rigoler. Élise, Marine Ter et Florian pour ces vancances à Europapark. Camille J., pour ta permanente bonne humeur et les soirées passées ensemble. Bénédicte, pour être une superbe amie, malgré le fait que l'on se voit rarement (genre une fois par an vers janvier je dirais). Louis R, qui partage mes passions et surtout avec qui j'ai passé 3 superbes années à l'ENS (enfin, on se comprend...). Étienne, pour ce super duo que l'on forme lorsqu'il s'agit de se détendre sur un jeu (arrête juste avec tes fortunes stp). Cédric, tu dors ? Timothée et Sidney, qui sont de formidables rencontres grâce aux jeux et à Louis. Gildas, pour la découverte de civ (c'est de ta faute tout ça) et pour ce super duo de tft. Tristan, (qui se rappelle de Tristan ?). Erwan, on ne s'est peut-être pas beaucoup rencontré mais tu restes un amis sur lequel je peux compter. Tout le groupe de l'ENS : Alain, Émilie, François, Pierre, Rémi, Sarah, Théo, Thomas, etc., j'ai été ravi de faire partie de cette promo et de vous avoir rencontrés. Lisa, évidemment pour ce chouchou, que je porte encore aujourd'hui (enfin, j'espère ne pas l'avoir oublié tout de même !). Élisa (et Luccio !) et François, pour ces nombreux fous rires. Youyou et Toitoi (ou l'inverse ?), pour les découvertes et soirées jeux, gros bisous d'amour Youyou. Lucie, même si Alice te préfère à moi, tu restes quelqu'un que je peux supporter. Valentine, mon contact fréquent, pour tes visites occasionnelles mais toujours super agréables (il faudra que tu me montres ta machine à arrêter le temps par contre). Benjamin le propriétaire pragmatique, ce n'est a priori pas qu'une qualité, mais tu en as tellement d'autres. Élora, pour ces fous rires, et pour les parties de belote. Bichon, pour être un très bon partenaire de belote (et plus récemment de lol apparemment). Léo, pour ta gentillesse (quoique je ne reçois pas les mails de Rennes donc je ne peux pas assurer cela). Anaïs, on s'est déjà vus quelque part non ? Je ne sais plus. Mes voisins préférés, pour les merveilleux moments passés ensemble. Vous êtes sûrs de vouloir rester à Rennes ?

Évidemment je remercie également la famille Bouillet, qui m'a accueilli à bras ouverts et qui forme maintenant comme une nouvelle famille pour moi. Mario, qui est une des personnes les plus accueillantes du monde (assieds toi avec nous stp), Pierrot, que j'aime un peu trop taquiner sur nos désaccords, Marie, ton enthousiasme me surprendra toujours, et Pierre, pour ta curiosité sur tous les sujets (de toute façon tu pourrais tout faire). Aussi merci à mes belles-nièces Ficelle, Madi et Taïga, c'est toujours un plaisir de vous voir et de vous voir grandir. Désolé plutôt que merci à mes adversaires Arnaud et Anne-So, je pense mener la course de beaucoup trop pour que vous puissiez me rattraper un jour (en plus aujourd'hui j'ai un cadeau pour Mario, et vous ?). Également merci à la famille un peu plus éloignée : Ywé, Camille et Chloé, c'est un immense plaisir de vous avoir rencontrés, et Mathilde avec qui je noue une relation particulière, n'en déplaise à certains.

Merci aussi à ma famille : Tata Guigui, pour ton enthousiasme à chaque bulletin de notes. Isa, pour les discussions très intéressantes passées ensembles. Tonton Loulou, pour être un tonton plutôt inattendu et très sympathique. Néna, tu débordes d'amour et de tendresse. Milou, pour ta gentillesse et tes attentions. Papa, tu m'as toujours laissé faire mes propres choix sans trop intervenir, tout en te réjouissant que je fasse des maths. Mimi, tu as toujours été un frère d'exception pour moi, un exemple jusque tard, et tu es probablement la raison principale pour laquelle j'aime autant les maths. Et enfin Rima, je n'arrive pas bien à voir si tu me soutiens ou non, j'imagine que tu préfèrerais qu'on passe plus de temps ensembles, mais peut-être comprendras-tu en grandissant.

Évidemment comment ne pas te remercier, Alice, pour m'avoir accompagné et soutenu pendant ces quatre ans, même dans les moments difficiles, je ne serais sûrement pas allé aussi loin sans toi. Je t'embrasse de tout mon cœur.

Enfin, j'aimerai remercier les personnes qui nous ont quittées pendant ces années, en particulier toi, qui nous a toujours aimés, toujours soutenus dans nos choix, même pendant tes difficultés. Tu as toujours été là pour moi. Tu me manques.

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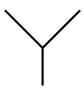
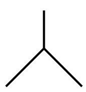
Introduction

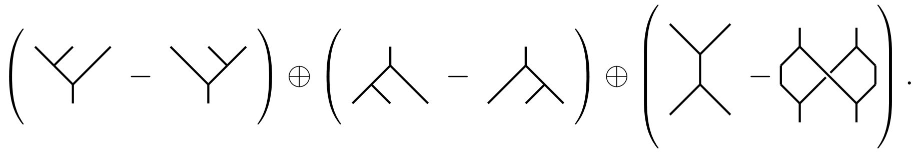
Contexte et motivations

Nous nous sommes intéressés dans cette thèse à certaines propérades et leur éventuelle koszulité ou koszulité à homotopie près. Les propérades sont des objets algébriques permettant d'encoder des structures algébriques de type bigèbres, là où les opérades ne permettent que d'encoder des algèbres. Le principal exemple de propérade est la propérade $\mathcal{B}$ encodant les bigèbres associatives. Ces dernières étant la structures de base des algèbres de Hopf, leur étude est naturellement importante. Cette propérade est notamment étudiée par M. Markl dans [Mar06], mais aussi par S. Merkulov et B. Vallette dans [MV09], ces derniers ayant montré qu'elle était Koszul à homotopie près.

D'autres exemples de propérades peuvent être cités, comme la propérade $\mathcal{B}i\mathcal{L}ie$ encodant les bigèbres de Lie, dont la koszulité a été montrée par B. Vallette dans [Val07, Corollaire 8.5], la propérade $\varepsilon\mathcal{B}$ encodant les bigèbres associatives infinitésimales, dont la koszulité a également été montrée par B. Vallette dans [Val07, Corollaire 8.5], ou encore la propérade $\mathcal{DP}ois$ encodant les bigèbres doubles Poissons, dont la koszulité a été montrée par J. Leray dans [Ler20, Théorème 5.11], utilisant la koszulité de la propérade $\mathcal{DL}ie$ encodant les bigèbres double Lie. Pour les deux premiers exemples, des équivalents diopéradiques existent, dont la koszulité a été démontré par W.L. Gan dans [Gan02, Corollaire+ 5.10].

Dans ce manuscrit, nous utiliserons la notation par graphes pour présenter des propérades. Par exemple pour les bigèbres associatives, nous définirons l'espace des générateurs

$E :=$  $\oplus$  , ainsi que l'espace des relations dans la propérade libre sur E, notée $\mathcal{F}(E)$, défini par

$R :=$ 

Enfin, nous définissons la propérade $\mathcal{B}$ par le quotient $\mathcal{F}(E)/(R)$, où (R) désigne l'idéal engendré par R. La koszulité d'une propérade, comme définie par B. Vallette dans [Val07], peut être vue comme la concentration de l'homologie de la construction bar de cette propérade en un certain sous-espace. Dans ce manuscrit, nous nous concentrerons sur une définition équivalente de la koszulité comme acyclicité d'un complexe. Nous démontrerons la koszulité ou la non koszulité de certaines propérades. Dans le premier cas, nous utiliserons une condition suffisante donnée par B. Vallette dans [Val07, Proposition 8.4]. Dans le second cas, nous utiliserons la définition équivalente de koszulité pour en déduire des conditions nécessaires de koszulité à mettre en défaut.

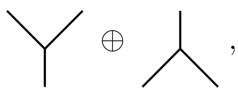
Le premier objectif de cette thèse était d'étudier les rack bigèbres, une structure permettant de généraliser les algèbres de Hopf au même titre que les racks généralisent les groupes et que les algèbres de Leibniz généralisent les algèbres de Lie. Les racks bigèbres sont notamment étudiées par C. Alexandre, M. Bordemann, S. Rivière et F. Wagemann dans [Ale+18b] et [Ale+18a]. Le but était d'étudier les propriétés de la propérade encodant cette structure afin d'éventuellement en déduire un modèle minimal, pour enfin en déduire la cohomologie des rack bigèbres.

Le second objectif était d'étudier une certaine famille de propérades afin de déduire si, dans un cas restreint, nous avions des relations entre confluence et koszulité. En effet, dans un cadre plus restreint comme celui des opérades, la confluence d'un système induit par la présentation d'une opérade implique sa koszulité grâce à une structure appelée opérade shuffle, que V. Dotsenko et A. Khoroshkin ont défini dans [DK10], inspirés par le critère PBW de la koszulité démontré par É. Hoffbeck dans [Hof08]. Cependant, dans le cadre des propérades, nous n'avons pas de structure similaire aidant à prouver que la confluence d'un système impliquerait la koszulité d'une propérade. C'est pourquoi nous voulions étudier dans un premier temps une famille de propérades, contenant plusieurs propérades connues comme étant Koszul, afin de déterminer lesquelles étaient Koszul ou non.

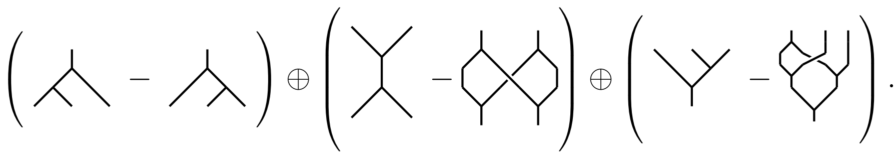
Les deux premiers chapitres servant à rappeler les bases respectivement de la théorie des représentations et des propérades, les deux suivants contiennent le travail fait sur ces deux objectifs. Dans les deux premiers chapitres, nous pouvons aussi trouver quelques résultats intermédiaires qui servent pour les deux suivants, mais qui avaient plutôt leurs places dans ces chapitres.

Résultats du troisième chapitre

Les rack bigèbres étant assez proches des bigèbres associatives dans leurs générateurs et relations, il était assez naturel de regarder ce qui avait déjà été fait par le passé sur la propérade $\mathcal{B}$ afin de s'en inspirer. Nous souhaitions aussi dans un premier temps ne regarder que la propérade des rack bigèbres non unitaires non counitaires, en effet cela simplifiait grandement l'étude des ces structures, et il est plutôt commun d'ignorer, au moins dans un premier temps, les unités et counités d'une structure lorsque l'on l'étudie via des opérades ou propérades. La propérade $\mathcal{R}ack\mathcal{B}$ encodant les rack bigèbres (non counitaires non unitaires) est encore une fois définie par les générateurs

$E :=$ 

avec les relations données par

$R :=$ 

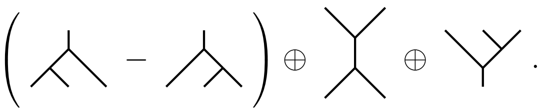
Les trois conditions à vérifier pour qu'une propérade $\mathcal{P} = \mathcal{F}(E)/(R)$ soit Koszul à homotopie près sont :

(i) la propérade quadratique $\mathcal{P}_2$ associée à $\mathcal{P}$ est Koszul,

(ii) les $\mathbb{S}$-bimodules $\mathcal{P}$ et $\mathcal{P}_2$ sont isomorphes,

(iii) il existe une graduation supplémentaire sur $\mathcal{P} = \bigoplus_\lambda \mathcal{P}_\lambda$ telle que $\mathcal{P}_\lambda$ est de dimension finie pour tout λ.

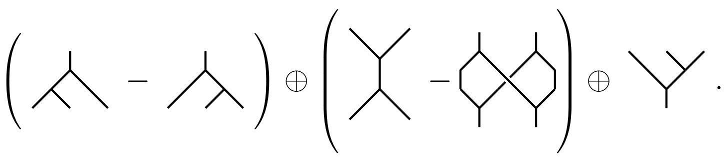
Le but était donc de s'inspirer de la démonstration de S. Merkulov et B. Vallette de koszulité à homotopie près de $\mathcal{B}$ dans [MV09]. Ainsi nous devions étudier dans un premier temps la propérade quadratique associée à $\mathcal{R}ack\mathcal{B}$, notée $\mathcal{R}ack\mathcal{B}_2$ et donnée par les générateurs E et les relations

$R_2 :=$ 

Grâce à un calcul de la propérade libre sur E donné en fin de Chapitre 2 et le critère de koszulité donné dans [Val07, Proposition 8.4], on démontre le théorème suivant (cf Théorème 3.2).

Théorème A. *La propérade $\mathcal{R}ack\mathcal{B}_2$ est Koszul.*

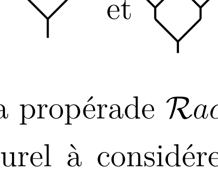
La deuxième étape était donc de montrer que les $\mathbb{S}$-bimodules $\mathcal{R}ack\mathcal{B}$ et $\mathcal{R}ack\mathcal{B}_2$ étaient isomorphes entre eux. Pour cela, l'idée était de s'inspirer de la démonstration faite dans le cas de $\mathcal{B}$ par B. Enriquez et P. Etingof dans [EE05, Proposition 6.2]. Nous voulions d'abord adapter cette démonstration dans le cas d'une propérade intermédiaire, en ajoutant dans un premier temps seulement la relation de bigèbres, c'est à dire pour la propérade $\mathcal{R}ack\mathcal{B}_1$ donnée par les générateurs E et les relations

$R_1 :=$ 

La démonstration de B. Enriquez et P. Etingof s'adaptant bien à ce cas, et la propérade $\mathcal{R}ack\mathcal{B}_1$ étant graduée par le nombre de chemins, cela a donné le résultat suivant (cf Théorème 3.6).

Théorème B. *La propérade $\mathcal{R}ack\mathcal{B}_1$ est Koszul à homotopie près.*

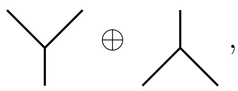
Néanmoins, la propérade $\mathcal{R}ack\mathcal{B}$ présente un double problème dans sa dernière relation. Le premier étant que dans la propérade $\mathcal{R}ack\mathcal{B}_2$, les opérations formelles



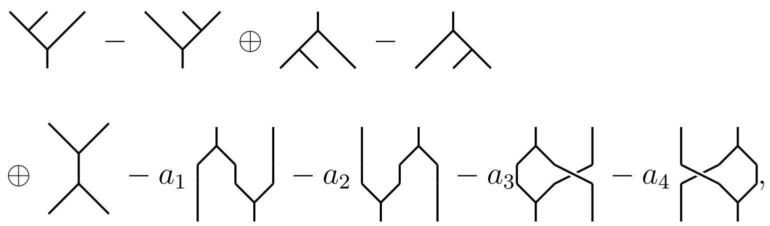
sont toutes deux nulles, mais dans la propérade $\mathcal{R}ack\mathcal{B}$ elles sont simplement identifiées. Ainsi le morphisme qui semble naturel à considérer, et qui est considéré dans les cas de $\mathcal{B}$ et $\mathcal{R}ack\mathcal{B}_1$, ne peut pas être bijectif. De plus, la dernière condition pour qu'une propérade soit Koszul à homotopie près ne peut pas être vérifiée aussi facilement que pour $\mathcal{B}$ et $\mathcal{R}ack\mathcal{B}_1$ car la dernière relation ne conserve pas le nombre de chemins. Ces deux difficultés posent la question de la koszulité à homotopie près de $\mathcal{R}ack\mathcal{B}$, et même de l'existence d'un modèle minimal pour cette propérade.

Résultats du quatrième chapitre

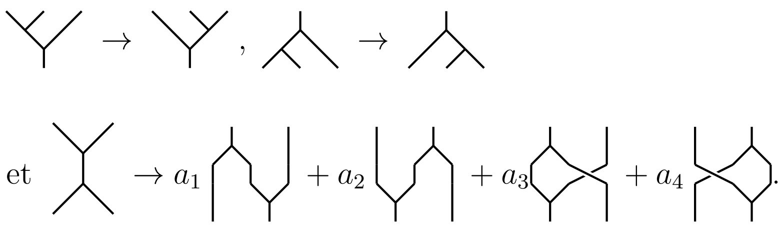
Dans ce dernier chapitre, nous nous intéressons à une famille de propérades à paramètres. Le but était de déterminer si la confluence d'un système induit par ces propérades (en poids 3) était équivalente à leur koszulité. La famille est donnée par quatre paramètres $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$, l'espace des générateurs

$E :=$ 

et l'idéal des relations R_a donné par

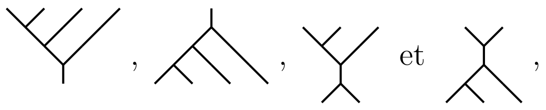


où on note $\between_a$ la dernière relation. On pose alors $\mathcal{P}_a = \mathcal{F}(E)/R_a$. Nous disons ici que $\mathcal{P}_a$ induit un système confluent si le système de réécriture donné par $\mathcal{F}^{(3)}(E)$ et les règles de réécriture induites par

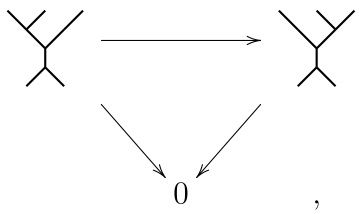


est confluent.

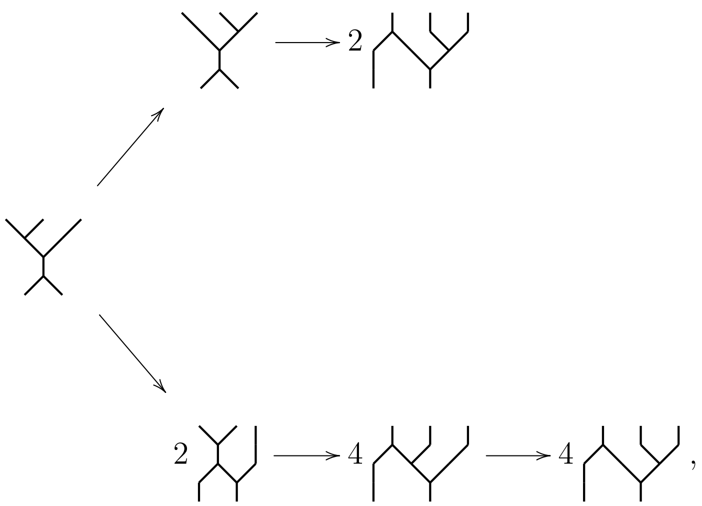
Les monômes critiques de ce système sont donnés par les graphes



mais on sait déjà que les deux premiers ne nous poseront pas de problème car ils sont indépendants de a et leur confluence est connue. Le dernier étant le symétrique horizontal du troisième, nous pouvons déterminer les relations que doit vérifier a pour que le dernier nous donne confluence on fonction de celles données par le troisième. Nous nous intéresserons donc seulement au troisième. Par exemple, pour les paramètres $a = 0$, nous avons le système de réécriture



ainsi, nous dirons dans ce cas que $\mathcal{P}_a$ induit un système confluent. Comme contre-exemple, pour le paramètre $a = (2, 0, 0, 0)$, nous avons le système de réécriture



donc dans ce cas, nous dirons que $\mathcal{P}_a$ induit un système non-confluent. Une fois trouvées les conditions sur a sous lesquelles $\mathcal{P}_a$ induit un système confluent, le but était de déterminer si ces propérades étaient Koszul, et si les autres ne l'étaient pas.

Comme étape intermédiaire, nous avons défini le morphisme $\varphi_a : \mathcal{A} \boxtimes \mathcal{C} \longrightarrow \mathcal{P}_a$ donné par l'inclusion des propérades $\mathcal{A}$ et $\mathcal{C}$ dans $\mathcal{P}_a$, encodant respectivement les algèbres associatives et les cogèbres coassociatives, suivi du produit de composition dans $\mathcal{P}_a$. En effet, d'après [Val07, Proposition 8.4], si ce morphisme est un isomorphisme en poids 3, alors $\mathcal{P}_a$ est Koszul. Nous avons donc énoncé la conjecture suivante (cf Conjecture 4.2).

Conjecture C. *Soit $a \in \mathbb{C}^4$, les assertions suivantes sont équivalentes :*

(i) La propérade $\mathcal{P}_a$ induit un système confluent.

(ii) Le morphisme φ_a est une bijection en poids 3.

(iii) La propérade $\mathcal{P}_a$ est Koszul.

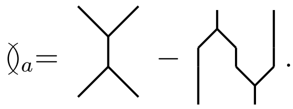
C'est ici qu'intervient le code `SageMath` qui accompagnera la plupart des résultats à suivre (voir [Néd] et Appendice A). Ce code a pour but de générer une propérade libre sur un nombre fini de générateurs, comme par exemple E, en générant poids par poids toute les manières d'ajouter un générateur aux éléments déjà générés. Une fois cette propérade libre générée, nous pouvons ensuite générer une famille génératrice d'un idéal dans cette propérade libre, et ainsi déterminer sa dimension, et donc la dimension de n'importe quelle propérade définie par générateurs et relations en un poids et biarité donnés. Grâce à ce code, nous avons pu déterminer le théorème suivant (cf Théorème 4.10).

Théorème D. *Soit $a \in \mathbb{C}^4$, les assertions suivantes sont équivalentes :*

(i) $\mathcal{P}_a$ induit un système confluent.

(ii) φ_a est un isomorphisme de $\mathbb{S}$-bimodules en poids 3.

De plus, en étudiant les propérades qui induisent un système confluent à isomorphisme près, nous pouvons remarquer qu'il y en a seulement trois : $\varepsilon\mathcal{B}$ étudiée par B. Vallette dans [Val07], la propérade $\frac{1}{2}\mathcal{B}$ étudiée par M. Markl et A. A. Voronov dans [MV10], et une dernière propérade donnée par



Pour étudier la conjecture, il suffit alors d'étudier les propérades qui induisent un système non-confluent et de déterminer s'il en existe une qui soit de Koszul ou non. Dans ce but, nous avons étudié le complexe de Koszul $\mathcal{P}_a \boxtimes \mathcal{P}_a^{\text{i}}$ afin d'étudier si la non confluence du système induit par $\mathcal{P}_a$ suffisait à prouver que ce complexe n'était pas acyclique.

M. Bremner, S. Madariaga et L. Peresi ont introduit dans [BMP16] une méthode "diviser pour mieux régner" afin d'étudier des représentations de groupes symétriques définies par des relations en étudiant chacune de leurs composantes isotypiques. Cette méthode a été illustrée par M. Bremner et V. Dotsenko dans [BD17] pour déterminer quelles étaient les opérades d'une famille à paramètres régulières. Nous avons donc étendu cette méthode aux représentations de produits de groupes symétriques afin de l'utiliser dans la famille de propérades à paramètres que nous étudiions. Le code `SageMath` a donc été étendu afin de pouvoir étudier des propérades cette fois comme représentations de groupes symétriques, et de pouvoir calculer des bases de Gröbner d'idéaux déterminantaux de matrices. On peut résumer la méthode utilisée en la suite d'implications suivante.

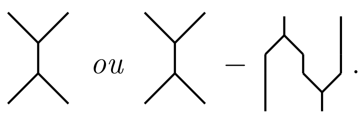
$$\begin{aligned}
\mathcal{P}_a \text{ est Koszul } &\Longrightarrow \mathcal{P} \boxtimes \mathcal{P}_a^{\text{i}} \text{ est acyclique } \Longrightarrow (\mathcal{P} \boxtimes \mathcal{P}_a^{\text{i}})^{(4)}(2,4) \text{ est acyclique} \\
&\Longrightarrow \text{ pour tout } (\lambda, \mu) \vdash (2,4), \text{ la composante isotypique} \\
&\qquad [(\mathcal{P} \boxtimes \mathcal{P}_a^{\text{i}})^{(4)}(2,4)]_{(\lambda,\mu)} \text{ est acyclique} \\
&\Longrightarrow \text{ pour tout } (\lambda, \mu) \vdash (2,4), \text{ la caractéristique d'Euler de} \\
&\qquad [(\mathcal{P} \boxtimes \mathcal{P}_a^{\text{i}})^{(4)}(2,4)]_{(\lambda,\mu)} \text{ est nulle.}
\end{aligned}$$

Malheureusement, cette suite d'implications semble trop forte pour prouver la conjecture. Nous avons tout de même pu énoncer des théorèmes partiels comme le suivant (cf Théorème 4.14).

Théorème E. *Pour $a = (a_1, 0, a_3, 0)$, les assertions suivantes sont équivalentes :*

(i) La propérade $\mathcal{P}_a$ est Koszul.

(ii) La relation $\between_a$ est donnée par l'une des formules suivantes, à isomorphisme près :



Afin de potentiellement pouvoir montrer toute la conjecture, plusieurs pistes ont commencées à être explorées, comme regarder le complexe de Koszul en d'autres biarités, regarder plus en détails les matrices qui donnent les multiplicités du complexe pour déterminer leur rang de manière plus précise, ou encore chercher directement un cycle dans le complexe qui ne serait pas un bord, notamment à partir de la non confluence. Mais pour le moment, aucune de ces directions n'ont abouti.

Conventions

Vector spaces. In this manuscript, every vector space will be over $\mathbb{C}$. Every result is also true in any algebraically closed field of characteristic zero. For V and W two vector spaces, if a $\mathbb{C}$-algebra R acts on V on the right and on W on the left, we denote by $V \otimes_R W$ the tensor product of V and W over R. We denote by $V \otimes W$ the tensor product of V and W over $\mathbb{C}$.

Symmetric groups. We denote by $\mathbb{S}_n$ the symmetric group over the set $\{1, \ldots, n\}$ and by $[\sigma(1), \ldots, \sigma(n)]$ any permutation $\sigma \in \mathbb{S}_n$, or by cycles with parentheses. For example, in $\mathbb{S}_4$, we have $[2, 3, 1, 4] = (1, 2, 3)$, or (123) if there is no ambiguity.

Groups. All groups considered will be written multiplicatively. For G a group, we denote by G^{op} the opposite group defined by the set G and the product $a \star b := ba$ for $a, b \in G$.

Tuples. For $(i_1, \ldots, i_n)$ an n-tuple of natural numbers, we will often denote by $\bar{i} := (i_1, \ldots, i_n)$. We denote by $\left|\bar{i}\right| := i_1 + \cdots + i_n$.

Partitions. For an integer $n \in \mathbb{N}$, we denote by $\lambda \vdash n$ any partition of n, that is any non increasing tuple $\bar{i} = (i_1, \ldots, i_k)$ such that $\left|\bar{i}\right| = n$, up to adding or removing zeros.

Chapter 1

Representation theory of products of symmetric groups

A properad being, in particular, an $\mathbb{S}$-bimodule, it is natural to consider it as a collection of representations of products of symmetric groups. In this chapter we will remind a few properties and theorems that can be used to study properads as representations. The advantage of this tool is to divide a big problem into smaller problems on every isotypic components, see [BD17, Section 5]. We have to introduce some notations and definitions to extend the case of $\mathbb{S}$-modules to the case of $\mathbb{S}$-bimodules, see Chapter 2 for more details on $\mathbb{S}$-bimodules.

First we give some definitions and notations around representations, and present the isotypic decomposition of the regular representation of the symmetric group $\mathbb{S}_n$. Then we present the two main tools we will use, Pieri's formula and representation matrices. The first one is a consequence of the Littlewood–Richardson rule, which gives a way to merge two irreducible representations of $\mathbb{S}_n$ and $\mathbb{S}_m$ into a representation of $\mathbb{S}_{n+m}$, giving its decomposition into isotypic components. The second one allows us to divide a question of dimensions into smaller questions of multiplicities. More details and definitions about representations of symmetric groups can be found in [FH13] and [Sag91].

1.1 Definitions and notations

Here we remind the basic definitions and notations we will use around representations. First of all, let us remind the definition of a representation.

Definition 1.1. Let G be a group, a *(left) representation* of G is a vector space V with a homomorphism $G \to \mathrm{GL}(V)$. We also say that V is a (left) G-module.

A *right representation* of G is a vector space W with a homomorphism $G^{\mathrm{op}} \to \mathrm{GL}(W)$. We also say that W is a right G-module.

For V a left G-module, we denote by V^{op} the corresponding right G^{op}-module. For W a right G-module, we denote by W^{op} the corresponding left G^{op}-module. This gives an equivalence between the categories of right and left G-modules.

Notation 1.2. (i) For every group G, we denote by $\mathbb{C}[G]$ the *regular representation* of G. As a vector space, $\mathbb{C}[G]$ has the elements of G as a basis and the left (resp. right) action of G on the basis is given by multiplication on the left (resp. right).

(ii) For H a subgroup of G and a left (resp. right) representation V of G, we have the structure of a left (resp. right) H-module on V. We denote this representation by ${}^{G}_{H}{\downarrow} V$ (resp. $V \downarrow^{G}_{H}$) and call it the *restricted representation* of V on H.

(iii) Moreover, for a left (resp. right) representation W of H, we have the structure of a left (resp. right) representation of G on $\mathbb{C}[G] \downarrow^{G}_{H} \otimes_H W$ (resp. $W \otimes_H {}^{G}_{H}{\downarrow} \mathbb{C}[G]$). We call this representation the *induced representation* of W on G and denote it by ${}^{G}_{H}{\uparrow} W$ (resp. $W \uparrow^{G}_{H}$).

(iv) For $n, m \in \mathbb{N}$ and two left (resp. right) representations V and W respectively of $\mathbb{S}_n$ and $\mathbb{S}_m$, we denote by $V \sqcup_l W$ (resp. $V \sqcup_r W$) the left (resp. right) representation of $\mathbb{S}_{n+m}$ given by $V \sqcup_l W := {}^{\mathbb{S}_{n+m}}_{\mathbb{S}_n \times \mathbb{S}_m}{\uparrow} (V \otimes W)$ (resp. $V \sqcup_r W := (V \otimes W) \uparrow^{\mathbb{S}_{n+m}}_{\mathbb{S}_n \times \mathbb{S}_m}$).

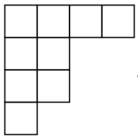
(v) For a left $\mathbb{S}_m$-module V and a right $\mathbb{S}_n$-module W, we denote by $V \boxtimes W$ the representation of $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$ given by the vector space $V \otimes W$ with the left action by $\mathbb{S}_m$ on V and the right action by $\mathbb{S}_n$ on W.

1.2 Young diagrams and isotypic decomposition of the symmetric group

Young diagrams and tableaux are a powerful tool to study representations of symmetric groups. Such diagrams correspond to irreducible representations of the corresponding symmetric group, and tableaux give the dimensions of these irreducible representations, and their multiplicities in the regular representation. Let us first remind the definitions of these tools. See for example [FH13, Lecture 4] or [Sag91, Chapter 2] for more informations on Young diagrams and tableaux.

To any partition $\lambda \vdash n$, we associate a Young diagram. For example, for the partition

$\lambda = (4, 2, 2, 1) \vdash 9$, the Young diagram associated to λ is



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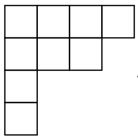
From now on, we will often denote a partition by its Young diagram. For any partition λ, we denote by λ' its conjugate partition, that is the partition obtained by taking the diagonal symmetry of the Young diagram. More precisely, for a partition $\lambda \vdash n$, we have $\lambda'_i = \mathrm{Card}\{j \mid \lambda_j \geq i\}$, and we get $\lambda' \vdash n$. For the partition above, we get $\lambda' = (4, 3, 1, 1)$ and the Young diagram



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Let $\lambda \vdash n$ and $\mu \vdash m$. We write $\lambda \leq \mu$ if $n \leq m$ and the Young diagram of λ is included in that of μ. In other words, $\lambda \leq \mu$ if and only if we can add boxes to the Young diagram of λ to get the Young diagram of μ.

Definition 1.3. A *tableau* is a Young diagram of a partition $\lambda \vdash n$ with the boxes numbered from 1 to n. We say that a tableau in *standard* if every row is increasing from left to right and every column is increasing from top to bottom.

Example 1.4. For $\lambda = (2, 1, 1) \vdash 4$, the standard tableaux of shape λ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

From now on, all vector spaces are considered of finite dimension. Most of the results below are true for any dimensions, but we will only consider finite dimensional representations in the next chapters. The following theorem may be the most important of this chapter, Pieri's formula and representation matrices make sense because of this result.

Theorem 1.5 ([FH13, Theorem 4.3])**.** *To any partition $\lambda \vdash n$, one can construct an irreducible representation V_λ of $\mathbb{S}_n$. Moreover, every irreducible representation of $\mathbb{S}_n$ is isomorphic to V_λ for some $\lambda \vdash n$.*

Examples 1.6. The irreducible representation $V_{(n)}$ is the trivial representation, and

$V_{(1^n)} =: \mathrm{sgn}_n$ is the signature representation. We have, for $\lambda \vdash n$,

$$V_\lambda \otimes V_{(1^n)} \simeq V_{\lambda'}.$$

Corollary 1.7. *Every representation V of $\mathbb{S}_n$ can be decomposed as follows :*

$$V = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus m_\lambda(V)},$$

where V_λ is given by Theorem 1.5 and $m_\lambda(V) \in \mathbb{N}$ is called the multiplicity of V_λ in V. We denote by $[V]_\lambda := V_\lambda^{\oplus m_\lambda(V)}$ the isotypic component of V along λ.

The following result is also very important when one wants to study the structure of a representation comparing it to the regular representation. For example, one can ask for an operad if it is regular, in the sense that in every arity it is given by the regular representation. This question is solved for a family of operads in [BD17].

Theorem 1.8 ([BMP16, Part 1])**.** *The isotypic decomposition into irreducibles of the regular representation $\mathbb{C}[\mathbb{S}_n]$ is given by*

$$\mathbb{C}[\mathbb{S}_n] = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus d_\lambda}$$

where V_λ is given by Theorem 1.5 and $d_\lambda := \dim(V_\lambda)$. The multiplicity d_λ is also the number of standard tableaux of shape λ.

Example 1.9. For $n = 3$, we have the partitions $\lambda_1 = (1, 1, 1)$, $\lambda_2 = (2, 1)$ and $\lambda_3 = (3)$. We have the standard tableaux

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ ,\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \text{ and }\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}.$$

Thus the decomposition of $\mathbb{C}[\mathbb{S}_3]$ is given by

$$\mathbb{C}[\mathbb{S}_3] = V_{\lambda_1} \oplus V_{\lambda_2}^{\oplus 2} \oplus V_{\lambda_3}.$$

1.3 Pieri's rules

In this section we remind general results on representations. Note that we can state these results for left or right modules, but in this section we consider only left modules.

Let V and W be respectively representations of $\mathbb{S}_n$ and $\mathbb{S}_m$. The following rule is very useful if one wants to understand the representation structure of $V \sqcup_l W$. If we know the isotypic decompositions of V and W, the Littlewood–Richardson numbers give us the decomposition of $V \sqcup_l W$. The proof of this rule can be found in [Sag91, Theorem 4.9.4].

Definition 1.10 (See [FH13, Appendix A] and [Sag91, Theorem 4.9.4])**.** Let $\lambda \vdash n$, $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash m$ and $\nu \vdash m+n$ be three partitions. We call Littlewood–Richardson tableau of type $(\lambda, \mu; \nu)$ any Young diagram of shape λ, completed into a Young diagram of shape ν by boxes filled by μ_1 1s, μ_2 2s, ..., μ_k ks such that each row is non decreasing and each column is strictly increasing, and such that if one takes the sequence formed by these numbers listed from right to left starting from the first row to the last one, one has at any point more ps than $p+1$s for $1 \leq p \leq k-1$.

Theorem 1.11 (Littlewood–Richardson's rule)**.** *For $n, m \in \mathbb{N}$, $\lambda \vdash n$ and $\mu \vdash m$, we have*

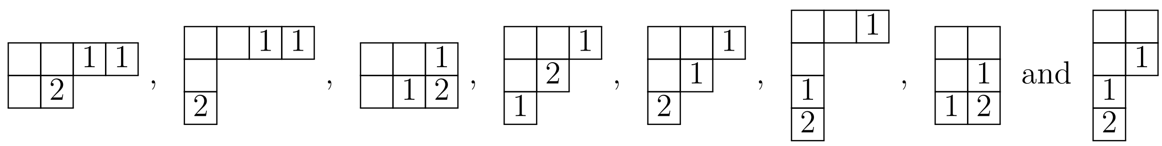
$$V_\lambda \sqcup_l V_\mu = \bigoplus_{\nu \vdash n+m} V_\nu^{\oplus N_{\lambda\mu}^\nu},$$

where the $N_{\lambda\mu}^\nu$ are the Littlewood–Richardson numbers, which are the number of Littlewood–Richardson tableaux of type $(\lambda, \mu; \nu)$.

Example 1.12. Littlewood–Richardson's rule allows us for example to compute

$$V_{(2,1)} \sqcup_l V_{(2,1)} = V_{(4,2)} \oplus V_{(4,1,1)} \oplus V_{(3,3)} \oplus V_{(3,2,1)}^{\oplus 2} \oplus V_{(3,1,1,1)} \oplus V_{(2,2,2)} \oplus V_{(2,2,1,1)}.$$

In fact, the corresponding Littlewood–Richardson tableaux are



Consequences of the Littlewood–Richardson rule are Pieri's rule and its dual form.

Theorem 1.13 (Pieri's rules)**.** *For $n, m \in \mathbb{N}$ and $\lambda \vdash n$, we have*

$$V_\lambda \sqcup_l V_{(m)} = \bigoplus_{\substack{\mu \vdash n+m \\ \mu \geq^c \lambda}} V_\mu,$$

the sum over the partitions $\mu \vdash n+m$ which can be obtained from λ by adding boxes to *its Young diagram, maximum one by column. We also have*

$$V_\lambda \sqcup_l V_{(1^m)} = \bigoplus_{\substack{\mu \vdash n+m \\ \mu \geq^l \lambda}} V_\mu,$$

the sum over the partitions $\nu \vdash n+m$ which can be obtained from λ by adding boxes to its Young diagram, maximum one by row.

From these rules one can deduce two corollaries, the first one allows us to compute the isotypic decomposition of any $\mathbb{S}_n$-module as an $\mathbb{S}_{n+1}$-module.

Corollary 1.14. *For $n \in \mathbb{N}$, $\lambda \vdash n$ and $\mu \vdash n+1$, we have*

$$m_\mu({}^{\mathbb{S}_{n+1}}_{\mathbb{S}_n}\uparrow (V_\lambda)) = \begin{cases} 1 & \text{if } \mu \geq \lambda \\ 0 & \text{else.} \end{cases}$$

Proof. Pieri's rule gives us

$${}^{\mathbb{S}_{n+1}}_{\mathbb{S}_n}\uparrow V_\lambda = {}^{\mathbb{S}_{n+1}}_{\mathbb{S}_n}\uparrow (V_\lambda \otimes \mathbb{C}) = V_\lambda \sqcup_l V_{(1)} = \bigoplus_{\substack{\mu \vdash n+1 \\ \mu \geq^c \lambda}} V_\mu$$

the sum over partitions $\mu \vdash n+1$ which can be obtained from λ by adding one box, that is $\mu \geq \lambda$.

□

And the second one allows us to compute any $\mathbb{S}_n$-module as an $\mathbb{S}_{n-1}$-module.

Corollary 1.15. *For $n \in \mathbb{N}$, $\lambda \vdash n$ and $\mu \vdash n-1$, we have*

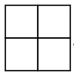
$$m_\mu({}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}}\downarrow (V_\lambda)) = \begin{cases} 1 & \text{if } \mu \leq \lambda \\ 0 & \text{else.} \end{cases}$$

Proof. This result comes from Frobenius reciprocity, see [FH13, Corollary 3.20], and Corollary 1.14. One consequence of Frobenius reciprocity is, for $\lambda \vdash n$ and $\mu \vdash n-1$,

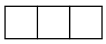
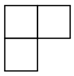
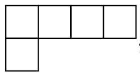
$$m_\mu({}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}}\downarrow (V_\lambda)) = m_\lambda({}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}}\uparrow (V_\mu)) = \begin{cases} 1 & \text{if } \mu \leq \lambda \\ 0 & \text{else} \end{cases}$$

□

Examples 1.16. For λ given by the Young diagram

,

and μ_1, μ_2 and μ_3 given respectively by the diagrams

 ,  and ,

we have

$$m_{\mu_1}((V_\lambda)\downarrow^{\mathbb{S}_4}_{\mathbb{S}_3}) = 0, m_{\mu_2}((V_\lambda)\downarrow^{\mathbb{S}_4}_{\mathbb{S}_3}) = 1 \text{ and } m_{\mu_3}((V_\lambda)\uparrow^{\mathbb{S}_5}_{\mathbb{S}_4}) = 0$$

1.4 Representation matrices

The properads we will study are defined by generators and relations. On the one hand we study the free properad over some generators by straight forward calculation, either by hand or by computer calculation. The space of relations, on the other hand, is harder to compute, in fact we can get a generating family with the same method, but computing the dimension leads to huge matrices that take too much time to study.

In the case of operads, in [BD17, Section 5], M. Bremner and V. Dotsenko present a method to "divide and conquer" the problem. The idea is to consider the space of relations in some arity n not as a vector space but as a representation of $\mathbb{S}_n$. Thus instead of a huge matrix corresponding to the entire space, they compute smaller representation matrices that correspond to each isotypic component. They used an algorithm due to J. M. Clifton in [Cli81] to compute matrices from permutations, that can be used to get multiplicities of representations of symmetric groups.

However, as we will see in Chapter 2, our space of relations in biarity (m, n) is a representation of $\mathbb{S}_m \times \mathbb{S}_n^{\text{op}}$, thus we must adapt this method to this case.

1.4.1 Representation matrices for $\mathbb{S}_n$-modules

Let $n \in \mathbb{N}$ and d_λ be the dimension of the irreducible representation V_λ for every $\lambda \vdash n$. We have an isomorphism of algebras

$$\varphi : \mathbb{C}[\mathbb{S}_n] \to \bigoplus_{\lambda \vdash n} \mathcal{M}_{d_\lambda}(\mathbb{C}),$$

where $\mathcal{M}_m(\mathbb{C})$ is the space of matrices of dimension $m \times m$ over $\mathbb{C}$, with module structure induced by φ. See for example [BMP16, Part 1] for the construction of this morphism. For every $\lambda \vdash n$, we denote by P_λ the composition of φ with the projection on $\mathcal{M}_{d_\lambda}(\mathbb{C})$, we have that P_λ is an isomorphism from $V_\lambda^{\oplus d_\lambda}$ to $\mathcal{M}_{d_\lambda}(\mathbb{C})$.

Let $\mathcal{P}$ be an operad given by generators E and relations $R : \mathcal{P} = \mathcal{F}(E)/(R)$. Suppose, for given $n, q \in \mathbb{N}$, that $\mathcal{F}(E)(n)$ is q copies of the regular representation, that is

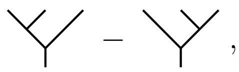
$$\mathcal{F}(E)(n) = \mathbb{C}[\mathbb{S}_n]^{\oplus q} = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus q d_\lambda},$$

and that $R(n)$ is generated, as an $\mathbb{S}_n$-module, by a finite number of elements in $\mathcal{F}(E)(n)$, say $x_1, \ldots, x_p$.

Example 1.17. This case corresponds, for example, to the case where $\mathcal{F}(E)(n)$ is given by q different trees, and R has p consequences in arity n. For example when

$$E = E(2) = \mathbb{C}[\mathbb{S}_2],$$

we get $\mathcal{F}(E)(4) = \mathbb{C}[\mathbb{S}_4]^{\oplus 5}$, and if R is generated by



$R(4)$ is generated, as an $\mathbb{S}_4$-module, by 5 elements.

The goal here is to compute the multiplicity $m_\lambda(R(n))$ of V_λ in $R(n)$ for any $\lambda \vdash n$. We will prove the following.

Proposition 1.18. *Let $R(n)$ be the $\mathbb{S}_n$-module generated by $x_1, \ldots, x_p$ in $\mathbb{C}[\mathbb{S}_n]^{\oplus q}$, and for every $1 \leq i \leq p$, $x_i = (x_i^{(1)}, \ldots, x_i^{(q)})$ their decompositions along $\mathbb{C}[\mathbb{S}_n]^{\oplus q}$. Then we have, for every $\lambda \vdash n$,*

$$m_\lambda(R(n)) = \operatorname{rk}\left(\left(P_\lambda(x_i^{(j)})\right)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}\right)$$

Moreover, we call the block matrix $\left(P_\lambda(x_i^{(j)})\right)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}$ the representation matrix of $R(n)$ for λ.

To prove this, we will need a lemma.

Lemma 1.19. *Let* $n \in \mathbb{N}$, $A_{i,j} \in \mathcal{M}_n(\mathbb{C}) =: \mathcal{M}$ *for every* $1 \leq i \leq p$ *and* $1 \leq j \leq q$. *We have*

$$\dim(\mathcal{M} \cdot (A_{1,1}, \ldots, A_{1,q}) + \cdots + \mathcal{M} \cdot (A_{p,1}, \ldots, A_{p,q})) = \operatorname{rk}\left((A_{i,j})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \right) \cdot n,$$

where $\mathcal{M} \cdot (A_{i,1}, \ldots, A_{i,q}) := \operatorname{Span}\{(NA_{i,1}, \ldots, NA_{i,q}), N \in \mathcal{M}\}$.

Proof. We denote by V the space

$$V := \mathcal{M} \cdot (A_{1,1}, \ldots, A_{1,q}) + \cdots + \mathcal{M} \cdot (A_{p,1}, \ldots, A_{p,q}).$$

Let us first remark that we have an isomorphism

$$\begin{aligned} \mathcal{M}^{\oplus q} &\simeq \mathcal{M}_{n,qn}(\mathbb{C}) \\ (A_1, \ldots, A_q) &\mapsto (A_1 | \cdots | A_q). \end{aligned}$$

Let us fix $1 \leq i \leq p$, we have

$$\mathcal{M} \cdot (A_{i,1}, \ldots, A_{i,q}) \simeq \operatorname{Span}\{(NA_{i,1} | \cdots | NA_{i,q}), N \in \mathcal{M}\}.$$

We now denote for $1 \leq j \leq q$ the rows of $A_{i,j}$ by $L_{1,j}, \ldots, L_{n,j}$:

$$A_{i,j} = \begin{pmatrix} L_{1,j} \\ \hline \vdots \\ \hline L_{n,j} \end{pmatrix}.$$

Thus the vector space $\mathcal{M} \cdot (A_{i,1}, \ldots, A_{i,q})$ is isomorphic to

$$\operatorname{Span}\left\{ \left(\begin{array}{c} N_{1,1}L_{1,1} + \cdots + N_{1,n}L_{n,1} \\ \hline \vdots \\ \hline N_{n,1}L_{1,1} + \cdots + N_{n,n}L_{n,1} \end{array} \middle| \cdots \middle| \begin{array}{c} N_{1,1}L_{1,q} + \cdots + N_{1,n}L_{n,q} \\ \hline \vdots \\ \hline N_{n,1}L_{1,q} + \cdots + N_{n,n}L_{n,q} \end{array} \right), N \in \mathcal{M} \right\}.$$

If we denote the rows of $(A_{i,1}|\cdots|A_{i,q})$ by $L_1^{(i)},\ldots,L_n^{(i)}$:

$$(A_{i,1}|\cdots|A_{i,q}) = \begin{pmatrix} L_1^{(i)} \\ \hline \vdots \\ \hline L_n^{(i)} \end{pmatrix},$$

we get

$$\begin{aligned}\mathcal{M}\cdot(A_{i,1},\ldots,A_{i,q}) &\simeq \mathrm{Span}\left\{\begin{pmatrix} N_{1,1}L_1^{(i)}+\cdots+N_{1,n}L_n^{(i)} \\ \hline \vdots \\ \hline N_{n,1}L_1^{(i)}+\cdots+N_{n,n}L_n^{(i)} \end{pmatrix}, N\in\mathcal{M}\right\} \\ &= \begin{pmatrix} \mathrm{Span}(L_1^{(i)},\ldots,L_n^{(i)}) \\ \hline \vdots \\ \hline \mathrm{Span}(L_1^{(i)},\ldots,L_n^{(i)}) \end{pmatrix}.\end{aligned}$$

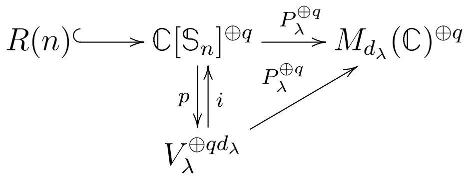
Finally, we have

$$\begin{aligned} V &= \begin{pmatrix} \mathrm{Span}(L_1^{(1)},\ldots,L_n^{(1)}) \\ \hline \vdots \\ \hline \mathrm{Span}(L_1^{(1)},\ldots,L_n^{(1)}) \end{pmatrix} + \cdots + \begin{pmatrix} \mathrm{Span}(L_1^{(p)},\ldots,L_n^{(p)}) \\ \hline \vdots \\ \hline \mathrm{Span}(L_1^{(p)},\ldots,L_n^{(p)}) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Span}(L_1^{(1)},\ldots,L_n^{(1)},\ldots,L_1^{(p)},\ldots,L_n^{(p)}) \\ \hline \vdots \\ \hline \mathrm{Span}(L_1^{(1)},\ldots,L_n^{(1)},\ldots,L_1^{(p)},\ldots,L_n^{(p)}) \end{pmatrix}.\end{aligned}$$

But the dimension of the space $\mathrm{Span}(L_1^{(1)},\ldots,L_n^{(1)},\ldots,L_1^{(p)},\ldots,L_n^{(p)})$ is $\mathrm{rk}\left((A_{i,j})_{\substack{1\le i\le p\\ 1\le j\le q}}\right)$, which concludes the proof.

□

Proof of Proposition 1.18. We have the following commutative diagram



with p and i respectively the projection and inclusion of $V_\lambda^{\oplus q d_\lambda}$ in $\mathbb{C}[\mathbb{S}_n]^{\oplus q}$. Then

$$\dim(P_\lambda^{\oplus q}(R(n)) = \dim(p(R(n))) = m_\lambda(R(n))d_\lambda.$$

The goal is now to compute $\dim(P_\lambda^{\oplus q}(R(n))$.

As $R(n)$ is generated by $x_1, \ldots, x_p$, $P_\lambda^{\oplus q}(R(n))$ is generated by $P_\lambda^{\oplus q}(x_1), \ldots, P_\lambda^{\oplus q}(x_p)$. But the action of $\mathbb{S}_n$ on $\mathcal{M}_{d_\lambda}(\mathbb{C})^{\oplus q}$ is given by the isomorphism $\varphi^{\oplus q}$. Moreover, the projection $P_\lambda^{\oplus q}$ of $\varphi^{\oplus q}$ on $\mathcal{M}_{d_\lambda}(\mathbb{C})^{\oplus q}$ is surjective, and

$$P_\lambda^{\oplus q}(R(n)) = \mathcal{M}_{d_\lambda}(\mathbb{C}) \cdot P_\lambda^{\oplus q}(x_1) + \cdots + \mathcal{M}_{d_\lambda}(\mathbb{C}) \cdot P_\lambda^{\oplus q}(x_p).$$

But because $P_\lambda^{\oplus q}(x_i) = (P_\lambda(x_i^{(1)}), \ldots, P_\lambda(x_1)(x_i^{(q)}))$, we have, by Lemma 1.19,

$$\dim(P_\lambda^{\oplus q}(R(n))) = \operatorname{rk}\left(\left(P_\lambda(x_i^{(j)})\right)_{\substack{1 \le i \le p \\ 1 \le j \le q}}\right) \cdot d_\lambda,$$

which proves Proposition 1.18.

□

As an example of application, we refer the reader to [BD17] where M. Bremner and V. Dotsenko classified the parametrized one-relation operads that are regular using this method. In their case, the matrix encoding all identities is 120×120 in the polynomial ring $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$, which may seem small enough to compute, but when one wants to compute determinental ideals and their Gröbner bases, this is way too much. The method using representation matrices turns this matrix into five matrices of sizes $5d_\lambda \times 5d_\lambda$ with $\lambda \vdash 4$, thus $d_\lambda = 1, 3, 2, 3, 1$, which they managed to compute.

Moreover, this method is not only faster than computing the complete matrix, but it also gives more information on the space of relations.

1.4.2 The case of $\mathbb{S}$-bimodules

Let $n, m \in \mathbb{N}$. Let us denote $\lambda \vdash m$ and $\mu \vdash n$ by $(\lambda, \mu) \vdash (m, n)$. The regular representation $\mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}]$ has the decomposition into isotypic components

$$\mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}] = \bigoplus_{(\lambda,\mu)\vdash(m,n)} (V_\lambda \boxtimes V_\mu^{\mathrm{op}})^{\oplus d_\lambda d_\mu}.$$

In fact we have the sequence of isomorphisms of modules

$$\begin{aligned}
\mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}] &\simeq \mathbb{C}[\mathbb{S}_m] \boxtimes \mathbb{C}[\mathbb{S}_n^{\mathrm{op}}] \\
&\simeq \left(\bigoplus_{\lambda\vdash m} V_\lambda^{\oplus d_\lambda}\right) \boxtimes \left(\bigoplus_{\mu\vdash n} (V_\mu^{\mathrm{op}})^{\oplus d_\mu}\right) \\
&= \bigoplus_{(\lambda,\mu)\vdash(m,n)} V_\lambda^{\oplus d_\lambda} \boxtimes (V_\mu^{\mathrm{op}})^{\oplus d_\mu} \\
&= \bigoplus_{(\lambda,\mu)\vdash(m,n)} (V_\lambda \boxtimes V_\mu^{\mathrm{op}})^{\oplus d_\lambda d_\mu}.
\end{aligned}$$

Remark 1.20. The representations $V_\lambda \boxtimes V_\mu^{\mathrm{op}}$ are irreducible and give the isomorphism classes of all irreducible representations of $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$.

Thus let us denote, for a representation V of $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$, by $m_{\lambda,\mu}(V)$ the multiplicity of $V_\lambda \boxtimes V_\mu^{\mathrm{op}}$ in V and by $[V]_{\lambda,\mu} := (V_\lambda \boxtimes V_\mu)^{\oplus m_{\lambda,\mu}(V)}$ the corresponding isotypic component. In this case we have an isomorphism of algebras

$$\varphi : \mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}] \to \bigoplus_{(\lambda,\mu)\vdash(m,n)} \mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C})$$

given by the composition

$$\begin{aligned}
\mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}] \xrightarrow{\sim} \mathbb{C}[\mathbb{S}_m] \otimes \mathbb{C}[\mathbb{S}_n] \xrightarrow{\sim} &\left(\bigoplus_{\lambda\vdash m} \mathcal{M}_{d_\lambda}(\mathbb{C})\right) \otimes \left(\bigoplus_{\mu\vdash n} \mathcal{M}_{d_\mu}(\mathbb{C})\right) \\
&= \bigoplus_{(\lambda,\mu)\vdash(m,n)} \mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C}).
\end{aligned}$$

Moreover, for any $n, m \in \mathbb{N}$, we have an isomorphism

$$\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C}) \simeq \mathcal{M}_{nm}(\mathbb{C})$$

given by the composition

$$\begin{aligned}\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C}) \to & (\mathbb{C}^n)^* \otimes \mathbb{C}^n \otimes (\mathbb{C}^m)^* \otimes \mathbb{C}^m \\ = & (\mathbb{C}^n)^* \otimes (\mathbb{C}^m)^* \otimes \mathbb{C}^n \otimes \mathbb{C}^m \to (\mathbb{C}^{nm})^* \otimes \mathbb{C}^{nm} \to \mathcal{M}_{nm}(\mathbb{C})\end{aligned}$$

which is exactly the Kronecker product of matrices.

Definition 1.21. For $A \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{M}_m(\mathbb{C})$, the Kronecker product $A \odot B$ is the block matrix in $\mathcal{M}_{nm}(\mathbb{C})$ given by

$$A \odot B = (A_{ij}B)_{1 \leq i,j \leq n} .$$

Property 1.22. *Let $n, m \in \mathbb{N}$, $M, A \in \mathcal{M}_n(\mathbb{C})$ and $N, B \in \mathcal{M}_m(\mathbb{C})$. We have*

$$(MA) \odot (NB) = (M \odot N)(A \odot B).$$

Proof. We have

$$\begin{aligned}(M \odot N)(A \odot B) &= (M_{i,j}N)_{i,j} (A_{i,j}B)_{i,j} \\ &= \left(\left(\sum_{k=1}^{n} M_{i,k}A_{k,j} \right) NB \right)_{i,j} \\ &= \left((MA)_{i,j} NB \right)_{i,j} \\ &= (MA) \odot (NB).\end{aligned}$$

□

We will denote by $P_{\lambda,\mu}$ the projection of φ on $\mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C})$. This is an isomorphism from $(V_\lambda \boxtimes V_\mu^{\mathrm{op}})^{d_\lambda d_\mu}$ to $\mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C})$.

As before, we consider a subspace $R(m,n)$ of $\mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}]^{\oplus q}$. For example, R can be the concequences of some relations in a free properad in a given biarity (see Chapter 2). Suppose that $R(m,n)$ is generated as an $\mathbb{S}_n \times \mathbb{S}_m^{\mathrm{op}}$-module by elements $x_1, \ldots, x_p \in \mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}]^{\oplus q}$. Our goal now is to compute the multiplicity $m_{\lambda,\mu}(R(m,n))$ of $V_\lambda \boxtimes V_\mu^{\mathrm{op}}$ in $R(m,n)$. We will prove the following.

Proposition 1.23. *Let $R(m,n)$ be the $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$-module generated by $x_1, \ldots, x_p \in \mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}]^{\oplus q}$, and for every $1 \leq i \leq p$, $x_i = (x_i^{(1)}, \ldots, x_i^{(q)})$ their decompositions along $\mathbb{C}[\mathbb{S}_m \times$*

$\mathbb{S}_n^{\mathrm{op}}]^{\oplus q}$. *Then we have, for every* $(\lambda, \mu) \vdash (m, n)$,

$$m_{\lambda,\mu}(R(m,n)) = \mathrm{rk}\left(\left(\sum_{k=1}^{N_{i,j}^{\lambda,\mu}} A_{i,j}^{(k)} \odot B_{i,j}^{(k)}\right)_{\substack{1\le i\le p\\ 1\le j\le q}}\right),$$

where

$$P_{\lambda,\mu}(x_i^{(j)}) = \sum_{k=1}^{N_{i,j}^{\lambda,\mu}} A_{i,j}^{(k)} \otimes B_{i,j}^{(k)} \in \mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C}).$$

Moreover, we call the matrix $\left(\sum_{k=1}^{N_{i,j}^{\lambda,\mu}} A_{i,j}^{(k)} \odot B_{i,j}^{(k)}\right)_{\substack{1\le i\le p\\ 1\le j\le q}}$ *the representation matrix of* $R(m,n)$ *for* (λ, μ).

Let us first prove a lemma.

Lemma 1.24. *Let* $n, m, N_1, \dots, N_p \in \mathbb{N}$, *and for* $1 \le i \le p$ *and* $1 \le k \le N_i$, *matrices* $A_i^{(k)} \in \mathcal{M}_n(\mathbb{C})$ *and* $B_i^{(k)} \in \mathcal{M}_m(\mathbb{C})$. *We have, as vector spaces,*

$$\begin{aligned}(\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C})) \cdot \left(\sum_{k=1}^{N_1} A_1^{(k)} \otimes B_1^{(k)}, \dots, \sum_{k=1}^{N_p} A_p^{(k)} \otimes B_p^{(k)}\right)\\ \simeq \mathcal{M}_{nm}(\mathbb{C}) \cdot \left(\sum_{k=1}^{N_1} A_1^{(k)} \odot B_1^{(k)}, \dots, \sum_{k=1}^{N_p} A_p^{(k)} \odot B_p^{(k)}\right),\end{aligned}$$

where

$$\begin{aligned}&(\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C})) \cdot \left(\sum_{k=1}^{N_1} A_1^{(k)} \otimes B_1^{(k)}, \dots, \sum_{k=1}^{N_p} A_p^{(k)} \otimes B_p^{(k)}\right)\\ &= \mathrm{Span}\left(\left(\sum_{k=1}^{N_1} (MA_1^{(k)}) \otimes (NB_1^{(k)}), \dots, \sum_{k=1}^{N_p} (MA_p^{(k)}) \otimes (NB_p^{(k)})\right), M \in \mathcal{M}_n(\mathbb{C}), N \in \mathcal{M}_m(\mathbb{C})\right).\end{aligned}$$

Proof.

$$
\begin{aligned}
&(\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C})) \cdot \left(\sum_{k=1}^{N_1} A_1^{(k)} \otimes B_1^{(k)}, \dots, \sum_{k=1}^{N_p} A_p^{(k)} \otimes B_p^{(k)} \right) \\
&= \operatorname{Span}\left(\left(\sum_{k=1}^{N_1} (MA_1^{(k)}) \otimes (NB_1^{(k)}), \dots, \sum_{k=1}^{N_p} (MA_p^{(k)}) \otimes (NB_p^{(k)}) \right), M \in \mathcal{M}_n(\mathbb{C}), N \in \mathcal{M}_m(\mathbb{C}) \right) \\
&\simeq \operatorname{Span}\left(\left(\sum_{k=1}^{N_1} (MA_1^{(k)}) \odot (NB_1^{(k)}), \dots, \sum_{k=1}^{N_p} (MA_p^{(k)}) \odot (NB_p^{(k)}) \right), M \in \mathcal{M}_n(\mathbb{C}), N \in \mathcal{M}_m(\mathbb{C}) \right) \\
&= \operatorname{Span}\left(\left(\sum_{k=1}^{N_1} (M \odot N)(A_1^{(k)} \odot B_1^{(k)}), \dots, \sum_{k=1}^{N_p} (M \odot N)(A_p^{(k)} \odot B_p^{(k)}) \right), M \in \mathcal{M}_n(\mathbb{C}), N \in \mathcal{M}_m(\mathbb{C}) \right) \\
&= \operatorname{Span}\left(\left(\sum_{k=1}^{N_1} M(A_1^{(k)} \odot B_1^{(k)}), \dots, \sum_{k=1}^{N_p} M(A_p^{(k)} \odot B_p^{(k)}) \right), M \in \mathcal{M}_{nm}(\mathbb{C}) \right) \\
&= \mathcal{M}_{nm}(\mathbb{C}) \cdot \left(\sum_{k=1}^{N_1} A_1^{(k)} \odot B_1^{(k)}, \dots, \sum_{k=1}^{N_1} A_p^{(k)} \odot B_p^{(k)} \right).
\end{aligned}
$$

□

Remark 1.25. Property 1.22 and Lemma 1.24 can also be proved by the fact that the Kronecker product corresponds to the tensor product of endomorphisms.

Proof of Proposition 1.23. We have the commutative diagram

$$
\begin{array}{ccccc}
R(m,n) & \hookrightarrow & \mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}]^{\oplus q} & \xrightarrow{P_{\lambda,\mu}^{\oplus q}} & (\mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C}))^{\oplus q} \\
 & & p \downarrow \uparrow i & \nearrow P_{\lambda,\mu}^{\oplus q} & \\
 & & (V_\lambda \boxtimes V_\mu)^{\oplus q d_\lambda d_\mu} & &
\end{array}
$$

Thus $\dim(P_{\lambda,\mu}(R(m,n))) = m_{\lambda,\mu}(R(m,n)) d_\lambda d_\mu$.

Now because $R(m,n)$ is generated as an $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$ module by $x_1, \dots, x_p$ and φ is a morphism of algebras, $P_{\lambda,\mu}^{\oplus q}(R(m,n))$ is generated by $P_{\lambda,\mu}^{\oplus q}(x_1), \dots, P_{\lambda,\mu}^{\oplus q}(x_p)$. But the action of $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$ on $(\mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C}))^{\oplus q}$ is given by $\varphi^{\oplus q}$ and the diagonal action on $\mathbb{C}[\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}]^{\oplus q}$, and $P_{\lambda,\mu}$ is a surjection, thus we have

$$P_{\lambda,\mu}^{\oplus q}(R(m,n)) = \sum_{i=1}^{p} (\mathcal{M}_{d_\lambda}(\mathbb{C}) \otimes \mathcal{M}_{d_\mu}(\mathbb{C})) \cdot (A_{i,1}^{(k)} \otimes B_{i,1}^{(k)}, \dots, A_{i,q}^{(k)} \otimes B_{i,q}^{(k)}).$$

By Lemma 1.24, we have

$$P_{\lambda,\mu}^{\oplus q}(R(m,n)) \simeq \sum_{i=1}^{p} \mathcal{M}_{d_\lambda d_\mu}(\mathbb{C}) \cdot (A_{i,1}^{(k)} \odot B_{i,1}^{(k)}, \ldots, A_{i,q}^{(k)} \odot B_{i,q}^{(k)}).$$

And finally, by Lemma 1.19, the dimension of this space is

$$\operatorname{rk}\left(\left(\sum_{k=1}^{N_{i,j}^{\lambda,\mu}} A_{i,j}^{(k)} \odot B_{i,j}^{(k)}\right)_{i,j}\right) d_\lambda d_\mu,$$

which proves Proposition 1.23.

□

1.4.3 A few properties of characters

For a represention V of G, we will denote by χ_V its character, that is the map

$$\begin{aligned} \chi_V : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \operatorname{Tr}(\rho_V(g)), \end{aligned}$$

where ρ_V is the structural morphism $\rho_V : G \longrightarrow \mathrm{GL}(V)$.

Properties 1.26. *Let V be a representation of G, we have, for every $g, h \in G$,*

$$\chi_V(g) = \chi_V(hgh^{-1}).$$

Moreover, a representation is characterized by its character.

For more properties on characters, one can see [FH13, Chapter 2].

Proposition 1.27. *For an irreducible representation $V_{\lambda,\mu}$ of $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$, we have*

$$V_{\lambda,\mu} \simeq V_{\lambda,\mu}^*$$

as representations, where V^* denotes the dual representation of V, given by the structural morphism $\rho^* : \mathbb{S}_n \to \mathrm{GL}(V^*)$ defined for $\sigma \in \mathbb{S}_n$ by

$$\rho^*(\sigma) := \rho(\sigma^{-1})^{\mathrm{T}},$$

where ρ is the structural morphism of V and f^{T} denotes the transposition of f.

Proof. We have $V_{\lambda,\mu} = V_\lambda \boxtimes V_\mu^{\mathrm{op}}$. Let us denote by $\rho_{\lambda,\mu}$, ρ_λ and ρ_μ the structure morphisms respectively of $V_{\lambda,\mu}$, V_λ and V_μ. We have, for $(\sigma,\tau) \in \mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$,

$$\rho_{\lambda,\mu}(\sigma,\tau) = \rho_\lambda(\sigma) \odot \rho_\mu(\tau).$$

Moreover, for a representation (V, ρ_V) of a symmetric group $\mathbb{S}_n$, the dual representation (V^*, ρ_{V^*}) is isomorphic to V. This is given by the fact that the character of V^* evaluated on $\sigma \in \mathbb{S}_n$ is the character of V evaluated on σ^{-1}, which is in the same conjugacy class as σ. Thus we have

$$\begin{aligned}
\chi_{V^*_{\lambda,\mu}}(\sigma,\tau) &= \chi_{(V_\lambda \boxtimes V_\mu)^*}(\sigma,\tau) \\
&= \chi_{V^*_\lambda}(\sigma)\chi_{V^*_\mu}(\tau), \text{ since } \operatorname{Tr}(A \odot B) = \operatorname{Tr}(A)\operatorname{Tr}(B) \\
&= \chi_{V_\lambda}(\sigma)\chi_{V_\mu}(\tau) \\
&= \chi_{V_{\lambda,\mu}}(\sigma,\tau).
\end{aligned}$$

□

Chapter 2

Introduction to properads

In this chapter we recall the main definitions and properties of properads. Properads are a generalization of operads (see [LV12]) and allow us to encode bialgebras instead of only algebras. In other terms we consider formal operations which take several inputs and have several outputs. This difference leads us to a lot of issues, some properties of operads do not hold true any longer for properads. For example we often study binary operads arity by arity, and get finite dimensions in every arity (when we do not have artiy 0 and arity 1 elements different from the identity), but in the case of free binary properads generated by a product μ and a coproduct Δ, the element $\mu \circ \Delta$ has biarity $(1, 1)$, and we can compose it with itself infinitely many times and get infinite dimension in every biarity. Some other results like the fact that confluence of a rewriting system implies the Koszul property are still open in the case of properads.

In the first section we just give the definitions we will need later about $\mathbb{S}$-bimodules and properads, most of these definitions are from [Val07]. In the second section, we study some properties and definitions of properads, such as infinitesimal composition product and Koszul theory for properads, again a lot of the definitions come from [Val07], but also from [MV09]. In the third section, we give examples of properads, with the structures they encode. In the last section we study the free binary properad on low weights, by straightforward calculation.

2.1 Properads

In the case of operads, we first define $\mathbb{S}$-modules to define operads as monoids in the category $\mathbb{S}$-mod of $\mathbb{S}$-modules (see [LV12, Chapter 5]). In the case of properads, we first define $\mathbb{S}$-bimodules and define properads as monoids in the category $\mathbb{S}$-bimod of $\mathbb{S}$-bimodules. There are several ways to define an associative product in this category, but we will focus on one way : the connected composition product that leads to properads.

2.1.1 $\mathbb{S}$-bimodules and graphs

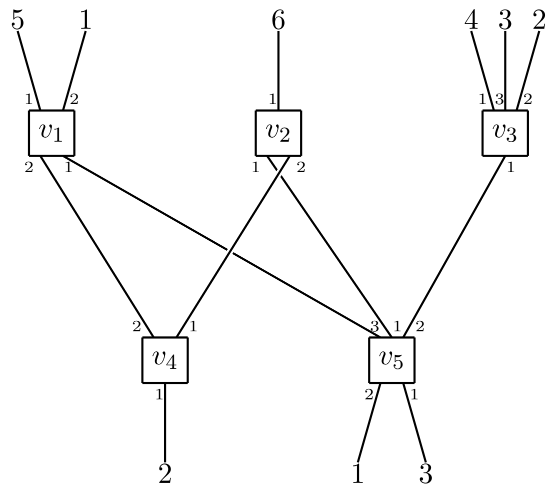
The $\mathbb{S}$-bimodules are just a generalization of $\mathbb{S}$-modules, because we want to encode operations with several outputs, we also need an action of the symmetric group on the outputs

Definition 2.1 ($\mathbb{S}$-bimodules, [Val07, Section 1])**.** An $\mathbb{S}$-*bimodule* P is a collection of vector spaces $(P(m,n))_{n,m\in\mathbb{N}}$ such that for every $n,m \in \mathbb{N}$, $P(m,n)$ is a left $\mathbb{S}_m$-module and a right $\mathbb{S}_n$-module such that these actions commute. An $\mathbb{S}$-*bimodule* morphism between P and Q is a collection of equivariant maps $f_{m,n} : P(m,n) \longrightarrow Q(m,n)$. These definitions form the category of $\mathbb{S}$-bimodules denoted by $\mathbb{S}$-bimod.

As in the case of $\mathbb{S}$-modules, we can think of elements of $P(m,n)$ for an $\mathbb{S}$-bimodule P as formal operations with m outputs and n inputs. The actions of the symmetric groups correspond to the permutation of outputs or inputs. We can also represent these elements as graphs, more precisely 1-level graphs, see the definition below, with m outputs and n inputs. We call (m,n) the biarity of such an element.

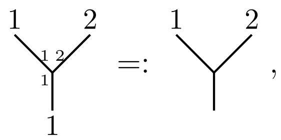
In this manuscript, a graph will denote a non-planar directed acyclic graph with a global flow (i.e. each vertex has inputs and outputs, and each edge connects an input to an output), from top to bottom, with a numbering of the inputs and outputs of each vertex, and a numbering of the inputs and outputs of the graph.

Figure 2.1 – Example of a directed acyclic graph with a global flow

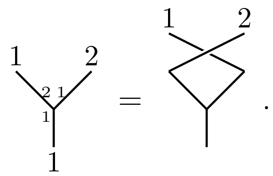


Remark 2.2. We will often, instead of numbering the inputs and outputs of each vertex, use some visual keys to differentiate them. For example, later we will consider graphs

with only two inputs and one output representing a product, the input on the left will represent the first input and the one on the right the second one. Moreover, when there is only one input or one output, we will not label it. For example, we have



with



Definition 2.3 (Levelled graph, [Val07, Section 1])**.** We call a *k-level graph* a connected graph for which the set of all vertices is partitioned in k levels that we index from top (1) to bottom (k), such that every edge connects an output of a level to an input of a strictly lower level.

Notation 2.4.

(i) We denote by $\mathcal{G}$ the set of connected graphs, and by $\mathcal{G}_{m,n}$ the subset of $\mathcal{G}$ with only graphs of biarity (m, n).

(ii) We denote by $\mathcal{G}^{(k)}$ the set of k-level graphs and by $\mathcal{G}^{(k)}_{m,n}$ the set of k-level graphs of biarity (m, n).

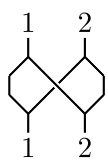
(iii) We denote by $\mathcal{G}^{\ell}$ the set of levelled graphs and by $\mathcal{G}^{\ell}_{m,n}$ the set of levelled graphs of biarity (m, n).

(iv) For a k-level graph G, we denote by $\mathcal{V}_i(G)$, for $1 \leq i \leq k$, the set of vertices of G in the ith level. For a graph G, we denote by $\mathcal{V}(G)$ the set of vertices of G.

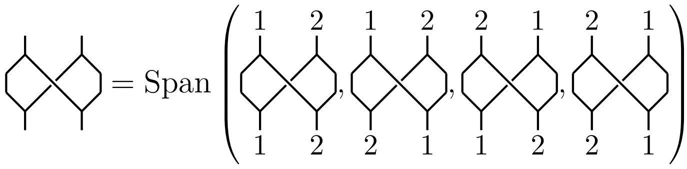
(v) For a vertex v of G, we denote by $\mathrm{In}(v)$ the set of inputs of v and by $\mathrm{Out}(v)$ the set of outputs of v.

Remark 2.5. We do not have an injection $\mathcal{G}^{\ell} \to \mathcal{G}$, a graph in $\mathcal{G}$ can be levelled in multiple ways, thus we have a choice of levelling for a given graph.

For example the following graph is a graph in $\mathcal{G}^{(2)}_{2,2}$ with 4 internal vertices :



We can see that for every $m, n \in \mathbb{N}^*$, $\mathrm{Span}(\mathcal{G}^{\ell}_{m,n})$ (resp. $\mathrm{Span}(\mathcal{G}_{m,n})$) is an $\mathbb{S}$-bimodule concentrated in biarity (m, n). For a graph G in $\mathcal{G}^{\ell}_{m,n}$ (resp. $\mathcal{G}_{m,n}$), we denote by $\mathrm{Bimod}(G)$ the sub-$\mathbb{S}$-bimodule of $\mathrm{Span}(\mathcal{G}^{\ell}_{m,n})$ (resp. $\mathrm{Span}(\mathcal{G}_{m,n})$) generated by G. We will also denote $\mathrm{Bimod}(G)$ by the underlying graph of G, without numbering the inputs and outputs. For example, as vector spaces we have



and the actions of $\mathbb{S}_2$ on the left and right are by permuting the outputs and inputs.

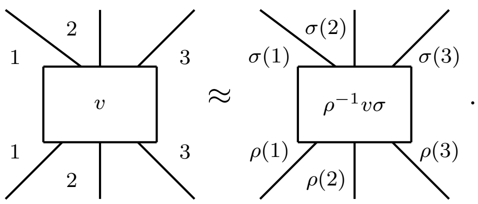
2.1.2 Properads

In order to define properads as monoids in the category $\mathbb{S}$-bimod, we need to define a monoidal structure on this category, that is, an associative product and a unit for this product (up to isomorphisms). Here we define the connected composition product $\boxtimes$ (denoted by $\boxtimes_c$ in [Val07, Section 1]). We first define $\boxtimes$ using graphs, and then we define it algebraically.

Definition 2.6 (Connected composition product, [Val07, Section 1])**.** Let P and Q be two $\mathbb{S}$-bimodules. Their *connected composition product* is defined by

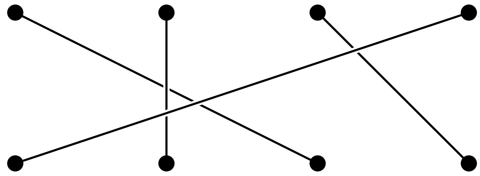
$$P \boxtimes Q := \bigoplus_{G \in \mathcal{G}^{(2)}} \left(\bigotimes_{v \in \mathcal{V}_2(G)} Q(|\mathrm{Out}(v)|, |\mathrm{In}(v)|) \otimes \bigotimes_{v \in \mathcal{V}_1(G)} P(|\mathrm{Out}(v)|, |\mathrm{In}(v)|) \right) / \approx,$$

where the relation $\approx$ is generated by



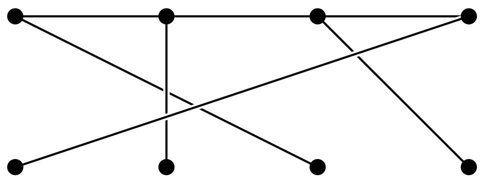
In order to define this product algebraically, we first need to define connected permutations and block permutations (see again [Val07, Section 1]). Let $\sigma \in \mathbb{S}_n$ be a permutation. We define the geometric representation of σ as the graph defined by the vertices

$(A_1, \dots, A_n, B_1, \dots, B_n)$ and the edges $((A_1, B_{\sigma(1)}), \dots, (A_n, B_{\sigma(n)}))$. For example the geometric representation of the permutation $[3, 2, 4, 1] \in \mathbb{S}_4$ is

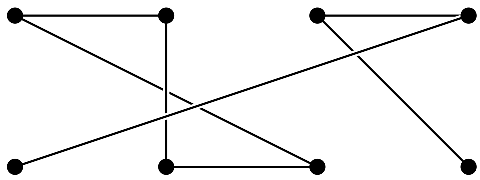


Definition 2.7 (Connected permutations, [Val07, Section 1])**.** Let $N \in \mathbb{N}$ be an integer. Let $\bar{i} = (i_1, \dots, i_a)$ and $\bar{j} = (j_1, \dots, j_b)$ be two tuples such that $|\bar{i}| = |\bar{j}| = N$. An $(\bar{i}, \bar{j})$-*connected permutation* is a permutation $\sigma \in \mathbb{S}_N$ such that the graphs obtained by adding the edges $(B_{i_1+\dots+i_{k-1}+l}, B_{i_1+\dots+i_{k-1}+l+1})$ for $1 \leq k \leq a$ and $1 \leq l \leq i_k - 1$ and the edges $(A_{j_1+\dots+j_{k-1}+l}, A_{j_1+\dots+j_{k-1}+l+1})$ for $1 \leq k \leq b$ and $1 \leq l \leq j_k - 1$ to the geometric representation of σ is connected. We denote by $\mathbb{S}^c_{\bar{i},\bar{j}}$ the set of $(\bar{i}, \bar{j})$-connected permutations.

For example, for the same permutation $\sigma := [3, 2, 4, 1] \in \mathbb{S}_4$, σ is $((1, 1, 1, 1), (4))$-connected because the graph



is connected. But σ is not $((1, 2, 1), (2, 2))$-connected because the graph



is not connected.

Notations 2.8. Let us give some notations (see also [Val07, Conventions]).

(i) For $\bar{i}$ an a-tuple, we denote by $\mathbb{S}_{\bar{i}}$ the product of symmetric groups $\mathbb{S}_{i_1} \times \dots \times \mathbb{S}_{i_a}$.

(ii) For P an $\mathbb{S}$-bimodule and $\bar{i}, \bar{j}$ two a-tuples, we denote by $P(\bar{i}, \bar{j})$ the tensor product $P(i_1, j_1) \otimes \dots \otimes P(i_a, j_a)$. We will denote by $(p_1, \dots, p_a)$ the element $p_1 \otimes \dots \otimes p_a$ in $P(\bar{i}, \bar{j})$.

(iii) Let τ be a permutation of $\mathbb{S}_a$ and $\bar{\imath} = (i_1, \ldots, i_a)$ be an a-tuple. We denote by $\tau_{\bar{\imath}}$ and call the block permutation associated to τ and $\bar{\imath}$ the permutation

$$[i_1+\cdots+i_{\tau^{-1}(1)-1}+1, \ldots, i_1+\cdots+i_{\tau^{-1}(1)}, \ldots, i_1+\cdots+i_{\tau^{-1}(a)-1}+1, \ldots, i_1+\cdots+i_{\tau^{-1}(a)}].$$

We can now define the connected composition product, using connected permutations and the previous notations.

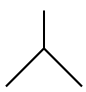
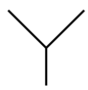
Proposition 2.9 (Algebraic connected composition product, [Val07, Proposition 1.5])**.** *Let P and Q be two $\mathbb{S}$-bimodules. Their connected composition product is isomorphic to the $\mathbb{S}$-bimodule given, for every $n, m \in \mathbb{N}$, by*

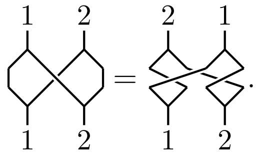
$$(Q \boxtimes P)(m,n) \simeq \left(\bigoplus_{N,\bar{l},\bar{k},\bar{j},\bar{\imath}} \mathbb{C}[\mathbb{S}_m] \otimes_{S_{\bar{l}}} Q(\bar{l},\bar{k}) \otimes_{S_{\bar{k}}} \mathbb{C}[\mathbb{S}^c_{\bar{k},\bar{j}}] \otimes_{S_{\bar{j}}} P(\bar{j},\bar{\imath}) \otimes_{S_{\bar{\imath}}} \mathbb{C}[\mathbb{S}_n] \right) / \sim,$$

where the direct sum runs over the integers $N \in \mathbb{N}$, the b-tuples $\bar{l}$, $\bar{k}$, the a-tuples $\bar{j}$, $\bar{\imath}$ such that $\left|\bar{l}\right| = m$, $\left|\bar{k}\right| = \left|\bar{j}\right| = N$ and $\left|\bar{\imath}\right| = n$, and the equivalence relation $\sim$ is the following. For $\sigma_1 \in \mathbb{S}_n$, $\sigma_2 \in S^c_{\bar{k},\bar{j}}$, $\sigma_3 \in \mathbb{S}_m$, $(q_1, \ldots, q_b) \in Q(\bar{l},\bar{k})$, $(p_1, \ldots, p_a) \in P(\bar{j},\bar{\imath})$, $\tau \in \mathbb{S}_a$ and $\rho \in \mathbb{S}_b$, we have

$$\begin{aligned} &\sigma_3 \otimes (q_1, \ldots, q_b) \otimes \sigma_2 \otimes (p_1, \ldots, p_a) \otimes \sigma_1 \\ &\sim \sigma_3 \rho_{\bar{l}}^{-1} \otimes \left(q_{\rho^{-1}(1)}, \ldots, q_{\rho^{-1}(b)}\right) \otimes \rho_{\bar{k}} \sigma_2 \tau_{\bar{j}} \otimes \left(p_{\tau(1)}, \ldots, p_{\tau(a)}\right) \otimes \tau_{\bar{\imath}}^{-1} \sigma_1 \end{aligned}$$

Let us give some explanations about this proposition. The permutations on the left and right correspond to permutations of outputs and inputs. The spaces $Q(\bar{l},\bar{k})$ and $P(\bar{j},\bar{\imath})$ contain the formal operations we want to compose. The space $\mathbb{C}[S^c_{\bar{k},\bar{j}}]$ corresponds, in terms of graphs, to every possible way to connect the elements that gives us a connected 2-level graph. Finally, the equivalence relation represents the fact that we can permute elements if we adapt the corresponding permutations. For example, for

$P =$  and $Q =$  , we have, in $Q \boxtimes P(2,2)$,



Let I be the $\mathbb{S}$-bimodule defined by $I(1,1) = \mathbb{C}$ and $I(m,n) = 0$ otherwise.

Proposition 2.10 ([Val07, Proposition 1.6])**.** *The connected composition product $\boxtimes$ is associative. In other words, for P, Q, R three $\mathbb{S}$-bimodules, there is an isomorphism of $\mathbb{S}$-bimodules*

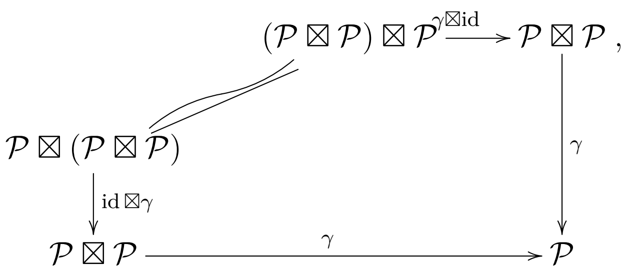
$$R \boxtimes (Q \boxtimes P) \simeq (R \boxtimes Q) \boxtimes P.$$

Moreover, ($\mathbb{S}$-bimod, $\boxtimes$, I) is a monoidal category.

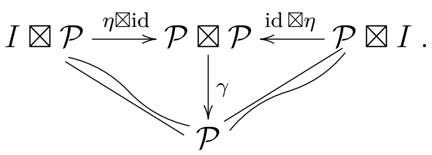
Definition 2.11 (Properad, [Val07, Section 2])**.** A *properad* is an monoid in the category ($\mathbb{S}$-bimod, $\boxtimes, I$). In other words, a properad is an $\mathbb{S}$-bimodule $\mathcal{P}$ together with $\mathbb{S}$-bimodule morphisms

$$\gamma : \mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$$
$$\eta : I \to \mathcal{P}$$

such that the following diagrams commute



and



The basic idea is that a properad is a collection of formal operations, with an identity given by η in biarity $(1,1)$ and a way to compose operations by γ which gives us another operation in the properad. That way we can define properads encoding structures of bialgebras, with relations coming from the definition of γ. For example, γ can give the same operation for two different compositions of elements, which leads to a relation between these two compositions.

A trivial example of properad would be the identity properad, given by I, concentrated in biarity $(1,1)$, given here by $\mathbb{C}$, and with the composition given by the product in $\mathbb{C}$. But if we want some more complex examples, we need more definitions.

2.1.3 Weight graded $\mathbb{S}$-bimodule and free properad over an $\mathbb{S}$-bimodule

Definition 2.12 (Weight graded $\mathbb{S}$-bimodule, [Val07, Section 2])**.** A *weight graded* $\mathbb{S}$-*bimodule* P is a direct sum over $\rho \in \mathbb{N}$ of $\mathbb{S}$-bimodules

$$P = \bigoplus_{\rho \in \mathbb{N}} P^{(\rho)}.$$

A morphism of weight graded $\mathbb{S}$-bimodules is a morphism of $\mathbb{S}$-bimodules that preserves this decomposition. The weight graded $\mathbb{S}$-bimodules form a category denoted gr-$\mathbb{S}$-bimod.

A weight graded $\mathbb{S}$-bimodule P such that $P^{(0)} = I$ is called connected, and we denote by $\overline{P}$ the subspace of P such that $P = I \oplus \overline{P}$.

For two weight graded $\mathbb{S}$-bimodules P and Q, one can define a weight on their connected composition product by the following : the weight of an element $\sigma_1 \otimes (q_1, \dots, q_b) \otimes \sigma_2 \otimes (p_1, \dots, p_b) \otimes \sigma_3$ is the sum of the weights of the elements q_1, …, q_b, p_1, …, p_a. This gives a monoidal structure on gr-$\mathbb{S}$-bimod. Thus one can define the notion of weight graded properads.

Definition 2.13. A *weight graded properad* is a monoid in the monoidal category of weight graded $\mathbb{S}$-bimodules.

Notation 2.14. Let P, Q be two weight graded $\mathbb{S}$-bimodules. We denote by

$$\underbrace{Q}_{\rho_2} \boxtimes \underbrace{P}_{\rho_1}$$

the subspace of $Q \boxtimes P$ generated by the elements of the form $\sigma_3 \otimes (q_1, \dots, q_b) \otimes \sigma_2 \otimes (p_1, \dots, p_b) \otimes \sigma_3$ such that the sum of the weights of p_1, …, p_a is equal to ρ_1 and the sum of the weights of $q_1, \ldots, q_b$ is equal to ρ_2.

Example 2.15. If $P^{(0)} = Q^{(0)} = 0$, the space

$$\underbrace{Q}_{1} \boxtimes \underbrace{P}_{1}$$

is generated by the elements of $Q \boxtimes P$ of the form

$$\sigma_1 \otimes (1 \otimes \cdots \otimes p \otimes \cdots \otimes 1) \otimes \sigma_2 \otimes (1 \otimes \cdots \otimes q \otimes \cdots \otimes 1) \otimes \sigma_3,$$

where $p \in P^{(1)}$ and $q \in Q^{(1)}$.

Definition 2.16 (Free properad, [Val08, Section 5])**.** Let E be an $\mathbb{S}$-bimodule, the free properad over E is the weight graded properad given by

$$\mathcal{F}(E) := \bigoplus_{k \in \mathbb{N}} \mathcal{F}^{(k)}(E),$$

with

$$\mathcal{F}^{(k)}(E) := \left(\bigoplus_{\substack{G \in \mathcal{G} \\ |\mathcal{V}(G)| = k}} \bigotimes_{v \in \mathcal{V}(G)} E(|\mathrm{Out}(v)|, |\mathrm{In}(v)|) \right) / \approx,$$

where $\approx$ is the same relation as in 2.6.

One can see Section 2.3 for examples of properads presented by generators and relations, that means properads given by $\mathcal{F}(E)/(R)$ with E an $\mathbb{S}$-bimodule, $R \subset \mathcal{F}(E)$ and (R) the properadic ideal generated by R, see [Val08] for the notion of an ideal of a monoid defined with respect to the monoidal category. We say that a properad of the form $\mathcal{F}(E)/(R)$ is quadratic if $R \subset \mathcal{F}^(2)(E)$.

2.2 Properties of properads

Here we state some properadic properties and definitions such as Koszul duality. We will use these definitions and results later in the manuscript.

2.2.1 Infinitesimal composition product

Let us first define infinitesimal composition product that will be used to compose formal operations by pairs in a certain way.

Definition 2.17. Let P_1, P_2, Q_1 and Q_2 be $\mathbb{S}$-bimodules. We denote by $(Q_1; Q_2) \boxtimes (P_1; P_2)$ the subspace of $(Q_1 \oplus Q_2) \boxtimes (P_1 \oplus P_2)$ generated by the elements of the form

$$\sigma_3 \otimes (q_1, \ldots, q_b) \otimes \sigma_2 \otimes (p_1, \ldots, p_a) \otimes \sigma_1 \tag{2.1}$$

such that exactly one element p_i is in P_2 and the others are in P_1, and exactly one element q_i is in Q_2 and the others are in Q_1.

More generally, if $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ are $\mathbb{S}$-bimodules, let us denote by $(Q_1; Q_2, \ldots, Q_m) \boxtimes (P_1; P_2, \ldots, P_n)$ the subspace of $(Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m) \boxtimes (P_1 \oplus P_2 \oplus \cdots \oplus P_n)$ generated by the elements of the form (2.1) such that for every $2 \leq k \leq n$, there is exactly one element p_i in P_k and the others are in P_1, and for every $2 \leq l \leq n$, there is exactly one element q_i in Q_l, and the others are in Q_1

Finally, if $\overline{a} = (a_2, \ldots, a_n)$ is an $(n-1)$-tuple and $\overline{b} = (b_2, \ldots, b_m)$ is an $(m-1)$-tuple, we denote by $(Q_1; Q_2, \ldots, Q_m) \boxtimes_{\overline{b},\overline{a}} (P_1; P_2, \ldots, P_n)$ the subspace of $(Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m) \boxtimes (P_1 \oplus P_2 \oplus \cdots \oplus P_n)$ generated by the elements of the form (2.1) such that for every $2 \leq k \leq n$, there is exactly a_k elements p_i in P_k and the others are in P_1, and for every $2 \leq l \leq n$, there is exactly b_l elements q_i in Q_l and the others are in Q_1.

Definition 2.18. Let P and Q be $\mathbb{S}$-bimodules. We define the *infinitesimal composition product* of P with Q by

$$Q \boxtimes_{(1)} P := (I; Q) \boxtimes (I; P).$$

More generally, if $P_2, \ldots, P_n, Q_2, \ldots, Q_m$ are $\mathbb{S}$-bimodules, we denote the *infinitesimal composition product* of $(P_2, \ldots, P_n)$ with $(Q_2, \ldots, Q_m)$ by

$$(Q_2, \ldots, Q_m) \boxtimes_{(1)} (P_2, \ldots, P_n) := (I; Q_2, \ldots, Q_m) \boxtimes (I; P_2, \ldots, P_n).$$

Finally, if $\overline{a} = (a_2, \ldots, a_n)$ is an $(n-1)$-tuple and $\overline{b} = (b_2, \ldots, b_m)$ is an $(m-1)$-tuple, we denote by

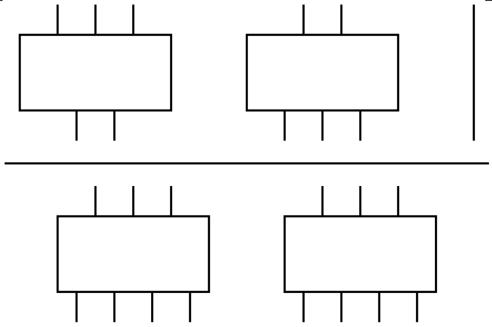
$$(Q_2, \ldots, Q_m) \boxtimes_{\overline{b},\overline{a}} (P_2, \ldots, P_n) := (I; Q_2, \ldots, Q_m) \boxtimes_{\overline{b},\overline{a}} (I; P_2, \ldots, P_n).$$

Remark 2.19. These definitions are inspired by the analogous definition for $\mathbb{S}$-modules in [LV12, Section 6.1], this is why we use the same notation.

Example 2.20. For example, if P and Q are connected $\mathbb{S}$-bimodules, we can look at infinitesimal composition of non trivial blocks of P and Q by looking at the space $\overline{P} \boxtimes_{(1)} \overline{Q}$.

Notation 2.21. If each of the $\mathbb{S}$-bimodules $P_2, \ldots, P_n$ and $Q_2, \ldots, Q_m$ is concentrated in one biarity, the infinitesimal composition product $(Q_2, \ldots, Q_m) \boxtimes_{\bar{b},\bar{a}} (P_2, \ldots, P_n)$ in a given biarity will be represented with vertices corresponding to the P_i's over vertices corresponding to the Q_i's. For example, if P_2 is concentrated in biarity $(2,3)$, P_3 is concentrated in biarity $(3,2)$ and Q is concentrated in biarity $(4,3)$, we will represent the infinitesimal composition product $((P_2, P_3) \boxtimes_{(1,1),(2)} (Q))(8,6)$ by Figure 2.2. The horizontal line in the middle corresponds to any connected permutation, not to be confused with the notation from [Mar06, Definition 19] which does not make sense here, and the vertical line corresponds to I.

Figure 2.2 – Representation of an infinitesimal composition product



Remark 2.22. We have, for an $\mathbb{S}$-bimodule E,

(i) $\mathcal{F}^{(0)}(E) = I$,

(ii) $\mathcal{F}^{(1)}(E) = E$,

(iii) Because a (connected) graph with two vertices has a unique 2-level graph structure, we have
$$\mathcal{F}^{(2)}(E) = E \boxtimes_{(1)} E,$$

(iv) For a higher weight k, one can compute
$$\left(\mathcal{F}^{(k-1)}(E) \boxtimes_{(1)} E\right) \oplus \left(E \boxtimes_{(1)} \mathcal{F}^{(k-1)}(E)\right),$$
which projects into $\mathcal{F}^{(k)}(E)$, then we can identify elements that are given by the same graph (with vertices labelled by elements of E but forgetting the levels).

With this definition, we state a very useful theorem which describes how to compose two formal operations defined by representations of symmetric groups. Because irreducible representations of product of symmetric groups are tensor products of irreducible representations of the symmetric groups, it is enough to consider tensor products of representations.

Theorem 2.23. *Let $k, l, m, n \in \mathbb{N}^*$ and*

(i) V be a left $\mathbb{S}_k$-module,

(ii) W be a right $\mathbb{S}_l$-module,

(iii) X be a left $\mathbb{S}_m$-module,

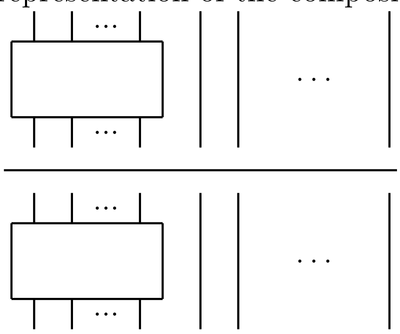
(iv) Y be a right $\mathbb{S}_n$-module.

Take $\mathcal{Q} := V \boxtimes W$ and $\mathcal{P} := X \boxtimes Y$. We have the isomorphism of $\mathbb{S}$-bimodules

$$(\mathcal{Q} \boxtimes_{(1)} \mathcal{P})(k+m-1, l+n-1) \simeq (V \sqcup_l \, {}^{\mathbb{S}_m}_{\mathbb{S}_{m-1}}\!\downarrow X) \boxtimes (W \downarrow^{\mathbb{S}_l}_{\mathbb{S}_{l-1}} \sqcup_r Y).$$

Remark 2.24. Here we take the specific biarity where exactly one edge will connect the unique non trivial operation of $\mathcal{P}$ with the unique non trivial operation of $\mathcal{Q}$. This is a specific type of dioperadic composition, see [Gan02, Section 1]. We use Notation 2.21 in Figure 2.3.

Figure 2.3 – Graphic representation of the composition of two operations



Proof. Let us denote by A the left side of Theorem 2.23. We have, by proposition 2.9,

$$A = \Bigg(\bigoplus_{\overline{k}, \overline{l}, \overline{m}, \overline{n}} \mathbb{C}[\mathbb{S}_{m+k-1}] \otimes_{\mathbb{S}_{\overline{k}}} (\mathbb{C} \otimes \cdots \otimes \mathcal{Q}(k,l) \otimes \cdots \otimes \mathbb{C})$$
$$\otimes_{\mathbb{S}_{\overline{l}}} \mathbb{C}[\mathbb{S}^c_{\overline{l},\overline{m}}] \otimes_{\mathbb{S}_{\overline{m}}} (\mathbb{C} \otimes \ldots, \mathcal{P}(m,n), \ldots, \mathbb{C}) \otimes_{\mathbb{S}_{\overline{n}}} \mathbb{C}[\mathbb{S}_{n+l-1}] \Bigg) / \sim$$

the sum over tuples $\overline{k} = (1,\dots,k,\dots,1)$, $\overline{l} = (1,\dots,l,\dots,1)$, $\overline{m} = (1,\dots,m,\dots,1)$, $\overline{n} = (1,\dots,n,\dots,1)$, with $\left|\overline{k}\right| = m+k-1$, $\left|\overline{l}\right| = |\overline{m}| = l+m-1$ and $|\overline{n}| = n+l-1$. But because of the relation $\sim$, we can take only the terms of the sum where $\mathcal{Q}(k,l)$ and $\mathcal{P}(m,n)$ appear in first place, thus removing the $\mathbb{C}$'s, we find

$$A \simeq \left(\mathbb{C}[\mathbb{S}_{m+k-1}] \otimes_{\mathbb{S}_k} \mathcal{Q}(k,l) \otimes_{\mathbb{S}_l} \mathbb{C}[\mathbb{S}^c_{(l,1,\dots,1),(m,1,\dots,1)}] \otimes_{\mathbb{S}_m} \mathcal{P}(m,n) \otimes_{\mathbb{S}_n} \mathbb{C}[\mathbb{S}_{n+l-1}]\right) / \sim',$$

where the relation $\sim'$ is given by

$$\sigma_1 \otimes q \otimes \sigma_2 \otimes p \otimes \sigma_3 \sim' \sigma_1(\mathrm{id}_k, \rho^{-1}) \otimes q \otimes (\mathrm{id}_l, \rho)\sigma_2(\mathrm{id}_m, \tau) \otimes p \otimes (\mathrm{id}_n, \tau^{-1})\sigma_3.$$

for $\rho \in \mathbb{S}_{m-1}$, $\tau \in \mathbb{S}_{l-1}$, and where we denoted for $\sigma \in \mathbb{S}_n$ and $\sigma' \in \mathbb{S}_m$ by (σ, σ') the corresponding permutation in $\mathbb{S}_{n+m}$. The first goal of this proof is to understand the bimodule $\mathbb{C}[\mathbb{S}^c_{(l,1,\dots,1),(m,1,\dots,1)}]$ as a left $\mathbb{S}_l$-module and a right $\mathbb{S}_m$-module. We have

$$\mathbb{S}^c_{(l,1,\dots,1),(m,1,\dots,1)} \simeq \{1,\dots,l\} \times \mathbb{S}_{l-1} \times \{1,\dots m\} \times \mathbb{S}_{m-1}.$$

Thus

$$\mathbb{C}[\mathbb{S}^c_{(l,1,\dots,1),(m,1,\dots,1)}] \simeq \mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}] \otimes \mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}].$$

Let us first define two maps, for $i \in \{1,\dots,l\}$:

$$\begin{aligned} \delta_i : \{1,\dots,l-1\} &\longrightarrow \{1,\dots,l\} \\ a &\longmapsto \begin{cases} a & \text{if } a < i \\ a+1 & \text{if } a \geq i \end{cases} \end{aligned}$$

and

$$\begin{aligned} s_i : \{1,\dots,l\} &\longrightarrow \{1,\dots,l-1\} \\ a &\longmapsto \begin{cases} a & \text{if } a < i \\ a-1 & \text{if } a \geq i \end{cases}. \end{aligned}$$

The left action of $\mathbb{S}_l$ on $\mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}]$ is given, for (e_i) the canonical basis of $\mathbb{C}^l$, $i \in \{1, \ldots, l\}$, $\sigma \in \mathbb{S}_l$ and $\rho \in \mathbb{S}_{l-1}$, by

$$\sigma \cdot (e_i \otimes \rho) = e_{\sigma(i)} \otimes s_{\sigma(i)} \sigma \delta_i \rho.$$

The structure on $\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]$ on the right is given by a similar formula. Let us now define another structure on $\mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}]$ given by an isomorphism with $\mathbb{C}[\mathbb{S}_l]$. We define again two maps.

$$\begin{aligned} u : \mathbb{S}_{l-1} &\longrightarrow \mathbb{S}_l \\ \sigma &\longmapsto [\sigma(1), \ldots, \sigma(l-1), l] \end{aligned}$$

and

$$\begin{aligned} d : \{\sigma \in \mathbb{S}_l \text{ such that } \sigma(l) = l\} &\longrightarrow \mathbb{S}_{l-1} \\ \sigma &\longmapsto [\sigma(1), \ldots, \sigma(l-1)]. \end{aligned}$$

We have a morphism of vector spaces

$$\begin{aligned} \varphi : \mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}] &\longrightarrow \mathbb{C}[\mathbb{S}_l] \\ e_i \otimes \rho &\longmapsto (i, i+1, \ldots, l) u(\rho), \end{aligned}$$

which is an isomorphism, with

$$\begin{aligned} \varphi^{-1} : \mathbb{C}[\mathbb{S}_l] &\longrightarrow \mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}] \\ \sigma &\longmapsto e_{\sigma(l)} \otimes d\left((l, \ldots, \sigma(l))\sigma\right). \end{aligned}$$

This isomorphism induces a left $\mathbb{S}_l$-module structure on $\mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}]$, given by

$$\begin{aligned}\sigma \star (e_i \otimes \rho) &= \varphi^{-1}(\sigma\varphi(e_i \otimes \rho))\\ &= \varphi^{-1}(\sigma(i,\ldots,l)u(\rho))\\ &= e_{\sigma(i)} \otimes d((l,\ldots,\sigma(i))\sigma(i,\ldots,l)u(\rho))\\ &= \sigma \cdot (e_i \otimes \rho).\end{aligned}$$

We can do the same computation for the right action of $\mathbb{S}_m$ and prove that, as $\mathbb{S}$-bimodules, we have

$$\mathbb{C}[\mathbb{S}^c_{(l,1,\ldots,1),(m,1,\ldots,1)}] \simeq \mathbb{C}[\mathbb{S}_l] \boxtimes \mathbb{C}[\mathbb{S}_m].$$

Now A is isomorphic to

$$\left(\mathbb{C}[\mathbb{S}_{m+k-1}] \otimes_{\mathbb{S}_k} V \otimes W \otimes_{\mathbb{S}_l} (\mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}]) \otimes (\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]) \otimes_{\mathbb{S}_m} X \otimes Y \otimes_{\mathbb{S}_n} \mathbb{C}[\mathbb{S}_{n+l-1}]\right) / \sim',$$

but the relation $\sim'$ can be cut in half saying that A is isomorphic to

$$\left(\mathbb{C}[\mathbb{S}_{m+k-1}] \otimes_{\mathbb{S}_k} V \otimes (\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]) \otimes_{\mathbb{S}_m} X\right) / \sim'_1 \otimes \left(W \otimes_{\mathbb{S}_l} (\mathbb{C}^l \otimes \mathbb{C}[\mathbb{S}_{l-1}]) \otimes Y \otimes_{\mathbb{S}_n} \mathbb{C}[\mathbb{S}_{n+l-1}]\right) / \sim'_2 .$$

Now let us study the first half

$$\begin{aligned}B &:= (\mathbb{C}[\mathbb{S}_{m+k-1}] \otimes_{\mathbb{S}_k} V \otimes (\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]) \otimes_{\mathbb{S}_m} X) / \sim'_1\\ &\simeq \mathbb{C}[\mathbb{S}_{m+k-1}] \otimes_{\mathbb{S}_k \times \mathbb{S}_{m-1}} (V \otimes ((\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]) \otimes_{\mathbb{S}_m} X)).\end{aligned}$$

The idea now is to understand the left action of $\mathbb{S}_{m-1}$ on $((\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]) \otimes_{\mathbb{S}_m} X)$. The action of $\mathbb{S}_{m-1}$ on $\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]$ is just the one on $\mathbb{C}[\mathbb{S}_{m-1}]$. We have the sequence of isomorphisms

$$((\mathbb{C}^m \otimes \mathbb{C}[\mathbb{S}_{m-1}]) \otimes_{\mathbb{S}_m} X) \simeq \mathbb{C}[\mathbb{S}_m] \otimes_{\mathbb{S}_m} X \simeq X.$$

The action of $\mathbb{S}_{m-1}$ on $\mathbb{C}[\mathbb{S}_m]$ in the second term is given by φ : let $\rho \in \mathbb{S}_m$ and $\sigma \in \mathbb{S}_{m-1}$, we have

$$\begin{aligned}\sigma \cdot \rho &= \varphi(\sigma\varphi^{-1}(\rho)) \\ &= \varphi(e_{\rho(m)} \otimes \sigma(m, \ldots \rho(m))\rho) \\ &= (\rho(m), \ldots, m)u(\sigma d((m, \ldots, \rho(m))\rho)).\end{aligned}$$

Thus we have $\sigma \cdot \mathrm{id} = u(\sigma)$, and the action of $\sigma \in \mathbb{S}_{m-1}$ on X is the action on ${}^{\mathbb{S}_m}_{\mathbb{S}_{m-1}}\!\downarrow X$. Finally, we get

$$B \simeq \mathbb{C}[\mathbb{S}_{m+k-1}] \otimes_{\mathbb{S}_k \times \mathbb{S}_{m-1}} (V \otimes {}^{\mathbb{S}_m}_{\mathbb{S}_{m-1}}\!\downarrow X) = V \sqcup_l {}^{\mathbb{S}_m}_{\mathbb{S}_{m-1}}\!\downarrow X.$$

We do the same for the second half and get Theorem 2.23.

□

A special case of this theorem is when V, W, X and Y are regular representations of respective symmetric groups.

Corollary 2.25. *If* $V = \mathbb{C}[\mathbb{S}_k]$, $W = \mathbb{C}[\mathbb{S}_l]$, $X = \mathbb{C}[\mathbb{S}_m]$ *and* $Y = \mathbb{C}[\mathbb{S}_n]$, *we have*

$$(\mathcal{Q} \boxtimes_{(1)} \mathcal{P})(m+k-1, n+l-1) = \mathbb{C}[\mathbb{S}_{m+k-1} \times \mathbb{S}^{\mathrm{op}}_{n+l-1}]^{\oplus lm}$$

Proof. This result is given by the fact that, as left $\mathbb{S}_l$-modules,

$${}^{\mathbb{S}_l}_{\mathbb{S}_{l-1}}\!\downarrow \mathbb{C}[\mathbb{S}_l] \simeq \mathbb{C}[\mathbb{S}_{l-1}]^{\oplus l}.$$

and as right $\mathbb{S}_m$-modules, we have

$$\mathbb{C}[\mathbb{S}_m] \downarrow^{\mathbb{S}_m}_{\mathbb{S}_{m-1}} \simeq \mathbb{C}[\mathbb{S}_{m-1}]^{\oplus m}$$

A proof of this property is given in the proof of Theorem 2.23.

□

2.2.2 Koszul duality

The main goal of this manuscript is to study koszulity and homotopy koszulity of properads presented by generators and relations. This property on properads and operads leads to many results and tools to study algebras over such properads and operads, such as homology, homotopy and minimal models.

Most results and definitions in this section come from [Val07, Section 7] for koszulity and [MV09, Section 5] for homotopy koszulity. Here, we will not recall all necessary definitions for the definition of a Koszul properad : the notion of coproperad, bar and cobar constructions, etc., see [Val07] for precise definitions. Moreover, we will give the definitions in the case of a properad $\mathcal{P}$ defined by $\mathcal{P} = \mathcal{F}(E)/(R)$, with E concentrated in weight 1. Notice that, in the general case, $\mathcal{P}$ is bigraded by the number of non-trivial vertices and by the total weight, but in the case of a properad with generators concentrated in weight 1, the gradings coincide. However, the free properad on the $\mathbb{S}$-bimodule $\mathcal{P}$ is bigraded with two (potentially different) gradings.

Definition 2.26 ([Val07, Section 7])**.** Let $\mathcal{P} = \mathcal{F}(E)/(R)$ with E concentrated in weight 1. The *reduced bar construction* of $\mathcal{P}$ is given by

$$\overline{\mathcal{B}}(\mathcal{P}) := \mathcal{F}^c(\Sigma\overline{\mathcal{P}}),$$

where Σ denotes the suspension of an $\mathbb{S}$-bimodule, see [Val07, Section 4.1.1], and $\mathcal{F}^c$ denotes the free coproperad over an $\mathbb{S}$-bimodule, see [Val07, Section 2.8], a coproperad being a comonoid in the category ($\mathbb{S}$-bimod, $\boxtimes, I$), see [Val07, Section 2.3]. Thus $\overline{\mathcal{B}}(\mathcal{P})$ is bigraded with the number s of non-trivial vertices and the total weight ρ. We will denote these gradings as follows

$$\overline{\mathcal{B}}^{(s)}(\mathcal{P})_d^{(\rho)},$$

with d the homological degree. Then we define the Koszul dual coproperad of $\mathcal{P}$ by

$$(\mathcal{P}^{¡})^{\rho} := H_{(\rho)}(\overline{\mathcal{B}}^{*}(\mathcal{P})^{(\rho)}, d_\theta)$$

where d_θ contracts edges between adjacent vertices using the properad structure of $\mathcal{P}$.

Proposition 2.27 ([Val07, Section 7])**.** *We have*

$$(\mathcal{P}^{i})^{(\rho)} = \ker\left(d_\theta : \overline{\mathcal{B}}^{(\rho)}(\mathcal{P})^{(\rho)} \to \overline{\mathcal{B}}^{(\rho-1)}(\mathcal{P})^{(\rho)}\right).$$

Moreover, if $\mathcal{P}$ is concentrated in degree 0*, we have*

$$(\mathcal{P}^{i})^{(\rho)}_{d} = \begin{cases} (\mathcal{P}^{i})^{(\rho)} & \text{if } d=\rho \\ 0 \text{ else} \end{cases}.$$

Definition 2.28 ([Val07, Section 7])**.** We say that $\mathcal{P}$ is *Koszul* if the natural inclusion $\mathcal{P}^{¡} \to \overline{\mathcal{B}}(\mathcal{P})$ is a quasi-isomorphism.

Remark 2.29. This definition of koszulity implies that the properad $\mathcal{P}$ is quadratic, see [Val07, Corollary 7.5].

One can see that for a Koszul properad $\mathcal{P}$, we have quasi-isomorphisms

$$\overline{\mathcal{B}}^{c}(\mathcal{P}^{¡}) \to \overline{\mathcal{B}}^{c}(\overline{\mathcal{B}}(\mathcal{P})) \to \mathcal{P},$$

the last one being the bar-cobar resolution, with $\overline{\mathcal{B}}^{c}$ the cobar construction, see [Val07, Section 4.2]. Thus we call the resolution $\overline{\mathcal{B}}^{c}(\mathcal{P}^{¡}) \to \mathcal{P}$ the Koszul resolution, which is minimal model of $\mathcal{P}$, see [Mar06, Section 1] for the definition of a minimal model. Moreover, we have the following

Theorem 2.30 ([Val07, Theorem 7.6])**.** *The morphism of weight graded connected dg-properads $\overline{\mathcal{B}}^{c}(\mathcal{P}^{i}) \to \mathcal{P}$ is a quasi-isomorphism if and only if $\mathcal{P}$ is Koszul.*

Now, we give definitions and results from [Val07, Section 7] on the Koszul dual properad of a quadratic properad.

Definition 2.31 (Czech dual, [Val07, Section 7])**.** Let E be an $\mathbb{S}$-bimodule, the *Czech dual* $E^{\vee}$ of E is the $\mathbb{S}$-bimodule defined for every $m,n \in \mathbb{N}$ by

$$E^{\vee}(m,n) = \mathrm{sgn}_{m} \otimes E(m,n)^{*} \otimes \mathrm{sgn}_{n},$$

where sgn_{m} is the signature representation of $\mathbb{S}_{m}$.

Theorem 2.32 (Koszul dual properad, [Val07, Corollary 7.12])**.** *Let $\mathcal{P} = \mathcal{F}(E)/(R)$ be a quadratic properad such that $\sum_{m,n} \dim(E(m,n))$ is finite. Then the Czech dual of the Koszul dual coproperad $(\mathcal{P}^{i})^{\vee}$ is endowed with the structure of a quadratic properad :*

$$\mathcal{P}^{!} := (\mathcal{P}^{i})^{\vee} = \mathcal{F}(\Sigma E^{\vee})/(\Sigma^{2} R^{\perp}).$$

Corollary 2.33. *If we have the decomposition into isotypic components of*

$$\mathcal{P}^{!}(m,n) = \sum_{(\lambda,\mu)\vdash(m,n)} V_{\lambda,\mu}^{\oplus m_{\lambda,\mu}(\mathcal{P}^{!}(m,n))},$$

then we have

$$\mathcal{P}^{¡}(m,n) = \sum_{(\lambda,\mu)\vdash(m,n)} V_{\lambda,\mu}^{\oplus m_{\lambda',\mu'}(\mathcal{P}^{!}(m,n))}.$$

In other words, $m_{\lambda,\mu}(\mathcal{P}^{!}(m,n)) = m_{\lambda',\mu'}(\mathcal{P}^{¡}(m,n))$.

Proof. We have the isomorphisms of modules

$$V_{\lambda,\mu}^{\vee} = sgn_m \otimes V_{\lambda,\mu}^{*} \otimes sgn_n \simeq sgn_m \otimes V_{\lambda,\mu} \otimes sgn_n \simeq V_{\lambda',\mu'},$$

where the middle one is given by Proposition 1.27. □

Since koszulity of a properad implies that it is quadratic, if a properad $\mathcal{P}$ has no quadratic presentation, it cannot be Koszul. Thus if we want to find a minimal model (if it exists) of a non-quadratic properad, we want to define a generalization of koszulity. That is what S. Merkulov and B. Vallette did in [MV09, Section 5].

Let E be an $\mathbb{S}$-bimodule concentrated in degree 0 and $R \subset \mathcal{F}(E)$. For $k \in \mathbb{N}^*$, let us denote by $\pi_k : \mathcal{F}(E) \longrightarrow \mathcal{F}^{(k)}(E)$ the natural projection. We set $R_k := \pi_k(R)$.

Definition 2.34 (Homotopy koszulity, [MV09, Section 5]). Let $\mathcal{P} = \mathcal{F}(E)/(R)$ be a properad. We say that the properad $\mathcal{P}$ is *homotopy Koszul* if :

(i) the quadratic properad $\mathcal{P}_2 = \mathcal{F}(E)/(R_2)$ is Koszul,

(ii) $\mathcal{P}$ and $\mathcal{P}_2$ are isomorphic as $\mathbb{S}$-bimodules,

(iii) There is on $\mathcal{P}$ an extra grading $\mathcal{P} = \bigoplus_\lambda \mathcal{P}_\lambda$ of finite-dimensional $\mathbb{S}$-bimodules.

Theorem 2.35 ([MV09, Theorem 40]). *If $\mathcal{P}$ is homotopy Koszul, it admits a minimal model of the form* $(\mathcal{F}(\Sigma^{-1}\overline{\mathcal{P}}_2^{¡}), \delta)$.

This theorem does not give the form of the differential, but in this manuscript it only serves as a motivation to prove homotopy koszulity of non-quadratic properads, we will not construct minimal models of such properads. For examples of minimal models of non-quadratic properads, one can see [Mar06] and [MV09, Section 9].

2.2.3 Methods to prove or refute (homotopy) koszulity

In this subsection, we define some tools to prove or refute koszulity of a quadratic properad. Note that these tools can thus be used to prove or refute homotopy koszulity of non-quadratic properads.

Koszul complex

In [Val07, Section 7], B. Vallette introduced the Koszul complex of a properad in order to prove koszulity of properads more easily than with the definition above or the fact that $\overline{\mathcal{B}}^c(\mathcal{P}^{\text{i}}) \to \mathcal{P}$ is a quasi-isomorphism. In this manuscript, the Koszul complex will be used to prove non koszulity of properads rather than koszulity. See [Val07, Section 7] for the definition of the Koszul complex.

Theorem 2.36 (Koszul criterion, [Val07, Theorem 7.8])**.** *Let $\mathcal{P}$ be a weight graded connected dg-properad. The following are equivalent :*

(i) The properad $\mathcal{P}$ is Koszul.

(ii) The Koszul complex $\mathcal{P} \boxtimes \mathcal{P}^{\text{i}}$ is acyclic.

(iii) The Koszul complex $\mathcal{P}^{\text{i}} \boxtimes \mathcal{P}$ is acyclic.

(iv) The morphism $\overline{\mathcal{B}}^c(\mathcal{P}^{\text{i}}) \to \mathcal{P}$ is a quasi-isomorphism.

Replacement rule

Here we take again the definitions from [Val07, Section 8]. Let V and W be two $\mathbb{S}$-bimodules, $R \subset \mathcal{F}^{(2)}(V)$, $S \subset \mathcal{F}^{(2)}(W)$ and

$$D \subset (V \boxtimes_{(1)} W) \oplus (W \boxtimes_{(1)} V).$$

We denote by $\mathcal{A}$ and $\mathcal{B}$ the properads given respectively by $\mathcal{F}(V)/(R)$ and $\mathcal{F}(W)/(S)$, and set

$$\mathcal{P} := \mathcal{F}(V \oplus W)/(R \oplus D_\lambda \oplus S).$$

Definition 2.37 (Replacement rule, [Val07, Section 8])**.** Let λ be a morphism of $\mathbb{S}$-bimodules

$$\lambda : W \boxtimes_{(1)} V \to V \boxtimes_{(1)} W$$

such that D is the image of

$$(\mathrm{id}, -\lambda) : W \boxtimes_{(1)} V \to (W \boxtimes_{(1)} V) \oplus (V \boxtimes_{(1)} W).$$

If the two morphisms

$$\underbrace{\mathcal{A}}_{1} \boxtimes \underbrace{\mathcal{B}}_{2} \to \mathcal{P} \text{ and } \underbrace{\mathcal{A}}_{2} \boxtimes \underbrace{\mathcal{B}}_{1} \to \mathcal{P}$$

are injective, we call λ a *replacement rule* and we denote D by D_λ.

Theorem 2.38 ([Val07, Proposition 8.4])**.** *Let $\mathcal{P}$ be a properad of the form*

$$\mathcal{P} = \mathcal{F}(V \oplus W)/(R \oplus D_\lambda \oplus S)$$

with $R \subset \mathcal{F}^{(2)}(V)$, $S \subset \mathcal{F}^{(2)}(W)$ and λ a replacement rule, and such that $\sum_{n,m} \dim((V \oplus W)(n,m))$ is finite and $\mathcal{A} := \mathcal{F}(V)/(R)$ and $\mathcal{B} := \mathcal{F}(W)/(S)$ are Koszul properads. Then $\mathcal{P}$ is Koszul.

This theorem proves koszulity of a properad, based on koszulity of two other properads. Most of the time, we use this theorem in the case where V is given by operations (with one output) and W is given by cooperations (with one input), thus $\mathcal{A}$ and $\mathcal{B}^{\mathrm{op}}$ are operads, where $\mathcal{B}^{\mathrm{op}}$ is the reversed $\mathbb{S}$-bimodule of $\mathcal{B}$ (see [Val07, Section 8]), and koszulity of $\mathcal{A}$ and $\mathcal{B}$ can be proved using operadic tools such as rewriting theory. Moreover, proving that λ is a replacement rule can be done computing dimensions of modules.

2.3 Examples of properads

For a vector space V, one can define the endomorphism properad End_V defined for every $m, n \in \mathbb{N}$ by $\mathrm{End}_V(m,n) = \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$, and with the composition product given by the composition of morphisms. For a properad $\mathcal{P}$ and a vector space V, we call structure of $\mathcal{P}$-algebra on V any morphism of properad $\mathcal{P} \to \mathrm{End}_V$. Thus we can define the category $\mathcal{P}$-alg of $\mathcal{P}$-algebras. We say that $\mathcal{P}$ encodes a structure of algebras if the category of $\mathcal{P}$-algebras is isomorphic to the category of these algebras.

In this section, we give examples of properads encoding well known structures of bialgebras, such as associative bialgebras, infinitesimal associative bialgebras, Lie bialgebras and double Poisson bialgebras. We also discuss on the (homotopy) Koszul property of these properads.

2.3.1 Associative bialgebras : $\mathcal{B}$

Definition 2.39. A *(non-unital non-counital) associative bialgebra* is the data (B, μ, Δ) of :

(i) a vector space B,

(ii) an associative product $\mu : B \otimes B \longrightarrow B$:

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu),$$

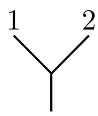
(iii) a coassociative coproduct $\Delta : B \longrightarrow B \otimes B$:

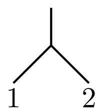
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$$

which is a morphism of associative algebras :

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (23) \circ (\Delta \otimes \Delta).$$

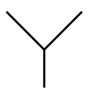
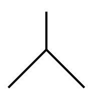
In order to define the properad $\mathcal{B}$ encoding (non-unital non-counital) associative bialgebras, what we need is the generating operations. Here we have a product of biarity $(1, 2)$ and a coproduct of biarity $(2, 1)$, which we will denote respectively by



and


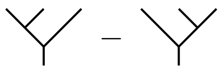
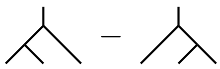
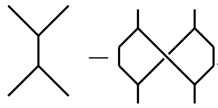
.

Because we do not ask for any symmetry on these operations, we will take as a generating space for the properad $\mathcal{B}$ the $\mathbb{S}$-bimodule

$E :=$  $\oplus$  ,

that corresponds to the regular representation of $\mathbb{S}_1 \times \mathbb{S}_2^{\mathrm{op}}$ in biarity $(1, 2)$ and the regular representation of $\mathbb{S}_2 \times \mathbb{S}_1^{\mathrm{op}}$ in biarity $(2, 1)$.

The relations will be given by associativity, coassociativity and the fact that Δ is a morphism of associative algebras (or equivalently, μ is a morphism of coassociative coalgebras). We will denote these relations by the following elements in $\mathcal{F}(E)$:

$R_{\text{Ass}} =$  , $R_{\text{CoAss}} =$  and $R_{\text{BiAss}} =$ .

Thus we can take $R = R_{\text{Ass}} \oplus R_{\text{CoAss}} \oplus R_{\text{BiAss}}$, and define the properad $\mathcal{B}$ by

$$\mathcal{B} := \mathcal{F}(E)/(R).$$

S. Merkulov and B. Vallette proved the following in [MV09, Proposition 41].

Theorem 2.40. *The properad $\mathcal{B}$ is homotopy Koszul.*

Thus there exists a minimal model of $\mathcal{B}$, see [MV09, Corollary 42], or [Mar06] for a detailled computation in small weights.

2.3.2 Infinitesimal associative bialgebras : $\varepsilon\mathcal{B}$

Definition 2.41. An *infinitesimal associative bialgebra* is the data (B, μ, Δ) of :

(i) a vector space B,

(ii) an associative product $\mu : B \otimes B \longrightarrow B$:

$$\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu),$$

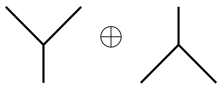
(iii) a coassociative coproduct $\Delta : B \longrightarrow B \otimes B$:

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,$$

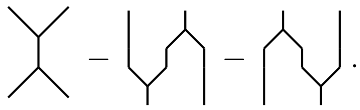
such that :

$$\Delta \circ \mu = (\mu \otimes \text{id}) \circ (\text{id} \otimes \Delta) + (\text{id} \otimes \mu) \circ (\Delta \otimes \text{id}).$$

By the same method as for $\mathcal{B}$, we define the properad $\varepsilon\mathcal{B}$ encoding infinitesimal associative bialgebras. We take the $\mathbb{S}$-bimodule

$E :=$ 

and we define the relation

$R_{\varepsilon\,\mathrm{BiAss}} =$ 

Thus we take $R = R_{\mathrm{Ass}} \oplus R_{\mathrm{CoAss}} \oplus R_{\varepsilon\,\mathrm{BiAss}}$ and

$$\varepsilon\mathcal{B} = \mathcal{F}(E)/(R).$$

B. Vallette proved the following theorem in [Val07, Corollary 8.5], using the replacement rule method.

Theorem 2.42. *The properad $\varepsilon\mathcal{B}$ is Koszul.*

2.3.3 Lie bialgebras : $\mathcal{B}i\mathcal{L}ie$

Definition 2.43. A *Lie bialgebra* is the data (L, b, δ) of :

(i) a vector space L,

(ii) a skew-symmetric Lie bracket $b : L \otimes L \longrightarrow L$:

$$\mu \circ (\mathrm{id} \otimes \mu) + \mu \circ (\mathrm{id} \otimes \mu) \circ (123) + \mu \circ (\mathrm{id} \otimes \mu) \circ (132) = 0,$$

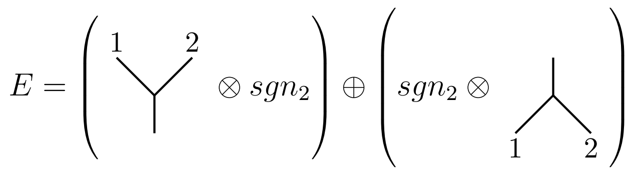
(iii) a skew-symmetric Lie cobracket $\delta : L \longrightarrow L \otimes L$:

$$(\mathrm{id} \otimes \Delta) \circ \Delta + (123) \circ (\mathrm{id} \otimes \Delta) \circ \Delta + (132) \circ (\mathrm{id} \otimes \Delta) \circ \Delta = 0$$

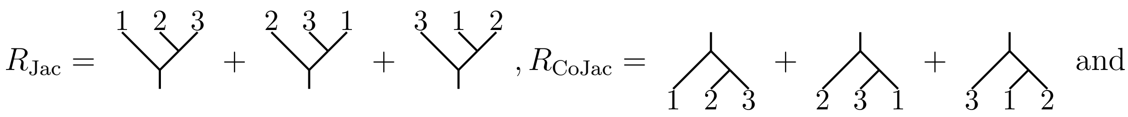
such that :

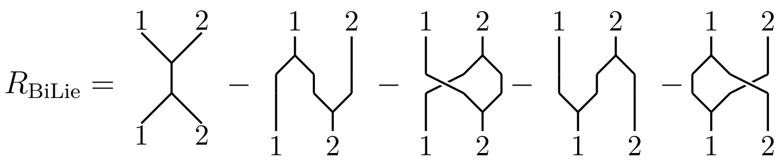
$$\delta \circ b = (\mathrm{id} \otimes b) \circ (\delta \otimes \mathrm{id}) + (\mathrm{id} \otimes b) \circ (12) \circ (\mathrm{id} \otimes \delta) + (b \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) + (b \otimes \mathrm{id}) \circ (23) \circ (\delta \otimes \mathrm{id}).$$

Here we take as a generating space



and the following elements in $\mathcal{F}(E)$:





and take $\mathcal{B}i\mathcal{L}ie = \mathcal{F}(E)/(R)$ with $R = R_{\text{Jac}} \oplus R_{\text{CoJac}} \oplus R_{\text{BiLie}}$.

Again, B. Vallette proved the following, also in [Val07, Corollary 8.5].

Theorem 2.44. *The properad $\mathcal{B}i\mathcal{L}ie$ is Koszul.*

2.3.4 Double Poisson algebras : $\mathcal{DP}ois$

As an example of a properad where the generators are not concentrated in biarities $(1, 2)$ and $(2, 1)$, J. Leray studied in [Ler20] the properad of double Poisson algebras. The notion of Double Poisson algebras has been introduced by M. Van den Bergh in [Ber07].

Definition 2.45. A *double Poisson algebra* is the data (P, μ, ψ) of :

(i) a vector space P,

(ii) an associative product $\mu : P \otimes P \longrightarrow P$:

$$\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu),$$

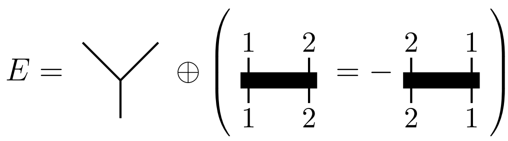
(iii) a skew-symmetric double bracket $\psi : P \otimes P \longrightarrow P \otimes P$ such that :

$$(\psi \otimes \text{id}) \circ (\text{id} \otimes \psi) + (123) \circ (\psi \otimes \text{id}) \circ (\text{id} \otimes \psi) \circ (123) + (132) \circ (\psi \otimes \text{id}) \circ (\text{id} \otimes \psi) \circ (132) = 0$$

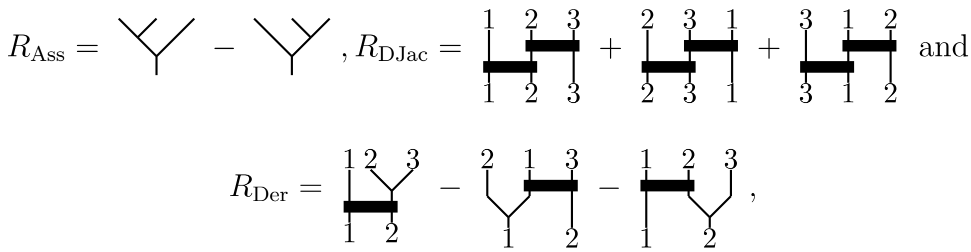
such that :

$$\psi \circ (\mathrm{id} \otimes \mu) = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) \circ (12) + (\mathrm{id} \otimes \mu) \circ (\psi \otimes \mathrm{id}).$$

For the properad $\mathcal{DP}ois$ encoding double Poisson algebras, we take as a generating space



and the relations



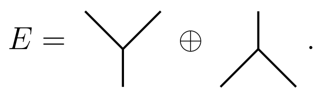
thus we define $R = R_{\mathrm{Ass}} \oplus R_{\mathrm{DJac}} \oplus R_{\mathrm{Der}}$ and $\mathcal{DP}ois = \mathcal{F}(E)/(R)$.

In [Ler20, Theorem 5.11], J. Leray proved the following.

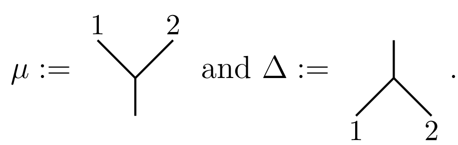
Theorem 2.46. *The properad $\mathcal{DP}ois$ is Koszul.*

2.4 Study of binary free properads in low weights

The goal of this section is to compute, in low weights, a simple example of a free properad, the one generated by a product and a coproduct without symmetries, so that



These results will be useful later to compute dimensions of spaces. Set



Remark 2.47. One can also use the `SageMath` script in [Néd] to compute these dimensions and graphs, see Appendix A.

2.4.1 Weight 2

We have
$$\mathcal{F}^{(2)}(E) = E \boxtimes_{(1)} E.$$
Thus, $\mathcal{F}^{(2)}(E)(m,n)$ is generated as a vector space by elements of the form
$$\sigma_3 \otimes (1,\ldots,e_2,\ldots,1) \otimes \sigma_2 \otimes (1,\ldots,e_1,\ldots,1) \otimes \sigma_1,$$
where $\sigma_1 \in \mathbb{S}_n$, $\sigma_3 \in \mathbb{S}_m$, $\sigma_2 \in \mathbb{S}^c_{((1,\ldots,\varepsilon_2,\ldots,1),(1,\ldots,\varepsilon_1,\ldots,1))}$, $(e_1,\varepsilon_1) \in \{(\Delta,2),(\mu,1)\}$ and $(e_2,\varepsilon_2) \in \{(\Delta,1),(\mu,2)\}$. In fact, one can rewrite every element of E using actions of $\mathbb{S}_2$ to get only μ's and Δ's. Let x be an element of this form and $N := |(1,\ldots,\varepsilon_1,\ldots,1)| = |(1,\ldots,\varepsilon_2,\ldots,1)|$. The strategy to compute $\mathcal{F}^{(2)}(E)$ will be the following.

(i) We first reason on N. We look for every $N \in \mathbb{N}$ such that there exists $(\varepsilon_1,\varepsilon_2) \in \{1,2\}^2$ such that $\mathbb{S}^c_{((1,\ldots,\varepsilon_1,\ldots,1),(1,\ldots,\varepsilon_2,\ldots,1))} \subset \mathbb{S}_N$ is not empty.

   Let us first remark that for $\bar{i}$ a tuple only composed by 1's, and $\bar{j}$ a tuple such that $\left|\bar{i}\right| = \left|\bar{j}\right| = N$, we have that $\mathbb{S}^c_{(\bar{i},\bar{j})}$ is not empty if and only if $\bar{j} = (N)$, and in this case $\mathbb{S}^c_{(\bar{i},\bar{j})} = \mathbb{S}_N$. Thus if $\varepsilon_1 = 1$ or $\varepsilon_2 = 1$, for $N \geq 3$,$\mathbb{S}^c_{(\bar{i},\bar{j})}$ is empty.

   Suppose now that $\varepsilon_1 = \varepsilon_2 = 2$. If $N \geq 4$, $\mathbb{S}^c_{((1,\ldots,\varepsilon_1,\ldots,1),(1,\ldots,\varepsilon_2,\ldots,1))}$ is empty. Thus N should be lower or equal to 3.

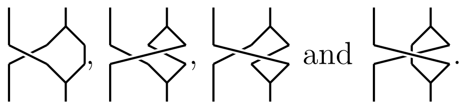
(ii) For each such N, we look for every $(\varepsilon_1,\varepsilon_2) \in \{1,2\}^2$ such that $\mathbb{S}^c_{((1,\ldots,\varepsilon_1,\ldots,1),(1,\ldots,\varepsilon_2,\ldots,1))}$ is not empty.

   For $N = 3$, we must have $\varepsilon_1 = \varepsilon_2 = 2$. For $N = 2$, we must either have $\varepsilon_1 = 2$ and $\varepsilon_2 = 1$, or $\varepsilon_1 = 1$ and $\varepsilon_2 = 2$, or $\varepsilon_1 = \varepsilon_2 = 2$. For $N = 1$, we must have $\varepsilon_1 = \varepsilon_2 = 1$.

(iii) Every such possibility gives us the elements $(1,\ldots,e_1,\ldots,1)$ and $(1,\ldots,e_2,\ldots,1)$ to consider, then we consider every possible connected permutation σ_2 between them.

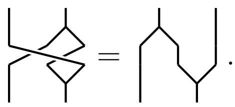
   Let us look at the case $N = 3$: the possible tuples are
$$((1,2),(1,2)),((1,2),(2,1)),((2,1),(1,2)),((2,1) \text{ and } (2,1)).$$

For example, $\mathbb{S}^c_{((1,2),(1,2))} = \{[2,1,3],[2,3,1],[3,1,2],[3,2,1]\}$. The corresponding graphs are the following



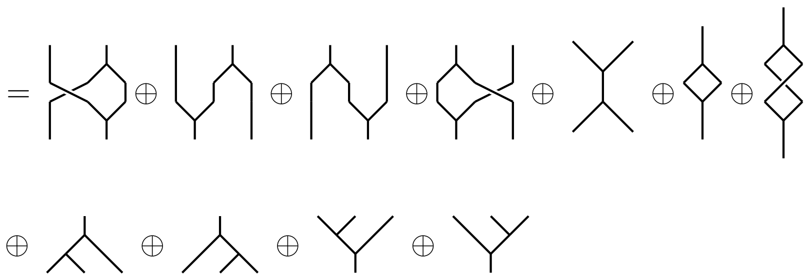
We do the same for every tuples.

(iv) We compute the equivalence relation $\sim$ in the algebraic definition of $\boxtimes$. Here we just give an example of identification thanks to the relation $\sim$:



(v) Finally, we consider every possible permutations for σ_1 and σ_3. We have

$\mathcal{F}^{(2)}(E) =$



Proposition 2.48. *We have*

(i) $\mathcal{F}^{(2)}(E)(2,2) = \mathbb{C}[\mathbb{S}_2 \times \mathbb{S}_2^{\mathrm{op}}]^{\oplus 5}$,

(ii) $\mathcal{F}^{(2)}(E)(1,1) = \mathbb{C}[\mathbb{S}_1 \times \mathbb{S}_1^{\mathrm{op}}]^{\oplus 2}$,

(iii) $\mathcal{F}^{(2)}(E)(1,3) = \mathbb{C}[\mathbb{S}_1 \times \mathbb{S}_3^{\mathrm{op}}]^{\oplus 2}$,

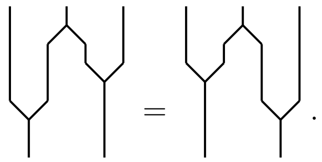
(iv) $\mathcal{F}^{(2)}(E)(3,1) = \mathbb{C}[\mathbb{S}_3 \times \mathbb{S}_1^{\mathrm{op}}]^{\oplus 2}$.

2.4.2 Weight 3

Here we first compute

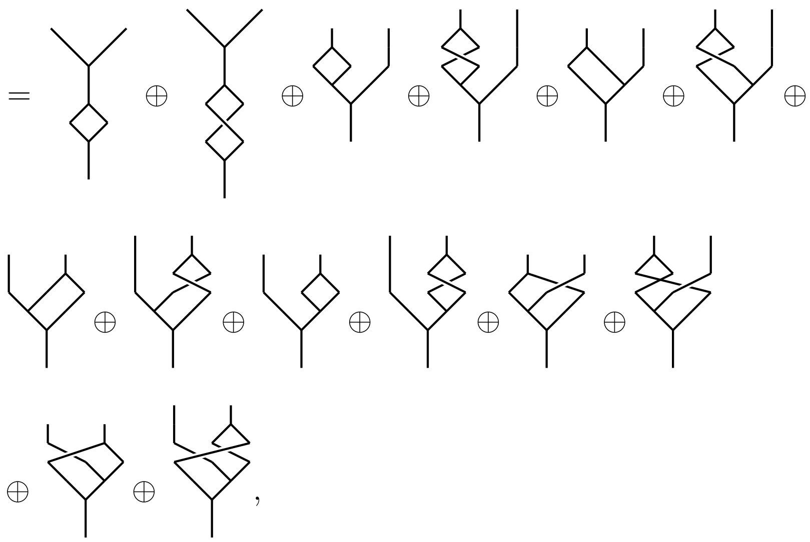
$$\left(E \boxtimes_{(1)} \mathcal{F}^{(2)}(E)\right) \oplus \left(\mathcal{F}^{(2)}(E) \boxtimes_{(1)} E\right)$$

and look at underlying graphs with labbelled vertices by elements of E. Other than this last extra step, the strategy is exactly the same. An example of identification one can have in this extra step is

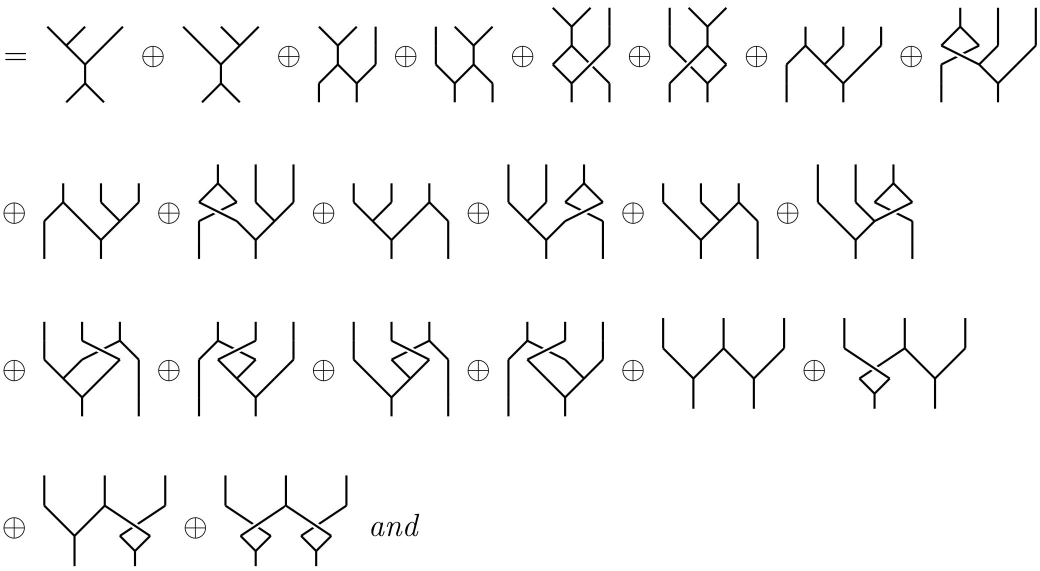


Proposition 2.49. *We have*

$$\mathcal{F}^{(3)}(E)(1,2) =$$

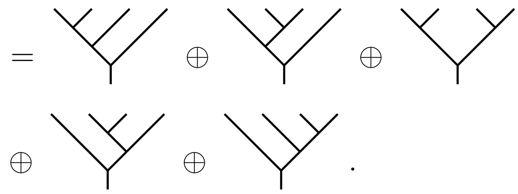


$\mathcal{F}^{(3)}(E)(2,3) =$



and

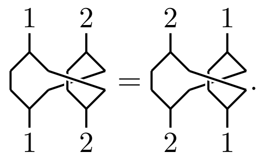
$\mathcal{F}^{(3)}(E)(1,4) =$



The spaces $\mathcal{F}^{(3)}(E)(2,1)$, $\mathcal{F}^{(3)}(E)(3,2)$ *and* $\mathcal{F}^{(3)}(E)(4,1)$ *are obtained by rotating the graphs of* $\mathcal{F}^{(3)}(E)(1,2)$, $\mathcal{F}^{(3)}(E)(2,3)$ *and* $\mathcal{F}^{(3)}(E)(1,4)$ *respectively upside down. Moreover, we have*

$$\begin{aligned}\mathcal{F}^{(3)}(E) =& \mathcal{F}^{(3)}(E)(1,2) \oplus \mathcal{F}^{(3)}(E)(2,1) \oplus \mathcal{F}^{(3)}(E)(2,3) \\ & \oplus \mathcal{F}^{(3)}(E)(3,2) \oplus \mathcal{F}^{(3)}(E)(1,4) \oplus \mathcal{F}^{(3)}(E)(4,1).\end{aligned}$$

Remark 2.50. In this case and in lower weights, we get only copies of regular representations, but looking in higher weights, the free properad on E can have symmetries. For example, we have the following equality in $\mathcal{F}^{(4)}(E)(2,2)$:



Chapter 3

The properad of rack bialgebras

The initial goal of this thesis was to study rack bialgebras, more specifically, the properad $\mathcal{R}ack\mathcal{B}$ of rack bialgebras. However this properad seems more complicated than expected and does not verify most of the properties we were looking for.

Groups, Lie algebras and Hopf algebras are naturally linked to each other, and there are generalizations of these structures. We have racks for groups, Leibniz algebras for Lie algebras, and the natural structure generalizing Hopf algebras and related to racks and Leibniz algebras is that of rack bialgebras. One can see [Ale+18b] and [Ale+18a] for more details on these structures.

In the first section we describe rack bialgebras and the properad $\mathcal{R}ack\mathcal{B}$. In the second section we define and study the quadratic properad associated to $\mathcal{R}ack\mathcal{B}$, which is the first step to show homotopy koszulity of a properad. In the last section, we discuss more about $\mathcal{R}ack\mathcal{B}$ and the fact that it may not be homotopy Koszul.

3.1 Rack bialgebras

Rack bialgebras are a generalization of Hopf algebras, in the sense that we weaken the associative product to a rack product. We will use here the Sweedler notation when there is no ambiguity, i.e. for a linear map $\Delta : R \longrightarrow R \otimes R$ on a vector space R and $x \in R$, we denote $\Delta(x)$ by $\sum_{(x)} x^{(1)} \otimes x^{(2)}$.

Definition 3.1. A *(non-unital non-counital non cocommutative) rack bialgebra* is a vector space R endowed with linear maps $\mu : R \otimes R \longrightarrow R$ and $\Delta : R \longrightarrow R \otimes R$ such that R is a coassociative coalgebra with respect to Δ, μ is a morphism of coalgebras and is self distributive for Δ. In other words, if we denote by $x \rhd y := \mu(x \otimes y)$, for $x, y, z \in R$, we have :

(i) coassociativity of Δ :

$$\sum_{(x)}\sum_{(x^{(1)})}(x^{(1)})^{(1)}\otimes(x^{(1)})^{(2)}\otimes x^{(2)}=\sum_{(x)}\sum_{(x^{(2)})}x^{(1)}\otimes(x^{(2)})^{(1)}\otimes(x^{(2)})^{(2)},$$

thus we denote this element by $\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$,

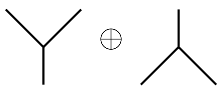
(ii) μ is a morphism of coalgebras :

$$\Delta(x \rhd y)) = \sum_{(x)}\sum_{(y)}(x^{(1)} \rhd y^{(1)}) \otimes (x^{(2)} \rhd y^{(2)}),$$

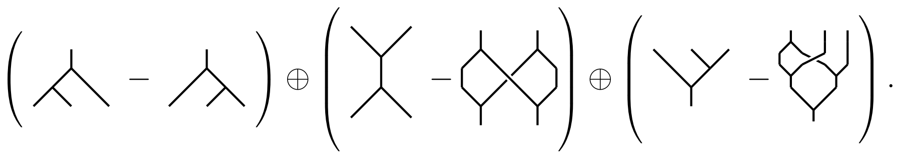
(iii) μ is self distributive for Δ :

$$(x \rhd (y \rhd z)) = \sum_{(x)}((x^{(1)} \rhd y) \rhd (x^{(2)} \rhd z)).$$

As in Section 2.3, we will construct the properad $\mathcal{R}ack\mathcal{B}$ encoding rack bialgebras from the definition of this structure. We set

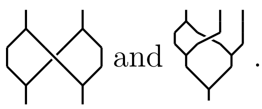
$E :=$ 

and

$R :=$ 

Thus we define the properad $\mathcal{R}ack\mathcal{B} := \mathcal{F}(E)/(R)$.

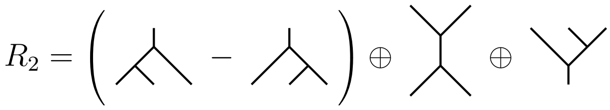
The main issue with this properad is that it is not quadratic because of the two graphs



This means that in order to find a minimal model of $\mathcal{R}ack\mathcal{B}$, one should try to determine if it is homotopy Koszul.

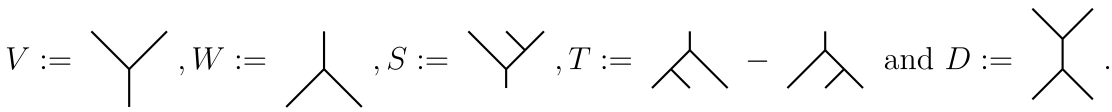
3.2 The quadratic properad $\mathcal{R}ack\mathcal{B}_2$

The quadratic properad $\mathcal{R}ack\mathcal{B}_2$ associated to $\mathcal{R}ack\mathcal{B}$ is given by generators E and relations



Theorem 3.2. *The properad $\mathcal{R}ack\mathcal{B}_2$ is Koszul.*

Proof. Here we can use a replacement rule, see Theorem 2.38, to study this properad. In fact, this properad has the form $\mathcal{R}ack\mathcal{B}_2 = \mathcal{F}(V \oplus W)/(S \oplus D \oplus T)$ where



Moreover, D is the image of

$$(\mathrm{id} - \lambda) : \underbrace{(I \oplus W)}_{1} \boxtimes \underbrace{(I \oplus V)}_{1} \longrightarrow (\underbrace{(I \oplus W)}_{1} \boxtimes \underbrace{(I \oplus V)}_{1}) \oplus (\underbrace{(I \oplus V)}_{1} \boxtimes \underbrace{(I \oplus W)}_{1})$$

with λ the zero $\mathbb{S}$-bimodule morphism

$$\lambda : \underbrace{(I \oplus W)}_{1} \boxtimes \underbrace{(I \oplus V)}_{1} \longrightarrow \underbrace{(I \oplus V)}_{1} \boxtimes \underbrace{(I \oplus W)}_{1}.$$

According to Theorem 2.38, in order to prove that $\mathcal{R}ack\mathcal{B}_2$ is Koszul, we only have to prove the following

(i) The properad $\mathcal{N} := \mathcal{F}(V)/(S)$ is Koszul.

(ii) The properad $\mathcal{C} := \mathcal{F}(W)/(T)$ is Koszul.

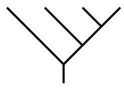
(iii) The morphism λ is a replacement rule, in other words the morphisms of $\mathbb{S}$-bimodules

$$\underbrace{\mathcal{N}}_{1} \boxtimes \underbrace{\mathcal{C}}_{2} \to \mathcal{R}ack\mathcal{B}_2 \text{ and } \underbrace{\mathcal{N}}_{2} \boxtimes \underbrace{\mathcal{C}}_{1} \to \mathcal{R}ack\mathcal{B}_2 \tag{3.1}$$

are injective.

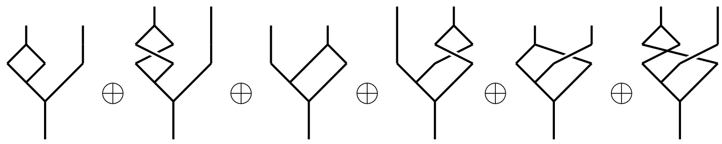
(iv) $\sum_{m,n} \dim((V \oplus W)(m,n))$ is finite.

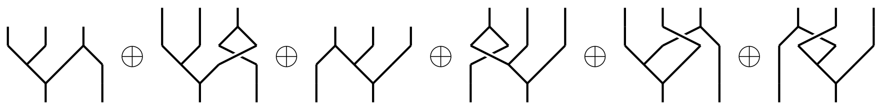
(iv) is clear. $\mathcal{C}$ is the reversed $\mathbb{S}$-bimodule of the Koszul operad of associative algebras $\mathcal{A}$, thus (ii) is true. $\mathcal{N}$ is an operad for which every critical pair of monomials is confluent (see [LV12, Theorem 8.3.1]), because they are all of the form

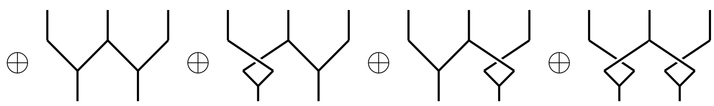
 ,

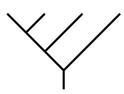
which only rewrites to 0. Thus by [LV12, Theorem 8.3.1], $\mathcal{N}$ is Koszul and (i) is true.

Thus we have to prove (iii). In order to do so, we can describe the weight 3 of $\mathcal{R}ack\mathcal{B}_2$ and the $\mathbb{S}$-bimodules $\underbrace{\mathcal{N}}_{1} \boxtimes \underbrace{\mathcal{C}}_{2}$ and $\underbrace{\mathcal{N}}_{2} \boxtimes \underbrace{\mathcal{C}}_{1}$ because these $\mathbb{S}$-bimodules are only sent to weight 3 elements of $\mathcal{R}ack\mathcal{B}_2$. By Section 2.4, one can determine a basis of the space $(R_2)^{(3)}$ of relations in weight 3 and find, thank's to the dimensions, that

$(\mathcal{R}ack\mathcal{B}_2)^{(3)}(1,2) =$  ,

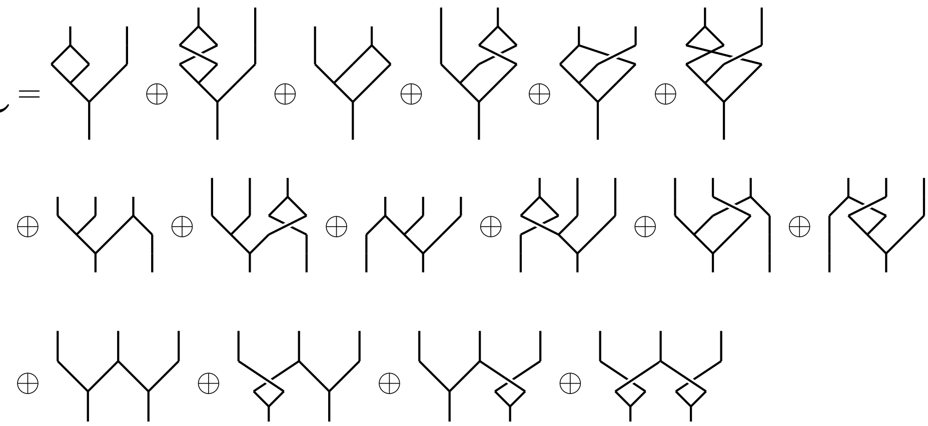
$(\mathcal{R}ack\mathcal{B}_2)^{(3)}(2,3) =$ 

 ,

$(\mathcal{R}ack\mathcal{B}_2)^{(3)}(1,4) =$  .

The spaces $(\mathcal{R}ack\mathcal{B}_2)^{(3)}(2,1)$, $(\mathcal{R}ack\mathcal{B}_2)^{(3)}(3,2)$ and $(\mathcal{R}ack\mathcal{B}_2)^{(3)}(4,1)$ are obtained by rotating these graphs upside down. By the same method as in Section 2.4, we have

$\underbrace{\mathcal{N}}_{2} \boxtimes \underbrace{\mathcal{C}}_{1} =$



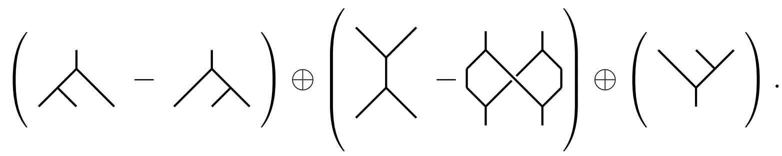
and $\underbrace{\mathcal{N}}_{1} \boxtimes \underbrace{\mathcal{C}}_{2}$ is obtained by rotating these graphs upside down. Thus we can see that the morphisms of (3.1) are injective because they send elements of the bases on different elements of the base of $\mathcal{R}ack\mathcal{B}_2$.

□

3.3 The case of $\mathcal{R}ack\mathcal{B}_1$

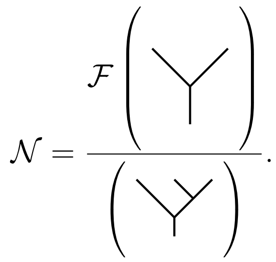
As we will see later, $\mathcal{R}ack\mathcal{B}$ is hard to study because of the self-distributivity relation, but first we can look at another properad which is very similar to the properad $\mathcal{B}$ of associative bialgebras. Let us define the properad $\mathcal{R}ack\mathcal{B}_1 = \mathcal{F}(E)/(R_1)$ where

$R_1 =$



In [EE05, Proposition 6.2], B. Enrinquez and P. Etingof show that the natural morphism of $\mathbb{S}$-bimodules between $\mathcal{A} \boxtimes \mathcal{C}$ and $\mathcal{B}$ is an isomorphism. Here the only difference between the properads $\mathcal{R}ack\mathcal{B}_1$ and $\mathcal{B}$ is the relation of associativity which is replaced by a right nilpotency of the operation of biarity $(1, 2)$. Thus we can adapt the proof from

[EE05] by replacing the operad $\mathcal{A}$ by the operad



We call right nilpotent algebras the algebras over $\mathcal{N}$. First we need some properties on right nilpotent algebras.

Proposition 3.3. *(i) The free right nilpotent algebra over a vector space E is the vector space $\mathcal{N}(E) := \bigoplus_{n\geq 1} E^{\otimes n}$ together with the map*

$$\mu : \mathcal{N}(E) \otimes \mathcal{N}(E) \longrightarrow \mathcal{N}(E)$$
$$x_1 \dots x_n \otimes y_1 \dots y_m \longmapsto \begin{cases} x_1 \dots x_n y_1 & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases}$$

(ii) If (N_1, μ_1), (N_2, μ_2) are two right nilpotent algebras, $N_1 \otimes N_2$ is a right nilpotent algebra with the operation $\mu = (\mu_1 \otimes \mu_2) \circ (1324)$

We deduce from this the following.

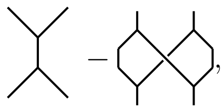
Proposition 3.4. *Let C be a coalgebra. Let $\mathcal{N}(C)$ be the free right nilpotent algebra over C. The cooperation $\Delta : C \longrightarrow C \otimes C$ can be extended to $\mathcal{N}(C)$, and this defines a structure of $\mathcal{R}ack\mathcal{B}_1$-bialgebra over $\mathcal{N}(C)$.*

We denote by $\boxtimes_{nc}$ the non connected composition product, which replaces in Definition 2.6 the set of connected graphs by the set of (not necessarily connected) graphs, and for $\mathcal{P}$ a properad, let us denote by $\mathcal{P}_{nc}$ the PROP (non connected) generated by $\mathcal{P}$, see [EE05, Section 2]. We are leaving the world of properads for the sake of the following proof because we will need to consider non connected elements in the proof of the following theorem. Let $i_{p,q} : (\mathcal{N}_{nc} \boxtimes_{nc} \mathcal{C}_{nc})(q,p) \longrightarrow (\mathcal{R}ack\mathcal{B}_1)_{nc}(q,p)$ be the composition of the inclusions of $\mathcal{N}_{nc}$ and $\mathcal{C}_{nc}$ in $(\mathcal{R}ack\mathcal{B}_1)_{nc}$ with the composition product of $(\mathcal{R}ack\mathcal{B}_1)_{nc}$.

Theorem 3.5. *$i_{p,q}$ is an isomorphism of $\mathbb{S}$-bimodules.*

Proof. Here we adapt the proof of [EE05, Proposition 6.2], which is made on several steps.

Step 1 : $i_{p,q}$ is surjective : Thanks to the relation



we can write every elements of $(\mathcal{R}ack\mathcal{B}_1)_{nc}$ with coproducts above products. Thus $i_{p,q}$ is surjective.

Step 2 : We define an isomorphism : Let us define a morphism

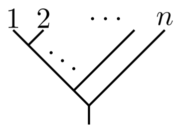
$$i : \bigoplus_{N_1,\dots,N_q>0} C_{N_1,\dots,N_q} \to (\mathcal{N}_{nc} \boxtimes_{nc} \mathcal{C}_{nc})(q,p),$$

where $C_{N_1,\dots,N_q} := \mathcal{C}_{nc}(N_1 + \cdots + N_q, p)$. For $x \in C_{N_1,\dots,N_q}$, we define $i(x)$ by

$$i(x) := \mathrm{id} \otimes (\mu^{N_1}, \dots, \mu^{N_q}) \otimes \mathrm{id} \otimes x \otimes \mathrm{id},$$

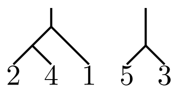
where for $n \in \mathbb{N}$, μ^n is the operation

$$\mu^n :=$$



For example, the image of

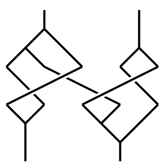
$x :=$



$\in C_{2,3}$

is

$i(x) =$



$\in (\mathcal{N}_{nc} \boxtimes_{nc} \mathcal{C}_{nc})(2,2).$

We can consider $\bigoplus_{N_1,\dots,N_q>0} C_{N_1,\dots,N_q}$ as an $\mathbb{S}$-bimodule with the left $\mathbb{S}_q$-module structure by permutation of the N_i's. Thus i is a morphism of $\mathbb{S}$-bimodules. Since every element of $(\mathcal{N}_{nc} \boxtimes_{nc} \mathcal{C}_{nc})(q,p)$ can be uniquely written

$$\mathrm{id} \otimes (\mu^{N_1}, \dots, \mu^{N_q}) \otimes \mathrm{id} \otimes x \otimes \mathrm{id},$$

i is an isomorphism of $\mathbb{S}$-bimodules. Thus, in order to show that $i_{p,q}$ is injective, it is enough to show that $i_{p,q} \circ i$ is injective. Fix $x \in C_{N_1,\dots,N_q}$ non zero, we will define a morphism of $\mathbb{S}$-bimodules $\alpha : \mathcal{R}ack\mathcal{B}_1(p,q) \to V$, with V a vector space, such that $(\alpha \circ i_{p,q} \circ i)(x)$ is not zero.

Step 3 : Definition of α : Let C be a coalgebra. By Proposition 3.4, $\mathcal{N}(C)$ is an $(\mathcal{R}ack\mathcal{B}_1)_{nc}$-bialgebra. Let

$$(\cdot)_{\mathcal{N}(C)} : \mathcal{R}ack\mathcal{B}_1(p,q) \longrightarrow \mathrm{Hom}(\mathcal{N}(C)^{\otimes p}, \mathcal{N}(C)^{\otimes q})$$

be the structural morphism of $\mathcal{N}(C)$ and $\varphi_{N_1\dots,N_q}$ be the morphism

$$\varphi_{N_1,\dots,N_q} : \mathrm{Hom}(\mathcal{N}(C)^p, \mathcal{N}(C)^q) \longrightarrow \mathrm{Hom}(C^p, C^{N_1+\cdots+N_q}),$$

which associates to a morphism $f : \mathcal{N}(C)^{\otimes p} \longrightarrow \mathcal{N}(C)^{\otimes q}$ the composition

$$C^{\otimes p} \hookrightarrow \mathcal{N}(C)^{\otimes p} \xrightarrow{f} \mathcal{N}(C)^{\otimes q} \twoheadrightarrow C^{\otimes N_1+\dots N_q}.$$

We define the morphism $\alpha : \mathcal{R}ack\mathcal{B}_1(p,q) \to \mathrm{Hom}(C^p, C^{N_1+\cdots+N_q})$ to be the composition of $(\cdot)_{\mathcal{N}(C)}$ with $\varphi_{N_1,\dots,N_q}$. We can see that the element $\alpha \circ i_{p,q} \circ i(x)$ is equal to the morphism x_C which is the image by the structural morphism $(\cdot)_C : \mathcal{C}_{nc} \longrightarrow \mathrm{End}_C$ of x. Now we need to find a coalgebra C (depending on x) such that this x_C is non zero.

Step 4 : Find a suitable coalgebra : x is a sum of elements of the form $\sigma_1 \otimes (\Delta^{k_1} \otimes \cdots \otimes \Delta^{k_p}) \otimes \sigma_2$ where $k_1 + \cdots + k_p = N_1 + \cdots + N_q$, $\sigma_1 \in \mathbb{C}[S_{N_1+\cdots+N_q}]$, $\sigma_2 \in \mathbb{C}[S_p]$ and Δ^k corresponds to the k-th composition of Δ with itself. For example, $\Delta^2 = (\Delta \otimes \mathrm{id}) \circ \Delta$. By coassociativity this notion does not depend on a choice. These elements can be rewritten into elements of the form $\sigma \otimes (\Delta^{k_1} \otimes \cdots \otimes \Delta^{k_p}) \otimes \mathrm{id}$ by shuffling the Δ^ks.

One can write

$$x = \sum_{k_1+\cdots+k_p = N_1+\cdots+N_q} x_{k_1,\dots k_p},$$

where $x_{k_1,\dots,k_p} := \sigma_{k_1,\dots,k_p} \otimes (\Delta^{k_1} \otimes \cdots \otimes \Delta^{k_p}) \otimes \mathrm{id}$ with $\sigma_{k_1,\dots,k_p} \in \mathbb{C}[S_{N_1+\cdots+N_q}]$. Since $x \neq 0$, there exists a tuple $(k_1, \dots, k_p)$ such that $x_{k_1,\dots,k_p} \neq 0$, thus such that $\sigma_{k_1,\dots,k_p} \neq 0$. Now take $(k_1, \dots, k_p)$ such a tuple. Let $C := \overline{T}^c(V)$ be the reduced cofree coalgebra on a vector space $V := \mathrm{Span}(v_1, \dots, v_{N_1+\cdots+N_q})$ of dimension $N_1 + \cdots + N_q$. Here the coproduct on V is the reduced coproduct $\overline{\Delta}$, which is the deconcatenation product without counits.

Finally, define the element

$$v := v_1 \dots v_{k_1} \otimes v_{k_1+1} \dots v_{k_1+k_2} \otimes \dots \otimes v_{k_1+\dots+k_{p-1}+1} \dots v_{k_1+\dots+k_p} \in C^p.$$

Since, for all $l > k$ and $u_1 \dots u_k \in V^{\otimes k}$, $\overline{\Delta}^l(u_1 \dots u_k) = 0$, we have

$$x_C(v) = (x_{k_1,\dots,k_p})_C(v).$$

For $\sigma \in \mathbb{S}_{N_1+\dots+N_q}$, we have

$$(\sigma \otimes (\Delta^{k_1} \otimes \dots \otimes \Delta^{k_p}) \otimes \mathrm{id})_C(v_{k_1,\dots,k_q}) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(N_1+\dots+N_q)}.$$

But since $\sigma_{k_1,\dots,k_p}$ is a non zero linear combination of permutations of $S_{N_1+\dots+N_q}$ which acts freely on the subspace $\mathrm{Span}(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(N_1+\dots+N_q)}, \sigma \in \mathbb{S}_{N_1+\dots+N_q})$ of C, $x_C(v_{k_1,\dots,k_q})$ is not zero, which completes the proof.

□

This proves the following.

Theorem 3.6. *The properad $\mathcal{R}ack\mathcal{B}_1$ is homotopy Koszul.*

Proof. We need three conditions :

(i) the associated quadratic properad $\mathcal{R}ack\mathcal{B}_2$ is Koszul,

(ii) the $\mathbb{S}$-bimodules $\mathcal{R}ack\mathcal{B}_1$ and $\mathcal{R}ack\mathcal{B}_2$ are isomorphic,

(iii) there exists an extra grading on $\mathcal{R}ack\mathcal{B}_1 = \bigoplus_\lambda \mathcal{R}ack\mathcal{B}_1(\lambda)$ such that $\mathcal{R}ack\mathcal{B}_1(\lambda)$ is a finite dimensional $\mathbb{S}$-bimodule for all λ.

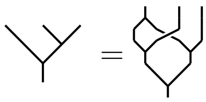
(i) has been proved. (ii) comes from Theorem 3.5 and the fact that the morphism $i_{p,q}$ preserves the number of connected components, thus it is an isomorphism of $\mathbb{S}$-bimodules between $\mathcal{R}ack\mathcal{B}_1$ and $\mathcal{R}ack\mathcal{B}_2$. For (iii), as $\mathcal{R}ack\mathcal{B}_1$ has a representation by generators and relations, we can take the graduation induced by the path grading, which is given by the number of different paths from any input to any output (see [MV10, Section 5]). This graduation is preserved by the relations.

□

3.4 Is $\mathcal{R}ack\mathcal{B}$ homotopy Koszul ?

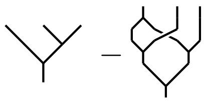
The main goal of this chapter was to compute a minimal model of $\mathcal{R}ack\mathcal{B}$ if it exists. M. Markl exhibited in [Mar06, section 8] a properad which admits no minimal model, thus the question of the existence of a minimal model for $\mathcal{R}ack\mathcal{B}$ can be asked. A sufficient condition for a properad to have a minimal model is to be homotopy Koszul, but we do not know if $\mathcal{R}ack\mathcal{B}$ is homotopy Koszul or not. In fact, even if $\mathcal{R}ack\mathcal{B}_2$ is Koszul, two conditions of this definition remain open :

(i) The natural morphism to consider between $\mathcal{R}ack\mathcal{B}_2$ and $\mathcal{R}ack\mathcal{B}$ is not an isomorphism because of the fact that the non zero element

 $\in \mathcal{R}ack\mathcal{B}$

has no non zero inverse image in $\mathcal{R}ack\mathcal{B}_2$. This does not prove that there is no isomorphism between these $\mathbb{S}$-bimodules, but finding one seems not easy.

(ii) The path grading that we take to prove that $\mathcal{B}$ and $\mathcal{R}ack\mathcal{B}_1$ are homotopy Koszul is not a grading on $\mathcal{R}ack\mathcal{B}$ because of the relation

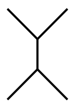
 ,

in fact the number of paths are respectively 3 and 4. This may be a way to prove the eventual non homotopy koszulity of $\mathcal{R}ack\mathcal{B}$.

Chapter 4

Associative and coassociative bialgebras

The goal of this chapter is to study a family of properads to check in this particular case if confluence of a system induced by a properad is equivalent or not to its koszulity. The case we are studying is the case of associative and coassociative bialgebras, with a relation which allows us to rewrite



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In this chapter, we use a `SageMath` script to compute dimensions of vector spaces. The idea behind the `SageMath` script is to generate all possible graphs starting from a list of given generators, then we can define relations and study the ideal generated by them, ask for dimensions, etc. See Appendix A and [Néd] for more details.

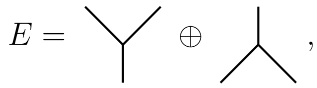
In the first section we describe the situation and state the main conjecture of the chapter, then we talk about some symmetries we can use to simplify the problem, and focus on confluence to get relations we will keep in mind for the rest of the chapter. In the second section, we study a specific morphism of $\mathbb{S}$-bimodules and prove that it is an isomorphism if and only if the properad induces a confluent system using two methods : an direct one and another using representation theory. In the last section, we study the koszulity of these properads looking at their Koszul complex.

4.1 Presentation of the problem

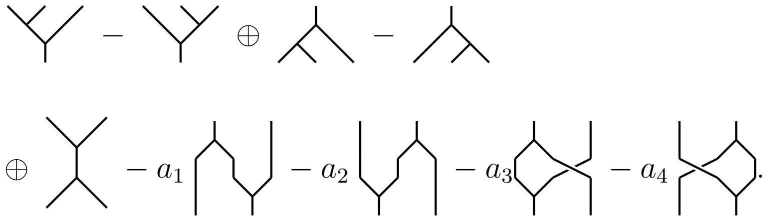
We start here with a properad generated by an associative product and a coassociative coproduct, together with one quadratic relation in biarity $(2, 2)$. This family of properads does not contain all possible properads encoding associative and coassociative bialgebras, but it has 4 parameters.

4.1.1 The properads encoding associative and coassociative bialgebras

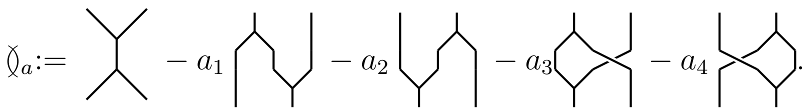
We will study the following family of properads. Let E be the $\mathbb{S}$-bimodule



$a = (a_1, a_2, a_3, a_4)$ be a 4-tuple in $\mathbb{C}$, and R_a be the ideal generated by

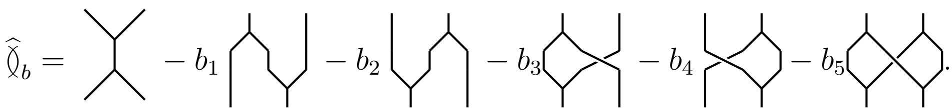


We denote by $\between_a$ the term



This notation is inspired from J-L. Loday in [Lod08]. Then we denote by $\mathcal{P}_a$ the properad $\mathcal{F}(E)/R_a$.

Remark 4.1. This family of properads contains known properads such as $\varepsilon\mathcal{B}$ (see [Val07]) and $\frac{1}{2}\mathcal{B}$ (see [MV10]). However, these properads are quadratic and with dioperadic relations, thus the properad $\mathcal{B}$ is not in this family, but we could take instead of 4-tuples the 5-tuples $b = (b_1, \ldots, b_5) \in \mathbb{C}^5$ and define



We denote by $\widehat{\mathcal{P}}_b$ the properad with generators E and relations generated by associativity, coassociativity and $\widehat{\between}_b$. This family is not quadratic and the quadratic properad associated to $\widehat{\mathcal{P}}_b$ is $\mathcal{P}_a$ with $a = (b_1, b_2, b_3, b_4)$, thus studying homotopy koszulity of properads of the form $\widehat{\mathcal{P}}_b$ starts by studying koszulity of properads of the form $\mathcal{P}_a$.

Let $\mathcal{A}$ be the properad (or operad) of associative algebras and $\mathcal{C}$ be the properad of coassociative coalgebras. Let us define the morphism of $\mathbb{S}$-bimodules

$$\varphi_a : \mathcal{A} \boxtimes \mathcal{C} \longrightarrow \mathcal{P}_a$$

defined by the composition

$$\mathcal{A} \boxtimes \mathcal{C} \hookrightarrow \mathcal{F}(E) \boxtimes \mathcal{F}(E) \to \mathcal{F}(E) \twoheadrightarrow \mathcal{P}_a.$$

We can state the following conjecture, which links ϕ_a to koszulity of $\mathcal{P}_a$ and confluence of the system it induces, where confluence is defined in Section 4.1.3.

Conjecture 4.2. *Let $a \in \mathbb{C}^4$, the following are equivalent :*

(i) the properad $\mathcal{P}_a$ induces a confluent system,

(ii) the morphism φ_a is a bijection in weight 3*,*

(iii) the properad $\mathcal{P}_a$ is Koszul.

In fact the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are true in the case of operads, but an algebra (thus an operad) can be Koszul but the system it induces be non confluent, for example the algebra

$$A = T(x, y, z)/(x^2 - yx, xz, zy)$$

induces a confluent system for the order giving the rule $x^2 \to yx$ but not for the one giving the rule $yx \to x^2$ (see [DR17]). Moreover, (ii) $\Rightarrow$ (iii) has been proven by B. Vallette in [Val07] (see Theorem 2.38) for properads in general.

4.1.2 Symmetries and isomorphisms

In order to simplify the situation, one can look at symmetries and isomorphisms to reduce the size of the family of properads we are looking at. What we call symmetries are the link we can find between some results using levelled graphs and the same result but looking at the horizontal symmetry of the graphs. For instance, the graphs representing coassociativity are obtained by symmetry from the graphs representing associativity. We are also looking at automorphisms of $\mathbb{S}$-bimodules on E, and look at what they induce on the free properad on E.

Symmetries

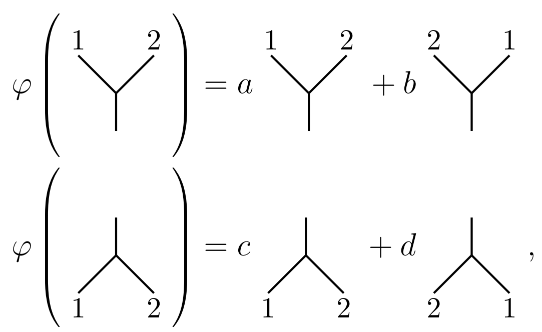
Most of the time, to compute dimensions, morphisms, etc., we will study $\mathcal{P}_a$ in some weight and biarity. But because the $\mathbb{S}$-bimodule E is stable by horizontal symmetry, so is $\mathcal{F}(E)$. Thus we can wonder if we can save computations. The relations of associativity and coassociativity are also horizontally symmetric to each other, and the horizontal symmetry of the relation $\between_a$ is $\between_{\tilde{a}}$ with $\tilde{a} := (a_2, a_1, a_3, a_4)$. Thus if we look at the reversed properad $\mathcal{P}_a^{\mathrm{op}}$ (see [Val07, Section 8]), we get the properad $\mathcal{P}_{\tilde{a}}$.

This remark will save a lot of computation for the next sections, because, for example, if we look at the dimension, depending on a, of $\mathcal{P}_a$ in some weight w and biarity (m, n), we have, as $\mathbb{S}_n \times \mathbb{S}_m^{\mathrm{op}}$-modules :

$$\mathcal{P}_a^{(w)}(n, m) \simeq (\mathcal{P}_a^{\mathrm{op}})^{(w)}(m, n) \simeq \left(\mathcal{P}_{\tilde{a}}^{(w)}(m, n)\right)^{\mathrm{op}}.$$

Isomorphisms

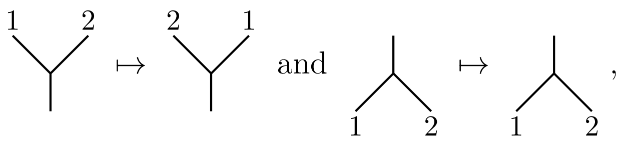
The $\mathbb{S}$-bimodule automorphisms of E are the morphisms $\varphi : E \longrightarrow E$ of the form :

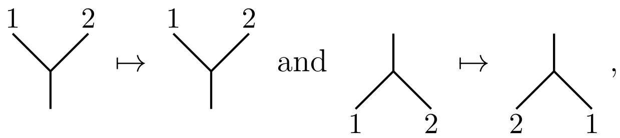


where $a^2 - b^2 \neq 0$ and $c^2 - d^2 \neq 0$. In fact in both biarities $(1, 2)$ and $(2, 1)$, the equivariant morphisms are given by matrices of the form

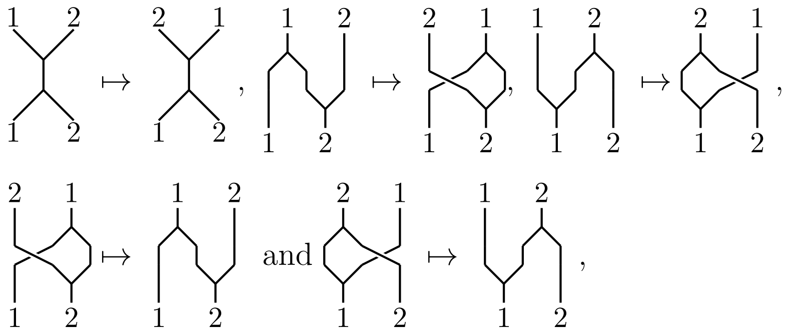
$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

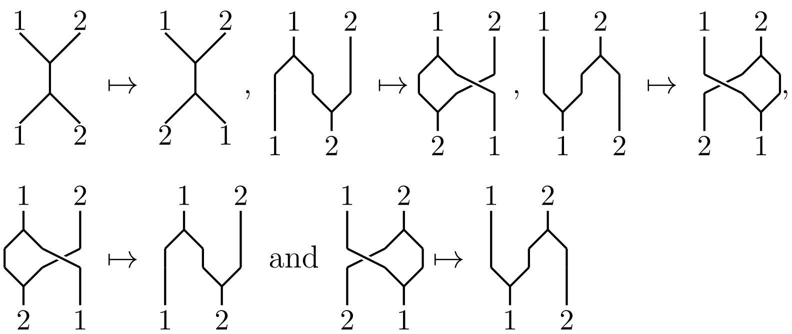
with $a, b \in \mathbb{C}$. Take for example

$\varphi^\tau :$ 

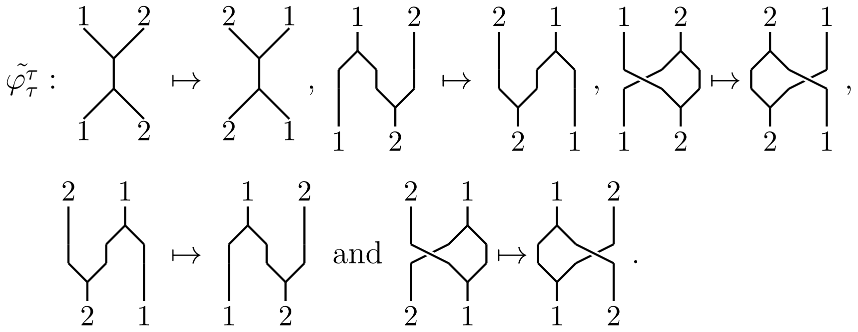
$\varphi_\tau :$ 

and $\varphi^\tau_\tau = \varphi_\tau \circ \varphi^\tau$. Thus, if for φ an automorphism of E, we denote by $\tilde{\varphi}$ the induced automorphism of the free properad $\mathcal{F}(E)$, we have

$\tilde{\varphi^\tau} :$ 

$\tilde{\varphi}_\tau :$ 

and



We will use these $\mathbb{S}$-bimodules automorphisms of $\mathcal{F}(E)$ to reduce the size of the family of properads later. For example, the properad $\mathcal{P}_{(1,0,0,0)}$ is isomorphic to the properads $\mathcal{P}_{(0,1,0,0)}$, $\mathcal{P}_{(0,0,1,0)}$ and $\mathcal{P}_{(0,0,0,1)}$, respectively with the isomorphisms $\tilde{\varphi}_\tau^\tau$, $\tilde{\varphi}_\tau$ and $\tilde{\varphi}^\tau$.

4.1.3 Confluence

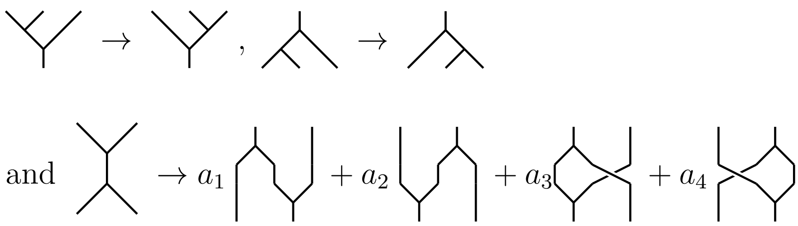
Here we state the conditions on a for $\mathcal{P}_a$ to induce a confluent system in the sense given below. Thus we can reformulate the conjecture with these conditions.

Definition 4.3 ([Mal19, Chapter 1])**.** A rewriting system is a data $(A, \to)$ of a set A and a binary relation $\to$ on A called the rewriting relation. We say, for $a, b \in A$, that a rewrites to b if there exists a sequence in A

$$a \to a_1 \to \cdots \to a_n \to b,$$

and we write $a \twoheadrightarrow b$. We say that the system $(A, \to)$ is confluent if for any tuple (a, b, c) in A such that $b \twoheadleftarrow a \twoheadrightarrow c$, there exists $d \in A$ such that $b \twoheadrightarrow d \twoheadleftarrow c$. See [Mal19] for more details on rewriting systems, confluence and linear rewriting.

We take as rewriting rules on the space $\mathcal{F}^{(2)}(E)$ the following ones

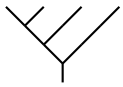
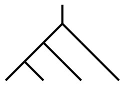
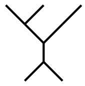
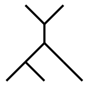


and take the induced ones on $\mathcal{F}^{(3)}(E)$.

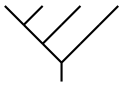
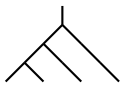
Definition 4.4. Let $a \in \mathbb{C}^4$. We say that $\mathcal{P}_a$ *induces a confluent system* (in weight 3) if the rewriting system $\mathcal{F}^{(3)}(E)$ with the rewriting relation given above is confluent.

Remark 4.5. This system terminates, thus here confluence is equivalent to convergence (See [Mal19]).

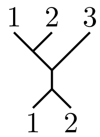
The critical monomials, which are the elements that can be rewritten two different ways, of this rewriting system are

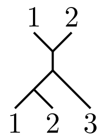
 ,  ,  and  .

We already know that starting from

 and  ,

we have confluence, these are known operadic cases. We only need to check what happens for



and


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but we will focus on the first one and only give the relations for the other one, because it is obtained from the first one by applying the horizontal symmetry. First we will give some examples.

Example 4.6. For $a = (0, 0, 0, 0)$, we have the rewriting diagram in Figure 4.1,where the top arrow is given by associativity, and the two others are given by the relation $\between_a$. Thus $\mathcal{P}_0$ induces a confluent system, but for $a = (2, 0, 0, 0)$, we have the rewriting diagram in Figure 4.2, where the top path is given by associativity then the relation $\between_a$, and the bottom path is given by the relation $\between_a$ two times followed by associativity. Thus is this case, $\mathcal{P}_a$ induces a non confluent system because the two elements at the end of each path cannot be rewritten anymore and are different.

Figure 4.1 – Rewriting diagram for $a = (0, 0, 0, 0)$

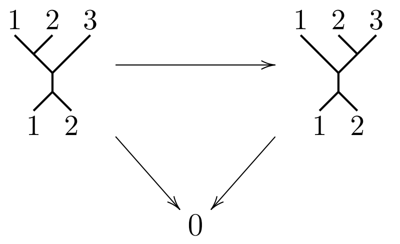
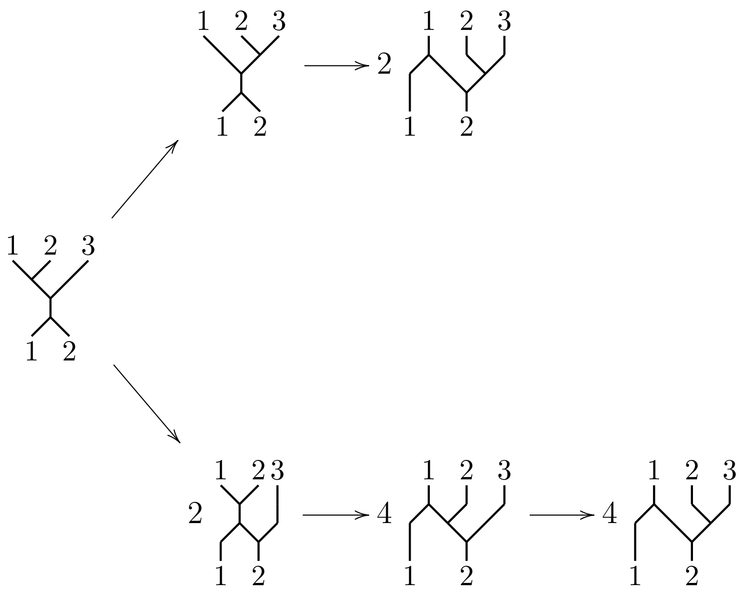


Figure 4.2 – Rewriting diagram for $a = (2, 0, 0, 0)$

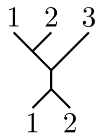


Proposition 4.7. *Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ and $\tilde{a} := (a_2, a_1, a_3, a_4)$. The properad $\mathcal{P}_a$ induces a confluent system if and only if*

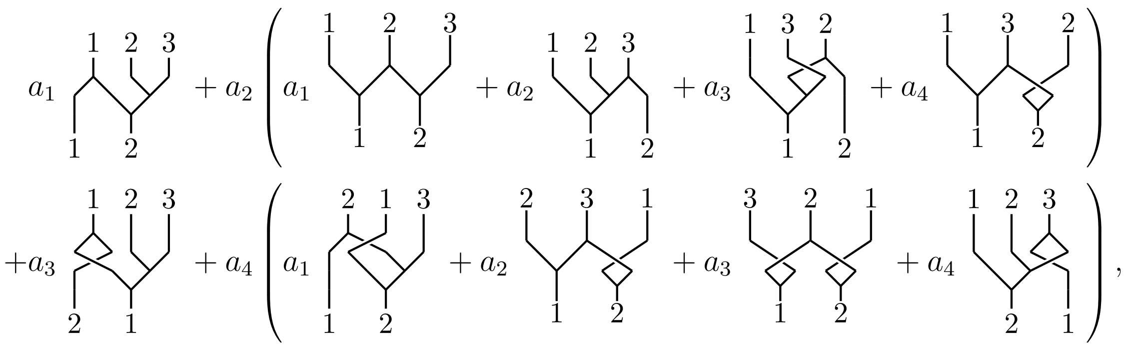
$$a \in C := \{a \in \mathbb{C}^4 | \forall i \in \{1, 2, 3, 4\}, a_i = a_i^2 \text{ and } a_2 a_4 = a_1 a_3 = 0\} \text{ and } \tilde{a} \in C.$$

This means that $\mathcal{P}_a$ induces a confluent system if and only if for all $i \in \{1, 2, 3, 4\}$, $a_i = a_i^2$ and $a_1 a_3 = a_1 a_4 = a_2 a_3 = a_2 a_4 = 0$.

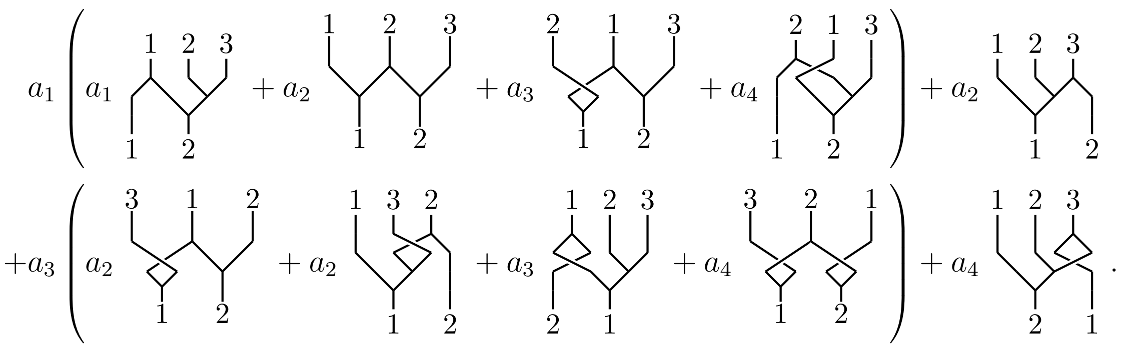
Proof. Doing the rewriting diagram for



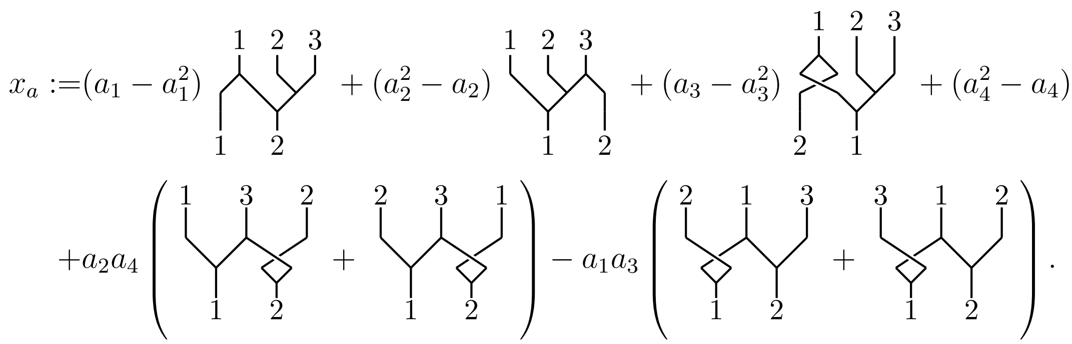
in the general case, at the end of one path, we get :



and at the end of the other path, we get



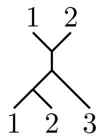
The difference between these two elements is



Thus this graph rewrites uniquely if and only if $x_a = 0$ in $\mathcal{F}^{(3)}(E)$. In other words, it rewrites uniquely if and only if a is in the set

$$C = \{a \in \mathbb{C}^4 | \forall i \in \{1,2,3,4\}, a_i = a_i^2 \text{ and } a_2a_4 = a_1a_3 = 0\}.$$

Starting from the graph



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the computation is the same but the roles of a_1 and a_2 are switched, this means that $\mathcal{P}_a$ induces a confluent system if and only if $a \in C$ and $\tilde{a} \in C$.

□

Remark 4.8. If we consider a 5-tuple b, we can check that $\widehat{\mathcal{P}}_b$ induces a confluent system if and only if

$$\begin{aligned} &(b_5 = 0 \text{ and } (b_1, b_2, b_3, b_4) \in C \text{ and } (b_2, b_1, b_3, b_4) \in C) \\ \text{or } &(b_5 \neq 0 \text{ and } b_1 = b_2 = b_3 = b_4 = 0). \end{aligned}$$

In other words, the only non-quadratic properad that induces a confluent system of the form $\widehat{\mathcal{P}}_b$ is $\mathcal{B}$, up to isomorphism (one can check that we can consider $b_5 = 1$ or 0).

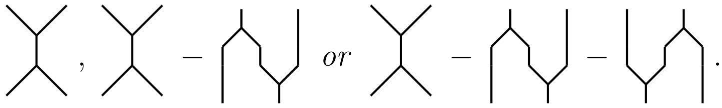
One can now state this new conjecture, which is equivalent to Conjecture 4.2.

Conjecture 4.9. *Let a be a 4-tuple in $\mathbb{C}$, then the following are equivalent :*

(i) the morphism φ_a is an isomorphism in weight 3,

(ii) the properad $\mathcal{P}_a$ is Koszul.

Moreover, both are true if and only if $\between_a$ is one of the following terms, up to isomorphism :



In fact, $\mathcal{P}_a$ induces a confluent system if and only if $\between_a$ is one of these terms, up to isomorphism of properads.

4.2 Morphism of $\mathbb{S}$-bimodules between $\mathcal{A} \boxtimes \mathcal{C}$ and $\mathcal{P}_a$

Let us recall that the morphism φ_a is defined as the composition

$$\mathcal{A} \boxtimes \mathcal{C} \hookrightarrow \mathcal{F}(E) \boxtimes \mathcal{F}(E) \to \mathcal{F}(E) \twoheadrightarrow \mathcal{P}_a.$$

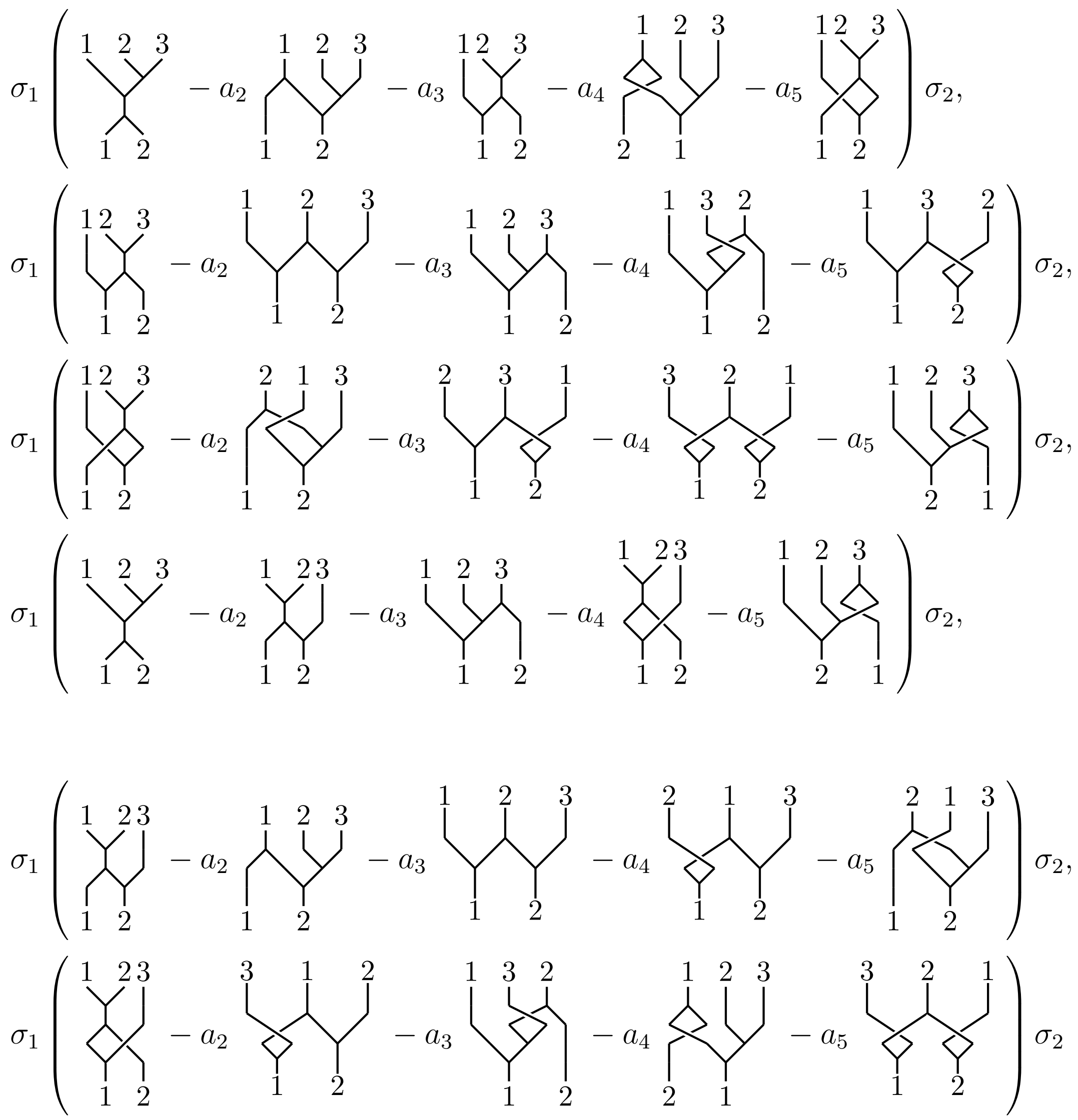
One can check that this morphism is surjective in weight 3. Thus the question is : is φ_a injective in weight 3 ? We only need to compute the dimensions of $\mathcal{A} \boxtimes \mathcal{C}$ and $\mathcal{P}_a$ in weight 3, biarity by biarity. The methods we will use in this section can be used to compute the biarities $(1,4)$, $(4,1)$, $(1,2)$ and $(2,1)$, and in these cases, φ_a is always injective.

Let us treat biarity $(2,3)$, then biarity $(3,2)$ will be studied by symmetry. Thanks to section 2.4, we have the dimensions of $\mathcal{A} \boxtimes \mathcal{C}$ and $\mathcal{F}(E)$ in weight 3 and biarity $(2,3)$, which are respectively 120 and 264. Thus we only need to calculate the dimension of $R_a^{(3)}(2,3)$. We will do so using two methods, the first one is quite direct, based on the fact that we know a generating family of $R_a^{(3)}(2,3)$, thus we can find its dimension. The second one will use representation theory, and will be way faster in terms of computations. Both of these proofs are accompagnied by the `Jupyter` notebooks `isobiarity23Method1.ipynb` and `isobiarity23Method2.ipynb` in [Néd] and Appendix A.

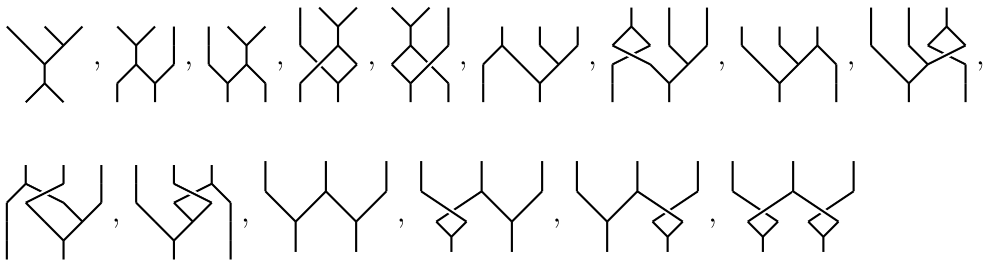
4.2.1 Direct proof

In order to make the computations easier, we decompose the morphism $\mathcal{F}(E) \twoheadrightarrow \mathcal{P}_a$ into the composition of the morphisms $\mathcal{F}(E) \twoheadrightarrow \mathcal{F}(E)/R$ and $\mathcal{F}(E)/R \twoheadrightarrow \mathcal{P}_a$ where R is the ideal generated by

and the second morphism is the quotient by the image $\overline{R_a}$ of R_a in $\mathcal{F}(E)/R$. A generating family of $\overline{R_a}^{(2)}(2,3)$ is the following :



for $\sigma_1 \in \mathbb{S}_2$ and $\sigma_2 \in \mathbb{S}_3$. Here we described a generating family in the basis of $(\mathcal{F}(E)/R)^{(3)}(2,3)$ given by the graphs



with every possible permutations above and below. The dimension of $(\mathcal{F}(E)/R)^{(3)}(2,3)$ is 180, thus we want to find conditions on a for the dimension of $\overline{R_a}^{(3)}(2,3)$ to be 60. Once the 72×180 matrix describing the generating family of $\overline{R_a}^{(2)}(2,3)$ in the canonical basis generated in `SageMath`, we can easily make some operations on rows to get an equivalent matrix of the form

$$\begin{pmatrix} I_{60} & * \\ 0 & B_a \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix and B_a is a 12×120 matrix with coefficients in $\mathbb{C}[a_1, a_2, a_3, a_4]$. For $i, j \in \{1, 2, 3, 4\}$, let us denote by $R_i := a_i^2 - a_i$ and $R_{i,j} := a_i a_j$. We define the following block matrices :

$$\tilde{I}_{2n} := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \ C := \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \text{ and } \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix},$$

with

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \Delta = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Thus $\mathrm{rk}(C) = \mathrm{rk}(D) = 6$. We have

$$B_a = \begin{pmatrix} -R_1 I_{12} & -R_3 \tilde{I}_{12} & R_2 I_{12} & R_4 \tilde{I}_{12} & 0_{12} & 0_{12} & 0_{12} & -R_{1,3} C & R_{2,4} D & 0_{12} \end{pmatrix}.$$

Here we can see we have 4 cases :

(i) If $a \in C$, then $B_a = 0$ and the dimension of $\overline{R_a}^{(2)}(2,3)$ is 60.

(ii) If there exists $i \in \{1,2,3,4\}$ such that $R_i \neq 0$, the dimension of $\overline{R_a}^{(2)}(2,3)$ is 72. We denote by $NC_{\text{norm}} = \{a \in \mathbb{C}^4 | \exists i \in \{1,2,3,4\}$ such that $R_i \neq 0\}$.

(iii) There is a finite number of remaining cases, because if for all $i \in \{1,2,3,4\}$, $R_i = 0$, it means that $a \in \{0,1\}^4$. If ($R_{1,3} = 0$ and $R_{2,4} \neq 0$) or ($R_{1,3} \neq 0$ and $R_{2,4} = 0$), i.e. if

$$a \in \{(1,1,1,0),(1,1,0,1),(1,0,1,1),(0,1,1,1),(1,0,1,0),(0,1,0,1)\} =: NC_{x,0},$$

the dimension of $\overline{R_a}^{(2)}(2,3)$ is 66.

(iv) There is only one tuple left here, the case where $a = (1,1,1,1)$, we denote by $NC_1 := \{(1,1,1,1)\}$. In this case, the dimension of $\overline{R_a}^{(2)}(2,3)$ is 70.

Thus we can prove a part of Conjecture 4.2.

Theorem 4.10. *Let $a \in \mathbb{C}^4$, the following are equivalent :*

(i) the properad $\mathcal{P}_a$ induces a confluent system,

(ii) the morphism φ_a is an isomorphism of $\mathbb{S}$-bimodules in weight 3.

Proof. We have

$$a \in C \Leftrightarrow \dim\left(\overline{R_a}^{(3)}(2,3)\right) = 60 \Leftrightarrow \dim\left((\mathcal{A} \boxtimes \mathcal{C})^{(3)}(2,3)\right) = \dim\left(\mathcal{P}_a^{(3)}(2,3)\right),$$

which is true if and only if φ_a is injective in weight 3 and biarity $(2,3)$.

On biarity $(3,2)$, one can use the same method or simply see that, as vector spaces,

$$\mathcal{P}_a^{(3)}(3,2) = (\mathcal{P}_a^{\text{op}})^{(3)}(2,3) = \mathcal{P}_{\tilde{a}}^{(3)}(2,3),$$

thus $\tilde{a} \in C$ if and only if φ_a is injective in weight 3 and biarity $(3,2)$. The same method shows that for every $a \in \mathbb{C}^4$, φ_a is injective in weight 3 and biarities $(1,4)$, $(4,1)$, $(1,2)$ and $(2,1)$. Finally we have that φ_a is injective in weight 3 if and only if $a \in C$ and $\tilde{a} \in C$. □

4.2.2 Proof of theorem 4.10 using representation theory

Here we use representation matrices, see Section 1.4, to compute the representation $R_a^{(3)}(2,3)$ of the group $\mathbb{S}_2 \times \mathbb{S}_3^{\text{op}}$. $R_a^{(3)}(2,3)$ is a subspace of $\mathcal{F}(E)^{(3)}(2,3) = \mathbb{C}[\mathbb{S}_2 \times \mathbb{S}_3^{\text{op}}]^{\oplus 22}$, thus we study $R_a^{(3)}(2,3)$ as a space of relations in 22 copies of the regular representation of $\mathbb{S}_2 \times \mathbb{S}_3^{\text{op}}$. We have a total of 13 relations. Thus we get for every $(\lambda,\mu) \vdash (2,3)$ a $13d_\lambda d_\mu \times 22 d_\lambda d_\mu$ representation matrix $C_{\lambda,\mu}(R_a^{(3)}(2,3))$ such that $m_{\lambda,\mu}(R_a^{(3)}(2,3)) = \text{rk}(C_{\lambda,\mu}(R_a^{(3)}(2,3)))$. For every $(\lambda,\mu) \vdash (2,3)$, we compute the partial Smith form (see [BD17]) of $C_{\lambda,\mu}(R_a^{(3)}(2,3))$ and get a matrix of the form

$$\begin{pmatrix} I_{k_{\lambda,\mu}} & * \\ 0 & B^a_{\lambda,\mu} \end{pmatrix}$$

We simplify the matrices, getting rid of zero and duplicate columns and rows, getting new matrices $SB^a_{\lambda,\mu}$ such that $\text{rk}(SB^a_{\lambda,\mu}) = \text{rk}(B^a_{\lambda,\mu})$. The matrices $SB^a_{\lambda,\mu}$ are given in the Tables 4.1 and 4.2, and the integers $k_{\lambda,\mu}$ are given in the Table 4.3.

Table 4.1 – $SB^a_{\lambda,\mu}$ depending on λ for $\mu = (2,1)$

$\lambda \backslash \mu$	$(2,1)$
$(1,1)$	$\begin{pmatrix} -2R_{2,4} & R_2 & R_2 & R_3 & 0 & -R_{1,3} & -R_1 & 0 & -R_4 & -R_4 \\ R_{2,4} & -R_2 & 0 & 0 & R_3 & -R_{1,3} & 0 & -R_1 & 0 & R_4 \end{pmatrix}$
(2)	$\begin{pmatrix} 2R_{2,4} & R_2 & R_2 & R_3 & 0 & R_{1,3} & R_1 & 0 & R_4 & R_4 \\ R_{2,4} & -R_2 & 0 & 0 & R_3 & R_{1,3} & 0 & R_1 & 0 & -R_4 \end{pmatrix}$

Table 4.2 – $SB^a_{\lambda,\mu}$ depending on λ for $\mu \neq (2,1)$

$\lambda \backslash \mu$	$(1,1,1)$	(3)
$(1,1)$	$\begin{pmatrix} -R_2 & R_3 & -R_1 & -R_4 \end{pmatrix}$	$\begin{pmatrix} 2R_{2,4} & -R_2 & R_3 & -2R_{1,3} & -R_1 & R_4 \end{pmatrix}$
(2)	$\begin{pmatrix} -R_2 & R_3 & R_1 & R_4 \end{pmatrix}$	$\begin{pmatrix} -2R_{2,4} & -R_2 & R_3 & 2R_{1,3} & R_1 & -R_4 \end{pmatrix}$

Table 4.3 – $k_{\lambda,\mu}$ depending on λ and μ

$\lambda \backslash \mu$	$(1,1,1)$	$(2,1)$	(3)
$(1,1)$	12	24	12
(2)	12	24	12

From this we compute $m_{\lambda,\mu}(R_a^{(3)}(2,3))$ depending on μ and a, see Table 4.4. Moreover, $(\mathcal{A} \boxtimes \mathcal{C})^{(3)}(2,3) = \mathbb{C}[\mathbb{S}_2 \times \mathbb{S}_3^{\text{op}}]^{\oplus 10}$, thus the only case where the $\mathbb{S}_2 \times \mathbb{S}_3^{\text{op}}$-module structures of $(\mathcal{A} \boxtimes \mathcal{C})^{(3)}(2,3)$ and $\mathcal{P}_a^{(3)}(2,3)$ are the same is when $a \in C$. We can also compute the dimension of $R_a^{(3)}(2,3)$ and check that it is 144 if and only if $a \in C$. This provides another proof of theorem 4.10.

Table 4.4 – $m_{\lambda,\mu}(R_a^{(3)}(2,3))$ depending on a and μ

$a \backslash \mu$	(3)	$(2,1)$	$(1,1,1)$
C	12	24	12
NC_{norm}	13	26	13
NC_1	12	26	13
$NC_{x,0}$	12	25	13

4.3 (Non)-koszulity

We can check that all $a \in C$ such that $\tilde{a} \in C$ provide a Koszul properad. In fact if $a = (0,0,0,0)$, we get the properad $\frac{1}{2}\mathcal{B}$ (see [MV10]), if $a = (1,1,0,0)$, we get $\varepsilon\mathcal{B}$ (see [Val07, Corollary 8.5]) and if $a = (1,0,0,0)$, Theorem 4.10 and Theorem 2.38 show that $\mathcal{P}_a$ is Koszul. Thus we can state the following.

Theorem 4.11. *Let $a \in \mathbb{C}^4$. The following are equivalent :*

(i) the properad $\mathcal{P}_a$ induces a confluent system,

(ii) the morphism φ_a is an isomorphism of $\mathbb{S}$-bimodules in weight 3.

Moreover, in that case, the properad $\mathcal{P}_a$ is Koszul.

Now the question is, if $a \notin C$ or $\tilde{a} \notin C$, can we show that $\mathcal{P}_a$ is not Koszul ? We have the following sequence of implications :

$$\begin{aligned}
\mathcal{P}_a \text{ is Koszul} &\implies \mathcal{P}_a \boxtimes \mathcal{P}_a^{\text{i}} \text{ is acyclic} \implies (\mathcal{P}_a \boxtimes \mathcal{P}_a^{\text{i}})^{(4)}(2,4) \text{ is acyclic} \\
&\implies \text{for all } (\lambda,\mu) \vdash (2,4), \text{ the isotypic component} \\
&\quad [(\mathcal{P}_a \boxtimes \mathcal{P}_a^{\text{i}})^{(4)}(2,4)]_{(\lambda,\mu)} \text{ is acyclic} \\
&\implies \text{for all } (\lambda,\mu) \vdash (2,4), \text{ the Euler characteristic of} \\
&\quad [(\mathcal{P}_a \boxtimes \mathcal{P}_a^{\text{i}})^{(4)}(2,4)]_{(\lambda,\mu)} \text{ is zero.}
\end{aligned}$$

The goal of this section is to compute the multiplicities of every chain of the Koszul complex of $\mathcal{P}_a$ in biarity $(2,4)$ and weight 4. We can look for multiplicities instead of dimensions because every map in the complex is equivariant. We look in weight 4 because in lower weights, this criterion is always true. We also look for biarity $(2, 4)$ because it seems to be the simplest one to study among the ones where we can find something interesting. We could look for biarity $(3, 3)$, or higher weights, the method would be the same but the computations would be too long.

For the `SageMath` computations, the file `Koszulcpxbiarity24.ipynb` in [Néd] accompanies this section, see Appendix A.

4.3.1 Koszul complex of the properad

Let us study the Koszul complex of $\mathcal{P}_a$ in weight 4 biarity $(2, 4)$:

$$0 \to (\mathcal{P}_a^{¡})^{(4)}(2,4) \to \underbrace{\mathcal{P}_a}_{1} \boxtimes \underbrace{\mathcal{P}_a^{¡}}_{3}(2,4) \to \underbrace{\mathcal{P}_a}_{2} \boxtimes \underbrace{\mathcal{P}_a^{¡}}_{2}(2,4) \to \underbrace{\mathcal{P}_a}_{3} \boxtimes \underbrace{\mathcal{P}_a^{¡}}_{1}(2,4) \to \mathcal{P}_a^{(4)}(2,4) \to 0.$$

Let us first introduce some notations.

Notations 4.12. Let $(\lambda, \mu) \vdash (2, 4)$. We set

(i) $M_{\lambda,\mu} := m_{\lambda,\mu}(\mathbb{C}[\mathbb{S}_2 \times \mathbb{S}_4^{\mathrm{op}}])$.

(ii) $X^a := (\mathcal{P}_a^{(3)}(2,3) \boxtimes_{(1)} E(1,2))(2,4)$ and $x^a_{\lambda,\mu} := m_{\lambda,\mu}(X^a)$.

(iii) $Y^a := (E(1,2) \boxtimes_{(1)} (\mathcal{P}_a^{¡})^{(3)}(2,3))(2,4)$ and $y^a_{\lambda,\mu} := m_{\lambda,\mu}(Y^a)$.

(iv) $(M\mathcal{P}_a)_{\lambda,\mu} := m_{\lambda,\mu}(\mathcal{P}_a^{(4)}(2,4))$ and $(M\mathcal{P}_a^{¡})_{\lambda,\mu} := m_{\lambda,\mu}((\mathcal{P}_a^{¡})^{(4)}(2,4))$.

(v) $(MR_a)_{\lambda,\mu} := m_{\lambda,\mu}(R_a^{(4)}(2,4))$ and $(MR_a^{\perp})_{\lambda,\mu} := m_{\lambda,\mu}((R_a^{\perp})^{(4)}(2,4))$

As the criterion we are looking for only involves multiplicities of the chains, we will study this complex degree by degree. Recall that the leftmost non zero chain is concentrated is degree 4, and the rightmost non zero chain is concentrated in degree 0.

Degree 0

Here the method is exactly the same as in Section 4.2.2, we have

$$(M\mathcal{P}_a)_{\lambda,\mu} = 93M_{\lambda,\mu} - (MR_a)_{\lambda,\mu}.$$

We compute the partial Smith form of the representation matrices for $R_a^{(4)}(2,4)$ and get for every $(\lambda,\mu) \vdash (2,4)$ a matrix of the form

$$C_{\lambda,\mu}(R_a^{(4)}(2,4)) := \begin{pmatrix} I_{k_{\lambda,\mu}} & * \\ 0 & B^a_{\lambda,\mu} \end{pmatrix}$$

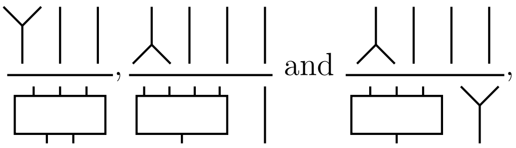
such that $(MR_a)_{\lambda,\mu} = k_{\lambda,\mu} + r^a_{\lambda,\mu}$ where $r^a_{\lambda,\mu} := \mathrm{rk}(B^a_{\lambda,\mu})$. Here the issue is that the representation matrices are too big to be studied as simply as in 4.2.2. We will study this case later.

Degree 1

Looking at the combinatorics with the weight and the biarity, we can see that

$$\begin{aligned}\underbrace{\mathcal{P}_a}_{3} \boxtimes \underbrace{\mathcal{P}_a^{\text{!`}}}_{1}(2,4) =& X^a \oplus (\mathcal{P}_a^{(3)}(1,4) \boxtimes_{(1)} E(2,1))(2,4) \\ & \oplus ((E(1,2), \mathcal{P}_a^{(2)}(1,3)) \boxtimes_{(1)} E(2,1))(2,4).\end{aligned}$$

According to Notation 2.21, these subspaces correspond respectively to the diagrams



the first space has multiplicity $x^a_{\lambda,\mu}$, which we will compute in 4.3.3, the second one is given by Corollary 2.25, and the last one can be computed with the same methods as in Theorem 2.23, it is equal to

$$\left(\bigoplus_{\substack{\overline{k}=(1,\ldots,2,\ldots,1) \\ |\overline{k}|=5}} \mathbb{C}[\mathbb{S}_2] \otimes (E(1,2) \otimes \mathcal{P}_a^{(2)}(1,3)) \otimes_{\mathbb{S}_2 \times \mathbb{S}_3} \mathbb{C}[\mathbb{S}^c_{(2,3),\overline{k}}] \otimes_{\mathbb{S}_{\overline{k}}} (\mathbb{C} \otimes \cdots \otimes E(2,1) \otimes \cdots \otimes \mathbb{C}) \otimes \mathbb{C}[\mathbb{S}_4] \right.$$

$$\left. \oplus \bigoplus_{\substack{\overline{k}=(1,\ldots,2,\ldots,1) \\ |\overline{k}|=5}} \mathbb{C}[\mathbb{S}_2] \otimes (\mathcal{P}_a^{(2)}(1,3) \otimes E(1,2)) \otimes_{\mathbb{S}_3 \times \mathbb{S}_2} \mathbb{C}[\mathbb{S}^c_{(3,2),\overline{k}}] \otimes_{\mathbb{S}_{\overline{k}}} (\mathbb{C} \otimes \cdots \otimes E(2,1) \otimes \cdots \otimes \mathbb{C}) \otimes \mathbb{C}[\mathbb{S}_4] \right) / \sim .$$

But again we can reorder the blocks and use the fact that the $\mathbb{S}$-bimodules involved in these arities are regular representations to get

$$\begin{aligned}&\Big(\mathbb{C}[\mathbb{S}_2]\otimes(\mathbb{C}[\mathbb{S}_2]\otimes\mathbb{C}[\mathbb{S}_3])\otimes_{\mathbb{S}_2\times\mathbb{S}_3}\mathbb{C}[\mathbb{S}^c_{(2,3),(2,1,1,1)}]\otimes_{\mathbb{S}_2}\mathbb{C}[\mathbb{S}_2]\otimes\mathbb{C}[\mathbb{S}_4]\Big)\ /\sim'\\ =&\Big(\mathbb{C}[\mathbb{S}_2]\otimes\mathbb{C}[\mathbb{S}^c_{(2,3),(2,1,1,1)}]\otimes\mathbb{C}[\mathbb{S}_4]\Big)\ /\sim'.\end{aligned}$$

Here we have

$$\mathbb{S}^c_{(2,3),(2,1,1,1)}\simeq\{1,2,3\}\times\{1,2\}\times\{1,2\}\times\mathbb{S}_3,$$

with a right action of $\mathbb{S}_3$ corresponding to the relation $\sim'$, which gives

$$\begin{aligned}((E(1,2),\mathcal{P}_a^{(2)}(1,3))\boxtimes_{(1)}E(2,1))(2,4)&=\mathbb{C}[\mathbb{S}_2]\otimes\mathbb{C}[\mathbb{S}^c_{(2,3),(2,1,1,1)}]\otimes_{\mathbb{S}_3}\mathbb{C}[\mathbb{S}_4]\\ &=\mathbb{C}[\mathbb{S}_2]\otimes\mathbb{C}^{12}\otimes\mathbb{C}[\mathbb{S}_4]\\ &=\mathbb{C}[\mathbb{S}_2\times\mathbb{S}_4^{\mathrm{op}}]^{\oplus 12}.\end{aligned}$$

Finally, we have

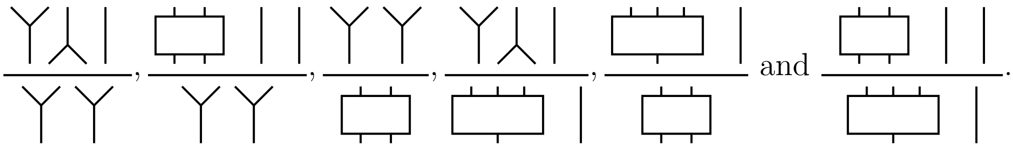
$$m_{\lambda,\mu}((\underbrace{\mathcal{P}_a}_{3}\boxtimes\underbrace{\mathcal{P}_a^{\mathrm{i}}}_{1})(2,4))=x^a_{\lambda,\mu}+20M_{\lambda,\mu}.$$

Degree 2

Similarly to degree 1, we can see that the space $\underbrace{\mathcal{P}_a}_{2}\boxtimes\underbrace{\mathcal{P}_a^{\mathrm{i}}}_{2}(2,4)$ is

$$\begin{aligned}&\Big(E(1,2)\boxtimes_{(2),(1,1)}(E(1,2),E(2,1))\Big)(2,4)\oplus\Big(E(1,2)\boxtimes_{(2),(1)}(\mathcal{P}_a^{\mathrm{i}})^{(2)}(2,2)\Big)(2,4)\\ \oplus&\Big(\mathcal{P}_a^{(2)}(2,2)\boxtimes_{(1),(2)}E(1,2)\Big)(2,4)\oplus\Big(\mathcal{P}_a^{(2)}(1,3)\boxtimes_{(1)}(E(1,2),E(2,1))\Big)(2,4)\\ \oplus&\Big(\mathcal{P}_a^{(2)}(2,2)\boxtimes_{(1)}(\mathcal{P}_a^{\mathrm{i}})^{(2)}(1,3)\Big)(2,4)\oplus\Big(\mathcal{P}_a^{(2)}(1,3)\boxtimes_{(1)}(\mathcal{P}_a^{\mathrm{i}})^{(2)}(2,2)\Big)(2,4).\end{aligned}$$

According to Notation 2.21, these subspaces correspond respectively to the diagrams



Thus, by Corollary 2.25 and the previous method, we have

$$m_{\lambda,\mu}((\underbrace{\mathcal{P}_a}_{3} \boxtimes \underbrace{\mathcal{P}_a^{\text{i}}}_{1})(2,4)) = 42M_{\lambda,\mu}.$$

Let us give some details on the computation of the first space. This space is equal to

$$\begin{aligned}&\Big(\mathbb{C}[\mathbb{S}_2] \otimes (E(1,2) \otimes E(1,2) \otimes_{\mathbb{S}_2\times\mathbb{S}_2} \mathbb{C}[\mathbb{S}^c_{(2,2),(1,1,2)}] \otimes_{\mathbb{S}_2} (E(1,2) \otimes E(2,1)) \otimes_{\mathbb{S}_2} \mathbb{C}[\mathbb{S}_4]\Big) / \sim' \\ = &\Big(\mathbb{C}[\mathbb{S}_2] \otimes \mathbb{C}[\mathbb{S}^c_{(2,2),(1,1,2)}] \otimes \mathbb{C}[\mathbb{S}_4]\Big) / \sim',\end{aligned}$$

where the relation $\sim'$ corresponds to the exchange between the two products of $E(1,2)$. We have

$$\mathbb{S}^c_{(2,2),(1,1,2)} \simeq \{1,2\} \times \{1,2\} \times \{1,2\} \times \mathbb{S}_2,$$

where an element $(a, b_1, b_2, \tau) \in \mathbb{S}^c_{(2,2),(1,1,2)}$ corresponds to the choice a of an output of the coproduct, an input b_1 of the first product which is linked to the output a, an input b_2 of the second product which is linked to the other output, and a permutation $\tau \in \mathbb{S}_2$ for the rest of the permutation. For example, $(2,1,2,(1,2))$ corresponds to the connected permutation $[3,2,4,1]$. This set is endowed with a left action of $\mathbb{S}_2$ given by

$$\sigma \cdot (a, b_1, b_2, \tau) := (\sigma \cdot a, b_{\sigma(1)}, b_{\sigma(2)}, \tau),$$

and a right action given by multiplication with τ. Thus, we have

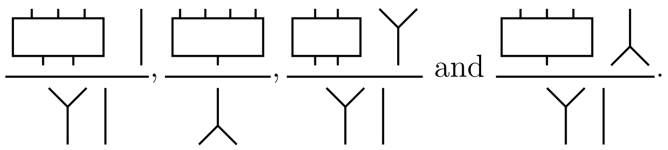
$$\begin{aligned}\Big(E(1,2) \boxtimes_{(2),(1,1)} (E(1,2), E(2,1))\Big)(2,4) &= \mathbb{C}[\mathbb{S}_2] \otimes_{\mathbb{S}_2} \mathbb{C}[\mathbb{S}^c_{(2,2),(1,1,2)}] \otimes \mathbb{C}[\mathbb{S}_4] \\ &= \mathbb{C}[\mathbb{S}_2] \otimes \mathbb{C}^8 \otimes \mathbb{C}[\mathbb{S}_4] \\ &= \mathbb{C}[\mathbb{S}_2 \times \mathbb{S}_4^{\text{op}}]^{\oplus 8}.\end{aligned}$$

Degree 3

We have

$$\underbrace{\mathcal{P}_a}_{2} \boxtimes \underbrace{\mathcal{P}_a^{\text{i}}}_{2}(2,4) = Y^a \oplus \Big(E(2,1) \boxtimes_{(1)} (\mathcal{P}_a^{\text{i}})^{(3)}(1,4)\Big)(2,4)$$
$$\oplus \Big(E(1,2) \boxtimes_{(1)} ((\mathcal{P}_a^{\text{i}})^{(2)}(2,2), E(1,2))\Big)(2,4)$$
$$\oplus \Big(E(1,2) \boxtimes_{(1)} ((\mathcal{P}_a^{\text{i}})^{(2)}(1,3), E(2,1))\Big)(2,4).$$

According to Notation 2.21, these subspaces correspond respectively to the diagrams

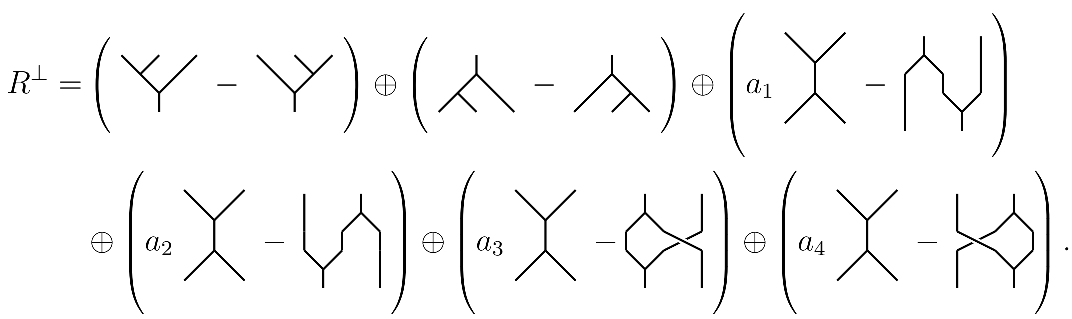


Thus, by Corollary 2.25 and the previous method, we have

$$m_{\lambda,\mu}((\underbrace{\mathcal{P}_a}_{1} \boxtimes \underbrace{\mathcal{P}_a^{\text{i}}}_{3})(2,4)) = y^a_{\lambda,\mu} + 9M_{\lambda,\mu}.$$

Degree 4

Using Corollary 2.33, we do the same here as for degree 0, but for the properad $\mathcal{P}_a^! = \mathcal{F}(\Sigma E)/(\Sigma^2 R^\perp)$, with



Thus we find

$$\begin{aligned}(M\mathcal{P}_a^{\mathrm{i}})_{\lambda,\mu} &= 93M_{\lambda,\mu} - m_{\lambda',\mu'}((R^{\perp})^{(4)}(2,4)) \\ &= 93M_{\lambda,\mu} - k^{\perp}_{\lambda',\mu'} - r^{a,\perp}_{\lambda',\mu'},\end{aligned}$$

where the partial Smith form of the representation matrices for $(R^{\perp})^{(4)}(2,4)$ and $(\lambda,\mu) \vdash (2,4)$ are of the form

$$\begin{pmatrix} I_{k^{\perp}_{\lambda,\mu}} & * \\ 0 & B^{a,\perp}_{\lambda,\mu} \end{pmatrix},$$

and $r^{a,\perp}_{\lambda,\mu} := \mathrm{rk}(B^{a,\perp}_{\lambda,\mu})$.

4.3.2 Computing values

One necessary condition for $\mathcal{P}_a$ to be Koszul is

$$(M\mathcal{P}_a^{\mathrm{i}})_{\lambda,\mu} + 42M_{\lambda,\mu} + (M\mathcal{P}_a)_{\lambda,\mu} = 9M_{\lambda,\mu} + y^a_{\lambda,\mu} + 20M_{\lambda,\mu},$$

which is equivalent to

$$199M_{\lambda,\mu} = x^a_{\lambda,\mu} + y^a_{\lambda,\mu} + k_{\lambda,\mu} + k^{\perp}_{\lambda',\mu'} + r^a_{\lambda,\mu} + r^{a,\perp}_{\lambda',\mu'}.$$

We know the values $M_{\lambda,\mu}$ from Theorem 1.5 and can compute $k_{\lambda,\mu}$ and $k^{\perp}_{\lambda',\mu'}$ with `SageMath`, and then $199M_{\lambda,\mu} - k_{\lambda,\mu} - k^{\perp}_{\lambda',\mu'}$ depending on μ (independant of λ), see Table 4.5.

Table 4.5 – Values of $M_{\lambda,\mu}$, $k_{\lambda,\mu}$, $k^{\perp}_{\lambda',\mu'}$ and $199M_{\lambda,\mu} - k_{\lambda,\mu} - k^{\perp}_{\lambda',\mu'}$ depending on μ

μ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$M_{\lambda,\mu}$	1	3	2	3	1
$k_{\lambda,\mu}$	73	219	146	219	73
$k^{\perp}_{\lambda',\mu'}$	92	276	184	276	92
$199M_{\lambda,\mu} - k_{\lambda,\mu} - k^{\perp}_{\lambda',\mu'}$	34	102	68	102	34

4.3.3 Computation of $x^a_{\lambda,\mu}$ and $y^a_{\lambda,\mu}$ via Pieri's rules

We denote, for $(\alpha,\beta)\vdash(2,3)$, $A^a_{\alpha,\beta} := m_{\alpha,\beta}(\mathcal{P}^{(3)}_a(2,3))$. Thus we have

$$\mathcal{P}^{(3)}_a(2,3) = \bigoplus_{(\alpha,\beta)\vdash(2,3)} V_{\alpha,\beta}^{\oplus A^a_{\alpha,\beta}} = \bigoplus_{(\alpha,\beta)\vdash(2,3)} (V_\alpha \boxtimes V_\beta)^{\oplus A^a_{\alpha,\beta}}.$$

According to Theorem 2.23, we have

$$X^a = \bigoplus_{(\alpha,\beta)\vdash(2,3)} \Big(V_\alpha \boxtimes (V_\beta \downarrow^{\mathbb{S}_3}_{\mathbb{S}_2} \sqcup_r \mathbb{C}[\mathbb{S}_2])\Big)^{\oplus A_{\alpha,\beta}}.$$

Thus by Pieri's rules :

$$\begin{aligned}
X^a &= \bigoplus_{(\alpha,\beta)\vdash(2,3)} \bigoplus_{\substack{\gamma\vdash 2\\ \gamma\leq\beta}} (V_\alpha \boxtimes (V_\gamma \sqcup_r \mathbb{C}[\mathbb{S}_2]))^{\oplus A_{\alpha,\beta}} \\
&= \bigoplus_{(\alpha,\beta)\vdash(2,3)} \bigoplus_{\substack{\gamma\vdash 2\\ \gamma\leq\beta}} \Big(\Big(V_\alpha \boxtimes (V_\gamma \sqcup_r V_{(1,1)})\Big) \oplus \Big(V_\alpha \boxtimes (V_\gamma \sqcup_r V_{(2)})\Big)\Big)^{\oplus A_{\alpha,\beta}} \\
&= \bigoplus_{(\alpha,\beta)\vdash(2,3)} \bigoplus_{\substack{\gamma\vdash 2\\ \gamma\leq\beta}} \left(\left(\bigoplus_{\substack{\delta\vdash 4\\ \delta\leq^l\gamma}} V_\alpha \boxtimes V_\delta\right) \oplus \left(\bigoplus_{\substack{\delta\vdash 4\\ \delta\leq^c\gamma}} V_\alpha \boxtimes V_\delta\right)\right)^{\oplus A_{\alpha,\beta}}.
\end{aligned}$$

In terms of Young diagrams, the partition δ is obtained from β by removing one box and adding two boxes, but not both on the same column for the first direct sum and not both on the same row for the second one. For every $\beta\vdash 3$ and $\delta\vdash 4$, we count the number of ways to get δ from β these ways in Table 4.6. We also remind the values of $A^a_{\alpha,\beta}$ from Section 4.2.2 for every $a\in\mathbb{C}^4$ and $(\alpha,\beta)\vdash(2,3)$, depending on a and β, see Table 4.7. Finally, we obtain the values of $x^a_{\lambda,\mu}$ depending on a and μ, see Table 4.8.

We denote now, for $(\alpha,\beta)\vdash(2,3)$, $B^a_{\alpha,\beta} := m_{\alpha,\beta}(\mathcal{P}^{\text{i}(3)}_a(2,3))$. We have

$$\mathcal{P}^{\text{i}(3)}_a(2,3) = \bigoplus_{(\alpha,\beta)\vdash(2,3)} V_{\alpha,\beta}^{\oplus B^a_{\alpha,\beta}} = \bigoplus_{(\alpha,\beta)\vdash(2,3)} (V_\alpha \otimes V_\beta)^{\oplus B^a_{\alpha,\beta}}.$$

We have, by Theorem 2.23,

$$Y^a = \bigoplus_{(\alpha,\beta)\vdash(2,3)} \Big((\mathbb{C} \sqcup_l {}^{\mathbb{S}_2}_{\mathbb{S}_1}\!\downarrow V_\alpha) \boxtimes (\mathbb{C}[\mathbb{S}_2] \downarrow^{\mathbb{S}_2}_{\mathbb{S}_1} \sqcup_r V_\beta\Big)^{\oplus B_{\alpha,\beta}}.$$

Table 4.6 – Number of ways to get δ from β by removing one box and adding two boxes, not on the same column (left), not on the same row (right)

$\beta\backslash\delta$	$\ydiagram{4}$	$\ydiagram{3,1}$	$\ydiagram{2,2}$	$\ydiagram{2,1,1}$	$\ydiagram{1,1,1,1}$
$\ydiagram{3}$	1 0	1 1	1 0	0 1	0 0
$\ydiagram{2,1}$	1 0	2 1	1 1	1 2	0 1
$\ydiagram{1,1,1}$	0 0	1 0	0 1	1 1	0 1

Table 4.7 – Values of $A^a_{\alpha,\beta}$ depending on a and β

$a \in \backslash\beta$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
C	10	20	10
NC_{norm}	9	18	9
NC_1	10	18	9
$NC_{x,0}$	10	19	9

Table 4.8 – $x^a_{\lambda,\mu}$ depending on a and μ

$a \in \backslash\mu$	$\ydiagram{4}$	$\ydiagram{3,1}$	$\ydiagram{2,2}$	$\ydiagram{2,1,1}$	$\ydiagram{1,1,1,1}$
C	30	90	60	90	30
NC_{norm}	27	81	54	81	27
NC_1	28	83	55	82	27
$NC_{x,0}$	29	86	57	85	28

Thus, by Pieri's rules,

$$\begin{aligned} Y^a &= \bigoplus_{(\alpha,\beta)\vdash(2,3)} ((\mathbb{C} \sqcup_l \mathbb{C}) \boxtimes (\mathbb{C} \sqcup_r V_\beta))^{\oplus 2B_{\alpha,\beta}} \\ &= \bigoplus_{\beta\vdash 3} (\mathbb{C}[\mathbb{S}_2] \boxtimes (\mathbb{C} \sqcup_r V_\beta))^{\oplus 4B_{\alpha,\beta}} \\ &= \bigoplus_{\beta\vdash 3} \bigoplus_{\substack{\gamma\vdash 4 \\ \gamma\geq\beta}} (\mathbb{C}[\mathbb{S}_2] \boxtimes V_\gamma)^{\oplus 4B_{\alpha,\beta}} . \end{aligned}$$

Here, in terms of Young diagrams, γ is obtained from β by adding one box. For every $\gamma \vdash 4$ and $\beta \vdash 3$, we count the number of ways to get γ from β this way in Table 4.9. We also have, by `SageMath` computations and Corollary 2.33, the values of $B^a_{\alpha,\beta}$ depending on β and a, see Table 4.10. Then we get the values of $y^a_{\lambda,\mu}$ depending on a and μ, see Table 4.11, and the values of $x^a_{\lambda,\mu} + y^a_{\lambda,\mu}$ depending on a and μ, see Table 4.12.

Table 4.9 – Number of ways to get γ from β adding one box

$\beta \backslash \gamma$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
(3)	1	1	0	0	0
$(2,1)$	0	1	1	1	0
$(1,1,1)$	0	0	0	1	1

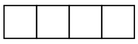
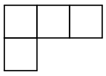
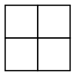
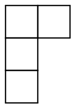
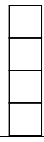
Table 4.10 – Values of $B^a_{\alpha,\beta}$ depending on a and β

$a \in \backslash\beta$	(3)	$(2,1)$	$(1,1,1)$
C	1	2	1
NC_{norm}	0	0	0
NC_1	0	0	1
$NC_{x,0}$	0	1	1

Table 4.11 – $y^a_{\lambda,\mu}$ depending on a and μ

$a \in \backslash\mu$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
C	4	12	8	12	4
NC_{norm}	0	0	0	0	0
NC_1	0	0	0	4	4
$NC_{x,0}$	0	4	4	8	4

Table 4.12 – $x^a_{\lambda,\mu} + y^a_{\lambda,\mu}$ depending on a and μ

$a \in \backslash\mu$					
C	34	102	68	102	34
NC_{norm}	27	81	54	81	27
NC_1	28	83	55	86	31
$NC_{x,0}$	29	90	61	93	32

4.3.4 Computation of $r^{a,\perp}_{\lambda',\mu'}$ via Gröbner basis

In order to find an upper bound of the rank of a polynomial matrix $M(x_1, \dots, x_l)$ of size $n \times m$, one can compute, for $1 \leq i \leq \min(n, m)$, the determinantal ideal $DI_i(M)$ (see [BD17, Section 3]), which is the ideal generated by all the minors of M of size i, and compute its Gröbner basis. The Gröbner basis is 0 if and only if for all $(z_1, \dots, z_l) \in \mathbb{C}^l$, $\text{rk}(M(z_1, \dots, z_l)) < i$. We can compute Gröbner bases for the polynomial matrix $B^{a,\perp}_{\lambda,\mu}$ in the variables (a_1, a_2, a_3, a_4), and find the values of $r^{a,\perp}_{\lambda',\mu'}$ depending on a and μ, see Table 4.13.

Table 4.13 – $r^{a,\perp}_{\lambda',\mu'}$ depending on a and μ

$a \in \backslash\mu$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
C	0	0	0	0	0
NC_{norm}	1	3	2	3	1
NC_1	1	3	2	3	0
$NC_{x,0}$	1	3	2	2	0

Thus, if $\mathcal{P}_a$ is Koszul, we have

$$r^a_{\lambda,\mu} = 199M_{\lambda,\mu} - k_{\lambda,\mu} - k^{\perp}_{\lambda',\mu'} - x^a_{\lambda,\mu} - y^a_{\lambda,\mu} - r^{a,\perp}_{\lambda',\mu'} =: \Sigma^a_{\lambda,\mu},$$

and we have the values of $\Sigma^a_{\lambda,\mu}$ depending on a and μ in Table 4.14.

4.3.5 Possible values of $r^a_{\lambda,\mu}$

For the case of $r^a_{\lambda,\mu}$, the problem is the size of the matrices. Most of the matrices are really big, thus they have many minors and require too much computations. However, we can use this method for some partitions $(\lambda, \mu) \vdash (2, 4)$, for example for $(\lambda, \mu) =$

Table 4.14 – $\Sigma^a_{\lambda,\mu}$ depending on a and μ

$a \in \backslash \mu$	(4)	(3, 1)	(2, 2)	(2, 1, 1)	(1, 1, 1, 1)
C	0	0	0	0	0
NC_{norm}	6	18	12	18	6
NC_1	5	16	11	13	11
$NC_{x,0}$	4	9	5	7	10

$((1,1),(1,1,1,1))$, we get

$$r^a_{\lambda,\mu} = \begin{cases} 0 & \text{if } a \in C \\ 3 & \text{if } a \in NC_1 \\ 2 & \text{if } a \in NC_{x,0} \end{cases} .$$

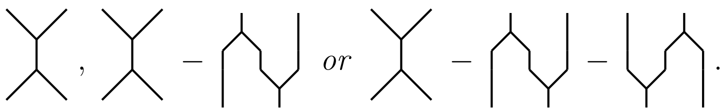
This proves the following.

Theorem 4.13. *Let $a = (a_1, a_2, a_3, a_4) \in \{0,1\}^4$. The following are equivalent :*

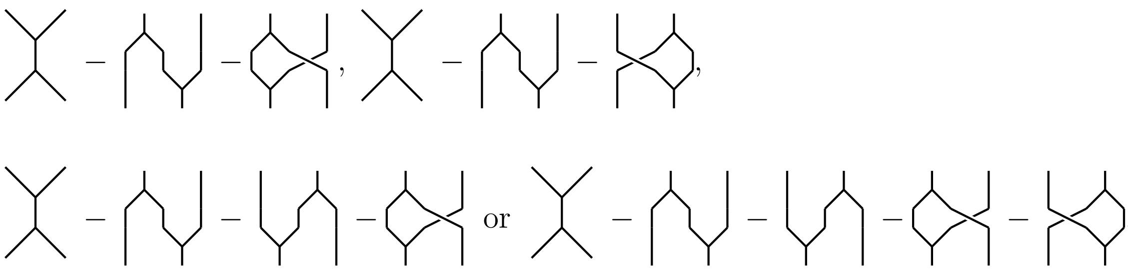
(i) the properad $\mathcal{P}_a$ induces a confluent system,

(ii) the properad $\mathcal{P}_a$ is Koszul.

Moreover, in this case, $\between_a$ is one of the following up to isomorphism :



Nevertheless, the subfamily of properads considered in this theorem is finite of size 16. Up to isomorphism, this family is of size 7, thus this theorem is based on koszulity of three different properads up to isomorphism, and proves non Koszulity of four different properads up to isomorphisms, the ones given by $\between_a$ equal to

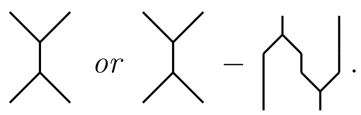


We use now the previous method via `SageMath` on the matrices $B^a_{\lambda,\mu}$ for $\lambda = (1,1)$ and $\mu = (1,1,1,1)$ and find that the determinantal ideals $DI_i(B^a_{\lambda,\mu})$ are equal to zero for $i \geq 7$, but not for $i = 6$. Thus there exists infinitely many $a \in \mathbb{C}^4$ such that $\mathrm{rk}(B^a_{\lambda,\mu}) = 6$, and thus there exists $a \in NC_{\mathrm{norm}}$ such that $\mathrm{rk}(B^a_{\lambda,\mu}) = 6$ (for example $a = (2,2,0,0)$). This means that the Euler characteristic can be zero for properads that induces a non confluent system. We have to keep in mind that this does not prove that Conjecture 4.2 is false, because of the sequence of implications we used so far, and the fact that we looked at particular partitions (λ,μ), but this makes it harder to prove. A way to provide results similar to what we want is to look at different families of properads with fewer parameters. For example, if we take $a_2 = a_4 = 0$ and look at the Gröbner basis for $i = 6$, we get 0, thus we have the following.

Theorem 4.14. *For $a = (a_1, 0, a_3, 0)$, the following are equivalent :*

(i) the properad $\mathcal{P}_a$ is Koszul,

(ii) the relation $\lozenge_a$ is one of the following, up to isomorphism :

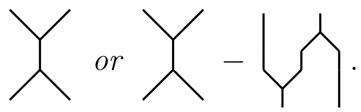


We can do the same for a similar family of properads.

Theorem 4.15. *For $a = (0, a_2, 0, a_4)$, the following are equivalent :*

(i) the properad $\mathcal{P}_a$ is Koszul,

(ii) the relation $\lozenge_a$ is one of the following, up to isomorphism :

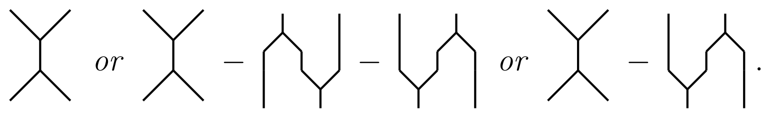


We can also try with more restrictions, in order to get the properad $\varepsilon\mathcal{B}$.

Theorem 4.16. *For $a = (a_1, 1, 0, 0)$, the following are equivalent :*

(i) the properad $\mathcal{P}_a$ is Koszul,

(ii) the relation $\between_a$ *is one of the following, up to isomorphism :*



These results all prove non koszulity of some properads of the form $\mathcal{P}_a$ that induces a non confluent system, but not all of them. In order to prove Conjecture 4.2, one can look in different weights and biarities. For example, one can look in biarity $(3,3)$ and weight 4, the spaces considered are bigger, thus the calculations by hand and by computer take more time, but we may find another stronger criterion on $\mathcal{P}_a$ to be Koszul. Another idea would be to study the other partitions $(\lambda,\mu) \vdash (2,4)$ to find potential obstructions. This does not require many more calculations but the problem is that we have to compute minors of big size in matrices of bigger size. We could also go step by step, considering for example $a_1 \neq 0$ (because the problem holds in NC_{norm}), thus we can divide by a_1 and simplify the matrix.

However these tools can be used to study families of parametrized properads in order to look at possibly Koszul properads, or to prove their non-koszulity. In the case of non-quadratic properads, one can use these tools to prove (non) homotopy koszulity of such properads by looking at their associated quadratic properads. However, the condition of $\mathbb{S}$-bimodule isomorphism between the properads and their associated quadratic properad seems not easy to refute.

Appendix A

SageMath SCRIPT

Here we present the idea behind the `SageMath` script used in Chapter 4. The script can be found in [Néd], together with the files `isobiarity23.ipynb` and `Koszulcpxbiarity24.ipynb` that accompagny Chapter 4, the file `readme.txt` that describes every module, and a file `tutorial.ipynb` that shows how to use the script on a simple example. In this appendix, we will focus on how the script works rather that describing the modules or showing how to actually use it. The script is meant to evolve, maybe even by changing the language.

A.1 Idea of constructions of properads and ideals

To construct a free properad over generators, we give a list of graphs which are generators in weight 1. In order to construct the graphs in weight n from weight $n-1$, we take every element in weight $n-1$ and look at every possible way we can link a generator to this element. This constructs a list of graphs, but possibly with duplicates. That is why we need to remove all duplicate graphs from this list and this is what take the most time in this construction. The time taken by this step is why we save all generated properads, and later ideals, in files via the `pickle` module.

The idea for the free ideal is the same, we give a list of relations, a relation being a list of couples of the form (graph, coefficient), and a free properad in which this ideal lives. Then we generate, step by step, the space of relations with the same method as for free properads.

In order to study the family of properads in Chapter 4, we generated a free properad over a product and a coproduct. In this properad, we generated an ideal with relations being associativity, coassociativity, and $\between_a$, a being encoded as polynomial variables. Thus we can compute a generating family of the space of relations in some weight and biarity and study it. The issue here is that the generating family in weight 4, the one that interests us, is way too big and we cannot ask for the rank depending on a that easily. Thus we need to use the divide and conquer method from M. Bremner and V. Dotsenko in [BD17].

A.2 The divide and conquer method

Here the idea was to encode the divide and conquer method, we use exactly the same method as in [BD17], but we just add the Kronecker product to the process because we work on properads instead of operads. The other difficulty was to get only one relation per orbit under the $\mathbb{S}_m \times \mathbb{S}_n^{\mathrm{op}}$ action, because the external outputs and inputs of similar graphs were in the same order. That is why we added to the function that removes duplicates a standardizing function, that uniformizes all graphs.

Once one relation per orbit chosen, we can compute the representation matrices of the space of relations in some weight and biarity and eventually compute the multiplicities of this representation in terms of a. In order to compute these multiplicities, one can use Gröbner bases or primary decompositions of the determinental ideals.

A.3 More uses and limits

This script can be used to compute free properads or properads by generators and relations weight by weight (if the space of relations is homogeneous). For a specific properad, one can easily compute any desired dimension for a weight lower or equal to 4. This may take a few hours for weight 4. For example, one can study the properad BiB^λ encoding balanced infinitesimal bialgebras, see [Que24, Section 2], or the properad V encoding V-gebras, see [LV23, Section 3].

However, there are some limits to this script. So far, handling symmetries on generators is not easy, for example for the properad V, one has to compute the free properad over a non-cocommutative bi-tensor, then compute by hand the consequences of cocommutativity of the bi-tensor and write down these additionnal relations, together with the other ones, in the space of relations. This increases the amount of calculations done by the script, thus slows it down a lot. Another limit of this script is that it is very long to compute weight 5 of a free properad. Maybe it can be optimized.

BIBLIOGRAPHY

[Ale+18a] Charles Alexandre, Martin Bordemann, Salim Riviere, and Friedrich Wagemann. *Algebraic deformation quantization of Leibniz algebras*. 2018. arXiv: 1810.04050 [math.AT]. URL: https://arxiv.org/abs/1810.04050.

[Ale+18b] Charles Alexandre, Martin Bordemann, Salim Riviere, and Friedrich Wagemann. « Structure theory of Rack-Bialgebras ». In: (2018). arXiv: 1412.5907 [math.QA]. URL: https://arxiv.org/abs/1412.5907.

[BD17] Murray Bremner and Vladimir Dotsenko. « Classification of Regular Parametrized One-relation Operads ». In: *Canadian Journal of Mathematics* 69.5 (Oct. 2017), pp. 992–1035. ISSN: 1496-4279. DOI: 10.4153/cjm-2017-018-3. URL: http://dx.doi.org/10.4153/CJM-2017-018-3.

[Ber07] Michel Van den Bergh. *Double Poisson algebras*. 2007. arXiv: math/0410528 [math.QA]. URL: https://arxiv.org/abs/math/0410528.

[BMP16] Murray R. Bremner, Sara Madariaga, and Luiz A. Peresi. « Structure theory for the group algebra of the symmetric group, with applications to polynomial identities for the octonions ». eng. In: *Commentationes Mathematicae Universitatis Carolinae* 57.4 (2016), pp. 413–452. URL: http://eudml.org/doc/287582.

[Cli81] Joseph M. Clifton. « A simplification of the computation of the natural representation of the symmetric group S_n ». In: *American mathematical society* 83.2 (1981), pp. 248–250.

[DK10] Vladimir Dotsenko and Anton Khoroshkin. « Gröbner bases for operads ». In: *Duke Mathematical Journal* 153.2 (June 2010). ISSN: 0012-7094. DOI: 10.1215/00127094-2010-026. URL: http://dx.doi.org/10.1215/00127094-2010-026.

[DR17] Vladimir Dotsenko and Soutrik Roy Chowdhury. « Anick resolution and Koszul algebras of finite global dimension ». In: *Communications in Algebra* 45.12 (Mar. 2017), pp. 5380–5383. ISSN: 1532-4125. DOI: 10.1080/00927872.2017.1307383. URL: http://dx.doi.org/10.1080/00927872.2017.1307383.

[EE05] Benjamin Enriquez and Pavel Etingof. « On the invertibility of quantization functors ». In: *Journal of Algebra* 289.2 (2005), pp. 321–345. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2005.01.056. URL: https://www.sciencedirect.com/science/article/pii/S0021869305002309.

[FH13] William Fulton and Joe Harris. *Representation theory. A first Course*. 0072-5285. Springer New York, 2013.

[Gan02] Wee Liang Gan. *Koszul duality for dioperads*. 2002. arXiv: math/0201074 [math.QA]. URL: https://arxiv.org/abs/math/0201074.

[Hof08] Éric Hoffbeck. *A Poincaré-Birkhoff-Witt criterion for Koszul operads*. 2008. arXiv: 0709.2286 [math.AT].

[Ler20] Johan Leray. « Protoperads II: Koszul duality ». en. In: *Journal de l'École polytechnique — Mathématiques* 7 (2020), pp. 897–941. DOI: 10.5802/jep.131. URL: http://www.numdam.org/articles/10.5802/jep.131/.

[Lod08] Jean-Louis Loday. *Generalized bialgebras and triples of operads*. 320. Astérisque, 2008.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*. 0072-7830. Springer, 2012.

[LV23] Johan Leray and Bruno Vallette. *Pre-Calabi–Yau algebras and homotopy double Poisson gebras*. 2023. arXiv: 2203.05062 [math.QA].

[Mal19] Philippe Malbos. *Lectures on Algebraic Rewritting*. Dec. 2019.

[Mar06] Martin Markl. « A resolution (minimal model) of the PROP for bialgebras ». In: *Journal of Pure and Applied Algebra* 205.2 (2006), pp. 341–374. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2005.07.007. URL: https://www.sciencedirect.com/science/article/pii/S0022404905001544.

[MV09] Sergei Merkulov and Bruno Vallette. « Deformation theory of representations of prop(erad)s II ». In: 2009.636 (2009), pp. 123–174. DOI: doi:10.1515/CRELLE.2009.084. URL: https://doi.org/10.1515/CRELLE.2009.084.

[MV10] Martin Markl and Alexander A. Voronov. « PROPped-Up Graph Cohomology ». In: *Tschinkel Y., Zarhin Y. (eds) Algebra, Arithmetic, and Geometry. Progress in Mathematics* 270 (2010). DOI: https://doi.org/10.1007/978-0-8176-4747-6_8.

[Néd] Silvère Nédélec. *script for free properads and properadic ideals*. https://github.com/knirf/Free-properads-and-properad-ideals.git.

[Que24] Alexandre Quesney. *Balanced infinitesimal bialgebras, double Poisson gebras and pre-Calabi-Yau algebras*. 2024. arXiv: 2312.14893 [math.QA].

[Sag91] Bruce E. Sagan. *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*. 0072-5285. Springer, 1991.

[Val07] Bruno Vallette. « A Koszul duality for props ». In: *Trans. Amer. Math. Soc.* 359 (2007), pp. 4865–4943. DOI: https://doi.org/10.1090/S0002-9947-07-04182-7.

[Val08] Bruno Vallette. « Free Monoid in Monoidal Abelian Categories ». In: *Applied Categorical Structures* 17.1 (Mar. 2008), pp. 43–61. ISSN: 1572-9095. DOI: 10.1007/s10485-008-9130-y. URL: http://dx.doi.org/10.1007/s10485-008-9130-y.

COLLEGE DOCTORAL PAYS DE LA LOIRE — ED MATHEMATIQUES STIC

Nantes Université

Titre : Sur la koszulité de certaines propérades

Mot clés : Propérades, théorie des représentations, algèbre homologique, théorie de l'homotopie, théorie de la réécriture

Résumé : Dans cette thèse nous nous intéressons à certaines propérades et leur éventuelle koszulité ou homotopie koszulité. Ces propriétés homologiques permettent notamment le calcul d'un modèle minimal et la (co)homologie des algèbres sur ces propérades. Nous rappelons d'abord dans les deux premiers chapitres les bases de la théorie des représentations et des propérades. Nous nous sommes dans un premier temps intéressés à la propérade $\mathcal{R}ack\mathcal{B}$ encodant les racks bigèbres, un type de structure généralisant les algèbres de Hopf au même titre que les racks et les algèbres de Leibniz généralisent respectivement les groupes et les algèbres de Lie. La propérade $\mathcal{R}ack\mathcal{B}$ étant particulière dans sa construction, nous avons d'abord montré l'homotopie koszulité d'une propérade intermédiaire. Dans un second temps, nous nous sommes intéressés à une famille de propérades à paramètres, encodant des bigèbres associatives et coassociatives munies d'une relation supplémentaire faisant intervenir le produit et le coproduit. Le but était de déterminer si la confluence, en un certain sens, de ces propérades était équivalente à leur koszulité.

Title: On Koszulity of some properads

Keywords: Properads, representation theory, homological algebra, homotopy theory, rewriting theory

Abstract: In this manuscript, we were interested in some properads and their eventual koszulity or homotopy koszulity. These homological properties lead to several powerful tools, such as minimal model and (co)homology of algebras over such properads. We remind in the first two chapters the basics on representation theory and properads. Then we study the properad $\mathcal{R}ack\mathcal{B}$ encoding rack bialgebras, a structure that generalizes the structure of Hopf algebras, just as racks and Lie algebras generalize respectively groups and Lie algebras. Because the properad $\mathcal{R}ack\mathcal{B}$ is particular in its presentation, we decided to prove the homotopy koszulity of an intermediate properad. In the final chapter, we studied a family of parametrized properads that encode associative and coassociative bialgebras with an extra relation involving the product and the coproduct. The goal was to determine if confluence, in a certain sense, was equivalent to koszulity for these properads.