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Abstract

We study the Carleson measures and the Toeplitz operators on the class of the so-called small weighted Bergman spaces, introduced recently by Seip. A characterization of Carleson measures is obtained which extends Seip's results from the unit disk of $\mathbb{C}$ to the unit ball of $\mathbb{C}^n$. We use this characterization to give necessary and sufficient conditions for the boundedness and compactness of Toeplitz operators. Finally, we study the Schatten p classes membership of Toeplitz operators for $1 < p < \infty$.

Furthermore, we also consider the Bergman type projection acting on L^∞ to the Bloch space $\mathcal{B}$ on $\mathbb{B}_n$. A characterization of radial weight so that the projection is bounded is obtained.

Finally, we investigate the weighted Fock spaces in one and several complex variables. We evaluate the dimension of these spaces in terms of the weight function extending and completing earlier results by Rozenblum–Shirokov and Shigekawa.

Chapter 1

Introduction

The thesis covers three groups of results concerning weighted spaces of analytic functions: the weighted Bergman spaces of analytic functions in the unit disc and unit ball in $\mathbb{C}^n$ and the Fock spaces of entire functions in $\mathbb{C}$ and $\mathbb{C}^n$. We are interested in geometric properties of these spaces (the dimension question) and the operator theoretic properties concerning the embedding operator (the Carleson measures), the Bergman projection operator (to the Bloch space) and the Toeplitz operator.

In Chapter 4 and 5 we deal with weighted Bergman spaces. In Chapter 4 we concentrate on the so called small (radial) Bergman spaces introduced by Seip in 2013. They constitute a class of Begman spaces interpolating, in a sense, between the standard Bergman space and the classical Hardy space. Extending the results by Seip (for $\mathbb{D}$), we describe the Carleson measures for the unit ball. Furthermore, we study the Toeplitz operators T_μ with measure symbol μ and describe when T_μ is bounded, compact, and is in the Schatten class for some classes of weights, in terms of the symbol μ. These results will be published in [23].

In Chapter 5 we extend the recent results of Peláez and Rättyä (from $\mathbb{D}$) and obtain a complete description of radial weights such that the corresponding Bergman projection operator acts boundedly from $L^\infty(\mathbb{B}_n)$ to the Bloch space.

In Chapter 6 we study non-radial weighted (Hilbert) Fock spaces $\mathcal{F}^2_\psi$. The question we are interested in here is when $\dim \mathcal{F}^2_\psi = \infty$ in terms of the weight ψ. In 2006, Rozenblum and Shirokov claimed that if ψ is subharmonic on $\mathbb{C}$ and $\Delta\psi(\mathbb{C}) = \infty$, then $\dim \mathcal{F}^2_\psi = \infty$. In fact, this statement is true only if $\Delta\psi$ has no point masses with masses larger than or equal to 4π. We correct the statement of Rozenblum and Shirokov, and obtain a criterion for the space $\mathcal{F}^2_\psi$ to be of infinite dimension. Furthermore, we calculate $\dim \mathcal{F}^2_\psi$ in terms of $\Delta\psi$. In the case of the Fock spaces on $\mathbb{C}^n$, we extend somewhat the theorem of Shigekawa (1991) and give a new sufficient condition for $\dim \mathcal{F}^2_\psi(\mathbb{C}^n) = \infty$. We also produce several examples that show how complicated is to produce a criterion for $\dim \mathcal{F}^2_\psi(\mathbb{C}^n) = \infty$.

The techniques we use include estimates of logarithmic potentials in the plane, the Bedford–Taylor solution for the complex Monge–Ampère problem, the Peláez–Rättyä estimates on the reproducing kernels for the weighted Bergman space, the Lelong number estimates for plurisubharmonic functions, and the Carleson embedding for the Hardy space in the polydisk.

1.1 Carleson measures and Toeplitz operators on small Bergman spaces on the ball

The original notion of Carleson measures was introduced by L. Carleson [6, 7] in his work on interpolating sequences and the corona problem on the algebra H^∞ of all bounded analytic functions on the unit disk. It then plays a crucial role in studying function spaces and operators acting on them. The Carleson measures on Bergman spaces were studied by Hastings [16], and later on by Luecking [25], and many others. For sampling and interpolation in large Bergman spaces, see Seip [38] and Borichev, Dhuez, Kellay [4]. Recently, Pau and Zhao [27] gave a characterization for Carleson measures and vanishing Carleson measures on the unit ball by using the products of functions in weighted Bergman spaces. In [29], Peláez and Rättyä gave a description of Carleson measures for A^2_ρ on unit disk when ρ satisfies that $\frac{1}{(1-r)\rho(r)}\int_r^1 \rho(t)dt$ is either equivalent to 1 or tends to ∞, and in [30] they then got a criterion for A^2_ρ on unit disk when $\rho \in \widehat{\mathcal{D}}$, which means $\int_r^1 \rho(s)ds \lesssim \int_{\frac{r+1}{2}}^1 \rho(s)ds$. In 2018, we have obtained a criterion for A^2_ρ on the unit ball for ρ belongs to the class S which was introduced by Seip in [39] in 2013. A close relationship between the class S and $\widehat{\mathcal{D}}$ will be presented specifically afterwards. A short time ago, in June 2019, Juntao Du, Songxiao Li, Xiaosong Liu, Yecheng Shi extended the description of Peláez and Rättyä offered in [29] to higher dimensions when $\rho \in \widehat{\mathcal{D}}$, see [12].

In [39], Seip gave a characterization of Carleson measures for A^2_ρ with $\rho \in S$ in the case $n = 1$. One of our main results, Theorem 4.2.1, extends this result to the case $n > 1$.

The Toeplitz operators acting on various spaces of holomorphic functions have been extensively investigated by a lot of authors, and the theory is especially well understood in the case of Hardy spaces or standard Bergman spaces (see [42], [43] and the references therein). For the Toeplitz operators on the Fock space see, for example, Fulsche, Hagger [14] and Schuster, Varolin [37]. In 1987, Luecking [26] was the first one to consider Toeplitz operators on Bergman spaces with measures as symbols, and some interesting results about Toeplitz operators acting on large Bergman spaces were obtained by Lin and Rochberg [24]. In this thesis, we will study the boundedness and compactness of T_μ on A^2_ρ, with $\rho \in S$.

We also study when our Toeplitz operators belong to the Schatten class. A description for the classical weighted Bergman spaces on the unit disk is given in [43, Chapter 7], and a description for the case of large Bergman spaces on the disk was obtained in 2015 by H. Arroussi, I. Park, and J. Pau (see [1]). In 2016, Peláez and Rättyä [31] gave an interesting characterization for the case of small Bergman spaces on unit disk, when the weight $\rho \in \widehat{\mathcal{D}}$.

For weights ρ in S^*, we obtain a characterization of the symbols of the Toeplitz operators in the Schatten classes $\mathcal{S}_p$. In [34], Peláez, Rättyä and Sierra gave a characterization for the case of dimension $n = 1$ when the weight is regular, that is $\rho^*(r) \asymp \rho(r)$. As an easy observation, our result is equivalent to their result when $n = 1$. We point out that our approach is completely different from that of [34], which does not seem to work in higher dimensions. On the other hand, for weights ρ in $S \setminus S^*$, this characterization fails. A counterexample was given in [34] and we will show this failure for all ρ in $S \setminus S^*$.

In this thesis, we restrict ourselves to the case $1 < p < \infty$. For the case $0 < p \leq 1$, the techniques we use should be modified.

1.2 Bergman type projections

Let ρ be a radial weight and X be a space of measurable functions on $\mathbb{B}_n$. The Bergman type projection P_ρ acting on X is given by

$$P_\rho f(z) = \int_{\mathbb{B}_n} K_\rho(z,w) f(w) \rho(w) dv(w), \qquad z \in \mathbb{B}_n, f \in X,$$

where $K_\rho(z,w)$ is the reproducing kernel of the weighted Bergman space A^2_ρ.

When ρ is the standard radial weight $\rho(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, the projection is denoted by P_α.

Projections play an important role in studying operator theory on spaces of analytic functions. Bounded analytic projections can also be used to establish duality relations and to obtain useful equivalent norms in spaces of analytic functions. Hence the boundedness of projections is an interesting topic which has been examined by many authors in recent years [8, 10, 11, 32, 33]. In [32], Peláez and Rättyä considered the projection P_{ρ_1} acting on $L^p_{\rho_2}(\mathbb{D})$, $1 \leq p < \infty$ when two weights ρ_1, ρ_2 are in the class $\mathcal{R}$ of so called regular weights. A radial weight ρ is regular if $\widehat{\rho}(r) \asymp (1-r)\rho(r), r \in (0,1)$. Recently, in 2019, they extended these results to the case where $\rho_1 \in \widehat{\mathcal{D}}$, ρ_2 is radial [33].

Chapter 5 is devoted to studying the projections acting on the space L^∞. In the case of standard radial weight, we have the following theorem.

Theorem A. *For any $\alpha > -1$, the Bergman type projection P_α is a bounded linear operator from L^∞ onto the Bloch space $\mathcal{B}$.*

See [42, Theorem 3.4] for a proof. This theorem is also valid for the case of one dimension [43, Theorem 5.2].

In [33], Peláez and Rättyä obtain an interesting result for one dimensional case.

Theorem B. *Let ρ be a radial weight. Then the projection $P_\rho : L^\infty(\mathbb{D}) \to \mathcal{B}(\mathbb{D})$ is bounded if and only if $\rho \in \widehat{\mathcal{D}}$.*

In this thesis, we extend this theorem to the case of several variables.

1.3 Dimension of the Fock type spaces

Let ψ be a plurisubharmonic function on $\mathbb{C}^n$. The weighted Fock space $\mathcal{F}^2_\psi$ is the space of entire functions f such that

$$||f||^2_\psi = \int_{\mathbb{C}^n} |f(z)|^2 e^{-\psi(z)} dV(z) < \infty$$

where dV is the volume measure on $\mathbb{C}^n$. Note that $\mathcal{F}^2_\psi$ is a closed subspace of $L^2(\mathbb{C}^n, e^{-\psi} dv)$ and hence is a Hilbert space endowed with the inner product

$$\langle f, g\rangle_\psi = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-\psi(z)} dV(z), \qquad f, g \in \mathcal{F}^2_\psi.$$

In Chapter 6 we study when the space $\mathcal{F}^2_\psi$ is of finite dimension depending on the weight ψ. This problem (at least for the case $n = 1$) is motivated by some quantum mechanics questions, especially in the study of zero modes, eigenfunctions with zero eigenvalues. In [36, Theorem 3.2], Rozenblum and Shirokov proposed a sufficient condition for the space $\mathcal{F}^2_\psi$ to be of infinite dimension, when ψ is a subharmonic function.

Theorem C. *Let ψ be a finite subharmonic function on the complex plane such that the measure $\mu = \Delta\psi$ is of infinite mass:*

$$\mu(\mathbb{C}) = \int_{\mathbb{C}} d\mu(z) = \infty. \tag{1.1}$$

Then the space $\mathcal{F}^2_\psi$ has infinite dimension.

We improve and extend somewhat this statement, give a necessary and sufficient condition on ψ for the space $\mathcal{F}^2_\psi$ to be of finite dimension, and calculate this dimension.

The situation is much more complicated in $\mathbb{C}^n$, $n \geq 2$. Shigekawa established in [40] (see also [15, Theorem 7.10] in a book by Haslinger), the following interesting result.

Theorem D. *Let $\psi : \mathbb{C}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let $\mu(z)$ denote the lowest eigenvalue of the Levi matrix*

$$L_\psi(z) = i\partial\bar{\partial}\psi(z) = \left(\frac{\partial^2\psi(z)}{\partial z_j \partial \overline{z_k}}\right)^n_{j,k=1}.$$

Suppose that

$$\lim_{|z|\to\infty} |z|^2\mu(z) = \infty. \tag{1.2}$$

Then $\dim(\mathcal{F}^2_\psi) = \infty$.

Note that the condition (1.2) is far from being necessary. A corresponding example is given in [15, Section 7]. In this thesis, we improve Theorem D by presenting a weaker condition for the dimension of the Fock space $\mathcal{F}^2_\psi$ to be infinite. Furthermore, we give several examples that show how far our condition is from being necessary.

1.4 Outline of the thesis

We will state our main results in Chapter 2. In Chapter 3, we introduce some notions and notation, recall some basic facts which will be used later on. The Carleson measures and Toeplitz operators on small Bergman spaces on the ball will be examined in Chapter 4, and the boundedness of Berman type projection acting on L^∞ is studied in Chapter 5. Finally, the study on the dimension of the Fock type spaces will be presented in Chapter 6.

CHAPTER 2

MAIN RESULTS

2.1 Chapter 4

Let ρ be a positive continuous and integrable function on $[0,1)$. We extend it to $\mathbb{B}_n$ by $\rho(z) = \rho(|z|)$, and call such ρ a radial weight function, or simply radial weight. We assume that

$$\int_0^1 x^{2n-1}\rho(x)dx = 1,$$

and consider the points $r_k \in [0,1)$ determined by the relation

$$\int_{r_k}^1 \rho(x)dx = 2^{-k}.$$

Denote by S the class of weights ρ such that

$$\inf_k \frac{1-r_k}{1-r_{k+1}} > 1. \tag{2.1}$$

This class of weights was introduced by Seip in [39]. We also introduce a subclass S^* of weights in S determined by the condition that $\rho^*(r) \lesssim \rho(r)$ for $r \in (0,1)$, where

$$\rho^*(r) = \frac{1}{1-r}\int_r^1 \rho(t)dt.$$

Denote by A^2_ρ the weighted Bergman space consisting of all f holomorphic functions on $\mathbb{B}_n$ such that

$$||f||^2_\rho = \int_{\mathbb{B}_n} |f(z)|^2\rho(z)dv(z) < \infty,$$

where dv is the normalized volume measure on $\mathbb{B}_n$.

For every nonnegative integer k, set

$$\Omega_k = \{z \in \mathbb{B}_n : r_k \leq |z| < r_{k+1}\},$$

and let μ_k be the measure defined by $\mu_k = \chi_{\Omega_k}\mu$ whenever a nonnegative Borel measure μ on $\mathbb{B}_n$ is given.

Throughout this text, the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a positive constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a space of functions or a set of numbers. If both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$, then we write $U(z) \asymp V(z)$.

Our results are following:

Theorem 2.1.1. *Let $\rho \in S$, and let μ be a finite positive Borel measure on $\mathbb{B}_n$. Then*

(i) μ is a Carleson measure for A^2_ρ if and only if each μ_k is a Carleson measure for the Hardy space H^2 with Carleson constant $C_{\mu_k}(H^2) \lesssim 2^{-k}$, $k \geq 0$.

(ii) μ is a vanishing Carleson measure for A^2_ρ if and only if

$$\lim_{k\to\infty} 2^k C_{\mu_k}(H^2) = 0.$$

Theorem 2.1.2. *Let $\rho \in S$, and let μ be a finite positive Borel measure on $\mathbb{B}_n$. Then*

(i) The Toeplitz operator T_μ is bounded on A^2_ρ if and only if μ is a Carleson measure for A^2_ρ.

(ii) The Toeplitz operator T_μ is compact on A^2_ρ if and only if μ is a vanishing Carleson measure for A^2_ρ.

For a measure μ on $\mathbb{B}_n$ and $\alpha > 0$, we define the function $\widehat{\mu}_\alpha$ by

$$\widehat{\mu}_\alpha(z) = \frac{2^k \mu\big(E(z,\alpha)\big)}{(1-|z|)^n}, \qquad z \in \Omega_k.$$

Here, $E(z,\alpha)$ is the Bergman metric ball.

Let $\widetilde{T_\mu}$ be the Berezin transform of T_μ, defined by

$$\widetilde{T_\mu}(z) = \langle T_\mu k_z^\rho, k_z^\rho\rangle_\rho, \qquad z \in \mathbb{B}_n,$$

where k_z^ρ is the normalized reproducing kernel of A_ρ^2. Set

$$d\lambda_\rho(z) = \frac{2^k \rho(z) dv(z)}{(1-|z|)^n}, \qquad z \in \Omega_k.$$

Theorem 2.1.3. *Let ρ be in S^*, μ be a finite positive Borel measure and $1 < p < \infty$. The following conditions are equivalent:*

(a) The Toeplitz operator T_μ is in the Schatten class $\mathcal{S}_p$.

(b) The function $\widetilde{T_\mu}$ is in $L^p(\mathbb{B}_n, d\lambda_\rho)$.

(c) The function $\widehat{\mu}_\alpha$ is in $L^p(\mathbb{B}_n, d\lambda_\rho)$ for sufficiently small $\alpha > 0$.

2.2 Chapter 5

In Chapter 5, we obtain a characterization of radial weight such that the Bergman type projection is bounded from L^∞ to $\mathcal{B}$.

Theorem 2.2.1. *Let ρ be a radial weight. Then the projection $P_\rho : L^\infty \to \mathcal{B}$ is bounded if and only if $\rho \in \widehat{\mathcal{D}}$.*

2.3 Chapter 6

Let ψ be a measurable function on $\mathbb{C}^n$. The weighted Fock space $\mathcal{F}_\psi^2$ is the space of entire functions f such that

$$\|f\|_\psi^2 = \int_{\mathbb{C}^n} |f(z)|^2 e^{-\psi(z)} dV(z) < \infty.$$

If $\psi : \mathbb{C} \to [-\infty, \infty)$ is a subharmonic function, denote by μ_ψ the corresponding Riez measure, $\mu_\psi = \Delta\psi$. Consider the class $\mathcal{M}^d$ of the positive σ-finite atomic measures with masses which are integer multiples by 4π. Given a σ-finite measure μ, consider the corresponding atomic measure μ^d,

$$\mu^d = \max\left\{\mu_1 \in \mathcal{M}^d : \mu_1 \leq \mu\right\}.$$

Denote $\mu^c = \mu - \mu^d$.

Our results about dimension of $\mathcal{F}^2_\psi$ are follows:

Theorem 2.3.1. *Let ψ be a subharmonic function on the complex plane. Then the Fock space $\mathcal{F}^2_\psi$ is finite-dimensional if and only if*

$$(\mu_\psi)^c(\mathbb{C}) < \infty. \tag{2.2}$$

If ψ is finite on $\mathbb{C}$, then we can write condition (6.4) *as $\mu_\psi(\mathbb{C}) < \infty$. Finally, if $(\mu_\psi)^c(\mathbb{C}) < \infty$, then*

$$\dim \mathcal{F}^2_\psi = \left\lceil \frac{(\mu_\psi)^c(\mathbb{C})}{4\pi} \right\rceil.$$

Theorem 2.3.2. *Let $\psi : \mathbb{C}^n \to \mathbb{R}$ be a C^2 smooth function. Given $M > 0$, consider $\psi_M(z) = M\log(|z|^2)$. Suppose that for every $M > 0$, the function $\psi - \psi_M$ is plurisubharmonic outside a compact subset of $\mathbb{C}^n$. Then $\dim \mathcal{F}^2_\psi = \infty$.*

Theorem 2.3.3. *Suppose that $\psi(z) = \psi(|z|^2)$ is a radial plurisubharmonic function of class C^2. Then $\dim \mathcal{F}^2_\psi = \infty$ if and only if*

$$\int_{\mathbb{C}^n} (dd^c\psi)^n = \infty.$$

CHAPTER 3

PRELIMINARIES

3.1 Some basic notation

Let $\mathbb{C}^n$ denote the n-dimensional complex Euclidean space. For any two points $z = (z_1, \ldots, z_2)$, $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we use the well-known notation

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n} \quad \text{and} \quad |z| = \sqrt{\langle z, z\rangle}.$$

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball and $\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}$ be the unit sphere in $\mathbb{C}^n$. Denote by $H(\mathbb{B}_n)$ the space of all holomorphic functions on the unit ball $\mathbb{B}_n$. Let dV be the volume measure on $\mathbb{C}^n$ and dv be the normalized volume measure on $\mathbb{B}_n$. The normalized surface measure on $\mathbb{S}_n$ will be denoted by $d\sigma$.

Given $a \in \mathbb{B}_n \setminus \{0\}$ and $r > 0$, let $\delta(a) = \sqrt{2(1-|a|)}$. Define $Q(a, r) \subset \mathbb{B}_n$ and $O(a, r) \subset \mathbb{S}_n$ as follows:

$$Q(a, r) = \{z \in \mathbb{B}_n : \sqrt{|1 - \langle a/|a|, z\rangle|} < r\},$$

$$O(a, r) = \{\zeta \in \mathbb{S}_n : \sqrt{|1 - \langle a/|a|, \zeta\rangle|} < r\}.$$

For simplicity of notation, we write Q_a instead of $Q\big(a, \delta(a)\big)$, O_a instead of $O\big(a, \delta(a)\big)$. Let φ_a denote the Möbius transformation on $\mathbb{B}_n$ that interchanges 0 and a, that is

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{1-|a|^2}\, P_a^{\perp}(z)}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B}_n,$$

where P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[a]$ generated by a, and $P_a^{\perp}$ is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [a]$. The Bergman metric ball $E(a,r)$ is defined by

$$E(a,r) = \{z \in \mathbb{B}_n : \beta(a,z) < r\},$$

where $\beta(a,z)$ is the Bergman metric given by

$$\beta(a,z) = \frac{1}{2}\log\frac{1+|\varphi_a(z)|}{1-|\varphi_a(z)|}, \qquad a,z \in \mathbb{B}_n.$$

We state here some auxiliary lemmas which can be found in [42].

Lemma 3.1.1. *The Bergman metric ball $E(0,r)$ is a Euclidean ball of radius $R = \tanh r$, centered at the origin, and*

$$E(a,r) = \varphi_a\Big(E(0,r)\Big).$$

Moreover, $v\Big(E(a,r)\Big) \asymp (1-|a|)^{n+1}$.

Lemma 3.1.2. *Suppose that c is real and $t > -1$. Then the integral*

$$I_{c,t}(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^t dv(w)}{|1-\langle z,w\rangle|^{n+1+t+c}}, \qquad z \in \mathbb{B}_n,$$

has the following asymptotic properties:

(a) If $c < 0$, then $I_{c,t}$ is bounded in $\mathbb{B}_n$.

(b) If $c = 0$, then

$$I_{c,t}(z) \asymp \log\frac{1}{1-|z|^2}$$

as $|z| \to 1^-$.

(c) If $c > 0$, then

$$I_{c,t}(z) \asymp (1-|z|^2)^{-c}$$

as $|z| \to 1^-$.

Lemma 3.1.3. *Suppose that N is a natural number, $a_l \in \mathbb{B}_n \setminus \{0\}$, $1 \leq l \leq N$,*

$$E = \bigcup_{l=1}^{N} O_{a_l}.$$

There exists a subsequence $\{l_i\}$, $1 \leq i \leq M$, such that

(a) $O_{a_{l_i}}$, $1 \leq i \leq M$, *are disjoint.*

(b) $O\Big(a_{l_i}, 3\delta(a_{l_i})\Big)$, $1 \leq i \leq M$, *cover E.*

3.2 Bergman spaces

Definition 3.2.1. An integrable function $\rho : \mathbb{B}_n \to (0, \infty)$ is called a *weight function*, or simply a *weight*. A weight ρ is called *radial* if $\rho(z) = \rho(|z|)$ for all $z \in \mathbb{B}_n$. The *(radial) weighted Bergman space* A^2_ρ is the space of functions f in $H(\mathbb{B}_n)$ such that

$$||f||^2_\rho = \int_{\mathbb{B}_n} |f(z)|^2 \rho(z) dv(z) < \infty.$$

Note that A^2_ρ is a closed subspace of $L^2(\mathbb{B}_n, \rho dv)$ and hence is a Hilbert space endowed with the inner product

$$\langle f, g \rangle_\rho = \int_{\mathbb{B}_n} f(z)\overline{g(z)}\rho(z) dv(z), \qquad f, g \in A^2_\rho.$$

When $\rho(z) = (1 - |z|^2)^\alpha,\ \alpha > -1$, we obtain the *standard weighted Bergman spaces* A^2_α. For numerous results on the Bergman space A^2_α, see [42].

Denote by K^ρ_z the *reproducing kernel* of A^2_ρ,

$$\langle f, K^\rho_z \rangle_\rho = f(z), \qquad f \in A^2_\rho, \quad z \in \mathbb{B}_n,$$

and the function $K_\rho(z, w)$ will be defined as

$$K_\rho(z, w) = K^\rho_w(z), \qquad z, w \in \mathbb{B}_n.$$

The normalized reproducing kernel will be denoted by k_z^ρ,

$$k_z^\rho = \frac{K_z^\rho}{\|K_z^\rho\|_\rho}, \qquad z \in \mathbb{B}_n.$$

If $\{e_k(z)\}$ is an orthonormal basis of A_ρ^2, then

$$K_\rho(z, w) = \sum_{k=1}^{\infty} e_k(z)\overline{e_k(w)}, \qquad z, w \in \mathbb{B}_n.$$

The reproducing kernel of the classical weighted Bergman space A_α^2 is given by

$$K^\alpha(z, w) = \frac{1}{(1 - \langle z, w\rangle)^{n+1+\alpha}}, \qquad z, w \in \mathbb{B}_n.$$

Definition 3.2.2. For $0 < p < \infty$, the *Hardy space* H^p is the space consisting of functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\int_{\mathbb{S}_n} |f(r\xi)|^p d\sigma(\xi)\right)^{\frac{1}{p}}.$$

3.3 Carleson measures

Let μ be a finite positive Borel measure on $\mathbb{B}_n$ and let X be a Hilbert space of analytic functions in $\mathbb{B}_n$.

Definition 3.3.1. We say that μ is a *Carleson measure* for X if there exists a positive constant C such that

$$\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \leq C\|f\|_X^2, \qquad f \in X.$$

It is clear that μ is a Carleson measure for X if and only if $X \subset L^2(\mathbb{B}_n, d\mu)$ and the identity operator Id : $X \to L^2(\mathbb{B}_n, d\mu)$ is bounded. The *Carleson constant* of μ for X, denoted by $\mathcal{C}_\mu(X)$, is the norm of this identity operator Id.

Definition 3.3.2. Suppose that μ is a Carleson measure for X. We say that μ is a *vanishing Carleson measure* for X if the identity operator Id is compact. That is,

$$\lim_{k\to\infty}\int_{\mathbb{B}_n}|f_k(z)|^2d\mu(z)=0$$

whenever $\{f_k\}$ is a bounded sequence in X which converges to 0 uniformly on compact subsets of $\mathbb{B}_n$.

We state here two results on the characterization of Carleson measures for classical Bergman spaces A^2_α and Hardy spaces H^2, which can be found in [42].

Theorem 3.3.3. *A positive Borel measure μ on $\mathbb{B}_n$ is a Carleson measure for A^2_α if and only if $\mu\Big(E(a,r)\Big)\lesssim(1-|a|^2)^{n+1+\alpha}$ for all $a\in\mathbb{B}_n$.*

Theorem 3.3.4. *A positive Borel measure μ on $\mathbb{B}_n$ is a Carleson measure for H^2 if and only if $\mu(Q_a)\lesssim(1-|a|)^n$ for all $a\in\mathbb{B}_n\setminus\{0\}$. Furthermore,*

$$C_\mu(H^2)\asymp\sup_{a\in\mathbb{B}_n\setminus\{0\}}\frac{\mu(Q_a)}{(1-|a|)^n}.$$

3.4 Schatten classes

Let H be a separable Hilbert space, and $0<p<\infty$.

Definition 3.4.1. The *Schatten class $\mathcal{S}_p$* is the space of all compact operators T on H for which the sequence $\{\lambda_k\}$ of the singular numbers of T belongs to the $p-$summable sequence space ℓ^p.

We usually call $\mathcal{S}_1$ the trace class and $\mathcal{S}_2$ the Hilbert–Schmidt class.

For $1\le p<\infty$, the class $\mathcal{S}_p$ is a Banach space with the norm

$$||T||_p=\left(\sum_k|\lambda_k|^p\right)^{\frac{1}{p}}.$$

Lemma 3.4.2 ([42]). *Suppose that T is a positive compact operator on H and $0<p<\infty$, then $T\in\mathcal{S}_p$ if and only if $T^p\in\mathcal{S}_1$. Moreover, $||T||_p^p=||T^p||_1$.*

See [42, Chapter 1] for more results on the Schatten classes.

3.5 Toeplitz operators

Definition 3.5.1. Given a function $\varphi \in L^\infty(\mathbb{B}_n)$, the *Toeplitz operator* T_φ on A^2_ρ with *symbol* φ is defined by

$$T_\varphi f = P(\varphi f), \qquad f \in A^2_\rho,$$

where $P : L^2(\mathbb{B}_n, \rho dv) \to A^2_\rho$ is the orthogonal projection onto A^2_ρ.

We can write T_φ as

$$T_\varphi f(z) = \int_{\mathbb{B}_n} K_\rho(z, w) f(w) \varphi(w) \rho(w) dv(w), \qquad z \in \mathbb{B}_n,$$

where $K_\rho(z, w)$ is the reproducing kernel for A^2_ρ.

The Toeplitz operators can also be defined for unbounded symbols or for finite measures on $\mathbb{B}_n$. In fact, given a finite positive Borel measure μ on $\mathbb{B}_n$, the Toeplitz operator $T_\mu : A^2_\rho \to A^2_\rho$ is defined as follows

$$T_\mu f(z) = \int_{\mathbb{B}_n} K_\rho(z, w) f(w) d\mu(w), \qquad z \in \mathbb{B}_n.$$

Note that

$$\langle T_\mu f, g \rangle_\rho = \int_{\mathbb{B}_n} f(z)\overline{g(z)} d\mu(z), \qquad f, g \in A^2_\rho.$$

3.6 Subharmonic functions and potentials on $\mathbb{C}$

In this section we are going to formulate several definitions and properties of subharmonic functions and potentials on $\mathbb{C}$. These results can be found in many books, we refer to [35] for more details.

Definition 3.6.1. Let U be an open subset of $\mathbb{C}$. A function $u : U \to [-\infty, \infty)$ is called *subharmonic* on U if

(i) u is upper semicontinuous, that is, the set $\{z \in U : u(z) < \alpha\}$ is open for every real number α.

(ii) u satisfies the local submean inequality, i.e. given $z \in U$, there exists $t > 0$ such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta, \qquad 0 \leq r < t.$$

Proposition 3.6.2. *Let u, v be subharmonic functions on an open subset U of $\mathbb{C}$. Then*

(i) $\max(u, v)$ *is subharmonic on U;*

(ii) $\alpha u + \beta v$ is subharmonic on U for all $\alpha, \beta \geq 0$;

(iii) e^u is subharmonic on U.

Example 3.6.3. If f is holomorphic on an open subset U of $\mathbb{C}$, then $\log|f|$ and $|f|^\alpha$, $\alpha > 0$, are subharmonic on U.

Proposition 3.6.4. *Let U be an open subset of $\mathbb{C}$, and $u \in C^2(U)$. Then u is subharmonic on U if and only if the Laplacian Δu is positive on U.*

Proposition 3.6.5. *Let u be a subharmonic function on a domain D in $\mathbb{C}$, with $u \not\equiv -\infty$ on D. Then u is locally integrable on D, i.e. $\int_K |u(z)|\, dV(z) < \infty$ for each compact subset K of D.*

Definition 3.6.6. Let u be a subharmonic function on a domain D in $\mathbb{C}$, with $u \not\equiv -\infty$ on D. The *generalized Laplacian* of u is the Radon measure Δu on D such that

$$\int_D \phi \Delta u = \int_D u \Delta \phi\, dV$$

for all $\phi \in C_c^\infty$, the space of all C^∞ functions $f : D \to \mathbb{R}$ whose support $\operatorname{supp} f$ is a compact subset of D.

Definition 3.6.7. Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. The *logarithmic potential* of μ is the function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ defined a.e. by

$$p_\mu(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \log|z - w|\, d\mu(w), \qquad z \in \mathbb{C}.$$

Proposition 3.6.8. *Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Then*

$$\Delta p_\mu = \mu.$$

Remark 3.6.9. Given $R > 0$, we consider the function $G = G_R$ as follows

$$G(z) = \frac{1}{2\pi}\int_{D(0,R)} \log|z-w| d\mu(w) + \frac{1}{2\pi}\int_{\mathbb{C}\setminus D(0,R)} \log\left|\frac{z-w}{w}\right| d\mu(w), \quad z \in \mathbb{C}.$$

Here and later on, $D(z,r) = \{w \in \mathbb{C} : |w-z| < r\}, z \in \mathbb{C}, r > 0$. We call G the *modified logarithmic potential* of the finite measure μ (not necessarily with compact support).

Since $G(z) = p_\mu(z) - \frac{1}{2\pi}\int_{\mathbb{C}\setminus D(0,R)} \log|w|\, d\mu(w)$, then G also satisfies

$$\Delta G = \Delta p_\mu = \mu.$$

Next, we state here a result by Hayman [17, Lemma 4], which will be used later. Let μ be a finite positive measure. Given $z \in \mathbb{C}, h > 0$, set

$$n(z,h) = \mu\big(D(z,h)\big) \quad \text{and} \quad N(z,h) = \int_{D(z,h)} \log\left|\frac{h}{w-z}\right| d\mu(w).$$

Lemma 3.6.10. *Let $z_0 \in \mathbb{C}, 0 < d < h/2$. There exists a set K of area at most πd^2 such that*

$$N(z,h/2) \le n(z_0,h)\log\frac{16h}{d}, \qquad z \in D(z_0,h/2)\setminus K.$$

3.7 Plurisubharmonic functions

Definition 3.7.1. Let Ω be an open subset of $\mathbb{C}^n$. A function $u : \Omega \to [-\infty,\infty)$ is called *plurisubharmonic* on Ω if

(i) u is upper semicontinuous, that is, the set $\{z \in \Omega : u(z) < \alpha\}$ is open for every real number α.

(ii) u satisfies the mean-value inequality

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta} b) \, d\theta$$

for all $a, b \in \mathbb{C}^n$ such that the disk $\{a + wb : |w| \leq 1\}$ is contained in Ω.

Example 3.7.2. If u is holomorphic on an open subset Ω of $\mathbb{C}^n$, then $\log |f|$ is plurisubharmonic on U.

Proposition 3.7.3. *Let Ω be an open subset of $\mathbb{C}^n$. Then a function $u \in C^2(\Omega)$ is plurisubharmonic if and only if its Levi form $L_u(z; b)$ is non-negative, i.e.,*

$$L_u(z; b) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) b_j \overline{b}_k \geq 0, \qquad z \in \Omega, b \in \mathbb{C}^n.$$

We refer to the book of Hörmander [18] for further properties of plurisubharmonic functions and the survey carried out by Kiselman [22] on the development of the theory of plurisubharmonic functions.

Let us recall here the Hörmander theorem [18, Theorem 4.4.4].

Theorem 3.7.4. *Let φ be a plurisubharmonic function in the pseudoconvex open set $\Omega \subset \mathbb{C}^n$. If $z_0 \in \Omega$ and $e^{-\varphi}$ is integrable in a neighborhood of z_0 one can find an analytic function f in Ω such that $f(z_0) = 1$ and*

$$\int_\Omega |f(z)|^2 (1 + |z|^2)^{-3n} e^{-\varphi(z)} \, dV(z) < \infty.$$

An open set $\Omega \subset \mathbb{C}^n$ is called *pseudoconvex* if there exists a continuous plurisubharmonic function u in Ω such that

$$\{z \in \Omega : u(z) < c\} \Subset \Omega$$

for every $c \in \mathbb{R}$. Note that a ball is pseudoconvex.

The following version of this result is given in [3, Section IV].

Theorem 3.7.5. *Let φ be a plurisubharmonic function in $\mathbb{C}^n$. Then there exists an entire function $f \not\equiv 0$ such that*

$$\int_{\mathbb{C}^n} |f(z)|^2 (1+|z|^2)^{-3n} e^{-\varphi(z)}\, dV(z) < \infty.$$

We denote as usual $d = \partial + \bar{\partial}$ and set $d^c = i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$.

A *complex current* of bidegree (p,q) (or bidimension $(n-p, n-q)$) is a differential form

$$T = \sum_{\substack{|J|=p \\ |K|=q}}{}' \, T_{JK} dz_J \wedge d\bar{z}_K,$$

where coefficients T_{JK} are distributions, the sum is taken only over increasing multi-indices J, K. We say that a current T of bidegree (p,p) is positive if

$$T \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p}$$

is positive for any $(1,0)$-forms $\alpha_1, \ldots, \alpha_{n-p} \in \mathbb{C}_{(1,0)}$. If $dT = 0$, then T is said to be closed.

Definition 3.7.6. Let u be a plurisubharmonic function on Ω, an open subset of $\mathbb{C}^n$, and T a closed positive current of bidimension (p,p). According to Bedford-Taylor [2] we define

$$dd^c u \wedge T = dd^c(uT),$$

where $dd^c(\cdot)$ is taken in the sense of distribution theory. Given locally bounded plurisubharmonic functions $u_1, \ldots, u_q$ on Ω, we define inductively

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T).$$

In particular, when u is a locally bounded plurisubharmonic function, one obtains a well defined positive measure $(dd^c u)^n$. If $u \in C^2(\Omega)$, then

$$(dd^c u)^n = n! 4^n \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) dV.$$

Theorem 3.7.7. *Let $\Omega \Subset \mathbb{C}^n$ be a smooth strongly pseudoconvex domain and let $f \in C(\partial\Omega)$ be a continuous function on the boundary. Then there exists a function u which is continuous on $\overline{\Omega}$, plurisubharmonic on Ω and solves the Dirichlet problem*

$$(dd^c u)^n = 0 \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega.$$

This theorem is the fundamental result on the solution of the Dirichlet problem for complex Monge-Ampère equations, see the papers of Bedford-Taylor [2] and of Demailly [9, Theorem 7.5].

3.8 Lelong number

Let u be a plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$ such that $u \not\equiv -\infty$. Then u is locally integrable with respect to the Lebesgue measure in Ω, and $\mu_u = \frac{1}{2\pi}\Delta u = \frac{1}{2\pi}\sum_{j=1}^n \frac{\partial^2 u}{\partial z_j \partial \overline{z_j}}$ is a positive Borel measure on Ω.

Definition 3.8.1. Let $a \in \Omega$. The *Lelong number* $\nu_u(a)$ of u at a is the limit

$$\nu_u(a) = \lim_{r\to 0} \frac{\mu_u\big(B(a,r)\big)}{\tau_{2n-2} r^{2n-2}},$$

where $\tau_{2n-2} = \pi^{n-1}/(n-1)!$ is the volume of the unit ball in $\mathbb{C}^{n-1}$, and $B(a,r) := \{z \in \mathbb{C}^n : |z-a| < r\}$ is the Euclidean ball of center a and radius $r > 0$ in $\mathbb{C}^n$.

Proposition 3.8.2. *The Lelong number of u at the point $a \in \Omega$ can also be expressed by the following formulas:*

$$\nu_u(a) = \lim_{r\to 0} \frac{1}{\log r} \int_{|\xi|=r} u(a + r\xi)\, d\sigma(\xi),$$
$$\nu_u(a) = \lim_{r\to 0} \frac{\sup_{|z|\le r} u(a+z)}{\log r}.$$

Example 3.8.3. If $u(z) = \log|z|$, $z \in \mathbb{C}^n$, then $\nu_u(0) = 1$.

See [19, 20, 41] for more details. In particular, if $\nu_u(a) < 2$, then e^{-u} is locally integrable with respect to V in a neighborhood of a.

We state here a result of Kiselman [21], which will be used later.

Theorem 3.8.4. *Let u be a plurisubharmonic function on an open subset Ω of $\mathbb{C}^n$ and K be a compact set in Ω. Then for each $0 < \alpha < \frac{2}{\sup_{z\in K}\nu_u(z)}$, there exists a positive constant C_α such that*

$$V(\{z \in K : u(z) \leq t\}) \leq C_\alpha e^{\alpha t}, \qquad t \in \mathbb{R}.$$

Chapter 4

Carleson measures and Toeplitz operators on small Bergman spaces on the ball

In this chapter, we study the Carleson measures and the Toeplitz operators on the class of the so-called small weighted Bergman spaces, introduced by Seip. A characterization of Carleson measures is obtained which extends Seip's results from the unit disk of $\mathbb{C}$ to the unit ball of $\mathbb{C}^n$. We use this characterization to give necessary and sufficient conditions for the boundedness and compactness of Toeplitz operators. Finally, we study the Schatten p classes membership of Toeplitz operators for $1 < p < \infty$.

4.1 Remark on the classes of weights

We recall that S is the class of radial weights ρ such that

$$\inf_k \frac{1-r_k}{1-r_{k+1}} > 1, \tag{4.1}$$

where $r_k \in [0,1)$ are determined by the relation

$$\int_{r_k}^1 \rho(x)dx = 2^{-k}.$$

This class was introduced by Seip in [39].

Lemma 4.1.1. *Let $\rho \in S$. Then we have an equivalent norm in the weighted Bergman space A^2_ρ as follows*

$$\|f\|^2_\rho \asymp \sum_{k=1}^{\infty} 2^{-k} \int_{\mathbb{S}_n} |f(r_k\xi)|^2 d\sigma(\xi), \qquad f \in A^2_\rho. \tag{4.2}$$

Proof. The conclusion follows from the fact that the function Φ_f,

$$\Phi_f(r) = \int_{\mathbb{S}_n} |f(r\xi)|^2 d\sigma(\xi)$$

is non-decreasing. □

We denote by $\widehat{\mathcal{D}}$ the class of doubling weights ρ, which means

$$\int_r^1 \rho(s)ds \lesssim \int_{\frac{r+1}{2}}^1 \rho(s)ds$$

for $r \in (0,1)$.

It is easy to see that $S \subset \widehat{\mathcal{D}}$.

Example 4.1.2. The functions

$$\rho(x) = (1-x)^{-\beta}, \qquad 0 < \beta < 1,$$

and

$$\rho(x) = (1-x)^{-1} \left(\log \frac{1}{1-x}\right)^{-\alpha}, \qquad 1 < \alpha < \infty,$$

belong to S and $\widehat{\mathcal{D}}$.

Lemma 4.1.3.

$$\left\{A^2_\rho : \rho \in S\right\} = \left\{A^2_\rho : \rho \in \widehat{\mathcal{D}}\right\}.$$

Proof. For $\rho \in S \cup \widehat{\mathcal{D}}$, we can find $\tilde{\rho} \in S \cap \widehat{\mathcal{D}}$ such that $A^2_\rho = A^2_{\tilde{\rho}}$. Indeed, by the monotonicity of the functions Φ_f, we obtain that if $h_{\rho_1} \gtrsim h_{\rho_2}$, then $A^2_{\rho_1} \subset A^2_{\rho_2}$, where $h_\rho(x) = \int_{1-x}^1 \rho(t)\,dt$. Correspondingly, if $h_{\rho_1} \asymp h_{\rho_2}$, then $A^2_{\rho_1} = A^2_{\rho_2}$. Now, if $\rho \in S$, then we can interpolate h_ρ linearly between the points $1 - r_k$, $k \geq 1$, to get $h_{\tilde{\rho}}$ such that $A^2_\rho = A^2_{\tilde\rho}$ and for some $c > 1$, $h_{\tilde\rho}(cx) \le 2h_{\tilde\rho}(x)$. Hence, $h_{\tilde\rho}(2x) \le dh_{\tilde\rho}(x)$ for some $d > 1$, and, thus, $\tilde\rho \in \widehat{\mathcal{D}}$. On the other hand, if $\rho \in \widehat{\mathcal{D}}$, then we can interpolate $\log h_\rho$ linearly between the points 2^{-k}, $k \ge 1$, to get $h_{\tilde\rho}$ such that $A^2_\rho = A^2_{\tilde\rho}$ and $h_{\tilde\rho}(dx) \le 2h_{\tilde\rho}(x)$ for some $d > 1$. Hence, $\tilde\rho \in S$. □

We introduce a subclass S^* of weights in S determined by the condition that $\rho^*(r) \lesssim \rho(r)$ for $r \in (0,1)$, where

$$\rho^*(r) = \frac{1}{1-r}\int_r^1 \rho(t)dt.$$

Example 4.1.4. The weights

$$\rho(x) = (1-x)^{-\beta}\left(\log\frac{1}{1-x}\right)^\alpha, \qquad 0 < \beta < 1, \alpha \in \mathbb{R}$$

belong to S^*, but the weights

$$\rho(x) = (1-x)^{-1}\left(\log\frac{1}{1-x}\right)^\alpha, \qquad \alpha < -1,$$

$$\rho(x) = (1-x)^{-1}\left(\log\frac{1}{1-x}\right)^{-1}\left(\log\log\frac{1}{1-x}\right)^\alpha, \qquad \alpha < -1,$$

do not belong to S^*.

4.2 Carleson measures

Let μ be a finite positive Borel measure on $\mathbb{B}_n$. We recall that μ is a Carleson measure for the Bergman space A^2_ρ if

$$\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \lesssim \|f\|^2_\rho, \qquad f \in A^2_\rho.$$

It is clear that μ is a Carleson measure for A^2_ρ if and only if $A^2_\rho \subset L^2(\mathbb{B}_n, d\mu)$ and the identity operator $\mathrm{Id} : A^2_\rho \to L^2(\mathbb{B}_n, d\mu)$ is bounded. The Carleson constant of μ for A^2_ρ, denoted by $\mathcal{C}_\mu(A^2_\rho)$, is the norm of this identity operator Id. Suppose that μ is a Carleson measure for A^2_ρ. We say that μ is a vanishing Carleson measure for A^2_ρ if the above identity operator Id is compact. That is, $\lim_{k\to\infty}\int_{\mathbb{B}_n}|f_k(z)|^2 d\mu(z) = 0$ whenever $\{f_k\}$ is a bounded sequence in A^2_ρ which converges to 0 uniformly on compact subsets of $\mathbb{B}_n$.

In [39], Seip gave a characterization of Carleson measures for A^2_ρ with $\rho \in S$ in the case $n = 1$. Our following result extends this result to the case $n > 1$.

We use the following notation. For every nonnegative integer k,

$$\Omega_k = \{z \in \mathbb{B}_n : r_k \le |z| < r_{k+1}\},$$

and let μ_k be the measure defined by $\mu_k = \chi_{\Omega_k}\mu$ whenever a nonnegative Borel measure μ on $\mathbb{B}_n$ is given.

Theorem 4.2.1. *Let $\rho \in S$, and let μ be a finite positive Borel measure on $\mathbb{B}_n$. Then*

(i) μ is a Carleson measure for A^2_ρ if and only if each μ_k is a Carleson measure for the Hardy space H^2 with Carleson constant $C_{\mu_k}(H^2) \lesssim 2^{-k}$, $k \ge 0$.

(ii) μ is a vanishing Carleson measure for A^2_ρ if and only if

$$\lim_{k\to\infty} 2^k C_{\mu_k}(H^2) = 0.$$

Theorem 4.2.1 (i) for the case $n = 1$ was obtained by Seip in [39].

4.3 Toeplitz operators

Given a finite positive Borel measure μ on $\mathbb{B}_n$, the Toeplitz operator $T_\mu : A^2_\rho \to A^2_\rho$ is defined as follows:

$$T_\mu f(z) = \int_{\mathbb{B}_n} K_\rho(z,w) f(w)\, d\mu(w), \qquad z \in \mathbb{B}_n.$$

The Toeplitz operators acting on various spaces of holomorphic functions have been extensively studied by many authors, and the theory is especially well understood in the case of Hardy spaces or standard Bergman spaces (see [42], [43] and the references therein). Luecking [26] was the first to study Toeplitz operators on Bergman spaces with measures as symbols, and some interesting results about Toeplitz acting on large Bergman spaces were obtained by Lin and Rochberg [24].

First, we are going study the boundedness and compactness of T_μ on A^2_ρ, with $\rho \in S$.

Theorem 4.3.1. *Let $\rho \in S$, and let μ be a finite positive Borel measure on $\mathbb{B}_n$. Then*

(i) The Toeplitz operator T_μ is bounded on A^2_ρ if and only if μ is a Carleson measure for A^2_ρ.

(ii) The Toeplitz operator T_μ is compact on A^2_ρ if and only if μ is a vanishing Carleson measure for A^2_ρ.

Next we study when our Toeplitz operators belong to the Schatten class. We refer to [43, Chapter 1] for a brief account on the Schatten classes. A description for the standard Bergman spaces on the unit disk was given (see [43, Chapter 7]), and a description for the case of large Bergman spaces on the disk was obtained in 2015 by H. Arroussi, I. Park, and J. Pau [1]. In 2016, Peláez and Rättyä [31] gave an interesting characterization for the case of small Bergman spaces on unit disk, where the weight $\rho \in \widehat{\mathcal{D}}$.

For weights ρ in S^*, we obtain a characterization of the symbols of the Toeplitz operators in the Schatten classes $\mathcal{S}_p$. In [34], Peláez, Rättyä and Sierra gave a characterization for the case of dimension $n = 1$ when the weight is regular, that is $\rho^*(r) \asymp \rho(r)$. As an easy observation, our result is equivalent to their result when $n = 1$. We point out that our approach is completely different from that of [34], which does not seem to work in higher dimensions. On the other hand, for regular weights ρ in $S \setminus S^*$, this characterization fails. A counterexample was given in [34].

For a measure μ on $\mathbb{B}_n$ and $\alpha > 0$, we define the function $\widehat{\mu}_\alpha$ by

$$\widehat{\mu}_\alpha(z) = \frac{2^k \mu\Big(E(z,\alpha)\Big)}{(1-|z|)^n}, \qquad z \in \Omega_k.$$

Let $\widetilde{T_\mu}$ be the Berezin transform of T_μ, defined by

$$\widetilde{T_\mu}(z) = \langle T_\mu k_z, k_z \rangle_\rho, \qquad z \in \mathbb{B}_n,$$

where k_z is the normalized reproducing kernels of A^2_ρ. Set

$$d\lambda_\rho(z) = \frac{2^k \rho(z) dv(z)}{(1-|z|)^n}, \qquad z \in \Omega_k.$$

Theorem 4.3.2. *Let ρ be in S^*, μ be a finite positive Borel measure and $1 < p < \infty$. The following conditions are equivalent:*

(a) The Toeplitz operator T_μ is in the Schatten class $\mathcal{S}_p$.

(b) The function $\widetilde{T_\mu}$ is in $L^p(\mathbb{B}_n, d\lambda_\rho)$.

(c) The function $\widehat{\mu}_\alpha$ is in $L^p(\mathbb{B}_n, d\lambda_\rho)$ for sufficiently small $\alpha > 0$.

4.4 Proof of Theorem 4.2.1

Lemma 4.4.1. *Let μ be a finite positive measure on $\mathbb{B}_n$. Then μ_k is a Carleson measure for H^2 if and only if $\mu_k(Q_a) \lesssim (1-|a|)^n$ for all $a \in \Omega_k$. Furthermore, $\mathcal{C}_{\mu_k}(H^2) \asymp \sup_{a \in \Omega_k} (1-|a|)^{-n} \mu_k(Q_a)$.*

Proof. Let $a \in \mathbb{B}_n \setminus \{0\}$. Then $a \in \Omega_l$ for some $l \geq 1$. If $l > k$, then $\mu_k(Q_a) = 0$ and there is nothing to prove. When $a \in \Omega_l$, $l \leq k$, we can cover $Q_a \setminus r_k \mathbb{B}_n$ by a finite family $\{Q_{a_l} : l \in \Lambda\}$ with $a_l \in \Omega_{k-1}$, where Λ is a finite index set. Applying Lemma 3.1.3 to the set $\{O_{a_l} : l \in \Lambda\}$, we get a subset Λ_0 of Λ such that $O_{a_l}, l \in \Lambda_0$, are disjoint and $O\Big(a_l, 3\delta(a_l)\Big), l \in \Lambda_0$, cover O_a. Moreover, it is easy to see that

$$Q_a \setminus r_k \mathbb{B}_n \subset \bigcup_{l \in \Lambda_0} Q\Big(a_l, 3\delta(a_l)\Big).$$

Then

$$\mu_k(Q_a) = \mu_k(Q_a \setminus r_k \mathbb{B}_n) \leq \sum_{l \in \Lambda_0} \mu_k\Big(Q\Big(a_l, 3\delta(a_l)\Big)\Big).$$

Since $a_l \in \Omega_{k-1}$, we have $\mu_k\Big(Q\big(a_l, 3\delta(a_l)\big)\Big) \lesssim (1-|a_l|)^n \asymp \sigma(O_{a_l})$. Hence

$$\mu_k(Q_a) \lesssim \sum_{l\in\Lambda_0} \sigma(O_{a_l}) = \sigma\Big(\bigcup_{l\in\Lambda_0} O_{a_l}\Big).$$

Finally,

$$\sigma\Big(\bigcup_{l\in\Lambda_0} O_{a_l}\Big) \lesssim \sigma(O_a) \asymp (1-|a|)^n.$$

Therefore $\mu_k(Q_a) \lesssim (1-|a|)^n$. This completes the proof. □

4.4.1 Proof of Part (i)

($\Longleftarrow$) Since μ_k are Carleson measures for H^2 with Carleson constants $\lesssim 2^{-k}$, the same holds for H^2 on the smaller ball $r_{k+2}\mathbb{B}_n$. Indeed, we just use the characterization of Carleson measures and the fact that if $Q\big(a, \delta(a)\big) \cap r_{k+2}^{-1}\Omega_k \neq \emptyset$, then $1-|a| \gtrsim 1-r_{k+2}$ and, hence, $r_{k+2}Q\big(a,\delta(a)\big) \subset Q\big(a, M\delta(a)\big)$ for some $M < \infty$ independent of a and k.

Therefore,

$$\int_{\Omega_k} |f(z)|^2 d\mu(z) \lesssim 2^{-k} \int_{\mathbb{S}_n} |f(r_{k+2}\xi)|^2 d\sigma(\xi)$$

for an arbitrary function f in A^2_ρ and for all k. Summing this estimate over all $k \geq 1$ we get

$$\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \lesssim \sum_{k=1}^{\infty} 2^{-k} \int_{\mathbb{S}_n} |f(r_{k+2}\xi)|^2 d\sigma(\xi) \asymp \|f\|_\rho^2.$$

($\Longrightarrow$) We just need to check that $\mu_k(Q_a) \lesssim 2^{-k}(1-|a|)^n$ when a is in Ω_k, $k \geq 0$. We use the test function

$$f_a(z) = (1 - \langle a, z\rangle)^{-\gamma} \tag{4.3}$$

with large γ. By (4.2), we have

$$\begin{aligned}\|f_a\|_\rho^2 &\asymp \sum_{j=1}^{\infty} 2^{-j} \int_{\mathbb{S}_n} \frac{1}{|1-\langle a, r_j\xi\rangle|^{2\gamma}} d\sigma(\xi)\\ &\asymp \sum_{j=1}^{\infty} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}}.\end{aligned}$$

Since $a \in \Omega_k$, the relation (4.1) yields that

$$\|f_a\|_\rho^2 \asymp 2^{-k}(1-|a|)^{-2\gamma+n}. \tag{4.4}$$

Indeed,

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}} &= \sum_{j\le k} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}} + \sum_{j>k} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}} \\
&\asymp \sum_{j\le k} \frac{2^{-j}}{(1-r_j)^{2\gamma-n}} + \sum_{j>k} \frac{2^{-j}}{(1-|a|)^{2\gamma-n}} \\
&\asymp \frac{2^{-k}}{(1-r_k)^{2\gamma-n}} + \frac{2^{-k}}{(1-|a|)^{2\gamma-n}} \\
&\asymp 2^{-k}(1-|a|)^{-2\gamma+n}.
\end{aligned}$$

On the other hand, for every z in Q_a, we have

$$\begin{aligned}
|1-\langle a,z\rangle| &= |(1-|a|)+|a|(1-\langle a/|a|,z\rangle)| \\
&\le (1-|a|)+|a||1-\langle a/|a|,z\rangle| \\
&< (1-|a|)+2|a|(1-|a|) \\
&\le 3(1-|a|).
\end{aligned}$$

Hence,

$$|f_a(z)| \gtrsim (1-|a|)^{-\gamma}, \qquad z \in Q_a. \tag{4.5}$$

Thus,

$$\int_{\mathbb{B}_n} |f_a(z)|^2 d\mu(z) \gtrsim (1-|a|)^{-2\gamma}\mu(Q_a \cap \Omega_k).$$

Since μ is a Carleson measure for A^2_ρ, we get

$$\mu(Q_a \cap \Omega_k) \lesssim 2^{-k}(1-|a|)^n.$$

This implies that μ_k is a Carleson measure for Hardy space H^2 with Carleson constant $C_{\mu_k}(H^2) \lesssim 2^{-k}$. □

4.4.2 Proof of Part (ii)

Suppose that μ is a vanishing Carleson measure for A^2_ρ. Given a in Ω_k, consider the function f_a defined by (4.3). By (4.4), $||f_a||^2_\rho \asymp 2^{-k}(1-|a|)^{-2\gamma+n}$. Set

$$h_a(z) = \frac{(1-\langle a, z\rangle)^{-\gamma}}{2^{-k/2}(1-|a|)^{-\gamma+n/2}}. \tag{4.6}$$

Then $||h_a||^2_\rho \asymp 1$ and by (4.5),

$$|h_a(z)|^2 \gtrsim \frac{2^k}{(1-|a|)^n}, \qquad z \in Q_a.$$

Since μ is a vanishing Carleson measure for A^2_ρ and h_a tends to 0 uniformly on compact subsets of the unit ball as $|a| \to 1$, we have

$$\lim_{|a|\to 1} \int_{\mathbb{B}_n} |h_a(z)|^2 d\mu(z) = 0.$$

Thus, $\displaystyle\sup_{a\in\Omega_k} \frac{2^k \mu_k(Q_a \cap \Omega_k)}{(1-|a|)^n} \to 0$ as $k \to \infty$. Hence, $\displaystyle\lim_{k\to\infty} 2^k \mathcal{C}_{\mu_k}(H^2) = 0$.

Conversely, let $\mu^r = \mu|_{\mathbb{B}_n \setminus \overline{r\mathbb{B}_n}}$, where $r\mathbb{B}_n = \{z \in \mathbb{B}_n : |z| < r\}$. Then $(\mu^r)_k \leq \mu_k$, $k \geq 1$ and $(\mu^r)_k = 0$ if $r_{k+1} \leq r$. Therefore, Part (i) of Theorem 4.2.1 implies that

$$\int_{\mathbb{B}_n} |h(z)|^2 d\mu^r(z) \leq C_r ||h||^2_\rho, \qquad h \in A^2_\rho,$$

where

$$C_r = \sup_{k\,:\,r_{k+1}>r} 2^k \mathcal{C}_{\mu_k}(H^2), \qquad \text{and} \qquad \lim_{r\to 1} C_r = 0. \tag{4.7}$$

Let $\{f_k\}$ be a bounded sequence in A^2_ρ converging uniformly to 0 on compact subsets of $\mathbb{B}_n$. Let $\varepsilon > 0$. By (4.7), there exists $r_0 \in (0,1)$ such that $C_r < \varepsilon$ for all $r \geq r_0$. Moreover, by the uniform convergence on compact subsets, we may choose $k_0 \in \mathbb{N}$ such that $|f_k(z)|^2 < \varepsilon$ for all $k \geq k_0$ and $z \in \overline{r_0\mathbb{B}_n}$. It follows that

$$\begin{aligned}
\int_{\mathbb{B}_n} |f_k(z)|^2 d\mu(z) &= \int_{\overline{r_0\mathbb{B}_n}} |f_k(z)|^2 d\mu(z) + \int_{\mathbb{B}_n \setminus \overline{r_0\mathbb{B}_n}} |f_k(z)|^2 d\mu(z) \\
&< \varepsilon \mu(\overline{r_0\mathbb{B}_n}) + \int_{\mathbb{B}_n} |f_k(z)|^2 d\mu^{r_0}(z) \\
&\leq \varepsilon \mu(\overline{r_0\mathbb{B}_n}) + C_{r_0} ||f_k||^2_\rho \\
&\leq \varepsilon C, \qquad k \geq k_0,
\end{aligned}$$

for some positive constant C. Hence, μ is a vanishing Carleson measure for A^2_ρ. $\square$

4.5 Proof of Theorem 4.3.1

4.5.1 Proof of Part (i)

($\Longrightarrow$) Given a in Ω_k, we define h_a by (4.6). Then

$$\|h_a\|_\rho^2 \asymp 1 \text{ and } |h_a(z)|^2 \gtrsim 2^k(1-|a|)^{-n}, \qquad z \in Q_a.$$

Consider the function

$$T_\mu^\#(a) = \langle T_\mu h_a, h_a \rangle_\rho = \int_{\mathbb{B}_n} |h_a|^2 d\mu(z). \tag{4.8}$$

Since T_μ is bounded, $A := \sup_{a\in\mathbb{B}_n} T_\mu^\#(a) < \infty$. Then

$$\begin{aligned} A &\geq \int_{\mathbb{B}_n} |h_a(z)|^2 d\mu(z) \geq \int_{\mathbb{B}_n} |h_a(z)|^2 d\mu_k(z) \\ &\geq \int_{Q_a} |h_a(z)|^2 d\mu_k(z) \gtrsim 2^k(1-|a|)^{-n} \mu_k(Q_a). \end{aligned} \tag{4.9}$$

Hence, $\mu_k(Q_a) \lesssim 2^{-k}(1-|a|)^n$ for every $a \in \Omega_k$. By Theorem 4.2.1 and Lemma 4.4.1, μ is a Carleson measure for A^2_ρ.

($\Longleftarrow$) For every $f, g \in A^2_\rho$ we have

$$\langle T_\mu f, g \rangle_\rho = \int_{\mathbb{B}_n} f(z)\overline{g(z)} d\mu(z).$$

Then by the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |\langle T_\mu f, g \rangle_\rho| &\leq \int_{\mathbb{B}_n} |f(z)||g(z)| d\mu(z) \\ &\leq \left(\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \right)^{\frac{1}{2}} \left(\int_{\mathbb{B}_n} |g(z)|^2 d\mu(z) \right)^{\frac{1}{2}}. \end{aligned}$$

Since μ is a Carleson measure for A^2_ρ, there exists a positive constant C such that

$$\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \leq C\|f\|_\rho^2,$$

and

$$\int_{\mathbb{B}_n} |g(z)|^2 d\mu(z) \leq C\|g\|^2_\rho.$$

Hence,

$$|\langle T_\mu f, g\rangle_\rho| \leq C\|f\|_\rho \|g\|_\rho \qquad \text{for all } f, g \in A^2_\rho.$$

Thus, T_μ is bounded on A^2_ρ. □

4.5.2 Proof of Part (ii)

We need the following auxiliary results.

Proposition 4.5.1. *Suppose that $f \in A^2_\rho$ with $\rho \in S$. Then*

$$|f(z)|^2 \leq \frac{C2^k}{(1-|z|)^n}\|f\|^2_\rho, \qquad z \in \Omega_k, k \geq 0, \tag{4.10}$$

where C is a positive constant independent of k and z.

Proof. Let $z \in \Omega_k$. Applying [42, Corollary 4.5] to the function $g(z) = f(r_{k+2}z)$ at the point $\frac{z}{r_{k+2}}$, we obtain

$$|f(z)|^2 \leq \int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2 \frac{(1-|z/r_{k+2}|^2)^n}{|1-\langle z/r_{k+2}, \zeta\rangle|^{2n}} d\sigma(\zeta).$$

By (4.1), $|1-\langle z/r_{k+2}, \zeta\rangle| \geq 1 - |\langle z/r_{k+2}, \zeta\rangle| \geq 1 - \frac{|z||\zeta|}{r_{k+2}} = 1 - |z|/r_{k+2} \gtrsim 1-|z|$ for $z \in \Omega_k, \zeta \in \mathbb{S}_n$. Thus,

$$\begin{aligned}
|f(z)|^2 &\lesssim \int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2 \frac{(1-|z|^2)^n}{(1-|z|)^{2n}} d\sigma(\zeta) \\
&\leq \frac{(1+|z|)^n}{(1-|z|)^n} \int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2 d\sigma(\zeta) \\
&\lesssim \frac{2^k}{(1-|z|)^n} 2^{-k} \int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2 d\sigma(\zeta) \\
&\leq \frac{2^k}{(1-|z|)^n} \sum_{j=1}^{\infty} 2^{-j} \int_{\mathbb{S}_n} |f(r_{j+2}\zeta)|^2 d\sigma(\zeta) \\
&\lesssim \frac{2^k}{(1-|z|)^n} \|f\|^2_\rho,
\end{aligned}$$

with constants independent of k and z. □

Corollary 4.5.2. *A sequence of functions* $\{f_k\} \subset A^2_\rho$ *converges to* 0 *weakly in* A^2_ρ *if and only if it is bounded in* A^2_ρ *and converges to* 0 *uniformly on each compact subset of* $\mathbb{B}_n$*.*

Proof of Part (ii) *of Theorem 4.3.1.* Suppose that T_μ is compact on A^2_ρ. We define h_a, $a \in \mathbb{B}_n$ by (4.6), and $T^{\#}_\mu$ by (4.8). Then $||h_a||^2_\rho \asymp 1$ and h_a converges uniformly to 0 on compact subsets of $\mathbb{B}_n$ as $|a| \to 1$. Since T_μ is compact, $T^{\#}_\mu(a) \to 0$ as $|a| \to 1$. By (4.9) this implies that

$$\sup_{a\in\Omega_k} \frac{2^k \mu_k(Q_a)}{(1-|a|)^n} \to 0 \text{ as } k \to \infty.$$

Hence,

$$\lim_{k\to\infty} 2^k C_{\mu_k}(H^2) = 0.$$

By Part (ii) of Theorem 4.2.1, μ is a vanishing Carleson measure for A^2_ρ.

Conversely, assume that μ is a vanishing Carleson measure for A^2_ρ. For every $h \in A^2_\rho$ we have

$$||T_\mu h||_\rho = \sup_{\substack{g\in A^2_\rho \\ ||g||_\rho \le 1}} |\langle T_\mu h, g\rangle_\rho|.$$

Furthermore,

$$\begin{aligned}
|\langle T_\mu h, g\rangle_\rho| &= \left|\int_{\mathbb{B}_n} h(z)\overline{g(z)}d\mu(z)\right| \le \int_{\mathbb{B}_n} |h(z)||g(z)|d\mu(z) \\
&\le \left(\int_{\mathbb{B}_n} |h(z)|^2 d\mu(z)\right)^{1/2} \left(\int_{\mathbb{B}_n} |g(z)|^2 d\mu(z)\right)^{1/2} \\
&\lesssim \left(\int_{\mathbb{B}_n} |h(z)|^2 d\mu(z)\right)^{1/2} ||g||_\rho.
\end{aligned}$$

The last inequality follows from the fact that μ is a Carleson measure for A^2_ρ. Therefore,

$$||T_\mu h||_\rho \lesssim \left(\int_{\mathbb{B}_n} |h(z)|^2 d\mu(z)\right)^{1/2}, \qquad h \in A^2_\rho.$$

Now, let $\{f_k\} \subset A^2_\rho$ be bounded and converge uniformly to 0 on compact subsets of $\mathbb{B}_n$. Since μ is a vanishing Carleson measure for A^2_ρ,

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k(z)|^2 d\mu(z) = 0.$$

It follows that $||T_\mu f_k||_\rho \to 0$ and hence T_μ is compact. □

4.6 Proof of Theorem 4.3.2

Proposition 4.6.1. *Let $K_\rho(z,w)$ be the reproducing kernel of A^2_ρ.*

(a) Let $k \geq 1$, $z \in \Omega_k$. Then

$$K_\rho(z,z) \asymp \frac{2^k}{(1-|z|)^n}. \tag{4.11}$$

(b) There exists $\alpha = \alpha(\rho) > 0$ such that for every $z \in \mathbb{B}_n$,

$$|K_\rho(z,w)|^2 \asymp K_\rho(z,z)K_\rho(w,w) \tag{4.12}$$

whenever $w \in E(z,\alpha)$.

Proof. (a) Fix $k \geq 1$. Given $z \in \Omega_k$, let L_z be the point evaluation at z on A^2_ρ. It is well-known that

$$K_\rho(z,z) = ||L_z||^2.$$

By Proposition 4.5.1,

$$||L_z||^2 \lesssim \frac{2^k}{(1-|z|)^n}.$$

Furthermore, choosing h_z by (4.6), we have $||h_z||_\rho \asymp 1$ and

$$|h_z(z)|^2 \gtrsim \frac{2^k}{(1-|z|)^n}.$$

Hence,

$$||L_z||^2 \gtrsim \frac{2^k}{(1-|z|)^n}.$$

Thus

$$K_\rho(z,z) \asymp \frac{2^k}{(1-|z|)^n}, \qquad z \in \Omega_k.$$

(b) In this proof, we use an argument of Lin and Rochberg in [24]. It is well-known that

$$|K_\rho(z,w)|^2 \leq K_\rho(z,z)K_\rho(w,w)$$

for all $z, w \in \mathbb{B}_n$.

For any fixed $z_0 \in \Omega_k$, consider the subspace $A^2_\rho(z_0)$ defined as

$$A^2_\rho(z_0) = \left\{ f \in A^2_\rho : f(z_0) = 0 \right\}.$$

Denote by $\mathcal{L}_{z_0}$ the one-dimensional subspace spanned by the function

$$k_{\rho,z_0}(z) = \frac{K_\rho(z, z_0)}{\sqrt{K_\rho(z_0, z_0)}}.$$

Then we have the orthogonal decomposition

$$A^2_\rho = A^2_\rho(z_0) \oplus \mathcal{L}_{z_0}.$$

Hence $K_\rho(z, w) = K_{\rho,z_0}(z, w) + \overline{k_{\rho,z_0}(w)} k_{\rho,z_0}(z)$, where K_{ρ,z_0} is the reproducing kernel of $A^2_\rho(z_0)$. Therefore,

$$K_\rho(z_0, w) = \overline{k_{\rho,z_0}(w)} k_{\rho,z_0}(z_0)$$

and

$$K_\rho(w, w) = K_{\rho,z_0}(w, w) + |k_{\rho,z_0}(w)|^2. \tag{4.13}$$

We are going to prove that there exists $\alpha > 0$ such that

$$K_{\rho,z_0}(w, w) < \frac{1}{2} K_\rho(w, w), \qquad w \in E(z_0, \alpha). \tag{4.14}$$

By (4.1), there exists $\alpha_1 > 0$ such that $E(z_0, \alpha) \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1}, 0 < \alpha < \alpha_1$. Hence, for every $f \in A^2_\rho(z_0)$ such that $\|f\|_\rho = 1$, by Proposition 4.5.1 we have

$$|f(w)|^2 \lesssim \frac{2^k}{(1 - |w|)^n} \asymp \frac{2^k}{(1 - |z_0|)^n} \tag{4.15}$$

whenever $w \in E(z_0, \alpha)$. Since $E(z_0, \alpha) = \varphi_{z_0}\Big(E(0, \alpha)\Big)$, we can rewrite (4.15) as

$$|f\Big(\varphi_{z_0}(\eta)\Big)|^2 \lesssim \frac{2^k}{(1 - |z_0|)^n} \tag{4.16}$$

whenever $\eta \in E(0, \alpha)$. Note that $f(z_0) = f\Big(\varphi_{z_0}(0)\Big) = 0$. Therefore, by the Schwarz lemma, we get

$$|f\Big(\varphi_{z_0}(\eta)\Big)|^2 \lesssim |\eta|^2 \frac{2^k}{(1 - |z_0|)^n} \asymp |\eta|^2 \frac{2^k}{(1 - |\varphi_{z_0}(\eta)|)^n}$$

whenever $\eta \in E(0,\alpha)$. This implies that there is a constant $C > 0$ such that

$$|f(\varphi_{z_0}(\eta))|^2 \le C|\eta|^2 \frac{2^k}{(1-|\varphi_{z_0}(\eta)|)^n}, \qquad \eta \in E(0,\alpha).$$

Thus, we can choose α so small that

$$|f(\varphi_{z_0}(\eta))|^2 < \frac{1}{2} K_\rho(\varphi_{z_0}(\eta), \varphi_{z_0}(\eta)), \qquad \eta \in E(0,\alpha).$$

This proves (4.14).

Now, from (4.13) and (4.14), we obtain that $|k_{\rho,z_0}(w)|^2 > \frac{1}{2}K_\rho(w,w)$ whenever $w \in E(z_0,\alpha)$. This means that

$$|K_\rho(w,z_0)|^2 > \frac{1}{2} K_\rho(z_0,z_0) K_\rho(w,w)$$

whenever $w \in E(z_0,\alpha)$, which completes our proof. □

Lemma 4.6.2. *Let T be a positive operator on A^2_ρ, and let $\tilde{T}$ be the Berezin transform of T, defined by*

$$\tilde{T}(z) = \langle T k_z, k_z \rangle_\rho, \qquad z \in \mathbb{B}_n.$$

(a) Let $0 < p \le 1$. If $\tilde{T} \in L^p(\mathbb{B}_n, d\lambda_\rho)$, then T is in $\mathcal{S}_p$.

(b) Let $p \ge 1$. If T is in $\mathcal{S}_p$, then $\tilde{T} \in L^p(\mathbb{B}_n, d\lambda_\rho)$.

Here, $d\lambda_\rho(z) = \dfrac{2^k \rho(z) dv(z)}{(1-|z|)^n}$ if $z \in \Omega_k$.

Proof. Note that

$$d\lambda_\rho(z) \asymp K(z,z)\rho(z)dv(z) = ||K_z||^2 \rho(z) dv(z).$$

The proof is similar to the proof of [1, Lemma 4.2]. The positive operator T is in $\mathcal{S}_p$ if and only if T^p is in the trace class $\mathcal{S}_1$. Fix an orthonormal basis $\{e_k\}$ of A^2_ρ. Since T^p is positive, it is in S_1 if and only if $\sum_k \langle T^p e_k, e_k\rangle_\rho < \infty$. Let $U = \sqrt{T^p}$. By Fubini's theorem, the reproducing property of K_z, and Parseval's identity, we have

$$\sum_k \langle T^p e_k, e_k \rangle_\rho = \sum_k ||U e_k||^2_\rho = \sum_k \int_{\mathbb{B}_n} |U e_k(z)|^2 \rho(z) dv(z)$$

$$
\begin{aligned}
&= \int_{\mathbb{B}_n} \left(\sum_k |Ue_k(z)|^2 \right) \rho(z)dv(z) = \int_{\mathbb{B}_n} \left(\sum_k |\langle Ue_k, K_z\rangle_\rho|^2 \right) \rho(z)dv(z) \\
&= \int_{\mathbb{B}_n} \left(\sum_k |\langle e_k, UK_z\rangle_\rho|^2 \right) \rho(z)dv(z) = \int_{\mathbb{B}_n} \|UK_z\|_\rho^2 \rho(z)dv(z) \\
&= \int_{\mathbb{B}_n} \langle T^p K_z, K_z\rangle_\rho \rho(z)dv(z) = \int_{\mathbb{B}_n} \langle T^p k_z, k_z\rangle_\rho \|K_z\|_\rho^2 \rho(z)dv(z) \\
&\asymp \int_{\mathbb{B}_n} \langle T^p k_z, k_z\rangle_\rho d\lambda_\rho(z).
\end{aligned}
$$

Hence, both (a) and (b) are the consequences of the well-known inequalities (see [43, Proposition 1.31])

$$
\begin{aligned}
\langle T^p k_z, k_z\rangle_\rho &\le \langle T k_z, k_z\rangle_\rho^p = \left(\tilde{T}(z)\right)^p, \qquad 0 < p \le 1, \\
\langle T^p k_z, k_z\rangle_\rho &\ge \langle T k_z, k_z\rangle_\rho^p = \left(\tilde{T}(z)\right)^p, \qquad p \ge 1.
\end{aligned}
$$

□

Lemma 4.6.3. *Let $\rho \in S^*$ and $z \in \Omega_k$. Then there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0)$ we have*

$$
|f(z)|^2 \lesssim \frac{2^k}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \rho(w)dv(w)
$$

for all $f \in H(\mathbb{B}_n)$.

Proof. Let $z \in \Omega_k$. For each $f \in H(\mathbb{B}_n)$, by the subharmonicity of the function $w \mapsto |f(w)|^2$ and the estimate $v\big(E(z,\alpha)\big) \asymp (1-|z|)^{n+1}$, we have

$$
|f(z)|^2 \lesssim \frac{1}{(1-|z|)^{n+1}} \int_{E(z,\alpha)} |f(w)|^2 dv(w).
$$

Clearly $1-|z| \asymp 1-|w|$ for $w \in E(z,\alpha)$. Hence,

$$
\begin{aligned}
|f(z)|^2 &\lesssim \frac{1}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \frac{1}{1-|w|} dv(w) \\
&= \frac{2^k}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \frac{2^{-k}}{1-|w|} dv(w). \qquad (4.17)
\end{aligned}
$$

By (4.1), for small α_0 we have $E(z,\alpha_0) \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1}$. Therefore, for every $\alpha \in (0,\alpha_0)$, we have $r_{k-1} < |w| < r_{k+2}$ for $w \in E(z,\alpha)$. Since $\int_{r_{k+2}}^1 \rho(t)dt = 2^{-k-2}$, we obtain $2^{-k} \lesssim \int_{|w|}^1 \rho(t)dt$ for every $w \in E(z,\alpha), \alpha \in (0,\alpha_0)$. Plugging this into (4.17) and using that $\rho^*(w) \lesssim \rho(w)$, we get

$$\begin{aligned} |f(z)|^2 &\lesssim \frac{2^k}{(1-|z|)^n}\int_{E(z,\alpha)}|f(w)|^2\rho^*(w)dv(w) \\ &\lesssim \frac{2^k}{(1-|z|)^n}\int_{E(z,\alpha)}|f(w)|^2\rho(w)dv(w). \end{aligned}$$

This completes the proof. □

Proof of Theorem 4.3.2. $(a) \Rightarrow (b)$. This follows from Lemma 4.6.2 (b).

$(b) \Rightarrow (c)$. By Proposition 4.6.1 (b), for sufficiently small $\alpha > 0$, we have

$$|K_z(w)|^2 \asymp \|K_z\|_\rho^2\|K_w\|_\rho^2, \qquad w \in E(z,\alpha), z \in \mathbb{B}_n.$$

Then by Proposition 4.6.1 (a), we get

$$\begin{aligned} \widetilde{T_\mu}(z) &= \int_{\mathbb{B}_n}|k_z(w)|^2d\mu(w) = \|K_z\|_\rho^{-2}\int_{\mathbb{B}_n}|K_z(w)|^2d\mu(w) \\ &\geq \|K_z\|_\rho^{-2}\int_{E(z,\alpha)}|K_z(w)|^2d\mu(w) \\ &\asymp \int_{E(z,\alpha)}\|K_w\|_\rho^2d\mu(w) \asymp \widehat{\mu}_\alpha(z). \end{aligned}$$

Since $\widetilde{T_\mu}$ is in $L^p(\mathbb{B}_n, d\lambda_\rho)$, $\widehat{\mu}_\alpha$ is also in $L^p(\mathbb{B}_n, d\lambda_\rho)$.

$(c) \Rightarrow (a)$. For every orthonormal basis $\{e_l\}$ of A^2_ρ, we have

$$\sum_l \langle T_\mu e_l, e_l\rangle_\rho^p = \sum_l\left(\int_{\mathbb{B}_n}|e_l(z)|^2d\mu(z)\right)^p. \tag{4.18}$$

By Lemma 4.6.3,

$$|e_l(z)|^2 \lesssim \frac{2^k}{(1-|z|)^n}\int_{E(z,\alpha)}|e_l(w)|^2\rho(w)dv(w), \qquad z \in \Omega_k.$$

By Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned} \int_{\mathbb{B}_n}|e_l(z)|^2d\mu(z) &\lesssim \int_{\mathbb{B}_n}|e_l(w)|^2\widehat{\mu}_\alpha(w)\rho(w)dv(w) \\ &\leq \left(\int_{\mathbb{B}_n}|e_l(w)|^2\widehat{\mu}_\alpha(w)^p\rho(w)dv(w)\right)^{1/p} \\ &\qquad \times \left(\int_{\mathbb{B}_n}|e_l(w)|^2\rho(w)dv(w)\right)^{1/q} \\ &= \left(\int_{\mathbb{B}_n}|e_l(w)|^2\widehat{\mu}_\alpha(w)^p\rho(w)dv(w)\right)^{1/p}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Thus, (4.18) implies that

$$\begin{aligned}\sum_l \langle T_\mu e_l, e_l\rangle_\rho^p &\lesssim \int_{\mathbb{B}_n}\left(\sum_l |e_l(w)|^2\right)\widehat{\mu}_\alpha(w)^p\rho(w)dv(w)\\ &= \int_{\mathbb{B}_n}\|K_w\|_\rho^2\widehat{\mu}_\alpha(w)^p\rho(w)dv(w)\\ &\asymp \int_{\mathbb{B}_n}\widehat{\mu}_\alpha(w)^p d\lambda_\rho(w) < \infty.\end{aligned}$$

This proves (a). □

Remark 4.6.4. Let $1<p<\infty$. In the case of large weighted Bergman spaces, Arroussi, Park and Pau proved in [1, Theorem 4.6] that

$$T_\mu \in \mathcal{S}_p \iff \widetilde{\mu}_\varepsilon(z) = \frac{\mu\Big(B(z,\varepsilon(1-|z|))\Big)}{(1-|z|)^{2n}} \text{ is in the corresponding weighted } L^p,$$

where $B(z,\varepsilon(1-|z|))$ is the Euclidean ball with center z and radius $\varepsilon(1-|z|)$. When the dimension $n=1$, we can see that $\widetilde{\mu}_\varepsilon$ is in L^p if and only if $\widehat{\mu}_\varepsilon$ is in L^p. However, for $n>1$, this equivalence is not true anymore.

Let us verify this. Choose $z_k \in \mathbb{B}_n$ such that $|z_k|$ tends to 1 sufficiently rapidly as $k\to\infty$. Consider

$$\mu = \sum_{k=1}^\infty c_k \chi_{B(z_k,\varepsilon)} \quad \text{and} \quad \mu^* = \sum_{k=1}^\infty c_k \chi_{B(z_k,3\varepsilon)},$$

where $c_k>0$ will be chosen later. We have

$$\mu \lesssim \widetilde{\mu}_\varepsilon \lesssim \mu^*$$

and

$$\sum_{k=1}^\infty c_k \frac{v\Big(B(z_k,\varepsilon)\Big)}{v\Big(E(z_k,\varepsilon)\Big)}\chi_{E(z_k,\varepsilon)} \lesssim \widehat{\mu}_\varepsilon \lesssim \sum_{k=1}^\infty c_k \frac{v\Big(B(z_k,\varepsilon)\Big)}{v\Big(E(z_k,\varepsilon)\Big)}\chi_{E(z_k,3\varepsilon)}.$$

Hence

$$\widetilde{\mu}_\varepsilon \in L^p \iff \sum_{k=1}^\infty c_k^p v\Big(B(z_k,\varepsilon)\Big) < \infty,$$

and

$$\widehat{\mu}_\varepsilon \in L^p \Longleftrightarrow \sum_{k=1}^{\infty} c_k^p \frac{\left(v\left(B(z_k,\varepsilon)\right)\right)^p}{\left(v\left(E(z_k,\varepsilon)\right)\right)^{p-1}} < \infty.$$

Since

$$\frac{c_k^p\left(v\left(B(z_k,\varepsilon)\right)\right)^p\left(v\left(E(z_k,\varepsilon)\right)\right)^{1-p}}{c_k^p v\left(B(z_k,\varepsilon)\right)} = \left(\frac{v\left(B(z_k,\varepsilon)\right)}{v\left(E(z_k,\varepsilon)\right)}\right)^{p-1}$$

$$\asymp (1-|z_k|)^{(n-1)(p-1)} \longrightarrow 0$$

as $k \to \infty$, we can choose c_k such that $\widehat{\mu}_\varepsilon \in L^p$ but $\widetilde{\mu}_\varepsilon \notin L^p$. On the other hand, one can easily see that $\widetilde{\mu}_\varepsilon \in L^p$ implies $\widehat{\mu}_\varepsilon \in L^p$.

Remark 4.6.5. When $\rho \in S \setminus S^*$, Theorem 4.3.2 does not remain valid anymore.

Let us denote

$$\varphi(x) = \int_{1-e^{-x}}^{1} \rho(s)ds, \qquad 0 < x < \infty.$$

Then φ is positive and $\lim\limits_{x\to\infty} \varphi(x) = 0$. Moreover,

$$\rho \in S \quad \text{if and only if} \quad \inf_k \left(\varphi^{-1}(2^{-k-1}) - \varphi^{-1}(2^{-k})\right) > 0.$$

In particular, ρ is in S if $|\varphi'| = O(\varphi)$ at ∞. On the other hand,

$$\rho \in S^* \quad \text{if and only if} \quad |\varphi'| \gtrsim \varphi \text{ at } \infty.$$

Now we consider

$$\rho(r) = \frac{1}{(1-r)}\left|\varphi'\left(\log\frac{1}{1-r}\right)\right|, \qquad 0 \le r < 1,$$

where φ is a differentiable positive function from $[0,\infty)$ to $[0,\infty)$ satisfying the following properties: $\lim\limits_{x\to\infty}\varphi(x) = 0$, $|\varphi'|$ decreases, $|\varphi'| = o(\varphi)$ at ∞, and $|\varphi'(t+1)| \ge \delta|\varphi'(t)|$ for $t > 0$, δ being a positive constant.

Claim 1: *Such ρ is in $S \setminus S^*$.*

Proof of Claim 1. Since $|\varphi'| = o(\varphi)$, $\frac{|\varphi'(x)|}{\varphi(x)} \leq 1$ for every $x \geq x_0$. For $\varepsilon \in (0, \log 2)$ we have

$$\int_x^{x+\varepsilon} \frac{|\varphi'(t)|}{\varphi(t)} dt \leq \varepsilon, \qquad x \geq x_0.$$

It follows that $\log \varphi(x) \leq \varepsilon + \log \varphi(x+\varepsilon)$. Hence

$$\varphi(x) \leq 2\varphi(x+\varepsilon), \qquad x \geq x_0.$$

Since φ is decreasing, φ^{-1} is decreasing. For every $0 < y \leq \frac{\varphi(x_0)}{2}$, set $x = \varphi^{-1}(2y) \geq x_0$. Then $y = \frac{\varphi(x)}{2} \leq \varphi(x+\varepsilon)$. Thus

$$\varphi^{-1}(y) \geq x + \varepsilon = \varphi^{-1}(2y) + \varepsilon.$$

Therefore, $\varphi^{-1}(2^{-k-1}) \geq \varepsilon + \varphi^{-1}(2^{-k})$ for $k \geq k_0$. It gives us that

$$e^{\varphi^{-1}(2^{-k-1})} \geq (1+\varepsilon) e^{\varphi^{-1}(2^{-k})}, \qquad k \geq k_0.$$

Hence

$$\inf_k \frac{1 - r_k}{1 - r_{k+1}} = \inf_k \frac{e^{\varphi^{-1}(2^{-k-1})}}{e^{\varphi^{-1}(2^{-k})}} > 1,$$

and $\rho \in S$. Clearly $\rho \notin S^*$. □

Claim 2: *We have*

$$\int_0^1 r^m \rho(r)\, dr \asymp \varphi(\log m), \qquad m \geq 2,$$

and

$$\int_{\mathbb{D}} |z|^{2m} d\mu_k(z) \asymp \frac{1}{m}, \qquad e^{\varphi^{-1}(2^{-k})} < m < e^{\varphi^{-1}(2^{-k-1})},$$

where $d\mu_k = \chi_{\Omega_k} dv, \ k \geq 0$.

Proof of Claim 2. Let us write

$$\int_0^1 r^m \rho(r)\, dr = \int_0^{1-\frac{1}{m}} r^m \rho(r)\, dr + \int_{1-\frac{1}{m}}^1 r^m \rho(r)\, dr.$$

Moreover,

$$\int_{1-\frac{1}{m}}^{1} r^m \rho(r)\,dr \asymp \int_{1-\frac{1}{m}}^{1} \rho(r)\,dr = \varphi(\log m).$$

On the other hand,

$$\begin{aligned}\int_0^{1-\frac{1}{m}} r^m \rho(r)\,dr &\lesssim \sum_{k=0}^{\log m} \int_{1-e^{-k}}^{1-e^{-k-1}} \frac{r^m}{1-r}\Big|\varphi'\Big(\log\frac{1}{1-r}\Big)\Big|\,dr\\ &\lesssim \sum_{k=0}^{\log m} \exp(-m\,e^{-k})|\varphi'(k)|.\end{aligned}$$

Let $A=\log m$. Since $|\varphi'(t+1)|\geq \delta|\varphi'(t)|$,

$$|\varphi'(k)| \leq \delta^{k-A}|\varphi'(A)| \lesssim e^{c(A-k)}|\varphi'(A)|$$

for all $0\leq k\leq A$, where c is a positive constant.

Hence

$$\begin{aligned}\int_0^{1-\frac{1}{m}} r^m \rho(r)\,dr &\lesssim \sum_{k=0}^{A} \exp(-e^{A-k})\exp\Big(c(A-k)\Big)|\varphi'(A)|\\ &= |\varphi'(A)|\sum_{j=0}^{A}\exp(cj-e^j) \lesssim |\varphi'(A)|\\ &= |\varphi'(\log m)| = o\Big(\varphi(\log m)\Big).\end{aligned}$$

Therefore, $\int_0^1 r^m\rho(r)\,dr \asymp \varphi(\log m)$.

Making a similar argument as above leads us to the desired result

$$\int_{\mathbb{D}} |z|^{2m} d\mu_k(z) \asymp \frac{1}{m}, \qquad e^{\varphi^{-1}(2^{-k})} < m < e^{\varphi^{-1}(2^{-k-1})}. \qquad \square$$

Now we are going to construct a measure μ on $\mathbb{D}$ such that $\widehat{\mu}_\alpha \in L^p(\mathbb{D}, d\lambda_\rho)$ for any $\alpha > 0$, but $T_\mu \notin \mathcal{S}_p$. Note that, for the sake of simplicity, we consider in this Remark only the case of dimension $n=1$.

Consider the orthonormal basis $(e_m)_{m\geq 0}$ of A^2_ρ,

$$e_m(z) = \frac{z^m}{\|z^m\|_\rho}.$$

Set $m_k = e^{\varphi^{-1}(2^{-k})}$. We obtain that

$$\int_{\mathbb{D}} |e_m(z)|^2 d\mu_k(z) \asymp \frac{1}{m}\,\frac{1}{\varphi(\log m)}, \qquad m_k < m < m_{k+1}.$$

Then,

$$\begin{aligned}
\sum_{m\geq 0} \langle T_{\mu_k} e_m, e_m\rangle_\rho^p &= \sum_{m\geq 0}\Big(\int_{\mathbb{D}} |e_m(z)|^2 d\mu_k(z)\Big)^p \\
&\gtrsim \sum_{m=m_k}^{m_{k+1}} \frac{1}{m^p}\,\frac{1}{\big(\varphi(\log m)\big)^p} := A_k.
\end{aligned}$$

Furthermore,

$$\widehat{\mu_{k,\alpha}}(z) \lesssim 2^k(1-|z|)\chi_{\tilde{\Omega}_k}(z),$$

where

$$\tilde{\Omega}_k = \{z\in\mathbb{D} : \operatorname{dist}(z,\Omega_k) < \alpha\}.$$

This implies that

$$\begin{aligned}
\int_{\mathbb{D}} \widehat{\mu_{k,\alpha}}^{\,p}\, d\lambda_\rho &\lesssim 2^{(p+1)k}\int_{\tilde{\Omega}_k} (1-|z|)^p\,\frac{\rho(|z|)}{1-|z|}\, dv(z) \\
&= 2^{(p+1)k}\int_{\tilde{\Omega}_k} (1-|z|)^{p-1}\rho(|z|)\, dv(z) \\
&\lesssim 2^{(p+1)k}\int_{r_k}^{r_{k+1}} (1-r)^{p-2}\Big|\varphi'\Big(\log\frac{1}{1-r}\Big)\Big|\, dr \\
&\asymp 2^{(p+1)k}\sum_{m=m_k}^{m_{k+1}}\int_{1-\frac{1}{m}}^{1-\frac{1}{m+1}} (1-r)^{p-2}\Big|\varphi'\Big(\log\frac{1}{1-r}\Big)\Big|\, dr \\
&\leq 2^{(p+1)k}\sum_{m=m_k}^{m_{k+1}} |\varphi'(\log m)|\int_{1-\frac{1}{m}}^{1-\frac{1}{m+1}} (1-r)^{p-2}\, dr \\
&\lesssim 2^{(p+1)k}\sum_{m=m_k}^{m_{k+1}} \frac{1}{m^p}\,|\varphi'(\log m)| \\
&\leq \sum_{m=m_k}^{m_{k+1}} \frac{1}{m^p}\,\frac{1}{\big(\varphi(\log m)\big)^p}\,\frac{|\varphi'(\log m)|}{\varphi(\log m)} := B_k.
\end{aligned}$$

The last estimate comes from the fact that $2^k = \frac{1}{\varphi(\log m_k)} \leq \frac{1}{\varphi(\log m)}$ for all m between m_k and m_{k+1}.

Since $|\varphi'| = o(\varphi)$ at ∞ and $m_k \to \infty$ as $k \to \infty$, we have $B_k = o(A_k)$. Hence there exist k_s such that $\sum_{s\geq 0} \left(\frac{B_{k_s}}{A_{k_s}}\right)^{\frac{1}{2}} < \infty$. Let

$$d\mu = \sum_{s\geq 0} \frac{1}{(A_{k_s}B_{k_s})^{\frac{1}{2p}}} d\mu_{k_s}.$$

Then

$$\begin{aligned}\sum_{m\geq 0} \langle T_\mu e_m, e_m\rangle_\rho^p &\geq \sum_{m\geq 0}\sum_{s\geq 0} \frac{1}{(A_{k_s}B_{k_s})^{\frac{1}{2}}} \langle T_{\mu_{k_s}} e_m, e_m\rangle_\rho^p \\ &\gtrsim \sum_{s\geq 0} \frac{1}{(A_{k_s}B_{k_s})^{\frac{1}{2}}} A_{k_s} = \sum_{s\geq 0} \left(\frac{A_{k_s}}{B_{k_s}}\right)^{\frac{1}{2}} = \infty,\end{aligned}$$

but

$$\begin{aligned}\int_{\mathbb{D}} \widehat{\mu_\alpha}^p \, d\lambda_\rho &\asymp \sum_{k\geq 0} \int_{\tilde{\Omega}_k} \widehat{\mu_\alpha}^p \, d\lambda_\rho \asymp \sum_{s\geq 0} \frac{1}{(A_{k_s}B_{k_s})^{\frac{1}{2}}} \int_{\tilde{\Omega}_{k_s}} \widehat{\mu_{k_s,\alpha}}^p \, d\lambda_\rho \\ &\lesssim \sum_{s\geq 0} \frac{1}{(A_{k_s}B_{k_s})^{\frac{1}{2}}} B_{k_s} = \sum_{s\geq 0} \left(\frac{B_{k_s}}{A_{k_s}}\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

Chapter 5

Bergman type projections

In this chapter, we study the Bergman type projection acting from L^∞ to the Bloch space $\mathcal{B}$ of $\mathbb{B}_n$, $n > 1$, and then provide a characterization of radial weight so that the projection is bounded.

5.1 Introduction and main result

Definition 5.1.1. Let us recall that the *Bloch space* of $\mathbb{B}_n$, denoted by $\mathcal{B}$, is the space of holomorphic functions f in $\mathbb{B}_n$ such that

$$\sup_{z\in\mathbb{B}_n}(1-|z|^2)|Rf(z)| < \infty,$$

where

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

is the *radial derivative* of f at $z \in \mathbb{B}_n$.

In the one dimensional case, the Bloch space consists of analytic functions f in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty,$$

and is denoted by $\mathcal{B}(\mathbb{D})$.

Definition 5.1.2. Let ρ be a radial weight and X be a space of measurable functions on $\mathbb{B}_n$. The Bergman type projection P_ρ acting on X is given by

$$P_\rho f(z) = \int_{\mathbb{B}_n} K_\rho(z,w) f(w)\rho(w) dv(w), \qquad z \in \mathbb{B}_n, f \in X,$$

where $K_\rho(z,w)$ is the reproducing kernel of the weighted Bergman space A^2_ρ.

When ρ is the standard radial weight $\rho(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, the projection is denoted by P_α.

A radial weight ρ belongs to the class $\widehat{\mathcal{D}}$ if $\widehat{\rho}(r) \lesssim \widehat{\rho}(\frac{1+r}{2})$ for all $r \in [0,1)$, where $\widehat{\rho}(r) = \int_r^1 \rho(s)ds$.

Projections play a crucial role in studying operator theory on spaces of analytic functions. Bounded analytic projections can also be used to establish duality relations and to obtain useful equivalent norms in spaces of analytic functions. Hence the boundedness of projections is an interesting topic which has been studied by many authors in recent years [8, 10, 11, 32, 33]. In [32], Peláez and Rättyä considered the projection P_{ρ_1} acting on $L^p_{\rho_2}(\mathbb{D})$, $1 \le p < \infty$ when two weights ρ_1, ρ_2 are in the class $\mathcal{R}$ of so called regular weights. A radial weight ρ is regular if $\widehat{\rho}(r) \asymp (1-r)\rho(r)$, $r \in (0,1)$. Recently, in 2019, they extended these results to the case where $\rho_1 \in \widehat{\mathcal{D}}$, ρ_2 is radial [33].

In this chapter, we are going to study the projections acting on the space L^∞. In the case of standard radial weight, we have the following result.

Theorem 5.1.3. *For any $\alpha > -1$, the Bergman type projection P_α is a bounded linear operator from L^∞ onto the Bloch space $\mathcal{B}$.*

See [42, Theorem 3.4] for a proof. This theorem is also valid for the case of one dimension [43, Theorem 5.2].

In [33], Peláez and Rättyä obtain an interesting result in the one dimensional case.

Theorem 5.1.4. *Let ρ be a radial weight. Then the projection $P_\rho : L^\infty(\mathbb{D}) \to \mathcal{B}(\mathbb{D})$ is bounded if and only if $\rho \in \widehat{\mathcal{D}}$.*

We extend this theorem to the case of several variables and obtain the following result.

Theorem 5.1.5. *Let ρ be a radial weight. Then the projection $P_\rho : L^\infty \to \mathcal{B}$ is bounded if and only if $\rho \in \widehat{\mathcal{D}}$.*

5.2 Some auxiliary lemmas

To prove Theorem 5.1.5 we need several auxiliary lemmas.

Lemma 5.2.1. *Let ρ be a radial weight. Then the following conditions are equivalent:*

(i) $\rho \in \widehat{\mathcal{D}}$;

(ii) There exist $C = C(\rho) > 0$ and $\beta_0 = \beta_0(\rho) > 0$ such that

$$\widehat{\rho}(r) \le C \left(\frac{1-r}{1-t}\right)^\beta \widehat{\rho}(t), \qquad 0 \le r \le t < 1,$$

for all $\beta \ge \beta_0$;

(iii) The asymptotic equality

$$\int_0^1 s^x \rho(s) ds \asymp \widehat{\rho}\left(1 - \frac{1}{x}\right), \qquad x \in [1, \infty),$$

is valid;

(iv) There exist $C_0 = C_0(\rho) > 0$ and $C = C(\rho) > 0$ such that

$$\widehat{\rho}(0) \le C_0 \widehat{\rho}(\frac{1}{2})$$

and $\rho_n \le C \rho_{2n}$ for all $n \in \mathbb{N}$.

This lemma can be found in [28].

Lemma 5.2.2. *If*

$$f(z) = \sum_{n=0}^{\infty} a_j z^j \in H^p, \qquad 0 < p \leq 2,$$

then

$$\sum_{j=0}^{\infty} (j+1)^{p-2} |a_j|^p \lesssim \|f\|_p^p.$$

Lemma 5.2.3. *Let $\{a_j\}$ be a sequence of complex numbers such that $\sum j^{q-2}|a_j|^q < \infty$ for some q, $2 \leq q < \infty$. Then the function $f(z) = \sum_{n=0}^{\infty} a_j z^j$ is in H^q, and*

$$\|f\|_q^q \lesssim \sum_{j=0}^{\infty} (j+1)^{q-2} |a_j|^q.$$

Two above lemmas are the classical Hardy-Littlewood inequalities, which can be found, for example, in Duren's book [13, Theorem 6.2 and 6.3].

Lemma 5.2.4. *Let ρ be a radial weight. Then the reproducing kernel $K_\rho(z,w)$ is given by*

$$K_\rho(z,w) = \frac{1}{2} \sum_{d=0}^{\infty} \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} \langle z, w \rangle^d, \qquad z, w \in \mathbb{B}_n,$$

where

$$\rho_s = \int_0^1 t^s \rho(t) dt, \qquad s \geq 1.$$

Proof. By the multinomial formula (see [42, (1.1)]), we have that

$$\langle z, w \rangle^d = \sum_{\beta \in \mathbb{N}^n, |\beta| = d} \frac{d!}{\beta!} z^\beta \bar{w}^\beta, \qquad z, w \in \mathbb{C}^n.$$

Hence, for $\alpha \in \mathbb{N}^n$, $|\alpha| = d$,

$$\int_{\mathbb{S}_n} \xi^\alpha \langle z, \xi \rangle^d d\sigma(\xi) = \sum_{\beta \in \mathbb{N}^n, |\beta| = d} \frac{d! z^\beta}{\beta!} \int_{\mathbb{S}_n} \xi^\alpha \bar{\xi}^\beta d\sigma(\xi), \quad z \in \mathbb{B}_n.$$

By Lemma 1.11 in [42],

$$\int_{\mathbb{S}_n} \xi^\alpha \bar{\xi}^\beta d\sigma(\xi) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \dfrac{\alpha!(n-1)!}{(d+n-1)!} & \text{if } \alpha = \beta, \end{cases}$$

and we obtain

$$\begin{aligned}\int_{\mathbb{S}_n} \xi^\alpha \langle z, \xi\rangle^d d\sigma(\xi) &= \frac{d!}{\alpha!} z^\alpha \int_{\mathbb{S}_n} \xi^\alpha \bar{\xi}^\alpha d\sigma(\xi) \\ &= \frac{d!}{\alpha!} \frac{\alpha!(n-1)!}{(d+n-1)!} z^\alpha \\ &= \frac{d!(n-1)!}{(d+n-1)!} z^\alpha, \qquad z \in \mathbb{B}_n.\end{aligned}$$

Therefore, for $\alpha \in \mathbb{N}^n$, $|\alpha| = d$ we have

$$\begin{aligned}\int_{\mathbb{B}_n} w^\alpha \langle z, w\rangle^d \rho(w) dv(w) &= 2n \int_0^1 t^{2n-1+2d} \rho(t) dt \int_{\mathbb{S}_n} \xi^\alpha \langle z, \xi\rangle^d d\sigma(\xi) \\ &= 2 \frac{d! n! \rho_{2n-1+2d}}{(d+n-1)!} z^\alpha, \qquad z \in \mathbb{B}_n.\end{aligned}$$

It follows that

$$z^\alpha = \frac{(d+n-1)!}{2 d! n! \rho_{2n-1+2d}} \int_{\mathbb{B}_n} w^\alpha \langle z, w\rangle^d \rho(w) dv(w), \qquad z \in \mathbb{B}_n. \tag{5.1}$$

Since $\rho(t) > 0, 0 < t < 1$, we have $\rho_s \geq C_\varepsilon (1-\varepsilon)^s$ for every $\varepsilon > 0$. Given $z \in \mathbb{B}_n$, we have

$$\begin{aligned}&\int_{\mathbb{B}_n} \left| \frac{1}{2} \sum_{d=0}^\infty \frac{(d+n-1)!}{d! n! \rho_{2n-1+2d}} \langle z, w\rangle^d \right|^2 \rho(w) dv(w) \\ &= \frac{1}{4} \sum_{d_1, d_2 \geq 0} \frac{(d_1+n-1)!(d_2+n-1)!}{d_1! d_2! (n!)^2 \rho_{2n-1+2d_1} \rho_{2n-1+2d_2}} \int_{\mathbb{B}_n} \langle z, w\rangle^{d_1} \langle w, z\rangle^{d_2} \rho(w) dv(w) \\ &= \frac{1}{4} \sum_{d_1, d_2 \geq 0} \frac{(d_1+n-1)!(d_2+n-1)!}{d_1! d_2! (n!)^2 \rho_{2n-1+2d_1} \rho_{2n-1+2d_2}} \int_{\mathbb{B}_n} \sum_{|\beta|=d_2} w^\beta \bar{z}^\beta \frac{d_2!}{\beta!} \langle z, w\rangle^{d_1} \rho(w) dv(w) \\ &= \frac{1}{2} \sum_{d \geq 0} \left(\frac{(d+n-1)!}{d! n!} \right) \frac{1}{\rho^2_{2n-1+2d}} \sum_{|\beta|=d} \frac{(d!)^2}{\beta!} \frac{n! \rho_{2n-1+2d}}{(d+n-1)!} z^\beta \bar{z}^\beta \\ &= \frac{1}{2} \sum_{d \geq 0} \frac{(d+n-1)!}{n! \rho_{2n-1+2d}} \sum_{|\beta|=d} \frac{z^\beta \bar{z}^\beta}{\beta!} = \frac{1}{2} \sum_{d \geq 0} \frac{(d+n-1)!}{d! n! \rho_{2n-1+2d}} |z|^{2d} < \infty.\end{aligned}$$

Thus, the function $w \mapsto \frac{1}{2} \sum_{d=0}^\infty \frac{(d+n-1)!}{d! n! \rho_{2n-1+2d}} \langle w, z\rangle^d$ belongs to A^2_ρ.

By (5.1) and by continuity, for every $f \in A^2_\rho(\mathbb{B}_n)$,

$$f(z) = \int_{\mathbb{B}_n} f(w) \left(\frac{1}{2} \sum_{d=0}^\infty \frac{(d+n-1)!}{d! n! \rho_{2n-1+2d}} \langle z, w\rangle^d \right) \rho(w) dv(w), \qquad z \in \mathbb{B}_n,$$

which implies our conclusion. □

5.3 Proof of Theorem 5.1.5

It suffices to consider only the case $n > 1$.

Proposition 5.3.1. *If $\rho \in \widehat{\mathcal{D}}$, then the projection $P_\rho : L^\infty \to \mathcal{B}$ is bounded, where P_ρ is defined by*

$$P_\rho\varphi(z) = \int_{\mathbb{B}_n} K_\rho(z,w)\varphi(w)\rho(w)dv(w), \qquad \varphi \in L^\infty, z \in \mathbb{B}_n.$$

Proof. We have

$$K_\rho(z,w) = \frac{1}{2}\sum_{d=0}^{\infty} \frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}}\langle z,w\rangle^d.$$

Hence, for a fixed $w \in \mathbb{B}_n$,

$$\begin{aligned}
RK_\rho(z,w) &= \sum_{j=1}^{n} z_j \frac{\partial K_\rho(z,w)}{\partial z_j} \\
&= \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\left(\frac{1}{2}\sum_{d=0}^{\infty}\frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}}\langle z,w\rangle^d\right) \\
&= \frac{1}{2}\sum_{j=1}^{n} z_j \sum_{d=0}^{\infty}\frac{(d+n-1)!}{d!n!\rho_{2n-1+2d}} d\bar{w}_j\langle z,w\rangle^{d-1} \\
&= \frac{1}{2}\sum_{d=1}^{\infty}\frac{(d+n-1)!}{(d-1)!n!\rho_{2n-1+2d}}\langle z,w\rangle^d \\
&= \frac{1}{2}\sum_{d=1}^{\infty}\frac{\Gamma(d+n)}{\Gamma(d)\Gamma(n+1)\rho_{2n-1+2d}}\langle z,w\rangle^d.
\end{aligned}$$

Now, given $\varphi \in L^\infty$, let

$$f(z) := P_\rho\varphi(z) = \int_{\mathbb{B}_n} K_\rho(z,w)\varphi(w)\rho(w)dv(w), \qquad z \in \mathbb{B}_n.$$

For all $z \in \mathbb{B}_n$ we have

$$\begin{aligned}
|Rf(z)| &= \left|\int_{\mathbb{B}_n} RK_\rho(z,w)\varphi(w)\rho(w)dv(w)\right| \\
&\le \int_{\mathbb{B}_n} |RK_\rho(z,w)||\varphi(w)|\rho(w)dv(w) \\
&\le ||\varphi||_\infty \int_{\mathbb{B}_n} |RK_\rho(z,w)|\rho(w)dv(w). \qquad (5.2)
\end{aligned}$$

Set

$$g(\lambda)=\sum_{d=1}^{\infty}\frac{\Gamma(d+n)}{\Gamma(d)}\frac{\lambda^{d-1}}{\rho_{2n-1+2d}},\qquad \lambda\in\mathbb{D}.$$

Since $\rho(t)>0, 0<t<1$, g is analytic in the unit disc. Then

$$RK_\rho(z,w)=\frac{\langle z,w\rangle}{2\Gamma(n+1)}g(\langle z,w\rangle). \tag{5.3}$$

Next we consider the reproducing kernel $K^1_\rho(z,w)$ of the Bergman space in the unit disc with the weight ρ. We have

$$K^1_\rho(z,w)=\frac{1}{2}\sum_{d=0}^{\infty}\frac{(z\overline{w})^d}{\rho_{2d+1}}.$$

Furthermore,

$$\begin{aligned}\frac{\partial^n}{\partial z^n}K^1_\rho(z,w)&=\frac{1}{2}\sum_{d=n}^{\infty}\frac{\Gamma(d+1)(z\overline{w})^{d-n}\overline{w}^n}{\Gamma(d-n+1)\rho_{2d+1}}\\&=\frac{1}{2}\sum_{s=1}^{\infty}\frac{\Gamma(s+n)}{\Gamma(s)}\frac{(z\overline{w})^{s-1}\overline{w}^n}{\rho_{2s+2n-1}}\\&=\frac{1}{2}g(z\overline{w})\overline{w}^n.\end{aligned}$$

By a result of Peláez and Rättyä ([32, Theorem 1 (ii)]), we have

$$\int_{\mathbb{D}}\left|\frac{\partial^n}{\partial z^n}K^1_\rho(z,w)\right|(1-|z|^2)^{n-2}dA(z)\asymp\int_0^{|w|}\frac{dt}{\widehat{\rho}(t)(1-t)^2},\qquad \frac{1}{2}\le|w|<1,$$

where $\widehat{\rho}(t)=\int_t^1\rho(s)ds$.

Thus,

$$\int_{\mathbb{D}}|g(z\overline{w})|(1-|z|^2)^{n-2}dA(z)\asymp\int_0^{|w|}\frac{dt}{\widehat{\rho}(t)(1-t)^2},\qquad \frac{1}{2}\le|w|<1.$$

Since g is analytic in the unit disc, we have

$$\int_{\mathbb{D}}|g(z\overline{w})|(1-|z|^2)^{n-2}dA(z)\lesssim 1+\int_0^{|w|}\frac{dt}{\widehat{\rho}(t)(1-t)^2},\qquad w\in\mathbb{D}. \tag{5.4}$$

Now, by (5.3), we have

$$\begin{aligned}\int_{\mathbb{B}_n}|RK_\rho(z,w)|\rho(w)dv(w)&\lesssim\int_{\mathbb{B}_n}|g(\langle z,w\rangle)|\rho(w)dv(w)\\&\asymp\int_0^1 r^{2n-1}\rho(r)\left(\int_{\mathbb{S}_n}|g(\langle rz,\xi\rangle)|d\sigma(\xi)\right)dr.\end{aligned}$$

By [42, Lemma 1.9] and the unitary invariance of $d\sigma$, we have

$$\int_{\mathbb{S}_n} |g(\langle rz, \xi\rangle)| d\sigma(\xi) \asymp \int_{\mathbb{D}} |g(r|z|\lambda)|(1-|\lambda|^2)^{n-2} dA(\lambda).$$

Thus, by (5.4) we obtain

$$\begin{aligned}
\int_{\mathbb{B}_n} |RK_\rho(z,w)|\rho(w)dv(w) &\lesssim \int_0^1 r^{2n-1}\rho(r)\left(1+\int_0^{r|z|}\frac{dt}{\widehat{\rho}(t)(1-t)^2}\right)dr \\
&\lesssim 1+\int_0^{|z|}\frac{1}{\widehat{\rho}(t)(1-t)^2}\left(\int_{t/|z|}^1 r^{2n-1}\rho(r)dr\right)dt \\
&\lesssim 1+\int_0^{|z|}\frac{\widehat{\rho}(t/|z|)}{\widehat{\rho}(t)}\frac{dt}{(1-t)^2}\lesssim\frac{1}{1-|z|}, \qquad z\in\mathbb{B}_n.
\end{aligned}$$

By (5.2) we obtain now that

$$|Rf(z)| \lesssim \|\varphi\|_\infty \frac{1}{1-|z|^2}, \qquad z \in \mathbb{B}_n,$$

and, hence,

$$\sup_{z\in\mathbb{B}_n}(1-|z|^2)|Rf(z)| \lesssim \|\varphi\|_\infty.$$

It is easy to see that

$$|f(0)| \lesssim \|\varphi\|_\infty.$$

Therefore, P_ρ is bounded. The Proposition 5.3.1 is proved. □

Proposition 5.3.2. *Suppose that the projection $P_\rho : L^\infty \to \mathcal{B}$ is bounded. Then $\rho \in \widehat{\mathcal{D}}$.*

Proof. Given $\xi \in \mathbb{S}_n$ and $w \in \mathbb{B}_n$, let us consider a function g given by

$$g(\lambda) = RK_\rho(\lambda\xi, w), \qquad \lambda \in \mathbb{D}.$$

Then

$$g(\lambda) = \sum_{d=1}^\infty c_d \langle \xi, w\rangle^d \lambda^d,$$

where $c_d = \dfrac{1}{2n}\dfrac{\Gamma(d+n)}{\Gamma(d)\Gamma(n)\rho_{2n-1+2d}}$. By the Hardy–Littlewood inequality (see Lemma 5.2.2) we have

$$\begin{aligned}
\sum_{d=1}^\infty \frac{c_d|\langle \xi, w\rangle|^d}{d+1} &\lesssim \int_0^{2\pi} |g(e^{i\theta})|\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} |RK_\rho(e^{i\theta}\xi, w)|\frac{d\theta}{2\pi}.
\end{aligned}$$

Integrating both sides of the above inequality over $\xi \in \mathbb{S}_n$ we obtain

$$\begin{aligned}\sum_{d=1}^{\infty} \frac{c_d}{d+1} \int_{\mathbb{S}_n} |\langle \xi, w\rangle|^d \, d\sigma(\xi) &\lesssim \int_{\mathbb{S}_n} \int_0^{2\pi} |RK_\rho(e^{i\theta}\xi, w)| \frac{d\theta}{2\pi} d\sigma(\xi) \\ &= \int_{\mathbb{S}_n} |RK_\rho(\xi, w)| \, d\sigma(\xi).\end{aligned}$$

By the unitary invariance of $d\sigma$ and [42, Lemma 1.9], we have

$$\begin{aligned}\int_{\mathbb{S}_n} |\langle \xi, w\rangle|^d \, d\sigma(\xi) &= |w|^d \int_{\mathbb{S}_n} |\xi_1|^d \, d\sigma(\xi) \\ &= (n-1)|w|^d \int_{\mathbb{D}} (1-|z|^2)^{n-2} |z|^d \, dA(z) \\ &= (n-1)\pi |w|^d \int_0^1 (1-t)^{n-2} t^{d/2} dt \\ &\asymp \frac{\Gamma(\frac{d}{2}+1)\Gamma(n)}{\Gamma(\frac{d}{2}+n)} |w|^d.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{S}_n} |RK_\rho(\xi, w)| \, d\sigma(\xi) &\gtrsim \sum_{d=1}^{\infty} \frac{c_d}{d+1} \frac{\Gamma(\frac{d}{2}+1)\Gamma(n)}{\Gamma(\frac{d}{2}+n)} |w|^d \\ &= \frac{1}{2n} \sum_{d=1}^{\infty} \frac{\Gamma(d+n)\Gamma(\frac{d}{2}+1)}{(d+1)\Gamma(d)\Gamma(\frac{d}{2}+n)\rho_{2n-1+2d}} |w|^d.\end{aligned}$$

Since

$$\frac{\Gamma(d+n)\Gamma(\frac{d}{2}+1)}{(d+1)\Gamma(d)\Gamma(\frac{d}{2}+n)} \asymp 1,$$

we get

$$\int_{\mathbb{S}_n} |RK_\rho(\xi, w)| \, d\sigma(\xi) \gtrsim \frac{1}{2n} \sum_{d=1}^{\infty} \frac{|w|^d}{\rho_{2n-1+2d}}, \qquad w \in \mathbb{B}_n.$$

Therefore, for $z \in \mathbb{B}_n$, we have

$$\begin{aligned}\int_{\mathbb{B}_n} |RK_\rho(z, w)| \rho(w) dv(w) &= 2n \int_0^1 r^{2n-1} \rho(r) \int_{\mathbb{S}_n} |RK_\rho(z, r\xi)| \, d\sigma(\xi) \, dr \\ &= 2n \int_0^1 r^{2n-1} \rho(r) \int_{\mathbb{S}_n} |RK_\rho(\xi, rz)| \, d\sigma(\xi) \, dr \\ &\gtrsim \sum_{d=1}^{\infty} \frac{|z|^d}{\rho_{2n-1+2d}} \int_0^1 r^{2n-1+d} \rho(r) dr \\ &= \sum_{d=1}^{\infty} \frac{\rho_{2n-1+d}}{\rho_{2n-1+2d}} |z|^d.\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{z\in\mathbb{B}_n}(1-|z|^2)\int_{\mathbb{B}_n}|RK_\rho(z,w)|\rho(w)dv(w) \\ &\gtrsim \sup_{z\in\mathbb{B}_n}(1-|z|)\sum_{d=1}^{\infty}\frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}}|z|^d \\ &\geq \sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{d=1}^{N}\frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}}\left(1-\frac{1}{N}\right)^d \\ &\gtrsim \sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{d=1}^{N}\frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}}.\end{aligned}$$

Since P_ρ is bounded from L^∞ to $\mathcal{B}$,

$$\sup_{z\in\mathbb{B}_n}(1-|z|^2)\int_{\mathbb{B}_n}|RK_\rho(z,w)|\rho(w)dv(w)<\infty.$$

Given $N\geq 2n$, we obtain that

$$1\gtrsim\frac{1}{4N-2n}\sum_{d=3N-n+1}^{4N-2n}\frac{\rho_{d+2n-1}}{\rho_{2d+2n-1}}\geq\frac{1}{4N}(N-n)\frac{\rho_{4N}}{\rho_{6N}},$$

and, hence,

$$\rho_{6N}\gtrsim\rho_{4N}.$$

If $8N\leq k<8N+8$, $N\geq 2n+8$, then

$$\rho_k\leq\rho_{8N}\lesssim\rho_{12N}\lesssim\rho_{18N}\leq\rho_{2k},$$

and by Lemma 5.2.1 we conclude that $\rho\in\widehat{\mathcal{D}}$. □

From Propositions 5.3.1 and 5.3.2, we obtain the conclusion of Theorem 5.1.5.

CHAPTER 6

DIMENSION OF THE FOCK TYPE SPACES

In this chapter, we study the weighted Fock spaces in one and several complex variables. We evaluate the dimension of these spaces in terms of the weight function extending and complete earlier results by Rozenblum–Shirokov and Shigekawa.

6.1 Introduction

Let ψ be a plurisubharmonic function on $\mathbb{C}^n$, $n \geq 1$. The weighted Fock space $\mathcal{F}^2_\psi$ is the space of entire functions f such that

$$||f||^2_\psi = \int_{\mathbb{C}^n} |f(z)|^2 e^{-\psi(z)} dV(z) < \infty,$$

where dV is the volume measure on $\mathbb{C}^n$.

Note that $\mathcal{F}^2_\psi$ is a closed subspace of $L^2(\mathbb{C}^n, e^{-\psi} dV)$ and hence is a Hilbert space endowed with the inner product

$$\langle f, g \rangle_\psi = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-\psi(z)} dV(z), \qquad f, g \in \mathcal{F}^2_\psi.$$

For numerous results on the Fock space on $\mathbb{C}$, see the book of Zhu [44].

In this chapter we study when the space $\mathcal{F}^2_\psi$ is of finite dimension depending on the weight ψ. This problem (at least for the case $n = 1$) is motivated by some quantum mechanics questions, especially in the study of zero modes, eigenfunctions with zero eigenvalues. In [36, Theorem 3.2], Rozenblum and Shirokov proposed a sufficient condition for the space $\mathcal{F}^2_\psi$ to be of infinite dimension, when ψ is a subharmonic function.

Theorem 6.1.1. *Let ψ be a finite subharmonic function on the complex plane such that the measure $\mu = \Delta\psi$ is of infinite mass:*

$$\mu(\mathbb{C}) = \int_{\mathbb{C}} d\mu(z) = \infty. \tag{6.1}$$

Then the space $\mathcal{F}^2_\psi$ has infinite dimension.

We improve and extend somewhat this statement in this chapter, give a necessary and sufficient condition on ψ for the space $\mathcal{F}^2_\psi$ to be of finite dimension, and calculate this dimension.

The situation is much more complicated in $\mathbb{C}^n$, $n \geq 2$. Shigekawa established in [40] (see also [15, Theorem 7.10] in a book by Haslinger), the following interesting result.

Theorem 6.1.2. *Let $\psi : \mathbb{C}^n \to \mathbb{R}$ be a C^∞ smooth function and let $\lambda_0(z)$ denote the lowest eigenvalue of the Levi matrix*

$$L_\psi(z) = i\partial\bar{\partial}\psi(z) = \left(\frac{\partial^2\psi(z)}{\partial z_j \partial \overline{z_k}}\right)_{j,k=1}^n.$$

Suppose that

$$\lim_{|z|\to\infty} |z|^2\lambda_0(z) = \infty. \tag{6.2}$$

Then $\dim(\mathcal{F}^2_\psi) = \infty$.

Note that the condition (6.2) is not necessary. A corresponding example is given in [15, Section 7]. In this chapter, we improve Theorem 6.1.2 by presenting a weaker condition for the dimension of the Fock space $\mathcal{F}^2_\psi$ to be infinite. Furthermore, we give several examples that show how far our condition is from being necessary.

The rest of this chapter is organized as follows. The case of dimension one is considered in Section 6.2, and the case of higher dimension is considered in Section Section 6.3.

6.2 The case of $\mathbb{C}$

Given a subharmonic function $\psi : \mathbb{C} \to [-\infty, \infty)$, denote by μ_ψ the corresponding Riez measure, $\mu_\psi = \Delta\psi$. Next, consider the class $\mathcal{M}^d$ of the positive σ-finite atomic measures with masses which are integer multiples by 4π. Given a σ-finite measure μ, consider the corresponding atomic measure μ^d,

$$\mu^d = \max\left\{\mu_1 \in \mathcal{M}^d : \mu_1 \le \mu\right\}.$$

Denote $\mu^c = \mu - \mu^d$, $\mu^d = \sum_k 4\pi\delta_{x_{k,\mu}}$.

Denote by $\mathcal{M}^c$ the class of the positive σ-finite measures μ such that $\mu^d = 0$. Note that if ψ is finite on the complex plane, then μ_ψ has no point masses and $\mu_\psi \in \mathcal{M}^c$. Furthermore, if $\mu_\psi \in \mathcal{M}^c$, then $e^{-\psi} \in L^1_{loc}(dV)$.

Lemma 6.2.1. *Let ψ, ψ_1 be two subharmonic functions such that $(\mu_\psi)^c = (\mu_{\psi_1})^c$. Then* $\dim \mathcal{F}^2_\psi = \dim \mathcal{F}^2_{\psi_1}$.

Proof. Let F, F_1 be two entire functions with the zero sets, correspondingly, $\left\{x_{k,\mu_\psi}\right\}$ and $\left\{x_{k,\mu_{\psi_1}}\right\}$ (taking into account the multiplicities). Then

$$\Delta \log|F|^2 = (\mu_\psi)^d, \Delta \log|F|^2 = (\mu_{\psi_1})^d,$$

and the functions $h = \psi - \log|F|^2 - \psi^c$, $h_1 = \psi_1 - \log|F_1|^2 - \psi_1^c$ are harmonic. Let $h = \operatorname{Re} H$, $h_1 = \operatorname{Re} H_1$ for some entire functions H, H_1.

Given an entire function f, we have

$$\begin{aligned} f \in \mathcal{F}^2_\psi &\iff \int_{\mathbb{C}} |f(z)|^2 e^{-\psi(z)}\, dV(z) < \infty \\ &\iff \int_{\mathbb{C}} |f(z)|^2 e^{-\psi^c(z) - h(z) - \log|F(z)|^2}\, dV(z) < \infty \\ &\iff \int_{\mathbb{C}} |f(z) e^{-H(z)/2}/F(z)|^2 e^{-\psi^c(z)}\, dV(z) < \infty \\ &\iff \int_{\mathbb{C}} |f(z) e^{-H(z)/2}/F(z)|^2 e^{-\psi_1^c(z)}\, dV(z) < \infty \end{aligned}$$

$$\iff \int_{\mathbb{C}} |f(z)e^{-H(z)/2}/F(z)|^2 e^{-\psi_1(z)+h_1(z)+\log|F_1(z)|^2}\, dV(z) < \infty$$
$$\iff \int_{\mathbb{C}} |f(z)e^{-H(z)/2+H_1(z)/2}F_1(z)/F(z)|^2 e^{-\psi_1(z)}\, dV(z) < \infty$$
$$\iff f\cdot \frac{F_1}{F}e^{-H/2+H_1/2} \in \mathcal{F}^2_{\psi_1}.$$

Thus, $\dim \mathcal{F}^2_\psi = \dim \mathcal{F}^2_{\psi_1}$. □

Lemma 6.2.2. *Let ψ be a subharmonic function such that $\mu_\psi \in \mathcal{M}^c$. If $\dim \mathcal{F}^2_\psi < \infty$, then $\mu_\psi(\mathbb{C}) < \infty$.*

For a proof, see the proof of [36, Theorem 3.2].

Lemma 6.2.3. *Let ψ be a subharmonic function. Then*

$$\dim \mathcal{F}^2_\psi \le \left\lceil \frac{\mu_\psi(\mathbb{C})}{4\pi} \right\rceil.$$

Here and later on, given a real number x, $\lceil x \rceil$ is the maximal integer smaller than x.

Proof. Set $\mu = \mu_\psi$ and consider a modified logarithmic potential G of the measure μ:

$$G(z) = \frac{1}{2\pi}\int_{D(0,2)} \log|z-w|\,d\mu(w) + \frac{1}{2\pi}\int_{\mathbb{C}\setminus D(0,2)} \log\left|\frac{z-w}{w}\right| d\mu(w)$$
$$= G_1(z) + G_2(z).$$

From now on, $D(z,r) = \{w \in \mathbb{C} : |w-z| < r\}$. Since $\Delta G = \mu = \Delta\psi$, by Lemma 6.2.1 we have $\dim \mathcal{F}^2_\psi = \dim \mathcal{F}^2_G$.

Next,

$$\left|G_1(z) - \frac{\mu\big(D(0,2)\big)}{2\pi}\log|z|\right| \le \frac{1}{2\pi}\int_{D(0,2)} \log\left|1-\frac{w}{z}\right| d\mu(w)$$
$$\le \frac{C}{|z|}, \qquad |z| \ge 4, \quad (6.3)$$

and

$$G_2(z) - \frac{\mu\big(\mathbb{C}\setminus D(0,2)\big)}{2\pi}\log|z|$$
$$= \frac{1}{2\pi}\int_{\mathbb{C}\setminus D(0,2)} \log\left|\frac{1}{z}-\frac{1}{w}\right| d\mu(w) \le 0, \qquad |z| \ge 4.$$

Thus,

$$G(z) \leq \frac{\mu(\mathbb{C})}{2\pi} \log(1+|z|) + \frac{C}{1+|z|}, \qquad z \in \mathbb{C}.$$

Now, given an entire function f, we have

$$f \in \mathcal{F}^2_\psi \Longrightarrow \int_{\mathbb{C}} |f(z)|^2 (1+|z|)^{-\mu(\mathbb{C})/(2\pi)}\, dV(z) < \infty.$$

By a Liouville type theorem, f is a polynomial of degree N such that

$$\int_1^\infty r^{2N} r^{-\mu(\mathbb{C})/(2\pi)} r\, dr < \infty.$$

Therefore, $N < -1 + \mu(\mathbb{C})/(4\pi)$. Thus $\dim \mathcal{F}^2_\psi \leq \left\lceil \frac{\mu(\mathbb{C})}{4\pi} \right\rceil$. □

Lemma 6.2.4. *Let ψ be a subharmonic function and suppose that $\mu_\psi \in \mathcal{M}^c$. Then*

$$\dim \mathcal{F}^2_\psi \geq \left\lceil \frac{\mu_\psi(\mathbb{C})}{4\pi} \right\rceil.$$

Proof. Set $\mu = \mu_\psi$ and choose $\varepsilon > 0$, $R > 1$ such that

$$\frac{\mu\big(D(0,R)\big)}{4\pi} > \left\lceil \frac{\mu(\mathbb{C})}{4\pi} \right\rceil + \frac{\varepsilon}{2}.$$

Next, increasing R, we can guarantee that

$$\mu\big(D(0,R)\big) > \mu(\mathbb{C}) - \frac{1}{2}.$$

Consider a modified logarithmic potential U of the measure μ :

$$U(z) = \frac{1}{2\pi} \int_{D(0,R)} \log|z-w|\, d\mu(w) + \frac{1}{2\pi} \int_{\mathbb{C}\setminus D(0,R)} \log\left|\frac{z-w}{w}\right| d\mu(w)$$
$$= U_1(z) + U_2(z).$$

Since $\Delta U = \mu = \Delta \psi$, by Lemma 6.2.1 we have $\dim \mathcal{F}^2_\psi = \dim \mathcal{F}^2_U$. Arguing as in (6.3), we get

$$U_1(z) \geq \frac{\mu\big(D(0,R)\big)}{2\pi} \log|z| - \frac{C}{|z|}, \qquad |z| \geq 2R.$$

Next, let $|z| \geq 2R$. Then

$$\begin{aligned}U_2(z) = \frac{1}{2\pi}\int_{\mathbb{C}\setminus(D(0,R)\cup D(z,|z|/2))} \log\left|\frac{z-w}{w}\right| d\mu(w) \\ + \frac{1}{2\pi}\int_{D(z,|z|/2)} \log\left|\frac{z-w}{w}\right| d\mu(w) \\ \geq C - \frac{1}{2\pi}\int_{D(z,|z|/2)} \log\left|\frac{z/2}{z-w}\right| d\mu(w) = C - U_3(z).\end{aligned}$$

Given $m \geq 1$, denote $A_m = \{z \in \mathbb{C} : 2^m R \leq |z| \leq 2^{m+1}R\}$. Fix $m \geq 1$ and $k \geq 1$ and apply Lemma 3.6.10 with $\nu = \mathbf{1}_{\mathbb{C}\setminus D(0,R)}\mu$, $2^m R \leq |z_0| < 2^{m+1}R$, $h = 2^{m-1}R$, $n(z_0, h) \leq 1/2$, and $d = 2^{m-k-1}R$ to get for some $C, C_1 > 0$, $\delta \in (0,1)$:

$$m_2\{z \in A_m : U_3(z) > C_1 + \delta k\} \leq C \cdot 2^{2m} R^2 2^{-2k}, \quad k \geq 1.$$

Hence,

$$\begin{aligned}\int_{\mathbb{C}} (1+|z|)^{-2-\varepsilon} e^{U_3(z)}\, dV(z) \leq C + C \sum_{m\geq 1}\sum_{k\geq 1} 2^{-(2+\varepsilon)m} e^{\delta k} \\ \times m_2\{z \in A_m : C_1 + \delta k \leq U_3(z) < C_1 + \delta(k+1)\} \\ \leq C + C \sum_{m\geq 1}\sum_{k\geq 1} 2^{-(2+\varepsilon)m} e^{\delta k} 2^{2m} R^2 2^{-2k} < \infty.\end{aligned}$$

Next, for every $0 \leq N \leq \left\lceil \frac{\mu(\mathbb{C})}{4\pi} \right\rceil - 1$ we have

$$\begin{aligned}\int_{\mathbb{C}} |z|^{2N} e^{-U(z)}\, dV(z) \leq C \int_{\mathbb{C}} |z|^{2N} (1+|z|)^{-\mu(D(0,R))/(2\pi)} e^{U_3(z)}\, dV(z) \\ \leq C \int_{\mathbb{C}} |(|1+|z|)^{-2-\varepsilon} e^{U_3(z)}\, dV(z) < \infty.\end{aligned}$$

Here we use the fact that $\mu_\psi \in \mathcal{M}^c$, hence e^{-U} is locally integrable.

Finally, we have

$$\dim \mathcal{F}^2_\psi \geq \left\lceil \frac{\mu(\mathbb{C})}{4\pi} \right\rceil.$$

□

Summing up Lemmata 6.2.1, 6.2.2, 6.2.3 and 6.2.4 we obtain the following result, extending and slightly correcting Theorem 6.1.1.

Theorem 6.2.5. *Let ψ be a subharmonic function on the complex plane. Then the Fock space $\mathcal{F}^2_\psi$ is finite-dimensional if and only if*

$$(\mu_\psi)^c(\mathbb{C}) < \infty. \tag{6.4}$$

If ψ is finite on $\mathbb{C}$, then we can write the condition (6.4) *as $\mu_\psi(\mathbb{C}) < \infty$. Finally, if $(\mu_\psi)^c(\mathbb{C}) < \infty$, then*

$$\dim \mathcal{F}^2_\psi = \left\lceil \frac{(\mu_\psi)^c(\mathbb{C})}{4\pi} \right\rceil.$$

6.3 The case of $\mathbb{C}^n$, $n > 1$

Theorem 6.3.1. *Let $\psi : \mathbb{C}^n \to \mathbb{R}$ be a C^2 smooth function. Given $M > 0$, consider $\psi_M(z) = M \log(|z|^2)$. Suppose that for every $M > 0$, the function $\psi - \psi_M$ is plurisubharmonic outside a compact subset of $\mathbb{C}^n$. Then* $\dim \mathcal{F}^2_\psi = \infty$.

Proof. We use the fundamental result of Bedford–Taylor [2] on the solutions of the Dirichlet problem for the complex Monge–Ampère equation. Given $M > 0$, choose $r_M > 0$ such that $\psi - \psi_M$ is plurisubharmonic on $\mathbb{C}^n \setminus \overline{\mathbb{B}_n(0, r_M)}$. Solving the Dirichlet problem for the complex Monge–Ampère equation on $\mathbb{B}_n(0, r_M)$ with the boundary conditions $(\psi - \psi_M)|_{\partial \mathbb{B}_n(0,r_M)}$, we obtain a function u_M. Set

$$\tilde{\psi}_M(z) = \begin{cases} (\psi - \psi_M)(z), & z \in \mathbb{C}^n \setminus \mathbb{B}_n(0, r_M), \\ u_M(z), & z \in \mathbb{B}_n(0, r_M). \end{cases}$$

Then $\tilde{\psi}_M$ is a continuous plurisubharmonic function on $\mathbb{C}^n$ (see Theorem 3.7.7).

Now, by Theorem 3.7.5, there exists an entire function $f \not\equiv 0$ such that

$$\int_{\mathbb{C}^n} |f(z)|^2 (1 + |z|^2)^{-3n} e^{-\tilde{\psi}_M(z)} \, dV(z) < \infty.$$

Hence, for every $0 \le k \le M - \dfrac{3n}{2}$ we have

$$\begin{aligned}\int_{\mathbb{C}^n}|f(z)|^2|z|^{2k}e^{-\psi(z)}\,dV(z) &\le C+\int_{\mathbb{C}^n\setminus\mathbb{B}_n(0,r_M)}|f(z)|^2|z|^{2k}e^{-\psi(z)}\,dV(z)\\ &= C+\int_{\mathbb{C}^n\setminus\mathbb{B}_n(0,r_M)}|f(z)|^2|z|^{2k}e^{-\psi_M(z)}e^{-(\psi(z)-\psi_M(z))}\,dV(z)\\ &\le C+\int_{\mathbb{C}^n\setminus\mathbb{B}_n(0,r_M)}|f(z)|^2|z|^{-3n}e^{-\widetilde{\psi}_M(z)}\,dV(z)<\infty.\end{aligned}$$

Since M is arbitrary, we have $\dim\mathcal{F}^2_\psi=\infty$. □

Remark 6.3.2. Theorem 6.1.2 is an immediate corollary of Theorem 6.3.1.

Indeed, an easy computation shows that if $f(z)=\varphi(|z|^2)$, $\varphi\in C^2\big((0,+\infty)\big)$ then

$$\frac{\partial^2 f(z)}{\partial z_j\partial\bar z_k}=\varphi''(|z|^2)\bar z_j z_k+\varphi'(|z|^2)\delta_{jk},$$

where δ_{jk} is the Kronecker delta symbol. This implies that

$$i\partial\bar\partial f(z)=\varphi'(|z|^2)I+\varphi''(|z|^2)z^*z,$$

where $z^*=\begin{bmatrix}\bar z_1\\ \dots\\ \bar z_n\end{bmatrix}$, $z^*z=\begin{bmatrix}\bar z_j z_k\end{bmatrix}_{j,k=1}^n$. Note also that the spectrum of the matrix $i\partial\bar\partial f(z)$ is

$$\sigma\big(i\partial\bar\partial f(z)\big)=\Big\{\varphi'(|z|^2),\varphi'(|z|^2)+|z|^2\varphi''(|z|^2)\Big\}. \tag{6.5}$$

The first eigenvalue has multiplicity $n-1$ and the second one has multiplicity 1.

Furthermore,

$$\begin{aligned}L_\psi(z)=i\partial\bar\partial\psi(z)=i\partial\bar\partial(\psi-\psi_M)(z)+\frac{M}{|z|^2}I-\frac{M}{|z|^4}z^*z\\ =L_{\psi-\psi_M}(z)+\frac{M}{|z|^2}I-\frac{M}{|z|^4}z^*z.\end{aligned}$$

Let $z\in\mathbb{C}^n$ and let $V=\begin{bmatrix}V_1\\ \dots\\ V_n\end{bmatrix}$ be a normalized eigenvector corresponding to an eigenvalue ν of $L_{\psi-\psi_M}(z)$. By the hypothesis of Theorem 6.1.2, for $|z|>r_M$ we have

$\lambda_0|z|^2 \geq M$, where λ_0 is the smallest eigenvalue of $L_\psi(z)$. Thus,

$$\nu = \langle L_{\psi-\psi_M}(z)V, V\rangle = \langle L_\psi(z)V, V\rangle - \frac{M}{|z|^2} + \frac{M}{|z|^4}\langle z^* zV, V\rangle$$
$$\geq \lambda_0 - \frac{M}{|z|^2} + \frac{M}{|z|^4}|zV|^2 \geq 0.$$

Therefore, $\psi - \psi_M$ is plurisubharmonic on $\mathbb{C}^n \setminus \overline{\mathbb{B}_n(0, r_M)}$, and we are in the conditions of Theorem 6.3.1. □

Now we give an easy example when Theorem 6.3.1 applies while Theorem 6.1.2 does not work.

Example 6.3.3. Set

$$\psi(z) = \varphi(|z|^2) = \left(\log(1 + |z|^2)\right)^{\frac{3}{2}}, \qquad z \in \mathbb{C}^n.$$

Then $\varphi(t) = \left(\log(1 + t)\right)^{\frac{3}{2}}, t > 0$.

Evidently, $\dim \mathcal{F}^2_\psi = \infty$. We will show that the condition (6.2) fails for ψ while the conditions of Theorem 6.3.1 are satisfied.

We have

$$\varphi'(t) = \frac{3}{2}\frac{1}{1+t}\left(\log(1 + t)\right)^{\frac{1}{2}},$$

and

$$\varphi''(t) = -\frac{3}{2}\frac{\left(\log(1 + t)\right)^{\frac{1}{2}}}{(1+t)^2} + \frac{3}{4(1+t)^2\left(\log(1+t)\right)^{\frac{1}{2}}}.$$

By (6.5), the eigenvalues of the matrix $L_\psi(z)$ are

$$\lambda_1(z) = \frac{3\left(\log(1 + |z|^2)\right)^{\frac{1}{2}}}{2(1 + |z|^2)},$$

and

$$\lambda_2(z) = \frac{3\left(\log(1 + |z|^2)\right)^{\frac{1}{2}}}{2(1 + |z|^2)^2} + \frac{3|z|^2}{4(1 + |z|^2)^2\left(\log(1 + |z|^2)\right)^{\frac{1}{2}}}$$
$$= \frac{3}{4}\frac{2\log(1 + |z|^2) + |z|^2}{(1 + |z|^2)^2\left(\log(1 + |z|^2)\right)^{\frac{1}{2}}}.$$

For $|z| \geq 2$, the smallest eigenvalue of matrix $L_\psi(z)$ is $\lambda_2(z)$ and

$$\lim_{|z|\to\infty} |z|^2\lambda_2(z) = 0.$$

Hence the condition (6.2) does not hold.

On the other hand, for $M > 0$, the eigenvalues of matrix $L_{\psi-\psi_M}(z)$ are

$$\alpha_1(z) = \lambda_1(z) - \frac{M}{|z|^2},$$

and

$$\alpha_2(z) = \lambda_2(z).$$

Since $\lim_{|z|\to\infty} |z|^2\lambda_1(z) = \infty$ and $\alpha_2(z) > 0$, $z \neq 0$, the conditions of Theorem 6.3.1 are satisfied. $\square$

In the rest of this chapter we show that in different situations the sufficient condition of Theorem 6.3.1 is not necessary for $\dim \mathcal{F}^2_\psi = \infty$.

Example 6.3.4. Set

$$\psi(z, w) = |z|^2 + 2\log(1 + |w|^2), \qquad w, z \in \mathbb{C}.$$

It is clear that $\dim \mathcal{F}^2_\psi = \infty$. Let us verify that for $M > 2$ the function $\psi - \psi_M$ is not plurisubharmonic at the points $(1, w)$, $w \in \mathbb{C}$.

We start with some easy computations:

$$\frac{\partial\psi}{\partial z} = \overline{z}, \quad \frac{\partial^2\psi}{\partial z\partial\overline{z}} = 1, \quad \frac{\partial^2\psi}{\partial z\partial\overline{w}} = 0,$$
$$\frac{\partial\psi}{\partial w} = \frac{2\overline{\psi}}{1+|w|^2}, \quad \frac{\partial^2\psi}{\partial w\partial\overline{z}} = 0, \quad \frac{\partial^2\psi}{\partial w\partial\overline{w}} = \frac{2}{(1+|w|^2)^2}.$$

Now, given $M > 0$ we have

$$L_{\psi-\psi_M}(z, w) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{(1+|w|^2)^2} \end{bmatrix} + \frac{M}{(|z|^2+|w|^2)^2}\begin{bmatrix} |z|^2 & \overline{z}w \\ z\overline{w} & |w|^2 \end{bmatrix} - \frac{M}{|z|^2+|w|^2} I$$
$$= \begin{bmatrix} 1 - \frac{M|w|^2}{(|z|^2+|w|^2)^2} & \frac{M\overline{z}w}{(|z|^2+|w|^2)^2} \\ \frac{Mz\overline{w}}{(|z|^2+|w|^2)^2} & \frac{2}{(1+|w|^2)^2} - \frac{M|z|^2}{(|z|^2+|w|^2)^2} \end{bmatrix},$$

and, hence,

$$\begin{aligned}
\det\left(L_{\psi-\psi_M}(z,w)\right) \\
&= \frac{2}{(1+|w|^2)^2} - \frac{M|z|^2}{(|z|^2+|w|^2)^2} - \frac{2M|w|^2}{(1+|w|^2)^2(|z|^2+|w|^2)^2} \\
&= \frac{2(|z|^2+|w|^2)^2 - M\left(|z|^2(1+|w|^2)^2 + 2|w|^2\right)}{(1+|w|^2)^2(|z|^2+|w|^2)^2} < 0
\end{aligned}$$

for $M > 2, z = 1$ and arbitrary w. Therefore, the conditions of Theorem 6.3.1 do not hold. □

In the following examples we evaluate the dimension of $\mathcal{F}^2_\psi$ and the applicability of our criterion in Theorem 6.3.1, for some concrete weight functions ψ and for ψ in some special classes.

Example 6.3.5. Let $k \geq 3$. Set $\psi(z) = |z_1^k + z_2^k|^2, z = (z_1, z_2) \in \mathbb{C}^2$. Given $M > 0$, we have

$$L_{\psi-\psi_M}(z) = \begin{bmatrix} k^2|z_1|^{2(k-1)} - \frac{M}{|z|^4}|z_2|^2 & k^2(z_1\overline{z_2})^{k-1} + \frac{M}{|z|^4}\overline{z_1}z_2 \\ k^2(\overline{z_1}z_2)^{k-1} + \frac{M}{|z|^4}z_1\overline{z_2} & k^2|z_2|^{2(k-1)} - \frac{M}{|z|^4}|z_1|^2 \end{bmatrix},$$

and, hence,

$$\begin{aligned}
\det\left(L_{\psi-\psi_M}(z)\right) \\
&= \left(k^2|z_1|^{2(k-1)} - \frac{M}{|z|^4}|z_2|^2\right)\left(k^2|z_2|^{2(k-1)} - \frac{M}{|z|^4}|z_1|^2\right) \\
&\quad - \left(k^2(z_1\overline{z_2})^{k-1} + \frac{M}{|z|^4}\overline{z_1}z_2\right)\left(k^2(\overline{z_1}z_2)^{k-1} + \frac{M}{|z|^4}z_1\overline{z_2}\right) \\
&= -\frac{k^2M}{|z|^4}\left(|z_1|^{2k} + |z_2|^{2k} + (z_1\overline{z_2})^k + (\overline{z_1}z_2)^k\right) \\
&= -\frac{k^2M}{|z|^4}\,|z_1^k + z_2^k|^2 < 0
\end{aligned}$$

when $z_1^k + z_2^k \neq 0$. Thus, for $M > 0$, the function $\psi - \psi_M$ is not plurisubharmonic outside a compact subset of $\mathbb{C}^2$.

Next we are going to verify that $\dim \mathcal{F}^2_\psi = \infty$.

We have

$$X := \int_{\mathbb{C}^2} e^{-|z_1^k+z_2^k|^2} dV(z_1,z_2) \asymp 4\int_0^\infty \int_{\mathbb{S}_2} r^3 e^{-r^{2k}|\zeta_1^k+\zeta_2^k|^2} d\sigma(\zeta_1,\zeta_2)\, dr$$
$$\asymp \int_{\mathbb{S}_2} |\zeta_1^k+\zeta_2^k|^{-4/k} d\sigma(\zeta_1,\zeta_2).$$

Given $\varepsilon > 0$, we consider the set

$$T_\varepsilon = \left\{(\zeta_1,\zeta_2) \in \mathbb{S}_2 : |\zeta_1^k+\zeta_2^k| < \varepsilon\right\}.$$

Given $(\zeta_1,\zeta_2) \in \mathbb{S}_2$ such that $|\zeta|_1 \ge |\zeta|_2$, set $\zeta_1 = \sqrt{\frac{1}{2}+r}\cdot e^{i\theta}$ and $\zeta_2 = \sqrt{\frac{1}{2}-r}\cdot e^{i\varphi}$, $r \ge 0$. If $(\zeta_1,\zeta_2) \in T_\varepsilon$, then $|\zeta_1|^2 - |\zeta_2|^2 < C\varepsilon$ for some constant $C = C(k) > 0$. Hence, $r \lesssim \varepsilon$. Next, since $|\zeta_1^k+\zeta_2^k| \lesssim \varepsilon$, we obtain $|\theta-\varphi| < C\varepsilon$. As a result, we obtain that

$$\sigma(T_\varepsilon) \lesssim \varepsilon^2.$$

Set

$$U_s = \left\{(\zeta_1,\zeta_2) \in \mathbb{S}_2 : 2^{-s} < |\zeta_1^k+\zeta_2^k| \le 2^{-s+1}\right\}.$$

Then

$$X \asymp \sum_{s=0}^\infty \int_{U_s} |\zeta_1^k+\zeta_2^k|^{-4/k} d\sigma(\zeta_1,\zeta_2) \lesssim \sum_{s=0}^\infty 2^{-2s}\, 2^{4s/k} = \sum_{s=0}^\infty 2^{-2s(1-(2/k))} < \infty$$

since $k \ge 3$. Thus $1 \in \mathcal{F}^2_\psi$.

In the same way, for every $\alpha > 0$ we get

$$\int_{\mathbb{C}^2} e^{-\alpha|z_1^k+z_2^k|^2} dV(z) < \infty.$$

Consider the entire function $f(z) = e^{\beta(z_1^k+z_2^k)^2}$, $0 < \beta < \frac{1}{2}$. Since

$$\int_{\mathbb{C}^2} |e^{\beta(z_1^k+z_2^k)^2}|^2 e^{-|z_1^k+z_2^k|^2}\, dV(z) = \int_{\mathbb{C}^2} e^{2\beta\,\mathrm{Re}((z_1^k+z_2^k)^2)} e^{-|z_1^k+z_2^k|^2}\, dV(z)$$
$$\le \int_{\mathbb{C}^2} e^{-(1-2\beta)|z_1^k+z_2^k|^2}\, dV(z) < \infty,$$

we conclude that $\dim \mathcal{F}^2_\psi = \infty$. □

Remark 6.3.6. Interestingly, $\mathcal{F}^2_\psi = 0$ if $k = 2$. Indeed, let $\psi(z_1, z_2) = |z_1^2 + z_2^2|^2$, $f \in \mathcal{F}^2_\psi$, $f(z_1, z_2) = (z_1^2 + z_2^2)^s g(z_1, z_2)$ for some $s \ge 0$, where $g(z_1, z_2)$ is not a multiple of $z_1^2 + z_2^2$. By the mean value property, for every $z_1 \in \mathbb{C} \setminus D(0, 10)$ we have

$$\begin{aligned}
&|g(z_1, iz_1)|^2 \\
&\quad \lesssim (1 + |z_1|)^2 \int_{D(iz_1, 2/(1+|z_1|)) \setminus D(iz_1, 1/(1+|z_1|))} |g(z_1, z_2)|^2 e^{-|z_1^2+z_2^2|^2}\, dv(z_2) \\
&\qquad \lesssim (1 + |z_1|)^2 \int_{D(iz_1, 2/(1+|z_1|)) \setminus D(iz_1, 1/(1+|z_1|))} |f(z_1, z_2)|^2 e^{-|z_1^2+z_2^2|^2}\, dv(z_2).
\end{aligned}$$

Hence,

$$\int_{\mathbb{C}} |g(z_1, iz_1)|^2 (1 + |z_1|)^{-2}\, dv(z_1) \lesssim \|f\|^2_\psi,$$

and by a Liouville type theorem, $g(z, iz) \equiv 0$. Analogously, $g(z, -iz) \equiv 0$. Set $h(z, w) = g(z - iw, iz - w)$. Then h is an entire function and $h(0, w) = h(w, 0) \equiv 0$. Hence, $h(z, w) = zwh_1(z, w)$ for another entire function h_1 and $g(z_1, z_2) = (z_1^2 + z_2^2) g_1(z_1, z_2)$ for some entire function g_1. This contradiction shows that $\mathcal{F}^2_\psi = 0$.

Extending the previous example to $\mathbb{C}^n$ with $n \ge 3$ requires a bit more work.

Example 6.3.7. Let $n \ge 3$, $k \ge n + 1$. Set

$$\psi(z) = |z_1^k + \cdots + z_n^k|^2, \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Let us verify that for $M > 0$, the function $\psi - \psi_M$ is not plurisubharmonic outside a compact subset of $\mathbb{C}^n$.

Set

$$A(z) = \frac{M}{|z|^4} \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \\ \vdots \\ \overline{z_n} \end{bmatrix} \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}.$$

We have

$$L_\psi(z) = k^2 \begin{bmatrix} |z_1|^{2(k-1)} & (z_1\overline{z_2})^{k-1} & \cdots & (z_1\overline{z_n})^{k-1} \\ (\overline{z_1}z_2)^{k-1} & |z_2|^{2(k-1)} & \cdots & (z_2\overline{z_n})^{k-1} \\ \vdots & \vdots & \cdots & \vdots \\ (\overline{z_1}z_n)^{k-1} & (\overline{z_2}z_n)^{k-1} & \cdots & |z_n|^{2(k-1)} \end{bmatrix}$$

$$= k^2 \begin{bmatrix} z_1^{k-1} \\ z_2^{k-1} \\ \vdots \\ z_n^{k-1} \end{bmatrix} \begin{bmatrix} \overline{z_1}^{k-1} & \overline{z_2}^{k-1} & \cdots & \overline{z_n}^{k-1} \end{bmatrix}.$$

Then

$$L_{\psi-\psi_M}(z) = L_\psi(z) + A(z) - \frac{M}{|z|^2} I.$$

The spectra of the matrices $L_\psi(z)$ and $A(z)$ are

$$\sigma_{L_\psi(z)} = \left\{ k^2\left(|z_1|^{2(k-1)} + |z_2|^{2(k-1)} + \cdots + |z_n|^{2(k-1)}\right), 0 \right\},$$

$$\sigma_{A(z)} = \left\{ \frac{M}{|z|^2}, 0 \right\}.$$

Let V be a unit vector in $\mathbb{C}^n$ orthogonal to $\begin{bmatrix} z_1^{k-1} \\ z_2^{k-1} \\ \vdots \\ z_n^{k-1} \end{bmatrix}$ and to $\begin{bmatrix} \overline{z_1} \\ \overline{z_2} \\ \vdots \\ \overline{z_n} \end{bmatrix}$. Then

$$\langle L_{\psi-\psi_M}(z)V, V\rangle = \langle L_\psi(z)V + A(z)V - \frac{M}{|z|^2}V, V\rangle = -\frac{M}{|z|^2} < 0.$$

Thus, for $M > 0$, the function $\psi - \psi_M$ is plurisubharmonic at no points of $\mathbb{C}^n \setminus \{0\}$.

Finally, let us verify that $\dim \mathcal{F}^2_\psi = \infty$. Set

$$X := \int_{\mathbb{C}^n} e^{-|z_1^k+\cdots+z_n^k|^2}\, dV(z)$$

$$\asymp \int_0^\infty \int_{\mathbb{S}_n} r^{2n-1} e^{-r^{2k}|\zeta_1^k+\cdots+\zeta_n^k|^2} d\sigma(\zeta_1, \cdots, \zeta_n)\, dr$$

$$\asymp \int_{\mathbb{S}_n} |\zeta_1^k + \cdots + \zeta_n^k|^{-2n/k} d\sigma(\zeta_1, \cdots, \zeta_n).$$

Given $\varepsilon > 0$, we consider the set

$$T_\varepsilon = \left\{(\zeta_1, \cdots, \zeta_n) \in \mathbb{S}_n : |\zeta_1^k + \cdots + \zeta_n^k| < \varepsilon\right\}.$$

Set

$$P(z) = \sum_{j=1}^{n} z_j^k, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Then the function $f = \log|P|$ is plurisubharmonic. We calculate the Lelong number of f at $a \in \mathbb{C}^n$,

$$\nu_f(a) = \lim_{r \to 0} \frac{\sup_{|z| \le r} f(a+z)}{\log r} \in [0, \infty].$$

If $f(a) \neq 0$, then $\nu_f(a) = 0$. Otherwise, let $a = (a_1, \dots, a_n) \neq 0$ and $f(a) = 0$. Without loss of generality, we can assume $a_1 \neq 0$. If $0 < r < \frac{|a_1|}{2}$, then

$$f\Big(a + (r, 0, \dots, 0)\Big) = \log|(a_1 + r)^k - a_1^k| = \log|k a_1^{k-1} r + O(r^2)|, \qquad r \to 0,$$

and hence, $\nu_f(a) = 1$. By Theorem 3.8.4, applied to $\Omega = 2\mathbb{B}_n$, $K = \overline{\mathbb{B}_n} \setminus \frac{1}{2}\mathbb{B}_n$, $1 < \alpha < 2$, we obtain

$$\begin{aligned} v\Big(\{z \in K : |P(z)| \le e^{-u}\}\Big) &= v\Big(\{z \in K : f(z) \le -u\}\Big) \\ &\le C_\alpha e^{-\alpha u}, \qquad u \ge 0. \end{aligned}$$

By homogeneity of P,

$$\sigma(T_\varepsilon) \le C\varepsilon^\alpha, \qquad \varphi > 0,$$

for some constant $C > 0$.

Arguing as in Example 6.3.5, we obtain first that $1 \in \mathcal{F}^2_\psi$ and then that $\dim \mathcal{F}^2_\psi = \infty$ for $k \ge n + 1$. □

At the end of this chapter, we consider two special classes of weight functions ψ: radial weight functions and the functions of the form $\psi(z_1, \dots, z_n) = \sum_{j=1}^n \psi_j(z_j)$.

Suppose that $\psi(z) = \varphi(|z|^2)$ is a radial plurisubharmonic function of class C^2. By the computations in Remark 6.3.2,

$$\frac{\partial^2\psi}{\partial z_j \partial \overline{z_k}}(z) = \varphi''(|z|^2)\overline{z_j} z_k + \varphi'(|z|^2)\delta_{jk}. \tag{6.6}$$

The action of the Monge–Ampère operator on ψ is

$$\begin{aligned}(dd^c\psi)^n &= 4n! \det\Big(\frac{\partial^2\psi}{\partial z_j \partial \overline{z_k}}\Big)\, dV \\ &= 4n!(\varphi'(|z|^2))^{n-1}(\varphi'(|z|^2) + |z|^2\varphi''(|z|^2))\, dV.\end{aligned}$$

Proposition 6.3.8. *Suppose that* $\psi(z) = \varphi(|z|^2)$ *is a radial plurisubharmonic function of class* C^2. *Then* $\dim \mathcal{F}^2_\psi = \infty$ *if and only if*

$$\int_{\mathbb{C}^n} (dd^c\psi)^n = \infty. \tag{6.7}$$

Proof. Since the spectrum of the matrix (6.6) consists of the eigenvalues $\varphi'(|z|^2)$ and $\varphi'(|z|^2) + |z|^2\varphi''(|z|^2)$, the first eigenvalue has multiplicity $n-1$ and the second one has multiplicity 1, we have $\varphi' \geq 0$, $\big(r\varphi'(r)\big)' \geq 0$ on $\mathbb{R}_+$. Furthermore, we have

$$\begin{aligned}\int_{\mathbb{C}^n} (dd^c\psi)^n = C\int_0^\infty \big(\varphi'(r^2)\big)^{n-1}\big(\varphi'(r^2) + r^2\varphi''(r^2)\big)\, dr^{2n} \\ = C\int_0^\infty d\Big(\big(r\varphi'(r)\big)^n\Big).\end{aligned}$$

Thus, (6.7) is equivalent to the relation $\lim_{r\to\infty} r\varphi'(r) = \infty$. Now, if $r\varphi'(r)$ is bounded on $\mathbb{R}_+$, then $\psi(z) = O(\log|z|)$, $|z| \to \infty$, and a version of the Liouville theorem shows that $\dim \mathcal{F}^2_\psi < \infty$. On the other hand, if $\lim_{r\to\infty} r\varphi'(r) = \infty$, then $\log|z| = o\big(\psi(z)\big)$, $|z| \to \infty$, and the polynomials belong to $\mathcal{F}^2_\psi$. Hence, $\dim \mathcal{F}^2_\psi = \infty$. □

For general C^2 plurisubharmonic functions, the radial case suggests the following question. Is it true that $\dim \mathcal{F}^2_\psi = \infty$ if and only if (6.7) holds? Our last example gives a negative answer to this question.

Example 6.3.9. Given subharmonic functions ψ_j on the complex plane, $1 \le j \le n$, set

$$\psi(z_1, \dots, z_n) = \sum_{j=1}^{n} \psi_j(z_j). \tag{6.8}$$

Claim: $\dim \mathcal{F}^2_\psi < \infty$ if and only if either $\max_j \dim \mathcal{F}^2_{\psi_j} < \infty$ or $\min_j \dim \mathcal{F}^2_{\psi_j} = 0$.

In one direction, by the Fubini theorem, if $\dim \mathcal{F}^2_\psi < \infty$, then $\max_j \dim \mathcal{F}^2_{\psi_j} < \infty$ or $\min_j \dim \mathcal{F}^2_{\psi_j} = 0$. In the opposite direction, it is clear that if $\min_j \dim \mathcal{F}^2_{\psi_j} = 0$, then $\mathcal{F}^2_\psi = 0$. It remains to verify that if $\max_j \dim \mathcal{F}^2_{\psi_j} < \infty$, then $\dim \mathcal{F}^2_\psi < \infty$.

First, suppose that $n = 2$, $\dim \mathcal{F}^2_{\psi_1} < \infty$, $N = \dim \mathcal{F}^2_{\psi_2} < \infty$. Fix a basis (g_k), $1 \le k \le N$, in the space $\mathcal{F}^2_{\psi_2}$ and choose a family of points (w_m), $1 \le m \le N$, such that $\det Q \neq 0$, where $Q = \Big(g_k(w_m)\Big)_{k,m=1}^{N}$.

Next, choose $f \in \mathcal{F}^2_\psi$. By the mean value property,

$$|f(z, w)|^2 \le \frac{1}{\pi} \int_{D(z,1)} |f(\zeta, w)|^2 \, dV(\zeta), \qquad z, w \in \mathbb{C}.$$

Therefore, for every $z \in \mathbb{C}$, the function $f(z, \cdot)$ belongs to $\mathcal{F}^2_{\psi_2}$, and, hence, we have

$$f(z, \cdot) = \sum_{k=1}^{N} a_k(z) g_k.$$

In the same way, the functions $f(\cdot, w_j)$, $1 \le j \le N$, belong to $\mathcal{F}^2_{\psi_1}$.

Next,

$$Q^{-1} \begin{bmatrix} F(z, w_1) \\ \vdots \\ F(z, w_N) \end{bmatrix} = \begin{bmatrix} a_1(z) \\ \vdots \\ a_N(z) \end{bmatrix}.$$

Hence, every a_j belongs to $\mathcal{F}^2_{\psi_1}$. Since $\dim \mathcal{F}^2_{\psi_1} < \infty$, we conclude that the space $\mathcal{F}^2_\psi$ has finite dimension. For $n \ge 2$ we can just use an inductive argument. This completes the proof of Claim.

Let us return to general ψ satisfying (6.8). Then

$$\int_{\mathbb{C}^n} (dd^c\psi)^n = C \int_{\mathbb{C}^n} \prod_{j=1}^{n} \Delta\psi_j(z_j) \, dV(z) = C \prod_{j=1}^{n} \int_{\mathbb{C}} \Delta\psi_j(z_j) \, dV(z_j).$$

Now, if $n = 2$, $\psi_1(z) = |z|^2$, $\Delta\psi_2(z) = \max(1 - |z|, 0)$, then

$$\int_{\mathbb{C}^n} (dd^c\psi)^n = \infty,$$

but $\mathcal{F}^2_\psi = 0$. Thus, Proposition 6.3.8 does not extend to general C^2-smooth plurisubharmonic functions.

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Résumé: Nous étudions les mesures de Carleson et les opérateurs de Toeplitz sur la classe des espaces de Bergman dite de petite taille, introduits récemment par Seip. On obtient une caractérisation des mesures de Carleson qui étend les résultats de Seip à partir du disque unité de $\mathbb{C}$ à la boule unité $\mathbb{B}_n$ de $\mathbb{C}^n$. Nous utilisons cette caractérisation pour donner les conditions nécessaires et suffisantes à la continuité et à la compacité des opérateurs de Toeplitz. Enfin, nous étudions l'appartenance des opérateurs Toeplitz aux classes de Schatten d'ordre p pour $1 < p < \infty$.

De plus, nous considérons également la projection de type Bergman agissant sur L^∞ à valeurs dans l'espace de Bloch $\mathcal{B}$ de la boule $\mathbb{B}_n$. Une caractérisation du poids radial pour que la projection soit continue est obtenue.

Enfin, nous examinons les espaces de Fock pondérés en une et plusieurs variables complexes. Nous évaluons la dimension de ces espaces en étendant et en complétant des résultats antérieurs obtenus par Rozenblum–Shirokov et Shigekawa.

Abstract: We study the Carleson measures and the Toeplitz operators on the class of the so-called small weighted Bergman spaces, introduced recently by Seip. A characterization of Carleson measures is obtained which extends Seip's results from the unit disk of $\mathbb{C}$ to the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. We use this characterization to give necessary and sufficient conditions for the boundedness and compactness of Toeplitz operators. Finally, we study the Schatten p classes membership of Toeplitz operators for $1 < p < \infty$.

Furthermore, we also consider the Bergman type projection acting on L^∞ to the Bloch space $\mathcal{B}$ on $\mathbb{B}_n$. A characterization of radial weight so that the projection is bounded is obtained.

Finally, we investigate the weighted Fock spaces in one and several complex variables. We evaluate the dimension of these spaces in terms of the weight function extending and completing earlier results by Rozenblum–Shirokov and Shigekawa.