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Aspects of Quantum Gravity

Aspects de Gravitation Quantique

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Résumé

La nature de la gravite quantique est une question ouverte importante en physique fondamentale dont la résolution nous permettrait de comprendre la structure la plus profonde de l'espace-temps et de la matière.

Cependant, jusqu'a' présent, il n'y a pas de solution complète a' la question de la gravité quantique, malgré de nombreux efforts et tentatives. La gravitation quantique en boucle est une approche particulière de la gravitation quantique indépendante du fond, inspirée par une formulation de la relativité générale en tant que théorie dynamique des connexions. La théorie contient deux branches, l'approche canonique et l'approche de mousses de spin. La manière canonique est basée sur la formulation hamiltonienne de l'action de premier ordre de la relativité générale suivant une quantification à la Dirac sur des algèbres de flux, tandis que les modelés de mousses de spin se présentent comme une formulation covariante de la gravite quantique, définie comme un modelé de somme d'état sur graphiques.

Cette thèse concerne principalement, sans toutefois s'y limiter, le problème de la gravite quantique dans le contexte de la gravite quantique en boucle. Les deux aspects fondamentaux et les conséquences physiques de la gravite à boucles sont étudies dans ce travail. La description de la gravite en termes de groupe de jauge non compacte $\mathfrak{su}(1,1)$ est étudiée a' la fois de manière canonique et sous forme de mousse de spin. Nous étudions l'invariance de Lorentz de la gravite quantique de la boucle, a' la fois dans l'approche canonique et dans le modelé de mousse de spin. Nous présentons une description de jauge $\mathfrak{su}(1,1)$ de la théorie de la gravite en quatre dimensions, au lieu de la description habituelle $\mathfrak{su}(2)$. Nous étudions la quantification de boucle au niveau cinématique, ou' nous avons constate que les aires de type espace ont des spectres discrets, tandis que les aires de type temps ont des spectres continus. Pour les aires de type espace, il n'y a pas d'anomalie entre la description $\mathfrak{su}(1,1)$ et la description $\mathfrak{su}(2)$. Dans le même temps, nous effectuons l'analyse semi-classique (asymptotique pour grand j) du modelé de mousse de spin de Conrady-Hnybida dans une situation très générale dans laquelle des tétraèdres de type temps avec des triangles de type temps sont pris en compte. Nous identifions la contribution dominante avec des géométries simplicales discrètes et nous retrouvons l'action de gravite de Regge.

Dans une seconde partie de cette thèse, je me suis penche sur le problème de la dynamique effective de la gravitation à boucle a' haute énergie en cosmologie et dans le contexte de la physique de trous noirs. Nous étudions le lien entre la gravite mimétique étendue, une classe de théories scalaires-tenseurs, et la dynamique effective de la cosmologie quantique à boucles ainsi que les modèles de trous noirs polymères sphériques inspirés de la gravite quantique à boucles. La comparaison entre les formulations mimétiques et hamiltoniennes de polymère nous fournit un guide pour comprendre l'absence de covariance dans les modèles de polymères non homogènes. En attendant, nous résolvons explicitement les équations d'Einstein modifiées dans le cadre de modèles de polymères effectifs a' symétrie sphérique. La métrique effective pour une géométrie de trou noir intérieure statique décrivant la région piégée est donnée. Les résultats obtenus dans cette partie ont des implications intéressantes pour la cosmologie sans singularité et les trous noirs qui méritent d'être approfondies.

Mots-clés : gravité quantique, gravité quantique en boucle, modèles de mousse de spin, limite semi-classique, dynamique effective, trous noirs, cosmologie quantique en boucle, gravité mimétique, théories scalaires-tenseurs

Abstract

The nature of quantum gravity is an important open question in fundamental physics whose resolution would allow us to understand the deepest structure of space-time and matter. However, up to now there is no complete solution to the question of quantum gravity, despite many efforts and attempts. Loop quantum gravity is a tentative background independent approach to quantum gravity inspired by a formulation of general relativity as a dynamical theory of connections. The theory contains two approaches, the canonical approach and the spin foam approach. The canonical way is based on the Hamiltonian formulation of the first order action of pure gravity following a dirac quantization of holonomy-flux algebras, whereas the spin foam model arises as a covariant formulation of quantum gravity, which is defined as a state sum model over certain graphs.

This thesis mostly involves, but not restricts to, the problem of quantum gravity in the context of loop quantum gravity. Both fundamental aspects and the physical consequences of loop gravity are investigated in this work. The description of gravity in terms of a non-compact gauge group $\mathfrak{su}(1,1)$ is studied in both the canonically as well as in the spin foam approach. We study the Lorentzian invariance of loop quantum gravity, in both the canonical approach and the spin foam model approach. We introduce an $\mathfrak{su}(1,1)$ gauge description of gravity theory in four dimensions, instead of the usual $\mathfrak{su}(2)$ description. We investigate the loop quantization at the kinematical level, where we find that space-like areas have discrete spectra, whereas time-like areas have continuous spectra. And we show that there is no anomaly between the $\mathfrak{su}(1,1)$ description and the $\mathfrak{su}(2)$ description of space-like areas. Meanwhile, we perform the semi-classical (large-j asymptotic) analysis of the spin foam model (Conrady-Hnybida extension) in the most general situation, in which timelike tetrahedra with timelike triangles are taken into account. We identify the dominant contribution to the discrete simplicial geometries and recover the Regge action of gravity.

On a second part of this thesis we focus on the problem of the high energy effective dynamics of loop gravity in cosmology and black holes through simplified models. We investigate the link between the family of extended Mimetic gravity, a class of scalar-tensor theories, and the effective dynamics of loop quantum cosmology as well as the spherical polymer black hole models inspired from loop quantum gravity. This comparison provides us with a guide to understand the absence of covariance in inhomogeneous polymer models. Futhermore, we solve the modified Einstein's equations explicitly in the framework of effective spherically symmetric polymer models. The effective metric for a static interior

Black Hole geometry describing the trapped region is given. The results obtained in this part lead to some interesting implications for singularity free cosmology and black holes which are worth pursuing further.

Keywords: quantum gravity, loop quantum gravity, spin foam models, semi-classical limit, effective dynamics, black holes, loop quantum cosmology, mimetic gravity, scalar-tensor theories

List of Publications

The thesis is based on the following papers, which are the result of my own work and of collaborations during my doctoral studies at Aix-Marseille University since Oct. 2016. [a]

1. "Asymptotic analysis of spin foam amplitude with timelike triangles"
   H. Liu and M. Han.
   Phys. Rev. D **99**, no. 8, 084040 (2019), arXiv:1810.09042 [gr-qc]
2. "Polymer Schwarzschild black hole: An effective metric"
   J. Ben Achour, F. Lamy, **H. Liu** and K. Noui.
   EPL **123**, no. 2, 20006 (2018), arXiv:1803.01152 [gr-qc]
3. "Non-singular black holes and the Limiting Curvature Mechanism: A Hamiltonian perspective"
   J. Ben Achour, F. Lamy, **H. Liu** and K. Noui.
   JCAP **1805**, no. 05, 072 (2018), arXiv:1712.03876 [gr-qc]
4. "Effective loop quantum cosmology as a higher-derivative scalar-tensor theory"
   D. Langlois, **H. Liu**, K. Noui and E. Wilson-Ewing.
   Class. Quant. Grav. **34**, no. 22, 225004 (2017), arXiv:1703.10812 [gr-qc]
5. "Gravity as an $\mathfrak{su}(1,1)$ gauge theory in four dimensions"
   H. Liu and K. Noui.
   Class. Quant. Grav. **34**, no. 13, 135008 (2017) ,arXiv:1702.06793 [gr-qc]

The following recent work and early investigations will not be directly addressed in this thesis:

1. "Stealth Schwarzschild-(A)dS black hole in DHOST theories after GW170817: Linear time-dependent scalar dressing"
   J. B. Achour and **H. Liu**.
   Phys. Rev. D **99**, no. 6, 064042 (2019), arXiv:1811.05369 [gr-qc]
2. "Semiclassical analysis of black holes in loop quantum gravity: Modeling Hawking radiation with volume fluctuations"
   P. Heidmann, **H. Liu** and K. Noui.
   Phys. Rev. D **95**, no. 4, 044015 (2017), arXiv:1612.05364 [gr-qc]

a. The complete list of publications can be found on INSPIRE HEP: http://inspirehep.net/author/profile/Hong.Guang.Liu.1.
All publications listed here are peer-reviewed papers.

3. “Distinguishing f(R) theories from general relativity by gravitational lensing effect”
H. Liu, X. Wang, H. Li and Y. Ma.
Eur. Phys. J. C **77**, no. 11, 723 (2017), arXiv:1508.02647 [gr-qc]

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Contents

Introduction

The problem of quantum gravity

Nowadays physics is mainly based on two different fundamental theories: General relativity (GR), and quantum physics (QP). Up to now, both theories are examined in their own way, by various experiments. On one hand, the Standard Model [1], which is a quantum field theory based on the general framework of quantum physics, successfully describes all particles and interactions we have so far directly observed (except gravity) [2]. On the other hand, general relativity, representing the gravitational laws as curved spacetime geometry, fulfills our observations on the large scale structure, for example, the precession of the perihelion of Mercury, the gravitational waves, and the gravitational lensing effect, etc. For a review of tests of GR we refer to [3].

Both theories have their own schemes: the quantum theories, related to the Planck constant $\hbar$, describe the quantum behavior at very small scales, where the $\hbar \to 0$ limit going to it's classical limit. On the contrary, GR describes the large scale spacetime dynamics, specified by the gravitational constant G, where $G \to 0$ is the asymptotic flat limit. The regime, where both gravity and quantum effects become strong, is called the "quantum gravity" scale. The scale related to such regime is usually referred to the Planck length

$$l_p = \sqrt{\hbar G} \tag{0.1}$$

which contains both coupling constant $\hbar$ and G. Note that here we take the light speed $c = 1$. The energy scale corresponding to such regime, is the Planck energy

$$E_p = \sqrt{\frac{\hbar}{G}} \sim 10^{19}\mathrm{Gev} \tag{0.2}$$

A natural question one may ask is that whether these two theories are compatible with each other on such "quantum gravity" scale. Actually such idea was born just after the built of GR and QP when we want to unify these two theories in an elegant way. Recently, there are growing number of experimental and theoretical reasons for a quantum gravity theory. Even through the planck energy is far beyond the energy scale we can produce, quantum gravity effect can appear at cosmological level, where we are collecting more and more observational data. For example, the very first moment of the Universe is expected to be understood with such a theory [4]. The dark energy [5] and inflation of the universe [6] are also closely related issues. Moreover, general relativity suffers from the presence of a singularity in which the theory ceases to be predictive, for example, inside the black hole, implying that Einstein's gravity is not a complete theory [7]. It is then expected a quantum gravity theory will allow us to resolve these singularities and explain observations.

However, up to date there is no complete answer to the question of quantum gravity, despite many efforts and attempts. Conceptually, GR describes the dynamics of the spacetime geometry itself, while quantum field theory assumes a fixed non-dynamical background theory, ignoring the back reaction between geometry and energy. For a consistent theory, the spacetime geometry should also inherit a quantum behavior. This implies a new understanding of the geometry and field theory. A common argument indicated by many theoretical hints is that, the continuous classical geometry is not fundamental, but actually emerges from quantum fundamental entitles obeying quantum dynamics, in an appropriate continuum limit, e.g. [8]. Thus, along this line, the main research scheme of quantum gravity can be summarized as the following

— Derive the fundamental theory: Identify the fundamental entitles and symmetries and define their quantum dynamics.
— Find the continuum limit (Renormalization of the theory). Describe the emergence of continuous geometry and fields from fundamental entitles.
— Extract possible predictions from the theory and check their consistency with known observations.

However, each step remains tough and open problems with a lot of technical difficulties due to the non-linear and the non Lie algebraic nature of the constraint system of GR, which is far different from QP.

There has been tremendous progress with various methods trying to attempt the quantization of gravity. They can be divided into two strategies: perturbative approach or non-perturbative approach. The perturbative approach assumes a background metric which is the classical solution of Einstein equation, and quantizes, by the standard methods, the perturbation modes which is the spin two particle called the graviton, as in e.g. asymptotic safety [9] and string theory. However, such approach suffers from the renomalizablity problems which requires a modification to the original gravity theory, e.g. to introduce extra symmetries such as supersymmetry [10]. The non-perturbative approach, instead, tries to give a background independent theory by keeping the general covariance of general relativity and quantize the whole spacetime. Examples include the causal dynamical triangulation [11, 12], quantum regge calculus [13, 14], causal sets [15], group field theory [16] and loop quantum gravity. However, it is difficult to choose an appropriate continuum limit in this approach such that classical GR emerges as its effective dynamics. For a general review of all these approaches, we refer to [17–19].

Over the last thirty years, two theories have distinguished themselves in this quest: string theory [20–23] and loop quantum gravity (LQG) [24–28]. They have each, in their own way, allowed us to describe the intimate nature of spacetime and to resolve the singularities. And they already lead to valuable results which change our usual notion of the spacetime. However, they are very different in how to approach the issue. For string theory, gravity is an effective theory that needs to be modified in order to bring it into the paradigm of quantum field theory. Conversely, for LQG, gravity is a fundamental theory and it is necessary to modify the quantification rules in order to adapt to GR. Both of them are based on solid and consistent mathematics. Thus, without experiments or observations, it is very difficult to refute one or the other of the approaches. It is hoped, however, to find methods to verify or refute them.

Loop quantum gravity and spin foam models

The thesis topic revolves, but not restricts to, the problem of quantum gravity in the context of the LQG. LQG is a background independent approach to the quantum gravity problem inspired by GR as a dynamical theory of connections. Its canonical approach is based on the Hamiltonian formulation of the first order gravity action [b] [24–26], where the quantization is performed on holonomy-flux algebras. A well-defined and consistent way to represent the constraints as operators on kinematical Hilbert space $\mathcal{H}_{kin}$ was developed, which leads to quantum constraint equations (quantum Einstein equation) and finally the physical Hilbert space $\mathcal{H}_{phy}$.

The spin foam model (SFM) arises as a covariant approach to quantum gravity, for a review, see [27, 29]. It is defined as a state sum model over some certain graphs, where a spin foam can be regarded as a Feynmann diagram with 5-valent vertices, corresponding to the quantum 4-simplices, as building blocks of the discrete quantum spacetime. SFM is proved to be closely related to topological quantum field theories (TQFT) and tensor networks (TN) .

Inspired by the LQG, similar loop quantization methods has been employed in symmetry reduced models, such as loop quantum cosmology (LQC) [30–33] and spherical polymer black hole models [34–37]. These symmetry reduced theories serve as a beautiful laboratory for testing the quantization procedure used in full theory. Meanwhile, they build the link to observations, for example, they bring in very interesting results with bouncing cosmology and singularity resolution of black holes. Moreover, there are other possible observational effects and predictions of the theory. Example include the Bekenstein-Hawking black hole entropy [38–41], the possible existence of Planck star [42–44], and quantum superpositions of spacetime as well as discrete time which may be detectable at a laboratory [45, 46]. However, generally speaking, the renormalization and continuum limit of the theory are still open questions, both for canonical approach and spin foam models, despite many recent investigations, e.g. [8, 47–53].

b. The theory can also be build from an extension of the ADM phase space which derived form the second order action of gravity

Summary of the thesis

The thesis is based on different works by me and my collaborators [54–59]. It is divided into two parts covering the canonical approach, the spinfoam models and the semi-classical symmetry reduced models for cosmology and black holes. Each part will begin with a short review chapter, followed by the related works. Here we give a brief summary of the contents in each part.

In part I, we concentrate on the covariant SL$(2,\mathbb{C})$ description of the loop quantum gravity and its semi-classical analysis. This part enrolls both the canonical approach and the spin foam model approach. There are two chapters after a very brief review to the canonical loop quantum gravity and the spin foam models. As a summary,

— In chapter 2 we present an $\mathfrak{su}(1,1)$ gauge description of gravity theory in four dimensions. A partial gauge-fixing is made to the Hamiltonian formulation of the first order action of pure gravity [60] which reduces $\mathfrak{sl}(2,\mathbb{C})$ to its sub-algebra $\mathfrak{su}(1,1)$. This case corresponds to a splitting of the spacetime $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ inherits an arbitrary Lorentzian metric. As a result, a parametrization of the phase space in terms of an $\mathfrak{su}(1,1)$ commutative connection and its associated conjugate electric field is found. A loop quantization is then discussed where the kinematical Hilbert space is on a given fixed graph Γ whose edges are colored with unitary representations of $\mathfrak{su}(1,1)$. It turns out that space-like areas have discrete spectra, whereas time-like areas have continuous spectra.

— In chapter 3, the semi-classical behavior of 4-dimensional spin foam amplitude is investigated for the extended spin foam model (Conrady-Hnybida extension) [61, 62] on a simplicial complex. The most general situation is under consideration, in which timelike tetrahedra with timelike triangles are taken into account. It turns out that the large j asymptotic behavior of such SFM is determined by critical configurations of the amplitude. The critical configurations that correspond to the Lorentzian simplicial geometries with timelike tetrahedra and triangles are identified. Their contributions to the amplitude are phases asymptotically, whose exponents equal to Regge action of gravity. It turns out that, if each 4-simplex contains both timelike and spacelike tetrahedra, the critical configurations will correspond to non-degenerate Lorentzian Regge geometries only, which excluds non-Regge like geometries appearing in EPRL/FK models [63, 64].

In Part II we concentrate on the effective dynamics of the symmetry reduced models in LQG. It contains three chapters after a brief review to scalar tensor theories and symmetry reduced models (loop quantum cosmology and spherical symmetric polymer models) inspired from LQG.

— In chapter 5, we reproduce the loop quantum cosmology (LQC) [65] effective dynamics with a recently introduced higher-derivative scalar-tensor theory (Mimetic gravity introduced by Chamseddine and Mukhanov). The theory leads to a modified Friedmann equation allowing for bouncing solutions [66]. As we note in the present work, this Friedmann equation turns out to reproduce exactly the loop quantum cosmology effective dynamics for a flat isotropic and homogeneous space-time. Here this result is generalized to a class of scalar-tensor theories, belonging to the family of mimetic gravity, which all reproduce the loop quantum cosmology (LQC) [65] effective dynamics for flat, closed and open isotropic and homogeneous space-times

— The non-singular black hole solutions in (extended) mimetic gravity with a limiting curvature are also revisited from a Hamiltonian point of view. The result is given in chapter 6. It turns out that the black hole has no singularity, due to the limiting curvature mechanism. Again a class of scalar-tensor theories, belonging to the family of extended mimetic gravity whose dynamics reproduces the general shape of the effective corrections of spherically symmetric polymer models in the context of LQG [67] is exhibited, but in an undeformed covariant manner. In that respect, extended mimetic gravity can be viewed as an effective covariant theory which naturally implements a covariant notion of point wise holonomy-like corrections similar in spirit to the ones used in polymer models. The difference between the mimetic and polymer Hamiltonian formulations provides us with a guide to understand the absence of covariance in inhomogeneous polymer models.

— In addition, in chapter 7 an effective metric is found for a static interior BH geometry describing the trapped region, in the framework of effective spherically symmetric polymer models (or equivalently, the (extended) mimetic gravity with a limiting curvature) with arbitrary anomaly free point-wise holonomy quantum correction function. For the simple case when the holonomy correction is the usual sine function used in the polymer literature, the interior metric describes a regular trapped region and presents strong similarities with the Reissner-Nordström metric, with a new inner horizon generated by quantum effects.

Part I.

Towards Lorentzian Invariance in Loop Quantum Gravity

1. Introduction and Overview

Sommaire

1.1. Introduction

Loop quantum gravity was founded on the observation by Ashtekar [68] that working only with the self-dual part (or equivalently the anti-self-dual part) of the Hilbert-Palatini action leads to a simplified parametrization of the phase space of pure gravity. Indeed, the canonical variables are very similar to those of the Yang-Mills gauge theory, there is no second class constraints, and the first class constraints associated to the local symmetries are polynomial functionals of the the canonical variables. However, the drawback of the Ashtekar's original approach is that the phase space becomes complex which forces one to impose reality conditions in order to recover the phase space of real general relativity. Of course the imposition of the reality conditions on the classical level, rather than the imposition on the quantum level, looses the beauty of the Ashtekar formulation, and recovers the standard Palatini formulation of general relativity, which we do not know how to quantize. Unfortunatelly, up to now no one knows how to go the other way around, by implementing the reality conditions after quantization of the Ashtekar theory. This difficulty motivated the work of Barbero [69] and, later on, Immirzi [70], who introduced a family of canonical transformations, parametrized by the so-called Barbero-Immirzi parameter γ, and leading to a canonical theory in terms of a real $\mathfrak{su}(2)$ connection kown as the Ashtekar-Barbero connection. The action that leads to this canonical formulation was finally found by Holst [71].

In fact, the Holst action is a first order formulation of gravity with a full $\mathfrak{sl}(2,\mathbb{C})$ internal symmetry and an explicit dependency on the parameter γ which appears as a coupling constant for a topological term. One uses a partial gauge fixing in this action in order to derive a canonical theory in terms of the Ashtekar-Barbero. The gauge choice is referring as the time gauge, and, by doing so, the Lorentz gauge algebra in the internal space is reduced to its rotational $\mathfrak{su}(2)$ subalgebra. Finally, Loop Quantum Gravity is a canonical quantization of this gauge fixed first order formulation of gravity which lead to a beautiful construction of the space of quantum geometry states on the kinematical level. At this stage, one may naturally ask the question whether the construction of Loop Quantum Gravity deeply relies on the time gauge or not. A more concrete question would be whether the physical predictions of Loop Quantum Gravity will be changed or not when one makes another partial gauge fixing or no gauge fixing at all in the Holst action prior to quantization. Indeed, the discreteness of the quantum geometry at the Planck scale predicted in Loop Quantum Gravity can be interpreted as a direct consequence of the compactness (via Harmonic analysis) of the residual symmetry group $SU(2)$ in the time gauge. These important problems have been studied quite a lot the last twenty years, but so far it is fair to say that no definitive conclusion have closed the debates.

Meanwhile, spin foam model arises as a covariant formulation of Loop Quantum Gravity (LQG), for a review, see [25, 27–29, 72]. A spin foam can be regraded as a Feynmann diagram with 5-valent vertices, corresponding to the quantum 4-simplices, as the building blocks of discrete quantum spacetime. The boundary of a 4-simplex contains 5 tetrahedra. The spin foam model is understood as a path integral formulation of the topological BF model with holst terms, where the gravity is recovered after imposing the so-called simplicity constraint at the quantum level. However, the solution space of the simplicity constraint is only accessible after a gauge fixing, similar to the difficulty in canonical approach. As one of the popular spin foam models, the Lorentzian Engle-Pereira-Rovelli-Livine/Freidel-Krasnov (EPRL/FK) model comes with a gauge-fixing within each tetrahedron such that in the local frame the timelike normal vector of the tetrahedron reads $u = (1, 0, 0, 0)$ in a 4D Minkowski spacetime with signature $(-1, 1, 1, 1)$, known as the "time-gauge". As a result, this model subjugates to the restriction that tetrahedra and triangles are all spacelike [63], in another words, the tetrahedra are all living in the Euclidean subspace. As a result, such spin foam models only correspond to a special class of 4D Lorentzian triangulations. However, in the extended spin foam model by Conrady and Hnybida, some tetrahedron normal vectors are chosen to be spacelike $u = (0, 0, 0, 1)$. Thus, the model contains timelike tetrahedra and triangles which live in 3D Minkowski subspaces [61, 73, 74]. Even though we do not know how to solve the simplicity constraint without any gauge fixing, but it looks that we have a consistent model with all possible Lorentzian triangulations. However, the issue of Lorentzian invariance is not fully addressed at the semi-classical level, thus whether the emergent gravity theory maintains the Lorentzian invariance is still in a doubt.

The semiclassical behavior of the spin foam model is determined by its large-j asymptotics. Recently there have been many investigations of large-j spin foams, in particular to the asymptotics of EPRL/FK model [50, 75–82], and the models with cosmological constant [83, 84]. It has been shown that, in large-j asymptotics, the spin foam amplitude is dominated by the contributions from critical configurations, which give the simplicial geometries and the discrete Regge action on a simplicial complex. The resulting geometries from the above analysis only contain spacelike tetrahedra and spacelike triangles. Recently, the asymptotics of the Hnybida-Conrady extended model with timelike tetrahedron was investigated [85]. The critical configurations of the extended model lead to simplicial geometries containing timelike tetrahedra. But the limitation is that all the triangles are still spacelike within each timelike tetrahedron. In all the examples of geometries in classical Lorentzian Regge calculus, and their convergence to smooth geometries [86–88], the Regge geometries contain timelike triangles. In order to have the Regge geometries emerge as the critical configurations from the spin foam model, we have to extend the semiclassical analysis to contain timelike triangles. Moreover, the amplitude may also contains critical configurations corresponding to nondegenerate split signature 4-simplices (e.g., Euclidean 4-simplices in Lorentzian EPRL model) and degenerate vector geometries, which breaks the Lorentzian invariance.

In this part, we investigate such Lorentzian invariance issue in both the canonical approach and the spin foam model. This part is organized as follows. We begin with a very brief review of the canonical loop quantum gravity based on the Hamiltonian formulation of the Holst action, and the spin foam model as well as its semi-classical limit. Then in chapter 2 we present a $\mathfrak{su}(1,1)$ gauge description of the gravity theory in four dimensions. We investigate the loop quantization at the kinematical level, where we found no anomaly between the $\mathfrak{su}(1,1)$ description and $\mathfrak{su}(2)$ time-gauge description of the space-like areas. Finally in chapter 3 we extend the semiclassical analysis of extended model to general situations, in which both timelike tetrahedra and timelike triangles are taken into account. We identify the dominate contribution with discrete simplicial geometries and Regge action of gravity. We find that for a vertex amplitude containing at least one timelike and one spacelike tetrahedron, critical configurations only give Lorentzian 4-simplices, while the split signature and degenerate 4-simplex do not appear.

1.2. Brief review on canoncial loop quantum gravity

This section is a brief review of the canonical approach of the loop quantum gravity which is based on the Hamiltonian formulation of the first order action of gravity, and the Dirac quantization is performed on "holonomy-flux algebras" We will focus on the canonical analysis of Holst action which we investigate in detail in chapter 2, following a very brief introduction to the loop quantization procedure. The detailed review of LQG is referring to [24–27, 89, 90].

1.2.1. Hamiltonian formulism of first order Lorentz-covariant gravity

In below we summarize the main results of the Hamiltonian analysis of the fully Lorentz invariant Holst action [71]. We start with recalling the main steps of the constraints analysis and present the solutions of the second class constraints proposed by Barros e Sa [91]. Then, we describe the parametrization of the Lorentz covariant phase space that gives the commonly used $su(2)$ Ashtekar-Barbero connection. The same parametrization will serve in chapter 2 to build the $\mathfrak{su}(1,1)$ connection.

1.2.1.1. Action and constraints analysis

The Holst action [71] is a generalization of the Hilbert-Palatini first order action with a Barbero-Immirzi parameter γ. In terms of the co-tetrad $e^I_\alpha(x)$ and the Lorentz connection one-form $\omega^{IJ}_\alpha(x)$, the corresponding Lagrangian density is

$$\mathcal{L}[e,\omega] = \frac{1}{2}\epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL} + \frac{1}{\gamma} e^I \wedge e^J \wedge F_{IJ}\,,$$

where $F[\omega] = d\omega + \omega\wedge\omega$ is the curvature two-form of the connection ω, ϵ_{IJKL} the fully antisymmetric symbol which defines an invariant non-degenerate bilinear form on $\mathfrak{sl}(2,\mathbb{C})$, and internal indices are lowered and raised with the flat metric η_{IJ} and its inverse η^{IJ}. The Holst action is equivalent to the Hilbert-Palatini action. Indeed, if the co-tetrad is not degenerated (i.e. if its determinant is not vanishing), the variation respect to ω is given as

$$D_\mu e^I_\nu = 0 \tag{1.1}$$

where D_μ is the covariant derivative

$$D_\mu e^I_\nu = \partial_\mu e^I_\nu + \omega^{IJ}_\mu e_{\nu J} \tag{1.2}$$

This equation is nothing else but the torsion free condition with which one can uniquely solve ω in terms of e and find that ω^{IJ}_μ are nothing but the components of the Levi-Civita connection. Plugging back this solution into the action eliminates the Barbero-Immirzi parameter γ by virtue of the Bianchi identities and leads to the second order Einstein-Hilbert action.

Now, we recall basic results on the canonical analysis of the Holst Lagrangian. For this purpose, it is convenient to introduce the notation

$$^{\gamma}\xi_{IJ} = \xi_{IJ} - \frac{1}{2\gamma}\epsilon_{IJKL}\,\xi^{KL}\,, \tag{1.3}$$

for any element $\xi \in \mathfrak{sl}(2,\mathbb{C})$, with the useful relation

$$^{\gamma}\xi_{IJ}\theta^{IJ} = \xi^{IJ\gamma}\theta_{IJ} \tag{1.4}$$

After performing a $3+1$ decomposition (based on a splitting $\mathcal{M} = \Sigma\times\mathbb{R}$ of the space-time) in order to distinguish between temporal and spatial coordinates (0 is the time label and small latin letters from the beginning of the alphabet $a, b, c, \cdots$ hold for spacial indices), a straightforward calculation leads to the following canonical expression of the Lagrangian density

$$\mathcal{L}[e,\omega] = {^{\gamma}\pi^a_{IJ}}\,\dot{\omega}^{IJ}_a - g^{IJ}\mathcal{G}_{IJ} - N\mathcal{H} - N^a\mathcal{H}_a, \tag{1.5}$$

where we have introduced the notations $\dot{\omega} = \partial_0\omega$ for the time derivative of ω, g^{IJ} for $-\omega^{IJ}_0$, N for the lapse function N, and N^a for the shift vector. All these functions are Lagrange multipliers which enforce respectively the Gauss, Hamiltonian, and diffeomorphism constraints

$$\mathcal{G}_{IJ} = D_a{^{\gamma}\pi^a_{IJ}}, \qquad \mathcal{H} = \pi^a_{IK}\pi^{bK}_{J}{^{\gamma}F^{IJ}_{ab}}, \qquad \mathcal{H}_a = \pi^b_{IJ}{^{\gamma}F^{IJ}_{ab}}. \tag{1.6}$$

These constraints are expressed in terms of the spatial connection components ω^{IJ}_a, and the canonical momenta defined by

$$\pi^a_{IJ} \equiv \epsilon_{IJKL}\,\epsilon^{abc}\,e^K_b\,e^L_c. \tag{1.7}$$

Since $\pi^a_{IJ} = -\pi^a_{JI}$ contains 18 components, and the co-tetrad has only 12 independent components, we need to impose 6 primary constraints often called the simplicity constraints

$$\mathcal{C}^{ab} = \epsilon^{IJKL}\pi^a_{IJ}\pi^b_{KL} \approx 0, \tag{1.8}$$

in order to parametrize the space of momenta in terms of the π variables instead of the co-tetrad variables. Classically, it is equivalent to work with the 12 components e_a^I or with the 18 components π^a_{IJ} constrained to satisfy the 6 relations $\mathcal{C}^{ab} \approx 0$. Hence, at this stage, the non-physical Hamiltonian phase space is parametrized by the 18 pairs of canonically conjugated variables $(\omega_a^{IJ}, \pi^a_{IJ})$, with the set of 10 constraints (1.6) to which we add the 6 constraints $\mathcal{C}^{ab} \approx 0$.

Studying the stability under time evolution of these "primary" constraints is rather standard and has been performed first for the Hilbert-Palatini action in [92] and for the Holst action in [91]. Here we will not reproduce all the steps of this analysis, but only focus on the structure of the second class constraints and their resolution. Details with our notations can be found in [93]. Notice first that in order to recover the 4 phase space degrees of freedom (per space-time points) of gravity, the theory needs to have secondary constraints, which in addition have to be second class. This is indeed the case. Technically, this comes from the fact that the algebra of constraints fails to close because the scalar constraint $\mathcal{H}$ does not commute weakly with the simplicity constraint $\mathcal{C}^{ab}$. Hence, requiring their stability under time evolution generates the following 6 additional secondary constraints

$$\mathcal{D}^{ab} = \epsilon_{IJMN}\, \pi^{cMN} \left(\pi^{aIK} D_c \pi^{bJ}{}_K + \pi^{bIK} D_c \pi^{aJ}{}_K\right) \approx 0. \tag{1.9}$$

The Dirac algorithm closes here with 18×2 phase space variables (parametrized by the components of π and ω), and 22 constraints $\mathcal{H}$, $\mathcal{H}_a$, $\mathcal{G}_{IJ}$, $\mathcal{C}^{ab}$ and $\mathcal{D}^{ab}$. Among these constraints, the first 10 are first class (up to adding second class constraints) as expected, and the remaining 12 are second class. One can check explicitly that $\mathcal{C}^{ab} \approx 0$ and $\mathcal{D}^{ab} \approx 0$ form a set of second class constraints (their associated Dirac matrix is invertible), and that the first class constraints generate the symmetries of the theory, namely the space-time diffeomorphisms and the Lorentz gauge symmetry. Finally, we are left with the expected 4 phase space degrees of freedom per spatial point:

$$18 \times 2(\text{dynamical variables}) - 10(\text{first class constraints}) \times 2 - 12(\text{second class constraints}).$$

We recover the two gravitational modes.

1.2.1.2. Parametrization of the phase space

Now since we have clarified the Hamiltonian structure of the theory, we are going to show how to solve the second class constraints following [91]. First, one writes the 18 components of π_a^{IJ} as

$$\pi^a_{0i} = 2E^a_i, \qquad \pi^a_{ij} = 2(E^a_i \chi_j - E^a_j \chi_i)\,, \tag{1.10}$$

where $\chi_i = e_i^a e_a^0$ (which encodes the deviation of the normal to the hypersurfaces from the time direction) and E_i^a (which corresponds to the usual densitized triad of loop gravity) are now twelve independent variables. Note that e_i^a is the inverse of e_a^i viewed as a 3×3 matrix. This is trivially a solution of the simplicity constraints (1.8) because somehow we have returned to the co-tetrad parametrization (1.7). Note that in such parameterization, the induced metric on the hypersurface Σ is given by

$$\det(q) q^{ab} = E_i^a \left((1 - \chi^k \chi_k) \delta^{ij} + \chi^i \chi^j \right) E_j^b \tag{1.11}$$

Then, we plug the solution (1.10) into the canonical term of the Lagrangian (1.5) which gives

$$^{\gamma}\pi^a_{IJ} \dot{\omega}^{IJ}_a = E_i^a \dot{A}_a^i + \zeta_i \dot{\chi}^i \qquad \text{where} \quad A_a^i = {}^{\gamma}\omega_a^{0i} + {}^{\gamma}\omega_a^{ij} \chi_j \quad \text{and} \qquad \zeta^i = {}^{\gamma}\omega_a^{ij} E_j^a \tag{1.12}$$

This result strongly suggests that the 18 components of the connection could be expressed in terms of the 12 independent variables (A_a^i, χ^i) when one solves the 6 secondary second class constraints. This is indeed the case and it can be seen by inverting the relation (1.12) as follows

$$^{\gamma}\omega_a^{0i} = A_a^i - {}^{\gamma}\omega_a^{ij} \chi_j, \qquad\qquad {}^{\gamma}\omega_a^{ij} = \frac{1}{2} \left(Q_a^{ij} - E_a^i \zeta^j - E_a^j \zeta^i \right), \tag{1.13}$$

where E_a^i is the inverse of E_i^a, and $Q_a^{ij} = Q_a^{ji}$ has a vanishing action on E_i^a. The explicit form of Q_a^{ij} can be obtained from $\mathcal{D}^{ab} \approx 0$ as shown in [91]. Furthermore, when $\gamma^2 \neq 1$, one can uniquely express ω in terms of ${}^{\gamma}\omega$ using the inverse of the map (1.3).

As a consequence, the phase space can be parametrized by the twelve pairs of canonical variables (A_a^i, E_i^a) and (χ_i, ζ^i) with the (non-trivial) Poisson brackets given by

$$\left\{ A_a^i(x), E_j^b(y) \right\} = \delta_j^i \delta_a^b \, \delta^3(x-y) \quad \text{and} \quad \left\{ \chi_i(x), \zeta^j(y) \right\} = \delta_i^j \, \delta^3(x-y). \tag{1.14}$$

Remark that if we work in the time gauge (i.e. $\chi = 0$), the variable A_a^i coincides exactly with the usual Ashtekar-Barbero connection.

1.2.1.3. First class constraints

It remains to express the first class constraints (1.6) in terms of the new phase space variables (7.3). This is an easy task using the defining relations (1.10) and (1.13). This was done by Barros e Sa. The constraints have quite a simple form except the Hamiltonian constraint whose expression is more involved: it can be found in [91] and we will only consider this constraint after partial gauging fixing. The vector constraint $\mathcal{H}_a$ takes the form

$$\begin{aligned}\mathcal{H}_a &= E^b\cdot(\partial_a A_b - \partial_b A_a) + \zeta\cdot\partial_a\chi + \frac{\gamma^2}{1+\gamma^2}\Big[(E^b\cdot A_b)(A_a\cdot\chi) - (E^b\cdot A_a)(A_b\cdot\chi)\\ &\quad +(A_a\cdot\chi)(\zeta\cdot\chi) - (A_a\cdot\zeta) + \frac{1}{\gamma}\left(E^b\cdot(A_b\times A_a) + \zeta\cdot(\chi\times A_a)\right)\Big],\end{aligned} \tag{1.15}$$

where $\cdot$ denotes the scalar product $\lambda\cdot\mu = \lambda_i\mu^i$ and $\times$ denotes the cross product $(\lambda\times\mu)^i = \epsilon^{ijk}\lambda_j\mu_k$ for any two pairs of vectors λ and μ in $\mathbb{R}^3$. Concerning, the Lorentz constraints $\mathcal{G}_{IJ}$, they can be split into its boost part $\mathcal{B}_i \equiv \mathcal{G}_{0i}$, and its rotational part $\mathcal{R}_i \equiv (1/2)\epsilon_i{}^{jk}\mathcal{G}_{jk}$ whose expressions are

$$\mathcal{B} = \partial_a\left(E^a - \frac{1}{\gamma}\chi\times E^a\right) - (\chi\times E^a)\times A_a + \zeta - (\zeta\cdot\chi)\chi\,, \tag{1.16a}$$

$$\mathcal{R} = -\partial_a\left(\chi\times E^a + \frac{1}{\gamma}E^a\right) + A_a\times E^a - \zeta\times\chi\,. \tag{1.16b}$$

One can check that these constraints satisfy indeed the Lorentz algebra

$$\{\mathcal{B}\cdot u, \mathcal{B}\cdot v\} = -\mathcal{R}\cdot u\times v, \quad \{\mathcal{R}\cdot u, \mathcal{R}\cdot v\} = \mathcal{R}\cdot u\times v, \quad \{\mathcal{B}\cdot u, \mathcal{R}\cdot v\} = \mathcal{B}\cdot u\times v, \tag{1.17}$$

where u and v are arbitrary vectors.

1.2.1.4. The time gauge and Ashtekar-Barbero variables

The very well-known “time" gauge refers to the condition $e^\mu_a n_\mu = 0$ where n_μ is a given as a time like vector $n_\mu = \delta^0_{mu}$. It is directly to check that, in such case, corresponding to $e^0_a \approx 0$, or equivalently $\chi \approx 0$. The condition $\chi\approx 0$ drastically simplifies the boost constraints which become equivalent to $\zeta - \partial_a E^a \approx 0$. The conditions $\chi\approx 0$ and $\zeta - \partial_a E^a \approx 0$ form a set of second class constraints that can be solved explicitly for χ and ζ. By doing so, the variables (χ,ζ) are eliminated from the theory, which breaks $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{su}(2)$. It corresponds to taking a slicing $\Sigma\times\mathbb{R}$ of the space-time where the hypersurfaces Σ are space-like, as one can see from the induced metric (1.11)

$$\det(q)q^{ab} = E^a_i\delta^{ij}E^b_j \tag{1.18}$$

In such gauge, the canonical pairs can be given as [60]

$$\mathcal{A}^i_a = -\gamma A^i_a = -\gamma^\gamma\omega^{0i}_a = \beta K^i + \Gamma^i, \qquad \mathcal{E}^a_i = E^a_i \tag{1.19}$$

where $\beta = -\gamma$ with non-trivial bracket

$$\{\mathcal{A}^i_a(x), \mathcal{E}^a_i(y)\} = \beta\delta^i_j\delta^a_b\delta^3(x-y) \tag{1.20}$$

Note that these fields take values on a spatial slice Σ, where K refers to the extrinsic curvature and $\Gamma = \frac{1}{2}\epsilon_{ijk}\omega_a^{jk}$. One can check that, in such gauge, the Gaussian constraint together with the second class constraint $\mathcal{D}$ given in (1.9) Γ is the spin connection compatible with $\mathcal{E}$. $(\mathcal{A}, \mathcal{E})$ are the well-known Ashetkar-Barbaro connection and the densitized triad of loop gravity [68–70]. The corresponding first class constraints (1.6) are now given by

$$
\begin{aligned}
\mathcal{G}_i &= D_a \mathcal{E}_i^a \\
\mathcal{H}_a &= E_i^b F_{ab}^i \\
\mathcal{H} &= \frac{1}{2\det(\mathcal{E})}\left(\epsilon^{ij}{}_k \mathcal{E}_i^a \mathcal{E}_j^b F_{ab}^k - (1+\gamma^2)\mathcal{E}_i^a \mathcal{E}_j^b K_{[a}^i K_{b]}^j\right) \\
&= \qquad \mathcal{H}_E \qquad + \qquad \mathcal{H}_L
\end{aligned}
\tag{1.21}
$$

where D_a is now the covariant derivative induced by $\mathfrak{su}(2)$ connection $\mathcal{A}$, F_{ab}^i is now the $\mathfrak{su}(2)$ curvature two form $F = D\mathcal{A} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ of connection $\mathcal{A}$, and we rescale the original $\mathcal{H}$ defined in (1.6), to have a density one scalar constraint. Note that the Hamiltonian constraint can be separated into two parts where we refer to the Euclidean part $\mathcal{H}_E$ and the Lorentzian part $\mathcal{H}_L$. One can check that, the constraints satisfies the following Dirac's hypersurface deformation algebra

$$
\begin{aligned}
\{C_V[N^a], C_H[N]\} &= -C_H[\mathcal{L}_{N^a} N] \\
\{C_V[N_1^a], C_V[N_2^b]\} &= C_V[\mathcal{L}_{N_1} N_2^a] \\
\{C_H[N_1], C_H[N_2]\} &= C_V[q^{ab}\left(N_1 \partial_b N_2 - N_2 \partial_b N_1\right)]
\end{aligned}
\tag{1.22}
$$

where $\mathcal{L}_N$ is the lie derivative along vector N, and C denotes the smeared constraints defined as

$$
C_G[\Lambda] = \int d^3x \Lambda^i \mathcal{G}_i, \qquad C_V[N^a] = \int d^3x N^a \mathcal{H}_a, \qquad C_H[N] = \int d^3x N \mathcal{H} \tag{1.23}
$$

with $\mathcal{G}_i$, $\mathcal{H}_a$ and $\mathcal{H}$ refers to gauss, vector and scalar constraint respectively as given in (1.21).

1.2.2. Loop quantization

In order to quantize the classical theories with complicated constraint algebras (GR for example), the quantization procedure will follows the Dirac canonical quantization programme [94] instead of the usual Fork space canonical quantization. The quantization procedure can be generally summarized as the following:

— Identify a set of (redundant) functions which fully parameterize the phase space. These functions is not necessarily to be gauge invariant. However, in order to perform the usual quantization we would like them to have simple algebras, e.g. the cotangent space to a group, which is the case for holonomy-flux algebra as we will show later.

— Find the kinematical Hilbert space as representations of the quantum algebra, by asking that the constraints can be represented as closed and densely defined operators on it.

— Induce the physical Hilbert space from kinematical Hilbert space, which corresponds to solve the constraint operators and find the generalized joint kernel.

— Identify the complete (gauge-invariant) observables which commute with all quantum constraints.

In the following we will summarize the dirac quantization procedure used in Loop Quantum Gravity based on holonomies-flux Algebras. We will focus on the construction of the kinematical Hilbert spaces and the spin network states in such space. We refer to [24–26, 89] for the detailed review, and [90, 95] for some brief reviews. Note that the construction here is based on the time gauge with $\mathfrak{su}(2)$ gauge group. The full Lorentzian or other non-compact construction is still an open question despite many attempts, e.g. [96–99]

1.2.2.1. Quantum Kinematics

As we shown in previous section, under time gauge, the phase space of Holst action is an cotangent bundle $\mathcal{M} = (\mathcal{A}, \mathcal{E}) = T^*(\mathcal{A})$ over the space of $\mathfrak{su}(2)$ connections. Inspired from the lattice gauge theory, instead of working with $\mathcal{A}$ and $\mathcal{E}$, we can choose to work with SU(2) holonomies

$$U_e := P \exp \int_e \mathcal{A} \quad \in \mathrm{SU}(2) \tag{1.24}$$

along some path e, and electric fluxes

$$E_S(f) := \int_S *\mathcal{E}_i f^i \tag{1.25}$$

across a two surface S with some $\mathfrak{su}(2)^*$ valued smearing function f. Here P denotes path ordering. According to the bracket (7.3), U_e and E_S satisfy

$$\{U_e, U'_e\} = 0, \quad \{E_{S,f}, U_e\} = \beta U_{e_1} f(S \cap e) U_{e_2} \tag{1.26}$$

where we assume e and S intersect transversely with each other at $S \cap e$ thus separate the path as $e = e_1 \circ e_2$.

In order to construct the kinematic Hilbert space, we take the Schrödinger type representation where states are functionals $\psi(U)$ over configuration space. Here U is regraded as a distributional connections as the generalization of U, which is defined by the map from a path e to SU(2) holonomies

$$\bar{U} : e \to \bar{U}_e \in \text{SU(2)} \tag{1.27}$$

The functions on $\bar{U}$ can be described by cylinder functions of a finite number of holonomies on some oriented graph Γ, where a cylindrical function is represented as $\psi(U(A)) = \psi_\Gamma(\{U_e(A)\}_{e\in\Gamma})$ [100]. Here e represent paths on the graph, called edges which intersect with each other in the endpoint only. The definition is generalized to different graphs where

$$Cyl : \cup_\Gamma Cyl_\Gamma / \sim \tag{1.28}$$

where two cylindrical functions $\psi_{\Gamma'} \sim \psi'_{\Gamma''}$ iff there exists a larger graph $\Gamma \supset \Gamma'', \Gamma'$ such that $\psi_\Gamma = \psi'_\Gamma$. The electric flux E acts as a smooth vector field on the space of cylindrical functions, which appears as the conjugate variables associated to cylindrical functions. This forms the classical holonomy-flux algebra which is the $*$-subalgebra of $T^*(Cyl)$.

As a nature representation of the above algebra, the holonomy operator will act multiplicatively and the flux vector fields act as derivation operators, more specific,

$$(\psi_1\psi_2)[U] = \psi_1(U)\psi_2(U), \qquad (E_{S,f}\psi)[U] = (X_{S,f}[\psi])[U] \tag{1.29}$$

The nature measure associated with such space is the Haar mearsure on SU(2). Hence, the Hilbert space related to the graph Γ is defined as

$$\mathcal{H}_\Gamma = L_2[\text{SU}(2)]^{\otimes L} \sim (\text{Fun}[\text{SU}(2)^{\otimes L}]; d\mu^{\otimes L}) \tag{1.30}$$

where L denotes the number of edges in Γ and $d\mu$ the Haar measure on SU(2). This concept can be extended to a collection of different graphs, where the measure is now Ashtekar-Lewandowski measure $d\mu_{AL}$. The idea is to find the new graph Γ contains both graphs and extend the dependence of cylindrical functions trivially to all edges in Γ. The measure is then the product of Haar measures on Γ. It is clear such kinematical Hilbert space is given as

$$\mathcal{H}_{kin} = L_2[\bar{U}, d\mu_{AL}] \tag{1.31}$$

by completing Cyl w.r.t. the inner product. Note that, the kinematical Hilbert space here contains a diffeomorphism invariant cyclic states, namely the state is invariant under the action of diffeomorphism and any states can be approximated by linear combination of products of operators act on this state. It has been proved that, the $\mathcal{H}_{kin}$ that contains a diffeomorphism invariant cyclic vector is uniquely defined [101, 102].

1.2.2.2. Spin network states

By applying the Peter-Weyl theorem to each edges of a a given graph Γ, cylindrical functions $\psi_\Gamma[A]$ over the graph can be formally decomposed as

$$\psi_\Gamma[U] \;=\; \sum_{j_1,\cdots,j_E} \mathrm{tr}\left(\tilde{f}(j_e)\bigotimes_{e=1}^{E}\pi_{s_e}(U_e)\right)\;, \tag{1.32}$$

where

$$\pi_s : SU(2) \to \mathrm{End}(V_j) \quad \text{and} \quad \tilde{f} \in \bigotimes_{e=1}^{N} V_{j_e}^{*}\,. \tag{1.33}$$

The summation runs over unitary irreducible representations of SU(2) labelled by spin j_e. V_{j_e} refers to the modulus of the representation, $V_{j_e}^*$ for its dual, and tr denotes the pairing between $\otimes_e V_{s_e}$ and its dual $\otimes_e V_{s_e}^*$. For a given graph Γ, clearly $\tilde{f}$ can be regraded as a function living on the vertices where the edges intersect with each other via their end points. With such decomposition, we can introduce a particular label on the graph Γ, where each edge carries a spin label of unitary representations j_e, and each vertex carries a tensor label refers to $\otimes_e V_{j_e}^*$. This forms a labelled graph which is called generalized spin networks. The associated states is called the (generalzied) spin network states. Figure 1.1 shows an example of the generalized spin network graph. Note that any two spin network states are orthogonal unless their spin network graph and corresponding labels are identical to each other up to trivial $j = 0$ extensions. The generalization of the decomposition to the whole $\mathcal{H}_{kin}$ can be constructed as patching together the fixed graph spin network basis which gives an orthonormal basis of $\mathcal{H}_{kin}$.

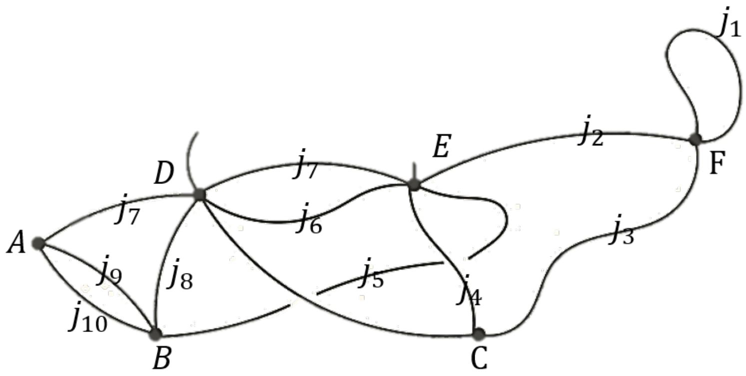


Figure 1.1. – A generalzied spin network graph, where $j_1 - j_{10}$ are the spin colors on edges and $A - F$ refer to the tensor at each vertex with a suitable dimension

1.2.2.3. Geometric operators

The spin network have a natural geometrical interpretation, where each edge is normal to a spacelike surface and each vertex related to a sub-region in Σ. It is possible to get the area and volume spectrum by acting area and volume operators on it. However, since we have not taken account the constraints, the physical meaning only valid in the case when they forms gauge invariant observables [103, 104].

We start with the area operator [100]. From (1.18) of the inverse metric q^{ab} that we contract twice with the normal n_a to a given surface S. This leads to the formula

$$\det(q)\, n^2 \;=\; (n_a \mathcal{E}^{ai})\, \delta_{ij}\, (\mathcal{E}^{bj} n_b)\,, \tag{1.34}$$

where $n^2 = n_a n_b g^{ab}$. Hence, the determinant of the induced metric h on the surface S is given by

$$\det(h) \;=\; (n_a \mathcal{E}^{ai})\, \delta_{ij}\, (\mathcal{E}^{bj} n_b)\,. \tag{1.35}$$

As a consequence, the action of the area operator $\hat{S}$, punctured by an edge e of the graph Γ colored by a representation s_e, on $\mathcal{H}_{kin}(\Sigma)$ is diagonal and its eigenvalue $S(s)$ is given by the equation

$$S(e) \;= \int_S \sqrt{|\det(h)|} = \;\gamma \ell_p^2 \sqrt{\pi_e(J_1^2 + J_2^2 + J_0^2)} \;=\; \gamma \ell_p^2 \sqrt{\pi_e(C)} \tag{1.36}$$

where $\pi_e(C)$ is identified with the unique eigenvalue of the Casimir tensor $C \equiv J_0^2 + J_1^2 + J_2^2$ in the representation j_e. Obviously, the evaluation $\pi_e(C)$ gives discrete values according to

$$\pi_{j_e}(C) = j_e(j_e + 1) \tag{1.37}$$

We deduce immediately that $S(e)^2$ is positive and discrete. As a consequence, the area operator of any space-like surface has a discrete spectrum.

The volume operator follows the same procedure as area operator, however, its form is more complicated. There is no explicit result of volume spectrums due to the complicated and non-diagonalized volume operator acting on each vertex of generalized spin network graph. However, there are plenty of studies, both analytical and numerical, shows that it has a discrete spectrum [105–110]. We shall see later the volume operator has an important role in the imposition of Hamiltonian constraint.

1.2.2.4. Gaussian constraint

Now following the Dirac quantization scheme, after the building of kinematic Hilbert space, we will impose the constraints to extract the physical Hilbert space. According to the canoncial analysis, the physical Hilbert space is restrict to the annihilation states of the quantum gaussian, vector and scalar constraints equations, namely

$$\hat{C}_G(\vec{\Lambda})\psi = \hat{C}_V(\vec{N})\psi = \hat{C}_H(N)\psi = 0 \tag{1.38}$$

where C_G,C_V and C_H correspond to smeared Gauss, diffeomorphism and Hamiltonian constraints given in (1.23)

Here clearly the Gauss constraint generate the $SU(2)$ gauge transformations on generalized holonomies

$$\bar{U} \to \bar{U}^{\Lambda}, \qquad \bar{U}_e^{\Lambda} = g_{e_b}(\Lambda)\bar{U}_e g_{e_f}(\Lambda) \tag{1.39}$$

where e_b and e_f correspond to the beginning and end point of the edge e. Such gauge transformation naturally induce an action on cylindrical functions ψ, namely,

$$\psi \to \psi^{\Lambda}, \qquad \psi^{\Lambda}(\bar{U}) = \psi(\bar{U}^{\Lambda^{-1}}) \tag{1.40}$$

In spin network basis, such gauge transformation only acts on the vertices, which corresponding to a transformation on $T_v \in \otimes_{e:v\subset\partial e} V_{j_e}^*$ in the representation j_e respectively, namely,

$$T \to T^{\Lambda} \in \otimes_{e:v\subset\partial e} V_{j_e}^*, \qquad T_v^{\Lambda} = \left(\otimes_{e:v\subset\partial e}\pi_{j_e}\right)(\Lambda_v) \circ T_v \tag{1.41}$$

The solution of the Gauss constraint is then the gauge invariant spin-network states which carries the $SU(2)$ invariant tensor

$$I_v \in \mathrm{Inv}_{SU(2)}(\otimes_{e:v\subset\partial e} V_{j_e}^*) \tag{1.42}$$

at each vertex. This leads to the gauge-invariant Hilbert space $\mathcal{H}_0$ which span by gauge invariant spin networks.

1.2.2.5. Diffeomorphsim Constraint

The action of diffeomorphism constraint on generalized holonomies can be expressed as:

$$\bar{U} \to \bar{U}^V, \qquad \bar{U}_e^V = \bar{U}_{V\circ e} \tag{1.43}$$

where $V \in \text{diff}(M) : M \to M$ is a diffeomorphism. As a result, the cylindrical functions transforms as $\psi^V(\bar{U}) = \psi(\bar{U}^{V^{-1}})$. Note that, this transformation induce a deformation on spin networks, produces linear combination of orthogonal spin network states thus is not continuous. As a consequence, there is no generators for these unitary diffeomorphism operators. To get exactly the state annihilated by the diffeomorphism constraint, we need to solving (1.38). However, unlike the gauss constraint, we can not directly impose the constraint on spin-network states, since they are not continuous. Instead, we consider the solution on the algebraic dual of the spin network states, which are simply linear functionals on spin networks. For a state $\Phi \in Cyl^*$, we can decompose it as

$$\Phi = \sum_S \Phi_S \psi_S * = \sum_S \Phi_S \langle \psi_S, \cdot \rangle \tag{1.44}$$

where $\psi_S^* \in Cyl^*$ denotes the dual of spin network ψ. The constraint equation is then

$$\Phi(\psi^V) = \Phi(\psi) \tag{1.45}$$

The elementary solution is given by the group averaging technique, which is given by

$$\Phi_S = \sum_{V_1 \in \text{diff}/\text{diff}_S} \frac{1}{|\text{GS}_S|} \sum_{V_2 \in \text{GS}_S} \langle V_1 \circ V_2 \circ \psi_S, \cdot \rangle \tag{1.46}$$

where $\text{GS}_S = \text{diff}_\Gamma/\text{Tdiff}_S$ donates gauge symmetries with $\text{Tdiff}_S \subset \text{diff}$ the identity on S and $\text{diff}_S \subset \text{diff}$ the diffeomorphisms mapping S to itself. Intuitively, these diff-invariant states can be regarded as states labelled by diff-equivalence class of spin net work states

$$\Phi_{[S]} = \sum_{S\in[S]} \langle \psi_S, \cdot \rangle \tag{1.47}$$

where $[S] = V \circ S; V \in \text{diff}(M)$ denotes the diff-equivalence class of S [a], which is know as generalized knot classes. Any solution is the linear combination of such elementary solution.

---

a. Note that, this is only a intuitive picture and one has to be careful since (1.45) admits a large kernel with "hourglass" spin-networks [26, 90]

The diff-invariant Hilbert space $\mathcal{H}_{\text{diff}}$ is then given as the span of $\Phi_{[S]}$, with the inner product defined as

$$\left\langle \Phi_{[S]}, \Phi_{[S']} \right\rangle = \Phi_{[S]}(\psi_{S'}) = \sum_{S \in [S]} \langle \psi_S, \psi_{S'} \rangle \,, \qquad\qquad S' \in [S'] \tag{1.48}$$

Noted that, $\mathcal{H}_{\text{diff}}$ is not separable because the set of singular knot classes has uncountably infinite cardinality [25].

1.2.2.6. Scalar constraint

Following the dirac procedure, the final step is to impose and solve the scalar constraint C_H in (1.23). However, the scalar constraint is much more difficult than the gauss and diffeomorphism constraint. One technical reason is that the constraint is highly non-polynomial (due to the inverse volume element $1/\sqrt{q} = 1/\det E$). A possible solution is to define a rescaled constraint $\sqrt{q}C_H$ and make use of the spatial diffeomorphsim invariance to construct a UV finite background independent operator. However, the Hamiltonian constraint do not commute with the diffeomorphism as we see in the constraint algebra (1.22). And as showing before, it is impossible to implement the infinitesimal diffeomorphsim constraint directly due to the non-continuity of the diffeomorphism action. Moreover, the constraint algebra (1.22) is not a Lie algebra. Thus it is difficult to find a well-defined anomaly free Hamiltonian operator on $\mathcal{H}_{kin}$. The problem was originally studied in [111], where a family of quantum hamiltonian constraints are proposed. However, the anomaly freeness of these operators remains an open question. Later on, a new technical is proposed in [112] which is called the Master Constraint Programme (MCP) . We shall not reproduce the steps in details but only sketch the main ideas.

In [111], Thiemann proposed a family of well defined Hamiltonian constraints. The quantization is based on the following ideas: The inverse volume element can be absorbed into brackets between connection $\mathcal{A}$ and volumes V_Σ of the spatial slice

$$\epsilon^{ijk}\epsilon_{abc}\frac{E^a_i E^b_j}{\det E} = \frac{1}{4\gamma}\{\mathcal{A}^k_c(x), V_\Sigma\} \tag{1.49}$$

The constraint operator is well defined via a bi-linear form on $\mathcal{H}_{\text{diff}} \in Cyl^*$. The curvature is given as the limit of holonomy around loops shrinking with regulator ϵ which denotes to coordinate length of a refined triangulation. When ϵ is small enough, the regulated definition of H_E will be independent of the regulator on $\mathcal{H}_{\text{diff}}$. In this process, the bracket enrolls the volume and the curvature can be approximated as

$$\epsilon e\{\mathcal{A}(x), V_\Sigma\} \approx -U_e^{-1}\{U_e, V_\Sigma\}, \qquad \epsilon^2 F \approx U_{\partial S} - \mathbb{I} \tag{1.50}$$

where e denote the path emanating in x, and S is a surface in Σ. The Lorentzian part which enrolls the extrinsic curvature can be expressed out once we have the form of Euclidean constraint via the relation

$$K_a^i E_i^a = \{H_E, V_\Sigma\} \tag{1.51}$$

There are plenty of studies for calculating the matrix elements of the Hamiltonian operator defined above, for example, [113–116]. The action of Hamiltonian constraint on $\psi_\Gamma \in Cyl$, roughly speaking is creating new edges between edges incident in a common vertex v, thus it is local around the vertices. The Hamiltonian constraint defined in such way is proved to be self-adjoint [117], and does admit a non-trivial kernel. However, there are several ambiguities in the definition of the constraints, for example, the ambiguity of different representation used to regularize the constraint [118]. Most importantly, whether such constraints are anomaly free is not being settled. The anomaly freeness and quantum deformation of the classical algebra (1.22) has been studied a lot in symmetry reduced models inspired by canonical loop quantum gravity [119–126], which we will mention in detail in part II.

In order to overcome the anomaly free problem, instead considering the infinite many Hamiltonian constraints, we can construct a single Master constraint [112] as

$$\boldsymbol{M} = \frac{1}{2}\int_M \frac{C_H^2}{\sqrt{q}} \tag{1.52}$$

which is a positive definite integral of density weight one integrand with Hamiltonian constraints. Thus $M = 0$ defines actually the same constraint surface as the hamiltonian constraint $C_H(N)$. Moreover one check the algebra becomes

$$\{\vec{C}_V(\vec{N}), \boldsymbol{M}\} = \{\boldsymbol{M}, \boldsymbol{M}\} = 0 \tag{1.53}$$

thus we can implement and solve $\boldsymbol{M}$ directly on $\mathcal{H}_{\text{diff}}$, and define the inner product which finally gives us the physical Hilbert space $\mathcal{H}_{phy}$. The Master constraint can be regarded as a Riemann sum over a triangulation Δ of M:

$$\boldsymbol{M} = \lim_{\epsilon\to 0} \sum_{\Delta\in\mathcal{K}} \left(\frac{H(\Delta)}{\sqrt{V(\Delta)}}\right)^2 = \lim_{\epsilon\to 0} \sum_{\Delta\in\mathcal{K}} \bar{C}(\Delta)C(\Delta) \tag{1.54}$$

where $V_\Delta = \int_\Delta \sqrt{q}$ is the volume of tetrahedra. The implementation of $\hat{\boldsymbol{M}}$ on $\mathcal{H}_{\text{diff}}$ is defined via a bi-linear form on $\mathcal{H}_{diff} \in Cyl^*$

$$Q_{\boldsymbol{M}}(\Psi, \Psi') = \sum_{[S]} \sum_{v\in V(\Gamma_S)} \overline{\Psi(\hat{C}_v^\dagger \psi_S)}\Psi'(\hat{C}_v^\dagger \psi_S) \tag{1.55}$$

Here $\Psi, \Psi' \in \mathcal{H}_{diff}$, $V(\Gamma)$ refers to all vertices in a spin network states Γ and S is a representative of $[S]$. The constraint C_v here takes the same form as Thiemann's constraint operator H as introduced before but replace V to $\sqrt{V}$. One can prove that (Theorem 10.6.1 in [25]), such positive defined bi-linear form is closeable and induces a unique, positive self-adjoint operator $\hat{\boldsymbol{M}}$ on $\mathcal{H}_{diff}$. Moreover, the point zero is contained in the point spectrum of $\hat{\boldsymbol{M}}$, namely, $\hat{\boldsymbol{M}}\Psi = 0$ implies $Q_M(\Psi, \Psi) = 0$, which implies $\Psi(\hat{C}_v^\dagger \psi_S) = 0$ for all $[S]$ and $v \in V(\Gamma_S)$.

The final step of the quantization is supplying a physical inner product and a separable physical Hilbert space. With the fact that $\mathcal{H}_{diff}$ can be decomposed as an uncountablely infinite, almost direct, sum of separable, invariant Hilbert spaces, we can directly apply the direct integral decomposition to define the physical Hilbert space, as mentioned in [25].

The Master Constraint procedure can be further generalized, such that one can add the diffeomorphism and even Gauss constraint inside to form a single constraint. Such consideration opens the possibility of a non-graph-changing Hamiltonian constraint defined on $\mathcal{H}_{kin}$ and leads to the semi-classical analysis of the model [127–129].

1.3. Brief review on spin foam models

The first order gravity we described before can be expressed as a constrained BF theory with gauge group G to be $\mathrm{Sl}(2,\mathbb{C})$ for Lorentzian or $Spin(4)$ for Euclidean in 4d. This is known as the Plebanski formulation of general relativity described by the action

$$S = \int_M \langle B \wedge \mathcal{F}[A]) \rangle + \Phi \tag{1.56}$$

where B is an $\mathfrak{g}$ valued two form (bivector) and $\mathcal{F}[A] = dA + A \wedge A$ is the curvature of connection A of $\mathfrak{g}$. $\langle \cdot, \cdot \rangle$ denote the G invariant inner product. Here the constraint Φ is imposed to make the two form $\mathcal{B}$ simple, and is called the simplicity constraint, which is given by

$$\Phi = \phi_{IJKL} B^{IJ} \wedge B^{KL} \tag{1.57}$$

One can immediately check that, the solution of the constraint Φ on the B^{IJ} leads to

$$B^{IJ} = \pm e^I \wedge e^J, \text{and} B^{IJ} = \pm * (e^I \wedge e^J) \tag{1.58}$$

which recovers the Holst action.

The covariant quantization of such constraint BF theory leads to spin foam models. We will briefly review the ideas here. Generally speaking, the spin foam model is a state sum model defined on some simplicial complex $\mathcal{K}$. It is proved that, the semi-classical limit of the spin foam model is given by a simplicial manifold, which is a triangulation of the spacetime manifold, with the action relates to Regge action [130–132] which is a discrete version of General Relativity. We will summarize the spin foam quantization procedure and introduce the commonly used Engle-Pereira-Rovelli-Livine/Freidel-Krasnov (EPRL/FK) model and Conrady-Hnybida Extension . We suggest [27, 29] for a detailed review of spin foam model.

1.3.1. Spin foam quantization

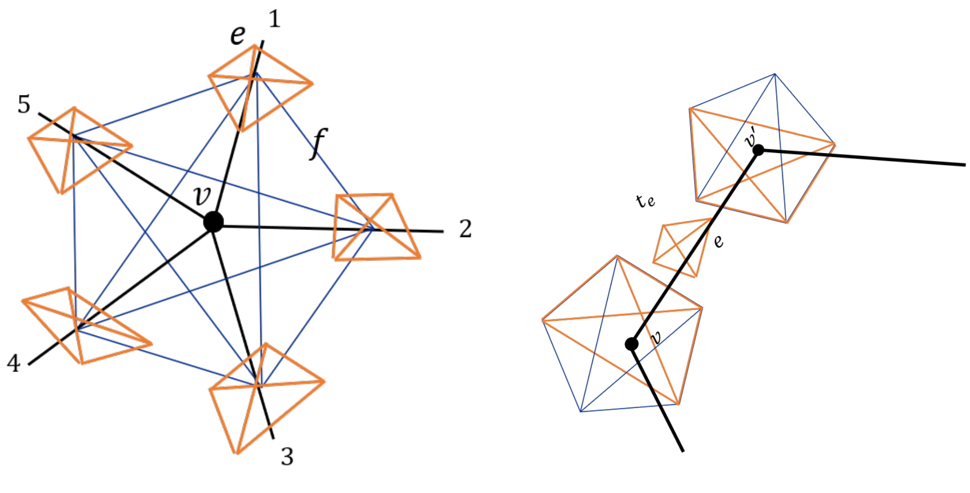
The path integral formula of the BF theory reads

$$\int \mathcal{D}B\mathcal{D}A \exp[-\mathrm{i}S_{BF}] = \int \mathcal{D}A\delta(\mathcal{F}[A]) \tag{1.59}$$

After we integrate out the bivector B the theory can be regarded as a second order theory.

The detailed meaning of the formal path integral expression above is achieved by replace the manifold M by an simplicial decomposition (triangulation) $\mathcal{K}$ of M [b]. We denote the dual complex associated to $\mathcal{K}$ as $\mathcal{K}*$. A simple example is the 4-dimesional simplicial manifold $\mathcal{K}$ consisting simplices σ_v, tetrahedra τ_e, triangles f, edges and vertices. Respectively v,e and f are labels for vertices, edges and faces on the dual graph $\mathcal{K}*$. A triangulation $\mathcal{K}$ and its dual complex $\mathcal{K}*$ can be obtained by gluing the basic building blocks, d dimensional simplices σ, with identifying pairs of their boundaries ($d-1$ dimensional simplices τ), as shown in Figure 1.2.

b. In general, one can take arbitrary cellular decomposition of the manifold, e.g. the decomposition using polyhedrons has been studies in [133, 134]



(a) (b)

Figure 1.2. – (a): A dual graph of a 4 simplex as the building blocks of the 4 diemensional trianglation, where black point reperents the vertex v; black lines refer to edges e which is dual to boundary tetrahedra τ_e reprsented by orange lines; and bule line represent faces f. (b) A glued graph contains two 4 simplcies sharing a common tetrahedron τ_e. The dual graph (black color) connected to each other along edge e.

With such triangulation, we can formally express the smearing of B field as a lie algebra $\mathfrak{g}$ element on each face f

$$B_f := \int_{f*} B \tag{1.60}$$

The connection A is also discretized as holonomies along edges e

$$g_e := P\exp \int_e \mathcal{A} \tag{1.61}$$

Now the phase space associated with manifold $\mathcal{K}$ becomes

$$P_{\mathcal{K}} = T^*\mathsf{SL}(2,\mathbb{C})^E, \qquad (B_f^{IJ}, h_f) \in T^*G \tag{1.62}$$

where E is the number of the dual faces f in $\mathcal{K}$, $h_f = \prod_{e\in\partial f} g_f \in G$ is the holonomy around the face f and $B_f^{IJ} \in \mathfrak{g}$ is the conjugate momenta. Then the path integral 1.59 reduces to

$$Z(\mathcal{K}) = \int \prod_{e\in\mathcal{K}*} dg_e \prod_{f\in\mathcal{K}*} \delta(h_f) \tag{1.63}$$

The integration measure for the group variable is the invariant measure on G. By using Peter-Weyl's theorem, the Dirac delta function $\delta(h_f)$ can be expressed as

$$\delta(h_f) = \sum_{\pi} d_\pi \mathrm{Tr}[\pi(h_f)] \tag{1.64}$$

where π denotes the unitary irreps of G. Thus (1.63) can be reformulated as

$$Z(\mathcal{K}) = \sum_{\{\pi_f\}} \prod_{f\in\mathcal{K}*} d_{\pi_f} \prod_{e\in\mathcal{K}*} I_e(\pi_f) \tag{1.65}$$

where $I_e = \int dg_e \otimes_{f:e\subset\partial f} \pi_f(g_e)$ stands for the group average for the states on faces f bounded by e.

When the manifold comes with a boundary ∂M, the triangulation $\mathcal{K}$ on M naturally induces a triangulation and a dual complex on ∂M, which we denote as $\partial\mathcal{K}$ and $\partial\mathcal{K}*$ respectively. The boundary state is then a function depends on the holonomies $U_e(A_\partial)$ which is the restriction of $U_e(A)$ on $\mathcal{K}*$. This boundary state can be described via spin network graph Γ with gauge group G, namely, it is a spin network state with gauge group G:

$$\psi(A_\partial) = \psi_\Gamma(U_e(A_\partial)) \in L_2[G^{\otimes L}/G^{\otimes N}] \tag{1.66}$$

where L refers to the number of links and N is the number of nodes on the boundary of $\partial\mathcal{K}*$. With the boundary state $\psi_\Gamma(U_e(A_\partial))$, one can now get the amplitude of BF theory associated to such state

$$Z_{BF} = \int \mathcal{D}A\delta(\mathcal{F}[A])\psi_\Gamma(U_e(A_\partial)) \tag{1.67}$$

which is nothing else but the evaluation of the spin network states over flat connections. A general boundary state for a 4-simplex, which is the building block of 4d simplicial manifold $\mathcal{K}$, is the spin network graph contains 5 nodes and 10 links, denoted as Γ_5. By decomposing $\mathcal{K}$ into 4-simplices with identified pairing intermediate boundaries, the partition function can be written as a product of 4-simplex amplitudes with a summation over intermediate boundary states.

$$Z(\mathcal{K}) = \sum_{\{\pi_f, i_e\}} \prod_{f\in\mathcal{K}*} d_{\pi_f} \prod_{v} A_v(\pi_f, i_e) \tag{1.68}$$

This formally gives a tensor network description where those intermediate boundaries are maximally entangled states as pointed in [51].

Instead of working with the original BF theory, we would like to start with the BF theory with the Holst term, which is

$$S = \int_M \langle^\gamma B \wedge \mathcal{F}[A]\rangle + \Phi \tag{1.69}$$

The phase space associated with manifold $\mathcal{K}$ are

$$P_{\mathcal{K}} = T^*\text{SL}(2,\mathbb{C})^L, \qquad (\Sigma_f^{IJ}, h_f) \in T^*\text{SL}(2,\mathbb{C}) \tag{1.70}$$

for a Lorentzian model, where L is the number of triangles, $h_f \in \text{SL}(2,\mathbb{C})$ is the holonomy along the edges and $\Sigma_f^{IJ} \in \mathfrak{sl}(2,\mathbb{C})$ is its conjugate momenta. Σ^{IJ} and B^{IJ} are related to each other by

$$\Sigma = (* + \gamma^{-1})B, \qquad B = \frac{\gamma}{1+\gamma^2}(1 - \gamma *)\Sigma \tag{1.71}$$

for $\gamma \neq \pm i$.

1.3.2. Simplicity constraint and EPRL/FK-CH Models

Since Z_{BF} is nothing else but the integration over space of flat connections, clearly there is no local degree of freedom in the BF theory. The local degree of freedom will appear after we impose the simplicity constraint which yielding the general relativity.

With the simplicial decomposition $\mathcal{K}$ of M, it has been proven that, a linear version of simplicity constraint can be employed

$$(u_e)^I B_{IJ} = \frac{\gamma}{1+\gamma^2}(u_e)^I((1-\gamma *)\Sigma_{f\,IJ}) = 0 \tag{1.72}$$

where u_e is a 4 normal vector associated to each tetrahedron t_e. Geometrically, the simplicity constraint implies that, each triangle f in tetrahedron t_e is associated with a simple bivector B_f.

The constraint operators is weakly imposed on the states, since the simplicity constraint do not commute among themselves

$$\langle\psi|\,(u_e)^I B_{IJ}\,|\psi\rangle = 0 \tag{1.73}$$

Such imposition of the simplicity constraint leads to the EPRL-CH model.

Usually a partial gauge fixing is taken to the above constraints, which corresponding to pick a special normal u for all of the tetrahedra $\forall_e,\ u_e = u$. As a result, the intertwiner associated with each tetrahedron defined above is replaced by the intertwiner of the stabilizer group $H \in G$. In a Euclidean model, one can fix the normal to be $u = (1,0,0,0)$ without losing genetic, while in a Lorentzian model, there are different choices with different normal subgroups:

— $u = (1,0,0,0)$, $H = SU(2)$, $G = \text{Spin}(4)$, Euclidean EPRL/FK models

— $u = (1,0,0,0)$, $H = SU(2)$, $G = \text{SL}(2,\mathbb{C})$, Lorentzian EPRL/FK models

— $u = (0,0,0,1)$, $H = SU(1,1)$, $G = \text{SL}(2,\mathbb{C})$, Conrady-Hnybida Extension

Such implementation corresponding to reduces the possible unitary irreps appears in the model. We will not going to details the calculation but summarize the result of restriction condition after imposing weakly the quantum simplicity constraint (3.9) [61, 63, 64] :

— $u = (1, 0, 0, 0)$, spacelike triangles

$$\rho = \gamma n, \qquad n = j \tag{1.74}$$

— $u = (0, 0, 0, 1)$, spacelike triangles

$$\rho = \gamma n, \qquad n = j \tag{1.75}$$

— $u = (0, 0, 0, 1)$, timelike triangles

$$\rho = -n/\gamma, \qquad s = \frac{1}{2}\sqrt{n^2/\gamma^2 - 1} \tag{1.76}$$

Here $(\rho \in \mathbb{R},\ n \in \mathbb{Z}/2)$ are labels of $\mathrm{SL}(2, \mathbb{C})$ irreps, $j \in \mathbb{N}/2$ is the label of $\mathrm{SU}(2)$ irreps or $\mathrm{SU}(1, 1)$ discrete series and $s \in \mathbb{R}$ is the label of $\mathrm{SU}(1, 1)$ continous series, we will give a brief introduction of $\mathrm{SU}(1, 1)$ and $\mathrm{SL}(2, \mathbb{C})$ representation theory in chapter 3 later. As a result, the area spectrum is given by

$$A_f = l_p^2 \begin{cases} \frac{n_f}{2} & \text{timelike triangle} \\ \gamma j_f & \text{spacelike triangle} \end{cases} \tag{1.77}$$

which is discretized and coincide with the result from canonical approach.

The spin foam amplitude can be expressed in the coherent state representation:

$$A_v(K) = \sum_{j_f} \prod_f \mu(j_f) \int_{\mathrm{SL}(2,\mathbb{C})} \prod_e dg_{\nu e} \prod_{(e,f)} \int_{S^2} dN_{ef} \left\langle \Psi_{\rho_f n_f}(N_{ef}) \right| D^{(\rho_f, n_f)}(g_{ev} g_{ve'}) \left| \Psi_{\rho_f n_f}(N_{e'f}) \right\rangle \tag{1.78}$$

where $|\Psi_\rho(N)\rangle$ refers to the peremlomov coherent states [135, 136] for subgroup H in ρ representation with N a unit normal on a sphere or a hyperboloid for spacelike or timelike surface respectively. The SFM on $\mathcal{K}$ then can be written in the integral representation

$$]Z(\mathcal{K}) = \sum_{\vec{J}} \prod_f d_{J_f} \int [\mathrm{d}X]\, e^{\sum_f J_f F_f[X]}, \tag{1.79}$$

where f are 2-faces in $\mathcal{K}$ colorred by half-integer spins J_f. Eq.(1.79) can be regarded as a universal integral expression of spin foam models, while different models have different variables X, functions $F_f[X]$ and measure $[dX]$. dJ is a face amplitude related to the dimension of the representation J_f. For EPRL/FK model and the Conrady-Hnybida extension, X and $F_f[X]$ are given by,

— Euclidean EPRL/FK model:

$$X \equiv \left(g_{ve}^{\pm}, \xi_{ef}\right) \tag{1.80}$$

including $(g^+_{ve}, g^-_{ve}) \in \mathrm{Spin}(4)$ at each pair of 4-simplex v and 3d boundary tetrahedron $e \subset \partial v$, and $\xi_{ef} \in \mathbb{C}^2$ at each pair of e and $f \subset \partial e$. ξ_{ef} is normalized by the $SU(2)$ Hermitian inner product $\langle \cdot | \cdot \rangle$ on $\mathbb{C}^2$. $F_f[X]$ in the exponent is a function of $g^{\pm}_{ve}, \xi_{ef}$ and independent of J_f:

$$F_f\,[X] = \sum_{v, f\subset v} \Big[(1-\gamma)\ln\langle \xi_{ef}\big|(g^-_{ve})^{-1} g^-_{ve'}\big|\xi_{e'f}\rangle + (1+\gamma)\ln\langle \xi_{ef}\big|(g^+_{ve})^{-1} g^+_{ve'}\big|\xi_{e'f}\rangle\Big]. \tag{1.81}$$

— Lorentzian EPRL model

$$X \equiv (g_{ve}, z_{vf}, \xi_{ef}) \tag{1.82}$$

including now $g_{ve} \in \mathrm{SL}(2,\mathbb{C})$, $z_{vf} \in \mathbb{CP}^1$ at each vertex $v \subset \partial f$ on dual face f. The normalized spinors $\xi_{ef} \in \mathbb{C}^2$ is normalized by the $SU(2)$ Hermitian inner product $\langle \cdot\,,\,\cdot \rangle$ and defined as

$$\xi^{\alpha} = v^{-1\dagger}\xi^{\alpha}_0, \qquad \text{with}\xi_0 = (1,0)^T, \quad v \in \mathrm{SU}(2) \tag{1.83}$$

Defining $Z_{vef} = g^{\dagger}_{ve} z_{vf}$, $F_f[X]$ is written as

$$F_f[X] = \sum_{v, f\subset v} \left(\ln \frac{\langle \xi_{ef}, Z_{vef}\rangle^2 \langle Z_{ve'f}, \xi_{e'f}\rangle^2}{\langle Z_{vef}, Z_{vef}\rangle \langle Z_{ve'f}, Z_{ve'f}\rangle} + i\gamma \ln \frac{\langle Z_{ve'f}, Z_{ve'f}\rangle}{\langle Z_{vef}, Z_{vef}\rangle} \right) \tag{1.84}$$

— Hnybida-Conrady extension - spacelike triangles f in timelike tetrahedra

$$X \equiv (g_{ve}, z_{vf}, \xi_{ef}) \tag{1.85}$$

Here again $g_{ve} \in \mathrm{SL}(2,\mathbb{C})$, $z_{vf} \in \mathbb{CP}^1$., and ξ are spinors defined as

$$\xi^{\alpha} = v^{-1\dagger}\xi^{\alpha}_0, \qquad \text{with} \quad \begin{cases} \xi^+_0 = (1,0)^T \\ \xi^-_0 = (0,1)^T \end{cases}, \quad v \in \mathrm{SU}(1,1) \tag{1.86}$$

which is normalized by $\mathrm{SU}(1,1)$ invaraint inner product $\langle \cdot, \cdot \rangle$. Function $F(X)$ is given by

$$F^{\pm}_f[X] = \sum_{v, f\subset v} \left(\mathrm{i}\gamma \ln \frac{\langle Z_{vef}, Z_{vef}\rangle}{\langle Z_{ve'f}, Z_{ve'f}\rangle} - \ln \frac{\langle \xi^{\pm}_{e'f}, Z_{ve'f}\rangle^2 \langle Z_{vef}, \xi^{\pm}_{ef}\rangle^2}{\langle Z_{vef}, Z_{vef}\rangle \langle Z_{ve'f}, Z_{ve'f}\rangle} \right) \tag{1.87}$$

where $Z_{vef} = g^{\dagger}_{ve} z_{vf}$.

— Hnybida-Conrady extension - timelike triangles f in timelike tetrahedra
We will going to study it in chapter 3

1.3.3. Semi-classical analysis of the model

The semi-classical regime of the model is defined through the limit $\hbar \to 0$ while keeping the area fixed. By the area spectrum, it is immediately to see this procedure amount to take all the spins $J_f = \Lambda j_f$ scale uniformly to the limit $\Lambda \to \infty$, which is refer to a large spin (large-j) limit. With the integral formulation of spin foam model using the coherent states (1.79), since all the action is proportional to the spin, the semi-classical analysis turns out to the usual asympotics analysis of the integral.

Here we will show the example with Hnybida-Conrady extension where all triangles f are spacelike. Such limit is investigated in [85]. These result will be used in our later derivation in chapter 3 for the semi-classical analysis of a mixed timelike tetrahedron, namely, the tetrahedron contains both timelike and spacelike triangles as its boundary. The analysis for the other models will follow the exactly same procedure and can be found on [77–80, 85, 137]

As in the usual asymptotic analysis, the critical points are determined by the equation of motion

$$\delta_{z_{vf}} F(X) = \delta_{v_{ef}} F(X) = \delta_{g_{ve}} F(x) = 0 \tag{1.88}$$

while the dominate part are given in the case

$$\mathrm{Re}(F(X)) = 0 \tag{1.89}$$

Solutions of above equation specify the critical configuration $X_0 \subset X$ where the amplitude is given as

$$I \sim \sum_{X_0} \frac{1}{\sqrt{H(X_0)}} \mathrm{e}^{\lambda \sum_f A_f F_f(X_0)} \tag{1.90}$$

One can prove that, the critical configurations X_0 of spin foam models are determined by the following equations

$$B_f(v) := G_{ve} B_{ef} g_{ev} = G_{ve'} B_{e'f} g_{e'v}, \tag{1.91}$$

$$\forall_{f:e\in\partial f}\ N_e(v) \cdot B_f(v) = 0, \tag{1.92}$$

$$\sum_{f:e\in\partial f} \epsilon_{ef}(v) A_f B_f(v) = 0 \tag{1.93}$$

where B_{ef} is a bivector specified by ξ_{ef} and $N_e(v) = g_{ve} u$ is a normalized vector. Here $G = \Psi(g) \in SO(1,3)$ is the defined via the map $\Psi : SL(2,\mathbb{C}) \to SO(1,3)$. These equations have a direct geometrical meaning: they implies 10 triangles at vertex v specified by the bivector $B_f(v)$ in M, which can form 5 tetrahedra. One can immediately check that, when arbitrarily 4 $N_e(v)$ out of 5 are independent, which we denotes the non-degenerate case, these equations is exactly the equations determines a 4-simplex up to an orientation. Thus the critical configurations of the spin foam model corresponding to a simplicial geometry. Furthermore, one can prove that, the action evaluated at the critical configuration leads to a phase, which is the Regge action on given simplicial geometry up to a sign

$$F(X_0) = \mathrm{i} r \sum_f A_f \epsilon_f + 2\pi k, \qquad r = \pm 1, \quad k \in \mathbb{N} \tag{1.94}$$

However, in EPRL models, the critical configurations also possibly contain the degenerate contributions, where all $N_e(v)$ at a vertex v are parallel to each other. Such contribution corresponds to a degenerate vector geometry or non-degenerate flipped signature 4 simplex, which can only be removed after fixing the boundary data.

2. Gravity as an $\mathfrak{su}(1,1)$ Gauge Theory in Four Dimensions

Sommaire

2.1. Introduction

We would like address in this chapter the question we mentioned at the beginning of the part: whether the construction and physical predictions of Loop Quantum Gravity are changed or not when one makes another partial gauge fixing or no gauge fixing at all. Most of the approaches to address this issue are based on attempts to quantize the Holst action without any partial gauge fixing, and then keeping the full Lorentz internal invariance of the theory. Now if one performs the canonical analysis of the $\mathfrak{sl}(2,\mathbb{C})$ Holst action, second class constraints appear simply because the connection has more components than the tetrad field. The appearance of second class constraints makes the classical analysis and then the quantization of the theory much more involved. In the analysis of constrained systems, there are two ways of dealing with second class constraints: one can either solve them explicitly, or implement them in the symplectic structure by working with the Dirac bracket. These two methods are totally equivalent. Using the Dirac bracket, Alexandrov and collaborators

[138–141] were able to construct a two-parameters family of Lorentz-covariant connections (which are diagonal under the action of the area operator, and transform properly under the action of spatial diffeomorphisms). Generically, these connections are non-commutative and therefore the theory becomes very difficult to quantize. The alternative route to deal with covariant connections was initiated by Barros e Sa in [91] who solved explicitly the second class constraints. In this approach, the phase space is parametrized by two pairs of canonical variables: the generalization (A, E) of the usual Ashtekar-Barbero connection and its conjugate densitized triad E; and a new pair of canonically conjugated fields (χ, ζ), where χ and ζ both take values in $\mathbb{R}^3$. Then, Barros e Sa expressed the remaining boost, rotation, diffeomorphism and scalar constraints in terms of these variables. The elegance of this approach is that it enables one to have a simple symplectic structure with commutative variables, and a tractable expression for the boost, rotation and diffeomorphism generators. Although the scalar constraint becomes more complicated, this structure is enough to study the kinematical structure of loop quantum gravity with a fully Lorentz invariance. This has precisely been done in [93, 142] where one constructed the unique spatial connection which is not only commutative but also transforms covariantly under the action of boosts and rotations. In fact, this connection coincides with the commutative Lorentz connection studied earlier in [141] and the one found in [143]. Furthermore, it has been shown to be gauge related to the Ashtekar-Barbero connection via a pure boost parametrized by the vector χ viewed as a velocity. Hence, the construction proposed in [93, 142] works only when $\chi^2 < 1$. Thus, the pairs of canonical variables formed with the $\mathfrak{sl}(2, \mathbb{C})$ connection and its conjugate electric field parametrize only a part of the fully covariant phase space of the Holst action.

This chapter enables us to explore the sector $\chi^2 > 1$ while studying a partial gauge fixing of the Holst action that reduces $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{su}(1, 1)$. Hence, we start with the Lorentz covariant parametrization of the Holst action found by Barros e Sa [91]. We find a partial gauge fixing which breaks the $\mathfrak{sl}(2, \mathbb{C})$ internal symmetry into $\mathfrak{su}(1, 1)$ and this is possible if and only if $\chi^2 > 1$. Such a partial gauge fixing corresponds to a canonical splitting of the space-time $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ is no more space-like (as it is the case in the usual Ashtekar-Barbero parametrization) but inherits a Lorentzian metric of signature is $(-, +, +)$. As a consequence, only three out of the initial six first class constraints remain after the partial gauge fixing, and they generate as expected the local $\mathfrak{su}(1, 1)$ gauge transformations. The other three constraints form with the three gauge fixing conditions a set of second class constraints that we solve explicitly. Then, we construct an $\mathfrak{su}(1, 1)$ connection which appears to be commutative in the sense of the Poisson bracket. This remarkable construction allows us to investigate the loop quantization of the theory and to build the kinematical Hilbert space on a given graph Γ whose edges are associated to $\mathrm{SU}(1, 1)$ holonomies. It is well-known that [144] the non-compactness of the gauge group prevents us from defining the projective limit of spin-networks and then the sum over all graphs of kinematical Hilbert space is ill-defined. Nonetheless, if one restricts the study to one given graph Γ, it is possible to define the action of the area operator and one easily finds that a space-like area has a discrete spectrum whereas the spectrum of a time-like area is continuous. In other words, if one considers a spin-network defined on a graph Γ dual to a discretization $\Delta = \Gamma^*$ of a $(2+1)$-dimensional manifold, edges e of Γ are colored with representations in the discrete series (resp. in the continuous series) if the dual face $f = e^*$ of Δ is space-like (resp. time-like). The spectrum of space-like areas is in total agreement with the one obtained in the usual Ashtekar-Barbero formalism for space-like surfaces. Note that there is a close relationship between our work here and the results obtained in [73, 97] in the framework of spin-foam models.

This chapter is organized as follows. In Section II, we present the partial gauge fixing that breaks $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{su}(1,1)$ before constructing the $\mathfrak{su}(1,1)$ connection and its associated electric field. In Section III we explore the kinematical quantization of the theory on a given graph and we compute the spectra of area operators which act unitarily in the kinematical Hilbert space. We conclude in Section IV with a brief summary of the most important results and a discussion on the consequences of this new parametrization for the description of black holes in Loop Quantum Gravity.

2.2. Gravity as an SU(1,1) gauge theory

In this section, we first show how to make a partial gauge fixing of the full Lorentz invariant Holst action which reduces the internal $\mathfrak{sl}(2,\mathbb{C})$ gauge symmetry to $\mathfrak{su}(1,1)$. At the same time, we keep the invariance under diffeomorphisms on Σ. In that case, we will see that the splitting of the space-time $\mathcal{M} = \Sigma \times \mathbb{R}$ is such that Σ is no more a space-like hypersurface as it is the case in the time gauge but inherits instead a Lorentzian structure. Then, we construct a parametrization of the phase space in terms of an $\mathfrak{su}(1,1)$ connection and its conjugate electric field which transforms in the adjoint representation of $\mathfrak{su}(1,1)$. Furthermore, we show that these variables are Darboux coordinates for the phase space, which paves the way towards a quantization of the theory explored in the following Section.

2.2.1. Breaking the internal symmetry: from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{su}(1,1)$

As we have already underlined in section ??, imposing the time gauge $\chi \approx 0$ in the fully covariant Holst action breaks the boost invariance and only the rotational parts of the constraints remain first class among the original 6 internal symmetries. Hence, we get an $\mathfrak{su}(2)$ invariant theory of gravity. In fact, we proceed in a very similar way to construct an $\mathfrak{su}(1,1)$ invariant theory from the

Holst action: we find a partial gauge fixing such that two components of the boosts constraints and one of the rotational constraints remain first class whereas the three others form with the gauge fixing conditions a second class system. Naturally, we consider a gauge fixing condition of the form

$$\mathcal{X} \equiv \chi - \chi_0 \approx 0 \tag{2.1}$$

where χ_0 is a fixed non-dynamical vector. Inspiring ourselves with what happens in the time gauge, we expect (2.1) to form a second class system with three out of the six constraints (1.16). These three second class components of the Lorentz generators are supposed to be

$$\mathcal{R} \cdot u \approx 0\,, \quad \mathcal{R} \cdot v \approx 0\,, \quad \mathcal{B} \cdot n \approx 0\,, \tag{2.2}$$

where u and v are two given normalized orthogonal vectors and $n = v \times u$. The reason is that we are left with two boosts and one rotations which are expected to reproduce (up to the addition of second class constraints) an $\mathfrak{su}(1,1)$ Poisson algebra. To derive the conditions for this to happen, we start rewriting (2.2) as a linear system of equations for ζ:

$$M\zeta = \begin{pmatrix} \zeta \cdot U \\ \zeta \cdot V \\ \zeta \cdot W \end{pmatrix} \approx \begin{pmatrix} \mathcal{R} \cdot u|_{\zeta=0} \\ \mathcal{R} \cdot v|_{\zeta=0} \\ \mathcal{B} \cdot n|_{\zeta=0} \end{pmatrix} \quad \text{with} \quad M \equiv \begin{pmatrix} {}^tU \\ {}^tV \\ {}^tW \end{pmatrix} \text{ and } \left\{ \begin{array}{lcl} U & \equiv & \chi \times u \\ V & \equiv & \chi \times v \\ W & \equiv & -n + (\chi \cdot n)\chi \end{array} \right. \tag{2.3}$$

The system admits an unique solution for ζ if and only if

$$\text{det}M = U \times V \cdot W = (1 - \chi^2)(\chi \cdot n)^2 \neq 0\,, \tag{2.4}$$

which implies that $\chi^2 \neq 1$ and $\chi \cdot n \neq 0$. When we assume this is the case, the solution ζ_0 can be easily expressed in terms of the components of χ_0, E and A inverting (2.3) as follows

$$\zeta_0 \;=\; M^{-1} \begin{pmatrix} \mathcal{R} \cdot u|_{\zeta=0} \\ \mathcal{R} \cdot v|_{\zeta=0} \\ \mathcal{B} \cdot n|_{\zeta=0} \end{pmatrix} = \frac{(\mathcal{B} \cdot n + \mathcal{R} \cdot \chi \times n)\,\chi - (1 - \chi^2)\mathcal{R} \times n}{(1 - \chi^2) n \cdot \chi}|_{\zeta=0}\,, \tag{2.5}$$

where we used the expression

$$M^{-1} \;=\; \frac{1}{U \times V \cdot W} \big(V \times W\,,\, W \times U\,,\, U \times V\big)\,. \tag{2.6}$$

Hence, the three constraints (2.2) are equivalent to the three conditions

$$\mathcal{Z} \equiv \zeta - \zeta_0(\chi_0, E, A) \approx 0\,. \tag{2.7}$$

Now, it becomes clear that the gauge fixing conditions $\mathcal{X} \approx 0$ (2.1) and the three constraints $\mathcal{Z} \approx 0$ form a second class system because their associated 6×6 Dirac matrix Δ

$$\Delta(x,y) \equiv \begin{pmatrix} X(x,y) & Y(x,y) \\ -{}^tY(x,y) & Z(x,y) \end{pmatrix} \text{ with } \begin{cases} X^i_j(x,y) & \equiv & \{\chi^i(x), \chi_j(y)\} = 0 \\ Y^i_j(x,y) & \equiv & \{\mathcal{X}^i(x), \mathcal{Z}_j(y)\} \; = \; \delta^j_i \delta^3(x-y) \\ Z^i_j(x,y) & \equiv & \{\mathcal{Z}^i(x), \mathcal{Z}_j(y)\} \end{cases} \tag{2.8}$$

is invertible whatever Z is. These two constraints allow to eliminate the variables χ and ζ from the phase space provided that one introduces the external non dynamical field χ_0.

We are left with three constraints from (1.16) which are required to satisfy an $\mathfrak{su}(1,1)$ Poisson algebra once one replaces χ by χ_0 and ζ by ζ_0. These constraints are denoted

$$\mathcal{J}_u \equiv \mathcal{B} \cdot u|_{\chi_0,\zeta_0}\,, \quad \mathcal{J}_v \equiv \mathcal{B} \cdot v|_{\chi_0,\zeta_0}\,, \quad \mathcal{J}_n \equiv \mathcal{R} \cdot n|_{\chi_0,\zeta_0}\,. \tag{2.9}$$

From now on, we will omit to mention the index 0 for χ to lighten the notations. However, χ has to be understood as an external non dynamical field, and not as the initial dynamical variable in the fully Lorentz invariant Holst action.

A long but standard calculation shows that the three constraints (2.9) form a closed Poisson algebra only when

$$u \cdot \chi = v \cdot \chi = 0\,. \tag{2.10}$$

This is equivalent to the condition that $\chi = \pm|\chi| n$ where $|\chi| \equiv \sqrt{\chi \cdot \chi}$ is the norm of χ. Without loss of generality, we choose $\chi = |\chi| n$. As a consequence, the partial gauge fixing (2.1) leaves the remaining three constraints (2.9) first class only when (2.10) is satisfied. In that case, the expressions of (2.9) simplify a lot and they can be written as

$$\mathcal{J}_0 \equiv \mathcal{J}_n = n \cdot \tilde{\mathcal{J}}\,, \quad \mathcal{J}_1 \equiv C\mathcal{J}_v = Cu \cdot \tilde{\mathcal{J}}, \quad \mathcal{J}_2 \equiv C\mathcal{J}_u = -Cv \cdot \tilde{\mathcal{J}}\,, \tag{2.11}$$

where $C = 1/\sqrt{|\chi^2 - 1|}$ is a normalization function and we introduced the vector field

$$\tilde{\mathcal{J}} \equiv -\frac{1}{\gamma}\Big(\partial_a E^a + \partial_a(E^a \times \chi) \times \chi\Big) + \tilde{A}_a \times E^a \tag{2.12}$$

given in terms of the $\mathfrak{su}(2)$-valued one form $\tilde{A}$ defined by

$$\tilde{A}_a = A_a - (A_a \cdot \chi)\chi - \partial_a \chi\,. \tag{2.13}$$

Finally, one shows that the constraints algebra reduces to the simple form

$$\{\mathcal{J}_0, \mathcal{J}_1\} = \mathcal{J}_2\,, \qquad \{\mathcal{J}_0, \mathcal{J}_2\} = -\mathcal{J}_1\,, \qquad \{\mathcal{J}_1, \mathcal{J}_2\} = \sigma \mathcal{J}_0\,, \tag{2.14}$$

where

$$\sigma \equiv \frac{1-\chi^2}{|1-\chi^2|} = \mathrm{sg}(1-\chi^2). \tag{2.15}$$

The function $\mathrm{sg}(x)$ denotes the sign of $x \neq 0$. As a consequence, the remaining three constraints form an $\mathfrak{su}(2)$ Poisson algebra when $\chi^2 < 1$ and an $\mathfrak{su}(1,1)$ Poisson algebra when $\chi^2 > 1$ (the case $\chi^2 = 1$ is excluded from the scope of our method and should be studied in a different way [a]). We can write the constraints algebra in the more compact form

$$\{\mathcal{J}_\alpha, \mathcal{J}_\beta\} = \epsilon_{\alpha\beta}{}^{\tau}\, \mathcal{J}_\tau \tag{2.16}$$

where $\alpha, \beta, \tau \in (0,1,2)$ and $\epsilon_{\alpha\beta\tau}$ is the totally antisymmetric symbol with $\epsilon_{012} = +1$. Furthermore, the indices are lowered and raised with the flat metric and its inverse $\mathrm{diag}(\sigma, +1, +1)$: it is the flat Euclidean metric $\delta_{\alpha\beta}$ when $\sigma = +1$ and the flat Minkowski metric $\eta_{\alpha\beta} \equiv \mathrm{diag}(-1,+1,+1)$ when $\sigma = -1$. Hence, as announced above, one recognizes respectively the $\mathfrak{su}(2)$ and the $\mathfrak{su}(1,1)$ Lie algebras.

Let us close this analysis with one remark. The gauge fixing condition (2.1) makes the three constraints (2.2) (which are first class in the full Lorentz invariant Holst action) second class. Hence, we have left two boosts and one rotation first class in order to get an $\mathfrak{su}(1,1)$ gauge symmetry at the end of the process. This is what we arrive at when $\chi^2 > 1$ but we obtain an $\mathfrak{su}(2)$ gauge symmetry when $\chi^2 < 1$ even though we kept two boosts among the remaining first class constraints. The reason is that, at the end of the gauge fixing process, the remaining first class constraints are non-trivial linear combinations of the six initial first class constraints and the gauge fixing conditions. Hence, they could form either an $\mathfrak{su}(1,1)$ or an $\mathfrak{su}(2)$ algebra. The two most important ingredients in our construction is that, first, we replace three out of the initial six first class constraints by constraints of the type (2.7) which fix ζ, and second we impose that the remaining constraints (when ζ and χ are replaced from $\mathcal{X} \approx 0$ and $\mathcal{Z} \approx 0$) form a closed Poisson algebra. In that respect, we could have considered the conditions $\mathcal{B}.u \approx \mathcal{B}.v \approx \mathcal{B}.n \approx 0$ instead of (2.2): we would have obtained another set of conditions fixing ζ and then, following the same strategy, we would have shown that the remaining three constraints are generators of a closed algebra

a. This case corresponds to a slicing of the space-time in a light like direction. Our analysis based on a partial gauge fixing can be adapted to that situation. Such a Hamiltonian description could provide us with a new formulation (eventually simpler) of gravity in the light front related to [145].

provided that (2.10) is satisfied. The remaining symmetry would have been $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$ depending on the sign of σ exactly as in the previous analysis.

2.2.2. On the space-time foliation

Let us discuss the reason why the sign σ of $(\chi^2 - 1)$ determines the signature of the symmetry algebra $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$. For that purpose, it is very instructive to study the properties of the metric g_{ab} induced on the hypersurface Σ whose expression is

$$g_{ab} \equiv e^I_a \eta_{IJ} e^J_b = e^i_a \gamma_{ij} e^j_b \quad \text{with} \quad \gamma_{ij} \equiv \delta_{ij} - \chi_i \chi_j \tag{2.17}$$

where we inverted the defining relation $\chi_i = e^a_i e^0_a$ to replace e^0_a by $e^i_a \chi_i$. It is immediate to notice that this formula is compatible with the expression of the inverse metric given in [91, 142]

$$\det(g)\, g^{ab} \;=\; (1-\chi^2)\, E^a_i \gamma^{ij} E^b_j \,, \quad \gamma^{ij} \equiv \delta_{ij} - \frac{\chi_i \chi_j}{1-\chi^2} \,, \tag{2.18}$$

due to the properties

$$E^a_i = \det(e) e^a_i \,, \quad \det(g) = (1-\chi^2)\det(e)^2 \,, \quad \gamma^{ij}\gamma_{jk} = \delta^i_k \,. \tag{2.19}$$

Thus, the identity (2.17) implies immediately that the metric induced on Σ has the same signature as γ_{ij}. This latter metric can be easily diagonalized and its eigenvalues/eigenvectors are easily obtained from

$$\gamma_{ij} u^j = u_i \text{ when } u \cdot \chi = 0 \,, \quad \text{and} \quad \gamma_{ij} \chi^j = (1-\chi^2)\chi_i \,. \tag{2.20}$$

Therefore, the signature of the metric depends on the sign of $(\chi^2 - 1)$: Σ is spacelike when $\chi^2 < 1$ whereas it inherits a Lorentzian metric when $\chi^2 > 1$. This clearly explains the presence of σ in the constraints algebra (2.15) and the nature of the gauge symmetry. When the symmetry algebra is $\mathfrak{su}(2)$, the space-time is foliated as usual into hypersurfaces orthogonal to a timelike vector whereas it is foliated in a space-like direction when the symmetry algebra is $\mathfrak{su}(1,1)$. This latest case is not conventional but it is the one we are interested in.

2.2.3. Phase space parametrization

From now on, we will mainly focus on the case $\chi^2 > 1$ which has never been studied so far (we will shortly discuss the case $\chi^2 < 1$ at the end of this Section). As the theory admits $\mathfrak{su}(1,1)$ as a gauge symmetry algebra, it is natural to look for a parametrization of the phase space adapted to this symmetry. More precisely, we look for conjugate variables which transform in a covariant way under the Poisson action of the $\mathfrak{su}(1,1)$ generators. In a first part, we exhibit an unique $\mathfrak{su}(1,1)$-valued connection which is commutative in the sense of the Poisson bracket. This connection is the $\mathfrak{su}(1,1)$ analogous of the generalized Ashtekar-Barbero connection defined for $\chi \neq 0$ in [93, 142] for instance. In a second part, we show that it is canonically conjugate to an electric field which transforms as a vector under the action of the first class constraints. Hence, the $\mathfrak{su}(1,1)$-connection together with its conjugate electric field provide us with a very useful and natural parametrization of the phase space. We finish with computing the action of the vectorial constraints on these variables which transform as expected under the action of the generators of diffeomorphisms.

2.2.3.1. The connection

Now, we address the problem of finding an $\mathfrak{su}(1,1)$ connection defined by

$$\mathcal{A} = \mathcal{A}^0 J_0 + \mathcal{A}^1 J_1 + \mathcal{A}^2 J_2 \quad \text{with} \quad [J_\alpha, J_\beta] = \epsilon_{\alpha\beta}{}^{\tau} J_\tau \tag{2.21}$$

which satisfies the following requirements. First, is constructed from the components of A (such that it is commutative in the sense of the Poisson bracket) and the non-dynamical vectors (χ, u and v) only. Second it transforms as

$$\delta_\varepsilon \mathcal{A} \;=\; d\varepsilon + [\mathcal{A}, \varepsilon]\,, \tag{2.22}$$

under the action of the gauge transformations where $\varepsilon = \varepsilon^\alpha(x) J_\alpha$ is an arbitrary $\mathfrak{su}(1,1)$-valued function on Σ. For this relation to make sense, we have to precise the definition of δ_ε in terms of the gauge generators. In particular, we have to establish the link between the parameter $\varrho \in \mathbb{R}^3$ entering in the smeared constraint $\tilde{\mathcal{J}}(\varrho)$ and the parameter ε defining the $\mathfrak{su}(1,1)$ infinitesimal gauge transformations of $\mathcal{A}$. From (2.11), it is natural to expect that

$$\delta_\varepsilon \mathcal{A} \;=\; \{\tilde{\mathcal{J}}(\varrho), \mathcal{A}\} \quad \text{with} \quad \varepsilon^0 = \varrho \cdot n,\;\; \varepsilon^1 = c_1 \varrho \cdot u,\;\; \varepsilon^2 = c_2 \varrho \cdot v\,, \tag{2.23}$$

where c_1 and c_2 are functions of χ. Now, the problem consists in finding the components of $\mathcal{A}$ and the functions c_1 and c_2 such that $\mathcal{A}$ transforms as an $\mathfrak{su}(1,1)$ connection under the action of the first class constraints.

We are going to propose an ansatz for $\mathcal{A}$. As the expressions of the gauge generators are simpler with $\tilde{A}$ instead of A itself, we also look for an $\mathfrak{su}(1,1)$ connection $\mathcal{A}$ written in terms of $\tilde{A}$. This is possible because, when $\chi^2 \neq 1$, $\tilde{A}$ can be uniquely expressed in terms of A and χ inverting the relation (2.13) as follows:

$$A_a \;=\; \tilde{A}_a + \partial_a \chi + \chi \cdot (\tilde{A}_a + \partial_a \chi) \frac{\chi}{1-\chi^2}\,. \tag{2.24}$$

Inspiring ourselves from the decomposition (2.11) of the first class constraints into $\mathfrak{su}(1,1)$ gauge generators, we propose the following form for the components of $\mathcal{A}$:

$$\mathcal{A}^0 = p_0\,(\tilde{A}\cdot n) + q_0\,, \quad \mathcal{A}^1 = p_1\,(\tilde{A}\cdot u) + q_1\,, \quad \mathcal{A}^2 = p_2\,(\tilde{A}\cdot v) + q_2\,, \tag{2.25}$$

where (p_0, p_1, p_2) are functions of χ whereas (q_0, q_1, q_2) are one-forms constructed from $d\chi$, du and dv only.

Hence, the problem reduces now in finding the functions (c_1, c_2) and (p_0, p_1, p_2) together with the one-forms (q_0, q_1, q_2) which solve the equations (2.23). These equations can be more explicitly written as

$$p_0\{\tilde{\mathcal{J}}(\varrho), (\tilde{A}\cdot n)\} = d(\varrho\cdot n) + c_1(p_2\,(\tilde{A}\cdot v) + q_2)\varrho\cdot u - c_2(p_1\,(\tilde{A}\cdot u) + q_1)\varrho \tag{2.26}$$
$$p_1\{\tilde{\mathcal{J}}(\varrho), (\tilde{A}\cdot u)\} = d(c_1\varrho\cdot u) + (p_2\,(\tilde{A}\cdot v) + q_2)\varrho\cdot n - c_2(p_0\,(\tilde{A}\cdot n) + q_0)\varrho \tag{2.27}$$
$$p_2\{\tilde{\mathcal{J}}(\varrho), (\tilde{A}\cdot v)\} = d(c_2\varrho\cdot v) - (p_1\,(\tilde{A}\cdot u) + q_1)\varrho\cdot n + c_1(p_0\,(\tilde{A}\cdot n) + q_0)\varrho \tag{2.28}$$

where each Poisson brackets on the l.h.s. are easily deduced from

$$\{\tilde{\mathcal{J}}(\varrho), \tilde{A}\} = -\frac{1}{\gamma}(1-\chi^2)d\varrho + \tilde{A}\times\varrho - \frac{1}{\gamma}\chi\times(d\chi\times\varrho) + (\tilde{A}\cdot\chi\times\varrho)\chi\,. \tag{2.29}$$

A straightforward calculations show that the previous system reduces to the following three sets of equations:

$$\begin{array}{ll} p_0(1-\chi^2) = c_1 p_2 = c_2 p_1 = -\gamma\,, & dn + c_1 q_2 u - c_2 q_1 v = 0\,,\\ p_1 = -p_2 = -c_2 p_0 = \gamma c_1/(\chi^2-1)\,, & d(c_1 u) + q_2 n - c_2 q_0 v + p_1[(u\cdot d\chi)\chi - (\chi\cdot d\chi)u]/\gamma = 0\,,\\ p_1 = -p_2 = c_1 p_0 = -\gamma c_2/(\chi^2-1)\,, & d(c_2 v) - q_1 n + c_1 q_0 u + p_2[(v\cdot d\chi)\chi - (\chi\cdot d\chi)v]/\gamma = 0\,. \end{array}$$

This is clearly an overcomplete set of conditions for the unkowns of the problem. However, an immediate analysis shows that (up to a simple sign ambiguity), the system admits an unique solution given by

$$p_0 = \frac{\gamma}{\chi^2-1}\,, \quad p_1 = \frac{\gamma}{\sqrt{\chi^2-1}}\,, \quad p_2 = -\frac{\gamma}{\sqrt{\chi^2-1}}\,, \tag{2.30}$$
$$q_0 = dv\cdot u\,, \quad q_1 = -\frac{1}{\sqrt{\chi^2-1}}v\cdot dn\,, \quad q_2 = -\frac{1}{\sqrt{\chi^2-1}}u\cdot dn\,, \tag{2.31}$$

with $c_1 = -c_2 = \sqrt{\chi^2-1}$.

As a conclusion, let us summarize the main results of this part. The theory admits an $\mathfrak{su}(1,1)$ gauge connection $\mathcal{A} = \mathcal{A}^0 J_0 + \mathcal{A}^1 J_1 + \mathcal{A}^2 J_2$ whose components are

$$\mathcal{A}^0 = \frac{\gamma}{\chi^2-1}\tilde{A}\cdot n + u\cdot dv\,, \tag{2.32}$$

$$\mathcal{A}^1 = \frac{1}{\sqrt{\chi^2-1}}\left(\gamma\tilde{A}\cdot u - v\cdot dn\right)\,, \tag{2.33}$$

$$\mathcal{A}^2 = -\frac{1}{\sqrt{\chi^2-1}}\left(\gamma\tilde{A}\cdot v + u\cdot dn\right)\,. \tag{2.34}$$

We have just proved that it transforms as follows

$$\delta_\varepsilon\mathcal{A} = \{\tilde{\mathcal{J}}(\varrho),\mathcal{A}\} = d\varepsilon + [\mathcal{A},\varepsilon] \quad \text{with} \quad \varrho = \varepsilon^0 n + \frac{\varepsilon^1 u - \varepsilon^2 v}{\sqrt{\chi^2-1}} \tag{2.35}$$

under the action of the first class constraints. Note that this transformation law is totally consistent with the fact that

$$\tilde{\mathcal{J}}(\varrho) = \mathcal{J}_0(\varepsilon^0) + \mathcal{J}_1(\varepsilon^1) + \mathcal{J}_2(\varepsilon^2)\,, \tag{2.36}$$

where the components of $\tilde{\mathcal{J}}$ are the smeared $\mathfrak{su}(1,1)$ generators introduced in (2.11).

Let us close this analysis with two remarks.
First, one can reproduce exactly the same analysis when $\chi^2 < 1$. In that case, one obtains an $\mathfrak{su}(2)$ connection whose expression is very similar to the previous one obtained for $\mathfrak{su}(1,1)$: everything happens as if one makes the replacement $\sqrt{\chi^2-1} \mapsto -\sqrt{1-\chi^2}$ in the components of the connection. The $\mathfrak{su}(2)$-valued connection is certainly related to the generalized Ashtekar-Barbero connection obtained in different ways [141–143]. In the limit $\chi \to 0$ with n constant, one recovers the usual Ashtekar-Barbero connection in the time-gauge written in the orthonormal basis $(n,-u,v)$:

$$\mathcal{A} = \mathcal{A}^0\, n - \mathcal{A}^1\, u + \mathcal{A}^2\, v = \gamma A\,. \tag{2.37}$$

Second, by construction, the limit $\chi \to 0$ does not exist for the $\mathfrak{su}(1,1)$-valued connection. The analogous of the time gauge is defined by the limit $|\chi| \to \infty$ where the direction n tends to a constant. Let us study this limit, and for simplicity, we assume that the direction n is constant. Starting from the relation

$$\tilde{A}^i_a = {}^\gamma\omega^{ij}_a\chi_j + {}^\gamma\omega^{0i}_a - {}^\gamma\omega^{0i}_a\chi_j\chi^i\,, \tag{2.38}$$

we obtain the following limits for the components of $\mathcal{A}$

$$\mathcal{A}^0_a \to -\gamma\, {}^\gamma\omega^{0i}_a n_i\,, \qquad \mathcal{A}^1_a \to \gamma\, {}^\gamma\omega^{ij}_a n_j u_i\,, \qquad \mathcal{A}^2_a \to -\gamma\, {}^\gamma\omega^{ij}_a n_j v_i\,. \tag{2.39}$$

One recognizes the components of the spin-connection in what we could call the “space-gauge" which would be defined by the choice $e^a_i n^i = 0$ (instead of $e^a_0 = 0$ for the usual time gauge). As a consequence, the limit $|\chi| \to \infty$ with n constant is well-defined and consists in a foliation of the space-time $\mathcal{M} = \Sigma \times \mathbb{R}$ where the slices Σ are orthogonal to the space-like vector $(0, n)$.

2.2.3.2. The electric field

We follow the same strategy to construct an electric field $\mathcal{E}$ which transforms as an $\mathfrak{su}(1,1)$ under the gauge transformations. More precisely, we are looking for $\mathcal{E} = \mathcal{E}^0 J_0 + \mathcal{E}^1 J_1 + \mathcal{E}^2 J_2$ which satisfies two conditions. First we require its components to be constructed from E, χ, u and v only and we consider the natural ansatz

$$\mathcal{E}^0 = r_0(E \cdot n)\,, \quad \mathcal{E}^1 = r_1(E \cdot u)\,, \quad \mathcal{E}^2 = r_2(E \cdot v)\,, \tag{2.40}$$

where (r_0, r_1, r_2) are functions of χ only. Second we require $\mathcal{E}$ to transform as a vector

$$\delta_\varepsilon \mathcal{E} \equiv \{\tilde{\mathcal{J}}(\varrho), \mathcal{E}\} = [\mathcal{E}, \varepsilon] \quad \text{with} \quad \varrho = \varepsilon^0 n + \frac{\varepsilon^1 u - \varepsilon^2 v}{\sqrt{\chi^2 - 1}}\,, \tag{2.41}$$

in adequacy with what has been done in the previous part for the connection. A simple calculation shows that these conditions implies necessarily

$$r_1 = \sqrt{\chi^2 - 1}\, r_0\,, \quad r_2 = -\sqrt{\chi^2 - 1}\, r_0\,, \tag{2.42}$$

where, at this point, r_0 is free because equations (2.41) form a linear system for the unknowns (r_0, r_1, r_2).

Let us close this analysis with three remarks.
First, the free parameter r_0 can be fixed requiring in addition that $\mathcal{E}$ is canonically conjugate to $\mathcal{A}$ according to

$$\{\mathcal{A}^1, \mathcal{E}^1\} = \{\mathcal{A}^2, \mathcal{E}^2\} = 1 \quad \text{and} \quad \{\mathcal{A}^0, \mathcal{E}^0\} = -1, \tag{2.43}$$

which easily leads to $r_0 = 1/\gamma$.

Second, it will be useful to express the (inverse of the) induced metric q^{ab} on Σ in terms of the $\mathfrak{su}(1,1)$-covariant electric field. A direct calculation shows that

$$\det(g)\, g^{ab} = -\gamma^2\, \mathcal{E}^{\alpha a}\, \eta_{\alpha\beta}\, \mathcal{E}^{\beta b}\,. \tag{2.44}$$

Note that this formula makes very clear that the metric g_{ab} is Lorentzian and its signature is $(-1, +1, +1)$ as we have already seen in a previous analysis (2.20).

The final remark concerns the $\mathfrak{su}(1,1)$ gauge generators $\mathcal{J}_\alpha$. It is immediate to see that one can express them in terms of $\mathcal{A}$ and $\mathcal{E}$ only as follows

$$\mathcal{J}_\alpha(x)\, J^\alpha \;=\; \partial_a \mathcal{E}^a(x) \,+\, [\mathcal{A}_a(x)\,,\, \mathcal{E}^a(x)]\,. \tag{2.45}$$

We recover the usual Gauss-like form of the constraints, and this expression makes very clear that $\mathcal{A}$ and $\mathcal{E}$ transforms respectively as a connection and a vector under the action of the gauge generators.

2.2.3.3. Transformations under diffeomophisms

As for the Ashtekar-Barbero connection (or its generalization), we do not expect $\mathcal{A}$ to be a fully space-time connection on $\mathcal{M}$. However, it must transform correctly under diffeomorphisms induced on the hypersurface Σ. To see this is indeed the case, we first need to identify the generators of diffeomorphisms on Σ. A direct calculation shows that they are given by the following linear combination of the $\mathfrak{su}(1,1)$ gauge generators and the vectorial constraints:

$$\tilde{\mathcal{H}}(N^a) \;\equiv\; \mathcal{H}(N^a) - \frac{\gamma}{(1+\gamma^2)\chi^2}\tilde{\mathcal{J}}(N^a\Omega_a) \quad \text{with} \quad \Omega_a \equiv \gamma\chi \times A_a - (A_a\cdot\chi)\chi\,, \tag{2.46}$$

which, after some calculations, reduces to

$$\begin{aligned}
\tilde{\mathcal{H}}(N^a) \;&=\; \int d^3x\, N^a \left(E^b\cdot(\partial_a A_b - \partial_b A_a) - A_a\cdot\partial_b E^b + \zeta_0\cdot\partial_a\chi\right) \\
&=\; \int d^3x\, N^a \eta_{\alpha\beta}\left(\mathcal{E}^{\alpha b}\cdot(\partial_a \mathcal{A}_b^\alpha - \partial_b\mathcal{A}_a^\alpha) - \mathcal{A}_a^\alpha\cdot\partial_b\mathcal{E}^{\alpha b}\right)\,.
\end{aligned} \tag{2.47}$$

Hence, it is immediate to see from this last expression that the constraints $\tilde{\mathcal{H}}(N^a)$ form the algebra of diffeomorphisms. Furthermore, their actions on $\mathcal{A}$ and $\mathcal{E}$ is exactly the lie derivative along the vector field N^a:

$$\{\tilde{\mathcal{H}}(N^a),\mathcal{A}_b\} = -\mathcal{L}_{N^a}\mathcal{A}_b\,, \quad \{\tilde{\mathcal{H}}(N^a),\mathcal{E}_b\} = -\mathcal{L}_{N^a}\mathcal{E}_b\,. \tag{2.48}$$

Thus, as announced above, $\mathcal{A}$ is an $\mathfrak{su}(1,1)$-valued connection on Σ.

2.3. On the quantization

We have now all the ingredients to start the quantization of gravity formulated in terms of the $\mathfrak{su}(1,1)$ gauge connection. Following the standard construction of Loop Quantum Gravity, we assume that quantum states are polymer states, and then we build the kinematical Hilbert space from holonomies of the connection along edges on Σ.

2.3.1. Quantum states on a fixed graph

As usual, to any closed graph $\Gamma \subset \Sigma$ with N nodes and E edges, one associates a kinematical Hilbert space $\mathcal{H}_{kin}(\Gamma)$ which is isomorphic to

$$\mathcal{H}_{kin}(\Gamma) \;\simeq\; \left(\text{Fun}[SU(1,1)^{\otimes E}]/SU(1,1)^{\otimes N};\, d\mu^{\otimes E}\right)\,, \tag{2.49}$$

where $d\mu$ is the Haar measure on SU$(1,1)$. Due to the non-compactness of the gauge group, such a Hilbert space needs a regularization to be well-defined (which consists basically in "dividing" by the infinite volume of the group). The details of the regularization of non-compact spin-networks has been well studied in [144]. However, it is well-known that the "projective sum" $\oplus_\Gamma \mathcal{H}_{kin}(\Gamma)$ on the space of all graphs on Σ is ill-defined and, up to our knowledge, no one knows how to construct a non-compact Ashtekar-Lewandowski measure. Thus, only the kinematical Hilbert space on a fixed graph Γ is mathematically well-defined and we limit the study of quantum states as elements of $\mathcal{H}_{kin}(\Gamma)$ only. Hence, a quantum state is a function $\psi_\Gamma[A] \equiv f(U_1, \cdots, U_E)$ of the holonomies

$$U_e \;\equiv\; P\exp\int_e \mathcal{A} \;\in SU(1,1) \tag{2.50}$$

along the edges e of Γ. The electric field $\mathcal{E}$ is promoted as an operator whose action on ψ_Γ is formally given by

$$\hat{\mathcal{E}}^a_i(x)\,\psi_\Gamma[A] \;=\; i\ell_p^2 \frac{\delta}{\delta \mathcal{A}^i_a(x)}\psi_\Gamma[A]\,, \tag{2.51}$$

where ℓ_p is the Planck length. Note that the flux of $\mathcal{E}$ across a surface is a well-defined operator on $\mathcal{H}_{kin}(\Gamma)$: it acts as a vector field on the space of SU$(1,1)$ functions.

The Peter-Weyl theorem implies that ψ_Γ can be formally decomposed as follows

$$\psi_\Gamma[A] \;=\; \sum_{s_1,\cdots,s_E} \text{tr}\left(\tilde{f}(s_e)\bigotimes_{e=1}^{E}\pi_{s_e}(U_e)\right) \tag{2.52}$$

where

$$\pi_s : SU(1,1) \to \text{End}(V_s) \quad \text{and} \quad \tilde{f} \in \bigotimes_{e=1}^{N} V^*_{s_e}\,. \tag{2.53}$$

The sum runs over unitary irreducible representations of SU$(1,1)$ labelled generically by s_e. We used the notation V_{s_e} for the modulus of the representation, $V^*_{s_e}$ for its dual, and tr denotes the pairing between $\otimes_e V_{s_e}$ and its dual $\otimes_e V^*_{s_e}$. Due to the gauge invariance of ψ_Γ, the Fourier modes $\tilde{f}$ are in fact SU$(1,1)$ intertwiners and the expression of $\psi[A]$ needs a regularization to be well-defined

[144]. Furthermore, unitary irreducible representations of SU$(1,1)$, which are classified into the two discrete series (both labelled with integers) and the continuous series (labelled with real numbers), are infinite dimensional (see [146] for a review on representations theory of $\mathfrak{su}(1,1)$).

2.3.2. Area operators

Thus, edges of SU$(1,1)$ spin-networks can be colored with discrete or real numbers. The geometrical interpretation is clear: these two different types of colors label edges which are normal to either time-like or space-like surfaces. To see how to link the representations to the time-like or space-like natures of the surfaces, we have to compute the spectrum of the area operators in terms of the quadratic Casimir of $\mathfrak{su}(1,1)$. For that purpose, we start with the expression (2.44) of the inverse metric g^{ab} that we contract twice with the normal n_a to a given surface S. This leads to the formula

$$\det(g)\, n^2 \;=\; -\gamma^2\, (n_a \mathcal{E}^{a\alpha})\, \eta_{\alpha\beta}\, (\mathcal{E}^{b\beta} n_b)\,, \tag{2.54}$$

where $n^2 = n_a n_b g^{ab}$. Hence, the determinant of the induced metric h on the surface S is given by

$$\det(h) \;=\; -\gamma^2 (n_a \mathcal{E}^{a\alpha})\, \eta_{\alpha\beta}\, (\mathcal{E}^{b\beta} n_b)\,. \tag{2.55}$$

As a consequence, the action of the area operator $\hat{S}$, punctured by an edge e of the graph Γ colored by a representation s_e, on $\mathcal{H}_{kin}(\Sigma)$ is diagonal and its eigenvalue $S(s)$ is given by the equation

$$S(e)^2 \;=\; -i^2 \gamma^2 \ell_p^4\, \pi_e(J_1^2 + J_2^2 - J_0^2) \;=\; \gamma^2 \ell_p^4\, \pi_e(C) \tag{2.56}$$

where $\pi_e(C)$ is identified with the unique eigenvalue of the Casimir tensor $C \equiv -J_0^2 + J_1^2 + J_2^2$ in the representation s_e. Obviously, the evaluation $\pi_e(C)$ depends on the nature discrete ($s_e = j_e \in \mathbb{N}$) or continuous ($s_e \in \mathbb{R}$) of the representation according to

$$\pi_{j_e}(C) = j_e(j_e+1) \quad \text{and} \quad \pi_{s_e}(C) = -(s_e^2 + \frac{1}{4})\,. \tag{2.57}$$

We deduce immediately that $S(e)^2$ is positive when e is colored with a discrete representation whereas $S(e)^2$ is negative when e is colored with a representation in the continuous series. As a consequence, the area operator of any space-like surface has a discrete spectrum and the area operator of any time-like surface has a continuous spectrum. Furthermore, the spectrum of space-like areas is in total agreement of the usual spectrum in Loop Quantum Gravity. Note that a very similar result has been recently derived in the context of twisted geometries

[74].

2.4. Discussion

In this chapter, we have formulated gravity as an $\mathrm{SU}(1,1)$ gauge theory. We have started with the Hamiltonian formulation of the fully Lorentz invariant Holst action on a space-time manifold of the form $\mathcal{M} = \Sigma \times \mathbb{R}$. Then we have considered a partial gauge fixing which reduces the internal $\mathfrak{sl}(2,\mathbb{C})$ gauge symmetry to $\mathfrak{su}(1,1)$. The 3-dimensional slice Σ inherits a Lorentzian metric of signature $(-,+,+)$. The partial gauge fixing relies on the introduction on an external non-dynamical vector field χ which measures the normal of the hypersurface Σ but it plays in fact no physical role at the end of the process.

Next we found that the phase space of the partially gauge fixed theory is well-parametrized by a pair $(\mathcal{A},\mathcal{E})$ formed with an $\mathfrak{su}(1,1)$-valued connection on Σ and its canonically conjugate electric field whose components can be identified to vectors in the flat (2+1) Minkowski space-time. The phase space comes with first class constraints: the Gauss constraints which generate $\mathfrak{su}(1,1)$ gauge transformations, the vectorial constraints which have been shown to generate diffeomorphisms on Σ and the usual scalar constraint that we have not studied in this work.

Finally, we have explored the quantization of the theory studying some aspects of the kinematical Hilbert space $\mathcal{H}_{kin}(\Gamma)$ on a fixed given graph Γ which lies on Σ. Due to the non-compactness of the gauge group $\mathrm{SU}(1,1)$, $\mathcal{H}_{kin}(\Gamma)$ needs a regularization to be well-defined and the projective sum over all possible graphs is not under control. This is why we restrict our study to the quantization on a fixed graph only. We compute the spectrum of the area operators acting on $\mathcal{H}_{kin}(\Gamma)$ and found that the spectrum is discrete for space-like surfaces and continuous for time-like surfaces. Furthermore, the usual quantization of the Holst action in the time-gauge ($\chi = 0$) and the new quantization presented here and based to another totally inequivalent partial gauge fixing ($\chi^2 > 1$) lead to exactly the same spectrum of the area operator (on space-like surfaces) at the kinematical level. This strongly suggests that the time gauge introduces no anomaly in the quantization of gravity, at least at the kinematical level, as it was already underlined in [142] in a different situation.

This formulation of gravity seems very interesting because it offers another point of view on the quantization of gravity in four dimensions. Now, we have a description of the kinematical quantum states of gravity not only on space-like surfaces Σ but also on time-like surfaces (only remains the description of the quantum states on null-surfaces, what we hope to study in the future). Hence, with those space-like and time-like kinematical quantum states, we are closer to have a fully covariant description of quantum gravity. In that respect, it would be very instructive to make a contact between these two canonical quantizations and spin-foam models for covariant quantum gravity. Furthermore, if we understand how to “connect" the time-like and the space-like kinematical quantum states, we could open a new and promising way towards a better understanding of the dynamics in Loop Quantum Gravity.

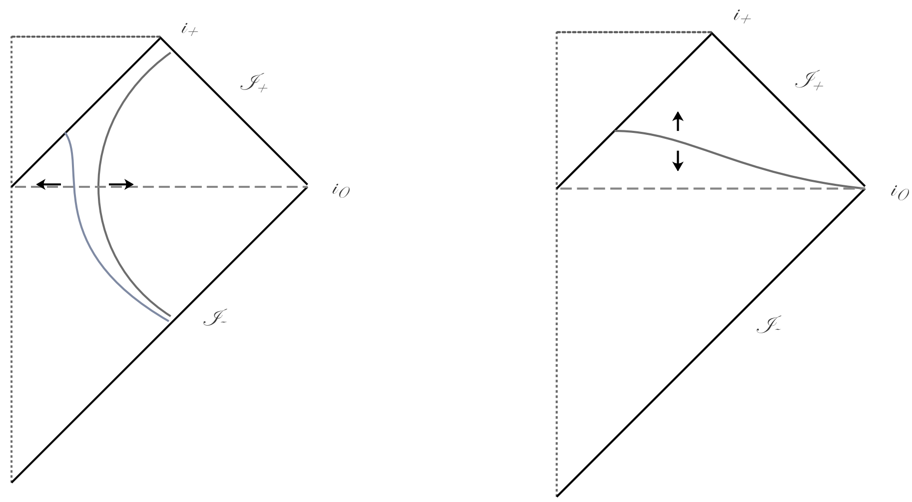


Figure 2.1. – Different Hamiltonian slicings of a spherical black hole space-time. The picture (b) represents the usual slicing in terms of space-like hypersurfaces which leads to the effective $SU(2)$ Chern-Simons description of the black hole: In that case, the horizon appears as a boundary of Σ. In the picture (a), we have represented two slicings of the black hole space-time where Σ are Lorentzian hypersurfaces: these gauge choices would lead to new descriptions of black holes in Loop Quantum Gravity. In particular, the slicing which does not cross the horizon is interesting in view of a holographic description of black holes in the frame of Loop Quantum Gravity.

It is also interesting to notice that the Hamiltonian constraint in the formalism where Σ is space-like becomes a component of the vectorial constraints in the formalism where Σ is time-like. The reverse is also true. As we know very well how to quantize the vectorial constraints on the kinematical Hilbert space, we think again that understanding the relation between these two Hamiltonian quantizations could lead us to a solution of the Hamiltonian constraint. We hope to study these questions related to the quantum dynamics in the future.

Beside, we deeply think that this new formulation will allow us to understand better the physics of quantum black holes in Loop Quantum Gravity. In the usual treatment [38–41, 147–150], black holes are considered as isolated horizons and they appear as boundary of a 3 dimensional space-like hypersurface Σ. Their effective dynamics has been shown to be governed by an $SU(2)$ Chern-Simons theory whose quantization leads to the construction and the counting of the quantum microstates for the black holes. With the $\mathfrak{su}(1,1)$ formulation of gravity, it is now possible to start a Hamiltonian quantization of gravity where Σ is time-like. Naturally, one would expect that quantizing black holes with space-like or time-like slices would lead to two equivalent descriptions of the black hole microstates. At first sight, we would say that, starting with a time-like slicing, one would get an SU$(1,1)$ Chern-Simons theory as an effective dynamics for the spherical black hole for instance. Thus, we can ask the question how an SU$(1,1)$ and an $SU(2)$ Chern-Simons theories could provide two equivalent Hilbert spaces when they are quantized. This may be possible when γ becomes complex and equal to $\pm i$ because, in that case, we expect the two gauge group of the Chern-Simons theories to become the same Lorentz group. This would give one more argument in favor of the analytic continuation procedure introduced and studied in [151–155]. However, this idea might be too naive because, on a time-like slicing, the black hole does not appear as a boundary anymore and a particle leaving on the slice Σ now cross the horizon and does not see any border. To finish, this new formulation of Loop Quantum Gravity opens the possibility to define a kind of "holographic" description for black holes in the framework of Loop Quantum Gravity as shown in the picture Fig. 2.1 above. We hope to study all these very intriguing aspects related to black holes in a future work

Appendix

2.A. "Time" vs. "Space" gauge in the Holst action

The very well-known "time" gauge refers to the condition e^0_a which breaks $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{su}(2)$ in the Holst action. It corresponds to taking a slicing $\Sigma \times \mathbb{R}$ of the space-time where the hypersurfaces Σ are space-like. In fact, one can easily generalize the time gauge by considering instead the condition $e^\mu_a n_\mu = 0$ where n_μ is a given fixed vector. When n_μ is time-like, the slices Σ are space-like (as for the time gauge where $n_\mu = \delta^0_\mu$) whereas the slices are time-like when n_μ is space-like. We want to study thus latter case in this appendix. To simplify the analysis, we assume (without loss of generality) that $n_\mu = \delta^3_\mu$.

We are going to show that the Hamiltonian analysis of the Holst action such a gauge leads to a phase space which corresponds to the limit $|\chi| \to \infty$ and $n_i \to \delta^3_i$. First, we notice that the only non vanishing components of π^a_{IJ} are $E^a_\alpha \equiv \pi^a_{\alpha 3}$ with $\alpha \in (0,1,2)$. It is immediate to check that the simplicity constraints $\mathcal{C}^{ab} \approx 0$ are satisfied. In this gauge, it is "natural" to choose the third direction to be the "time" parameter because of the slicing. Hence, the "symplectic" term (in the third direction) in the Holst action involves only the component ${}^\gamma\omega^{\alpha 3}_a$ of the spin-connection (with $\alpha \in (0,1,2)$ and $a \in (0,1,2)$ also) according to the formula

$$ {}^{\gamma}\pi^{a}_{IJ}\,\partial_3\omega^{IJ}_{a} = E^{a}_{\alpha}\partial_3 A^{\alpha}_{a}\,, \qquad \text{where} \quad A^{\alpha}_{a} \equiv {}^{\gamma}\omega^{\alpha 3}_{a}\,. \tag{2.58} $$

Hence, the connection A is clearly the variable canonically conjugate to E. Finally, one shows that the resolution of the second class constrains $\mathcal{D}^{ab} \approx 0$ leads to the following expression for the gauge generators

$$ \mathcal{J}_0 \;=\; -\frac{1}{\gamma}\partial_a E^{a0} - A^{1}_{a}E^{a2} + A^{2}_{a}E^{a1}\,, \tag{2.59} $$

$$ \mathcal{J}_1 \;=\; -\frac{1}{\gamma}\partial_a E^{a2} + A^{0}E^{1} - A^{1}E^{0}\,, \tag{2.60} $$

$$ \mathcal{J}_2 \;=\; \frac{1}{\gamma}\partial_a E^{a}1 - A^{0}E^{2} + A^{2}E^{0}\,. \tag{2.61} $$

They satisfy the constraints algebra

$$ \{\mathcal{J}_0,\mathcal{J}_1\} \;=\; \mathcal{J}_2\,, \qquad \{\mathcal{J}_0,\mathcal{J}_2\} \;=\; -\mathcal{J}_1\,, \qquad \{\mathcal{J}_1,\mathcal{J}_2\} \;=\; -\mathcal{J}_0\,, \tag{2.62} $$

which is nothing by the $\mathfrak{su}(1,1)$ algebra. At this point, it is not difficult to see that the associated covariant connection has the following components

$$ \mathcal{A}^{0}_{a} = -\gamma\;{}^{\gamma}\omega^{03}_{a}\,, \qquad \mathcal{A}^{1}_{a} = \gamma\;{}^{\gamma}\omega^{23}_{a}\,, \qquad \mathcal{A}^{2}_{a} = -\gamma\;{}^{\gamma}\omega^{13}_{a}\,. \tag{2.63} $$

We recover as announced the same expression of the $\mathfrak{su}(1,1)$-valued connection in the limit $|\chi| \to \infty$ (2.39) a part that we have interchanged the components 0 and 3 of space-time indices.

3. Asymptotics of Spin Foam Models with Timelike Triangles

Sommaire

3.1. Introduction

In this cahpter we extend the semiclassical analysis of extended model to general situations, in which we take into account both timelike tetrahedra and timelike triangles. Our work is motivated by the examples of geometries in classical Lorentzian Regge calculus, and their convergence to smooth geometries [86–88]. In all examples the Regge geometries contain timelike triangles. In order to have the Regge geometries emerge as critical configurations from spin foam model, we have to extend the semiclassical analysis to contain timelike triangles.

In our analysis, we first derive the large-j integral form of the extended spin foam model with coherent states for timelike triangles. The large-j asymptotic analysis is based on the stationary phase approximation of the integral. The asymptotics of the integral is a sum of contributions from critical configurations.

Before coming to our main result, we would like to mention some key assumptions for the validity of the result: The following results are valid when we assume every timelike tetrahedron containing at least one spacelike and one timelike triangle. It is the case in all Regge geometry examples mentioned above. Our results also apply to some special cases when all triangles in a tetrahedron are timelike. Moreover all tetrahedra in our discussion are assumed to be non-degenerate. Here we don't consider the critical configurations with a degenerate tetrahedron. Finally, the Hessian evaluated at every critical configuration is assumed to be a non-degenerate matrix.

The main result is summarized as follows: Firstly for a single 4-simplex and its vertex amplitude, it is important to have boundary data satisfy the length matching condition and orientation matching condition. Namely, (1) among the 5 tetrahedra reconstructed by the boundary data (by Minkowski Theorem), each pair of them are glued with their common triangles matching in shape (match their 3 edge lengths), and (2) all tetrahedra have the same orientation. The amplitude has critical configurations only if these 2 conditions are satisfied, otherwise the amplitude is suppressed asymptotically, The critical configurations have geometrical interpretations as geometrical 4-simplices, which may generally have one of three possible signatures: Lorentzian, split, or degenerate.

— When the 4-simplex has Lorentzian signatures: The contribution at the critical configuration is given by a phase, whose exponent is Regge action with a sign related to orientations, i.e. the vertex amplitude gives asymptotically

$$A_v \sim N_+ \mathrm{e}^{\mathrm{i}S_\Delta} + N_- \mathrm{e}^{-\mathrm{i}S_\Delta} \tag{3.1}$$

up to an overall phase depending on the boundary coherent state. The Regge action in the 4-simplex reads $S_\Delta = \sum_f A_f \theta_f$ with A_f the area of triangle f. θ_f relates to the dihedral angle Θ_f by $\theta_f = \pi - \Theta_f$. The area spectrum is different between timelike and spacelike triangles in a timelike tetrahedron.

$$A_f = \begin{cases} \frac{n_f}{2} & \text{timelike triangle} \\ \gamma j_f & \text{spacelike triangle} \end{cases} \tag{3.2}$$

$n_f \in \mathbb{Z}_+$ satisfies the simplicity constraint $n_f = \gamma s_f$ where $s_f \in \mathbb{R}_+$ labels the continuous series irreps of SU$(1,1)$. $j_f \in \mathbb{Z}_+/2$ labels the discrete series irreps of SU$(1,1)$. $N_\pm$ are geometric factors depend on the lengths and orientations of the reconstructed 4 simplex.

— The reconstructed 4-simplices have split signatures: The vertex amplitude gives asymptotically

$$A_v \sim N_+ \mathrm{e}^{\mathrm{i}\gamma^{-1}S_\Delta} + N_- \mathrm{e}^{-\mathrm{i}\gamma^{-1}S_\Delta} \tag{3.3}$$

up an overall phase. Here $S_\Delta = \sum_f A_f \theta_f$ where θ_f is a boost dihedral angle.

— The reconstructed 4-simplices are degenerate (vector geometry) and there is a single critical point. The asymptotical vertex amplitude is given by a phase depending on the boundary coherent states.

It is important to remark that for a vertex amplitude containing at least one timelike and one spacelike tetrahedron, critical configurations only give Lorentzian 4-simplices, while the split signature and degenerate 4-simplex do not appear.

The last 2 cases only appear when all tetrahedra are timelike in a vertex amplitude. The situation is similar to Lorentzian EPRL/FK model, where the Euclidean signature and degenerate 4-simplex appear because all tetrahedra are spacelike.

Our analysis is generalized to the spin foam amplitude on a simplicial complex $\mathcal{K}$ with many 4-simplices. We identify the critical configurations corresponding to simplicial geometries with all 4-simplices being Lorentzian and globally oriented. The configurations come in pairs, corresponding to opposite global orientations. Each pair gives the following asymptotic contribution to the spin foam amplitude (up to an overall phase)

$$N_+ \mathrm{e}^{\mathrm{i}S_{\mathcal{K}}} + N_- \mathrm{e}^{-\mathrm{i}S_{\mathcal{K}}} \tag{3.4}$$

where

$$S_{\mathcal{K}} = \sum_{f \text{ bulk}} A_f \varepsilon_f + \sum_{f \text{ boundary}} A_f \left(\theta_f + p_f \pi\right) \tag{3.5}$$

is the Regge action on the simplicial complex, up to a boundary term with $p_f \in \mathbb{Z}$ (p_f is the number of 4-simplices sharing f minus 1). The additional boundary term $p_f A_f \pi$ doesn't affect the Regge equation of motion. Here the simplicial geometries and Regge action generally contain timelike tetrahedra and timelike triangles. ε_f is the deficit angle. ε_f and θ_f at timelike triangles are given by

$$\varepsilon_f = 2\pi - \sum_f \Theta_f(v), \quad \theta_f = \pi - \sum_f \Theta_f(v) \tag{3.6}$$

$\Theta_f(v)$ is the dihedral angle within the 4-simplex at v. It is a rotation angle between spacelike normals of tetrahedra, because the tetrahedra sharing a timelike triangle are all timelike.

To obtain (3.4), we have assumed each bulk triangle is shared by an even number of 4-simplices. This assumption is true in many important examples of classical Regge calculus.

This chapter is organized as follows. In section 3.2, we write the coherent states for timelike triangles in large j approximation and express the spin foam amplitude in terms of the coherent states. In section 3.3, we derive and analyze the critical equations. The critical equations are reformulated in geometrical form for a timelike tetrahedron containing both spacelike and timelike triangles. Then in section 3.4, we reconstruct nondegenerate simplicial geometries from critical configurations. In section 3.5, the critical configurations for degenerate geometries are analyzed. Finally in section 3.7, we derive the difference between phases evaluated at pairs of critical configurations corresponding to opposite orientated simplicial geometries.

3.2. Spinfoam amplitude in terms of SU(1,1) continuous coherent states

The spin foam models are defined as a state sum model over simplicial manifold $\mathcal{K}$ and it's dual, which consists of simplices σ_v, tetrahedra τ_e, triangles f, edges and vertices (v,e and f are labels for vertices, edges and faces on the dual graph respectively). A triangulation is obtained by gluing simplices σ with pairs of their boundaries (tetrahedrons τ). The phase space associated with manifold $\mathcal{K}$ are

$$P_{\mathcal{K}} = T^*\mathrm{SL}(2,\mathbb{C})^L, \qquad (\Sigma_f^{IJ}, h_f) \in T^*\mathrm{SL}(2,\mathbb{C}) \tag{3.7}$$

for a Lorentzian model, where L is the number of triangles, $h_f \in \mathrm{SL}(2,\mathbb{C})$ is the holonomy along the edges and $\Sigma_f^{IJ} \in \mathfrak{sl}(2,\mathbb{C})$ is its conjugate momenta. h_f can be decomposed as

$$h_f = \prod_{v\subset\partial f} g_{ev} g_{ve'} \tag{3.8}$$

where $g_{ve} \in \mathrm{SL}(2,\mathbb{C})$ and $g_{ev} = {g_{ve}}^{-1}$. Σ_f^{IJ} is subject to the simplicity constraint

$$\frac{\gamma}{1+\gamma^2}(u_e)^I((1-\gamma*)\Sigma_{f\,IJ}) = 0 \tag{3.9}$$

where u_e is a 4 normal vector associated to each tetrahedron t_e, γ is a real number known as the Immirizi parameter, and $*$ is the Hodge dual operator. Geometrically, the simplicity constraint implies that, each triangle f in tetrahedron t_e is associated with a simple bivector

$$B_f = \frac{\gamma}{1+\gamma^2}(1-\gamma*)\Sigma_f \tag{3.10}$$

The state sum is defined over all quantum states of the physical Hilbert space on a given $\mathcal{K}$, given as

$$Z(\mathcal{K}) = \sum_J \prod_f \mu_f(J_f) \prod_v A_v(J_f, i_e) \tag{3.11}$$

Here $J = \vec{j}_f$ represents the combination of labels of the $\mathrm{SL}(2,\mathbb{C})$ irreps associated to each triangle. i_e is the intertwiner associated with each tetrahedron

$$i_e \in \mathrm{Inv}_G[V_{J_1} \otimes \cdots \otimes V_{J_4}] \tag{3.12}$$

which impose the gauge invariance. The vertex amplitude $A_v(J_f, i_e)$ associated with each 4 simplex σ_v captures the dynamics of the model, while the face amplitude $\mu_f(J_f)$ is a weight for the J sum.

Usually a partial gauge fixing is taken to the above models, which corresponding to pick a special normal u for all of the tetrahedra $\forall_e,\ u_e = u$. As a result, the intertwiners associated with each tetrahedron defined above is replaced by the intertwiners of the the stabilizer group $H \in G$. There are two different gauge fixing:

— $u = (1, 0, 0, 0)$, $H = SU(2)$, EPRL/FK models

— $u = (0, 0, 0, 1)$, $H = SU(1, 1)$, Conrady-Hnybida Extension

which, after impose the quantum simplicity constraint (3.9) lead to the following conditions [61, 63, 64]

— $u = (1, 0, 0, 0)$, spacelike triangles

$$\rho = \gamma n, \qquad n = j \tag{3.13}$$

— $u = (0, 0, 0, 1)$, spacelike triangles

$$\rho = \gamma n, \qquad n = j \tag{3.14}$$

— $u = (0, 0, 0, 1)$, timelike triangles

$$\rho = -n/\gamma, \qquad s = \frac{1}{2}\sqrt{n^2/\gamma^2 - 1} \tag{3.15}$$

Here $(\rho \in \mathbb{R},\ n \in \mathbb{Z}/2)$ are labels of $\mathrm{SL}(2, \mathbb{C})$ irreps, $j \in \mathbb{N}/2$ is the label of $\mathrm{SU}(2)$ irreps or $\mathrm{SU}(1, 1)$ discrete series and $s \in \mathbb{R}$ is the label of $\mathrm{SU}(1, 1)$ continous series, we will give a brief introduction of $\mathrm{SU}(1, 1)$ and $\mathrm{SL}(2, \mathbb{C})$ representation theory later. As a result, the area spectrum is given by

$$A_f = \begin{cases} \frac{n_f}{2} & \text{timelike triangle} \\ \gamma j_f & \text{spacelike triangle} \end{cases} \tag{3.16}$$

The spin foam vertex amplitude can be expressed in the coherent state representation:

$$A_v(K) = \sum_{j_f} \prod_f \mu(j_f) \int_{\mathrm{SL}(2,\mathbb{C})} \prod_e dg_{\nu e} \prod_{(e,f)} \int_{S^2} dN_{ef} \left\langle \Psi_{\rho_f n_f}(N_{ef}) \right| D^{(\rho_f, n_f)}(g_{ev} g_{ve'}) \left| \Psi_{\rho_f n_f}(N_{e'f}) \right\rangle \tag{3.17}$$

Here N is the unit vector in a sphere or hyperbolid which labels the coherent states $|\Psi_{\rho n}\rangle$ of $\mathrm{SL}(2, \mathbb{C})$ in the unitary irrep $\mathcal{H}_{(\rho,n)}$. By $\mathrm{SU}(1, 1)$ decomposition of $\mathrm{SL}(2, \mathbb{C})$ unitary irrep, $\mathrm{SL}(2, \mathbb{C})$ irrep is isomorphic to a direct sum of irreps of $\mathrm{SU}(1, 1)$. The area of timelike triangles is related to $\mathrm{SU}(1, 1)$ spin s and the Immirzi parameter γ by $A_f = \gamma\sqrt{s^2 + 1/4}$ which is consistent with the spectrum from canonical approach [61, 156]. However, the solution of quantum simplicity constraint (3.9 on timelike triangles induced a Y-map where the physical Hilbert space $\mathcal{H} \in \mathcal{H}_{(\rho,n)}$ is isomorphic to continuous series of $\mathrm{SU}(1, 1)$ with spin s fixed by (3.15). As a result, the area spectrum is now given by

$$A_f = \gamma\sqrt{s^2 + 1/4} = \frac{n_f}{2} \tag{3.18}$$

which is quantized.

In the following, we first give a brief introduction of the SU$(1,1)$ and SL$(2,\mathbb{C})$ representation theory. Then we write the SL$(2,\mathbb{C})$ states explicitly using continuous SU$(1,1)$ coherent states in terms of spinor variables. Finally we derive the integral from of spin foam amplitude on timelike triangles with a spin foam action.

3.2.1. Representation theory of SL$(2,\mathbb{C})$ and SU$(1,1)$ group

SL$(2,\mathbb{C})$ group has 6 generators J^i and K^i with commutation relation

$$\begin{gathered}[J^i, J^i] = \epsilon^{ij}_k J^k, \quad [J^i, K^j] = \epsilon^{ij}_k K^k, \\ [K^i, K^j] = -\epsilon^{ij}_k J^k\end{gathered} \tag{3.19}$$

The unitary representations of the group are labelled by pairs of numbers ($\rho \in \mathbb{R}, n \in \mathbb{Z}_+$) from the two Casimirs

$$\begin{aligned}C_1 &= 2(\vec{J}^2 - \vec{K}^2) = \frac{1}{2}(n^2 - \rho^2 - 4) \\ C_2 &= -4\vec{J}\cdot\vec{K} = n\rho\end{aligned} \tag{3.20}$$

The Hilbert space $\mathcal{H}_{(\rho,n)}$ of unitary irrep of SL$(2,\mathbb{C})$ can be represented as a space of homogeneous functions $F : \mathbb{C}^2\backslash\{0\} \to \mathbb{C}$ with the homogeneity property

$$F(\beta z_1, \beta z_2) = \beta^{i\rho/2+n/2-1}\beta^{*i\rho/2-n/2-1}F(z_1, z_2) \tag{3.21}$$

The inner product in $\mathcal{H}_{(\rho,n)}$ is given by

$$\langle F_1|F_2\rangle = \int_{\mathbb{CP}_1} \pi((F_1)^* F_2\omega) \tag{3.22}$$

where $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}_1$. ω is the SL$(2,\mathbb{C})$ invariant 2-form defined by

$$\omega = \frac{\mathrm{i}}{2}(z_2\mathrm{d}z_1 - z_1\mathrm{d}z_2) \wedge (\bar{z}_2\mathrm{d}\bar{z}_1 - \bar{z}_1\mathrm{d}\bar{z}_2) \tag{3.23}$$

SU$(1,1)$ group is a subgroup of SL$(2,\mathbb{C})$ with generators $\vec{F} = (J^3, K^1, K^2)$. $\vec{F}$ and $\vec{G} = \mathrm{i}\vec{F} = (K^3, -J^1, -J^2)$ transform as Minkowski vectors under SU$(1,1)$. The Casimir reads $Q = (J^3)^2 - (K^1)^2 - (K_2)^2$. The unitary representation of SU$(1,1)$ group is usually built from the eigenstates of J^3 which is labelled by

j, m:

$$\langle jm|jm'\rangle = \delta_{mm'} \tag{3.24}$$

where m is the eigenvalue of J^3 and j related to the eigenvalues of the Casimir Q.

The unitary irrep of SU(1,1) contains two series: the discrete series and continuous series. For the discrete series, one has

$$Q\,|jm\rangle = j(j+1)\,|jm\rangle\,, \quad \text{with } j = -\frac{1}{2}, -1, -\frac{3}{2}, ... \tag{3.25}$$

The eigenvalue m of J^3 takes the values

$$m = -j, -j+1, -j+2.... \quad \text{or} \quad m = j, j-1, j-2.... \tag{3.26}$$

The Hilbert spaces of spin j are denoted by $\mathcal{D}_j^{\pm}$ with $m \gtrless 0$. For the continuous series, Q takes continuous value

$$Q\,|jm\rangle = j(j+1)\,|jm\rangle \tag{3.27}$$

where $j = -1/2 + \mathrm{i}s$ and s is a real number $s \in \mathbb{R}_+$. Thus in continuous case, we can use s instead of j to represent the spin. The eigenvalues m takes the values

$$m = 0, \pm 1, \pm 2, ... \qquad \text{or} \qquad m = \pm\frac{1}{2}, \pm\frac{3}{2}, ... \tag{3.28}$$

The irreps of this series are denoted by C_s^{ϵ} where $\epsilon = 0, 1/2$ corresponding to the integer m and half-integer m respectively.

Instead of $|jm\rangle$, one may also choose the generalized continuous eigenstates $|j\lambda\sigma\rangle$ of K^1 as the basis of the irrep Hilbert space [157]:

$$\langle j\lambda'\sigma'|j\lambda\sigma\rangle = \delta(\lambda - \lambda')\delta_{\sigma\sigma'} \tag{3.29}$$

where $\sigma = 0, 1$ distinguish the two-fold degeneracy of the spectrum and λ here is a real number. For continuous series irreps, Casimir Q takes

$$Q\,|j\lambda\sigma\rangle = j(j+1)\,|j\lambda\sigma\rangle = -\left(s^2 + \frac{1}{4}\right)|j\lambda\sigma\rangle\,. \tag{3.30}$$

3.2.2. Unitary irreps of SL$(2,\mathbb{C})$ and the decomposition into SU$(1,1)$ continuous state

The Hilbert space $\mathcal{H}_{(\rho,n)}$ can be decomposed as a direct sum of irreps of SU$(1,1)$. The decomposition can be derived from the homogeneity property and the Plancherel decomposition of SU$(1,1)$. As shown in [62], the functions F in the SL$(2,\mathbb{C})$ Hilbert space satisfying (3.21) can be described by pairs of functions f^{α} : SU$(1,1) \rightarrow$

$\mathbb{C}, \alpha = \pm 1$ via

$$F(z_1, z_2) = \sqrt{\pi}(\alpha\langle z, z\rangle)^{i\rho/2-1} f^\alpha(v^\alpha(z_1, z_2)), \tag{3.31}$$

where v^α is the induced SU$(1,1)$ matrix

$$v^\alpha = \begin{cases} \frac{1}{\sqrt{\langle z,z\rangle}} \begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix}, & \alpha = 1 \\ \frac{1}{\sqrt{-\langle z,z\rangle}} \begin{pmatrix} \bar{z}_2 & \bar{z}_1 \\ z_1 & z_2 \end{pmatrix}, & \alpha = -1 \end{cases} \tag{3.32}$$

with $\langle z, z\rangle = z^\dagger \sigma_3 z = \bar{z}_1 z_1 - \bar{z}_2 z_2$ being SU$(1,1)$ invariant inner product. Here α is a signature

$$\alpha = \begin{cases} 1, & |z_1| > |z_2| \\ -1, & |z_1| < |z_2| \end{cases} \tag{3.33}$$

Then $\mathcal{H}_{(\rho,n)}$ is isomorphic to the Hilbert space $L^2(\text{SU}(1,1)) \oplus L^2(\text{SU}(1,1))$ with inner product

$$\Big\langle (f_1^+, f_1^-) \Big| (f_2^+, f_2^-) \Big\rangle = \sum_\alpha \int dv (f_1^\alpha(v))^* f_2^\alpha(v) \tag{3.34}$$

where dv is the SU(1,1) measure.

The function f in SU$(1,1)$ continuous series representations with continuous basis reads

$$f^\alpha_{j\lambda}(z) = \begin{cases} \sqrt{2j+1}(D^j_{n/2,\lambda}(v(z)), 0), & \alpha = 1 \\ \sqrt{2j+1}(0, D^j_{-n/2,\lambda}(v(z))), & \alpha = -1 \end{cases} \tag{3.35}$$

Noticed that here we assume $s \neq 0$. $D^j_{m\lambda}$ is the Wigner matrix with mixed basis (3.24) and (3.29)

$$D^j_{m\lambda\sigma}(v) = \langle j, m|\, v(z)\, |j, \lambda, \sigma\rangle \tag{3.36}$$

Recall the quantum simplicity constraint (3.15),

$$\rho = -n/\gamma, \qquad s = \frac{1}{2}\sqrt{n^2/\gamma^2 - 1} \tag{3.37}$$

Asymptotically, when $s \gg 1$, we have

$$\rho \sim -2s \sim -\frac{n}{\gamma} \tag{3.38}$$

Since n is discrete, s and ρ are also discrete.

3.2.3. Derivation of representation matrix

Now we will derive the wigner matrix of continuous series in unitary irreps of SU(1,1) group in the large s approximation. We begin with the introduction of the wigner matrix of continuous series given in [158]. Then by transformations of hypergeometric functions and saddle point approximation we obtain the representation matrix in large s limit.

3.2.3.1. Wigner matrix

First let us introduce the parametrization of the $\mathrm{SU}(1,1)$ group element v:

$$v(z) = \mathrm{e}^{\mathrm{i}\phi J^3}\mathrm{e}^{\mathrm{i}tK^2}\mathrm{e}^{\mathrm{i}uK^1} = \begin{pmatrix} v_1 & v_2 \\ \bar{v}_2 & \bar{v}_1 \end{pmatrix} \tag{3.39}$$

where

$$v_1 = \mathrm{e}^{\frac{\mathrm{i}\phi}{2}}\left(\cosh\left(\frac{t}{2}\right)\cosh\left(\frac{u}{2}\right) - \mathrm{i}\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{u}{2}\right)\right) \tag{3.40}$$

$$v_2 = e^{\frac{\mathrm{i}\phi}{2}}\left(\mathrm{i}\cosh\left(\frac{u}{2}\right)\sinh\left(\frac{t}{2}\right) - \cosh\left(\frac{t}{2}\right)\sinh\left(\frac{u}{2}\right)\right) \tag{3.41}$$

Note that the generators defined here is complex version of what we used in the main part. In this parametrization, the wigner matrix which defined as

$$D^j_{m\lambda\sigma}(v) = \langle j,m|\, v\, |j\lambda\sigma\rangle \tag{3.42}$$

can be expressed by [158]

$$D^j_{m\lambda\sigma} = \mathrm{e}^{\mathrm{i}m\phi}d^j_{m\lambda\sigma}\mathrm{e}^{\mathrm{i}\lambda u} = \mathrm{e}^{\mathrm{i}m\phi}S^j_{m\lambda\sigma}\Big(T^j_{m\lambda}F^j_{m,\mathrm{i}\lambda}(\beta) - (-1)^\sigma T^j_{-m\lambda}F^j_{-m,\mathrm{i}\lambda}(\bar{\beta})\Big)\mathrm{e}^{\mathrm{i}\lambda u} \tag{3.43}$$

where

$$F^j_{m,\mathrm{i}\lambda}(\beta) = (1-\beta)^{(m-\mathrm{i}\lambda)/2}\beta^{(m+\mathrm{i}\lambda)/2}{}_2F_1\left(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta\right) \tag{3.44}$$

$$T^j_{m\lambda} = \frac{1}{\Gamma(-m-j)\Gamma(m+1+\mathrm{i}\lambda)} \tag{3.45}$$

Here ${}_2F_1(a,b,c,z)$ refers to Gaussian hypergeometric function, and $\Gamma(z)$ is the Gamma function. Normalization factor $S^j_{m\lambda\sigma}$ reads

$$S^j_{m\lambda\sigma} = \sqrt{\frac{\Gamma(m-j)}{\Gamma(m+j+1)}}\frac{2^{j-1}\Gamma(-j+\mathrm{i}\lambda)}{\mathrm{i}^\sigma\sin(\pi/2(-j-\mathrm{i}\lambda+\sigma))} \tag{3.46}$$

with $\beta = (1-\mathrm{i}\sinh(t))/2$.

Above equation (3.43) can be written in terms of normalized spinors $v =$

(v_1, v_2) in SU$(1,1)$ inner product $\langle v, v\rangle = 1$. According to the parametrization, we have

$$v_1 + v_2 = \mathrm{e}^{-\frac{u}{2}+\frac{\mathrm{i}\phi}{2}}\left(\cosh\left(\frac{t}{2}\right) + \mathrm{i}\sinh\left(\frac{t}{2}\right)\right), \quad v_1 - v_2 = \mathrm{e}^{\frac{u}{2}+\frac{\mathrm{i}\phi}{2}}\left(\cosh\left(\frac{t}{2}\right) - \mathrm{i}\sinh\left(\frac{t}{2}\right)\right) \tag{3.47}$$

Wigner matrix D can be written in terms of v and $\bar{v}$

$$D^j_{m\lambda\sigma} = S^j_{m\lambda\sigma}\Big(T^j_{m\lambda}F^j_{m,\mathrm{i}\lambda}(v) - (-1)^\sigma T^j_{-m\lambda}F^j_{-m,\mathrm{i}\lambda}(\bar{v})\Big) \tag{3.48}$$

with

$$\begin{aligned} F^j_{m,\mathrm{i}\lambda}(v) =& 2^{-m}(v_1+v_2)^{(m-\mathrm{i}\lambda)}(v_1-v_2)^{(m+\mathrm{i}\lambda)}\times \\ & {}_2F_1\left(-j+m, j+m+1; m+\mathrm{i}\lambda+1; (\bar{v}_1+\bar{v}_2)(v_1-v_2)/2\right) \end{aligned} \tag{3.49}$$

3.2.3.2. Asymptotics of Gauss hypergeometric function

According to (3.43), we need to evaluate the hypergeometric function

$$\begin{aligned} &{}_2F_1\left(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta\right), \\ &{}_2F_1\left(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta\right) \end{aligned} \tag{3.50}$$

The function itself is complicated. However, we only need the asymptotics behavior with $j \sim m \sim \lambda \gg 1$ in our case. According to (3.35), m is chosen to be $n/2$ which related to $j = -1/2 + \mathrm{i}s$ by simplicity constraint (3.15). Correspondingly, λ is also chosen to be related to s.

First we would like to transform the original function to a more convenient form. According to the transformation properties of hypergeometric function, we have

$$ {}_2F_1\left(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta\right) = (1-\beta)^{-m+\mathrm{i}\lambda} {}_2F_1\left(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta\right) \tag{3.51} $$

$$ \begin{aligned} &{}_2F_1\left(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta\right) \\ &\qquad = (\beta)^{m+\mathrm{i}\lambda} {}_2F_1\left(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; -m+\mathrm{i}\lambda+1; 1-\beta\right) \end{aligned} \tag{3.52} $$

$$ \begin{aligned} &\frac{\sin(\pi(-m+\mathrm{i}\lambda))}{\pi\Gamma(m+\mathrm{i}\lambda+1)} {}_2F_1\left(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta\right) \\ =&\beta^{-m-\mathrm{i}\lambda} \frac{{}_2F_1\left(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; m-\mathrm{i}\lambda+1; 1-\beta\right)}{\Gamma(m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} \\ &\quad - (1-\beta)^{-m+\mathrm{i}\lambda} \frac{{}_2F_1\left(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; -m+\mathrm{i}\lambda+1; 1-\beta\right)}{\Gamma(-m+\mathrm{i}\lambda+1)\Gamma(-j+m)\Gamma(j+m+1)} \end{aligned} \tag{3.53} $$

$$ \begin{aligned} &\frac{\sin(\pi(m+\mathrm{i}\lambda))}{\pi\Gamma(-m+\mathrm{i}\lambda+1)} {}_2F_1\left(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta\right) \\ =&(1-\beta)^{m-\mathrm{i}\lambda} \frac{{}_2F_1\left(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; -m-\mathrm{i}\lambda+1; \beta\right)}{\Gamma(-m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} \\ &\quad - (\beta)^{m+\mathrm{i}\lambda} \frac{{}_2F_1\left(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta\right)}{\Gamma(m+\mathrm{i}\lambda+1)\Gamma(-j-m)\Gamma(j-m+1)} \end{aligned} \tag{3.54} $$

From (3.52) and (3.53), we have

$$ \begin{aligned} &{}_2F_1\left(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta\right) = \Gamma(-m+\mathrm{i}\lambda+1)\Gamma(-j+m)\Gamma(j+m+1)\times \\ &\left(-\frac{(\beta)^{m+\mathrm{i}\lambda}\sin(\pi(-m+\mathrm{i}\lambda))}{\pi\Gamma(m+\mathrm{i}\lambda+1)} {}_2F_1\left(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta\right)\right. \\ &\left.+\frac{(1-\beta)^{m-\mathrm{i}\lambda}}{\Gamma(m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} {}_2F_1\left(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; m-\mathrm{i}\lambda+1; 1-\beta\right)\right) \end{aligned} \tag{3.55} $$

Similarly, from (3.51) and (3.54), we have

$$ \begin{aligned} &{}_2F_1\left(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta\right) = \Gamma(m+\mathrm{i}\lambda+1)\Gamma(-j-m)\Gamma(j-m+1)\times \\ &\left(-\frac{(1-\beta)^{-m+\mathrm{i}\lambda}\sin(\pi(m+\mathrm{i}\lambda))}{\pi\Gamma(-m+\mathrm{i}\lambda+1)} {}_2F_1\left(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; -m+\mathrm{i}\lambda+1; 1-\beta\right)\right. \\ &\left.+\frac{\beta^{-m-\mathrm{i}\lambda}}{\Gamma(-m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} {}_2F_1\left(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; -m-\mathrm{i}\lambda+1; \beta\right)\right) \end{aligned} \tag{3.56} $$

Then in terms of (3.51) and (3.55), the function $d^j_{m\lambda\sigma}$ can be written as

$$\begin{aligned}d^j_{m\lambda\sigma}(\beta) =&S^j_{m\lambda\sigma}\Bigg[(1+(-1)^\sigma\tan(\pi(-m+\mathrm{i}\lambda)))\\ &\times\frac{(1-\beta)^{(-m+\mathrm{i}\lambda)/2}\beta^{(m+\mathrm{i}\lambda)/2}{}_2F_1\left(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;m+\mathrm{i}\lambda+1;\beta\right)}{\Gamma(-m-j)\Gamma(m+\mathrm{i}\lambda+1)}\\ &-(-1)^\sigma\frac{\beta^{(-m-\mathrm{i}\lambda)/2}(1-\beta)^{(m-\mathrm{i}\lambda)/2}{}_2F_1\left(j-\mathrm{i}\lambda+1,-j-\mathrm{i}\lambda;m-\mathrm{i}\lambda+1;1-\beta\right)}{\Gamma(m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)\Gamma^{-1}(j+m+1)}\Bigg]\end{aligned} \tag{3.57}$$

Now we only need to evaluate the hypergeometric function ${}_2F_1\left(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;m+\mathrm{i}\lambda+1;\beta\right)$, since ${}_2F_1\left(j-\mathrm{i}\lambda+1,-j-\mathrm{i}\lambda;m-\mathrm{i}\lambda+1;1-\beta\right)$ is nothing else but the complex conjugation of the previous one. Similar, start from (3.52) and (3.56), we have

$$\begin{aligned}d^j_{m\lambda\sigma}(\beta) =&S^j_{m\lambda\sigma}\Bigg[(-\tan(\pi(m+\mathrm{i}\lambda))-(-1)^\sigma)\\ &\times\frac{(1-\beta)^{(-m+\mathrm{i}\lambda)/2}\beta^{(m+\mathrm{i}\lambda)/2}{}_2F_1\left(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;-m+\mathrm{i}\lambda+1;1-\beta\right)}{\Gamma(m-j)\Gamma(-m+\mathrm{i}\lambda+1)}\\ &+\frac{\beta^{(-m-\mathrm{i}\lambda)/2}(1-\beta)^{(m-\mathrm{i}\lambda)/2}{}_2F_1\left(j-\mathrm{i}\lambda+1,-j-\mathrm{i}\lambda;-m-\mathrm{i}\lambda+1;\beta\right)}{\Gamma(-m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)\Gamma^{-1}(j-m+1)}\Bigg]\end{aligned} \tag{3.58}$$

Clearly the two expression obey the relation $d^j_{m\lambda\sigma}(\beta)=-(-1)^\sigma d^j_{-m\lambda\sigma}(\bar{\beta})$

From (3.57), we need the large s approximation of the hypergeometric function ${}_2F_1\left(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;m+\mathrm{i}\lambda+1;\beta\right)$. Here we will only concentrate on the the parameters such that $m=n/2=\gamma s$ and $\lambda\sim s$ are satisfied. In this choice, all the parameters will scale together with s. A choice of λ is $\lambda=-s$. The generalization to parameters where m and λ scales with Λ but takes different value is straight forward. Noted the smearing of λ requires to calculate $\lambda=-s_0+\epsilon$ where $\epsilon\ll\lambda$.

For simplicity, we will transform the original function as

$$\begin{aligned}&{}_2F_1\left(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;m+\mathrm{i}\lambda+1;\beta\right)\\ =&(1-\beta)^{-1/2}{}_2F_1\left(j+\mathrm{i}\lambda+1,j+m+1;m+\mathrm{i}\lambda+1;\frac{\beta}{\beta-1}\right)\qquad\text{with }\lambda=-s,m=\gamma s,\gamma>0\\ =&(1-\beta)^{-1/2}{}_2F_1\left(\frac{1}{2},\frac{1}{2}+(\gamma+\mathrm{i})s;(\gamma-\mathrm{i})s+1;\frac{\beta}{\beta-1}\right)\end{aligned} \tag{3.59}$$

We will use the integral representation for Hypergeometric functions [159]:

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(1+b-c)\Gamma(c)}{2\pi\mathrm{i}\Gamma(b)} \int_0^{1+} \frac{t^{b-1}(t-1)^{c-b-1}}{(1-zt)^a} dt, \quad \text{if } c-b \notin N \ \& \ \mathrm{Re}(b) > 0 \tag{3.60}
$$

The validity region for these equations is $|\arg(1-z)| < \pi$. In (3.60), the integration path is the anti-clockwise loop that starts and ends at $t=0$, encircles the point $t=1$, and excludes the point $t=1/z$. In our case, we have $\mathrm{Re}(c-b) = 1/2$ and $\mathrm{Re}(b) = 1/2 + m = 1/2 + \gamma s$ which satisfy the requirement. Thus with (3.60) we rewrite the original hypergeometric function as

$$
{}_2F_1\left(\frac{1}{2}, \frac{1}{2} + (\gamma + \mathrm{i})s; (\gamma - \mathrm{i})s + 1; \frac{\beta}{\beta - 1}\right) = \frac{G(s)}{2\pi\mathrm{i}} \int_0^{1+} dt f(t,\beta) \mathrm{e}^{s\Psi(t)} \tag{3.61}
$$

where $\Psi(t)$ and $f(t,\beta)$ are

$$
\Psi(t) = (\gamma + \mathrm{i}) \ln t - 2\mathrm{i} \ln(t-1), \quad f(t,\beta) = \left(t(t-1)(1 - \frac{\beta t}{\beta - 1})\right)^{-\frac{1}{2}} \tag{3.62}
$$

and $G(s)$ is

$$
G(s) = \frac{\Gamma(\frac{1}{2} + 2\mathrm{i}s)\Gamma((\gamma - \mathrm{i})s + 1)}{\Gamma(\frac{1}{2} + (\gamma + \mathrm{i})s)} \sim \frac{\sqrt{2\pi(\gamma - \mathrm{i})s}((\gamma - \mathrm{i}))^{(\gamma - \mathrm{i})s}(2\mathrm{i})^{2\mathrm{i}s}}{((\gamma + \mathrm{i}))^{(\gamma + \mathrm{i})s}} \tag{3.63}
$$

Here we use the asymptotic formula of Γ functions

$$
\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} \mathrm{e}^{-z} \tag{3.64}
$$

Note that $|G(s)| \sim \sqrt{s} \exp(-\pi s)$. We will see later the contribution form $\exp(-\pi s)$ will cancel the contribution form $|\exp(s\Psi(t))|$ at the saddle point t_0.

Clearly when $\beta/(\beta-1) \neq 1$, we have three branch points $t=0$, $t=1$ and $t=(\beta-1)/\beta$ for $f(t,z)$ and two branch points $t=0$ and $t=1$ for $\Psi(t)$. The branch cuts for $\Psi(t)$ on the real axis are given by $(-\infty, 0]$ and $(0,1]$, which can be seen in Fig. 1. We need to exclude the point $t_\beta = (\beta-1)/\beta$ from the path.

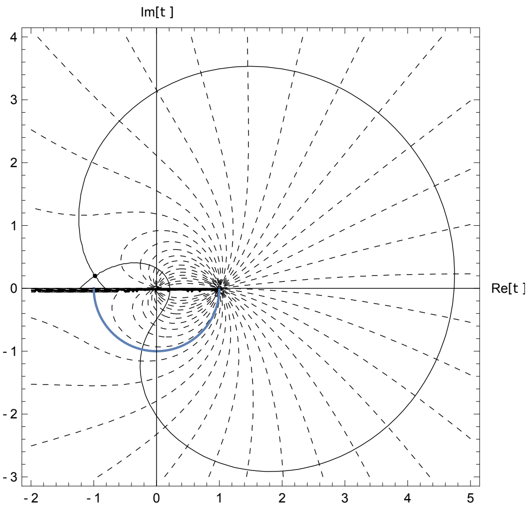


Figure 1. – The value of $\mathrm{Re}(\Phi(t))$ (dash line) and the steepest decent and ascend path (black line) over the t-complex plane for $\gamma = 0.1$. The blue line shows the position of possible poles t_β of f

There is one saddle point t_0 given by the solution of the equation $\Psi'(t) = 0$

$$t_0 = \frac{\gamma + \mathrm{i}}{\gamma - \mathrm{i}} \tag{3.65}$$

consequently, at the saddle point $\mathrm{Re}(\Psi(t_0)) = \pi$. The steepest decent and ascend curves are shown in Fig. 1. The original integration path then can be deformed as the steepest decent curve and two equal real part curve of $\Psi(t)$.

The corresponding value at the saddle point t_0 reads

$$\mathrm{e}^{s\Psi(t_0)} = \left(\frac{\gamma+\mathrm{i}}{\gamma-\mathrm{i}}\right)^{(\gamma+\mathrm{i})s}\left(\frac{2\mathrm{i}}{\mathrm{i}+\gamma}\right)^{-2\mathrm{i}s}, \quad f(t_0,\beta) = \frac{(2\mathrm{i})^{\mathrm{i}\epsilon}}{\sqrt{2\mathrm{i}}}\left(\frac{\gamma-\mathrm{i}(1-2\beta)}{1-\beta}\right)^{-\frac{1}{2}-\mathrm{i}\epsilon}\left(\frac{\gamma+\mathrm{i}}{(\gamma-\mathrm{i})^3}\right)^{-\frac{1}{2}} \tag{3.66}$$

and

$$\Phi''(t_0) = \frac{-\mathrm{i}(\gamma-\mathrm{i})^3}{2(\gamma+\mathrm{i})}, \quad \alpha = arg(n\Psi''(t_0)) = \frac{\pi}{2} - \arg\left(\frac{\mathrm{sgn}(\gamma+\mathrm{i})}{\mathrm{sgn}(\gamma-\mathrm{i})^3}\right), \quad \theta = \frac{\pi-\alpha}{2} \tag{3.67}$$

Then by the saddle point approximation we have

$$\begin{aligned} I = \frac{G(n)}{2\pi\mathrm{i}}\int_C dt f(t)\mathrm{e}^{s\Psi(t)} &\sim \frac{\mathrm{e}^{s\Psi(t_0)+\mathrm{i}\theta}}{\sqrt{n}}\left(f(t_0)\sqrt{\frac{2\pi}{|\Phi''(t_0)|}} + \mathcal{O}(s^{-1})\right), \quad as \;\; s\to\infty \\ &\sim \sqrt{\gamma-\mathrm{i}}\left(\frac{\gamma-\mathrm{i}(1-2\beta)}{1-\beta}\right)^{-1/2} + \mathcal{O}(s^{-1/2}) \end{aligned} \tag{3.68}$$

Note that the generalization to $\lambda = -s_0 + \delta$ or $s = s_0 + \delta$ leads to a modification with $\left(\frac{\gamma - \mathrm{i}(1-2\beta)}{1-\beta}\right)^{-\mathrm{i}\delta}$.

We also need to consider the branch point $t_\beta = (\beta-1)/\beta$. When it lives outside the contour C, the integration over contour C is exactly the path required by (3.60). Thus in this case we get the asymptotics of the hypergeometric function with usual saddle point method as (3.68). However, when $(\beta-1)/\beta$ inside the contour, we need to deform the contour to exclude the branch point and the branch cut due to $(\beta-1)/\beta$. A possible way is we choose the branch cut along one of the steepest decent path start at $(1-\beta)/\beta$, and deform the contour C exclude the branch point and branch cut, which may gives a non-trivial contribution to the asymptotic expansion. Since $t_\beta = (\beta-1)/\beta$ is a $1/2$ order branch point, according to [160], in this case, the contribution comes from branch point is given by

$$\begin{aligned} I_1 &\sim 2\sqrt{\pi}\frac{G(n)}{2\pi \mathrm{i}}\mathrm{e}^{s\Psi(t_\beta)} f(t_\beta,\beta)\left(t_\beta - \frac{\beta-1}{\beta}\right)^{\frac{1}{2}}\left(\frac{1}{s|\Psi'(t_\beta)|}\right)^{\frac{1}{2}} + \mathcal{O}(s^{-1/2}) \\ &\sim (1-\beta)^{(\gamma+\mathrm{i})s}\,\beta^{(-\gamma+\mathrm{i})s}\frac{\sqrt{2(\gamma-\mathrm{i})}(-1)^{\gamma s}2^{2\mathrm{i}s}((\gamma-\mathrm{i}))^{(\gamma-\mathrm{i})s}}{((\gamma+\mathrm{i}))^{(\gamma+\mathrm{i})s}}\sqrt{\frac{1-\beta}{|-\mathrm{i}(1-2\beta)+\gamma|}} + \mathcal{O}(s^{-1/2}) \end{aligned} \tag{3.69}$$

Since the asymptotics contribution contains power of s in terms of $\mathrm{e}^{s\Psi(t)}$, the full asymptotics of the function will comes from the largest $Re(\Psi(t))$ of t_0 and t_β. In our case, t_β is in the negative imaginary half plane

$$t_\beta = \frac{\beta-1}{\beta} = \frac{\bar{\beta}}{\beta} \tag{3.70}$$

And it is easy to show

$$\mathrm{Re}(\Psi(t_\beta)) = \begin{cases} -\pi, & t<0 \\ \pi & t>0 \end{cases} \tag{3.71}$$

When $t > 0$, the contribution from t_β is lower than t_0 in arbitrary order after multiply by power s, and the final result is given by (3.68). The contribution form the branch point only exist when $\sinh(t) + \gamma < \epsilon_0 < 0$ and the contribution reads

$$I = I_0 - I_1 \tag{3.72}$$

And in this case the final asymptotics is given by the sum of (3.68) and (3.69). A special case is when the branch point locates near the critical point $|t_0 - t_\beta| \le \epsilon_0$

, where the result is

$$
\begin{aligned}
I \sim & \frac{G(n)}{2\pi\mathrm{i}}\left(\frac{\pi \mathrm{e}^{\mathrm{i}\pi(-1/4+\theta/2)}}{\Gamma(1/4)} f(t_0)\left(t_0-\frac{\beta-1}{\beta}\right)^{\frac{1}{2}}\left(\frac{2}{s|\Psi''(t_0)}\right)^{-\frac{1}{4}} \mathrm{e}^{s\Psi(t_0)}+\mathcal{O}(s^{-3/4})\right)\\
\sim & \frac{2\sqrt{\pi} s^{1/4}}{\Gamma(1/4)}(-\mathrm{i}(\gamma-\mathrm{i})(\gamma+\mathrm{i}))^{1/4}+\mathcal{O}(s^{-1/4})
\end{aligned}
\tag{3.73}
$$

Note that, for the continuos of the approximation on β, we have $\epsilon_0 \sim s^{-1/2}$. Fig (2) shows the error level of above asymptotics result when $s=100$.

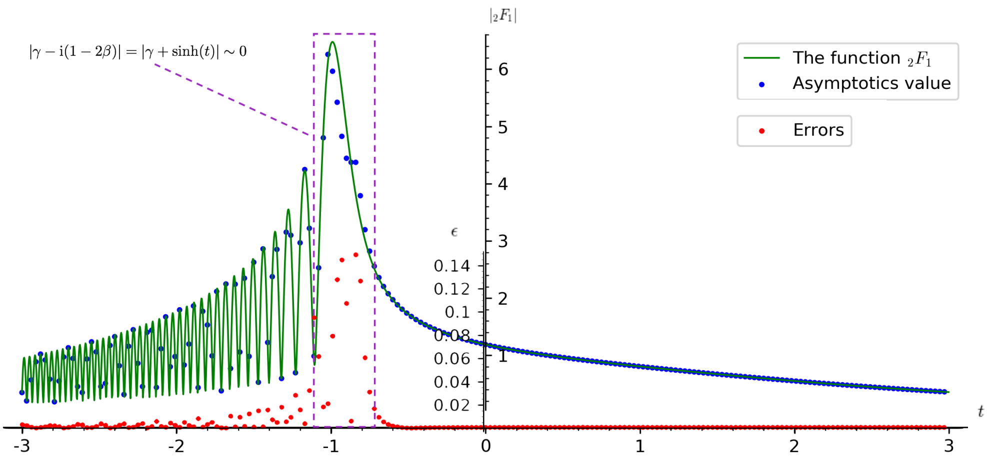


Figure 2. – The function $_2F_1\,(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;m+\mathrm{i}\lambda+1;\beta)$ as shown in (3.59) and it's asymptotics result I given as (3.68), (3.72) and (3.73) respectively, with $t\in[-3,3]$, $s=100$, $\gamma=1$. The absolute error is defined as $\epsilon=|(|I|-|_2F_1|)|/|_2F_1|$.

Now we can write out the final result, according to (3.59), we have

$$
{}_2F_1\,(j-\mathrm{i}s+1,-j-\mathrm{i}s;n/2-\mathrm{i}s+1;\beta(t)) \sim \frac{\sqrt{\gamma-\mathrm{i}}(1+\mathrm{i})}{\sqrt{2(\mathrm{i}\gamma+(1-2\beta))}}+\mathcal{O}(s^{-1/2})
\tag{3.74}
$$

From (3.57), for $\sinh(t)>-\gamma$ we have

$$
\begin{aligned}
d{0}^{j}_{n/2,-\mathrm{i}\lambda,\sigma} \sim & S^{j}_{m\lambda\sigma}\left(\frac{1}{\sqrt{s(\gamma-\mathrm{i}(1-2\beta))}}+\mathcal{O}(s^{-1})\right)\left(\frac{(1-(-1)^{\sigma}\mathrm{i})\,(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}\right.\\
& \left.-(-1)^{\sigma}\frac{\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\right)
\end{aligned}
\tag{3.75}
$$

where we use the approximation

$$\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1)\sim 2\pi\sqrt{(\gamma-\mathrm{i})s}(-(\gamma+\mathrm{i})s)^{-(\gamma+\mathrm{i})s}((\gamma-\mathrm{i})s)^{(\gamma-\mathrm{i})s}\mathrm{e}^{2\mathrm{i}s} \tag{3.76}$$

$$\Gamma(\frac{n}{2}-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)\Gamma^{-1}(j+m+1)\sim\sqrt{2}\pi\sqrt{(\gamma+\mathrm{i})s}(-2\mathrm{i}s)^{-2\mathrm{i}s}\mathrm{e}^{2\mathrm{i}s} \tag{3.77}$$

for $\sinh(t)<-\gamma$, the contribution from the extra branch point reads

$$\begin{aligned} d_{1}{}^{j}_{n/2,-\mathrm{i}s,\sigma}\sim&S^{j}_{m\lambda\sigma}\left(\frac{\sqrt{2}}{\sqrt{s|\gamma-\mathrm{i}(1-2\beta)|}}+\mathcal{O}(s^{-1})\right)\left(\frac{(1-(-1)^{\sigma}\mathrm{i})\,(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}}{\sqrt{2}\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\right.\\ &\left.-(-1)^{\sigma}\frac{\sqrt{2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}\right) \end{aligned} \tag{3.78}$$

One check the final result is approximately

$$\begin{aligned} d^{j}_{n/2,-\mathrm{i}s,\sigma}=d_{0}{}^{j}_{n/2,-\mathrm{i}s,\sigma}-d_{1}{}^{j}_{n/2,-\mathrm{i}s,\sigma}\sim S^{j}_{m\lambda\sigma}\left(\frac{1}{\sqrt{s|\gamma-\mathrm{i}(1-2\beta)|}}++\mathcal{O}(s^{-1})\right)\times\\ \left(\frac{(1-(-1)^{\sigma}\mathrm{i})\,(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}-(-1)^{\sigma}\frac{\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\right) \end{aligned} \tag{3.79}$$

When $|\gamma-\mathrm{i}(1-2\beta)|<\epsilon$, which means the branch point near the saddle point, we have

$$\begin{aligned} d^{j}_{n/2,-\mathrm{i}s,\sigma}\sim&S^{j}_{m\lambda\sigma}\left(\frac{2\sqrt{\pi}(-\mathrm{i}(1+\gamma^{2}))^{1/4}s^{1/4}}{\Gamma(1/4)\sqrt{s}}+\mathcal{O}(s^{-3/4})\right)\left(\frac{(1-(-1)^{\sigma}\mathrm{i})\,(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}\right.\\ &\left.-(-1)^{\sigma}\frac{\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\right) \end{aligned} \tag{3.80}$$

3.2.3.3. full representation matrix

According to (3.48), now we can write out D matrix in terms of group elements v:

$$D_{m,\lambda}(z) = \frac{S^j_{m,\lambda,\sigma}}{\sqrt{s_0}}\left(\frac{H(|\gamma+\mathrm{Im}(\bar{v}_1 v_2))|-\epsilon)}{\sqrt{|\gamma+\mathrm{Im}(\bar{v}_1 v_2))|}} + H(\epsilon - |\gamma+\mathrm{Im}(\bar{v}_1 v_2))|)\frac{2\sqrt{\pi}(1+\gamma^2)^{1/4}s_0^{1/4}}{\sqrt{\pi}\Gamma(1/4)}\right)$$
$$\left(T^j_{+\sigma}\left(\frac{v_1-v_2}{\sqrt{2}}\right)^{m+\mathrm{i}\lambda}\left(\overline{\frac{v_1-v_2}{\sqrt{2}}}\right)^{-m+\mathrm{i}\lambda} - T^j_{-\sigma}\left(\frac{v_1+v_2}{\sqrt{2}}\right)^{m-\mathrm{i}\lambda}\left(\overline{\frac{v_1+v_2}{\sqrt{2}}}\right)^{-m-\mathrm{i}\lambda}\right) + \mathcal{O}(s^{-3/4}) \tag{3.81}$$

where H is the Heaviside step function

$$H(x)\begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases} \tag{3.82}$$

ϵ is defined as

$$\epsilon = \frac{\Gamma(1/4)^2}{4\pi\sqrt{(1+\gamma^2)s}} \tag{3.83}$$

such that D is continuous for v. Note that the contribution from $|\gamma+\mathrm{Im}(\bar{v}_1 v_2))| < \epsilon$ is actually a regulator of the $1/2$ order singular points because of $|\gamma+\mathrm{Im}(\bar{v}_1 v_2))|$. In the inner product this regulator naturally arises as the asymptotics with $1/2$ order singular points. In this sense, we can ignore the regulator since we are only interested in the inner product in the amplitude. The constant is given by

$$T^j_{+\sigma} = \frac{1-(-1)^\sigma \mathrm{i}}{\Gamma(-m-j)\Gamma(m-j)} \tag{3.84}$$

$$T^j_{-\sigma} = \frac{(-1)^\sigma}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} \tag{3.85}$$

with S given in (3.46). In the asymptotics limit, we have

$$S^j_{m\lambda\sigma}\overline{S}^j_{m\lambda\sigma} \sim \frac{\pi}{2\cosh(2\pi s)}, \tag{3.86}$$

$$T^j_1\overline{T}^j_1 \sim \frac{2\cos(\pi(-m-\mathrm{i}s))\cos(\pi(m-\mathrm{i}s))}{\pi^2} \sim \frac{\cosh(2\pi s)}{\pi^2}, \quad \text{when} \quad s \gg 1 \tag{3.87}$$

$$T^j_2\overline{T}^j_2 \sim \frac{\cosh(2\pi s)}{\pi^2}. \tag{3.88}$$

where we use the asymptotic approximation of Gamma function

$$\lim_{z\to\infty}\frac{\Gamma(z+\epsilon)}{\Gamma(z)z^\epsilon} = 1 \tag{3.89}$$

Form the parity property of representation matrix, we have

$$D^{\sigma j}_{-m,\lambda}(v) = -(-1)^{\sigma} \mathrm{e}^{-\mathrm{i}\pi m} D^{\sigma j}_{m,\lambda}(\bar{v}) \tag{3.90}$$

3.2.4. Representation of decomposed SL$(2,\mathbb{C})$ continuous state

One see that, with the representation matrix of continuous series of SU$(1,1)$, and some transformations of hypergeometric function and asymptotic analysis, we prove that when $n \gg 1$ and $\lambda = -s$,

$$\begin{aligned} D^{j}_{\frac{n}{2},-s}(v) = \frac{1}{\sqrt{s|\gamma + \mathrm{Im}(\bar{v}_1 v_2)|}} \times \\ \left(\tilde{T}^{j}_{+\sigma} \left(\frac{v_1 - v_2}{\sqrt{2}} \right)^{\frac{n}{2}-\mathrm{i}s} \left(\overline{\frac{v_1 - v_2}{\sqrt{2}}} \right)^{-\frac{n}{2}-\mathrm{i}s} \right. \\ \left. - \tilde{T}^{j}_{-\sigma} \left(\frac{v_1 + v_2}{\sqrt{2}} \right)^{\frac{n}{2}+\mathrm{i}s} \left(\overline{\frac{v_1 + v_2}{\sqrt{2}}} \right)^{-\frac{n}{2}+\mathrm{i}s} \right) \end{aligned} \tag{3.91}$$

where $\sqrt{2}\tilde{T}^{j}_{\pm\sigma} = \sqrt{2} S^{j}_{n/2,-s,\sigma} / T^{j}_{\pm\sigma}$ are some phases: $\tilde{T}_{\pm}\overline{\tilde{T}_{\pm}} = 1/2$ [a]. The detailed definition of $S^{j}_{n/2,-s,\sigma}$ and $T^{j}_{\pm\sigma}$ are given in (3.46) and (3.84).

The $m = -n/2$ case in (3.35) can be obtained by the relation

$$D^{\sigma j}_{-m,\lambda}(v) = -(-1)^{\sigma} \mathrm{e}^{-\mathrm{i}\pi m} D^{\sigma j}_{m,\lambda}(\bar{v}) \tag{3.92}$$

When $\alpha = 1$, we would like to write elements of $v^{\alpha} \in \mathrm{SU}(1,1)$ introduced in (3.32) as

$$\frac{v_1 - v_2}{\sqrt{2}} = \frac{\langle \bar{z}, l_0^+ \rangle}{\sqrt{\langle z, z \rangle}}, \quad \frac{v_1 + v_2}{\sqrt{2}} = \frac{\langle \bar{z}, l_0^- \rangle}{\sqrt{\langle z, z \rangle}}. \tag{3.93}$$

where

$$l_0^{\pm} = \frac{1}{\sqrt{2}}(n_1 \pm n_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{3.94}$$

Notice that, $\langle l_0^+, l_0^+ \rangle = \langle l_0^-, l_0^- \rangle = 0$, $\langle l_0^-, l_0^+ \rangle = 1$, thus they form a null basis in $\mathbb{C}^2$. Similarly, for $\alpha = -1$, we have

$$\frac{v_1 - v_2}{\sqrt{2}} = -\frac{\langle l_0^+, \bar{z} \rangle}{\sqrt{-\langle z, z \rangle}}, \qquad \frac{v_1 + v_2}{\sqrt{2}} = \frac{\langle l_0^-, \bar{z} \rangle}{\sqrt{-\langle z, z \rangle}} \tag{3.95}$$

With this notation, we finally obtain

a. Here we ignore the regulator in (3.81) for the zero points of $|\gamma + \mathrm{Im}(\bar{v}_1 v_2)|$ since it will appear naturally as the integration contribution from this $1/2$ singularity in the inner product. One can check Appendix 3.2.3 for details.

$$F^{(\rho,n)}_{-s,\sigma,\alpha}(z) = \frac{\sqrt{\pi}\alpha^{n/2+\sigma+1}}{\sqrt{s}\sqrt{\alpha\langle z,z\rangle}\sqrt{|\alpha(\gamma-\mathrm{i})\langle z,z\rangle + 2\mathrm{i}\alpha\langle \bar{z},l_0^-\rangle\langle l_0^+,\bar{z}\rangle|}}$$
$$\times\left(\tilde{T}^j_{+\sigma}(\alpha\langle z,z\rangle)^{i\rho/2+\mathrm{i}s}(\langle l_0^+,\bar{z}\rangle\langle\bar{z},l_0^+\rangle)^{-\mathrm{i}s}\left(\frac{\langle\bar{z},l_0^+\rangle}{\langle l_0^+,\bar{z}\rangle}\right)^{\frac{n}{2}}\right. \tag{3.96}$$
$$\left.-\tilde{T}^j_{-\sigma}(\alpha\langle z,z\rangle)^{i\rho/2-\mathrm{i}s}(\langle l_0^-,\bar{z}\rangle\langle\bar{z},l_0^-\rangle)^{\mathrm{i}s}\left(\frac{\langle\bar{z},l_0^-\rangle}{\langle l_0^-,\bar{z}\rangle}\right)^{\frac{n}{2}}\right)$$

One can check the homogeneity property (3.21):

$$F(\lambda z) = \lambda^{m+\mathrm{i}\rho/2-1}\bar{\lambda}^{-m+\mathrm{i}\rho/2-1}F(z) \tag{3.97}$$

The coherent state is built from the reference state $\lambda = -s$, and we choose $\sigma = 1$, according to [73],

$$\Psi^{(\rho,n)}_{\tilde{g},\alpha}(z) = D^{(\rho,n)}(\tilde{g})F^{(\rho,n)}_{-s,1,\alpha}(z) = \frac{\sqrt{\mathrm{i}\pi}\tilde{S}^j_{m,-s,\sigma}\alpha^{-2\mathrm{i}s+m}}{\sqrt{|\langle z,z\rangle|}\sqrt{|(\gamma-\mathrm{i})\langle z,z\rangle + 2\mathrm{i}\langle\bar{z},l^-\rangle\langle l^+,\bar{z}\rangle|}}\times$$
$$\left(\tilde{T}^j_{+1}\langle z,z\rangle^{i\rho/2+\mathrm{i}s}(\langle l^+,\bar{z}\rangle\langle\bar{z},l^+\rangle)^{-\mathrm{i}s}\left(\frac{\langle\bar{z},l^+\rangle}{\langle l^+,\bar{z}\rangle}\right)^{\frac{n}{2}} - \tilde{T}^j_{-1}\langle z,z\rangle^{i\rho/2-\mathrm{i}s}(\langle l^-,\bar{z}\rangle\langle\bar{z},l^-\rangle)^{\mathrm{i}s}\left(\frac{\langle\bar{z},l^-\rangle}{\langle l^-,\bar{z}\rangle}\right)^{\frac{n}{2}}\right) \tag{3.98}$$

where $\tilde{g}\in SU(1,1)$, and $l^\pm = \tilde{g}^{-1\dagger}l_0^\pm$ is defined though

$$\langle l_0^\pm,\overline{\tilde{g}^t z}\rangle = \langle\tilde{g}^{-1\dagger}l_0^\pm,\bar{z}\rangle = \langle l^\pm,\bar{z}\rangle \tag{3.99}$$

3.2.5. Spinform amplitude

Now we can write down explicitly the inner product between the coherent states appearing in the amplitude (3.17) by inserting (3.98) and using (3.22):

$$\left\langle\Psi^{(\rho_f,n_f)}_{\tilde{g}_{e'f}\delta}\middle| D^{(\rho_f,n_f)}(g_{ve'}g_{ev})\middle|\Psi^{(\rho_f,n_f)}_{\tilde{g}_{ef}\delta}\right\rangle = \sum_\alpha\int_{CP_1}\omega_{z_{vf}}\Psi^{(\rho_f,n_f)}_{\tilde{g}_{e'f}\delta\alpha}\left(g^t_{ve'}z_{vf}\right)\overline{\Psi^{(\rho_f,n_f)}_{\tilde{g}_{ef}\delta\alpha}}\left(g^t_{ev}z_{vf}\right)$$
$$= \int_{CP_1/\langle Z,Z\rangle=0}\frac{\omega_{z_{vf}}}{h_{vef}h_{ve'f}}\left(N_{f+}\mathrm{e}^{S_{vf+}} + N_{f-}\mathrm{e}^{S_{vf-}} + N_{fx+}\mathrm{e}^{S_{vfx+}} + N_{fx-}\mathrm{e}^{S_{vfx-}}\right) \tag{3.100}$$

where N are some normalization factors, ω is the SL$(2,\mathbb{C})$ invariant measure defined in (3.23). The exponents read

$$S_{vf\pm} = S_{ve'f\pm} - S_{vef\pm}, \quad S_{vfx\pm} = S_{ve'f\pm} - S_{vef\mp} \tag{3.101}$$

with

$$S_{vef\pm} = s_f \left[\gamma \ln \frac{\langle Z_{vef}, l^{\pm}_{ef} \rangle}{\langle l^{\pm}_{ef}, Z_{vef} \rangle} \mp \mathrm{i} \ln \langle Z_{vef}, l^{\pm}_{ef} \rangle \langle l^{\pm}_{ef}, Z_{vef} \rangle + \mathrm{i}(-1 \pm 1) \ln \langle Z_{vef}, Z_{vef} \rangle \right] \tag{3.102}$$

where $Z_{vef} = g^{\dagger}_{ve} \bar{z}_{vf}$. $l^{\pm}_{ef}$ here is defined as $l^{\pm} = v(N_{ef})^{-1\dagger} l^{\pm}_0$ with $l^{\pm}_0$ defined in (3.94), and $v(N_{ef}) \in$ SU$(1,1)$ which encoding the unit normal. $\langle Z_{ve'f}, Z_{ve'f} \rangle$ has the same sign as $\langle Z_{vef}, Z_{vef} \rangle$. The integrand is invariant under the following gauge transformations:

$$g_{ve} \to g_v g_{ve}, \qquad z_{vf} \to \lambda_{vf} (g^T_v)^{-1} z_{vf} \tag{3.103}$$

$$g_{ve} \to s_{ve} g_{ve}, \qquad s_{ve} = \pm 1 \tag{3.104}$$

$$g_{ve} \to g_{ve} v_e, \qquad l^{\pm}_{ef} \to v_e l^{\pm}_{ef}, \tag{3.105}$$

where $g_v \in$ SL$(2,\mathbb{C})$, $v_e \in$ SU$(1,1)$, and $\lambda_{vf} \in \mathbb{C} \setminus \{0\}$.

It's worth to point out that both $S_{vf\pm}$ and $S_{vfx\pm}$ are purely imaginary, and they are all proportional to s_f which will be uniform scaled later to derive the asymptotics. The real valued function h is given by

$$h_{vef} = |\langle Z_{vef}, Z_{vef} \rangle| \sqrt{\left| \gamma - \mathrm{i} + \frac{2 \langle l^{-}_{ef}, Z_{vef} \rangle \langle Z_{vef}, l^{+}_{ef} \rangle}{\langle Z_{vef}, Z_{vef} \rangle} \right|} \tag{3.106}$$

h_{vef} can be 0 when we integrate over z on $\mathbb{CP}_1$ and SL$(2,\mathbb{C})$ group elements g in (3.17), and the zeros of h are exactly the points where we define the principle value, i.e. at $\langle Z, Z \rangle = 0$. However, as shown later in 3.3.1later , the singularities due to h are of half order thus the final integral is remain finite at these points.

3.3. Analysis of critical points

As we shown above, the actions $S_{vf\pm}$ and $S_{vfx\pm}$ are pure imaginary, and they are proportional to s_f. Thus we can use stationary phase approximation to evaluate the amplitude in the semi-classical limit where s is uniformly scaled by a factor $\Lambda \to \infty$. Note that the denominator h defined by (3.106) in (3.100) contains $1/2$ order singular point at $\langle Z, Z \rangle = 0$, as shown in following sections. Then the integral is of the following type

$$I = \int \mathrm{d}x \frac{1}{\sqrt{x - x_0}} g(x) \mathrm{e}^{\Lambda S(x)} \tag{3.107}$$

Here g is an analytic function which does not scale with Λ. There are two different asymptotic equations for such type integral according to the critical point x_c located exactly at the branch point x_0 or away from it. According to [161], if x_c located exactly at x_0, the leading order contribution will locate at the critical points (which is also the branch points), and the asymptotic expansion is given by

$$I \sim g(x_c)\frac{\pi e^{i\pi(\mu-2)/8}}{\Gamma(3/4)}\left(\frac{2}{\Lambda|\det H(x_c)|}\right)^{1/4} e^{\Lambda S(x_c)} \tag{3.108}$$

where $H(x_c)$ is the Hessian matrix at x_c, and $\mu = \operatorname{sgn}\det H(x_c)$.

As we explain in the following sections, the critical points of Eq.(3.100) are always located at the branch points, when every tetrahedron containing the timelike triangle f also contain at least one spacelike triangle. It is quite generic to have every tetrahedron contain both timelike and spacelike triangles in a simplicial geometry. In addition, in case that we consider tetrahedra with all triangles timelike, for a single vertex amplitude, the critical point is again located at the branch points, when the boundary data give the closed geometrical boundary of a 4-simplex (i.e. the tetrahedra at the boundary are glued with shape matching). We don't consider the possibility other than (3.108).

3.3.1. Analysis of singularities and corresponding stationary phase approximation

In this appendix we concentrate on the analysis of singularities appears in the denominator of the integrand of vertex amplitude.

3.3.1.1. Analysis of singularities

For simplicity, we consider one vertex case for some v mainly. As we show, the amplitude enrolls the integration in the form

$$I = \int \prod_e dg_{ve} \int \prod_f \Omega_{vf} \prod_f \frac{1}{h_{vef}h_{ve'f}} e^{S_{vf}} \tag{3.109}$$

where h is a real valued function

$$h_{vef} = |\langle Z_{vef}, Z_{vef}\rangle| \sqrt{\left|\gamma - i(1 - \frac{2\langle l^-_{ef}, Z_{vef}\rangle\langle Z_{vef}, l^+_{ef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle})\right|} \tag{3.110}$$

Here each dual face is determined by two edges $f = (e, e')$. Note that the square root part inside h_{vef} is the spinor representation for the square root term inside the wigner d matrix:

$$\left|\gamma - \mathrm{i}(1 - \frac{2\langle l_{ef}, Z_{vef}\rangle\langle Z_{vef}, l_{ef}^{+}\rangle}{\langle Z_{vef}, Z_{vef}\rangle})\right| = |\gamma + \mathrm{Im}(v_1\bar{v}_2)| \tag{3.111}$$

The zero sets of h is given by $\langle Z_{vef}, Z_{vef}\rangle = 0$ or $|\gamma + \mathrm{Im}(v_1\bar{v}_2)| = 0$.

We can rewrite the original $\langle Z_{vef}, Z_{vef}\rangle$ as

$$\langle Z_{vef}, Z_{vef}\rangle = 2\,\mathrm{Re}(\langle l_{ef}^{-}, Z_{vef}\rangle\langle Z_{vef}, l_{ef}^{+}\rangle) = \mathrm{Re}(f) \tag{3.112}$$

where we define f as

$$f := 2\langle l_{ef}^{-}, Z_{vef}\rangle\langle Z_{vef}, l_{ef}^{+}\rangle \tag{3.113}$$

In this notation h_{vef} becomes

$$h_{vef} = |\,\mathrm{Re}(f)|\sqrt{|\gamma + \frac{\mathrm{Im}(f)}{\mathrm{Re}(f)}|} = |f||\cos(\phi_f)|\sqrt{|\gamma + \tan(\phi_f)|} \tag{3.114}$$

Suppose the function f are linearly independent to each other. This requirement is the same as require the boundary tetrahedron $l_{ef}^{\pm}$ is non degenerate. In this case, we can define a coordinate transformation among the set of the original coordinates $(z, g) \to (\mathrm{Re}(f), \mathrm{Im}(f), z', g')$. The coordinate transformation only transfer among the number of f variables and leaves the left invariant, e.g. we only transfer 40 variables in one vertex case and leave the other 4 invariant. The elements of Jacobian matrix of the transformation $J(f)$ is given by

$$\frac{\partial(\mathrm{Re}(f_{vef}))}{\partial z} = \overline{\frac{\partial(\mathrm{Re}(f_{vef}))}{\partial \bar{z}}} = \delta_z\langle Z_{vef}, Z_{vef}\rangle = (g_{ve}\eta Z_{vef})^T \tag{3.115}$$

$$\frac{\partial(\mathrm{Im}(f_{vef}))}{\partial z} = \mathrm{i}(\delta_z\langle Z_{vef}, Z_{vef}\rangle - 2\delta_z\langle l_{ef}^{-}, Z_{vef}\rangle\langle Z_{vef}, l_{ef}^{+}\rangle = \mathrm{i}((g_{ve}\eta Z_{vef} - 2g\eta l_{ef}^{+}\langle l_{ef}^{-}, Z_{vef}\rangle)^T \tag{3.116}$$

$$\frac{\partial(\mathrm{Re}(f_{vef}))}{\partial g} = \delta_g\langle Z_{vef}, Z_{vef}\rangle = \langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle \tag{3.117}$$

$$\begin{aligned}\frac{\partial(\mathrm{Im}(f_{vef}))}{\partial g} &= \mathrm{i}(\langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle \\ &\quad - 2\langle l_{ef}^{-}, Z_{vef}\rangle\langle L^\dagger Z_{vef}, l_{ef}^{+}\rangle - 2\langle l_{ef}^{-}, L^\dagger Z_{vef}\rangle\langle Z_{vef}, l_{ef}^{+}\rangle)\end{aligned} \tag{3.118}$$

where L represents generators of SL$(2,\mathbb{C})$. Note that $\delta_g\langle Z_{vef}, Z_{vef}\rangle$ is zero when L are SU$(1,1)$ generators. However, the Jacobian is non zero in general, e.g. in one vertex case of vertex v, we have the non-trivial contribution from terms like

$$\begin{aligned}&\partial_{g_1}(13,14,15), \quad \partial_{g_2}(21,24,25), \quad \partial_{g_3}(31,32,35), \quad \partial_{g_4}(42,43,45), \\ &\partial_z(12,23,34,41,51,52,53,54)\end{aligned} \tag{3.119}$$

where 12 is the representation of ef label in terms of numbers labelling edges and corresponding faces (e_1, e_2). Apart from those 0 in (3.117), other zeros of matrix elements only possible when $Z = \zeta l^{\pm}$. The Jacobian matrix in this case is given by ($Z = \zeta l^{+}$ as an example),

$$\frac{\partial(\mathrm{Re}(f_{vef}))}{\partial z} = \overline{\frac{\partial(\mathrm{Re}(f_{vef}))}{\partial \bar{z}}} = (g_{ve}\eta Z_{vef})^T \tag{3.120}$$

$$\frac{\partial(\mathrm{Im}(f_{vef}))}{\partial z} = \overline{\frac{\partial(\mathrm{Im}(f_{vef}))}{\partial \bar{z}}} = -\mathrm{i}(g_{ve}\eta Z_{vef})^T \tag{3.121}$$

$$\frac{\partial(\mathrm{Re}(f_{vef}))}{\partial g} = \langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle = \begin{cases} 0, & L = F \\ 2\langle Z_{vef}, L^\dagger Z_{vef}\rangle, & L = \mathrm{i}F \end{cases} \tag{3.122}$$

$$\frac{\partial(\mathrm{Im}(f_{vef}))}{\partial g} = \mathrm{i}(\langle Z_{vef}, L^\dagger Z_{vef}\rangle - \langle L^\dagger Z_{vef}, Z_{vef}\rangle) = \begin{cases} 2\mathrm{i}\langle Z_{vef}, L^\dagger Z_{vef}\rangle, & L = F \\ 0, & L = \mathrm{i}F \end{cases} \tag{3.123}$$

Clearly the Jacobian matrix is still well defined and leads to non zero Jacobian.

After this coordinate transformation, the original integration becomes

$$I = \prod_v \int \frac{\Omega'}{J(f)} \prod_{e,f} \mathrm{d}\,\mathrm{Re}(f_{vef})\mathrm{d}\,\mathrm{Im}(f_{vef}) \prod_f \frac{e^{S_{vf}}}{|\mathrm{Re}(f_{vef})||\mathrm{Re}(f_{ve'f})|\sqrt{|\gamma + \frac{\mathrm{Im}(f_{vef})}{\mathrm{Re}(f_{vef})}|}\sqrt{|\gamma + \frac{\mathrm{Im}(f_{ve'f})}{\mathrm{Re}(f_{ve'f})}|}} \tag{3.124}$$

With a further polar coordinate transformation

$$\rho_{vef} = \sqrt{\mathrm{Re}(f_{vef})^2 + \mathrm{Im}(f_{vef})^2}, \qquad \phi_{vef} = \arg(f_{vef}) \ \in \ [0, \pi/2) \tag{3.125}$$

whose Jacobian is given by

$$J^1_{vef} = \frac{1}{\rho_{vef}} \tag{3.126}$$

The Jacobian is well defined except on the points where $|f| = 0$. After the coordinates transformation, we have

$$I = \int \Omega' \prod_{e,f} \int \mathrm{d}\rho_{vef} \int_0^{\pi/2} \mathrm{d}\phi_{vef} \frac{1}{J(\rho,\phi)} \prod_f \frac{e^{S_{vf}}}{|\cos(\phi_{vef})||\cos(\phi_{ve'f})|\sqrt{|\gamma + \tan(\phi_{vef})||\gamma + \tan(\phi_{ve'f})|}} \tag{3.127}$$

Clearly all possible singular points are $1/2$ order. The singular points due to $|\gamma + \tan(\phi_{ve'f})|$ and due to $|\cos(\phi_{ve'f})|$ are separated. The integration respects to ρ does not have singularities.

3.3.1.2. Multidimensional Stationary phase approximation

In appendix 3.2.3, we already use the saddle point approximation when there is a branch point appearing in the non-scaled function $g(x)$. When adapting to the stationary phase approximation, for the $1/2$ order singular point locates exactly at the critical point, the result is the following:

$$I = \int \frac{g(x)}{\sqrt{x}} \mathrm{e}^{\Lambda S(x)} \sim g(x_c) \frac{\pi \mathrm{e}^{\mathrm{i}\pi(\mu-2)/8}}{\Gamma(3/4)} \left(\frac{2}{\Lambda |S''(x_c)|} \right)^{1/4} \mathrm{e}^{\Lambda S(x_c)} \tag{3.128}$$

where $\Lambda \sim \infty$ and S is pure imaginary. Note that the dominate part here is given by the $-1/4$ order of Λ instead of $-1/2$ as in the asymptotic formula without singularities. The regulator appears in (3.81) is exactly this $1/4$ order difference.

However, this asymptotic formula only hold for single variable integral. We will generalize this single variable approximation to multi variables case. Recall Fubini's theorem:

Theorem 3.3.1. *Let $w = f(x_1, x_2, \ldots, x_n)$ be a n variable valued complex function. If the integral of f on the domain $B = \prod_i^n I_n$ where I_n are intervals in $\mathbb{R}$ is absolutely convergent:*

$$\int_B |f(x_1, x_2, \ldots, x_n)| d(x_1, x_2, \cdots, x_2) < \infty, \tag{3.129}$$

then the multiple integral will give the same result as the iterated integral,

$$\int_{A \times B} |f(x, y)| d(x, y) = \int_A (\int_B f(x, y) dy) dx = \int_B (\int_A f(x, y) dx) dy \tag{3.130}$$

The result is independent of the iterate order.

Here from (3.127) we have the integral in the form

$$I = \int d^n x \prod_{i=1}^{j} (x_i)^{-1/2} g(x) \mathrm{e}^{\mathrm{i}S(x)} \tag{3.131}$$

where $S(x) \in \mathbb{R}$, $x \in \mathbb{R}^n$, $j < n$ and $g(x)$ is analytic. $j < n$ illustrates the fact that only in a subspace of the total variables space will have singularities. Then in a closed region M where the stationary phase points (solutions of $\delta S = 0$) exists, we have

$$\int_M d^n x | \prod_i (x_i)^{-1/2} g(x) \mathrm{e}^{\mathrm{i}S(x)} | \sim \int_M d^n x | \prod_i (x_i)^{-1/2} \tilde{g}(x) | < \infty \tag{3.132}$$

From Fubini's theorem, we then can write the multi-dimensional integral as iterated integral. For the original variables, since the singularities exist only in a subspace of the total variables space, we can always perform a coordinate transformation, such that variables with singularities are separated from those do not have, as we show in (3.127). Then the final result is given by performing the stationary phase approximation iteratively. Each step one may use the usual stationary phase approximation or the one with singularities. The lowest order of the total integration is given by picking lowest order approximation of each single integration.

However, due to technical reason, we would like to derive the saddle point equations directly from $S(x)$ instead of evaluate it iteratively. According to the approximation, each single valued integral is dominated by the phase $S(x_0)$ where x_0 is the solution of saddle point equation $\delta_x S(x) = 0$. Then iteratively, the saddle points are given by

$$\begin{aligned}
&\delta_{x_1} S(x_1, x_2, \ldots, x_n) = 0, \\
&\delta_{x_2} S(x_1^0, x_2, \ldots, x_n) = \left(\delta_{x_1} S(x) \frac{\partial x_1^0}{\partial x_2} + \delta_{x_2} S(x)\right)\Big|_{x_1 = x_1^0} = \delta_{x_2} S(x)|_{x_1=x_1^0} = 0, \\
&\qquad\qquad\qquad\qquad\vdots \\
&\delta_{x_n} S(x_1^0, x_2^0, \ldots, x_n) = \delta_{x_n} S(x)|_{x_1=x_1^0, x_2=x_2^0, \ldots, x_{n-1}=x_{n-1}^0} = 0
\end{aligned} \tag{3.133}$$

where $x_i^0(x_{i+1}, \ldots, x_n)$ is the solution of the corresponding equation of motion $\delta_{x_i}(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}, \ldots, x_n)$ respect to x_i. As one can see from (3.133), the above equation of motion is nothing else but we solve the original equation of motion $\{E_n = \delta S(x)\}$ iteratively. Thus they have the same solutions. The saddle points given by the two method will coincide with each other. Note that, for variables whose saddle points near the singularities, the induced measure which contains second derivatives of the action will given in the order $1/4$ instead of $1/2$ for those do not have singularities. As a result, there is no general Hessian term as in the previous EPRL approximation, and the measure is more involved as some special functions of second derivatives of the action. As a result finally we have order $I \sim g(\Lambda)\Lambda^{-a/2-b/4}$ for b variables have singular points.

3.3.2. Equation of Motion

Since both $S_{vf\pm}$ and $S_{vfx\pm}$ are purely imaginary, their critical points, or namely critical configurations, are solutions of equations of motion. The equations of motion are given by variations of S's respects to spinors z, $\mathrm{SU}(1,1)$ group elements v and $\mathrm{SL}(2,\mathbb{C})$ group elements g.

Before calculating the variation, we would like to introduce a decomposition of spinor Z. We first introduce following lemmas:

Lemma 3.3.2. *Given a specific l^+ satisfying $\langle l^+, l^+\rangle = 0$, there exist $\tilde{l}^-$, s.t. $\langle l^+, \tilde{l}^-\rangle = 1, \langle \tilde{l}^-, \tilde{l}^-\rangle = 0$. For two elements $\tilde{l}_1^-$ and $\tilde{l}_2^-$ satisfying the condition, they are related by*

$$\tilde{l}_1^- = \tilde{l}_2^- + \mathrm{i}\eta l^+, \qquad \eta \in R \tag{3.134}$$

This is easy to proof since $\langle \tilde{l}^- + \mathrm{i}\eta l^+, \tilde{l}^- + \mathrm{i}\eta l^+\rangle = \eta^2\langle l^+, l^+\rangle + \langle \tilde{l}_2^-, \tilde{l}_2^-\rangle - \mathrm{i}\eta\langle l^+, \tilde{l}^-\rangle + \mathrm{i}\eta\langle \tilde{l}^-, l^+\rangle$ and $\langle l^+, \tilde{l}^- + \mathrm{i}\eta l^+\rangle = \langle l^+, \tilde{l}^-\rangle + \mathrm{i}\eta\langle l^+, l^+\rangle$.

Lemma 3.3.3. *For a given l^+ and $\tilde{l}^-$ defined by Lemma 3.3.2, l^+ and $\tilde{l}^-$ form a null basis in two dimensional spinors space.*

This lemma is proved by using the fact that given l^+ and $\tilde{l}^-$, there exists a $\mathrm{SU}(1,1)$ element $\tilde{g}$, such that $l^+ = \tilde{g}l_0^+$ and $\tilde{l}^- = \tilde{g}l_0^-$, and the fact that l_0^+ and l_0^- forms a null basis.

With Lemma 3.3.3, for a given l^+ or l^-, we have

Theorem 3.3.4. *For given l^+ and $\tilde{l}^-$ defined by Lemma 3.3.2, spinor Z_{vef} always can be decomposed as*

$$Z_{vef} = \zeta_{vef}(\tilde{l}_{ef}^{\mp} + \alpha_{vef} l_{ef}^{\pm}) \tag{3.135}$$

where $\zeta_{vef} \in \mathbb{C}$ and $\alpha_{vef} \in \mathbb{C}$.

At the vertex v, from the action S_{vef+} (S_{vef-}), we only have l^+ (l^-) enters the action, thus we can choose arbitrarily $\tilde{l}_{vef}^{\mp}$ to form a basis. By Lemma 3.3.2, we can always write $\tilde{l}'^{\mp}_{vef} = \tilde{l}'^{\mp}_{vef} + \mathrm{i}\,\mathrm{Im}(\alpha_{vef}) l_{ef}^{\pm}$ s.t.,

$$Z_{vef} = \zeta_{vef}(\tilde{l}_{vef}^{\mp} + \mathrm{Re}(\alpha_{vef}) l_{ef}^{\pm}). \tag{3.136}$$

$\mathrm{Im}(\alpha)$ is basis dependent. It is easy to check that if we replace Z inside the action (3.101) by the decomposition (3.135), the action is independent of $\mathrm{Im}(\alpha)$, which means that $\mathrm{Im}(\alpha)$ is a gauge freedom.

We will drop the tilde on $\tilde{l}$ in the following. One should keep in mind that we have the freedom to choose the l^- (l^+) such that for some vertices v, $\mathrm{Im}(\alpha_{vef}) = 0$.

From the decomposition of Z_{vef}, there is naturally a constraint. By the fact $Z_{vef} = g_{ve}^{\dagger} \bar{z}_{vf}$, we have

$$\bar{z}_{vf} = g_{ve}^{-1\dagger}\, Z_{vef} = g_{ve'}^{-1\dagger}\, Z_{ve'f}. \tag{3.137}$$

In terms of decomposition of Z_{vef}

$$g_{ve}^{-1\dagger}\,(l_{ef}^{\pm} + \alpha_{vef} l_{ef}^{\mp}) = \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}^{-1\dagger}\,(l_{e'f}^{\pm} + \alpha_{ve'f} l_{vef}^{\mp}) \tag{3.138}$$

This can be written as

$$g_{ve}\ J\,(l_{ef}^{\pm} + \alpha_{vef} l_{ef}^{\mp}) = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'}\ J\,(l_{e'f}^{\pm} + \alpha_{ve'f} l_{ef}^{\mp}) \tag{3.139}$$

where we used the anti-linear map J:

$$J(a,b)^T = (-\bar{b}, \bar{a}), \quad JgJ^{-1} = -JgJ = g^{-1\dagger} \tag{3.140}$$

3.3.2.1. variation respect to z

From the definition of SU$(1,1)$ inner product, for arbitrary spinor u we have

$$\begin{aligned}\delta_{\bar{z}}\langle u, Z\rangle &= \delta_{\bar{z}}(u^\dagger \eta g^\dagger \bar{z}) = (g\eta u)^\dagger \delta\bar{z}, \\ \delta_z\langle Z, u\rangle &= \delta_z((g^\dagger \bar{z})^\dagger \eta u) = (\delta z)^T (g\eta u)\end{aligned} \tag{3.141}$$

Then it is straight forward to see the variation of S_{vef} leading to

$$\delta_{\bar{z}} S_{vef\pm} = (\frac{n_f}{2} \pm \mathrm{i}s_f)\frac{(g_{ve}\eta l^\pm_{ef})^\dagger}{\langle l^\pm_{ef}, Z_{vef}\rangle} - \mathrm{i}(\rho_f \pm s_f)\frac{(g_{ve}\eta Z_{vef})^\dagger}{\langle Z_{vef}, Z_{vef}\rangle} \tag{3.142}$$

and

$$\delta_z S = -\overline{\delta_{\bar{z}} S} \tag{3.143}$$

which comes from the fact that S is pure imaginary. With the definition of S_{vf} in (3.101), after inserting the decomposition, we obtain the following equations

$$\delta S_{vf+} = (\gamma - \mathrm{i})s_f(\frac{g_{ve}\eta l^+_{ef}}{\bar{\zeta}_{vef}} - \frac{g_{ve'}\eta l^+_{e'f}}{\bar{\zeta}_{ve'f}}) = 0 \qquad \text{with } Z = \zeta(l^- + \alpha l^+) \tag{3.144}$$

$$\delta S_{vf-} = -\mathrm{i}s_f(\frac{g_{ve}\eta n_{vef}}{\mathrm{Re}(\alpha_{vef})\bar{\zeta}_{vef}} - \frac{g_{ve'}\eta n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f})\bar{\zeta}_{ve'f}}) = 0 \qquad \text{with } Z = \zeta(l^+ + \alpha l^-) \tag{3.145}$$

$$\delta S_{vfx+} = -(\gamma - \mathrm{i})s_f\frac{g_{ve'}\eta l^+_{e'f}}{\bar{\zeta}_{ve'f}} - \mathrm{i}s_f\frac{g_{ve}\eta n_{vef}}{\mathrm{Re}(\alpha_{vef})\bar{\zeta}_{vef}} = 0 \quad \text{with } Z_{e'} = \zeta(l^- + \alpha l^+) \;\&\; Z_e = \zeta(l^+ + \alpha l^-) \tag{3.146}$$

$$\delta S_{vfx-} = (\gamma - \mathrm{i})s_f\frac{g_{ve}\eta l^+_{vef}}{\bar{\zeta}_{vef}} + \mathrm{i}s_f\frac{g_{ve'}\eta n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f})\bar{\zeta}_{ve'f}} = 0 \quad \text{with } Z_e = \zeta(l^- + \alpha l^+) \;\&\; Z_{e'} = \zeta(l^+ + \alpha l^-) \tag{3.147}$$

where

$$n_{vef} := l^+_{ef} + \mathrm{i}\left(\gamma\,\mathrm{Re}(\alpha_{vef}) + \mathrm{Im}(\alpha_{vef})\right) l^-_{ef} \tag{3.148}$$

Note that n_{vef} here satisfies Lemma. 3.3.3 and can form a basis with l^-_{ef} given in S_{vef-}.

3.3.2.2. variation respect to SU$(1,1)$ group elements v_{ef}

Since $l^\pm = v^{-1\dagger} l^\pm_0$ with $v \in$ SU$(1,1)$, the variation respect to $l^\pm$ is the variation respect to the SU$(1,1)$ group element v. If we considering a small perturbation of v which is given by $v' = v\mathrm{e}^{-\epsilon_i F^i}$, where F^i are generators of SU$(1,1)$ group, we have $v'^{-1} = \mathrm{e}^{\epsilon_i F^i} v^{-1}$. The variation is then given by

$$\delta v^{-1} = \epsilon_i F^i v^{-1}, \quad \delta v^{-1\dagger} = \epsilon_i v^{-1\dagger} (F^i)^\dagger \tag{3.149}$$

Thus for arbitrary spinor u, we have

$$\begin{aligned} \delta\langle u, m\rangle &= \delta\langle u, v^{-1\dagger} m_0\rangle = \epsilon^i \langle u, v^{-1\dagger} F_i^\dagger m_0\rangle \\ \delta\langle m, u\rangle &= \delta\langle v^{-1\dagger} m_0, u\rangle = \epsilon^i \langle v^{-1\dagger} F_i^\dagger m_0, u\rangle \end{aligned} \tag{3.150}$$

When $S_{ef} = S_{vef\pm} - S_{v'ef\pm}$, the variation reads

$$\begin{aligned} \delta S =& \epsilon^i (\frac{n_f}{2} \mp \mathrm{i} s_f) \left(\frac{\langle Z_{v'ef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm \rangle}{\langle Z_{v'ef}, l_{ef}^\pm \rangle} - \frac{\langle Z_{vef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm \rangle}{\langle Z_{vef}, l_{ef}^\pm \rangle} \right) \\ &+ \epsilon^i (\frac{n_f}{2} \pm \mathrm{i} s_f) \left(\frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{vef} \rangle}{\langle l_{ef}^\pm, Z_{vef} \rangle} - \frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{v'ef} \rangle}{\langle l_{ef}^\pm, Z_{v'ef} \rangle} \right) \end{aligned} \tag{3.151}$$

While $S_{ef} = S_{vef\pm} - S_{v'ef\mp}$, we have

$$\begin{aligned} \delta S =& \epsilon^i (\frac{n_f}{2}) \left(\frac{\langle Z_{vef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm \rangle}{\langle Z_{vef}, l_{ef}^\pm \rangle} - \frac{\langle Z_{v'ef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\mp \rangle}{\langle Z_{v'ef}, l_{ef}^\mp \rangle} + \frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{vef} \rangle}{\langle l_{ef}^\pm, Z_{vef} \rangle} - \frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\mp, Z_{v'ef} \rangle}{\langle l_{ef}^\mp, Z_{v'ef} \rangle} \right) \\ &+ \epsilon^i s_f \left(\frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{vef} \rangle}{\langle l_{ef}^\pm, Z_{vef} \rangle} + \frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{v'ef} \rangle}{\langle l_{ef}^\pm, Z_{v'ef} \rangle} + \frac{\langle Z_{v'ef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\mp \rangle}{\langle Z_{v'ef}, l_{ef}^\mp \rangle} + \frac{\langle Z_{vef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm \rangle}{\langle Z_{vef}, l_{ef}^\pm \rangle} \right) \end{aligned} \tag{3.152}$$

Since $F^i = 1/2(\mathrm{i}\sigma_3, \sigma_1, \sigma_2,)$ is SU$(1,1)$ generators, we have

$$(F^0)^\dagger l_0^\pm = \tfrac{\mathrm{i}}{2\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \tfrac{\mathrm{i}}{2} l_0^\mp \tag{3.153}$$

$$(F^1)^\dagger l_0^\pm = \tfrac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \pm \tfrac{1}{2} l_0^\pm \tag{3.154}$$

$$(F^2)^\dagger l_0^\pm = \tfrac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \mp \tfrac{1}{2} l_0^\mp \tag{3.155}$$

Then in the first case we only left with one equation, which reads

$$0 = (\frac{n_f}{2} \mp \mathrm{i} s_f) \left(\frac{\langle Z_{v'ef}, \mathrm{i} l_{ef}^\mp \rangle}{\langle Z_{v'ef}, l_{ef}^\pm \rangle} - \frac{\langle Z_{vef}, \mathrm{i} l_{ef}^\mp \rangle}{\langle Z_{vef}, l_{ef}^\pm \rangle} \right) + (\frac{n_f}{2} \pm \mathrm{i} s_f) \left(\frac{\langle \mathrm{i} l_{ef}^\mp, Z_{vef} \rangle}{\langle l_{ef}^\pm, Z_{vef} \rangle} - \frac{\langle \mathrm{i} l_{ef}^\mp, Z_{v'ef} \rangle}{\langle l_{ef}^\pm, Z_{v'ef} \rangle} \right) \tag{3.156}$$

After inserting the decomposition $Z = \zeta(l^{\mp} + \alpha l^{\pm})$ correspondingly, we get

$$\begin{aligned} 0 =&(\frac{n_f}{2} \mp \mathrm{i}s_f)\,(\bar{\alpha}_{v'ef} - \bar{\alpha}_{vef}) + (\frac{n_f}{2} \pm \mathrm{i}s_f)\,(\alpha_{v'ef} - \alpha_{vef}) \\ =&2\mathrm{i}s_f\gamma\,\mathrm{Re}(\alpha_{v'ef} - \alpha_{vef}) \pm 2\mathrm{i}s_f\,\mathrm{Im}(\alpha_{vef} - \alpha_{v'ef}) \end{aligned} \tag{3.157}$$

The solution reads

$$\gamma\,\mathrm{Re}(\alpha_{vef}) \mp \mathrm{Im}(\alpha_{vef}) = \gamma\,\mathrm{Re}(\alpha_{v'ef}) \mp \mathrm{Im}(\alpha_{v'ef}) \tag{3.158}$$

Here $\mathrm{Im}(\alpha)$ is the decomposition of Z respect to $l^{\mp}_{ef}$ specified by v_{ef}. Note that in this case, we only have $l^{+}_{ef}(l^{-}_{ef})$ in the action, thus there is an ambiguity of v_{ef}. However, changing v_{ef} corresponds to adding the same constant to both $\mathrm{Im}(\alpha_v)$ and $\mathrm{Im}(\alpha'_v)$, thus the relation is kept unchange. After absorbing $\mathrm{Im}(\alpha)$ into $\tilde{l}$ by a redefinition, the equation actually tells us that,

$$\tilde{l}^{\mp}_{vef} - \tilde{l}^{\mp}_{v'ef} = \pm\gamma(\mathrm{Re}(\alpha_{vef}) - \mathrm{Re}(\alpha_{v'ef}))l^{\pm}_{ef} \tag{3.159}$$

which fixes the transformation of $\tilde{l}_{vef}$ between vertices and removes the ambiguity between different vertices v in the bulk. With this redefinition, it is easy to see that n_{vef} defined in (3.148) satisfies $n_{vef} = n_{ve'f}$, thus we ignore the v variable and define

$$n_{ef} := n_{vef} = n_{v'ef} \tag{3.160}$$

In the mixing case there will be two different equations for F_2 and F_3, which leads to

$$0 =\frac{n_f}{2}\left(\mathrm{Re}\,\frac{\langle Z_{v'ef}, l^{\pm}\rangle}{\langle Z_{v'ef}, l^{\mp}_{ef}\rangle} - \mathrm{Re}\,\frac{\langle Z_{vef}, l^{\mp}\rangle}{\langle Z_{vef}, l^{\pm}_{ef}\rangle}\right) \pm \mathrm{i}s_f\left(\mathrm{i}\,\mathrm{Im}\,\frac{\langle Z_{v'ef}, l^{\pm}\rangle}{\langle Z_{v'ef}, l^{\mp}_{ef}\rangle} + \mathrm{i}\,\mathrm{Im}\,\frac{\langle Z_{vef}, l^{\mp}\rangle}{\langle Z_{vef}, l^{\pm}_{ef}\rangle}\right) \tag{3.161}$$

$$0 =\frac{n_f}{2}\left(\mathrm{Re}\,\frac{\langle Z_{v'ef}, l^{\pm}\rangle}{\langle Z_{v'ef}, l^{\mp}_{ef}\rangle} + \mathrm{Re}\,\frac{\langle Z_{vef}, l^{\mp}\rangle}{\langle Z_{vef}, l^{\pm}_{ef}\rangle}\right) \pm \mathrm{i}s_f\left(\mathrm{i}\,\mathrm{Im}\,\frac{\langle Z_{v'ef}, l^{\pm}\rangle}{\langle Z_{v'ef}, l^{\mp}_{ef}\rangle} - \mathrm{i}\,\mathrm{Im}\,\frac{\langle Z_{vef}, l^{\mp}\rangle}{\langle Z_{vef}, l^{\pm}_{ef}\rangle}\right) \tag{3.162}$$

The equations give the solution

$$\begin{aligned} \gamma\,\mathrm{Re}(\alpha_{v'ef}) \pm \mathrm{Im}(\alpha_{v'ef}) = 0, \qquad &\text{with} \qquad Z_{v'ef} = \zeta_{v'ef}(l^{\pm}_{ef} + \alpha_{v'ef}l^{\mp}_{ef}) \\ \gamma\,\mathrm{Re}(\alpha_{vef}) \mp \mathrm{Im}(\alpha_{vef}) = 0, \qquad &\text{with} \qquad Z_{vef} = \zeta_{vef}(l^{\mp}_{ef} + \alpha_{vef}l^{\pm}_{ef}) \end{aligned}$$

Here l^{+} and l^{-} completely fix the group element v. α corresponds to the decomposition of Z with these l^{+} and l^{-}. The n_{vef} in this case is simply $n_{vef} = l^{+}_{ef}$.

3.3.2.3. variation respect to $\mathbf{SL}(2,\mathbb{C})$ elements g

With the small perturbation of g which is given by $g' = g\mathrm{e}^L$, the variation of $\mathrm{SL}(2,\mathbb{C})$ group element g is given by

$$\delta g = gL, \quad \delta g^\dagger = -L^\dagger g^\dagger \tag{3.163}$$

where L is a linear combination of $\mathrm{SL}(2,\mathbb{C})$ generators, $L = \epsilon_i F^i + \tilde{\epsilon}_i G^i = (\epsilon_i + \mathrm{i}\tilde{\epsilon_i})F^i$. Here Fs are $\mathrm{SU}(1,1)$ lie algebra generators defined as above, and we use the fact that in spin $1/2$ representation $G = \mathrm{i}F$. 1 Then for arbitrary u, we have

$$\begin{aligned}\delta\langle u, Z\rangle &= \delta\langle u, g^\dagger \bar{z}\rangle = \langle u, L^\dagger g^\dagger \bar{z}\rangle = \langle u, L^\dagger Z\rangle \\ \delta\langle Z, u\rangle &= \delta\langle g^\dagger \bar{z}, u\rangle = (L^\dagger g^\dagger \bar{z})^\dagger \eta u = \langle L^\dagger Z, u\rangle.\end{aligned} \tag{3.164}$$

The variation leads to

$$\begin{aligned}\delta S = \sum_f \epsilon_{ef}(v) \Bigg(-(\frac{n_f}{2} \mp \mathrm{i}s_f) \left(\frac{\langle L^\dagger Z_{vef}, l^\pm_{ef}\rangle}{\langle Z_{vef}, l^\pm_{ef}\rangle} \right) + (\frac{n_f}{2} \pm \mathrm{i}s_f) \left(\frac{\langle l^\pm_{ef}, L^\dagger Z_{vef}\rangle}{\langle l^\pm_{ef}, Z_{vef}\rangle} \right) \\ -\mathrm{i}(\rho_f \pm s_f) \left(\frac{\langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle} \right) \Bigg)\end{aligned} \tag{3.165}$$

where $\epsilon_{ef}(v) = \pm 1$ is determined according to the face orientation is consistent to the edge e or opposite (up to a global sign). We have

$$\epsilon_{ef}(v) = -\epsilon_{e'f}(v), \quad \epsilon_{ef}(v) = -\epsilon_{ef}(v'). \tag{3.166}$$

We write $\epsilon_{ef}(v) = +1$ in the following for simplicity, and recover general ϵ at the end of the derivation.

From the property of $\mathrm{SU}(1,1)$ generator,

$$\eta F \eta = -F^\dagger \tag{3.167}$$

we have

$$\langle F^\dagger Z, u\rangle = -Z^\dagger F \eta u = -Z^\dagger \eta F^\dagger u = -\langle Z, F^\dagger u\rangle \tag{3.168}$$

Then (3.165) can be written as

$$\begin{aligned}&\sum_f (\frac{n_f}{2} \mp \mathrm{i}s_f) \left(\frac{\langle Z_{vef}, F^\dagger l^\pm_{ef}\rangle}{\langle Z_{vef}, l^\pm_{ef}\rangle} \right) \\ &\qquad + (\frac{n_f}{2} \pm \mathrm{i}s_f) \left(\frac{\langle l^\pm_{ef}, F^\dagger Z_{vef}\rangle}{\langle l^\pm_{ef}, Z_{vef}\rangle} \right) = 0\end{aligned} \tag{3.169}$$

and

$$\sum_f -(\frac{n_f}{2} \mp \mathrm{i}s_f)\frac{\langle Z_{vef}, F^\dagger l^\pm_{ef}\rangle}{\langle Z_{vef}, l^\pm_{ef}\rangle} + (\frac{n_f}{2} \pm \mathrm{i}s_f)\frac{\langle l^\pm_{ef}, F^\dagger Z_{vef}\rangle}{\langle l^\pm_{ef}, Z_{vef}\rangle} \\ - 2\mathrm{i}(\rho_f \pm s_f)\frac{\langle Z_{vef}, F^\dagger Z_{vef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle} = 0 \tag{3.170}$$

After inserting the decomposition of Z and solution of simplicity constraint, we have the following equations: For both $S_\pm$, (3.169) becomes

$$\begin{aligned} 0 &= \delta_F S_\pm \\ &= \mp 2\mathrm{i}\sum_f s_f \langle l^\mp_{ef} \mp \mathrm{i}(\gamma Re(\alpha_{vef}) \mp \mathrm{Im}(\alpha_{vef}))l^\pm_{ef}, F^\dagger l^\pm_{ef}\rangle \end{aligned} \tag{3.171}$$

(3.170) will leads to different equations for different actions $S_\pm$ due to the appearance of $\langle Z_{vef}, F^\dagger Z_{vef}\rangle$ term. The variation of S_+ reads

$$\begin{aligned} 0 &= \delta_G S_+ \\ &= -2\gamma\sum_f s_f \langle l^-_{ef} - \mathrm{i}(\frac{1}{\gamma}\mathrm{Re}(\alpha_{vef}) - \mathrm{Im}(\alpha_{vef}))l^+_{ef}, F^\dagger l^+_{ef}\rangle, \end{aligned} \tag{3.172}$$

while the variation of S_- reads

$$\delta_G S_- = 2\mathrm{i}\sum_f s_f \frac{\langle n_{vef}, F^\dagger n_{vef}\rangle}{Re(\alpha_{vef})} + 2\gamma\sum_f s_f \langle n_{vef}, F^\dagger l^-_{ef}\rangle. \tag{3.173}$$

3.3.2.4. summary

As a summary, after we introduce the decomposition of Z as (3.135):

$$Z_{vef} = \zeta_{vef}(\tilde{l}^\mp_{ef} + \alpha_{vef} l^\pm_{ef}) \tag{3.174}$$

and a spinor n as (3.148)

$$n_{vef} := l^+_{ef} + \mathrm{i}\left(\gamma\,\mathrm{Re}(\alpha_{vef}) + \mathrm{Im}(\alpha_{vef})\right) l^-_{ef} \tag{3.175}$$

the equation of motion is given by the following equations

— parallel transport equations

$$S_{vf+}: \frac{g_{ve}\eta l^+_{ef}}{\bar{\zeta}_{vef}} = \frac{g_{ve'}\eta l^+_{e'f}}{\bar{\zeta}_{ve'f}}, \qquad g_{ve}^{-1\dagger}\,(l^-_{ef} + \alpha_{vef} l^+_{ef}) = \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}^{-1\dagger}\,(l^-_{e'f} + \alpha_{ve'f} l^+_{vef}) \tag{3.176}$$

$$S_{vf-}: \frac{g_{ve}\eta n_{vef}}{\mathrm{Re}(\alpha_{vef})\bar{\zeta}_{vef}} = \frac{g_{ve'}\eta n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f})\bar{\zeta}_{ve'f}}, \qquad g_{ve}^{-1\dagger}\,(l^+_{ef} + \alpha_{vef} l^-_{ef}) = \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}^{-1\dagger}\,(l^+_{e'f} + \alpha_{ve'f} l^-_{vef}) \tag{3.177}$$

$$S_{vfx+}: \frac{g_{ve}\eta n_{vef}}{\mathrm{Re}(\alpha_{vef})\bar{\zeta}_{vef}} = -(1+\mathrm{i}\gamma)\frac{g_{ve'}\eta l^+_{e'f}}{\bar{\zeta}_{ve'f}}, \quad g_{ve}^{-1\dagger}\,(l^+_{ef} + \alpha_{vef} l^-_{ef}) = \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}^{-1\dagger}\,(l^-_{e'f} + \alpha_{ve'f} l^+_{vef}) \tag{3.178}$$

$$S_{vfx-}: -(1+\mathrm{i}\gamma)\frac{g_{ve}\eta l^+_{vef}}{\bar{\zeta}_{vef}} = \frac{g_{ve'}\eta n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f})\bar{\zeta}_{ve'f}} \quad g_{ve}^{-1\dagger}\,(l^-_{ef} + \alpha_{vef} l^+_{ef}) = \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}^{-1\dagger}\,(l^+_{e'f} + \alpha_{ve'f} l^-_{vef}) \tag{3.179}$$

Here $S_{vf\pm} = S_{ve'f\pm} - S_{vef\pm}$, $S_{vfx\pm} = S_{ve'f\pm} - S_{vef\mp}$ with $S_{vef\pm}$ is the action given in (3.102), the same for $S_{ef\pm}$ and $S_{efx\pm}$.

— vertcies relations

$$S_{ef\pm}: \qquad \gamma\,\mathrm{Re}(\alpha_{vef}) \mp \mathrm{Im}(\alpha_{vef}) = \gamma\,\mathrm{Re}(\alpha_{v'ef}) \mp \mathrm{Im}(\alpha_{v'ef}) \tag{3.180}$$
$$S_{ef\pm x}: \quad \gamma\,\mathrm{Re}(\alpha_{vef}) \mp \mathrm{Im}(\alpha_{vef}) = \gamma\,\mathrm{Re}(\alpha_{v'ef}) \pm \mathrm{Im}(\alpha_{v'ef}) = 0 \tag{3.181}$$

— closure constraints

$$0 = -2\mathrm{i}\sum_{f\ /\mathrm{w}\ S_{+(x)}} s_f\langle l^-_{ef} - \mathrm{i}(\gamma\,\mathrm{Re}(\alpha_{vef}) - \mathrm{Im}(\alpha_{vef}))l^+_{ef}, F^\dagger l^+_{ef}\rangle + 2\mathrm{i}\sum_{f\ /\mathrm{w}\ S_{-(x)}} s_f\langle n_{ef}, F^\dagger l^-_{ef}\rangle \tag{3.182}$$

$$\begin{aligned}0 = &-2\gamma\sum_{f\ /\mathrm{w}\ S_{+(x)}} s_f\langle l^-_{ef} - \mathrm{i}(\frac{1}{\gamma}\mathrm{Re}(\alpha_{vef}) - \mathrm{Im}(\alpha_{vef}))l^+_{ef}, F^\dagger l^+_{ef}\rangle \\ &+ 2\sum_{f\ /\mathrm{w}\ S_{-(x)}} \mathrm{i}s_f\frac{\langle n_{ef}, F^\dagger n_{ef}\rangle}{\mathrm{Re}(\alpha_{vef})} + \gamma s_f\langle n_{vef}, F^\dagger l^-_{ef}\rangle\end{aligned} \tag{3.183}$$

3.3.3. Analysis of critical equations in bivector representation

3.3.3.1. Bivector representation

For given spinors l^- and l^+, there is a 3-vector v^i associated to them

$$v^i = 2\langle l^+, F^i l^-\rangle \tag{3.184}$$

From which we can define a $\mathrm{SU}(1,1)$ valued bivector in spin-$\frac{1}{2}$ representation

$$V = 2\langle l^+, F^i l^- \rangle F^i = -2(l^+)^\dagger (F^i)^\dagger \eta l^- F^i = -\frac{1}{2}(l^+)^\dagger \sigma_i \eta l^- \sigma_i = -\eta l^- \otimes (l^+)^\dagger + \frac{1}{2}\langle l^+, l^- \rangle I_2 \tag{3.185}$$

where we use the fact $\eta F \eta = -F^\dagger$ and the completeness of pauli matrix. Since $\langle l^-, F l^+ \rangle = -\langle l^+, F l^- \rangle$,

$$V = -2\langle l^-, F^i l^+ \rangle F_i = \eta l^+ \otimes (l^-)^\dagger - \frac{1}{2}\langle l^+, l^- \rangle I_2 \tag{3.186}$$

From the fact

$$K^i = -K_i = J^{0i}, \qquad J^i = J_i = \frac{1}{2}\epsilon^{0i}{}_{jk} J^{jk} \tag{3.187}$$

where $J^i = *K^i$. We have in spin $1/2$ representation $* \to \mathrm{i}$ and $J^i = \mathrm{i}K^i$. The bivector can be encoded into SL$(2,\mathbb{C})$ bivector that in spin-1 representation reads

$$V^{IJ} = \begin{pmatrix} 0 & -v^1 & -v^2 & 0 \\ v^1 & 0 & v^0 & 0 \\ v^2 & -v^0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.188}$$

Then $(*V)^{IJ}$ reads

$$(*V)^{IJ} = \begin{pmatrix} 0 & 0 & 0 & v^0 \\ 0 & 0 & 0 & -v^2 \\ 0 & 0 & 0 & v^1 \\ -v^0 & v^2 & -v^1 & 0 \end{pmatrix} = (v^I_{ef} \wedge u^J) \tag{3.189}$$

where the encoded 4-vector $v^I_{ef} := (v^0, -v^2, v^1, 0)$, $u^I = (0,0,0,1)$. Clearly one can see that

$$v^I = \mathrm{i}(\left\langle l^- \right| \hat\sigma^I \left| l^+ \right\rangle + u^I) \tag{3.190}$$

where $\hat\sigma = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)$.

Since $\langle l^+, F^i l^- \rangle = \langle l_0^+, v^\dagger F^i v^{-1\dagger} l_0^- \rangle$, in this sense, v_i is nothing else but the SO$(1,2)$ rotation of 3 vector $v_0 = (0,0,1)$ with group element $v^{-1\dagger}$.

Similarly, we can define

$$W^\pm = 2\mathrm{i}\langle l^\pm, F^i l^\pm \rangle F^i = -\mathrm{i}\eta l^\pm \otimes (l^\pm)^\dagger \tag{3.191}$$

with

$$W^{\pm IJ} = w^{\pm I}_{ef} \wedge u^J, \qquad w^{\pm I} := \left\langle l^\pm \right| \hat\sigma^I \left| l^\pm \right\rangle \tag{3.192}$$

Here $w^{\pm I}_{ef}$ is a null vector $w^{\pm I}_{ef} w^\pm_{ef\,I} = 0$.

We introduce $\mathrm{SO}(1,3)$ group elements G given by

$$G_{ve} = \pi(g_{ve}) \tag{3.193}$$

where $\pi : \mathrm{SL}(2,\mathbb{C}) \rightarrow \mathrm{SO}(1,3)$. Since the action (3.101) is invariant under the transformation $g_{ve} \rightarrow \pm g_{ve}$, two group elements related to g_{ve} are gauge equivalent if they satisfy

$$\tilde{G}_{ve} = G_{ve} I^{s_{ve}}, \qquad s_{ve} = \{0,1\} \tag{3.194}$$

where I is the inversion operator. With this gauge transformation, we can always assume $G_{ve} \in \mathrm{SO}_+(1,3)$.

Now we will analysis and reformulate the critical point equations we get in Sec. 3.3 in bivector representation. The analysis is done for all possible actions appearing in the amplitude (3.100).

3.3.3.2. S_{vf+} case

From (3.139) and (3.144) in S_{vf+} case,

$$g_{ve}\eta l^+_{ef} = \frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}} g_{ve'}\eta l^+_{e'f} \qquad\qquad g_{ve}\ J\ \tilde{Z}_{vef} = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'}\ J\ \tilde{Z}_{vef} \tag{3.195}$$

we have

$$g_{ve}\eta l^+_{ef} \otimes (l^-_{ef} + \alpha_{vef} l^+_{ef})^\dagger g_{ev} = g_{ve'}\eta l^+_{ef} \otimes (l^-_{e'f} + \alpha_{ve'f} l^+_{e'f})^\dagger g_{e'v} \tag{3.196}$$

with the fact that $\langle l^+, l^+\rangle = 0$ and $\langle l^-, l^+\rangle = 1$. With (3.185), the above equation can be written as

$$g_{ve}(V_{ef} + \mathrm{i}\bar{\alpha}_{vef} W^+_{ef}) g_{ev} = g_{ve'}(V_{e'f} + \mathrm{i}\bar{\alpha}_{ve'f} W^+_{e'f}) g_{e'v} \tag{3.197}$$

In spin-1 representation, this equation reads

$$g_{ve}(V_{ef} + (\mathrm{Im}(\alpha_{vef}) + \mathrm{Re}(\alpha_{vef}) *) W^+_{ef}) g_{ev} = g_{ve'}(V_{e'f} + (\mathrm{Im}(\alpha_{ve'f}) + \mathrm{Re}(\alpha_{ve'f}) *) W^+_{e'f}) g_{e'v} \tag{3.198}$$

We can define a bivector X_{vef}

$$X_{vef} = V_{ef} + (\mathrm{Im}(\alpha_{vef}) + \mathrm{Re}(\alpha_{vef}) *) W^+_{ef} \tag{3.199}$$

Easy to check X is a simple bivector which can be expressed as

$$X = *(v + \mathrm{Im}(\alpha) w^+) \wedge (u - \mathrm{Re}(\alpha) w^+) = *(\tilde{v} \wedge \tilde{u}) \tag{3.200}$$

Here by the definition of v and w, we have

$$\tilde{v}^I = (\tilde{v}^0, -\tilde{v}^2, \tilde{v}^1, 0) \ , \qquad \tilde{w}^I = (w^{+0}, -w^{+2}, w^{+1}, 0) \ , \tag{3.201}$$

where

$$\tilde{v}^i = -2\langle l^- + \mathrm{i}\,\mathrm{Im}(\alpha) l^+, F^i l^+\rangle, w^{+i} = 2\mathrm{i}\langle l^+, F^i l^+\rangle \tag{3.202}$$

One can check $\tilde{v}^I \tilde{v}_I = \tilde{u}^I \tilde{u}_I = 1$, thus X is timelike. (3.198) implies

$$(G_{ve}\tilde{v}_{vef}) \wedge (G_{ve}\tilde{u}_{vef}) = (G_{ve'}\tilde{v}_{ve'f}) \wedge (G_{ve'}\tilde{u}_{ve'f}). \tag{3.203}$$

which reminds us define

$$X_f(v) := G_{ve} X_{vef} G_{ev} = G_{ve'} X_{ve'f} G_{e'v} \tag{3.204}$$

Noted that, from this equation, we have

$$(G_{ve}u)_I X_f^{IJ}(v) = -\,\mathrm{Re}(\alpha_{vef})(G_{ve}w^+_{ef})^J \tag{3.205}$$

which is 0 only when $\mathrm{Re}(\alpha_{vef}) = 0$.

Go back to equations we get from the variation respecting to g, clearly (3.171) and (3.172) can be written as

$$\sum_f \epsilon_{ef}(v)\langle l^- + \mathrm{i}\,\mathrm{Im}(\alpha) l^+, F^i l^+\rangle = 0 \tag{3.206}$$

$$\sum_f \epsilon_{ef}(v)\,\mathrm{Re}(\alpha)\langle l^+, F^i l^+\rangle = 0 \tag{3.207}$$

In terms of 4 vectors $\tilde{v}$ and w, these equation reads

$$\sum_f \epsilon_{ef}(v) G_{ve}\tilde{v}_{vef} = 0 \qquad \sum_f \epsilon_{ef}(v)\,\mathrm{Re}(\alpha_{vef}) G_{ve} w^+_{ef} = 0 \tag{3.208}$$

where $\tilde{v}$ is defined by (3.201). Then we can write (3.208) as

$$\sum_f \epsilon_{ef}(v) X_f(v) = 0 \tag{3.209}$$

which is a closure condition to bivectors.

3.3.3.3. S_{vf-} case

In this case, from (3.139) and (3.145) we have

$$g_{ve}\ \eta n_{vef} = \frac{\bar{\zeta}_{vef}\,\mathrm{Re}(\alpha_{vef})}{\bar{\zeta}_{ve'f}\,\mathrm{Re}(\alpha_{ve'f})} g_{ve'}\ \eta n_{ve'f} \tag{3.210}$$

$$g_{ve}\ J\ \tilde{Z}_{vef} = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'}\ J\ \tilde{Z}_{vef} \tag{3.211}$$

where $n_{ef} := l^+_{ef} + \mathrm{i}(\gamma\,\mathrm{Re}(\alpha_{vef}) + \mathrm{Im}(\alpha_{vef}))l^-_{ef}$. Note with equation (3.163), we see n does not change for different vertex v: $n_{ef}(v) = n_{ef}(v')$. n defined here satisfies the relation in Lemma 3.3.2, thus according to Lemma 3.3.3, $\{n, l^-\}$ forms a null basis. With n and l^-, $\tilde{Z}$ can be rewritten as

$$Z = l^+ + \alpha l^- = n + (1 - \mathrm{i}\gamma)\,\mathrm{Re}(\alpha) l^- \tag{3.212}$$

This leads to the tensor product equation

$$g_{ve}\frac{\eta n_{ef}}{\mathrm{Re}(\alpha_{ef})} \otimes (n_{ef} + (1 - \mathrm{i}\gamma)\,\mathrm{Re}(\alpha_{vef}) l^-_{ef})^\dagger g_{ev} = (e \to e') \tag{3.213}$$

The right part of above equation means exchange all the e in left part to e'.

In terms of bivector variables, according to (3.185), we have

$$g_{ve}(V_{ef} + \frac{(\mathrm{i} - \gamma)W^+_{ef}}{(1+\gamma^2)\,\mathrm{Re}(\alpha_{vef})}) g_{ev} = (e \to e') \tag{3.214}$$

Noted now V is the space-like bivector generated by n with l^- and W^+ is null bivector generated by n with itself. Again bivector $X_{vef} := V_{ef} - (\gamma - *)W_{ef}/((1+\gamma^2)\,\mathrm{Re}(\alpha))$ is a simple bivector. X_{vef} can be written as

$$X_{vef} = *((v_{ef} - \frac{\gamma}{(1+\gamma^2)\,\mathrm{Re}(\alpha_{vef})} w^+_{ef}) \wedge (u - \frac{\gamma}{(1+\gamma^2)\,\mathrm{Re}(\alpha_{vef})} w^+_{ef})) = *(\tilde{v}_{vef} \wedge \tilde{u}_{vef}) \tag{3.215}$$

where

$$\tilde{v}^I = (\tilde{v}^0, -\tilde{v}^2, \tilde{v}^1, 0), \qquad w^{+i} = 2\mathrm{i}\langle n, F^i n\rangle \tag{3.216}$$

Here

$$\tilde{v}^i = 2\langle n, F^i(l^- - \frac{\mathrm{i}\gamma n}{(1+\gamma^2)\,\mathrm{Re}(\alpha)})\rangle, w^{+i} = 2\mathrm{i}\langle n, F^i n\rangle \tag{3.217}$$

$\tilde{v}^I \tilde{v}_I = \tilde{u}^I \tilde{u}_I = 1$ implies X is timelike.

Then (3.213) leads to

$$X_f(v) := G_{ve} X_{vef} G_{ev} = G_{ve'} X_{ve'f} G_{e'v} \tag{3.218}$$

which is the parallel transport of X between edge e and e'. With (3.215), we can write $X_f(v)$ as

$$X_f(v) = G_{ve}\tilde{v}_{vef}) \wedge (G_{ve}\tilde{u}_{vef} \tag{3.219}$$

Note here again we have

$$(G_{ve}u)_I X^{IJ}_{vf} = -\frac{1}{(1+\gamma^2)\,\mathrm{Re}(\alpha)}(G_{ve}w^+_{ef})^J \tag{3.220}$$

which is some null vector and can not be 0.

Form (3.171) and (3.173), we have the following equations of motion from variation respecting to g

$$\sum_f \epsilon_{ef}(v)\langle n, F^\dagger(l^- - \frac{\mathrm{i}\gamma n}{(1+\gamma^2)\,\mathrm{Re}(\alpha)})\rangle = 0 \qquad \sum_f \epsilon_{ef}(v)\frac{\langle n, F^\dagger n\rangle}{\mathrm{Re}(\alpha)} = 0 \tag{3.221}$$

In terms of 4-vectors,

$$\sum_f \epsilon_{ef}(v) G_{ve} v_{ef} = 0 \qquad \sum_f \epsilon_{ef}(v)\frac{G_{ve}w^+_{ef}}{\mathrm{Re}(\alpha)} = 0 \tag{3.222}$$

which leads to

$$\sum_f \epsilon_{ef}(v) X_f(v) = 0 \tag{3.223}$$

3.3.3.4. S_{vfx} case

We will use S_{vfx-} as an example, the S_{vfx+} will be exactly the same but switch e and e' here. From the critical point equations (3.139) and (3.146),

$$\begin{gathered}(\gamma - \mathrm{i})s_f\frac{g_{ve}\eta l^+_{vef}}{\bar{\zeta}_{vef}} = -\mathrm{i}s_f\frac{g_{ve'}\eta n_{ve'f}}{\bar{\zeta}_{ve'f}\,\mathrm{Re}(\alpha_{ve'f})}\,, \\ g_{ve}\bar{\zeta}_{vef}J\;(l^-_{ef} + \alpha_{vef}l^+_{ef}) = g_{ve'}\bar{\zeta}_{ve'f}J\;(l^+_{e'f} + \alpha_{ve'f}l^-_{e'f})\end{gathered} \tag{3.224}$$

With the equation (3.163) from the variation respecting to $\mathrm{SU}(1,1)$ group elements v_{ef}, in this case $n = l^+$, and $\tilde{Z}_{ve'f}$ can be written as $\tilde{Z}_{ve'f} = l^+_{e'f} + (1 - \mathrm{i}\gamma)\,\mathrm{Re}(\alpha_{ve'f})l^-_{e'f}$.

The tensor product between the two equations leads to

$$\begin{aligned}&(\mathrm{i}\gamma+1)g_{ve}(\eta l^+_{ef}\otimes(l^-_{ef})^\dagger + \bar{\alpha}_{vef}\eta l^+_{ef}\otimes(l^+_{ef})^\dagger)g_{ev} = g_{ve'}\;\eta n_{ve'f}\otimes(\frac{n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f})} + (1-\mathrm{i}\gamma)l^-_{e'f})^\dagger\;g_{e'v} \\ &= g_{ve'}\;(\frac{\eta n_{ve'f}\otimes n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f})} + (1+\mathrm{i}\gamma)\eta n_{ve'f}\otimes(l^-_{e'f})^\dagger)\;g_{e'v}\end{aligned} \tag{3.225}$$

In bivector representation

$$g_{ve}(V_{ef}+\mathrm{i}\bar{\alpha}_{vef}W^{+}_{ef})g_{ev}=g_{ve'}\;(V_{e'f}+\frac{(\mathrm{i}-\gamma)W^{+}_{ve'f}}{\mathrm{Re}(\alpha_{e'f})(1+\gamma^{2})})\;g_{e'v} \tag{3.226}$$

Easily to see one recovers the corresponding bivectors in $S_{vf\pm}$ case respectively. Thus the equation implies

$$X_{f}(v):=g_{ve}X_{vef}g_{ev}=g_{ve'}X_{ve'f}g_{e'v} \tag{3.227}$$

with X_{vef} defined by (3.200) and $X_{ve'f}$ defined by (3.215). The closure constraint, in these case, are the combination of corresponding equation in (3.208) or (3.222) according to their representations in S_{+} or S_{-}. Then we still have

$$\sum_{f}\epsilon_{ef}(v)X_{f}(v)=0 \tag{3.228}$$

3.3.3.5. Summary

As a summary, given any solution to the critical equations, we can define a bivector

$$\begin{aligned}X_{vef}&=-2\mathrm{i}\langle l^{-},F^{i}l^{+}\rangle F_{i}-\mathrm{i}\bar{\alpha}_{vef}\langle l^{+},F^{i}l^{+}\rangle F_{i}\\&=V_{ef}-(\mathrm{Im}(\alpha_{vef})+\mathrm{Re}(\alpha_{vef})*)W^{+}_{ef}\end{aligned} \tag{3.229}$$

or

$$\begin{aligned}X_{vef}&=-2\mathrm{i}\langle n,F^{\dagger i}l^{-}\rangle F_{i}-\frac{\mathrm{i}+\gamma}{(1+\gamma^{2})\,\mathrm{Re}(\alpha_{vef})}\langle n,F^{\dagger i}n\rangle F_{i}\\&=-V_{ef}-\frac{1-\gamma*}{(1+\gamma^{2})\,\mathrm{Re}(\alpha_{vef})}W^{+}_{ef}\end{aligned} \tag{3.230}$$

corresponding to their action is composited by S_{vef+} or S_{vef-}. Here V_{ef} is a spacelike bivector and W_{ef} is a null bivector. In spin-1 representation, we can express the above bivector as

$$X^{IJ}_{ef}=(*)(\tilde{v}^{I}_{vef}\wedge\tilde{u}^{J}_{vef}) \tag{3.231}$$

where

$$\tilde{v}_{vef}=\begin{cases}v_{ef}-\mathrm{Im}(\alpha_{vef})w^{+}_{ef}, & S_{vef+}\\ v_{ef}-\frac{\gamma}{(1+\gamma^{2})\,\mathrm{Re}(\alpha_{vef})}w^{+}_{ef}, & S_{vef-}\end{cases} \tag{3.232}$$

$$\tilde{u}_{vef}=\begin{cases}u+\mathrm{Re}(\alpha_{vef})w^{+}_{ef}, & S_{vef+}\\ u+\frac{1}{(1+\gamma^{2})\,\mathrm{Re}(\alpha_{vef})}w^{+}_{ef}, & S_{vef-}\end{cases} \tag{3.233}$$

with

$$v_{ef} = \begin{cases} -2\mathrm{i}\langle l^-_{ef}, F^i l^+_{ef}\rangle, & S_{vef+} \\ -2\mathrm{i}\langle n_{ef}, F^i l^-_{ef}\rangle & S_{vef-} \end{cases}, \tag{3.234}$$

$$w^+_{ef} = \begin{cases} 2\langle l^+_{ef}, F^i l^+_{ef}\rangle, & S_{vef+} \\ 2\langle n_{ef}, F^i n_{ef}\rangle & S_{vef-} \end{cases} \tag{3.235}$$

The bivector X_{vef} satisfies the parallel transport equation:

$$g_{ve}X_{vef}g_{ve}^{-1} = g_{ve'}X_{ve'f}g_{ve'}^{-1} \tag{3.236}$$

This corresponds to

$$X_f(v) := g_{ve}X_{vef}g_{ev} = v^I_{ef}(v) \wedge N^I_e(v) \tag{3.237}$$

where

$$v^I_{ef}(v) := G_{ve}\tilde{v}_{vef}, \qquad N^I_e(v) = G_{ve}\tilde{u}_{vef} \tag{3.238}$$

The closure constraint in terms of the bivector variable then reads

$$2\sum_f \gamma\epsilon_{ef}(v)s_f X_f(v) = \sum_f \epsilon_{ef}(v)B_f(v) = 0 \tag{3.239}$$

where $B_f = 2\gamma s_f X_f = n_f X_f$ with $B_f^2 = -n_f^2$. Note that the closure constraint is composed by two independent equations enrolling $\tilde{v}$ and w^+

$$\begin{aligned} &\sum_f \epsilon_{ef}(v)\tilde{v}_{vef} = 0, \\ &\begin{cases} \sum_f \epsilon_{ef}(v)\,\mathrm{Re}(\alpha_{vef})w^+_{ef} = 0, & S_{vef+} \\ \sum_f \epsilon_{ef}(v)(\mathrm{Re}(\alpha_{vef})^{-1}w^+_{ef} = 0, & S_{vef-} \end{cases} \end{aligned} \tag{3.240}$$

3.3.4. Timelike tetrahedron containing both spacelike and timelike triangles

The timelike tetrahedron in a generic simplicial geometry contains both spacelike and timelike triangles. For spacelike triangles, the irreps of $\mathrm{SU}(1,1)$ are in the discrete series, in contrast to the continuous series used in timelike triangles. The simplicity constraint is also different from (3.15). This leads to different face actions on triangles with different signature, and the total action is expressed by the sum of these actions. The action on spacelike triangle and corresponding critical point equations have already been derived in [85]. The results are reviewed in Appendix ??.

The variations with respect to z_{vf} and v_{ef} give equations of motions (3.236) for timelike triangles and (??) for spacelike triangles respectively. In addition, for timelike triangles, solutions should satisfy (3.158), (3.163) or (3.163).

The variation respect to SL$(2,\mathbb{C})$ group element g_{ve} involves all faces connected to e, which may include both spacelike and timelike triangles. In general, from (3.171 - 3.173) and (??-??), the action including different types of triangles gives

$$
\begin{aligned}
&-2\mathrm{i}\sum_{f\text{ /w }S_{+(x)}} s_f\langle l^-_{ef}-\mathrm{i}(\gamma\,\mathrm{Re}(\alpha_{vef})-\mathrm{Im}(\alpha_{vef}))l^+_{ef},F^\dagger l^+_{ef}\rangle\\
&+2\mathrm{i}\sum_{f\text{ /w }S_{-(x)}} s_f\langle n_{ef},F^\dagger l^-_{ef}\rangle-2\sum_{f\text{ /w }S_{\mathrm{sp}}} j_f\langle\xi^\pm_{ef},F^\dagger\xi^\pm_{ef}\rangle=0
\end{aligned}
\tag{3.241}
$$

$$
\begin{aligned}
&-2\gamma\sum_{f\text{ /w }S_{+(x)}} s_f\langle l^-_{ef}-\mathrm{i}(\frac{1}{\gamma}\,\mathrm{Re}(\alpha_{vef})-\mathrm{Im}(\alpha_{vef}))l^+_{ef},F^\dagger l^+_{ef}\rangle\\
&+2\sum_{f\text{ /w }S_{-(x)}} \mathrm{i}s_f\frac{\langle n_{ef},F^\dagger n_{ef}\rangle}{\mathrm{Re}(\alpha_{vef})}+\gamma s_f\langle n_{vef},F^\dagger l^-_{ef}\rangle+2\mathrm{i}\gamma\sum_{f\text{ /w }S_{\mathrm{sp}}} j_f\langle\xi^\pm_{ef},F^\dagger\xi^\pm_{ef}\rangle=0
\end{aligned}
\tag{3.242}
$$

Summation of the two equations leads to

$$
(1+\gamma^2)\sum_{f\text{ /w }S_{+(x)}} s_f\,\mathrm{Re}(\alpha_{vef})\langle l^+_{ef},F^i l^+_{ef}\rangle+\sum_{f\text{ /w }S_{-(x)}} s_f\frac{\langle n_{ef},F^i n_{ef}\rangle}{\mathrm{Re}(\alpha_{vef})}=0 \tag{3.243}
$$

This equation only involves timelike triangles. Since ${w^{+i}}_{ef}=\langle l^+_{ef},F^i l^+_{ef}\rangle$ (or $w^{+i}_{ef}=\langle n_{ef},F^i n_{ef}\rangle$ in $S_{-(x)}$ case) are null vectors, the above equation implies summing over null vectors equal to 0. In a tetrahedron contains both timelike and spacelike triangles, the number of timelike triangles, which is also the number of null vectors here, is less than 4. If one has less than 4 null vectors sum to 0 in 4-dimensional Minkowski space, then they are either trivial or colinear. The only possibility to have a nondegenerate tetrahdron from (3.243) is that all the timelike faces are in the action S_+ and set $\mathrm{Re}(\alpha)=0$. The solution reads

$$
\mathrm{Re}(\alpha_{vef})=0\qquad\&\qquad\forall_{f\in t_e},\ S_f=S_{+(x)} \tag{3.244}
$$

which means in order to have critical point, the action associated to each triangle f of the tetrahedron t_e must be S_+ or S_{+x}, the other actions do not have stationary point. The closure constraint is now given by (3.241) minus (3.242)

$$
\begin{aligned}
&-2\mathrm{i}\sum_{f\text{ /w }S_{+(x)}} s_f\langle l^-_{ef}+\mathrm{i}\,\mathrm{Im}(\alpha_{vef})l^+_{ef},F^i l^+_{ef}\rangle=0\\
&-2\sum_{f\text{ /w }S_{sp}} j_f\langle\xi^\pm_{ef},F^i\xi^\pm_{ef}\rangle=0
\end{aligned}
\tag{3.245}
$$

1 The parallel transport equations for timelike triangles still keep the same form as (3.144-3.146). After we impose condition (3.244), the parallel transport equa-

tion becomes

$$
\begin{aligned}
& g_{ve} l_{ef}^{+} \otimes (l_{ef}^{-} + \mathrm{i}\,\mathrm{Im}(\alpha_{vef}) l_{ef}^{+})^{\dagger} g_{ev} \\
& \qquad = g_{ve'} l_{e'f}^{+} \otimes (l_{e'f}^{-} + \mathrm{i}\,\mathrm{Im}(\alpha_{ve'f}) l_{e'f}^{+})^{\dagger} g_{e'v}
\end{aligned}
\tag{3.246}
$$

One recognize the same composition of spinors $l_{ef}^{-} + \mathrm{i}\,\mathrm{Im}(\alpha_{vef}) l_{ef}^{+}$ in (3.245) and (3.246). This is exactly the spinor satisfying Lemma (3.3.2). Recall (3.158), coming from the variation respect to $SU(1,1)$ group elements v_{ef}, we have

$$
\mathrm{Im}(\alpha_{vef}) = \mathrm{Im}(\alpha_{v'ef}) \tag{3.247}
$$

in S_{+} case or $\mathrm{Im}(\alpha_{vef}) = 0$ in S_{x+} case respectively. However, recall for S_{+} case, there is an ambiguity in defining $\tilde{l}^{-}$ and $\mathrm{Im}(\alpha)$ from lemma 3.3.2. This ambiguity does not change the action, and gives the same vector $v^{i} = \langle \tilde{l}_{ef}^{-}, F^{i} l_{ef}^{+} \rangle$. Thus we can always remove the $\mathrm{Im}(\alpha_{vef})$ by a redefinition of l_{ef}^{-}, which does not change the geometric form of the critical equations. With (3.247), this redefinition will extended to both end points of the edge e. Thus we always make the choice that $\mathrm{Im}(\alpha_{vef}) = 0$ and drop all $\mathrm{Im}(\alpha_{vef})$ terms in (3.245) and (3.246)

In bivector representation, we can build bivectors for timelike triangles,

$$
X_{ef} = *(v_{ef} \wedge u), \tag{3.248}
$$

with v_{ef} a normalized vector defined by $v_{ef}^{I} = \mathrm{i}(\left\langle l_{ef}^{+} \middle| \hat{\sigma}^{I} \middle| l_{ef}^{-} \right\rangle - u^{I})$. The parallel transportation equation implies we can define a bivector $X_f(v)$ independent of e

$$
X_f(v) = G_{ve} X_{ef} G_{ev} \tag{3.249}
$$

Clearly in this case we have

$$
N_e \cdot X_f(v) = 0, \qquad \text{with} \;\; N_e = G_{ve} u \tag{3.250}
$$

For spacelike triangles, the bivector is defined in (??). One see they have exactly the same form as in the timelike case and follow the same condition, except now $v_{ef}^{I} = \left\langle \xi_{ef}^{\pm} \middle| \hat{\sigma}^{I} \middle| \xi_{ef}^{\pm} \right\rangle - \left\langle \xi_{ef}^{\pm} \middle| \xi_{ef}^{\pm} \right\rangle u^{I}$ instead. With bivectors X_{ef} and X_f, (3.245) becomes (after recover the sign factor $\epsilon_{ef}(v)$)

$$
\sum_{f \text{ /w } S_{+(x)}} \epsilon_{ef}(v) s_f X_f(v) - \sum_{f \text{ /w } S_{sp}} \epsilon_{ef}(v) j_f X_f(v) = 0 \tag{3.251}
$$

In summary, the critical equations for a timelike tetrahedron with both timelike and spacelike triangles imply a nondegenerate tetrahedron geometry only when timelike triangles have action $S_{+(x)}$. Suppose we have a solution (j_f, g_{ve}, z_{vf}), one can define bivectors

$$
B_{ef} = 2A_f X_{ef} = 2A_f * (v_{ef} \wedge u) \tag{3.252}
$$

where

$$v_{ef}^I = \left\{ \begin{array}{ll} -\mathrm{i}(\left\langle l_{ef}^+ \right| \sigma^I \left| l_{ef}^- \right\rangle - u^I) & \text{for timelike triangle} \\ \left\langle \xi_{ef}^\pm \right| \sigma^I \left| \xi_{ef}^\pm \right\rangle - \langle \xi^\pm, \xi^\pm \rangle u^I & \text{for spacelike case} \end{array} \right. , \tag{3.253}$$

and

$$A_f = \left\{ \begin{array}{ll} \gamma s_f = n_f/2 & \text{for timelike triangle} \\ \gamma j_f = \gamma n_f/2 & \text{for spacelike triangle} \end{array} \right. \tag{3.254}$$

We define $B_{ef}(v)$ as

$$B_f(v) := G_{ve} B_{ef} G_{ev} \tag{3.255}$$

The critical point equations imply

$$B_{ef}(v) = B_{e'f}(v) = B_f(v) \tag{3.256}$$

$$N_e \cdot B_f(v) = 0 \tag{3.257}$$

$$\sum_{f \in t_e} \epsilon_{ef}(v) B_f(v) = 0 \tag{3.258}$$

where $N_e^I = G_{ve} u^I$, $\epsilon_{ef}(v) = \pm 1$ and changes it's sign when exchanging vertex and edge variables.

3.3.5. Tetrahedron containing only timelike triangles

Starting from the critical equations derived above, we can see what happens when all faces appear inside the closure constrain is timelike. For simplicity, we will use S_+ action as an example, the other cases will follow similar properties as they can be written in similar forms as S_+.

Suppose we have a solution to critical equations with all the face actions being S_+. As we have shown above, the solution satisfies two closure constraints,

$$\sum_f s_f (v_{ef} + \mathrm{Im}(\alpha_{vef}) w_{ef}^+) = 0, \tag{3.259}$$

$$\sum_f s_f \,\mathrm{Re}(\alpha_{vef}) w_{ef}^+ = 0 \tag{3.260}$$

Clearly here we have family of solutions generated by the continuous transformations

$$\begin{array}{l} \mathrm{Re}(\alpha_{vef}) \to \tilde{C}_{ve} \,\mathrm{Re}(\alpha_{vef}), \\ \mathrm{Im}(\alpha_{vef}) \to \mathrm{Im}(\alpha_{vef}) + C_{ve} \,\mathrm{Re}(\alpha_{vef}) \end{array} \tag{3.261}$$

In other words, the closure constraint only fixes α up to C_{ve} and $\tilde{C}_{ve}$.

Back to the bivectors inside the parallel transportation equation, it is easy to see, the bivector can be rewritten as

$$X = V + (\mathrm{Im}(\alpha) + \mathrm{Re}(\alpha)*)W^+ = X^0 + \mathrm{Re}(\alpha)(C + \tilde{C}*)W^+ \tag{3.262}$$

where $X_0 = V + \mathrm{Im}(\alpha^0_{vef})$ for some given $\mathrm{Im}(\alpha^0_{vef})$. Suppose we have a solution to some fixed C and $\tilde{C}$, the parallel transported bivector then reads

$$G_{ve}X_{ef}G_{ev} = G_{ve}X^0_{ef}G_{ev} + \mathrm{Re}(\alpha)(C + \tilde{C}*)G_{ve}W^+_{ef}G_{ev} = *((G_{ve}\tilde{v}_{vef}) \wedge (G_{ve}\tilde{u}_{vef})) \tag{3.263}$$

From the fact that in spin-$1/2$ representation $* \to \mathrm{i}$, we define $c := C + \mathrm{i}\tilde{C}$.

From the parallel transported vector $\tilde{v}_f := G_{ve}\tilde{v}_{vef}$ and $\tilde{u}_f := G_{ve}\tilde{u}_{vef}$, one can determine a null vector $\tilde{w}_f$ related to face $f = (e, e')$ uniquely up to a scale by

$$\tilde{w}_f.\tilde{v}_f = \tilde{w}_f.\tilde{u}_f = 0 \tag{3.264}$$

From the definition of $\tilde{v}$ and $\tilde{u}$, we see that $w_{ef}.\tilde{u}_{vef} = w_{ef}.\tilde{v}_{vef} = 0$ and the same relation for e'. Since $G \in \mathrm{SO}_+(1,3)$ which preserves the inner product, we then have

$$\tilde{w}_f \propto G_{ve}w_{ef} \propto G_{ve'}w_{e'f} \tag{3.265}$$

Suppose a solution to critical equations determines a geometrical 4-simplex up to scaling and reflection with normals $N_e(v) = G_{ve}u$ (Appendix 3.A for the geometrical interpretation of the critical solution. We suppose the solution is non-degenerate here. The degenerate case will be discussed in Sec. 3.5). From this 4-simplex, we can get its boundary tetrahedron with faces normals $v^g_{ef}(v) = G_{ve}v^s_{ef}$. For two edges e and e' belong to the same face f, N_e and $N_{e'}$ determine uniquely a null vector (up to scaling), which is perpendicular to N_e and $N_{e'}$. Then from (3.264) and (3.265), the vector is proportional to $\tilde{w}_f$. Then it implies that,

$$v^s_{ef} = \tilde{v}_{ef} + d_{ef}w_{ef} \tag{3.266}$$

The tetrahedra determined by v^s_{ef} (by Minkowski Theorem) satisfy the length matching condition, which further constrain d_{ef}. 10 d_{ef}'s are over-constrained by 20 length matching conditions. $d_{ef} = 0$ corresponds to a solution if the boundary data (relating to $\tilde{v}_{ef}$) also satisfy the length matching condition. We have the parallel transportation equation:

$$g_{ve}X^0_{ef}g_{ev} + d_{ef}g_{ve}W^+_{ef}g_{ev} = g_{ve'}X^0_{e'f}g_{e'v} + d_{e'f}g_{ve'}W^+_{e'f}g_{e'v} \tag{3.267}$$

However, from (3.263) we know that

$$g_{ve}X^0_{ef}g_{ev} + \mathrm{Re}(\alpha_{ef})c_{ve}g_{ve}W^+_{ef}g_{ev} = g_{ve'}X^0_{e'f}g_{e'v} + \mathrm{Re}(\alpha_{e'f})c_{ve'}g_{ve'}W^+_{e'f}g_{e'v} \tag{3.268}$$

which means

$$(\mathrm{Re}(\alpha_{vef})c_{ve} - d_{ef})g_{ve}W^+_{ef}g_{ev} = (\mathrm{Re}(\alpha_{ve'f})c_{ve'} - d_{e'f})g_{ve'}W^+_{e'f}g_{e'v} \tag{3.269}$$

They are 10 complex equations, with 5 complex c_{ve}, thus again give an over-constrained system.

A special case is that the boundary data itself satisfy the length matching condition. In this case, $d_{ef} = 0$ correspond to a critical solution. It can be further proved that (3.269) with $d_{ef} = 0$ implies

$$\forall_e \, c_{ve} = 0 \tag{3.270}$$

The condition is nothing else but (3.244), and it is easy to see that in this case the critical equations reduce to (3.252 - 3.256).

3.4. Geometric interpetation and reconstruction

The critical solutions of spinfoam action are shown to satisfy certain geometrical bivector equations, we would like to compare them with a discrete Lorentzian geometry. The general construction of a discrete Lorentzian geometry and the relation with critical solutions for spacelike triangles were discussed in detail in [79] and [85]. We will see that our solutions, which include timelike triangles, can be applied to a similar reconstruction procedure. We demonstrate the detailed analysis in Appendix 3.A. The main result is summarized here. The result is valid when every timelike tetrahedron contains both spacelike and timelike triangles. It is also valid for tetrahedra containing only timelike triangles in the special case with Eq.(3.270).

The following condition at a vertex v implies the nondegenerate 4-simplex geometry:

$$\prod_{e1,e2,e3,e4=1}^{5} \det(N_{e1}, N_{e2}, N_{e3}, N_{e4}) \neq 0 \tag{3.271}$$

which means any 4 out of 5 normals are linearly independent. Since $N_e = G_{ve}u$, the above non-degeneracy condition is a constraint on G_{ve}. Here $u = (0, 0, 0, 1)$ or $u = (1, 0, 0, 0)$ for a timelike or spacelike tetrahedron.

Then we can prove that satisfying the nondegeneracy condition, each solution $B_{ef}(v)$ at a vertex v determines a geometrical 4-simplex uniquely up to shift and inversion. The bivectors $B^{\Delta}_{ef}(v)$ of the reconstructed 4-simplex satisfy

$$B^{\Delta}_{ef}(v) = r(v) B_{ef}(v) \tag{3.272}$$

where $r(v) = \pm 1$ relates to the 4-simplex (topological) orientation defined by an ordering of tetrahedra. The reconstructed normals are determined up to a sign

$$N^{\Delta}_{ve} = (-1)^{s_{ve}} N_{ve} \tag{3.273}$$

We can prove that for a vertex amplitude, the solution exists only when the boundary data determines tetrahedra that are glued with length-matching (the pair of glued triangles have their edge-lengths matched).

Given the boundary data, we can determines geometric group elements $G^{\Delta} \in O(1,3)$ from reconstructed normals N^{Δ}. Then it can be shown that, after one choose s_v and s_{ve}, such that

$$\forall_e \ \det\ G^{\Delta}_{ve} = (-1)^{s_v} = r(v). \tag{3.274}$$

G^{Δ}_{ve} relates to G_{ve} by

$$G_{ve} = G^{\Delta}_{ve} I^{s_{ve}} (IR_u)^{s_v} \tag{3.275}$$

where R_N is the reflection respecting to normalized vector N defined as

$$(R_N)^I_J = \mathbb{I}^I_J - \frac{2N^I N_J}{N \cdot N} \tag{3.276}$$

The choice of $s_{ve} = \pm 1$ corresponds to a gauge freedom and is arbitrary here. Condition 3.274 is called the orientation matching condition, which essentially means that the orientations of 5 boundary tetrahedra determined by the boundary condition are required to be the same.

For a vertex amplitude, the non-degenerate geometric critical solutions exist if and only if the length matching condition and orientation matching condition are satisfied. Up to gauge transformations, there are two gauge inequivalent solutions which are related to each other by a reflection respect to any normalized 4 vector e_α (this reflection is referred to as the parity transformation in e.g. [77–80])

$$\tilde{B}_{ef}(v) = R_{e_\alpha}(B_{ef}(v)), \qquad \tilde{s}_v = s_v + 1 \tag{3.277}$$

which means

$$\tilde{G}_{ve} = R_{e_\alpha} G_{ve} (IR_N) \tag{3.278}$$

Geometrically the second one corresponds to the reflected simplex. These two critical solutions correspond to the same 4-simplex geometry, but associates to different sign of the oriented 4-simplex volume $V(v)$. $\mathrm{sgn}(V(v))$ is referred to as the (geometrical) orientation of the 4-simplex [b], which shouldn't be confused with $r(v)$. This result generalizes [85] to the spin foam vertex amplitude containing timelike triangles.

The reconstruction can be extended to simplicial complex $\mathcal{K}$ with many 4-simplices, in which some critical solutions of the full amplitude correspond to nondegenerate Lorentzian simplicial geometries on $\mathcal{K}$ (see Appendix 3.A). But similar to the situation in [79, 80], 4-simplices in $\mathcal{K}$ may have different $\mathrm{sgn}(V(v))$. We may divide the complex $\mathcal{K}$ into sub-complexes, such that each sub-complex is globally orientated, i.e. the sign of the orientated volume $\mathrm{sgn}(V)$ is a constant.

b. $\mathrm{sgn}(V(v))$ is a discrete analog of the volume element compatible to the metric in smooth pseudo-Riemannian geometry.

Then we have the following result:

For critical solutions corresponding to simplicial geometries with all 4-simplices globally oriented, picking up a pair of them corresponding to opposite global orientations, they satisfy

$$\tilde{G}_f = \begin{cases} R_{u_e} G_f(e) R_{u_e} & \text{internal faces} \\ I^{r_{e1}+r_{e0}} R_{u_{e1}} G_f(e_1, e_0) R_{u_{e0}} & \text{boundary faces} \end{cases} \tag{3.279}$$

where $G_f = \prod_{v \subset \partial f} G_{e'v} G_{ve}$ is the face holonomy. We will use this result to derive the phase difference of their asymptotical contributions to the spin foam amplitude. Note that, the asymptotic formula of the spinfoam amplitude is given by summing over all possible configuration of orientations.

3.5. Split signature and degenerate 4 simplex

This section discusses the critical solutions that violate the non-degeneracy condition (3.271). We refer to these solutions as degenerate solutions. If the non-degeneracy condition is violated, then in each 4-simplex, all five normals N_e of tetrahedra t_e are parallel, since we only consider nondegenerate tetrahedra [85]. When it happens with all t_e timelike (or spacelike), with the help of gauge transformation $G_{ve} \to GG_{ve}$, we can write $N_e(v) = G_{ve}u,\ u = (0,0,0,1)$, where all the group variables $G_{ve} \in \mathrm{SO}_+(1,2)$. However, when the vertex amplitude contains at least one timelike and one spacelike tetrahedron, the non-degeneracy condition (3.271) cannot be violated since timelike and spacelike normals certainly cannot be parallel. Therefore the solutions discussed in this section only appear in the vertex amplitude with all tetrahedra timelike. Moreover, these degenerate solutions appears when the boundary data are special, i.e. correspond to the boundary of a split signature 4-simplex or a degenerate 4-simplex, as we see in a moment.

When the tetrahedron contains both timelike and spacelike triangles, the closure constraint (3.243) concerning w involves at most 3 null vectors, which directly leads to $\mathrm{Re}(\alpha_{vef}) = 0$ as the only solution. For degenerate solutions, the bivector $X_f(v) = g_{ve} X_{ef} g_{ev}$ in (3.249) becomes

$$X_f(v) = *G_{ve}(v_{ef} \wedge u) G_{ev} = G_{ve} v_{ef} \wedge u = v^g_{vef} \wedge u \tag{3.280}$$

The parallel transportation equation (3.256) becomes

$$v^g_f(v) = v^g_{ve} = v^g_{ve'} = 2A_f G_{ve} v_{ef}. \tag{3.281}$$

Thus, the degenerate critical solutions satisfy

$$v^g_f(v) = v^g_{ve} = v^g_{ve'}, \qquad \sum_f \epsilon_{ef}(v) v^g_f(v) = 0 \tag{3.282}$$

and the collection of vectors $v_f^g(v)$ is referred to as a vector geometry in [77].

In the case that all triangles in a tetrahedron are timelike, we use S_{vf+} as an example. The degeneracy implies $G_{ve}u = G_{ve'}u = u$,. The parallel transportation equation (3.263) becomes

$$(G_{ve}\tilde{v}_{vef} - G_{ve'}\tilde{v}_{ve'f}) \wedge u = c_{ve}\,\mathrm{Re}(\alpha_{vef})G_{ve}w_{ef}^+ \wedge u - c_{ve'}\,\mathrm{Re}(\alpha_{ve'f})G_{ve'}w_{e'f}^+ \wedge u. \tag{3.283}$$

$c_{ve} = C_{ve} + \mathrm{i}\tilde{C}_{ve}$ is the factor which solves the closure constrain with a given normalization of $\mathrm{Re}(\alpha_{vef})$, e.g. $\sum_f \mathrm{Re}(\alpha_{vef}) = 1$ as shown in (3.261). (3.283) directly leads to

$$G_{ve}(\tilde{v}_{vef} + C_{ve}\,\mathrm{Re}(\alpha_{vef})w_{ef}) = G_{ve'}(\tilde{v}_{ve'f} + C_{ve}\,\mathrm{Re}(\alpha_{vef})w_{ef}) \tag{3.284}$$

$$\tilde{C}_{ve}\,\mathrm{Re}(\alpha_{vef})G_{ve}w_{ef} = \tilde{C}_{ve'}\,\mathrm{Re}(\alpha_{ve'f})G_{ve'}w_{e'f} \tag{3.285}$$

Notice that from (3.284), since w_{ef} is null and $w_{ef} \cdot v_{ef} = 0$, we have

$$G_{ve}w_{ef} \propto G_{ve'}w_{e'f}. \tag{3.286}$$

It implies that (3.285) is only a function of $\tilde{C}$. However, at a vertex v, there are only 5 independent $\tilde{C}$ variables out of 10 equations. Thus (3.285) are over constrained equations and give 5 consistency condition for G_{ve} unless $\tilde{C} = 0$.

Actually one can show that, there is no solution when $\tilde{C} \neq 0$. We give the proof here. For simplicity, we only focus on a single 4-simplex.

Suppose we have solutions to above equations with $\tilde{C} \neq 0$, then the following equations hold according to (3.284), (3.285) and the closure constraint (3.208)

$$\begin{aligned} v_f^g(v) &= v_{ef}^g(v) = v_{e'f}^g(v), \quad \sum_{f \subset t_e} \epsilon_{ef}(v) v_{ef}^g(v) = 0\,, \\ w_f^g(v) &= w_{ef}^g(v) = w_{e'f}^g(v), \quad \sum_{f \subset t_e} \epsilon_{ef}(v) w_{ef}^g(v) = 0, \end{aligned} \tag{3.287}$$

where

$$\begin{aligned} v_{ef}^g(v) &= G_{ve}\tilde{v}_{ef} + C_i\,\mathrm{Re}(\alpha_{vef})G_{ve}w_{ef} \\ w_{ef}^g(v) &= \tilde{C}_i\,\mathrm{Re}(\alpha_{vef})G_{ve}w_{ef} \end{aligned} \tag{3.288}$$

Suppose v^g satisfy the length matching condition. From above equations, $\tilde{v}_{ef}^g = v_{ef}^g + a w_{ef}^g$ with arbitrary real number a are also solutions. This means $\tilde{v}^g$ should also satisfy the length matching condition. However the transformation from v to $v + aw$ changes the edge lengths of the tetrahedron, and the length matching condition gives constraint to a. This conflict with the fact that a is arbitrary to form the solution. It means that we can not have a solution with $\tilde{C} \neq 0$ and length matching condition satisfied.

Thus, when boundary data satisfies the length matching condition, the only possible solution of (3.285) is $\tilde{C}_{ve}=0$. This corresponds to $\mathrm{Re}(\alpha)=0$ thus only possible with action S_+. One recognizes that this is the same condition as in the case of tetrahedron with both timelike and spacelike triangles, e.g. (3.244). In this case C_{ve} thus $\mathrm{Im}(\alpha)$ can be uniquely determined by the closure and length matching condition. The critical point equations again becomes (3.281) and (3.282)

In the end of this section, we introduce some relations between the vector geometry and non-degenerate split signature 4-simplex. As shown in Appendix 3.A.6, the vector geometries in 3 dimensional subspace V can be map to the split signature space M' with signature $(-,+,+,-)$ (flip the signature of $u=(0,0,0,1)$), with the map $\Phi^{\pm}:\wedge^2 M^{4\prime}\to V$ for bivectors B,

$$\Phi^{\pm}(B)=(\mp B-*'B)\cdot' u. \tag{3.289}$$

$\Phi^{\pm}$ naturally induced a map from $g\in$ SO$(2,2)$ to the subgroup $h\in$ SO$(1,2)$, defined by

$$\Phi^{\pm}(gBg^{-1})=\Phi^{\pm}(g)\Phi^{\pm}(B) \tag{3.290}$$

If the vertex amplitude has the critical solutions being a pair of non-gauge-equivalent vector geometries $\{G^{\pm}_{ve}\}$, they are equivalent to a pair of non-gauge-equivalent $\{G_{ve}\in SO(M')\}$ satisfying the nondegenerate condition. One of the non-degenerate $\{G_{ve}\}$ satisfies $G^{\pm}_{ve}=\Phi^{\pm}(G_{ve})$, while the other $\{\tilde{G}_{ve}\}$ satisfies

$$\Phi^{\pm}(\tilde{G})=\Phi^{\pm}(R_u G R_u)=\Phi^{\mp}(G) \tag{3.291}$$

When the vector geometries are gauge equivalent, the corresponding geometric $SO(M')$ solution is degenerate. In this case the reconstructed 4 simplex is degenerate and the 4 volume is 0.

3.6. Summary of geometries

We summarize all possible reconstructed geometries corresponding to critical configurations of Conrady-Hnybida extended spin foam model (include EPRL model) here. We first introduce the length matching condition and orientation matching condition for the boundary data. Namely, (1) among the 5 tetrahedra reconstructed by the boundary data (by Minkowski Theorem), each pair of them are glued with their common triangles matching in shape (match their 3 edge lengths), and (2) all tetrahedra have the same orientation. The amplitude will be suppressed asymptotically if orientation matching condition is not satisfied.

For given boundary data satisfies length matching condition and orientation matching condition, we may have the following reconstructed 4 simplex geometries corresponding to critical configurations of Conrady-Hynbida model:

— Lorentzian $(-+++)$ 4 simplex geometry: reconstructed by boundary data which may contains
  — both timelike and spacelike tetrahedra,
  — all tetrahedra being timelike.
  — all tetrahedra being spacelike.

— Split signature $(-++-)$ 4 simplex geometry: This case is only possible when every boundary tetrahedron are timelike.

— Euclidean signature $(++++)$ 4 simplex geometry: This case is only possible when every boundary tetrahedron are spacelike.

— Degenerate 4 simplex geometry: This case is only possible when all boundary tetrahedron are timelike or all of them are spacelike.

When length matching condition is not satisfied, we might still have one gauge equivalence class of solutions which determines a single vector geometry. This solution exists again only when all boundary tetrahedron are timelike or all of them are spacelike.

Our analysis is generalized to a simplicial complex $\mathcal{K}$ with many 4-simplices. A most general critical configuration of Conrady-Hnybida model may mix all the types of geometries on the entire $\mathcal{K}$. One can always make a partition of $\mathcal{K}$ into sub-regions such that in each region we have a single type of reconstructed geometry with boundary. However, this may introduce nontrivial transitions between different types of geometries through boundary shared by them as suggested in [79]. It is important to remark that, if we take the boundary data of each 4 simplex to contain at least one timelike and one spacelike tetrahedron, critical configurations will only give Lorentzian 4-simplices.

3.7. Phase difference

In this section, we compare the difference of the phases given by a pair of critical solutions with opposite (global) $\mathrm{sgn}(V)$ orientations on a simplical complex $\mathcal{K}$. Recall that the amplitude is defined with SU(1,1) and SU(2) coherent states at the timelike and spacelike boundary. When we define the coherent state, we have a phase ambiguity from K_1 direction in SU(1,1) (or J_3 direction in SU(2)), thus the action is determined up to this phase. Thus the phase difference ΔS is the essential result in the asymptotic analysis of spin foam vertex amplitude. The phase difference at a spacelike triangle has already been discussed in [85], we only focus on timelike triangles here.

Given a timelike triangle f, in Lorentzian signature, the normals N_e and $N_{e'}$ are spacelike and span a spacelike plane, while in split signature they form a timelike surface. The dihedral angles Θ_f at f are defined as follows: In Lorentzian signature, the dihedral angle is $\Theta_f = \pi - \theta_f$ where

$$\cos\theta_f = N_e^\Delta \cdot N_{e'}^\Delta, \qquad \theta_f \in (0,\pi) \tag{3.292}$$

While in split signature, the boost dihedral angle θ_f is defined by

$$\cosh\theta_f = |N_e^\Delta \cdot' N_{e'}^\Delta|, \qquad \theta_f \gtreqless 0 \;\text{ while }\; N_e^\Delta \cdot' N_{e'}^\Delta \gtreqless 0; \tag{3.293}$$

3.7.1. Lorentzian signature solutions

As we shown before, when every tetrahedron has both timelike and spacelike triangles, the critical solutions only comes from S_+. So we focus on S_+ action.
From the action (3.101), after inserting the decomposition (3.135), we find

$$\begin{aligned} S_{vf+} =& \frac{n_f}{2}\ln\frac{\zeta_{vef}\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}\zeta_{ve'f}} - \mathrm{i}s_f\ln\frac{\zeta_{ve'f}\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}\zeta_{vef}} = -2\mathrm{i}\gamma s_f(\arg(\zeta_{ve'f}) - \arg(\zeta_{vef}) - 2\mathrm{i}s\ln\frac{|\zeta_{ve'f}|}{|\zeta_{vef}|} \\ =& -2\mathrm{i}s_f(\theta_{e'vef} + \gamma\phi_{e'vef}) \end{aligned} \tag{3.294}$$

where θ and ϕ are defined by

$$\begin{aligned} \theta_{e'vef} &:= \ln\frac{|\zeta_{ve'f}|}{|\zeta_{vef}|}, \\ \phi_{e'vef} &:= \arg(\zeta_{ve'f}) - \arg(\zeta_{vef}) \end{aligned} \tag{3.295}$$

The face action at a triangle dual to a face f then reads

$$S_f = \sum_{v\in\partial f} S_{vf} = -2\mathrm{i}s_f\left(\sum_{v\in\partial f}\theta_{e'vef} + \gamma\sum_{v\in\partial f}\phi_{e'vef}\right) \tag{3.296}$$

We start the analysis from faces dual to boundary triangles (boundary faces) and then going to internal faces.

3.7.1.1. Boundary faces

For critical configurations solving critical equations (we keep $\mathrm{Im}(\alpha) = 0$ by redefinition of l^-_{ef}), they satisfy

$$g_{ve}\eta l^+_{ef} = \frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}} g_{ve'}\eta l^+_{e'f} \tag{3.297}$$

$$g_{ve}Jl^-_{ef} = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'}^{-1\dagger}Jl^-_{e'f}. \tag{3.298}$$

We then have

$$G_f(e_1,e_0)\ \ \eta l^+_{e_0 f} \tag{3.299}$$
$$= \mathrm{e}^{-\sum_{v\in p_{e_1e_0}}\theta_{e'vef}+\mathrm{i}\sum_{v\in p_{e_1e_0}}\phi_{e'vef}}\ \ \eta l^+_{e_1 f}$$
$$G_f(e_1,e_0)\ \ Jl^-_{e_0 f} \tag{3.300}$$
$$= \mathrm{e}^{\sum_{v\in p_{e_1e_0}}\theta_{e'vef}-\mathrm{i}\sum_{v\in p_{e_1e_0}}\phi_{e'vef}}\ \ Jl^-_{e_1 f}$$

where $G_f(e_1,e_0)$ is the product of edge holonomy along the path $p_{e_0e_1}$

$$G_f(e_1,e_0) := g_{e_1v_1}...g_{e'v_0}g_{v_0e_0} \tag{3.301}$$

Suppose we have holonomies G and $\tilde{G}$ from the pair of critical solutions with global $\mathrm{sgn}(V)$ orientation, then one can see

$$\tilde{G}^{-1}G\ \eta l^+_{e_0 f} \tag{3.302}$$
$$= \mathrm{e}^{-\sum_{v\in p_{e_1e_0}}\Delta\theta_{e'vef}+\mathrm{i}\sum_{v\in p_{e_1e_0}}\Delta\phi_{e'vef}}\ \eta l^+_{e_0 f}$$
$$\tilde{G}^{-1}G\ Jl^-_{e_0 f} \tag{3.303}$$
$$= \mathrm{e}^{\sum_{v\in p_{e_1e_0}}\Delta\theta_{e'vef}-\mathrm{i}\sum_{v\in p_{e_1e_0}}\Delta\phi_{e'vef}}\ Jl^-_{e_0 f}$$

For a single 4-simplex, the above equations read

$$(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}(g_{e'v}g_{ve})\ \ \eta l^+_{e_0 f} = \frac{\bar{\zeta'}_{vef}}{\bar{\zeta'}_{ve'f}}\frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}}\ \ \eta l^+_{ef} \tag{3.304}$$
$$= \mathrm{e}^{-\Delta\theta_{e'vef}+\mathrm{i}\Delta\phi_{e'vef}}\ \ \eta l^+_{ef}$$
$$(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}(g_{e'v}g_{ve})\ \ Jl^-_{e_0 f} = \frac{\bar{\zeta'}_{ve'f}}{\bar{\zeta'}_{vef}}\frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}}\ \ Jl^-_{e_0 f} \tag{3.305}$$
$$= \mathrm{e}^{\Delta\theta_{e'vef}-\mathrm{i}\Delta\phi_{e'vef}}\ \ Jl^-_{e_0 f}$$

which lead to

$$g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}g_{e'v}\quad g_{ve}\ \eta l^+_{e_0 f} \tag{3.306}$$
$$= \mathrm{e}^{-\Delta\theta_{e'vef}+\mathrm{i}\Delta\phi_{e'vef}}\quad g_{ve}\ \eta l^+_{e_0 f}$$
$$g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}g_{e'v}\quad g_{ve}\ Jl_{e_0 f} \tag{3.307}$$
$$= \mathrm{e}^{\Delta\theta_{e'vef}-\mathrm{i}\Delta\phi_{e'vef}}\quad g_{ve}\ Jl^-_{e_0 f}$$

We can define an operator T_{ef} by

$$T_{ef} := \eta l^+_{ef}\otimes(l^-_{ef})^\dagger = \left|\eta l^+_{ef}\right\rangle\left\langle l^-_{ef}\right| \tag{3.308}$$

From the facts $\left\langle l_{ef}^{-} \middle| \eta l_{ef}^{+} \right\rangle = \langle l_{ef}^{-}, l_{ef}^{+} \rangle = 1, \left\langle l_{ef}^{-} \middle| J l_{ef}^{-} \right\rangle = 0$, the action of this operator leads to

$$\begin{aligned} T_{ef} \left| \eta l_{ef}^{+} \right\rangle &= \left| \eta l_{ef}^{+} \right\rangle \left\langle l_{ef}^{-} \middle| \eta l_{ef}^{+} \right\rangle = \left| \eta l_{ef}^{+} \right\rangle \\ T_{ef} \left| J l_{ef}^{-} \right\rangle &= 0 \end{aligned} \tag{3.309}$$

From the definition of (3.229) (with $\alpha = 0$), by using (3.186) and (3.191), one then see

$$X_{ef} \left| \eta l_{ef}^{+} \right\rangle = \frac{1}{2} \left| \eta l_{ef}^{+} \right\rangle, \quad X_{ef} \left| J l_{ef}^{-} \right\rangle = -\frac{1}{2} \left| J l_{ef}^{-} \right\rangle \tag{3.310}$$

Then we have

$$2X_f \;\; g_{ve} \left| \eta l_{ef}^{+} \right\rangle = 2 g_{ve} X_{ef} g_{ev} g_{ve} \left| \eta l_{ef}^{+} \right\rangle = g_{ve} 2 X_{ef} \left| \eta l_{ef}^{+} \right\rangle = g_{ve} \left| \eta l_{ef}^{+} \right\rangle \tag{3.311}$$

$$2X_f \;\; g_{ve} \left| J l_{ef}^{-} \right\rangle = 2 g_{ve} X_{ef} g_{ev} g_{ve} \left| J l_{ef}^{-} \right\rangle = g_{ve} 2 X_{ef} \left| J l_{ef}^{-} \right\rangle = -g_{ve} \left| J l_{ef}^{-} \right\rangle . \tag{3.312}$$

From (3.306) and (3.307), it is easy to see

$$g_{ve} (\tilde{g}_{e'v} \tilde{g}_{ve})^{-1} g_{e'v} = \mathrm{e}^{-2\Delta\theta_{e'vef} X_f + 2\mathrm{i}\Delta\phi_{e'vef} X_f} \tag{3.313}$$

For a general simplicial complex with boundary, given a boundary face f with two edges e_0 and e_1 connecting to the boundary, and v is the bulk end-point of e_0 if we define

$$G_f(e_1, e_0) = G_f(v, e_1)^{-1} g_{ve_0} \tag{3.314}$$

It can be proved that

$$G_f(v, e_1) X_{e_1 f} G_f(v, e_1)^{-1} = g_{ve_0} X_{e_0 f} g_{e_0 v} \tag{3.315}$$

which is the generalization of the parallel transportation equation within a single 4-simplex. Then we can apply the same derivation as the single-simplex case by replacing $g_{ve'} \to G(v, e_1)$, which leads to

$$g_{ve} \tilde{G}_f(e_1, e_0)^{-1} G_f(e_1, e_0) g_{ev} = \mathrm{e}^{-2 \sum_{v \in \partial f} \Delta\theta_{e'vef} X_f + 2\mathrm{i} \sum_{v \in \partial f} \Delta\phi_{e'vef} X_f} . \tag{3.316}$$

3.7.1.2. Internal faces

The discussion of internal face f is similar to the boundary case, we have

$$G_f \; \eta l_{ef}^{+} = \mathrm{e}^{-\sum_{v \in \partial f} \theta_{e'vef} + \mathrm{i} \sum_{v \in \partial f} \phi_{e'vef}} \;\; \eta l_{ef}^{+} \tag{3.317}$$

$$G_f \; J l_{ef}^{-} = \mathrm{e}^{\sum_{v \in \partial f} \theta_{e'vef} - \mathrm{i} \sum_{v \in \partial f} \phi_{e'vef}} \;\; J l_{ef}^{-} \tag{3.318}$$

where G_f is the face holonomy

$$G_f := \overleftarrow{\prod_{v\in\partial f}} g_{e'v} g_{ve}. \tag{3.319}$$

By the action of bivector X_{ef} in (3.310),

$$\begin{aligned}
&\mathrm{e}^{-\sum_{v\in\partial f}\theta_{e'vef}2X_{ef}+\mathrm{i}\sum_{v\in\partial f}\phi_{e'vef}2X_{ef}}\left|\eta l^+_{ef}\right\rangle & (3.320)\\
&\quad=\mathrm{e}^{-\sum_{v\in\partial f}\theta_{e'vef}+\mathrm{i}\sum_{v\in\partial f}\phi_{e'vef}}\left|\eta l^+_{ef}\right\rangle\\
&\mathrm{e}^{-\sum_{v\in\partial f}\theta_{e'vef}2X_{ef}+\mathrm{i}\sum_{v\in\partial f}\phi_{e'vef}2X_{ef}}\left|J\, l^-_{ef}\right\rangle & (3.321)\\
&\quad=\mathrm{e}^{\sum_{v\in\partial f}\theta_{e'vef}-\mathrm{i}\sum_{v\in\partial f}\phi_{e'vef}}\left|J\, l^-_{ef}\right\rangle
\end{aligned}$$

Compare to (3.317) and (3.318), we see that

$$G_f = \mathrm{e}^{-\sum_{v\in\partial f}\theta_{e'vef}2X_{ef}+\mathrm{i}\sum_{v\in\partial f}\phi_{e'vef}2X_{ef}} \tag{3.322}$$

Given G_f and $\tilde{G}_f$ from a pair of critical solutions with opposite $\mathrm{sgn}(V)$ orientation, we find

$$g_{ve}\tilde{G}_f^{-1}G_f g_{ev} = \mathrm{e}^{-2\sum_{v\in\partial f}\Delta\theta_{e'vef}X_f+2\mathrm{i}\sum_{v\in\partial f}\Delta\phi_{e'vef}X_f} \tag{3.323}$$

3.7.1.3. Phase difference

For a pair of globally orientated (constant $\mathrm{sgn}(V)$) critical solutions with opposite orientation, from (3.296) we have

$$\Delta S_f = -2\mathrm{i}s_f\left(\sum_{v\in\partial f}\Delta\theta_{e'vef}+\gamma\sum_{v\in\partial f}\Delta\phi_{e'vef}\right) \tag{3.324}$$

where $\Delta\theta$ and $\Delta\phi$ are determined by

$$g_{ve}\tilde{G}_f^{-1}G_f g_{ev} = \mathrm{e}^{-2\sum_{v\in\partial f}\Delta\theta_{e'vef}X_f+2\mathrm{i}\sum_{v\in\partial f}\Delta\phi_{e'vef}X_f} \tag{3.325}$$

$G_f \equiv G_f(e_1, e_0)$ if f is a boundary face. Since $\gamma s_f = n_f/2 \in \mathbb{Z}/2$, we may restrict

$$\sum_{v\in\partial f}\Delta\phi_{e'vef} \in [-\pi,\pi]. \tag{3.326}$$

because ΔS_f is an exponent.

After projecting to $\mathrm{SO}_+(1,3)$,

$$g_{ve}\tilde{G}_f^{-1}G_f g_{ev} \to G_{ve}\tilde{G}_f^{-1}G_f G_{ev}, \qquad \mathrm{i}\to * \tag{3.327}$$

For spacelike normal vector $u = (0,0,0,1)$, from it is easy to see G and $\tilde{G}$ are related by

$$\tilde{G} = R_{e_0} G R_u I \in SO_+(1,3) \tag{3.328}$$

and

$$\tilde{G}_f = R_{u_e} G_f R_{u_e} \tag{3.329}$$

for both internal and boundary triangles f. The equation then leads to

$$G_{ve}\tilde{G}_f^{-1} G_f G_{ev} = G_{ve} R_u G_f^{-1} R_u G_f G_{ev} = R_{N_e} R_{N_{e'}} \tag{3.330}$$

for both internal and boundary triangles f. N_e and $N_{e'}$ here are given by

$$N_e = G_{ve} u, \qquad N_{e'} = G_{ve}(G_f^{-1} u), \tag{3.331}$$

thus $N_{e'}$ is the parallel transported vector along the face.

Therefore in both internal case and boundary case, we have

$$R_{N_e} R_{N_{e'}} = \mathrm{e}^{-2\sum_{v\in\partial f} \Delta\theta_{e'vef} X_f + 2*\sum_{v\in\partial f} \Delta\phi_{e'vef} X_f} \tag{3.332}$$

On the other hand, from the fact that, $R_N = G R_u G$, and the fact that $G^{\Delta}_{ve} = G I^{s_{ve}} (I R_u)^s_v$, we have

$$R_{N_e} R_{N'_e} = R_{N^{\Delta}_e} R_{N^{\Delta}_{e'}} \tag{3.333}$$

Since $R_{N^{\Delta}}$ is a reflection respect to spacelike normal N^{Δ}, we have (see Appendix 3.B)

$$R_{N^{\Delta}_e} R_{N^{\Delta}_{e'}} = \mathrm{e}^{2\theta_f \frac{N^{\Delta}_e \wedge N^{\Delta}_{e'}}{|N^{\Delta}_e \wedge N^{\Delta}_{e'}|}} \tag{3.334}$$

where f is the triangle dual to the face determined by edges e and e'. $\theta_f \in [0,\pi]$ satisfies $N^{\Delta}_e \cdot N^{\Delta}_{e'} = \cos(\theta_f)$. From the geometric reconstruction,

$$B_f = n_f X_f = -\frac{1}{Vol^{\Delta}} r W^{\Delta}_e W^{\Delta}_{e'} * (N^{\Delta}_{e'} \wedge N^{\Delta}_e), \tag{3.335}$$

Since $|B_f|^2 = -n_f^2$, we have

$$\left|\frac{1}{Vol^{\Delta}} r W^{\Delta}_e W^{\Delta}_{e'}\right| |N^{\Delta}_{e'} \wedge N^{\Delta}_e| = n_f \tag{3.336}$$

Thus

$$X_f = \frac{B_f}{n_f} = \sigma_f \frac{*(N^{\Delta}_{e'} \wedge N^{\Delta}_e)}{|(N^{\Delta}_{e'} \wedge N^{\Delta}_e)|} \tag{3.337}$$

where $\sigma_f = -r\mathrm{sign}(W^{\Delta}_{e'} W^{\Delta}_e)$. Since N_e and $N_{e'}$ are both spacelike, we have $\sigma_f = -r$. Keep in mind that r is the orientation and is a constant sign on the (sub-)triangulation. Therefore

$$e^{2r\sum_{v\in\partial f}\Delta\theta_{e'vef}\frac{*(N^\Delta_{e'}\wedge N^\Delta_e)}{|N^\Delta_{e'}\wedge N^\Delta_e|}+2r\sum_{v\in\partial f}\Delta\phi_{e'vef}\frac{N^\Delta_{e'}\wedge N^\Delta_e}{|N^\Delta_{e'}\wedge N^\Delta_e|}} = e^{2\theta_f\frac{N^\Delta_e\wedge N^\Delta_{e'}}{|N^\Delta_e\wedge N^\Delta_{e'}|}} \tag{3.338}$$

which implies

$$\begin{aligned}&\sum_{v\in\partial f}\Delta\theta_{e'vef}=0,\\ &-r\sum_{v\in\partial f}\Delta\phi_{e'vef}=\theta_f \mod \pi\end{aligned} \tag{3.339}$$

The phase difference is then

$$\Delta S_f = 2\mathrm{i}rA_f\theta_f \mod \mathrm{i}\pi \tag{3.340}$$

where $A_f=\gamma s_f=n_f/2\in\mathbb{Z}/2$ is the area spectrum of the timelike triangle.

The $i\pi$ ambiguity relates to the lift ambiguity from $G_f\in\mathrm{SO}^+(1,3)$ to $\mathrm{SL}(2,\mathbb{C})$. Some ambiguities may be absorbed into gauge transformations $g_{ve}\to -g_{ve}$. Firstly we consider a single 4-simplex, (3.339) reduces to $\Delta\theta_{e'vef}=0$ and $\Delta\phi_{e'vef}=-\theta_f$ mod π (Here we use the notation that we move the orientation r from $\Delta\phi$ in (3.339) to the definition of ΔS. Keep in mind ΔS always depends on the orientation r). However it is shown in Appendix 3.C that this ambiguity can indeed be absorbed into the gauge transformation of g_{ve}, i.e. if we fix the gauge,

$$\Delta\phi_{e'vef}=-\theta_f(v) \mod 2\pi, \tag{3.341}$$

where $\theta_f(v)$ is the angle between tetrahedron normals in the 4-simplex at v. Although this fixing of lift ambiguity only applies to a single 4-simplex, it is sufficient for us to obtain ΔS_f^Δ unambiguously. Applying (3.341) to the case with many 4-simplices

$$\sum_{v\in\partial f}\Delta\phi_{e'vef}=-\sum_{v\in\partial f}\theta_f(v) \mod 2\pi \tag{3.342}$$

Since $\theta_f(v)$ relates to the dihedral angle $\Theta_f(v)$ by $\theta_f(v)=\pi-\Theta_f(v)$, for an internal f, $\sum_{v\in\partial f}\Delta\phi_{e'vef}$ relates to the deficit angle $\varepsilon_f=2\pi-\sum_{v\in\partial f}\Theta_f(v)$ by

$$\sum_{v\in\partial f}\Delta\phi_{e'vef}=(2-m_f)\pi-\varepsilon_f \mod 2\pi \tag{3.343}$$

where m_f is the number of $v\in\partial f$. Similarly, for a boundary f, $\sum_{v\in\partial f}\Delta\phi_{e'vef}$ relates to the deficit angle $\theta_f=\pi-\sum_{v\in\partial f}\Theta_f(v)$ by

$$\sum_{v\in\partial f}\Delta\phi_{e'vef}=(1-m_f)\pi-\theta_f \mod 2\pi \tag{3.344}$$

As a result, the total phase difference is

$$\exp(\Delta S_f) = \exp\Big\{2\mathrm{i}r\sum_{f \text{ bulk}} A_f\left[(2-m_f)\pi - \varepsilon_f\right] \\ +2\mathrm{i}r\sum_{f \text{ boundary}} A_f\left[(1-m_f)\pi - \theta_f\right]\Big\} \tag{3.345}$$

The exponent is a Regge action when all bulk m_f are even, i.e. every internal f has even number of vertices. Obtaining Regge calculus only requires all bulk m_f's to be even, while boundary m_f's can be arbitrary, since the boundary terms $A_f(1-m_f)\pi$ doesn't affect the Regge equation of motion.

The above phase difference is for a general simplicial complex, the result for a single 4-simplex is simply given by removing the bulk terms and letting all boundary $m_f = 1$.

3.7.1.4. Determine the phase for bulk triangles

For the internal faces in the bulk, we can determine the phase at critical point uniquely.

Recall (3.322, the holonomy $G_f(v) = g_{ve}G_f(e)g_{ev}$ at vertex v reads

$$G_f(v) = \mathrm{e}^{-\sum_{v\in\partial f}\theta_{e'vef}2X_f(v)+\mathrm{i}\sum_{v\in\partial f}\phi_{e'vef}2X_f(v)} \tag{3.346}$$

Recall (3.441) as we shown in Appendix 3.A, for edges $E_{l1}(v)$ and $E_{l1}(v)$ of the triangle f in the frame of vertex v,

$$\begin{aligned} G_f(v)E_{l1}(v) &= \mu E_{l1}(v), \\ G_f(v)E_{l2}(v) &= \mu E_{l2}(v) \end{aligned} \tag{3.347}$$

where $\mu = (-1)^{\sum_{e\subset\partial f} s_e} = \pm 1$. Here s_e is defined as $s_e = s_{ve} + s_{v'e} + 1$ for edge $e = (v, v')$ with $s_{ve} \in \{0, 1\}$. With edges $E_{l1}(v)$ and $E_{l1}(v)$, the bivector $X_f(v)$ at vertex v can be expressed as

$$X_f(v) = \frac{*(N_{e'}(v)\wedge N_e(v))}{|N_{e'}(v)\wedge N_e(v)|} = \frac{E_{l1}(v)\wedge E_{l2}(v)}{|E_{l1}(v)\wedge E_{l2}(v)|} \tag{3.348}$$

From (3.347) and (3.348), with the fact that $\mathrm{e}^{X_f(v)}$ is a boost, one immediately see $\mu_e = 1$ and

$$G_f(v) = \mathrm{e}^{\mathrm{i}\sum_{v\in\partial f}\phi_{e'vef}2X_f(v)} = \mathrm{e}^{2r\sum_{v\in\partial f}\phi_{e'vef}\frac{N^{\Delta}_{e'}\wedge N^{\Delta}_{e}}{|N^{\Delta}_{e'}\wedge N^{\Delta}_{e}|}} \tag{3.349}$$

where we use (3.337). As we proved in Appendix 3.B, there exists spacelike normalized vector $\tilde{N}$ in the plane span by N_e and $N_{e'}$ such that

$$G_f(v) = R_N R_{\tilde{N}} \tag{3.350}$$

From (3.329),

$$\begin{aligned} G_{ve}\tilde{G}_f(e)G_f(e)G_{ev} &= G_{ve}R_u G_f(e) R_u G_f(e) G_{ev} \\ &= R_N G_f(v) R_N G_f(v) \end{aligned} \tag{3.351}$$

Then it is straightforward to show

$$\begin{aligned} G_{ve}\tilde{G}_f(e)G_f(e)G_{ev} &= R_N G_f(v) R_N G_f(v) \\ =& R_N R_N R_{\tilde{N}} R_N R_N R_{\tilde{N}} = R_{\tilde{N}} R_{\tilde{N}} = 1 \end{aligned} \tag{3.352}$$

Thus

$$\mathrm{e}^{2\sum_{v\in\partial f}(\tilde{\phi}_{e'vef}+\phi_{e'vef})*X_f} = 1 \tag{3.353}$$

which leads to

$$\sum_{v\in\partial f}(\tilde{\phi}_{e'vef}+\phi_{e'vef}) = 0 \mod \pi \tag{3.354}$$

The π ambiguity here relates to the lift ambiguity again. Note that, fixing of lift ambiguity to these 4-simplices sharing the triangle f as in the Appendix 3.C leads to $g_{ve}\tilde{G}_f(e)G_f(e)g_{ev} = 1$. Then we have

$$\sum_{v\in\partial f}(\tilde{\phi}_{e'vef}+\phi_{e'vef}) = 0 \mod 2\pi \tag{3.355}$$

where the π ambiguity is fixed. Combine with (3.343), we have

$$\begin{aligned} \sum_{v\in\partial f}\phi_{e'vef} &= -\sum_{v\in\partial f}\tilde{\phi}_{e'vef} \\ &= \frac{(2-m_f)\pi-\varepsilon_f}{2} \mod \pi \end{aligned} \tag{3.356}$$

As a result, the total phase for bulk triangles is

$$\exp(S_f) = \exp\left\{ir\sum_{f \text{ bulk}} A_f\left[(2-m_f)\pi-\varepsilon_f\right]\right\} \tag{3.357}$$

Again, the exponent is a Regge action when all bulk m_f are even, i.e. every internal f has even number of vertices.

Note that, the above derivation assumes a uniform orientation $\mathrm{sgn}(V)$, but the asymptotic formula of the spinfoam amplitude is given by summing over all possible configurations of orientations. As suggested by [79], at a critical solution, one can make a partition of $\mathcal{K}$ into sub-regions such that each region has a uniform orientation, so that the above derivation can be applied.

3.7.2. split signature solutions

In this subsection, we focus on a single 4-simplex. We consider a pair of the degenerate solutions $g^{\pm}_{ve}$ which can be reformulated as non-degenerate solutions in the flipped signature space $(-++-)$ here. When degenerate solutions are gauge equivalent, there exists only a single critical point, then there is a single phase depending on boundary coherent states.

Since (3.316) and (3.323) hold for all $\mathrm{SL}(2,\mathbb{C})$ elements which solve critical equations, they also hold for degenerate solutions $g^{\pm}_{ve}$. Thus from (3.313), we have

$$\begin{aligned} g^{\pm}_{ev}g^{\mp}_{ev}g^{\mp}_{ve'}g^{\pm}_{e'v} &= \mathrm{e}^{\mp 2\Delta\theta_{e'vef}X^{\pm}_{f} \pm 2\mathrm{i}\Delta\phi_{e'vef}X^{\pm}_{f}} \\ &= \mathrm{e}^{\mp 2\Delta\theta_{e'vef}X^{\pm}_{f}} \end{aligned} \tag{3.358}$$

Notice that since all $g^{\pm}_{ve} \in \mathrm{SU}(1,1) \subset \mathrm{SL}(2,\mathbb{C})$, we have $2\Delta\phi_{e'vef} = 0 \mod 2\pi$ ($*X^{\pm}_{f}$ generates rotations in v^{g}-u plane).

From (3.455), we have

$$\Phi^{\pm}(g_{ev}\tilde{g}_{ev}\tilde{g}_{ve'}g_{e'v}) = \Phi^{\pm}(g_{ev})\Phi^{\pm}(\tilde{g}_{ev})\Phi^{\pm}(\tilde{g}_{ve'})\Phi^{\pm}(g_{e'v}) = g^{\pm}_{ev}g^{\mp}_{ev}g^{\mp}_{ve'}g^{\pm}_{e'v} \tag{3.359}$$

Since $\tilde{G}_{ve} = R_u G_{ve} R_u$, we have

$$\Phi^{\pm}(R_{N_e}R_{N_{e'}}) = G^{\pm}_{ev}G^{\mp}_{ev}G^{\mp}_{ve'}G^{\pm}_{e'v} \tag{3.360}$$

For X_f in flipped signature space M', from the definition of $\Phi^{\pm}$ in (3.289), we have

$$\Phi^{\pm}(*'X_f) = \pm\Phi^{\pm}(X_f) = \pm v^{g\pm}_{ef} = \pm\Phi^{\pm}(X^{\pm}_{f}) \tag{3.361}$$

where we know $X^{\pm}_{f} = v^{g\pm}_{ef} \wedge u$ in degenerate case, and $X^{\pm}_{f}$ can be regarded as bivectors in $\mathrm{so}(V) \sim \wedge^2 V$. Then we have

$$\Phi^{\pm}(\mathrm{e}^{2\Delta\theta_{e'vef}*'X_f}) = \mathrm{e}^{\mp 2\Delta\theta_{e'vef}X^{\pm}_{f}} \tag{3.362}$$

where we identify the $\mathrm{SO}(1,2)$ acting on V to the one acting on M'.

Therefore, $\Delta\theta$ contribution to the phase difference in degenerate solutions $\{g^{\pm}\}$ is identified to the $\Delta\theta$ written in flipped signature solutions $\{g\}$ satisfying $\Phi^{\pm}(g) = g^{\pm}$. $\Delta\theta$ is given by

$$R_{N_e}R_{N_{e'}} = \mathrm{e}^{2\Delta\theta_{e'vef}*'X_f} \tag{3.363}$$

where X_f is the bivector from flipped signature solutions

$$X_f = \frac{B_f}{n_f} = -r\frac{*'(N^{\Delta}_{e'} \wedge N^{\Delta}_{e})}{|*`(N^{\Delta}_{e'} \wedge N^{\Delta}_{e})|} \tag{3.364}$$

From the fact that geometrically,

$$R_{N_e} R_{N_{e'}} = R_{N_e^\Delta} R_{N_{e'}^\Delta} = \mathrm{e}^{2\theta_f \frac{N_e^\Delta \wedge N_{e'}^\Delta}{|N_e^\Delta \wedge N_{e'}^\Delta|}}, \tag{3.365}$$

where $\theta_f \in \mathbb{R}$ is a boost dihedral angle. We have

$$-r\Delta\theta_{e'vef} = \theta_f, \qquad 2\Delta\phi_{e'vef} = 0 \mod 2\pi \tag{3.366}$$

the phase difference is

$$\Delta S_f^\Delta = 2\mathrm{i} r s_f \theta_f = 2\mathrm{i} r \frac{1}{\gamma} A_f \theta_f \mod \pi\mathrm{i} \tag{3.367}$$

We can again fix the πi ambiguity by using the method in Appendix 3.C. There is no ambiguity in θ_f since it is a boost angle. As a result,

$$\exp(\Delta S_f) = \exp\left(2\mathrm{i} r \frac{1}{\gamma} A_f \theta_f\right) \tag{3.368}$$

The generalization to simplicial complex is similar to the non-degenerate case, by substituting every g and $\tilde{g}$ there with $g^\pm$.

3.8. Discussion

The present work studies the large-j asymptotics limit of spin foam amplitude with timelike triangles in a most general configuration on a 4d simplicial manifold with many 4-simplices. It turns out the asymptotics of spin foam amplitude is determined by critical configurations of the corresponding spinfoam action on the simplicial manifold. The critical configurations have geometrical interpretations as different types of geometries in separated subregions: Lorentzian $(-+++)$ 4-simplices, split $(--++)$ 4-simplices or degenerate vector geometries. The configurations come in pairs which corresponding to opposite global orientations in each subregion. In each sub-complex with globally oriented 4-simplices coming with the same signature, the asymptotic contribution to the spinfoam amplitude is an exponential of Regge action, up to a boundary term which does not affect the Regge equation of motion.

An important remark is that, for a vertex amplitude containing at least one timelike and one spacelike tetrahedron, critical configurations only give Lorentzian 4-simplices, while Euclidean and degenerate vector geometries do not appear. In all known examples of Lorentzian Regge calculus, the geometries are corresponding to such configuration, for example, the Sorkin triangulation [162] where each 4-simplex containing 4 timelike tetrahedra and 1 spacelike tetrahedron. Since such configuration only gives Regge-like critical configurations which is supposed to be the result of simplicity constraint in spin foam models [29], the result could open a new and promising way towards a better understanding of the imposition of simplicity constraint. Furthermore, Such configuration also naturally inherits the causal structure to spin foam models, which may open the possibility to build the connection between spin foam models and causal sets theory [15] or causal dynamical triangulation theories [11, 12].

With this work, the asymptotics of Conrady-Hnybida spin foam model, with arbitrary timelike or spacelike non-degenerate boundaries, is now complete. In the present work we mainly concentrate on the case where each tetrahedron contains both timelike and spacelike triangles, which is the case in all Regge calculus geometry examples. The geometrical interpretation of the case where tetrahedron containing only timelike triangles is much more complicated and we only identify its critical configurations on special cases with the boundary data satisfies length matching condition and orientation matching condition. Further investigation is needed for all possible critical configurations in such case.

Moreover, in the present analysis we do not give the explicit form of measure factors of the asymptotics formula, which is important for the evaluation of the spin foam propagator and amplitude. The measure factor in EPRL model is related to the Hessian matrix at the critical configuration [163, 164]. However, the measure factor for the triangulation with timelike triangles is a much more complicated function of second derivatives of the action, due to the appearance of singularities. A further study of such kind multidimensional stationary phase approximation, in particular, the derivation of the measure factor would be interesting.

The present work opens the possibility to have Regge geometries in Lorentzian Regge calculus emerges as critical configurations from spin foam model, which may leads to a semi-classical effective description of spin foam model. Especially, this may lead to a effective equation of motion for symmetry reduced models, e.g., FLRW cosmology or black holes, from the semi-classical limit of spin foam models.

Appendix

3.A. Geometric interpretation and reconstruction

In this appendix we summarize the geometric reconstruction theorems for tetrahedron with spacelike triangles only in [77–80, 85], and extend them to general tetrahedron may contains also timelike triangles. We start with a single simplex σ_v corresponding to a vertex v, and then generalize the result to general simplicial manifold with many simplices. For simplicity, we introduce a short hand notation for a single simplex σ_v:

$$N_i := N_{e_i}(v) \qquad B_{ij}^G = -B_{ji}^G = \epsilon_{e_ie_j}(v)B_{e_ie_j}(v) \qquad B_{ij}^G = *(v_{ij}^G \wedge N_i) \tag{3.369}$$

where e_ie_j represents the face determined by the dual edge e_i and e_j, and $i = 0, 1, ..., 4$, and v_{ij} here is the trianlges normal scaled with the area : $v_{ij}^2 = \pm 4A_{ij}^2$.

Note that here we will assume our boundary data to be a geometric boundary data, which means they satisfy length matching condition and orientation matching condition. The detailed meaning of these conditions will become clear later. The geometric boundary data is necessary to get a Regge like geometric solution. For non-geometric boundary data, there will be at most one solution up to gauge equivalence, which is an analogy to the result in EPRL model [77, 78].

3.A.1. Non-degenerate condtion and classification of the solution

To begin with, we would like to introduce the non-degenerate condition. We will first consider non-degenerate simplices and then move to degenerate case. For the boundary data, non-degenerate means for a boundary tetrahedron any 3 out of 4 face normal vectors n_{ef} span a 3-dimensional space. With non-degenerate boundary data, for any 3 different edges i, j, k in a 4 simplex one of the following holds

— $N_{ei} = \pm N_{ej}$ and $N_{ej} = \pm N_{ek}$,

— $N_{ei} \neq N_{ej}$

The first case can be further proved that leads to all N_i are parallel by using the closure constraint of B_{ij}. This result was first proved in [77] and later by [85].

The only non-degenerate case is then specify by the following non-degeneracy condition

$$\prod_{e1,e2,e3,e4=0}^{5} \det(N_{e1}, N_{e2}, N_{e3}, N_{e4}) \neq 0 \tag{3.370}$$

which means any 4 out of 5 normals are linear independent and span a 4 dimensional Minkowski space. Since $N_e(v) = g_{ve}N^0$, it is easy to see the non-degenerate condition is actually a constraint on $\{g_{ve}\}$.

3.A.2. Nondegenerate geometry on a 4-simplex

For simplicity, we start with one 4-simplex σ_v in 4 dimensional Minkowski space $M = R^4$ here. For each 4-simplex σ_v dual to the vertex v, we associate it with a reference frame. In this reference frame, the 5 vertices of the 4-simplex $[p_0, p_1, p_2, p_3, p_4]$ have the coordinates $p_i : (x_i^I) = (x_i^0, x_i^1, x_i^2, x_i^3)$. Based on these coordinates, we introduce vectors y_i, a as well as covector A in an auxiliary space

R^5,

$$y_i = (x_i^I, 1)^T, \qquad \text{and} \quad a = (0, ..., 0, 1)^T, \quad A = a^T \tag{3.371}$$

We define the $k+1$ vector in R^5

$$\tilde{V}_{\alpha_0,...,\alpha_k} = y_{\alpha_0} \wedge ... \wedge y_{\alpha_k} \tag{3.372}$$

where $\alpha_i \in \{0, \cdots, 5\}$. With covector A, for k-vectors Ω in R^5 satisfying $A \llcorner \Omega = 0$, we can identify it with a k-vector in M. For example, since $A \llcorner A \llcorner \tilde{V}_{\alpha_0,...,\alpha_5} = 0$, we then induce a 4-vector in M from $\tilde{V}_{\alpha_0,...,\alpha_5}$,

$$V_{\alpha_0,...,\alpha_5} = A \llcorner \tilde{V}_{\alpha_0,...,\alpha_k} = (y_{\alpha_1} - y_{\alpha_0}) \wedge ... \wedge (y_{\alpha_5} - y_{\alpha_0}) \tag{3.373}$$

This vector is actually $4!$ times the volume 4-vector of 4-simplex:

$$V_{\alpha_0,...,\alpha_4} = (x_{\alpha_1} - x_{\alpha_0}) \wedge ... \wedge (x_{\alpha_4} - x_{\alpha_0}) = E_{\alpha_1\alpha_0} \wedge ... \wedge E_{\alpha_5\alpha_0} \tag{3.374}$$

$E^I_{\alpha_i\alpha_0} = x^I_{\alpha_i} - x^I_{\alpha_0}$ is the edge vector related to the oriented edge $l_{\alpha_i\alpha_0} = [p_{\alpha_i}, p_{\alpha_0}]$. Notice that the volume 4-vector comes with a sign respecting to the order of points.

We further define 3-vector and bivector by skipping some points

$$V_i = (-1)^i V_{0...\hat{\imath}...4} \tag{3.375}$$

$$B_{ij} = A \llcorner \tilde{V}_{0...\hat{\imath}...n} = \begin{cases} (-1)^{i+j+1} V_{0...\hat{\imath}...\hat{\jmath}...4} & i < j \\ (-1)^{i+j} V_{0...\hat{\jmath}...\hat{\imath}...4} & i > j \end{cases} \tag{3.376}$$

where $\hat{\imath}$ means omitting i_{th} elements. We have the following properties for V_i and B_{ij}

$$\sum_i V_i = 0, \tag{3.377}$$

$$B_{ij} = -B_{ij} m \qquad \forall_i \sum_{j \neq i} B_{ij} = 0, \tag{3.378}$$

One can further check that B_{ij} can be written as

$$B_{ij} = \frac{1}{2} (-1)^{\mathrm{sgn}(\sigma)} \epsilon^{ijkmn} E_{mk} \wedge E_{nk} \tag{3.379}$$

And one has $B_{ij}^2 = \pm 4 A_{ij}^2$ with A_{ij} is the area of the corresponding spacelike or timelike triangles in non-degenerate case.

Suppose the volume 4-vector of 4-simplex $V_{0,...,4}$ is non-degenerate. In this case any 4 out of 5 y_i are linearly independent. One can introduce a dual basis $\hat{y}_i$ and $\tilde{y}_i$ defined by

$$\hat{y}_i \lrcorner y_j = \delta_{ij}, \qquad \hat{y}_i = \tilde{y}_i + \mu_i A, \quad \tilde{y}_i \lrcorner a = 0 \tag{3.380}$$

with properties

$$\sum_i \hat{y}_i = A, \qquad \sum_i \tilde{y}_i = 0 \tag{3.381}$$

$\tilde{y}_i$ here can be regarded as covectors belong to M. With $\tilde{y}_i$, we have

$$V_i = -\tilde{y}_i \lrcorner V_{0\ldots4}, \quad B_{ij} = \tilde{y}_j \lrcorner \tilde{y}_i \lrcorner V_{0\ldots4} \tag{3.382}$$

Thus covectors $\tilde{y}_i$ are conormal to subsimplices V_i. And by using Hodge star, we have

$$V_i = -Vol * \tilde{y}_i, \qquad B_{ij} = -Vol * (\tilde{y}_j \wedge \tilde{y}_i) \tag{3.383}$$

where the volume $Vol > 0$ is the absolute value of the oriented 4-volume

$$V_4 := \det(V_{0,\ldots,4}) = \mathrm{sgn}(V_4) Vol \tag{3.384}$$

It can be shown that

$$\frac{1}{V_4} = \epsilon^{ijkl} \det(\tilde{y}_i, \tilde{y}_j, \tilde{y}_k, \tilde{y}_l) \tag{3.385}$$

and the co-frame vector E_{ij} is given by

$$E_{ij} = V_4 \epsilon_{ijklm}(v) * (\tilde{y}^k \wedge \tilde{y}^l \wedge \tilde{y}^m) \tag{3.386}$$

If the subsimplices V_i are non-degenerate, by introducing normalized vectors N_i, we can write $\tilde{y}_i$ as

$$\tilde{y}_i = \frac{1}{Vol} W_i N_i, \quad N_i \cdot N_i = t_i, \; W_i > 0 \tag{3.387}$$

where $t_i = \pm 1$ distinguish spacelike or timelike normals respectively. This leads to

$$B_{ij} = -\frac{1}{Vol} W_i W_j * (N_j \wedge N_i), \qquad \sum_i W_i N_i = 0 \tag{3.388}$$

In order to make the normal out-pointing, we redefine the normalized normal vectors N_i by

$$N_i^{\Delta} = -t_i N_i, \quad W_i^{\Delta} = -t_i W_i \qquad \sum_i W_i^{\Delta} N_i^{\Delta} = 0 \tag{3.389}$$

such that N_i^{Δ} are out-pointing.

3.A.3. Reconstruct geometry from non-degenerate critical points

We begin with the reconstruction of normals. Recall in critical point equations (3.256), normals N_e satisfying

$$\forall_{f\in t_e}\eta_{IJ}{N_e}^I B_f(v)^{JK}=0 \tag{3.390}$$

If there is another normal vector N satisfy the same condition for some edge e, easy to see we have

$$\forall_{f\in t_e}\quad B_f(v)\sim *(N\wedge N_e) \tag{3.391}$$

which means for an edge e, B_{ef} are proportional to each other. This clearly contrary to the fact that we have a non degenerate solution. Thus, for given bivectors which are the solution of the critical point equation, if we require a vector N satisfies

$$\forall_{f\in t_e}\eta_{IJ}N^I B_f(v)^{JK}=0 \tag{3.392}$$

for a edge tetrahedron t_e, we then have $N=\pm N_e$ after normalization. The condition (3.392) is sufficient and necessary.

Considering a 4-simplex σ_v at some vertex v, the critical point equation (3.256) can be written in short hand notation we introducing in (3.369) as

$$B_f(v)=B^{\{G\}}_{ij}=-B^G_{ji},\quad N_i\llcorner B^{\{G\}}_{ij}=0,\qquad \sum_j B^{\{G\}}_{ij}=0 \tag{3.393}$$

Now we give normalized vectors N_i satisfying non-degenerate condition. If we require the bivectors satisfy (3.393), they are uniquely determined up to a constant $\lambda\in R$

$$B'_{ij}=\lambda W_iW_j*(N_j\wedge N_i) \tag{3.394}$$

Here $W_i\in R$ are non zero and determined by

$$\sum_i W_iN_i=0 \tag{3.395}$$

The proof is stated first in [79] and later [85]. Note that the bivector B_{ij} is independent of the choice of signature of normal vectors N since the sign of W and N will change simultaneously. λ can be fixed up to a sign by the normalization of B'_{ij}

$$|B_f|^2=-4\gamma^2s_f^2=-4A_f^2 \tag{3.396}$$

Then it can be proved that non-degenerate geometric solution determines 4 simplex specified by bivectors B^Δ uniquely up to shift and inversion such that

$$B^\Delta_{ij}=rB^{\{G\}}_{ij} \tag{3.397}$$

where $r = \pm 1$ is the geometric Plebanski orientation. The construction can be done as follows. With given 5 normals N_i, we take any 5 planes orthogonal to N_i. With the non-degeneracy condition, they cut out a 4 simplex Δ' which is uniquely determined up to shifts and scaling. According to (3.388) and (3.394), bivectors of the reconstructed 4 simplex $B_{ij}^{\Delta'}$ related to B_{ij} as

$$B_{ij}^{\Delta'} = \lambda B_{ij}^{\{G\}} \tag{3.398}$$

Then the identity of the normalization will determines the scaling up to a sign

$$B_{ij}^{\{G\}} = rB_{ij}^{\Delta'} = -\frac{1}{Vol} r W_i^{\Delta} W_j^{\Delta} * (N_j^{\Delta} \wedge N_i^{\Delta}) \tag{3.399}$$

where Vol is the $4!$ volume of the 4-simplex.

Let us move to the boundary tetrahedron. Since G_e is a SO$(1,3)$ rotation, it action then keeps the shape of tetrahedrons. Thus the tetrahedron with bivectors $B_{ij} = *(v_{ij} \wedge u_i)$ has the same shape with the tetrahedron with face bivectors $B_{ij}^{\{G\}} = G_i * (v_{ij} \wedge u_i)$. For given v_{ij}, when the boundary data is non-degenerate, we can cut out a tetrahedron with planes perpendicular to v_{ij} in the 3 dimensional Minkowski space orthogonal to u. Clearly, the face bivectors of this tetrahedron satisfy

$$B_{ij} = \lambda'_{ij} * (v_{ij} \wedge u) \tag{3.400}$$

with λ'_{ij} arbitrary real number. However, from the closure constraint, we have

$$\sum_{j:j\neq i} B'_{ij} = *(\sum_{j:j\neq i} \lambda'_{ij} v_{ij}) \wedge u = 0 \tag{3.401}$$

Since $\forall_j \; v_{ij}.u = 0$, the above closure equation implies

$$\sum_{j:j\neq i} \lambda'_{ij} v_{ij} = 0 \tag{3.402}$$

which according closure with v_{ij} leads to

$$\exists_{\lambda} : \lambda'_{ij} = \lambda \tag{3.403}$$

Thus, for every edge e_i, there exists a tetrahedron determined uniquely up to inversion and translation with face bivectors

$$B_{ij} = r_i (v_{ij} \wedge u) \tag{3.404}$$

in the subspace perpendicular to N_i with $r_i = \pm 1$.

The edge lengths of the tetrahedron is then determined uniquely by v_{ij}. We denote ${l^i_{jk}}^2$ the signed square lengths of the edge between faces ij and ik. The length matching condition can be expressed as

$$l_{(ijk)}^2 := {l_{jk}^i}^2 = {l_{ik}^j}^2 = {l_{ij}^k}^2 \tag{3.405}$$

The non-degenerate solution exists if and only if the lengths satisfy length matching condition. In case when length matching condition is satisfied, we can write $l_{(ijk)}^2$ using the missing indices different from i,j,k as $l_{(ml)}^2$, with this notation, one introduce lengths Gram matrix of the 4 simplex

$$G^l = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & l_{01}^2 & \cdots & l_{04}^2 \\ 1 & l_{10}^2 & 0 & \cdots & l_{24}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & l_{40}^2 & l_{41}^2 & \cdots & 0 \end{pmatrix} \tag{3.406}$$

The signature of G^l corresponds to the signature of reconstructed 4 simplex. We denote the signature as (p,q). Based on G^l is degenerate or not, we have

— If G^l is non degenerate, then there exist a unique up to rotation, shift and reflection non degenerate 4 simplex with signature (p,q). There are two non-equivalent 4 simplex up to rotations and shift. The normals of two reconstructed 4 simplices $\{N_i\}$ and $\{N_i'\}$ are related by

$$N_i' = (-1)^{s_i} G N_i = G I^{s_i} N_i \tag{3.407}$$

— If G^l is degenerate, then there exist a unique up to rotation and shift degenerate 4 simplex with signature (p,q). The 4 volume in this case is 0.

The signature here is related to the signature of boundary tetrahedron. For all boundary tetrahedra being timelike, the possible signatures are Lorentzian $(-+++)$, split $(-++-)$ or degenerate $(-++0)$. For all boundary tetrahedra being spacelike, the possible signatures are Lorentzian $(-+++)$, Euclidean $(++++)$ or degenerate $(0+++)$. For boundary data contains both spacelike and timelike tetrahedra, the only possible reconstructed 4 simplex is in Lorentzian signature $(-+++)$.

3.A.4. Gauge equivalent class of solutions

Suppose we have a non-degenerate geometric boundary data and the 4 volume is non-degenerate, then we can reconstruct geometric non-degenerate 4-simplex up to orthogonal transformations. Suppose we have this reconstructed 4-simplex with geometric bivectors B_{ij}^{Δ} with normals N_i^{Δ}. From these normals, we can introduce

$$v_{ij}^{\Delta} = -\frac{1}{Vol}\left(W_i^{\Delta} W_j^{\Delta} N_j^{\Delta} - \frac{W_i^{\Delta} W_j^{\Delta} N_i^{\Delta} \cdot N_j^{\Delta}}{(N_i^{\Delta})^2} N_i^{\Delta}\right) \tag{3.408}$$

Easy to check that $v_{ij}^{\Delta} \cdot N_i^{\Delta} = 0$ and $B_{ij}^{\Delta} = *(v_{ij}^{\Delta} \wedge N_i^{\Delta})$. Thus these are nothing else but normals of faces of the ith tetrahedron recovered from bivectors B_{ij}^{Δ}. Easy to check that we have

$$v_{ij}^{\Delta} \cdot v_{ik}^{\Delta} = v_{ij} \cdot v_{ik} \tag{3.409}$$

by the fact that $B_{ij}^{\Delta} \cdot B_{ik}^{\Delta} = B_{ij} \cdot B_{ik}$. We can introduce group elements $G_i^{\Delta} \in O$ for each i satisfy

$$G_i^{\Delta} u = N_i^{\Delta}, \qquad \forall_{j:j\neq i}\, G_i^{\Delta} v_{ij} = v_{ij}^{\Delta} \tag{3.410}$$

Note that there are only 4 independent conditions out of 5.

We would like compare these group elements G_i^{Δ} obtained from B_{ij}^{Δ} with G_i from critical point solution. From reconstruction of bivectors and normals, we know that

$$B_{ij}^{\Delta} = (-1)^s B_{ij}^{\{G\}}, \qquad N_i = (-1)^{s_i} N_i^{\Delta} \tag{3.411}$$

where $(-1)^s$ with $s \in \{0, 1\}$ and $s_i \in \{0, 1\}$. The condition leads to

$$\begin{aligned} &* (G_i v_{ij} \wedge N_i) = B_{ij}^{\{G\}} = (-1)^s B_{ij}^{\Delta} \\ &= (-1)^s * (v_{ij}^{\Delta} \wedge N_i^{\Delta}) = *((-1)^{s+s_i} v_{ij}^{\Delta} \wedge N_i) \end{aligned} \tag{3.412}$$

Since $N_i \cdot v_{ij}^{\Delta} = N_i \cdot G_i v_{ij} = 0$, we have

$$G_i v_{ij} = (-1)^{s+s_i} v_{ij}^{\Delta}, \qquad G_i N = (-1)^{s_i} N_i^{\Delta} \tag{3.413}$$

which implies

$$G_i = G_i^{\Delta} I^{s_i} (I R_N)^s \tag{3.414}$$

For $G_i \in$ SO, we have $\det G_i = 1$, then from (3.414)

$$\det G_i^{\Delta} = (-1)^s \tag{3.415}$$

Since there is only one reconstructed 4 simplex up to rotations from O, thus two G^{Δ} solutions are related by

$$G_i^{\Delta'} = G G_i^{\Delta}, \quad G \in O \tag{3.416}$$

which means

$$\forall_i \frac{\det G_i^{\Delta'}}{\det G_i^{\Delta}} = \det G \tag{3.417}$$

This condition reminds us to introduce an orientation matching condition for boundary data where the reconstructed 4 simplex have

$$\forall_i \ \det G_i^{\Delta} = r \qquad r \in \{-1, 1\} \tag{3.418}$$

We call the boundary data as the geometric boundary data if it satisfy the length matching condition and orientation matching condition.

After we choose reconstructed 4 simplex, we have fixed the value of s by

$$r = (-1)^s \tag{3.419}$$

and it is Plebanski orientation. However s_i is still arbitrary.

With (3.414) and (3.415), we can identify the geometric solution and reconstructed 4-simplices. Up to SO rotations, there are two reconstructed 4 simplices. The two classes of simplices solutions are related by reflection respect to any normalization 4 vector e_α

$$B_{ij}^{\tilde{G}} = R_{e_\alpha}(B_{ij}^{\{G\}}), \qquad s' = s + 1 \tag{3.420}$$

which means

$$\tilde{G}_i = R_{e_\alpha} G_i (I R_u) \in \mathrm{SO}(1,3) \tag{3.421}$$

With the gauge choice that $G_i \in \mathrm{SO}_+(1,3)$, we can rewrite (3.421) as

$$\tilde{G}_i = R_{e_0} I^{r_i} G_i R_u \tag{3.422}$$

such that $\tilde{G}_i \in \mathrm{SO}_+(1,3)$. It is direct to see $r_i = 0$ for u timelike and $r_i = 1$ for u spacelike.

3.A.5. Simplicial manifold with many simplices

The above interpretation and reconstruction are with in single 4-simplex case. Now we will generalize the result to simplicial manifold with many simplices. We will consider two neighboring 4 simplices where there corresponding center v and v' are connected by a dual edge $e = (v, v')$. For a short hand notation, we will use prime to represent the parallel transported bivector and normals from simplex with center v' to v, e.g. $N_i' = G_{vv'} N_i(v')$. We denote the edge $e = (v, v')$ as e_0.

Since $N_e(v) = G_{ve} u$ and $N_e(v') = G_{v'e} u$, we have $N_e(v) = G_{vv'} N_e(v')$ for $G = (v, v')$. From the reconstruction theorem, with (3.411), we have

$$N_0^\Delta = (-1)^{s_0 + s'_0} N'^\Delta_0 \tag{3.423}$$

From the parallel transport equation $X_f(v) = g_{vv'} X_f(v') g_{v'v}$, with the fact $\epsilon_{ef}(v) = -\epsilon_{ef}(v')$, we have

$$B_{0i}^{\{G\}} = -r(v) \frac{1}{Vol} W_i^\Delta W_0^\Delta * (N_i^\Delta \wedge N_0^\Delta) = r(v') \frac{1}{Vol'} W'^\Delta_i W'^\Delta_0 * (N'^\Delta_i \wedge N'^\Delta_0) \tag{3.424}$$

where B_{0i}^Δ is the geometric bivector corresponding to the triangle f dual to face determined by e, e_i, e_i'. Now similar to (3.408), we can define

$$v_{0i}^{\Delta}(v) = -\frac{1}{Vol}\left(W_0^{\Delta}(v)W_i^{\Delta}(v)N_i^{\Delta}(v) - \frac{W_0^{\Delta}(v)W_i^{\Delta}(v)N_0^{\Delta}(v)\cdot N_i^{\Delta}(v)}{(N_0^{\Delta}(v))^2}N_0^{\Delta}(v)\right) \tag{3.425}$$

which satisfies $v_{0i}^{\Delta}(v)\cdot N_0^{\Delta}(v) = 0$. The geometrical group elements $\Omega_{vv'}^{\Delta} \in O(1,3)$ is defined from

$$v_{0i}^{\Delta}(v) = \Omega_{vv'}^{\Delta} v_{0i}^{\Delta}(v'), \quad N_0^{\Delta}(v) = \Omega_{vv'}^{\Delta} N_0^{\Delta}(v') \tag{3.426}$$

(3.424) now reads

$$B_{0i}^{\{G\}} = r(v) * (v_{0i}^{\Delta}(v) \wedge N_0^{\Delta}(v)) = -r(v') * (G_{vv'} v_{0i}^{\Delta}(v') \wedge G_{vv'} N_0^{\Delta}(v')) \tag{3.427}$$

From (3.423) and (3.427), with the fact that, $v_{0i}^{\Delta}(v)\cdot N_0^{\Delta}(v) = G_{vv'}v_{0i}^{\Delta}(v')\cdot G_{vv'}N_0^{\Delta}(v') = 0$, we have

$$v_{0i}^{\Delta}(v) = -(-1)^{s_0+s_0'} r(v) r(v') G_{vv'} v_{0i}^{\Delta}(v'), \quad N_0^{\Delta}(v) = (-1)^{s_0+s_0'} G_{vv'} N_0^{\Delta}(v') \tag{3.428}$$

Compare with (3.426),

$$\Omega_{vv'}^{\Delta} = G_{vv'} I I^{s_0+s_0'} (I R_{N_0(v')})^{s+s'}, \qquad \det \Omega_{vv'}^{\Delta} = (-1)^{s+s'} \tag{3.429}$$

where s and s' is determined by $(-1)^s = r(v)$ and $(-1)^{s'} = r(v')$. Note that, from the fact $N_0(v') = G_0(v')u = I^{s_0'} N_0^{\Delta}(v')$, and $R_N = G R_u G^{-1}$, we have $R_{N_0^{\Delta}} = R_{N_0}$. One can check that the (3.429) can be written as

$$\Omega_{vv'}^{\Delta} = I I^{s_0+s_0'} I^{s+s'} G_{ve} R_u^{s+s'} G_{ev'} = I G_{ve}^{\Delta} G_{ev'}^{\Delta} \tag{3.430}$$

which coincide with the geometric solution for single simplex. Note that, after fixing a pair of compatible values of s and s', another pair of compatible values are given by $s+1$ and $s'+1$ due to the common tetrahedron t_e shared by two 4 simplices. This is nothing else but reflecting simtounesly every 4 simplex connects with each other. Then according to (3.421), these two possible non gauge equivalent solutions are related by

$$\tilde{G}_f = \begin{cases} R_{u_e} G_f(e) R_{u_e} & \text{internal faces} \\ I^{r_{e1}+r_{e0}} R_{u_{e1}} G_f(e_1, e_0) R_{u_{e0}} & \text{boundary faces} \end{cases} \tag{3.431}$$

where $G_f = \prod_{v\subset\partial f} G_{e'v} G_{ve}$ is the face holonomy.

For a simplicial manifold, we will introduce the consistent orientation. For two 4 simplex σ_v and $\sigma_{v'}$ share a same tetrahedron t_e, we say they are consistently oriented if their orientation satisfies $[p_0, p_1, p_2, p_3, p_4]$ and $-[p_0, p_1, p_2, p_3, p_4]$. Therefore we have $\epsilon^{01234}(v) = -\epsilon^{01234}(v')$ for the orientation in (3.379). The orientated volume then contains a minus sign in V'.

From (3.423) and (3.424), we have

$$N'^{\Delta}_i = -(-1)^{s_0+s'_0} r(v) r(v') \frac{W^{\Delta}_i W^{\Delta}_0 Vol'}{W'^{\Delta}_i W'^{\Delta}_0 Vol} N^{\Delta}_i + a_i N^{\Delta}_0 \tag{3.432}$$

where a_i are some coefficients s.t. $\sum_i W'^{\Delta}_i N'^{\Delta}_i = -W'^{\Delta}_0 N'^{\Delta}_0$. We introduce $\tilde{y}$ where $\tilde{y}_i = \frac{1}{Vol} W^{\Delta}_i N^{\Delta}_i$, then

$$B^G_{0i} = -r(v) Vol * (\tilde{y}_i \wedge \tilde{y}_0), \quad \tilde{y}'_i = -(-1)^{s_0+s'_0} r(v) r(v') \frac{W^{\Delta}_0}{W'^{\Delta}_0} \tilde{y}_i + \tilde{a}_i \tilde{y}_0 \tag{3.433}$$

where $\tilde{a}_i$ are coefficients s.t. $\sum_i \tilde{y}_i = -\tilde{y}_0$. We then have

$$-\frac{1}{V'} = \det\left(\tilde{y}'_0, \tilde{y}'_1, \tilde{y}'_2, \tilde{y}'_3\right) = (-r(v) r(v'))^3 \left(\frac{W^{\Delta}_0}{W'^{\Delta}_0}\right)^2 \frac{Vol}{Vol'} \det(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = -\tilde{r}(v) \tilde{r}(v') \left(\frac{W^{\Delta}_0}{W'^{\Delta}_0}\right)^2 \tag{3.434}$$

where we define $\tilde{r}(v) = r(v)\mathrm{sgn}(V(v))$. The equation results in $\tilde{r}(v) = \tilde{r}(v') = \tilde{r}$. Therefore $\tilde{r} = \mathrm{sgn}(V(v)) r(v)$ is a global sign on the entire triangulation after we choose compatible orientation. The equation also implies $|W^{\Delta}_0| = |W'^{\Delta}_0|$. With the fact that normal vector N^{Δ}_0 and N'^{Δ}_0 are in the same type (spacelike or timelike), we have $W^{\Delta}_0 = W'^{\Delta}_0$. Thus (3.432) leads to

$$N'^{\Delta}_i = -(-1)^{s_0+s'_0} \mathrm{sgn}(VV') \frac{W^{\Delta}_i W^{\Delta}_0 Vol'}{W'^{\Delta}_i W'^{\Delta}_0 Vol} N^{\Delta}_i + a_i N^{\Delta}_0 = \mu_e N^{\Delta}_i + a_i N^{\Delta}_0 \tag{3.435}$$

where we define a sign factor $\mu_e := -(-1)^{s_0+s'_0}\mathrm{sgn}(VV')$. One can see that, for a edge E_{lm} in the tetrahedron t_e shared by σ_v and $\sigma_{v'}$, we have

$$E'_{lm} = V' \epsilon_{lmjk}(v') * (\tilde{y}'^j \wedge \tilde{y}'^k \wedge \tilde{y}'^0) = \mu_e V \epsilon_{lmjk}(v) * (\tilde{y}^j \wedge \tilde{y}^k \wedge \tilde{y}^0) = \mu_e E_{lm} \tag{3.436}$$

The equation thus implies the co-frame vectors on all edges of tetrahedron t_e at neighboring vertices v and v' are related by

$$E_l(v) = \mu_e G_{vv'} E_l(v') \tag{3.437}$$

Since $E_l(v') \perp N_0(v')$, the relation is a direct consequence of (3.429) with the fact $\tilde{r}(v) = \tilde{r}(v') = \tilde{r}$. This relation shows that, the vectors E in a tetrahedron shared by two 4 simplices σ_v and $\sigma_{v'}$ satisfies

$$g_{l_1 l_2} := \eta_{IJ} E^I_{l_1}(v) E^J_{l_2}(v) = \eta_{IJ} E^I_{l_1}(v') E^J_{l_2}(v') \tag{3.438}$$

where $g_{l_1 l_2}$ is the induced metric on the tetrahedron and it is independent of v. If the oriented volume of these two neighboring 4-simplices are come with the same signature, i.e. $\mathrm{sgn}(V(v)) = \mathrm{sgn}(V(v'))$, We can associated a reference frame in each 4 simplex σ_v and the frame transformation is given by $\Omega_{vv'} = \mu_e G_{vv'} \in$ SO$(1,3)$. The matrix $\Omega_{e=(v,v')}$ is a discrete spin connection compatible with the co-frame then. Note that, since $\tilde{r}(v) = r(v)\mathrm{sgn}(V(v))$ is a global sign, globally orienting $\mathrm{sgn}(V(v))$ will make $r = r(v)$ a global orientation on the dual face.

Let us go back to the original geometric rotation $\Omega^{\Delta}_{vv'}$. Suppose we orient consistently all pairs of 4 simplices on the simplicial complex $\mathcal{K}$. We then choose a sub-complex with boundary such that, with in it the oriented volume $\mathrm{sgn}(V)$ is a constant. Then for the holonomy along edges of an internal face, we have

$$\Omega^{\Delta}_f(v) = \Omega^{\Delta}_{v_0v_n}\Omega^{\Delta}_{v_nv_{n-1}}\cdots\Omega^{\Delta}_{v_1v_0} = I^n I^{s_{0n}+s_{n,n-1}+\cdots+s_{10}} G_{v_0v_n}G_{v_nv_{n-1}}\cdots G_{v_1v_0} = \mu_e G_f(v) \tag{3.439}$$

while for a boundary face,

$$\Omega^{\Delta}_f(v_n,v_0) = \Omega_{v_nv_{n-1}}\cdots\Omega_{v_1v_0} = I^n I^{s_{n,n-1}+\cdots+s_{10}} G_{v_0v_n}G_{v_nv_{n-1}}\cdots G_{v_1v_0} = \mu_e G_f(v_n,v_0) \tag{3.440}$$

where n is the number of internal edges belong to the face f. Here $\mu_e = I^n\prod_{e\in f} I^{s_e} = \pm 1$, $s_{e=(v,v')} = s_{ve} + s_{ve'}$ is independent from orientation.

Suppose the edges of the triangle due to face f is given by $E_{l1}(v)$ and $E_{l2}(v)$, then from (3.437) and (3.439-3.440), we have

$$G_f(v)E_l(v) = \mu_e E_l(v), \text{or} \qquad G_f(v_n,v_0)E_l(v_0) = \mu_e E_l(v_n) \tag{3.441}$$

For the normals $N_0(v)$ and $N_1(v)$ which othrognal to the triangle due to f, from (3.435) and (3.439-3.440), we have

$$G_f(v)N_1(v)^{\Delta} = aN_0(v)^{\Delta} + bN_1(v)^{\Delta}, G_f N_1(v)\cdot E_{l1}(v) = G_f N_1(v)\cdot E_{l2}(v) = 0, \tag{3.442}$$

For boundary faces with boundary tetrahedron t_{e_n} and t_{e_0}, similarly, we have

$$G_f(v_n,v_0)N_{e0}(v_0)\cdot E_{l1}(v_n) = G_f(v_n,v_0)N_{e0}(v_0)\cdot E_{l2}(v_n) = 0 \tag{3.443}$$

3.A.6. Flipped signature solution and vector geometry

Now let us consider degenerate case, where the 4 volume is 0 and G_i can be gauge fixed to its subgroup $G_i \in SO(1,2)$ for timelike tetrahedron. In this case, the 4-normals of boundary tetrahedra are then gauge fix to be $\forall_i\ N_i = u$. We can introduce a auxiliary space $M^{4'}$ with metric $g'_{\mu\nu}$ from M^4 by flipping the norm of u

$$g'_{\mu\nu} = g_{\mu\nu} - 2u_\mu u_\nu \tag{3.444}$$

where $g_{\mu\nu}$ is the metric in M^4. We will use prime to all the operations in $M^{4'}$. For the norm of u, we have

$$t = u\cdot u, \qquad t' = -t = u\cdot' u \tag{3.445}$$

Notice that for the subspace V orthogonal to u, the restriction of both scalar product coincide. Thus for vectors in V we can use both scalar product. The Hodge dual operation satisfies $*'^2 = -*^2 = t = -t'$.

For the subspace V, we can introduce maps $\Phi^\pm$

$$\Phi^\pm : \Lambda^2 M^{4'} \to V, \quad \Phi^\pm(B) = t'(\pm B - t' *' B) \cdot' u = (\mp B + *'B) \cdot' u \tag{3.446}$$

where B is a bivector in $M^{4'}$. Clear for a vector $v \in V$, we have

$$\Phi^\pm(*'(v \wedge u)) = v \tag{3.447}$$

The map $\Phi^\pm$ naturally induce a map from $G \in \mathrm{SO}(2,2)$ to the subgroup $h \in \mathrm{SO}(1,2)$, which defined by

$$\Phi^\pm(GBG^{-1}) = \Phi^\pm(G)\Phi^\pm(B) \tag{3.448}$$

where

$$\Phi^\pm(G) \in O(V) \tag{3.449}$$

Easy to see when $G = h \in \mathrm{SO}(1,2)$, we have $\Phi^\pm(h) = h$. And one can further prove that the condition is sufficient and necessary as shown in [85].

Clearly for given bivectors $B_{ij}^{\{G\}} = G_i * (v_{ij} \wedge u)$ in M', if $B_{ij}^{\{G\}} = -B_{ji}^{\{G\}}$, we have

$$v_{ij}^{\{G\}\pm} = -v_{ji}^{\{G\}\pm}, \;\; v_{ij}^{\{G\}\pm} = \Phi^\pm(G)v_{ij} = \Phi^\pm(B_{ij}^{\{G\}}) \tag{3.450}$$

and the closure $\sum_i B_{ij}^g = 0$ leads to

$$\sum_i v_{ij}^{\{G\}\pm} = 0 \tag{3.451}$$

One can prove the condition is necessary. In other words, if we have $g_i^\pm$ such that $v_{ij}^{\{G\}\pm} = -v_{ji}^{\{G\}\pm}$, we can always build unique $G_i \in SO(M')$ (up to I^{s_i}) which constitute a $SO(M')$ solution.

In summary we see that there is an 1-1 correspondence between

— pair of two non-gauge equivalent vector geometries,

— geometric $SO(M')$ non-degenerate solution.

The two vector geometries are obtained from $SO(M')$ solutions $\{g_{ve}\}$ as $g_{ve}^\pm = \Phi^\pm(g_{ve})$. This is the flipped signature case for a Gram matrix with given geometric boundary data. For example, with all boundary tetrahedra timelike, the signature of reconstructed non-degenerate 4 simplex is split $(-++-)$.

From the reconstruction for non-degenerate solutions, we have the orientation matching condition for the geometric group elements $G^{\Delta\pm} \in O(V)$ where

$$G_i^{\Delta\pm} v_{ij} = v_{ij}^{\Delta\pm}, \qquad v_{ij}^{\Delta\pm} = \Phi^\pm(B_{ij}^\Delta) \tag{3.452}$$

One can show that, in flipped signature case, this condition becomes

$$\det G_{ve}^{\Delta} = \det G_{ve}^{\Delta\pm} \tag{3.453}$$

Since the critical point solutions are in 1-1 correspondence with reconstructed 4 simplices up to reflection and shift. As a direct result from (3.421), for non-degenerate boundary data satisfying length matching condition and orientation matching condition, there are two gauge inequivalent solutions corresponding to reflected 4 simplices which are related by

$$\tilde{G} = R_u G R_u \tag{3.454}$$

where $\tilde{G}$ and G represent two gauge equivalent series. Two non-equivalent geometric $SO(M')$ non-degenerate solutions then satisfy

$$\Phi^{\pm}(\tilde{G}) = \Phi^{\pm}(R_u G R_u) = \Phi^{\mp}(g) \tag{3.455}$$

Finally, when the $SO(M')$ solution is degenerate, we can assume $N_i = u$ by gauge transformations. In this case, we see $\Phi^{+}(G) = \Phi^{-}(G) = h$. Thus the vector geometries are gauge equivalent. The inverse is also true. When the vector geometries are gauge equivalent, we have $\Phi^{+}(G) = \Phi^{-}(G)$, which means there exists G_i (uniquely up to gauge transformations) such that after gauge transformations $N_i = G_i u = u$. This corresponds to the degenerate reconstructed 4 simplex with zero 4-volume.

3.B. Derivation of rotation with dihedral angles

In this appendix, we prove the following equation

$$R_{N_i} R_{N_j} = \Omega_{ij} = \mathrm{e}^{2\theta_{ij} \frac{N_i \wedge N_j}{|N_i \wedge N_j|}} \tag{3.456}$$

which is used in Sec. 3.7. For two normalized spacelike vector N_i, N_j, $N_i^I N_{iI} = N_j^J N_{jJ} = 1$, compatible with (3.292) and (3.293), we have

$$N_i^I N_{jI} = \cos\theta_{ij}, \tag{3.457}$$

$$|N_j \wedge N_i|^2 = -|*N_j \wedge N_i|^2 = \sin^2(\theta_{ij}) \tag{3.458}$$

For N_i, N_j are timelike and the signature of plane span by $N_i \wedge N_j$ is mixed in flipped signature case, we have

$$N_i^I N_{jI} = \cosh\theta_{ij}, \tag{3.459}$$

$$|N_j \wedge N_i|^2 = |*' N_j \wedge N_i|^2 = -\sinh^2(\theta_{ij}) \tag{3.460}$$

Now from

$$(R_N)^I_J = I - \frac{2N^I N_J}{N \cdot N} = I - 2tN^I N_J \tag{3.461}$$

where we define $t := N^I N_I$. Easy to see for a vector v in $N_i \wedge N_j$ plane,

$$\begin{aligned} R_{N_i} R_{N_j} v &= (I - 2tN_i^K N_{iI})(I - 2tN_j^I N_{jJ})v^J \\ &= v - 2t(N_i \cdot v)N_i - 2t(N_j \cdot v)N_j + 4(N_i \cdot N_j)(N_j \cdot v)N_i \end{aligned} \tag{3.462}$$

which leads to

$$R_{N_i} R_{N_j} - R_{N_j} R_{N_i} = 4(N_i \cdot N_j) N_i \wedge N_j \tag{3.463}$$

$$\text{Tr}(R_{N_i} R_{N_j}) = 4(N_i \cdot N_j)^2 - 2 \tag{3.464}$$

Let us introduce spacetime rotations $\Omega \in SO_{\pm}(1,3)$. For connected components in Lorentzian group, two group elements Ω and Ω' are equal is they satisfy

$$\Omega - \Omega^{-1} = \Omega' - \Omega'^{-1}, \qquad \text{Tr}(\Omega) = \text{Tr}(\Omega') \tag{3.465}$$

The space rotation can be written using bivectors as

$$\Omega_{ij} = \mathrm{e}^{2\theta_{ij} \frac{N_i \wedge N_j}{|N_i \wedge N_j|}} = \cos(2\theta_{ij}) + \sin(2\theta_{ij}) \frac{N_i \wedge N_j}{|N_i \wedge N_j|} \tag{3.466}$$

and for spacelike normal vectors we have

$$\Omega_{ij} - \Omega_{ji} = 2\sin(2\theta_{ij}) \frac{N_i \wedge N_j}{|N_i \wedge N_j|} = 4(N_i \cdot N_j)(N_i \wedge N_j) \tag{3.467}$$

$$\text{Tr}(\Omega_{ij}) = 2\cos(2\theta_{ij}) = 2(2\cos^2(\theta_{ij}) - 1) = 4(N_i \cdot N_j)^2 - 2 \tag{3.468}$$

while for timelike normal vectors span a mixed signature plane, Ω is a boost,

$$\Omega_{ij} = \mathrm{e}^{2\theta_{ij} \frac{N_i \wedge N_j}{|N_i \wedge N_j|}} = \cosh(2\theta_{ij}) + \sinh(2\theta_{ij}) \frac{N_i \wedge N_j}{|N_i \wedge N_j|} \tag{3.469}$$

with

$$\Omega_{ij} - \Omega_{ji} = 2\sinh(2\theta_{ij}) \frac{N_i \wedge N_j}{|N_i \wedge N_j|} = 4(N_i \cdot N_j)(N_i \wedge N_j) \tag{3.470}$$

$$\text{Tr}(\Omega_{ij}) = 2\cosh(2\theta_{ij}) = 2(2\cosh^2(\theta_{ij}) - 1) = 4(N_i \cdot N_j)^2 - 2 \tag{3.471}$$

Notice that here $|N_i \wedge N_j|$ is defined as

$$|N_i \wedge N_j| = \sqrt{||N_i \wedge N_j|^2|}, \tag{3.472}$$

Thus in both case we have

$$R_{N_i}R_{N_j} = \Omega_{ij} = \mathrm{e}^{2\theta_{ij}\frac{N_i\wedge N_j}{|N_i\wedge N_j|}} \tag{3.473}$$

where θ_{ij} is angle between normals and related to the dihedral angle by (3.292) and (3.293).

3.C. Fix the ambiguity in the action

In this appendix we show how to choose the $\mathrm{SL}(2,\mathbb{C})$ lift to fix the ambiguity in the action. Note that here we only fix the ambiguity for single 4-simplex σ_v with boundary data, where the deficit angle $\Theta_f = \theta_f$ is the angle between normals. The ambiguity (in one 4 simplex σ_v with boundary) which due to odd n_f can be expressed as

$$\Delta S - \Delta S^{\Delta} = \mathrm{i}r \sum_{f:n_f\ odd} \begin{matrix} \Delta\phi - \Theta_f & \text{non degenerate case} \\ \Delta\phi & \text{split signature case} \end{matrix} \tag{3.474}$$

The procedure we use here is an extension of the one used for spacelike triangles in [85].

3.C.0.1. non-degenerate case

Suppose we have a non-degenerate solutions $\{G^0_{ve} \in SO(1,3)\}$ with normals v^0_{ef} of triangles of non-degenerate boundary tetrahedra. The area of these triangles is given by spins $\gamma s^0_f = \frac{n^0_f}{2}$. Define the following continuos path

$$G_{ve}(t), \quad v_{ef}(t), \quad u(t) = u = (0,0,0,1)^T, \tag{3.475}$$

where $\forall e G^0_{ve} = G_{ve}(0), v^0_{ef} = v_{ef}(0)$. Such that
- $\forall t \in [0,1]$, $\{G_{ve}(t)\}$ is a solution of critical point equations with boundary data where the normals of triangles of boundary tetrahedra are $v_{ef}(t)$,
- $\forall t \neq 1$ boundary data is non-degenerate, and $v_{ef}(1) \neq 0$,
- $\forall t \neq 1$ solution $\{G_{ve}(t)\}$ is non-degenerate,
- for $t = 1$, pair of solutions $\{G_{ve}(t)\}$ and $\{\tilde{g}_{ve}(t) = R_{e_\alpha} g_{ve}(t) R_u\}$ are gauge equivalent.

In this path, the function

$$f(t) = \sum_{f:n_f\ odd} \Delta\phi_{eve'f}(t) - r\Theta_f(t) \mod 2\pi \tag{3.476}$$

takes values in $\{0,\pi\}$ and changing continuously with the phase the difference from stationary points determined by $\{G_{ve}(t)\}$ and $\{\tilde{G}_{ve}(t) = R_{e_\alpha} G_{ve}(t) R_u\}$.

Thus $f(t)$ is a constant. Since at $t = 1$, we have two geometric solutions are gauge equivalent to each other, which means the lifts $g_{ve}, \tilde{g}_{ve}$ of solutions satisfy

$$\forall_e \; \tilde{g}_{ve} = (-1)^{r_{ve}} g g_{ve}, \qquad r_{ve} = \{0, 1\} \tag{3.477}$$

From (3.313),

$$(-1)^{r_{ve}+r_{ve'}} = g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1} g_{e'v} = \mathrm{e}^{-2\Delta\theta_{e'vef} X_f + 2\mathrm{i}\Delta\phi_{e'vef} X_f} \tag{3.478}$$

which leads to $\Delta\phi_{eve'f}(1) = (r_{ve} + r_{ve'})\pi \mod 2\pi$ since we have $(2X_f)^2 = 1$. We shall consider a subgraph of spin network which contains those odd n links. The subgraph has even valence nodes. Thus we can decompose into Euler cycles. In those cycles every link of odd n will appears exactly once. For a Euler cycle consisting edges with odd n, every edge will be counted twice, thus we have

$$\sum_{e \in cycle} \Delta\phi_{eve'f}(1) = \sum_{e \in cycle} 2 r_{ve}\pi = 0 \mod 2\pi \tag{3.479}$$

Also, from the fact that two geometrical solution is gauge equivalent $\forall_e \; \tilde{G}_{ve} = G G_{ve}$, we have $R_{N_e} R_{N_{e'}} = G_{ve}(\tilde{G}_{e'v}\tilde{G}_{ve})^{-1} G_{e'v} = 1$, thus

$$\Theta_f(1) = \tilde{r}_f \pi \mod 2\pi, \qquad \tilde{r}_f = \tilde{r}_{ve} + \tilde{r}_{ve'} \in \{0, 1\} \; . \tag{3.480}$$

which can be fixed again using Euler cycles as for $\Delta\phi$.

The path can be achieved by deforming solutions in the following way: First choose a timelike plane with simple normalized bivector V at some vertex v satisfies

$$\forall_f V \wedge *B_f \neq 0 \; . \tag{3.481}$$

The path is made by contracting the two directions in $*V$, and we donate the $t = 1$ as the limit for contracting directions to 0. From above condition we have $\lim_{t \to 1} B_f$ exist and keep nonzero. The dual action of the shrinking on geometric normal vectors N^{Δ} also have a limit which is their normalized components lying in $*V$ plane (after normalization). By suitable definition of boundary data, we can assume $G_{ve}(1) = \lim_{\to 1} G_{ve}(t)$ exist. Now we end up with a highly degenerate 4-simplex which contained in a 2d plane and all bivectors are proportional to V.

3.C.0.2. split signature case

The treatment concerning degenerate solutions following the similar method. Start form the non-degenerate boundary data, where normals of triangles of boundary tetrahedra are given by v^0_{ef} and area of these triangles are related to spins $n_f/2$. Suppose from these boundary data, we can reconstruct a non-degenerate 4-simplex in flipped signature space M'. In this case, we have two non-gauge equivalent solutions $\{g_{ve}^{\pm}\}$. We define the following path

$$g_{ve}^{\pm}(t), \quad v_{ef}(t), \quad u(t) = u = (0,0,0,1)^T, \tag{3.482}$$

where $\forall e g_{ve}^{0\pm} = g_{ve}^{\pm}(0), v_{ef}^{0} = v_{ef}(0)$. The path satisfies

- $\forall t \in [0,1]$, $\{g_{ve}^{\pm}(t)\}$ are solutions of critical point equation with boundary data given by $v_{ef}(t)$,
- $\forall t \in [0,1]$ boundary data is non-degenerate, e.g. the boundary tetrahedron is non-degenerate,
- $\forall t \neq 1$ solutions $\{g_{ve}^{\pm}\}$ are non-gauge equivalent thus we have a non-degenerate reconstructed 4-simplex in M'
- for $t = 1$, the reconstructed 4-simplex is degenerate in M'.

Now the constant function $f(t) \in \{0, \pi\}$ reads

$$f(t) = \sum_{f: n_f \ odd} \Delta\phi_{eve'f}(t) \mod 2\pi \tag{3.483}$$

Following the same argument in non-degenerate case, we have for the lifts

$$g_{ve}^{+}(1) = (-1)^{r_{ve}} g_{ve}^{-}(1) \tag{3.484}$$

Based on the same consideration using Euler cycles, we have

$$f(1) = \sum_{f: n_f \ odd} \Delta\phi_{eve'f}(t) = 0 \mod 2\pi \tag{3.485}$$

Thus we have

$$\Delta S^0 - \Delta S^{\Delta 0} = 0 \mod 2\pi \tag{3.486}$$

The path is built by the following way: We choose a spacelike normal such that, in flipped signature space

$$\forall_f N \wedge B_f \neq 0. \tag{3.487}$$

The path is then made by contracting in the direction of N in the flipped space M'. The contraction leads to a continuos path of non-degenerate solutions in M' until $t = 1$ where the 4-simplex is degenerate.

Part II.

Effective Dynamics of Cosmology and Black Hole Models in Loop Quantum Gravity

4. Introduction and Overview

Sommaire

4.1. Introduction

Understanding the very early universe and the centre of black holes remains a fascinating open question in cosmology. In the context of general relativity, an expanding universe containing "standard" matter fields (which satisfy the null energy condition) is generically associated with an initial singularity, where the space-time curvature becomes infinite. Similar singularities also present inside the black holes. In this sense, classical general relativity is a incomplete theory and fails to explain these phenomenal. When the value of the curvature approaches the Planck scale, quantum gravity effects are expected to become physically important and could prevent the formation of space-time singularities. This is exactly what happens in the context of loop quantum cosmology (LQC) [30–33] where quantum gravity effects are repulsive, in opposition to attractive classical gravity, and lead to a bouncing universe.

However, quantizing gravity might not be necessary to resolve the initial cosmological singularity and one could envisage modifications of gravity at high curvature so that the singularity in general relativity is replaced by a bounce in a modified gravity theory (see [4, 165] for recent reviews). Of course, these two approaches to avoid the singularity could be two sides of the same coin if the classical equations derived from modified gravity can be interpreted as an effective description of the quantum behaviour.

Scalar-tensor theories provide a very large class of models for modified gravity theories. Among these, higher-order scalar-tensor theories, whose Lagrangians contain not only first order but also second order derivatives of the scalar field, have attracted a lot of attention lately. Allowing for higher-order time derivatives in the Lagrangian is potentially dangerous as this could lead to higher-order equations of motion which may require extra initial conditions and thus introduce an additional degree of freedom, known as the Ostrogradsky ghost, because it leads to an Ostrogradsky instability [166, 167]. It is however possible to find higher-order scalar-tensor theories that contain a single scalar degree of freedom (in addition to the usual tensorial modes associated with gravity) by imposing some restrictions on the initial Lagrangian. Initially, it was believed that a theory of this type was necessarily characterized by second order Euler-Lagrange equations, thus pointing to Horndeski theories [168] (see also [169]). In fact, requiring second order equations of motion turns out to be restrictive and a much larger class of theories, dubbed Degenerate Higher-Order Scalar-Tensor (DHOST) theories, has been recently identified, showing that the absence of an extra unstable scalar mode is compatible with higher order Euler-Lagrange equations [170–177]. These theories could provide an interesting arena to construct models for the early universe, as well as late-time cosmology. Depending on whether corrections to general relativity appear at high-curvature scales and/or at large scales and low curvatures, the second order derivatives of the scalar field then correspond to ultraviolet and/or infrared corrections, and in particular high-curvature corrections can in some cases act as an ultraviolet cutoff like those that arise in a number of approaches to quantum gravity.

Among DHOST theories, one can distinguish a special family of scalar-tensor theories that share properties similar to those of mimetic gravity. Mimetic gravity is a higher order scalar-tensor theory which admits, in addition to the usual invariance under diffeomorphisms, a conformal invariance (which can be generalized to a conformal-disformal invariance). Mimetic gravity was introduced in [178] as a model for dark matter (see also [179]). More recently, this model has been shown to admit (as a number of other scalar-tensor theories) non-singular bouncing cosmologies [66, 180–183].

In the present work, we concentrate on the general mimetic gravity theories. We note that the special Lagrangian proposed in [66] corresponding classical equations of motion for a cosmological background are exactly the same as the so-called effective equations of loop quantum cosmology. This result suggests that it may be possible to describe loop quantum gravity at an effective level in some appropriate regimes as a higher-derivative scalar-tensor theory. The main purpose of this section is to highlight this relation between mimetic gravity and loop quantum cosmology and generalize the result also to loop quantum black holes. The relation nonetheless provides a proposal for an effective description of loop quantum gravity in terms of higher-order scalar-tensor theories. Such an effective description is potentially very interesting, especially as it may give important insights into the relation between the quantization of gravity à la loop and the usual perturbative quantization techniques. Moreover, since the effective quantum corrections from Mimetic theory always appears in a covariant manner, while polymer loop quantum black hole is difficult to keep the spatial covariance (in a non-homogeneous model). The difference between the mimetic effective corrections and the polymer loop quantum black hole ones provides an interesting guide to understand the lack of covariance of polymer black hole models.

This part is organized as follows. In the following chapter, we give a short presentation of (degenerate) higher-order scalar-tensor theories and we present some basic properties of mimetic gravity, which can be seen as a particular example of these theories. we also provide a brief review of loop quantum cosmology and polymer black holes in Sec. 4.3. We then show in chapter 5 that there exists a family of DHOST mimetic actions $S[g_{\mu\nu}, \phi]$ which all reduce to $S[a, N]$ for homogeneous and isotropic space-time. These actions generalize the model proposed recently by Chamseddine and Mukhanov in [66] and can be viewed as a proposal for an effective description of loop quantum gravity. The result is then generalized to polymer black hole cases in chapter 6, where a class of scalar-tensor theories, belonging to the family of extended mimetic gravity whose dynamics reproduces the general shape of the effective corrections of spherically symmetric polymer models in the context of LQG [184] is exhibited, but in an undeformed covariant manner. Finally in chapter 7, an effective metric is found for a static interior BH geometry describing the trapped region, in the framework of effective spherically symmetric polymer models.

4.2. Brief Review on Higher-Order Scalar-Tensor Theories

In this section, we briefly review the main aspects of degenerate higher-order scalar-tensor (DHOST) theories . Their Lagrangian depends on a metric $g_{\mu\nu}$ and on a scalar field ϕ, including its first and second derivatives, $\nabla_\mu \phi \equiv \phi_\mu$ and $\nabla_\mu \nabla_\nu \phi \equiv \phi_{\mu\nu}$:

$$S[g_{\mu\nu}, \phi] \;=\; \int d^4x \, \sqrt{-g} \, \mathcal{L}(\phi, \phi_\mu, \phi_{\mu\nu}; g_{\mu\nu}) \, . \tag{4.1}$$

In general, such theories propagate an extra degree of freedom in addition to the usual scalar mode and the two tensor modes of the metric (assuming a linear dependence on the Riemann tensor [a]), see 4.5 below for an explicit example of such an action. This additional degree of freedom leads to instabilities (at least at the quantum level) and is known as an Ostrogradsky ghost [166, 167].

However, it is possible to find higher-order scalar-tensor theories that do not contain any Ostrogradsky ghost by imposing appropriate degeneracy conditions on the Lagrangian, thus defining DHOST theories. DHOST theories whose Lagrangian is at most cubic in $\phi_{\mu\nu}$ have already been classified [177]. In principle, this classification could be generalized to higher powers of $\phi_{\mu\nu}$.

Below, we first recall the main properties of DHOST theories and then concentrate on mimetic theories, which form a special family within DHOST theories.

4.2.1. Evading the Ostrogradsky Instability

Starting with a Lagrangian with second derivatives of ϕ is unusual in physics. Generically, the corresponding equation of motion for ϕ is fourth order in time derivatives, which means that more than two initial conditions (per space point) are required to fully specify the evolution. This signals the presence of an extra degree of freedom in the theory.

However, there exist special Lagrangians with higher-order derivative terms for which the Euler-Lagrange equations remain second order. This is precisely the property verified by Horndeski theories in the context of scalar-tensor theories. It is even possible to find Lagrangians leading to third or fourth order Euler-Lagrange equations but without the dangerous extra scalar mode. Examples of scalar-tensor theories of this type are the so-called beyond Horndeski theories, later encompassed in the DHOST theories. All these models are degenerate, a property which can also be seen in other contexts [185–188].

By construction, DHOST theories satisfy some degeneracy conditions so that they contain at most one scalar degree of freedom. To implement this degeneracy, it is useful to work with a Hamiltonian formulation, based on the usual (3+1) ADM-decomposition of the metric on a space-time of the form $\Sigma \times \mathbb{R}$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + q_{ab}N^aN^b & q_{ab}N^b \\ q_{ab}N^a & q_{ab} \end{pmatrix}, \tag{4.2}$$

where q_{ab} is the induced metric on the space slice Σ, N is the lapse function and N^a the shift vector. In this framework, the action 4.1 explicitly depends on second time derivatives of the scalar field and takes the general form [b] (up to

a. If the Lagrangian is not linear in the Riemann tensor, then the theory can admit up to 8 degrees of freedom, most of them being unstable.

b. We do not consider theories which involve, after a (3+1)-decomposition, second time derivatives of the metric components which are not total derivatives. Such theories are expected to propagate Ostrograsky ghosts that cannot be removed.

boundary terms that we neglect)

$$S[g_{\mu\nu},\phi] \;=\; \int d^4x\, N\sqrt{q}\,\mathcal{L}(q_{ab},K_{ab},N,N^a;\phi,A_*,\dot{A}_*)\,, \tag{4.3}$$

with

$$K_{ab}\equiv\frac{1}{2N}\left(\dot{q}_{ab}-D_aN_b-D_bN_a\right)\,,\qquad A_*\equiv\frac{\dot{\phi}-N^a\partial_a\phi}{N}\,, \tag{4.4}$$

where D_a is the covariant derivative compatible with q_{ab}. For simplicity, we use the same notation for the Lagrangian densities $\mathcal{L}$ in the covariant 4.1 and the non-covariant 4.3 versions of the action, even though they are not strictly speaking the same function.

To perform the Hamiltonian analysis [173], it is convenient to use the auxiliary variable A_* as an independent variable, so that all second time derivatives of ϕ are absorbed in $\dot{A}_*$. This procedure thus introduces a new pair of variables, A_* and its conjugate momentum, which a priori describes an extra scalar degree of freedom. However, it is still possible that the theory propagates no more than one scalar degree of freedom if there exist constraints (in addition to the usual four constraints associated with space-time diffeomorphism invariance) so that the effective number of physical degrees of freedom is reduced. The existence of a primary constraint is equivalent to the requirement that the Hessian matrix of 4.3 (whose coefficients are the second derivatives of the action with respect to velocities of the fields) is degenerate. This property of degeneracy of the Hessian matrix has been used systematically to construct DHOST Lagrangians, initially with a quadratic dependence on $\phi_{\mu\nu}$ [172] and, more recently, with a cubic dependence [177]. Note that the primary constraint is usually of the second-class type and imposing its time conservation leads to a secondary constraint, which is also second-class. Both constraints thus eliminate the dangerous extra degree of freedom [173]. The special case where the primary constraint is first-class, signalling an additional local symmetry of the action, is seen in the mimetic models, which will be discussed in the next subsection.

All the DHOST theories that have been identified can be written in the form

$$\begin{aligned} S[g,\phi] \;=\; \int d^4x\,\sqrt{-g}\,\Big[&f_2(X,\phi)\,R+C_{(2)}^{\mu\nu\rho\sigma}\,\phi_{\mu\nu}\,\phi_{\rho\sigma} \\ &+f_3(X,\phi)\,G_{\mu\nu}\phi^{\mu\nu}+C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}\,\phi_{\mu\nu}\,\phi_{\rho\sigma}\,\phi_{\alpha\beta}\Big]\;, \end{aligned} \tag{4.5}$$

where the functions f_2 and f_3 depend only on the scalars ϕ and $X\equiv\phi_\mu\phi^\mu$; R and $G_{\mu\nu}$ denote, respectively, the usual Ricci scalar and Einstein tensor associated with the metric $g_{\mu\nu}$. The tensors $C_{(2)}$ and $C_{(3)}$ are the most general tensors constructed from the metric $g_{\mu\nu}$ and the first derivative of the scalar field ϕ_μ. It is easy to see that the quadratic terms can be rewritten as

$$C_{(2)}^{\mu\nu\rho\sigma}\,\phi_{\mu\nu}\,\phi_{\rho\sigma} = \sum_{A=1}^{5} a_A(X,\phi)\,L_A^{(2)}\,, \tag{4.6}$$

with the elementary quadratic Lagrangians (i.e., terms quadratic in $\phi_{\mu\nu}$ or $\Box\phi$; since ϕ_μ terms only contain one derivative, they do not contribute to the order of DHOST terms) given by

$$\begin{aligned}
&L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}\,, \qquad L_2^{(2)} = (\Box\phi)^2\,, \qquad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu\,,\\
&L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \qquad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2\,.
\end{aligned} \tag{4.7}$$

In a similar fashion, the cubic terms can be written as

$$C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}\,\phi_{\mu\nu}\,\phi_{\rho\sigma}\,\phi_{\alpha\beta} = \sum_{A=1}^{10} b_A(X,\phi)\,L_A^{(3)}\,, \tag{4.8}$$

with the elementary cubic Lagrangians being

$$\begin{aligned}
&L_1^{(3)} = (\Box\phi)^3\,, \quad L_2^{(3)} = (\Box\phi)\,\phi_{\mu\nu}\phi^{\mu\nu}\,, \quad L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho\,,\\
&L_4^{(3)} = (\Box\phi)^2\,\phi_\mu\phi^{\mu\nu}\phi_\nu\,, \quad L_5^{(3)} = \Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma\,,\\
&L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\,, \quad L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda\,,\\
&L_9^{(3)} = \Box\phi\,(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,, \quad L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3\,.
\end{aligned} \tag{4.9}$$

In general, as explained above, theories with an action of the form 4.1 contain two tensor modes and two scalar modes, one of which leads to an Ostrogradsky instability. DHOST theories correspond to specific restrictions of the functions a_A and b_A, so that the Hessian matrix is degenerate. One finds that these theories contain at most one propagating scalar mode. DHOST theories include Horndeski and beyond-Horndeski theories but many other higher-order scalar-tensor theories as well.

4.2.2. Mimetic theories

Originally, mimetic gravity was introduced by Chamseddine and Mukhanov as a scalar-tensor theory defined by the usual Einstein-Hilbert action [178]

$$S_{CM}[\tilde{g}_{\mu\nu},\phi] \;=\; S_{EH}[g_{\mu\nu}] \;\equiv\; \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R[g_{\mu\nu}] \tag{4.10}$$

where the metric $g_{\mu\nu}$ is related to $\tilde{g}_{\mu\nu}$ and ϕ by the non-invertible conformal transformation

$$g_{\mu\nu} \;\equiv\; -\tilde{X}\,\tilde{g}_{\mu\nu} \qquad \text{with} \qquad \tilde{X} \;\equiv\; \tilde{g}^{\mu\nu}\phi_\mu\phi_\nu\,. \tag{4.11}$$

It is immediate to see that the metric $g_{\mu\nu}$ is invariant under a local conformal transformation of the metric $\tilde{g}_{\mu\nu}$ while ϕ is left unchanged

$$\tilde{g}_{\mu\nu} \longmapsto \Omega(x)\,\tilde{g}_{\mu\nu}\,, \qquad \phi \longmapsto \phi\,. \tag{4.12}$$

Here Ω is an arbitrary function on the space-time. Hence, the action 4.10 is invariant under this same local transformation provided that the coupling to matter (if there is any) is defined with respect to $g_{\mu\nu}$.

There exist two different equivalent reformulations of the mimetic action 4.10 as it was emphasized in [189]. The observation that

$$X \equiv g^{\mu\nu}\phi_\mu\phi_\nu \;=\; -\frac{1}{\tilde{X}}\tilde{g}^{\mu\nu}\phi_\mu\phi_\nu \;=\; -1\,, \tag{4.13}$$

enables us to replace the action 4.10 by the following one

$$S^{(1)}_{CM}[g_{\mu\nu},\phi;\lambda] \;\equiv\; S_{EH}[g_{\mu\nu}] + \int d^4x\,\sqrt{-g}\,\lambda(X+1)\,, \tag{4.14}$$

where the equation of motion for λ reproduces exactly the condition 4.13. It follows that the actions 4.10 and 4.14 are classically equivalent.

In the second reformulation of mimetic gravity, one considers the action written in terms of the metric $\tilde{g}_{\mu\nu}$ and ϕ. Using the transformation law of the Ricci tensor under a conformal transformation of the metric, the action 4.10 can be rewritten as the following higher-derivative scalar-tensor theory

$$S^{(2)}_{CM}[\tilde{g}_{\mu\nu},\phi] \;\equiv\; \frac{1}{16\pi G}\int d^4x\,\sqrt{-\tilde{g}}\,\left(\tilde{X}R[\tilde{g}_{\mu\nu}] + \frac{6}{\tilde{X}}\tilde{\phi}_\mu{}^\nu\tilde{\phi}^{\mu\rho}\phi_\nu\phi_\rho\right)\,, \tag{4.15}$$

where $\tilde{\phi}_{\mu\nu} = \tilde{\nabla}_\mu\phi_\nu$ with $\tilde{\nabla}$ being the covariant derivative compatible with $\tilde{g}_{\mu\nu}$, and indices are lowered and raised with $\tilde{g}_{\mu\nu}$ and its inverse. The action 4.15 is clearly of the form 4.6, where the only non-trivial coefficients are

$$f_2 \;=\; \frac{\tilde{X}}{16\pi G} \qquad \text{and} \qquad a_4 \;=\; \frac{3}{8\pi G\tilde{X}}\,. \tag{4.16}$$

As expected, this theory is degenerate and can be shown to belong to the class Ia (according to the classification given in [176]) which corresponds to Lagrangians which are obtained from a conformal-disformal transformation of the quadratic Horndeski action. This is in total agreement with the fact that the mimetic gravity action has been obtained from a (non-invertible) conformal transformation of the Einstein-Hilbert action (which can be viewed as a particular case of Horndeski theory). A Hamiltonian analysis of the mimetic action 4.14 (and generalizations thereof) can be found in [190].

We close this short review with a remark concerning generalizations of mimetic gravity. First of all, one can generalize the action 4.10 assuming that $g_{\mu\nu}$ is now a general non-invertible conformal-disformal transformation of $\tilde{g}_{\mu\nu}$ [191], i.e.,

$$g_{\mu\nu} \equiv A(\tilde{X},\phi)\tilde{g}_{\mu\nu} + B(\tilde{X},\phi)\phi_\mu\phi_\nu \quad \text{with} \quad \frac{\partial}{\partial\tilde{X}}(A+\tilde{X}B) = 0\,. \tag{4.17}$$

In that case, it is easy to see that the local conformal invariance of $\tilde{g}_{\mu\nu}$ has been generalized to invariance under the local symmetry

$$\delta\tilde{g}_{\mu\nu} = \alpha(x)\tilde{g}_{\mu\nu} + \beta(x)\phi_\mu\phi_\nu \quad \text{with} \quad (A - \tilde{X}A_{\tilde{X}})\alpha(x) = \tilde{X}^2 A_{\tilde{X}}\beta(x)\,, \tag{4.18}$$

where $\alpha(x)$ and $\beta(x)$ are functions on the space-time and $A_{\tilde{X}} \equiv \partial_{\tilde{X}}A$. Hence, we obtain in this way an action $S[\tilde{g}_{\mu\nu},\phi]$ that is invariant under the symmetry 4.18, and therefore the theory is degenerate. In fact, given any higher-order scalar-tensor action $S[g_{\mu\nu},\phi]$, the action defined by

$$\tilde{S}[\tilde{g}_{\mu\nu},\phi] \equiv S[g_{\mu\nu},\phi] \tag{4.19}$$

where $g_{\mu\nu}$ is defined by 4.17 is necessarily degenerate. This family of actions provides a large generalization of mimetic gravity theories.

4.2.2.1. Limiting curvature hypothesis

The initial physical motivation for considering the theory (??) has been to propose an alternative to cold dark matter in the universe. Actually, it reproduces exactly the results of a model introduced earlier by Mukohyama in [192]. Later on, the original proposal of [178] has been extended, in adding a potential $V(\phi)$ for the scalar field in (??) and has been shown to provide potentially interesting models for both the early universe and the late time cosmology [179]. Finally, more recently, mimetic gravity has been applied to construct non-singular cosmologies and non-singular black holes in [66, 193]. Physically, the idea is very simple and consists in finding higher-order scalar-tensor Lagrangians (in the family of generalized mimetic gravity) in such a way that their Euler-Lagrange equations impose an upper limit on the Ricci scalar $\mathcal{R}$: this is called the limiting curvature hypothesis. If this is the case, $\mathcal{R}$ never diverges and one could expect to resolve in that way some divergences which appear in classical general relativity.

Such a hypothesis can be implemented concretely and, in some situations, it is sufficient to transform the original cosmological singularity into a non-singular bounce in the context of homogeneous and isotropic space-times [66], and also to remove the black-hole singularity in the context of spherically symmetric space-times. The limiting curvature hypothesis was extended very recently in [194] to construct theories whose equations of motion impose upper limits not only on the Ricci scalar but also on any invariant constructed from the Riemann tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ such as the Ricci tensor squared $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, the Weyl tensor squared $C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, or invariants involving derivatives of the Riemann. However, as the authors pointed out in [194], these extended Lagrangians contain higher-order derivatives of the metric (which is not the case for the original mimetic gravity) and presumably suffer from Ostrogradski instabilities generically [195]. As a result, one should consider these theories of gravity with a limiting curvature as effective descriptions which are physically valuable up to some energy scale only.

4.3. Brief review on symmetry reduced models in loop quantum gravity

In this section, we brief review the symmetry reduced models inspired by loop quantum gravity. These models are based on the Ashetekar-Barbero connection variables and adapt a loop (flux-holonomy) quantization. The effective dynamics are obtained by expressing the classical field in terms of holonomies, which leads to bouncing solutions in cosmology and black holes. We will mainly concentrate on the loop quantum cosmology, which is the homogeneous symmetry reduce models for cosmological case. The spherical case follows the same idea, and leads to several polymer models. For dedicated reviews of LQC and black hole models, see, e.g., [30–33, 35].

4.3.1. Loop quantum cosmology and its effective dynamics

LQC is a proposal for quantizing cosmological space-times using the variables and the non-perturbative techniques of LQG. More specifically, as in the full theory, LQC is based on the Ashtekar-Barbero connection variables, and the elementary variables to be promoted to fundamental operators in the quantum theory are holonomies along edges and fluxes across surfaces.

4.3.1.1. Hamiltonian framework with Ashtekar-Barbero variables

The Ashtekar-Barbero variables are related to the metric and extrinsic curvature as follows. We first introduce the densitized triads $E^a_i = \sqrt{q}e^a_i$ (where q is the determinant of the spatial metric and the triads e^a_i are related to the inverse of the spatial metric by $q^{ab} = e^a_i e^b_j \delta^{ij}$). The conjugate variable to E^a_i is the

$\mathfrak{su}(2)$-valued Ashtekar-Barbero connection

$$A_a^i = \Gamma_a^i + \gamma K_a^i \,, \tag{4.20}$$

where $K_a^i = K_{ab} e_i^b$ encodes the extrinsic curvature, Γ_a^i is the spin-connection such that

$$D_a e_i^b \;\equiv\; \partial_a e_i^b - \Gamma_{ac}^b e_i^c + \epsilon_{ij}{}^k \Gamma_a^j e_k^b \;=\; 0 \,, \tag{4.21}$$

with Γ_{ac}^b being the usual Christoffel symbols on the spatial slice, and γ is the real-valued Barbero-Immirzi parameter. The symplectic structure of gravity in the Ashtekar-Barbero variables is given by the Poisson bracket

$$\{A_a^i(x)\,,\, E_j^b(y)\} \;=\; 8\pi G\gamma\, \delta_a^b\, \delta_j^i\, \delta^{(3)}(x-y). \tag{4.22}$$

For simplicity, the matter field is often assumed to be a scalar ψ, with a Lagrangian

$$L_\psi = \int \mathrm{d}^4x\, \sqrt{-g}\left[-\tfrac{1}{2}(\partial\psi)^2 - V(\psi)\right]. \tag{4.23}$$

In the Hamiltonian framework, ψ comes with a conjugate momentum π_ψ such that

$$\{\psi, \pi_\psi\} = 1 \qquad \text{and} \qquad \pi_\psi \equiv \frac{\partial L_\psi}{\partial \dot{\psi}} = \frac{\sqrt{q}}{N}\,\dot{\psi}\,. \tag{4.24}$$

The lapse function N in the metric is not a dynamical variable and, in the Hamiltonian formulation, plays the role of a Lagrange multiplier that enforces that the scalar constraint $\mathcal{H}$ vanish, with $\mathcal{H}$ given by [25, 69, 196]

$$\mathcal{H} = -\frac{E_i^a E_j^b}{16\pi G\gamma^2\sqrt{q}}\,\epsilon^{ij}{}_k\left(F_{ab}{}^k - (1+\gamma^2)\Omega_{ab}{}^k\right) + \frac{\pi_\psi^2}{2\sqrt{q}} + \sqrt{q}\,V(\psi), \tag{4.25}$$

where $F_{ab}{}^k = 2\partial_{[a}A_{b]}^k + \epsilon_{ij}{}^k A_a^i A_b^j$ is the field strength of the Ashtekar-Barbero connection, while the tensor $\Omega_{ab}{}^k = 2\partial_{[a}\Gamma_{b]}^k + \epsilon_{ij}{}^k \Gamma_a^i \Gamma_b^j$ measures the spatial curvature.

In the particular case of a spatially flat FLRW space-time, with metric

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\vec{x}^2\,, \tag{4.26}$$

the densitized triads E_i^a are given by

$$E_i^a = p\left(\frac{\partial}{\partial x^i}\right)^a \qquad \text{with} \qquad p = a^2\,, \tag{4.27}$$

and the Ashtekar-Barbero connection can be written as

$$A_a^i = c\,(\mathrm{d}x^i)_a \qquad \text{with} \qquad c = \frac{\gamma\,\dot{a}}{N}\,, \tag{4.28}$$

since $\Gamma_a^i = 0$ because of the space-time symmetries. Therefore, the gravitational sector of the phase space is two-dimensional, and the symplectic structure, inherited from 4.22, reduces to [c]

$$\{c, p\} = \frac{8\pi G\gamma}{3}. \tag{4.29}$$

Moreover, the scalar constraint (4.25) becomes

$$\mathcal{H} = -\frac{3}{8\pi G\gamma^2}p^{1/2}\,c^2 + \frac{\pi_\psi^2}{2p^{3/2}} + p^{3/2}V(\psi). \tag{4.30}$$

Note that $\Omega_{ab}{}^k = 0$ for the spatially flat FLRW space-time. The dynamical evolution of any observable $\mathcal{O}$ is then given by the smeared Hamiltonian constraint according to

$$\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}t} = \{\mathcal{O}, \mathcal{C}_H\} \qquad \text{with} \qquad \mathcal{C}_H = \int \mathrm{d}^3\vec{x}\,N\mathcal{H}\,. \tag{4.31}$$

Note that the spatial diffeomorphism and Gauss constraints cannot contribute to the smeared Hamiltonian constraint since they identically vanish for the choice of variables 4.27 and 4.28. Also, the spatial integral of $N\mathcal{H}$ is trivial in FLRW space-times since every term is independent of position due to homogeneity. One can use the relation 4.31 with $\mathcal{O} = p$ to recover the usual Friedmann equation.

4.3.1.2. Quantum theory

The quantization of such a theory (assuming now for simplicity that $V(\psi) = 0$) following the standard Wheeler-de Witt procedure will give a quantum cosmology where the classical big-bang singularity is not resolved in any meaningful sense. Indeed, sharply-peaked wave packets closely follow the classical (singular) solutions, and the expectation value of, e.g., the energy density of ψ can become arbitrarily large [30].

The situation is markedly different in LQC for the reason that the fundamental operators of the theory are holonomies and areas, not operators corresponding to the connection A_a^i itself. For this reason, in LQC it is not possible to directly

c. As a technical remark, note that for the symplectic structure to be well-defined in such a space-time, it is necessary to restrict integrals to some finite region, called the fiducial cell. This restriction on the integrals acts as an infrared regulator which should be removed once the dynamics are determined by taking the limit of the fiducial cell going to the entire spatial manifold. For simplicity, here we choose the fiducial cell such that its volume with respect to the metric $\mathrm{d}\mathring{s}^2 = \mathrm{d}\vec{x}^2$ is 1.

promote the symmetry-reduced scalar constraint 4.30 to an operator in the quantum theory since there is no operator corresponding to $\hat{c}$. Instead, it is necessary to go back one step and construct an operator corresponding to 4.25.

This can be done in two parts. First, since the p contained in the E^a_i corresponds to an area, it can directly be promoted to be an operator. Second, the $\mathfrak{su}(2)$-valued field strength can be expressed in terms of holonomies in the same fashion as in lattice gauge theories,

$$F_{ab} \simeq \frac{h_{\Box_{ab}} - \mathbb{I}}{Ar_{\Box}}, \tag{4.32}$$

where $h_{\Box_{ab}}$ is the holonomy of the connection A_a around a loop in the a–b plane, $\mathbb{I}$ is the identity and $Ar_{\Box}$ is the area of that loop. In a lattice gauge theory, one would be interested in the limit of the right-handside of 4.32 when $Ar_{\Box} \to 0$, in which case the relation 4.32 becomes exact. However, this is not natural in LQC, since the spectrum of the area operator in LQG is discrete and has a minimum non-zero eigenvalue

$$\Delta = 4\sqrt{3}\pi\gamma\ell_{\rm Pl}^2 \tag{4.33}$$

where $\ell_{\rm Pl}$ is the Planck length. Therefore, what is done in LQC is to express F_{ab} in terms of the holonomy of A_a around a loop with this minimal area Δ.

More specifically, given the symmetries of the FLRW space-time, the loop $\Box_{ab}$ is assumed to be a square loop in the a–b plane. The holonomy of A_a along edges parallel transported by the vectors $(\partial/\partial x^i)^a$ is easily evaluated. Since the Ashtekar-Barbero connection $A_a = A^i_a\tau_i$ is $\mathfrak{su}(2)$-valued [d], it is necessary to choose a representation in which to calculate the holonomy. This is usually chosen to be the $j = 1/2$ representation. This is not only the simplest non-trivial representation, but also corresponds to the smallest excitation Δ of area possible in LQG, which is precisely the area that has been chosen for the loop $\Box_{ab}$ to have from physical grounds, as argued in the previous paragraph.

In the $j = 1/2$ representation, the τ_i can be chosen to be the Pauli matrices (up to a factor of $i/2$ in order to have the correct normalization as shown in footnote d). Hence, the holonomy of A_a along a path parallel to $(\partial/\partial x^i)^a$ and of length ℓ

d. The elements τ_i form a basis of the $\mathfrak{su}(2)$ Lie algebra satisfying

$$[\tau_i, \tau_j] = 2\epsilon_{ij}{}^k\tau_k \quad \text{and} \quad \mathrm{tr}(\tau_i\tau_j) = -2\delta_{ij}\,.$$

We used the notation $\epsilon_{ij}{}^k$ for the totally antisymmetric symbol with $\epsilon_{123} = +1$, indices are raised and lowered with δ_{ij}, and Tr denotes the trace in the 2-dimensional fundamental representation (known as the Killing form). In the spin-1/2 representation, the elements τ_i are represented by the 2-dimensional matrices

$$\tau_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

with respect to the fiducial metric $\mathrm{d}\mathring{s}^2 = \mathrm{d}\vec{x}^2$, is (no sum over i)

$$h_i(\ell) = \exp\left(\int_0^\ell \mathrm{d}x^i A_a \left(\frac{\partial}{\partial x^i}\right)^a\right) = \cos\frac{\ell c}{2}\mathbb{I} + 2\sin\frac{\ell c}{2}\tau_i\,. \tag{4.34}$$

Note that the fiducial metric allows to identify internal (Lie algebra) indices $i, j, k, \cdots$ with space indices $a, b, c, \cdots$ so that, from now on, we will use the same notation $i, j, k, \cdots$ to label indifferently internal and space directions. An important point here is that the length ℓ is measured with respect to the fiducial metric, and the physical length of the edge along which h_i is evaluated is given by $a\,\ell$ where a is the scale factor. Therefore, requiring that the physical length of h_i be $\sqrt{\Delta}$ corresponds to setting $\sqrt{p}\,\ell = \sqrt{\Delta} \Rightarrow \ell = \sqrt{\Delta/p}$ where we used $a = \sqrt{p}$. Then, the holonomy $h_{\Box_{ij}}$ around a square loop in the x^i–x^j plane with a physical area equal to Δ 4.33 is

$$h_{\Box_{ij}} = h_j(\bar{\mu})^{-1}h_i(\bar{\mu})^{-1}h_j(\bar{\mu})h_i(\bar{\mu}), \tag{4.35}$$

where $\bar{\mu} = \sqrt{\Delta/p}$, and $Ar_\Box = \bar{\mu}^2$.

Now, the holonomy 4.35 can be defined in the quantum theory, and therefore can be used as the operator corresponding to $\hat{F}_{ab}$. Then, given the field strength operator, it is now easy to define operators corresponding to the scalar constraint, and this completes the quantum theory. For the precise details concerning the Hilbert space and the Hamiltonian constraint operator (which are not necessary here for our purposes), see, e.g., [30, 32].

The resulting quantum theory resolves the big-bang singularity in a precise sense: first, there is an upper bound on the operator corresponding to the matter energy density, and second, the states corresponding to singular space-times (i.e., $p = a^2 = 0$) decouple from non-singular states under the action of the Hamiltonian constraint operator and thus an initial state which is non-singular will always remain non-singular. These important differences from the Wheeler-de Witt theory arise from expressing the field strength operator in terms of the holonomy of the connection around a loop of area Δ rather than directly promoting c to be an operator.

Furthermore, for the case that the scalar field ψ is massless, it provides a good relational clock and it is possible to speak of the relational evolution of the wave function $\Psi(p,\psi)$ with respect to ψ. In particular, it is possible to construct an "initial" state $\Psi(p,\psi_o)$ at some "initial" relational time ψ_o and to evolve it using the Hamiltonian constraint operator. One especially interesting possibility is to construct a wave packet sharply peaked around a classical configuration at a low curvature scale when general relativity can be trusted, and then evolve the wave packet towards the high-curvature regime. An important result in this case is that the wave packet remains sharply peaked throughout its entire evolution, assuming (i) it is initially sharply peaked and (ii) that the expectation value of p

always remains large compared to l_p^2. For such states, the full quantum dynamics are extremely well approximated by an effective theory.

4.3.1.3. Effective dynamics

The effective dynamics of LQC are obtained by expressing the classical field strength F_{ab} in terms of the holonomy around a square loop of area Δ as we did in the previous section, and then treating the resulting $\mathcal{H}$ classically. It is easy to verify that the LQC effective scalar constraint is

$$\mathcal{H}^{\text{eff}} = -\frac{3\,p^{3/2}}{8\pi G\Delta\gamma^2}\,\sin^2\bar{\mu}c + \frac{\pi_\psi^2}{2p^{3/2}} + p^{3/2}V(\psi). \tag{4.36}$$

In this way, the effective theory captures the physics corresponding to the discrete nature of geometry in LQC (specifically, the existence of the area gap Δ), but ignores the effect of quantum fluctuations since $\mathcal{H}^{\text{eff}}$ is treated classically [e].

Using 4.31 with 4.36, it is easy to derive the time derivative of p:

$$\dot{p} = 2N\frac{p}{\gamma\sqrt{\Delta}}\sin\bar{\mu}c\cos\bar{\mu}c\,. \tag{4.37}$$

Moreover, the constraint 4.36 implies

$$\sin^2\bar{\mu}c = \frac{\rho}{\rho_c}, \tag{4.38}$$

where ρ is the matter energy density,

$$\rho = \frac{\pi_\psi^2}{2p^3} + V(\psi)\,, \tag{4.39}$$

and ρ_c is a constant defined by

$$\rho_c = \frac{3}{8\pi G\gamma^2\Delta} = \frac{\sqrt{3}}{32\pi^2\gamma^3}\rho_{\text{Pl}}\,, \tag{4.40}$$

with ρ_{Pl} being the Planck density. Combining the above relations yields the effective Friedmann equation

$$H^2 = \left(\frac{\dot{p}}{2Np}\right)^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho}{\rho_c}\right)\,, \tag{4.41}$$

e. This is a good approximation for states that are initially sharply peaked (i.e., all expectation values satisfy $\langle\hat{\mathcal{O}}^2\rangle \approx \langle\hat{\mathcal{O}}\rangle^2$ which means that quantum fluctuations are negligeable) so long as the spatial volume of the space-time is much larger than the Planck volume since for these states quantum fluctuations do not grow significantly and hence always remain negligeable [197].

where $H = \dot{a}/(Na)$ is the Hubble parameter. It can also be checked that the continuity equation in effective LQC is the same as in classical general relativity.

As can easily be seen from the LQC effective Friedmann equation, the big-bang singularity of general relativity is replaced by a bounce that occurs when the energy density of ψ reaches the critical energy density $\rho_c \propto \rho_{\mathrm{Pl}}$ 4.40. This bounce clearly originates from the quantum geometry of LQG, and in the limit of $\Delta \to 0$, the classical Friedmann equation is recovered. Numerical simulations of the dynamics generated by the LQC Hamiltonian constraint operator for sharply-peaked wave functions have explicitly shown that the effective equations do indeed provide an excellent approximation to the full quantum dynamics, even around and at the bounce point . Thus, it is clear that the bounce occurs due to the discrete geometry of LQG, irrespective of quantum fluctuations.

Note that the bounce occurs when the energy density is of the order of the Planck scale but the volume of the spatial slice at the bounce time can be much larger than the Planck volume. Consequently, as long as the bounce occurs at a spatial volume much larger than l_p^3, the effective theory can be trusted at all times for sharply-peaked states. Note that this condition is automatically satisfied for non-compact FLRW space-times since their spatial volume is always infinite (so long as the scale factor remains non-vanishing, which is always true in LQC).

In conclusion, the LQC effective dynamics is a powerful tool which significantly simplifies calculations of quantum gravity effects in semi-classical cosmological states. Clearly, it would be very helpful if it were possible to develop an effective theory that would hold more generally, for instance for all states in LQG that have nice semi-classical properties and whose geometric observables of interest concern (spatial) regions that are much larger than l_p^3. For this reason, we explore scalar-tensor theories that are able to reproduce the LQC effective dynamics for cosmological geometries.

4.3.2. Effective description of quantum black holes in loop quantum gravity

The first descriptions of quantum black holes in LQG were introduced in [198, 199]. They allowed to obtain a complete description of the black hole microstates whose counting is very well-known to reproduce the Bekenstein-Hawking formula at the semi-classical limit. Even though important issues concerning the role of the Barbero-Immirzi parameter in the counting procedure remain unsolved, (see [200–204] for some proposals to overcome them), this result has been an important success for LQG. Unfortunately, these studies do not say much about the dynamical "geometry" of the quantum black holes in the context of LQG: microstates are described in terms of algebraic structures (intertwiners between representations) and recovering the geometrical content of the quantum black holes at the semi-classical limit is a very complicated task which required coarse-graining schemes still under development. One way to attack the problem would be to find, exactly as it was done in LQC, an effective description of spherically symmetric solutions and see how quantum gravity effects modifies Einstein equations in this sector.

Such a program, which relies on a polymer quantization of spherically symmetric geometries, was initiated in [34] and further developed in [205–208]. The effective dynamics of the interior black hole was explored in [209, 210] and its construction follows very closely what is done in LQC, the black hole interior geometry being a homogenous anisotropic cosmology. However, the treatment of the whole space-time and its inhomogenous exterior geometry remains more challenging. While the loop quantization of the inhomogeneous vacuum exterior geometry was worked out in details in [35, 184], an effective theory of this quantum geometry is not yet available (see [35] for a review on the quantization of the full spherically symmetric geometry). Moreover, going beyond the vacuum case and including matter has also proved to be very challenging, mainly because the standard holonomy corrections spoil generally the covariance of such models [211, 212]. For this reason, studying an effective inhomogenous gravitational collapse including the quantum corrections from loop quantum gravity is up to now out of reach with the standard techniques. Another way to understand the geometry of quantum black holes (at the semi-classical limit) would be to “guess" the modifications induced by quantum gravity effects in mimicking LQC as it was done in [42] where the notion of “Planck stars" has been introduced. Some phenomenological aspects of these potentially new astrophysical objects have been studied in [42, 44, 213]. While the heuristic idea of the Planck stars is fascinating, a quantum theory of spherically symmetric geometry coupled to matter from which such effective description emerged still has to be built.

In this section, we give a brief review on the formulation of the classical spherically symmetric Ashtekar-Barbero phase space and then we describe the so-called holonomy corrected phase space for inhomogenous models. As explained, this leads to an effective polymer model which has been the starting point for a quantization of spherically symmetric geometries using the LQG techniques [184]. This phase space, which describes vacuum spherically symmetric gravity, has only global degrees of freedom. The quantum effective corrections are introduced via holonomy corrections which ensure the existence of an anomaly-free (albeit deformed) Dirac's hypersurface deformation algebra [211, 214]. In that sense, the resulting polymer phase space describes a vacuum geometry with somehow a deformed notion of covariance. We refer to [34, 37, 205–210, 215] for the construction and effective dynamics of homogenous anisotropic black hole models which will not be directly addressed in this thesis.

4.3.2.1. The classical vacuum spherically symmetric Ashtekar-Barbero phase space

Starting from the full Ashtekar-Barbero phase space, one can impose spherical symmetry to reduce it. This symmetry reduction has been presented in [216] and we shall not reproduce the different steps in details but only sketch the main lines.

The spherically symmetric Ashtekar-Barbero phase space is parametrized by two pairs of canonical conjugate variables (after having gauge fixed the Gauss constraint) given by

$$\{K_x(u), E^x(v)\} = \frac{2\kappa}{\beta}\,\delta(u-v)\,, \qquad \{K_\phi(u), E^\phi(v)\} = \frac{\kappa}{\beta}\,\delta(u-v)\,, \tag{4.42}$$

where $1/\beta$ is the Barbero-Immirzi parameter and $\kappa = 8\pi G$ with G the newton constant. Note the u and v are radial coordinates and $\delta(u-v)$ denotes the usual one-dimensional delta distribution.

The first pair corresponds to the inhomogenous component of the connection with its associated electric field, while the second one is the angular contribution built from the angular components of the connection. As usual in the LQG litterature, x denotes the radial direction as shown in the expression of the induced metric

$$ds^2 \equiv \gamma_{rr} dr^2 + \gamma_{\theta\theta} d\Omega^2 \equiv \frac{(E^\phi)^2}{|E^x|} dx^2 + |E^x| d\Omega^2\,. \tag{4.43}$$

The first class constraints generating the gauge symmetries (i.e. spatial diffeomorphisms and time reparametrization) are

$$D[N^x] = \frac{1}{2\kappa}\int_\Sigma dx\; N^x \left[2E^\phi K_\phi' - K_x (E^x)'\right]\,, \tag{4.44}$$

$$\mathcal{H}[N] = -\frac{1}{2\kappa}\int_\Sigma dx\; N|E^x|^{-1/2}\left[E^\phi K_\phi^2 + 2K_\phi K_x E^x + (1-\Gamma_\phi^2)E^\phi + 2\Gamma_\phi' E^x\right]\,, \tag{4.45}$$

where the spin-connection component which shows up in the expression of $\mathcal{H}$[N] is given by $\Gamma_\phi \equiv -(E^x)'/(2E^\phi)$. The two constraints satisfy obviously the hypersurface deformation algebra of general relativity (after reduction to the spherically symmetric geometries)

$$\{D[N^x], D[M^x]\} = D[\mathcal{L}_N M]\,, \tag{4.46}$$

$$\{D[N^x], \mathcal{H}[M]\} = -\mathcal{H}[\mathcal{L}_N M]\,, \tag{4.47}$$

$$\{\mathcal{H}[N], \mathcal{H}[M]\} = D[\gamma^{xx}(N\partial_x M - M\partial_x N)]\,, \tag{4.48}$$

where we used the same notations as in the equations (6.1) and (6.2) in the introduction.

4.3.2.2. The holonomy corrected polymer phase space

Following the strategy used to study polymer models at the effective level, one introduces holonomy corrections (called loop corrections) to work with holonomies instead of the connection itself. These effective corrections originate from a general regularization procedure borrowed from Loop Quantum Gravity [f]. However, several new difficulties show up when considering an inhomogenous background such as vacuum spherically symmetric gravity compared to homogenous cosmological backgrounds. In particular, one has to ensure that the loop regularization does not generate anomalies in the first class constraints algebra.

The usual regularization consists in making the replacement

$$K_\phi \longrightarrow f(K_\phi)\,, \tag{4.49}$$

directly in the expression of the Hamiltonian constraint, where the function f encodes quantum gravity effects at the effective level. Such a procedure is obviously not unique and suffers from ambiguities. Furthermore, there could be as many different functions f as many K_ϕ that appear in the Hamiltonian constraint. Yet, a standard choice (in the LQG litterature) is

$$f(K_\phi) \;=\; \frac{\sin\left(\rho K_\phi\right)}{\rho}\,, \tag{4.50}$$

where ρ is a new fundamental (quantum gravity) scale such that the limit $\rho \to 0$ reproduces the classical phase space. Notice that this choice is motivated by the polymerization obtained in LQC, from the computation of the non local curvature operator using $\mathrm{SU}(2)$ holonomies (one can refer to [30] for more details). Moreover, the scale ρ is a constant which corresponds in the terminology of LQC to taking the μ_0-scheme which has been debated a lot and has been replaced by the so-called $\bar{\mu}$-scheme. A first attempt to introduce a $\bar{\mu}$-scheme in spherically symmetric loop models has been presented in [217]. In that case, f is now a function $f(K_\phi, E^x)$ of K_ϕ and E^x. It was shown that the resulting quantum corrections are not periodic anymore if one requires anomaly-freeness of the first class constraints algebra, challenging the possibility to interpret such corrections as resulting fundamentally from $\mathrm{SU}(2)$ holonomy corrections.

With this standard regularization (4.49), the holonomy corrected first class

f. Note that the full regularization implies additional corrections: the triad corrections associated to the regularization of the inverse volume term. Yet, most of the investigations only implement one type of corrections while ignoring the second type for practical reasons. Even more, among the holonomy corrections, only the point wise quantum corrections are taken into account. Beyond the technical aspect, the motivation for focusing on the holonomy corrections only comes from the observation that in the cosmological sector, such corrections are enough to obtain a singularity resolution mechanism.

constraints take the form

$$D[N^x] = \frac{1}{2\kappa}\int_\Sigma dx\ N^x\left[2E^\phi K_\phi' - K_x(E^x)'\right], \tag{4.51}$$

$$\mathcal{H}[N] = -\frac{1}{2\kappa}\int_\Sigma dx\ N|E^x|^{-1/2}\left[E^\phi f_1(K_\phi) + 2f_2(K_\phi)K_xE^x + (1-\Gamma_\phi^2)E^\phi + 2\Gamma_\phi' E^x\right], \tag{4.52}$$

where the two functions f_1 and f_2 regularize the two K_ϕ arguments which appear in the expression of the classical Hamiltonian constraint. The corresponding Poisson algebra is

$$\{D[N^x], D[M^x]\} = D[\mathcal{L}_N M], \tag{4.53}$$

$$\{D[N^x], \mathcal{H}[M]\} = -\mathcal{H}[\mathcal{L}_N M], \tag{4.54}$$

$$\{\mathcal{H}[N], \mathcal{H}[M]\} = D[\beta(K_\phi)\gamma^{xx}(N\partial_x M - M\partial_x N)], \tag{4.55}$$

provided that the two corrections functions satisfy the differential equation

$$f_2 = \frac{1}{2}\frac{df_1}{dK_\phi}. \tag{4.56}$$

This implies in turn that the deformation of the Dirac's algebra is given by

$$\beta(K_\phi) = \frac{df_2}{dK_\phi}. \tag{4.57}$$

Interestingly, at the bounce, the sign of the function $\beta(K_\phi)$ changes (when one takes the usual $\sin$ polymerization function (4.50)) and the constraints algebra becomes effectively euclidean. This observation has suggested a possible signature change of the space-time in the very quantum region. This point is still debated and has received more attention in the context of the deformed algebra approach to the cosmological perturbations in LQC [119, 120]. See [121–125] for a general discussion on the conceptual and technical consequences of this deformation in early cosmology, and [126] in the context of black hole. Finally, note that the above deformation disappears when working with the self-dual Ashtekar variables. Indeed, thanks to the the self-dual formulation, one can introduce point wise holonomy corrections with a $\bar{\mu}$-scheme without affecting the Dirac's algebra which keeps its classical form. This was shown both for spherically symmetric gravity coupled to a scalar field and for the unpolarized Gowdy model, which both exhibit local degrees of freedom [218, 219].

In the case of a real Ashtekar-Barbero spherically symmetric polymer phase space, the usual choice for an effective description of spherically symmetric geometries in LQG corresponds to taking

$$f_1(K_\phi) = \frac{\sin^2(\rho K_\phi)}{\rho^2}\,, \qquad f_2(K_\phi) = \frac{\sin(2\rho K_\phi)}{2\rho}\,, \tag{4.58}$$

which obviously satisfies (4.56).

5. Effective loop Quantum Cosmology as a Higher-derivative scalar-tensor Theory

Sommaire

5.1. Introduction

In this chapter, we concentrate on the specific mimetic gravity Lagrangian proposed in [66]. We note that the corresponding classical equations of motion for a cosmological background are exactly the same as the so-called effective equations of loop quantum cosmology. This result suggests that it may be possible to describe loop quantum gravity at an effective level in some appropriate regimes as a higher-derivative scalar-tensor theory. The main purpose of this chapter is to highlight this relation between mimetic gravity and loop quantum cosmology. While this relation has only been established at the cosmological level, it nonetheless provides a proposal for an effective description of loop quantum gravity in terms of higher-order scalar-tensor theories. Such an effective description is potentially very interesting, especially as it may give important insights into the relation between the quantization of gravity à la loop and the usual perturbative quantization techniques.

The chapter is organized as follows. In the following section, we show how the loop quantum cosmology effective dynamics can be derived from an action principle $S[a, N]$ with a Lagrangian (invariant under time reparametrizations) that depends on the scale factor a and on the lapse function N, for all isotropic cosmologies. (This calculation is already known for the spatially flat case [220, 221].) In Sec. 5.3 we show that there exists a family of DHOST mimetic actions $S[g_{\mu\nu}, \phi]$ which all reduce to $S[a, N]$ for homogeneous and isotropic space-time. These actions generalize the model proposed recently by Chamseddine and Mukhanov in [66] and can be viewed as a proposal for an effective description of loop quantum gravity. We conclude in Sec. 5.4 with a discussion.

5.2. Loop quantum cosmology from mimetic gravity

The goal now is to find a family of modified gravity theories that, when restricted to the spatially flat FLRW space-time, reproduce precisely the LQC effective Friedmann equation 4.41. In fact, one such modified gravity theory with precisely this property has already been found [220, 221] (see also [222] for an $f(R)$ modified gravity theory whose dynamics are a good approximation to the LQC Friedmann equation). Here, we will generalize these earlier results to a whole class of scalar-tensor theories, and in Sec. 5.2.1 we extend these results to the case of non-vanishing spatial curvature.

To begin, we look for an action $S[a, N, \psi]$ where the dynamical variables are the scale factor $a(t)$, the lapse function $N(t)$ and a field $\psi(t)$ that represents the matter content of the universe. The action is of course invariant under time reparametrizations. Afterwards, we will construct a class of covariant actions of modified gravity which reduce to $S[a, N, \psi]$ when the metric is fixed by the flat FLRW metric 4.26. We assume that the field ψ is a massless scalar field minimally coupled to gravity. Hence, the modified action of gravity we are looking for takes the form

$$\int d^4x \sqrt{|g|} \left(\frac{1}{16\pi G} R - \frac{1}{2} g^{\mu\nu} \psi_\mu \psi_\nu + \cdots \right) , \tag{5.1}$$

where the remaining (so far unknown) part does not involve the matter content represented here by ψ. As a consequence, on an FLRW space-time, the previous action reduces to

$$S[a, N, \psi] \;=\; \int dt \left(-\frac{3a\dot{a}^2}{8\pi G N} + a^3 \frac{\dot{\psi}^2}{2N} + N a^3 \mathcal{L}\left(a, \frac{\dot{a}}{N}\right) \right) \tag{5.2}$$

where the unknown function $\mathcal{L}$ has to be fixed in such a way that $S[a, N, \psi]$ reproduces the LQC effective dynamics. The fact that $\mathcal{L}$ depends on $\dot{a}/N$ (rather than on $\dot{a}$ and N separately) is a consequence of requiring invariance under time reparametrization. Furthermore, the Lagrangian does not involve non-trivial higher derivatives (which cannot be eliminated from the action with integrations by parts) of the scale factor, otherwise the associated classical equations of motion would (necessarily) be higher order, hence they would not reproduce the LQC effective dynamics.

As can be seen from 4.36, up to an overall prefactor the effective Hamiltonian constraint of loop quantum cosmology is expressed only in terms of the combination $c/\sqrt{p}$, which classically corresponds to the Hubble rate $H = \dot{a}/(Na)$ (up to the prefactor $\gamma\sqrt{\Delta}$), as can be seen from the definitions 4.27 and 4.28. This suggests restricting the function $\mathcal{L}$ to be to the form

$$\mathcal{L}\left(a, \frac{\dot{a}}{N}\right) \;=\; \mathcal{F}\,(H)\;. \tag{5.3}$$

A Hamiltonian analysis of the action 5.2 with $\mathcal{L} = \mathcal{F}(H)$ clarifies the link with LQC. Due to the invariance under time reparametrization, the lapse N is still a Lagrange multiplier and the only non-trivial pairs of canonically conjugate pairs of variables are

$$\{a, \pi_a\} \;=\; 1 \;=\; \{\psi, \pi_\psi\}\,. \tag{5.4}$$

The Lagrangian is clearly not degenerate and the momenta are given in terms of the velocities by

$$\pi_a \;= a^2\left[-\frac{3H}{4\pi G} + a^2\mathcal{F}'(H)\right]\,, \qquad \pi_\psi = \frac{a^3}{N}\dot{\psi}\,, \tag{5.5}$$

where $\mathcal{F}'$ is the derivative of the function $\mathcal{F}$. The shape of the LQC effective Hamiltonian suggests the ansatz

$$\pi_a \;=\; \alpha\, a^n \arcsin\left(\beta\,\frac{\dot{a}}{Na}\right)\,, \tag{5.6}$$

where n, α and β are constants to be fixed. The condition that the momentum π_a should be approximately given by the classical result $\pi_a = -3a\dot{a}/4\pi GN$ at low curvatures (or small H) sets $n = 2$ and further requires that $\alpha\beta = -3/4\pi G$. Then, for this ansatz 5.6 for π_a, $\mathcal{F}$ must be

$$\mathcal{F}(H) \;=\; \alpha\, H \arcsin(\beta H) + \frac{\alpha}{\beta}\sqrt{1-\beta^2H^2} + \frac{3H^2}{8\pi G} - \frac{\alpha}{\beta}\,, \tag{5.7}$$

where the integration constant has been fixed so that $\mathcal{F}(0) = 0$, which means that one recovers the standard general relativity action in the low curvature regime.

Given this explicit form for the Lagrangian, one finds for the Hamiltonian density

$$\mathcal{H} \;=\; a^3\left(\frac{\pi_\psi^2}{2a^6} - \frac{8\pi G}{3}\alpha^2\sin^2\left(\frac{\pi_a}{2\alpha a^2}\right)\right)\,. \tag{5.8}$$

Using the same procedure as in the previous subsection, one easily finds that the modified Friedman equation is given by

$$H^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho}{\rho_c}\right) \quad \text{with} \quad \rho = \frac{\pi_\psi^2}{2a^6} \quad \text{and} \quad \rho_c = \frac{8\pi G}{3}\alpha^2 . \tag{5.9}$$

This coincides with the LQC effective dynamics provided

$$\alpha = \frac{3}{8\pi G\gamma\sqrt{\Delta}}. \tag{5.10}$$

Hence, α is determined by Newton's constant and the Barbero-Immirzi parameter. We recover the result of Chamseddine and Mukhanov from a Hamiltonian point of view. Let us emphasize that exactly the same function has been found in a rather different context much earlier in [220, 221] from a Lagrangian point of view.

Before showing the large class of scalar-tensor theories whose Hamiltonian constraint reduces to 5.8 for spatially flat FLRW space-times, we will show that this result can be extended to allow for non-vanishing spatial curvature.

5.2.1. Spatial curvature

It is possible to generalize the previous procedure to the case of a spatially curved FLRW space-time. In classical general relativity, one gets an additional contribution from the Einstein-Hilbert term coming due to the 3-dimensional curvature ${}^3R$ evaluated in a non-flat homogeneous and isotropic space-time. Hence, we start with the action

$$S_k[a,N,\psi] = \int dt\left(-\frac{3a\dot{a}^2}{8\pi GN} + a^3\frac{\dot{\psi}^2}{N} + \frac{3Nka}{8\pi G} + Na^3\mathcal{L}_k\left(a,\frac{\dot{a}}{N}\right)\right), \tag{5.11}$$

where k denotes the usual spatial curvature parameter. As in previous subsection, the extra term $\mathcal{L}_k$ (such that $\mathcal{L}_0 = \mathcal{L}$) contains the (yet unknown) additional terms that are necessary to obtain the LQC effective dynamics in place of the classical general relativity ones.

5.2.1.1. Curved Cosmology in Mimetic Gravity

We now assume that the curvature dependence can be taken into account by simply adding to $\mathcal{L}$ obtained in the flat case a new term that depends on the scale factor but not on its derivatives:

$$\mathcal{L}_k\left(a,\frac{\dot{a}}{N}\right) = \mathcal{F}(H) - \frac{3}{8\pi G}V_k(a). \tag{5.12}$$

The potential-like term V_k must vanish for $k = 0$, in order to recover the results of the flat case. The overall normalization is chosen for later convenience.

Since the two new terms in the action 5.11 depend only on a (and not on $\dot{a}$), it is straightforward to generalize the Hamiltonian analysis of the flat case. The new Hamiltonian constraint is given by

$$\mathcal{H} = a^3 \left(\rho - \rho_c \sin^2\left(\frac{\pi_a}{2\alpha a^2}\right) - \frac{3k}{8\pi G a^2} + \frac{3V_k(a)}{8\pi G} \right), \qquad \rho = \frac{\pi_\psi^2}{2a^6}, \tag{5.13}$$

and then the modified Friedmann equation becomes

$$H^2 = \left(\frac{8\pi G}{3}\rho - \frac{k}{a^2} + V_k(a) \right) \left(1 - \frac{1}{\rho_c} \left[\rho - \frac{3k}{8\pi G a^2} + \frac{3V_k(a)}{8\pi G} \right] \right). \tag{5.14}$$

One can now compare this modified Friedmann equation, based on the very simple ansatz 5.12, with the LQC effective Friedmann equation obtained for spatially curved FLRW space-times.

5.2.1.2. Curved Cosmology in Effective LQC and Quantization Ambiguities

Before making this comparison, it is important to review a quantization ambiguity which arises when performing the loop quantization of a homogeneous space-time with non-vanishing spatial curvature: this quantization ambiguity concerns the precise quantity that is to be expressed in terms of Planck-length holonomies. Due to this quantization ambiguity, there exist three quantization prescriptions that give slightly different effective theories, which can each be compared to 5.14.

To be more specific concerning this quantization ambiguity, for closed FLRW space-times there exists a direct generalization of the procedure followed for the spatially flat space-time reviewed above in Sec. 4.3.1.2, i.e., to express the field strength in terms of the holonomy of the Ashtekar-Barbero connection around a closed loop [223, 224]. This is known as the 'F' loop quantization. The 'F' loop quantization is not possible for the open FLRW space-time, nor for spatially curved Bianchi space-times (the problem is that, when expressing the field strength in this fashion, the resulting function is not almost-periodic in the connection, and so cannot be promoted to be an operator in the quantum theory, see [225–227] for details). For the Bianchi space-times, an alternative way forward is to express the connection itself (rather than the field strength) in terms of a Planck-length holonomy, as proposed in [226, 227]; this alternative quantization is known as the 'A' loop quantization. However, for the open FLRW space-time, neither the 'F' nor the 'A' loop quantizations are viable. Instead, it has been proposed that the 'K' quantization, where one considers 'holonomies' of the extrinsic curvature rather than of the Ashtekar-Barbero connection, may provide a reasonable approximation to a proper loop quantization [225]. Interestingly, the

'F', 'A', and 'K' loop quantizations are all possible in the closed FLRW model, and it is possible to compare the resulting effective theories that result from each of these quantization prescriptions [228, 229], with the surprising result that the 'K' quantization in fact appears to provide a better approximation to the 'F' loop quantization than the 'A' loop quantization does. (All three quantizations are also possible for the spatially flat FLRW space-time and also Bianchi type I models, but in both cases all three quantizations turn out to be exactly equivalent.) Therefore, for the closed FLRW space-time it is possible to compare the effective theories resulting from the 'F', 'A' and 'K' quantization prescriptions to 5.14, while for the open space-time only the effective theory resulting from the 'K' quantization is known.

Let us start with the 'F' loop quantization. In this case, the LQC effective Friedmann equation is [223, 228]

$$H^2 = \left(\frac{8\pi G}{3}\rho - \frac{k}{a^2} - \frac{k}{\gamma^2 a^2} + \frac{1}{\Delta\gamma^2}\sin^2\frac{\sqrt{k\Delta}}{a}\right) \\ \times \left(1 - \frac{1}{\rho_c}\left[\rho - \frac{3k}{8\pi G a^2} - \frac{3k}{8\pi G\gamma^2 a^2} + \frac{3}{8\pi G\Delta\gamma^2}\sin^2\frac{\sqrt{k\Delta}}{a}\right]\right), \tag{5.15}$$

which is precisely of the form of 5.14 with

$$V_k(a) = -\frac{k}{\gamma^2 a^2} + \frac{k}{\Delta\gamma^2}\sin^2\frac{\sqrt{k\Delta}}{a}. \tag{5.16}$$

Note that it is not guaranteed that the effective Friedmann equation has the general form of 5.14 for some $V_k(a)$, and therefore this result is encouraging because it indicates that the same modified gravity theory can be used to describe the LQC effective dynamics of both spatially flat and closed FLRW space-times (assuming the 'F' quantization for the closed FLRW space-time).

Now, while it is not known how to perform a proper loop quantization for the open FLRW space-time, it appears reasonable to assume that the effective dynamics for such a loop quantization would also have the form 5.14 where $V_k(a)$ is 5.16 with the only difference that now k is negative. If this is indeed the case, then the resulting LQC effective Friedmann equation for the open FLRW space-time would be

$$H^2 = \left(\frac{8\pi G}{3}\rho + \frac{|k|}{a^2} + \frac{|k|}{\gamma^2 a^2} - \frac{1}{\Delta\gamma^2}\sinh^2\frac{\sqrt{|k|\Delta}}{a}\right) \\ \times \left(1 - \frac{1}{\rho_c}\left[\rho + \frac{3|k|}{8\pi G a^2} + \frac{3|k|}{8\pi G\gamma^2 a^2} - \frac{3}{8\pi G\Delta\gamma^2}\sinh^2\frac{\sqrt{|k|\Delta}}{a}\right]\right). \tag{5.17}$$

Moving on to the 'A' loop quantization, the effective equation for the closed

FLRW space-time turns out to have a form which is different than 5.14 [228]. This provides an explicit example that shows that, in general, generic modifications to the Friedmann equations cannot be written in the form 5.14. It is only in some special cases that it will be possible to describe quantum gravity effects via a mimetic scalar-tensor theory, as is the case for the 'F' loop quantization.

Finally, for the 'K' loop quantization the effective Friedmann equation for the closed and open FLRW space-times is [225, 229]

$$H^2 = \left(\frac{8\pi G}{3}\rho + \frac{k}{a^2}\right) \times \left(1 - \frac{1}{\rho_c}\left[\rho + \frac{3k}{8\pi G a^2}\right]\right), \tag{5.18}$$

where $k > 0$ for a closed universe and $k < 0$ for an open universe. Interestingly, in this case we find that the effective Friedmann equation is again of the form 5.14, this time with $V_k(a) = 0$. There are three important points here. First, for the 'K' quantization, which can be performed for both open and closed FLRW space-times, the effective theory can be understood to come from a scalar-tensor theory, and the same theory describes equally well space-times with positive or negative spatial curvature. Second, in agreement with earlier results [228, 229], we find that the 'K' quantization is more similar to the 'F' quantization than the 'A' quantization is, in that its resulting effective Friedmann equations can be described by a scalar-tensor theory. And third, the 'K' quantization does not require a $V_k(a)$ term in the action 5.11. Since a non-vanishing $V_k(a)$ term breaks Lorentz invariance (as we shall discuss in more detail in the following section), it is quite interesting that the effective theory of the 'K' quantization in fact corresponds to a modified gravity theory which is Lorentz invariant.

To summarize, there is a quantization ambiguity that arises in LQC when considering spatially curved homogeneous space-times. There exist 'F', 'A' and 'K' quantizations for the closed FLRW space-time, while only the 'K' quantization is known for the open topology. For closed FLRW space-time, the effective equations resulting from the 'F' quantization prescription (but not the 'A' quantization) can be understood to come from a mimetic gravity theory with the action 5.11, for a particular choice of $V_k(a)$. This mimetic gravity theory in fact provides a candidate effective theory for a proper loop quantization of the open FLRW space-time. Finally, the effective equations for the 'K' quantization of FLRW space-times (for all k) can also be understood to follow from a mimetic gravity action of the form 5.11, in this case with $V_k(a) = 0$.

In short, these results show that the LQC effective equations (with the exception of the 'A' quantization) for homogeneous and isotropic cosmologies are identical to the Friedmann equations of a particular mimetic gravity theory.

5.3. Effective loop quantum gravity and mimetic gravity

In this section, we address the question of finding a covariant action of modified gravity that reduces to

$$S[a,N,\psi] = \int dt N a^3 \left(\frac{\dot{\psi}^2}{2N^2} - \frac{\rho_c}{2}\left[\beta H \arcsin(\beta H) + \sqrt{1-\beta^2 H^2} - 1\right]\right) , \quad \beta^2 = \frac{3}{2\pi G \rho_c} \tag{5.19}$$

in a spatially flat homogeneous and isotropic space-time (the non-flat case will be discussed below), as found in the previous section. Of course, such a condition is not very restrictive and one can expect that many different covariant actions could yield the same cosmological action.

Because of the non-linearity of the Lagrangian in the Hubble parameter H, it is not possible to find an $f(R)$ theory which exactly reproduces 5.19 in the cosmological sector, although an approximate construction has been found in [222] (where one considers $f(R)$ theories à la Palatini). The reason is that the Ricci scalar, given by

$$R \;=\; 6\left(\frac{1}{a}\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{a}}{N}\right) + \left(\frac{\dot{a}}{Na}\right)^2\right) , \tag{5.20}$$

involves second derivatives of the scale factor and this prevents us from recovering a Lagrangian $f(R)$ that depends only on a and $\dot{a}$ in the cosmological sector.

Another possibility would be to consider an action involving nonlinear combinations of the Riemann tensor. It was shown in [221] that a Lagrangian containing contractions of the Ricci tensor of the form $R_{\mu_1}{}^{\mu_2} R_{\mu_2}{}^{\mu_3} \cdots R_{\mu_n}{}^{\mu_1}$ indeed reduces to 5.19 for spatially flat FLRW space-times. However, the explicit expression of this Lagrangian is very cumbersome and it does not appear to be suited for calculations away from the homogeneous and isotropic sector. Furthermore, as the Lagrangian involves higher powers of the Ricci tensor, the theory will propagate more degrees of freedom than the usual two tensor modes, and these additional degrees of freedom will lead to Ostrogradsky instabilities.

Following [66], we will try to explore scalar-tensor actions, in particular mimetic theories. Following the results recalled in the previous sections, we look for a covariant action of the form

$$S[g_{\mu\nu},\phi] \;=\; \int d^4x\, \sqrt{-g}\left(\frac{fR}{16\pi G} + \mathcal{L}_\phi(\phi,\phi_\mu,\phi_{\mu\nu}) + \lambda(X+1) - \frac{1}{2}g^{\mu\nu}\psi_\mu\psi_\nu\right) \tag{5.21}$$

where f is a function of ϕ only and $\mathcal{L}_\phi$ is a scalar function which depends on ϕ and its first and second derivatives ϕ_μ and $\phi_{\mu\nu}$ only.

As we said above, the solution of our problem is far from being unique and one can find an infinite class of solutions with few restrictions on the Lagrangian $\mathcal{L}_\phi$.

Here, we want to propose the simplest class of solutions which generalize the Chamseddine-Mukhanov model, and we restrict ourselves to higher-derivative Lagrangians for the scalar field of the form

$$\mathcal{L}_\phi \;=\; \mathcal{L}_\phi\left(\phi, L_1^{(2)}, L_2^{(2)}, L_3^{(2)}, L_4^{(2)}, L_5^{(2)}\right) \tag{5.22}$$

where the $L_A^{(2)}$ are the five elementary quadratic Lagrangians introduced in 4.7. Of course, one could also consider further generalizations including the cubic elementary Lagrangians.

Now, we find the conditions that f and $\mathcal{L}_\phi$ must satisfy for the action 5.21 to reduce to 5.2 when the metric $g_{\mu\nu}$ is 4.26 and the fields ϕ and ψ depend on time only. In this case,

$$L_1^{(2)} \;=\; \left(\mathcal{D}_t^2\phi\right)^2 + 3\left(\mathcal{D}_t\phi\,\frac{\mathcal{D}_t a}{a}\right)^2 , \quad L_2^{(2)} = \left[\frac{1}{a^3}\mathcal{D}_t\left(a^3\,\mathcal{D}_t\phi\right)\right]^2 , \tag{5.23}$$

$$L_3^{(2)} \;=\; -\frac{1}{a^3}\left(\mathcal{D}_t\phi\right)^2\mathcal{D}_t^2\phi\,\mathcal{D}_t(a^3\mathcal{D}_t\phi)\,, \tag{5.24}$$

$$L_4^{(2)} \;=\; -\left(\mathcal{D}_t\phi\,\mathcal{D}_t^2\phi\right)^2 , \quad L_5^{(2)} = \left[\left(\mathcal{D}_t\phi\right)^2\mathcal{D}_t^2\phi\right]^2 , \tag{5.25}$$

where we have introduced the notation

$$\mathcal{D}_t\varphi \equiv \frac{\dot\varphi}{N} \tag{5.26}$$

for any time-dependent function $\varphi(t)$.

The advantage to consider mimetic theories is that the time derivative of the scalar field is automatically normalized since we have, in the cosmological background,

$$X \;=\; -\left(\frac{\dot\phi}{N}\right)^2 \;=\; -1\,. \tag{5.27}$$

As consequence, in the cosmological context, the elementary quadratic Lagrangians simplify drastically:

$$L_1^{(2)} = 3\left(\frac{\dot a}{Na}\right)^2 , \; L_2^{(2)} = 9\left(\frac{\dot a}{Na}\right)^2 , \; L_3^{(2)} = L_4^{(2)} = L_5^{(2)} = 0\,. \tag{5.28}$$

Due to the form of 5.2, the functions f and $\mathcal{L}_\phi$ are necessarily independent of ϕ. All these ingredients allow us to conclude immediately that the action 5.21 reproduces the LQC effective dynamics when the Lagrangian $\mathcal{L}_\phi$ takes the form

$$\mathcal{L}_\phi(L_1^{(2)}, L_2^{(2)}, L_3^{(2)}, L_4^{(2)}, L_5^{(2)}) \;=\; \left(\sum_{i=1}^{5}\alpha_i L_i^{(2)}\right) + U(L_1^{(2)}, L_2^{(2)}, L_3^{(2)}, L_4^{(2)}, L_5^{(2)}) \tag{5.29}$$

with the conditions

$$\alpha_1 + 3\alpha_2 = \frac{f}{8\pi G}\,, \quad U(3H^2, 9H^2, 0, 0, 0) = \frac{\rho_c}{2}\left(1 - \sqrt{1-\beta^2 H^2} - \beta H \arcsin(\beta H)\right) \tag{5.30}$$

for any value of H. The mimetic gravity model proposed by Chamseddine and Mukhanov clearly belongs to this class and consists in taking $f = 24\pi G\alpha_2 = 1$, with all other $\alpha_A = 0$, and U is assumed to be a function of $L_2^{(2)} = (\Box\phi)^2$ only. However, this result shows that there exist different possibilities to get the LQC effective dynamics. Nonetheless, in a theory of mimetic gravity with the condition $X + 1 = 0$, the Lagrangians $L_3^{(2)}$, $L_4^{(2)}$ and $L_5^{(2)}$ are always vanishing, not only in the cosmological sector. This is an immediate consequence of the fact that $\phi_{\mu\nu}\phi^\mu = 0$. Thus, one can restrict the function $\mathcal{L}_\phi$ to depend only on the first two arguments without loss of generality.

Let us close this section with a discussion on the case of a non-flat cosmology. The only difference with what has been done so far is that the scalar-tensor action, when evaluated for a spatially curved FLRW space-time, must include the additional $V_k(a)$ term that arises in 5.12 (at least for the 'F' loop quantization, but not for the 'K' quantization, as explained in 5.2.1.2). Such a term can be obtained by adding to the effective covariant action 5.21 with $\mathcal{L}_\phi$ given by 5.29 a new potential term

$$-\int d^4x \sqrt{|g|}\,\mathcal{V}(g_{\mu\nu}, \phi) \tag{5.31}$$

which reproduces exactly $-V_k(a)$ when evaluated on a curved FLRW space-time. As in the previous situation, we obviously do not expect the solution for $\mathcal{V}$ to be unique. However, we can exhibit some properties $\mathcal{V}$ must have. The fact that $V_k(a)$ is a highly non-linear function (even non-polynomial) of the scale factor a which does not depend on its time derivatives implies that $\mathcal{V}$ cannot be constructed from the full space-time components of the Riemann tensor only (with no contractions with derivatives of ϕ) nor from the elementary scalar-tensor Lagrangians $L_A^{(2)}$. The reason is that, when evaluated on a curved FLRW space-time, such terms necessarily produce derivatives of the scale factor which cannot be removed by integration by parts. Cubic (or even higher-order) scalar-tensor terms as $L_A^{(3)}$ produce the same problem. Only the components of the 3-dimensional Riemann tensor are functions of a only when evaluated on a curved FLRW solution. This can be easily illustrated with the 3-dimensional Ricci scalar ${}^3R$ which reduces to

$${}^3R = -\frac{6k}{a^2}\,. \tag{5.32}$$

From this point of view, the potential $\mathcal{V}(g_{\mu\nu}, \phi)$ can be constructed from the 3-dimensional Riemann tensor, in which case it breaks the Lorentz invariance. Furthermore, as V_k can be expanded as the following series

$$V_k(a) \;=\; \sum_{n>0} v_n \left(\frac{6k}{a^2}\right)^n \;\equiv\; \tilde{V}(-\frac{6k}{a^2})\,, \tag{5.33}$$

one can choose for the potential $\mathcal{V}_k$ a function which depends only on ${}^3R$ according to

$$\mathcal{V}(g_{\mu\nu},\phi) \;=\; \tilde{V}({}^3R)\,. \tag{5.34}$$

Interestingly, similar potential are considered in Hořava-Lifshitz-type models of gravity to construct renormalizable theories of gravity. Note that, using the Stückelberg method, it is nonetheless possible to render $\mathcal{V}$ fully covariant using the scalar field ϕ as a covariant clock. The price to pay is that the covariant version of $\mathcal{V}$ would involve terms like $R_{\mu\nu\rho\sigma}\phi^\mu\phi^\nu\phi^\rho\phi^\sigma$ in the action which would most probably introduce many unhealthy degrees of freedom in the theory.

5.4. Perspectives

In this chapter, we have constructed a family of higher-derivative scalar-tensor theories that possess the property to reproduce exactly the effective dynamics of loop quantum cosmology for flat, closed and open homogeneous and isotropic space-times, leading to bouncing solutions. This family thus generalizes the particular model considered by Chamseddine and Mukhanov for the spatially flat case [66].

An important question is whether this family, identified only at the level of the homogeneous and isotropic dynamics, contains a theory that could fully represent an effective description of the full loop quantum gravity (LQG). Indeed, a study limited to homogeneous cosmologies is too restrictive to be conclusive and one should go beyond homogenous and isotropic solutions to test further the interest of the theories we have identified. In particular, it would interesting to investigate whether some of them can adequately describe anisotropic space-times, cosmological perturbations, black holes, and more.

Concerning anisotropic space-times, it should be stressed that the equations of motion for the Bianchi I space-time derived in the specific mimetic theory of [66] do not coincide with the LQC effective equations given in [230, 231], even if their qualitative dynamics are quite similar (see, e.g., [232]). It would thus be worth studying whether another theory among the family identified in the present work is able to reproduce these Bianchi I equations. We leave this question for future work.

It would also be interesting to extend our study to include linear cosmological perturbations around the FLRW background. Indeed, in the presence of perturbative inhomogeneities a number of important qualitative similarities have already been noticed between some modified gravity theories and the LQC effective dynamics [233]. One might hope that there is in fact an exact correspondence for at least one mimetic theory of gravity. Another point is that, as shown in [183], ghost and gradient instabilities arise when considering cosmological perturbations for the action of the mimetic theory proposed in [66] (note that these instabilities are different from the higher derivative ghosts which mimetic gravity is safe from as a DHOST theory); it would be interesting to see if this is the case for all actions in the mimetic family, or if there exist some mimetic actions without ghost or gradient instabilities for cosmological perturbations.

Beyond the natural extensions discussed above, it should also be mentioned that an alternative description of the LQC effective dynamics offers a complementary framework that could provide new insights. One such example is the question of a possible signature change in LQC: due to a modification in the Dirac algebra of the (effective) constraints of LQC, it has been suggested that the signature of the metric may change from Lorentzian to Euclidean around the bounce point, see [234] for details. While this question is difficult to address in LQC due to the gauge-fixings that are necessary before quantization, the metric clearly remains Lorentzian at all times in the mimetic theories considered here and this could suggest that there is no signature change in LQC.

Finally, this result suggests some new links between LQG, Hořava-Lifshitz gravity, and non-commutative geometry. As seen in Sec. 5.2.1.2, for some (but not all) versions of LQC it is necessary to add a Lorentz-violating term in order to recover the correct effective Friedmann equations in the presence of scalar curvature, along the lines of what is done in Hořava-Lifshitz gravity. As Hořava-Lifshitz gravity is known to be perturbatively renormalizable, it is possible that this effective theory may be renormalizable as well. In addition, the mimetic condition 4.13 has been argued to arise naturally in non-commutative geometry when requiring the quantization of the three-dimensional volume [235]. It is intriguing that it is precisely modified gravity theories that satisfy this condition which give the LQC effective dynamics for isotropic cosmologies. An exploration in greater depth of the links between these different approaches to the problem of quantum gravity may provide important new insights.

6. Non-singular Black Holes and the Limiting Curvature Mechanism: A Hamiltonian Perspective

Sommaire

6.1. Introduction

It was realized in [55, 236] that the original theory of mimetic gravity with a limiting curvature introduced in [66] reproduces exactly the effective Loop Quantum Cosmology bounce (LQC) [30] as a homogeneous and isotropic solution [a]. Naturally, such an observation raises the question of whether one could consider mimetic gravity with a limiting curvature as an effective description (up to some energy scale) of the full theory of Loop Quantum Gravity (LQG), even out of the cosmological sector.

Nonetheless, there are at least two obstacles for such a scenario to be possible. On the one hand, LQC is not a subsector of the full theory of LQG but rather a homogenous symmetry reduced model which is quantized importing the technics of the full LQG theory. Despite some preliminary results concerning the embedding of LQC into LQG [237–241], this aspect remains up to now poorly understood. Therefore the relation established between LQC and mimetic gravity does not say much a priori about an eventual link between LQG and mimetic gravity. On the other hand, even without considering the embedding of LQC into LQG, one could wonder whether the result obtained in [55, 236] can be generalized to less symmetric situations, such as spherically symmetric [184] or Gowdy loop models [242]. Such symmetry reduced models exhibit non perturbative inhomogeneities, and as such, are highly non trivial to quantize using the LQG technics. The reason is that, contrary to the homogenous cosmological sector, the invariance under spatial diffeomorphisms survives the symmetry reduction. Therefore, the loop quantization of such inhomogenous models faces the additional difficulty of keeping the spatial covariance (in the sense of the symmetry reduced model) in the quantization. More precisely, the issue of the covariance takes the form of requiring (at the quantum level) an anomaly free Dirac's hypersurface deformation algebra (DHDA), which is the algebra of the first class constraints generating the invariance under diffeomorphisms. Classically, this algebra is generated by the Hamiltonian constraint $\mathcal{H}$, the vectorial constraints D_a satisfying

$$\{D[U^a], D[V^b]\} = D[\mathcal{L}_U V^a]\,, \quad \{D[U^a], \mathcal{H}[N]\} = -\mathcal{H}[\mathcal{L}_U N]\,, \tag{6.1}$$
$$\{\mathcal{H}[N], \mathcal{H}[M]\} = D[\gamma^{ab}\Big(N\partial_b M - M\partial_b N\Big)]\,, \tag{6.2}$$

where we have used the following standard notations: $U = U^a\partial_a$ and $V = V^a\partial_a$ are spatial vectors, N and M are scalars, $D[U^a]$ and $\mathcal{H}[N]$ are the smeared vectorial and Hamiltonian constraints respectively, γ^{ab} is the inverse spatial metric and $\mathcal{L}_U$ denotes the usual Lie derivative along the vector U. In the quantum theory, this algebra is modified. Indeed, introducing point-wise holonomy corrections to regularize the first class constraints, as required by the loop quantization

a. Notice that a very similar construction has been considered much earlier in [221] in an attempt to reproduce the effective dynamics of LQC from an $f(R)$ theory.

procedure, has very important consequences. The fate of the DHDA under the introduction of the holonomy corrections has been studied recently in a series of papers [211, 212, 243–245]. A generic result is that for system having no local degrees of freedom, such as vacuum spherically symmetric gravity [211], or Gowdy model with local rotational symmetry [243], the DHDA exhibits a deformed notion of covariance in the sense that (6.2) is replaced by

$$\{\mathcal{H}[N], \mathcal{H}[M]\} = \beta(\tilde{K}) D[\gamma^{ab}\Big(N\partial_b M - M\partial_b N\Big)] \, , \tag{6.3}$$

where $\beta(\tilde{K})$ is a "deformation", called sometimes the "sign change deformation", which depends generically on the homogeneous component of the extrinsic curvature $\tilde{K}$. For systems having local degrees of freedom, the situation is even worse since the DHDA has been shown not to be a close algebra anymore.

All this seems to lead us to the conclusion that it is not possible to provide an effective description of such loop quantized phase space using a covariant scalar tensor theory such as [66, 194], since these theories share the same (undeformed) DHDA as classical General Relativity (augmented with a scalar degree of freedom). A first notorious complication in such model is that the quantum corrections are implemented at the Hamiltonian level and going back to a Lagrangian formulation is often very complicated (see [246] for earlier investigations on this aspect). Nonetheless, the situation is not totally hopeless. There are at least two reasons to think that one can circumvent these apparent obstacles we have discussed above. The first one relies on the observation that the difficulties related to the deformation of the DHDA seems to be inherent to the fact of working with the real Ashtekar-Barbero variables. Recent results have shown that such difficulties disappear when working with the initial self-dual variables, at least in the case of spherical symmetry gravity coupled to matter and unpolarized Gowdy models [218, 219]. In this situation, the DHDA keeps indeed its classical form without any deformation even if some holonomy corrections, including a $\bar{\mu}$-scheme, have been taken into account. Therefore, this opens the possibility of describing these self-dual loop symmetry reduced models using a covariant scalar-tensor theory. The second one relies on the fact that up to now, the polymer construction of black hole models (as well as inhomogeneous cosmology and Gowdy system) has focused only on the minimal loop regularization, namely the point-wise holonomy corrections. Yet, only few investigations have considered together the additional corrections inherent to LQG: the triad corrections related to the inverse volume regularization, as well as the holonomy of the inhomogeneous component of the connection. Therefore, it is not clear if the deformation (6.3) will not be removed when one takes into account these additional corrections.

Facing the problem of the deformed notion of covariance in polymer models inspired from LQG, we adopt the following strategy. As shown in [55, 236], mimetic gravity with a limiting curvature is a covariant scalar-tensor theory which reproduces in the cosmolgical sector the effective dynamics of LQC. Thus, we consider this theory as a potential guide to obtain an effective Lagrangian description of point wise holonomy-like corrections which maintain the covariance beyond the cosmological sector. As a first study, we focus our attention on the spherically symmetric sector and derive its Hamiltonian formulation. As expected, it leads to effective quantum corrections similar to the ones introduced in the polymer framework, but in a covariant manner. The difference between the mimetic effective corrections and the polymer ones provides an interesting guide to understand the lack of covariance of polymer black hole models.

The chapter is organized as follows. In section II, we recall some important aspects of the extended theories of mimetic gravity and we illustrate the limiting curvature mechanism in the context of black holes. In section III, we revisit the non-singular black hole solution described in [66] from a Hamiltonian point of view. In particular, we find a parametrization of the phase space which makes the resolution of the equations of motion simpler than in the Lagrangian formulation. Furthermore, we exhibit the explicit form of the Hamiltonian constraint for mimetic gravity with a limiting curvature in the case of a spherically symmetric space-time. In section IV, we compare the Hamiltonian formulation of this theory of mimetic gravity with those obtained in LQG from a “loop" regularization of the usual Hamiltonian constraint of gravity. Contrary to what happens in the cosmological sector, the theory of mimetic gravity proposed in [66] does not reproduce the effective dynamics of spherically symmetric LQG. However, we exhibit a theory in the class of extended mimetic gravity whose dynamics reproduces the general shape of the effective corrections of spherically symmetric polymer models, but in an undeformed covariant manner. In that respect, extended mimetic gravity can be viewed as an effective covariant theory which naturally implements a covariant notion of point wise holonomy-like corrections similar in spirit to the ones used in polymer models. The difference between the mimetic and polymer Hamiltonian formulations provides us with a guide to understand the lack of covariance in inhomogeneous polymer models.

6.2. Mimetic gravity and the limiting curvature hypothesis

In this section, we review some aspects of extended theories of mimetic gravity [247, 248]. Extended mimetic gravity generalizes the original proposal of [178]: it is still conformally invariant and propagates generically only one scalar in addition to the usual two tensorial modes. Then, we recall how to implement the limiting curvature hypothesis in this context and how the proposal of [66] leads indeed to a non-singular black hole solution.

6.2.1. Extended mimetic gravity

Let us start by discussing the original theory of mimetic gravity defined by the action (4.14)

$$S[g_{\mu\nu},\phi,\lambda] \;\equiv\; \int d^4x\,\sqrt{-g}\,\left[\frac{1}{2}\mathcal{R}+\lambda(g^{\mu\nu}\phi_\mu\phi_\nu\pm 1)\right]\,, \tag{6.4}$$

where the sign (± 1) is for the moment arbitrary and will be fixed hereafter: the $+$ (resp. $-$) sign implies that ϕ^μ are the components of a timelike (resp. spacelike) vector. The dynamical variables are the metric $g_{\mu\nu}$ with signature $(-1,+1,+1,+1)$, the scalar field ϕ and the extra variable λ.

6.2.1.1. Equations of motion

The equations of motion are easily computed, and the Euler-Lagrange equations for λ, ϕ and $g_{\mu\nu}$ are respectively given by

$$X\pm 1 \;=\; 0\,, \qquad \nabla_\mu(\lambda\phi^\mu) \;=\; 0\,, \qquad G_{\mu\nu} \;=\; T_{\mu\nu}\,, \tag{6.5}$$

where the stress-energy tensor for the “mimetic" matter field is

$$T_{\mu\nu} \;=\; -2\lambda\phi_\mu\phi_\nu+\lambda(X\pm 1)g_{\mu\nu}\,. \tag{6.6}$$

Let us recall that we used the same notations as in the introduction for the gradient $\phi_\mu\equiv\nabla_\mu\phi$ and the kinetic energy $X\equiv\phi^\mu\phi_\mu$. The trace of the last equation in (6.5) allows us to express the variable λ in terms of the Ricci scalar according to

$$\lambda \;=\; \pm\frac{1}{2}\mathcal{R}\,. \tag{6.7}$$

Substituting this expression for λ into the first two equations in (6.5) leads to the following system of equations for the scalar field and the metric

$$\nabla_\mu(\mathcal{R}\phi^\mu) \;=\; 0\,, \qquad G_{\mu\nu}\pm\mathcal{R}\phi_\mu\phi_\nu \;=\; 0\,. \tag{6.8}$$

Notice that the first equation (for the scalar field) is not independent from Einstein equations. Indeed, the conservation of the Einstein tensor $\nabla^\mu G_{\mu\nu}$ together with the mimetic condition necessarily imply that

$$0 \;=\; \nabla^\mu G_{\mu\nu}=\nabla^\mu(\mathcal{R}\phi_\mu\phi_\nu)=\nabla^\mu(\mathcal{R}\phi_\mu)\phi_\nu+\mathcal{R}\phi^\mu\phi_{\mu\nu}=\nabla^\mu(\mathcal{R}\phi_\mu)\phi_\nu\Longrightarrow\nabla_\mu(\mathcal{R}\phi^\mu)=0 \tag{6.9}$$

Hence, the equations of motion (6.5) are equivalent to

$$G_{\mu\nu}\pm\mathcal{R}\phi_\mu\phi_\nu \;=\; 0\,, \quad X\pm 1 \;=\; 0\,, \quad \lambda \;=\; \pm\frac{1}{2}\mathcal{R}\,. \tag{6.10}$$

These equations have been solved for cosmological space-times in [178].

6.2.1.2. Generalization: extended mimetic gravity

The mimetic action (6.4) can be generalized to the form [248]

$$S[g_{\mu\nu}, \phi, \lambda] \equiv \int d^4x \sqrt{-g} \left[\frac{f(\phi)}{2} \mathcal{R} + L_\phi(\phi, \chi_1, \cdots, \chi_p) + \lambda(X \pm 1) \right], \quad (6.11)$$

where f is an arbitrary function of ϕ, L_ϕ depends on ϕ and χ_n which are variables constructed with second derivatives of the scalar field according to

$$\chi_n \equiv \mathrm{Tr}([\phi]^n) \equiv \sum_{\mu_1, \cdots, \mu_n} \phi^{\mu_2}_{\mu_1} \phi^{\mu_3}_{\mu_2} \cdots \phi^{\mu_n}_{\mu_{n-1}} \phi^{\mu_1}_{\mu_n} . \quad (6.12)$$

Here, we have used the notation $[\phi]$ for the matrix whose coefficients are $[\phi]_{\mu\nu} \equiv \phi_{\mu\nu}$. Indices are lowered and raised by the metric and its inverse. One can show that this extended action defines the most general mimetic gravity like theory [247, 248] which propagates at most three degrees of freedom (one scalar in addition to the usual two tensorial modes). Notice that mimetic gravity has recently been generalized in [249] to actions with higher derivatives of the metric (which propagate more than three degrees of freedom out of the unitary gauge). Due to the mimetic condition $X \pm 1 = 0$, any X dependency in f or L_ϕ can be removed. More precisely, as it was shown in [248], if one starts with an action (6.11) where f and L_ϕ depend also on X (and eventually its derivatives $\partial_\mu X$), the associated equations of motion are equivalent to the equations of motion obtained from the same action where f and L_ϕ are evaluated to $X = \mp 1$ (and eventually $\partial_\mu X = 0$).

The Euler-Lagrange equations for (6.11) can be easily obtained in full generality. But, for simplicity, we assume that f is a constant (and thus independent of ϕ) which can be fixed to $f = 1$. Deriving the action with respect to ϕ and $g_{\mu\nu}$ respectively leads to the equations

$$\frac{\partial L_\phi}{\partial \phi} - 2\nabla^\mu(\lambda\phi_\mu) + \sum_{n=1}^{p} n \, \nabla^{\mu\nu} \left([\phi]^{n-1}_{\mu\nu} \frac{\partial L_\phi}{\partial \chi_n} \right) = 0 \quad \text{and} \quad G_{\mu\nu} = T_{\mu\nu} , \quad (6.13)$$

where $[\phi]^n$ is the power n of the matrix $[\phi]$ with the convention $[\phi]^0_{\mu\nu} \equiv g_{\mu\nu}$, and now the stress-energy tensor reads

$$T_{\mu\nu} = -2\lambda\phi_\mu\phi_\nu + \lambda(X \pm 1)g_{\mu\nu} + T^{(\phi)}_{\mu\nu} \quad \text{with} \quad (6.14)$$

$$T^{(\phi)}_{\mu\nu} \equiv L_\phi \, g_{\mu\nu} + \sum_{n=1}^{p} n \left\{ -2 \frac{\partial L_\phi}{\partial \chi_n} [\phi]^n_{\mu\nu} + \nabla^\alpha \left[\frac{\partial L_\phi}{\partial \chi_n} \left([\phi]^{n-1}_{\alpha\mu} \phi_\nu + [\phi]^{n-1}_{\alpha\nu} \phi_\mu - [\phi]^{n-1}_{\mu\nu} \phi_\alpha \right) \right] \right\} \quad (6.15)$$

To get rid of λ in the Einstein equation, we proceed as in the previous case. First, we take the trace of the second equation in (6.13) to express λ in terms of ϕ and $g_{\mu\nu}$

$$\lambda = \mp\frac{1}{2}(\mathcal{R}+T^{(\phi)}), \quad T^{(\phi)} = 4L_\phi - \sum_{n=1}^{p} n\left\{2\frac{\partial L_\phi}{\partial \chi_n}\chi_n + \nabla^\alpha\left[\phi_\alpha \frac{\partial L_\phi}{\partial \chi_n}\chi_{n-1}\right]\right\} \tag{6.16}$$

Then we substitute this expression in the two equations above (6.13). Furthermore, the equation for the scalar field can be obtained from the conservation of the stress-energy tensor, and thus is not independent from Einstein equations.

Hence, as the trace of Einstein equations is trivially satisfied (the trace has been used to determine λ), the equations of motion are equivalent to the mimetic condition and (the traceless part of) Einstein equations only:

$$X \pm 1 = 0, \qquad G_{\mu\nu} = \pm(\mathcal{R}+T^{(\phi)})\,\phi_\mu\phi_\nu + T^{(\phi)}_{\mu\nu}. \tag{6.17}$$

Solutions to these equations have been studied in the context of cosmology [66] and black holes [193] with a particular choice for L_ϕ which makes the solutions nonsingular. Here, we focus on black hole solutions and we are going to see how one can choose L_ϕ to resolve the black hole singularity.

6.2.2. Black hole with a limiting curvature

The non-singular black hole introduced by Chamseddine and Mukhanov in [193] is a “static" spherically symmetric solution of the general mimetic action (6.11) where L_ϕ is a function of χ_1 only defined by

$$L_\phi(\chi_1) = \frac{2}{3}\rho_m\, f(x), \quad x = \frac{\chi_1}{\sqrt{\rho_m}}, \quad f(x) \equiv 1 + \frac{1}{2}x^2 - \sqrt{1-x^2} - x\arcsin x \tag{6.18}$$

where ρ_m defines a new energy scale in the theory. This expression of L_ϕ seems to be an ad hoc choice a priori, but it leads to very appealing nonsingular cosmological and black hole solutions. Notice that, in the cosmological sector, the equation of motion of the scale factor reproduces exactly the effective dynamics of LQC as it was pointed out in [55, 236].

Let us now consider a spherically symmetric space-time only, and let us explain physically why (6.18) produces non-singular black hole solutions. For that purpose, we start writing the metric in Schwarzschild coordinates

$$ds^2 = -F(R)\,dT^2 + \frac{1}{F(R)}\,dR^2 + R^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{6.19}$$

where F is a function of R only. In usual general relativity (with no modifications), Einstein equations lead to the Schwarzschild solution where $F(R) = 1 - 2m/R$, m being the mass of the black hole. Thus, there is an event horizon at $R = 2m$ and one can distinguish between the outside ($F > 0$ or $R > 2m$) and the inside of the black hole ($F < 0$ or $R < 2m$). The singularity occurs inside the black hole where the curvature becomes arbitrary large (in the limit $R \to 0$).

It was shown in the original paper [193] that the action (6.11) with the field Lagrangian (6.18) reproduces correctly the Schwarzschild metric far from the high curvature regions (compared with the scale ρ_m). In particular, spherically symmetric solution (6.19) possesses an event horizon, very similar to the Schwarzschild horizon. Thus one can still define a region inside (or behind) the horizon ($F < 0$) and a region outside the horizon ($F > 0$).

Concerning the scalar field, let us start assuming that it depends on R and T for purposes of generality. We will shortly reduce ourselves to the case of a static scalar field $\phi(R)$. The scalar field satisfies the mimetic condition which reads

$$-\frac{1}{F(R)}\left(\frac{\partial\phi}{\partial T}\right)^2 + F(R)\left(\frac{\partial\phi}{\partial R}\right)^2 \pm 1 \;=\; 0\,. \tag{6.20}$$

This equation allows to resolve the scalar field ϕ in terms of the geometry $F(R)$. A simple class of solutions of this partial differential equation can be obtained from the ansatz

$$\phi(R,T) \;=\; qT + \psi(R)\,, \tag{6.21}$$

where q is a constant and ψ satisfies

$$\left(\frac{d\psi}{dR}\right)^2 \;=\; \frac{q^2 \mp F}{F^2}\,. \tag{6.22}$$

Notice that similar ansatz were considered in [250, 251] to find black holes and stars solutions in the context of Horndeski (or beyond Horndeski) theories.

It is clear that the equation (6.22) admits a solution only if the condition $q^2 \mp F \geq 0$ is fulfilled. As a result, in the static case (where $q = 0$), one cannot find any global spherically symmetric solution for the space-time. Indeed, the condition $\pm F \leq 0$ implies that only the action with a $+$ (resp. $-$) sign could lead to a description of the region inside (resp. outside) the black hole. Only a non-static solution for the scalar field ($q \neq 0$) could enable us to describe a fully static spherically symmetric space-time. However, we will proceed as in [193]: we will restrict ourselves to the region inside the black hole (we expect the limiting curvature hypothesis to affect mainly the regions inside the black hole), we choose a mimetic action with a $+$ sign, and we will argue how this is enough to resolve indeed the singularity. From a phenomenological point of view, we could interpret the action (6.11) with (6.18) as an effective description of general relativity in a region (inside the black hole) where the curvature becomes high (with respect to the scale ρ_m). Such a modification could result from quantum gravity effects for instance [55].

When the scalar field is static, the mimetic condition reduces to a simple differential equation

$$\left(\frac{d\phi}{dR}\right)^2 = -\frac{1}{F}, \tag{6.23}$$

in the region (behind the horizon) where $F \leq 0$ (with appropriate boundary conditions). The form of L_ϕ (the presence of $\arcsin(x)$ or $\sqrt{1-x^2}$ with $x = \chi_1/\sqrt{\rho_m}$ for instance) imposes that the scalar field ϕ must satisfy the condition

$$|\chi_1| \leq \sqrt{\rho_m} \implies |\frac{d}{dR}\left(R^2\sqrt{-F}\right)| \leq \sqrt{\rho_m}R^2\,. \tag{6.24}$$

If one naively substitutes the Schwarzschild solution in this inequality, one gets the condition that

$$\rho \equiv \frac{m}{R^3} \leq \frac{2}{9}\rho_m\,, \tag{6.25}$$

which can be interpreted by the fact that the density inside the black hole is bounded from above. Hence, one would expect the singularity to be resolved. This has been shown to be indeed the case in [193] from a resolution of the equations of motion. We are going to reproduce this result in the next section from a Hamiltonian point of view.

6.3. Hamiltonian description

In this section, we perform the Hamiltonian analysis of the mimetic action with a limiting curvature. We first introduce the ADM parametrization for the metric and we solve the mimetic condition to integrate out the scalar field ϕ. Then, we start the Hamiltonian analysis and we find a nice parametrization of the phase space such that the Hamiltonian and vectorial constraints take a rather simple form. Finally, we resolve the Hamilton equations far behind the horizon, and we recover that the solution is indeed non-singular. As we are going to see, the Hamiltonian point of view leads to a simpler analysis of the equations of motion than the Lagrangian point of view, as it was done in [193].

6.3.1. 3+1 decomposition: metric and scalar field

In this subsection, we start introducing the tools which are necessary to perform the Hamiltonian analysis of the theory restricted to spherically symmetric geometries. Notice that the “radial" and “time" coordinates (in the Schwarzschild parametrization) exchange their roles when one crosses the horizon. In particular, the time coordinate behind the horizon would correspond to the radial coordinate in the Schwarzschild parametrization.

6.3.1.1. ADM decomposition for spherical space-time

We start with the usual ADM decomposition of the (non-static) spherically symmetric metric inside the black hole:

$$ds^2 = -N^2 dt^2 + \gamma_{rr}(dr + N^r dt)^2 + \gamma_{\theta\theta} d\Omega^2 , \quad d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2 , \tag{6.26}$$

where $N(r,t)$ is the lapse function, $N^r(r,t)$ is the radial component of the shift vector and

$$\gamma \equiv \text{diag}\left[\gamma_{rr}(r,t), \gamma_{\theta\theta}(r,t), \gamma_{\theta\theta}(r,t)\sin^2\theta\right] \tag{6.27}$$

are the non-vanishing components of the (spherically symmetric) induced metric on the three dimensional space-like hypersurface. In the following, we will use the standard notations $\gamma^{rr} \equiv \gamma_{rr}^{-1}$ and $\gamma^{\theta\theta} \equiv \gamma_{\theta\theta}^{-1}$ for the components of the inverse metric γ^{-1}.

6.3.1.2. Resolution of the mimetic condition to integrate out the scalar field

Concerning the scalar field, we assume that it depends on time t only (which corresponds to a “static" solution from the point of view of an observer outside the horizon). The mimetic condition $X + 1 = 0$ implies that the lapse function necessarily depends on t according to

$$\dot{\phi}(t)^2 = N(t)^2 . \tag{6.28}$$

Without loss of generality, we take the solution $\dot{\phi} = +N$ that we substitute in the action in order to integrate out the scalar field ϕ. To do so, we also need to compute $\chi_1 = \Box\phi$ in terms of the metric variables.

Thus, we start by computing second derivatives of the scalar field $\phi_{\mu\nu}$, and an easy calculation shows that the only non-vansihing components of $\phi_{\mu\nu}$ are

$$\phi_{tt} = -(N^r)^2 K_{rr} , \quad \phi_{rt} = -N^r K_{rr} , \quad \phi_{ii} = -K_{ii} , \tag{6.29}$$

where $i \in \{r, \theta, \varphi\}$ labels spatial coordinates, and K_{ij} are the components of the extrinsic curvature

$$K_{ij} \equiv \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i) , \tag{6.30}$$

with D_i being the covariant derivative compatible with the spatial metric γ_{ij} (6.27). To go further, one has to compute explicitly the components of the extrinsic curvature. Only the diagonal components are non-trivial with

$$K_{rr} = \frac{1}{2N}\left(\dot{\gamma}_{rr} - \gamma'_{rr}N^r - 2\gamma_{rr}(N^r)'\right) \, , \qquad K_{\theta\theta} = \frac{K_{\varphi\varphi}}{\sin^2\theta} = \frac{1}{2N}\left(\dot{\gamma}_{\theta\theta} - \gamma'_{\theta\theta}N^r\right) \tag{6.31}$$

As usual, dot and prime denote respectively time and radial derivatives. We have now all the ingredients to start the Hamiltonian analysis of the theory.

6.3.2. Hamiltonian analysis

Using the Gauss-Codazzi relation, the action (6.4) with $f(\phi) = 1$ and L_ϕ given by (6.18) reduces to the form

$$S \;=\; \int dt\, d^3x\, N\sqrt{\gamma}\left[\frac{1}{2}(K_{ij}K^{ij} - K^2 + R) + L_\phi(K)\right] , \tag{6.32}$$

where we have substituted the solution of the mimetic constraint for the scalar field (6.29), and R is the three-curvature whose expression in terms of the metric components is

$$R \;=\; \frac{2}{\gamma_{\theta\theta}}\left[1 - \left(\frac{\gamma'_{\theta\theta}}{\sqrt{\gamma_{\theta\theta}\gamma_{rr}}}\right)^2 + \frac{2}{\gamma_{rr}}\left(\frac{\gamma'_{\theta\theta}}{\sqrt{\gamma_{\theta\theta}\gamma_{rr}}}\right)'\right] . \tag{6.33}$$

To simplify the Hamiltonian analysis, it is more convenient to introduce a new parametrization of the metric in terms of the variables α and γ defined by

$$\left\{\gamma \;\equiv\; \gamma_{rr}\gamma_{\theta\theta}^2 \, , \quad \alpha \;\equiv\; \frac{\gamma_{rr}}{\gamma_{\theta\theta}}\right\} \Longleftrightarrow \left\{\gamma_{rr} = (\gamma\alpha^2)^{1/3} \, , \quad \gamma_{\theta\theta} = \left(\frac{\gamma}{\alpha}\right)^{1/3}\right\} . \tag{6.34}$$

With these new variables, the action simplifies and becomes

$$S \;=\; \int dt\, d^3x\, \sqrt{\gamma}\left[\frac{1}{12N}\left(A^2 - B^2\right) + NL_\phi\left(\frac{B}{2N}\right)\right] \tag{6.35}$$

where A and B depend respectively on the variables (α, N^r) and (γ, N^r) according to

$$A \equiv \frac{\dot{\alpha}}{\alpha} - \frac{\alpha'}{\alpha}N^r - 2(N^r)' \, , \qquad B \equiv \frac{\dot{\gamma}}{\gamma} - \frac{\gamma'}{\gamma}N^r - 2(N^r)' \, . \tag{6.36}$$

Notice that R can be expressed explicitly in terms of α and γ, what we will not do here because its explicit form is not needed for our purposes.

It is clear from this expression of the Lagrangian (6.35) that N and N^r are Lagrange multipliers, and there are only two pairs of conjugate variables defined by the Poisson brackets

$$\{\alpha(u), \pi_\alpha(v)\} \;=\; \delta(u-v) \;=\; \{\gamma(u), \pi_\gamma(v)\}\,. \tag{6.37}$$

Momenta are easily computed and are given in terms of the velocities by

$$\pi_\alpha \;=\; \frac{1}{3}\frac{\sqrt{\gamma}}{N\alpha}A\,, \qquad \pi_\gamma \;=\; -\frac{1}{3}\sqrt{\frac{\rho_m}{\gamma}}\,\arcsin\left(\frac{B}{2N\sqrt{\rho_m}}\right)\,. \tag{6.38}$$

Inverting these relations to obtain velocities in terms of momenta is immediate

$$\frac{\dot\alpha}{\alpha} \;=\; 3\frac{N\alpha}{\sqrt{\gamma}}\pi_\alpha + \frac{\alpha'}{\alpha}N^r + 2(N^r)'\,, \tag{6.39}$$

$$\frac{\dot\gamma}{\gamma} \;=\; -2N\sqrt{\rho_m}\sin\left(3\sqrt{\frac{\rho_m}{\gamma}}\pi_\gamma\right) + \frac{\gamma'}{\gamma}N^r + 2(N^r)'\,. \tag{6.40}$$

From these expressions, one easily deduces the expression of the Hamiltonian

$$H \;=\; \int dr\;(N\mathcal{H} + N^r D_r) \tag{6.41}$$

with the Hamiltonian and vectorial constraints respectively given by

$$\mathcal{H} \;\equiv\; \frac{3}{\sqrt{\gamma}}\alpha^2\pi_\alpha^2 - \sqrt{\gamma}\left[\frac{4}{3}\rho_m\sin^2\left(\frac{3}{2}\sqrt{\frac{\gamma}{\rho_m}}\pi_\gamma\right) + \frac{1}{2}R\right]\,, \tag{6.42}$$

$$D_r \;\equiv\; -\left(\gamma'\pi_\gamma + 2\gamma\pi_\gamma' + \alpha'\pi_\alpha + 2\alpha\pi_\alpha'\right)\,. \tag{6.43}$$

Hence, time derivative of any functionnal $\mathcal{O}$ of the phase space variables is computed from the Poisson bracket

$$\dot{\mathcal{O}}(t,r) \;=\; \{\mathcal{O}; H\}\,, \tag{6.44}$$

from which one easily deduces the equations of motion. To these equations, one adds the two constraints $\mathcal{H} \approx 0$ and $D_r \approx 0$ to integrate completely the system. Notice that $\approx$ denotes the weak equality in the phase space.

Notice that the Hamiltonian constraint can be written as

$$\rho \equiv \frac{9\alpha^2}{4\gamma}\pi_\alpha^2 - \frac{3}{8}R \;\approx\; \rho_m\,\sin^2\left(\frac{3}{2}\sqrt{\frac{\gamma}{\rho_m}}\pi_\gamma\right)\,, \tag{6.45}$$

where ρ will be interpreted as the energy density of the mimetic scalar field, as we are going to argue later. The density ρ is necessarily bounded by ρ_m.

6.3.3. Spherically "static" solutions deeply inside the black hole

To go further, we restrict ourselves to spherically static solutions which correspond, in the black hole interior, to having a time dependency only.

6.3.3.1. Equations of motion

In that case, the vectorial constraint $D_r = 0$ is strongly satisfied and the expression of the 3-dimensional Ricci scalar reduces to $R = 2(\alpha/\gamma)^{1/3}$. Hence, after some simple calculations, one shows that the equations of motion for the phase space variables are

$$\dot{\alpha} = \frac{6\alpha^2\pi_\alpha}{\sqrt{\gamma}}\,, \qquad \dot{\pi}_\alpha = -\frac{6\alpha\pi_\alpha^2}{\sqrt{\gamma}} + \frac{1}{3}\frac{\gamma^{1/6}}{\alpha^{2/3}}\,, \qquad \dot{\gamma} = -2\gamma\sqrt{\rho_m}\,\sin\left(3\sqrt{\frac{\gamma}{\rho_m}}\pi_\gamma\right) \tag{6.46}$$

$$\dot{\pi}_\gamma = \frac{3}{2}\frac{\alpha^2\pi_\alpha^2}{\gamma^{2/3}} + \frac{\rho_m}{3\sqrt{\gamma}}\sin^2\left(\frac{3}{2}\sqrt{\frac{\gamma}{\rho_m}}\pi_\gamma\right) + \frac{\rho_m}{2}\pi_\gamma \sin\left(3\sqrt{\frac{\gamma}{\rho_m}}\pi_\gamma\right) - \frac{\alpha^{1/3}}{6\gamma^{5/3}} \tag{6.47}$$

where we have fixed the lapse to the value $N = 1$ for simplicity (which corresponds to a redefinition of the time variables). These equations are highly non-linear and very cumbersome to solve. Even though we do not expect to find explicit solutions, we are going to present some interesting properties they satisfy.

First of all, the equation for γ (6.46) together with the Hamiltonian constraint (6.45) leads to the so-called master equation in [193] given by

$$\left(\frac{\dot{\gamma}}{4\gamma}\right)^2 = \rho\left(1 - \frac{\rho}{\rho_m}\right)\,, \quad \text{with} \quad \rho = \frac{1}{4}\left[\frac{1}{4}\left(\frac{\dot{\alpha}}{\alpha}\right)^2 - 3\left(\frac{\alpha}{\gamma}\right)^{1/3}\right]\,. \tag{6.48}$$

In [193], the same equation has been obtained from the Lagrangian point of view in terms of the variable $\epsilon \equiv (4/3)\rho$ and the constant $\epsilon_m \equiv (4/3)\rho_m$. We recover an equation very similar to the one satisfied by the scale factor in the framework of effective LQC [55], which leads to a non-singular scenario for early universe cosmology. Here, we have a similar dynamics for γ which is also bounded from below: this leads to a non-singular black hole solution [193] as it can be easily seen from the expression of the Kretschmann tensor $\mathcal{K} = 4(\alpha/\gamma)^{1/3}$: it is bounded when γ does not vanish, provided that α does not tend to infinity neither.

6.3.3.2. Deep inside the black hole: resolution of the singularity

To understand better the dynamics of the variables α and γ, let us consider the regime where the 3-dimensional scalar curvature R becomes negligible in the expression of the energy density (6.45). This hypothesis implies the condition

$$\frac{\alpha^2\pi_\alpha^2}{\gamma} \gg R \quad \Longleftrightarrow \quad \left(\frac{\dot{\alpha}}{\alpha}\right)^2 \gg \left(\frac{\alpha}{\gamma}\right)^{1/3} . \tag{6.49}$$

It has been shown in [193] that this condition is satisfied deeply inside the black hole (in the region where the quantum effects are supposed to be important). In this situation, the equations for α and π_α become

$$\dot{\alpha} = \frac{6\alpha^2\pi_\alpha}{\sqrt{\gamma}} , \qquad \dot{\pi}_\alpha = -\frac{6\alpha\pi_\alpha^2}{\sqrt{\gamma}} , \tag{6.50}$$

which can be integrated exactly. First, α can be expressed in terms of γ using the relations:

$$\sqrt{\gamma}\,\frac{\dot{\alpha}}{\alpha} \;=\; C \quad \Longleftrightarrow \quad \alpha(t) \;=\; \alpha_0 \,\exp\left(C\int_0^t \frac{du}{\sqrt{\gamma(u)}}\right) , \tag{6.51}$$

where C and α_0 are integration constants. Then, the equation for γ (6.48) simplifies and becomes

$$\dot{\gamma}^2 \;=\; C^2\,(\gamma - \gamma_m) , \qquad \gamma_m \equiv \frac{C^2}{16\rho_m} , \tag{6.52}$$

from which we deduce immediately that γ is bounded from below by γ_m. Furthermore, one easily integrates this equation and obtains an exact expression for γ (in the regime where R is negligible in the expression of $\mathcal{H}_0$) given by

$$\gamma(t) \;=\; \gamma_m(1 + 4\rho_m t^2) , \tag{6.53}$$

assuming that $\gamma(0) = \gamma_m$. This expression coincides completely with Eq.(64) of [193] when the constant C has been fixed to $C = 3r_g$ (r_g being the Schwarzchild radius). Finally, substituting this expression in (6.51), one obtains

$$\alpha(t) \;=\; a\,\exp\left[2\,\sinh^{-1}(2\sqrt{\rho_m}t)\right] , \tag{6.54}$$

where a is a new constant. Hence, we recover exactly the solution found in [193] where a has been fixed to

$$a \;=\; \frac{64}{9}\,\rho_m . \tag{6.55}$$

As a conclusion, the singularity is clearly avoided and replaced by a bounce very similar to the LQC bounce.

Let us recall that the analysis of this subsection is valid only when the condition

(6.49) is fulfilled. Using the relation (6.51), the validity condition can be more explicitly given by

$$C^6 \gg \alpha\,\gamma^2 \quad \Longleftrightarrow \quad 18^2 r_g^2 \rho_m \gg (1+4\rho_m t^2)^2 \,\exp\left[2\,\sinh^{-1}(2\sqrt{\rho_m}t)\right]. \quad (6.56)$$

Thus, $\rho_m t^2 \ll 1$ is sufficient for this condition to be satisfied provided that

$$Q \;\equiv\; r_g^2 \rho_m \gg 1\,, \quad (6.57)$$

which is obviously the case deep inside the black hole. However, this is not necessary. Indeed, in the regime where $\rho_m t^2 \gg 1$, the previous condition gives

$$\rho_m t^3 \;\ll\; r_g\,, \quad (6.58)$$

which is less restrictive than $\rho_m t^2 \ll 1$. Hence, the solution (deep inside the black hole) given by (6.53) and (6.54) is valid when t is sufficiently small according to (6.58).

6.3.3.3. Comparing with the Schwarzschild solution

To conclude this analysis of the geometry deep inside the black hole, let us make a comparison with the usual Schwarzschild solution. When expressed in terms of the parameters α and γ, the Schwarzschild black hole is defined by (see [193] for instance)

$$\alpha_s \;=\; \frac{1-\tau^2}{r_g^2\tau^6}\,, \quad \gamma_s \;=\; r_g^4(1-\tau^2)\tau^6 \quad \text{with} \quad \frac{t}{r_g} \;\equiv\; \arcsin\tau - \tau\sqrt{1-\tau^2}\,. \quad (6.59)$$

In the regime (6.58), we necessarily have $t/r_g \ll Q^{-1/3} \ll 1$, hence $\tau \ll 1$ and then

$$\tau^3 \;\approx\; \frac{3t}{2r_g}\,. \quad (6.60)$$

Thus, the Schwarzschild solution, to compare the regularized solution with, reduces to

$$\alpha_s(t) \;\approx\; \frac{4}{9t^2}\,, \quad \gamma_s(t) \;\approx\; \frac{9}{4}r_g^2 t^2\,, \quad (6.61)$$

in the region where the curvature is high. We see that both α and γ^{-1} are regularized compared to the classical Schwarzschild solution as shown in Fig. (1): this is the effect of the limiting curvature. However, close to the horizon (which corresponds to $\tau \approx 1$), one recovers the Schwarzschild solution because the limiting curvature effect becomes negligible.

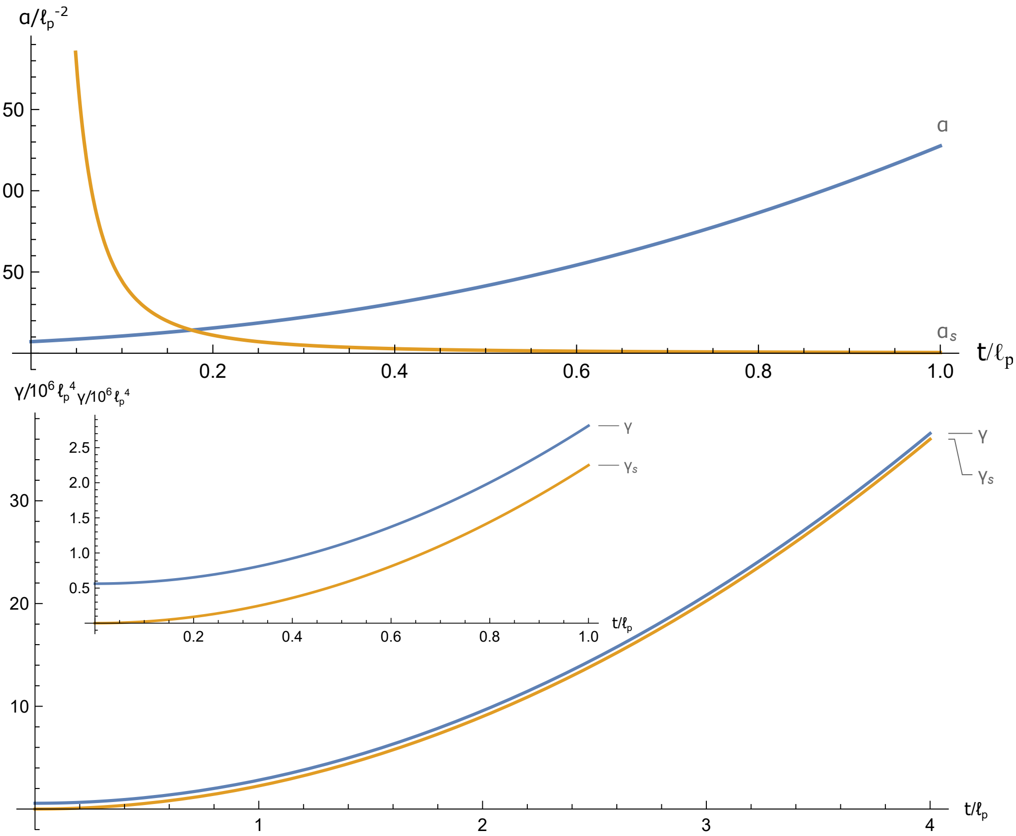


Figure 1. – These two graphs show the solutions $\alpha(t)$ and $\gamma(t)$ compared to the Schwarzschild solution $\alpha_s(t)$ and $\gamma_s(t)$ in the deep quantum regime (6.58). In particular, we see that the limiting curvature effect makes the functions $\alpha(t)$ and $1/\gamma(t)$ no more divergent when $t \to 0$, which regularizes the singularity. In these plots, we work in the Planck unit with $r_g = 1000\ell_p$ and $\rho_m = \ell_p^{-2}$, where ℓ_p is the Planck length. In that case, the deep quantum regime condition (6.58) becomes $t \ll 10\ell_p$.

6.4. On the relation with effective polymer black holes

In [55, 236], it was shown that extended mimetic gravity with (6.18) reproduces the dynamics of effective LQC when restricted to homogenous and isotropic cosmology. A natural question is whether this relation extends itself to the spherically symmetric sector. We explore this question in this section where we compare the Hamiltonian of the non-singular black hole with limiting curvature with the standard polymer model of vacuum spherically symmetric gravity in terms of Ashtekar-Barbero variables studied in [184]. While the relation between the Chamseddine-Mukhanov mimetic theory with limiting curvature and

LQC is shown not to hold beyond homogenous geometries, one can propose new mimetic theories to describe a generalized regularization for polymer models which provides us with an undeformed notion of covariance as well as a natural inbuilt $\bar{\mu}$-scheme.

6.4.1. Comparing mimetic and polymer black holes

In this section, we compare the Hamiltonian structures of LQG and mimetic gravity with a limiting curvature. Following the introduction section 4.3.2 of this part, we can write the effective constraints of spherically symmetric LQG as follows:

$$\mathcal{H} = |E^x|^{-1/2}\left[E^\phi \frac{\sin^2(\rho K_\phi)}{\rho^2} + 2\frac{\sin(2\rho K_\phi)}{2\rho}K_x E^x + (1-\Gamma_\phi^2)E^\phi + 2\Gamma'_\phi E^x\right], \tag{6.62}$$

$$D_x = 2E^\phi K'_\phi - K_x (E^x)' . \tag{6.63}$$

To make a comparison with mimetic gravity in the spherical sector, we need to reformulate the mimetic phase space (6.37) and the mimetic constraints (6.42) in terms of the LQG variables. Let us start with the phase space variables. First of all, the coordinates α and γ are easily related to the components E^ϕ and E^x of the electric field as follows:

$$\alpha = \gamma_{rr}\gamma_{\theta\theta}^{-1} = (E^\phi)^2(E^x)^{-2}, \quad \gamma = \gamma_{rr}\gamma_{\theta\theta}^2 = (E^\phi)^2 E^x . \tag{6.64}$$

To obtain the relation between the momenta (π_α, π_γ) and the components (K_ϕ, K_x) of the extrinsic curvature, we transform the symplectic potential Θ of the LQG phase space

$$\Theta \equiv \beta\left[K_\phi \delta E^\phi + \frac{1}{2}K_x \delta E^x\right], \tag{6.65}$$

as follows

$$\Theta = \beta\left[K_\phi \delta(\alpha^{\frac{1}{6}}\gamma^{\frac{1}{3}}) + \frac{1}{2}K_x \delta(\alpha^{-\frac{1}{3}}\gamma^{\frac{1}{3}})\right], \tag{6.66}$$

$$= \beta\left[\frac{K_\phi}{3E^\phi E^x} + \frac{K_x}{6(E^\phi)^2}\right]\delta\gamma + \beta\left[\frac{K_\phi (E^x)^2}{6E^\phi} - \frac{K_x (E^x)^3}{6(E^\phi)^2}\right]\delta\alpha . \tag{6.67}$$

Due to the equality $\Theta = \pi_\gamma \delta\gamma + \pi_\alpha \delta\alpha$, we deduce immediately the useful relations

$$\pi_\alpha = \beta\left[\frac{K_\phi (E^x)^2}{6E^\phi} - \frac{K_x (E^x)^3}{6(E^\phi)^2}\right], \quad \pi_\gamma = \beta\left[\frac{K_\phi}{3E^\phi E^x} + \frac{K_x}{6(E^\phi)^2}\right]. \tag{6.68}$$

Substituting (6.64) and (6.68) in (6.42), one immediately obtains the expressions of the mimetic Hamiltonian constraint in terms of the LQG variables

$$\mathcal{H} = \frac{\beta^2}{12}\frac{(K_\phi E^\phi - K_x E^x)^2}{E^\phi\sqrt{E^x}} - \frac{4\rho_m}{3}E^\phi\sqrt{E^x}\sin^2\left(\frac{\beta}{6}\frac{2K_\phi E^\phi + K_x E^x}{\sqrt{\rho_m}E^\phi\sqrt{E^x}}\right) - \frac{1}{2}E^\phi\sqrt{E^x}R\,. \tag{6.69}$$

Hence, the Hamiltonian constraints of mimetic gravity and LQG do not not coincide. The identification we have noticed for homogeneous and isotropic backgrounds does not extend to spherically symmetric geometries. While only the variable K_ϕ is polymerized in the polymer Hamiltonian (using a constant scale ρ), it is a rather complicated combination of the different fields that enters in the sine function in mimetic gravity, namely

$$\mathcal{A} = \tilde{\rho}\Big(K_\phi + \frac{E^x}{E^\phi}K_x\Big) \qquad \text{with} \qquad \tilde{\rho} = \frac{\beta}{3\sqrt{\rho_m E^x}}\,. \tag{6.70}$$

Notice that in this last quantity, $\tilde{\rho}$ depends on the phase space field E^x, and provides a natural $\bar{\mu}$-scheme.

The result obtained in this section is not surprising. Indeed, since the effective polymer model has a deformed notion of covariance compared with General Relativity, one does not expect to find a covariant effective action which would reproduce its Hamiltonian formulation. However, one can use mimetic gravity with a limiting curvature as a guide to build undeformed covariant notion of pointwise holonomy like corrections. Therefore, while we do not expect to reproduce any existing inhomogeneous polymer models from these scalar-tensor theories (albeit in the cosmological sector where the issue of covariance disappears), we want to build new extended mimetic Lagrangians which reproduce “as closely as possible" the Hamiltonian of the current spherically symmetric polymer models. The difference between the mimetic and polymer Hamiltonian formulations will then provide an interesting guide to understand the absence of covariance in inhomogeneous polymer black holes. The next section is devoted to this task.

6.4.2. Polymer Hamiltonians from extended mimetic gravity

In this section, we present a method to reconstruct, from an extended mimetic gravity Lagrangian, a Hamiltonian constraint supplemented with quantum corrections which has a form very similar to a given polymer effective theory.

6.4.2.1. General construction

Our starting point is again the most general form for the extended mimetic action (6.11)

$$S[g_{\mu\nu},\phi,\lambda] \;\equiv\; \int d^4x\,\sqrt{-g}\,\left[\frac{f(\phi)}{2}\,\mathcal{R}+\,L_\phi(\phi,\chi_1,\cdots,\chi_p)\,+\,\lambda(X\pm 1)\right]\,, \quad (6.71)$$

where f is an arbitrary function of ϕ, and L_ϕ depends on ϕ and χ_n defined in (6.12). For simplicity, we choose $f=1$ and we assume that L_ϕ does not depend on ϕ.

Thanks to the mimetic constraint, in the “static" case where the scalar field depends on time t only, one can relate directly the second order derivative of the scalar field $\phi_{\mu\nu}$ to the extrinsic curvature of the 3-hypersurface Σ of the ADM decomposition. Using (6.29), one can rewrite the variable χ_n as

$$\chi_n=(-1)^n\sum_{i_1,\dots,i_n}K^{i_2}_{i_1}K^{i_3}_{i_2}\dots K^{i_n}_{i_{n-1}}K^{i_1}_{i_n}\,. \quad (6.72)$$

In the spherically symmetric case, K^r_r, K^θ_θ and K^φ_φ ($\propto K^\theta_\theta$) are the only non-trivial components of K^ν_μ. In this simplified context, it is more convenient to introduce the following combinations of these components

$$X=K^r_r+K^\theta_\theta=\frac{\dot{E}^\phi-(N^rE^\phi)'}{NE^\phi},\qquad Y=K^\theta_\theta=\frac{\dot{E}^x-N^rE^{x\prime}}{2NE^x}\,, \quad (6.73)$$

which involve separately the velocities of E^ϕ and E^x. This will be very useful for the Hamiltonian analysis. Their covariant form can be given in terms of $\chi_1=\Box\phi$ and $\chi_2=\phi_{\mu\nu}\phi^{\mu\nu}$ only as follows

$$X=\frac{2}{3}\Box\phi+\frac{1}{6}\sqrt{6\phi_{\mu\nu}\phi^{\mu\nu}-2(\Box\phi)^2}\,,\quad Y=\frac{1}{3}\Box\phi-\frac{1}{6}\sqrt{6\phi_{\mu\nu}\phi^{\mu\nu}-2(\Box\phi)^2}\,. \quad (6.74)$$

Hence, we can view the χ_i variables as functions of only the two variables X and Y, and then we can reformulate the general action (6.71) as

$$S=\int dtd^3x\;N\;E^\phi\sqrt{E^x}\left[-(2XY-Y^2)+\tilde{L}_\phi(X,Y)+\frac{1}{2}R\right]\,, \quad (6.75)$$

where $\tilde{L}_\phi(X,Y)=L_\phi(\chi_1,\cdots,\chi_p)$. This expression of the action is much more suitable for a Hamiltonian analysis. First, we compute the momenta conjugated to the variables E^x and E^ϕ

$$
\begin{aligned}
\pi_x &= \frac{\delta Y}{\delta \dot{E}^x}\frac{\delta L}{\delta Y} = \frac{E^\phi}{2\sqrt{E^x}}(-2(X+Y)+\frac{\partial \tilde{L}_\phi}{\partial Y})\,, && (6.76)\\
\pi_\phi &= \frac{\delta X}{\delta \dot{E}^\phi}\frac{\delta L}{\delta X} = \sqrt{E^x}(-2Y+\frac{\partial \tilde{L}_\phi}{\partial X})\,, && (6.77)
\end{aligned}
$$

where L denotes the full Lagrangian of the theory. Notice that in the classical limit, where $\tilde{L}_\phi(X,Y)\to 0$, the momenta become "classical" and reduce to the expected form

$$
\pi_x \to \pi_x^c = -\frac{E^\phi}{\sqrt{E^x}}(X+Y) \quad \text{and} \quad \pi_\phi \to \pi_\phi^c = -2\sqrt{E^x}Y\,. \tag{6.78}
$$

To compute the Hamiltonian, one has to invert the relations (6.76) and (6.77) in order to express the velocities in terms of the momenta. However, it is not always possible to have an explicit inversion and then to have an explicit form of the Hamiltonian.

Our goal is to find a function $\tilde{L}_\phi(X,Y)$ which has the right semi-classical limit and which reproduces the general shape of the quantum correction appearing in the polymer model as shown in (4.52). Since such corrections only affect the angular component of the extrinsic curvature in the polymer model, we focus on π_ϕ in (6.77), and we require that there exists a function f such that

$$
Y = -f\left(\frac{\pi_\phi}{\sqrt{E^x}}\right). \tag{6.79}
$$

Remark that such a quantum correction has a natural inbuilt $\bar{\mu}$-scheme through its dependency in E^x to keep the covariance. Since the function f does not depend on X, we can conclude that the general function $\tilde{L}_\phi(X,Y)$ which reproduces the polymer Hamiltonian takes the form:

$$
\tilde{L}_\phi(X,Y) = 2XY + Xf^{-1}(-Y) + g(Y)\,, \tag{6.80}
$$

where $g(Y)$ is a function of Y only which satisfies $g(Y)\to 0$ in the classical limit. Indeed, this choice of $\tilde{L}_\phi(X,Y)$ leads to the action (6.75)

$$
S = \int dtd^3x\; N\; E^\phi\sqrt{E^x}\left[\,Y^2 + Xf^{-1}(Y) + g(Y) + \frac{1}{2}R\,\right]\,, \tag{6.81}
$$

whose Hamiltonian (after a few calculations) reads

$$
\begin{aligned}
H &= N\left[2\, E^x\, \pi_x\, Y + E^\phi\, \pi_\phi\, X - \; E^\phi\sqrt{E^x}\left(Y^2 + X f^{-1}(Y) + g(Y) + \frac{1}{2}R \right)\right] \\
&= -\frac{N}{\sqrt{E^x}}\; \left[2E^x\pi_x f\left(\frac{\pi_\phi}{\sqrt{E^x}}\right) + E^\phi\left(f^2\left(\frac{\pi_\phi}{\sqrt{E^x}}\right) - E^x g(Y)\right) + \frac{1}{2}E^\phi E^x R\right] \\
&= -\beta^2\frac{N}{\sqrt{E^x}}\; \left[\frac{2}{\beta}K_x E^x f\left(\frac{2\beta K_\phi}{\sqrt{E^x}}\right) + E^\phi\left(\frac{1}{\beta^2}f^2\left(\frac{2\beta K_\phi}{\sqrt{E^x}}\right) - E^x g(Y)\right) + \frac{1}{2\beta^2}E^\phi E^x R\right] \,,
\end{aligned}
\tag{6.82}
$$

where we have used the relations (6.68). Notice that one can express the three-curvature R as in (4.52) but this is not needed here. As a conclusion, one can immediately identify this Hamiltonian with the polymer model Hamiltonian (4.52) with the conditions that

$$
f_1(E^x, K_\phi) = \frac{1}{\beta^2}f^2\left(\frac{2\beta K_\phi}{\sqrt{E^x}}\right) - E^x g(Y) \qquad \text{and} \qquad f_2(E^x, K_\phi) = \frac{1}{\beta}f\left(\frac{2\beta K_\phi}{\sqrt{E^x}}\right) \,. \tag{6.83}
$$

In that way, one obtains an extended theory of mimetic gravity which reproduces the general shape of the holonomy corrections considered in the polymer model, with an additional natural $\bar{\mu}$-scheme.

Let us now quickly look at the cosmological limit of the previous action. In that case, all the χ_i variables can be expressed in terms of $\Box\phi$ only, and we have

$$
X = 2Y = \frac{2}{3}\Box\phi \,. \tag{6.84}
$$

As a consequence, the mimetic potential takes the very simple form

$$
L_\phi = \frac{4}{9}\left(\Box\phi\right)^2 + \frac{2}{3}\Box\phi\, f^{-1}\left(\frac{2}{3}\Box\phi\right) + g\left(\frac{1}{3}\Box\phi\right) \,. \tag{6.85}
$$

Having presented the general formulation, we can now present concrete example for the corrections functions and look for the action which reproduces the holonomy corrections of some special case treated in the litterature.

6.4.2.2. A first example: the standard anomaly free sine function corrections

As a first example, let us consider the standard holonomy corrections given by sine functions which satisfy the anomaly free constraint (4.56):

$$
f_1(K_\phi) = \frac{\sin(\rho K_\phi)}{\rho}\,, \qquad f_2(K_\phi) = \frac{1}{2}\frac{df_1(K_\phi)}{dK_\phi}\,. \tag{6.86}
$$

Here the scale ρ is a constant. In our case, quantum corrections are derived from a covariant theory, and we always obtain a dynamical scale (where ρ has been changed to $\rho \to 2\rho\beta/\sqrt{E^x}$) leading to

$$f_1(E^x, K_\phi) = \frac{\sqrt{E^x}}{2\beta\rho} \sin\left(\frac{2\beta\rho}{\sqrt{E^x}} K_\phi\right), \qquad f_2(E^x, K_\phi) = \frac{E^x}{\beta^2\rho^2} \sin^2\left(\frac{\beta\rho}{\sqrt{E^x}} K_\phi\right). \tag{6.87}$$

Here, the $\bar{\mu}$-scheme has been somehow imposed from the covariance property the (symmetry reduced) underlying theory.

From the previous construction, we can explicitly derive the mimetic Lagrangian which reproduces such holonomy-like corrections (6.87). Using (6.83), one can first write the angular component of the extrinsic curvature $Y = -K^\theta_\theta$ as follows

$$Y = -f\left(\frac{2\beta K_\phi}{\sqrt{E^x}}\right) = -\beta f_2(E^x, K_\phi). \tag{6.88}$$

Then, one easily obtains the expressions of the functions f and g entering in the definition of $\tilde{L}_\phi$

$$f^{-1}(-Y) = \frac{1}{\rho} \arcsin(-2\rho Y), \qquad g(Y) = -Y^2 + \frac{1}{2\rho^2}(1 - \sqrt{1 - 4\rho^2 Y^2}), \tag{6.89}$$

which finally leads to the Lagrangian

$$\tilde{L}_\phi = 2XY - Y^2 + \frac{X}{\rho} \arcsin(-2\rho Y) + \frac{1}{2\rho^2}\left(1 - \sqrt{1 - 4\rho^2 Y^2}\right). \tag{6.90}$$

The symmetry reduction to the cosmological sector leads

$$L_\phi = \frac{1}{3}\Box\phi^2 - \frac{2}{3\rho}\Box\phi \arcsin\left(\frac{2}{3}\rho\Box\phi\right) + \frac{1}{2\rho^2}\left(1 - \sqrt{1 - \frac{4}{9}\rho^2\Box\phi^2}\right). \tag{6.91}$$

As expected, (6.91) does not coincide with the extended mimetic Lagrangian initially proposed by Chamseddine and Mukhanov (6.18) which has been shown to reproduce the LQC dynamics [55, 236].

6.4.2.3. A second example: the Tibrewala's effective corrections

Hence, we can ask the question whether one can find a different Lagrangian which both reproduces the holonomy-like corrections and coincides with the Chamseddine and Mukhanov Lagrangian in the cosmological sector. For that purpose, let us consider another example of effective quantum corrections first derived in [217]. Such corrections were obtained by investigating the possibility to implement covariantly point wise holonomy-like corrections, depending both on E^x and K_ϕ, on the Reissner-Norstrom phase space. These effective corrections are given by the functions

$$f_2(K_\phi) = \frac{\sqrt{E^x}\sin\!\left(2\beta\rho K_\phi/\sqrt{E^x}\right)}{2\beta\rho}\,, \tag{6.92}$$

$$f_1(K_\phi) = \frac{3E^x\sin^2(\beta\rho K_\phi/\sqrt{E^x})}{\beta^2\rho^2} - 2K_\phi\frac{\sqrt{E^x}\sin\!\left(2\beta\rho K_\phi/\sqrt{E^x}\right)}{2\beta\rho}\,. \tag{6.93}$$

Following exactly the same strategy as in the previous example, we first find the functions f and g which are given here by

$$f^{-1}(-Y) = \frac{1}{\rho}\arcsin(-2\rho Y)\,, \tag{6.94}$$

$$g(Y) = \frac{3}{2\rho^2}(1-\sqrt{1-4\rho^2Y^2}) - Y^2 - \frac{\arcsin(2\rho Y)}{\rho}Y\,. \tag{6.95}$$

Then, we deduce immediately the mimetic Lagrangian

$$\tilde{L}_\phi = 2XY - Y^2 + \frac{X+Y}{\rho}\arcsin(-2\rho Y) + \frac{3}{2\rho^2}(1-\sqrt{1-4\rho^2Y^2})\,. \tag{6.96}$$

Interestingly, the symmetry reduction to the cosmological sector leads to the Lagrangian

$$L_\phi = \frac{1}{3}\Box\phi^2 - \frac{1}{\rho}\Box\phi\arcsin\!\left(\frac{2}{3}\rho\Box\phi\right) + \frac{3}{2\rho^2}\left(1-\sqrt{1-\frac{4}{9}\rho^2\Box\phi^2}\right) \tag{6.97}$$

which coincides exactly with the Chamseddine-Mukhanov Lagrangian, the one that reproduces the LQC dynamics (6.18).

6.5. Discussion

In this chapter, we start by revisiting the regular black hole interior solution obtained in the context of mimetic gravity [193] from a Hamiltonian perspective. We introduced a suitable parametrization of the static spherically symmetric metric which allowed us to perform a complete Hamiltonian analysis of the theory. We wrote the Hamiltonian equations and we showed that the determinant γ of the spatial metric admits an evolution equation very similar to the modified Friedmann equation obtained in effective LQC (6.48). Thus, the black hole has a bounded energy density which prevents the existence of a singularity. Finally, we solved the Hamilton equations in the regime deep inside the black hole and we made a comparison with the standard Schwarzschild solution, showing explicitly that the limiting curvature mechanism indeed resolves the singularity.

Then, motivated by the recent finding that the initial extended mimetic theory reproduces exactly the effective dynamics of LQC, we have investigated whether this result survives in the spherically symmetric sector. We have argued that this cannot be “naively" the case for a very general reason: the spherically symmetric polymer model fails to be covariant, and there is a priori no hope to reproduce this phase space and its effective quantum corrections from a covariant theory such as mimetic gravity. From a more general perspective, the polymer treatment of inhomogeneous backgrounds (cosmological perturbations, spherical symmetry, Gowdy system) is well known to suffer from a lack of covariance. The holonomy corrections usually introduced during the regularization break the Dirac's hypersurface deformation algebra, or at best, lead to its deformation.[211, 212, 243–245]. It becomes then crucial to understand either the conceptual and technical consequences of such deformation, either to understand how to cure it. In this work we have presented a new strategy towards the second problem based on mimetic gravity. Since this theory provides a covariant Lagrangian which reproduces exactly the effective LQC dynamics in the cosmological sector, it provides an interesting tool to derive covariant polymer-like Hamiltonian models beyond the cosmological sector to be compared with the existing ones. As such, we view mimetic theory as a guide to derive covariant notion of point-wise holonomy-like corrections in polymer models.

With this issue of covariance in mind, we have presented a general procedure to construct mimetic Lagrangians which admit a Hamiltonian formulation very similar in spirit to existing polymer models of black holes, but which is fully covariant. Then we have applied our procedure to two examples of polymers black hole models: the standard anomaly-free sine corrections model and the Tribrewala's $\bar{\mu}$-scheme corrections model. From the differences between the mimetic and polymer Hamiltonian formulations, one can extract several insights which could be useful when trying to build undeformed covariant polymer black hole models based on the real Ashtekar-Barbero variables. First, the covariant mimetic formulation always contains a $\bar{\mu}$-scheme, i.e. holonomy-like corrections which depend both on K_ϕ and E^x and not only on K_ϕ. As such, the dynamical nature of the polymer scale ρ is a necessary ingredient to maintain the covariance in presence of effective quantum corrections of the polymer type. However, the $\bar{\mu}$-scheme, while necessary, is not sufficient since previous attempts to include it in spherically symmetric models still lead to a deformed notion of covariance [217] (see [252] for a similar conclusion in the context of cosmological perturbations). Therefore additional ingredients are required to maintain the covariance in such effective polymer models. The second difference between the two approaches concerns the object which is polymerized. In the standard polymer approach to the spherically symmetric background, one polymerizes the connection K_ϕ while the limiting curvature mechanism of mimetic theory suggests that the covariance requires to polymerize a more complicated combination of the canonical variables, given by (6.70). The choice of connection is crucial in this procedure, and the Ashetkar-Barbero connection K_ϕ might not be the suitable one to consider [b].

From a more general point of view, our study suggests that the current procedure developed in spherically symmetric polymer models still lacks some crucial ingredients to provide a consistent covariant effective framework. However, let us emphasize again that the polymer models we have considered only involve the minimal loop corrections, i.e. the so called point-wise holonomy corrections. Therefore, it is still possible to improve construction of polymer effective actions by adding the loop corrections. It could well be that the so called triad corrections, associated to the loop regularization of the inverse volume term, conspire with the holonomy corrections to provide an undeformed algebra of first class constraints. This remains to be checked. Moreover, the spherically symmetric sector raises the question of how to concretely deal with the holonomy of the inhomogeneous component of the connection K_x. Despite some preliminary work on this question, a consistent and tractable implementation of this extended holonomy is still to be understood. Therefore, our results might suggest that polymer models built on solely point-wise holonomy corrections (and with the real Ashtekar-Barbero variables) could not admit covariant effective action beyond the cosmological sector, and motive us to look for a generalization of the current regularization used in such polymer models to implement consistently an undeformed notion of covariance (see [253, 254] for some recent proposals in black hole and cosmology in this direction).

b. It is already known that this connection does not transform as a true space-time connection under the action of the scalar constraint, and as such, does not provide a good candidate to ensure a fully covariant quantum description when implementing the dynamics. It might be that the same problem emerges already at the effective level in these polymer models.

7. Polymer Schwarzschild Black Hole: An Effective Metric

Sommaire

7.1. Introduction

The extraordinary recent detections of gravitational waves (GW) by the LIGO and the LIGO/Virgo collaborations allowed us to "hear" black holes for the first time, a century after Schwarzschild predicted their existence from Einstein equations. These detections have opened a new window on black holes and we hope to learn much more on these fascinating astrophysical objects in a near future. So far, the observations of GW emitted by binaries of black holes or neutron stars are in total agreement with the predictions of general relativity. However, when the GW detectors become more sensitive and allow probing deeper the "very strong" gravity regime at the merger, one will possibly measure deviations from Einstein gravity.

Perhaps, the main reason to expect gravity to be modified is the existence of singularity theorems in classical gravity. The presence of such singularity is believed to be pathological and to indicate a breakdown of the classical theory which should be modified and regularized by quantum gravity effects. However, how quantum gravity regularizes precisely black holes singularities is still unknown simply because a complete theory of quantum gravity is still missing. Faced with such an important difficulty, one has instead proposed candidates for regular metrics with the requirements that they are non-singular modifications of the classical black hole metric and they are physically reasonable. The Hayward [255] or more recently the Planck star metrics [213] are typical examples. Hence, the regular metrics could be interpreted as effective quantum geometries. From this point of view, it is natural to think that they could be recovered from a semi-classical limit of a black hole quantum geometry. In practice, this is an extremely difficult problem since it will require the development of suitable coarse-graining technics of the underlying quantum geometry, a major challenge in non-perturbative approach to quantum gravity such as Loop Quantum Gravity.

One way to circumvent this difficulty would be to construct and classify (from first principles) effective theories of quantum gravity as one does for studying in a systematic way dark energy for instance. See [244, 256, 257] for efforts along this line. In that way, one could write a modified gravity action (or modified Einstein equations) which takes into account quantum corrections, and then study the spherically symmetric sector and look for black hole solutions. Of course, these solutions are expected to be regular and to predict new physical phenomena which could be in principle observable. In the framework of loop quantum cosmology [30], one knows how to construct and classify effective quantum Friedmann equations (depending on the choice of the spin-j representation which labels the holonomy corrections, as well as the choice of regularization scheme). See [241] for details on this classification. It is well-known that they lead to a regular cosmology with no more initial singularity. However, the effective description of loop quantum black holes is much less understood, the challenge being to generalize the technics applied in LQC to the inhomogeneous black hole background. Indeed, in this inhomogeneous case, one has to make sure that the effective corrections do no generate anomalies in the algebra of first class constraints, and thus do not spoil covariance. Taking care of this potential covariance issue, one can obtain modified Einstein's equation for polymer black holes [211, 214, 217]. Their resolution for the simple vacuum modified Schwarzschild interior has not been investigate yet. In this work, we fill this gap.

In the polymer framework, the effective corrections are introduced at the phase space level, in the hamiltonian constraint. In the treatment of interior black holes, several regularization schemes have been developed. Models such as [34, 205] and more recently [209, 210, 258, 259] make use of the homogeneity of the interior geometry to introduce a regularization very similar to cosmological polymer models. Yet, the exterior black hole geometry is inhomogeneous, and the modified Einstein's equations obtained in [34, 205, 209, 210, 258, 259] hold only for the interior geometry. In this chapter, we adopt a different strategy. We consider the full inhomogeneous geometry, and introduce the polymer regularization satisfying the anomaly freedom conditions of [211], paying thus attention to the underlying covariance of the effective approach. Only after, we reduce the problem to the interior homogeneous geometry. The advantage is that we obtain one and only one set of modified Einstein's field equations valid for the whole black hole geometry (both exterior and interior regions), i.e

Eq.(7.15)-(7.16). The modified field's equation for interior region are then simply obtained by suitable gauge fixing.

Following thus the approach of [211], the quantum corrections of the effective Hamiltonian constraint, induced by the regularization, are parametrized by a single real valued function $f(x)$ of one phase space variable x. This is a consequence of the requirement that the deformed symmetry algebra (generated by the effective Hamiltonian and vectorial constraint) remains closed so that there is no anomalies. See Eq. (7.9) and discussion below. However, even though there is a standard choice for $f(x)$ in loop quantum gravity, the precise definition of the "regularization" function $f(x)$ is in fact ambiguous. For this reason, it is important to study the effective corrected Einstein equations for an arbitrary function $f(x)$, as initiated in [211].

In this chapter, we consider the effective theory introduced in [211] and we solve explicitly the effective Einstein equations for static spherically symmetric interior space-times. More precisely, we focus on the static region inside the horizon, where quantum gravity effects are supposed to become important, and we find an explicit form of the effective metric in this region for an arbitrary deformation function $f(x)$. Surprisingly, the effective metric can be simply expressed in terms of $f(x)$, and then we can easily deduce the conditions for the black hole to be non-singular as one wishes. We apply our result to the case where $f(x)$ is the standard deformation function used in loop quantum gravity (7.13), and we show that the black hole presents strong similarities with the Reissner-Nordström space-time. The interior effective geometry inherits an inner horizon due to the non perturbative quantum gravity effects. Equipped with this new interior effective geometry, we explore then the possibility to extend our black hole solution to the whole space-time (outside the trapped region) and we discuss the challenge to perform coordinate transformation in this model with deformed covariance. Finally, we apply the strategy developed in [260] to obtain a well defined invariant line element under the deformed symmetry and we show that the main novelty is a transition between Lorentzian to Euclidean signature deep inside the interior region.

7.2. Covariant polymer phase space regularization

Let us first present the effective Einstein equations obtained in loop quantum gravity for spherically symmetric black holes and justify our choice of regularization.

We start with an ADM parametrization of the metric

$$ds^2 = -N^2\, dt^2 + q_{rr}\,(dr + N^r dt)^2 + q_{\theta\theta}\, d\Omega^2 \tag{7.1}$$

where each function N, N^r, q_{rr} and $q_{\theta\theta}$ depends on the radial and time coordinates (t,r), and $d\Omega^2$ is the metric on the unit two-sphere. Following the Ashtekar-Barbero construction, it is more convenient to express the metric components q_{rr} and $q_{\theta\theta}$ in terms of the components of the electric field E^r and E^ϕ as follows

$$q_{rr} \equiv \frac{(E^\phi)^2}{E^r}\,, \qquad q_{\theta\theta} \equiv E^r\,. \tag{7.2}$$

Hence, the phase space is parametrized by two pairs of conjugate fields defined by the Poisson brackets

$$\{K_\phi(r), E^\phi(s)\} = \delta(r-s)\,,\ \{K_r(r), E^r(s)\} = 2\delta(r-s) \tag{7.3}$$

where we have fixed for simplicity the Newton constant and the Barbero-Immirzi parameter to 1. The variables K_ϕ and K_r are $\mathfrak{su}(2)$ connections.

As usual, the lapse function N and the shift vector N^r are Lagrange multipliers which enforce respectively the Hamiltonian and vectorial constraints,

$$H = \frac{E^\phi}{2\sqrt{E^r}}(1+K_\phi^2-\Gamma_\phi^2) + \sqrt{E^r}(K_\phi K_r + \partial_r\Gamma_\phi) \tag{7.4}$$
$$V = 2E^\phi\partial_r K_\phi - K_r\partial_r E^r\,, \tag{7.5}$$

where $\Gamma_\phi \equiv -\partial_r E^r/2E^\phi$ is linked to the Levi-Civita connection. These constraints are first class, they generate diffeomorphisms restricted to spherically symmetric space-times, and they satisfy the closed Poisson algebra

$$\{H[N], V[N_1^r]\} = -H[N_1^r\partial_r N]\,, \tag{7.6}$$
$$\{V[N_1^r], V[N_2^r]\} = V[N_1^r\partial_r N_2^r - N_2^r\partial_r N_1^r]\,, \tag{7.7}$$
$$\{H[N_1], H[N_2]\} = V[q^{rr}(N_1\partial_r N_2 - N_2\partial_r N_1)]\,, \tag{7.8}$$

where $H[N]$ and $V[N_1^r]$ are the smeared constraints.

In this chapter, we focus on the effective dynamic obtained from the anomaly free loop regularization which is introduced prior quantization. Concretely, we keep the phase space parametrization (7.3) unchanged and we modify the expression of the constraint (7.4). As we consider solely point-wise holonomy corrections of K_ϕ here, only the dependency of the Hamiltonian constraint on K_ϕ is modified according to

$$H = \frac{E^\phi}{2\sqrt{E^r}}[1+f(K_\phi)-\Gamma_\phi^2] + \sqrt{E^r}[g(K_\phi)K_r + \partial_r\Gamma_\phi] \tag{7.9}$$

where the functions f and g are not fixed yet. The requirement of anomaly freedom of the Dirac's algebra requires then that

$$g(x) = f'(x)/2 \tag{7.10}$$

In that case, the Poisson bracket between Hamiltonian constraints (7.8) is deformed according to

$$\{H[N_1], H[N_2]\} = V[\beta(K_\phi)\, q^{rr}(N_1 N_2' - N_2 N_1')] \tag{7.11}$$

where the deformation function $\beta(K_\phi)$ is given by

$$\beta(x) = f''(x)/2 \tag{7.12}$$

as initially derived in [211, 214]. Such a deformation is a generic feature of holonomy corrected symmetry reduced models of gravity [245]. The other two brackets (7.6) and (7.7) are unchanged. Moreover, K_r is not modified in our regularization since it can be completely remove from the scalar constraint by a simple redefinition of the constraints, as shown in [184]. Consequently, the regularization of K_r doesn't play any role in the classical regularization and can be safely ignored at this step. The holonomies of K_r will nevertheless be crucial in the quantum theory when introducing the one dimensional spin network defining the kinematical Hilbert space. See [184] for more details.

Finally, our regularization is restricted to the μ_0-scheme, as in [184], since introducing holonomy corrections within the $\bar{\mu}$-scheme, i.e $K_\phi \to f(K_\phi, E^x)$, and requiring at the same time the anomaly freedom of the effective Dirac's algebra generates inconsistencies as shown in [217]. Therefore, the standard improved dynamics used in polymer cosmological models cannot be generalized as it stands to such inhomogeneous spherically symmetric polymer models. See [209, 210, 258, 259] for a recent alternative strategy. This concludes our justifications for our classical regularization of the phase space.

7.3. Effective Einstein's equations

Hence, as it was emphasized in the introduction, the regularization induced by holonomy corrections inspired from loop quantum gravity is parametrized by the sole function $f(x)$. The explicit expression of this effective correction remains ambiguous. Nonetheless, as we require naturally that $f(x)$ reproduces the classical behavior in the low curvature regime, we must have $f(x) \approx x^2$ when $x \ll 1$. In the literature, the usual choice is

$$f(x) = \frac{\sin^2(\rho x)}{\rho^2}\,, \tag{7.13}$$

where ρ is a deformation real parameter that tends to zero at the classical limit. The presence of a trigonometric function is reminiscent from the $SU(2)$ gauge invariance in loop quantum gravity: roughly, one replaces the "connection" variable K_ϕ by a point-wise "holonomy-like" variable $\sin(\rho K_\phi)/\rho$. Note that (7.13) is associated to the computation of the regularization of the connection (or its curvature) in term of holonomies within the $j = 1/2$ fundamental representation of SU(2). Yet, one could obtain more complicated trigonometric functions by evaluating this regularization in another j-representation of SU(2), as done for polymer cosmological models in [241]. Therefore, keeping f(x) general in our resolution allows to keep track of this ambiguity of the polymer regularization.

Now, we have all the ingredients to compute the effective Einstein equations for deformed spherically symmetric space-times. They are given by the Hamilton equations

$$\dot{F} = \{F, H[N] + V[N^r]\} \, , \tag{7.14}$$

for F being one of the four phase space variables (7.3). The time evolutions of the electric field components simply read

$$\dot{E}^r = N\sqrt{E^r} f'(K_\phi) + N^r \partial_r E^r \, , \tag{7.15}$$

$$\dot{E}^\phi = \frac{N}{2}\left[\sqrt{E^r} K_r f''(K_\phi) + \frac{E^\phi}{\sqrt{E^r}} f'(K_\phi)\right] + \partial_r (N^r E^\phi) \, . \tag{7.16}$$

The expression of $\dot{K}_\phi$ is more involved and thus we do not report it here. The component K_r can be obtained by solving the Hamiltonian constraint (7.9).

7.3.1. Outside the black hole

Note that for $N^r = 0$ and static geometry, and upon using the standard loop effective corrections (7.13), equation (7.15) implies that the angular extrinsic curvature is quantized as

$$K_\phi = \frac{n\pi}{2\rho} \qquad \text{with} \qquad n \in \mathbb{N} \tag{7.17}$$

Hence, for $n \neq 0$, the resulting geometry has a divergent extrinsic curvature K_ϕ in the semi-classical limit, i.e when $\rho \to 0$. It implies that outside the hole, the only consistent inhomogeneous static solution is the classical Schwarzschild's one, i.e

$$K_\phi = 0 \tag{7.18}$$

corresponding to $n = 0$. Therefore, the effective loop corrections introduced above do not allow to have a modified Schwarzschild geometry outside the hole when looking for a static exterior solution. This shortcoming is intimately related to the lack of a proper $\bar{\mu}$-scheme in the present regularization. It is expected that once a fully consistent $\bar{\mu}$-scheme will be implemented, i.e with a polymer scale $\rho(E^x)$ running with the geometry, potential modifications of the exterior geometry could show up.

7.3.2. Inside the black hole: static ansatz

We turn now to the interior problem. We are interested in solving these equations inside a "static" black hole. As the role of the variables r and t changes when one crosses the horizon, this corresponds to considering time-dependent fields only. In that case, the effective Einstein equations dramatically simplify and read

$$\dot{E}^r = N\sqrt{E^r} f'(K_\phi)\,, \tag{7.19}$$

$$\dot{E}^\phi = \frac{N}{2}\left[\sqrt{E^r} K_r f''(K_\phi) + \frac{E^\phi}{\sqrt{E^r}} f'(K_\phi)\right]\,, \tag{7.20}$$

$$\dot{K}_\phi = -\frac{N}{2\sqrt{E^r}}\left[1 + f(K_\phi)\right]\,, \tag{7.21}$$

together with the Hamiltonian constraint

$$f'(K_\phi)E^r K_r + [1 + f(K_\phi)]E^\phi \;=\; 0\,, \tag{7.22}$$

from where we easily get the dynamics of K_r.

7.3.2.1. General algorithm

Now, we are going to solve these equations explicitly for any function f. As we are going to show, it is very convenient to fix the lapse function $N(t)$ (by a gauge fixing) such that

$$N\; f'(K_\phi) \;=\; 2\,. \tag{7.23}$$

In that case, the equation (7.19) for E^r decouples completely from the other variables and can be easily integrated to

$$E^r(t) \;=\; t^2 + a\,, \tag{7.24}$$

where a is an integration constant that we fix to $a = 0$ (in order to recover the Schwarzschild solution at the classical limit). Another important consequence of the gauge choice (7.23) is that the equation (7.21) for K_ϕ also decouples and takes the very simple form

$$\frac{f'(K_\phi)}{1 + f(K_\phi)}\,\dot{K}_\phi \;=\; -\frac{1}{t}\,. \tag{7.25}$$

It can be immediately integrated to the form

$$f(K_\phi) = \frac{r_s}{t} - 1\,, \tag{7.26}$$

where r_s is an integration constant with the dimension of a length. As we are going to see later on, $t = r_s$ corresponds to the location of the black hole (outer) horizon. Hence, K_ϕ is easily obtained by inverting the function $f(x)$. Indeed, when f is monotonous, it admits a global reciprocal function f^{-1}, otherwise the reciprocal function is defined locally. Then, the expression of E^ϕ follows immediately. Indeed, if one substitutes K_r from (7.22) into (7.20), one obtains the following equation for E^ϕ

$$\frac{\dot{E}^\phi}{E^\phi} = \frac{1}{t}\left(1 - \frac{[1+f(K_\phi)]f''(K_\phi)}{[f'(K_\phi)]^2}\right), \tag{7.27}$$

which can be easily integrated to

$$E^\phi \;=\; b\frac{f'(K_\phi)}{1+f(K_\phi)}, \tag{7.28}$$

where b is a new integration constant that will be fixed later. The remaining variable K_r is given immediately from the Hamiltonian constraint (7.22) together with (7.24) and (7.28). Hence, we have integrated explicitly and completely the modified Einstein equations in the region inside a “static" spherically symmetric black hole where the effective metric is

$$ds^2 = -\frac{1}{F(t)}dt^2 + \left(\frac{2b}{r_s}\right)^2 F(t)dr^2 + t^2 d\Omega^2, \tag{7.29}$$

with $F(t)$ related to $f(x)$ by

$$F(t) = \frac{1}{4}\left[f' \circ f^{-1}(\frac{r_s}{t} - 1)\right]^2 = \left[2\frac{df^{-1}}{dx}(\frac{r_s}{t} - 1)\right]^{-2} \quad . \tag{7.30}$$

In the region where $t \approx r_s$, quantum gravity effects are negligible and the metric should reproduce the Schwarzschild metric. We see immediately in (7.29) that a necessary condition for this to be the case is that

$$2b = r_s \tag{7.31}$$

This fixes the constant b. Furthermore, in such a regime, we know that $f(x) \approx x^2$, then $f^{-1}(x) \approx \sqrt{x}$, hence

$$F(t) \approx |r_s/t - 1| \tag{7.32}$$

As a consequence, we recover the expected classical metric with r_s being the Schwarzschild radius. However, while the metric smoothly matches the Schwarzschild metric at the outer horizon, the extrinsic curvature does not, leaving a gluing which is not $\mathcal{C}^1$.

7.3.2.2. Inverse problem

Before studying concrete examples, let us consider a converse situation where a deformed metric $g_{\mu\nu}$ of the form (7.29) is given. Then, one asks the question whether one can find a deformation function $f(x)$ such that the deformed metric $g_{\mu\nu}$ is a solution of the effective Einstein equations. The answer is positive and $f(x)$ can be obtained immediately by inverting the relation (7.30) between $F(t)$ and $f(x)$ as follows

$$f^{-1}(x) \;=\; \frac{1}{2}\int_0^x du \left|F\left(\frac{r_s}{1+u}\right)\right|^{-1/2}. \tag{7.33}$$

As the function $f^{-1}(x)$ is monotonic, one can invert this relation and define the deformation function $f(x)$ without ambiguity. This can be done for the Hayward metric for instance, even though in that case $f^{-1}(x)$ is defined as an integral, and thus $f(x)$ is implicit.

7.3.3. Example: the standard $j = 1/2$ sine correction

To illustrate this result, let us consider some interesting physical situations. First, the case where there is no quantum deformation corresponds to $f(x) = x^2$. As we have just said above, we recover immediately the Schwarzschild metric.

Then, let us study the more interesting case where $f(x)$ is the usual function considered in polymer black hole models (7.13). In that case, the reciprocal function is

$$f^{-1}(x) = \frac{\arcsin(\rho\sqrt{x})}{\rho}\,, \tag{7.34}$$

which is defined for $x \leq 1/\rho^2$ only. As a consequence, the effective metric for a black hole is of the form (7.29) with

$$F(t) \;=\; \left(\frac{r_s}{t} - 1\right)\left(1 + \rho^2 - \rho^2\frac{r_s}{t}\right)\,, \tag{7.35}$$

which is defined for $t \leq r_s$ a priori. At this point, we can make several interesting remarks. First, one recovers the Schwarzschild metric, when t approaches r_s. Then, in addition to the usual outer horizon (located at $t = r_s$), the metric has an inner horizon located at

$$t = \frac{\rho^2 r_s}{1+\rho^2} \tag{7.36}$$

The computation of the Ricci and Kretschmann scalars shows that there is no curvature singularity inside the trapped region. One can naturally extend this solution outside the trapped region by using a generalized advanced time coordinate v such that

$$dv = dr + dt/F(t) \tag{7.37}$$

and

$$ds^2 = F(t)dv^2 - 2dvdt + t^2 d\Omega^2 \tag{7.38}$$

The metric and inverse metric are regular when $F(t) = 0$. This allows to define the expansion of null radial outgoing geodesics, leading to

$$\theta_+ = F(t)/t \tag{7.39}$$

Hence the zeros of F correspond to the locus of the horizons (inner and outer), and the region comprised between them is trapped. The Ricci scalar

$$R \approx -2\rho^2/t^2 \tag{7.40}$$

diverges at $t = 0$, which is the locus of a timelike singularity as in Reissner-Nordström's (RN) black hole. Our metric is actually very similar to this solution, and leads to the same Penrose diagram. However, a main difference is that an outer horizon (at $t = r_s$) is always present in our geometry, while naked singularities appear for super-extreme RN black holes.

In the end, this naive extension is not satisfactory since it does not allow recovering Schwarzschild's solution in the classical region ($r \gg r_s$), except if the parameter ρ becomes r-dependent and tends to zero, which would drastically modify the equations of motion [217].

7.3.4. Signature change from covariance

Moreover, while the extension of the metric outside the trapped region is natural for a standard RN solution, it is not clear whether the extension is allowed or not in our context. The reason is that the deformation of the Hamiltonian constraint (7.11) modifies the invariance of the effective theory under time reparametrizations. Then, if we believe that such a deformed symmetry is the right one and it is no longer given by usual diffeomorphisms (what can be discussed), we could not perform an arbitrary time redefinition as we did to extend the metric outside the trapped region. The deformed symmetry has recently been analyzed in great details in [260]. It was realized that the effective metric which is invariant under these deformed transformations is slightly different from (7.1) where N has to be rescaled according to

$$N^2 \longrightarrow \beta(K_\phi)\, N^2 , \tag{7.41}$$

where β, which has been introduced in (7.11), is explicitly given, in our case, as a function of time by

$$\beta(t) \equiv \beta(K_\phi(t)) = 1 - 2\rho^2 \left(\frac{r_s}{t} - 1 \right) . \tag{7.42}$$

With this new invariant metric, the function $g_{tt}(t)$ acquires a zero in the trapped region which corresponds to a transition between a lorentzian and an euclidean signature within the trapped region at

$$t = \frac{2\rho^2 r_s}{1 + 2\rho^2} \tag{7.43}$$

Such a transition was studied in more detail in [260]. Starting from another gauge choice, namely $K_\phi = \pi/(2\rho)$, and solving the field equations (7.19-7.22) deep inside the black hole, it was shown that the geometry is regular. Yet, the lorentzian to euclidean transition rises new difficulties concerning for instance the fate of matter inside this trapped region, since the standard evolution equations become elliptic [126]. Similar aspects were encountered in the context of the perturbations analysis in loop quantum cosmology known as the deformed algebra approach [261].

7.4. Discussion

In this work, we have solved explicitly a large class of modified Einstein equations arising in the effective polymer approach to black holes. We have adopted a different strategy than existing interior Schwarzschild models such as [209, 210, 258, 259]. We first consider the full polymer regularization of the inhomogeneous geometry consistent with covariance, and only then reduce the problem to the interior homogeneous geometry. By doing this, we ensures that the regularization of the hamiltonian constraint does not generate any anomalies and thus, that we still have the right number of degrees of freedom at the effective level. This point is ignored in [209, 210, 258, 259] and the regularization introduced in these models is different, since there are no anomaly freedom condition to constrain it. Consequently, the effective metric obtained here and the one presented in [209, 210, 258, 259] are very different.

Focusing on the usual deformation considered in polymer models studied by Gambini and Pullin, we have found a black hole (interior) solution whose structure shows strong similarities with the Reissner-Nordström black hole. The main novelty due to the quantum gravity effect is the appearance of an inner horizon, while the expected Schwarzschild solution is recovered when one approaches the outer horizon, albeit not smoothly. This last point is a consequence of the lack of a proper $\bar{\mu}$-scheme regularization in the Gambini-Pullin model.

Strictly speaking, we obtained a solution only inside in a trapped region, valid for $t \in [t_-, t_+]$ and the question of its extension in the whole space-time deserves to be study carefully. In particular, we see that the naive extension outside the trapped surface does not allow recovering Schwarzschild's solution in the classical region ($r \gg r_s$), except if the parameter ρ becomes r-dependent and tends to zero. This underlines the limitation of the current model to have a consistent semi-classical limit. A generalization of the current regularization is required to account for a $\overline{\mu}$-scheme, as already emphasized in [217] and more recently in [56]. See also [219] for a more recent proposal including such $\overline{\mu}$-scheme in polymer black holes using self dual variables.

Our results open interesting theoretical and phenomenological directions to follow. First, it would be interesting to include additional effective corrections such as the triad corrections affecting the intrinsic geometry, which are usually considered separately. An even more challenging step is to go beyond static geometries and study dynamical black holes, an open issue up to now in the polymer framework. See [219, 253] for recent proposals in this direction. This is particularly important for understanding Hawking radiation as well as quantum gravitational collapse and eventually bouncing and black hole to white hole transitions scenarios. Finally, it would be interesting to use this model to investigate possible quantum gravity modifications of the structure inside astrophysical objects. We plan to address these important questions in the future.

Conclusion

Here we give a general conclusion and remarks for the topics described in the thesis. We will summarize the results obtained in the present work, and most importantly, we will point out their limitations and possible future research directions. For a detailed discussion and remarks, we refer to the concluding sections in each chapter.

In this thesis, we investigate different aspects of quantum gravity, mostly in the context of loop quantum gravity and spin foam models. Both fundamental aspects and the physical consequences of loop gravity are investigated, where in Part I we focus on the Lorentzian invariance of the theory, and in Part II we investigate the effective dynamics description of the loop gravity in symmetry reduced models. These two parts are naturally related to each other via a well established semi-classical and continuum analysis of the theory, which is currently still under investigation. Our investigation achieves the following results:

- We formulated the gravity as an $\mathrm{SU}(1,1)$ gauge theory, whose phase space is well-parametrized by a pair $(\mathcal{A},\mathcal{E})$ formed with an $\mathfrak{su}(1,1)$-valued connection and its canonically conjugate electric field. We further explore the quantization of the $\mathrm{SU}(1,1)$ gauge gravity theory at kinematical level and compute the area spectrum. It turns out that the spectrum is discrete for space-like surfaces and continuous for time-like surfaces. The area spectrum is exactly the same (on space-like surfaces) at the kinematical level resulting from usual quantization in time-gauge.
- We derive the integration formulation for the Conrady-Hnybida extended spin foam amplitude with timelike triangles, based on $\mathfrak{su}(1,1)$ coherent states with continuous series representation. We then perform the semi-classical analysis to the spin foam partition function on a simplicial complex, with the most general configuration in which timelike tetrahedra with timelike triangles are taken into account. It turns out that
  - The large-j asymptotic behavior is determined by the critical configurations of the amplitude.
  - The critical configuration corresponds to simplicial geometry, which may contains in general nondegenerate Lorentzian 4-simplices, nondegenerate split signature 4-simplices and degenerate vector geometries.
  - The critical contributions to the amplitude are the asymptotic phases, whose exponents equal to the Regge action of gravity (in nondegenerate case)

    - Vertex amplitudes containing at least one timelike and one spacelike tetrahedra only give Lorentzian 4-simplices, while the split signature or degenerate 4-simplex does not appear.
- Inspired from the mimetic theory first introduced by Chamseddine and Mukhanov [178, 262–264], we revisit the extended Mimetic gravity, a family of higher-derivative scalar-tensor theories, from a Hamiltonian perspective.
    - We show explicitly how limiting curvature mechanism in the context of extended Mimetic gravity resolves the singularity.
    - We found a family of extended Mimetic gravity that possess the property to reproduce exactly the effective dynamics of loop quantum cosmology for flat, closed and open homogeneous and isotropic space-times, leading to bouncing solutions.
    - We present a general procedure to construct mimetic Lagrangians which admit a Hamiltonian formulation very similar in spirit to existing spherically symmetrical polymer models of black holes in the context of LQG, but in an fully covariant manner. The difference provides us with a guide to understand the absence of covariance in inhomogeneous polymer models.
- We give an effective metric of black hole interior in the framework of effective spherically symmetric polymer models inspired by Loop Quantum Gravity. Starting from the anomaly free polymer regularization of one phase space variable for inhomogeneous spherically symmetric geometry, and then reducing to the homogeneous interior problem, we provide an alternative treatment to existing polymer interior black hole models which focus directly on the interior geometry, ignoring covariance issue when introducing the polymer regularization. We solve explicitly the modified Einstein equations obtained for a static interior black hole geometry and find the effective metric describing the trapped region inside the black hole for any polymer regularization. We investigate the explicit form of the effective metric in $j = 1/2$ case, where the regularization function is the usual sine function used in the polymer literature. For this case, the interior metric describes a regular trapped region and presents strong similarities with the Reissner-Nordström metric, with a new inner horizon generated by quantum effects. We discuss the gluing of our interior solution to the exterior Schwarzschild metric and the challenge to extend the solution outside the trapped region due to covariance requirement.

These result opens interesting theoretical and phenomenological directions. Below we list some interesting topics with possible questions to investigate.

Area spectrum for timelike surfaces: Continuous vs. discrete

As we show, the quantization of the SU$(1,1)$ gauge gravity theory at kinematical level leads to a continuous spectrum for time-like surfaces. However, such spectrum in the full Lorentzian invariant spin-foam model is discrete for time-like surfaces, as shown in [61]. This may due to the fact that the space gauge introduces anomaly in the quantization at the kinematical level. Or this may relate to a possible tension between spin foam models and canonical approach. Such question is related again to the question whether the construction of LQG deeply relies on the partial gauge fixing or not. There are naturally several topics one can investigate, for example,

— Is the area operator in $\mathfrak{su}(1,1)$ case a gauge-invariant complete Dirac observable [103, 104, 265, 266] in some situation, e.g., on black hole horizons? Otherwise we may not be able to take the area spectrum physically.

— Is there a relation between physical Hilbert space and spin foam models on timelike boundaries? What is the difference between previous studies [267, 268]?

— What is the the connection between "spacelike" state and "timelike" state at kinematical level?

— A comparison study for quantum black microstates with space-like slices [39, 40, 149] and time-like slices.

— Can we study in detail the full dirac quantization procedure of $\mathfrak{su}(1,1)$ model, more importantly, express the scalar constraint or Master constraint on kinematical Hilbert space?

— Can we study the weak rotation of spin foam models to investigate the relation between Euclidean and Lorentzian models?

These may help us get a better understanding how the Lorentzian covariance is manifested with gauge fixing, and finally understand the dynamics of loop quantum gravity.

Null hypersurface and null boundaries

Now, we have a description of the kinematical quantum states of gravity not only on space-like surfaces but also on time-like surfaces. The only remaining description is the quantum states about null-surfaces. The description on null-surfaces can be constructed in both the canonical and spin foam approach

— In the canonical analysis on a null foliation, e.g. the one given as [145], is there a gauge fixing procedure exist similar in SU(2) and SU$(1,1)$ case but now with a gauge group E_2 or $E_2 \otimes D$, which can produce a simple coordinate of the phase space such that it is possible to perform the loop quantization.

— Can we take a null gauge to the quantum simplicity constraint in spin foam models such that one get the spin foam partition function with null boundaries? A preliminary study [269] has been taken on null kinematics.

Such a null description will be interesting since it could lead to a better understanding of the Lorentzian invariance, the dynamics of isolated horizon and the casual structure in the quantum gravity theory.

Flatness problem in Spin foam models

It has been argued that, the classical limit of EPRL/FK SFM only describe flat geometries, so-called "flatness" problem [270–272]. The flatness is a consequence of summing over all spins, and as mentioned in [272], it is tightly related to the using of $\mathrm{SU}(2)$ boundaries. It would be interesting to investigate the following questions:

- In Conrady-Hnybida extended model we can have both $\mathrm{SU}(2)$ and $\mathrm{SU}(1,1)$ boundaries together, and we proved that such boundaries only lead to Lorentzian non-degenerate 4 simplices in the semi-classical limit. It is natural to ask if the "flatness" problem still appear with the configuration contains both timelike and spacelike boundaries for each 4 simplex. If the answer is no, what does this imply to the graph dependence of the spin foam models?
- Is there any nature extension of the theory, for example, analytic continuation of parameters, such that the "flatness" problem can be resolved. There seems to be a strong evidence in the continuum limit approach proposed by [50, 51, 81].

Renormalization, phase diagram, and continuum limit of spin foam models
Investigation of the renormalizability and continuum limit is a crucial step in the quantum gravity program. And it is the basis for studying the emergent physics and the effective dynamics. However, There is still an open question in spin foam models despite an increasing attentions recently, e.g. [8, 47–53]. The main questions are the following:

- How can we define and study the renormalization group flow [273] of spin foam models and their continuum phase diagram? Since spin foam model can be regarded as special tensor network models, the renormalization method used in tensor network models [274, 275] maybe helpful, as pointed out in [**Bahr**, 8].
- Can we identify the phases appear in the phase diagram which have possible geometrical interpretations? Since in the EPRL model both Lorentzian, Euclidean and degenerate vector geometries appear in the asymptotics configuration, one expect there are phase transitions between these asymptotics geometries. It is also interesting to know how different causal configuration (timelike or spacelike surfaces for example) influence those geometries. Moreover, there are also possible phase transition between lager-j and small-j regime, as indicated in [276].
- Can we define a controlled continuum limit (a coarse-grain scheme) of spin foam model, such that it reproduce the classical continuous geometry? Is there any nature extension of the theory to make such definition easier, e.g. the one given by [50, 51, 81] or [134].

Emergent gravity from spin foams and the effective dynamics
With a suitable continuum limit along the semi-classical limit procedure defined for spin foam model, in principle we can study how gravity emerge from the quantum geometry and investigate their effective dynamics. There will be sev-

eral possible topics related. For example,

— With our semi-classical limit for the extended model which now possible with configuration used in usual Lorentzian Regge calculus model, can we directly apply the deformed continuum limit taken in [50, 51], to get the effective dynamics of spin foam model in the symmetry reduced case, for example, the cosmological case, similar to [277], and black holes?

— Are the effective dynamics from the Lorentzian model coincide with the wick rotated one from Euclidean model?

— Can we derive the entanglement entropy from full spin foam partition function? In principle such entropy would contains the dynamical information thus one may expect we could emerge gravity from it. In this sense one expect an area law, as indicated in [278, 279]. However, it seems the kinematical studies only get the logarithm law e.g. as shown in [280]. Is this problem related to the "flatness" problem?

Edge holonomies in spherical symmetric polymer models

As we shown in chapter 6, mimetic theory can be taken as a guide to derive a consistent covariant effective framework for black holes. However, it is hard to resolve the covariance, or understand consequences of deformation, without an exact form of edge holonomies appears in the effective Einstein equations. The black hole dynamics (including matter) of the polymer model, is also closely related to such problem. However, those edge holonomies, instead of point holonomy, are less understood in polymer models. The following investigations may help us understand

— Can we have a family of covariant scalar-tensor theories such that they can describe the homogeneous model of BH interior, e.g. [34, 209, 210, 258, 259]? Such a theory may lead us to a possible formulation beyond homogenous model. It can also help us understand better the possible $\bar{\mu}$-scheme and the anomaly free condition in homogeneous models.

— Can we get hints for the possible correction for non-homogeneous $\mathfrak{su}(2)$ edge holonomies, for examples, those described in [184], from $\mathfrak{su}(1,1)$ theory of homogeneous exterior BH model [37], by compare the vector constraint in $\mathfrak{su}(2)$ theory with scalar constraint from homogeneous $\mathfrak{su}(1,1)$ theory?

— Are there possible generalizations of the current regularization used in polymer models to implement consistently , e.g., the ideas given in [254, 281]?

— Can we finally have an effective black hole dynamics from these model, such that one is possible to study the hawking radiation, gravitational collapse and bouncing scenarios? As an example, [219, 281] provide some proposals in this direction.

We hope in the near future we can find answers to those questions. This will definitely lead us to a more complete understanding to the "quantum gravity" problem.

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