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Conditions d'existence des processus déterminantaux et permanentaux

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Résumé

Nous établissons des conditions nécessaires et suffisantes d'existence et d'infinie divisibilité pour des processus ponctuels α-déterminantaux et, lorsque α est positif, pour leur intensité sous-jacente (en tant que processus de Cox). Dans le cas où l'espace est fini, ces distributions correspondent à des lois binomiales, négatives binomiales et gamma multidimensionnelles. Nous étudions de façon approfondie ces deux derniers cas avec un noyau non nécessairement symétrique.

Mots clés : processus déterminantal, processus permanental, processus α-déterminantal, infiniment divisible, complèment monotone, processus de fermion, processus de boson, vecteur permanental, loi négative binomiale multidimensionnelle, loi gamma multidimensionnelle, M-matrice, inverse de M-matrice, vecteur gaussien, cycles de matrice, permanent, déterminant.

Abstract

We establish necessary and sufficient conditions for the existence and infinite divisibility of α-determinantal processes and, when α is positive, of their underlying intensity (as Cox process). When the space is finite, these distributions correspond to multidimensional binomial, negative binomial and gamma distributions. We make an in-depth study of these last two cases with a non necessarily symmetric kernel.

Keywords: determinantal process, permanental process, α-determinantal process, α-permanental process, infinitely divisible, complete monotonicity, fermion process, boson process, permanental vector, multivariate negative binomial distribution, multivariate gamma distribution, M-matrix, inverse M-matrix, Gaussian vector, matrix cycles, permanent, determinant.

Plan

1 Introduction générale **11**

1.1 Préliminaires 11

1.2 Contexte 13

1.2.1 Définitions avec un espace d'état fini 13

1.2.2 Liens entre loi d'intensité permanentale et d'autres lois . . 14

1.2.3 Définition d'un processus α-determinantal 15

1.2.4 Lien entre loi binomiale négative multivariée et processus α-déterminantal 23

1.2.5 Divisibilité infinie 23

1.2.6 Formules de développement en séries entières 24

1.2.7 Fonctions complètement/absolument monotones 25

1.2.8 Lien avec la conjecture de Shirai et Takahashi sur l'α-déterminant d'une matrice semi-définie positive 26

1.2.9 Lien avec la conjecture de corrélation gaussienne 28

1.3 Résultats présentés dans la thèse 30

1.3.1 Conditions d'existence de processus α-déterminantaux . . 30

1.3.2 Etude approfondie du cas où l'espace d'état est fini 31

1.4 Gaussian moment product conjecture and perspectives 34

2 Existence conditions for α-determinantal processes 39

2.1 Introduction 39

2.2 Preliminaries 40

2.3 Main results 43

2.4 Fredholm determinant expansion 46

2.5 α- permanental process ($\alpha > 0$) 49

2.6 α- determinantal process ($\alpha < 0$) 51

2.7 Infinite divisibility 64

3 Advanced study of the case of permanental distributions with finite state space 67

3.1 Introduction 67

3.2 Notation, cycles and permanents 71

3.2.1 Positive cycles and symmetric cycles 73

3.2.2 Permanent of matrices with rows and columns repetition . 74

3.3 NSC to be β-positive for all $\beta > 0$ 77

3.4 Links between β-permanentality and β-positivity 80

3.5 NSC to be β-permanental for all $\beta > 0$ 85

3.6 Dimension 3 89

3.7 Beyond dimension 3 92

3.7.1 Positive answer to the two questions 92

3.7.2 Some conditions for 1-positivity 99

Bibliography **109**

Chapitre 1

Introduction générale

1.1 Préliminaires

L'objet de nos travaux est une classe particulière de processus ponctuels : les processus α-déterminantaux.
Les processus ponctuels sont des variables aléatoires, dont chaque réalisation correspond à un ensemble de points isolés à l'intérieur d'un espace d'état. Typiquement cet espace d'état peut être un espace euclidien de dimension finie ($\mathbb{R}^d$), ou plus généralement un espace Polonais. Au sein des processus ponctuels, le plus connu est assurément le processus de Poisson, dont chaque point peut être tiré indépendamment des autres. L'intensité est la mesure qui donne, pour une zone donnée, le nombre moyen de points.
Les mathématiciens et physiciens se sont également intéressés à des processus ponctuels ayant des propriétés de répulsion ou d'attraction, car ils permettent de simuler des phénomènes où la dépendance joue un rôle. Par exemple, dans un réseau de téléphonie mobile, il peut être intéressant de modéliser la position des stations de base de même fréquence par un processus ponctuel avec une propriété de répulsion. A l'inverse, pour modéliser des phénomènes de défaut de crédit ou de faillite d'entreprise, il peut être intéressant d'utiliser un processus ponctuel avec une propriété d'attraction : la survenance d'une faillite d'entreprise est peut–être le signe que le secteur va mal et donc le risque qu'il y ait d'autres faillites augmente.
A noter que le terme processus est en quelque sorte un abus de langage lorsque l'on décrit un phénomène aléatoire où le temps ne joue aucun rôle, mais comme l'usage est bien établi dans le cadre des processus ponctuels, on conservera cette

terminologie, que le temps joue un rôle ou non.

Les processus déterminantaux (ou processus de fermion), introduits par Macchi dans les années 70 ([Mac75]) permettent de simuler les propriétés de répulsion entre les points. La terminologie "processus de fermion" est plutôt utilisée dans le cadre de la physique, car ces processus rappellent le comportement répulsif des fermions (principe d'exclusion de Pauli). En mathématiques, on utilise souvent la terminologie *processus déterminantal*, qui caractérise le fait que la fonction de corrélation (également appelée intensité jointe) s'exprime comme le déterminant d'un noyau. Une des raisons de leur intérêt est leur lien avec les matrices aléatoires. Par exemple, la loi de valeurs propres d'une matrice aléatoire GUE (Gaussian Unitary Ensemble) est celle d'un processus déterminantal.
Les processus ponctuels permanentaux, ou processus de boson (voir [MM06]) permettent de simuler des propriétés d'attraction entre les points. On peut généralement (mais pas toujours) les voir comme un cas particulier de processus de Cox, introduits par David Cox en 1955. Ces processus peuvent être simulés en superposant deux aléas : tout d'abord on tire au hasard une intensité (mesure aléatoire) selon une certaine loi. Une fois l'intensité obtenue, on tire les points selon la loi d'un processus de Poisson ayant cette mesure intensité. La terminologie processus ponctuel permanental caractérise le fait que la fonction de corrélation s'exprime comme le permanent d'un noyau.
A noter que cette intensité aléatoire est appelée par certains auteurs processus permanental (voir par exemple N. Eisenbaum, H. Kaspi dans [EK09] ou H. Kogan, M.B. Marcus et J. Rosen dans [KM10]). Ces auteurs et d'autres se sont intéressés à ces processus pour leur lien avec le temps local de processus de Markov, liens connus sous l'appellation générique de *théorèmes d'isomorphisme* (Dynkin [Dyn83], Eisenbaum et Kaspi [EK09]). Ces processus seront appelés *processus d'intensité permanentale*, lorsque des confusions sont possibles avec les processus ponctuels permanentaux.
Les processus α-déterminantaux forment une famille de processus ponctuels qui inclut les processus de Poisson ($\alpha = 0$), les processus déterminantaux ($\alpha = -1$) et les processus ponctuels permanentaux ($\alpha = 1$). Généralement, lorsque α est strictement négatif, on a répulsion entre les points, et la répulsion est d'autant plus forte que α est éloigné de zéro. Lorsque α est strictement positif, on a attraction entre les points et l'attraction est d'autant plus forte que α est grand. Ces processus ont été décrits par Shirai et Takahashi dans [ST03]. Lorsque α est positif, on peut aussi utiliser la terminologie processus (ponctuel) α-permanental (ou tout simplempent, par abus de langage, processus permanantal), au lieu de

processus α-déterminantal.

Dans cette thèse, nous établissons des conditions nécessaires et suffisantes d'existence et d'infinie divisibilité de processus α-déterminantaux. Lorsque α est positif, nous nous intéressons également aux conditions d'existence et d'infinie divisibilité de l'intensité de ces processus, en tant que processus de Cox. Nous avons étudié de façon plus approfondie le cas où α est positif et l'espace d'état fini.

1.2 Contexte

Le but de cette section est de présenter les principaux concepts qui seront utilisés dans la suite. Notre objectif n'est pas d'être exhaustif mais de donner des définitions claires, quelques résultats essentiels liés à ces définitions, et d'expliquer sommairement d'où viennent les résultats que nous utilisons.

1.2.1 Définitions avec un espace d'état fini

Nous commençons par donner les définitions dans le cadre d'un espace d'état fini. Des définitions dans le cadre d'un espace d'état plus général (espace Polonais) sont données plus loin dans l'introduction et au Chapitre 2.

Un vecteur aléatoire $\boldsymbol{X} = (X_d, \ldots, X_d)$, dont les coordonnées sont des réels positifs, est dit d'intensité (β-)permanentale lorsque sa fonction génératrice des moments peut être écrite de la façon suivante :

$$\mathbb{E}\left(\exp\left(\sum_{i=1}^{d} z_i X_i\right)\right) = \det(I - ZA)^{-\beta} \tag{1.1}$$

où $Z = \mathrm{diag}(z_1, \ldots, z_d)$, β est un réel positif ou nul, $A = (a_{ij})_{1\leq i,j\leq d}$ est une matrice carré d'ordre d et I est la matrice identité d'ordre d. On dira aussi que ce vecteur suit une loi (β-)permanentale. A noter que cette loi n'existe que pour $\beta > 0$, raison pour laquelle on a fait cette supposition.

Lorsque ce vecteur aléatoire existe, on dira que la matrice A est β-permanentale. On dira aussi que la matrice A est un noyau de cette loi permanentale (ou du vecteur (β-)permanental associé). Notons que ce noyau n'est pas unique pour une loi permanentale donnée.

On dira qu'un vecteur aléatoire $\boldsymbol{X} = (X_1, \ldots, X_d)$, à coordonnées entières, suit une loi binomiale négative multivariée de noyau A, lorsque sa fonction génératrice des probabilités peut s'écrire de la façon suivante :

$$\mathbb{E}\left(\prod_{i=1}^{d} z_i^{X_i}\right) = \det(I - A)^{\beta} \det(I - ZA)^{-\beta}$$

avec les mêmes notations que pour la définition précédente.

On dira qu'un vecteur aléatoire $\boldsymbol{X} = (X_1, \ldots, X_d)$, à coordonnées entières, suit une loi binomiale multivariée de noyau A, lorsque sa fonction génératrice des probabilités peut s'écrire de la façon suivante :

$$\mathbb{E}\left(\prod_{i=1}^{d} z_i^{X_i}\right) = \det(I + A)^{-\beta} \det(I + ZA)^{\beta}$$

toujours avec les mêmes conventions que pour la définition précédente.

1.2.2 Liens entre loi d'intensité permanentale et d'autres lois

En dimension $d = 1$, un vecteur d'intensité β-permanentale est une variable aléatoire de loi gamma.

En dimension $d > 1$, une loi permanentale est une loi gamma multivariée (Vere-Jones dans [VeJ97]).

Pour $\beta = \frac{1}{2}$ et lorsque le noyau A est réel symétrique semi-défini positif, un vecteur d'intensité $\frac{1}{2}$-permanentale est de la forme $(X_1^2, \ldots, X_d^2)$, où $(X_1, \ldots, X_d)$ est une vecteur gaussien centré de matrice de covariance $2A$.

Pour $\beta = \frac{n}{2}$ avec n entier naturel et toujours lorsque le noyau A est réel symétrique semi-défini positif, les coordonnées d'un vecteur d'intensité $\frac{n}{2}$-permanentale correspondent à une somme de carrés de gaussiennes.

Toujours lorsque la matrice A est réelle symétrique semi-définie positive, les coordonnées d'un vecteur d'intensité β-permanentale correspondent aux coefficients diagonaux d'une matrice de Wishart :

- la fonction génératrice des moments d'un vecteur $\boldsymbol{X}$ d'intensité permanentale peut s'écrire : $\mathbb{E}\left(\exp\left(\sum_{i=1}^{d} z_i X_i\right)\right) = \det(I - ZA)^{-\beta}$, où z est un vecteur de dimension d et $Z = \mathrm{diag}(z)$

- la fonction génératrice des moments d'une matrice de Wishart W peut s'écrire : $\mathbb{E}(\exp(\mathrm{tr}(\theta W)) = \det(I - \theta A)^{-\beta}$, où θ est une matrice carrée d'ordre d.

A noter que, pour A réelle symétrique semi-définie positive de dimension $d \times d$, les lois de Wishart existent si et seulement si $\beta > \frac{d-1}{2}$ ou 2β entier. Il est clair que l'existence d'une loi de Wishart pour une matrice A et un réel positif β donné implique l'existence d'un vecteur d'intensité β-permanentale. La réciproque est fausse : en effet, toujours pour A réelle symétrique semi-définie positive, la loi d'intensité β-permanentale existe au moins pour $\beta > \frac{d-2}{2}$ ou 2β entier (voir [Roy14]).

1.2.3 Définition d'un processus α-determinantal

Processus ponctuel

On considère un espace Polonais E localement compact. Une configuration localement finie sur E est une mesure de Radon sur E à valeurs entières. On peut aussi identifier une configuration avec un ensemble $\{(M, \alpha_M) : M \in F\}$, où F est un sous-ensemble dénombrable de points de E, et pour chaque point M, α_M est un entier qui correspond à la multiplicité du point M. On dit que M est un point multiple lorsque $\alpha_M \geq 2$.

Soit λ une mesure de Radon sur E et soit $\mathcal{X}$ l'espace des configurations localement finies sur E. On munit l'espace $\mathcal{X}$ de la topologie vague des mesures, c'est à dire la plus petite topologie telle que, pour toute fonction f réelle continue à support compact, la fonction

$$\mathcal{X} \ni \xi \mapsto \langle f, \xi \rangle = \sum_{x \in \xi} f(x) = \int f \mathrm{d}\xi$$

soit continue.

On note $\mathcal{B}(\mathcal{X})$ la tribu borélienne correspondante.

Un processus ponctuel sur E est une mesure aléatoire à valeurs dans $\mathcal{X}$. On dit que c'est un processus ponctuel simple si la probabilité d'avoir une configuration avec au moins un point multiple est nulle.

Densités de Janossy

Pour un sous-ensemble mesurable relativement compact $\Lambda \subset E$, les densités de Janossy d'un processus ponctuel ξ par rapport à une mesure de référence λ (qu'on supposera toujours être une mesure de Radon) sont les fonctions (quand elles existent) $j_n^\Lambda : E^n \to \mathbb{R}_+$ pour $n \in N$, telles que

$$\begin{aligned} &j_n^\Lambda(x_1, \dots, x_n) = n!\,\mathbb{P}(\xi(\Lambda) = n)\,\pi_n^\Lambda(x_1, \dots, x_n) \\ &j_0^\Lambda(\emptyset) = \mathbb{P}(\xi(\Lambda) = 0), \end{aligned}$$

où π_n^Λ est la densité par rapport à $\lambda^{\otimes n}$ des n-uplets (ordonnés) $(x_1, \dots, x_n)$, obtenus en tirant d'abord ξ, conditionnellement au fait d'avoir n points dans Λ, puis en choisissant uniformément un ordre entre les points.

Pour $\Lambda_1, \dots, \Lambda_n$ sous-ensembles mesurables deux à deux disjoints inclus dans Λ,

$$\int_{\Lambda_1 \times \cdots \times \Lambda_n} j_n^\Lambda(x_1, \dots, x_n)\lambda(dx_1)\dots\lambda(dx_n)$$

est la probabilité qu'il y ait exactement un point dans chaque sous-ensemble Λ_i $(1 \leq i \leq n)$, et aucun autre point ailleurs.

Pour une fonction positive ou nulle f de support relativement compact $\Lambda \subset E$, on a la formule suivante

$$\mathbb{E}(f(\xi)) = f(\emptyset)\,j_0^\Lambda(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f(x_1, \dots, x_n)\,j_n^\Lambda(x_1, \dots, x_n)\lambda(dx_1)\dots\lambda(dx_n).$$

Fonctions de corrélation

Pour $n \in \mathbb{N}$ et $a \in \mathbb{R}$, on note $a^{(n)} = \prod_{i=0}^{n-1}(a - i)$.

Les fonctions de corrélation (appelées aussi intensités jointes) d'un processus ponctuel ξ par rapport à une mesure de référence λ sont (lorsqu'elles existent) les fonctions $\rho_k : E^k \to \mathbb{R}_+$ vérifiant :

$$\mathbb{E}\left(\prod_{i=1}^{d} \xi(\Lambda_i)^{(n_i)}\right) = \int_{\Lambda_1^{n_1} \times \dots \times \Lambda_d^{n_d}} \rho_n(x_1, \cdots, x_n)\lambda(dx_1)\dots\lambda(dx_n).$$

pour tous $\Lambda_1, \ldots, \Lambda_d$ sous-ensembles de E, mesurables bornés deux à deux disjoints, et pour tous $n_1, \ldots, n_d$ entiers, avec $n = \sum_{i=1}^{d} n_i$.

Lorsque pour tout sous-ensemble Λ de E, mesurable borné, $\xi(\Lambda)$ a une queue exponentielle, les fonctions de corrélation déterminent alors de façon unique le processus ponctuel ξ, (voir [Len73] et [Len75]. Les processus ponctuels considérés dans cette thèse entrent tous dans ce cadre.

Intuitivement, pour un processus ponctuel simple, $\rho_n(x_1, \ldots, x_n)\lambda(dx_1)\ldots\lambda(dx_n)$ est la probabilité infinitésimale d'avoir au moins un point dans le voisinage de chaque x_i (chaque voisinage ayant le volume infinitésimal $\lambda(dx_i)$ autour de x_i).

α-déterminant et β-permanent

On note $\det_\alpha$ l'α-déterminant d'une matrice, à savoir, pour une matrice $A = (a_{ij})_{1\leq i,j\leq n}$:

$$\det_\alpha A = \sum_{\sigma\in\Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$

où Σ_n est l'ensemble des permutations de $\{1, \ldots, n\}$ et $\nu(\sigma)$ est le nombre de cycles de la permutation σ.

On note per_β le β-permanent d'une matrice, à savoir, pour une matrice $A = (a_{ij})_{1\leq i,j\leq n}$:

$$\mathrm{per}_\beta A = \sum_{\sigma\in\Sigma_n} \beta^{\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$

On suit ici les conventions utilisées par Shirai et Takahashi dans [ST03] et Vere-Jones dans [VeJ88] et [VeJ97], mais on a remplacé α par β dans le cas du β-permanent, car en général β correspond à $1/\alpha$, ce que l'on peut voir à travers la formule suivante

$$\det_\alpha A = \alpha^n \,\mathrm{per}_{1/\alpha} A$$

Pour $\alpha = \beta = 1$, l'α-déterminant et le β-permanent sont tous deux égaux au permanent usuel.

Pour $\alpha = -1$, l'α-déterminant est égal au déterminant usuel et

$$\mathrm{per}_{-1} A = (-1)^n \det A$$

Processus α-déterminantal

Soient α un réel et K un noyau défini de E^2 dans $\mathbb{R}$ ou $\mathbb{C}$. Un processus α-déterminantal de noyau K est défini (quand il existe) comme un processus ponctuel ayant pour fonctions de corrélation par rapport à une mesure de référence λ :

$$\rho_n(x_1, \dots, x_n) = \det{}_\alpha(K(x_i, x_j))_{1 \leq i,j \leq n} \qquad (n \in \mathbb{N})$$

Nous excluons le cas de processus ponctuels presque sûrement réduits à la configuration vide.

Le cas $\alpha = -1$ correspond à un processus déterminantal et le cas $\alpha = 1$ à un processus ponctuel permanental, dans leurs acceptions usuelles. Le cas $\alpha = 0$ correspond à un processus ponctuel de Poisson. On supposera que $\alpha \neq 0$.

Comme indiqué précédemment, lorsque $\alpha > 0$, un processus ponctuel α-déterminantal a généralement des propriétés d'attraction entre les points et lorsque $\alpha < 0$, il a généralement des propriétés de répulsion entre les points. Plus précisément, on a les propriétés suivantes concernant les fonctions de corrélation, lorsque le noyau est hermitien semi-défini positif :

$$\rho_n(x_1, \dots, x_n) \geq \rho_1(x_1) \dots \rho_1(x_n) \text{ lorsque } \alpha = 1/m > 0 \tag{1.2}$$
$$\rho_n(x_1, \dots, x_n) \leq \rho_1(x_1) \dots \rho_1(x_n) \text{ lorsque } \alpha = -1/m < 0 \tag{1.3}$$

où m est un entier strictement positif (voir [ST03], Proposition 4.3).

Dans le cas où le noyau est réel symétrique semi-défini positif, on a même

$$\rho_n(x_1, \dots, x_n) \geq \rho_1(x_1) \dots \rho_1(x_n) \text{ lorsque } \alpha = 2/m > 0 \tag{1.4}$$

Pour $\alpha = 2$, ceci est une conséquence d'un cas particulier, démontré initialement par Frenkel [Fre07], de l'inégalité des produits de moments gaussiens (voir section 1.4). Pour $\alpha = 2/m$, on peut montrer l'inégalité (1.4) par convolution à partir du cas $\alpha = 2$, comme le font Shirai et Takahashi pour montrer le cas $\alpha = 1/m$ à partir du cas $\alpha = 1$ (dans [ST03], preuve de leur Proposition 4.3).

On peut cependant noter que, pour les autres valeurs de $\alpha > 0$, il existe des noyaux réels symétriques semi-définis positifs, tels que le processus α-déterminantal correspondant existe, mais l'inégalité (1.4) n'est pas garantie.

Lorsque $\alpha > 0$, on utilisera parfois l'appellation processus (ponctuel) α-permanental à la place de processus α-déterminantal, afin de rappeler le lien avec le processus ponctuel permanental lorsque $\alpha = 1$.

Opérateur intégral de classe trace

Un opérateur $\mathcal{K}$ sur un espace de Hilbert H est dit de classe trace lorsque l'on a la propriété suivante :

$$\|\mathcal{K}\|_1 := \sum_{i \in I} (|\mathcal{K}| e_i, e_i) < \infty$$

où $|\mathcal{K}| = (\mathcal{K}^* \mathcal{K})^{1/2}$ et $(e_i)_{i \in I}$ est une base orthonormale de H. Cette propriété ne dépend pas de la base orthonormale choisie. Un opérateur de classe trace est nécessairement de Hilbert-Schmidt, donc compact, donc continu.

Sa trace est alors définie par

$$\operatorname{tr} \mathcal{K} = \sum_{i \in I} (\mathcal{K} e_i, e_i)$$

La trace ne dépend pas non plus de la base orthonormale choisie.

Un opérateur $\mathcal{K}$ sur l'espace de Hilbert $L^2(E)$ est un opérateur intégral s'il existe une fonction $K : E^2 \to \mathbb{C}$ telle que

$$\mathcal{K} f(x) = \int K(x, y) f(y) d\lambda(y)$$

K est appelé le noyau de l'opérateur $\mathcal{K}$.

Déterminant de Fredholm

De même que l'on a étendu la notion de trace à la dimension infinie, on va étendre le concept de déterminant.

On introduit la notion de déterminant de Fredholm pour un opérateur de classe trace défini sur l'espace de Hilbert H.

On considère $H^{\otimes n} = H \otimes \cdots \otimes H$ le produit tensoriel de H répété n fois :

$$H^{\otimes n} = \mathrm{Vect}\{\varphi_1 \otimes \cdots \otimes \varphi_n : \varphi_1, \ldots, \varphi_n \in H\}$$

Par convention $H^{\otimes 0} = \mathbb{C}$ (ou $\mathbb{R}$ si ce dernier est le corps de base).

On définit le produit scalaire sur $H^{\otimes n}$ par

$$\langle \varphi_1 \otimes \cdots \otimes \varphi_n, \psi_1 \otimes \cdots \otimes \psi_n \rangle = \prod_{i=1}^{n} \langle \varphi_i, \psi_i \rangle$$

Si $(e_i)_{i \in I}$ est une base orthonormale de H, $(e_{i_1} \otimes \cdots \otimes e_{i_n})_{(i_1,\ldots,i_n) \in I^n}$ est une base orthonormale de $H^{\otimes n}$

On considère l'action naturelle du groupe symétrique Σ_n sur $H^{\otimes n}$. On note ϕ_σ, l'application correspondante pour un élement σ de Σ_n.

En particulier

$$\phi_\sigma(\varphi_1 \otimes \cdots \otimes \varphi_n) = \varphi_{\sigma(1)} \otimes \cdots \otimes \varphi_{\sigma(n)}$$

Le sous-espace antisymétrique de $H^{\otimes n}$ est défini par

$$\mathcal{A}H^{\otimes n} = \{x \in H^{\otimes n} : \forall \sigma \in \Sigma_n,\ \phi_\sigma(x) = \mathrm{sgn}(\sigma)\, x\}$$

Pour $x \in H^{\otimes n}$, on pose

$$\wedge^n(x) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \phi_\sigma(x)$$

On note

$$\varphi_1 \wedge \cdots \wedge \varphi_n = \wedge^n(\varphi_1 \otimes \cdots \otimes \varphi_n) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma)\, \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(n)}$$

On a en particulier

$$\langle \varphi_1 \wedge \cdots \wedge \varphi_n, \psi_1 \wedge \cdots \wedge \psi_n \rangle = \det(\langle \varphi_i, \psi_j \rangle)_{1 \le i,j \le n}$$

Si I est totalement ordonné, $(e_{i_1} \wedge \cdots \wedge e_{i_n})_{i_1 < \cdots < i_n}$ forme une base de $\mathcal{A}H^{\otimes n}$.

Pour un opérateur $\mathcal{K}$ sur H, on note

$$\wedge^n(\mathcal{K}) = (\mathcal{K} \otimes \cdots \otimes \mathcal{K})|_{\mathcal{A}H^{\otimes n}}$$

Par convention, $\wedge^0(\mathcal{K})$ est l'application identité sur le corps de base $\mathbb{C}$ (ou $\mathbb{R}$).

Lorsque $\mathcal{K}$ est de classe trace, comme $|\wedge^n(\mathcal{K})| = \wedge^n(|\mathcal{K}|)$, on a

$$\begin{aligned}\|\wedge^n(\mathcal{K})\|_1 &= \sum_{k_1<\cdots<k_n} \det(\langle|\mathcal{K}|e_{k_i}, e_{k_j}\rangle)_{1\le i,j\le n}\\ &= \frac{1}{n!}\sum_{(k_1,\ldots,k_n)\in\Delta_n} \det(\langle|\mathcal{K}|e_{k_i}, e_{k_j}\rangle)_{1\le i,j\le n}\end{aligned}$$

où $\Delta_n = \{(k_1,\ldots,k_n) \in I^n : \forall (i,j) \in [\![1,n]\!]^2, i \neq j \implies k_i \neq k_j\}$.

Si on choisit $(e_i)_{i\in I}$ comme étant une base de fonctions propres de $|\mathcal{K}|$ et si les $(\nu_i)_{i\in I}$ sont les valeurs propres associées, on obtient :

$$\|\wedge^n(\mathcal{K})\|_1 = \frac{1}{n!}\sum_{(k_1,\ldots,k_n)\in\Delta_n}\prod_{i=1}^n |\nu_{k_i}| \le \frac{1}{n!}\sum_{k_1,\ldots,k_n}\prod_{i=1}^n |\nu_{k_i}| = \frac{1}{n!}\|\mathcal{K}\|_1^n$$

On en déduit que la série $\sum_{n=0}^{\infty} \mathrm{tr}(\wedge^n(\mathcal{K}))$ converge normalement pour $\| \ \|_1$.

Le déterminant de Fredholm pour un opérateur de classe trace $\mathcal{K}$ sur l'espace de Hilbert H est défini par

$$\mathrm{Det}(I+\mathcal{K}) = \sum_{n=0}^{\infty} \mathrm{tr}(\wedge^n(\mathcal{K}))$$

En notant $(\lambda_i)_{0\le i<n}$ les valeurs propres non nulles de l'opérateur $\mathcal{K}$, répétées en tenant compte de leur multiplicité algébrique, on a les propriétés suivantes :

$$\begin{aligned}\mathrm{Det}(I+\mathcal{K}) &= \prod_{i\in I}(1+\lambda_i)\\ \mathrm{Det}(I+\mathcal{K}) &= \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\mathrm{tr}(\mathcal{K}^n)\right) \text{ si } \|\mathcal{K}\| < 1\end{aligned}$$

Si $\mathcal{K}$ est un opérateur de classe trace et $\mathcal{L}$ désigne un opérateur continue, on a

$$\mathrm{Det}(\mathcal{I}+\mathcal{K}\mathcal{L}) = \mathrm{Det}(\mathcal{I}+\mathcal{L}\mathcal{K})$$

Si $\mathcal{K}$ et $\mathcal{L}$ sont des opérateurs de classe trace, on a

$$\mathrm{Det}((\mathcal{I}+\mathcal{K})(\mathcal{I}+\mathcal{L})) = \mathrm{Det}(\mathcal{I}+\mathcal{K})\,\mathrm{Det}(\mathcal{I}+\mathcal{L})$$

$\mathcal{K} \mapsto \mathrm{Det}(\mathcal{I}+\mathcal{K})$ est une fonctionnelle analytique définie sur l'espace des opérateurs de classe trace muni de la $\| \; \|_1$. En particulier, elle est continue pour cette norme.

Fonctionnelle de Laplace

On considère un noyau $K : E^2 \to \mathbb{C}$ définissant un opérateur intégral $\mathcal{K}$ sur $L^2(E)$ localement de classe trace (c'est à dire tel que pour tout compact $\Lambda \subset E$, $\mathcal{K}_\Lambda = p_\Lambda \mathcal{K} p_\Lambda$ est de classe trace, où p_Λ désigne la projection orthogonal de $L^2(E)$ sur le sous espace $L^2(\Lambda)$).

Un processus α-determinantal de noyau K a la fonctionnelle de Laplace suivante

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[\exp\left(-\int_E f d\xi\right)\right] = \mathrm{Det}\left(\mathcal{I}+\alpha\mathcal{K}[1-e^{-f}]\right)^{-1/\alpha}$$

où f est une fonction positive ou nulle à support compact définie sur E, $\mathcal{K}[1-e^{-f}]$ signifie $\sqrt{1-e^{-f}}\mathcal{K}\sqrt{1-e^{-f}}$, $\mathcal{I}$ est l'opérateur identité sur $L^2(E)$ et Det est le déterminant de Fredholm.

Cette formule donnant la fonctionnelle de Laplace peut également servir de définition alternative à celle qui a été donnée à l'aide des fonctions de corrélation.

On peut trouver des précisions sur le lien entre les fonctions de corrélation et la fonctionnelle de Laplace d'un processus α-determinantal dans le chapitre 4 de [ST03].

On a pour tout $\Lambda \subset E$ relativement compact

$$\mathbb{E}(\xi(\Lambda)) = \int_\Lambda \rho_1(x) d\lambda(x) = \mathrm{tr}\,\mathcal{K}_\Lambda$$

1.2.4 Lien entre loi binomiale négative multivariée et processus α-déterminantal

Une loi binomiale multivariée dont la fonction génératrice de probabilités vaut

$$\mathbb{E}\left(\prod_{i=1}^{d} z_i^{X_i}\right) = \det(I+A)^{-\beta}\det(I+ZA)^{\beta} \qquad (\beta > 0)$$

est la loi d'un processus $-1/\beta$-déterminantal de noyau $\beta A(I+A)^{-1}$, avec comme espace d'état $E = [\![1, d]\!]$ fini.

De même, une loi négative binomiale multivariée dont la fonction génératrice de probabilités vaut

$$\mathbb{E}\left(\prod_{i=1}^{d} z_i^{X_i}\right) = \det(I-A)^{\beta}\det(I-ZA)^{-\beta} \qquad (\beta > 0)$$

est la loi d'un processus $1/\beta$-permanental de noyau $\beta A(I-A)^{-1}$, avec comme espace d'état $E = [\![1, d]\!]$ fini.

Dans les deux cas, X_i est un entier naturel correspondant au nombre de points en i (point multiple lorsque $X_i \geq 2$).

1.2.5 Divisibilité infinie

La question de la divisibilité infinie des vecteurs dont les coordonnées sont des carrés de coordonnées de vecteurs gaussiens centrés a été posée pour la première fois par Paul Lévy en 1948. Cette question est longuement restée en suspens. Griffiths y a apporté une réponse dans [Gri84].

En section 3.5, nous étendons le résultat de Griffiths aux vecteurs permanentaux de noyau non a priori symétrique et sans supposer que les coefficients du noyau sont tous non nuls.

Griffiths et Milne ont aussi traité le cas de la divisibilité infinie d'une loi binomiale négative multivariée dans [GM87]. Cependant, leur résultat n'est correct que si l'on fait l'hypothèse de symétrie du noyau ou de l'absence de coefficients nuls. En section 3.3, nous donnons une preuve valable sans ces hypothèses.

1.2.6 Formules de développement en séries entières

On a la formule suivante, due à Vere-Jones [VeJ88]

$$\det(I - ZA)^{-\beta} = \sum_{n_1,\ldots,n_d=0}^{\infty} \prod_{i=1}^{d} \frac{z_i^{n_i}}{n_i!} \operatorname{per}_\beta A[n_1,\ldots,n_d] \tag{1.5}$$

où

- $A = ((a_{ij})_{1\leq i,j\leq d})$ est une matrice carrée d'ordre d
- $A[n_1,\ldots,n_d] = (A_{ij})_{1\leq i,j\leq d}$, où A_{ij} est la matrice de taille $n_i \times n_j$ dont tous les coefficients sont égaux à a_{ij}. On dira que $A[n_1,\ldots,n_d]$ est une matrice dérivée de la matrice A. Par la suite, on notera $\boldsymbol{n} = (n_1,\ldots,n_d)$ et $A[\boldsymbol{n}] = A[n_1,\ldots,n_d]$.

Cette formule a été publiée concomitamment et indépendamment par Foata et Zeilberger [FZ88].

C'est une généralisation du "Macmahon Master Theorem" (1916), qui correspond au cas $\beta = 1$.

On voit à travers cette formule que, pour A de rayon spectral strictement inférieur à 1, l'existence d'une loi binomiale négative ayant pour fonction génératrice des probabilités $(z_1,\ldots,z_d) \mapsto \det(I-A)^{\beta}\det(I-ZA)^{-\beta}$ équivaut à :

$$\operatorname{per}_\beta A[n_1,\ldots,n_d] \geq 0$$

pour tous entiers $n_1,\ldots,n_d$, ce qui correspond à une infinité de conditions sur les dérivées de la matrice A.

Compte tenu de l'analogie entre déterminant et déterminant de Fredholm, il n'est pas surprenant qu'il existe une généralisation de cette formule au déterminant de Fredholm. Shirai et Takahashi ont montré dans [ST03] la formule suivante:

Pour un opérateur $\mathcal{K}$, de noyau K et de classe trace tel que $\|\alpha\mathcal{K}\| < 1$ ($\alpha \in \mathbb{R}^*$), on a

$$\det(\mathcal{I} - \alpha\mathcal{K})^{-1/\alpha} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{E^n} \det_\alpha(K(x_i,x_j)_{1\leq i,j\leq n}\lambda(dx_1)\ldots\lambda(dx_n).$$

Lorsque $\alpha \in \{-1/m; m \in \mathbb{N}\}$, la formule est valable sans la condition $\|\alpha\mathcal{K}\| < 1$.

La formule (2.3) que l'on donne dans le chapitre 2 est une généralisation simple de cette formule.

1.2.7 Fonctions complètement/absolument monotones

Une fonction f à valeurs dans $\mathbb{R}$ ou $\mathbb{C}$, définie sur un ouvert connexe $\Omega \subset \mathbb{R}^d$, est dite complètement monotone dans la direction du cône convexe ouvert $C \subset \mathbb{R}^d$ si elle est C^∞ sur Ω et vérifie, pour tout $k \geq 0$, tous vecteurs $u_1, \ldots, u_k \in C$ et tout $x \in \Omega$:

$$(-1)^k D_{u_1} \ldots D_{u_k} f(x) \geq 0$$

où D_u représente la dérivée directionnelle en direction de u. Par convention, le cas limite $k = 0$ correspond à $f \geq 0$.

Une fonction f à valeurs dans $\mathbb{R}$ ou $\mathbb{C}$, définie sur un ouvert connexe $\Omega \subset \mathbb{R}^d$, est dite absolument monotone dans la direction du cône convexe ouvert $C \subset \mathbb{R}^d$ si elle est C^∞ et vérifie, pour tout $k \geq 0$, tous vecteurs $u_1, \ldots, u_k \in C$ et tout $x \in \Omega$:

$$D_{u_1} \ldots D_{u_k} f(x) \geq 0.$$

$\mathbb{R}^d$ peut être remplacé par un espace euclidien V de dimension d par isomorphisme d'espaces vectoriels.

Une fonction analytique définie sur un ouvert connexe $\Omega \subset \mathbb{C}^d$ sera dite complètement monotone (resp. absolument monotone) en direction d'un cône convexe ouvert $C \in \mathbb{R}^d$ si sa restriction à $\Omega \cap \mathbb{R}^d$ est complètement monotone en direction de C (resp. absolument monotone en direction de C).

De façon immédiate, on a l'équivalence suivante :
f (définie sur un ouvert $\Omega \subset \mathbb{R}^d$) est complètement monotone en direction du cône convexe ouvert C
ssi
$x \mapsto f(-x)$ (définie sur un ouvert $-\Omega = \{x \in \mathbb{R}^d : -x \in \Omega\} \subset \mathbb{R}^d$) est absolument monotone en direction du cône convexe ouvert C.

En général, par défaut, $C = (\mathbb{R}_+^*)^d$. Pour les matrices carrées (réelles, resp. complexes) d'ordre n (isomorphe à $\mathbb{R}^{n^2}$, resp. $\mathbb{C}^{n^2}$), C sera par défaut l'ensemble des matrices définies positives d'ordre n (dans $\mathbb{R}$, resp. $\mathbb{C}$).

On considère un ouvert $\Omega \subset V$, où V est un espace euclidien et un cône convexe ouvert $C \subset V$. On suppose que $\Omega + C = \Omega$. Soit $C^* = \{l \in V^* : \langle l, x\rangle \geq 0\}$. D'après le théorème de Bernstein-Hausdorff-Widder-Choquet, une fonction f définie sur Ω est complètement monotone ssi il existe une mesure positive sur C^* telle que

$$f(x) = \int_{C^*} e^{-\langle l,x\rangle} d\mu(l)$$

Donc, à une constante près, cela revient à dire que toute fonction complètement monotone peut s'exprimer sous la forme de la transformée de Laplace d'une variable aléatoire.

La preuve de ce théorème repose essentiellement sur le fait que tout élément f extrémal (c'est à dire appartenant à une génératrice extrémale) du cône convexe des fonctions complètement monotones est de la forme $f(x) = Ke^{-\langle l,x\rangle}$, où $l \in C^*$ et $K \in \mathbb{R}_+$ (voir la preuve en détail dans [Cho69]).

La conséquence de ce théorème est qu'une fonction f analytique, absolument monotone, dont le domaine de définition contient $(\mathbb{R}_-^*)^d$, admet au voisinage de n'importe quel point de $(\mathbb{R}_-^*)^d$ un développement en série entière dont tous les coefficients sont positifs ou nuls : pour tout $\alpha \in (\mathbb{R}_-^*)^d$, on peut écrire $f(z+\alpha) = \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} a_{n_1,\ldots,n_d} z_1^{n_1} \ldots z_d^{n_d}$, où les $a_{n_1,\ldots,n_d}$ sont des réels positifs ou nuls. La réciproque est vraie, à savoir qu'une fonction f analytique dont le domaine de définition contient $(\mathbb{R}_-^*)^d$ et qui admet en tout point de $(\mathbb{R}_-^*)^d$ un développement en série entière à coefficients positifs ou nuls, est absolument monotone.

On peut trouver des précisions sur les fonctions complètement/absolument monotones notamment dans [SS14], [Cho69] et [Wid41].

1.2.8 Lien avec la conjecture de Shirai et Takahashi sur l'α-déterminant d'une matrice semi-définie positive

Il est connu que le déterminant et le permanent d'une matrice hermitienne semi-définie positive sont positifs ou nuls. Ce résultat peut s'étendre facilement aux $\pm m$-permanents et $\pm 1/m$-déterminants ($m \in \mathbb{N}^*$), en utilisant la formule (1.5), avec $\beta = \pm 1$ et $\beta = \pm m$, et en utilisant le fait qu'une série entière à coefficients positifs ou nuls conserve cette propriété lorsqu'on l'élève à une puissance entière.

Cette remarque entraîne l'existence des processus $1/m$-permanentaux pour tout noyau K hermitien semi-défini positif (car $\mathcal{K}(I+\mathcal{K}/m)^{-1}$ est encore semi-défini positif). Elle entraîne également l'existence des processus $-1/m$-déterminantaux pour tout noyau K hermitien semi-défini positif dont le spectre est inclus dans $[0,m]$ ($\mathcal{K}(I-\mathcal{K}/m)^{-1}$ est encore semi-défini positif lorsque le spectre de $\mathcal{K}$ est inclus dans [0,m[et par passage à la limite si $\mathcal{K}$ possède une valeur propre égale à m).

Dans le cas symétrique réel semi-défini positif, on a de plus le fait que les $m/2$-permanents et $2/m$-déterminants ($m \in \mathbb{N}^*$) sont positifs ou nuls. Ceci implique l'existence de processus $m/2$-permanentaux pour tout noyau symétrique réel semi-défini positif. On peut trouver des précisions à ce sujet dans [ST03].

La positivité des $m/2$-permanents (ou des $2/m$-déterminants), pour $m \in \mathbb{N}^*$, peut se justifier par la formule suivante :

$$\mathbb{E}\left(\prod_{i=1}^{d} X_i^2\right) = \det{}_2 A, \tag{1.6}$$

où $\boldsymbol{X} = (X_d, \ldots, X_d)$ est un vecteur gaussien centré de matrice de covariance A. Par conséquent, le $\det_2$, et donc le $\mathrm{per}_{1/2}$, et, par convolution (voir [ST03] Proposition 4.3), les $\det_{2/m}$ et $per_{m/2}$ ($m \in \mathbb{N}^*$) d'une matrice semi-définie positive sont positifs ou nuls.

Dans le cas complexe, on a une formule analogue, mais qui correspond au permanent, à savoir :

$$\mathbb{E}\left(\prod_{i=1}^{d} |X_i|^2\right) = \mathrm{per}\, A = \det{}_1 A, \tag{1.7}$$

où $\boldsymbol{X} = (X_d, \ldots, X_d)$ est un vecteur gaussien centré complexe circulaire symétrique de matrice de covariance A, hermitienne semi-définie positive (ou ce qui revient au même de matrice de pseudo-covariance 0 et covariance A).

La question posée par Shirai et Takahashi en 2003 dans [ST03] peut se formuler de la façon suivante : pour quels $\beta > 0$ a-t-on $\mathrm{per}_\beta A \geq 0$ pour toute matrice A réelle symétrique semi-définie positive. Ils conjecturaient que c'était vrai pour tout $\beta \geq 1/2$, ce qui aurait entrainé l'existence de processus α-permanentaux pour tout $\alpha \in]0,2]$ et pour n'importe quel noyau semi-défini positif.

Peter Bränden a apporté une réponse négative à cette conjecture en 2014 dans [Brä12] en utilisant un résultat de l'article de Scott et Sokal [SS14], à savoir :

La fonction $A \mapsto (\det A)^{-\beta}$ est complètement monotone sur l'ensemble des matrices réelles symétriques définies positives d'ordre d (dans la direction du cône des matrices réelles symétriques définies positives d'ordre d) ssi 2β entier ou $\beta \geq \frac{d-1}{2}$.

Leur résultat entraîne l'existence des processus $1/\beta$-permanentaux pour tout noyau symétrique réel semi-défini positif de dimension $d \times d$, dès lors que 2β est un entier naturel ou $\beta \geq \frac{d-1}{2}$. Inversement, pour les autres $\beta > 0$, on peut toujours trouver un noyau symétrique réel semi-défini positif (de dimension $\frac{\lceil 2\beta+1\rceil(\lceil 2\beta+1\rceil+1)}{2}$ et de rang inférieur ou égal à $\lceil 2\beta+1\rceil$), tel qu'il n'existe pas de processus $1/\beta$-permanental avec ce noyau.

Ils montrent également que, dans le cas complexe, la fonction $A \mapsto (\det A)^{-\beta}$ est complètement monotone sur l'ensemble des matrices hermitiennes définies positives d'ordre d (dans la direction du cône des matrices hermitiennes définies positives d'ordre d) ssi β entier ou $\beta \geq d-1$.

De la même façon, ce résultat entraîne l'existence des processus $1/\beta$-permanentaux pour tout noyau hermitien semi-défini positif de dimension $d \times d$, dès lors que β est un entier naturel ou $\beta \geq d-1$. Inversement, pour les autres $\beta > 0$, on peut toujours trouver un noyau hermitien semi-défini positif (de dimension $(\lceil \beta+1\rceil)^2$ et de rang inférieur ou égal à $\lceil \beta+1\rceil$), tel qu'il n'existe pas de processus $1/\beta$-permanental avec ce noyau.

Comme indiqué dans la partie 1.2.2, pour A réelle symétrique semi-définie positive de dimension $d\times d$, la loi d'intensité β-permanentale existe au moins pour $\beta \geq \frac{d-2}{2}$ ou 2β entier (voir [Roy14]). Cela entraîne que le β-permanent de telles matrices est positif ou nul pour $\beta \geq \frac{d-2}{2}$ ou 2β entier. On peut aisément adapter le raisonnement de Royen pour montrer que, dans le cas d'une matrice hermitienne semi-définie positive de dimension $d\times d$, son β-permanent est positif ou nul lorsque $\beta \geq d-2$ ou β entier.

Enfin, il est intéressant de noter que, dans le cas particulier de matrices hermitiennes semi-définies positives de dimension inférieure ou égale à 5, une conséquence de l'article de Frenkel [Fre10] est que leur β-permanent est toujours positif ou nul lorsque $\beta \geq 1$.

1.2.9 Lien avec la conjecture de corrélation gaussienne

La conjecture de corrélation gaussienne est une conjecture assez connue et restée longtemps non résolue. Elle a été posée pour la première fois par Dunnett et Sobel en 1955 dans un cas particulier et en 1972 par Das Gupta, Eaton, Olkin, Perlman et Savage dans sa forme générale. Elle peut se formuler de la façon suivante. Soit $\mathbb{P}$ la loi de probabilité d'un vecteur gaussien centré quelconque sur $\mathbb{R}^d$. Pour tous ensembles convexes symétriques (par rapport à l'origine) $C_1, C_2 \subset \mathbb{R}^d$, on a alors

$$\mathbb{P}(C_1 \cap C_2) \geq \mathbb{P}(C_1)\mathbb{P}(C_2).$$

Cette conjecture se formule de façon équivalente :

$$\mathbb{P}\left(\bigcap_{i=1}^{d} A_i\right) \geq \mathbb{P}\left(\bigcap_{i=1}^{k} A_i\right)\mathbb{P}\left(\bigcap_{i=k+1}^{d} A_i\right)$$

pour $A_i = \{X_i^2 \leq x_i\}, x_1, \dots, x_d \geq 0, 1 \leq k \leq d$.

Elle a été prouvée par Thomas Royen en 2014 dans [Roy14].

La preuve de Royen généralise la conjecture à des distributions gamma multivariées (distributions d'intensité β-permanentale avec un noyau défini positif).

On peut en fait généraliser aisément le résultat de Royen. Pour tout $(Y_1, \dots, Y_d)$ vecteur β-permanental ($\beta > 0$) de noyau semi-défini positif, pour toutes fonctions $F_1, \dots, F_d$ mesurables, définies de $\mathbb{R}_+$ dans $\mathbb{R}_+$ et décroissantes, et pour tout $k \in [\![1, d]\!]$, on a la propriété suivante :

$$\mathbb{E}\left(\prod_{i=1}^{d} F_i(Y_i)\right) \geq \mathbb{E}\left(\prod_{i=1}^{k} F_i(Y_i)\right)\mathbb{E}\left(\prod_{i=k+1}^{d} F_i(Y_i)\right).$$

Plus précisément, si $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ est semi-définie positive et $A^\tau = \begin{pmatrix} A_{11} & \tau A_{12} \\ \tau A_{21} & A_{22} \end{pmatrix}$, avec A_{11} et A_{22} matrices carrées et $\tau \in [0, 1]$,

et si $(Y_1, \dots, Y_d)$ et $(Y_1^\tau, \dots, Y_d^\tau)$ sont respectivement des vecteurs permanentaux de noyaux A et A^τ, la preuve de Royen permet de voir que

$$\mathbb{E}\left(\prod_{i=1}^{d} F_i(Y_i^\tau)\right)$$

est croissante lorsque τ passe de 0 à 1.

La preuve de Royen utilise essentiellement la formule de transformée de Laplace (1.1) et le fait que le déterminant de toute sous-matrice principale de A^τ est une fonction décroissante de τ sur $[0, 1]$.

1.3 Résultats présentés dans la thèse

1.3.1 Conditions d'existence de processus α-déterminantaux

Plusieurs auteurs ont déjà établi des conditions nécessaires et suffisantes d'existence de processus α-déterminantaux dans des cas particuliers.

Macchi dans [Mac75] et Soshnikov dans son article de synthèse [Sos00] ont donné une condition nécessaire et suffisante d'existence d'un processus déterminantal ($\alpha = -1$) dans le cas d'un noyau auto-adjoint.

La même condition a aussi été établie d'une façon différente par Hough, Krishnapur, Peres et Virág dans [HKPV06] dans le cas $\alpha = -1$. Ils ont aussi donné une condition suffisante d'existence dans le cas $\alpha = 1$ et pour un noyau auto-adjoint.

Dans le cas particulier où les configurations de points sont dans un espace d'état de cardinal fini, l'article de Vere-Jones [VeJ97] donne des conditions nécessaires et suffisantes pour toute valeur de α. Cependant, dans le cas non symétrique, ces conditions soulèvent certains problèmes que nous décrivons dans la section suivante.

Shirai et Takahashi ont aussi donné des conditions suffisantes d'existence de processus α-determinantaux pour toute valeur de α.

Nous donnons une condition nécessaire et suffisante dans le cas où α est positif, ce qui correspond au Théorème 2.1.

Par ailleurs, dans le cas où α est négatif, nous étendons le résultat de Shirai et Takahashi au cas des noyaux non auto-adjoints et montrons que la condition obtenue est aussi nécessaire (Théorèmes 2.4 et 2.5).

Nous montrons également, que, toujours dans le cas où α est négatif, $-1/\alpha$ doit nécessairement être entier pour qu'un processus α-déterminantal existe. Ceci avait été noté par Vere-Jones dans [VeJ88] dans le cas de configurations de points dans un espace d'état de cardinal fini. Nous montrons que tout processus α-déterminantal peut se décomposer comme somme de processus déterminantaux indépendants et identiquement distribués (i.i.d.) (la réciproque, à savoir que la somme de n processus déterminantaux i.i.d. permet d'obtenir un processus $-1/n$-déterminantal étant évidente).

Pour toute valeur de α, nous donnons aussi une condition nécessaire et suffisante d'infinie divisibilité pour un processus α-déterminantal, condition qui sera précisée dans le chapitre suivant, dans le cadre d'un espace d'état fini.

1.3.2 Etude approfondie du cas où l'espace d'état est fini

Nous nous intéressons ici au cas où l'espace d'état est fini. Les matrices considérées dans cette partie sont à coefficients réels.

Les distributions permanentales et les distributions multidimensionnelles négatives binomiales auxquelles on s'intéresse ont été étudiées notamment par Griffiths [Gri84], Griffiths et Milne [GM87] et Vere-Jones [VeJ88].

Un processus d'intensité permanentale peut se définir comme un vecteur aléatoire $(X_1, \ldots, X_d)$ à coordonnées réelles positives ayant pour transformée de Laplace

$$\mathbb{E}\left(\exp\left(-\frac{1}{2}\sum_{i=1}^{d} z_i X_i\right)\right) = \det(I + ZA)^{-\beta} \tag{1.8}$$

où I is la matrice identité de taille $d \times d$, Z est la matrice diagonale $\mathrm{diag}((z_i)_{1\leq i\leq d})$, $A = (a_{ij})_{1\leq i,j\leq d}$ et β est un nombre réel strictement positif. Une distribution d'intensité permanentale correspond à la loi de ce processus.

Une matrice A est dite **β-permanentale** si un tel vecteur aléatoire existe.

Une matrice A est dite **β-positive** si tous les coefficients de la décomposition en série entière en $z_1^{n_1}...z_d^{n_d}$ de $\det(I - ZA)^{-\beta}$ sont positifs ou nuls (voir Section 3.2 (3.5) pour une formulation équivalente).

Si le rayon spectral d'une matrice $d \times d$ β-positive A est strictement plus petit que 1, il existe un vecteur aléatoire $(X_1, X_2, ..., X_d)$ ayant une loi négative binomiale multidimensionnelle, tel que sa fonction génératrice satisfasse :

$$\mathbb{E}\left(z_1^{X_1}...z_d^{X_d}\right) = \det(I - A)^{\beta} \det(I - ZA)^{-\beta}. \tag{1.9}$$

Cette loi négative binomiale multivariée est la loi d'un processus ponctuel α-permanental ξ (voir [ST03]) d'indice $\alpha = \frac{1}{\beta}$ et de noyau $\beta A(I - A)^{-1}$ par rapport à la mesure de référence $\sum_{k=1}^{d} \delta_k$ (ξ a la même loi que $\sum_{k=1}^{d} X_k\, \delta_k$).

Deux matrices A et B distinctes peuvent définir la même distribution d'intensité permanentale (ou la même distribution négative binomiale multidimensionnelle).

On dira que A et B sont **effectivement équivalentes** si pour chaque matrice diagonale $d \times d$ Z :

$$\det(I + ZA) = \det(I + ZB).$$

Par exemple, deux matrices A et DAD^{-1} (dites **diagonalement semblables**), où D est une matrice diagonale inversible, sont effectivement équivalentes. La réciproque est fausse, à savoir qu'il existe des matrices effectivement équivalentes mais que ne sont pas diagonalement semblables (voir [HL84] et [Loe86])

Une condition nécessaire et suffisante simple pour qu'une matrice A soit β-positive pour tout $\beta > 0$ a été établie par Griffiths et Milne [GM87], mais elle n'est vraie que dans les cas où A est symétrique ou bien A a des coefficients tous non nuls. En section 3.3, nous donnons un contre-exemple et établissons une condition nécessaire et suffisante valable dans tous les cas. Si la matrice est irréductible, on constate que le critère de Griffiths et Milne s'applique, comme cela est démontré dans la section 3.3.

Après avoir corrigé le critère de Griffiths et Milne, nous avons regardé dans quelle mesure les résultats construits à partir de leur critère initial étaient affectés. En particulier, la condition nécessaire et suffisante de Vere-Jones pour qu'une matrice soit β-permanentale pour tout $\beta > 0$, qui a été établie grâce au critère de Griffiths et Milne, peut facilement être corrigée. En section 3.5, nous corrigeons également l'argument de [EK09] qui montre que (à effective équivalence près) les matrices inversibles, β-permanentales pour tout β, sont des inverses de M-matrices (une matrice inversible est une M-matrice si tous ses coefficients non diagonaux sont négatifs ou nuls et si tous les coefficients de son inverse sont positifs ou nuls). Dans le cas symétrique, cette caractérisation a été établie précédemment par Bapat [Bap89]. Nous étendons aussi cette caractérisation à des matrices non inversibles.

La section 3.5 repose sur la section 3.4 qui établit diverses relations entre les propriétés de β-permanentalité et de β-positivité. De fait, ces deux notions sont très liées. Par exemple, il est facile de voir que β-permanentalité implique β-positivité.

La question de la description de la classe de matrices β-permanentales pour tout β est donc entièrement résolue. La classe des matrices β-positives pour tout β, est également bien décrite. On peut noter que les éléments de ces deux classes correspondent à des distributions de probabilité infiniment divisibles. Nous nous sommes alors intéressés à la question de la description des matrices β-permanentales et des matrices β-positives pour un β fixé.

Grâce aux résultats de Vere-Jones dans [VeJ88], on sait qu'une matrice symétrique β-permanentale est nécessairement semi-définie positive. Réciproquement, pour A symétrique semi-définie positive et $\beta = 1/2$, un vecteur β-permanental de noyau A a même loi que $(X_1^2, .., X_d^2)$ avec $(X_1, ..., X_d)$ vecteur gaussien centré de matrice de covariance $2A$. Par conséquent, A doit être 1/2-permanentale et plus généralement $n/2$-permanentale pour tout entier n. Cependant, comme indiqué en section 1.2.8, pour tout $\beta > 0$ tel que 2β n'est pas entier, il existe des matrices symétriques semi-définies positives, non β-positives et donc non β-permanentales (voir le travail de Bränden [Brä12] utilisant les résultats de Scott et Sokal [SS14]). Ce résultat résoud une conjecture posée par Shirai et Takahashi dans [ST03] et [Shi07] (voir explications dans la partie 1.2.8).

Jusqu'à présent, les seules matrices β-permanentales connues (à effective équivalence près) étaient soit symétriques semi-définies positives, soit des inverses de M-matrices. Kogan et Marcus [KM12] ont montré que si une matrice β-permanentale inversible de dimension 3 n'est pas effectivement équivalente à une matrice symétrique, alors elle est diagonalement semblable à l'inverse d'une M-matrice. Dans la section 3.6, nous établissons un résultat analogue pour les matrices β-positives : en dimension 3, une matrice irreductible et β-positive qui n'est pas diagonalement semblable à une matrice symétrique, est nécessairement diagonalement semblable à une matrice à coefficients tous positifs ou nuls. En section 3.7, nous répondons à la question posée par Kogan et Marcus dans le cas où la dimension est supérieure à 3 : existe-t-il (à effective équivalence près) des matrices inversibles et β-permanentales qui ne soient ni symétriques définies positives, ni des inverses de M-matrices ? Grâce aux résultats de la section 3.4, nous réduisons la question à la recherche de matrices 1-positives qui ne sont effectivement équivalentes ni à une matrice symétrique, ni à une matrice à coefficients tous positifs ou nuls. En fait, nous exhibons une famille de telles matrices et nous pouvons donc donner une réponse positive à la question de Kogan et Marcus.

Dans la partie 3.7.2, nous établissons des conditions nécessaires pour qu'une matrice soit β-positive pour un $\beta > 0$ donné. Ces conditions devraient être utiles pour trouver la forme générale de ces matrices. En particulier, nous montrons que des matrices irréductibles β-permanentales doivent satisfaire une condition restrictive simple : leurs coefficients nuls sont symétriques par rapport à la diagonale principale de la matrice (Théorème 3.19).

1.4 Gaussian moment product conjecture and perspectives

As mentionned in section 1.2.6, the existence of a multivariate negative binomial distributions with kernel A is related to the positivity of the β-permanent of the matrix A and its derivatives $A[\boldsymbol{n}] = A[n_1, \ldots, n_d]$.

One might be interested in stronger inequalities, such as

$$\det_\alpha A[\boldsymbol{n}] \geq \det_\alpha \operatorname{diag}(A)[\boldsymbol{n}] \tag{1.10}$$

where $\operatorname{diag}(A)$ denotes $\operatorname{diag}(a_{11}, \ldots, a_{dd})$. We are especially interested in the case where the matrix A is positive semi-definite (hermitian or real symmetric), which will be assumed in the following of this section.

The case $\alpha = 2$ for a real symmetric positive semi-definite matrix A is still unsolved in general. In this case, inequality (1.10) is what is called the Gaussian moment product conjecture. Because of (1.6), another formulation of this conjecture is

$$\mathbb{E}(\prod_{i=1}^{d} X_i^{2n_i}) \geq \prod_{i=1}^{d} \mathbb{E}(X_i^{2n_i}) \tag{1.11}$$

where $(X_1, \ldots, X_d)$ denotes a real Gaussian vector with covariance matrix A.

This inequality has been first conjectured by Frenkel in [Fre07] for $n_1 = \cdots = n_d$, then in full generality by Li and Wei in [LW09].

The reason why Frenkel was interested in this inequality was another open problem, called the real linear polarization constant problem.
In a real Hilbert space $\mathcal{H}$ of dimension at least d, the problem is to find the best constant c such that, for any nomalized vectors $x_1, \ldots, x_d \in \mathcal{H}$,

$$\max\{\langle v, x_1\rangle_{\mathcal{H}} \ldots \langle v, x_d\rangle_{\mathcal{H}} : v \in \mathcal{H}, \|v\|_{\mathcal{H}} = 1\} \geq c \tag{1.12}$$

We have $c \leq d^{-d/2}$ because $d^{-d/2}$ is attained when $x_1, \ldots, x_d$ are orthogonal. It has been conjectured by Benítez, Sarantopoulos and Tonge in [BST98] that $c = d^{-d/2}$.

As noted by Frenkel in [Fre07] (see also the proof in [MNPP16] section 3.3), inequality (1.10) in the case $\alpha = 2$ and A real symmetric positive semi-definite (or equivalently inequality (1.11)) implies the solution to the real linear polarization constant problem.

When the Hilbert space is complex, the question has been solved by Arias-de-Reyna in [Ari98], by using inequality (1.10) in the case $\alpha = 1$ and A hermitian positive semi-definite, which is a consequence of the Lieb permanent inequality [Lie66]

$$\operatorname{per}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \geq \operatorname{per} A_{11} \operatorname{per} A_{22}. \tag{1.13}$$

Remark that techniques we use in part 3.7 to prove that the permanent of certain matrices are nonnegative, are related to those used in the proof of inequality (1.13) by Djokovic in [Dok69].

In the special case, $n_1 = \cdots = n_d = 1$ and A real symmetric positive semi-definite, inequality (1.11) has been proved first by Frenkel [Fre07], using hafnian, introduced by Caianiello in 1953 and defined as follows

$$\operatorname{haf} B = \frac{1}{2^n n!} \sum_{\sigma \in \Sigma_{2n}} \prod_{i=1}^{n} a_{2i-1\,2i} \tag{1.14}$$

for a $2n \times 2n$ real symmetric matrix B. Frenkel uses the fact that

$$\operatorname{haf} A[2n_1, \dots, 2n_d] = \det{}_2 A[n_1, \dots, n_d] = \mathbb{E}\left(\prod_{i=1}^{d} X_i^{2n_i}\right) \tag{1.15}$$

where $(X_1, \dots, X_d)$ denotes a real Gaussian vector with covariance matrix A.

This special case has allowed Frenkel to improve the previously known lower bound in (1.12) up to $c = (1.91d)^{-d/2}$ (see [Fre07]), but not to completely solve the real linear polarization constant problem, which corresponds to the lower bound $c = d^{-d/2}$.

Inequality (1.11) in the case $n_1 = \cdots = n_d = 1$ has been reproved by Maliciet, Nourdin, Pecatti et Poly in [MNPP16], using Ornstein Uhlenbeck processes and multidimensional Hermite polynomials. They show a more general property, which implies that, for $(X_1^\tau, \dots, X_d^\tau)$ Gaussian vector with covariance matrix $A^\tau = (a_{ij}^\tau)_{1 \leq i,j \leq d} = (1-\tau)\operatorname{diag}(A) + \tau(A - \operatorname{diag}(A))$ and by denoting H_n for the n^{th} (probabilists) Hermite polynomial, the function

$$\tau \mapsto \mathbb{E}\left(\prod_{i=1}^{d} H_{n_i}(X_i^\tau)^2\right)$$

is increasing on $[0, 1]$.

For $n_1 = \dots n_d = 1$, they retrieve the result proved by Frenkel.

For any nonnegative integers $n_1, \dots, n_d$, the monomial $X_i^{n_i}$ does not correspond in general to the Hermite polynomial H_{n_i} (they are identical only for $n_i = 1$). Especially, $(X^n)_{n\in\mathbb{N}}$ is not a sequence of orthogonal polynomials for the standard Gaussian measure. Consequently, the method used by Maliciet, Nourdin, Pecatti and Poly in [MNPP16] can not be applied to prove inequality (1.11) for general positive integers $n_1, \dots, n_d$.

Note that the Gaussian moment product conjecture also implies inequality (1.10) for $2/\alpha$ positive integer and A real symmetric positive semi-definite. This can be proved by convolution (see [ST03] Proposition 4.3) or from the following formula

$$\operatorname{per}_{n\beta} A = \sum_{q=0}^{\infty} \binom{n}{q} \sum_{(I_1 \dots I_q)} \operatorname{per}_{\beta} A \tag{1.16}$$

where $(I_1, \dots, I_q)$ are ordered partitions of $[d]$ in q disjoints subsets (see [Chu97] ou [Fre10]).
In the same way, when A is hermitian positive semi-definite, proven inequality (1.10) in the case $\alpha = 1$ implies inequality (1.10) in the cases where $1/\alpha$ is a positive integer.

In the special case where $\alpha \geq 1$, A hermitian positive semi-definite with dimension $d \leq 5$, and $n_1 = \dots = n_d = 1$, Frenkel has proved that the inequality (1.10) is true (see [Fre10]).

A consequence of the Gaussian moment product inequality proved in the case $n_1 = \dots = n_d = 1$ is that the correlation inequality (1.4) is true for $m = 1$, and therefore, by convolution, for all positive integer m, namely $2/m$-permanental point processes have attractive properties for real symmetric positive semi-definite kernels.

It is interesting to see similarities and differences between the Gaussian correlation conjecture proved by Royen and the Gaussian moment product conjecture. As indicated in the previous chapter, from Royen's proof, we can see that, for

$F_1, \dots F_d$ measurable decreasing functions, defined from $\mathbb{R}_+$ to $\mathbb{R}_+$,

$$\mathbb{E}\left(\prod_{i=1}^{d} F_i\left((X_i^{\tau})^2\right)\right)$$

in increasing when τ goes from 0 to 1. Gaussian moment product conjecture correspond to functions $F_i(X) = X^{n_i}$ which are special increasing functions. Because of this difference, the sum of nonnegative terms that appear in the Royen's proof will become a sum of terms having alternating signs and we cannot deal with the sign of this sum.

Finally we present a simple alternative formulation of the Gaussian moment product conjecture, using coefficients of polynomials.

For $\boldsymbol{k} = (k_1, \dots, k_d)^t$ and $\boldsymbol{x} = (x_1, \dots, x_d)^t$, we define $\boldsymbol{k}! = \prod_{i=1}^d k_i!$ and $\boldsymbol{x}^{\boldsymbol{k}} = \prod_{i=1}^d x_i^{k_i}$. For a polynomial P in d variables $x_1, \dots, x_d$, we denote by $[\boldsymbol{x}^{\boldsymbol{k}}]\,P$ the coefficient of $x_1^{k_1} \dots x_d^{k_d}$ in $P(x_1, \dots, x_d)$.

The hafnian of a matrix $A[\boldsymbol{k}]$ such that $k_1 + \dots + k_d = 2n$ can be written

$$\operatorname{haf} A[\boldsymbol{k}] = \frac{\boldsymbol{k}!}{2^n n!}\,[\boldsymbol{x}^{\boldsymbol{k}}]\,(\boldsymbol{x}^t A \boldsymbol{x})^n$$

For $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in \{-1; 1\}^d$, we set $\boldsymbol{x}_{\boldsymbol{\epsilon}} = (\epsilon_1 x_1, \dots, \epsilon_d x_d)$ and for any function f from $\{-1; 1\}^d$ to $\mathbb{R}$ or $\mathbb{C}$, $\mathbb{E}_{\boldsymbol{\epsilon}}(f(\boldsymbol{\epsilon})) = \frac{1}{2^d}\sum_{\boldsymbol{\epsilon}\in\{-1;1\}} f(\boldsymbol{\epsilon})$. The Gaussian moment product conjecture can be rewritten in a simple way as follows

$$[\boldsymbol{x}^{\boldsymbol{2n}}]\ \mathbb{E}_{\boldsymbol{\epsilon}}\left((\boldsymbol{x}_{\boldsymbol{\epsilon}}^t A \boldsymbol{x}_{\boldsymbol{\epsilon}})^n\right) \geq [\boldsymbol{x}^{\boldsymbol{2n}}]\ \mathbb{E}_{\boldsymbol{\epsilon}}\left(\boldsymbol{x}_{\boldsymbol{\epsilon}}^t A \boldsymbol{x}_{\boldsymbol{\epsilon}}\right)^n$$

The fact that the inequality $\mathbb{E}_{\boldsymbol{\epsilon}}\left((\boldsymbol{x}_{\boldsymbol{\epsilon}}^t A \boldsymbol{x}_{\boldsymbol{\epsilon}})^n\right) \geq \mathbb{E}_{\boldsymbol{\epsilon}}\left(\boldsymbol{x}_{\boldsymbol{\epsilon}}^t A \boldsymbol{x}_{\boldsymbol{\epsilon}}\right)^n$ is true is well known (consequence of Jensen's inequality). The difficulty is to prove the inequality for each coefficient of even powers term.

Chapter 2

Existence conditions for α-determinantal processes

In this chapter, we give necessary and sufficient conditions for existence and infinite divisibility of α-determinantal processes. For that purpose we use results on negative binomial and ordinary binomial multivariate distributions.

2.1 Introduction

Several authors have already established necessary and sufficient conditions for existence of α-determinantal processes.

Macchi in [Mac75] and Soshnikov in its survey paper [Sos00] gave a necessary and sufficient condition for determinantal processes with self-adjoint kernels, which corresponds to the case $\alpha = -1$.

The same condition has also been established in a different way by Hough, Krishnapur, Peres and Virág in [HKPV06] in the case $\alpha = -1$. They have also given a sufficient condition of existence in the case $\alpha = 1$ and self-adjoint kernel.

In the special case where the configurations are on a finite space, the paper of Vere-Jones [VeJ97] provides necessary and sufficient conditions for any value of α.

Finally, Shirai and Takahashi have given sufficient conditions for the existence of an α-determinantal process for any values of α. However, in the case $\alpha > 0$, their sufficient condition (Condition B) in [ST03] does not work for the following example: the space is reduced to a single point space and the reference measure λ is a unit point mass. With their notations, the two kernels K and J_α are respectively reduced to two real numbers k and j_α, with

$$j_\alpha = \frac{k}{1 + \alpha k}$$

We can choose $\alpha > 0$ and $k < 0$ such that $j_\alpha > 0$. Under these assumptions, Condition B is fulfilled but the obtained point process has a negative correlation function ($\rho_1(x) = k$), which has to be excluded, since a correlation function is an almost everywhere nonnegative function.

We are going to strengthen Condition B of Shirai and Takahashi and obtain a necessary and sufficient condition in the case $\alpha > 0$. This is presented in Theorem 2.1.

Besides, in the case $\alpha < 0$, we extend the result of Shirai and Takahashi to the case of non self-adjoint kernels and show that the obtained condition is also necessary (Theorems 2.4 and 2.5). Moreover, we show that $-1/\alpha$ is necessarily an integer. This has been noticed by Vere-Jones in [VeJ88] in the case of configurations on a finite space.

We also give a necessary and sufficient condition for the infinite divisibility of an α-determinantal process for all values of α.

The main results are presented in Section 2.3. Section 2.2 introduces the needed notation. In Section 2.4, we write a multivariate version of a Shirai and Takahashi formulae on Fredholm determinant expansion. Sections 2.5 and 2.6 present the proofs of the results concerning respectively the cases $\alpha > 0$ and $\alpha < 0$. The proofs concerning infinite divisibility are presented in Section 2.7.

2.2 Preliminaries

Let E be a locally compact Polish space. A locally finite configuration on E is an integer-valued positive Radon measure on E. It can also be identified with a set $\{(M, \alpha_M) : M \in F\}$, where F is a countable subset of E with no accumulation points (i.e. a discrete subset of E) and, for each point in F, α_M is a non-null integer that corresponds to the multiplicity of the point M (M is a multiple point if $\alpha_M \geq 2$).

Let λ be a Radon measure on E. Let $\mathcal{X}$ be the space of the locally finite configurations of E. The space $\mathcal{X}$ is endowed with the vague topology of measures, i.e. the smallest topology such that, for every real continuous function f with compact support, defined on E , the mapping

$$\mathcal{X} \ni \xi \mapsto \langle f, \xi \rangle = \sum_{x \in \xi} f(x) = \int f \mathrm{d}\xi$$

is continuous. Details on the topology of the configuration space can be found in [AKR98].

We denote by $\mathcal{B}(\mathcal{X})$ the corresponding σ-algebra. A point process on E is a random variable with values in $\mathcal{X}$. We do not restrict ourselves to simple point processes, as the configurations in $\mathcal{X}$ can have multiple points.

For a real (or complex) function f defined on E, we denote by $\|f\|_\infty$ its supremum norm when it is bounded and by $\|f\|$ its L_2-norm when f belongs to $L^2(E)$.

For a $n \times n$ matrix $A = (a_{[ij]})_{1 \leq i,j \leq n}$, set:

$$\det_\alpha A = \sum_{\sigma \in \Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where Σ_n is the set of all permutations on $\{1, \ldots, n\}$ and $\nu(\sigma)$ is the number of cycles of the permutation σ.

For a relatively compact set $\Lambda \subset E$, the Janossy densities of a point process ξ w.r.t. a Radon measure λ are functions (when they exist) $j_n^\Lambda : E^n \to [0, \infty)$ for $n \in N$, such that

$$j_n^\Lambda(x_1, \ldots, x_n) = n! \, \mathbb{P}(\xi(\Lambda) = n) \, \pi_n^\Lambda(x_1, \ldots, x_n)$$
$$j_0^\Lambda(\emptyset) = \mathbb{P}(\xi(\Lambda) = 0),$$

where π_n^Λ is the density with respect to $\lambda^{\otimes n}$ of the ordered set $(x_1, \ldots, x_n)$, obtained by first sampling ξ, given that there are n points in Λ, then choosing uniformly an order between the points.

For $\Lambda_1, \dots, \Lambda_n$ disjoint subsets included in Λ, $\int_{\Lambda_1 \times \dots \times \Lambda_n} j_n^\Lambda(x_1, \dots, x_n)\lambda(dx_1) \dots \lambda(dx_n)$ is the probability that there is exactly one point in each subset Λ_i $(1 \leq i \leq n)$, and no other point elsewhere.

We recall that we have the following formula, for a nonnegative mesurable function f with support in a relatively compact set $\Lambda \subset E$:

$$\mathbb{E}(f(\xi)) = f(\emptyset)\, j_0^\Lambda(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f(x_1, \dots, x_n)\, j_n^\Lambda(x_1, \dots, x_n)\lambda(dx_1) \dots \lambda(dx_n).$$

For $n \in \mathbb{N}$ and $a \in \mathbb{R}$, we denote $a^{(n)} = \prod_{i=0}^{n-1}(a - i)$.

The correlation functions (also called joint intensities) of a point process ξ w.r.t. a Radon measure λ are functions (when they exist) $\rho_n : E^n \to [0, \infty)$ for $n \geq 1$, such that for any family of mutually disjoint relatively compact subsets $\Lambda_1, \dots, \Lambda_d$ of E and for any non-null integers $n_1, \dots, n_d$ such that $n_1 + \dots + n_d = n$, we have

$$\mathbb{E}\left(\prod_{i=1}^{d} \xi(\Lambda_i)^{(n_i)}\right) = \int_{\Lambda_1^{n_1} \times \dots \times \Lambda_d^{n_d}} \rho_n(x_1, \dots, x_n)\lambda(dx_1), \dots, \lambda(dx_n).$$

Intuitively, for a simple point process, $\rho_n(x_1, \dots, x_n)\lambda(dx_1) \dots \lambda(dx_n)$ is the infinitesimal probability that there is at least one point in the vicinity of each x_i (each vicinity having an infinitesimal volume $\lambda(dx_i)$ around x_i), $1 \leq i \leq n$.

Let α be a real number and K a kernel from E^2 to $\mathbb{R}$ or $\mathbb{C}$. An α-determinantal point process, with kernel K with respect to λ (also called α-permanental point process) is defined, when it exists, as a point process with the following correlation functions $\rho_n, n \in \mathbb{N}$ with respect to λ:

$$\rho_n(x_1, \dots, x_n) = \det_\alpha(K(x_i, x_j))_{1 \leq i,j \leq n}.$$

We denote by $\mu_{\alpha,K,\lambda}$ the probability distribution of such a point process.

We exclude the case of a point process almost surely reduced to the empty configuration.

The case $\alpha = -1$ corresponds to a determinantal process and the case $\alpha = 1$ to a permanental process. The case $\alpha = 0$ corresponds to the Poisson point process. We suppose in the following that $\alpha \neq 0$.

We will always assume that the kernel K defines a locally trace class integral operator $\mathcal{K}$ on $L^2(E,\lambda)$. Under this assumption, one obtains an equivalent definition for the α-determinantal process, using the following Laplace functional formula:

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[\exp\left(-\int_E f\,d\xi\right)\right] = \mathrm{Det}\left(\mathcal{I}+\alpha\mathcal{K}[1-e^{-f}]\right)^{-1/\alpha} \tag{2.1}$$

where f is a compactly-supported nonnegative function on E, $\mathcal{K}[1-e^{-f}]$ stands for $\sqrt{1-e^{-f}}\mathcal{K}\sqrt{1-e^{-f}}$, $\mathcal{I}$ is the identity operator on $L^2(E,\lambda)$ and Det is the Fredholm determinant. Details on the link between the correlation function and the Laplace functional of an α-determinantal process can be found in the chapter 4 of [ST03]. Some explanations and useful formula on the Fredholm determinant are given in chapter 2.1 of [ST03].

For a subset $\Lambda \subset E$, set: $\mathcal{K}_\Lambda = p_\Lambda \mathcal{K} p_\Lambda$, where p_Λ is the orthogonal projection operator from $L^2(E,\lambda)$ to the subspace $L^2(\Lambda,\lambda)$.

For two subsets $\Lambda, \Lambda' \subset E$, set: $\mathcal{K}_{\Lambda\Lambda'} = p_\Lambda \mathcal{K} p_{\Lambda'}$, and denote by $K_{\Lambda\Lambda'}$ its kernel. We have for any $x, y \in E$, $K_{\Lambda\Lambda'}(x,y) = \mathbb{1}_\Lambda(x)\,\mathbb{1}_{\Lambda'}(y)\,K(x,y)$.

When $\mathcal{I}+\alpha\mathcal{K}$ (resp. $\mathcal{I}+\alpha\mathcal{K}_\Lambda$) is invertible, $\mathcal{J}_\alpha$ (resp. $\mathcal{J}_\alpha^\Lambda$) is the integral operator defined by: $\mathcal{J}_\alpha = \mathcal{K}(\mathcal{I}+\alpha\mathcal{K})^{-1}$ (resp. $\mathcal{J}_\alpha^\Lambda = \mathcal{K}_\Lambda(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1}$) and we denote by J_α (resp. J_α^Λ) its kernel. Note that $\mathcal{J}_\alpha^\Lambda$ is not the orthogonal projection of $\mathcal{J}_\alpha$ on $L^2(\Lambda,\lambda)$.

2.3 Main results

Theorem 2.1. *For $\alpha > 0$, there exists an α-permanental process with kernel K iff:*

- $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda) \geq 1$, *for any compact set $\Lambda \subset E$*
- $\det_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0$, *for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1,\ldots,x_n) \in \Lambda^n$.*

Remark 2.2. Even when E is a finite set, note that the second condition of Theorem 2.1 consists in an infinite number of computations. Finding a simpler condition, that could be checked in a finite number of steps is still an open problem.

Theorem 2.3. *For $\alpha > 0$, if an α-permanental process with kernel K exists, then:*

$$\operatorname{Spec} \mathcal{K}_\Lambda \subset \{z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{2\alpha}\} \text{ , for any compact set } \Lambda \subset E.$$

We remark that this condition is equivalent to

$$\operatorname{Spec} \mathcal{J}_\alpha^\Lambda \subset \{z \in \mathbb{C} : |z| < \frac{1}{\alpha}\} \text{ , for any compact set } \Lambda \subset E$$

Theorem 2.4. *For $\alpha < 0$ and $\mathcal{K}$ an integral operator such that $\mathcal{I} + \alpha\mathcal{K}_\Lambda$ is invertible, for any compact set $\Lambda \subset E$, an α-determinantal process with kernel K exists iff the two following conditions are fulfilled:*

(i) $-1/\alpha \in \mathbb{N}$

(ii) $\det(J_\alpha^\Lambda(x_i, x_j))_{1\le i,j\le n} \ge 0$, *for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1, \dots, x_n) \in \Lambda^n$.*

The arguments developed in the proof of Theorem 2.4 shows that actually $(ii) \implies (i)$. Consequently, Condition (ii) is itself a necessary and sufficient condition. It also implies that $\operatorname{Det}(\mathcal{I} + \beta\mathcal{K}_\Lambda) > 0$ for any $\beta \in [\alpha, 0]$ and any compact $\Lambda \subset E$.

Theorem 2.5. *For $\alpha < 0$ and $\mathcal{K}$ an integral operator such that for some compact set $\Lambda_0 \subset E$, $\mathcal{I} + \alpha\mathcal{K}_{\Lambda_0}$ is not invertible, an α-determinantal process with kernel K exists iff:*

(i') $-1/\alpha \in \mathbb{N}$

(ii') $\det(J_\beta^\Lambda(x_i, x_j))_{1\le i,j\le n} \ge 0$, *for any $n \in \mathbb{N}$, any $\beta \in (\alpha, 0)$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1, \dots, x_n) \in \Lambda^n$.*

As in Theorem 2.4, we also have $(ii') \implies (i')$ and Condition (ii') is itself a necessary and sufficient condition.

Note that $\mathcal{I} + \alpha\mathcal{K}_{\Lambda_0}$ is not invertible if and only if there is almost surely at least one point in Λ_0.

Corollary 2.6. *For m a positive integer, the existence of a $(-1/m)$-determinantal process with kernel K is equivalent to the existence of a determinantal process with the kernel $\dfrac{K}{m}$.*

Corollary 2.7. *For $\alpha < 0$ and $\mathcal{K}$ a self-adjoint operator, an α-determinantal process with kernel K exists iff:*

- $-1/\alpha \in \mathbb{N}$
- $\operatorname{Spec}\mathcal{K} \subset [0, -1/\alpha]$

This result is well known in the case $\alpha = -1$ (see for example Hough, Krishnapur, Peres and Virág in [HKPV06]).

The sufficient part of this necessary and sufficient condition corresponds to condition A in [ST03] of Shirai and Takahashi.

Theorem 2.8. *For $\alpha < 0$, an α-determinantal process in never infinitely divisible.*

Theorem 2.9. *For $\alpha > 0$, an α-determinantal process is infinitely divisible iff*

- $\operatorname{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1$*, for any compact set* $\Lambda \subset E$
- $\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1}\prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0$*, for any* $n \in \mathbb{N}$*, any compact set* $\Lambda \subset E$ *and* $\lambda^{\otimes n}$*-a.e.* $(x_1, \ldots, x_n) \in \Lambda^n$*.*

This theorem gives a more general condition for infinite-divisibility of an α-permanental process than the condition given by Shirai and Takahashi in [ST03].

Theorem 2.10. *For $\mathcal{K}$ a a real symmetric locally trace class operator and $\alpha > 0$, an α-permanental process is infinitely divisible iff*

- $\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1$, *for any compact set* $\Lambda \subset E$
- $J_\alpha^\Lambda(x_1, x_2) \ldots J_\alpha^\Lambda(x_{n-1}, x_n) J_\alpha^\Lambda(x_n, x_1) \geq 0$, *for any* $n \in \mathbb{N}$, *any compact set* $\Lambda \subset E$ *and* $\lambda^{\otimes n}$*-a.e.* $(x_1, \ldots, x_n) \in \Lambda^n$.

Following Griffiths and Milne's remark in [GM87], when an α-permanental process with kernel K exists and is infinitely divisible, we can replace J_Λ^α by $|J_\Lambda^\alpha|$ and obtain an α-permanental process with the same probability distribution.

Remark 2.11. In Theorem 2.1, 2.9 and 2.10 , the condition

$$\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1, \text{ for any compact set } \Lambda \subset E$$

can be replaced by

$$\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) > 0, \text{ for any compact set } \Lambda \subset E.$$

2.4 Fredholm determinant expansion

In [ST03], Shirai and Takahashi have proved the following formula

$$\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{K})^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{E^n} \det_\alpha(K(x_i, x_j))_{1\leq i,j\leq n} \lambda(dx_1) \ldots \lambda(dx_n) \quad (2.2)$$

for a trace class integral operator $\mathcal{K}$ with kernel K and for $z \in \mathbb{C}$ such that $\|\alpha z\mathcal{K}\| < 1$. In the case where the space E is finite, this formula is also given by Shirai in [Shi07].

As $z \mapsto \mathrm{Det}(\mathcal{I} - \alpha z\mathcal{K})$ is analytic on $\mathbb{C}$ and $z \mapsto z^{-1/\alpha}$ is analytic on $\mathbb{C}^*$, we obtain that $z \mapsto \mathrm{Det}(\mathcal{I} - \alpha z\mathcal{K}_{\Lambda,\alpha})^{-1/\alpha}$ is analytic on $\{z \in \mathbb{C} : \mathcal{I} - \alpha z\mathcal{K}_{\Lambda,\alpha} \text{ invertible}\}$.

Therefore, the formula can be extended to the open disc D, centered in 0 with radius $R = \sup\{r \in \mathbb{R}_+ : \forall z \in \mathbb{C}, |z| < r \Rightarrow \mathcal{I} - \alpha z\mathcal{K} \text{ is invertible}\}$.

D is the open disc of center 0 and radius $1/\|\alpha\mathcal{K}\|$, if the operator $\mathcal{K}$ is self-adjoint, but it can be larger if $\mathcal{K}$ is not self-adjoint.

As remarked by Shirai and Takahashi, the formula (2.2) is valid for any $z \in \mathbb{C}$ if $-1/\alpha \in \mathbb{N}$.

The following proposition extends (2.2) to a multivariate case.

Proposition 2.12. *Let $\Lambda \subset E$ be a relatively compact set, $\Lambda_1, \dots \Lambda_d$ mutually disjoint subsets of Λ and $\mathcal{K}$ a locally trace class integral operator with kernel K. We have the following formula*

$$\begin{aligned} \operatorname{Det}\left(\mathcal{I} - \alpha \sum_{k=1}^{d} z_k\, \mathcal{K}_{\Lambda_k \Lambda}\right)^{-1/\alpha} \\ = \sum_{n_1,\dots,n_d=0}^{\infty} \left(\prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!}\right) \int_{\Lambda_1^{n_1}\times\dots\times\Lambda_d^{n_d}} \det_\alpha (K(x_i,x_j))_{1\le i,j\le n}\, \lambda(dx_1)\dots\lambda(dx_n) \end{aligned} \tag{2.3}$$

for any $z_1, \dots, z_d \in \mathbb{C}$, such that $\mathcal{I} - \alpha\gamma \sum_{k=1}^{d} z_k \mathcal{K}_{\Lambda_k\Lambda}$ is invertible for any complex number γ satisfying $|\gamma| < 1$ (n denotes $n_1 + \dots + n_d$).

Proof. We apply the formula (2.2) to the class trace operator $\sum_{k=1}^{d} z_k \mathcal{K}_{\Lambda_k\Lambda}$ and we use the multilinearity property of the α-determinant of a matrix with respect to its rows.

We obtain

$$
\begin{aligned}
&\operatorname{Det}\left(\mathcal{I}-\alpha\sum_{k=1}^{d} z_k\mathcal{K}_{\Lambda_k\Lambda}\right)^{-1/\alpha} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{E^n}\det{}_\alpha\left(\sum_{k=1}^{d} z_k K_{\Lambda_k\Lambda}(x_i,x_j)\right)_{1\le i,j\le n}\lambda(dx_1)\dots\lambda(dx_n) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{E^n}\sum_{k_1,\dots k_n=1}^{d}\det{}_\alpha\left(z_{k_i}\mathbb{1}_{\Lambda_{k_i}}(x_i)\mathbb{1}_{\Lambda}(x_j)K(x_i,x_j)\right)_{1\le i,j\le n}\lambda(dx_1)\dots\lambda(dx_n) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\dots k_n=1}^{d}\int_{\Lambda_{k_1}\times\dots\times\Lambda_{k_n}}\det{}_\alpha\left(z_{k_i}K(x_i,x_j)\right)_{1\le i,j\le n}\lambda(dx_1)\dots\lambda(dx_n) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\dots k_n=1}^{d}\left(\prod_{i=1}^{n} z_{k_i}\right)\int_{\Lambda_{k_1}\times\dots\times\Lambda_{k_n}}\det{}_\alpha\left(K(x_i,x_j)\right)_{1\le i,j\le n}\lambda(dx_1)\dots\lambda(dx_n)
\end{aligned}
$$

where we have used the fact that $K_{\Lambda_k\Lambda}(x_i,x_j)=\mathbb{1}_{\Lambda_k}(x_i)\,\mathbb{1}_{\Lambda}(x_j)K(x_i,x_j)$ for the equality between the first and the second line.

As the value of the α-determinant of a matrix is unchanged by simultaneous interchange of its rows and its columns, the product $z_1^{n_1}\dots z_d^{n_d}$ where $n_1+\dots n_d=n$, will be repeated $\binom{n}{n_1\dots n_d}$ times. This gives the desired formula.

□

For a relatively compact set $\Lambda\subset E$ and $\Lambda_1,\dots,\Lambda_d$ mutually disjoint subsets of Λ, the computation of the Laplace functional of an α-determinantal process for the function $f:(z_1,\dots,z_d)\mapsto -\sum_{k=1}^{d}(\log z_k)\mathbb{1}_{\Lambda_k}$, with $z_1,\dots,z_d\in(0,1]$ gives thanks to (2.1):

$$
\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[\prod_{k=1}^{d} z_k^{\xi(\Lambda_k)}\right]=\operatorname{Det}\left(\mathcal{I}+\alpha\sum_{k=1}^{d}(1-z_k)\,\mathcal{K}_{\Lambda_k\Lambda}\right)^{-1/\alpha} \tag{2.4}
$$

which is the probability generating function (p.g.f.) of the finite-dimensional random vector $(\xi(\Lambda_1),\dots,\xi(\Lambda_d))$.

For $\alpha<0$, the formula (2.4) reminds the multivariate binomial distribution p.g.f. and for $\alpha>0$, the multivariate negative binomial distribution p.g.f., given by Vere-Jones in [VeJ97], in the special case where the space E is finite.

2.5 α- permanental process ($\alpha > 0$)

Proof of Theorem 2.1. We first prove that the conditions are necessary. We suppose that there exists an α-permanental process with $\alpha > 0$, kernel K defining the locally trace class integral operator $\mathcal{K}$.

By taking $d = 1$ in the formula (2.4), we have

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left(z^{\xi(\Lambda)}\right) = \mathrm{Det}\left(\mathcal{I} + \alpha(1-z)\,\mathcal{K}_\Lambda\right)^{-1/\alpha}$$

for any compact set $\Lambda \subset E$ and $z \in (0, 1]$.

Thus, $\mathrm{Det}(\mathcal{I} + \alpha(1-z)\mathcal{K}_\Lambda) \geq 1$ for $z \in (0, 1]$. By continuity (as $z \mapsto \mathrm{Det}(\mathcal{I} + (1-z)\mathcal{K}_\Lambda)$ is indeed analytic on $\mathbb{C}$), we obtain that $\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \geq 1$, which is the first condition. This implies that for any compact set $\Lambda \subset E$, $\mathcal{I} + \alpha\mathcal{K}_\Lambda$ is invertible. Hence $\mathcal{J}_\alpha^\Lambda$ exists and we have, for any nonnegative function f, with compact support included in Λ

$$\begin{aligned}
&\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left(\prod_{x\in\xi} e^{-f(x)}\right) = \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}[1-e^{-f}])^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda(1-e^{-f}))^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}\,\mathrm{Det}(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda e^{-f})^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^{n} e^{-f(x_i)}\right) \mathrm{det}_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1\leq i,j\leq n}\lambda(dx_1)\dots\lambda(dx_n)
\end{aligned} \tag{2.5}$$

where we have used for the equality between the first and the second line the fact that $\mathrm{Det}(\mathcal{I} + \mathcal{A}\mathcal{B}) = \mathrm{Det}(\mathcal{I} + \mathcal{B}\mathcal{A})$, for any trace class operator $\mathcal{A}$, and any bounded operator $\mathcal{B}$.

As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

$$\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1\leq i,j\leq n} \geq 0 \;\; \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in E^n$$

$j_{\alpha,n}^\Lambda(x_1, \dots, x_n) = \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha}\,\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1\leq i,j\leq n}$ is the Janossy density.

Conversely, if we assume $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha} > 0$ and $\det_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\le i,j\le n} \ge 0$ for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1,\dots,x_n) \in \Lambda^n$, the Janossy density will be correctly defined and, on any compact set Λ, we get the existence of a point process ξ_Λ with kernel K_Λ (see Proposition 5.3.II. in [DVJ03a] - here the normalization condition is automatic by chosing $f = 0$ in (2.5)).

The restriction of a point process η, defined on $\Lambda' \subset E$, to a subspace $\Lambda \subset \Lambda'$ is the point process denoted $\eta|_\Lambda$, obtained by keeping the points in Λ and deleting the points in $\Lambda' \backslash \Lambda$.

For any compact sets $\Lambda, \Lambda' \subset E$, such that $\Lambda \subset \Lambda'$, ξ_Λ and $\xi_{\Lambda'}|_\Lambda$ have the same Laplace functional, because we have for any nonnegative function f, with compact support included in Λ:

$$\begin{aligned}\mathbb{E}\left(\exp\left(-\int_\Lambda f d\xi_{\Lambda'}|_\Lambda\right)\right) &= \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_{\Lambda'}[1-e^{-f}])^{-1/\alpha}\\ &= \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_{\Lambda}[1-e^{-f}])^{-1/\alpha}\\ &= \mathbb{E}\left(\exp\left(-\int_\Lambda f d\xi_\Lambda\right)\right).\end{aligned}$$

Therefore, ξ_Λ and $\xi_{\Lambda'}|_\Lambda$ have the same probability distribution. We say that the family $(\mathcal{L}(\xi_\Lambda))$, Λ compact set included in E, is consistent.

Then we can obtain a point process on the complete space E by the Kolmogorov existence theorem for point processes (see Theorem 9.2.X in [DVJ03b] with $P_k(A_1,\dots,A_k;n_1,\dots,n_k) = \mathbb{P}\left(\xi_{\cup_{i=1}^k A_i}(A_1) = n_1,\dots,\xi_{\cup_{i=1}^k A_i}(A_k) = n_k\right)$: as $\xi_{\cup_{i=1}^k A_i}$ is a point process, it follows that the properties (i), (iii), (iv) are fulfilled ; (ii) is fulfilled because the family $(\mathcal{L}(\xi_\Lambda))$, Λ compact set included in E, is consistent).

As we used, in this second part of the proof, only the fact that $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha} > 0$ (instead of $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha} \ge 1$), the assertion in remark 2.11 is also proved. $\square$

Proof of Theorem 2.3. We suppose there exists an α-permanental process with $\alpha > 0$, kernel K defining the locally trace class integral operator $\mathcal{K}$.

Then, following the proof of the preceding theorem, we get that, for all $z \in [0,1]$

$$\mathrm{Det}(\mathcal{I}+\alpha(1-z)\mathcal{K}_\Lambda) = \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)\,\mathrm{Det}(\mathcal{I}-\alpha z\mathcal{J}_\alpha^\Lambda) > 0.$$

As the power series of $\mathrm{Det}(\mathcal{I}-\alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha}$ has all its terms nonnegative,

$$|(\mathrm{Det}(\mathcal{I}-\alpha z\mathcal{J}_\Lambda^\alpha)^{-1/\alpha}| \le (\mathrm{Det}(\mathcal{I}-\alpha\,|z|\,\mathcal{J}_\Lambda^\alpha)^{-1/\alpha}.$$

If z_0 is a complex number with minimum modulus such that $(\mathrm{Det}(\mathcal{I}-\alpha z_0 \mathcal{J}_\Lambda^\alpha) = 0$, by analycity of $z \mapsto \mathrm{Det}(\mathcal{I}-\alpha z \mathcal{J}_\alpha^\Lambda)$ on $\mathbb{C}$ and $z \mapsto z^{-1}$ on $\mathbb{C}^*$, $\mathrm{Det}(\mathcal{I}-\alpha z \mathcal{J}_\alpha^\Lambda)^{-1/\alpha}$ converges for $|z| < |z_0|$ and diverges for $z = z_0$. Thus the series diverges in $z = |z_0|$ and $|z_0| > 1$. This means that the series converges for $|z| \leq 1$ thus, in this case, $\mathrm{Det}(\mathcal{I} - \alpha z \mathcal{J}_\alpha^\Lambda) > 0$.

This implies the necessary condition: $\mathrm{Spec}\, \mathcal{J}_\alpha^\Lambda \subset \{z \in \mathbb{C} : |z| < \dfrac{1}{\alpha}\}$.

As ν eigenvalue of $\mathcal{K}$ is equivalent to $\dfrac{\nu}{1+\alpha\nu}$ eigenvalue of $\mathcal{J}$, and as, $\mathcal{K}$ and $\mathcal{J}$ being compact operators, their non-null spectral values are their eigenvalues, we get the other equivalent necessary condition:

$$\mathrm{Spec}\, \mathcal{K}_\Lambda \subset \{z \in \mathbb{C} : \mathrm{Re}\, z > -\frac{1}{2\alpha}\}.$$

$\square$

2.6 α- determinantal process ($\alpha < 0$)

We recall the following remark, already made for example in [HKPV06].

Remark 2.13. If we define kernels only $\lambda^{\otimes 2}$-almost everywhere, there can be problems when we consider only the diagonal terms, as $\lambda^{\otimes 2}\{(x,x) : x \in \Lambda\} = 0$. For example, in the formula

$$\mathrm{tr}\, K_\Lambda = \int_\Lambda K(x,x)\lambda(dx),$$

$\mathrm{tr}\, K_\Lambda$ is not uniquely defined. To avoid this problem, we write the kernel K_Λ as follows:

$$K_\Lambda(x,y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \overline{\psi_k}(y)$$

where $(\varphi_k)_{k\in\mathbb{N}}$, $(\psi_k)_{k\in\mathbb{N}}$ are orthonormal basis in $L^2(\Lambda, \lambda)$ and $(a_k)_{k\in\mathbb{N}}$ is a sequence of nonnegative real number, which are the singular values of the operator $\mathcal{K}_\Lambda$.

The functions φ_k and ψ_k, $k \in \mathbb{N}$, are defined λ-almost everywhere, but this gives then a unique value for the expression of type

$$\int_{\Lambda^n} F(K(x_i,x_j)_{1\leq i,j\leq n})\, G(x_1, \dots, x_n)\lambda(dx_1)\dots\lambda(dx_n)$$

where F is an arbitrary complex function from $\mathbb{C}^{n^2}$ and G is an arbitrary complex function from Λ^n.

With this remark, the quantities that appear with $F = \det_\alpha$ are well defined.

Lemma 2.14. *Let K be a kernel defined as in Remark 2.13 and defining a trace class integral operator $\mathcal{K}$ on $L^2(\Lambda, \lambda)$, where Λ is a non-λ-null compact set included in the locally compact Polish space E, λ be a Radon measure, n an integer and α a real number. Let F be a continuous fonction from $\mathbb{C}^{n^2}$ to $\mathbb{C}$. The three following assertions are equivalent*

(i) $F(K(x_i, x_j)_{1\leq i,j\leq n}) \geq 0$ $\lambda^{\otimes n} - a.e.(x_1, \ldots, x_n) \in \Lambda^n$

(ii) there exists a set $\Lambda' \subset \Lambda$ such that $\lambda(\Lambda \backslash \Lambda') = 0$ and $F((K(x_i, x_j))_{1\leq i,j\leq n}) \geq 0$ for any $(x_1, \ldots, x_n) \in (\Lambda')^n$

(iii) there exists a version of K such that $F((K(x_i, x_j))_{1\leq i,j\leq n}) \geq 0$ for any $(x_1, \ldots, x_n) \in \Lambda^n$

Proof. (i) is clearly a consequence of (ii). We assume now that (i) is satisfied and we denote by N the $\lambda^{\otimes n}$-null set of n-tuples $(x_1, \ldots, x_n) \in \Lambda^n$ such that $F((K(x_i, x_j))_{1\leq i,j\leq n}) < 0$. As in remark 2.13, we write the kernel K as follows

$$K(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \overline{\psi_k}(y) = \langle (\sqrt{a_k}\varphi_k)_{k\in\mathbb{N}}(x) | (\sqrt{a_k}\psi_k)_{k\in\mathbb{N}}(y) \rangle$$

where $(\varphi_k)_{k\in\mathbb{N}}$, $(\psi_k)_{k\in\mathbb{N}}$ are orthonormal basis in $L^2(\Lambda, \lambda)$, $(a_k)_{k\in\mathbb{N}}$ is a sequence of nonnegative real number, which are the singular values of the operator $\mathcal{K}$ and $\langle .|. \rangle$ denote the inner product in the Hilbert space $l_2(\mathbb{C})$.

As $\mathcal{K}$ is trace class, we have $\sum_{k=0}^{\infty} a_k < \infty$. Hence:

$$\sum_{k=0}^{\infty} a_k |\varphi_k(x)|^2 < \infty \text{ and } \sum_{k=0}^{\infty} a_k |\psi_k(x)|^2 < \infty \ \lambda\text{-a.e. } x \in \Lambda$$

From Lusin's theorem, there exists an increasing sequence $(A_p)_{p\in\mathbb{N}}$ of compact sets included in Λ such that, for any $p \in \mathbb{N}$

$$(\sqrt{a_k}\varphi_k)_{k\in\mathbb{N}} \text{ and } (\sqrt{a_k}\psi_k)_{k\in\mathbb{N}} \text{ are continuous from } A_p \text{ to } l_2(\mathbb{C}) \text{ and } \lambda(\Lambda \backslash A_p) < \frac{1}{p}$$

Therefore the kernel $K : (x,y) \mapsto \left\langle (\sqrt{a_k}\varphi_k)_{k\in\mathbb{N}}(x) | (\sqrt{a_k}\psi_k)_{k\in\mathbb{N}}(y) \right\rangle$ is continuous on A_p^2.

As E is a Polish space, it can be endowed with a distance that we denote by d. We consider the sets

$$A'_p = \{x \in A_p : \forall r > 0, \lambda(B(x,r) \cap A_p) > 0\}$$
$$B_{p,n} = \{x \in A_p : \lambda(B(x,1/n) \cap A_p) = 0\}$$

where $B(x,r)$ is the open ball in E of radius r centered at x and n is an integer.

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $B_{p,n}$ converging to $x \in A_p$. Then we have, when $d(x,x_k) < 1/n$,

$$\lambda(B(x, 1/n - d(x,x_k) \cap A_p) \leq \lambda(B(x_k, 1/n) \cap A_p) = 0$$

Therefore $\lambda(B(x,1/n) \cap A_p) = 0$ and $x \in B_{p,n}$: $B_{p,n}$ is closed, thus compact (as it is included in the compact set A_p).

The set of open balls $\{B(x,1/n) : x \in B_{p,n}\}$ is a cover of $B_{p,n}$. Then, by compactness, $B_{p,n}$ can be covered by a finite numbers of such balls. As the intersections of A_p and any such a ball is a λ-null set, we get $\lambda(B_{p,n}) = 0$.

Hence we have: $\lambda(A'_p) = \lambda\left(A_p \backslash \cup_{n\in\mathbb{N}} B_{p,n}\right) = \lambda(A_p) > \lambda(\Lambda) - 1/p$.

Let $(x_1, \dots, x_n) \in (A'_p)^n$. If $(x_1, \dots, x_n) \notin N$, then $F((K(x_i,x_j))_{1\leq i,j\leq n}) \geq 0$.

Otherwise $(x_1, \dots, x_n) \in N$. For any $i \in [\![1,n]\!]$ and any $r > 0$, we have

$\lambda(A_p \cap B(x_i,r)) > 0$, then $\lambda^{\otimes n}(A_p^n \cap B((x_1,\dots,x_n),r)) = \lambda^{\otimes n}(\prod_{i=1}^{n}(A_p \cap B(x_i,r))) > 0$.

where $B((x_1,\dots,x_n),r)$ denotes the open ball of radius r centered at x, in E^n endowed with the distance $d((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max_{1\leq i\leq n} d(x_i,y_i)$.

Then, as $\lambda^{\otimes n}(N) = 0$, for any $q \in \mathbb{N}$, there exists $(y_1^{(q)}, \dots, y_n^{(q)}) \in A_p^n \cap B((x_1,\dots,x_n),1/q) \backslash N$ and thus $(y_1^{(q)}, \dots, y_n^{(q)})$ converge to $(x_1,\dots,x_n)$ when $q \to \infty$.

As $(y_1^{(q)}, \dots, y_n^{(q)}) \notin N$, $F((K(y_i^{(q)}, y_j^{(q)}))_{1\leq i,j\leq n}) \geq 0$.

As K is continuous on A_p^2 and F is continuous on $\mathbb{C}^{n^2}$, we have that the function $(x_1, \ldots, x_n) \mapsto F((K(x_i, x_j))_{1 \le i,j \le n})$ is continuous on A_p^n. Hence we have: $F((K(x_i, x_j))_{1 \le i,j \le n}) \ge 0$.

Therefore, in all cases, if $(x_1, \ldots, x_n) \in (A'_p)^n$, $F((K(x_i, x_j))_{1 \le i,j \le n}) \ge 0$.

As $(A_p)_{p \in \mathbb{N}}$ is an increasing sequence, it is the same for $(A'_p)_{p \in \mathbb{N}}$. Hence we have: $\cup_{p \in \mathbb{N}} (A'_p)^n = \left(\cup_{p \in \mathbb{N}} A'_p \right)^n$.

We obtain:

$$F((K(x_i, x_j))_{1 \le i,j \le n}) \ge 0 \text{ for any } (x_1, \ldots x_n) \in \left(\cup_{p \in \mathbb{N}} A'_p \right)^n$$

As $\lambda(\Lambda \backslash \left(\cup_{p \in \mathbb{N}} A'_p \right)) = 0$, we finally obtain (ii) with $\Lambda' = \cup_{p \in \mathbb{N}} A'_p$.

We obtained that (i) and (ii) are equivalent conditions.

(i) is clearly a consequence of (iii). Assume now (ii). We will define a version K_1 of K satisfying the condition (iii).

As $\lambda(\Lambda) \ne 0$, $\Lambda' \ne \emptyset$. We set an arbitrary $x_0 \in \Lambda'$.

For $(x, x') \in \Lambda^2$, we define, $y = x$ if $x \in \Lambda'$, $y = x_0$ if $x \in \Lambda \backslash \Lambda'$, $y' = x'$ if $x' \in \Lambda'$, $y' = x_0$ if $x' \in \Lambda \backslash \Lambda'$ and $K_1(x, x') = K(y, y')$.

For $(x_1, \ldots, x_n) \in \Lambda^n$, we define, for $1 \le i \le n$, $y_i = x_i$ if $x_i \in \Lambda'$ and $y_i = x_0$ if $x_i \in \Lambda \backslash \Lambda'$. Then we have, $F((K_1(x_i, x_j))_{1 \le i,j \le n}) = F((K(y_i, y_j))_{1 \le i,j \le n}) \ge 0$ and K_1 is a version of K satisfying the condition (iii).

$\square$

Remark 2.15. Let $F_n, n \in \mathbb{N}$, be continuous functions from $\mathbb{C}^{n^2}$ to $\mathbb{C}$. For any non-$\lambda - null$ compact set Λ, the condition:

(i) $F_n((J_\alpha^\Lambda(x_i, x_j))_{1 \le i,j \le n}) \ge 0$, for any $n \in \mathbb{N}$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$

can always be replaced by the equivalent conditions:

(ii) there exists a set $\Lambda' \subset \Lambda$ such that $\lambda(\Lambda \backslash \Lambda') = 0$ and $F_n((J_\alpha^\Lambda(x_i, x_j))_{1 \leq i,j \leq n}) \geq 0$, for any $n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in (\Lambda')^n$.

or:

(iii) there exists a version of the kernel J such that $F_n((J_\alpha^\Lambda(x_i, x_j))_{1 \leq i,j \leq n}) \geq 0$, for any $n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in \Lambda^n$.

Proof. The proof of (ii) $\implies$ (iii) is done in the same way as in Lemma 2.14. The other parts of the proof are a direct application of Lemma 2.14. □

Proof that (i) is necessary in Theorem 2.4. This has been mentioned by Vere-Jones in [VeJ97] for the multivariate binomial probability distribution, which corresponds to a determinantal process with E being finite. To our knowledge, this has not been proved in other cases.

We consider the $n \times n$ matrix 1_n, whose elements are all equal to one.

We have: $\prod_{j=0}^{n-1}(1 + j\alpha) = 1 + \sum_{k=1}^{n-1} \sum_{1 \leq j_1 < \dots < j_k \leq n-1} j_1 \dots j_k \ \alpha^k$

We will show by induction on n that the number of permutations in Σ_n having $n-k$ cycles for $k \neq 0$ is $a_{nk} = \sum_{1 \leq j_1 < \dots < j_k \leq n-1} j_1 \dots j_k$: this is true for $n = 2$ and $k = 1$. Assume it is true for a given $n \in \mathbb{N}^*$ and for any $k \in [\![1, n-1]\!]$. If we consider the permutations $\sigma \in \Sigma_{n+1}$ having $n+1-k$ cycles ($0 \leq k \leq n$), we have 2 cases:
- either $\sigma(n+1) = n+1$: there is exactly a_{nk} permutations corresponding to this case (with the convention $a_{nn} = 0$, for the case $k = n$),
- or $\sigma(n+1) \neq n+1$. Then, if we denote $\tau_{n+1\,\sigma(n+1)}$ the transposition in Σ_{n+1} that exchange $n+1$ and $\sigma(n+1)$, $\tau_{n+1\,\sigma(n+1)} \circ \sigma$ is a permutation having $n+1$ as fixed point and $n+1-k$ other cycles (with elements in $[\![1, n]\!]$): there is exactly $na_{n\,k-1}$ permutations corresponding to this case.

Then we have

$$\begin{aligned} a_{n+1\ n+1-k} &= a_{nk} + n a_{n\,k-1} \\ &= \sum_{1\le j_1<\cdots<j_k\le n-1} j_1\ldots j_k + \sum_{\substack{1\le j_1<\cdots<j_{k-1}\le n-1 \\ j_k=n}} j_1\ldots j_k \\ &= \sum_{1\le j_1<\cdots<j_k\le n} j_1\ldots j_k \end{aligned}$$

which is what we expected.

Thus: $\det_\alpha 1_n = \prod_{j=0}^{n-1}(1+j\alpha)$.

If $\alpha < 0$ but $-1/\alpha \notin \mathbb{N}$, there exists therefore $n \in \mathbb{N}$ such that $\det_\alpha 1_n < 0$.

We suppose that there exists an α-determinantal process with $\alpha < 0$ but $-1/\alpha \notin \mathbb{N}$ and kernel K. Then we have $\det_\alpha(K(x_i,x_j))_{1\le i,j\le n} \ge 0$ $\lambda^{\otimes n}$ -a.e. $(x_1\ldots,x_n) \in E^n$.

As we exclude the case of a point process having no point almost surely and there is a sequence of compact sets Λ_p such that $\cup_{p\in\mathbb{N}}\Lambda_p = E$, there exists a compact set $\Lambda \in E$ such that

$$\mathbb{E}(\xi(\Lambda)) = \int_\Lambda K(x,x)\lambda(dx) > 0.$$

Applying Lemma 2.14, we get that there exist a version K_1 of the kernel K such that $\det_\alpha(K_1(x_i,x_j))_{1\le i,j\le n} \ge 0$ for any $(x_1\ldots,x_n) \in \Lambda^n$. We also have:

$$\int_\Lambda K(x,x)\lambda(dx) = \int_\Lambda K_1(x,x)\lambda(dx) > 0.$$

Hence there exists $x_0 \in \Lambda$ such that $K_1(x_0,x_0) > 0$.

For $(x_1,\ldots,x_n) = (x_0,\ldots,x_0)$, we get:

$$\det_\alpha(K_1(x_i,x_j))_{1\le i,j\le n} = K(x_0,x_0)^n \det_\alpha 1_n < 0$$

which is a contradiction. Therefore if $\alpha < 0$ and an α-determinantal process exists, then α must be in $\{-1/m : m \in \mathbb{N}\}$.

□

We consider a $d \times d$ square matrix A. If $n_1, \dots, n_d$ are d nonnegative integers, $A[n_1, \dots, n_d]$ is the $(n_1 + \dots + n_d) \times (n_1 + \dots + n_d)$ square matrix composed of the block matrices A_{ij}:

$$A[n_1, \dots, n_d] = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1d} \\ A_{21} & A_{22} & \dots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \dots & A_{dd} \end{pmatrix},$$

where A_{ij} is the $n_i \times n_j$ matrix whose elements are all equal to a_{ij} $(1 \leq i, j \leq d)$.

Lemma 2.16. *Given a $d \times d$ square matrix A, the following assertions are equivalent*

(i) $\det_{-1/m} A[n_1, \dots, n_d] \geq 0$, $\forall n_1, \dots, n_d \in \mathbb{N}$

(ii) $\det_{-1/m} A[n_1, \dots, n_d] \geq 0$, $\forall n_1, \dots, n_d \in \{0, \dots, m\}$

(iii) $\det A[n_1, \dots, n_d] \geq 0$, $\forall n_1, \dots, n_d \in \mathbb{N}$

(iv) $\det A[n_1, \dots, n_d] \geq 0$, $\forall n_1, \dots, n_d \in \{0, 1\}$

Proof. If there exists $k \in [\![1, d]\!]$ such that $n_k > 1$, the matrix $A[n_1, \dots, n_d]$ has at least two identical rows and its determinant is null. So it is clear that (iii) and (iv) are equivalent.

We have:

$$\det(I + ZA)^m = \sum_{n_1, \dots, n_d = 0}^{\infty} m^{n_1 + \dots n_d} \left(\prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!} \right) \det_{-1/m} A[n_1, \dots, n_d] \tag{2.6}$$

where $Z = \operatorname{diag}(z_1, \dots, z_d)$ and $z_1, \dots, z_d$ are d complex numbers. It is a special case of the formula (2.3) with $\alpha = -1/m$, finite space $E = [\![1, d]\!]$ and reference measure λ atomic, where each point of E has measure 1, $\Lambda_k = \{k\}$, for $k \in [\![1, d]\!]$, $\Lambda = E$. Indeed, $ZA = \sum_{k=1}^{d} z_k A_k$, where A_k is the $d \times d$ square matrix having the same k^{th} row as A and the other rows with all elements equal to 0. The matrix A corresponds to the operator $\mathcal{K}$, the matrix A_k corresponds to the operator $\mathcal{K}_{\Lambda_k \Lambda}$. Formula (2.6) also corresponds to the one given by Vere-Jones in [VeJ88].

We also have for $m = 1$:

$$\det(I + ZA) = \sum_{n_1, \dots, n_d = 0}^{1} \left(\prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!} \right) \det A[n_1, \dots, n_d]. \tag{2.7}$$

as $\det A[n_1, \dots, n_d] = 0$ if there exists $k \in [\![1, d]\!]$ such that $n_k > 1$.

(i) is equivalent to the fact that the multivariate power series (2.6) has all its coefficients nonnegative.

(iii) is equivalent to the fact that the multivariate power series (2.7) has all its coefficients nonnegative.

The power series (2.6) being the m^{th} power of the power serie (2.7), if there exists $k \in [\![1, d]\!]$ such that $n_k > m$, the coefficient of $\prod_{k=1}^{d} z^{n_k}$ is null. Therefore, (i) is equivalent to (ii).

For the same reason, we also have that (i) is a consequence of (iii).

Conversely, following Vere-Jones in [VeJ97], we can show by induction on the order of the matrix A, that the fact that the power series (2.6) has all its coefficients nonnegative implies that the power series (2.7) has all its coefficient non negative.

This proves the equivalence between (i) and (iii).

□

Proposition 2.17. *Let $\alpha < 0$ and $\mathcal{K}$ be an integral operator such that $\mathcal{I} + \alpha\mathcal{K}_\Lambda$ is invertible, for any compact set $\Lambda \subset E$. An α-determinantal process with kernel K exists iff:*

$$\det_\alpha(J_\alpha^\Lambda(x_i, x_j))_{1 \le i,j \le n} \ge 0, \textit{ for any } n \in \mathbb{N}, \textit{ and any compact set } \Lambda \quad \lambda^{\otimes n}\textit{-a.e. } (x_1, \dots, x_n) \in \Lambda^n \tag{2.8}$$

Condition (2.8) implies that $-\dfrac{1}{\alpha} \in \mathbb{N}$ *and* $\mathrm{Det}(\mathcal{I} + \beta\mathcal{K}) > 0$ *for any* $\beta \in [\alpha, 0]$.

Proof. We assume that there exists an α-determinantal process ξ with kernel K.

We already proved that it is necessary to have $-1/\alpha \in \mathbb{N}$.

By taking $d = 1$ in the formula (2.4), we have

$$\mathbb{E}\left(z^{\xi(\Lambda)}\right) = \mathrm{Det}\,(\mathcal{I} + \alpha(1-z)\,\mathcal{K}_\Lambda)^{-1/\alpha}$$

for any compact set $\Lambda \subset E$ and $z \in (0,1]$.

Then $\mathrm{Det}\,(\mathcal{I} + \alpha(1-z)\,\mathcal{K}_\Lambda) > 0$ for $z \in (0,1]$, and by continuity, $\mathrm{Det}\,(\mathcal{I} + \alpha\,\mathcal{K}_\Lambda) \geq 0$. As we assumed that $\mathcal{I}+\alpha K_\Lambda$ is invertible, we have necessarily $\mathrm{Det}\,(\mathcal{I} + \alpha\,\mathcal{K}_\Lambda) > 0$.

For any nonnegative function f, with compact support included in Λ

$$\begin{aligned}
&\mathbb{E}\left(\prod_{x\in\xi} e^{-f(x)}\right) = \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}[1-e^{-f}])^{-1/\alpha} \\
&\quad = \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}\,\mathrm{Det}(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda e^{-f})^{-1/\alpha} \\
&\quad = \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \left(\prod_{i=1}^{n} e^{-f(x_i)}\right) \det{}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \lambda(dx_1)\ldots\lambda(dx_n)
\end{aligned}$$

As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

$$\det{}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0 \;\; \lambda^{\otimes n}\text{-a.e. } (x_1,\ldots,x_n) \in E^n$$

Conversely, we assume that the condition

> $\det_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0$, for any $n \in \mathbb{N}$, $\lambda^{\otimes n}$-a.e. $(x_1,\ldots,x_n) \in \Lambda^n$ and any compact set Λ.

is fulfilled. We have

$$\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \det{}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \lambda(dx_1)\ldots\lambda(dx_n)$$

As $-1/\alpha \in \mathbb{N}$, this formula is valid for any $z \in \mathbb{C}$. Then we obtain for $z = 1$, $\mathrm{Det}(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda)^{-1/\alpha} \geq 0$.

We also have $(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda)(\mathcal{I} + \alpha\mathcal{K}_\Lambda) = (\mathcal{I} + \alpha\mathcal{K}_\Lambda)(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda) = \mathcal{I}$.

Then $\mathrm{Det}(\mathcal{I} - \alpha \mathcal{J}_\alpha^\Lambda) > 0$ and $\mathrm{Det}(\mathcal{I} + \alpha \mathcal{K}_\Lambda) > 0$.

Thus the Janossy density is correctly defined and, on any compact set Λ we get the existence of a point process with kernel K and reference mesure λ.

Then it can be extended to the complete space E by the Kolmogorov existence theorem (see Theorem 9.2.X in [DVJ03b]).

$\square$

Proof of Theorem 2.4. For any $m \in \mathbb{N}$, applying Lemma 2.16, we have for any compact set Λ

$$\det_{-1/m}(J^\Lambda_{-1/m}(x_i, x_j))_{1 \le i,j \le n} \ge 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \dots, x_n) \in \Lambda^n$$

is equivalent to

$$\det(J^\Lambda_{-1/m}(x_i, x_j))_{1 \le i,j \le n} \ge 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \dots, x_n) \in \Lambda^n$$

Now, assume we only have

$$\det_{-1/m}(J^\Lambda_{-1/m}(x_i, x_j))_{1 \le i,j \le n} \ge 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n.$$

By lemma 2.14, for each $n \in \mathbb{N}$, there exists a set $\Lambda'_n \subset \Lambda$ such that $\lambda(\Lambda \backslash \Lambda'_n) = 0$ and $\det_{-1/m}(J^\Lambda_{-1/m}(x_i, x_j))_{1 \le i,j \le n} \ge 0$ for any $(x_1 \dots, x_n) \in (\Lambda'_n)^n$.

If $\Lambda' = \cap_{n \in \mathbb{N}} \Lambda'_n$, we have $\lambda(\Lambda \backslash \Lambda') = 0$ and $\det_{-1/m}(J^\Lambda_{-1/m}(x_i, x_j))_{1 \le i,j \le n} \ge 0$ for any $n \in \mathbb{N}$ and $(x_1 \dots, x_n) \in (\Lambda')^n$.

Then, by Lemma 2.16, we have: $\det(J^\Lambda_{-1/m}(x_i, x_j))_{1 \le i,j \le n} \ge 0$, for any $n \in \mathbb{N}$ and $(x_1 \dots, x_n) \in (\Lambda')^n$.

Therefore, we have

$$\det(J^\Lambda_{-1/m}(x_i, x_j))_{1 \le i,j \le n} \ge 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n.$$

The converse is done through a similar proof, using Lemma 2.14 and 2.16.

Thus, we obtain:

$$\det_\alpha(J^\Lambda_\alpha(x_i, x_j))_{1 \le i,j \le n} \ge 0, \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e. } (x_1, \dots, x_n) \in \Lambda^n$$

is equivalent to

$$\det(J_\alpha^\Lambda(x_i,x_j))_{1\le i,j\le n} \ge 0, \text{ for any } n\in\mathbb{N},\ \lambda^{\otimes n}\text{-a.e. } (x_1,\dots,x_n)\in\Lambda^n$$

Theorem 2.4 is then a consequence of Proposition 2.17.

□

Proof of Theorem 2.5. We assume that there exists ξ an α-determinantal process with kernel K.

For $p\in(0,1)$, let ξ_p be the process obtained by first sampling ξ, then independently deleting each point of ξ with probability $1-p$.

Computing the correlation functions, we obtain that ξ_p is an α-determinantal process with kernel pK.

Thus we get from Theorem 2.4 that the conditions of the theorem must be fulfilled.

Conversely, we assume that these conditions are fulfilled. We obtain from Theorem 2.4 that an α-determinantal process ξ_p with kernel pK exists, for any $p\in(0,1)$.

We consider a sequence $(p_k)\in(0,1)^{\mathbb{N}}$ converging to 1 and a compact Λ.

$$\mathbb{E}(\exp(-t\xi_{p_k}(\Lambda)) = \mathrm{Det}(\mathcal{I}+\alpha p_k K_\Lambda(1-e^{-t}))^{-1/\alpha} \underset{k\to\infty}{\longrightarrow} \mathrm{Det}(\mathcal{I}+\alpha K_\Lambda(1-e^{-t}))^{-1/\alpha}$$

As $t\mapsto \mathrm{Det}(\mathcal{I}+\alpha K_\Lambda(1-e^{-t}))^{-1/\alpha}$ is continuous in 0, $(\mathcal{L}(\xi_{p_k}(\Lambda)))_{k\in\mathbb{N}}$ converge weakly. Thus $(\mathcal{L}(\xi_{p_k}(\Lambda)))_{k\in\mathbb{N}}$ is tight.

$\Gamma\subset\mathcal{X}$ is relatively compact if and only if, for any compact set $\Lambda\subset E$, $\{\xi(\Lambda):\xi\in\Gamma\}$ is bounded.

Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact sets such that $\cup_{n\in\mathbb{N}}\Lambda_n = E$.

As, for any $n\in\mathbb{N}$, $(\mathcal{L}(\xi_{p_k}(\Lambda_n)))_{k\in\mathbb{N}}$ is tight, we have that, for any $\epsilon>0$ and $n\in\mathbb{N}$, there exists $M_n>0$ such that for any $k\in\mathbb{N}, \mathbb{P}(\xi_{p_k}(\Lambda_n)>M_n)<\epsilon\,2^{-n-1}$

Let $\Gamma=\{\gamma\in\mathcal{X}:\forall n\in\mathbb{N}, \gamma(\Lambda_n)\le M_n\}$. It is a compact set and for any $k\in\mathbb{N}, \mathbb{P}(\xi_{p_k}\in\Gamma^c)<\epsilon$.

Therefore, $(\mathcal{L}(\xi_{p_k}))_{k\in\mathbb{N}}$ is tight. As E is Polish, $\mathcal{X}$ is also Polish (endowed with the Prokhorov metric). Thus there is a subsequence of $(\mathcal{L}(\xi_{p_k}))_{k\in\mathbb{N}}$ converging weakly to the probability distribution of a point process ξ. By unicity of the distribution of an α-determinantal process for given kernel and reference measure, ξ must be an α-determinantal process with kernel K, which gives the existence.

$\square$

Lemma 2.18. *Let $\mathcal{J}$ be a trace class self-adjoint integral operator with kernel J. We have*

$$\det(J(x_i, x_j))_{1\leq i,j\leq n} \geq 0 \textit{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\textit{-a.e.}(x_1, \dots, x_n) \in \Lambda^n$$

if and only if

$$\operatorname{Spec} \mathcal{J} \subset [0, \infty)$$

Proof. If we assume that the operator $\mathcal{J}$ is positive, the kernel can be written as follows:

$$J(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \overline{\varphi_k}(y)$$

where $a_k \geq 0$ for $k \in \mathbb{N}$.

Hence:

$$\det(J(x_i, x_j))_{1\leq i,j\leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1, \dots, x_n) \in \Lambda^n$$

Conversely, assume that

$$\det(J(x_i, x_j))_{1\leq i,j\leq n} \geq 0 \text{ for any } n \in \mathbb{N}, \lambda^{\otimes n}\text{-a.e.}(x_1, \dots, x_n) \in \Lambda^n.$$

From formula (2.2) with $\alpha = -1$, we have then for any z $\in \mathbb{C}$

$$\operatorname{Det}(\mathcal{I} + z\mathcal{J}) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{E^n} \det(J(x_i, x_j))_{1\leq i,j\leq n} \lambda(dx_1) \dots \lambda(dx_n). \tag{2.9}$$

As $\mathcal{J}$ is assumed to be self-adjoint, its spectrum is included in $\mathbb{R}$. Thanks to (2.9), it is impossible to have an eigenvalue in $\mathbb{R}_-^*$, as the power series has all its coefficients real nonnegative and the first coefficient ($n = 0$) is real positive. Hence $\operatorname{Spec} \mathcal{J} \subset [0, \infty)$.

$\square$

Proof of Corollary 2.7. We assume: $-1/\alpha \in \mathbb{N}$ and $\operatorname{Spec} \mathcal{K} \subset [0, -1/\alpha]$. Then we have, as $\mathcal{K}$ is self-adjoint, that for any compact set Λ, $\operatorname{Spec} \mathcal{K}_\Lambda \subset [0, -1/\alpha]$. Then $\operatorname{Det}(\mathcal{I} + \beta \mathcal{K}_\Lambda) > 0$ for any $\beta \in (\alpha, 0]$.

If $\mathcal{I} + \alpha K_\Lambda$ is invertible for any compact set $\Lambda \subset E$, we have $\operatorname{Spec} J_\alpha^\Lambda \subset [0, \infty)$ and J_α^Λ is a trace class self adjoint operator for any compact set Λ.

Then, applying Lemma 2.18, we get that

$$\det(J(x_i, x_j))_{1 \le i,j \le n} \ge 0 \text{ for any } n \in \mathbb{N}, \text{ compact set } \Lambda \text{ and } \lambda^{\otimes n}\text{-a.e.}(x_1, \ldots, x_n) \in \Lambda^n$$

Using Theorem 2.4, we get the existence of an α-determinantal process with kernel K.

When there exists a compact set Λ_0 such that $\mathcal{I} + \alpha K_{\Lambda_0}$ is not invertible, by the same line of proof, we obtain the announced result, using Theorem 2.5.

Conversely, we assume that there exists an α-determinantal process with kernel K.

Then, from Theorem 2.4 or 2.5, we get that $-1/\alpha \in \mathbb{N}$.

If $\mathcal{I} + \alpha K_\Lambda$ is invertible for any compact set $\Lambda \subset E$, we have $\operatorname{Spec} J_\alpha^\Lambda \subset [0, \infty)$, using Theorem 2.4 and lemma 2.18. Then $\operatorname{Spec} K_\Lambda \subset [0, -1/\alpha) \subset [0, -1/\alpha]$, for any compact set Λ.

If there exists a compact set Λ_0 such that $\mathcal{I} + \alpha K_{\Lambda_0}$ is not invertible, we have $\operatorname{Spec} J_\beta^\Lambda \subset [0, \infty)$ for any compact set Λ and any $\beta \in (\alpha, 0)$, using Theorem 2.5 and lemma 2.18. Then $\operatorname{Spec} K_\Lambda \subset [0, -1/\beta)$ for any $\beta \in (\alpha, 0)$. Therefore $\operatorname{Spec} K_\Lambda \subset [0, -1/\alpha]$ for any compact set Λ.

As K is self-adjoint, this implies in both cases that $\operatorname{Spec} K \subset [0, -1/\alpha]$.

□

Remark 2.19. Using the known result in the case $\alpha = -1$ (see for example Hough, Krishnapur, Peres and Virág in [HKPV06]) and corollary 2.6, one obtains a direct proof of Corollary 2.7.

2.7 Infinite divisibility

Proof of Theorem 2.8. For $\alpha < 0$, we have proved that it is necessary to have $-1/\alpha \in \mathbb{N}$. If an α-determinantal process was infinitely divisible, with $\alpha < 0$, it would be the sum of N i.i.d αN-determinantal processes for any $N \in \mathbb{N}^*$, as it can be seen for the Laplace functional formula (2.1). This would imply that $-1/(N\alpha) \in \mathbb{N}$, for any $N \in \mathbb{N}^*$, which is not possible. Therefore, an α-determinantal process with $\alpha < 0$ is never infinitely divisible. □

Some charactization on infinite divisibility have also been given in [EK09] in the case $\alpha > 0$.

Proof of Theorem 2.9. For $\alpha > 0$, assume that $\mathrm{Det}(\mathcal{I} + \alpha K_\Lambda) \geq 1$ and

$$\sum_{\sigma \in \Sigma_n : \nu(\sigma) = 1} \prod_{i=1}^{n} J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

for any compact set $\Lambda \subset E$, $n \in \mathbb{N}$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$. Then we have:

$$\begin{aligned}\sum_{\sigma \in \Sigma_n : \nu(\sigma) = k} \prod_{i=1}^{n} J_\alpha^\Lambda(x_i, x_{\sigma(i)}) &= \sum_{\substack{\{I_1, \ldots, I_k\} \\ \text{partition of } [\![1,n]\!]}} \sum_{\substack{\sigma_1 \in \Sigma(I_1), \ldots, \sigma_k \in \Sigma(I_k): \\ \nu(\sigma_1) = \cdots = \nu(\sigma_k) = 1}} \prod_{q=1}^{k} \prod_{i \in I_q} J_\alpha^\Lambda(x_i, x_{\sigma_q(i)}) \\ &= \sum_{\substack{\{I_1, \ldots, I_k\} \\ \text{partition of } [\![1,n]\!]}} \prod_{q=1}^{k} \left(\sum_{\substack{\sigma \in \Sigma(I_q): \\ \nu(\sigma) = 1}} \prod_{i \in I_q} J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \right) \geq 0,\end{aligned}$$

for any compact set $\Lambda \subset E$, $n \in \mathbb{N}$, $k \in [\![1, n]\!]$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$, where, for a finite set I, $\Sigma(I)$ denotes the set of all permutations on I.

Then, for any $N \in \mathbb{N}^*$ and any compact set $\Lambda \in E$, $\det_{N\alpha}(J_\alpha^\Lambda(x_i, x_j)/N)_{1 \leq i,j \leq n} \geq 0$. From Theorem 2.1, we get that there exists a $(N\alpha)$-permanental process with kernel K/N. This means that an α-permanental process with kernel K is infinitely divisible.

Conversely, if we assume an α-permanental process with kernel K is infinitely divisible, we get the existence of a $N\alpha$-permanental process with kernel K/N, for any $N \in \mathbb{N}^*$.

From Theorem 2.1, we have that $\mathrm{Det}(\mathcal{I} + \alpha K_\Lambda) \geq 1$ for any compact set $\Lambda \in E$.

We also have

$$\frac{1}{(N\alpha)^{n-1}}\det_{N\alpha}(J_\alpha^\Lambda(x_i,x_j))_{1\le i,j\le n}\ge 0,$$

for any $N\in\mathbb{N}^*$, any $n\in\mathbb{N}$, any compact set $\Lambda\in E$ and $\lambda^{\otimes n}$-a.e. $(x_1,\dots,x_n)\in\Lambda^n$.

When N tends to ∞, we obtain:

$$\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1}\prod_{i=1}^n J_\alpha^\Lambda(x_i,x_{\sigma(i)})\ge 0,$$

which is the desired result.

□

Proof of Theorem 2.10. We use the argument of Griffiths in [Gri84] and Griffiths and Milne in [GM87]. Assume

$$\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1}\prod_{i=1}^n J_\alpha^\Lambda(x_i,x_{\sigma(i)})\ge 0,$$

for any $n\in\mathbb{N}$ and any $(x_1,\dots,x_n)\in\Lambda^n$.

The condition $J_\alpha^\Lambda(x_1,x_2)\dots J_\alpha^\Lambda(x_{n-1},x_n)J_\alpha^\Lambda(x_n,x_1)\ge 0$ is satisfied for the elementary cycles, i.e. cycles such that $J_\alpha^\Lambda(x_i,x_j)=0$ if $i<j+1$ and $(i\neq 1$ or $j\neq n)$. Then it can be extended to any cycle by induction, using $J_\alpha^\Lambda(x_i,x_j)=J_\alpha^\Lambda(x_j,x_i)$.

With Lemma 2.14, we can then extend the proof to the case where

$$\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1}\prod_{i=1}^n J_\alpha^\Lambda(x_i,x_{\sigma(i)})\ge 0,$$

for any $n\in\mathbb{N}$ and $\lambda^{\otimes n}$-a.e. $(x_1,\dots,x_n)\in\Lambda^n$.

□

Remark 2.20. Note that the argument from Griffiths and Milne in [Gri84] and [GM87] is only valid for real symmetric matrices.

Chapter 3

Advanced study of the case of permanental distributions with finite state space

Existence conditions of permanental distributions are deeply connected to existence conditions of multivariate negative binomial distributions. The aim of this chapter is double-fold. It answers several questions generated by recent works on this subject, but it also goes back to the roots of this field and fixes existing gaps in older papers concerning conditions of infinite divisibility for these distributions.

3.1 Introduction

Permanental distributions and the class of multivariate negative binomial distributions that we are interested in, have been originally considered by Griffiths (1984) [Gri84], Griffiths and Milne (1987) [GM87] and Vere-Jones (1997) [VeJ88]. The recent renew of interest for these distributions mainly comes from their connections with the distribution of the local time process of Markov processes. These connections are known under the generic name of "isomorphism theorems". The first one is due to Dynkin (1983). To exploit the more recent isomorphism theorem of Eisenbaum and Kaspi (2009)[EK09], it was necessary to have a better understanding of the family of permanental distributions. Several authors have since made progress in that direction: Marcus and Rosen [MR13], Kogan and

Marcus [KM12], Eisenbaum [Eis13], [Eis14], [Eis16].

The aim of this paper is double. It answers several questions generated by [KM12] and [Eis16] but it also goes back to the roots of the subject and fixes an existing gap in [GM87]. To briefly describe our main results, we first remind reader of the following basic definitions. All the considered matrices are real.

A **permanental distribution** is the law of a nonnegative random vector

$(X_1, X_2, .., X_d)$ with Laplace transform

$$\mathbb{E}\left[\exp\{-\frac{1}{2}\sum_{i=1}^{d} z_i X_i\}\right] = \det(I + ZA)^{-\beta} \tag{3.1}$$

where I is the $d \times d$-identity matrix, Z is the diagonal matrix $\mathrm{diag}((z_i)_{1\le i\le d})$, $A = (a_{ij})_{1\le i,j\le d}$ and β is a fixed positive number.

A matrix A is said to be β**-permanental** if such a random vector exists.

A matrix A is said to be β**-positive** if the multivariate Taylor series expansion in $z_1^{n_1}...z_d^{n_d}$ of $\det(I - ZA)^{-\beta}$ has only nonnegative coefficients (see Section 2 (3.5) for an equivalent formulation).

If the spectral radius of a β-positive definite $d\times d$-matrix A is strictly smaller than 1, there exists a nonnegative random vector $(X_1, X_2, ..., X_d)$ with a multivariate negative binomial distribution such that its probability generating function satisfies:

$$\mathbb{E}[z_1^{X_1}...z_d^{X_d}] = \det(I - A)^{\beta} \det(I - ZA)^{-\beta}. \tag{3.2}$$

We mention that this multivariate negative binomial distribution corresponds to an α-permanental point process (see [ST03]) ζ with index $\alpha = \frac{1}{\beta}$ and kernel $\beta A(I - A)^{-1}$ with respect to the measure $\sum_{k=1}^{d} \delta_k$ (ζ has the same law as $\sum_{k=1}^{d} X_k\, \delta_k$).

Note that distinct matrices A and B may define the same permanental distributions (or the same multivariate negative binomial distribution): one says that A and B are **effectively equivalent** if for every Z in $\mathbb{R}^d$:

$$\det(I + ZA) = \det(I + ZB).$$

For example, A and DAD^{-1}, for D nonsingular diagonal matrix, are effectively equivalent. In this case, they are said to be **diagonally similar**. However, the converse is not true: there exist matrices that are effectively equivalent but not diagonally similar (see [HL84] and [Loe86]).

A necessary and sufficient condition for a matrix A to be β-positive for all $\beta > 0$ has been established by Griffiths and Milne [GM87]. In Section 3.3, we give a counterexample and correct their criteria. The gap in their proof comes from the negligence of the occurrence of zero entries in the considered matrices. Actually, this negligence has no consequence in the case where the matrices are symmetric but becomes problematic when they are not. It neither has consequences when the matrices are irreducible, but this claim requires a proof that is also given in Section 3.3.

Once Griffiths' and Milne's criterion was fixed, we checked whether the existing results based on their initial criterion were still true. In particular, Vere-Jones NSC for a matrix to be β-permanental for all $\beta > 0$, which is formulated thanks to Griffiths' and Milne's criterion, can easily be fixed. In Section 3.5, we also fix the argument in [EK09] which shows that (up to effective equivalence) the nonsingular, β-permanental for every β, matrices are inverse M-matrices (a nonsingular matrix A is an M-matrix if A has no positive off-diagonal entry and A^{-1} has no negative entry). In the symmetric case, this characterization has been established previously by Bapat [Bap89]. Moreover, we extend this characterization to singular matrices.

Section 3.5 relies on Section 3.4 which establishes various relations between the properties of β-permanentality and β-positivity. Indeed, they are deeply connected. For example one can easily see that β-permanentality implies β-positivity.

Hence the question of the description of the class of matrices that are β-permanental for all β is completely solved. The class of matrices that are β-positive for all β, is well described as well. Note that elements of these two classes correspond to infinitely divisible distributions. The question remains of the description for a fixed β of the β-permanental matrices and the β-positive matrices.

We mention a consequence of Vere-Jones results [VeJ88]: a β-permanental symmetric matrix is necessarily positive semi-definite. Conversely, for A symmetric positive semi-definite and $\beta = 1/2$, (3.1) corresponds to the distribution of $(X_1^2, .., X_d^2)$ with $(X_1, ..., X_d)$ centered Gaussian vector with covariance A. Consequently, A must be 1/2-permanental and more generally $n/2$-permanental for every positive integer n. However, for every $\beta > 0$ such that 2β is not an integer, there exist symmetric positive semi-definite matrices that are not β-positive and, therefore, not β-permanental (see the work of Bränden [Brä12] based on Scott and Sokal [SS14]). This result solves a conjecture set by Shirai and Takahashi [ST03], [Shi07].

The only known permanental matrices (up to effective equivalence) are symmetric positive semi-definite matrices or inverse M-matrices. Kogan and Marcus [KM12] have shown that if a nonsingular 3-dimensional permanental matrix is not effectively equivalent to a symmetric matrix then it is diagonally similar to an inverse M-matrix. In Section 3.6, we establish an analogous result for β-positive matrices: in dimension 3, an irreducible β-positive matrix which is not diagonally similar to a symmetric matrix, is necessarily diagonally similar to a matrix with no negative entry. In Section 3.7, we answer the question raised by Kogan and Marcus in the case of dimension greater than 3: Do there exist (up to effective equivalence) nonsingular irreducible permanental matrices that are not symmetric positive definite, nor inverse M-matrices? Thanks to the results of Section 3.4, we reduce the question to the search of 1-positive matrices that are not effectively equivalent to a symmetric matrix nor to a matrix with no negative entry. We actually exhibit families of such matrices and can hence give a positive answer to the question of Kogan and Marcus. This result seems quite surprising in view of [Eis16] according to which, a permanental matrix whose 3×3-principal submatrices are not effectively equivalent to symmetric matrices, is necessarily an inverse M-matrix.

So far, one is not able to give a precise description of β-permanental matrices nor of β-positive matrices. However, we establish in Section 3.7, some necessary conditions for a matrix to be β-positive for a given β (see Section 3.7.2), that might help to find the general form of these matrices. We also establish that irreducible β-permanental matrices must satisfy a restrictive condition: their zero entries are symmetric (Theorem 3.19).

All the sections rely on a preliminary section (Section 3.2) where the needed notation are introduced and preliminary results on cycles of matrices are established, together with a general formula on permanents of matrices with rows and columns repetition.

3.2 Notation, cycles and permanents

For I, J finite sets having the same cardinality, $\Sigma(I,J)$ denotes the set of the bijections from I to J, $\Sigma(I)$ denotes the set of the permutations of I [i.e. $\Sigma(I) = \Sigma(I,I)$] and Σ_d denotes $\Sigma([\![d]\!])$, where $[\![d]\!] = \{1, 2, .., d\}$.

The β-permanent of a $d \times d$-matrix $A = (a_{ij})_{1 \le i,j \le d}$ is defined by

$$\operatorname{per}_\beta A = \sum_{\sigma \in \Sigma_d} \beta^{\nu(\sigma)} \prod_{i=1}^{d} a_{i\sigma(i)}, \tag{3.3}$$

where $\nu(\sigma)$ is the number of cycles of the permutation σ.

In particular, $\operatorname{per}_1 A$ is the permanent of A and $\operatorname{per}_{-1} A = (-1)^d \det(A)$.

To a given $d \times d$-matrix $A = (a_{ij})_{1 \le i,j \le d}$, one associates square matrices with rows and columns repetition by setting for $n_1, \ldots, n_d, n'_1, \ldots, n'_d$ $2d$ nonnegative integers such that $\sum_{i=1}^d n_i = \sum_{i=1}^d n'_i$:

$$A[n_1, \ldots, n_d | n'_1, \ldots, n'_d] = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1d} \\ A_{21} & A_{22} & \ldots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \ldots & A_{dd} \end{pmatrix},$$

where for $1 \le i, j \le d$, A_{ij} is the $n_i \times n'_j$ matrix whose elements are all equal to a_{ij}.

We write $A[n_1, \ldots, n_d]$ for $A[n_1, \ldots, n_d | n_1, \ldots, n_d]$.

With this notation, one can reformulate the definition of β-positivity as it has first been enunciated by Vere-Jones [VeJ88]. Indeed, Vere-Jones [VeJ88] has established that for $\beta > 0$:

$$\det(I - ZA)^{-\beta} = \sum_{n_1, \ldots, n_d = 0}^{\infty} \prod_{i=1}^{d} \frac{z_i^{n_i}}{n_i!} \operatorname{per}_\beta A[n_1, \ldots, n_d], \tag{3.4}$$

which allows to see that A is β-positive iff

$$\text{for every } n_1, \ldots, n_d, \quad \operatorname{per}_\beta A[n_1, \ldots, n_d] \ge 0. \tag{3.5}$$

For $I = \{i_1, \dots i_k\}$ and $J = \{j_1, \dots j_k\}$, with $1 \leq i_1 < \cdots < i_k \leq d$ and $1 \leq j_1 < \cdots < j_k \leq d$, we denote by $A[I \times J]$ the $k \times k$ submatrix of A such that its (r, s) entry is the (i_r, j_s) entry of A. If $I = J$, $A[I]$ denotes $A[I \times I]$.

For k in $\{1, 2, .., d\}$, $A^{(k)}$ denotes the $(d-1)\times(d-1)$ principal submatrix obtained by removing the k^{th} row and k^{th} column from A.

We also need to define $\bar{A}^{(k)}$ the $(d-1) \times (d-1)$ matrix $(a_{ik}a_{kj})_{i,j \in [\![d]\!] \setminus \{k\}}$.

A **nonnegative matrix** is a matrix such that all its entries are nonnegative.

The cardinal of a finite set I, is denoted by $|I|$ or $\#I$.

We denote by sgn the sign function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

For a $d \times d$ real matrix A, $G(A)$ is the directed graph with vertex-set $[\![d]\!]$ and edge-set $\{(i, j) \in [\![d]\!]^2 : a_{ij} \neq 0\}$. A bidirectional edge between two vertices is a couple of edges joining theses two vertices in both ways. A cycle of A is a finite sequence $(i_1, i_2, ..., i_n)$ of $[\![d]\!]$ such that $a_{i_1 i_2} a_{i_2 i_3} .. a_{i_{n-1} i_n} a_{i_n i_1} \neq 0$. For $a_{i_1 i_1} \neq 0$, (i_1) is a cycle of A.

In a cycle $(i_1, \dots, i_n)$, the index k in i_k has to be understood modulo n (e.g., $i_{n+1} = i_1$). Similarly, for a permutation σ in Σ_n, the integers are written modulo n (e.g., $\sigma(n + i) = \sigma(i)$ and $\sigma(i) + n = \sigma(i)$).

A cycle $(i_1, \dots, i_n)$ is said to be **semi-elementary** if

- it is simple ($i_1, \dots, i_n$ are distinct vertices)
- two vertices i_k, i_l that are not neighbours in the cycle (i.e. $k \neq l + 1$ and $l \neq k + 1$) are not linked through a bidirectional edge (i.e. either $a_{i_k i_l} = 0$ or $a_{i_l i_k} = 0$).

A cycle $(i_1, \dots, i_n)$ is **elementary** if

- it is simple
- two vertices i_k, i_l that are not neighbours in the cycle are not linked ($a_{i_k i_l} = a_{i_l i_k} = 0$).

For A symmetric matrix, semi-elementary cycles are elementary.

3.2.1 Positive cycles and symmetric cycles

Two square matrices A and B are **signature similar** if $A = DBD^{-1}$ with D diagonal matrix with all its diagonal entries in $\{-1, +1\}$. In this section, we give a NSC for an irreducible matrix A to be signature similar to a nonnegative matrix. We also give a NSC for A to be diagonally similar to a symmetric matrix.

A cycle $(i_1, \ldots, i_n)$ of a matrix $A = (a_{ij})_{1 \le i,j \le d}$ is said to be **positive** if

$$\prod_{k=1}^{n} a_{i_k i_{k+1}} > 0,$$

and **negative** if $\prod_{k=1}^{n} a_{i_k i_{k+1}} < 0$.

A cycle $(i_1, \ldots, i_n)$ of A is said to be **symmetric** if

$$\prod_{k=1}^{n} a_{i_k i_{k+1}} = \prod_{k=1}^{n} a_{i_{k+1} i_k}.$$

The following lemma is due to Maybee (Theorem 4.1 in [May65]).

Lemma 3.1. *Let $A = (a_{ij})_{1 \le i,j \le d}$ and $B = (b_{ij})_{1 \le i,j \le d}$ be two irreducible matrices. Then A and B are diagonally similar iff :*
$G(A) = G(B)$, and for any cycle $(i_1, \ldots, i_n)$: $\prod_{k=1}^{n} a_{i_k i_{k+1}} = \prod_{k=1}^{n} b_{i_k i_{k+1}}$.

Since a matrix $A = (a_{ij})_{1 \le i,j \le d}$ which is diagonally similar to a nonnegative matrix is also signature similar to the matrix $(|a_{ij}|)_{1 \le i,j \le d}$, one obtains the following lemma.

Lemma 3.2. *An irreducible matrix is signature similar to a nonnegative matrix iff all its cycles are positive.*

If the zero entries of the matrix are symmetric, one can remove the irreducibility condition from Lemma 3.2. This can be seen by decomposing the matrix into a direct sum of irreducible matrices. Hence one obtains the following lemma.

Lemma 3.3. *A matrix with all its zero entries symmetric is signature similar to a nonnegative matrix iff all its cycles are positive.*

Assume that a matrix $A = (a_{ij})_{1\leq i,j\leq d}$ is diagonally similar to a symmetric matrix, then A is also diagonally similar to the matrix $(\sqrt{a_{ij}a_{ji}})_{1\leq i,j\leq d}$ (by assumption: $a_{ij}a_{ji} \geq 0$ for every i, j). This remark leads to the following lemma.

Lemma 3.4. *An irreducible matrix is diagonally similar to a symmetric matrix iff all its cycles are symmetric.*

3.2.2 Permanent of matrices with rows and columns repetition

We need to establish the following formulas for the arguments of Section 7.

Lemma 3.5. *Let B be a $n \times n$ matrix written as the following block matrix:*

$$B = \left(B_{ij}\right)_{1\leq i,j\leq d}$$

where for every (i,j), B_{ij} is an $n_i \times n'_j$ matrix and $n_1, \dots, n_d, n'_1, \dots, n'_d$ nonnegative integers such that $n_1 + \cdots + n_d = n'_1 + \cdots + n'_d = n$. Then we have

$$\operatorname{per} B = \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \; \sum_{\substack{|I_{ij}|=|J_{ij}|=k_{ij} \\ \cup_i J_{ij} = [\![n'_i]\!] \\ \cup_j I_{ij} = [\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} B_{ij}[I_{ij} \times J_{ij}] \right) \tag{3.6}$$

Proof. Denote by b_{ij} the (i,j)-entry of B. One has: $\operatorname{per} B = \sum_{\sigma \in \Sigma_n} \prod_{i=1}^{n} b_{i\sigma(i)}$.

For a subset I of $\mathbb{R}$ and a real number k, we define: $I - k = \{i - k : i \in I\}$ and $I + k = \{i + k : i \in I\}$. For each σ in Σ_n, we define $I_{ij} = [\![n_i]\!] \cap$

$\left(\sigma^{-1}([\![n'_i]\!] + \sum_{q=1}^{j-1} n'_q) - \sum_{q=1}^{j-1} n_q\right)$ and $J_{ij} = \left(\sigma([\![n_i]\!] + \sum_{q=1}^{i-1} n_q) - \sum_{q=1}^{i-1} n_q\right) \cap [\![n'_j]\!]$. Then we have

$$\begin{aligned}
\operatorname{per} B &= \sum_{\substack{\sqcup_i J_{ij} = [\![n'_j]\!] \\ \sqcup_j I_{ij} = [\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} B_{ij}[I_{ij} \times J_{ij}] \right) \\
&= \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \sum_{\substack{|I_{ij}| = |J_{ij}| = k_{ij} \\ \cup_i J_{ij} = [\![n'_j]\!] \\ \cup_j I_{ij} = [\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} B_{ij}[I_{ij} \times J_{ij}] \right)
\end{aligned}$$

where $\sqcup$ means disjoint union. $\square$

Corollary 3.6. *Let $n_1, \ldots, n_d, n'_1, \ldots, n'_d$ be nonnegative integers, such that $n_1 + \cdots + n_d = n'_1 + \cdots + n'_d \geq 1$. We have the following formula for a matrix with repetition of rows and columns :*

$$\operatorname{per} A[n_1, \ldots, n_d | n'_1, \ldots, n'_d] = \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \left(\prod_{i,j=1}^{d} a_{ij}^{k_{ij}} \frac{\prod_{i=1}^{d} n_i!\, n'_i!}{\prod_{i,j=1}^{d} k_{ij}!} \right) \tag{3.7}$$

Proof. We set $B = A[n_1, \ldots, n_d | n'_1, \ldots, n'_d]$. B_{ij} denotes the $n_i \times n'_j$ matrix whose elements are all equal to a_{ij} $(1 \leq i, j \leq d)$.

Applying formula (3.6) to B, we obtain

$$\begin{aligned}
&\operatorname{per} A[n_1, \ldots, n_d | n'_1, \ldots, n'_d] \\
&\quad = \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \left(\prod_{i,j=1}^{d} a_{ij}^{k_{ij}} k_{ij}! \right) \\
&\quad \times \#\{(I_{ij}, J_{ij})_{1 \leq i,j \leq d} : |I_{ij}| = |J_{ij}| = k_{ij}; \cup_i J_{ij} = [\![n'_j]\!]; \cup_j I_{ij} = [\![n_i]\!]\}
\end{aligned}$$

We have

$$\begin{aligned}
&\#\{(I_{ij}, J_{ij})_{1 \leq i,j \leq d} : |I_{ij}| = |J_{ij}| = k_{ij}; \cup_i J_{ij} = [\![n'_j]\!]; \cup_j I_{ij} = [\![n_i]\!]\} \\
&\quad = \#\{(I_{ij})_{1 \leq i,j \leq d} : |I_{ij}| = k_{ij}; \cup_j I_{ij} = [\![n_i]\!]\} \times \#\{(J_{ij})_{1 \leq i,j \leq d} : |J_{ij}| = k_{ij}; \cup_i J_{ij} = [\![n'_j]\!]\} \\
&\quad = \left(\prod_{i=1}^{d} \binom{n_i}{(k_{ij})_j} \right) \left(\prod_{j=1}^{d} \binom{n'_j}{(k_{ij})_i} \right),
\end{aligned}$$

where $\binom{n_i}{(k_{ij})_j} = \binom{n_i}{k_{i1} \ldots k_{id}}$ and $\binom{n_j}{(k_{ij})_i} = \binom{n'_j}{k_{1j} \ldots k_{dj}}$ denotes the multinomial coefficients.

Hence, one obtains

$$\begin{aligned}
&\operatorname{per} A[n_1,\ldots,n_d|n'_1,\ldots,n'_d] \\
&\quad = \sum_{\substack{\sum_i k_{ij}=n'_j \\ \sum_j k_{ij}=n_i}} \left(\prod_{i,j=1}^{d} a_{ij}^{k_{ij}} k_{ij}!\right) \left(\prod_{i=1}^{d} \binom{n_i}{(k_{ij})_j}\right) \left(\prod_{j=1}^{d} \binom{n'_j}{(k_{ij})_i}\right),
\end{aligned}$$

which leads to (3.7). □

For a polynomial P in d variables $x_1,\ldots,x_d$, we denote by $[x_1^{n_1},\ldots,x_d^{n_d}]\,P(x_1,\ldots,x_d)$ the coefficient of the $x_1^{n_1},\ldots,x_d^{n_d}$ term.

Corollary 3.7. *Let $n_1,\ldots,n_d,n'_1,\ldots,n'_d$ be integers, such that $n_1+\cdots+n_d = n'_1+\cdots+n'_d \geq 1$. We have the following formula for a matrix with repetition of rows and columns :*

$$\operatorname{per} A[n_1,\ldots,n_d|n'_1,\ldots,n'_d] = \left(\prod_{i=1}^{d} n'_i!\right) \left[\prod_{j=1}^{d} x_j^{n'_j}\right] \prod_{i=1}^{d} \left(\sum_{j=1}^{d} a_{ij}\, x_j\right)^{n_i} \tag{3.8}$$

Proof. Using the multinomial theorem, we have

$$\prod_{i=1}^{d}\left(\sum_{j=1}^{d} a_{ij}x_j\right)^{n_i} = \prod_{i=1}^{d}\left(\sum_{\sum_j k_{ij}=n_i} \binom{n_i}{(k_{ij})_j} \left(\prod_{j=1}^{d} a_{ij}^{k_{ij}}\, x_j^{k_{ij}}\right)\right)$$

Then

$$\left[\prod_{j=1}^{d} x_j^{n'_j}\right] \prod_{i=1}^{d}\left(\sum_{j=1}^{d} a_{ij}\, x_j\right)^{n_i} = \sum_{\substack{\sum_j k_{ij}=n_i \\ \sum_i k_{ij}=n'_j}} \prod_{i=1}^{d} \binom{n_i}{(k_{ij})_j} \left(\prod_{i,j=1}^{d} a_{ij}^{k_{ij}}\right)$$

By multiplying by $\left(\prod_{i=1}^{d} n'_i!\right)$ and using formula (3.7), we obtain the expected formula.

□

3.3 NSC to be β-positive for all $\beta > 0$

According to Griffiths and Milne [GM87], a $d \times d$-matrix A is β-positive for all positive β iff

(i) For any i, j in $[\![d]\!]$: $a_{ii} \geq 0$, and if $i \neq j$: $a_{ij}a_{ji} \geq 0$.
(ii) For any elementary cycle $(i_1, \ldots, i_n)$ of $A + A^t$: $\prod_{k=1}^{n}(a_{i_k i_{k+1}} + a_{i_{k+1} i_k}) \geq 0$.

Set $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and note that B is a counter-example. Indeed, B is clearly β-positive for all $\beta > 0$, but $(b_{12} + b_{21})(b_{23} + b_{32})(b_{31} + b_{13}) < 0$.

The problem with the proof of the above criterion is located in [GM87] page 18, line 15: a cycle of $A + A^t$ may not correspond to a cycle of A. Under Condition (i), for a given subset $i_1, \ldots, i_n$ of $[\![d]\!]$, $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_n i_1} \geq 0$ does not necessarily imply that $(a_{i_1 i_2} + a_{i_2 i_1})(a_{i_2 i_3} + a_{i_3 i_2}) \ldots (a_{i_n i_1} + a_{i_1 i_n}) \geq 0$. Nevertheless when A is symmetric or when all its entries are nonzero, this implication is correct.

Hence under the additional assumption that A is symmetric or A has no zero entry, Griffiths and Milne's criterion is correct. For the remaining cases, we present below two ways to fix the argument of [GM87]. Either we extend Condition (ii) to semi-elementary cycles (Theorem 3.8), either we assume that the matrix A is irreducible (Corollary 3.10). This enables us to give a complete answer to the question of β-positivity for all $\beta > 0$.

Theorem 3.8. *A matrix A is β-positive for all $\beta > 0$ iff the semi-elementary cycles of A are positive.*

Proof. Sufficiency Assume that for any semi-elementary cycle $(i_1, \ldots, i_n)$ of A

$$\prod_{k=1}^{n} a_{i_k i_{k+1}} > 0. \tag{3.9}$$

Then, we have (3.9) for any simple cycle $(i_1, \ldots, i_n)$ of A. Indeed, it is true for $n \leq 3$, as in this case the cycle must be semi-elementary. For $n > 3$, we make an induction proof. Assume that for any $n' \in [\![n-1]\!]$ and any cycle $(j_1, \ldots, j_{n'})$ of A: $\prod_{k=1}^{n'} a_{j_k j_{k+1}} > 0$. If $(i_1, \ldots, i_n)$ is a semi-elementary cycle, $\prod_{k=1}^{n} a_{i_k i_{k+1}} > 0$ by (3.9). If not, there exist distinct p and q, not neighbours in the cycle, $1 \leq p+1 < q \leq n$, and such that $a_{i_p i_q} \neq 0$ and $a_{i_q i_p} \neq 0$. Hence, we have:

$$\prod_{k=1}^{n} a_{i_k i_{k+1}} = \frac{1}{a_{i_p i_q} a_{i_q i_p}} \left(\left(\prod_{k \in [\![p,q-1]\!]} a_{i_k i_{k+1}} \right) a_{i_q i_p} \right) \left(\left(\prod_{k \in [\![1,n]\!] \setminus [\![p,q-1]\!]} a_{i_k i_{k+1}} \right) a_{i_p i_q} \right).$$

Note that (i_p, i_q) is an elementary cycle, hence $a_{i_p i_q} a_{i_q i_p} > 0$.
$(\prod_{k \in [\![p,q-1]\!]} a_{i_k i_{k+1}}) a_{i_q i_p}$ and $(\prod_{k \in [\![1,n]\!] \setminus [\![p,q-1]\!]} a_{i_k i_{k+1}}) a_{i_p i_q}$ are positive by induction hypothesis. Consequently $(i_1, i_2, ..i_n)$ is also positive.

Since any cycle is the finite union of simple cycles, the above argument works for any cycle. Hence any cycle (simple or not) is positive.

For $i_1, \dots, i_n$ in $[\![d]\!]$, either $(i_1, \dots, i_n)$ is a cycle or there exists $k \in [\![n]\!]$ such that $a_{i_k i_{k+1}} = 0$ (if $k = n$, i_{n+1} still denotes i_1). Hence, in both cases, we have $\prod_{k=1}^{n} a_{i_k i_{k+1}} \geq 0$.

For any $\sigma \in \Sigma_n$, $\prod_{k=1}^{n} a_{i_k i_{\sigma(k)}}$ is the product of $\nu(\sigma)$ terms, each term corresponding to a cycle of σ (which is not necessarily a cycle of A). Thanks to the above, one obtains: $\prod_{k=1}^{n} a_{i_k i_{\sigma(k)}} \geq 0$. Consequently, for any $n_1, \dots, n_d \in \mathbb{N}$ and $\beta > 0$, $\mathrm{per}_\beta\, A[n_1, \dots, n_d]$ is a sum of nonnegative terms and, therefore, is nonnegative.

This implies thanks to (3.5), that A is β-positive for all $\beta > 0$.

Necessity Assume that A is β-positive for all $\beta > 0$. Then, for any n in $\mathbb{N}^*$, $i_1, \dots, i_n$ in $[\![d]\!]$, we have $\sum_{\sigma \in \Sigma_n} \beta^{\nu(\sigma)} \prod_{k=1}^{n} a_{i_k i_{\sigma(k)}} \geq 0$. Dividing by β, and letting β tends to 0, one obtains

$$\sum_{\sigma \in \Sigma_n : \nu(\sigma)=1} \prod_{k=1}^{n} a_{i_k i_{\sigma(k)}} \geq 0, \tag{3.10}$$

We show now by induction the following property for every $n > 0$:
$P(n)$: For any simple cycle $(i_1, \dots, i_n)$ of A, $\mathrm{sgn}\left(\prod_{k=1}^{n} a_{i_k i_{k+1}}\right) = 1$.

$P(1)$ is true. Fix $n > 1$ and assume $P(p)$ is true for all $p \in [\![n-1]\!]$.

Let $(i_1, \dots, i_n)$ be a simple cycle. If there is no other simple cycle whose set of vertices is $\{i_1, \dots, i_n\}$, then we get directly (3.9) from (3.10).

Otherwise, there is another simple cycle $(j_1, \dots, j_n)$ having $\{i_1, \dots, i_n\}$ for set of vertices ($\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\}$).

Suppose that $\left(\prod_{k=1}^{n} a_{i_k i_{k+1}}\right)$ and $\left(\prod_{k=1}^{n} a_{j_k j_{k+1}}\right)$ have opposite signs. Without loss of generality, suppose that:

$$\operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_k i_{k+1}}\right) = 1 = -\operatorname{sgn}\left(\prod_{k=1}^{n} a_{j_k j_{k+1}}\right). \tag{3.11}$$

As $\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\}$, there exist σ in Σ_n such that $(j_1, \dots, j_n) = (i_{\sigma(1)}, \dots, i_{\sigma(n)})$.

For r, s in $[\![n]\!]$ such that $r > s$, and $(u_q)_{1 \le q \le n}$ sequence of real numbers, we use the following convention:

$$\prod_{q=r}^{s} u_q = \prod_{q=r}^{n} u_q \times \prod_{q=1}^{s} u_q. \tag{3.12}$$

Fix k in $[\![n]\!]$,

- either $\sigma(k+1) = \sigma(k) + 1$, and since $\operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_k i_{k+1}}\right) = 1$, one obtains:

$$\operatorname{sgn}(a_{i_{\sigma(k)} i_{\sigma(k+1)}}) = \prod_{q=\sigma(k+1)}^{\sigma(k)-1} \operatorname{sgn}(a_{i_q i_{q+1}}) \tag{3.13}$$

- either $\sigma(k+1) \neq \sigma(k) + 1$, and in this case one obtains (3.13) by induction hypothesis.

Consequently we have:

$$\operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_{\sigma(k)} i_{\sigma(k+1)}}\right) = \operatorname{sgn}\left(\prod_{k=1}^{n} \prod_{q=\sigma(k+1)}^{\sigma(k)-1} a_{i_q i_{q+1}}\right) = \operatorname{sgn}\left(\prod_{k=0}^{n-1} \prod_{q=\sigma(n-k+1)}^{\sigma(n-k)-1} a_{i_q i_{q+1}}\right)$$

Using (3.12), for each k, $(i_q, \sigma(n-k+1) \le q \le \sigma(n-k))$ is made of one piece of the cycle $(i_q, 1 \le q \le n)$. Besides note that the first piece ($k = 0$) starts at the index $i_{\sigma(n-k+1)} = i_{\sigma(1)}$, and that the last piece ($k = n-1$) ends at the index $i_{\sigma(n-k)} = i_{\sigma(1)}$. Hence there exists a positive integer r such that

$$\operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_{\sigma(k)} i_{\sigma(k+1)}}\right) = \operatorname{sgn}\left[\left(\prod_{q=\sigma(1)}^{\sigma(1)-1} \operatorname{sgn}(a_{i_q i_{q+1}})\right)^r\right] = \operatorname{sgn}\left[\prod_{q=1}^{n} \operatorname{sgn}(a_{i_q i_{q+1}})\right]^r = 1.$$

Consequently, we have

$$\operatorname{sgn}\left(\prod_{k=1}^{n} a_{j_k j_{k+1}}\right) = \operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_{\sigma(k)} i_{\sigma(k+1)}}\right) = 1,$$

which contradicts (3.11). Hence $\left(\prod_{k=1}^{n} a_{i_k i_{k+1}}\right)$ and $\left(\prod_{k=1}^{n} a_{j_k j_{k+1}}\right)$ have the same sign.

Using (3.10), we have a nonnegative sum of terms having the same sign. Therefore each term of the sum is nonnegative.

Hence $P(n)$ is true for all n, which establishes the necessity part. □

Theorem 3.9. *An irreducible matrix A is β-positive for all $\beta > 0$ iff it is signature similar to a nonnegative matrix.*

Proof. In the sufficiency part of the proof of Theorem 3.8, we have shown that all the semi-elementary cycles of A are positive iff all the cycles of A are positive. Theorem 3.9 is hence a consequence of Theorem 3.8 and Lemma 3.2. □

Corollary 3.10. *An irreducible matrix A is β-positive for all $\beta > 0$ iff the elementary cycles of $A + A^t$ are positive and for all $i, j \in [\![d]\!]$, $a_{ij}a_{ji} \geq 0$.*

Proof. Assume that A is β-positive for all $\beta > 0$. Thanks to Theorem 3.9, $A + A^t$ is signature similar to a nonnegative matrix. Hence all the elementary cycles of $A + A^t$ are positive. Besides thanks to Proposition 3.7 (ii) in [VeJ97], we also have for all $i, j \in [\![d]\!]$, $a_{ij}a_{ji} \geq 0$.

Conversely, if all the elementary cycles of $A + A^t$ are positive, then so are the cycles of $A + A^t$ (as $A + A^t$ is symmetric). If additionally, for all i, j in $[\![d]\!]$, $a_{ij}a_{ji} \geq 0$, the sign of any semi-elementary cycle of A is the sign of the corresponding cycle in $A + A^t$ and therefore, it is positive. Hence, by Theorem 3.8, A is β-positive. □

3.4 Links between β-permanentality and β-positivity

Remember that for a fixed $\beta > 0$, a $d \times d$-matrix A is β-permanental if $\det(I + ZA)^{-\beta}$ is the Laplace transform of a nonnegative random vector. Vere-Jones has obtained the following NSC for the realization of β-permanentality (Proposition 4.5 in [VeJ97]):

Fix $\beta > 0$. A matrix A is β-permanental iff for every $\alpha \geq 0$, $\det(I + \alpha A) > 0$ and $A(I + \alpha A)^{-1}$ is β-positive.

By continuity, the proposition below reformulates this NSC.

Proposition 3.11. *Fix $\beta > 0$. A matrix A is β-permanental iff for every $\alpha \geq 0$, $(I + \alpha A)$ is nonsingular and $A(I + \alpha A)^{-1}$ is β-positive.*

To establish his NSC, Vere-Jones notes that

$$\det(I - (Z - \alpha I)A)^{-\beta} = \det(I + \alpha A)^{-\beta} \det(I - ZA(I + \alpha A)^{-1})^{-\beta}, \quad (3.14)$$

and actually bases his proof on the following result that we will use several times.

Proposition 3.12. *Fix $\beta > 0$. A matrix A is β-permanental iff for every $\alpha \geq 0$, the multivariate power series $\det(I - (Z - \alpha I)A)^{-\beta}$ in $z_1, \dots, z_d$ contains only nonnegative coefficients.*

To justify the result presented in Proposition 3.12, Vere-Jones refers to a "multivariate analogue of Feller's complete monotonicity property for Laplace transform". But this result can also be seen as a consequence of Bernstein-Haussdorf-Widder-Choquet theorem (presented as Theorem 2.2 in [SS14]) and first proved by Choquet ([Cho69] Théorème 10).

The two next lemmas present stability properties for β-permanental matrices and β-positive matrices.

Theorem 3.13. *Fix $\beta > 0$.*

(i) *If A is β-positive matrix then for any $\gamma \geq 0$, $A + \gamma I$ is also β-positive.*

(ii) *If A is β-permanental matrix then for any $\gamma \geq 0$, $A + \gamma I$ is also β-permanental.*

Proof. (i) Let A be a β-positive matrix and γ a nonnegative real number. We have

$$\det(I - Z(A + \gamma I))^{-\beta} = \det(I - \gamma Z)^{-\beta} \det(I - Z(I - \gamma Z)^{-1}A)^{-\beta}$$

This power series is both product and composition of power series with nonnegative coefficients. Therefore it has only nonnegative coefficients and we can conclude that $A + \gamma I$ is β-positive.

(ii) The proof is similar to the previous one. For A β-permanental matrix and $\gamma > 0$, we have for any $\alpha \geq 0$

$$\begin{aligned}
&\det(I-(Z-\alpha)(A+\gamma I))^{-\beta} \qquad (3.15)\\
&\quad = \det((1+\gamma\alpha)I-\gamma Z)^{-\beta}\det(I-(Z-\alpha)((1+\alpha\gamma)I-\gamma Z)^{-1}A)^{-\beta}\\
&\quad = \det((1+\gamma\alpha)I-\gamma Z)^{-\beta}\det\left(I-\left(\sum_{k=1}^{\infty}\frac{\gamma^{k-1}}{(1+\alpha\gamma)^{k+1}}Z^k-\frac{\alpha}{1+\alpha\gamma}\right)A\right)^{-\beta}
\end{aligned}$$

Since A is β-permanental, this power series is both product and composition of power series with nonnegative coefficients (Proposition 3.12). Hence it has only nonnegative coefficients and one concludes that $A+\gamma I$ is β-permanental. □

The following lemma is a generalization of Lemma 2.5 in [KM12], which corresponds to (ii) with $\sigma \in \left]-\frac{1}{a_{kk}}, 0\right]$.

Lemma 3.14. *Fix $\beta > 0$.*

(i) *If a matrix A is β-positive, the matrix $A^{(k)} + \sigma\bar{A}^{(k)}$ is also β-positive, for any $\sigma \geq 0$.*

(ii) *If a matrix A is β-permanental, the matrix $A^{(k)}+\sigma\bar{A}^{(k)}$ is also β-permanental, for any $\sigma \geq -\frac{1}{a_{kk}}$ if $a_{kk} \neq 0$ and for any real σ if $a_{kk} = 0$.*

Proof. Without loss of generality we assume: $k = d$.

If $\bar{A}^{(d)} = 0$, then (i) and (ii) are obviously satisfied. We hence suppose that $\bar{A}^{(d)} \neq 0$.

(i) Consider the matrix $I - ZA$. For each i in $[\![d-1]\!]$, we add to the i^{th} row, $\frac{z_i a_{id}}{1-z_d a_{dd}}$ times the last row. The determinant of the obtained matrix is unchanged and the $d-1$ first entries of the last columns of this matrix are 0. Therefore, we have

$$\begin{aligned}
\det(I-ZA)^{-\beta} &= \det((\delta_{ij}-z_i a_{ij})_{1\leq i,j\leq d})^{-\beta}\\
&= (1-z_d\,a_{dd})^{-\beta}\det\left(\left(\delta_{ij}-z_i a_{ij}+\frac{z_i a_{id}}{1-z_d a_{dd}}(-z_d\,a_{dj})\right)_{1\leq i,j\leq d-1}\right)^{-\beta}\\
&= (1-z_d\,a_{dd})^{-\beta}\det\left(\left(\delta_{ij}-z_i\left(a_{ij}+\frac{z_d}{1-z_d a_{dd}}a_{id}\,a_{dj}\right)\right)_{1\leq i,j\leq d-1}\right)^{-\beta}
\end{aligned}$$

Denote by $Z^{(d)}$ the matrix $\mathrm{diag}(z_1,\dots,z_{d-1})$, to obtain

$$\det(I-ZA)^{-\beta} = (1-z_d\,a_{dd})^{-\beta}\,\det\left(I-Z^{(d)}\left(A^{(d)}+\frac{z_d}{1-z_d a_{dd}}\bar{A}^{(d)}\right)\right)^{-\beta} \tag{3.16}$$

For a $d\times d$ matrix M, define: $\|M\| = \sup_{x\in\mathbb{R}^d\setminus\{0\}} \dfrac{\|Mx\|}{\|x\|}$, where for any y in $\mathbb{R}^d$, $\|y\|$ denotes its Euclidean norm.

Fix $\sigma>0$. Set $R_d = \dfrac{\sigma}{1+\sigma a_{dd}}$ and for i in $[\![d-1]\!]$, $R_i = \dfrac{1}{\|A^{(d)}\|+2\|\sigma\bar{A}^{(d)}\|}$.

Then for $z_1,\dots,z_d\in\mathbb{C}^d$ such that $|z_i|\le R_i$, $1\le i\le d$, we have:

$$1-z_d\,a_{dd}\neq 0$$

and

$$\det\left(I-Z^{(d)}\left(A^{(d)}+\frac{z_d}{1-z_d a_{dd}}\bar{A}^{(d)}\right)\right)\neq 0$$

(indeed $\left\|Z^{(d)}\left(A^{(d)}+\dfrac{z_d}{1-z_d a_{dd}}\bar{A}^{(d)}\right)\right\|<1$).

This implies that for Z such that $|z_i|\le R_i$, $1\le i\le d$, the power series expansion of $\det(I-ZA)^{-\beta}$ converges. As A is β-positive, all the coefficients of this power series are nonnegative.

If we choose $z_d=R_d$, we get that all the coefficients of the power series expansion of $\det\left(I-Z^{(d)}\left(A^{(d)}+\sigma\bar{A}^{(d)}\right)\right)$ are nonnegative.

Consequently $A^{(d)}+\sigma\bar{A}^{(d)}$ is β-positive.

(ii) The identity (3.16) is still available. Since A is β-permanental, the function: $(z_1,z_2,..,z_d)\mapsto\det(I-ZA)^{-\beta}$, is *absolutely monotone* on the half space $\{z_1,\dots,z_d\in\mathbb{C}:\ \mathrm{Re}(z_1)<0,\dots,\mathrm{Re}(z_d)<0\}$. Equivalently the function $(z_1,z_2,..,z_d)\mapsto\det(I+ZA)^{-\beta}$ is completely monotone on $\{z_1,\dots,z_d\in\mathbb{C}:\ \mathrm{Re}(z_1)>0,\dots,\mathrm{Re}(z_d)>0\}$.

For $\sigma>-\dfrac{1}{a_{dd}}$, set $z_d=\dfrac{\sigma}{1+\sigma\,a_{dd}}$ (we adopt the convention: $-\frac{1}{a_{dd}}=-\infty$ when $a_{dd}=0$).

Hence, thanks to (3.16), the function $(z_1,...,z_{d-1})\mapsto\det\left(I-Z^{(d)}\left(A^{(d)}+\sigma\bar{A}^{(d)}\right)\right)^{-\beta}$ is absolutely monotone. Consequently $A^{(d)}+\sigma\bar{A}^{(d)}$ is β-permanental.

If $a_{dd} \neq 0$, this result can be extended to the case $\sigma = -\dfrac{1}{a_{dd}}$ by continuity. $\square$

Theorem 3.15. *For a fixed $\beta > 0$, let A be a β-positive matrix with spectral radius ρ. Then for every $r > \rho$, the matrix $(rI - A)^{-1}$ is β-permanental.*

Proof. For any $\alpha \geq 0$,

$$\det\left(I - (Z-\alpha)(rI-A)^{-1}\right)^{-\beta} = \det(rI-A)^{\beta}\det((\alpha+r)I - Z - A)^{-\beta}$$
$$= \det(rI-A)^{\beta}\det\left((\alpha+r)I - Z\right)^{-\beta}\det\left(I - \left((\alpha+r)I-Z\right)^{-1}A\right)^{-\beta},$$

which is both product and composition of power series with nonnegative coefficients. Hence it is a power series with nonnegative coefficients.

This is true for any $\alpha \geq 0$. Consequently thanks to Proposition 3.12, the matrix $(rI - A)^{-1}$ is β-permanental. $\square$

The proposition below shows that if a matrix A satisfies some stronger than β-positivity property, then for a big enough positive ρ, $A + \rho I$ is β-permanental.

Proposition 3.16. *For $\beta, \gamma > 0$, let A be a $d \times d$-matrix such that the multivariable Taylor series expansion in $z_1^{n_1}, \dots, z_d^{n_d}$ of $\det(I - (Z - \gamma I)A)^{-\beta}$ contains only nonnegative coefficients and is defined for all $|z_1|, \dots, |z_d| \leq \gamma$. Then there exists $\rho > 0$ such that the matrix $\rho I + A$ is β-permanental.*

Proof. For $0 \leq \gamma' \leq \gamma$, the power series $\det(I - (Z - \gamma' I)A)^{-\beta}$ contains only nonnegative coefficients, also. One sets $\rho = \gamma^{-1}$ and easily checks, using an argument similar to (3.15) that $\det(I - (Z-\alpha)(\rho I + A))^{-\beta}$ is the product of two power series with only nonnegative coefficients.

Thanks to Proposition 3.12, the matrix $\rho I + A$ is hence β-permanental. $\square$

3.5 NSC to be β-permanental for all $\beta > 0$

A NSC for a covariance matrix to be β-permanental for all $\beta > 0$, has been established by Griffiths in [Gri84]. Bapat [Bap89] has then shown that for nonsingular matrices, this NSC characterizes symmetric inverse M-matrices. Eisenbaum and Kaspi [EK09] have then extended Bapat's result to the nonsymmetric case, but they use Griffiths and Milne's criterion and neglect the occurrence of zero entries. Vere-Jones (Proposition 4.7 in [VeJ97]) has extended Griffiths NSC [Gri84] to the nonsymmetric case and uses also Griffiths and Milne's criterion. However Proposition 4.7 in [VeJ97] makes sense only under the additional assumption that the matrix $\mathrm{adj}(A)$ has no zero entry. Besides this assumption implies that: $\mathrm{rank}(A) \geq \dim(A) - 1$.

Under the assumption of irreducibility, Theorem 3.17 below contains the result of [EK09] and extends it to singular matrices.

Theorem 3.17. *An irreducible matrix A is β-permanental for all $\beta > 0$ iff A is signature similar to an element in the closure of the inverse M-matrices.*

Proof. **Step 1:** Assume that A is nonsingular. Thanks to Vere-Jones characterization , if A is β-permanental for every β, then for every $\alpha \geq 0$, $I + \alpha A$ is invertible and $A(I + \alpha A)^{-1}$ is β-positive, for all $\beta > 0$.
Since A is irreducible and invertible, then A^{-1} is irreducible and $A^{-1} + \alpha I$ is also irreducible for any $\alpha \geq 0$. As $I + \alpha A$ is invertible, $(A^{-1} + \alpha I)$ is also invertible and we have $(A^{-1} + \alpha I)^{-1} = A(I + \alpha A)^{-1}$ is irreducible for any $\alpha \geq 0$. Using Theorem 3.9 for the matrix $A(I + \alpha A)^{-1}$, one obtains that for every $\alpha \geq 0$, that $I + \alpha A$ is invertible and $A(I + \alpha A)^{-1}$ is signature similar to a nonnegative matrix. Set $B = A^{-1}$. There exists an irreducible matrix P with positive diagonal entries and $c > 0$ such that $B = cI - P$. One has: $I - \dfrac{P}{c+\alpha} = (c+\alpha)^{-1}A^{-1}(I + \alpha A)$. Hence for any $\alpha \geq 0$, $I - \dfrac{P}{c+\alpha}$ is invertible and $\left(I - \dfrac{P}{c+\alpha}\right)^{-1}$ is signature similar to a matrix with nonnegative entries.

For α big enough: $\left(I - \dfrac{P}{c+\alpha}\right)^{-1} = I + \dfrac{P}{c+\alpha} + \dfrac{F(\alpha)}{(c+\alpha)^2}$, where F is a bounded function in the vicinity of $+\infty$.

Choose now α_0 big enough such that

$$\min_{p_{ij}\neq 0} |p_{ij}| > \max_{i,j} \frac{|F_{ij}(\alpha_0)|}{c+\alpha_0}. \tag{3.17}$$

There exists a signature matrix S_{α_0} such that $S_{\alpha_0}\left(I - \dfrac{P}{c+\alpha_0}\right)^{-1} S_{\alpha_0}$ has nonnegative entries. Hence $S_{\alpha_0} P S_{\alpha_0} + \dfrac{S_{\alpha_0} F(\alpha_0) S_{\alpha_0}}{c+\alpha_0}$ has nonnegative off-diagonal entries. From (3.17), we know that: $\min_{(S_{\alpha_0} P S_{\alpha_0})_{ij}\neq 0} |(S_{\alpha_0} P S_{\alpha_0})_{ij}| > \max_{i,j} \dfrac{|(S_{\alpha_0} F(\alpha_0) S_{\alpha_0})_{ij}|}{c+\alpha_0}$. This implies that all the entries of $S_{\alpha_0} P S_{\alpha_0}$ are nonnegative.

Let λ_0 be the Perron-Frobenius eigenvalue of $S_{\alpha_0} P S_{\alpha_0}$. It is also an eigenvalue of P. If $\lambda_0 \geq c$, then for $\alpha = \lambda_0 - c$, one obtains that $A^{-1}(I+\alpha A) = B + \alpha I = \lambda_0 I - P$ is not invertible. This contradicts our assumption (i). Therefore one must have $\lambda_0 < c$.

This implies that $S_{\alpha_0} B S_{\alpha_0} = cI - S_{\alpha_0} P S_{\alpha_0}$ is a nonsingular M-matrix. Consequently, A is signature similar to an inverse M-matrix.

The converse is a consequence of Theorem 3.15. Hence Theorem 3.17 is established for nonsingular matrices.

Step 2: Assume that A is singular. There exists γ_0 such that for all $\gamma \in]0, \gamma_0[$, $A + \gamma I$ is invertible. Assume now that A is β-permanental for all β, then thanks to Theorem 3.13 (ii), for all $\gamma > 0$, $A + \gamma I$ is β-permanental for all $\beta > 0$. Hence for every $\gamma \in]0, \gamma_0[$, $A + \gamma I$ is signature similar to an inverse M-matrice.

We want to prove that there exists a signature matrix S such that for every $\gamma \in]0, \gamma_0[$, $S(A+\gamma I)S$ is an inverse M-matrix. For any $\gamma \in]0, \gamma_0[$, we denote by S_γ the signature matrix such that $S_\gamma(A+\gamma I)S_\gamma = S_\gamma A S_\gamma + \gamma I$ is an inverse M-matrix. Set: $\gamma_n = \gamma_0/n$. The sequence (S_{γ_n}) is a sequence of signature matrices. As the set of signature matrices with fixed size d is finite, there exists a signature matrix S such that $\{k \in \mathbb{N}^* : S_{\gamma_k} = S\}$ is infinite. Call this set J. The sequence $(SAS + \gamma_k I)_{k\in J}$ is a sequence of inverse M-matrices and converges to SAS.

Conversely, assume that there exists a signature matrix S and a sequence $(A_n)_{n\in\mathbb{N}}$ of inverse M-matrices such that SA_nS converges to A. If the simple limit of a sequence of Laplace transforms is continuous, then it is itself a Laplace transform. Consequently A is also β-permanental for all $\beta > 0$. $\square$

Remark 3.18. It follows from Theorem 1 in [FM88], and from the fact that a principal submatrix of an inverse M-matrix is still an inverse M-matrix, that if an irreducible matrix A belongs to the closure of inverse M-matrices, it can be written as follows:

$$A = D_1\, P\, B[n_1, \ldots, n_d]\, P^t\, D_2$$

where D_1, D_2 are diagonal matrices with positive diagonal entries, P is a permutation matrix, B is a $d \times d$ inverse M-matrix and $n_1, \ldots, n_d$ are positive integers.

Therefore, a matrix A is β-permanental for all $\beta > 0$, iff A has the above form.

Thanks to Theorem 3.17, we can establish the following theorem which represents a constraining necessary condition for an irreducible matrix to be β-permanental.

Theorem 3.19. *:*

(i) *Let A be an irreducible matrix. If A is β-permanental for all $\beta > 0$, then A has no zero entry.*

(ii) *For a fixed $\beta > 0$, let A be an irreducible β-permanental $d \times d$ matrix. Then the zero entries of A are symmetric, that is, for any i, j in $[\![d]\!]$, $a_{ij} = 0 \iff a_{ji} = 0$.*

Proof. (i) Denote by d the dimension of A. Thanks to Theorem 3.17, A is signature equivalent to $B = (b_{ij})_{1 \le i,j \le d}$ element of the closure of inverse M-matrices. The matrix B is also irreducible. For any given $i, j \in [\![d]\!]$, we show now that $b_{ij} > 0$.

An inverse M-matrix has the path product property (see [JS11] and [JS01]), i.e., if B is an inverse M-matrix, we have, for any integers $n \ge 3$ and $i_1, \ldots, i_n \in [\![d]\!]$:

$$\frac{\prod_{k=1}^{n-1} b_{i_k i_{k+1}}}{\prod_{k=2}^{n-1} b_{i_k i_k}} \le b_{i_1 i_n}$$

By continuity, for any matrix B in the closure of inverse M-matrices, one has

$$\prod_{k=1}^{n-1} b_{i_k i_{k+1}} \le b_{i_1 i_n} \prod_{k=2}^{n-1} b_{i_k i_k} \tag{3.18}$$

As B is irreducible, we chose $n, i_1, \ldots, i_n$ such that $i_1, \ldots, i_n$ is a path in $G(B)$ from i to j. Hence we have: $\prod_{k=1}^{n-1} b_{i_k i_{k+1}} > 0$. Using (3.18), we obtain: $b_{ij} =$

$b_{i_1 i_n} > 0$. We have proven that any irreducible matrix belonging to the closure of inverse M-matrices is entrywise positive. Consequently the matrix B has no zero entry, which implies that A also has no zero entry.

(ii) We prove our claim by induction on d. For $d = 1, 2$, it is obviously true. For $d = 3$, by Corollary 3.24, A is either diagonally similar to a symmetric positive semi-definite matrix, or to an element of the closure of inverse M-matrices (we mention that the proof of Corollary 3.24 does not make use of Theorem 3.19). In the first case, our claim is clearly true. In the second case, according to part (i) of the theorem, A has no zero entry.

Now, we consider an arbitrary integer $d \geq 4$ and we assume that the claim of the theorem is valid for any $p \times p$ matrix, with $p \in [\![d-1]\!]$. Let A be an irreducible β-permanental $d \times d$ matrix and suppose that there exists i, j in $[\![d]\!]$ such that $a_{ij} = 0$ and $a_{ji} \neq 0$. We want to find a contradiction.

Choose k in $[\![d]\!]$ such that $k \neq i$ and $k \neq j$. By Lemma 3.14, for any $x > 0$, $A^{(k)} + x\bar{A}^{(k)}$ is a β-permanental $(d-1) \times (d-1)$ matrix. For $x > 0$ small enough, $A^{(k)} + x\bar{A}^{(k)}$ is also irreducible. Using the induction hypothesis, the zero entries of $A^{(k)} + x\bar{A}^{(k)}$ are symmetric, for $x > 0$ small enough. The (i, j)-entry of this matrix is: $a_{ij} + x a_{ik} a_{kj} = x a_{ik} a_{kj}$. Its (j, i)-entry is $a_{ji} + x a_{jk} a_{ki}$. For $x > 0$ small enough, this last entry is nonzero. Hence, by symmetry: $x a_{ik} a_{kj} \neq 0$. Therefore, we have: $a_{ji} a_{ik} a_{kj} \neq 0$, which implies that the principal 3×3 submatrix $A[\{i, j, k\} \times \{i, j, k\}]$ of A is irreducible. As A is β-permanental, $A[\{i, j, k\} \times \{i, j, k\}]$ is also β-permanental. By the induction hypothesis for $p = 3$, the zero entries of $A[\{i, j, k\} \times \{i, j, k\}]$ must be symmetric, which is a contradiction with the hypothesis $a_{ij} = 0$ and $a_{ji} \neq 0$. Therefore $a_{ij} = 0$ implies $a_{ji} = 0$. Hence our claim is established for every d. $\square$

Theorem 3.20. *Fix $\beta_0 > 0$. Let A be an irreducible β_0-permanental matrix. The matrix A is β-permanental for all $\beta > 0$, iff for any $\sigma \geq 0$, every 3×3 principal submatrix of $A(I + \sigma A)^{-1}$ is β-permanental for all $\beta > 0$.*

The above result is a direct consequence of the following proposition.

Proposition 3.21. *Fix $\beta_0 > 0$. Let A be an irreducible β_0-permanental matrix. If any 3×3 principal submatrix of A is β-permanental for any $\beta > 0$, then A is β-positive for any $\beta > 0$.*

Proof of Proposition 3.21. Using Theorem 3.9 and Lemma 3.2, we have to show that any cycle $(i_1, \ldots, i_n)$ of A is positive, that is

$$a_{i_1 i_2} a_{i_2 i_3} .. a_{i_{n-1} i_n} a_{i_n i_1} > 0 \tag{3.19}$$

Since A is β_0-permanental, we have (see Proposition 3.7 in [VeJ97]): $a_{ii} \geq 0$ for any i, and $a_{ij} a_{ji} \geq 0$ for $i \neq j$. Hence (3.19) is satisfied for $n = 1$ and $n = 2$.

For $n = 3$, note that since (i_1, i_2, i_3) is a cycle, $A[\{i_1, i_2, i_3\}]$ is irreducible. Besides $A[\{i_1, i_2, i_3\}]$ is β-permanental for all $\beta > 0$. Thanks to Theorem 3.17 (ii), $A[\{i_1, i_2, i_3\}]$ is hence diagonally similar to an element of the closure of inverse M-matrices. Consequently $A[\{i_1, i_2, i_3\}]$ is diagonally similar to a nonnegative matrix. This implies that: $a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_1} \geq 0$, and (3.19) is satisfied for $n = 3$.

For $n > 3$, we make an induction proof. Assume that (3.19) is satisfied for n ($n \geq 3$), we show that (3.19) is satisfied for $n + 1$.

Let $(i_1, i_2, ..., i_{n+1})$ be a cycle of A. We have: $a_{i_1 i_2} \ldots a_{i_n i_{n+1}} a_{i_{n+1} i_1} \neq 0$. Thanks to Theorem 3.19, we know that $a_{i_1 i_2}$, $a_{i_2 i_1}$, $a_{i_2 i_3}$ and $a_{i_3 i_2}$ are not equal to 0. Consequently the matrix $A[\{i_1, i_2, i_3\}]$ is irreducible. But $A[\{i_1, i_2, i_3\}]$ is also β-permanental for all $\beta > 0$. Using Theorem 3.19 (i), we know that $A[\{i_1, i_2, i_3\}]$ has no zero entry. This implies that $a_{i_1 i_3} a_{i_3 i_1} \neq 0$, and we can write:

$$a_{i_1 i_2} \ldots a_{i_n i_{n+1}} a_{i_{n+1} i_1} = \frac{1}{a_{i_1 i_3} a_{i_3 i_1}} (a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_1})(a_{i_1 i_3} a_{i_3 i_4} \ldots a_{i_n i_{n+1}} a_{i_{n+1} i_1})$$

which is a product of three positive terms, by the induction hypothesis. □

3.6 Dimension 3

In [KM12], Kogan and Marcus establish a NSC for a nonsingular 3×3-matrix to be β-permanental for a fixed $\beta > 0$. Here we establish a NSC for a 3×3-matrix to be β-positive for a fixed β. In the corollary below, we also extend their result to singular matrices.

Theorem 3.22. *Fix $\beta > 0$. Let A be an irreducible 3×3-matrix which is not diagonally similar to a symmetric matrix. Then A is β-positive if and only if it is signature similar to a nonnegative matrix.*

Theorem 3.22 does not consider irreducible 3×3-matrices which are diagonally similar to a symmetric matrix. In Section 3.7.2, we give some conditions for such matrices to be 1-positive. These matrices are not necessarily positive semi-definite.

Proof. Suppose that A is β-positive and is not signature similar to a nonnegative matrix. We know by assumption that A is neither diagonally similar to a symmetric matrix. By Lemma 3.4, we have

$$a_{12}\,a_{23}\,a_{31} \neq a_{13}\,a_{32}\,a_{21} \tag{3.20}$$

and by Lemma 3.2, we have: $a_{12}\,a_{23}\,a_{31} < 0$ or $a_{13}\,a_{32}\,a_{21} < 0$. We may assume without loss of generality that:

$$a_{12}\,a_{23}\,a_{31} < 0. \tag{3.21}$$

Besides by Lemma 3.14, $A^{(3)} + \sigma \bar{A}^{(3)}$ must be β-positive, for all $\sigma \geq 0$, which implies by Proposition 3.7 in [VeJ97] that:

$$(a_{12} + \sigma\, a_{13}\,a_{32})(a_{21} + \sigma\, a_{23}\,a_{31}) \geq 0, \forall \sigma \geq 0. \tag{3.22}$$

Because of (3.20), one cannot have $a_{12} = a_{13}a_{32} = 0$, or $a_{21} = a_{23}a_{31} = 0$.

Set: $\sigma_0 = -\frac{a_{21}}{a_{23}a_{31}}$. Note that $\sigma_0 \geq 0$. If $\sigma_0 > 0$, then, because of (3.22), one has for every $\varepsilon > 0$: $a_{23}a_{31}(a_{12}+(\sigma_0+\varepsilon)\,a_{13}\,a_{32}) \geq 0$ and $a_{23}a_{31}(a_{12}+(\sigma_0-\varepsilon)\,a_{13}\,a_{32}) \leq 0$. Therefore we must have: $a_{12} + \sigma_0\,a_{13}\,a_{32} = 0$. This condition is equivalent to: $a_{12}\,a_{23}\,a_{31} = a_{13}\,a_{32}\,a_{21}$, which contradicts the assumption of Theorem 3.22. Hence: $\sigma_0 = 0$, which means that $a_{21} = 0$. But thanks to (3.22), this implies that for every $\sigma > 0$:

$$a_{12}a_{23}a_{31} + \sigma\, a_{13}\,a_{32}a_{23}\,a_{31} \geq 0$$

Letting σ tend to 0, one obtains: $a_{12}a_{23}a_{31} \geq 0$, which contradicts (3.21).

Consequently A must be signature similar to a nonnegative matrix.

Conversely, if A is signature similar to a nonnegative matrix, then clearly thanks to (3.4), A is β-positive for any given $\beta > 0$. $\square$

Remark 3.23. Thanks to Proposition 3.11, Kogan and Marcus NSC can be easily deduced from Theorem 3.22. Conversely, thanks to Theorem 3.15, one can also deduce Theorem 3.22 from Kogan and Marcus NSC.

Corollary 3.24. *Fix $\beta > 0$. Let A be an irreducible 3×3 matrix. If A is β-permanental, then :*
- either A is diagonally similar to a positive semi-definite symmetric matrix.
- or A is signature similar to an element of the closure of inverse M-matrices.

Proof. Assume that the matrix A is β-permanental. The eigenvalues of A have nonnegative real part because $z \mapsto \det(I - zA)^{-\beta}$ must be analytic for $\mathrm{Re}(z) < 0$ (see Vere-Jones in [VeJ97], Proposition 4.6). If A is diagonally similar to a symmetric matrix, then A must have real nonnegative eigenvalues only. Hence A is diagonally similar to a positive semi-definite symmetric matrix.

Thanks to Vere-Jones characterization (Proposition 3.11), for all $\alpha \geq 0$, $I + \alpha A$ is invertible and $A(I + \alpha A)^{-1}$ is β-positive. If A is not diagonally similar to a symmetric matrix, then $A(I+\alpha A)^{-1}$ is neither diagonally similar to a symmetric matrix. Indeed, we have: $A(I + \alpha A)^{-1} = \alpha^{-1} I - \alpha^{-1}(I + \alpha A)^{-1}$. Then using Theorem 3.22 for every $\alpha \geq 0$, $A(I+\alpha A)^{-1}$ is signature similar to a matrix with nonnegative entries. Consequently, for every $\alpha \geq 0$, $A(I+\alpha A)^{-1}$ is β'-positive for any $\beta' > 0$. Hence, thanks to Vere-Jones characterization, A is β'-permanental for every $\beta' > 0$. Therefore, by Theorem 3.17 (ii) A is signature similar to an element of the closure of inverse M-matrices. □

Remark 3.25. The permanent of an M-matrix is always nonnegative (see, e.g., [BN66], [Gib66] or [GN03]) and consequently it is so for any principal submatrix of an M-matrix. It is hence natural to ask whether M-matrices are 1-positive.

Fix $d \geq 3$, and consider a $d \times d$ M-matrix, $A = (a_{ij})_{1 \leq i,j \leq d}$, with no zero off-diagonal entry and not diagonally similar to a symmetric matrix. Then, thanks to Lemma 3.4, A has a nonsymmetric cycle. The lengh of a nonsymmetric cycle is always strictly greater than 2. Since the off-diagonal entries of A are nonzero, one can always choose three vertices i, j, k in this cycle such that they form a nonsymmetric cycle : $a_{ij}a_{jk}a_{ki} \neq a_{ji}a_{kj}a_{ik}$.

If for some $\beta > 0$, A was β-positive, then its 3×3-principal submatrix corresponding to the vertices i, j and k, would be β-positive, also. Since this principal submatrix is not diagonally similar to a symmetric matrix and has no zero entry, it should be, according Theorem 3.22, signature similar to a nonnegative matrix. This last claim can not be true. We conclude that such an M-matrix A cannot be β-positive, whatever the value of $\beta > 0$.

3.7 Beyond dimension 3

Fix $\beta > 0$. In view of the results of the previous section, it is natural to ask whether in dimension $d > 3$, there exists an irreducible β-positive $d \times d$-matrix which is not diagonally similar to a symmetric matrix nor signature similar to a nonnegative matrix.

This question is the analogue for β-positive matrices of the question generated by Kogan and Marcus work [KM12] on β-permanental 3×3-matrices. Namely, does there exist an irreducible β-permanental $d \times d$-matrix which is not diagonally similar to a symmetric positive semi-definite matrix nor signature similar to an element of the closure of inverse M-matrices?

We consider matrices that can be written as follows :

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.23}$$

where

- A_{11} is a $p \times p$ $(1 \leq p \leq d-1)$ symmetric positive semi-definite square matrix
- A_{12} and A_{21} are (not necessarily square) rank 1 matrices such that the block matrix $(A_{12}\,, (A_{21})^t)$ can be written $(\gamma_j C_i)_{\substack{1\leq i\leq p \\ 1\leq j\leq 2d-2p}}$, where $\gamma_1, \ldots, \gamma_{2d-2p}$ are nonnegative real numbers.
- A_{22} is a square nonnegative matrix.

We make use of these matrices to answer positively to the two questions (Section 3.7.1) but also to find some necessary conditions for a given matrix to be 1-positive (Section 3.7.2).

3.7.1 Positive answer to the two questions

Theorem 3.26. *The matrices that satisfy (3.23), are 1-positive.*

Proof. We prove Theorem 3.26 in two steps.

Step 1 - We show that if for any A_{11} symmetric positive semi-definite matrix, any rank 1 matrices A_{12} and A_{21} such that $A_{12} = (A_{21})^t$:

$$\operatorname{per}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \geq 0, \tag{3.24}$$

then Theorem 3.26 is proved.

Indeed, first note that if A has the form (3.23), then for any $n_1, \dots, n_d \in \mathbb{N}$ such that $n_1 + \dots + n_d \geq 1$, $A[n_1, \dots, n_d]$ has also the form (3.23). Hence thanks to (3.4), to show that any matrix A satisfying (3.23) is 1-positive, it is sufficient to show that for any matrix A satisfying (3.23) $\operatorname{per} A \geq 0$.

Assume (3.24). Let A be a matrix having the form (3.23). As we can exchange simultaneously rows and columns without changing the value of the permanent, we have

$$\operatorname{per} A = \operatorname{per}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \operatorname{per}\begin{pmatrix} A_{11} & \alpha_1 C & \dots & \alpha_q C & 0 \\ \beta_1 C^t & & & & \\ \vdots & & \mathrm{A}_{22} & & \\ \beta_{q'} C^t & & & & \\ 0 & & & & \end{pmatrix}$$

where C is a nonzero column vector, q, q' are positive integers, $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_{q'}$ are positive real numbers and 0 are zero matrices with the appropriate dimension (having no column - when A_{12} has no zero column - and/or no row - when A_{21} has no zero row).
One obtains: $\operatorname{per} A = \left(\prod_{i=1}^{q} \alpha_i\right)\left(\prod_{i=1}^{q'} \beta_i\right) \operatorname{per} B$, with

$$B = (b_{ij})_{1\leq i,j\leq d} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & C & \dots & C & 0 \\ C^t & & & & \\ \vdots & & \mathrm{A}'_{22} & & \\ C^t & & & & \\ 0 & & & & \end{pmatrix}$$

where $B_{11} = A_{11}$, A'_{22} is a block square matrix obtained from A_{22} by first dividing its i^{th} row by β_i for $1 \leq i \leq q'$, then dividing its j^{th} column by α_j for $1 \leq j \leq q$.

We show now that $\operatorname{per} B \geq 0$.

For $I \subset [\![d]\!]$, I^c denotes $[\![d]\!]\backslash I$

We have

$$\begin{aligned}
\operatorname{per} B = \sum_{\sigma\in\Sigma_d}\prod_{i=1}^{d} b_{i\,\sigma(i)} &= \sum_{\substack{I,J\subset[\![p+1,d]\!]:\\ |I|=|J|}} \sum_{\substack{\sigma\in\Sigma_d:\\ \{i>p:\sigma(i)>p\}=I=\sigma^{-1}(J)}} \prod_{i\in I} b_{i\,\sigma(i)} \prod_{i\in I^c} b_{i\,\sigma(i)} \\
&= \sum_{\substack{I,J\subset[\![p+1,d]\!]:\\ |I|=|J|}} \left(\sum_{\sigma\in\Sigma(I,J)}\prod_{i\in I} b_{i\,\sigma(i)}\right)\left(\sum_{\substack{\sigma\in\Sigma(I^c,J^c):\\ \sigma(I^c\cap[\![p+1,d]\!])\subset[\![p]\!]}}\prod_{i\in I^c} b_{i\,\sigma(i)}\right) \\
&= \sum_{\substack{I,J\subset[\![p+1,d]\!]:\\ |I|=|J|}} \operatorname{per} B[I\times J]\ \operatorname{per}\begin{pmatrix} B_{11} & B_{12}\\ B_{21} & 0\end{pmatrix}[I^c\times J^c]
\end{aligned}$$

Remark that A'_{22} is a nonnegative matrix. Hence, for all $I, J \subset [\![p+1,d]\!]$:

$\operatorname{per} B[I\times J] \geq 0$.

If $I, J \subset [\![p+1,d]\!]$, then $[\![p]\!] \subset I^c$ and $[\![p]\!] \subset J^c$. Set: $K = \begin{pmatrix} B_{11} & B_{12}\\ B_{21} & 0\end{pmatrix}[I^c\times J^c]$.

In case K satisfies the assumption of (3.24) (e.g., when $q = q'$ and $I = J$), then $\operatorname{per} K \geq 0$. But it might happen that for some choices of (I, J), K does not satisfy the assumption of (3.24). When it is so, either K contains a zero row or a zero column, and hence: $\operatorname{per} K = 0$.

We finally obtain $\operatorname{per} B \geq 0$, which completes Step 1.

Step 2 - We show now that (3.24) is true.

Let A be a square matrix satisfying

$$A = (a_{ij})_{1\leq i,j\leq d} = \begin{pmatrix} A_{11} & A_{12}\\ A_{21} & 0\end{pmatrix}$$

with A_{11} symmetric positive semi-definite $p\times p$-matrix, A_{12}, A_{21} such that $A_{21} = (A_{12})^t$ and 0 zero square matrix.

We have to show that $\operatorname{per} A \geq 0$.

$$\begin{aligned}
\operatorname{per} A = \sum_{\sigma\in\Sigma_d}\prod_{i=1}^{d} a_{i\sigma(i)} &= \sum_{\substack{I,J\subset[\![p]\!]:\\ |I|=|J|}} \sum_{\substack{\sigma\in\Sigma_d:\\ \{i\le p:\sigma(i)\le p\}=I=\sigma^{-1}(J)}} \prod_{i\in I} a_{i\,\sigma(i)} \prod_{i\in I^c} a_{i\,\sigma(i)} \\
&= \sum_{\substack{I,J\subset[\![p]\!]:\\ |I|=|J|}} \left(\sum_{\sigma\in\Sigma(I,J)}\prod_{i\in I} a_{i\,\sigma(i)}\right)\left(\sum_{\substack{\sigma\in\Sigma(I^c,J^c):\\ \sigma(I^c\cap[\![p]\!])\subset[\![p+1,d]\!]}}\prod_{i\in I^c} a_{i\,\sigma(i)}\right) \\
&= \sum_{\substack{I,J\subset[\![p]\!]:\\ |I|=|J|}} \operatorname{per} A[I\times J]\ \operatorname{per}\begin{pmatrix} 0 & A_{12}\\ A_{21} & 0\end{pmatrix}[I^c\times J^c]
\end{aligned}$$

For $I,J\subset[\![p]\!]$, if $|[\![p]\!]\backslash I|(=|[\![p]\!]\backslash J|)\neq d-p$, per $\begin{pmatrix} 0 & A_{12}\\ A_{21} & 0\end{pmatrix}[I^c\times J^c]=0$. Hence we have, if $2p<d$, $\operatorname{per} A=0$ and if $2p\geq d$:

$$\operatorname{per} A = \sum_{\substack{I,J\subset[\![p]\!]:\\ |I|=|J|=2p-d}} \operatorname{per} A_{11}[I\times J]\ \operatorname{per} A_{12}[([\![p]\!]\backslash I)\times[\![d-p]\!]]\operatorname{per} A_{21}[[\![d-p]\!]\times([\![p]\!]\backslash J)]$$

which leads to

$$\operatorname{per} A = \sum_{\substack{I,J\subset[\![p]\!]:\\ |I|=|J|=2p-d}} \operatorname{per} A_{11}[I\times J]\ \operatorname{per} A_{12}[([\![p]\!]\backslash I)\times[\![d-p]\!]]\operatorname{per} A_{12}[([\![p]\!]\backslash J)\times[\![p]\!]] \tag{3.25}$$

The case $I=J=\emptyset$ being trivial, assume that: $2p>d$, and set: $k=2p-d$. In view of (3.25), to show that $\operatorname{per} A\geq 0$, it is sufficient to prove that for any positive semi-definite $p\times p$-matrix B, the matrix $(\operatorname{per} B[I\times J])_{I,J\subset[\![p]\!]:|I|=|J|=k}$ is positive semi-definite.

To establish the latest, we remind some fundamental results of linear algebra (see for example Bathia's book [Bat97], pp.12-19).

The k-fold tensor product space of $\mathbb{R}^d$, denoted $\otimes^k(\mathbb{R}^p)$, is endowed with the inner product

$$\langle x_1\otimes\cdots\otimes x_k|y_1\otimes\cdots\otimes y_k\rangle = \langle x_1,y_1\rangle\ldots\langle x_k,y_k\rangle$$

for $x_1,\ldots,x_k,y_1,\ldots,y_k\in\mathbb{R}^p$, where $\langle .,.\rangle$ denotes the usual inner product on $\mathbb{R}^p$.

For any positive semi-definite real $p \times p$ matrix B, consider $\otimes^k B$, the k-fold tensor product of B on $\otimes^k(\mathbb{R}^p)$. It is a linear operator on $\otimes^k(\mathbb{R}^p)$ defined as follows

$$\otimes^k B(x_1 \otimes \cdots \otimes x_k) = Bx_1 \otimes \cdots \otimes Bx_k$$

for $x_1, \ldots, x_k \in \mathbb{R}^p$.

If $\epsilon_1, \ldots, \epsilon_p$ is an orthonormal eigenvector base of B in $\mathbb{R}^p$ and if $\lambda_1, \ldots, \lambda_p$ are its corresponding eigenvalues, $(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_k})_{1 \le i_1, \ldots, i_k \le p}$ is an orthonormal eigenvector base of $\otimes^k B$ in $\otimes^k(\mathbb{R}^p)$ and $(\lambda_{i_1} \ldots \lambda_{i_k})_{1 \le i_1, \ldots, i_k \le p}$ are its corresponding eigenvalues. Consequently, since B is positive semi-definite, $\otimes^k B$ is also positive semi-definite.

For $x_1, \ldots, x_k \in \mathbb{R}^p$, their symmetric tensor product $x_1 \vee \cdots \vee x_k$ is defined by

$$x_1 \vee \cdots \vee x_k = \frac{1}{\sqrt{k!}} \sum_{\sigma \in \Sigma_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

Denote by $(e_1, \ldots, e_p)$ the canonical base of $\mathbb{R}^p$ and for $I = \{i_1, \ldots, i_k\} \subset [\![p]\!]$, set: $e_I = e_{i_1} \vee \cdots \vee e_{i_k}$. In $\otimes^k(\mathbb{R}^p)$, we consider the subspace F spanned by the orthonormal family $(e_I)_{I \subset [\![p]\!] : |I| = k}$. This family is actually an orthonormal base of F. Let p_F denote the orthogonal projection from $\otimes^k(\mathbb{R}^p)$ onto F, then $p_F \circ (\otimes^k B)$ is represented by a positive semi-definite matrix in this base.

For $I, J \subset [\![p]\!]$ such that $|I| = |J| = k$, the (I, J) entry of this matrix is equal to $\langle e_I, (\otimes^k B) e_J \rangle$. We have:

$$\begin{aligned}
\langle e_I, (\otimes^k B) e_J \rangle &= \langle e_{i_1} \vee \cdots \vee e_{i_k} | Be_{j_1} \vee \cdots \vee Be_{j_k} \rangle \\
&= \frac{1}{k!} \sum_{\sigma, \sigma' \in \Sigma_p} \langle e_{i_{\sigma(1)}}, Be_{j_{\sigma'(1)}} \rangle \ldots \langle e_{i_{\sigma(k)}}, Be_{j_{\sigma'(k)}} \rangle \\
&= \sum_{\sigma \in \Sigma_p} \langle e_{i_1}, Be_{j_{\sigma(1)}} \rangle \ldots \langle e_{i_k}, Be_{j_{\sigma(k)}} \rangle \\
&= \operatorname{per} B[I \times J]
\end{aligned}$$

This proves that $(\operatorname{per} B[I \times J])_{I, J \subset [\![p]\!] : |I| = |J| = k}$ is positive semi-definite. Therefore $\operatorname{per} A \ge 0$ and Theorem 3.26 is proved. □

The following proposition provides a positive answer to the first question.

Proposition 3.27. *Let A be a square matrix satisfying the following condition:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.26}$$

where A_{11} *is a symmetric positive semi-definite matrix with at least one off-diagonal negative entry,* A_{12} *and* A_{21} *are entrywise positive matrices such that* $A_{12} = (A_{21})^t$ *has rank* 1*, and* A_{22} *is a nonsymmetric square nonnegative matrix.*

Then A *is* 1*-positive and is not diagonally similar to a symmetric matrix nor to a nonnegative matrix.*

Proof. Since A satisfies (3.26), A satisfies also (3.23). Hence thanks to Theorem 3.26, we know that A is 1-positive.

Denote by d the dimension of A and by p the dimension of A_{11}.
As A_{11} has at least an off-diagonal negative entry, there exists i, j in $[\![p]\!]$, with $i \neq j$, such that $a_{ij} = a_{ji} < 0$. As A_{22} is not symmetric, there exist k, l in $[\![p+1, d]\!]$, with $k \neq l$, such that $a_{kl} \neq a_{lk}$. Note that (i, j, k, l) is a nonsymmetric cycle of A. Consequently A is not diagonally similar to a symmetric matrix.
A_{22} is a nonnegative matrix and $a_{kl} \neq a_{lk}$, thus we have either $a_{kl} > 0$ or $a_{lk} > 0$. Hence, either (i, j, k, l) or (i, j, l, k) is a negative cycle. It follows that the matrix A is not diagonally similar to a nonnegative matrix. □

The following theorem answers positively to the second question.

Theorem 3.28. *For every* $d \geq 4$*, there exists nonsingular 1-permanental* $d \times d$ *matrices that are not diagonally similar to a symmetric matrix, nor diagonally similar to an inverse M-matrix.*

Proof. Let A be a matrix satisfying (3.26). Let r be a positive real number greater than the spectral radius of A. From Theorem 3.15, $(rI - A)^{-1}$ is 1-permanental.

By Proposition 3.27, A is not diagonally similar to a symmetric matrix, hence neither is $(rI - A)^{-1}$.

Assume that $(rI - A)^{-1}$ is diagonally similar to an inverse M-matrix. Then there exists a nonsingular diagonal matrix D, a positive real number c and a nonnegative matrix Q such that: $(rI - A)^{-1} = D^{-1}(cI - Q)^{-1}D$. One obtains: $DAD^{-1} = (r - c)I + Q$.

This implies that all the off-diagonal entries of DAD^{-1} are nonnegative. Besides, the diagonal entries of DAD^{-1} have the same sign as those of A, and thus they are nonnegative. Consequently all the entries of DAD^{-1} are nonnegative.

This leads to a contradiction, as, by Proposition 3.27, A is not diagonally similar to a matrix with nonnegative entries.

Therefore, $(rI - A)^{-1}$ is not diagonally similar to an inverse M-matrix. $\square$

Corollary 3.29. *Let A be a square matrix satisfying the following condition:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.27}$$

where the matrix A_{11} is symmetric, the matrices A_{12} and A_{21} have positive entries only and such that the block matrix $(A_{12}, (A_{21})^t)$ has rank 1, and A_{22} is a square matrix with positive entries only.

Then there exists $\gamma > 0$ such that $A + \gamma I$ is 1-permanental.

The above corollary of Theorem 3.26 gives another way to answer positively to the second question. Indeed, choose A satisfying both (3.27) and (3.26), and choose $\gamma > 0$ such that $A + \gamma I$ is 1-permanental. Then thanks to Proposition 3.27, $A + \gamma I$ is not diagonally equivalent to a symmetric matrix nor to a nonnegative matrix. This last property implies that $A + \gamma I$ can not be diagonally equivalent to an inverse M-matrix.

Proof of Corollary 3.29. First note that for $\gamma > 0$ sufficiently big, $A_{11} + \gamma I$ is symmetric positive definite. Hence we can assume in this proof that A_{11} is symmetric positive definite.

We just have to show that such a matrix A satisfies the assumption of Proposition 3.16. Thanks to (3.14), it is sufficient to check that for $\alpha > 0$ small enough, $A(\alpha) = A(I + \alpha A)^{-1}$ is 1-positive. Since matrices with the form (3.27) are 1-positive, it is hence sufficient to prove that $A(\alpha)$ has also the form (3.27) for $\alpha > 0$ small enough.

First assume that A is nonsingular. For $\alpha > 0$ small enough $I + \alpha A$ is nonsingular and $A(I + \alpha A)^{-1} = (A^{-1} + \alpha I)^{-1}$.

For any real nonsingular matrix B such that $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, with B_{11} nonsingular symmetric matrix, B_{12} and B_{21} matrices such that the block matrix $(B_{12}, (B_{21})^t)$ has rank 1, we have:

$(B^{-1})_{11}$ *is a symmetric matrix, and* $((B^{-1})_{12}, ((B^{-1})_{21})^t)$ *has rank* 1. *(*)*

Indeed, denote by $(B')^{-1}$ the Schur complement of B_{11}, i.e. $(B')^{-1} = B_{22} - B_{21}(B_{11})^{-1}B_{12}$. Then it well known that: $B^{-1} = \begin{pmatrix} (B^{-1})_{11} & (B^{-1})_{12} \\ (B^{-1})_{21} & (B^{-1})_{22} \end{pmatrix}$, with

$$\begin{aligned}
(B^{-1})_{11} &= (B_{11})^{-1} + (B_{11})^{-1}B_{12}B'B_{21}(B_{11})^{-1} \\
(B^{-1})_{12} &= -(B_{11})^{-1}B_{12}B' \\
(B^{-1})_{21} &= -B'B_{21}(B_{11})^{-1}
\end{aligned}$$

As $\operatorname{rank}(B_{12}, (B_{21})^t) = 1$, we can write: $(B_{12})_{ij} = \alpha_j C_i$ and $(B_{21})_{ij} = \beta_i C_j$. One obtains: $B_{12}B'B_{21} = \left(C_i C_j \sum_{kl} \alpha_k \beta_l b'_{kl}\right)_{i,j}$, which is symmetric. Therefore $(B^{-1})_{11}$ is symmetric.

Besides: $\operatorname{rank}(B_{12}, (B_{21})^t) = 1$ and $((B^{-1})_{21})^t = -(B_{11})^{-1}(B_{21})^t(B')^t$.

Hence $((B^{-1})_{12}, ((B^{-1})_{21})^t)$ has rank 1.

We can remove the assumption that B_{11} is invertible by continuity (consider $B + \epsilon I$ instead, and let ϵ tend to 0).

We make use of the fact that B satisfies *(*)* in two cases. First the case $B = A$, then the case $B = A^{-1} + \alpha I$. This proves that $A(\alpha) = (A^{-1} + \alpha I)^{-1}$ has the form $\begin{pmatrix} A(\alpha)_{11} & A(\alpha)_{12} \\ A(\alpha)_{21} & A(\alpha)_{22} \end{pmatrix}$, with $A(\alpha)_{11}$ symmetric invertible square matrix, $A(\alpha)_{12}$ and $A(\alpha)_{21}$ matrices such that the block matrix $(A(\alpha)_{12}, (A(\alpha)_{21})^t)$ has rank 1.
As $A(\alpha)$ tends to A when α tends to 0, for α small enough, $A(\alpha)_{12}$, $A(\alpha)_{12}$ and $A(\alpha)_{22}$ contain only positive entries and $A(\alpha)_{11}$ is symmetric positive definite. Hence $A(\alpha)$ has the form (3.27).

In case A is singular, one can use the previous argument for $A + \varepsilon I$ (where $\varepsilon > 0$, small enough) instead of A and let then ε tend to 0. □

3.7.2 Some conditions for 1-positivity

The following proposition shows that Theorem 3.26 is no longer valid if one removes the assumption that A_{11} is positive semi-definite, even if one assumes instead that A_{11} is 1-positive.

Proposition 3.30. *Let $A = (a_{ij})_{1\leq i,j\leq 3}$ be a symmetric 3×3-matrix such that A is not signature similar to a nonnegative matrix and $a_{33} = 0$. Then A is 1-positive iff $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is positive semi-definite.*

Proof. Thanks to Lemma 3.3, since A is not signature similar to a nonnegative matrix, A has a negative cycle. As A is symmetric, one hence must have: $a_{12}a_{23}a_{31} = a_{13}a_{32}a_{21} < 0$.

Assume that A is 1-positive. A is diagonally similar to $\begin{pmatrix} a_{11}/a_{13}^2 & a_{12}/(a_{13}a_{23}) & 1 \\ a_{12}/(a_{13}a_{23}) & a_{22}/a_{23}^2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

To show that $\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ is positive semi-definite, is equivalent to show that

$\begin{pmatrix} a_{11}/a_{13}^2 & a_{12}/(a_{13}a_{23}) \\ a_{12}/(a_{13}a_{23}) & a_{22}/a_{23}^2 \end{pmatrix}$ is positive semi-definite.

Hence we may assume without loss of generality that $a_{13} = a_{31} = a_{23} = a_{32} = 1$ and $a_{12} = a_{21} < 0$.

Thanks to (3.7), we have:

$$\operatorname{per} A[n_1, n_2, n_3] = \sum_{\substack{\sum_i k_{ij} = n_j \\ \sum_j k_{ij} = n_i}} \left(\prod_{i,j=1}^{3} a_{ij}^{k_{ij}} \frac{\prod_{i=1}^{3} (n_i!)^2}{\prod_{i,j=1}^{3} k_{ij}!} \right)$$

for $(n_1, n_2, n_3) \in \mathbb{N}^3 \backslash \{(0,0,0)\}$.

As $a_{33} = 0$, only the terms of this sum with $k_{33} = 0$ are nonzero. We choose to take $n_3 = n_1 + n_2 - 1$. For this choice, the above sum contains only four terms, corresponding to

- $k_{11} = 1, k_{12} = k_{21} = k_{22} = k_{33} = 0, k_{13} = k_{31} = n_1 - 1, k_{23} = k_{32} = n_2$
- $k_{12} = 1, k_{11} = k_{21} = k_{22} = k_{33} = 0, k_{13} = n_1 - 1, k_{31} = n_1, k_{23} = n_2, k_{32} = n_2 - 1$
- $k_{21} = 1, k_{11} = k_{12} = k_{22} = k_{33} = 0, k_{13} = n_1, k_{31} = n_1 - 1, k_{23} = n_2 - 1, k_{32} = n_2$
- $k_{22} = 1, k_{11} = k_{12} = k_{21} = k_{33} = 0, k_{13} = k_{31} = n_1, k_{23} = k_{32} = n_2 - 1$

Therefore we have

$$\operatorname{per} A[n_1, n_2, n_1 + n_2 - 1] = ((n_1 + n_2 - 1)!)^2 (n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22})$$

for any $(n_1, n_2) \in \mathbb{N}^2 \backslash \{(0,0)\}$.

As A is 1-positive, we get that $n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} \geq 0$, for any $(n_1, n_2) \in \mathbb{N}^2 \backslash \{(0,0)\}$. By dividing by any positive integer, we obtain that it is also true for any $(n_1, n_2) \in \mathbb{Q}_+^2 \backslash \{(0,0)\}$ and by continuity it is also true for any $n_1, n_2 \in \mathbb{R}_+$. As $a_{12} < 0$, it is true for any $n_1, n_2 \in \mathbb{R}$.

This proves that $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is positive semi-definite.

Conversely, if the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is positive semi-definite, then by Theorem 3.26, A is 1-positive. □

The following proposition shows that Theorem 3.26 is no longer valid if one removes the assumption that $\operatorname{rank}(A_{12}, (A_{21})^t) = 1$.

Proposition 3.31. *Consider the following matrix A:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.28}$$

where A_{11} is a symmetric $p \times p$-matrix with negative off-diagonal entries only ($p \geq 2$), A_{12} and A_{21} are matrices with positive entries only, such that $A_{12} = (A_{21})^t$, and A_{22} is a nonnegative $(d-p) \times (d-p)$-matrix such that none of its principal 2×2-submatrices is symmetric ($d - p \geq 2$) .

Then, if A is 1-positive, it is necessary that A_{12} has rank 1.

Proof. Assume that $A = (a_{ij})_{1 \leq i,j \leq d}$ is 1-positive and that A_{12} has not rank 1. Since $A_{12} \neq 0$, $\operatorname{rank}(A_{12})$ must be strictly greater than greater than 1. Hence there exist $i, j \in [\![p]\!]$, with $i \neq j$, and $k, l \in [\![p+1, d]\!]$, with $k \neq l$, such that

$$\operatorname{rank} \begin{pmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{pmatrix} > 1$$

But $A[\{i,j,k,l\}]$ is still 1-positive. We are going to obtain a contradiction by showing that this fact implies that

$$\operatorname{rank}\begin{pmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{pmatrix} \leq 1.$$

Set: $B = A[\{i,j,k,l\}] = (b_{qr})_{1\leq q,r\leq 4} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$,

where B_{11}, B_{12}, B_{21} and B_{22} are 2×2 submatrices.

According to the assumptions, we have: $b_{12} = b_{21} < 0$, and $b_{34} \neq b_{43}$. Without loss of generality, we may assume $b_{34} \neq 0$ (otherwise we exchange the indexes 3 and 4).

We also have that $B_{12} = (B_{21})^t$ is an entrywise positive matrice.

Besides $B^{(4)}$ is 1-positive and irreducible. Note that the cycle $(1,2,3)$ of $B^{(4)}$ is a negative cycle. Thanks to Theorem 3.22 and Lemma 3.2, one concludes that $B^{(4)}$ must be diagonally similar to a symmetric matrix. This implies $b_{23}b_{31} = b_{13}b_{32}$. Using the same argument for $B^{(3)}$, we obtain: $b_{24}b_{41} = b_{14}b_{42}$.

From Lemma 3.14, $B^{(4)} + x\bar{B}^{(4)}$ is also 1-positive for any $x > 0$.

For x small enough, the cycle $(1,2,3)$ of $B^{(4)} + x\bar{B}^{(4)}$ is negative. Then, from Theorem 3.22 and Lemma 3.2, this cycle must be symmetric. This implies that for x small enough

$$(b_{12}+xb_{14}b_{42})(b_{23}+xb_{24}b_{43})(b_{31}+xb_{34}b_{41}) = (b_{13}+xb_{14}b_{43})(b_{32}+xb_{34}b_{42})(b_{21}+xb_{24}b_{41})$$

and hence for every real x.

We have two polynomials that must have the same roots with same multiplicity, which leads to

$$\left\{\frac{b_{14}b_{42}}{b_{12}}, \frac{b_{24}b_{43}}{b_{23}}, \frac{b_{34}b_{41}}{b_{31}}\right\} = \left\{\frac{b_{14}b_{43}}{b_{13}}, \frac{b_{34}b_{42}}{b_{32}}, \frac{b_{24}b_{41}}{b_{21}}\right\}$$

where the equality is between multisets (the multiplicity is taken into account).

As $\dfrac{b_{14}b_{42}}{b_{12}} = \dfrac{b_{24}b_{41}}{b_{21}}$ and $\dfrac{b_{24}b_{43}}{b_{23}} \neq \dfrac{b_{34}b_{42}}{b_{32}}$ (indeed $b_{43} \neq b_{34}$ and $(b_{24}, b_{23}) = (b_{42}, b_{32})$), one shows that:

$$\frac{b_{24}b_{43}}{b_{23}} = \frac{b_{14}b_{43}}{b_{13}} \quad \text{and} \quad \frac{b_{34}b_{41}}{b_{31}} = \frac{b_{34}b_{42}}{b_{32}}.$$

Hence, one obtains: $\dfrac{b_{23}}{b_{13}} = \dfrac{b_{24}}{b_{14}}$, which implies that

$$\operatorname{rank}\begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} = 1.$$

□

We have mentioned that Theorem 3.22 does not consider 3×3-matrices which are diagonally similar to a symmetric matrices. The proposition below shows that symmetric 1-positive matrices are not necessarily positive semi-definite.

Proposition 3.32. *For $\alpha, \beta, \gamma > 0$, define the matrix*

$$A = \begin{pmatrix} 1 & -\alpha & \beta \\ -\alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}, \tag{3.29}$$

If $\alpha \leq 1$, or $\beta \leq 1$, or $\gamma \leq 1$, then A is 1-positive.

Proof. If $\alpha \leq 1$, the proposition is a special case of Theorem 3.26 with $d = 3$ and $a_{12} = a_{21} = -\alpha$, $a_{13} = a_{31} = \beta$, $a_{23} = a_{32} = \gamma$ and $A_{22} = (1)$.

Besides, the roles of α, β and γ are symmetric. Indeed, a matrix in the form (3.29) is diagonally similar to $\begin{pmatrix} 1 & \alpha & -\beta \\ \alpha & 1 & \gamma \\ -\beta & \gamma & 1 \end{pmatrix}$. Then by exchanging together the first and the second rows and the first and the second column, we can see that the parts of α and β can be exchanged. In the same way the parts of α and γ can be exchanged. The parts of β and γ can also be exchanged by simply intertwining together the second and third rows and the second and third columns.
Hence, if we have $\beta \leq 1$ or $\gamma \leq 1$, we obtain the same conclusion: A is 1-positive. □

The following theorem is an extension of Theorem 3.26 which corresponds to the case $n_1 = dim(A_{11})$ and $n_i = 1$, for every i in $\{2, 3, ..., d\}$.

Theorem 3.33. *Let B be a $n \times n$ written as the following block matrix:*

$$B = \left(B_{ij}\right)_{1 \leq i,j \leq d}$$

where for every i, j in $[\![d]\!]$, B_{ij} is a $n_i \times n_j$ matrix (the nonnegative integers $n_1, \ldots, n_d$ are such that $n_1 + \cdots + n_d = n$).
If B satisfies the three following conditions:

(i) For any i in $[\![d]\!]$, B_{ii} is positive semi-definite.

(ii) For any i, j in $[\![d]\!]$, if $i \neq j$ then B_{ij} is a nonnegative matrix.

(iii) For any i in $[\![d]\!]$, the $n_i \times 2(n - n_i)$ matrix written with $2(d-1)$ blocks of columns $(B_{ij}, (B_{ji})^t)_{\substack{1 \leq j \leq d \\ j \neq i}}$ has rank 1.

then B is 1-positive.

Proof. Denote by b_{ij} the (i, j)-entry of B.

Step 1: We first establish Theorem 3.33 under the assumption that none of the off-diagonal blocks of B has a zero entry (i.e., for every i, j, $i \neq j$, B_{ij} has no zero entry).

We prove that there exists a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ with positive diagonal entries, such that $B = DCD$, with $C = (c_{ij})_{1 \leq i,j \leq n} = \Big(C_{ij}\Big)_{1 \leq i,j \leq d}$, where:

(1) For every i, j in $[\![d]\!]$, C_{ij} is a $n_i \times n_j$-matrix.

(2) For every i in $[\![d]\!]$, C_{ii} is positive semi-definite.

(3) For any i, j in $[\![d]\!]$, if $i \neq j$ then all the entries of C_{ij} are equal and positive.

For $1 \leq l \leq n_1$, set: $d_l = b_{n_1+1\, l}$. For $n_1 < l \leq n$ set: $d_l = b_{1l}$. Define C by:

$$C = D^{-1} B D^{-1}.$$

For $1 \leq l \leq n_1$: $c_{n_1+1, l} = \dfrac{1}{b_{1, n_1+1}}$, which is positive and does not depend on l.

For $n_1 < l \leq n$: $c_{1\, l} = \dfrac{1}{b_{n_1+1, 1}}$, which is positive and does not depend on l.

As for all $i \in [\![d]\!]$, the $n_i \times 2(n - n_i)$ matrix made of $2(d-1)$ blocks of columns $(B_{ij}, (B_{ji})^t)_{\substack{1 \leq j \leq d \\ j \neq i}}$ has rank 1, hence $(C_{ij}, (C_{ji})^t)_{\substack{1 \leq j \leq d \\ j \neq i}}$ has rank 1 as well.

Since $c_{n_1+1,i}$ $(1 \le i \le n_1)$ is positive and does not depend on i and $(C_{1j}, (C_{j1})^t)_{2\le j\le d}$ has rank 1, then for any $j \in [\![2, n]\!]$, C_{1j} has identical rows with positive entries and C_{j1} has identical columns with positive entries.

As $c_{1\,l}$, for $n_1 < l \le n$, is positive and does not depend on l and as, for any $i \in [\![2, d]\!]$, $(C_{ij}, (C_{ji})^t)_{\substack{1\le j\le d\\ j\ne i}}$ has rank 1, and this implies that, for any $j \in [\![n]\!]\backslash\{i\}$, C_{ij} has identical rows with positive entries and C_{ji} has identical columns with positive entries.

Finally, for any i, j in $[\![d]\!]$ such that $i \ne j$, C_{ij} has identical rows and columns, hence all its entries are identical and positive. As B_{ii} (i in $[\![d]\!]$) is positive semi-definite, so is C_{ii}. Therefore C satisfies (1), (2) and (3).

To prove that the matrix B is 1-positive, it is sufficient to prove that C is 1-positive. Since C satisfies (2) and (3), so does $C[n_1, \dots, n_n]$. Consequently it is sufficient to prove that $\operatorname{per} C \ge 0$.

For any $i \ne j$ $(1 \le i, j \le d)$, denote by γ_{ij} the constant value of the entries of the matrix C_{ij}.

From (3.6), we have

$$\begin{aligned}
\operatorname{per} C &= \sum_{\substack{\sum_i k_{ij}=n_j\\ \sum_j k_{ij}=n_i}} \; \sum_{\substack{|I_{ij}|=|J_{ij}|=k_{ij}\\ \cup_i J_{ij}=[\![n_j]\!]\\ \cup_j I_{ij}=[\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} C_{ij}[I_{ij} \times J_{ij}] \right) \\
&= \sum_{\substack{\sum_i k_{ij}=n_j\\ \sum_j k_{ij}=n_i}} \left(\prod_{\substack{i,j=1\\ i\ne j}}^{d} k_{ij}!\,(\gamma_{ij})^{k_{ij}} \right) \sum_{\substack{I_{ii},J_{ii}\subset[\![n_i]\!]\\ |I_{ii}|=|J_{ii}|=k_{ii}\\ 1\le i\le d}} \; \sum_{\substack{\forall i\ne j, |I_{ij}|=|J_{ij}|=k_{ij}\\ \cup_{i\ne j} J_{ij}=[\![n_j]\!]\backslash J_{jj}\\ \cup_{j\ne i} I_{ij}=[\![n_i]\!]\backslash I_{ii}}} \left(\prod_{i=1}^{d} \operatorname{per} C_{ii}[I_{ii} \times J_{ii}] \right)
\end{aligned}$$

For fixed k_{ij} $(1 \le i, j \le d)$,

$$\Delta = \#\{(I_{ij}, J_{ij})_{\substack{1\le i,j\le d\\ i\ne j}} : \forall i \ne j, |I_{ij}| = |J_{ij}| = k_{ij}\,;\, \forall j, \cup_{i\ne j} J_{ij} = [\![n_j]\!]\backslash J_{jj}\,;\, \forall i, \cup_{j\ne i} I_{ij} = [\![n_i]\!]\backslash I_{ii}\}$$

does not depend on the choice of I_{ii}, J_{ii} with the conditions $|I_{ii}| = |J_{ii}| = k_{ii}$

$(1 \leq i \leq d)$. Hence we have

$$\begin{aligned}
\operatorname{per} C &= \sum_{\substack{\sum_i k_{ij}=n_j \\ \sum_j k_{ij}=n_i}} \left(\Delta \prod_{\substack{i,j=1 \\ i \neq j}}^{d} k_{ij}! \, (\gamma_{ij})^{k_{ij}} \right) \sum_{\substack{I_{ii}, J_{ii} \subset [\![n_i]\!] \\ |I_{ii}|=|J_{ii}|=k_{ii} \\ 1 \leq i \leq d}} \left(\prod_{i=1}^{d} \operatorname{per} C_{ii}[I_{ii} \times J_{ii}] \right) \\
&= \sum_{\substack{\sum_i k_{ij}=n_j \\ \sum_j k_{ij}=n_i}} \left(\Delta \prod_{\substack{i,j=1 \\ i \neq j}}^{d} k_{ij}! \, (\gamma_{ij})^{k_{ij}} \right) \left(\prod_{i=1}^{d} \left(\sum_{\substack{I,J \subset [\![n_i]\!] \\ |I|=|J|=k_{ii}}} \operatorname{per} C_{ii}[I \times J] \right) \right),
\end{aligned}$$

which is nonnegative, because for any i in $[\![d]\!]$ $(\operatorname{per} C_{ii}[I \times J])_{I,J \subset [\![n_i]\!] : |I|=|J|=k_{ii}}$ is positive semi-definite (see the argument developed in Step 2 of the proof of Theorem 3.26).

Step 2: To relax the assumption of no zero entry in the off-diagonal blocks, we show now that B is the limit as ϵ tends to 0 of matrices B_ϵ such that for every $\epsilon > 0$: $B_\epsilon = ((B_\epsilon)_{ij})_{1 \leq i,j \leq d}$, where the matrix $(B_\epsilon)_{ij}$ has the same size as B_{ij} and

- For any i in $[\![d]\!]$, $(B_\epsilon)_{ii}$ is positive semi-definite.
- For any i, j in $[\![d]\!]$, if $i \neq j$ then $(B_\epsilon)_{ij}$ has only positive entries.
- For any i in $[\![d]\!]$, the $n_i \times 2(n - n_i)$ matrix written with $2(d-1)$ blocks of columns $((B_\epsilon)_{ij}, ((B_\epsilon)_{ji})^t)_{\substack{1 \leq j \leq d \\ j \neq i}}$ has rank 1.

Apart from being positive semi-definite, the matrices $B_{ii}, i \in [\![d]\!]$ have no part in the proof. Hence without loss of generality, we may assume that for i in $[\![d]\!]$, $B_{ii} = 0$,

Assume than B has at least one zero entry in an off-diagonal block.

From B, we now build a matrix $B_\epsilon^{(1)}$ that has a number of zero entries strictly smaller than the number of zero entries of B, satisfies (i) and (ii) and such that $(B_\epsilon^{(1)})_{\epsilon > 0}$ converges to B as ϵ tends to 0.

There exist $i_0, j_0 \in [\![d]\!]$ with $i_0 \neq j_0$ such that $B_{i_0 j_0}$ has a zero entry, denote by (k_0, l_0) the indices in B of this zero entry : $b_{k_0 l_0} = 0$.

We are always in one of the three following cases:
case 1 - $b_{k_0 l} = b_{l k_0} = 0, \forall l \in [\![n]\!]$

case 2 - $b_{kl_0} = b_{l_0k} = 0, \forall k \in [\![n]\!]$
case 3 - $B_{i_0j_0} = 0$
Indeed, suppose that we are not in case 1 nor in case 2, then as $\text{rank}(B_{i_0j}, (B_{ji_0})^t)_{\substack{j\in[\![d]\!]\\ j\neq i_0}} = \text{rank}(B_{j_0j}, (B_{jj_0})^t)_{\substack{j\in[\![d]\!]\\ j\neq j_0}} = 1$, we have $b_{kl_0} = 0$ for all $k \in [\![1 + \sum_{q=1}^{i_0-1} n_q, \sum_{q=1}^{i_0} n_q]\!]$ and $b_{k_0l} = 0$ for all $l \in [\![1 + \sum_{q=1}^{j_0-1} n_q, \sum_{q=1}^{j_0} n_q]\!]$. We also have that there exists k in $[\![d]\!]$ such that $b_{kl_0} \neq 0$ or $b_{l_0k} \neq 0$. As $\text{rank}(B_{j_0j}, (B_{jj_0})^t)_j = 1$, this implies $B_{i_0j_0} = 0$, which is case 3.

If we are in case 1, there exists k in $[\![n_{i_0}]\!]$ such that the k^{th} row of the matrix $(B_{i_0j}, (B_{ji_0})^t)_{1\leq j\leq d}$ is nonzero. $B_\epsilon^{(1)}$ is obtained from B by replacing its k_0^{th} row by the $(\sum_{q=1}^{i_0-1} n_q + k)^{th}$ row of B multiplied by ϵ, and its k_0^{th} column by the $(\sum_{q=1}^{i_0-1} n_q + k)^{th}$ column of B multiplied by ϵ. With this definition, it is easy to verify that $B_\epsilon^{(1)}$ has the properties (i) and (ii), and the number of its zero entries is strictly smaller than the number of zero entries of B.

If we are in case 2, we do a similar construction, with l_0 instead of k_0.

If we are in case 3, $B_\epsilon^{(1)}$ is obtained from B by replacing the submatrix $B_{i_0j_0}(= 0)$ by $\epsilon K_{i_0} \times (K_{j_0})^t$, where K_{i_0} is a nonzero column of the matrix written with blocks of columns $(B_{i_0j}, (B_{ji_0})^t)_{1\leq j\leq d}$ and K_{j_0} is a nonzero column of the matrix written with blocks of columns $(B_{j_0j}, (B_{jj_0})^t)_{1\leq j\leq d}$. Note that $B_\epsilon^{(1)}$ satisfies Conditions (i) and (ii), and that the number of its zero entries is strictly smaller than the number of zero entries of B.

Moreover, in each case, $B_\epsilon^{(1)}$ tends to B when ϵ tends to 0.

The submatrices of $B_\epsilon^{(1)}$, $(B_\epsilon^{(1)})_{ij}, 1 \leq i,j \leq d$, are defined such that $B_\epsilon^{(1)} = ((B_\epsilon^{(1)})_{ij})_{1\leq i,j\leq d}$ and for every i,j, $(B_\epsilon^{(1)})_{ij}$ has the same size as the submatrix B_{ij}.

Define by induction $B_\epsilon^{(p)}$ from $B_\epsilon^{(p-1)}$, exactly as $B_\epsilon^{(1)}$ has been defined from B. This construction requires that $B_\epsilon^{(p-1)}$ has at least one zero entry in an off-diagonal block. We stop the construction at the first index $p_o \geq 1$ such that none of the off-diagonal block of $B_\epsilon^{(p_o)}$ has a zero entry. Set then $B_\epsilon = B_\epsilon^{(p_o)}$.

The matrix B_ϵ satisfies the three announced points, and as such is 1-positive thanks to Step 1. Since B_ϵ tends to B as ϵ tends to 0, B is 1-positive. □

Corollary 3.34. *Let B be a matrix satisfying all the assumptions of Theorem 3.33. Assume moreover that for any $i \neq j$, B_{ij} has no zero entry. Then there* *exists* $\gamma > 0$ *such that* $B + \gamma I$ *is 1-permanental.*

Proof. To use Proposition 3.16, thanks to (3.14), it is sufficient to prove that for $\alpha > 0$ small enough, $B(\alpha) = B(I + \alpha B)^{-1}$ verifies also Conditions (i), (ii) and (iii) of Theorem 3.33.

As in the proof of Corollary 3.29, for $\alpha > 0$ small enough, $B(\alpha)_{11}$ is symmetric and the matrix written by block of columns $(B(\alpha)_{1j}, (B(\alpha)_{j1})^t)_{2\leq j\leq d}$ has rank 1. Similarly, for any i in $[\![d]\!]$ and for $\alpha > 0$ small enough, $B(\alpha)_{ii}$ is symmetric and the matrix written by block of columns $(B(\alpha)_{ij}, (B(\alpha)_{ij})^t)_{\substack{1\leq j\leq d \\ j\neq i}}$ has rank 1. Then, as $B(\alpha)$ tends to B when α tends to 0, $B(\alpha)$ fulfills Conditions (i), (ii) and (iii) for α small enough.

Corollary 3.34 is hence a consequence of Theorem 3.33 and Proposition 3.16. $\square$

We want now to apply the construction of the previous chapter to regularity structures related with subcritical SPDEs. First we have to select subsets of labelled trees (or forests): indeed the set of all labelled trees is far too big and the requirement that the homogeneities be bounded below is violated since we can take arbitrary powers of a tree with negative homogeneity. However on suitable (admissible) subspaces we can define regularity structures which satisfy the properties of Definition 1.2.1.

Bibliography

[AKR98] S. Albeverio, Y. G. Kondratiev, M. Röckner: *Analysis and geometry on configuration spaces*, J. Funct. Anal. **154**, no. 2, 444–500 (1998).

[AGZ10] G.W. Anderson, A. Guionnet, O. Zeitouni: *An introduction to random matrices*, Cambridge university press (2003).

[Ari98] J. Arias-de-Reyna. *Gaussians variables, polynomials and permanents*, Lin. Alg. Appl. **285**, 107–114 (1998).

[Bap89] R.B. Bapat: *Infinite divisibility of multivariate gamma distributions and M-matrices*, The Indian Journal of Statistics **51**, Series A, Pt. 1, pp. 73-78 (1989).

[Bat97] R. Bathia: *Matrix analysis*, Springer, New York (1997).

[BST98] C. Benítem, Y. Sarantopolous, A.M. Tonge. *Lower bounds for norms of products of polynomials*, Math. Proc. Camb. Phil. Soc. **124**, 35-408 (1998).

[Brä12] P. Bränden: *Solution to two problems on permanents*, Linear Algebra and its Applications **436**, 53–58 (2012).

[BN66] R.A. Brualdi, M. Newman: *Proof of a permanental inequality*, Quart. J. Math. Oxford, Ser. (2) **17** 234—238 (1966).

[Cho69] G. Choquet: *Deux exemples classiques de représentation intégrale*, Enseignement Math. (2) 15, 3-75 (1969).

[Chu97] W.C. Chu: *Determinant, permanent and MacMahon Master Theorem*, Linear Algebra Appl. **255**, 171–183 (1997).

[DVJ03a] D.J. Daley, D. Vere-Jones: *An introduction to the theory of point processes - Vol. I: Elementary theory and methods*, 2^{nd} ed., Springer, New York (2003).

[DVJ03b] D.J. Daley, D. Vere-Jones: *An introduction to the theory of point processes - Vol. II: General theory and structures*, 2^{nd} ed., Springer, New York (2008).

[Dok69] D.Z. Djokovic, Simple proof of a theorem on permanents, Glasgow Math. J. **10**, 52–54 (1969).

[Dyn83] E.B. Dynkin: *Local times and quantum fields*, Seminar on Stochastic processes, 64-84, Birkhauser (1983).

[Eis13] N. Eisenbaum: *Inequalities for permanental processes*, Electron. J. Probab. **18**, no. 99, 1-15 (2013).

[Eis14] N. Eisenbaum: *Characterization of positively correlated squared Gaussian processes*, Annals of Probab. **42**, 559–575 (2014).

[Eis16] N. Eisenbaum: *Permanental vectors with nonsymmetric kernels*, Annals of Probab. **45**, no. 1, 210–224 (2017).

[EK09] N. Eisenbaum, H. Kaspi: *On permanental processes*, Stochastic Process. Appl. **119**, no. 5, 1401-1415 (2009).

[FM88] M. Fiedler and T.L. Markham: *A characterization of the closure of inverse M-matrices*, Linear Algebra Appl. **105**, 209–223 (1988).

[FZ88] D. Foata and D. Zeilberger: *Laguerre polynomials, weighted derangements, and positivity*, SIAM J. Discrete Math. **1**, 425-–433 (1988).

[Fre07] P.E. Frenkel: *Pfaffians, hafnians and products of real linear functionals*, Math. Res. Lett. **15**, no. 2, 351—358 (2007).

[Fre10] P.E. Frenkel: *Remarks on the α-permanent*, Math. Res. Lett. **17**, no. 4, 795–802 (2010)

[Gib66] P.M. Gibson: *A short proof of an inequality for the permanent function*, Proc. Amer. Math. Soc. **17**, 535–536 (1966).

[GN03] A. Goldberger, M. Neuman: *Multilinear functional inequalities, and other multilinear functions of nonnegative matrices and M-matrices*, Linear Algebra Appl. **369**, 295-310 (2003).

[Gri84] R.C. Griffiths: *Characterization of infinitely divisible multivariate gamma distributions*, J. Multivariate Anal. **15**, 13–20 (1984).

[GM87] R.C. Griffiths, R.K. Milne: *A class of infinitely divisible multivariate negative binomial distributions*, J. Multivariate Anal. **22**, 13–23 (1987).

[HKPV06] J.B. Hough, M. Krishnapur, Y. Peres, B. Virág: *Determinantal processes and independence*, Probab. Surv. **3**, 206–229 (2006).

[HL84] D.J. Hartfiel, R. Loewy: *On matrices having equal corresponding principal minors*, Linear Algebra and Appl. **58**, 147–167 (1984).

[JS11] C.R. Johnson and R. Smith: *Inverse M-matrices, II*, Linear Algebra Appl. **435**, 953–983 (2011).

[JS01] C.R. Johnson and R. Smith: *Path product matrices*, Linear and Multilinear Algebra **330**, 177–191 (2001).

[KM12] H. Kogan and M. B. Marcus: *Permanental vectors*, Stochastic processes and their applications **122**, 1226–1247 (2012).

[KM10] H. Kogan, M. B. Marcus and J. Rosen: *Permanental processes*, Comm. on Stochastic Analysis **5**, 81–102 (2011).

[Len73] A. Lenard: *Correlation functions and the uniqueness of the state in classical statistical mechanics*, Comm. Math. Phys. **30**, 35-–44 (1973).

[Len75] A. Lenard: *States of classical statistical mechanical systems of infinitely many particles I-II*, Arch. Rational Mech. Anal. **59**, 219—256 (1975).

[Lie66] E.H. Lieb: *Proofs of some conjectures on permanents*, J. Math. Mech. **16** 127–134 (1966).

[LW09] W. Li and A. Wei: *A Gaussian inequality for expected absolute products*, J. Theoret. Probab. **25**(1), 92—99, (2012).

[Loe86] R. Loewy: *Principal minors and and diagonal similarity of matrices*, Linear Algebra and Appl. **78**, 23–64 (1986).

[Mac75] O. Macchi: *The coincidence approach to stochastic point processes*, Adv. in Appl. Probab. **7**, 83–122 (1975).

[MNPP16] D. Malicet, I. Nourdin, G. Peccati, G. Poly. *Squared chaotic random variables: New moment inequalities with applications*, J. Funct. Anal. **270**, 649–670 (2016).

[MR13] M. Marcus and J. Rosen: *A sufficient condition for the continuity of permanental processes with applications to local times of Markov processes*, Ann. Prob. **41**, 671-698 (2013).

[May65] J. Maybee: *Remarks on the theory of cycles in matrices*, Purdue University, Department of Statistics (1965).

[MM06] P. Mccullagh, J. Møller: *The permanental processes*, Adv. Appl. Prob. **38** (2006)

[Roy14] T. Royen: *A simple proof of the Gaussian correlation conjecture extended to multivariate gamma distributions*, Far East J. Theor. Statist. **48**, no. 2, 139-145 (2014)

[SS14] A.D. Scott and A.D. Sokal: *Complete monotonicity for inverse powers of some combinatorially defined polynomials*, Acta Math. 213, 323-392 (2014).

[Shi07] T. Shirai: *Remarks on the positivity of α-determinants*, Kyushu J. Math. **61**, 169–189 (2007).

[ST03] T. Shirai, Y. Takahashi, *Random point fields associated with certain Fredholm determinants. I: fermion, Poisson and boson point processes*, J. Funct. Anal. **205**, no. 2, 414–463 (2003).

[Sos00] A. Soshnikov: *Determinantal random point fields*, Russian Math. Surveys **55**, 923–975 (2000).

[Sos02] A. Soshnikov: *Gaussian limit for determinantal random point fields*, Ann. Prob. **30**, 171-187 (2002).

[VeJ88] D. Vere-Jones: *A generalization of permanents and determinants*, Linear Algebra Appl. **111**, 119-124 (1988).

[VeJ97] D. Vere-Jones: *Alpha-permanents and their applications to multivariate gamma, negative binomial and ordinary binomial distributions*, New Zealand J. Math. **26**, 125-149 (1997).

[Wid41] D.V. Widder: *The Laplace Transform*, Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., (1941).