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Instabilité et croissance des normes de Sobolev pour certaines EDP hamiltoniennes

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MATHEMATIQUES ORSAY

Instabilité et croissance des normes de Sobolev pour certaines EDP hamiltoniennes

Résumé. Cette thèse est consacrée à l'étude de solutions globales et régulières de certaines EDP hamiltoniennes, du point de vue de la croissance de leurs normes de Sobolev. Un tel phénomène traduit une modification de la répartition de l'énergie dans l'espace des fréquences, appelée parfois « turbulence faible ». On étudie d'abord une équation d'évolution non-linéaire où intervient un laplacien fractionnaire, et l'on prouve des estimées a priori sur la vitesse de croissance des normes de Sobolev. On introduit ensuite une équation où de telles estimées sont optimales : une équation de Szegő, intégrable, avec une non-linéarité quadratique, et où certaines solutions régulières croissent à vitesse exponentielle tout en restant bornées dans l'espace d'énergie. On classifie les ondes progressives de cette équation de Szegő quadratique, et l'on met en évidence l'instabilité d'une partie d'entre elles. Enfin, on exhibe pour cette équation une hiérarchie de lois de conservation, qui permet d'étudier plus précisément les solutions rationnelles turbulentes.

Mots-clés. EDP hamiltonienne, complète intégrabilité, transfert d'énergie vers les hautes fréquences, équation de Szegő, paire de Lax, ondes progressives

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Instability and growth of Sobolev norms for certain Hamiltonian PDEs

Abstract. In this thesis we study global smooth solutions of certain Hamiltonian PDEs, in order to capture the possible growth of their Sobolev norms. Such a phenomenon is typical for what is sometimes called "weak turbulence" : a change in the distribution of energy between Fourier modes. We first study a nonlinear evolution equation involving a fractional Laplacian, and we prove a priori estimates on the growth of Sobolev norms. We then introduce an equation where these estimates turn out to be optimal : an integrable Szegő equation with a quadratic nonlinearity, which admits exponentially growing smooth solutions that remain bounded in the energy space. We classify the traveling wave solutions of this quadratic Szegő equation, and show that some of them are unstable. Eventually we find a hierarchy of conservation laws for this equation, which leads us into a deeper study of rational turbulent solutions.

Keywords. Hamiltonian PDE, complete integrability, high-frequency energy transfer, Szegő equation, Lax pair, traveling waves

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Chapitre 1

Introduction

La présente introduction se donne pour but de replacer dans leur contexte mathématique les résultats de cette thèse. On trouvera ces derniers énoncés dans la Proposition 1.1, le Théorème 1, les Théorèmes 5, 6, 7 et 8, ainsi que la Proposition 2.6.

1 Panorama général

1.1 La question de la croissance des normes de Sobolev

Pendant un demi-siècle, la théorie mathématique des équations aux dérivées partielles d'évolution non-linéaires s'est consacrée en priorité à l'étude de l'existence de solutions, locales ou globales, dans des espaces fonctionnels bien choisis, et aussi vastes que possible. Étant donnés une équation dépendant de la variable temporelle $t \in \mathbb{R}$, et un état « initial » dans un espace fonctionnel X, existe-t-il une fonction du temps à valeurs dans X qui coïncide avec l'état initial pour $t = t_0$, et qui obéisse à l'équation (en un sens à préciser), que ce soit à tout instant, ou au moins sur un intervalle de temps borné ? Grâce aux outils de l'analyse fonctionnelle, des notions de solution faible ont pu voir le jour, permettant de définir des trajectoires dans des espaces de fonction de basse régularité, et soulevant des questions d'unicité encore loin d'être résolues (comme en témoignent les recherches autour de l'équation de Navier-Stokes, par exemple).

Néanmoins, les avancées de cette théorie ont permis d'envisager sans présomption d'autres types de questionnements, en particulier celui du comportement qualitatif des solutions, une fois leur existence établie. À ce titre, une question naturelle est celle de l'instabilité : peut-on partir d'un état initial « sage » (typiquement, une fonction lisse, à support compact, type « bosse »), et observer au fil du temps l'apparition de phénomènes nouveaux (typiquement, l'apparition de fortes oscillations, ou de pics de plus en plus grands) ?

Pour donner une formulation plus précise à cette question, plaçons-nous dans un cadre volontairement restreint, inspiré des problèmes qu'a formulés Bourgain en l'an 2000 [8, Section 5]. Soit $d \in \{1, 2, 3\}$, et considérons l'équation de Schrödinger avec non-linéarité cubique défocalisante :

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta_{\mathbb{T}^d} u = |u|^2 u, \\ \qquad\qquad u(0) = u_0 \in X, \end{cases} \tag{1.1}$$

où $u : t \in \mathbb{R} \mapsto u(t) \in X$, où X désigne un espace de fonctions à valeurs complexes sur le tore de dimension d noté $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$, et où

$$\Delta_{\mathbb{T}^d} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

est le laplacien usuel. Dans les années 80 et 90, les théories d'existence ont permis d'établir (voir [9] pour $d = 2$ et [6] pour $d = 3$) que si l'état initial $u_0 \in C^\infty(\mathbb{T}^d, \mathbb{C})$, alors la solution u est définie de manière unique et garde sa régularité C^∞ pour tout $t \in \mathbb{R}$. Dès lors, la question de l'apparition d'instabilité peut être transcrite de la manière suivante : existe-t-il des topologies sur $C^\infty(\mathbb{T}^d)$ où l'orbite de u est bornée, et d'autres où elle ne l'est pas ? En particulier, il est bien connu que les solutions lisses de l'équation (1.1) obéissent à une loi de conservation de l'énergie :

$$\frac{1}{2}\int_{\mathbb{T}^d} |\nabla u(t)|^2_{\mathbb{C}^d} + \frac{1}{4}\int_{\mathbb{T}^d} |u(t)|^4 = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u_0|^2_{\mathbb{C}^d} + \frac{1}{4}\int_{\mathbb{T}^d} |u_0|^4, \quad \forall t \in \mathbb{R}. \tag{1.2}$$

En introduisant la topologie H^s, $s \in \mathbb{R}$, via la norme de Sobolev d'indice s définie pour $v \in C^\infty$ par

$$\|v\|^2_{H^s} := \sum_{k \in \mathbb{Z}^d} (1 + |k|^2_{\mathbb{C}^d})^s |\hat{v}(k)|^2,$$

où $\hat{v}(k)$ désigne le k^{e} coefficient de Fourier de v, nous voyons donc que les solutions lisses de (1.1), grâce à la conservation de l'énergie, restent bornées dans la topologie H^1. D'où la question de Bourgain :

Question (de la croissance des normes de Sobolev)**.** Soit $s > 1$. Existe-t-il un état initial $u_0 \in C^\infty(\mathbb{T}^d)$ tel que la solution correspondante vérifie $\|u(t)\|_{H^s} \to +\infty$ lorsque $t \to \pm\infty$?

Cette question appelle de nombreux commentaires :

- *Signification physique.* En quoi la croissance d'une norme H^s relève-t-elle d'un phénomène d'instabilité tel que nous l'avons esquissé ? Nous empruntons l'explication à Tao [60, Paragraphe 3.9] : supposons qu'il existe une certaine solution de (1.1) notée u telle que, pour un $s > 1$ et une suite de temps $\{t_n\}$ tendant vers l'infini, l'on ait $\|u(t_n)\|_{H^s} \geq n$. Puisque la norme H^1, elle, demeure bornée, on a donc à la fois

  $$\sum_{k \in \mathbb{Z}^d} (1 + |k|^2_{\mathbb{C}^d}) |\widehat{u(t_n)}(k)|^2 \leq C,$$
  $$\sum_{k \in \mathbb{Z}^d} (1 + |k|^2_{\mathbb{C}^d})^s |\widehat{u(t_n)}(k)|^2 \geq n^2,$$

  ce qui signifie qu'en se répartissant selon les différents modes, la suite $\widehat{u(t_n)}$ se concentre petit à petit sur les modes pour lesquels $(1 + |k|^2_{\mathbb{C}^d})^s \gg 1 + |k|^2_{\mathbb{C}^d}$ (c'est-à-dire tels que $|k|_{\mathbb{C}^d} \gg 1$), de façon à faire grandir la seconde somme tout en garantissant que la première reste bornée. Une réponse positive à la question de Bourgain relève donc de ce que les physiciens appellent la *turbulence d'onde* (ou turbulence faible) : un phénomène qui se traduit par une transition de l'énergie vers les hautes fréquences, et donc, qualitativement, l'apparition de « petites échelles », d'oscillations dont la longueur d'onde devient infime en regard de la longueur d'onde typique de u_0.

- *Régularité.* Pour simplifier l'exposition, nous avons eu recours, dans la formulation de la question, à un degré de régularité C^∞, qui permet de définir toutes les normes de Sobolev à chaque instant. Néanmoins, la question de la croissance des normes de Sobolev n'est pas réservée à ce type de donnée initiale, et nous pouvons l'étendre à tout niveau de régularité si nous établissons que l'équation considérée, comme c'est le cas pour (1.1), satisfait une propriété de *persistance de la régularité* : le flot dans l'espace d'énergie H^1 est globalement défini (remarquons que, dans cet espace, l'énergie (1.2) est bien définie, grâce à l'injection de Sobolev $H^1(\mathbb{T}^d) \hookrightarrow L^4(\mathbb{T}^d)$, valable pour $d \in [\![1,4]\!]$), et si $u_0 \in H^s$ pour un certain $s \in]1, s_*]$, alors la solution au sens H^1 reste dans H^s pour tout temps. Pour une telle solution, on peut alors se demander ce qu'il advient des normes $H^{s'}$, avec $1 < s' \leq s$, et si une croissance de ces normes est possible.

D'autre part, si l'on peut définir des solutions globales sous l'espace d'énergie, c'est-à-dire pour un H^s avec $s < 1$, alors la conservation de l'énergie (1.2) n'a plus de sens (puisque l'énergie n'est plus définie), mais la question de la croissance des normes de Sobolev demeure, et peut être posée pour les normes $H^{s'}$ avec $0 < s' \leq s$, en remarquant que la norme L^2 (ou H^0) des solutions de (1.1) est conservée :

$$\frac{1}{2}\int_{\mathbb{T}^d} |u(t)|^2 = \frac{1}{2}\int_{\mathbb{T}^d} |u_0|^2, \quad \forall t \in \mathbb{R}. \tag{1.3}$$

- *Vitesse de croissance.* La question de la croissance des normes de Sobolev peut être prise « à l'envers » : supposons qu'une solution $u(t)$ de (1.1) croisse en norme H^s pour un $s > 1$; quelles sont alors les vitesses de croissance possibles ? On parle alors d'estimées *a priori* sur les solutions. En utilisant les propriétés de dispersion du laplacien (sur lesquelles nous reviendrons plus loin), Bourgain [7] et Staffilani [58] ont montré qu'il existe une constante C dépendant seulement de $\|u_0\|_{H^1}$ et de d, telle que si $u_0 \in H^s$ pour un certain $s > 1$, alors

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s}(1+|t|)^{C(s-1)}, \quad \forall t \in \mathbb{R}.$$

  On parle alors de croissance *au plus polynomiale* : c'est une borne supérieure sur le taux de croissance des normes de Sobolev. Bien sûr, les méthodes utilisées pour prouver un tel résultat sont de type « condition nécessaire », et ne disent pas si une telle borne est possible à atteindre en pratique. D'ailleurs, Bourgain [8] conjecture plutôt que

$$\|u(t)\|_{H^s} \ll \|u_0\|_{H^s}(1+|t|)^{\varepsilon}, \quad \forall \varepsilon > 0,$$

  lorsque $|t| \to +\infty$, ce que l'on peut qualifier de croissance *sous-polynomiale.*

- *Croissance intermittente.* Un autre raffinement que l'on peut apporter à la question de la croissance des normes concerne sa permanence : si l'on remplace $\|u(t)\|_{H^s} \to +\infty$ par

$$\limsup_{|t|\to+\infty} \|u(t)\|_{H^s} = \infty, \tag{1.4}$$

  on autorise alors divers comportements selon la suite de temps considérée, et une variété beaucoup plus grande de comportements dynamiques qualitatifs. En particulier, le phénomène (1.4) peut coexister avec $\liminf_{|t|\to+\infty} \|u(t)\|_{H^s} < \infty$. Dans ce cas, le long d'une même trajectoire, l'énergie initialement emmagasinée dans les basses fréquences commence par migrer vers les hautes fréquences, faisant croître momentanément la norme H^s (on parle alors de *cascade ascendante* — en anglais *low-to-high frequency cascade*), puis elle est ensuite transférée à nouveau vers les basses fréquences (on parle cette fois de *cascade descendante* — en anglais *high-to-low frequency cascade*), et l'échange d'énergie se poursuit de la sorte entre les différents modes de Fourier.

- *Généricité.* Dans la formulation que nous en avons donnée ci-dessus, la question de la croissance des normes se présente d'emblée comme une question d'existence : celle d'un état initial u_0 donnant naissance à tel ou tel type de comportement asymptotique. On sait qu'en mathématique, l'existence d'un objet est difficile à prouver abstraitement, c'est-à-dire lorsque l'objet ne peut être simplement défini au moyen d'une formule, ou de façon constructive. Cette difficulté est généralement contournée par des raisonnements de généricité peu intuitifs de prime abord : on prouve qu'en un certain sens, un phénomène se produit « fréquemment », ce qui signifie donc qu'il se produit au moins une fois. Il faut donc s'attendre à ce qu'en répondant à la question de Bourgain, ce type de raisonnement apparaisse : il peut s'agir de définir une mesure de probabilité sur $X = C^\infty(\mathbb{T}^d)$ et de montrer que la croissance des normes arrive presque sûrement ; ou encore, en utilisant la structure d'espace de Fréchet de X, de montrer que l'ensemble des données initiales satisfaisant les propriétés recherchées constitue un G_δ-dense de X.

1.2 Le formalisme hamiltonien

Dans les formules (1.2) et (1.3), il est fait mention de lois de conservation qui peuvent sembler presque miraculeuses. Derrière elles se dissimule en fait une structure d'équation dite « structure hamiltonienne » qu'il convient maintenant de détailler, pour donner à la question de Bourgain son cadre conceptuel complet.

Notons $H = L^2(\mathbb{T}^d)$, l'espace des fonctions de carré intégrable et à valeurs complexes, muni de sa structure d'espace de Hilbert séparable via le produit scalaire hermitien

$$(f|g) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} f(e^{ix_1}, \dots, e^{ix_d}) \overline{g(e^{ix_1}, \dots, e^{ix_d})} dx_1 \dots dx_d,$$

pour tous $f, g \in H$. H jouit également d'une structure d'espace de Hilbert réel via le produit scalaire $\mathrm{Re}(\cdot|\cdot)$. On munit enfin H d'une structure symplectique donnée par la 2-forme constante et non dégénérée $\omega(f,g) := \mathrm{Im}(f|g)$. Considérons maintenant une fonctionnelle $\mathcal{H} : D \subseteq H \to \mathbb{R}$, définie sur un sous-espace D dense dans H, et supposons que $\mathcal{H}$ est différentiable au sens de Fréchet, c'est-à-dire que pour tout $u \in D$, il existe une forme $\mathbb{R}$-linéaire continue sur D, notée $d\mathcal{H}(u)$, telle que

$$\lim_{t \to 0} \frac{\mathcal{H}(u+th) - \mathcal{H}(u)}{t} = \langle d\mathcal{H}(u), h \rangle, \quad \forall h \in D.$$

Comme D est dense dans H, $d\mathcal{H}(u)$ se prolonge de manière unique à H entier, et donc se représente de manière unique, pour la structure de Hilbert réel, par un vecteur, que l'on note $\nabla\mathcal{H}(u)$: pour tout $h \in H$, $\langle d\mathcal{H}(u), h \rangle = \mathrm{Re}(\nabla\mathcal{H}(u)|h)$. À présent, comme ω est non dégénérée, on peut définir aussi le gradient symplectique de $\mathcal{H}$ (ou champ hamiltonien) défini de manière unique par la relation

$$\langle d\mathcal{H}(u), h \rangle = \omega(h, X_{\mathcal{H}}(u)), \quad \forall h \in H, \forall u \in D.$$

On voit en particulier, vu l'expression de ω, que $X_{\mathcal{H}}(u) = -i\nabla\mathcal{H}(u)$.

Par définition, l'évolution hamiltonienne associée à $\mathcal{H}$ est donnée par les courbes intégrales du champ $X_{\mathcal{H}}$, soit

$$\dot{u} = X_{\mathcal{H}}(u), \tag{1.5}$$

ou encore $i\dot{u} = \nabla\mathcal{H}(u)$, où ici et dans la suite, le point désignera toujours une dérivée par rapport au temps. Par exemple, si $D = H^2(\mathbb{T}^d)$ et

$$\mathcal{H}(u) = \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2_{\mathbb{C}^d} + \frac{1}{4} \int_{\mathbb{T}^d} |u|^4 \tag{1.6}$$

sur D, alors on voit en différentiant $\mathcal{H}$ que $\nabla\mathcal{H}(u) = -\Delta u + |u|^2 u$, et l'on retrouve l'équation de Schrödinger (1.1).

Par ailleurs, si u est une solution « assez régulière » de (1.5), notre système hamiltonien type, alors on peut calculer

$$\frac{d}{dt}\mathcal{H}(u) = \langle d\mathcal{H}(u), \dot{u}\rangle = \omega(\dot{u}, X_{\mathcal{H}}(u)) = 0,$$

puisque $\dot{u} = X_{\mathcal{H}}(u)$ et que ω est alternée. Donc $\mathcal{H}$ est une loi de conservation (au moins formellement) pour les solutions de (1.5), et on l'appelle généralement « l'énergie » ou « le hamiltonien ». Dans le cas de (1.6), dire que $H^1(\mathbb{T}^d)$ est l'espace d'énergie, comme on l'a fait plus haut, c'est dire qu'il s'agit du plus grand espace où $\mathcal{H}$ est défini partout.

De façon plus générale, si $\mathcal{F}$ est une fonctionnelle définie sur $D' \subseteq H$, réelle, différentiable, et *compatible* avec $\mathcal{H}$ (au sens où $D \subseteq D'$ ou $D' \subseteq D$), l'évolution de $\mathcal{F}$ le long des trajectoires du système hamiltonien associé à $\mathcal{H}$ est (formellement) donnée par

$$\frac{d}{dt}\mathcal{F}(u) = \langle d\mathcal{F}(u), \dot{u}\rangle = \omega(X_{\mathcal{H}}(u), X_{\mathcal{F}}(u)).$$

Définition. Si $\mathcal{H} : D \to \mathbb{R}$ et $\mathcal{F} : D' \to \mathbb{R}$ sont deux fonctionnelles réelles, différentiables et compatibles, alors on définit une nouvelle fonctionnelle notée $\{\mathcal{H}, \mathcal{F}\}$, et nommée le *crochet de Poisson de $\mathcal{H}$ et $\mathcal{F}$*, par $\{\mathcal{H}, \mathcal{F}\}(u) = \omega(X_{\mathcal{H}}(u), X_{\mathcal{F}}(u))$ pour $u \in D \cap D'$.

Le calcul précédent montre que $\mathcal{F}$ est une loi de conservation pour toutes les trajectoires du système hamiltonien associé à $\mathcal{H}$ si et seulement si $\{\mathcal{H}, \mathcal{F}\} = 0$. Comme cette condition est symétrique, puisque $\{\mathcal{F}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{F}\}$, on en déduit le théorème de Noether : $\mathcal{F}$ est une loi de conservation pour le système hamiltonien associé à $\mathcal{H}$ si et seulement si $\mathcal{H}$ est une loi de conservation pour le système hamiltonien associé à $\mathcal{F}$. Dans ce cas, on dit que $\mathcal{H}$ et $\mathcal{F}$ commutent.

Ce résultat permet de déduire des invariances de la fonctionnelle $\mathcal{H}$ de nouvelles lois de conservation. Reprenons l'exemple du hamiltonien de l'équation de Schrödinger (1.6). Ce hamiltonien vérifie, pour tout $\theta \in \mathbb{T}$, $\mathcal{H}(e^{-2i\theta}u) = \mathcal{H}(u)$, c'est-à-dire que $\mathcal{H}$ est une loi de conservation pour le système hamiltonien d'équation

$$i\dot{u} = 2u.$$

Il apparaît que ce système est associé au hamiltonien $Q(u) = \int_{\mathbb{T}^d} |u|^2$, qui est tel que $X_Q(u) = -2iu$. De la sorte, le théorème de Noether implique que Q est une loi de conservation pour le système hamiltonien associé à $\mathcal{H}$, et l'on retrouve (1.3), que l'on appelle *conservation de la masse.*

Remarque 1. Une autre invariance de $\mathcal{H}$ est utile à signaler dans le cas de (1.6) : pour $v \in H$ et $\alpha \in \mathbb{T}^d$, notons $\tau_\alpha v : x \in \mathbb{T}^d \mapsto v(e^{i(x-2\alpha)})$. On a $\tau_\alpha v \in H$, et de plus, $\mathcal{H}(\tau_\alpha u) = \mathcal{H}(u)$ pour tout $\alpha \in \mathbb{T}^d$ et $u \in D$, de sorte que $\mathcal{H}$ est une loi de conservation pour l'équation

$$\dot{u} = 2\frac{\partial u}{\partial x_j} =: 2\partial_{x_j} u,$$

où $j = 1, \ldots, d$ est fixé. Ce système est une équation de transport linéaire, qui provient du hamiltonien $M_j(u) := (i\partial_{x_j} u|u)$. Donc la j^{e} impulsion M_j est une loi de conservation pour les solutions de (1.1).

À l'aide de ce formalisme, nous pouvons étendre la question de Bourgain. Soit $\mathcal{H}$ une fonctionnelle comme ci-dessus (réelle et densément définie dans $H = L^2(\mathbb{T}^d)$), et $\mathcal{F}_k$, $1 \leq k \leq K$ une collection de fonctionnelles compatibles deux à deux, compatibles avec le hamiltonien $\mathcal{H}$ et commutant avec lui. On fait les hypothèses suivantes :

(H1) il existe un « espace d'énergie », c'est-à-dire un $s_0 \in [0, +\infty[$ maximal tel que $\mathcal{H}$ et $\mathcal{F}_k$, $k \in [\![1, K]\!]$, sont globalement définies sur $H^{s_0}(\mathbb{T}^d)$, avec pour tout $u \in H^{s_0}$,

$$\|u\|_{H^{s_0}} \leq f(\mathcal{H}(u), \mathcal{F}_1(u), \dots, \mathcal{F}_K(u)),$$

où $f : \mathbb{R}^{1+K} \to [0, +\infty[$ est une certaine fonction.

(H2) l'équation $\dot{u} = X_{\mathcal{H}}(u)$ admet un flot global sur H^{s_0}, le long des orbites duquel $\mathcal{H}$ et $\mathcal{F}_k$, $k \in [\![1, K]\!]$ sont conservés ;

(H3) la régularité additionnelle H^s persiste pour tout $s > s_0$ (au sens donné plus haut).

La question de la croissance des normes de Sobolev devient donc :

Question. Soit $s > s_0$. Existe-t-il une donnée initiale $u_0 \in H^s$ telle que la solution de (1.5) issue de u_0 au sens de (H2) soit non bornée dans une topologie $H^{s'}$, avec $s_0 < s' \leq s$?

Remarque 2. Dans toute cette partie, on peut accéder à un niveau de généralité plus grand en remplaçant $\mathbb{T}^d$ par d'autres variétés, et en particulier, par n'importe quelle variété riemannienne M, où l'opérateur de Laplace-Beltrami permet également de définir une échelle d'espaces de Sobolev. Notons aussi que le cadre hamiltonien peut être transposé à d'autres espaces de Hilbert H et à d'autres 2-formes ω non-dégénérées. C'est ce que l'on fait en particulier dans l'étude de l'équation de Korteweg-de Vries (KdV), dont nous ne parlerons pas ici, mais sur laquelle on trouvera un exposé complet dans [37].

Remarque 3. En règle générale, nous nous restreindrons toutefois à des fonctions définies sur des variétés compactes (ou cylindriques — voir [32], que nous évoquons plus loin), car les études de l'équation de Schrödinger (1.1) sur $\mathbb{R}^d$, $d \geq 2$ montrent que le phénomène dominant dans ce cas est le *scattering* (ou en français, la *diffusion*) : les ondes ayant « plus de place » dans l'espace euclidien, elles ont tendance à s'étaler, et par conséquent, les effets non-linéaires induits par le terme cubique deviennent négligeables. On prouve qu'asymptotiquement, pour (1.1) et $d \in \{2, 3, 4\}$ (voir [14, 39, 57], les solutions dans $\dot{H}^{\frac{d-2}{2}}(\mathbb{R}^d)$ convergent vers une solution de l'évolution libre

$$i\frac{\partial u}{\partial t} + \Delta_{\mathbb{R}^d} u = 0.$$

1.3 Réponse négative en dimension 1

Revenons à l'équation de Schrödinger cubique (1.1) et à la question originelle de Bourgain. Au moment où celui-ci l'a énoncée, les mathématiciens Zakharov et Shabat avaient déjà fourni une première réponse [71], dans le cas du cercle ($d = 1$), et également celui de la droite (en remplaçant $\mathbb{T}$ par $\mathbb{R}$). Expliquons leur résultat, qui est de grande importance : commençons par associer à chaque $v \in C^\infty(\mathbb{T}, \mathbb{C})$ deux opérateurs (non bornés) sur $L^2(\mathbb{T}, \mathbb{C}^2)$ (que l'on identifie à l'ensemble des couples de fonctions L^2 scalaires), en posant

$$L_v := \begin{bmatrix} i\partial_x & v \\ \bar{v} & -i\partial_x \end{bmatrix},$$
$$B_v := \begin{bmatrix} 2i\partial_x^2 - i|v|^2 & v' + 2v\partial_x \\ \bar{v}' + 2\bar{v}\partial_x & -2i\partial_x^2 + i|v|^2 \end{bmatrix}.$$

On remarque que L_v est autoadjoint de domaine $H^1(\mathbb{T}, \mathbb{C}^2)$, et que B_v est anti-adjoint de domaine $H^2(\mathbb{T}, \mathbb{C}^2)$. Si nous supposons à présent que u est une solution de l'équation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 2|u|^2 u \tag{1.7}$$

(où le coefficient 2 est une convention de normalisation), avec $\forall t \in \mathbb{R}$, $u(t) \in C^\infty(\mathbb{T}, \mathbb{C})$, alors des calculs purement algébriques permettent d'établir l'identité suivante :

$$\frac{d}{dt} L_{u(t)} = B_{u(t)} L_{u(t)} - L_{u(t)} B_{u(t)}, \tag{1.8}$$

que l'on peut écrire de manière plus synthétique grâce à un crochet de commutation : $\partial_t L_u = [B_u, L_u]$. De (1.8), on peut déduire que pour tout $t \in \mathbb{R}$, il existe un opérateur unitaire $U(t)$ sur $L^2(\mathbb{T}, \mathbb{C}^2)$ tel que $U(t)^* L_{u(t)} U(t) = L_{u(0)}$. En particulier, les propriétés spectrales de $L_{u(t)}$ sont conservées au cours du temps. Partant de ce constat, il est possible de montrer qu'il existe en réalité une infinité de fonctionnelles $\mathcal{E}_p$, définies sur $H^p(\mathbb{T})$ pour $p \geq 1$ entier, telles que $\{\mathcal{H}, \mathcal{E}_p\} = 0$ sur H^p pour tout p, et qui « contrôlent » la norme H^p au sens où il existe une fonction $f_p : \mathbb{R}^{1+p} \to [0, +\infty[$ telle que $\|v\|_{H^p} \leq f_p(\mathcal{H}(v), \mathcal{E}_1(v), \ldots, \mathcal{E}_p(v))$. Ainsi, dans le cas de l'équation de Schrödinger cubique en dimension 1, l'existence d'une infinité de lois de conservation permet de prouver qu'une solution de régularité H^p, $p \in \mathbb{N} \setminus \{0\}$ reste uniformément bornée dans H^p (et donc dans les espaces de régularité inférieure).

L'idée d'associer à la solution d'une certaine EDP une courbe dans un espace d'opérateurs, et d'étudier l'équation qui régit celle-ci, remonte à Lax, à la fin des années 1960, dans le cadre de son étude de l'équation de KdV [43]. Une paire d'opérateurs (L, B) dépendant d'un paramètre u et qui vérifient l'identité (1.8) lorsque $t \mapsto u(t)$ est solution d'une certaine EDP, est aujourd'hui appelée une *paire de Lax*.

Le résultat ci-dessus a été récemment généralisé par Kappeler, Schaad et Topalov [38], qui ont prouvé que pour n'importe quel $s \in \mathbb{R}$, $s \geq 1$, une solution de régularité H^s reste bornée dans H^s. Koch et Tataru [42] ont étendu cette conclusion à des solutions de basse régularité (dans H^s, avec $s > -\frac{1}{2}$).

Pourquoi n'observe-t-on pas de croissance des normes de Sobolev pour l'équation (1.7) en dimension 1 ? On peut en donner une explication intuitive. Si l'on considère d'abord l'équation de Schrödinger linéaire

$$i\dot{u} + \Delta_{\mathbb{T}} u = 0,$$

on sait, grâce à un passage dans l'espace de Fourier, que la solution en est donnée par l'action d'un opérateur pseudo-différentiel défini sur L^2 :

$$u(t) = e^{it\Delta} u(0),$$

ce qui signifie que si $k \in \mathbb{Z}$, alors $\widehat{u(t)}(k) = e^{-it|k|^2} \widehat{u(0)}(k)$ pour tout $t \in \mathbb{R}$. Cet opérateur $e^{it\Delta}$ est appelé le *propagateur linéaire* ; c'est une isométrie de L^2, et même de tous les espaces H^s. Si l'on revient maintenant à l'équation non-linéaire (1.7), et si l'on suppose que u en est solution, alors en appliquant le propagateur linéaire inverse, par une sorte de méthode de « variation de la constante », on trouve

$$\frac{d}{dt} e^{-it\Delta} u(t) = -2i e^{-it\Delta} (|u(t)|^2 u(t)),$$

soit en intégrant entre 0 et t et en appliquant le propagateur dans l'autre sens :

$$u(t) = e^{it\Delta} u(0) - 2i \int_0^t e^{i(t-s)\Delta} (|u(s)|^2 u(s)) ds,$$

c'est-à-dire que l'on écrit la solution u comme la somme de la solution linéaire et d'un terme d'interaction. Si l'on examine maintenant les fréquences, en notant $v_k(t)$ le k^{e} coefficient de

Fourier de $e^{-it\Delta}u(t)$, on trouve

$$v_k(t) = \widehat{u(0)}(k) - 2i \sum_{\substack{(k_1,k_2,k_3)\in\mathbb{Z}^3 \\ k_1-k_2+k_3=k}} \int_0^t e^{-is(k_1^2-k_2^2+k_3^2-k^2)} v_{k_1}(s)\overline{v_{k_2}(s)}v_{k_3}(s)ds. \tag{1.9}$$

Dans cette expression, il apparaît que l'interaction non-linéaire entre les coefficients de Fourier est pondérée par une phase oscillante $e^{-is\Phi}$, où $\Phi = \Phi(k_1, k_2, k_3, k) = k_1^2 - k_2^2 + k_3^2 - k^2$. Si nous considérons des données petites, et faisons l'ansatz $v = \varepsilon w$, puis faisons dans l'intégrale de (1.9) le changement de variable $\varepsilon^2 \cdot s = s'$, la phase devient $e^{-is'\Phi/\varepsilon^2}$. Heuristiquement, quand $\varepsilon \ll 1$, par une sorte d'argument « à la Riemann-Lebesgue », on peut imaginer que l'interaction non-linéaire, obtenue en intégrant cette phase, sera d'autant plus faible que la phase sera grande, et donc qu'à l'ordre principal en ε, les propriétés dynamiques du système sont dictées par les interactions des quadruplets de modes qui annulent la phase Φ. Si nous restreignons la somme dans (1.9) aux quadruplets d'entiers qui annulent Φ, nous obtenons ce que l'on appelle le *système résonnant*.

Supposons donc que $(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4$ est une *résonance*, c'est-à-dire un quadruplet d'entiers qui vérifient

$$\begin{cases} k_1^2 - k_2^2 + k_3^2 - k_4^2 = 0, \\ k_1 - k_2 + k_3 - k_4 = 0. \end{cases} \tag{1.10}$$

Si $k_1 \neq k_2$, alors $k_1 - k_2 = k_4 - k_3 \neq 0$, et $(k_1 - k_2)(k_1 + k_2) = (k_4 - k_3)(k_4 + k_3)$ peut être simplifié en $k_1 + k_2 = k_3 + k_4$, et l'on trouve alors $k_1 = k_4$. Autrement dit, les seules résonances dans (1.9) sont les résonances dites *triviales*, qui consistent simplement en une permutation des modes, soit $k_1 \in \{k_2, k_4\}$. En reprenant (1.9), et en « oubliant » artificiellement les termes non résonnants, nous trouvons l'équation d'évolution suivante :

$$v_k(t) = v_k(0) - 2i \int_0^t \left(2 \sum_{\ell\in\mathbb{Z}} |v_\ell(s)|^2 - |v_k(s)|^2 \right) v_k(s)ds. \tag{1.11}$$

Cela implique que $\dot{v_k} = -2i(2\|v\|_{L^2}^2 - |v_k|^2)v_k$, et donc que $|v_k|^2$ est constant. Le système résonnant (1.11) conserve donc le module des coefficients de Fourier, ce qui proscrit toute croissance de norme de Sobolev dans ce cadre. Pour peu que le système résonnant soit bien à même de décrire le comportement qualitatif des solutions de l'équation (1.7), on comprend donc pourquoi les solutions lisses de l'équation de Schrödinger en dimension 1 restent bornées dans tous les espaces de Sobolev.

Puisque le cas le plus simple de la question originelle de Bourgain est traité par la négative, et dans la perspective du développement ci-dessus au sujet des résonances, plusieurs directions de recherche se présentent pour tenter de complexifier l'équation sans la rendre inaccessible à tout effort théorique :

- *Changer la dimension de l'espace*, et considérer $\mathbb{T}^2$, $\mathbb{T}^3$, ou d'autres variétés. Dans le cas de $\mathbb{T}^2$, par exemple, l'équation aux résonances (1.10) devient $|k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2 = 0$, avec $k_j \in \mathbb{Z}^2$ pour $j = 1, \ldots, 4$, et $k_1 - k_2 + k_3 - k_4 = 0$, où le symbole $|\cdot|$ est mis pour la norme euclidienne d'un vecteur du plan. Dans ce cas, les résonances triviales existent certes toujours, mais si $k_1 \notin \{k_2, k_4\}$, alors on voit que $k_1 - k_2 = k_4 - k_3$, et grâce à la première équation, $(k_1 - k_2) \cdot (k_3 - k_2) = 0$, où le point désigne le produit scalaire standard dans $\mathbb{R}^2$. Autrement dit, (k_1, k_2, k_3, k_4) est une résonance non triviale si et seulement si ces quatres points forment un rectangle dans le plan (éventuellement réduit à une simple ligne si $k_2 = k_4$ ou $k_1 = k_3$). Nous voyons donc que le système résonnant est beaucoup plus riche que dans le cas du tore simple, et que l'on peut espérer qu'il encode cette fois des dynamiques turbulentes.

- *Changer la non-linéarité*, et examiner par exemple des équations de Schrödinger quintiques, c'est-à-dire l'équation
$$i\partial_t u + \Delta u = |u|^4 u, \tag{1.12}$$
où $u : \mathbb{R}_t \times \mathbb{T}_x \to \mathbb{C}$. Dans ce cas, deux termes s'ajoutent à l'équation (1.10), pour donner
$$\begin{cases} k_1^2 - k_2^2 + k_3^2 - k_4^2 + k_5^2 - k_6^2 = 0, \\ k_1 - k_2 + k_3 - k_4 + k_5 - k_6 = 0. \end{cases}$$
Cette équation, nettement plus compliquée que (1.10), admet des solutions non-triviales, comme $(3, 1, 0, 4, 3, 1)$ par exemple.

- *Changer l'opérateur* Δ, et tester d'autres relations de dispersion. C'est le laplacien qui est responsable des termes quadratiques dans l'équation aux résonances (1.10). L'association d'un mode et de l'action d'un opérateur sur les ondes monochromatiques associées est appelée *relation de dispersion*. Dans le cas de l'opérateur $-\Delta$ sur $\mathbb{T}$, cette relation est précisément donnée par $k \mapsto k^2$. En changeant l'opérateur $-\Delta$, on changerait donc la relation de dispersion, et aussi l'équation des résonances. L'idée la plus naturelle est de remplacer le laplacien par une de ses puissances dites « fractionnaires », c'est-à-dire $(\sqrt{-\Delta})^\alpha$, où $0 < \alpha < 2$ est n'importe quel réel (et pas nécessairement un rationnel !). On note $|D| := \sqrt{-\Delta}$ l'opérateur pseudo-différentiel qui multiplie les modes par la valeur absolue de la fréquence, de sorte que l'équation devient
$$i\partial_t u - |D|^\alpha u = |u|^2 u, \tag{1.13}$$
et la première équation de (1.10) s'écrit à présent $|k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k_4|^\alpha = 0$. La nouvelle relation de dispersion est alors $k \mapsto |k|^\alpha$.

Nous nous proposons dans la suite de passer en revue quelques tentatives dans ces diverses directions.

1.4 Perspectives de recherche

En dimension supérieure

Plusieurs résultats d'importance ont été obtenus pour l'équation de Schrödinger en dimensions 2 et 3. La première avancée notable remonte à 2010, et est due à Terence Tao et ses collaborateurs [12] : considérant l'équation de Schrödinger cubique défocalisante sur $\mathbb{T}^2$, ils ont montré que pour tout $s > 1$, tout $0 < \varepsilon \ll 1$, et tout $K \gg 1$, il existe un temps $T \in \mathbb{R}_+$ et une donnée initiale $u_0 \in C^\infty(\mathbb{T}^2)$ tels que la solution correspondante vérifie
$$\|u(0)\|_{H^s} = \|u_0\|_{H^s} \leq \varepsilon,$$
et
$$\|u(T)\|_{H^s} \geq K.$$
Autrement dit, pour l'équation (1.1) sur $\mathbb{T}^2$, il est possible de faire grandir arbitrairement les normes de Sobolev, en temps fini, sans qu'il soit possible néanmoins de prédire le « futur » de la solution concernée (peut-être, après avoir atteint une grande norme au temps $t = T$, la solution retourne-t-elle dans une petite boule centrée en 0 dans H^s). Encore moins peut-on

dire, à la lumière de ce résultat, s'il existe des trajectoires non bornées dans une topologie H^s.

Pour parvenir à prouver cela, les auteurs ont utilisé deux ingrédients : d'une part, une étude approfondie du système résonnant, pour lequel ils ont construit, grâce à un ingénieux agencement de rectangles de modes résonnants (cf. supra), des solutions dont l'énergie circule d'un mode à l'autre en direction des hautes fréquences ; d'autre part, un argument de *formes normales* qui permet de formaliser le passage au système résonnant, et de montrer que celui-ci est une bonne approximation, au moins dans un certain régime, de la dynamique du système complet.

Le système résonnant a ensuite été étudié plus amplement par Hani [31] :

$$v_k(t) = v_k(0) - 2i \sum_{\substack{(k_1,k_2,k_3)\in(\mathbb{Z}^2)^3 \\ k_1-k_2+k_3=k \\ |k_1|^2-|k_2|^2+|k_3|^2=|k|^2}} \int_0^t v_{k_1}(s)\overline{v_{k_2}(s)}v_{k_3}(s)ds. \tag{1.14}$$

Hani montre qu'en ne considérant que ce système, le phénomène d'instabilité s'étend à un ensemble de données initiales D_0 nulles en dehors de certains modes bien choisis : pour $\phi \in D_0$, il a démontré que pour s, ε, K comme ci-dessus, il existe $u(t)$ une solution lisse de (1.14) et un temps $T > 0$ tel que $\|u(0) - \phi\|_{H^s} \leq \varepsilon$ et $\|u(T)\|_{H^s} \geq K$. Ce résultat a une conséquence remarquable : notons $S(t)u_0$ la solution de (1.14) issue de u_0 prise au temps $t \in \mathbb{R}$. Si nous posons

$$\mathcal{F}_N = \left\{ u_0 \in \overline{D_0} \,\middle|\, \sup_{t>0} \|S(t)u_0\|_{H^s} \leq N \right\},$$

où $N \in \mathbb{N}$ et $\overline{D_0}$ désigne l'adhérence de D_0 dans H^s, et

$$\mathcal{F} = \left\{ u_0 \in \overline{D_0} \,\middle|\, \sup_{t>0} \|S(t)u_0\|_{H^s} < +\infty \right\},$$

alors $\mathcal{F} = \bigcup_{N\geq 0} \mathcal{F}_N$, et comme chaque $\mathcal{F}_N$ est fermé dans H^s (par continuité du flot dans H^s, $s > 1$) et d'intérieur vide dans $\overline{D_0}$ (grâce au résultat précédent), alors par le théorème de Baire, $\mathcal{F}$ est d'intérieur vide dans $\overline{D_0}$, ce qui signifie qu'il existe des trajectoires non-bornées dans H^s.

En revanche, il n'est pas possible à ce jour de montrer que ces trajectoires persistent lorsqu'on ne limite plus la somme (1.14) aux seuls quadruplets de résonances (ou de quasi-résonances), et que l'on considère l'équation de Schrödinger dans son intégralité.

Dans la suite du résultat de Tao *et al.* [12], un raffinement dû à Guardia et Kaloshin [29, 30] permet de donner des bornes quantifiées sur le temps T. Cela nous mène au second résultat d'ampleur, dû à Hani, Pausader, Tzvetkov et Visciglia [32] : ces auteurs ont considéré l'équation de Schrödinger (1.1) sur $\mathbb{R} \times \mathbb{T}^2$. Ajouter une dimension réelle permet en effet de recourir aux propriétés diffusives de l'équation de Schrödinger sur des variétés non-compactes, et celles-ci permettent de négliger légitimement, en temps long, les fréquences non-résonnantes sur $\mathbb{T}^2$. De la sorte, le système résonnant sur $\mathbb{T}^2$ fournit une approximation du système entier valable en grand temps, ce qui permet aux auteurs d'exhiber des données initiales petites et régulières qui donnent naissance à des trajectoires non-bornées dans un certain espace H^s de haute régularité. Ce résultat est le premier du genre pour une équation de Schrödinger. La croissance observée demeure très lente.

De façon plus générale, on trouve dans ces schémas de preuve la tension conceptuelle omniprésente, dans le domaine de l'étude de l'instabilité, entre l'étude d'un système simple pour lequel des calculs sont possibles et des résultats connus (ici, l'équation de Schrödinger résonnante (1.14) sur $\mathbb{T}^2$), et l'extension de ces résultats à des systèmes dont le sens physique est plus avéré, mais dont la complexité interdit une approche directe (ici, l'équation de Schrödinger « complète » (1.1)).

Avec une non-linéarité différente

Considérons à présent l'équation de Schrödinger quintique (1.12). En dimension 1, Grébert et Thomann [28] ont étudié les résonances non-triviales du type

$$\mathcal{A} = \{(n, n+k, n+3k, n+k, n+3k, n+4k) \mid n, k \in \mathbb{Z}, k \neq 0\}$$

(ainsi que l'ensemble des permutations des sites pairs et des sites impairs de ces sextuplets), en montrant qu'il s'agit des plus simples possibles, au sens où n'importe quel sextuplet résonnant non trivial fait au moins intervenir 4 modes différents, et que toutes les résonances à 4 modes sont comprises dans l'énumération ci-dessus. En appliquant un raisonnement de forme normale au système, les auteurs prouvent qu'étant donné un sextuplet $\mathcal{S} \in \mathcal{A}$, il existe une solution de (1.12) qui, à l'ordre principal, échange périodiquement (avec période $\tau \gg 1$) son énergie entre les sites pairs et les sites impairs de $\mathcal{S}$ pendant une durée de l'ordre de $\tau^{3/2}$ (indépendante du choix de $\mathcal{S}$). Grâce à un argument de théorie KAM, Haus et Procesi [34] ont montré que certaines de ces solutions existent pour tout temps.

Un tel phénomène est typiquement non-linéaire, et ne peut se produire pour l'équation avec non-linéarité cubique, puisque dans ce cas, le système résonnant conserve le module des modes de Fourier. Il ne s'agit pas, pour l'instant, d'instabilité ou de croissance de norme, car le phénomène observé est périodique, et les solutions sont bornées dans tous les H^s, $s > 1$, mais il met du moins en évidence un phénomène de transfert d'énergie entre différents modes, prototype d'un transfert vers les hautes fréquences tel que nous l'évoquions plus haut, et qu'on espère pouvoir l'observer un jour dans ce cadre.

Notons enfin que dans le cas de l'équation quintique sur $\mathbb{T}^2$, Haus et Procesi [33] montrent un résultat d'instabilité semblable à celui de Tao et ses collaborateurs pour le système cubique.

En modifiant la dispersion

Passons à présent à l'équation de Schrödinger fractionnaire (1.13) sur $\mathbb{T}$. L'idée de considérer ce système (en fait, une version légèrement plus générale de ce système) remonte à Majda et ses collaborateurs [47], dont les simulations numériques suggèrent l'existence de turbulence pour certaines solutions. Zakharov *et al.* [70] ont également étudié le système et confirmé l'opportunité de l'étudier. Le hamiltonien associé à (1.13) est

$$\mathcal{H}_\alpha(u) = \frac{1}{2}(|D|^\alpha u|u) + \frac{1}{4}\int_{\mathbb{T}} |u|^4, \tag{1.15}$$

et l'on voit donc que l'espace d'énergie associé à l'équation fractionnaire est $H^{\alpha/2}(\mathbb{T})$.

Dans le cas où $\alpha > 1$, la fonction $f_\alpha : x \mapsto |x|^\alpha$ est strictement convexe sur $\mathbb{R}$. On peut en déduire que l'équation $|k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k_4|^\alpha = 0$ n'a pas de solution si $k_1 - k_2 + k_3 - k_4 = 0$ et $\{k_1, k_3\} \neq \{k_2, k_4\}$. En effet, si $k_1 \neq k_2$, alors notons $h := k_1 - k_2 = k_4 - k_3$, en supposant sans perte de généralité que $h > 0$. On a

$$\frac{f_\alpha(k_2 + h) - f_\alpha(k_2)}{h} = \frac{f_\alpha(k_3 + h) - f_\alpha(k_3)}{h}.$$

Or la fonction $x \mapsto h^{-1}(f_\alpha(x + h) - f_\alpha(x))$ est strictement décroissante sur $\mathbb{R}$. Donc nécessairement, $k_2 = k_3$, et par suite $k_1 = k_4$. Les seules résonances sont donc triviales.

En revanche, le cas $\alpha = 1$ est nettement différent. L'équation correspondante,

$$i\partial_t u - |D|u = |u|^2 u, \tag{1.16}$$

est aussi appelée *équation de demi-onde*, car l'opérateur $i\partial_t - |D|$ peut servir à factoriser le d'alembertien : $-\partial_{tt} + \Delta = (i\partial_t - |D|) \circ (i\partial_t + |D|)$. Gérard et Grellier [17] ont étudié les résonances pour l'équation de demi-onde, et ont prouvé que

$$\begin{cases} |k_1| - |k_2| + |k_3| - |k_4| = 0, \\ k_1 - k_2 + k_3 - k_4 = 0, \end{cases}$$

si et seulement si $\{k_1, k_3\} \neq \{k_2, k_4\}$ ou si tous les k_j sont de même signe. Autrement dit, l'équation résonnante associée à (1.16) est en réalité un système découplé de deux équations qui ne prennent en compte respectivement que les modes positifs et négatifs de l'état au temps t. Définissons donc le projecteur de Szegő sur les modes positifs, noté Π :

$$\Pi\left(\sum_{k=-\infty}^{+\infty} a_k e^{ikx}\right) = \sum_{k=0}^{+\infty} a_k e^{ikx}.$$

L'équation résonnante, si l'on fait abstraction des résonances triviales, s'écrit

$$i\partial_t u + i\partial_x u = \Pi(|u|^2 u),$$

où u est une fonction concentrée sur les modes positifs. Le changement de variable $v(t,x) = u(t, x+t)$ permet de simplifier l'équation, et l'on trouve

$$i\partial_t v = \Pi(|v|^2 v).$$

Cette équation s'appelle l'équation de Szegő cubique, et nous reviendrons abondamment sur elle par la suite. Qu'il nous suffise de dire ici qu'en utilisant certaines propriétés d'instabilité de cette équation résonnante et une méthode de forme normale, Gérard et Grellier, dans l'article cité ci-dessus, parviennent à montrer un résultat « à la Tao » : pour tout $s > 1/2$, tout $0 < \varepsilon \ll 1$, et tout $K \gg 1$, il existe une solution régulière $u(t)$ de (1.16) et un temps $T > 0$ tels que $\|u(0)\|_{H^s} \leq \varepsilon$ et $\|u(T)\|_{H^s} \geq K$.

Ce résultat a été notablement amélioré dans le cas de l'équation (1.16) sur la droite $\mathbb{R}$, et en changeant le signe devant la non-linéarité (cas focalisant). Grâce à l'identification d'un phénomène d'interaction turbulente entre deux ondes progressives pour l'équation de Szegő cubique (sur $\mathbb{R}$), Gérard, Lenzmann, Pocovnicu et Raphaël [24] ont prouvé qu'il existe des solutions régulières de l'équation de demi-onde focalisante telles que $\|u(0)\|_{H^s} \leq \varepsilon$ et $\|u(t)\|_{H^s} \geq K$ pour tout $t \geq T$, ce qui donne une information dans ce cas sur le devenir ultérieur de la solution croissante — sans toutefois trancher la question de la divergence de la norme H^s.

Revenons à l'équation fractionnaire (1.13). La particularité du cas $\alpha = 1$ se lit également dans les propriétés du propagateur linéaire défini sur $L^2(\mathbb{T})$: si nous notons $t \mapsto e^{-it|D|^\alpha} u_0$ la solution de

$$\begin{cases} i\dfrac{\partial u}{\partial t} = |D|^\alpha u, \\ u(0) = u_0 \in L^2(\mathbb{T}), \end{cases}$$

alors il est possible de prouver [13, 61] que si $\alpha \neq 1$ et $\frac{2}{3} < \alpha \leq 2$, et si $u_0 \in C^\infty(\mathbb{T})$, alors

$$\int_0^1 \|e^{-it|D|^\alpha} u_0\|_{L^4}^4 dt \leq C_{\alpha,\gamma} \|u_0\|_{H^\gamma}^4, \qquad \forall \gamma > \frac{1}{4} - \frac{\alpha}{8}, \tag{1.17}$$

où $C_{\alpha,\gamma} > 0$ est une constante qui ne dépend que de α et γ. Une telle estimée est appelée une *inégalité de Strichartz*. Sa preuve est directement reliée aux propriétés de la relation de dispersion, et notamment au fait que la fonction $x \mapsto |x|^\alpha$ a une dérivée seconde qui vaut $\alpha(\alpha-1) \cdot |x|^{\alpha-2}$, et qui, dès lors, ne s'annule jamais si $\alpha \neq 1$. L'estimée (1.17) nous apprend que le propagateur linéaire possède un effet régularisant : en effet, l'injection de Sobolev (optimale) $H^{1/4} \hookrightarrow L^4$ montre à elle seule que

$$\|e^{-it|D|^\alpha} u_0\|_{L^4} \leq \|e^{-it|D|^\alpha} u_0\|_{H^{\frac{1}{4}}} = \|u_0\|_{H^{\frac{1}{4}}},$$

donc l'inégalité

$$\int_0^1 \|e^{-it|D|^\alpha} u_0\|_{L^4}^4 dt \leq C \|u_0\|_{H^{\frac{1}{4}}}^4 \tag{1.18}$$

est toujours vraie. L'inégalité (1.17) montre que lorsque $\alpha \neq 1$, on peut faire un peu mieux : même si $u_0 \notin H^{1/4}$, mais que u_0 apparatient à un espace de Sobolev de moindre régularité, (1.17) prouve que $e^{-it|D|^\alpha} u_0 \in L^4$ pour presque tout $t \in [0,1]$. En revanche, il est prouvé dans [17] que pour $\alpha = 1$, aucune inégalité meilleure que (1.18) n'est vraie.

On appelle *dispersives* les équations où une non-linéarité et un opérateur linéaire interviennent concurremment, et où ce dernier satisfait des estimées de Strichartz du type (1.17). Comme la présence de dispersion atténue les effets non-linéaires, puisqu'elle fournit de l'intégrabilité y compris dans des espaces de basse régularité, on conçoit qu'elle s'oppose plutôt, en revanche, à un phénomène de croissance de normes de Sobolev, ou de turbulence faible. Suivant [7, 10, 58], le but du chapitre 2 de cette thèse est de prouver les résultats suivants :

Proposition 1.1. *Pour $\frac{2}{3} < \alpha \leq 2$, l'équation (1.13) vérifie les hypothèses (H1), (H2) et (H3) ci-dessus. Le hamiltonien $\mathcal{H}_\alpha$ est donné par (1.15), et $Q(u)$ et $M(u) := -(i\partial_x u|u)$ sont également conservés. L'espace d'énergie est $H^{\alpha/2}$.*

Théorème 1. *Soit $u_0 \in C^\infty(\mathbb{T})$, et $u : t \mapsto u(t)$ la solution de (1.13) telle que $u(0) = u_0$. Soit $s > \alpha/2$. Notons $\bar{s} := \alpha + \lceil s - \alpha \rceil$.*

- *si $\alpha \in]\frac{2}{3}, 1[\cup]1, 2[$, alors il existe une constante A ne dépendant que de s et α, et une constante C ne dépendant que de $\|u_0\|_{H^{\bar{s}}}$, s et α telles que*

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^A.$$

- *si $\alpha = 1$ (et donc si u est solution de l'équation de demi-onde), alors il existe une constante B ne dépendant que de s et de $\|u_0\|_{H^{1/2}}$, et une constante C ne dépendant que de $\|u_0\|_{H^{\bar{s}}}$ et de s telles que*

$$\|u(t)\|_{H^s} \leq Ce^{B|t|^2}.$$

Ces résultats sont des estimées a priori, du même type que celles de Bourgain et Staffilani, évoquées plus haut, et ce sont les plus précises connues à ce jour. Bien qu'on ne puisse raisonnablement penser qu'elles sont optimales, elles semblent confirmer qu'il sera plus aisé d'observer des phénomènes de croissance de normes de Sobolev dans le cadre de l'équation de demi-onde ($\alpha = 1$) ou dans les systèmes approchants (type Szegő cubique), plutôt que dans le cas d'équations où la dispersion entre en jeu.

Notons aussi que dans le cas où $1 \leq \alpha < 2$, la croissance au plus polynomiale peut se prouver de façon simple, sans même recourir aux inégalités de Strichartz, mais simplement par un raisonnement d'énergie modifié, qui tire directement parti des propriétés de la fonction $x \mapsto |x|^\alpha$, et que l'on trouve également dans toute une série d'articles [50, 52, 65].

2 Autour de l'équation de Szegő cubique

À présent, nous souhaiterions présenter à grands traits l'étude de certains systèmes modèles non-dispersifs (parmi lesquels, les systèmes résonnants évoqués plus haut), en mettant entre parenthèses, pour le moment, la question de leur pouvoir d'approximation de systèmes plus pertinents d'un point de vue physique. Nous centrons cette partie sur l'équation de Szegő cubique, qui est aujourd'hui la mieux comprise de ces équations « cœur ».

2.1 Les variables action-angle et la turbulence générique

Présentation et théorème de turbulence

Notons L^2_+ le sous-espace fermé de $L^2(\mathbb{T})$ constitué des séries de Fourier dont les modes négatifs sont nuls :

$$L^2_+ = \left\{ f \in L^2(\mathbb{T}) \,\middle|\, \widehat{f}(n) = 0,\ \forall n < 0 \right\}.$$

Le projecteur orthogonal $\Pi : L^2 \to L^2_+$, déjà introduit plus haut, est le projecteur de Szegő. On restreint à L^2_+ le produit scalaire $(\cdot|\cdot)$ sur L^2, ainsi que la forme symplectique ω. Si G est un sous-espace vectoriel de L^2, on notera $G_+ := G \cap L^2_+$.

On appelle aussi $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ le disque unité ouvert de $\mathbb{C}$. L'espace L^2_+ peut être identifié à l'espace de Hardy $\mathbb{H}^2(\mathbb{D})$, c'est-à-dire l'espace des fonctions holomorphes sur $\mathbb{D}$ et telles que

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^2 d\theta < \infty. \tag{2.1}$$

En effet, étant donné $f = \sum_{n=0}^{\infty} \widehat{f}(n) e^{inx}$, on peut poser $f_{\mathbb{D}} := \sum_{n=0}^{\infty} \widehat{f}(n) z^n$, qui est bien holomorphe en tout $z \in \mathbb{D}$, et qui vérifie (2.1), grâce à la formule de Parseval. Inversement, si g est holomorphe sur $\mathbb{D}$ et vérifie (2.1), en notant $g(z) = \sum_{n=0}^{\infty} g_n z^n$, on a nécessairement $\sum |g_n|^2 < +\infty$. Dans la suite, on utilise indifféremment les deux notations $u(e^{ix})$ ou $u(z)$ pour $u \in L^2_+$.

Considérons à présent le hamiltonien suivant :

$$\mathcal{H}_{\text{cub}}(u) = \frac{1}{4} \int_{\mathbb{T}} |u|^4, \quad \forall u \in L^6_+.$$

Pour $h \in L^2_+$ et $u \in L^6_+$, on a $\langle d\mathcal{H}_{\text{cub}}(u), h \rangle = \mathrm{Re}(|u|^2 u | h)$, et comme $\nabla \mathcal{H}_{\text{cub}}(u)$ doit être un vecteur de L^2_+, on trouve $\nabla \mathcal{H}_{\text{cub}}(u) = \Pi(|u|^2 u)$. Pour la structure symplectique induite sur L^2_+, on obtient $X_{\mathcal{H}_{\text{cub}}}(u) = -i\Pi(|u|^2 u)$, et donc l'équation associée au hamiltonien $\mathcal{H}_{\text{cub}}$ est l'équation de Szegő cubique, qui s'écrit :

$$i\partial_t u = \Pi(|u|^2 u). \tag{2.2}$$

Grâce au théorème de Noether, les invariances de la fonctionnelle $\mathcal{H}_{\text{cub}}$ donnent deux autres lois de conservation a priori : la masse Q et le moment M, définis comme précédemment par

$$Q(u) = \frac{1}{2} \int_{\mathbb{T}} |u|^2,$$
$$M(u) = (Du|u), \quad D := -i\partial_x.$$

Remarquons que puisque u n'a que des modes positifs, $Q(u) + M(u) \simeq \|u\|^2_{H^{1/2}}$. Par ailleurs, en dimension 1, $H^{1/2}$ s'injecte dans tous les L^p, avec $1 \leq p < +\infty$. Cela prouve qu'un espace d'énergie au sens de (H1) ci-dessus doit être $H^{1/2}_+$. Effectivement, dans le premier article consacré à l'équation de Szegő [16], Gérard et Grellier prouvent le résultat suivant :

Proposition 2.1. *Soit $u_0 \in H^{1/2}_+$. Alors il existe une unique fonction $u \in C(\mathbb{R}, H^{1/2}_+)$ solution de (2.2) et telle que $u(0) = u_0$. Pour tout $T > 0$, l'application $u_0 \in H^{1/2}_+ \mapsto u \in C([-T,T], H^{1/2}_+)$ est continue. De plus, la régularité additionnelle persiste : si $u_0 \in H^s_+$ pour un certain $s > 1/2$, alors $u \in C(\mathbb{R}, H^s_+)$.*

Les hypothèses (H1)-(H3) sont réunies pour que soit posée la question de la croissance des normes de Sobolev pour les solutions de l'équation de Szegő. Le réponse la plus aboutie en ce sens est la suivante :

Théorème 2 ([21])**.** *Il existe un G_δ dense de l'ensemble $C^\infty_+(\mathbb{T})$ tel que si $u_0 \in G_\delta$, alors la solution $u(t)$ de (2.2) telle que $u(0) = u_0$ vérifie les propriétés suivantes :*

- *il existe une suite de temps $\{\bar{t}_n\}$ tendant vers l'infini, telle que $\forall s > \frac{1}{2}$, $\forall M \in \mathbb{N}$,*

$$\frac{\|u(\bar{t}_n)\|_{H^s}}{|\bar{t}_n|^M} \xrightarrow[n\to+\infty]{} +\infty;$$

- *il existe une suite de temps $\{\underline{t}_n\}$ tendant vers l'infini, telle que*

$$u(\underline{t}_n) \xrightarrow[n\to+\infty]{} u_0 \qquad \text{dans } C^\infty_+.$$

Par ailleurs, $\forall u_0 \in C^\infty_+$, si $u(t)$ est la solution issue de u_0, alors pour tout $s > \frac{1}{2}$, il existe une constante $C_s > 0$ telle que $\|u(t)\|_{H^s} \leq C_s \|u_0\|_{H^s} e^{C_s \|u_0\|^2_{H^s} |t|}$ pour tout $t \in \mathbb{R}$.

Le Théorème 2 affirme donc que génériquement, au sens de Baire, les solutions régulières de (2.2) ont des normes H^s qui, le long d'une suite de temps, croissent plus rapidement que n'importe quel polynôme en temps (croissance *sur-polynomiale*). En ce sens, l'estimée a priori en exponentielle est « presque » optimale. Cependant, cette croissance des normes est intermittente, au sens où selon une autre suite de temps, la solution se rapproche arbitrairement près de u_0 dans toutes les topologies H^s. De la sorte, on voit que les orbites de l'équation de Szegő cubique donnent à voir génériquement des phénomènes de cascade ascendante et de cascade descendante.

Remarque 4. La généricité au sens de Baire, dans le cas du Théorème 2, provient d'un raisonnement similaire à celui de Hani exposé plus haut, en écrivant l'ensemble des solutions qui obéissent à des bornes polynomiales comme une union dénombrable de fermés d'intérieur vide. On peut aussi se demander ce qu'il adviendrait des conclusions du théorème si l'on remplaçait la notion de généricité de Baire par une autre notion, par exemple, en munissant certains ensembles de données initiales d'une mesure de probabilité. La question est ouverte, et rien n'empêcherait que l'on obtienne des conclusions drastiquement différentes (comme des trajectoires presque sûrement bornées dans tous les H^s_+).

Remarque 5. Étendre les résultats du Théorème 2 à l'équation de demi-onde cubique (1.16) serait d'un grand intérêt, qu'il s'agisse de l'existence de croissance sur-polynomiale (ou tout du moins, l'existence d'orbites non-bornées dans H^s, $s > \frac{1}{2}$), ou de l'estimée a priori exponentielle. Ce travail reste à ce jour à accomplir. Haiyan Xu [69] a néanmoins implémenté la stratégie de Hani, Pausader, Tzvtekov et Visciglia [32] dans le cadre de l'équation de « guide d'onde » suivante :

$$i\partial_t u + \Delta_x u - |D|_y u = |u|^2 u, \qquad (t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}.$$

En utilisant, comme sus-mentionné, la dispersion qu'engendre le laplacien sur $\mathbb{R}$, et en s'appuyant sur le Théorème 2, l'auteure parvient à prouver l'existence d'orbites non-bornées pour cette équation.

Pour démontrer le Théorème 2, on voit, dans les grandes lignes, qu'il faut approcher toute donnée initiale par une donnée dont la solution associée a crû plus vite qu'une puissance du temps à un certain temps $\bar{t}$ et est revenue à proximité de son point de départ à un autre temps $\underline{t}$. Pour ce faire, nous allons voir que d'adverses dans le cas de l'équation de Schrödinger, les lois de conservation multiples qui existent également pour l'équation de Szegő cubique permettent de faire des calculs explicites, sans pour autant entraver le phénomène de croissance des normes de Sobolev, et mènent ainsi au résultat d'approximation que nous venons d'esquisser.

Intégrabilité et calculs explicites

La structure sous-jacente à l'équation de Szeő cubique est proche de celle de l'équation de Schrödinger non-linéaire en dimension 1 : il s'agit d'une, et même de deux paires de Lax. Les opérateurs qui jouent le rôle de L_u dans ce cadre sont appelés *opérateurs de Hankel*, et nous en donnons à présent la définition.

Pour $u \in H^{1/2}_+$, posons

$$H_u : \begin{cases} L^2_+ \longrightarrow L^2_+ \\ h \longmapsto \Pi(u\bar{h}) \end{cases}$$

l'opérateur de Hankel de symbole u. Il s'agit d'un opérateur $\mathbb{C}$-antilinéaire, et pour tout $h \in L^2_+$, on a $\|H_u(h)\|_{L^2} \le \|u\|_{H^{1/2}}\|h\|_{L^2}$. Dans la base hilbertienne canonique de L^2_+, H_u est donné par la matrice infinie suivante :

$$\begin{pmatrix} \widehat{u}(0) & \widehat{u}(1) & \widehat{u}(2) & \cdots \\ \widehat{u}(1) & \widehat{u}(2) & \widehat{u}(3) & \cdots \\ \widehat{u}(2) & \widehat{u}(3) & \widehat{u}(4) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

soit une matrice dont les anti-diagonales sont constantes. On démontre par ailleurs que H_u^2 est un opérateur $\mathbb{C}$-linéaire, à trace (donc compact), autoadjoint et positif. Le spectre de H_u^2 peut donc s'écrire comme une suite décroissante de valeurs propres qui tendent vers 0. Nous les notons

$$\rho_1^2(u) \ge \rho_2^2(u) \ge \cdots \ge \rho_n^2(u) \ge \cdots \longrightarrow 0.$$

Un autre opérateur est l'*opérateur de Toeplitz* de symbole b. Pour $b \in L^\infty$, on pose $T_b : h \in L^2_+ \mapsto \Pi(bh) \in L^2_+$. L'opérateur T_b est clairement borné sur L^2_+, et son adjoint est $(T_b)^* = T_{\bar{b}}$. La matrice de T_b dans la base canonique est

$$\begin{pmatrix} \widehat{b}(0) & \widehat{b}(-1) & \widehat{b}(-2) & \cdots \\ \widehat{b}(1) & \widehat{b}(0) & \widehat{b}(-1) & \cdots \\ \widehat{b}(2) & \widehat{b}(1) & \widehat{b}(0) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

c'est-à-dire que ce sont cette fois les diagonales qui sont constantes. Parmi ces opérateurs, l'un joue un rôle spécial : l'opérateur $S := T_{e^{ix}}$, que l'on appelle *décalage à droite* (*shift* en anglais). Son adjoint S^* est le décalage à gauche (accompagné d'une troncature, pour laisser l'espace L^2_+ invariant). Pour $u \in H^{1/2}_+$, on définit à présent l'*opérateur de Hankel décalé* K_u par les formules équivalentes suivantes :

$$K_u := S^* H_u = H_u S = H_{S^* u}.$$

On peut vérifier que K_u^2 possède la même théorie spectrale que H_u^2. Une relation unit par ailleurs ces deux opérateurs :

$$K_u^2(h) = H_u^2(h) - (h|u)u, \quad \forall h \in L_+^2.$$

En notant $\{\sigma_j^2(u)\}_j$ la suite décroissante des valeurs propres de K_u^2, il est possible d'en déduire la propriété d'entrelacement suivante :

$$\rho_1^2(u) \geq \sigma_1^2(u) \geq \rho_2^2(u) \geq \sigma_2^2(u) \geq \cdots \geq \rho_n^2(u) \geq \sigma_n^2(u) \geq \cdots \longrightarrow 0. \tag{2.3}$$

La paire de Lax pour l'équation de Szegő cubique s'énonce ainsi :

Théorème 3 ([16, 18])**.** *Soit $t \mapsto u(t)$ une solution de (2.2) dans H_+^s, pour un certain $s > \frac{1}{2}$. Alors on a les identités suivantes :*

$$\frac{d}{dt}H_u = [B_u, H_u],$$
$$\frac{d}{dt}K_u = [C_u, K_u],$$

où B_u et C_u sont des opérateurs $\mathbb{C}$-linéaires sur L_+^2, bornés, anti-adjoints, donnés par

$$B_u := -iT_{|u|^2} + \frac{i}{2}H_u^2,$$
$$C_u := -iT_{|u|^2} + \frac{i}{2}K_u^2.$$

Comme dans le cas de l'équation de Schrödinger cubique en dimension 1, ces identités, établies de manière purement algébrique, permettent de prouver que pour toute solution $t \mapsto u(t)$ de (2.2) dans l'espace d'énergie $H_+^{1/2}$, les opérateurs $H_{u(t)}^2$ (resp. $K_{u(t)}^2$) restent unitairement équivalents à $H_{u(0)}^2$ (resp. $K_{u(0)}^2$). En particulier, pour tout $t \in \mathbb{R}$,

$$\forall j \geq 1, \qquad \rho_j^2(u(t)) = \rho_j^2(u(0)), \quad \sigma_j^2(u(t)) = \sigma_j^2(u(0)),$$

ou autrement dit, $\{\mathcal{H}_{\text{cub}}, \rho_j^2\} = \{\mathcal{H}_{\text{cub}}, \sigma_j^2\} = 0$. On peut aller plus loin et démontrer [18] que pour tous $j, j' \geq 1$, on a

$$\{\rho_j^2, \rho_{j'}^2\} = \{\sigma_j^2, \sigma_{j'}^2\} = \{\rho_j^2, \sigma_{j'}^2\} = 0.$$

On dit que les ρ_j^2 et les σ_k^2 sont *en involution*.

Nous sommes donc, pour l'équation de Szegő cubique, dans une situation comparable à celle de l'équation de Schrödinger cubique en dimension 1 : il existe en réalité une infinité de fonctionnelles qui commutent avec $\mathcal{H}_{\text{cub}}$. Pourtant, une différence majeure subsiste : pour tout $u \in C^\infty(\mathbb{T})$ et $j \geq 1$, on a

$$\rho_j^2(u) \leq \rho_1^2(u) \leq \operatorname{Tr} H_u^2 \leq C\|u\|_{H^{1/2}}^2,$$

et de même pour $\sigma_j^2(u)$, ce qui signifie qu'ici, les lois de conservation en involution *ne contrôlent pas* la régularité de u au-delà de l'espace d'énergie, et donc, elles ne suffisent pas cette fois à empêcher l'apparition de phénomènes de croissance des normes de Sobolev, comme c'était le cas pour Schrödinger.

Une autre conséquence importante du Théorème 3 est l'existence de variétés de dimension finie, plongées dans l'espace d'énergie $H_+^{1/2}$ et stables par le flot de (2.2). Introduisons leur définition :

Définition. Soit $d \in \mathbb{N}$. Si $d = 2N$ (resp. $d = 2N + 1$) pour un certain $N \in \mathbb{N}$, nous notons $\mathcal{V}(d)$ l'ensemble des fonctions de $H^{1/2}_+$ qui s'écrivent sous la forme

$$u(z) = \frac{A(z)}{B(z)}, \quad z \in \mathbb{D},$$

où A (resp. B) est un polynôme complexe de degré au plus $N - 1$ (resp. N), et B (resp. A) est un polynôme complexe de degré *exactement* N, avec de plus les conditions suivantes : A et B sont premiers entre eux, $B(0) = 1$, et B n'a pas de racine dans $\overline{\mathbb{D}}$, le disque unité fermé de $\mathbb{C}$.

Un théorème ancien, dû à Kronecker (cf. [16]), stipule que

Proposition 2.2. *Pour tout $d \in \mathbb{N}$, $\mathcal{V}(d)$ est exactement l'ensemble des symboles $u \in H^{1/2}_+$ tels que*

$$\operatorname{rk} H^2_u + \operatorname{rk} K^2_u = d.$$

On remarque, à cause de (2.3) par exemple, que lorsque H_u est de rang fini, on a $\operatorname{rk} K^2_u \in \{\operatorname{rk} H^2_u, \operatorname{rk} H^2_u - 1\}$, de sorte que les rangs de H^2_u et de K^2_u coïncident sur les $\mathcal{V}(2N)$, et diffèrent de 1 sur les $\mathcal{V}(2N + 1)$. Par ailleurs, pour tout $d \geq 0$, $\mathcal{V}(d)$ est inclus dans $C^\infty_+(\mathbb{T})$.

Le Théorème 3 entraîne que toute donnée initiale rationnelle, pour l'équation de Szegő cubique, reste rationnelle au cours du temps. Plus précisément,

Corollaire 2.3. *Pour tout $d \in \mathbb{N}$, l'ensemble $\mathcal{V}(d)$ est invariant par le flot de l'équation de Szegő cubique.*

En écrivant les polynômes A et B avec leurs coefficients, on voit que $\mathcal{V}(d)$ possède une structure de *variété kählerienne* de dimension d, c'est-à-dire que c'est une variété complexe munie d'une métrique hermitienne $\tilde{h}$ (en l'occurrence, la restriction du produit scalaire à l'espace tangent à $\mathcal{V}(d)$), et telle que la 2-forme $\mathbb{R}$-linéaire $\operatorname{Im} \tilde{h}$ est fermée (c'est le cas ici, puisque ω est constante). Comme cette 2-forme est automatiquement non-dégénérée, on en déduit que les variétés kähleriennes de dimension d, et $\mathcal{V}(d)$ en particulier, peuvent être vues comme des variétés symplectiques de dimension réelle $2d$.

Or le théorème fameux d'Arnold-Liouville [3, 46] donne un résultat puissant d'intégrabilité pour les systèmes hamiltoniens sur des variétés symplectiques M de dimension finie (c'est-à-dire, rappelons-le, les variétés munies d'une 2-forme différentielle fermée et non-dégénérée). Si $F \in C^\infty(M)$, on peut définir comme ci-dessus le champ de vecteur X_F (en représentant la différentielle dF via la 2-forme ω non-dégénérée), le système hamiltonien associé à F comme le flot du champ de vecteur X_F, et le crochet de Poisson entre F et une autre fonction lisse G comme $\{F, G\}(m) = \langle dG(m), X_F(m)\rangle$ pour $m \in M$. Le théorème d'Arnold-Liouville se place dans l'hypothèse où $\dim M = 2d$, et où l'on peut trouver $F_1, \dots, F_d$, d fonctions lisses sur M, indépendantes (au sens où pour tout $m \in M$, les vecteurs $X_{F_1}(m), \dots, X_{F_d}(m)$ forment une base de $T_m M$ l'espace tangent à M en m) et en involution (au sens où $\{F_j, F_k\} = 0$ pour tous $1 \leq j, k \leq d$). S'il existe $f = (f_1, \dots, f_d) \in \mathbb{R}^d$ tel que l'ensemble de niveau

$$\{m \in M \mid (F_1(m), \dots, F_d(m)) = (f_1, \dots, f_d)\} =: \mathbb{T}(f)$$

est compact, connexe et non-vide, alors $\mathbb{T}(f)$ est difféomorphe au tore $\mathbb{T}^d$. De plus, il existe V un voisinage de $\mathbb{T}(f)$, U un ouvert de $\mathbb{R}^d$, et Φ un difféomorphisme de V vers $U \times \mathbb{T}^d$ tels que pour tout $g \in \mathbb{R}^d$ assez proche de f, $\mathbb{T}(g) \subseteq V$ et

$$\Phi(\mathbb{T}(g)) = \left\{ (I_1(g), \dots, I_d(g), \varphi_1, \dots, \varphi_d) \;\middle|\; (\varphi_1, \dots, \varphi_d) \in \mathbb{T}^d \right\},$$

et tels que pour tout $j \in [\![1,d]\!]$, il existe une fonction $\varpi_j : \mathbb{R}^d \to \mathbb{R}^d$ telle que le flot du champ de vecteur hamiltonien X_{F_j} est transporté par Φ sur la solution des équations

$$\begin{cases} \dot{I} = 0, \\ \dot{\varphi} = \varpi_j(I_1, \dots, I_d). \end{cases}$$

Dit de façon plus simple, étant donné un système hamiltonien associé (disons) à une fonction F_1, et d lois de conservation $F_1, \dots, F_d$ pour ce système, indépendantes et en involution, les ensembles de niveau compacts et connexes de l'application $m \in M \mapsto (F_1(m), \dots, F_d(m)) \in \mathbb{R}^d$ sont difféomorphes à des tores sur lesquels la trajectoire du système hamiltonien est une ligne droite qui s'enroule à vitesse constante. Les variables $(I, \varphi) \in \mathbb{R}^d \times \mathbb{T}^d$ s'appellent des *variables actions-angles*, et le théorème d'Arnold-Liouville est un théorème d'intégrabilité, dans le sens où il fournit des coordonnées dans lesquelles l'évolution du système peut être calculée explicitement.

Pour revenir à l'équation de Szegő cubique (2.2), le théorème d'Arnold-Liouville invite à chercher des variables actions-angles sur $\mathcal{V}(d)$. Elles sont construites dans [18] pour des données génériques dans $\mathcal{V}(d)$. Considérons $u \in \mathcal{V}(2N)$ tel que les valeurs propres de H_u^2 et K_u^2 vérifient une version forte de (2.3) :

$$\rho_1^2(u) > \sigma_1^2(u) > \rho_2^2(u) > \sigma_2^2(u) > \dots > \rho_N^2(u) > \sigma_N^2(u) > 0. \tag{2.4}$$

Alors si e_j est un vecteur propre de H_u^2 associé à ρ_j^2 (pour un $j \in [\![1,N]\!]$), on a également $H_u(e_j) \in \ker(H_u^2 - \rho_j^2 I)$, mais cet espace est une droite, grâce à l'hypothèse d'entrelacement renforcée. Donc il existe $\lambda \in \mathbb{C}$ tel que $H_u(e_j) = \lambda e_j$. En appliquant H_u à cette dernière égalité, on voit que $|\lambda| = \rho_j$. Soit donc $\varphi_j \in \mathbb{T}$ tel que

$$H_u(e_j) = \rho_j e^{i\varphi_j} e_j.$$

On définit de même un angle $\theta_k \in \mathbb{T}$ associé à σ_k^2. Pour un u vérifiant l'hypothèse (2.4), on a ainsi construit un $2d$-uplet $(\rho_1^2, \dots, \rho_N^2, \sigma_1^2, \dots, \sigma_N^2, \varphi_1, \dots, \varphi_N, \theta_1, \dots, \theta_N) \in \mathbb{R}_+^d \times \mathbb{T}^d$.

Inversement, étant donné un $2d$-uplet comme ci-dessus, dont les d premières composantes vérifient (2.4), on définit une matrice $\mathscr{C}(z)$ de taille $N \times N$, avec $z \in \mathbb{D}$ comme paramètre, dont le coefficient (j,k) vaut

$$\mathscr{C}(z)_{jk} = \frac{\rho_j e^{i\varphi_j} - \sigma_k e^{i\theta_k} z}{\rho_j^2 - \sigma_k^2}.$$

Alors, pour tout $z \in \mathbb{D}$, la matrice $\mathscr{C}(z)$ est inversible, et si nous notons $u(z)$ la somme de tous les coefficients de $\mathscr{C}(z)^{-1}$, alors $u \in \mathcal{V}(d)$, et les valeurs propres de H_u^2 (resp. K_u^2) sont exactement les ρ_j^2 (resp. σ_k^2).

Cette formule spectrale inverse permet de définir une application Φ comme ci-dessus : si nous notons $\mathcal{V}_>(d)$ l'ouvert dense de $\mathcal{V}(d)$ dont les éléments vérifient (2.4), et $(\mathbb{R}_+^d)_>$ le sous-ensemble de $\mathbb{R}_+^d$ dont les éléments vérifient eux aussi une inégalité de type (2.4), alors l'application

$$\Phi : \begin{cases} \mathcal{V}_>(d) \longmapsto (\mathbb{R}_+^d)_> \times \mathbb{T}^d \\ \quad u \longmapsto (\rho_1^2, \dots, \rho_N^2, \sigma_1^2, \dots, \sigma_N^2, \varphi_1, \dots, \varphi_N, \theta_1, \dots, \theta_N) \end{cases}$$

est un difféomorphisme. Enfin, si $t \mapsto u(t)$ est une solution de l'équation de Szegő cubique dans $\mathcal{V}_>(d)$, alors pour tous $1 \leq j, k \leq N$,

$$\frac{d\rho_j^2}{dt} = 0, \quad \frac{d\sigma_k^2}{dt} = 0,$$

$$\frac{d\varphi_j}{dt} = \rho_j^2, \quad \frac{d\theta_k}{dt} = \sigma_k^2.$$

Les $(\rho_j^2, \sigma_k^2, \varphi_j, \theta_k)$ sont donc bien des variables actions-angles au sens du théorème d'Arnold-Liouville. Un tel système hamiltonien, qui peut se restreindre à des variétés invariantes de dimension quelconque où la dynamique est intégrable au sens de Liouville, est appelé un système *complètement intégrable.*

Terminons ce paragraphe en mentionnant que l'application Φ peut être en fait étendue à l'espace $H_+^{1/2}$ tout entier [21], sans propriété de généricité (2.4), de sorte que l'on dispose de variables actions-angles globales dans l'espace d'énergie (et même dans un espace plus gros, appelé VMO_+). On peut en déduire, en particulier, que toutes les variétés $\mathcal{V}(d)$ sont *stables* dans $H_+^{1/2}$, au sens où pour tout $\varepsilon > 0$, il existe un $\delta > 0$ tel que si $u_0 \in H_+^{1/2}$ et $d(u_0, \mathcal{V}(d)) \leq \delta$, alors la solution $u(t)$ issue de u_0 vérifie

$$\forall t \in \mathbb{R}, \quad d(u(t), \mathcal{V}(d)) \leq \varepsilon,$$

où nous avons noté $d(\cdot, A)$ la distance à l'ensemble $A \subseteq H_+^{1/2}$ au sens de la norme.

Le phénomène de la « marguerite »

Indiquons à présent comment les calculs explicites, permis par l'intégrabilité et les formules ci-dessus, mettent en évidence les phénomènes de turbulence à l'origine du Théorème 2. Ceux-ci proviennent du fait que l'application Φ décrite plus haut devient singulière lorsque les valeurs propres de H_u^2 et K_u^2 se rapprochent.

Montrons-le sur un exemple simple, dont la description se trouve déjà dans [16], mais qui est exploité de façon vraiment systématique dans [21] : pour un paramètre $0 < \varepsilon \ll 1$ fixé, considérons le jeu de variables suivantes

$$\begin{array}{ccc} \rho_1^2 = (1+\varepsilon)^2, & \sigma_1^2 = 1, & \rho_2^2 = (1-\varepsilon)^2, \\ \varphi_1 = 0, & \theta_1 = 0, & \varphi_2 = \pi, \end{array}$$

toutes les autres valeurs étant fixées à 0. D'après le paragraphe précédent, cela correspond à une matrice

$$\mathscr{C}_\varepsilon(z) = \begin{pmatrix} \dfrac{1+\varepsilon - z}{(1+\varepsilon)^2 - 1} & \dfrac{1}{1+\varepsilon} \\ \dfrac{-(1-\varepsilon) - z}{(1-\varepsilon)^2 - 1} & -\dfrac{1}{1-\varepsilon} \end{pmatrix},$$

et conduit à une fonction $u_\varepsilon \in \mathcal{V}(3)$ explicite :

$$u_\varepsilon(z) = \frac{2z(1-\varepsilon^2) + 3\varepsilon}{2 - \varepsilon z},$$

telle que $H_{u_\varepsilon}^2$ a $(1+\varepsilon)^2$ et $(1-\varepsilon)^2$ pour valeurs propres, et $K_{u_\varepsilon}^2$ a seulement 1. Notons que $u_\varepsilon(z)$ possède un unique pôle en $z = \frac{2}{\varepsilon}$, loin du cercle unité $\mathbb{S}^1$ de $\mathbb{C}$.

À présent, nous modifions les variables de la manière suivante :

$$\begin{array}{ccc} \rho_1^2 = (1+\varepsilon)^2, & \sigma_1^2 = 1, & \rho_2^2 = (1-\varepsilon)^2, \\ \varphi_1 = 0, & \theta_1 = 0, & \varphi_2 = 0, \end{array}$$

Seul l'angle φ_2 a changé, mais maintenant, la matrice

$$\tilde{\mathscr{C}_\varepsilon}(z) = \begin{pmatrix} \dfrac{1+\varepsilon - z}{(1+\varepsilon)^2 - 1} & \dfrac{1}{1+\varepsilon} \\ \dfrac{(1-\varepsilon) - z}{(1-\varepsilon)^2 - 1} & \dfrac{1}{1-\varepsilon} \end{pmatrix}$$

correspond à une fonction

$$v_\varepsilon(z) = \frac{2+\varepsilon^2 - 2z(1-\varepsilon^2)}{2-(2-\varepsilon^2)z},$$

dont l'unique pôle est cette fois en $z = \frac{2}{2-\varepsilon^2}$, c'est-à-dire très proche de $\mathbb{S}^1$. Un simple calcul montre qu'à cause de ce pôle, pour tout $s > \frac{1}{2}$, on a

$$\|v_\varepsilon\|_{H^s_+} \simeq \left(\frac{1}{\varepsilon}\right)^{2s-1} \gg 1.$$

Considérons maintenant la solution $t \mapsto u(t)$ de l'équation de Szegő cubique (2.2) dont la donnée initiale est u_ε. Alors, grâce à la description de l'évolution en termes de variables actions-angles, nous savons que $\varphi_1(t) = t(1+\varepsilon)^2$ et $\varphi_2(t) = \pi + t(1-\varepsilon)^2$. Il existe ainsi un temps t_ε tel que $\varphi_1(t_\varepsilon) = \varphi_2(t_\varepsilon)$:

$$t_\varepsilon = \frac{\pi}{4\varepsilon}.$$

Alors, lorsque ces deux angles sont en phase, on se retrouve dans la situation décrite pour v_ε, et donc $\|u(t_\varepsilon)\|_{H^s_+} \simeq (\frac{1}{\varepsilon})^{2s-1} \simeq (t_\varepsilon)^{2s-1}$ pour $s > \frac{1}{2}$. Ainsi, pour chaque $\varepsilon > 0$, la solution que nous considérons est bornée dans toutes les normes H^s, mais la borne explose lorsque $\varepsilon \to 0$. De plus, la croissance observée est seulement polynomiale en temps, mais on peut en fait rendre la puissance de t_ε arbitrairement grande en faisant augmenter le nombre des actions qui se « rétractent » sur 1. Enfin, cette croissance est seulement transitoire, puisqu'à $t = 2t_\varepsilon$, les angles φ_1 et φ_2 sont à nouveau en opposition de phase.

En approchant toute donnée initiale dans C^∞_+ par une donnée rationnelle générique (*i.e.* une donnée appartenant à un certain $\mathcal{V}_>(d)$), puis en ajoutant à la liste de ses actions un appendice de la forme

$$\cdots > \delta(1+\varepsilon\xi_1) > \delta(1+\varepsilon\eta_1) > \cdots > \delta(1+\varepsilon\xi_L) > 0,$$

où δ, $\varepsilon > 0$, et où $\xi_1 > \eta_1 > \cdots > \xi_{L-1} > \eta_{L-1} > \xi_L > 0$ sont $2L-1$ nombres réels, on trouve une donnée initiale arbitrairement proche de celle dont on est parti (si δ est assez petit), et qui donne naissance à une solution dont les normes H^s (pour $s > \frac{1}{2}$) croissent plus vite que $t_\varepsilon^{(L-1)(2s-1)}$ tout en revenant régulièrement proches de leur point de départ. C'est ainsi que le théorème de Baire peut être appliqué et le Théorème 2 démontré.

Un exemple explicite de ce phénomène d'instabilité a été donné dans [20], lorsqu'on choisit pour donnée initiale $\varphi_0(z) = z+\varepsilon \in \mathcal{V}(3)$, et $\varepsilon \ll 1$. La solution de l'équation de Szegő cubique avec une telle donnée est, on le sait, de la forme

$$\varphi(t,z) = \frac{a(t)z + b(t)}{1-p(t)z},$$

avec $a, b, p \in \mathbb{C}$, $|p| < 1$ et $a + bp \neq 0$. Les coefficients de cette homographie peuvent être calculés explicitement en fonction du temps. La figure 1.1 représente l'évolution de $p(t)$ en fonction du temps, et il apparaît bien que le pôle de $\varphi(t,\cdot)$ s'approche régulièrement du cercle unité $\mathbb{S}^1$, et tout aussi régulièrement, retourne à l'infini lorsque $p(t)$ s'annule.

Conséquences diverses de l'intégrabilité

Citons également pêle-mêle d'autres conséquences importantes de l'intégrabilité pour l'équation de Szegő cubique, sur lesquelles nous aurons à revenir plus loin :

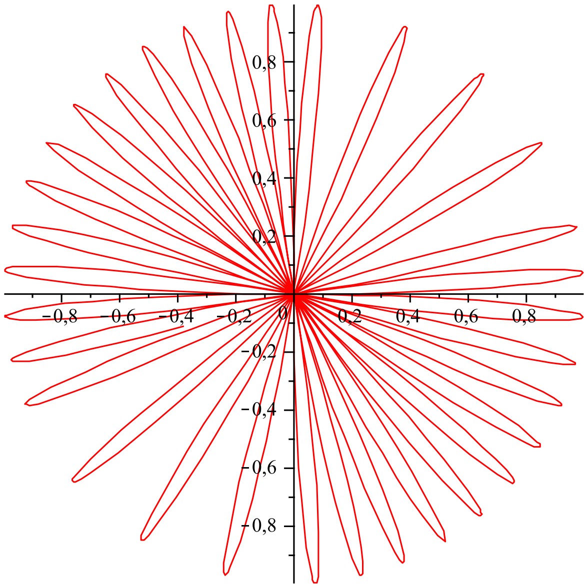

FIGURE 1.1: L'évolution de $p(t)$ lorsque la donnée initiale est $z + \varepsilon \in \mathcal{V}(3)$

- On peut être surpris que les solutions de l'équation de Szegő cubique exigent de la régularité, dans la mesure où le seul opérateur différentiel qui apparaît dans (2.2) est le projecteur de Szegő Π, qui est d'ordre 0. Or on prouve [16] qu'il n'existe pas, sur H^s_+, de flot uniformément continu sur les bornés, quel que soit $s < \frac{1}{2}$.

  D'autre part, l'équation sans Π
  $$i\partial_t u = |u|^2 u$$
  est en réalité une équation différentielle ordinaire posée en chaque point $x \in \mathbb{T}$, qui se résout explicitement en $u(t,x) = e^{-it|u(0,x)|^2}u(0,x)$. L'équation est donc bien posée sur L^∞, alors que ce n'est pas le cas non plus pour Szegő (2.2) (cf. [22]). En revanche, grâce à la paire de Lax du Théorème 3, on peut en fait prouver [22] que l'équation de Szegő est bien posée sur $BMO_+(\mathbb{T})$, où $BMO(\mathbb{T})$ est l'espace de John et Nirenberg, c'est-à-dire le sous-espace de $L^1_{\text{loc}}(\mathbb{T})$ constitué des fonctions f telles que
  $$\sup_{I \subseteq \mathbb{T}} \frac{1}{|I|} \int_I \left| f - \frac{1}{|I|} \int_I f \right| < +\infty,$$
  où le supremum est pris sur l'ensemble des intervalles $I \subseteq \mathbb{T}$. C'est un espace un plus gros que L^∞_+, dont il s'avère qu'il est en fait égal à $\Pi(L^\infty)$, ou encore à l'espace des symboles $u \in L^2_+$ tels que H_u soit continu sur L^2_+.

- Une autre caractéristique des équations intégrables est de posséder de multiples *ondes progressives*. Il s'agit de solutions de (2.2) qui n'évoluent qu'au travers des invariances du hamiltonien $\mathcal{H}_{\text{cub}}$, c'est-à-dire des solutions $t \mapsto u(t)$ dans BMO_+ telles qu'il existe $c, \omega \in \mathbb{R}$ tels que pour tout $t \in \mathbb{R}$,
  $$u(t,z) = e^{-i\omega t}u(0, ze^{-ict}), \quad \forall z \in \mathbb{D}. \tag{2.5}$$
  Ces solutions peuvent être classifiées (cf. [16]). On démontre que $u(t)$ est une onde stationnaire (*i.e.* de la forme (2.5) avec $c = 0$) si et seulement si $u(0)$ est une fonction

intérieure (c'est-à-dire une fonction de L^2_+ telle que $|u(0)|^2 \equiv 1$ sur $\mathbb{T}$). Ensuite, si $u(t)$ est une onde progressive avec $c \neq 0$, alors $u(0) \in C^\infty_+$, et il existe un entier $N \in \mathbb{N} \setminus \{0\}$, un entier $0 \leq \ell \leq N-1$, ainsi que α, $p \in \mathbb{C}$ avec $0 < |p| < 1$ tels que

$$u(0,z) = \frac{\alpha z^\ell}{1 - pz^N}, \quad \forall z \in \mathbb{D}.$$

Les tores de dimension 2 que parcourent ces ondes sont tous instables [18], excepté celui qui correspond à l'état fondamental $z \mapsto \frac{\alpha}{1-pz}$, qui n'est autre que $\mathcal{V}(2)$, et donc est stable, d'après le résultat évoqué plus haut.

2.2 D'autres modèles non-dispersifs

Les découvertes autour de l'équation de Szegő cubique (2.2) permettent d'envisager l'étude d'autres modèles, dans l'espoir de mettre en lumière des phénomènes et mécanismes inédits de croissance de normes de Sobolev. Nous nous proposons à présent d'en passer en revue quelques uns.

Une perturbation linéaire de l'équation de Szegő cubique

Haiyan Xu [67, 68] a mené l'étude d'une équation de Szegő cubique perturbée par un terme linéaire :

$$i\partial_t u = \Pi(|u|^2 u) + \alpha \int_{\mathbb{T}} u, \tag{2.6}$$

où $\alpha \in \mathbb{R}$ est fixé. Xu prouve que la paire de Lax pour H_u ne survit pas à cette perturbation, au contraire de celle pour K_u. Grâce à un calcul dans $\{u \in H^{1/2}_+ \mid \operatorname{rk} K_u^2 = 1\} = \mathcal{V}(2) \cup \mathcal{V}(3)$, que le flot de (2.6) préserve donc, il est possible prouver que lorsque $\alpha > 0$, la solution $t \mapsto u(t)$ de (2.6) telle que $u(0) = z + \sqrt{\alpha}$ vérifie

$$\|u(t)\|_{H^s} \simeq e^{(2s-1)\sqrt{\alpha}\cdot|t|}, \quad \forall s > \tfrac{1}{2},$$

lorsque $t \to \pm\infty$. Au vu de la deuxième partie du Théorème 2, dont on peut montrer qu'elle reste vraie dans le cas de (2.6), cette croissance est optimale, et c'est le premier exemple d'une solution d'un système hamiltonien dont les normes croissent à vitesse exponentielle. Cet exemple ne présente pas, en revanche, le phénomène d'intermittence qui est générique pour l'équation de Szegő non pertubée.

Là encore, la complète intégrabilité est d'une aide fondamentale, car elle permet le calcul explicite de la solution issue de $z + \sqrt{\alpha}$. Dans [68], Xu a trouvé d'autres solutions turbulentes (avec croissance exponentielle) dans des variétés $\mathcal{V}(2N) \cup \mathcal{V}(2N+1)$, $N \geq 2$, mais on ne sait toujours pas décrire ces solutions de façon exhaustive, et encore moins si cette croissance exponentielle des normes de Sobolev est générique dans C^∞_+.

L'équation de Szegő sur la droite réelle

Nous avons déjà évoqué l'étude de [24] sur l'équation de demi-onde sur $\mathbb{R}$; ce résultat prend sa source dans le travail d'Oana Pocovnicu, qui a considéré l'équation de Szegő sur la droite, avec un projecteur donné sur $L^2(\mathbb{R})$ par

$$\Pi\left(\int_{-\infty}^{+\infty} \widehat{f}(\xi) e^{ix\xi} d\xi\right) = \int_0^{+\infty} \widehat{f}(\xi) e^{ix\xi} d\xi.$$

On note naturellement $L^2_+(\mathbb{R}) = \Pi(L^2(\mathbb{R}))$, ainsi que $G_+ := G \cap L^2_+$ pour $G \subseteq L^2(\mathbb{R})$. Pocovnicu a montré que, de même que dans le cas de $\mathbb{T}$, un opérateur de Hankel de symbole

$u \in H^{1/2}_+(\mathbb{R})$ peut être défini sur $L^2_+(\mathbb{R})$ par la formule $H_u(h) = \Pi(u\bar{h})$, et que si $t \mapsto u(t)$ est une solution lisse de l'équation de Szegő cubique sur $\mathbb{R}$ (*i.e.* telle que $i\partial_t u = \Pi(|u|^2 u)$), alors

$$\frac{d}{dt} H_u = [B_u, H_u], \quad B_u := -iT_{|u|^2} + \tfrac{i}{2} H_u^2.$$

En revanche, il n'y a pas d'équivalent naturel de K_u sur la droite, donc pas de deuxième paire de Lax.

L'espace $L^2_+(\mathbb{R})$ s'identifie cette fois à l'ensemble des fonctions holomorphes sur le demi-plan de Poincaré $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, et dont la trace au bord est L^2. On peut faire un lien entre $L^2_+(\mathbb{R})$ et $L^2_+(\mathbb{T})$ grâce à la bijection biholomorphe

$$\omega : \mathbb{C}_+ \longrightarrow \mathbb{D}, \ z \longmapsto \frac{z-i}{z+i}.$$

Pour $f \in L^2_+(\mathbb{R})$, on pose

$$Uf(z) = \frac{2i\sqrt{\pi}}{1-z} f(\omega^{-1}(z)), \quad \forall z \in \mathbb{D},$$

de sorte que $Uf \in L^2_+(\mathbb{T})$, avec $\|Uf\|_{L^2(\mathbb{T})} = \|f\|_{L^2(\mathbb{R})}$. Pour autant, l'image par U d'une solution de l'équation de Szegő sur $\mathbb{R}$ n'est pas une solution de l'équation de Szegő cubique (2.2) sur le cercle, d'où l'intérêt d'une étude systématique de l'équation sur la droite réelle.

Effectivement, une première différence apparaît dans l'étude des ondes progressives [54] : la seule qu'admet l'équation de Szegő cubique sur $\mathbb{R}$, au sens de (2.5), est donnée, aux invariances près de l'équation, par

$$u(0,z) = \frac{C}{z+i}, \quad \forall z \in \mathbb{C}_+, \tag{2.7}$$

où $C \in \mathbb{C}$. Les tores de dimension 2 engendrés par ces fonctions sont stables. Par ailleurs, ils sont stables en grand temps par une petite perturbation de l'équation via un potentiel Toeplitz [55].

D'autre part, grâce à des variables actions-angles généralisées, Pocovnicu montre [53] qu'il existe des solutions turbulentes dans l'ensemble $\{u \mid \operatorname{rk} H_u^2 = 2\}$, dont les normes de Sobolev croissent polynomialement, en $|t|^{2s-1}$, pour $s > \frac{1}{2}$. Cette croissance est donc moins rapide que celle observée pour (2.6), mais elle a en commun avec elle d'être « univoque », c'est-à-dire en cascade de fréquence purement ascendante. Enfin, ce phénomène peut être interprété comme le résultat de l'interaction non-linéaire entre deux ondes progressives (du type (2.7)), dont l'une « pousse » l'autre vers l'infini (cf. [24]).

L'équation LLL

L'équation de plus bas niveau de Landau cubique (en anglais, *cubic lowest Landau level equation*), ou équation LLL, est donnée également par le hamiltonien

$$\mathcal{H}^{LLL}_{\text{cub}}(u) = \frac{1}{4}\int_{\mathbb{C}} |u(z)|^4 d\lambda(z),$$

où λ désigne la mesure de Lebesgue sur $\mathbb{C}$, et où u est un élément de l'espace de Bargmann-Fock

$$\mathcal{E} := \left\{ u \in L^2(\mathbb{C}) \,\middle|\, u(z) = e^{-\frac{|z|^2}{2}} f(z), \ \text{où } f \text{ fonction entière sur } \mathbb{C} \right\}.$$

En notant Π le projecteur orthogonal de $L^2(\mathbb{C})$ sur $\mathcal{E}$, et pour la structure symplectique induite par $\omega(u,v) = \mathrm{Im}(\int_{\mathbb{C}} u\bar{v}\, d\lambda)$, l'équation associée à $\mathcal{H}_{\text{cub}}^{LLL}$ s'écrit de manière identique à l'équation de Szegő cubique :

$$i\partial_t u = \Pi(|u|^2 u).$$

Pourtant, dans ce cadre fonctionnel, l'équation LLL possède un comportement différent. Gérard, Germain et Thomann [15] prouvent que l'équation est globalement bien posée sur $\mathcal{E}$, et qu'elle propage la régularité additionnelle, encodée par la décroissance spatiale des données. De plus, la croissance des normes de Sobolev est au plus polynomiale en temps — donc moins vigoureuse, potentiellement, que la croissance sur-polynomiale observée pour l'équation de Szegő cubique sur $\mathbb{T}$.

Une question intéressante et ouverte sur l'équation LLL est celle de l'existence d'une paire de Lax, inconnue à ce jour.

Le flot conforme sur $\mathbb{S}^3$

Dans leur étude de l'équation des ondes sur le cylindre d'Einstein, c'est-à-dire la variété $\mathbb{R} \times \mathbb{S}^3$ munie de la métrique lorentzienne

$$g = -dt^2 + dx^2 + (\sin x)^2 d\varpi^2,$$

où $x \in [-\pi, \pi]$, et où ϖ est la métrique riemannienne usuelle sur la sphère unité $\mathbb{S}^2$, Bizoń et ses collaborateurs [4] se rapportent à l'étude d'un système résonnant qui s'écrit, dans l'espace de Fourier, sous la forme

$$i\dot{\alpha}_n = \sum_{j=0}^{+\infty} \sum_{k=0}^{n+j} \left[\frac{\min(n,j,k,n+j-k)+1}{n+1} \right] \overline{\alpha_j} \alpha_k \alpha_{n+j-k}, \quad n \in \mathbb{N}. \tag{2.8}$$

C'est cette équation que l'on appelle le flot conforme sur $\mathbb{S}^3$. Dans (2.8), les facteurs, devant les produits de trois termes, sont souvent égaux à 1 (lorsque les fréquences qui interagissent sont grandes), et on peut penser que cette équation, qualitativement, doit partager certaines propriétés avec

$$i\dot{\alpha}_n = \sum_{j=0}^{+\infty} \sum_{k=0}^{n+j} \overline{\alpha_j} \alpha_k \alpha_{n+j-k}, \quad n \in \mathbb{N},$$

qui n'est autre que l'équation de Szegő cubique (2.2) sur $\mathbb{T}$. Cette similarité avec l'équation de Szegő donne l'idée de chercher des solutions de (2.8) sous forme de fractions rationnelles, *i.e.*

$$\alpha_n = (b + an) \cdot p^n, \quad \forall n \in \mathbb{N},$$

où a, b, $p \in \mathbb{C}$ avec $|p| < 1$, et dépendent du temps. Cet ansatz est consistant, et prouve donc l'existence d'une variété complexe de dimension 3 invariante par le flot de (2.8).

L'étude des ondes progressives et de leur stabilité est également commencée dans [5], mais l'existence d'une paire de Lax pour (2.8) n'est pas encore établie.

L'équation des *half-wave maps*

Une équation à mi-chemin entre l'équation de Szegő cubique et l'équation de demi-onde cubique a récemment connu un regain d'intérêt : il s'agit de l'équation des *half-wave maps*, qui apparaît naturellement en physique comme une limite continue de chaîne de spins (cf. [72]). Son cadre hamiltonien, légèrement différent des précédents, mérite qu'on s'y arrête.

Dans tout ce paragraphe, on considère des fonctions à valeurs dans la sphère $\mathbb{S}^2$, que l'on voit comme un sous-ensemble de $\mathbb{R}^3$. Pour $y_1, y_2 \in \mathbb{R}^3$, on note $y_1 \cdot y_2$ le produit scalaire réel standard entre y_1 et y_2. À présent, si $u : \mathbb{R} \to \mathbb{S}^2$, et si h_1, h_2 sont des fonctions sur $\mathbb{R}$ telles que $\forall x \in \mathbb{R}$, $h_1(x), h_2(x) \in T_{u(x)}\mathbb{S}^2$ l'espace tangent à $\mathbb{S}^2$ en $u(x)$, c'est-à-dire plus simplement telles que $h_1(x) \perp u(x)$ et $h_2(x) \perp u(x)$, alors on pose

$$\omega_u(h_1, h_2) := \int_{\mathbb{R}} \det\left[u(x), h_1(x), h_2(x)\right] dx.$$

On définit ainsi une forme symplectique sur l'espace (par exemple) des fonctions à valeurs dans $\mathbb{S}^2$ et dont chaque coordonnée est dans $L^3(\mathbb{R})$. Cette forme, contrairement aux cas précédemment examinés, n'est pas constante sur l'espace des phases, mais dépend du point u où on la considère.

Définissons l'opérateur $|\nabla|$, par analogie avec (1.16), dans l'espace de Fourier :

$$\widehat{(|\nabla| f)}(\xi) = |\xi| \widehat{f}(\xi), \quad \xi \in \mathbb{R},$$

pour une fonction $f \in L^2(\mathbb{R})$. Lorsque $u : \mathbb{R} \to \mathbb{S}^2$, $x \mapsto (u_1(x), u_2(x), u_3(x))$, la fonction $|\nabla| u$ désigne simplement $(|\nabla| u_1, |\nabla| u_2, |\nabla| u_3)$. On considère donc l'énergie suivante :

$$\mathcal{H}(u) := \frac{1}{2} \int_{\mathbb{R}} (u \cdot |\nabla| u),$$

définie sur les fonctions dont chaque composante est dans $\dot{W}^{1,6}(\mathbb{R})$. Si nous différentions $\mathcal{H}$ en u,

$$\langle d\mathcal{H}(u), h\rangle = \int_{\mathbb{R}} (h \cdot |\nabla| u)$$

où $h(x) \perp u(x)$ pour tout $x \in \mathbb{R}$. Introduisons artificiellement des termes dans cette expression, en utilisant le fait que $\|u(x)\|_{\mathbb{R}^3} = 1$ et que $h(x) \cdot u(x) = 0$:

$$\begin{aligned}
\langle d\mathcal{H}(u), h\rangle &= \int_{\mathbb{R}} (h \cdot [(u \cdot u)|\nabla| u - (u \cdot |\nabla| u) u]) \\
&= \int_{\mathbb{R}} (h \cdot [(u \wedge |\nabla| u) \wedge u]) \\
&= \int_{\mathbb{R}} \det[u, h, u \wedge |\nabla| u] \\
&= \omega_u(h, X_{\mathcal{H}}(u)),
\end{aligned}$$

avec $X_{\mathcal{H}}(u) = u \wedge |\nabla| u$. L'équation des *half-wave maps* est l'équation associée à $\mathcal{H}$ pour la structure symplectique ω_u, et s'énonce donc :

$$\partial_t u = u \wedge |\nabla| u. \tag{2.9}$$

La théorie de Cauchy de l'équation (2.9) est encore largement ignorée, mais on peut classifier [44] toutes ses ondes progressives d'énergie finie, *i.e.* les solutions de (2.9) de la forme $Q(x - vt)$, pour un $v \in \mathbb{R}$, telles que $\mathcal{H}(Q) < +\infty$, ou encore que $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{S}^2)$, l'ensemble des fonctions dont toutes les composantes sont dans $\dot{H}^{1/2}$. Toutes celles qui sont non-triviales sont de vitesse $|v| < 1$, et elles sont toutes issues d'une application semi-harmonique, *i.e.* telle que $v = 0$ et

$$Q \wedge |\nabla| Q \equiv 0,$$

à laquelle on applique une transformation de Lorentz de paramètre $|v| < 1$. Enfin, les applications semi-harmoniques sont toutes données par des produits de Blaschke finis.

L'abondance des ondes progressives pour (2.9) semble relever d'un phénomène d'intégrabilité, et effectivement, Gérard et Lenzmann [23] ont trouvé une paire de Lax pour l'équation, où les opérateurs qui interviennent sont des opérateurs de Hankel matriciels. Par conséquent, on sait que les solutions rationnelles de (2.9) restent rationnelles tant qu'elles existent. C'est là un signe encourageant pour l'étude de cette équation des *half-wave maps*, ainsi que pour l'étude de l'équation de demi-onde cubique, car la présence de l'opérateur pseudo-différentiel $|\nabla|$ rend ces deux équations relativement parentes. Notons enfin que les résultats que nous venons d'évoquer restent vrais lorsque les fonctions considérées sont définies sur $\mathbb{T}$ au lieu de $\mathbb{R}$.

2.3 Le cas de l'équation de Szegő quadratique

Présentons à présent le contenu des chapitres 3 à 5 de cette thèse : ils sont consacrés à l'étude d'un nouveau système modèle non-dispersif, inspiré par l'équation de Szegő cubique et se plaçant dans le même cadre d'étude qu'elle, mais où la non-linéarité est seulement quadratique.

Sur $L^4_+(\mathbb{T})$, définissons

$$\mathcal{H}_{\mathrm{quad}}(u) = \frac{1}{2}\left|\int_{\mathbb{T}} |u|^2 u\right|^2 .$$

Ce hamiltonien combine la propriété de faire intervenir seulement des intégrales cubiques en u (on en attend bien donc une équation quadratique) et d'être partout différentiable, au contraire de la norme L^3. Posons $J = J(u) := \int_{\mathbb{T}} |u|^2 u$. On a alors, pour $h \in L^2_+$,

$$\langle d\mathcal{H}_{\mathrm{quad}}(u), h\rangle = \mathrm{Re}(h|2J|u|^2 + \bar{J}u^2),$$

si bien que le système associé à $\mathcal{H}_{\mathrm{quad}}$ s'écrit

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2. \tag{2.10}$$

Comme $|J|$ est (formellement) conservé, les facteurs J et $\bar{J}$ sont réduits à une phase variable ; c'est pourquoi on appelle (2.10) l'*équation de Szegő quadratique*.

Remarquons que, tout comme le hamiltonien relatif à l'équation de Szegő cubique $\mathcal{H}_{\mathrm{cub}}$, la fonctionnelle $\mathcal{H}_{\mathrm{quad}}$ jouit des deux invariances par rotation d'angle θ à l'arrivée ($u \to e^{i\theta}u$) et par translation d'un angle $\alpha \in \mathbb{T}$ au départ ($u \to u(\cdot - \alpha)$). Cela induit les deux lois de conservation définies plus haut, la masse Q et le moment M. L'espace d'énergie pour (2.10) est donc $H^{1/2}$, et comme pour l'équation de Szegő cubique, on peut montrer qu'il existe un flot sur $H^{1/2}_+(\mathbb{T})$ qui propage la régularité (satisfaisant donc les hypothèse (H1)-(H3) décrites plus haut). Dans (2.10), on aurait pu se dispenser du facteur J en choisissant pour hamiltonien

$$\tilde{\mathcal{H}}_{\mathrm{quad}}(u) = \mathrm{Re}\left(\int_{\mathbb{T}} |u|^2 u\right),$$

dont le champ hamiltonien correspondant est tout simplement $X_{\tilde{\mathcal{H}}} = -2i\Pi(|u|^2) - iu^2$. Cependant, $\tilde{\mathcal{H}}_{\mathrm{quad}}$ n'admet plus l'invariance par rotation $u \to e^{i\theta}u$, et la norme L^2 n'est pas conservée pour l'équation $\dot{u} = X_{\tilde{\mathcal{H}}}(u)$. On peut d'ailleurs montrer [62] que toutes les solutions non nulles de cette équation explosent en temps fini.

Dans le chapitre 3, qui reprend [62], on montre que l'équation (2.10) admet une paire de Lax associée à l'opérateur de Hankel décalé K_u :

Théorème 4. *Soit $t \mapsto u(t)$ une solution de (2.10) dans H^s_+, pour $s > \frac{1}{2}$. Alors*

$$\frac{d}{dt}K_u = [B_u, K_u],$$

où $B_u = -i(T_{J\bar{u}} + T_{\bar{J}u})$ est un opérateur borné et anti-adjoint sur L^2_+.

Ce théorème a plusieurs conséquences, dont la première s'obtient en étudiant aussi l'évolution de l'opérateur H_u, qui n'obéit pas à une paire de Lax, mais à une identité algébrique qui prouve que son rang est conservé au cours du temps (cf. [64, Corollaire 2.2]) :

Corollaire 2.4. *Le flot de l'équation quadratique (2.10) laisse invariants les ensembles $\mathcal{V}(d)$, où $d \in \mathbb{N}$.*

En conséquence, l'équation de Szegő quadratique admet les mêmes variétés invariantes que l'équation de Szegő cubique.

La seconde conséquence, dans l'esprit de [22], est l'existence d'un flot pour (2.10) sur BMO_+, qui possède la double particularité (essentiellement liée à la nature quadratique de l'équation) de propager la régularité H^s quel que soit $s \geq 0$, et de posséder des estimées a priori exponentielles en temps à tous les niveaux de régularité :

Théorème 5. *Soit $u_0 \in BMO_+(\mathbb{T})$. Alors il existe une unique solution de (2.10) dans $C(\mathbb{R}, L^2_+) \cap C_{w*}(\mathbb{R}, BMO_+)$ (où la continuité faible-$*$ est entendue au sens de la dualité L^1_+–BMO_+) telle que $u(0) = 0$.*

De plus, si $R > 0$, on note $\mathcal{B}(R)$ l'ensemble des $v \in BMO_+$ tels que $\|v\|_{BMO} \leq R$, et on munit cet ensemble de la norme L^2. Alors pour tout $T > 0$, l'application

$$\mathcal{B}(R) \longrightarrow C([-T,T], L^2_+),\ u_0 \longmapsto u,$$

où u est l'unique solution au sens précédent, est lipschitzienne.

Enfin, s'il existe aussi $s > 0$ tel que $u_0 \in H^s \cap BMO_+$, alors la solution ci-dessus demeure dans H^s pour tout temps, et il existe une constante C qui ne dépend que de la norme BMO_+ de u_0, et telle que

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{C|t|}.$$

Bien sûr, la persistence de la régularité additionnelle à ce niveau de basse régularité (sous l'espace d'énergie) rend également pertinente la question de la croissance des normes de Sobolev dans ce cadre.

De plus, un raisonnement semblable à celui de [67] permet de montrer que si u est une solution dans $\mathcal{V}(2) \cup \mathcal{V}(3)$ de la forme

$$u(t,z) = b(t) + \frac{c(t)z}{1 - p(t)z}, \quad \forall z \in \mathbb{D},$$

avec $b, c, p \in \mathbb{C}$, $|p| < 1$, et $c \neq 0$, alors les normes de Sobolev de $u(t)$ croissent si et seulement si

$$\mathcal{H}_{\text{quad}}(u(0)) = \frac{1}{2}Q(u(0))^3. \tag{2.11}$$

Dans ce cas, $|p(t)| \to 1$ exponentiellement vite, et la croissance des normes H^s est exponentielle pour tout $s > \frac{1}{2}$.

Ici, une condition de croissance telle que (2.11) présente l'intérêt d'être homogène en $u(0)$. Cela permet en particulier de trouver des solutions turbulentes arbitrairement proches de 0, et donc d'énoncer sans grande peine, pour (2.10), un résultat « à la Tao » : pour tout $s > \frac{1}{2}$, tout $\varepsilon \ll 1$ et tout $K \gg 1$, il existe une solution $t \mapsto u(t)$ régulière et un temps $T > 0$, tels que $\|u(0)\|_{H^s} \leq \varepsilon$ et $\|u(T)\|_{H^s} \geq K$.

Dans le chapitre 4, tiré de [64], on donne une classification complète des ondes progressives d'énergie non nulle (*i.e.* non constantes en temps), selon une méthode qui se rapproche de [16]. On montre le théorème suivant :

Théorème 6 (Classification des ondes progressives). *L'état initial $v_0(z) \in H^{1/2}_+$ donne naissance à une onde progressive,* i.e. *une solution de (2.10) de la forme (2.5) avec $(\omega, c) \neq (0,0)$ si et seulement s'il existe $\lambda, p \in \mathbb{C}$ avec $|p| < 1$ et un entier $N \geq 1$ tels que l'un des deux points suivants soit vérifié :*

(i)

$$v_0(z) = \frac{\lambda}{1 - pz^N},$$

auquel cas

$$\omega = |\lambda|^4 \frac{3 - |p|^2}{(1 - |p|^2)^3} \quad \text{and} \quad c = \frac{|\lambda|^4}{N} \frac{1}{(1 - |p|^2)^2};$$

(ii)

$$v_0(z) = -\lambda \frac{1 + |p|^2}{1 - |p|^2} + \frac{\lambda}{1 - pz^N},$$

auquel cas

$$\omega = |\lambda|^4 |p|^4 \frac{(1 + 5|p|^2)(3 + 5|p|^2)}{(1 - |p|^2)^4} \quad \text{and} \quad c = -\frac{|\lambda|^4}{N} |p|^4 \frac{3 + 5|p|^2}{(1 - |p|^2)^3}.$$

Comme dans le cas de l'équation de Szegő cubique, on montre aussi que sur BMO_+, il n'y a pas plus d'ondes progressives avec $c \neq 0$, mais que des ondes stationnaires (telles que $c = 0$) s'ajoutent à celles qui sont listées dans le Théorème 6 : elles sont de la forme $v_0 = \lambda \Pi(\mathbb{1}_{\mathcal{B}})$, où $\lambda \in \mathbb{C}$, et où $\mathcal{B}$ est un borélien quelconque de $\mathbb{T}$.

Comme complément du Théorème 6, nous montrons que les ondes du type ***(ii)*** sont instables dans $H^{1/2}_+$, et que l'onde progressive $z \mapsto \frac{\lambda}{1-pz}$ est un état fondamental, donc est stable.

Citons également un résultat intermédiaire qui mérite d'être souligné. Il s'agit d'un résultat d'invariance, déjà prouvé dans [68] dans le contexte de l'équation de Szegő cubique :

Proposition 2.5. *Soit $u_0(z) \in BMO_+(\mathbb{T})$, et $u(t,z)$ la solution de l'équation de Szegő quadratique dans BMO_+ telle que $u(0,\cdot) = u_0$. Soit $\Psi(z)$ une fonction intérieure sur $\mathbb{D}$, c'est-à-dire une fonction telle que $|\Psi(e^{ix})| = 1$ pour tout $x \in \mathbb{T}$. Alors $\tilde{u}_0(z) := u_0(z\Psi(z))$ appartient à BMO_+, et la solution de (2.10) issue de $\tilde{u}_0$ dans BMO_+ est donnée par $u(t, z\Psi(z))$.*

À travers cette proposition, on voit en fait que les ondes progressives énumérées dans le Théorème 6 se déduisent l'une de l'autre par cet argument d'invariance (avec $\Psi(z) = z^N$, $N \geq 0$). Elles sont donc moins nombreuses que pour l'équation de Szegő cubique, ce qui traduit peut-être une intégrabilité moins forte (corroborée par l'existence de solutions rationnelles turbulentes, un phénomène proscrit dans le cas cubique — cf. [20, Theorem 2]).

Dans le chapitre 5, enfin, issu de [63], nous poursuivons l'étude des solutions rationnelles turbulentes dont un premier exemple était trouvé dans $\mathcal{V}(3)$. Pour ce faire, nous progressons dans le résultat d'intégrabilité en mettant en évidence d'autres lois de conservation. Soit $u \in H^{1/2}_+$. Notons

$$\sigma_1^2(u) \geq \sigma_2^2(u) \geq \cdots \geq \sigma_k^2(u) \geq \cdots$$

la liste des valeurs propres de K_u^2 classées par ordre décroissant. Pour $k \geq 1$, on introduit

$$\begin{aligned} u_k^K &:= \mathbb{1}_{\{\sigma_k^2(u)\}}(K_u^2)(u), \\ w_k^K &:= \mathbb{1}_{\{\sigma_k^2(u)\}}(K_u^2)(\Pi(|u|^2)), \end{aligned}$$

qui représentent respectivement la projection orthogonale de u et de $\Pi(|u|^2)$ sur l'espace de dimension finie $F_u(\sigma_k) := \ker(K_u^2 - \sigma_k^2(u)I)$. Posons enfin

$$\ell_k := \frac{1}{\operatorname{tr}(\mathbb{1}_{\{\sigma_k^2(u)\}}(K_u^2))} \left[(2Q + \sigma_k^2)\|u_k^K\|_{L^2}^2 - \|w_k^K\|_{L^2}^2 \right].$$

Théorème 7. *On a les identités suivantes, pour tout $j, k \geq 1$, valables sur l'ensemble des éléments $u \in H^{1/2}_+$ pour lesquels toutes les valeurs propres de K_u^2 sont simples :*

$$\{\ell_j, \ell_k\} = 0, \quad \{\ell_j, \sigma_k^2\} = 0, \quad \{\sigma_j^2, \sigma_k^2\} = 0.$$

De plus, les $\{\ell_k,\ k \geq 1\}$ sont des lois de conservation pour l'équation de Szegő quadratique.

Comme dans [68], on en déduit une condition nécessaire pour la croissance des normes des solutions rationnelles :

Proposition 2.6. *Supposons que $u_0 \in \mathcal{V}(d)$ pour un $d \in \mathbb{N}$, et supposons qu'il existe un $s_0 > \frac{1}{2}$ tel que la solution $u(t)$ de (2.10) issue de u_0 ne soit pas bornée dans H^{s_0}. Alors $u(t)$ n'est bornée dans aucun H^s, $s > \frac{1}{2}$, et de plus, il existe $k \geq 1$ tel que*

$$\sigma_k^2(u_0) \neq 0 \quad \textit{et} \quad \ell_k(u_0) = 0.$$

Cette condition nécessaire permet de procéder à l'analyse des solutions turbulentes de $\mathcal{V}(4)$, car elle fournit l'équivalent de (2.11) :

Théorème 8. *Soit $u_0 \in \mathcal{V}(4)$. Alors il existe $s_0 > \frac{1}{2}$ tel que la solution de (2.10) issue de u_0 ne soit pas bornée dans H^{s_0} si et seulement si $K_{u_0}^2$ admet deux valeurs propres simples $\sigma_1^2 > \sigma_2^2$, et si*

$$\mathcal{H}_{\text{quad}}(u_0) = \frac{1}{2} Q(u_0)^2 (Q(u_0) + \sigma_2^2).$$

Dans ce cas, pour tout $s > \frac{1}{2}$, il existe des constantes $C_s, C_s' > 0$ telles que pour tout $t \in \mathbb{R}$,

$$\frac{1}{C_s} e^{C_s'|t|} \leq \|u(t)\|_{H^s} \leq C_s e^{C_s'|t|}.$$

Ce résultat s'interprète donc, à la lumière de la classification des ondes progressives, comme la description de l'interaction turbulente entre deux ondes progressives, puisqu'asymptotiquement, les solutions du théorème précédent s'écrivent sous la forme exacte suivante :

$$u(t, z) = \frac{\alpha(t)}{1 - p(t)z} + \frac{\beta(t)}{1 - q(t)z}, \quad \forall z \in \mathbb{D},$$

avec α, β, p, $q \in \mathbb{C}$, et $|p|$, $|q| < 1$, et avec de plus

$$|\alpha(t)| \to 0, \quad 1 - |p(t)| = Ce^{-\tau|t|}, \quad |\beta(t)| \to \beta_\infty > 0, \quad |q(t)| \to q_\infty \in (0, 1).$$

3 Questions et perspectives

Dans la dernière partie de cette introduction, nous voudrions consigner quelques questions intéressantes et encore ouvertes au terme de ce travail de thèse.

1. *Généricité de la croissance exponentielle pour l'équation de Szegő quadratique.* Pour ce faire, il faudrait obtenir une description plus précise des ensembles de niveau des lois de conservation $(\{\sigma_j^2\}_j, \{\ell_k\}_k)$, notamment en trouvant des angles associés à ces (potentielles) actions. On sait déjà que ces angles seraient parfois des angles généralisés, car certains ensembles de niveau ne sont pas compacts dans $H^{1/2}_+$. C'est le cas, par exemple, des solutions turbulentes dans $\mathcal{V}(3)$ (resp. $\mathcal{V}(4)$), qui quittent tout compact de $H^{1/2}$: en effet, on peut voir que les seules valeurs d'adhérence faibles de leur orbite $\{u(t)\}$ sont des constantes (resp. des éléments de $\mathcal{V}(2)$), donc que l'orbite ne possède aucune point d'accumulation pour la topologie forte dans $H^{1/2}$. Pour le dire autrement, de telles orbites ne sont pas quasi-périodiques.

2. *Lien entre l'équation de Szegő quadratique et une équation de nature plus physique.* Il serait intéressant de savoir si l'équation de Szegő quadratique se trouve reliée à une équation de type « demi-onde ». Il semble naturel de considérer le hamiltonien $\mathcal{H}(u) = \frac{1}{2}(|D|u|u) + \frac{1}{2}|(u^2|u)|^2$, qui correspond à l'équation

$$i\partial_t u - |D|u = 2\left(\int_{\mathbb{T}} |u|^2 u\right)|u|^2 + \left(\int_{\mathbb{T}} |u|^2 \bar{u}\right) u^2, \tag{3.1}$$

mais l'analyse des résonances montre que l'équation de Szegő quadratique ne correspond qu'à une petite partie d'entre elles. Dès lors, peut-on corriger l'équation de demi-onde (3.1) pour que ses seules résonances non triviales mènent à l'équation de Szegő quadratique (2.10) ?

3. *Estimées exponentielles pour l'équation de demi-onde cubique.* Les meilleures estimées a priori connues à ce jour pour la croissance des normes de Sobolev des solutions de l'équation de demi-onde cubique (1.16) sont les estimées en $e^{C|t|^2}$ prouvées au chapitre 2 de cette thèse. Ces bornes sont nettement moins satisfaisantes que celles dont on dispose grâce au Théorème 2 pour l'équation de Szegő cubique (2.2), qui sont simplement en $e^{C'|t|}$. Ces dernières sont établies par un argument « à la Gronwall », en montrant que les solutions assez régulières de l'équation (2.2) vérifient

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty} < +\infty. \tag{3.2}$$

Il suffirait de montrer cela pour l'équation de demi-onde, et les estimées exponentielles suivraient de la même façon. Or, on ne sait pour le moment comment montrer (3.2) sans recourir aux propriétés des opérateurs de Hankel, et à la paire de Lax, qui fait cruellement défaut dans le cas de la demi-onde (1.16). Pour autant, il suffirait de montrer un résultat plus faible que (3.2), comme par exemple que si $t \mapsto u(t)$ est une solution régulière de l'équation de demi-onde, alors $\int_0^T \|u(t)\|_{L^\infty} dt = \mathcal{O}(T)$. Un tel résultat semble moins inaccessible que (3.2), car il a précisément été prouvé par Lindblad et Tao [45] (avec même un $o(T)$ au lieu du $\mathcal{O}(T)$) dans le cas de l'équation des ondes semi-linéaire en dimension 1.

4. *Construction de solutions turbulentes pour l'équation de demi-onde cubique sur* $\mathbb{T}$. Le résultat de [17] montre que les solutions de l'équation de demi-onde cubique, lorsque leur donnée initiale est de taille $\varepsilon \ll 1$ dans H^s, $s > 1$, sont correctement approchées par les solutions de l'équation de Szegő pendant un temps de l'ordre de $\varepsilon^{-2}|\log \varepsilon|$, ce qui représentente une amélioration d'un facteur $|\log \varepsilon|$ par rapport au temps caractéristique au-delà duquel les

effets non-linéaires se font sentir, et qui vaut ici ε^{-2} puisque la non-linéarité est cubique. Or n'importe quel résultat d'approximation valable sur un intervalle $[0, \varepsilon^{-2-\beta}]$, pour n'importe quel $\beta > 0$, permettrait de mettre en évidence des solutions à la croissance nettement plus significative pour l'équation de demi-onde (1.16). On peut penser que le facteur $|\log \varepsilon|$ est insuffisant car le résultat est trop général, et valable pour toutes les données de taille ε. Dès lors, est-il possible de construire des solutions particulières de (1.16), partant d'une certaine donnée initiale à modes positifs *bien choisie*, et qu'approxime de façon plus satisfaisante la solution de l'équation de Szegő cubique correspondant à la même donnée — on pense typiquement à la solution « marguerite » évoquée au paragraphe 2.1 ? Pour ce faire, on pourrait recourir à une méthode de type « optique géométrique », en construisant la solution par briques successives de plus en plus petites, dont la première serait précisément la solution de l'équation de Szegő.

5. *Détermination d'ondes progressives et de tores invariants pour l'équation de demi-onde cubique.* Dans [24], il est démontré qu'il existe des ondes progressives de célérité $c \in [1-\delta, 1[$, $\delta \ll 1$, pour l'équation de demi-onde cubique focalisante sur $\mathbb{R}$ qui, correctement normalisées, convergent lorsque $c \to 1$ vers l'état fondamental de l'équation de Szegő cubique, soit $z \mapsto \frac{\alpha}{1-pz}$. Un même résultat demeure valable sur $\mathbb{T}$. De plus, une observation très simple montre que si $\Psi \in L^2_+$ est une fonction intérieure sur $\mathbb{T}$, alors $e^{-it}\Psi(e^{i(x-t)})$ est solution de (1.16), et est une onde progressive de vitesse 1. Est-il possible de classifier les ondes progressives pour l'équation de demi-onde cubique (1.16) ? Pour ce faire, on peut espérer s'appuyer, comme dans [64], sur une paire de Lax approchée vérifiée par les solutions de (1.16) : si $t \mapsto u(t)$ est solution de l'équation de demi-onde cubique dans H^s, $s > \frac{1}{2}$, et si nous notons $u^+ := \Pi(u)$ et $u^- := (I - \Pi)(u)$, nous trouvons

$$\begin{cases} i\dfrac{d}{dt}K_{u^+} = (D + \dfrac{1}{2}I + T_{|u|^2} - \dfrac{1}{2}K^2_{u^+})K_{u^+} + K_{u^+}(D + \dfrac{1}{2}I + T_{|u|^2} - \dfrac{1}{2}K^2_{u^+}) + T_u K_{\overline{u^-}} T_{\bar{u}}, \\ -i\dfrac{d}{dt}K_{\overline{u^-}} = (D + \dfrac{1}{2}I + T_{|u|^2} - \dfrac{1}{2}K^2_{\overline{u^-}})K_{\overline{u^-}} + K_{\overline{u^-}}(D + \dfrac{1}{2}I + T_{|u|^2} - \dfrac{1}{2}K^2_{\overline{u^-}}) + T_{\bar{u}} K_{u^+} T_u. \end{cases}$$

Semblablement, il est intéressant de se demander si les tores de dimension supérieure ou égale à 3 pour l'équation de Szegő persistent également pour l'équation de demi-onde cubique. On s'attendrait cette fois à utiliser l'arsenal de la théorie KAM.

Chapitre 2

On the growth of Sobolev norms of solutions of the fractional defocusing NLS equation on the circle

Abstract

This paper is devoted to the study of large time bounds for the Sobolev norms of the solutions of the following fractional cubic Schrödinger equation on the torus :

$$i\partial_t u = |D|^\alpha u + |u|^2 u, \quad u(0,\cdot) = u_0,$$

where α is a real parameter. We show that, apart from the case $\alpha = 1$, which corresponds to a half-wave equation with no dispersive property at all, solutions of this equation grow at a polynomial rate at most. We also address the case of the cubic and quadratic half-wave equations.

1 Introduction

In the study of Hamiltonian partial differential equations, understanding the large time dynamics of solutions is an important issue. In usual cases, the conservation of the Hamiltonian along trajectories enables to control one Sobolev norm of the solution (in the so-called *energy space*), but when solutions are globally defined and regular, higher norms could grow despite the conservation laws, reflecting an energy transfer to high frequencies. Even for notorious equations, such as the nonlinear Schrödinger equation on manifolds, it is an old problem to know whether such instability occurs [8], and often still an open question.

Let us start for instance from the particular case of the defocusing Schrödinger equation on the torus of dimension one, with a cubic nonlinearity :

$$i\partial_t u = -\partial_x^2 u + |u|^2 u, \quad u(0,\cdot) = u_0. \tag{1.1}$$

Here, u is a function of time $t \in \mathbb{R}$ and of space variable $x \in \mathbb{T}$, and $u_0 \in H^1(\mathbb{T})$. Equation (1.1) is Hamiltonian, and because of the energy conservation, its trajectories are bounded is $H^1(\mathbb{T})$. But it is also well-known that (1.1) is integrable (see [27], [71]), with conservation laws ensuring that if u_0 belongs to $H^s(\mathbb{T})$ for some $s \in \mathbb{N}\backslash\{0\}$, then the solution u remains bounded in H^s (this is even true for any real $s \geq 1$ [38]).

In order to track down large time instability for Hamiltonian systems, Majda, McLaughlin and Tabak [47] suggested to replace the Laplacian in (1.1) by a whole family of pseudo-differential operators : the operators $|D|^\rho$ (sometimes written as $(\sqrt{-\partial_x^2})^\rho$) for real ρ. Recall that if $w = \sum_{k\in\mathbb{Z}} w_k e^{ikx}$ is a function on the torus, then

$$|D|^\rho w = \sum_{k\in\mathbb{Z}} |k|^\rho w_k e^{ikx}.$$

So we consider the following fractional Schrödinger equation :

$$i\partial_t u = |D|^\alpha u + |u|^2 u, \quad u(0,\cdot) = u_0, \tag{1.2}$$

where α is any positive number. If $\alpha = 2$, we recognize the classical Schrödinger equation (1.1). In the case $\alpha = 1$, (1.2) is a non-dispersive equation, called the (defocusing) "half-wave" equation :

$$i\partial_t u = |D| u + |u|^2 u, \quad u(0,\cdot) = u_0. \tag{1.3}$$

This half-wave equation has been studied by Gérard and Grellier in [17]. In particular, they show that the dynamics of (1.3) is related to the behaviour of the solutions of a toy model equation, called the cubic Szegő equation :

$$i\partial_t u = \Pi_+(|u|^2 u), \quad u(0,\cdot) = u_0, \tag{1.4}$$

where $\Pi_+ := \mathbf{1}_{D\geq 0}$ is the projection onto nonnegative Fourier modes. In a more precise way, equation (1.4) appears to be the completely resonant system of (1.3).

All the equations (1.2) derive from the Hamiltonian $\mathcal{H}_\alpha(u) := \frac{1}{2}(|D|^\alpha u, u) + \frac{1}{4}\|u\|_{L^4}^4$ for the symplectic structure endowed by the form $\omega(u,v) = \Im m(u,v)$, where $(u,v) := \int_{\mathbb{T}} u\bar{v}$ denotes the standard inner product on $L^2(\mathbb{T})$. The functional $\mathcal{H}_\alpha$ is therefore conserved along trajectories. Gauge invariance as well as translation invariance also imply the existence of two other conservation laws for equation (1.2) :

$$Q(u) := \frac{1}{2}\|u\|_{L^2}^2$$
$$M(u) := (Du, u), \quad \text{where } D := -i\partial_x,$$

i.e. the mass and the momentum respectively. Starting from these observations, it has been proved that for $\alpha = 1$, equation (1.3) admits a globally defined flow in H^s with $s \geq \frac{1}{2}$ (see [17]). In the case of the half-wave equation, the Brezis-Gallouët inequality [9] also ensures that H^s-norms of solutions grow at most like $e^{\exp B|t|}$, for some constant $B > 0$ depending on s and on the initial data.

The question of the large time instability of global solutions of (1.2) thus naturally arises : is it possible to find smooth initial data whose corresponding orbits are not bounded in some H^s space, or at least not polynomially bounded [1] ?

The cubic Szegő equation discloses this kind of instability, as recently shown in [19], [21] : for generic smooth initial data, the corresponding solution of the Szegő equation in H^s is polynomially unbounded, for any $s > \frac{1}{2}$. Therefore it is reasonable to think that the same statement should hold for the half-wave equation (1.3), though such a result seems far beyond our reach at the current stage of the theory. Nevertheless the theorem we prove in this paper gives an a priori bound for all solutions of the half-wave equation :

Theorem 1. *Let $u_0 \in \mathcal{C}^\infty(\mathbb{T})$, and $t \mapsto u(t)$ the solution of the half-wave equation (1.3) such that $u(0) = u_0$. Given any integer $n \geq 0$, we have*

$$\|u(t)\|_{H^{1+n}} \leq C e^{B|t|^2}, \quad \forall t \in \mathbb{R}, \tag{1.5}$$

where B can be chosen as $B_n \|u_0\|^8_{H^{1/2}}$ with $B_n > 0$ depending only on n, and where C can be chosen to depend only on n and $\|u_0\|_{H^{1+n}}$.

The bound appearing in (1.5) is an improvement the "double exponential bound" mentioned above. But as a matter of fact, finding any explicit non-trivial solution of (1.2) is still an open problem, and nothing is known about the optimality of (1.5). Solution with rapidly growing H^s-norms could perfectly well exist. H. Xu [67] typically proved the existence of exponentially growing solutions for a perturbation of the Szegő equation. Also striking is the result of Hani–Pausader–Tzvetkov–Visciglia [32], in the context of the Schrödinger equation, as well as its recent counterpart in [69].

Notice that in the case of the Szegő equation, the best bound quantifying the growth of Sobolev norms of solutions is $e^{B|t|}$: it was obtained by Gérard and Grellier in [16, section 3]. Hence (1.5) is likely to be improved, but recall that, as far as we know, the only way of proving the simple exponential bound for Szegő solutions makes use of the Lax pair structure associated with the equation. Elementary methods would only give an $e^{B|t|^2}$ bound (see Appendix A). Unfortunately, such a Lax pair structure apparently does not exist as regards the half-wave equation.

Even so, the proof of (1.5) in theorem 1 suggests that we could get a simple exponential bound, instead of $e^{B|t|^2}$, if we could deal with a less than cubic nonlinearity, say quadratic. Putting an L^3-norm in the energy $\mathcal{H}_1$, instead of the L^4 one, would give rise to a nonlinearity of the form $|u|u$, but the singularity at the origin may lead to solutions less regular than their initial data. It is possible to avoid this phenomenon by considering a system of two equations rather than a single scalar one :

$$\begin{cases} i\partial_t u_1 = |D|u_1 + u_2\overline{u_1}, \\ i\partial_t u_2 = |D|u_2 + \dfrac{u_1^2}{2}, \end{cases} \tag{1.6}$$

with $(u_1, u_2)|_{t=0} = (u_1^0, u_2^0)$. System (1.6) only involves (analytic) quadratic nonlinearities. It happens that Schrödinger systems of that kind frequently appear in physics : they are

1. We say that a solution $t \mapsto u(t)$ is *polynomially bounded* in H^s if there are positive constants C and A (not depending on time) such that for all $t \in \mathbb{R}$, $\|u(t)\|_{H^s} \leq C(1 + |t|)^A$.

closely linked with the SHG (Second-Harmonic Generation) theory in optics, and the study of propagation of solitons in so-called $\chi^{(2)}$ (or quadratic) media or materials (for a review, see e.g. [41, section 4]). Quadratic systems are also relevant in fluid mechanics, to describe the interaction between long nonlinear waves in fluid flows [26]. From a mathematical point of view, the interest in quadratic systems is more recent [35].

In the case of system (1.6), we prove the following theorem :

Theorem 2. *Let $(u_1^0, u_2^0) \in \mathcal{C}^\infty(\mathbb{T}) \times \mathcal{C}^\infty(\mathbb{T})$, and $t \mapsto (u_1(t), u_2(t))$ the solution of (1.6) such that $(u_1(0), u_2(0)) = (u_1^0, u_2^0)$. Given any integer $n \geq 0$, we have*

$$\|u_1(t)\|_{H^{1+n}},\ \|u_2(t)\|_{H^{1+n}} \leq Ce^{B'|t|}, \quad \forall t \in \mathbb{R}, \tag{1.7}$$

where B' can be chosen as $B'_n(\|u_1^0\|^2_{H^{1/2}} + \|u_2^0\|^2_{H^{1/2}})$ with $B'_n > 0$ depending only on n, and where C can be chosen to depend only on n and on the sum $\|u_1^0\|_{H^{1+n}} + \|u_2^0\|_{H^{1+n}}$.

Let us now return to equation (1.2). When $\alpha \neq 1$, (1.2) has dispersive properties. Using them for $\alpha > 1$ and proving some Strichartz estimate for the operator $e^{-it|D|^\alpha}$, Demirbas, Erdoğan and Tzirakis show in [13] that (1.2) is globally well-posed in the energy space $H^{\frac{\alpha}{2}}$ (and even below), with a method relying on Bourgain's high-low frequency decomposition.

Still for $\alpha > 1$, a naive calculation leads to an exponential bound for H^s-norms of solutions, *i.e.* a bound of the form $e^{A|t|}$. On the other hand, results such as Bourgain's [7] or Staffilani's [58] suggest that because of dispersion, solutions should be polynomially bounded. The polynomial growth of solutions of (1.2) for $\alpha > 1$ is also announced to be true in [13]. Indeed we establish a theorem also involving (part of) the case $\alpha < 1$:

Theorem 3. *Let $\alpha \in (\frac{2}{3}, 1) \cup (1, 2)$, and $u_0 \in \mathcal{C}^\infty(\mathbb{T})$. There exist a unique $u \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{C}^\infty(\mathbb{T}))$ solution of (1.2) with $u(0) = u_0$. Furthermore, this solution satisfies*

$$\|u(t)\|_{H^{\alpha+n}} \leq C_1(1 + C_2|t|)^A, \quad \forall t \in \mathbb{R},\ \forall n \in \mathbb{N}, \tag{1.8}$$

where

$$A := \begin{cases} \dfrac{2n+\alpha}{\alpha-1} & \text{when } \alpha \in (1,2), \\ \dfrac{2n(23\alpha-2)}{(2\alpha-1)(3\alpha-2)} + \dfrac{10\alpha}{3\alpha-2} & \text{when } \alpha \in (\frac{2}{3}, 1), \end{cases}$$

and C_1 and C_2 are positive constants.

In the case when $\alpha \in (1, 2)$, C_1 depends only on $\|u_0\|_{H^{\alpha+n}}$, and C_2 can be chosen as $C_{\alpha,n}\|u_0\|^\kappa_{H^{\alpha/2}}$ with $\kappa := 4 + \frac{\alpha-1}{2n+\alpha}$ and $C_{\alpha,n} > 0$ a constant which only depends on α and n.

Here again, it is not known whether (1.8) is optimal or not. Theorem 3, of course, does not prevent solutions of (1.2) from large time blow-up, and proving that some of them go to infinity in a certain topology, even at a very low rate, would be a big step forward.

Combining theorem 1 and 3 thus indicate that $\alpha = 1$ is an isolate point in the family of equations (1.2). Notice that, when $\alpha < 1$, theorem 3 includes the existence of a flow, which had not been proved so far. As for the condition $\alpha > \frac{2}{3}$, it appears to be convenient in the proof for technical reasons ; but since the heart of our work is to prove that the case $\alpha = 1$ is more likely to disclose weak turbulence phenomena than other cases, we postpone discussions and comments concerning the relevance of the value $\frac{2}{3}$ until Appendix B.

The proof of theorems 1, 2 and 3 is based on an idea which appeared in [65] and was later developped by Ozawa and Visciglia in [50] : in this last paper, the authors use a *modified energy method*, in order to sharpen H^1-estimates and thus prove well-posedness for the half-wave equation with quartic nonlinearity. To put it shortly, the idea consists in introducing a nonlinear energy which is in fact a perturbation of the norm one wishes to bound. The perturbation does not modify the size of the norm, but induces simplifications while differentiating, so that time-differentiation behaves like an anti-self-adjoint operator.

In the sequel of this paper, we begin by proving theorems 1, 2, and the first part of theorem 3, which can be done by elementary means. Then we address the case of $\alpha < 1$, using Bourgain spaces as in [6], [10], [11] — a theory which we fully develop for the convenience of the reader.

We were about to finish this paper when we were informed of a work by Planchon and Visciglia also applying the modified energy method to solutions of nonlinear Schrödinger equations on certain Riemaniann manifolds, and for every power nonlinearity.

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2 The case $\alpha \geq 1$

Throughout this section, we suppose that $\alpha \in [1, 2)$. We fix $u_0 \in \mathcal{C}^\infty(\mathbb{T})$, and we study $t \mapsto u(t)$, the associated solution of (1.2) (or (1.3), depending on the value of α).

2.1 The modified energy method

Fix $n \in \mathbb{N}$. The following lemma gathers some standard inequalities of which we will make an extensive use.

Lemma 2.1. *There exist an absolute constant $C > 0$, a constant C_α depending on α and a constant C_n depending on n such that, for every $t \in \mathbb{R}$,*

(i) $\|u(t)\|_{H^{\alpha/2}} \leq C\|u_0\|^2_{H^{\alpha/2}}$,

(ii) if $\alpha > 1$, $\|u(t)\|_{L^\infty} \leq C_\alpha \|u_0\|^2_{H^{\alpha/2}}$,

(iii) if $\alpha = 1$, $\|u(t)\|_{L^\infty} \leq C_n \|u_0\|^2_{H^{1/2}} \sqrt{\log\left(1 + \dfrac{\|u(t)\|^2_{H^{1+n}}}{C^2\|u_0\|^4_{H^{1/2}}}\right)}$.

We justify briefly these inequalities : (i) derives from the conservation of $Q + \mathcal{H}_\alpha$ together with the Sobolev embeddings in dimension one, and (ii) is a consequence of (i) and the injection $H^{\alpha/2} \hookrightarrow L^\infty$. As for (iii), it follows from the classical Brezis-Gallouët inequality : for $s > \frac{1}{2}$, and $w \in H^s(\mathbb{T})$,

$$\|w\|_{L^\infty} \leq C_s \|w\|_{H^{1/2}} \left[\log\left(1 + \frac{\|w\|_{H^s}}{\|w\|_{H^{1/2}}}\right)\right]^{1/2} \tag{2.1}$$

(see e.g. [16]). Here, to infer (iii), we begin by squaring the ratio of the two norms, and we then take into account the fact that the function $x \mapsto x\sqrt{\log(1 + \frac{1}{x^2})}$ is increasing.

To prove the estimates we have in mind, we are going to establish an inequality between the $H^{\alpha+n}$-norm of the solution and its derivative, and apply a Gronwall lemma. As announced, we define for this purpose a well-chosen nonlinear functional [2] :

$$\mathcal{E}_{\alpha,n}(u) := \|u\|_{L^2}^2 + \||D|^{\alpha+n}u\|_{L^2}^2 + 2\Re e(|D|^{\alpha+n}u, |D|^n(|u|^2u)) - \tfrac{1}{2}\||D|^{\frac{\alpha}{2}+n}(|u|^2)\|_{L^2}^2. \quad (2.2)$$

Roughly speaking, $\mathcal{E}_{\alpha,n}$ is a perturbation of the square of the $H^{\alpha+n}$-norm of u (the first two terms) by the means of two corrective quantities. First of all, let us show that the latter do not substantially modify the size of $\|u\|_{H^{\alpha+n}}^2$. To turn this into a rigorous statement, we begin by restricting ourselves to intervals of time on which $\|u\|_{H^{\alpha+n}}$ is larger than a certain constant $\mathcal{M}$ depending on $\|u_0\|_{H^{\alpha/2}}$, and we show that on such intervals,

$$\frac{1}{2}\|u\|_{H^{\alpha+n}}^2 \leq \mathcal{E}_{\alpha,n}(u) \leq 2\|u\|_{H^{\alpha+n}}^2, \quad (2.3)$$

for a suitable choice of $\mathcal{M}$ which we precise later.

Set $J_1(u) := 2\Re e(|D|^{\alpha+n}u, |D|^n(|u|^2u))$. We can write [3]

$$\begin{aligned}|J_1(u)| &\lesssim \||D|^{\alpha+n}u\|_{L^2}\||D|^n(|u|^2u)\|_{L^2}\\ &\lesssim \|u\|_{H^{\alpha+n}}\|u\|_{L^\infty}^2\|u\|_{H^n},\end{aligned}$$

where we used the tame estimates for products in H^s, for $s \geq 0$. Then interpolate H^n between $H^{\alpha+n}$ and $H^{\alpha/2}$ (or just bound the L^2-norm by a constant if $n = 0$), and using lemma 2.1, get

$$|J_1(u)| \lesssim \|u\|_{H^{\alpha+n}}^{2-\varepsilon_{\alpha,n}}\|u_0\|_{H^{\alpha/2}}^{2\varepsilon_{\alpha,n}}\|u\|_{L^\infty}^2, \quad \text{where } \varepsilon_{\alpha,n} := \min\left(1, \frac{2\alpha}{2n+\alpha}\right).$$

On the other side, introducing $J_2(u) := -\||D|^{\frac{\alpha}{2}+n}(|u|^2)\|_{L^2}^2$, we similarly obtain :

$$|J_2(u)| \lesssim \|u\|_{L^\infty}^2\|u\|_{H^{\frac{\alpha}{2}+n}}^2 \lesssim \|u\|_{H^{\alpha+n}}^{2-\varepsilon_{\alpha,n}}\|u_0\|_{H^{\alpha/2}}^{2\varepsilon_{\alpha,n}}\|u\|_{L^\infty}^2.$$

In sight of (ii) and (iii), all these estimates show that $J_1(u)$ and $J_2(u)$ are of lower order than $\|u\|_{H^{\alpha+n}}^2$. More precisely, if we now set, for real x,

$$g_\alpha(x) := \begin{cases} x^{\varepsilon_{\alpha,n}/2} & \text{if } \alpha > 1, \\ \dfrac{x^{\varepsilon_{1,n}/2}}{\log\left(1 + \frac{x^2}{C^2\|u_0\|_{H^{1/2}}^4}\right)} & \text{if } \alpha = 1, \end{cases}$$

we see that it suffices to request for instance that $g_\alpha(\|u\|_{H^{\alpha+n}}) \gg \|u_0\|_{H^{\alpha/2}}^{4+2\varepsilon_{\alpha,n}}$, which holds true whenever $\|u\|_{H^{\alpha+n}}$ is greater than a certain $\mathcal{M}$. As a conclusion, (2.3) is proved on intervals of the form $[T^*, T]$ which satisfy

$$\|u(t)\|_{H^{\alpha+n}} \geq \tilde{\mathcal{M}} := \max\{\mathcal{M}, 2\|u_0\|_{H^{\alpha+n}}\}, \, \forall t \in [T^*, T], \quad \text{and} \quad \|u(T^*)\|_{H^{\alpha+n}} = \tilde{\mathcal{M}}.$$

Now we study the evolution of $\mathcal{E}_{\alpha,n}(u)$ on $[T^*, T]$. As the L^2-norm of u is conserved, we denote by $J_0(u) := \||D|^{\alpha+n}u\|_{L^2}^2$, and compute at once :

$$\frac{d}{dt}J_0(u) = 2\Re e(|D|^{\alpha+n}\dot{u}, |D|^{\alpha+n}u),$$

2. From now on, the time-dependence of the terms will always be implicit. In addition, we will always restrict ourselves to nonnegative times $t \geq 0$, since it is possible to reverse the evolution of (1.2) via the transformation $u(t) \leftrightarrow \bar{u}(-t)$.

3. The symbol $\lesssim$ is understood as refering to constants depending only on n, or absolute constants, whose explicit form is not particularly meaningful.

where the dot refers to the time-derivative, and commutes with $|D|^\rho$ for all ρ. According to equation (1.2), $|D|^\alpha u = i\dot{u} - |u|^2 u$, so

$$\frac{d}{dt} J_0(u) = 2\Im m(|D|^{\alpha+n}\dot{u}, |D|^n \dot{u}) - 2\Re e(|D|^{\alpha+n}\dot{u}, |D|^n(|u|^2 u)).$$

Because of the imaginary part, the first term of this sum is zero. As for the second one, it combines with the time-derivative of $J_1(u)$, and we thus have

$$\begin{aligned}\frac{d}{dt}[J_0 + J_1](u) &= 2\Re e(|D|^{\alpha+n}u, |D|^n(|u|^2 u)^{\cdot}) \\ &= 2\Re e\left(\partial_x^n |D|^\alpha u, \partial_x^n(\dot{u}|u|^2 + u[|u|^2]^{\cdot})\right).\end{aligned}$$

Applying Leibniz formula, we get three terms (or only two when $n = 0$) :

$$\begin{aligned}&2\Re e(\partial_x^n |D|^\alpha u, (\partial_x^n \dot{u})|u|^2) \\ &+ 2\Re e\left(\partial_x^n |D|^\alpha u, \sum_{k=1}^{n} \binom{n}{k} [(\partial_x^{n-k}\dot{u})\partial_x^k(|u|^2) + (\partial_x^k u)\partial_x^{n-k}(|u|^2)^{\cdot}\right) \\ &+ 2\Re e(\partial_x^n |D|^\alpha u, u\partial_x^n(|u|^2)^{\cdot}).\end{aligned}$$

Each of these terms has to be estimated. The first one and the third one are more tricky, since all the time- and space-derivative are concentrated on the same function.

First term : A simplification fortunately occurs. Rewrite

$$\partial_x^n |D|^\alpha u = i\partial_x^n \dot{u} - \partial_x^n(|u|^2 u),$$

and observe that $\Re e(i\partial_x^n \dot{u}, (\partial_x^n \dot{u})|u|^2) = 0$. The first term then equals $-2\Re e(\partial_x^n(|u|^2 u), (\partial_x^n \dot{u})|u|^2)$. Let Q_1 be this new quantity. Assuming that $n \geq 1$, we can bound

$$|Q_1| \lesssim \||u|^2 u\|_{H^n} \|u\|_{L^\infty}^2 \|\dot{u}\|_{H^n} \lesssim \|u\|_{L^\infty}^4 \|u\|_{H^n}(\|u\|_{H^{\alpha+n}} + \|u\|_{L^\infty}^2 \|u\|_{H^n}).$$

Indeed, because of the equation, we have $\|\dot{u}\|_{H^s} \lesssim \|u\|_{H^{\alpha+s}} + \|u\|_{L^\infty}^2 \|u\|_{H^s}$ for any $s \geq 0$, so that $\|\dot{u}\|_{H^n} \lesssim \|u\|_{H^{\alpha+n}}$ (using again the property of the interval $[T^*, T]$). Hence $|Q_1| \lesssim \|u\|_{H^{\alpha+n}}^{2-\varepsilon_{\alpha,n}} \|u_0\|_{H^{\alpha/2}}^{2\varepsilon_{\alpha,n}} \|u\|_{L^\infty}^4$ with the same $\varepsilon_{\alpha,n}$ as above (notice that it is true even if $n = 0$).

Second term : As announced, we suppose here that $n \geq 1$, and fix a $k \in \{1, \cdots, n\}$. We must estimate $Q_2^{(k)} := 2\Re e(\partial_x^n |D|^\alpha u, (\partial_x^{n-k}\dot{u})\partial_x^k(|u|^2))$. Using the Sobolev embedding $H^{1/4} \hookrightarrow L^4$ as well as tame estimates again, write

$$\begin{aligned}|Q_2^{(k)}| &\lesssim \|\partial_x^n |D|^\alpha u\|_{L^2} \|\partial_x^{n-k}\dot{u}\|_{L^4} \|\partial_x^k(|u|^2)\|_{L^4} \\ &\lesssim \|u\|_{H^{\alpha+n}} \|u\|_{H^{\alpha+n-k+\frac{1}{4}}} \|u\|_{L^\infty} \|u\|_{H^{k+\frac{1}{4}}}.\end{aligned}$$

Interpolate the H^s-norms between $H^{\alpha/2}$ et $H^{\alpha+n}$, and get finally

$$|Q_2^{(k)}| \lesssim \|u\|_{H^{\alpha+n}}^{2-\theta(\alpha,n)} \|u_0\|_{H^{\alpha/2}}^{2+2\theta(\alpha,n)} \|u\|_{L^\infty},$$

where $\theta(\alpha, n) := \frac{\alpha-1}{2n+\alpha}$.

In the same way, $Q_2'^{(k)} := 2\Re e(\partial_x^n |D|^\alpha u, (\partial_x^k u)\partial_x^{n-k}(|u|^2)^{\cdot})$ can be proven to be controlled by the same quantity (with the same exponents).

Third term : This term is the most delicate. We have

$$\begin{aligned}2\Re e(\partial_x^n |D|^\alpha u, u\partial_x^n(|u|^2)^{\cdot}) &= 2\Re e\left(\bar{u}\partial_x^n |D|^\alpha u, \partial_x^n(|u|^2)^{\cdot}\right) \\ &\simeq 2\Re e\left(\partial_x^n(\bar{u}|D|^\alpha u), \partial_x^n(|u|^2)^{\cdot}\right),\end{aligned}$$

where the $\simeq$ sign means that the equality is true up to terms of order $\|u\|_{H^{\alpha+n}}^{2-\theta(\alpha,n)}\|u_0\|_{H^{\alpha/2}}^{2+2\theta(\alpha,n)}\|u\|_{L^\infty}$ (which we control in the same way as for the second term). Thus we would like to estimate $Q_3 := 2\Re e(|D|^n(\bar{u}|D|^\alpha u), |D|^n(|u|^2)^\cdot)$, but as $|u|^2$ is real, it appears, expanding the real part, that $Q_3 = (|D|^n(\bar{u}|D|^\alpha u + u|D|^\alpha \bar{u}), |D|^n(|u|^2)^\cdot)$, which combines with the time-derivative of $J_2(u)$, and finally leads to the following expression :

$$\left(|D|^n\left[\bar{u}|D|^\alpha u + u|D|^\alpha \bar{u} - |D|^\alpha(\bar{u}u)\right], |D|^n(|u|^2)^\cdot\right). \tag{2.4}$$

Now we have a very simple Leibniz lemma on the operator $|D|^\alpha$, which we will prove in section 2.2 :

Lemma 2.2. *Let $\alpha \in [1,2)$. For any integer $n \in \mathbb{N}$, there is a constant $C_n > 0$ depending only on n, such that for all function $u \in H^{\alpha+n}(\mathbb{T})$,*

$$\|\bar{u}|D|^\alpha u + u|D|^\alpha \bar{u} - |D|^\alpha(\bar{u}u)\|_{H^n} \le C_n \|u\|_{H^{\alpha/2}}^{1+\frac{\alpha-1}{2n+\alpha}} \|u\|_{H^{\alpha+n}}^{1-\frac{\alpha-1}{2n+\alpha}}.$$

Such a result is better than the crude L^∞-$H^{\alpha+n}$ estimate, because of the exponent of the $H^{\alpha+n}$-norm (which is strictly less than 1 as soon as $\alpha > 1$).

Consequently, expression (2.4) is controlled by $\|u\|_{H^{\alpha+n}}^{2-\theta(\alpha,n)}\|u_0\|_{H^{\alpha/2}}^{2+2\theta(\alpha,n)}\|u\|_{L^\infty}$ as well. To sum up, only the second and the third term really matter, whence

$$\left|\frac{d}{dt}\mathcal{E}_{\alpha,n}(u)\right| \lesssim \|u\|_{H^{\alpha+n}}^{2-\theta(\alpha,n)}\|u_0\|_{H^{\alpha/2}}^{2+2\theta(\alpha,n)}\|u\|_{L^\infty}.$$

First, assume $\alpha > 1$. In this situation, we can incorporate the L^∞-norm into the $H^{\alpha/2}$ one (see (ii) of lemma 2.1), so for $t \in [T^*, T]$,

$$\left|\int_{T^*}^t \frac{d}{d\tau}\mathcal{E}_{\alpha,n}(u)d\tau\right| \le \int_{T^*}^t \left|\frac{d}{d\tau}\mathcal{E}_{\alpha,n}(u)\right| d\tau \lesssim \|u_0\|_{H^{\alpha/2}}^{4+2\theta(\alpha,n)} \int_{T^*}^t \|u(\tau)\|_{H^{\alpha+n}}^{2-\theta(\alpha,n)} d\tau.$$

Furthermore, remembering our estimates (2.3) on $\mathcal{E}_{\alpha,n}(u)$,

$$\left|\int_{T^*}^t \frac{d}{d\tau}\mathcal{E}(u)d\tau\right| = |\mathcal{E}(u)(t) - \mathcal{E}(u)(T^*)| \ge \frac{1}{2}\|u(t)\|_{H^{\alpha+n}}^2 - 2\|u(T^*)\|_{H^{\alpha+n}}^2.$$

Let $f(t) := \|u(t)\|_{H^{\alpha+n}}^2$. The above calculation ensures that for some $C_{\alpha,n} > 0$ depending on α and n,

$$f(t) \le 4f(T^*) + C_{\alpha,n}\|u_0\|_{H^{\alpha/2}}^{4+2\theta(\alpha,n)} \int_{T^*}^t f(\tau)^{1-\frac{1}{2}\theta(\alpha,n)} d\tau.$$

Now $\theta(\alpha,n)$ is positive. A "Gronwall's lemma" argument (which is also known as "Osgood's lemma") thus proves that $f(t) \le 4f(T^*)(1 + C|t - T^*|)^A$ for $t \in [T^*, T]$. Notice that the value of $f(T^*)$, *i.e.* of $\tilde{\mathcal{M}}$, only depends on the value of $\|u_0\|_{H^{\alpha+n}}$. Moreover, this inequality remains true even for t outside any interval of type $[T^*, T]$, so it globally holds and the first part of theorem 3 is proved. At last, the constant A can be set to $\frac{2}{\theta(\alpha,n)}$, *i.e.* $\frac{4n+2\alpha}{\alpha-1}$, which implies the statement.

It remains to consider the case $\alpha = 1$. This time, $\theta(1,n) = 0$ for any n, and in addition, the L^∞-norm of u is not bounded by a constant anymore. Using part (iii) of lemma 2.1, and going on as in the previous case with an auxiliary function $g(t) = \|u(t)\|_{H^{1+n}}^2 / C^2\|u_0\|_{H^{1/2}}^4$, we find,

$$g(t) \le 4g(T^*) + C_n\|u_0\|_{H^{1/2}}^4 \int_{T^*}^t g(\tau)\sqrt{\log(1 + g(\tau))}d\tau,$$

for all $t \in [T^*, T]$. Osgood's lemma then yields $g(t) \le Cg(T^*)e^{B|t|^2}$, and the proof of theorem 1 is complete.

2.2 A Leibniz lemma

We eventually turn to the

Proof of lemma 2.2. Let $u = \sum_{k\in\mathbb{Z}} u_k e^{ikx} \in H^{\alpha+n}(\mathbb{T})$. We intend to control the H^n-norm of $F_\alpha(u) := \bar{u}|D|^\alpha u + u|D|^\alpha \bar{u} - |D|^\alpha(\bar{u}u)$. A straightforward computation yields

$$F_\alpha(u) = \sum_{k=-\infty}^{+\infty} e^{ikx} \left(\sum_{l=-\infty}^{+\infty} (|l|^\alpha + |k-l|^\alpha - |k|^\alpha)\, u_l \overline{u_{l-k}} \right).$$

The key idea is to replace $|l|^\alpha + |k-l|^\alpha - |k|^\alpha$ by a more symmetric coefficient, and then to recognize a convolution product. More precisely, define a continuous function φ of the real variable $x \in \mathbb{R} \setminus \{0,1\}$ with

$$\varphi(x) := \frac{|x|^\alpha + |1-x|^\alpha - 1}{|x|^{\frac{\alpha}{2}}|1-x|^{\frac{\alpha}{2}}},$$

and $\varphi(0) = \varphi(1) = 0$. For every l, $k \in \mathbb{Z}$ with $k \neq 0$, we then have $|l|^\alpha + |k-l|^\alpha - |k|^\alpha = \varphi(\frac{l}{k})|l|^{\frac{\alpha}{2}}|k-l|^{\frac{\alpha}{2}}$, which is true even if $k = 0$, once we have assigned $\varphi(\pm\infty) = 2$, by convention.

Now, to show that φ is bounded on $\mathbb{R}$, it suffices to check that it is bounded near 0 and $-\infty$, and then to invoke the symmetry of φ around $x = \frac{1}{2}$. And actually, since $\alpha \leq 2$, $\lim_{x\to 0}\varphi(x) = 0$ and $\lim_{x\to-\infty}\varphi(x) = 2$. Studying the variations of φ even show that $\|\varphi\|_{L^\infty(\mathbb{R})} = 2$, and hence is independent of α.

As a consequence,

$$\begin{aligned}
&\sum_{k=-\infty}^{+\infty} |k|^{2n} \left| \sum_{l=-\infty}^{+\infty} (|l|^\alpha + |k-l|^\alpha - |k|^\alpha) u_l \overline{u_{l-k}} \right|^2 \\
&\leq 2 \sum_{k=-\infty}^{+\infty} |k|^{2n} \left| \sum_{l=-\infty}^{+\infty} |l|^{\frac{\alpha}{2}}|k-l|^{\frac{\alpha}{2}}|u_l||\overline{u_{l-k}}| \right|^2 \\
&\leq 2^{2n} \sum_{k=-\infty}^{+\infty} \left[\left| \sum_{l=-\infty}^{+\infty} |l|^{\frac{\alpha}{2}+n}|k-l|^{\frac{\alpha}{2}}|u_l||\overline{u_{l-k}}| \right|^2 + \left| \sum_{l=-\infty}^{+\infty} |l|^{\frac{\alpha}{2}}|k-l|^{\frac{\alpha}{2}+n}|u_l||\overline{u_{l-k}}| \right|^2 \right],
\end{aligned}$$

because of the inequality $|k|^n \leq 2^{n-1}(|l|^n + |k-l|^n)$, satisfied for any $n \geq 1$, any k and l.

When $n = 0$, the result is rather immediate, since

$$\sum_{k=-\infty}^{+\infty} \left| \sum_{l=-\infty}^{+\infty} |l|^{\frac{\alpha}{2}}|k-l|^{\frac{\alpha}{2}}|u_l||\overline{u_{l-k}}| \right|^2 = \left\| \left| |D|^{\frac{\alpha}{2}}\tilde{u} \right|^2 \right\|_{L^2}^2 = \left\| |D|^{\frac{\alpha}{2}}\tilde{u} \right\|_{L^4}^4,$$

where $\tilde{u} := \sum_{k\in\mathbb{Z}} |u_k| e^{ikx}$. Using as always the embedding $L^4 \hookrightarrow H^{1/4}$, and interpolating between $\alpha/2$ and α, we get the result.

From here on, we suppose $n \neq 0$. We shall deal with the first part of the above sum (the last one follows identically). We consider two sequences $v := (|l|^{\frac{\alpha}{2}+n}|u_l|)_{l\in\mathbb{Z}}$ and $w := (|l|^{\frac{\alpha}{2}}|\overline{u_{-l}}|)_{l\in\mathbb{Z}}$. With these notations,

$$\sum_{k=-\infty}^{+\infty} \left| \sum_{l=-\infty}^{+\infty} |l|^{\frac{\alpha}{2}+n}|k-l|^{\frac{\alpha}{2}}|u_l||\overline{u_{l-k}}| \right|^2 = \|v \star w\|_{\ell^2}^2.$$

By Schur's lemma, $\|v \star w\|_{\ell^2} \leq \|v\|_{\ell^2}\|w\|_{\ell^1}$. But $\|v\|_{\ell^2} \leq \|u\|_{H^{\frac{\alpha}{2}+n}}$. As for $\|w\|_{\ell^1}$, write

$$\begin{aligned}\sum_{k=-\infty}^{+\infty} |w_k| &= \sum_{|k|\leq N} |w_k| + \sum_{|k|>N} |w_k| \\ &\leq \left(\sum_{|k|\leq N} |w_k|^2\right)^{\frac{1}{2}} \sqrt{2N+1} + \left(\sum_{|k|>N} |w_k|^2(1+|k|^2)\right)^{\frac{1}{2}} \left(\sum_{|k|>N} \frac{1}{1+|k|^2}\right)^{\frac{1}{2}} \\ &\leq \sqrt{2N+1}\|w\|_{\ell^2} + \sqrt{\frac{2}{N}}\|w\|_{h^1}.\end{aligned}$$

Taking the infimum over $N \in \mathbb{N}$, we finally get $\|w\|_{\ell^1} \leq \sqrt{\|w\|_{\ell^2}\|w\|_{h^1}}$. In our case, $\|w\|_{\ell^2} \leq \|u\|_{H^{\frac{\alpha}{2}}}$, and similarly $\|w\|_{h^1} \leq \|u\|_{H^{\frac{\alpha}{2}+1}}$.

Now we can conclude :

$$\begin{aligned}\|F_\alpha(u)\|_{H^n}^2 &\lesssim \|u\|_{H^{\frac{\alpha}{2}+n}}^2 \|u\|_{H^{\frac{\alpha}{2}}} \|u\|_{H^{\frac{\alpha}{2}+1}} \\ &\lesssim \|u\|_{H^{\frac{\alpha}{2}}}^{2\left(1+\frac{\alpha-1}{2n+\alpha}\right)} \|u\|_{H^{\alpha+n}}^{2\left(1-\frac{\alpha-1}{2n+\alpha}\right)},\end{aligned}$$

which corresponds to the statement. □

Remark 1. Lemma 2.2 probably takes the best advantage of the form of (2.4). There is still a kind of "virial" identity which holds for any value of α :

$$\frac{d}{dt}|u|^2 = i\left(u|D|\bar{u} - \bar{u}|D|u\right),$$

but the inequality $\|u|D|\bar{u} - \bar{u}|D|u\|_{L^2} \lesssim \|u\|_{H^{1/2}}\|u\|_{H^1}$, for instance, is false. As a counter-exemple, choose

$$u_N(x) = \frac{1}{\sqrt{\log N}} \sum_{n=1}^{N} \frac{e^{inx}}{n},$$

and let $N \to +\infty$.

2.3 Quadratic half-wave equations

We come to the system of equations (1.6) and the proof of (1.7). To see the Hamiltonian structure of (1.6), choose $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ as a phase space, endowed with the inner product $\langle (u_1, u_2), (v_1, v_2)\rangle := (u_1, v_1) + (u_2, v_2)$. Taking the imaginary part of $\langle \cdot, \cdot \rangle$ as our symplectic form, we infer from a simple calculation that the Hamiltonian $\tilde{\mathcal{H}}(u_1, u_2) := \frac{1}{2}[(|D|u_1, u_1) + (|D|u_2, u_2) + \int_{\mathbb{T}} \Re e(u_1^2\overline{u_2})]$ is associated to the system (1.6).

Notice that the functional $\tilde{\mathcal{H}}$ is invariant under the flow $(u_1, u_2) \mapsto (e^{i\theta}u_1, e^{2i\theta}u_2)$, with θ varying in $\mathbb{R}$. It follows then from the Noether theorem that $\tilde{\mathcal{Q}}(u_1, u_2) := \|u_1\|_{L^2}^2 + 2\|u_2\|_{L^2}^2$ is a conservation law for the system (1.6). As a consequence, the L^2-norms of u_1, u_2 stay bounded along the flow lines. In addition, the conservation of $\tilde{\mathcal{H}}$ as well as $\tilde{\mathcal{Q}}$ claims the uniform boundedness of $\|u_1\|_{H^{1/2}}$ and $\|u_2\|_{H^{1/2}}$ with respect to time.

Immediately, we get, for each $n \geq 0$,

$$\|u_1\|_{H^{1+n}}, \|u_2\|_{H^{1+n}} \lesssim e^{B|t|^2},$$

where $B > 0$ is independent of time : this follows from a straightforward application of inequality (iii) in lemma 2.1. But now, set $F(t) := (\|u_1\|_{H^{1+n}}^2 + \|u_2\|_{H^{1+n}}^2)$. Here we won't repeat the details of section 2.1, but we suppose from the beginning that F is "big enough", and we compute

$$\begin{aligned}
&\frac{d}{dt}\left[\||D|^{1+n}u_1\|_{L^2}^2 + \||D|^{1+n}u_2\|_{L^2}^2\right] \\
&= 2\Re e\left[(|D|^{1+n}\dot{u}_1, |D|^{1+n}u_1) + (|D|^{1+n}\dot{u}_2, |D|^{1+n}u_2)\right] \\
&= -2\Re e\left[(|D|^{1+n}\dot{u}_1, |D|^n(u_2\overline{u_1})) + \left(|D|^{1+n}\dot{u}_2, |D|^n\left(\frac{u_1^2}{2}\right)\right)\right].
\end{aligned}$$

Then, correcting the initial quantity with terms of lower order than $F(t)$, we rather estimate

$$A := 2\Re e\left[(|D|^{1+n}u_1, |D|^n(\dot{u}_2\overline{u_1} + u_2\dot{\overline{u_1}})) + (|D|^{1+n}u_2, |D|^n(\dot{u}_1 u_1))\right].$$

Now apply the Leibniz formula :

$$\begin{aligned}
A &= 2\Re e\left[(\partial_x^n|D|u_1, (\partial_x^n\dot{u}_2)\overline{u_1}) + (\partial_x^n|D|u_2, (\partial_x^n\dot{u}_1)u_1)\right] \\
&\quad + 2\Re e\left(\partial_x^n|D|u_1, \sum_{k=0}^{n-1}\binom{n}{k}[(\partial_x^k\dot{u}_2)\partial_x^{n-k}\overline{u_1} + (\partial_x^{n-k}u_2)\partial_x^k\dot{\overline{u_1}}\right) \\
&\quad + 2\Re e\left(\partial_x^n|D|u_2, \sum_{k=0}^{n-1}\binom{n}{k}(\partial_x^k\dot{u}_1)\partial_x^{n-k}u_1\right) \\
&\quad + 2\Re e(\partial_x^n|D|u_1, u_2\partial_x^n\dot{\overline{u_1}}).
\end{aligned} \tag{2.5}$$

We aim at showing that each of these terms is controlled by $F(t)$.

Exactly as before, we have a cancellation occuring in the first line of (2.5) : replace $\dot{u}_1$ by $i|D|u_1$, and $\dot{u}_2$ by $i|D|u_2$ (here again the nonlinearities can be neglected), and observe that

$$2\Im m\left[(\partial_x^n|D|u_1, (\partial_x^n|D|u_2)\overline{u_1}) + (\partial_x^n|D|u_2, (\partial_x^n|D|u_1)u_1)\right] = 0.$$

Concerning the second and the third line of (2.5), straightforward Sobolev estimates and interpolation inequalities are enough to conclude.

As for the fourth line of (2.5), where all time- and space-derivatives concentrate on the same function, we need an equivalent of lemma 2.2 for the operator $|D|^{1/2}$, namely :

Lemma 2.3 (Kenig-Ponce-Vega, see [40]). *For f, $g : \mathbb{T} \to \mathbb{C}$, we have*

$$\|f|D|^s g + g|D|^s f - |D|^s(fg)\|_{L^p} \lesssim \||D|^{s_1}f\|_{L^{p_1}}\||D|^{s_2}g\|_{L^{p_2}},$$

provided that $0 < s < 1$, $s = s_1 + s_2$ and s_1, $s_2 \geq 0$, and on the other side, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, with p, p_1, $p_2 \in (1, +\infty)$.

With this lemma, we can write

$$2\Re e(\partial_x^n|D|u_1, u_2\partial_x^n\dot{\overline{u_1}}) = 2\Re e(|D|^{1/2}(\partial_x^n u_1), (|D|^{1/2}u_2)\partial_x^n\dot{\overline{u_1}}) \tag{2.6}$$
$$+ 2\Re e(|D|^{1/2}(\partial_x^n u_1), u_2|D|^{1/2}(\partial_x^n\dot{\overline{u_1}})) \tag{2.7}$$
$$+ 2\Re e(|D|^{1/2}(\partial_x^n u_1), |D|^{1/2}\left[u_2\partial_x^n\dot{\overline{u_1}}\right] - (|D|^{1/2}u_2)\partial_x^n\dot{\overline{u_1}} - u_2|D|^{1/2}(\partial_x^n\dot{\overline{u_1}})). \tag{2.8}$$

We estimate separately

$$\begin{aligned}
|(2.6)| &\lesssim \||D|^{1/2}\partial_x^n u_1\|_{L^4}\||D|^{1/2}u_2\|_{L^4}\|\partial_x^n\dot{\overline{u_1}}\|_{L^2} \\
&\lesssim \|u_1\|_{H^{\frac{3}{4}+n}}\|u_2\|_{H^{\frac{3}{4}}}\|u_1\|_{H^{1+n}} \\
&\lesssim (\|u_1^0\|_{H^{1/2}}^2 + \|u_2^0\|_{H^{1/2}}^2)F(t).
\end{aligned}$$

On the other hand, (2.7) can be rewritten as $2\Re e(|D|^{1/2}(\partial_x^n \dot{u_1})|D|^{1/2}(\partial_x^n u_1), u_2)$, *i.e.*

$$\frac{d}{dt}\left[\Re e\left((\partial_x^n |D|^{1/2} u_1)^2, u_2\right)\right] - \Re e\left((\partial_x^n |D|^{1/2} u_1)^2, \dot{u_2}\right).$$

Thus, perturbing the initial quantity by a term of lower order than $F(t)$, it is enough to control

$$\left|\Re e\left((\partial_x^n |D|^{1/2} u_1)^2, |D| u_2\right)\right| \leq \|u_1\|^2_{H^{\frac{3}{4}+n}} \|u_2\|_{H^1} \lesssim (\|u_1^0\|^2_{H^{1/2}} + \|u_2^0\|^2_{H^{1/2}}) F(t).$$

Eventually, using L^4-$L^{\frac{4}{3}}$ duality, and lemma 2.3 with $s = \frac{1}{2}$, $s_1 = \frac{1}{2}$, $s_2 = 0$, and $p = \frac{4}{3}$, $p_1 = 4$, $p_2 = 2$, we infer that

$$|(2.8)| \lesssim \||D|^{1/2} \partial_x^n u_1\|_{L^4} \||D|^{1/2} u_2\|_{L^4} \|\partial_x^n \dot{\overline{u_1}}\|_{L^2} \lesssim (\|u_1^0\|^2_{H^{1/2}} + \|u_2^0\|^2_{H^{1/2}}) F(t),$$

as above.

To sum up, there exist a constant C_n depending only on n, such that, for all times $t \in \mathbb{R}$, $F(t) \leq 4F(0) + C_n(\|u_1^0\|^2_{H^{1/2}} + \|u_2^0\|^2_{H^{1/2}}) \int_0^t F(s) ds$, which means, by Gronwall's lemma, that

$$F(t) \leq 4F(0) e^{C_n(\|u_1^0\|^2_{H^{1/2}} + \|u_2^0\|^2_{H^{1/2}})|t|}.$$

This tells us that $\|u_1\|_{H^{1+n}}$ and $\|u_2\|_{H^{1+n}}$ grow at most exponentially, and theorem 2 is proved.

3 Dispersion estimates and Bourgain spaces

We come back to equation (1.2) and to the end of the proof of theorem 3. We have to deal now with the case when $\alpha < 1$. In this case, the boundedness of the $H^{\alpha/2}$-norm of the solutions is not enough to get a pointwise control of their L^∞-norm. In other terms, we need to prove a Strichartz estimate for solutions of (1.2).

3.1 The Strichartz estimate

From now on, α is fixed, with $\frac{2}{3} < \alpha < 1$. For $u_0 \in \mathscr{D}'(\mathbb{T})$ and $t \in \mathbb{R}$, denote by

$$S(t)u_0 := e^{-it|D|^\alpha} u_0 = \sum_{k=-\infty}^{+\infty} \widehat{u_0}(k) e^{i(kx - |k|^\alpha t)},$$

the solution of the homogeneous equation $i\partial_t u = |D|^\alpha u$, with value u_0 at time $t = 0$.

We are also going to use the Littlewood-Paley decomposition. For this purpose, let ψ be a nonnegative $\mathcal{C}^\infty$ function on $\mathbb{R}_+$, such that $\psi > 0$ on $\left[\frac{1}{2} + \frac{1}{10}, 2 - \frac{1}{10}\right]$ and $\psi \equiv 0$ outside $\left]\frac{1}{2}, 2\right[$. Without loss of generality, we can assume that $\sum_{j=1}^{+\infty} \psi(2^{-j}x) \equiv 1$ on $[2, +\infty)$. Let then $u \in \mathscr{D}'(\mathbb{T})$, and $N = 2^j$ for some integer $j \geq 1$. We define

$$\Delta_N u := \psi\left(\frac{|D|}{N}\right) u = \sum_{|k| \in \left[\frac{N}{2}, 2N\right]} \psi\left(\frac{|k|}{N}\right) \hat{u}(k) e^{ikx},$$

and $\Delta_1 u := u - \sum_{j \geq 1} \Delta_{2^j} u$. In the sequel, capital letters will always refer to dyadic integers, and we will use the simplified notation $\sum_N$ for a sum over all dyadic integers, starting from 1.

Let us recall a few facts about the Littlewood-Paley decomposition :

$$u = \sum_N \Delta_N u,$$
$$\|u\|_{H^s}^2 \simeq \sum_N N^{2s} \|\Delta_N u\|_{L^2}^2, \tag{3.1}$$

By the $\simeq$ sign, we just mean that the two quantities, as norms, are equivalent.
We can now state our Strichartz lemma :

Lemma 3.1 (Strichartz inequality). *There exist a constant $C_\alpha > 0$, depending on α, such that, for every $u \in L^2(\mathbb{T})$ and every $N = 2^j$,*

$$\|e^{-it|D|^\alpha} \Delta_N u\|_{L^4((0,1)_t, L^\infty(\mathbb{T}))} \leq C_\alpha \|u\|_{L^2} N^{\frac{1}{2}-\frac{\alpha}{4}}. \tag{3.2}$$

This lemma also has a non-localized version :

Corollary 3.2. *For every $\gamma > \frac{1}{2} - \frac{\alpha}{4}$ and every $u \in H^\gamma(\mathbb{T})$, we have*

$$\|e^{-it|D|^\alpha} u\|_{L^4((0,1)_t, L^\infty(\mathbb{T}))} \leq C_{\alpha,\gamma} \|u\|_{H^\gamma}.$$

Proof of the corollary. Using the triangle inequality, (3.2), and the Cauchy-Schwarz inequality, we can write that

$$\|e^{-it|D|^\alpha} u\|_{L^4L^\infty} = \left\| \sum_N e^{-it|D|^\alpha} \Delta_N u \right\|_{L^4L^\infty} \leq \sum_N \|e^{-it|D|^\alpha} \Delta_N u\|_{L^4L^\infty} \leq C \sum_N N^{\frac{1}{2}-\frac{\alpha}{4}} \|\Delta_N u\|_{L^2}$$
$$\leq C \left(\sum_N N^{2\gamma} \|\Delta_N u\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_N N^{-2\left(\gamma - \left(\frac{1}{2} - \frac{\alpha}{4}\right)\right)} \right)^{\frac{1}{2}} \leq C_{\alpha,\gamma} \|u\|_{H^\gamma},$$

where the last bound comes from (3.1). □

Proof of lemma 3.1. To proove the Strichartz inequality, we proceed in a quite usual manner : we begin by showing a dispersion estimate, and to conclude, we apply a TT^*-argument, combined with the Hardy-Littlewood-Sobolev inequality.

Let $N = 2^j$, $j \geq 1$, be a dyadic integer [4]. For $t \in \mathbb{R}$, we have

$$\begin{aligned} e^{-it|D|^\alpha} \Delta_N u(x) &= \sum_{k \in \mathbb{Z}} \widehat{\Delta_N u}(k) e^{i(kx - |k|^\alpha t)} \\ &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} \psi\left(\frac{|k|}{N}\right) u(y) e^{i(k(x-y) - |k|^\alpha t)} dy \\ &=: (u *_x \kappa_N)(x,t), \end{aligned}$$

where κ_N stands for the following kernel :

$$\kappa_N(x,t) := \sum_{k \in \mathbb{Z}} \psi\left(\frac{|k|}{N}\right) e^{i(kx - |k|^\alpha t)}.$$

4. For $N = 1$, (3.2) holds trivially.

Our first step will be to estimate $\|\kappa_N(\cdot,t)\|_{L^\infty(\mathbb{T})}$ for fixed $t \in (-1,1)$, $t \neq 0$. Applying the Poisson summation formula to the function $F_{x,t}(y) := \psi(|y|/N)e^{i(yx-|y|^\alpha t)}$, which is $\mathcal{C}^\infty$ and compactly supported, we have

$$\kappa_N(x,t) = \sum_{k\in\mathbb{Z}} F_{x,t}(k) = \sum_{n\in\mathbb{Z}} \widehat{F_{x,t}}(2\pi n) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} N\psi(|\xi|)e^{i[N\xi(x-2\pi n)-N^\alpha t|\xi|^\alpha]}d\xi,$$

and to study $\|\kappa_N\|_{L^\infty}$, we naturally restrict ourselves to $x \in (-\pi,\pi]$.

The integrals above will be estimated by a stationnary phase result called the Van der Corput lemma (see [59]) :

Lemma 3.3 (Van der Corput)**.** *Let φ, $\Psi : \mathbb{R} \to \mathbb{C}$ be two smooth functions, with Ψ compactly supported on $\mathbb{R}$. Suppose in addition that there exist $A > 0$ such that $|\varphi''| \geq A$ on* supp(Ψ)*. Then*

$$\left|\int_{\mathbb{R}} e^{i\varphi(x)}\Psi(x)dx\right| \leq \frac{C}{\sqrt{A}}\int_{\mathbb{R}} |\Psi'(x)|dx,$$

where $C > 0$ is an absolute constant.

In our case, the phase reads $N\xi(x-2\pi n) - N^\alpha t|\xi|^\alpha$, and we denote it by $\phi_n(\xi)$. Compute $\phi_n'(\xi) = N(x-2\pi n) - \alpha N^\alpha t\xi|\xi|^{\alpha-2}$. In particular, $\phi_n'(\xi) = 0$ if and only if

$$\operatorname{sgn}(\xi)\,(N|\xi|)^{1-\alpha} = \frac{\alpha t}{x-2\pi n}. \tag{3.3}$$

Because of ψ cutting all frequences below $\frac{1}{2}$, and because of the condition $1-\alpha > 0$, we have $(N|\xi|)^{1-\alpha} \geq 1$ on supp(ψ), whereas $|t\alpha| \leq 1$. Furthermore, when $n \neq 0$, $|x-2\pi n|^{-1} \leq \pi^{-1}$: in that case, (3.3) cannot hold.

So suppose first $n \neq 0$. Then $\|\phi_n'\|_{L^\infty(\mathrm{supp}(\psi))} \geq N|x-2\pi n| - N^\alpha 2^{1-\alpha}$, and integrating by parts, it is easy to estimate the integral

$$I_n := \int_{\mathbb{R}} N\psi(|\xi|)e^{i\phi_n(\xi)}d\xi = \int_{\mathbb{R}} N\frac{d}{d\xi}\left(\frac{1}{i\phi_n'(\xi)}\frac{d}{d\xi}\left(\frac{\psi(|\xi|)}{i\phi_n'(\xi)}\right)\right)e^{i\phi_n(\xi)}d\xi.$$

Thus, we have $|I_n| \leq CN\|\phi_n'\|_{L^\infty}^{-2}\|\psi\|_{H^2}$, where C is proportionnal to the size of supp(ψ). Finally, we sum on n :

$$\left|\sum_{n\neq 0} I_n\right| \leq \frac{C}{N}\sum_{n\neq 0}\left(|x-2\pi n| - \left(\frac{2}{N}\right)^{1-\alpha}\right)^{-2} \leq \frac{C}{N}\sum_{n\neq 0}(2\pi|n|-\pi-1)^{-2} \leq \frac{\tilde{C}}{N}, \tag{3.4}$$

with $\tilde{C}$ just depending on ψ.

The only difficult part, then, is I_0, because (3.3) could be satisfied. At this point, we apply lemma 3.3, and calculate $\phi_0''(\xi) = \alpha(1-\alpha)N^\alpha t|\xi|^{\alpha-2}$. It is clear that $|\phi_0''(\xi)| \geq \alpha(1-\alpha)N^\alpha|t|2^{2-\alpha}$ on supp(ψ), so we have

$$|I_0| \leq C\frac{N^{1-\frac{\alpha}{2}}}{\sqrt{|t|}}, \tag{3.5}$$

where $C > 0$ is a constant depending on α and ψ.

So far, (3.4) and (3.5) show that there exist a constant $C > 0$, depending only on α and ψ, such that $\|\kappa_N(\cdot,t)\|_{L^\infty} \leq C|t|^{-1/2}N^{1-\alpha/2}$ for all $t \in (-1,1)$, $t \neq 0$. In particular, for fixed t, considering $S(t)\Delta_N$ as an operator mapping $L^1(\mathbb{T})$ to $L^\infty(\mathbb{T})$, we have

$$\|S(t)\Delta_N\|_{L^1\to L^\infty} \leq C\frac{N^{1-\frac{\alpha}{2}}}{\sqrt{|t|}}. \tag{3.6}$$

Now comes the TT^*-argument. Define a linear operator $T : u \mapsto S(t)\Delta_N u$. We want to prove that T maps $L^2(\mathbb{T})$ into $L^4((0,1)_t, L^\infty(\mathbb{T}))$, as well as to bound its norm. To this end, we rather study the operator TT^*, where $T^* : L^{4/3}((0,1)_t, L^1(\mathbb{T})) \to L^2(\mathbb{T})$ is (a restriction of) the adjoint of T. We can find T^* explicitely. Let $g \in L^{4/3}L^1$: for $u \in L^2(\mathbb{T})$,

$$\iint_{(0,1)\times\mathbb{T}} [S(t)\Delta_N u(x)]\overline{g(t,x)}dxdt = \left(u, \int_0^1 S(-s)\Delta_N g(s,x)\right)_{L^2(\mathbb{T})}.$$

Thus, $TT^*(g)(t,x) = \int_0^1 \Delta_N S(t-s)\Delta_N g(s,x)ds$, and by (3.6), for all $t \in (0,1)$,

$$\|TT^*(g)(t,\cdot)\|_{L^\infty(\mathbb{T})} \leq C\int_0^1 \frac{N^{1-\frac{\alpha}{2}}}{\sqrt{|t-s|}}\|g(s,\cdot)\|_{L^1(\mathbb{T})}ds.$$

In the integral of the left hand side, we recognize a convolution product between $t \mapsto \|g(t,\cdot)\|_{L^1(\mathbb{T})}$ and the function $\omega : t \mapsto |t|^{-1/2}$. The Hardy-Littlewood-Sobolev inequalities guarantee that the convolution with ω maps $L^{4/3}((0,1)_t)$ to $L^4((0,1)_t)$. In other terms, $\|TT^*(g)\|_{L^4L^\infty} \leq CN^{1-\alpha/2}\|g\|_{L^{4/3}L^1}$, which implies that the operator norm of T is bounded :

$$\|T\|_{L^2 \to L^4((0,1)_t, L^\infty(\mathbb{T}))} \leq CN^{\frac{1}{2}-\frac{\alpha}{4}}.$$

This finishes the proof of lemma 3.1. □

Remark 2. Notice that the results of lemma 3.2 and corollary 3.2 remain true, with the same constants, when replacing $(0,1)_t$ by any time interval of length 1. This follows from the fact that $S(t)$ is an isometry in any $H^s(\mathbb{T})$, $s \geq 0$.

3.2 Bourgain spaces and embedding results

The Strichartz estimate of corollary 3.2 will enable us to prove the local well-posedness of equation (1.2) in a certain Hilbert space, usually called a Bourgain space, which we are now going to define.

Definition. Let $u : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$, and s, $b \in \mathbb{R}$.

- We say that $u \in H^{s,b}$ if for all $t \in \mathbb{R}$, $u(t) \in H^s(\mathbb{T})$, and if in addition, the function $t \mapsto \|u(t)\|_{H^s(\mathbb{T})}$ belongs to $H^b(\mathbb{R})$. Then, the norm $\|u\|_{H^{s,b}}$ is just the H^b-norm of $t \mapsto \|u(t)\|_{H^s(\mathbb{T})}$.

- We say that $u \in X^{s,b}_\alpha$ if the function $v : (t,x) \mapsto S(-t)u(t,x)$ belongs to $H^{s,b}$. Then, we define $\|u\|_{X^{s,b}_\alpha} := \|v\|_{H^{s,b}} = \|S(-t)u(t,x)\|_{H^{s,b}}$.

The space $X^{s,b}_\alpha$ is called a *Bourgain space*. Explicitely,

$$\|u\|^2_{X^{s,b}_\alpha} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}(1+|k|^2)^s(1+|\tau+|k|^\alpha|^2)^b|\mathcal{F}u(\tau,k)|^2d\tau, \tag{3.7}$$

where $\mathcal{F}u(\cdot,k)$ stands for the Fourier transform of $\widehat{u(\cdot,k)}$ with respect to time, *i.e.* the Fourier transform in both time- and space-variables.

Bourgain spaces are very convenient for several reasons. Playing on the two exponents s and b, we begin by showing two embedding results :

Lemma 3.4. *For any $b > \frac{1}{4}$, we have $\|u\|_{L^4(\mathbb{R}_t, L^2(\mathbb{T}))} \lesssim \|u\|_{X^{0,b}_\alpha}$.*

Proof. Assuming $u \in X_\alpha^{0,2b}$, write $\mathcal{F}u(\tau,k) = \int_{\mathbb{R}} \widehat{u(t,k)} e^{-it\tau} dt$, so by the inverse Fourier transform and the Cauchy-Schwarz inequality (observing that $4b > 1$),

$$|\widehat{u(t,k)}|^2 = \left| \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}u(\tau,k) e^{it\tau} d\tau \right|^2 \\ \leq C \left(\int_{\mathbb{R}} |\mathcal{F}u(\tau,k)|^2 (1 + |\tau + |k|^\alpha|^2)^{2b} d\tau \right) \left(\int_{\mathbb{R}} \frac{d\tau}{(1 + |\tau + |k|^\alpha|^2)^{2b}} \right).$$

Summing over $k \in \mathbb{Z}$, we find $\|u(t)\|_{L^2(\mathbb{T})}^2 \leq C_b \|u\|_{X_\alpha^{0,2b}}^2$, or equivalently $\|u\|_{L^\infty(\mathbb{R}_t, L^2(\mathbb{T}))} \leq C\|u\|_{X_\alpha^{0,2b}}$. But the equality $\|u\|_{L^2(\mathbb{R}_t, L^2(\mathbb{T}))} = \|u\|_{X_\alpha^{0,0}}$ also follows from (3.7) and the Parseval formula. Interpolating between these two statements gives the result. □

The following lemma is a consequence of the Strichartz inequality.

Lemma 3.5. *For any $b > \frac{1}{2}$ and $\gamma > \frac{1}{2} - \frac{\alpha}{4}$, we have $\|u\|_{L^4(\mathbb{R}_t, L^\infty(\mathbb{T}))} \lesssim \|u\|_{X_\alpha^{\gamma,b}}$.*

Proof. Let $u \in X_\alpha^{\gamma,b}$, and $v := S(-t)u$. Suppose at first that $t \mapsto u(t,\cdot)$ is supported on an interval I_t of length 1. Thus it is possible to apply corollary 3.2 directly : indeed,

$$\|u\|_{L^4(\mathbb{R}_t, L^\infty)} = \|S(t)v\|_{L^4(I_t, L^\infty)} = \left\| S(t) \int_{\mathbb{R}} \hat{v}(\tau) e^{it\tau} d\tau \right\|_{L^4(I_t, L^\infty)} \leq \int_{\mathbb{R}} \|S(t)\hat{v}(\tau)\|_{L^4(I_t, L^\infty)} d\tau \\ \leq C \int_{\mathbb{R}} \|\hat{v}(\tau)\|_{H^\gamma} d\tau \leq C \left(\int_{\mathbb{R}} \|\hat{v}(\tau)\|_{H^\gamma}^2 (1 + |\tau|^2)^b d\tau \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{d\tau}{(1 + |\tau|^2)^b} \right)^{\frac{1}{2}} \leq C_b \|v\|_{H^{\gamma,b}}.$$

Since $\|v\|_{H^{\gamma,b}} = \|u\|_{X_\alpha^{\gamma,b}}$, this finishes the first part of the proof.

Now, we remove the special assumption on u. By simple construction, it is possible to find a function $\vartheta \in \mathcal{C}_0^\infty((0,1))$, such that $0 \leq \vartheta \leq 1$ on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} \vartheta(t - n/2) = 1$, for all $t \in \mathbb{R}$. We have

$$\|u\|_{L^4(\mathbb{R}_t, L^\infty(\mathbb{T}))}^4 = \int_{\mathbb{R}} dt \left\| \sum_{n\in\mathbb{Z}} u(t,\cdot) \vartheta\left(t - \frac{n}{2}\right) \right\|_{L^\infty(\mathbb{T})}^4 \leq \int_{\mathbb{R}} dt \left| \sum_{n\in\mathbb{Z}} \vartheta\left(t - \frac{n}{2}\right) \|u(t,\cdot)\|_{L^\infty(\mathbb{T})} \right|^4 \\ = \int_{\mathbb{R}} dt \sum_{\substack{n_1\in\mathbb{Z} \\ n_2,n_3,n_4 \in \{n_1-1, n_1, n_1+1\}}} \vartheta\left(t - \frac{n_1}{2}\right) \vartheta\left(t - \frac{n_2}{2}\right) \vartheta\left(t - \frac{n_3}{2}\right) \vartheta\left(t - \frac{n_4}{2}\right) \|u(t,\cdot)\|_{L^\infty(\mathbb{T})}^4 \\ \leq C \sum_{n_1\in\mathbb{Z}} \left\| \vartheta\left(t - \frac{n_1}{2}\right) u(t,\cdot) \right\|_{L^4(\mathbb{R}_t, L^\infty(\mathbb{T}))}^4,$$

thanks to the elementary inequality : $abcd \leq \frac{1}{4}(a^4 + b^4 + c^4 + d^4)$. To each term of the sum, we apply the first part of the proof, and we find, using the embedding $\ell^4(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$:

$$\|u\|_{L^4(\mathbb{R}_t, L^\infty(\mathbb{T}))} \lesssim \left(\sum_{n\in\mathbb{Z}} \left\| \vartheta\left(t - \frac{n}{2}\right) S(-t)u(t,\cdot) \right\|_{H^{\gamma,b}}^4 \right)^{\frac{1}{4}} \\ \lesssim \left(\sum_{n\in\mathbb{Z}} \left\| \vartheta\left(t - \frac{n}{2}\right) S(-t)u(t,\cdot) \right\|_{H^{\gamma,b}}^2 \right)^{\frac{1}{2}} \\ \lesssim \|S(-t)u(t,\cdot)\|_{H^{\gamma,b}}.$$

The very last bound comes from the following classical lemma :

Lemma 3.6. *For any* $b \in [0,1]$, *any function* $w \in H^b(\mathbb{R})$, *any smooth* ϑ *as above, the following norms are equivalent :*

$$\|w\|_{H^b(\mathbb{R})} \simeq \left(\sum_{n\in\mathbb{Z}} \|\vartheta(\cdot - n/2)w(\cdot)\|^2_{H^b(\mathbb{R})} \right)^{\frac{1}{2}}.$$

□

3.3 The nonlinear estimate

To solve (1.2) locally in time, we need to introduce a restriction space. For $T > 0$, let $X^{s,b}_\alpha(T)$ be the set of all functions u defined on $[-T,T]$ such that there exist a function $\tilde{u} \in X^{s,b}_\alpha$ with $\tilde{u}|_{[-T,T]} \equiv u$. Endowed with the norm

$$\|u\|_{X^{s,b}_\alpha(T)} := \inf \left\{ \|\tilde{u}\|_{X^{s,b}_\alpha} \middle| \tilde{u} \in X^{s,b}_\alpha,\ \tilde{u} \equiv u \text{ on } [-T,T] \right\},$$

$X^{s,b}_\alpha(T)$ is a Banach space, so we can apply Picard's fixed-point theorem in $X^{s,b}_\alpha(T)$. From here on, we strictly follow the scheme of the proof of [10, Theorem 3].

Fix a function $\varphi \in C^\infty_0(\mathbb{R})$, such that $\varphi(t) = 1$ when $|t| \leq 1$. Then for $T \in (0,1]$, a solution of (1.2) on the interval $[-T,T]$ is a fixed point of the application

$$\mathcal{K} : u \longmapsto \varphi(t)S(t)u_0 - i\varphi\left(\tfrac{t}{T}\right) \int_0^t S(t-s)\left[|u(s)|^2 u(s)\right] ds. \tag{3.8}$$

Thanks to the definition of Bourgain spaces, estimating the first part of $\mathcal{K}$ is elementary : $\|\varphi(t)S(t)u_0\|_{X^{s,b}_\alpha} = \|\varphi\|_{H^b(\mathbb{R}_t)}\|u_0\|_{H^s(\mathbb{T})}$. All the difficulty lies in the nonlinear part, and we now turn to our central proposition.

Proposition 3.7. *Let* $\gamma > \frac{1}{2} - \frac{\alpha}{4}$, $b > \frac{1}{2}$, $b' > \frac{1}{4}$ *such that* $b + b' < 1$. *Then*

$$\|u_1\overline{u_2}u_3\|_{X^{\gamma,-b'}_\alpha} \leq C\|u_1\|_{X^{\gamma,b}_\alpha}\|u_2\|_{X^{\gamma,b}_\alpha}\|u_3\|_{X^{\gamma,b}_\alpha}.$$

Proof. We prove this proposition thanks to a duality argument. Since $(X^{\sigma,\beta}_\alpha)' = X^{-\sigma,-\beta}_\alpha$ for any σ, $\beta \in \mathbb{R}$, it is sufficent to show that, for a given $u_0 \in X^{-\gamma,b'}_\alpha$, we have

$$\left| \int_{\mathbb{R}\times\mathbb{T}} u_1\overline{u_2}u_3\overline{u_0} \right| \leq \|u_0\|_{X^{-\gamma,b'}_\alpha} \prod_{j=1}^3 \|u_j\|_{X^{\gamma,b}_\alpha}. \tag{3.9}$$

In fact, we restrict the proof of (3.9) to the case of smooth functions u_j, u_0, with compact support in time — and then conclude by density of such functions. To start with, we introduce new functions w_0, w_1, w_2, w_3 by

$$\mathcal{F}w_j(\tau,k) := (1+|k|^2)^{\frac{\gamma}{2}}(1+|\tau+|k|^\alpha|^2)^{\frac{b}{2}}\mathcal{F}u_j(\tau,k), \quad \text{for } j \in \{1,2,3\},$$
$$\mathcal{F}w_0(\tau,k) := (1+|k|^2)^{-\frac{\gamma}{2}}(1+|\tau+|k|^\alpha|^2)^{\frac{b'}{2}}\mathcal{F}u_0(\tau,k),$$

and we can replace the right hand side of (3.9) by $\|w_0\|_{L^2(\mathbb{R}\times\mathbb{T})} \prod_{j=1}^3 \|w_j\|_{L^2(\mathbb{R}\times\mathbb{T})}$.

To estalish (3.9), we perform a time- and space-localization. By L_0, L_j, N_0, N_j, we refer to dyadic integers, and $L := (L_0, L_1, L_2, L_3)$ will concern time, whereas $N := (N_0, N_1, N_2, N_3)$ will hint at space. Define, for $j \in \{1,2,3\}$,

$$u_j^{L_jN_j}(t,x) := \frac{1}{2\pi} \sum_{|n|\in[N_j,2N_j[} e^{inx} \int_{L_j \leq |\tau+|n|^\alpha| < 2L_j} (1+|n|^2)^{-\frac{\gamma}{2}}(1+|\tau+|n|^\alpha|^2)^{-\frac{b}{2}}\mathcal{F}w_j(\tau,n)e^{it\tau}d\tau,$$

$$u_0^{L_0N_0}(t,x) := \frac{1}{2\pi}\sum_{|n|\in[N_0,2N_0[} e^{inx}\int_{L_0\le|\tau+|n|^\alpha|<2L_0}(1+|n|^2)^{\frac{\gamma}{2}}(1+|\tau+|n|^\alpha|^2)^{-\frac{b'}{2}}\mathcal{F}w_0(\tau,n)e^{it\tau}d\tau.$$

A simple calculus shows that, for $j\in\{1,2,3\}$ first,

$$\begin{aligned}\|u_j^{L_jN_j}\|^2_{X^{\sigma,\beta}_\alpha} &= C\sum_{|n|\simeq N_j}\int_{|\tau+|n|^\alpha|\simeq L_j}(1+|n|^2)^{\sigma-\gamma}(1+|\tau+|n|^\alpha|^2)^{\beta-b}|\mathcal{F}w_j(\tau,n)|^2d\tau\\ &\lesssim L_j^{2(\beta-b)}N_j^{2(\sigma-\gamma)}\underbrace{\sum_{|n|\simeq N_j}\int_{|\tau+|n|^\alpha|\simeq L_j}|\mathcal{F}w_j(\tau,n)|^2d\tau}_{=:\,c_j(L_j,N_j)^2},\end{aligned}$$

where $c_j(L_j,N_j)$ satisfies $\sum_{L_j}\sum_{N_j}c_j(L_j,N_j)^2\le\|w_j\|^2_{L^2(\mathbb{R}\times\mathbb{T})}$. (Here, and in all the sequel, the summation over L_j or N_j means the summation over all dyadic integers.) We have a similar result for $u_0^{L_0N_0}$, so finally, for any σ, $\beta\in\mathbb{R}$,

$$\|u_j^{L_jN_j}\|_{X^{\sigma,\beta}_\alpha}\lesssim L_j^{\beta-b}N_j^{\sigma-\gamma}c_j(L_j,N_j), \tag{3.10}$$
$$\|u_0^{L_0N_0}\|_{X^{\sigma,\beta}_\alpha}\lesssim L_0^{\beta-b'}N_0^{\sigma+\gamma}c_0(L_0,N_0). \tag{3.11}$$

Now we are going to estimate

$$I(L,N) := \left|\int_{\mathbb{R}\times\mathbb{T}} u_1^{L_1N_1}\overline{u_2^{L_2N_2}}u_3^{L_3N_3}\overline{u_0^{L_0N_0}}\right|.$$

Notice that integrating on $\mathbb{T}$ implies that $I(L,N)=0$ unless $N_0\le 2(N_1+N_2+N_3)$. From now on, we suppose that this condition is fulfilled. Moreover, the proof below does not take into account the precise role of the u_j's, neither the conjugate bar, so we can assume that $N_1\ge N_2\ge N_3$.

Using the Hölder inequalities, lemmas 3.4 and 3.5, and finally (3.10) and (3.11), choosing any $\beta\in(\frac12,b)$, $\beta'\in(\frac14,b')$, $\gamma'\in(\frac12-\frac{\alpha}{4},\gamma)$, we bound $I(L,N)$:

$$\begin{aligned}I(L,N) &\le \|u_1^{L_1N_1}\|_{L^4(\mathbb{R}_t,L^2(\mathbb{T}))}\cdot\|u_2^{L_2N_2}\|_{L^4(\mathbb{R}_t,L^\infty(\mathbb{T}))}\cdot\|u_3^{L_3N_3}\|_{L^4(\mathbb{R}_t,L^\infty(\mathbb{T}))}\cdot\|u_0^{L_0N_0}\|_{L^4(\mathbb{R}_t,L^2(\mathbb{T}))}\\ &\le \|u_1^{L_1N_1}\|_{X^{0,\beta'}_\alpha}\cdot\|u_2^{L_2N_2}\|_{X^{\gamma',\beta}_\alpha}\cdot\|u_3^{L_3N_3}\|_{X^{\gamma',\beta}_\alpha}\cdot\|u_0^{L_0N_0}\|_{X^{0,\beta'}_\alpha}\\ &\le \frac{(L_0L_1)^{\beta'}(L_2L_3)^\beta}{L_0^{b'}(L_1L_2L_3)^b}(N_2N_3)^{\gamma'}\frac{N_0^\gamma}{(N_1N_2N_3)^\gamma}\prod_{j=0}^3 c_j(L_j,N_j)\\ &\le L_0^{-\varepsilon_0}L_1^{-\varepsilon_1}(L_2L_3)^{-\varepsilon'}(N_2N_3)^{-\eta'}\left(\frac{N_0}{N_1}\right)^\gamma\prod_{j=0}^3 c_j(L_j,N_j),\end{aligned}$$

where ε_0, ε_1, ε', η' are some positive constants. Consequently, we can sum on L_2,N_2,L_3,N_3, making use of the bound $c_j(L_j,N_j)\le\|w_j\|_{L^2(\mathbb{R}\times\mathbb{T})}$ for $j\in\{2,3\}$. Then, by Cauchy-Schwarz,

$$\begin{aligned}\sum_{L_0,L_1}\sum_{L_2,N_2,L_3,N_3}I(L,N) &\lesssim \left(\frac{N_0}{N_1}\right)^\gamma\|w_2\|_{L^2}\|w_3\|_{L^2}\left(\sum_{L_0}c_0(L_0,N_0)L_0^{-\varepsilon_0}\right)\left(\sum_{L_1}c_1(L_1,N_1)L_1^{-\varepsilon_1}\right)\\ &\lesssim \left(\frac{N_0}{N_1}\right)^\gamma K_0(N_0)^{\frac12}K_1(N_1)^{\frac12}\|w_2\|_{L^2}\|w_3\|_{L^2},\end{aligned}$$

introducing $K_j(N_j) := \sum_{L_j} c_j(L_j,N_j)^2$ for $j \in \{0,1\}$. It remains to sum on N_0, N_1, remembering that $N_0 \leq 6N_1$, and using Cauchy-Schwarz again :

$$\begin{aligned}\sum_{L,N} I(L,N) &\lesssim \sum_{\ell=-3}^{+\infty} \sum_{N_0} \left(\frac{N_0}{2^\ell N_0}\right)^\gamma K_0(N_0)^{\frac{1}{2}} K_1(2^\ell N_0)^{\frac{1}{2}} \|w_2\|_{L^2}\|w_3\|_{L^2} \\ &\lesssim \sum_{\ell=-3}^{+\infty} 2^{-\gamma\ell} \left(\sum_{N_0} K_0(N_0)\right)^{\frac{1}{2}} \left(\sum_{N_0} K_1(2^\ell N_0)\right)^{\frac{1}{2}} \|w_2\|_{L^2}\|w_3\|_{L^2} \\ &\lesssim \|w_0\|_{L^2}\|w_1\|_{L^2}\|w_2\|_{L^2}\|w_3\|_{L^2}.\end{aligned}$$

Hence (3.9) is proven, and so is proposition 3.7. □

Remark 3. The condition $b+b' < 1$ has not been used in the proof, except for the fact that $b' < b$; but it will be crucial in the next proposition.

3.4 Local and global well-posedness

We are now ready to state our

Proposition 3.8. *Let $\gamma > \frac{1}{2} - \frac{\alpha}{4}$ and $\frac{1}{2} < b < 1$. If $u_0 \in H^\gamma(\mathbb{T})$, there exist $T_0 > 0$, depending only on $\|u_0\|_{H^\gamma}$, such that the problem (1.2) admits a unique solution $u \in X_\alpha^{\gamma,b}(T)$ for all $T \leq T_0$. This solution satisfies $\|u\|_{X_\alpha^{\gamma,b}(T)} \leq C\|u_0\|_{H^\gamma(\mathbb{T})}$, where $C > 0$ is an absolute constant.*

If in addition u_0 belongs to $H^s(\mathbb{T})$ for some $s > \gamma$, then $u(t) \in H^s(\mathbb{T})$ for all $t \in [-T_0, T_0]$.

Proof. We intend to show that the functional $\mathcal{K}$, defined in (3.8), is a contraction in some ball of the space $X_\alpha^{\gamma,b}(T)$, for well-chosen T.

Since the first part of $\mathcal{K}$ has been previously bounded, we turn to

$$M_2(u) := \left\|\varphi\left(\tfrac{t}{T}\right)\int_0^t S(t-s)\left[|u(s)|^2u(s)\right]ds\right\|_{X_\alpha^{\gamma,b}(T)} = \left\|\varphi\left(\tfrac{t}{T}\right)\int_0^t S(-s)\left[|u(s)|^2u(s)\right]ds\right\|_{H^{\gamma,b}}.$$

Here, we take advantage of the regularizing property of time-integration. Let $U \in \mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{T})$, and b, b' as in proposition 3.7. Set $G(t,x) := \varphi\left(\frac{t}{T}\right)\int_0^t U(s,x)ds$. For fixed $k \in \mathbb{Z}$, a lemma of Ginibre [25, lemma (3.11)] guarantees that

$$\|\hat{G}(t,k)\|_{H^b(\mathbb{R}_t)} = \left\|\varphi\left(\tfrac{t}{T}\right)\int_0^t \hat{U}(s,k)ds\right\|_{H^b(\mathbb{R}_t)} \lesssim T^{1-b-b'}\|\hat{U}(t,k)\|_{H^{-b'}(\mathbb{R}_t)},$$

with an implicit constant which only depends on φ, b, b'. Squaring this identity, we find

$$\int_{\mathbb{R}} (1+|\tau|^2)^b |\mathcal{F}(G)(\tau,k)|^2 d\tau \lesssim T^{1-b-b'} \int_{\mathbb{R}} (1+|\tau|^2)^{-b'} |\mathcal{F}(U)(\tau,k)|^2 d\tau.$$

Eventually, multiply by $(1+|k|^2)^\gamma$ and sum over k to get $\|G\|_{H^{\gamma,b}} \lesssim T^{1-b-b'}\|U\|_{H^{\gamma,-b'}}$, which remains true, by approximation, for less regular functions.

As a consequence,

$$M_2(u) \lesssim T^{1-b-b'}\|S(-t)\left[|u(t)|^2u(t)\right]\|_{H^{\gamma,-b'}} = T^{1-b-b'}\||u|^2u\|_{X_\alpha^{\gamma,-b'}} \lesssim T^{1-b-b'}\|u\|^3_{X_\alpha^{\gamma,b}},$$

by proposition 3.7. This calculation is valid for any $\tilde{u} \in X_\alpha^{\gamma,b}$ such that $\tilde{u} = u$ on $[-T, T]$. Thus, we proved that

$$\|\mathcal{K}(u)\|_{X_\alpha^{\gamma,b}(T)} \leq \|\varphi\|_{H^b(\mathbb{R}_t)} \|u_0\|_{H^\gamma(\mathbb{T})} + CT^{1-b-b'} \|u\|^3_{X_\alpha^{\gamma,b}(T)}. \tag{3.12}$$

So $\mathcal{K}$ stabilizes the ball B centered at the origin, of radius $\tilde{C}\|u_0\|_{H^\gamma}$ (for some arbitrary $\tilde{C} > \|\varphi\|_{H^b(\mathbb{R}_t)}$), provided that $T \leq T_0$ with

$$T_0 := \left(\frac{\tilde{C} - \|\varphi\|_{H^b(\mathbb{R}_t)}}{\tilde{C}^3 C \|u_0\|^2_{H^\gamma}} \right)^{\frac{1}{1-b-b'}}.$$

Reasoning in the same way, by means of the identity

$$|u|^2 u - |v|^2 v = u(\overline{u-v})u + (u-v)\bar{v}(u+v),$$

we show that for $T \leq T_0$ [5], there exist a positive constant $c < 1$ such that $\|\mathcal{K}(u) - \mathcal{K}(v)\|_{X_\alpha^{\gamma,b}(T)} \leq c\|u - v\|_{X_\alpha^{\gamma,b}(T)}$ for all u, $v \in B$. Hence $\mathcal{K} : B \to B$ is a contraction, and has a fixed point, also called u. Since $u \in B$, we have $\|u\|_{X_\alpha^{\gamma,b}(T)} \leq \tilde{C}\|u_0\|_{H^\gamma(\mathbb{T})}$.

To prove the uniqueness of u, notice first, in view of (3.12), that any other fixed point of $\mathcal{K}$ in $X_\alpha^{\gamma,b}(T)$, as a function of some $X_\alpha^{\gamma,b}(\tilde{T})$ for some smaller $\tilde{T} \leq T$, lies in the ball centered at 0 and of radius $\tilde{C}\|u_0\|_{H^\gamma}$, so equals u in that space, by unicity of the fixed point. Now let $v \in X_\alpha^{\gamma,b}(T)$ be another solution of (1.2). Observe that, because $b > \frac{1}{2}$, both u and v are continuous functions from $[-T, T]$ to $H^\gamma(\mathbb{T})$. Define $T_1 := \sup\{t \in [0, T] \mid u(t) = v(t)\}$, and suppose $T_1 < T$. Then, translating time, and restarting the equation with $u(T_1) = v(T_1)$ as an initial data, we get a contradiction, by the previous remark.

Finally, if $u_0 \in H^s(\mathbb{T})$ for $s > \gamma$, and if u is the associated solution in $X_\alpha^{\gamma,b}(T_0)$, let us show that $u(t) \in H^s(\mathbb{T})$ for all $|t| \leq T_0$. It is crucial to see, modifying slightly the proof of proposition 3.7, that whenever $s > \gamma > \frac{1}{2} - \frac{\alpha}{4}$, and b, b' as above,

$$\|u_1 \overline{u_2} u_3\|_{X_\alpha^{s,-b'}} \leq C \sum_{j=1}^{3} \left(\|u_j\|_{X_\alpha^{s,b}} \prod_{k \neq j} \|u_k\|_{X_\alpha^{\gamma,b}} \right).$$

Given $\tilde{u}$ in the intersection of the ball of radius $\tilde{C}\|u_0\|_{H^s}$ in the space $X_\alpha^{s,b}(T)$, and of the ball of radius $C'\|u_0\|_{H^\gamma}$ in the space $X_\alpha^{\gamma,b}(T_0)$, (3.12) becomes, for $T \leq T_0$:

$$\begin{aligned}
\|\mathcal{K}(\tilde{u})\|_{X_\alpha^{s,b}(T)} &\leq \|\varphi\|_{H^b(\mathbb{R}_t)} \|u_0\|_{H^s(\mathbb{T})} + CT^{1-b-b'} \|\tilde{u}\|^2_{X_\alpha^{\gamma,b}(T)} \|\tilde{u}\|_{X_\alpha^{s,b}(T)} \\
&\leq \left(\|\varphi\|_{H^b(\mathbb{R}_t)} + C\tilde{C}C'^2 T^{1-b-b'} \|u_0\|^2_{H^\gamma(\mathbb{T})} \right) \|u_0\|_{H^s(\mathbb{T})}.
\end{aligned}$$

This shows that if T is chosen small enough, *regardless* of the size of $\|u_0\|_{H^s}$, $\mathcal{K}$ stabilizes the set we described above. The same can be done while estimating $\|\mathcal{K}(u) - \mathcal{K}(v)\|_{X_\alpha^{s,b}(T)}$, and $\mathcal{K}$ therefore has a fixed point $\tilde{u}$ in some $X_\alpha^{s,b}(T)$. Obviously, since $X_\alpha^{s,b}(T) \hookrightarrow X_\alpha^{\gamma,b}(T)$, we have $\tilde{u} \equiv u$ on $[-T, T]$. Repeating this argument after translating time and restarting the equation from $\tilde{u}(T) = u(T)$, the claim is proved. □

The next corollary follows as an immediate consequence, and uses explicitly the condition $\alpha > \frac{2}{3}$:

5. Or possibly a fixed fraction of T_0.

Corollary 3.9. *Let $u_0 \in \mathcal{C}^\infty(\mathbb{T})$. Then (1.2) admits a unique global solution $u \in \mathcal{C}(\mathbb{R}, \mathcal{C}^\infty(\mathbb{T}))$.*

Besides, let $b \in (\frac{1}{2}, 1)$. For any $\gamma \in (\frac{1}{2} - \frac{\alpha}{4}, \frac{\alpha}{2}]$, there exist $T_0^{(\gamma)}$, $C_\gamma > 0$, such that for any $t \in \mathbb{R}$,

$$\|u(\cdot - t)\|_{X_\alpha^{\gamma,b}(T_0^{(\gamma)})} \leq C_\gamma \|u_0\|_{H^\gamma(\mathbb{T})}. \tag{3.13}$$

Proof. It suffices to notice that

$$\alpha > \tfrac{2}{3} \iff \tfrac{1}{2} - \tfrac{\alpha}{4} < \tfrac{\alpha}{2},$$

so there exist $T_0 > 0$, only depending on $\|u_0\|_{H^{\alpha/2}}$, such that (1.2) can be solved locally in $X_\alpha^{\alpha/2,b}(T_0)$. But the energy $\mathcal{H}_\alpha$ and the mass Q are conserved along the trajectory, so $\|u(t)\|_{H^{\alpha/2}}$ remains bounded by $2(\mathcal{H}_\alpha + Q)(u_0)$. This proves that the solution is global, and we have $u(t) \in \mathcal{C}^\infty(\mathbb{T})$ for all $t \in \mathbb{R}$ because of the second part of the previous proposition. (3.13) also follows. □

3.5 End of the proof of theorem 3

All the needed results are gathered : now we can study the growth of the Sobolev norms of the solutions of (1.2) for $\alpha \in (\frac{2}{3}, 1)$.

We begin with the H^α-norm, introducing as in (2.2) the modified energy :

$$\mathcal{E}_\alpha(u) := \|u\|_{L^2}^2 + \||D|^\alpha u\|_{L^2}^2 + \underbrace{2\Re e(|D|^\alpha u, |u|^2 u)}_{=:J_1(u)} - \underbrace{\tfrac{1}{2}(|D|^\alpha(|u|^2), |u|^2)}_{=:J_2(u)},$$

when u is a solution of (1.2). J_1 and J_2 are of lower order than $\|u\|_{H^\alpha}^2$, so that when $\|u\|_{H^\alpha}$ is big enough, arguing as in section 2.1, we have $\frac{1}{2}\|u\|_{H^\alpha}^2 \leq \mathcal{E}_\alpha(u) \leq 2\|u\|_{H^\alpha}^2$.

Let us study the evolution of $\mathcal{E}_\alpha(u)$. The L^2-norm is conserved, so we directly pass on to

$$\frac{d}{dt}\||D|^\alpha u\|_{L^2}^2 = 2\Re e(|D|^\alpha \dot{u}, |D|^\alpha u) = -2\Re e(|D|^\alpha \dot{u}, |u|^2 u).$$

This combines with the derivative of $J_1(u)$, and gives rise to two terms :

$$\frac{d}{dt}[\, \||D|^\alpha u\|_{L^2}^2 + J_1(u)] = 2\Re e(|D|^\alpha u, \dot{u}|u|^2) + 2\Re e(|D|^\alpha u, u(|u|^2)^\cdot) =: Q_1(u) + Q_2(u).$$

Thanks to the equation, a simplification occurs : $Q_1(u) = -2\Im m(|D|^\alpha u, |u|^4 u)$. The Sobolev embedding $H^{2/5} \hookrightarrow L^{10}$ and interpolation between $H^{\alpha/2}$ and H^α then allows to bound

$$|Q_1(u)| \lesssim \|u\|_{H^\alpha}^{2-\varepsilon}, \quad \text{where } \varepsilon := \tfrac{6}{\alpha}\left(\alpha - \tfrac{2}{3}\right) > 0.$$

On the other hand, $Q_2(u)$ combines with the derivative of $J_2(u)$:

$$Q_2(u) + \tfrac{d}{dt}J_2(u) = (\bar{u}|D|^\alpha u + u|D|^\alpha \bar{u} - |D|^\alpha(\bar{u}u), (|u|^2)^\cdot).$$

Since $(|u|^2)^\cdot = i(u|D|^\alpha \bar{u} - \bar{u}|D|^\alpha u)$, we bound $\|(|u|^2)^\cdot\|_{L^2} \lesssim \|u\|_{L^\infty}\|u\|_{H^\alpha}$. As for the other side of the scalar product, we appeal to lemma 2.3, since $\alpha < 1$:

$$\|\bar{u}|D|^\alpha u + u|D|^\alpha \bar{u} - |D|^\alpha(\bar{u}u)\|_{L^2} \lesssim \||D|^{\frac{\alpha}{2}} u\|_{L^4}^2.$$

At this point, we have a Gagliardo-Nirenberg inequality :

Lemma 3.10. *For any $p > 2$, $s > 0$, there exist $C > 0$ such that*

$$\||D|^s f\|_{L^p} \leq C\left(\|f\|_{L^\infty} + \left\||D|^{s\frac{p}{2}} f\right\|_{L^2}^{\frac{2}{p}} \|f\|_{L^\infty}^{1-\frac{2}{p}}\right).$$

for every function $f : \mathbb{T} \to \mathbb{R}$.

Choose a real $\gamma \in (\frac{1}{2} - \frac{\alpha}{4}, \frac{\alpha}{2})$, for instance $\gamma = \frac{2+\alpha}{8}$, and apply lemma 3.10 with $f = |D|^{\frac{\alpha}{2}-\gamma} u$, $p = 4$ and $s = \gamma$. Thus

$$\||D|^{\frac{\alpha}{2}} u\|_{L^4}^2 \lesssim \||D|^{\frac{\alpha}{2}+\gamma} u\|_{L^2} \||D|^{\frac{\alpha}{2}-\gamma} u\|_{L^\infty}$$

— the other terms can be neglected.

All these calculations lead to the following fact : there exist a small $\theta > 0$ (which can be chosen to be $\frac{3\alpha-2}{8\alpha}$), and constants $C_1, C_2 > 0$ such that for all $t \in \mathbb{R}$,

$$\|u(t)\|_{H^\alpha}^2 \leq C_1 \|u_0\|_{H^\alpha}^2 + C_2 \int_0^t \|u(\tau)\|_{H^\alpha}^{2-2\theta} \|u(\tau)\|_{L^\infty} \||D|^{\frac{\alpha}{2}-\gamma} u(\tau)\|_{L^\infty} d\tau.$$

Denoting by $f(t)$ the right hand side of this inequality, and assuming that $t \geq 0$ without loss of generality, this implies that

$$\begin{aligned}
f(t)^\theta - f(0)^\theta &= \int_0^t \frac{f'(\tau) d\tau}{f(\tau)^{1-\theta}} \\
&\leq C_2 \int_0^t \|u(\tau)\|_{L^\infty} \||D|^{\frac{\alpha}{2}-\gamma} u(\tau)\|_{L^\infty} d\tau \\
&\leq C_2 \sqrt{t} \cdot \|u\|_{L^4([0,t],L^\infty)} \||D|^{\frac{\alpha}{2}-\gamma} u\|_{L^4([0,t],L^\infty)} \\
&\leq C_2 \sqrt{t} \left(\sum_{k=0}^{\lceil \frac{t}{T_0^{(\gamma)}} \rceil} \|u(\cdot - kT_0^{(\gamma)})\|_{X_\alpha^{\gamma,b}(T_0^{(\gamma)})} \right) \left(\sum_{k=0}^{\lceil \frac{t}{T_0^{(\alpha/2)}} \rceil} \|u(\cdot - kT_0^{(\alpha/2)})\|_{X_\alpha^{\frac{\alpha}{2},b}(T_0^{(\alpha/2)})} \right) \\
&\leq C_2 C_\gamma C_{\frac{\alpha}{2}} \|u_0\|_{H^\gamma} \|u_0\|_{H^{\frac{\alpha}{2}}} \sqrt{t} \left(\left\lceil \frac{t}{T_0^{(\gamma)}} \right\rceil + 1 \right) \cdot \left(\left\lceil \frac{t}{T_0^{(\alpha/2)}} \right\rceil + 1 \right),
\end{aligned}$$

where we fixed a real $b \in (\frac{1}{2}, 1)$, and used the localized version of lemma 3.5, as well as (3.13). This achieves to show that the H^α-norm of the solution of (1.2) grows at most polynomially, with the power of t being less than $\frac{5}{4\theta}$, hence than $\frac{10\alpha}{3\alpha-2}$.

The end of the proof crucially relies on this first step. Indeed, to estimate the evolution of the $H^{\alpha+n}$-norm of u, with $n \geq 1$, we follow exactly the same scheme as for the proof of theorems 1 and 3 in section 2.1. Each time the L^∞-norm of u appears, we bound it by $\|u\|_{H^\alpha}$ (recall that $\alpha > \frac{1}{2}$). Besides, we do not interpolate the H^s-norms between $H^{\alpha/2}$ and $H^{\alpha+n}$ anymore, but between H^α and $H^{\alpha+n}$.

The only difference is that we need a new (and, to some extent, rougher) version of lemma 2.2 :

Lemma 3.11. *Let $\frac{1}{2} < \alpha < 1$. For any integer $n \geq 1$, there is a constant $C_n > 0$ (independent of α) such that for all function $u \in H^{\alpha+n}(\mathbb{T})$,*

$$\|\bar{u}|D|^\alpha u + u|D|^\alpha \bar{u} - |D|^\alpha(\bar{u}u)\|_{H^n} \leq C\|u\|_{H^\alpha}^{1+\frac{1}{n}(\alpha-\frac{1}{2})} \|u\|_{H^{\alpha+n}}^{1-\frac{1}{n}(\alpha-\frac{1}{2})}.$$

Proof. Denote by $\mathcal{L}$ the left hand side of the inequality we intend to prove. Writing $u = \sum_{k\in\mathbb{Z}} u_k e^{ikx}$, we clearly have

$$\mathcal{L}^2 = \sum_{k=-\infty}^{+\infty} |k|^{2n} \left| \sum_{l=-\infty}^{+\infty} (|l|^\alpha + |k-l|^\alpha - |k|^\alpha) u_l \overline{u_{l-k}} \right|^2$$
$$\lesssim \sum_{k=-\infty}^{+\infty} \left[\left| \sum_{l=-\infty}^{+\infty} |l|^n |k-l|^\alpha |u_l| |\overline{u_{l-k}}| \right|^2 + \left| \sum_{l=-\infty}^{+\infty} |l|^\alpha |k-l|^n |u_l| |\overline{u_{l-k}}| \right|^2 \right],$$

where we used the elementary inequality $|k|^n \leq 2^{n-1}(|l|^n + |k-l|^n)$ and the triangle inequality associated to the concave function $x \mapsto x^\alpha$ on $\mathbb{R}_+$. Now, define $\tilde{u} := \sum_{k\in\mathbb{Z}} |u_k| e^{ikx}$, so that $\mathcal{L} \lesssim \| |D|^n \tilde{u} \cdot |D|^\alpha \overline{\tilde{u}} \|_{L^2} \lesssim \|u\|_{H^{n+1/4}} \|u\|_{H^{\alpha+1/4}}$. Interpolating these norms between H^α and $H^{\alpha+n}$ leads to the result. □

All of this proves that there exist a small $\theta' > 0$, and constants C_1, $C_2 > 0$ such that for all $t \in \mathbb{R}$,

$$\|u(t)\|_{H^{\alpha+n}}^2 \leq C_1 \|u_0\|_{H^{\alpha+n}}^2 + C_2 \int_0^t \|u(\tau)\|_{H^{\alpha+n}}^{2-2\theta'} \|u(\tau)\|_{H^\alpha}^{2+2\theta'} d\tau.$$

This holds with $\theta' = \frac{1}{2n}(\alpha - \frac{1}{2})$. On the other hand, we know that for some $A > 0$, $\|u(\tau)\|_{H^\alpha} \lesssim (1+|\tau|)^A$ for all $\tau \in \mathbb{R}$. By Osgood's lemma, theorem 3 is then fully established.

A Growth of Sobolev norms for the Szegő equation : an elementary bound

Let $u_0 \in \mathcal{C}^\infty(\mathbb{T})$ with only nonnegative frequencies (which we denote by $u_0 \in \mathcal{C}_+^\infty(\mathbb{T})$), and consider $t \mapsto u(t,x)$ the solution of the cubic Szegő equation (1.4) starting from u_0 at time $t = 0$: u satisfies $i\partial_t u = \Pi_+(|u|^2 u)$, and for all $t \in \mathbb{R}$, $u(t)$ also belongs[6] to $\mathcal{C}_+^\infty(\mathbb{T})$. The purpose of this section is to give an elementary proof of the following estimate, which is the counterpart of theorem 1 :

Proposition A.1. *For all $n \in \mathbb{N}$, there exist positive constants C and B such that*

$$\|u(t)\|_{H^{1+n}} \leq C e^{B|t|^2}. \tag{A.1}$$

C is a constant depending on n and $\|u_0\|_{H^{1+n}}$, whereas B can be chosen equal to $B_n \|u_0\|_{H^{1/2}}^8$ (here, B_n depends only on n, not on the considered solution).

We recall here that, though (A.1) is not the best bound available, it is the best one we can prove without resorting to the Lax pair formalism. The proof below only relies on the boundedness of trajectories in the space $H^{1/2}(\mathbb{T})$ (which is due to the conservation of mass and momentum). It also uses a standard fact about Hankel operators.

Definition. Let $v \in H_+^{1/2}(\mathbb{T})$ (*i.e.* v has vanishing negative frequencies). The Hankel operator of symbol v is the following $\mathbb{C}$–antilinear operator :

$$H_v : \begin{cases} L_+^2(\mathbb{T}) \longrightarrow L_+^2(\mathbb{T}) \\ \quad h \longmapsto \Pi_+(v\bar{h}). \end{cases}$$

6. All the claims in this section come from [16] and are proven there.

Proposition A.2. *For $h \in L^2_+(\mathbb{T})$, we have $\|H_v(h)\|_{L^2} \leq \|v\|_{H^{1/2}}\|h\|_{L^2}$.*

Proof. Expand $h = \sum_{k\geq 0} h_k e^{ikx}$ and $v = \sum_{k\geq 0} v_k e^{ikx}$. With these notations,

$$\Pi_+(v\bar{h}) = \sum_{k\geq 0} e^{ikx}\left(\sum_{l\geq k} v_l \overline{h_{l-k}}\right).$$

A simple application of the Cauchy-Schwarz inequality gives

$$\|\Pi_+(v\bar{h})\|_{L^2}^2 \leq \sum_{k\geq 0}\left(\sum_{l\geq 0}|v_{l+k}|^2\right)\left(\sum_{l\geq 0}|h_l|^2\right) = \|h\|_{L^2}^2 \sum_{\tilde{k}\geq 0}(1+\tilde{k})|v_{\tilde{k}}|^2,$$

which is the yielded result. □

Since the embedding $H^{1/2}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ fails, proposition A.2 is an improvement of the standard L^∞-L^2 estimate of a product in L^2, similarly to lemma 2.2. It is the key of the

Proof of proposition A.1. We compute, for $n \geq 0$,

$$\frac{d}{dt}\|\partial_x^{1+n}u\|_{L^2}^2 = 2\Re e(\partial_x^{1+n}\dot{u}, \partial_x^{1+n}u) = 2\Im m(\partial_x^{1+n}(|u|^2u), \partial_x^{1+n}u).$$

The last equality comes from the equation, and the fact that the frequencies of u are non-negative (so we get rid of Π_+ here). Then, using Leibniz rule, we expand $\partial_x^{1+n}(|u|^2u) = |u|^2\partial_x^{1+n}u + \sum_{k=1}^n \binom{1+n}{k}(\partial_x^k|u|^2)(\partial_x^{1+n-k}u) + u\partial_x^{1+n}|u|^2$. The first term cancels because of the imaginary part, and the "crossed terms" are easily estimated thanks to Sobolev injections and interpolation inequalities, as in the proof of theorem 1. As for the last term, we have

$$(u\partial_x^{1+n}|u|^2, \partial_x^{1+n}u) = (\Pi_+(u\partial_x^{1+n}|u|^2), \partial_x^{1+n}u) = (H_u(\partial_x^{1+n}|u|^2), \partial_x^{1+n}u),$$

because $|u|^2$ is real. So, by proposition A.2, and then inequality (2.1),

$$\frac{d}{dt}\|u\|_{H^{1+n}}^2 \lesssim \|u\|_{H^{1+n}}^2\|u\|_{H^{1/2}}\|u\|_{L^\infty} \lesssim \|u\|_{H^{1+n}}^2\|u\|_{H^{1/2}}^2\sqrt{\log(1+\|u\|_{H^{1+n}}^2)},$$

and the proof is complete, once we have recalled that $\|u\|_{H^{1/2}} \leq C_0\|u_0\|_{H^{1/2}}^2$. □

B Some comments on the threshold $\alpha = \frac{2}{3}$

In this section, we discuss the bound $\alpha = \frac{2}{3}$ that naturally appears in the proof of theorem 3.

The crucial point, in the proof above, is to know which Strichartz estimate we are able to establish. In particular, following the strategy of [10], we would like to know for which values of the parameter γ the inequality

$$\|e^{-it|D|^\alpha}u(x)\|_{L^4((0,1)_t, L^4(\mathbb{T}_x))} \leq C\|u\|_{H^\gamma} \tag{B.1}$$

holds, whenever $u \in H^\gamma(\mathbb{T})$. If true, (B.1) would imply that equation (1.2) is well-posed in $H^{\alpha/2}$, provided that $2\gamma < \frac{\alpha}{2}$, and we could then adapt the arguments we develop in section 3.5 to prove that solutions are polynomially bounded.

Looking back at (3.2), and interpolating the $L^4((0,1)_t, L^\infty(\mathbb{T}_x))$ estimate we obtained with the trivial $L^\infty((0,1)_t, L^2(\mathbb{T}_x))$ one, we find a bound for $\|e^{-it|D|^\alpha}u(x)\|_{L^8((0,1)_t, L^4(\mathbb{T}_x))}$, so (B.1) is proved with $\gamma = \frac{1}{4} - \frac{\alpha}{8}$. The condition $2\gamma < \frac{\alpha}{2}$ exactly means that $\alpha > \frac{2}{3}$.

However, scaling heuristics suggest that the natural value of γ should rather be $\gamma_0 := \frac{1}{4} - \frac{\alpha}{4}$. Here, the condition $2\gamma_0 < \frac{\alpha}{2}$ would enable us to extend the conclusions of theorem 3 until $\alpha = \frac{1}{2}$.

In [13], by a different method, Demirbas, Erdoğan and Tzirakis also prove a Strichartz estimate in the case $\alpha > 1$, but their result corresponds to ours (notice that in their work, they use other notations : what they call α is in fact half ours).

So far, we don't know if (B.1) can be proved with $\gamma = \gamma_0$. Usual counter-examples (such as localized functions) only confirm that the scaling exponent γ_0 is the best we can hope. Besides, the difficulty is not due to the particular framework of the torus, since when $\alpha < 1$ the speed of propagation of the waves (*i.e.* the group velocity) is finite anyway.

Chapitre 3

Optimal bounds for the growth of Sobolev norms of solutions of a quadratic Szegő equation

Abstract

In this paper, we study a quadratic equation on the one-dimensional torus :

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2, \quad u(0,\cdot) = u_0,$$

where $J = \int_{\mathbb{T}} |u|^2 u \in \mathbb{C}$ has constant modulus, and Π is the Szegő projector onto functions with nonnegative frequencies. Thanks to a Lax pair structure, we construct a flow on $BMO(\mathbb{T}) \cap \operatorname{Im}\Pi$ which propagates H^s regularity for any $s > 0$, whereas the energy level corresponds to $s = 1/2$. Then, for each $s > 1/2$, we exhibit solutions whose H^s norm goes to $+\infty$ exponentially fast, and we show that this growth is optimal.

1 Introduction

Quantifying the growth of Sobolev norms of solutions to a Hamiltonian PDE is a good way of understanding how fast the energy moves from lower to higher frequencies and conversely, a phenomenon which is typical of what is known as *wave turbulence*. Upper bounds on such a growth are usually known [7, 52, 58, 61], but it is very difficult to find out whether they are optimal or not, unless explicit growing solutions can be found.

A setting combining Hamiltonian properties, explicit computations, and growth of Sobolev norms has been succesfully introduced in 2010 by Gérard and Grellier [16]. The authors designed a simple toy model, called the *cubic Szegő equation*, which turns out to be completely integrable, in the sense that action-angle variables can be defined, making computations possible (though not obvious). On the other hand, this system discloses instability, and it is possible to prove the existence of generic turbulent orbits, with Sobolev norms growing more than polynomially in time and oscillating back and forth [21]. This toy model eventually enlightens the large-time behaviour of non-integrable hamiltonian systems, for which it enables to find turbulent solutions through a study of resonances [69] (see also [31, 32] for the same ideas in the case of the nonlinear Schrödinger equation on $\mathbb{T}^2$). Note that the cubic Szegő equation also looks similar to physically relevant equations, such as the one studied in [4]. However, in all these situations, the optimality of the bounds on the growth of solutions remains to be established.

As observed in [61], a priori estimates are much simpler to prove when nonlinearities are only quadratic. This gives the idea of exploiting the framework of the cubic Szegő equation in an apparently simpler case, where we only take a *cubic* Hamiltonian (instead of quartic). More specifically, we choose our phase space to be the closed subset of $L^2(\mathbb{T})$, called $L^2_+(\mathbb{T})$, consisting of all functions with only nonnegative frequencies :

$$L^2_+(\mathbb{T}) = \left\{ u \in L^2(\mathbb{T}) \,\middle|\, \hat{u}(k) = 0,\, \forall k < 0 \right\}.$$

L^2 is endowed with its standard Hilbert structure :

$$(f|g) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix})\overline{g(e^{ix})}dx.$$

We denote by $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ the orthogonal projection onto L^2_+. Π is usually called the *Szegő projector*. For any subspace G of $L^2(\mathbb{T})$, we will use the notation $G_+ := G \cap L^2_+$.

Remark 4. The space $L^2_+(\mathbb{T})$ is also called the Hardy space on the disc, and can be identified with the space of holomorphic functions on the open unit disc $\mathbb{D} \subset \mathbb{C}$ whose trace on the boundary $\partial\mathbb{D}$ is in L^2.

$$u(x) = \sum_{n=0}^{\infty} \hat{u}(n)e^{inx} \overset{\sim}{\longmapsto} u(z) = \sum_{n=0}^{\infty} \hat{u}(n)z^n, \quad \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^2 d\theta < \infty.$$

In the sequel, we shall use either notation to designate the points of our phase space.

On a dense subset of L^2_+, we can define a functional E, that will be taken as our Hamiltonian (our energy) :

$$E(u) := \frac{1}{2}\left| \int_{\mathbb{T}} |u|^2 u \right|^2 = \frac{1}{2}|(u^2|u)|^2,$$

With respect to the natural symplectic structure given by the 2-form $\omega(f,g) := \operatorname{Im}(f|g)$, the Hamiltonian equation deriving from E reads $\partial_t u = X_E(u)$, where $X_E(u)$ is the vector field taking values in L^2_+ and satisfying $dE(u)\cdot h = \omega(h, X_E(u))$ for all $h \in L^2_+$. We thus find the following PDE :

$$\partial_t u = -i\left[2(u^2|u)\Pi(|u|^2) + (u|u^2)u^2\right],$$

Let us here simplify the notation, calling J the factor $(u^2|u)$ (as a reminiscence of the J_3 of Szegő hierarchy introduced in [16]), so that $E = \frac{1}{2}|J|^2$ and the evolution of u is given by

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2. \tag{1.1}$$

Because of the invariances of the Hamiltonian E, we know that there are at least three conservation laws for smooth solutions of (1.1) :

$$\begin{aligned} Q(u) &:= (u|u), && \text{(the mass)}\\ M(u) &:= (Du|u),\ D := -i\partial_x, && \text{(the momentum)}\\ E(u) &= \tfrac{1}{2}|(u^2|u)|^2 = \tfrac{1}{2}|J|^2. && \text{(the energy)} \end{aligned}$$

In particular, the modulus of J is conserved by the flow, which makes system (1.1) look like a "quadratic Szegő equation". In fact, we will show the existence of infinitely many conservation laws for equation (1.1), proving that it is associated with a Lax pair structure (see Theorem 5 below).

Notice also that the $H^{1/2}$ norm of a smooth solution remains bounded, since for $u \in L^2_+$,

$$(Q+M)(u) = \sum_{n\geq 0}(1+n)|\hat{u}(n)|^2 \simeq \|u\|^2_{H^{1/2}}.$$

This is what makes $H^{1/2}_+(\mathbb{T})$ a natural space to define a flow. We will call it the *energy space*.

However, it turns out that we can even define solutions below the $H^{1/2}$ regularity, which is consistent with the fact that equation (1.1) does not involve any derivative in the x-variable. Following the approach of [22], and using the Lax pair structure, we will prove that (1.1) admits a flow on $BMO_+(\mathbb{T}) := BMO \cap L^2_+(\mathbb{T})$, where BMO denotes the usual space of John and Nirenberg of functions such that

$$\sup_{I\subset\mathbb{T}} \frac{1}{|I|}\int_I \left| f - \frac{1}{|I|}\int_I f \right| < +\infty,$$

where the supremum is taken on intervals I of $\mathbb{T}$.

The main theorem of this paper is the following :

Theorem 4. *Let $s \geq 0$ be any nonnegative real number. Let $u_0 \in BMO_+ \cap H^s_+(\mathbb{T})$.*

(i) Then there exists a unique $u \in C(\mathbb{R}, H^s_+) \cap C_{w*}(\mathbb{R}, BMO_+)$ solution to

$$i\partial_t u = 2J\Pi(|u|^2) + \overline{J}u^2, \quad u(0) = u_0.$$

This solution stays bounded in the BMO norm, and (when $s \geq \frac{1}{2}$) in the $H^{1/2}$ norm.

In addition, for each $t \in \mathbb{R}$ and $R > 0$, the mapping $u_0 \mapsto u(t)$ is Lipschitz continuous on the ball $\mathcal{B}_{BMO}(R) := \{v \in BMO_+(\mathbb{T}) \mid \|v\|_{BMO} \leq R\}$ endowed with the L^2 distance.

(ii) Let $s' \in (0, s]$ (if $s < \frac{1}{2}$), or $s' \in (\frac{1}{2}, s]$ (if $s \geq \frac{1}{2}$). Then there are constants $C, B > 0$ such that

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^{s'}} \leq Ce^{B|t|}.$$

B can be taken to depend only on s' and on $\|u_0\|_{BMO}$ (when $s' < 1$), on $\|u_0\|_{H^{1/2}}$ (when $s' = 1$) or on $\|u_0\|_{H^{s'}}$ (when $s' > 1$).

Moreover, in the case when $s \geq \frac{1}{2}$, these estimates are optimal, i.e. *for each $s \geq \frac{1}{2}$, there exists $u_0 \in BMO_+ \cap H^s$ such that $\|u(t)\|_{H^{s'}}$ grows exponentially fast for each $s' \in (\frac{1}{2}, s]$.*

Note that when $s \geq \frac{1}{2}$, $H^s_+(\mathbb{T}) \subset BMO_+(\mathbb{T})$. In particular, Theorem 4 yields that equation (1.1) is well-posed in the energy space.

Furthermore, when $s = 0$, $BMO_+ \cap H^0(\mathbb{T}) = BMO_+(\mathbb{T})$, hence the above theorem states the existence of a flow on BMO_+ which propagates additional regularity. This phenomenon strongly hinges on the quadratic nature of equation (1.1). As shown in [22], it is also possible to define a flow on BMO_+ for the cubic Szegő equation, but in that case it is not known whether it preserves H^s regularity or not, when $0 < s < 1/2$.

Let us turn to the optimality of the a priori estimates. As we mentioned before, the existence of turbulent trajectories can be shown thanks to explicit computations relying on the integrability of the equation. Indeed, a consequence of the Lax pair structure will be that the set

$$\mathcal{L}(1) := \left\{ u(x) = b + \frac{ce^{ix}}{1 - pe^{ix}} \,\middle|\, b, c, p \in \mathbb{C}, c \neq 0, |p| < 1 \right\}$$

is stable by the flow of (1.1). On this set, (1.1) will reduce to a system of coupled ODEs, from which we will be able to derive a necessary and sufficient condition for norm explosion. We prove the following proposition :

Proposition 1.1. *Suppose that u is a solution of (1.1) such that $u(0) = u_0 \in \mathcal{L}(1)$. Then the following statements are equivalent :*

(i) There exists $s > \frac{1}{2}$ such that $\|u(t)\|_{H^s}$ is unbounded as $t \to \pm\infty$.

(ii) For all *$s > \frac{1}{2}$, there exists $C_s, B_s > 0$ such that $\|u(t)\|_{H^s} \sim_{t\to\pm\infty} C_s e^{B_s|t|}$.*

(iii) The energy and the mass of u_0 satisfy the relation

$$E = \frac{1}{2} Q^3. \tag{1.2}$$

This proposition, which will prove the last part of Theorem 4, calls for several comments :

- On $\mathcal{L}(1)$, there is a dichotomy for solutions of (1.1) : either u remains bounded in every H^s, $s \geq 0$, either it blows up in each H^s topology, $s > \frac{1}{2}$, at an exponential rate (but remaining bounded in $H^{1/2}$ and below, of course, since $\mathcal{L}(1)$ is made of C^∞ functions). The question whether such a dichotomy remains true for smooth general solutions, or even on other stable finite-dimensional manifolds (described in Proposition 2.5), is widely open.

- Even if the equation we study is only quadratic, hence "less nonlinear" in a way than the cubic Szegő equation, it appears to be more turbulent, and the example of the turbulent solutions that we exhibit does not display the phenomenon of *backward energy cascade* (as opposed to the behaviour described in [21, Theorem 1]). In addition, regarding the cubic Szegő flow, it is known [20] that all trajectories that belong to a finite-dimensional stable manifold are bounded in every H^s, $s > \frac{1}{2}$, and the same phenomenon seems to occur in the case of the more physical situation studied in [4]. However, the dichotomy of Proposition 1.1 looks very similar to what Haiyan Xu observed for a *linear perturbation* of the cubic Szegő equation. The proof of Proposition 1.1 heavily relies on the technique she developed in [67].

- Condition (1.2) for norm explosion is only expressed in terms of conservation laws of the flow. Therefore, it is compatible with the autonomous nature of equation (1.1). But more striking is that (1.2) is *homogeneous* : if it is satisfied by u, then it is also satisfied by λu, $\lambda \in \mathbb{C}$. This allows to construct blow-up solutions with arbitrary small initial data in H^s, when $s > \frac{1}{2}$ is given, in great contrast with the situation described by H. Xu in [67].

This paper is organized as follow : in section 2, we point out the Lax pair structure of the quadratic Szegő equation ; in section 3, we prove the existence of the flow of (1.1) on $BMO_+(\mathbb{T})$; finally, in section 4, we prove the turbulence results contained in Proposition 1.1.

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2 The Lax pair structure

2.1 Smooth solutions

We begin by proving a useful lemma :

Lemma 2.1 (Smooth solutions)**.** *Let $s > \frac{1}{2}$, and $u_0 \in H^s_+(\mathbb{T})$. Then there exists a unique function solution $u \in C^\infty(\mathbb{R}, H^s_+(\mathbb{T}))$ such that $u(0) = u_0$ and*

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2.$$

We also have $\forall t \in \mathbb{R}$, $M(u(t)) = M(u_0)$, $Q(u(t)) = Q(u_0)$ and $E(u(t)) = E(u_0)$.

In the sequel, we will refer to these solutions as *smooth* solutions.

Proof of the lemma. It is well-known that H^s_+ is an algebra whenever $s > \frac{1}{2}$. In addition, $|J| \leq \|u\|^3_{L^3} \leq C\|u\|^3_{H^{1/2}}$. By a fixed point argument, we thus get the existence of a time $T > 0$ only depending on $\|u_0\|_{H^s}$ for which there is a unique solution of (1.1) starting from u_0 in $C([-T, T], H^s_+)$. We observe that M, Q and E are conserved along these local-in-time solutions.

The global existence comes from a simple energy estimate. For u solution as above, we have

$$\frac{d}{dt}\|D^s u\|^2_{L^2} \leq C\|u\|_{H^s}(\||u|^2\|_{H^s} + \|u^2\|_{H^s}) \leq C\|u\|^2_{H^s}\|u\|_{L^\infty}, \tag{2.1}$$

and with the help of the Brezis-Gallouët inequality (see [9, 16]) stating that there is $C_s > 0$ such that

$$\|u\|_{L^\infty} \leq C_s\|u\|_{H^{1/2}}\sqrt{\log\left(1 + \frac{\|u\|_{H^s}}{\|u\|_{H^{1/2}}}\right)},$$

and also using the boundedness of the $H^{1/2}$ norm of local solutions along time, we get $\frac{d}{dt}\|u\|^2_{H^s} \leq C\|u\|^2_{H^s}\sqrt{\log(1 + C'\|u\|^2_{H^s})}$. Integrating this inequality yields

$$\|u(t)\|_{H^s} \leq Ce^{Bt^2}.$$

This is enough to prove that $\|u(t)\|_{H^s}$ does not become infinite in finite time, so that local solutions constructed above are in fact global. Global uniqueness then follows from the local argument. □

2.2 The shifted Hankel operators

As in [18], let us now introduce three classes of operators acting on L^2_+. For a given $u \in H^{1/2}_+$, we define the *Hankel operator* of symbol u :

$$H_u : \begin{cases} L^2_+ \longrightarrow L^2_+, \\ \quad h \longmapsto \Pi(u\bar{h}), \end{cases}$$

and for $b \in L^\infty(\mathbb{T})$, we define as well the *Toeplitz operator* of symbol b by

$$T_b : \begin{cases} L^2_+ \longrightarrow L^2_+, \\ \quad h \longmapsto \Pi(bh). \end{cases}$$

Observe that T_b is a $\mathbb{C}$-linear operator, whereas H_u is $\mathbb{C}$-antilinear. The adjoint of T_b is $(T_b)^* = T_{\bar{b}}$, and we have $(H_u(h_1)|h_2) = (H_u(h_2)|h_1)$, for any h_1, $h_2 \in L^2_+$. A special Toeplitz operator is the *shift operator*, defined by $S := T_{e^{ix}} = T_z$.

Observe that, for $u \in H^{1/2}_+$, we have the following identity :

$$H_u S = S^* H_u = H_{S^* u}.$$

This gives rise to the definition of the third class of operators : the *shifted Hankel operator* of symbol u is the operator

$$K_u := S^* H_u.$$

The following proposition sums up important properties of K_u^2 :

Proposition 2.2. *For any $u \in H^{1/2}_+$, K_u^2 is a $\mathbb{C}$-linear positive self-adjoint operator on L^2_+. Moreover, K_u^2 is trace class (hence compact).*

The same proposition also holds for H_u^2, but will not be needed it here, because K_u turns out to be of more importance in the study of equation (1.1). Let us just mention that the operator norm of H_u in $\mathcal{L}(L^2_+)$ satisfies $\|H_u\| \le \|u\|_{H^{1/2}}$. (A proof of this elementary fact can be found *e.g.* in [61, Appendix A].)

We can now state the main theorem of this section :

Theorem 5 (Lax pair for K_u)**.** *Let $s > \frac{1}{2}$. Assume u is a smooth solution of (1.1) in H^s, as in Lemma 2.1. Then we have a Lax pair identity :*

$$\frac{d}{dt} K_u = B_u K_u - K_u B_u, \tag{2.2}$$

where $B_u := -i(T_{\bar{J}u} + T_{J\bar{u}})$ is a well-defined anti-self-adjoint operator on L^2_+.

Proof. Let $h \in L^2_+$. We write

$$i \frac{d}{dt} K_u(h) = \Pi(\bar{z}(i\dot{u})\bar{h}) = \Pi(2J\bar{z}|u|^2\bar{h}) + \Pi(\bar{J}\bar{z}u^2\bar{h}).$$

Note that the projector Π disappeared in the first term, because $\Pi(2J\bar{z}(I - \Pi)(|u|^2)\bar{h}) = 0$.

Compute in two ways :

$$\Pi(J\bar{z}|u|^2\bar{h}) = \Pi(\bar{z}u\overline{\bar{J}uh}) = \Pi(\bar{z}u\overline{\Pi(\bar{J}uh)}) = K_u T_{\bar{J}u}(h)$$

because $uh = \Pi(uh)$, and

$$\Pi(J\bar{z}|u|^2\bar{h}) = \Pi(J\bar{u}\bar{z}u\bar{h}) = \Pi(J\bar{u}\Pi(\bar{z}u\bar{h})) = T_{J\bar{u}}K_u(h),$$

because $\Pi(J\bar{u}(I-\Pi)(\bar{z}u\bar{h})) = 0$.

For the other term, we need an elementary lemma, which can be proved by the simple use of Fourier expansion :

Lemma 2.3. *Given $f \in L^2(\mathbb{T})$, the following identity holds :*

$$(I-\Pi)(\bar{z}f) = \bar{z}\overline{\Pi(\bar{f})}.$$

Thus,

$$\begin{aligned}\Pi(\bar{J}\bar{z}u^2\bar{h}) &= \Pi(\bar{J}u\Pi(\bar{z}u\bar{h})) + \Pi(\bar{J}u(I-\Pi)(\bar{z}u\bar{h}))\\ &= T_{\bar{J}u}K_u(h) + \Pi(\bar{J}\bar{z}u\overline{\Pi(\bar{u}h)})\\ &= T_{\bar{J}u}K_u(h) + K_uT_{J\bar{u}}(h).\end{aligned}$$

Summing up, and introducing the self-adjoint operator $A_u := T_{\bar{J}u} + T_{J\bar{u}}$, we have proved that

$$i\frac{d}{dt}K_u = A_uK_u + K_uA_u.$$

Now, multiply this identity by $-i$, and set $B_u := -iA_u$. Using the fact that K_u is $\mathbb{C}$-antilinear, we finally get

$$\frac{d}{dt}K_u = [B_u, K_u] = B_uK_u - K_uB_u,$$

which is the claim. □

Corollary 2.4. *The eigenvalues $\{\sigma_k^2\}_{k\geq 1}$ (repeated with multiplicity) of K_u^2 are conservation laws of equation (1.1).*

Proof. From (2.2), we get $\frac{d}{dt}K_u^2 = B_uK_u^2 - K_u^2B_u$. Now consider the system

$$\begin{cases} U'(t) = B_uU(t), \quad t \in \mathbb{R}\\ U(0) = I,\end{cases}$$

where the unknown $U(t)$ belongs to $\mathcal{L}(L^2_+)$. The existence of a global-in-time solution is ensured by the Cauchy-Lipschitz theorem for *linear* ordinary differential equations. Besides, $U(t)$ is a unitary operator for all time (because of the skew-adjointness of B_u). An easy computation then shows that $\frac{d}{dt}(U(t)^*K_u^2U(t)) = 0$, hence K_u^2 remains unitarily equivalent to $K_{u(0)}^2$, and its eigenvalues are conserved. □

Remark 5. The evolution of H_u can also be computed :

$$\frac{d}{dt}H_u = B_uH_u - H_uB_u + i\bar{J}(u|\cdot)u.$$

This is not a Lax pair : an extra term appears from Lemma 2.3, because of the zero mode. Nevertheless, if we consider the Hamiltonian E defined on functions on $\mathbb{R}$, instead of $\mathbb{T}$, with a projector Π given by

$$\Pi\left(\int_{-\infty}^{+\infty}\hat{f}(\xi)e^{ix\xi}d\xi\right) = \int_0^{+\infty}\hat{f}(\xi)e^{ix\xi}d\xi,$$

we can also make sense of equation (1.1), and this time, we would have

$$\frac{d}{dt}H_u = B_uH_u - H_uB_u,$$

with B_u defined as above. This suggests to start a program such as the one developed by Oana Pocovnicu [53, 54] for the cubic Szegő equation on the line.

2.3 Two consequences : finite dimensional invariant manifolds and L^∞ bounds

One of the main consequences of Theorem 5 is the existence of finite dimensional invariant manifolds for the flow of (1.1).

Definition. Let N be a nonnegative integer. We set

$$\mathcal{L}(N) := \left\{ u \in H^{1/2}_+(\mathbb{T}) \,\middle|\, \operatorname{rk} K_u = N \right\}.$$

It happens that we have a complete description of $\mathcal{L}(N)$.

Proposition 2.5 (see [67]). *Fix $N \in \mathbb{N}$. The set $\mathcal{L}(N)$ is exactly the set of all rational functions u which can be written in the form*

$$u(z) = \frac{A(z)}{B(z)}, \quad z \in \mathbb{D},$$

where A and B are complex polynomials of degree at most N, satisfying $\deg A = N$ or $\deg B = N$, $A \wedge B = 1$, $B(0) = 1$, and B having no root in the closed unit disc $\bar{\mathbb{D}}$.

We see that $\mathcal{L}(N)$ is a manifold of complex dimension $2N+1$, and that each $u \in \mathcal{L}(N)$ belongs to $C^\infty(\mathbb{T})$, hence $u \in H^s_+(\mathbb{T})$ for any $s > \frac{1}{2}$.

Corollary 2.6. *For each $N \in \mathbb{N}$, the flow of (1.1) preserves the manifold $\mathcal{L}(N)$.*

Proof. Since $\operatorname{rk} K_u = \operatorname{rk} K_u^2$, the rank of K_u is given by the number of non-zero eingenvalues in the list $\{\sigma_k^2\}_{k\geq 1}$. By the preceding corollary, this number is constant for the evolution related to the Hamiltonian E. □

Now we turn to another consequence, arguing as in [21] :

Corollary 2.7. *Let $s > 1$ and $u_0 \in H^s_+$. Then the solution u of (1.1) starting from u_0 satisfies*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty} < +\infty.$$

In addition, for any $s' \in (\frac{1}{2}, s]$, there exists $C, B > 0$ such that

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^{s'}} \leq C e^{B|t|}.$$

Proof. By Peller's theorem [51, Theorem 1.1 p. 232], we know that

$$\operatorname{Tr}|H_u| \sim \|u\|_{B^1_{1,1}} = \sum_{j\in\mathbb{N}} 2^j \|\Delta_j u\|_{L^1},$$

where $|H_u| = \sqrt{H_u^2}$, Tr is the trace norm, and $\Delta_j u$ is the j-th dyadic piece of u. On the other hand, when $s > 1$, $H^s \hookrightarrow B^1_{1,1}$, so $\operatorname{Tr}|K_u| = \operatorname{Tr}|H_{S^*u}|$ remains finite for all time. In fact,

$$\operatorname{Tr}|K_u| = \sum_{k\geq 1} \sqrt{\sigma_k^2},$$

so by Corollary 2.4, it is even conserved by the flow. As $B^1_{1,1} \hookrightarrow W$, the Wiener algebra, we find that $\sup_{t\in\mathbb{R}} \|S^*u(t)\|_W \leq C \sup_{t\in\mathbb{R}} \operatorname{Tr}|K_u| = C \operatorname{Tr}|K_{u_0}| \leq C'\|u_0\|_{H^s} < +\infty$. But naturally $|\hat{u}(0)| \leq \sqrt{Q}$, so that it is also bounded along time. Consequently, $\|u\|_W = |\hat{u}(0)| + \|S^*u\|_W$ remains bounded, and so does $\|u\|_{L^\infty}$ of course.

As for the proof of the a priori estimate, it goes as in Lemma 2.1, but since $\|u\|_{L^\infty}$ can be replaced by a constant in the r.h.s. of (2.1), Gronwall's lemma leads to a simple exponential bound. □

Remark 6. Here, we see how crucial the control of the L^2 norm is, because it enables to bound the mean of u. Taking simply as a Hamiltonian the functional $\tilde{E}(u) = \mathrm{Re}(u^2|u) = \mathrm{Re}\, J$ (instead of $E = \frac{1}{2}|J|^2$) would break one of the symmetries of the equation, and the associated evolution equation would read $i\partial_t u = 2\Pi(|u|^2) + u^2$ (*i.e.* no J factor). This equation also admits a Lax pair for K_u, but all non zero solutions blow up in finite time because of the zero mode. Indeed, writing $(u|1) = x + iy$ with $x, y \in \mathbb{R}$, the above equation implies that

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = y^2 - x^2 - 2Q, \end{cases}$$

and the fact that $Q \geq x^2 + y^2$ leads to $\dot{y} \leq -y^2 - 3x^2 \leq -y^2$. Besides, we can assume that $y_0 := y(t=0) \neq 0$, since $\dot{y}(t=0) \leq -Q < 0$. Thus, y must blow up in finite time $T \in [-\frac{1}{|y_0|}, \frac{1}{|y_0|}]$.

2.4 The case of H^1 data

Before going further, we show that an elementary argument can treat the case of H^1 initial data, in the spirit of [61].

Lemma 2.8. *Let $u_0 \in H^1_+$, and u be the solution of (1.1) starting from u_0. Then for any $s' \in (\frac{1}{2}, 1]$, there exists constants $C, B > 0$ such that*

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^{s'}} \leq Ce^{B|t|}.$$

We can choose $C = \|u_0\|_{H^1}$ and $B = B_0(2s'-1)\sqrt{2E}\|u_0\|_{H^{1/2}}$, where B_0 is some universal constant.

Proof. Since $\|u\|_{H^{1/2}}$ is a bounded quantity, it is enough to show the bound in the case when $s' = 1$, by interpolation. Since the L^2 norm of u is a constant of motion, we can study the evolution of the H^1 norm of u simply by computing :

$$\begin{aligned} \frac{d}{dt}\|Du(t)\|^2_{L^2} &= 2\,\mathrm{Re}(D\dot{u}|Du) = 2\,\mathrm{Im}(2JD(|u|^2) + \bar{J}D(u^2)|Du) \\ &= 4\,\mathrm{Im}(JuD\bar{u} + J\bar{u}Du + \bar{J}uDu|Du) \\ &= -4\,\mathrm{Im}(J\Pi(u\overline{Du})|Du) + 4\,\mathrm{Im}\left(J\bar{u} + \bar{J}u\big||Du|^2\right). \end{aligned}$$

Because of the imaginary part, the second term cancels out. Observing that $\Pi(u\overline{Du}) = H_u(Du)$, we claim that the modulus of the first term is controlled by

$$4|J| \cdot \|\Pi(u\overline{Du})\|_{L^2}\|Du\|_{L^2} \leq 4\sqrt{2E}\|u\|_{H^{1/2}}\|Du\|^2_{L^2}.$$

Indeed $\|H_u\| \leq \|u\|_{H^{1/2}}$ as mentioned above. Since the $H^{1/2}$ norm of u remains uniformly bounded along trajectories, this shows that the H^1 norm of u grows at most exponentially in time. □

3 The flow on BMO_+

The purpose of this section is to take advantage of the Lax pair structure in order to prove, as in [22], that the flow of (1.1) can be extended to $BMO_+(\mathbb{T})$. This space can be defined as the intersection of the BMO space of John and Nirenberg with L^2_+, or equivalently as the image of $L^\infty(\mathbb{T})$ through Π :

$$BMO_+(\mathbb{T}) = \{\Pi(b) \mid b \in L^\infty(\mathbb{T})\}.$$

The norm is then given by

$$\|u\|_{BMO} = \inf\{\|b\|_{L^\infty} \mid \Pi(b) = u\}.$$

Lastly, BMO_+ is isometric to the dual of L^1_+ (*i.e.* of L^1 functions of the torus with vanishing negative frequencies), and thus can also be equipped with the corresponding weak-$*$ topology.

Clearly, if $u = \Pi(b)$ for some $b \in L^\infty$, then for any $h \in L^2_+$, $\Pi(u\bar{h}) = \Pi(\Pi(b)\bar{h}) = \Pi(b\bar{h})$, so that $\|H_u(h)\|_{L^2} \leq \|b\|_{L^\infty}\|h\|_{L^2}$, hence H_u is continuous on L^2_+. Nehari [49] proved that the converse is true : H_u is a continuous operator of L^2_+ if and only if $u \in BMO_+$; in that case, we have $\|H_u\| = \|u\|_{BMO}$.

The only other fact we will use about BMO_+ is that it is continuously embedded in L^p for any $p < \infty$.

Before turning to the existence of a flow map on BMO_+, we would like to prove two crucial lemmas that we will use repeatedly in the sequel.

Lemma 3.1. *Let u be a smooth*[1] *solution of* (1.1)*, with $u(0) = u_0$. Then $\forall t \in \mathbb{R}$,*

$$\|u_0\|^2_{BMO} - \|u_0\|^2_{L^2} \leq \|u(t)\|^2_{BMO} \leq \|u_0\|^2_{BMO} + \|u_0\|^2_{L^2}.$$

Proof. From the Lax pair and the proof of Corollary 2.4, we know that along the evolution, K_u remains unitary equivalent to K_{u_0}. In particular, their operator norms are equal. Since $K_u = H_{S^*u}$, we then know that $\|S^*u\|_{BMO}$ is a conserved quantity. Now, for $h \in L^2_+$,

$$\|H_{S^*u}(h)\|^2_{L^2} = \|S^*H_u(h)\|^2_{L^2} = \|H_u(h)\|^2_{L^2} - |(u|h)|^2,$$

and taking the supremum over h with $\|h\| = 1$, we get

$$\|S^*u\|^2_{BMO} \leq \|u\|^2_{BMO} \leq \|S^*u\|^2_{BMO} + \|u\|^2_{L^2}.$$

As $\|u\|^2_{L^2}$ is also conserved, this gives the result. □

Now we can state the main lemma, where the quadratic nature of (1.1) is the most striking :

Lemma 3.2 (Lipschitz estimate in L^2)**.** *Let u and v be two smooth solutions of* (1.1)*, with $u(0) = u_0$ and $v(0) = v_0$. Then there exists B depending only on $\|u_0\|_{BMO}$ and $\|v_0\|_{BMO}$, such that $\forall t \in \mathbb{R}$,*

$$\|u(t) - v(t)\|_{L^2} \leq e^{B|t|}\|u_0 - v_0\|_{L^2}. \tag{3.1}$$

Proof. Write (1.1) as $i\dot{u} = F(u)$, and compute

$$\frac{d}{dt}\|u - v\|^2_{L^2} = 2\,\mathrm{Im}(F(u) - F(v)|u - v) = 2\int_0^1 \mathrm{Im}\,(dF(w_\theta)\cdot(u - v)|u - v)\,d\theta,$$

1. in the sense of Lemma 2.1, as always.

where $w_\theta := \theta u + (1-\theta)v$, for $\theta \in [0,1]$. We have

$$dF(w)\cdot h = \big(4(h||w|^2) + 2(w^2|h)\big)\,\Pi(|w|^2) + \big(2(|w|^2|h) + (h|w^2)\big)\,w^2 + 2T_{\overline{J_w}w + J_w\overline{w}}(h) + 2J_w H_w(h).$$

Here, $J_w := (w^2|w)$. Note that the operator $T_{\overline{J_w}w+J_w\overline{w}}$ is self-adjoint. Hence its contribution is cancelled by the imaginary part, and

$$\frac{d}{dt}\|u-v\|_{L^2}^2 = \int_0^1 8\,\mathrm{Im}\,\big((w_\theta^2|u-v)(|w_\theta|^2|u-v)\big) + 4\,\mathrm{Im}\,(J_{w_\theta}H_{w_\theta}(u-v)|u-v)\,d\theta.$$

Straightforward Cauchy-Schwarz inequalities then yield

$$\left|\frac{d}{dt}\|u-v\|_{L^2}^2\right| \le \int_0^1 \big(8\|w_\theta\|_{L^4}^4\|u-v\|_{L^2}^2 + 4\|w_\theta\|_{L^3}^3\|w_\theta\|_{BMO}\|u-v\|_{L^2}^2\big)\,d\theta.$$

Bounding the L^p norms of w_θ with the BMO norm of u and v, and using the preceding lemma, we find a constant B depending on $\|u_0\|_{BMO}$ and $\|v_0\|_{BMO}$ only, and such that

$$\left|\frac{d}{dt}f(t)\right| \le Bf(t), \quad f(t) := \|u(t)-v(t)\|_{L^2}^2.$$

Solving the usual Gronwall inequality leads to the statement of the lemma. □

Constructing the flow on BMO_+ simply consists in adjusting the strategy of [22].

Proposition 3.3. *For every $u_0 \in BMO_+$, there exists a unique $u \in C(\mathbb{R}, L^2_+)\cap C_{w*}(\mathbb{R}, BMO_+)$ solution to*

$$i\partial_t u = 2J\Pi(|u|^2) + \overline{J}u^2, \quad u(0) = u_0.$$

This solution stays bounded in the BMO norm : for all $t\in\mathbb{R}$,

$$\|u_0\|_{BMO}^2 - \|u_0\|_{L^2}^2 \le \|u(t)\|_{BMO}^2 \le \|u_0\|_{BMO}^2 + \|u_0\|_{L^2}^2.$$

In addition, for each $t\in\mathbb{R}$ and $R>0$, the mapping $u_0 \mapsto u(t)$ is Lipschitz on the ball $\mathcal{B}_{BMO}(R) := \{v \in BMO_+(\mathbb{T}) \mid \|v\|_{BMO} \le R\}$ endowed with the L^2 topology.

Proof. Let $u_0 \in BMO_+$. We first construct a sequence of smooth functions u_0^n such that

$$\|u_0^n - u_0\|_{L^2} \longrightarrow 0,$$
$$\|u_0^n\|_{BMO} \le \|u_0\|_{BMO}, \quad \forall n \in \mathbb{N}.$$

To do so, let $r<1$, and denote by $P_r(x) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{inx}$ the Poisson kernel. For any $v \in BMO_+$, we have $\|P_r * v\|_{L^2} \le \|v\|_{L^2}$. In addition, $P_r * v \in C^\infty_+$, and satisfies $\|P_r*v - v\|_{L^2} \to 0$ as $r \to 1^-$. Finally, for $h \in L^2_+$, we have

$$H_{P_r*v}(h) = P_r * (H_v(P_r * h)),$$

which proves that $\|P_r * v\|_{BMO} \le \|v\|_{BMO}$. Thus we can choose $u_0^n = P_{r_n} * u_0$, where $\{r_n\}$ is any sequence of elements of $(0,1)$ converging to 1.

Now, we denote by u^n the smooth solution of (1.1) such that $u^n(0) = u_0^n$. Fix $T>0$. By Lemma 3.2, the sequence $\{u^n\}$ is a Cauchy sequence in $C([-T,T], L^2_+)$, so $\{u^n\}$ converges locally uniformly to a function $u \in C(\mathbb{R}, L^2_+)$. In fact, $\{u^n\}$ also converges locally uniformly in any L^p_+ (for $2<p<\infty$), since

$$\sup_{t\in[-T,T]}\|u^n(t)-u^m(t)\|_{L^p} \le \sup_{t\in[-T,T]}\|u^n(t)-u^m(t)\|_{L^2}^{\frac{1}{p}}\|u^n(t)-u^m(t)\|_{L^{2p-2}}^{1-\frac{1}{p}}$$
$$\le C(\|u_0\|_{BMO})e^{CT/p}\|u_0^n - u_0^m\|_{L^2}^{\frac{1}{p}}.$$

This allows to pass to the limit in the following expression :

$$u^n(T) = u_0^n - i\int_0^T \left(2J_{u^n(s)}\Pi(|u^n(s)|^2) + \overline{J_{u^n(s)}}(u^n(s))^2\right) ds,$$

so that u is a solution of (1.1). Furthermore, if $t \in \mathbb{R}$ is fixed, then $\forall n \in \mathbb{N}$, $\|u^n(t)\|_{BMO}^2 \leq \|u_0^n\|_{BMO}^2 + \|u_0^n\|_{L^2}^2 \leq \|u_0\|_{BMO}^2 + \|u_0\|_{L^2}^2$. Hence, $\{u^n(t)\}$ is a bounded sequence in BMO_+, and any of its weak-$*$ cluster point must be $u(t)$, so $u(t) \in BMO_+$ and $u^n(t) \overset{*}{\rightharpoonup} u(t)$ as $n \to \infty$. By the principle of uniform boundedness applied to the functionals $\{\Lambda_{u^n(t)} \mid n \in \mathbb{N}\}$, where $\Lambda_{u^n(t)} : h \in L^1_+ \mapsto (u^n(t)|h)$, we have $\|u(t)\|_{BMO}^2 \leq \sup_n \|u^n(t)\|_{BMO}^2 \leq \|u_0\|_{BMO}^2 + \|u_0\|_{L^2}^2$. Finally, the weak-$*$ continuity of u is easy to check, now that we have a uniform bound for the BMO norm of u.

The next step consists in proving uniqueness of solutions in $X := C(\mathbb{R}, L^2_+) \cap C_{w*}(\mathbb{R}, BMO_+)$ via an extension of Lemma 3.2 to non-smooth solutions. This will also imply that the mapping $u_0 \mapsto u(t)$ is Lipschitz on balls of BMO_+. So let $u_0 \in BMO_+$, and let u be the solution of (1.1) constructed above. Suppose that v is another solution starting from u_0 in X, in the sense that it satisfies $v(0) = u_0$ and $\forall t \in \mathbb{R}$,

$$v(t) = u_0 - i\int_0^t \left(2J_{v(s)}\Pi(|v(s)|^2) + \overline{J_{v(s)}}(v(s))^2\right) ds. \tag{3.2}$$

Fix $T > 0$, and let us show that $u = v$ on $[-T, T]$. Observe that, by the principle of uniform boundedness again applied to $\{\Lambda_{v(t)} \mid t \in [-T, T]\}$, v is locally strongly bounded in BMO, so the mapping $[-T, T] \to L^2_+$, $t \mapsto 2J_{v(t)}\Pi(|v(t)|^2) + \overline{J_{v(t)}}(v(t))^2$ is (strongly) continuous. So (3.2) can be differentiated, and the same holds for u. Thus, we can differentiate $f(t) := \|u(t) - v(t)\|_{L^2}^2$. The rest of the argument of the proof of Lemma 3.2 remains valid (in particular, the formula for $f'(t)$ is still true), so $f(t) \leq f(0)e^{B|t|}$ on $[-T, T]$, which proves that $u = v$ on $[-T, T]$, therefore on $\mathbb{R}$.

As a consequence, we know that any solution of (1.1) in X is globally bounded in BMO_+, so if u and v are two such solutions, we do have a constant $B > 0$ only depending on $\|u(0)\|_{BMO}$ and $\|v(0)\|_{BMO}$ such that (3.1) holds for all time.

The last consequence of uniqueness of solutions is the generalization of Lemma 3.1 to solutions of (1.1) in X. Indeed, from the above construction, we know that for such solutions the L^2 norm is conserved. Besides, for any $t \in \mathbb{R}$, we have shown that $\|u(t)\|_{BMO}^2 \leq \|u_0\|_{BMO}^2 + \|u_0\|_{L^2}^2$. Now restart a new solution in X whose initial value is $u(t)$, and call it v. Uniqueness implies that for all $s \in \mathbb{R}$, $v(s) = u(t+s)$. Hence $\|v(-t)\|_{BMO}^2 = \|u_0\|_{BMO}^2 \leq \|v(0)\|_{BMO}^2 + \|v(0)\|_{L^2}^2 = \|u(t)\|_{BMO}^2 + \|u_0\|_{L^2}^2$. This finishes to prove that

$$\forall t \in \mathbb{R}, \quad \left|\|u(t)\|_{BMO}^2 - \|u_0\|_{BMO}^2\right| \leq \|u_0\|_{L^2}^2.$$

□

The stability properties of the flow of (1.1) on BMO_+ can be deduced from the extended version of Lemma 3.2.

Proposition 3.4 (Propagation of low regularity)**.** *Let $0 < s < \frac{1}{2}$. Let $u_0 \in BMO_+ \cap H^s$ and $s' \in (0, s]$. Then the solution u of (1.1) such that $u(0) = u_0$ satisfies*

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^{s'}} \leq \|u_0\|_{H^{s'}} e^{B|t|},$$

where B only depends on $\|u_0\|_{BMO}$ and on s. In particular, u stays in H^s for all time.

Remark 7. As recalled in the introduction, such a result is not known in the case of the cubic Szegő equation. From [22, Theorem 3], it is not clear whether additional regularity propagates or not : at least, it cannot shrink more than exponentially fast.

Proof of Proposition 3.4. For $u \in BMO_+$ and $y \in \mathbb{R}$, we denote by $\tau_y u$ the function given by $\tau_y u(e^{ix}) = u(e^{i(x-y)})$. From the invariances of the equation, if $t \mapsto u(t)$ is a BMO solution to (1.1) starting from u_0, then $t \mapsto \tau_y u(t)$ is the solution starting from $\tau_y u_0$. Thus we can apply inequality (3.1) to u and $\tau_y u$: for all $t \in \mathbb{R}$,

$$\|(u - \tau_y u)(t)\|_{L^2} \leq e^{B|t|}\|u_0 - \tau_y u_0\|_{L^2}.$$

As for any $v \in H^s$,

$$\|v\|_{H^s}^2 \simeq \|v\|_{L^2}^2 + \int_0^1 \frac{\|v - \tau_y v\|_{L^2}^2}{|y|^{1+2s}} dy, \tag{3.3}$$

it suffices to integrate the preceding inequality in y, and we get the result. □

Remark 8. Proposition 3.4 can be extended to other Besov spaces $B^s_{2,q}$, for $q \in [1, +\infty)$, since we have

$$\|v\|_{B^s_{2,q}}^q \simeq \|v\|_{L^2}^q + \int_0^1 \frac{\|v - \tau_y v\|_{L^2}^q}{|y|^{1+qs}} dy.$$

A last consequence of (3.1) is that it enables to get the full picture of the maximal growth rate of Sobolev norms of smooth solutions of (1.1).

Proposition 3.5. *Let $\frac{1}{2} < s < 1$, $u_0 \in H^s_+(\mathbb{T})$, and u the solution of (1.1) such that $u(0) = u_0$. Then for any $s' \in (\frac{1}{2}, s]$, the $H^{s'}$ norm of u grows at most exponentially in time.*

Proof. The proof is the same as for Proposition 3.4, since (3.3) holds for any $s \in (0, 1)$. □

4 Turbulent trajectories : an explicit computation in $\mathcal{L}(1)$

In this section, we follow the intuition of [67] : we study the flow of (1.1) on the manifold $\mathcal{L}(1)$, and we find the necessary and sufficient condition for weak turbulence to occur as stated in Proposition 1.1.

In view of Proposition 2.5, any element $u \in \mathcal{L}(1)$ can be written as

$$u(z) = b + \frac{cz}{1 - pz}, \quad z \in \mathbb{D}, \tag{4.1}$$

for some b, c, $p \in \mathbb{C}$ satisfying also $|p| < 1$ and $c \neq 0$. Let us rephrase the evolution of (1.1) in these coordinates (b, c, p).

If u is of the form (4.1),

$$J = \int_{\mathbb{T}} |u|^2 u = \frac{1}{2\pi}\int_0^{2\pi} \left(b + \frac{ce^{ix}}{1 - pe^{ix}}\right)^2 \left(\bar{b} + \frac{\bar{c}}{e^{ix} - \bar{p}}\right) dx,$$

and by the residue theorem,

$$J = |b|^2 b + \frac{2b|c|^2}{1 - |p|^2} + \frac{|c|^2 c\bar{p}}{(1 - |p|^2)^2}. \tag{4.2}$$

Similarly, we compute the three main conservation laws :

$$Q = |b|^2 + \frac{|c|^2}{1-|p|^2},$$
$$M = \frac{|c|^2}{(1-|p|^2)^2},$$
$$E = \frac{1}{2}|b|^6 + 2|b|^4\frac{|c|^2}{1-|p|^2} + \frac{|b|^2|c|^2}{(1-|p|^2)^2}\left(\mathrm{Re}(b\bar{c}p) + 2|c|^2\right) + 2\frac{|c|^4}{(1-|p|^2)^3}\,\mathrm{Re}(b\bar{c}p) + \frac{1}{2}\frac{|p|^2|c|^6}{(1-|p|^2)^4}.$$

By a partial fraction decomposition, we also know that

$$\Pi(|u|^2) = \Pi\left(\left(b + \frac{cz}{1-pz}\right)\left(\bar{b} + \frac{\bar{c}}{z-\bar{p}}\right)\right) = |b|^2 + \frac{|c|^2}{1-|p|^2} + \left(\frac{p|c|^2}{1-|p|^2} + \bar{b}c\right)\frac{z}{1-pz},$$

so that equation (1.1) on $\mathcal{L}(1)$ finally reads

$$\begin{cases} i\dot{p} = c\bar{J}, \\ i\dot{c} = 2bc\bar{J} + 2\bar{b}cJ + \dfrac{2Jp|c|^2}{1-|p|^2}, \\ i\dot{b} = b^2\bar{J} + 2|b|^2J + \dfrac{2J|c|^2}{1-|p|^2}, \end{cases}$$

with J given by (4.2).

Now we can turn to the

Proof of Proposition 1.1. We begin by proving that *(i)* implies *(iii)*. Suppose that there exists $\underline{s} > \frac{1}{2}$, as well as a sequence of times $\{t_n\}_{n\in\mathbb{N}}$, such that $\|u(t_n)\|_{H^{\underline{s}}} \to \infty$ as $n \to \infty$. In terms of coordinates (b, c, p), since $|b|$ is bounded (by Q), this means that

$$\sum_{k=1}^{\infty} |c(t_n)|^2|p(t_n)|^{2k}(1+k^2)^{\underline{s}} \underset{n\to+\infty}{\longrightarrow} \infty.$$

Since $|c|$ is a bounded function (by $\sqrt{M}$, for instance), we see that necessarily, $|p(t_n)| \to 1$ when $n \to \infty$. Now, in view of the expression of the momentum, $|c(t_n)| = \sqrt{M}\cdot(1-|p(t_n)|^2)$ goes to 0. As a consequence, writing

$$Q = |b(t_n)|^2 + \sqrt{M}|c(t_n)|,$$

we see that $|b(t_n)|^2 \to Q$ as $n \to \infty$. If we move on to the energy, we get

$$E = \frac{1}{2}|J(t_n)|^2 = \frac{1}{2}\left||b(t_n)|^2b(t_n) + 2\sqrt{M}b(t_n)|c(t_n)| + Mc(t_n)\overline{p(t_n)}\right|^2,$$

so that $\frac{1}{2}|b(t_n)|^6 \to E$ as $n \to \infty$. Hence $E = \frac{1}{2}Q^3$.

To conclude the proof, it suffices to show that *(iii)* implies *(ii)*, so we assume (1.2). Developing the expression of Q, we get in (b, c, p) coordinates :

$$|b|^4 + \frac{|b|^2|c|^2}{1-|p|^2} + 2\,\mathrm{Re}\left(\frac{b\bar{c}p}{1-|p|^2}\right)\left(|b|^2 + \frac{2|c|^2}{1-|p|^2}\right) = \frac{|c|^4(1-2|p|^2)}{(1-|p|^2)^3},$$

or equivalently, using the conservation laws :

$$Q(Q - |c|\sqrt{M}) + 2(Q + |c|\sqrt{M})\,\mathrm{Re}\left(\frac{b\bar{c}p}{1-|p|^2}\right) = 2|c|^2M - |c|M\sqrt{M}. \tag{4.3}$$

We would like to show that $|p(t)| \to 1$, or that $|c(t)| \to 0$ as $t \to +\infty$ (the negative times are treated in the same way). First of all, notice that c is never zero, because u must stay in $\mathcal{L}(1)$. Consequently, $t \mapsto |c(t)|$ is a smooth function of time.

Let us compute, from the equation on c,

$$\frac{d|c|}{dt} = \frac{1}{|c|}\operatorname{Re}(\bar{c}\dot{c}) = \frac{2|c|}{1-|p|^2}\left[|b|^2 + \frac{2|c|^2}{1-|p|^2}\right]\operatorname{Im}(b\bar{c}p) = 2|c|(Q+|c|\sqrt{M})\operatorname{Im}\left(\frac{b\bar{c}p}{1-|p|^2}\right),$$

so that, by (4.3),

$$\begin{aligned}\left(\frac{1}{|c|}\frac{d|c|}{dt}\right)^2 &= 4(Q+|c|\sqrt{M})^2\left[\operatorname{Im}\left(\frac{b\bar{c}p}{1-|p|^2}\right)\right]^2\\ &= 4(Q+|c|\sqrt{M})^2\frac{|b|^2|c|^2|p|^2}{(1-|p|^2)^2} - \left(2|c|^2M - |c|M\sqrt{M} - Q(Q-|c|\sqrt{M})\right)^2\\ &= 4(Q+|c|\sqrt{M})^2(Q-|c|\sqrt{M})(M-|c|\sqrt{M}) - (2M|c|^2+\sqrt{M}(Q-M)|c| - Q^2)^2\end{aligned}$$

Expanding this polynomial in $|c|$, one realizes that the terms in $|c|^4$ and $|c|^3$ cancel out. In the end,

$$\left(\frac{1}{|c|}\frac{d|c|}{dt}\right)^2 = -M(M+Q)^2|c|^2 + 2Q^2\sqrt{M}(M-Q)|c| + Q^3(4M-Q) = \mathcal{P}(|c|\sqrt{M}), \quad (4.4)$$

where $\mathcal{P}(X) := -(M+Q)^2X^2 + 2Q^2(M-Q)X + Q^3(4M-Q)$. The discriminant of $\mathcal{P}$ is

$$\Delta = Q^4(M-Q)^2 + (M+Q)^2Q^3(4M-Q) = 8M^2Q^4 + 4M^3Q^3 > 0,$$

so $\mathcal{P}$ has two distinct roots :

$$r_\pm := \frac{Q^2(M-Q) \pm 2MQ\sqrt{2Q^2+MQ}}{(M+Q)^2}.$$

Observe that $\mathcal{P}$ takes nonnegative values between r_- and r_+ only. In view of the differential equation (4.4), and since $|c| > 0$, we know that $\mathcal{P}(x)$ must be nonnegative at least for one $x > 0$. Therefore we must have $r_+ > 0$, and thus

$$\begin{aligned}&2M\sqrt{2Q+M} > \sqrt{Q}(Q-M)\\ &\Leftrightarrow (Q \le M) \text{ or } (8M^2Q + 4M^3 > Q^3 - 2Q^2M + QM^2)\\ &\Leftrightarrow (Q \le M) \text{ or } ((4M-Q)(Q+M)^2 > 0)\end{aligned}$$

So this proves that as a consequence of (1.2), we must have $Q < 4M$.

It follows that $r_- < 0$. Indeed, the product r_+r_- is given by to coefficients of $\mathcal{P}$, namely

$$r_+r_- = \frac{Q^3(4M-Q)}{-(M+Q)^2} < 0.$$

Therefore we can factorize $\mathcal{P}(X) = (M+Q)^2(X-r_-)(r_+-X)$, and we find

$$\left(\frac{d|c|}{dt}\right)^2 = (M+Q)^2|c|^2(|c|\sqrt{M}-r_-)(r_+-|c|\sqrt{M}).$$

Suppose that for some time $t_0 \in \mathbb{R}$,

$$\frac{d|c|}{dt}(t_0) = +(M+Q)|c(t_0)|\sqrt{(|c(t_0)|\sqrt{M}+|r_-|)(r_+-|c(t_0)|\sqrt{M})},$$

then $|c|$ increases and the equality holds true until $|c|$ reaches the value $\frac{r_+}{\sqrt{M}}$. This must happen in finite time, since for $t \geq t_0$,

$$\frac{d|c|}{dt}(t) \geq (M+Q)|c(t_0)|\sqrt{|r_-|} \cdot \sqrt{r_+ - |c(t)|\sqrt{M}},$$

so that there exists $K > 0$ such that $\frac{d}{dt}\sqrt{r_+ - |c(t)|\sqrt{M}} \leq -K$. Without loss of generality, we assume that $|c(t)| = \frac{r_+}{\sqrt{M}}$ at $t = 0$. Furthermore,

$$\frac{d^2|c|}{dt^2}(0) = -\frac{1}{2}\sqrt{M}(M+Q)\frac{r_+^2}{M}(r_+ + |r_-|) < 0,$$

thus $|c|$ must decrease immediately after time 0. This proves that for all $t \geq 0$,

$$\frac{d|c|}{dt} = -(M+Q)|c|\sqrt{(|c|\sqrt{M} + |r_-|)(r_+ - |c|\sqrt{M})}.$$

It appears that this equation can be integrated. To simplify notations, set $f := |c|\sqrt{M}$. We then have

$$\int_{f(0)}^{f(t)} \frac{df}{f\sqrt{f+|r_-|}\sqrt{r_+ - f}} = -(M+Q)t.$$

Making the change of variables $y = \sqrt{\frac{r_+ - f}{f + |r_-|}}$ in the integral yields, for some constant $C \in \mathbb{R}$,

$$\frac{1}{\sqrt{|r_-|r_+}} \log\left(\frac{2}{1 + \sqrt{\frac{|r_-|}{r_+}}\sqrt{\frac{r_+ - f(t)}{f(t)+|r_-|}}} - 1\right) = C - (M+Q)t.$$

In view of the coefficients of $\mathcal{P}$, we know that $|r_-|r_+ = Q^3(4M-Q)(Q+M)^{-2}$, so we set $\kappa := Q^{3/2}\sqrt{4M-Q}$, and we finally have

$$|c(t)| = \frac{f(t)}{\sqrt{M}} = \frac{C'}{|r_-|(1+Ce^{-\kappa t})^2 + r_+(1-Ce^{-\kappa t})^2} e^{-\kappa t}.$$

In the end, this proves that $|c(t)| \sim_{t\to+\infty} Ce^{-\kappa t}$. In view of our preliminary remark, we have in fact $|c(t)| \sim_{t\to\pm\infty} Ce^{-\kappa|t|}$. As a consequence, $|p(t)|$ goes to 1 in both time directions.

It remains to compute the H^s norm of u for $s > \frac{1}{2}$. Expanding u in Fourier series, and noticing again that b is bounded, we only need to estimate the sum

$$|c(t)|^2 \sum_{k=1}^{\infty} |p(t)|^{2k} \cdot k^{2s}$$

as $t \to \pm\infty$. It is a classical result that the power series $\sum_{k=1}^{\infty} x^k k^\alpha$ is equivalent to $C(1-x)^{-(1+\alpha)}$ when $x \to 1^-$. So

$$\|u(t)\|_{H^s}^2 \sim C\frac{|c|^2}{(1-|p|^2)^{1+2s}} = CM^{\frac{1}{2}+s}|c|^{1-2s},$$

i.e. $\|u(t)\|_{H^s}^2 \sim C_s e^{(2s-1)\kappa|t|}$, which was the claim. □

Remark 9. The coefficient κ we find in the proof of Theorem 1.1 is very similar to the one of the a priori estimate of Lemma 2.8. Indeed, because of assumption (1.2),

$$\kappa = Q^{3/2}\sqrt{4M-Q} = \sqrt{2E} \cdot \sqrt{4M-Q},$$

and the factor $\sqrt{4M-Q}$ is “not far” from $\|u_0\|_{H^{1/2}}$.

Chapitre 4

Classification of traveling waves for a quadratic Szegő equation

Abstract

We give a complete classification of the traveling waves of the following quadratic Szegő equation :

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2, \quad u(0,\cdot) = u_0,$$

and we show that they are given by two families of rational functions, one of which is generated by a stable ground state. We prove that the other branch is orbitally unstable.

1 Introduction

This paper is devoted to the continuation of the study of the following Hamiltonian PDE :

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2, \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{T}_x \to \mathbb{C}$, the factor $J = J(u) := \int_{\mathbb{T}} |u|^2 u \in \mathbb{C}$, and Π is the Szegő projector onto nonnegative frequencies :

$$\Pi\left(\sum_{k=-\infty}^{+\infty} u_k e^{ikx}\right) = \sum_{k\geq 0} u_k e^{ikx}.$$

The framework of such an evolution equation was first explored by Gérard and Grellier in their work on the so-called *cubic Szegő equation* :

$$i\partial_t u = \Pi(|u|^2 u). \tag{1.2}$$

In a vast series of papers [16, 18, 20, 21], these authors have shown that equation (1.2) enjoys very rich dynamical properties, which can be summarized in two words : "integrability" and "turbulence". Such a discovery paves the way for the study of a various range of other non-dispersive equations, among which the conformal flow on $\mathbb{S}^3$ [4, 5] or the cubic lowest Landau level (LLL) equation [15]. In this spirit, our aim would be to understand the behavior of solutions of equation (1.1), which shares some features with (1.2) but is far less understood by now.

Let us underline some basic facts about (1.1). To stick to the notations of [21], we denote by $L^2_+(\mathbb{T}) = \Pi(L^2(\mathbb{T}))$ the Hardy space on the disc (or simply L^2_+), and if G is a subspace of $L^2(\mathbb{T})$, G_+ will designate the intersection $G \cap L^2_+$. It is well-known that elements of L^2_+ can either be considered as Fourier series with only nonnegative modes, or as holomorphic functions $u(z)$ on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ satisfying

$$\sup_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^2 d\theta < +\infty.$$

The usual scalar product on L^2 can be restricted to L^2_+, with the convention that for $u, v \in L^2_+$, we have $(u|v) = \int_{\mathbb{T}} u\bar{v}$ (with respect to the normalized Lebesgue measure $d\theta/2\pi$). Through the standard symplectic form on L^2_+ given by $\omega(u, v) := \operatorname{Im}(u|v)$, we can see (1.1) as the Hamiltonian flow associated to the functional

$$E(u) := \frac{1}{2}|J|^2 = \frac{1}{2}\left|\int_{\mathbb{T}} |u|^2 u\right|^2.$$

Since the energy E is conserved along flow lines, the modulus of J remains constant, so (1.1) can be regarded as a quadratic system, and we call it therefore the *quadratic Szegő equation*. The invariance of E under phase rotation ($u \to e^{-i\theta}u$) and phase translation ($u \to u(ze^{-i\alpha})$) yields two other (formal) conservation laws : the mass $Q(u) := \int_{\mathbb{T}} |u|^2$ and the momentum $M(u) := (-i\partial_x u|u)$.

In [62], several properties of (1.1) were enlightened, relying on the crucial fact that (1.1) admits a *Lax pair*. Firstly, as in the case of the cubic Szegő equation [22], equation (1.1) admits a well-defined flow on $BMO_+(\mathbb{T})$, where $BMO(\mathbb{T})$ stands for the space of functions with *bounded mean oscillation* introduced by John and Nirenberg. This flow propagates every additional regularity, in the sense that if the initial datum also belongs to $H^s_+(\mathbb{T})$ for some $s > 0$, then the corresponding solution will stay in H^s_+ for all time. We thus recover a flow on the natural energy space $H^{1/2}_+$ where E, Q and M are all well-defined. Secondly, we showed the existence of finite-dimensional submanifolds of L^2_+ consisting of rational functions of z which are stable by the flow of (1.1), and where weak turbulence phenomena can occur, in contrast with what happens for the cubic Szegő equation (1.2). More precisely, we have proved that there is an invariant manifold of homographies starting of which the solutions of (1.1) in $C^\infty_+(\mathbb{T})$ are bounded in the $H^{1/2}$ norm, but grow exponentially fast in the H^s topology for any $s > 1/2$.

The purpose of the present work is to investigate another typical feature of integrable systems : we intend to give a complete classification of all traveling waves that are solutions of equation (1.1) in $H^{1/2}_+$, and to discuss their stability. Recall that a traveling wave is a solution $v(t,z)$ of the Cauchy problem associated to (1.1) for which there exists $\omega, c \in \mathbb{R}$ such that

$$v(t,z) = e^{-i\omega t} v_0(ze^{-ict}), \quad \forall t \in \mathbb{R},\ \forall z \in \mathbb{D}, \tag{1.3}$$

where $v_0 = v(0,\cdot)$ is the initial state. We refer to ω as the *pulsation*, and to c as the *velocity*.

Hereafter, we only look for such solutions with $c \neq 0$ or $\omega \neq 0$, for the set of steady solutions ($\omega = c = 0$) coincides with the subset $\{J(v) = 0\}$ of $H^{1/2}_+$ which seems hard to characterize (see Appendix A for a discussion about that point).

Theorem 6 (Classification of traveling waves)**.** *The initial state $v_0(z) \in H^{1/2}_+$ gives rise to a traveling wave solution of (1.1) with $(\omega, c) \neq (0,0)$ if and only if it is a constant, or if there exists $\lambda, p \in \mathbb{C}$ with $0 < |p| < 1$ and an integer $N \geq 1$ such that one of the following holds :*

(i)

$$v_0(z) = \frac{\lambda}{1 - pz^N},$$

in which case

$$\omega = |\lambda|^4 \frac{3 - |p|^2}{(1-|p|^2)^3} \quad \textit{and} \quad c = \frac{|\lambda|^4}{N} \frac{1}{(1-|p|^2)^2};$$

(ii)

$$v_0(z) = -\lambda \frac{1+|p|^2}{1-|p|^2} + \frac{\lambda}{1-pz^N},$$

in which case

$$\omega = |\lambda|^4 |p|^4 \frac{(1+5|p|^2)(3+5|p|^2)}{(1-|p|^2)^4} \quad \textit{and} \quad c = -\frac{|\lambda|^4}{N}|p|^4 \frac{3+5|p|^2}{(1-|p|^2)^3}.$$

Remark 10. Traveling waves with $c = 0$ and $\omega \neq 0$ are called *standing waves*. In particular, Theorem 6 asserts that there are no standing waves for (1.1) in $H^{1/2}_+$ apart from constants. However, if we remove the condition on the regularity of v_0 and only ask that $v_0 \in BMO_+$, then we will prove that v_0 is a standing wave if and only if

$$v_0 = \lambda \cdot \Pi(\mathbb{1}_{\mathcal{B}}),$$

where $\lambda \in \mathbb{C}$ and $\mathcal{B}$ is any Borel subset of $\mathbb{T}$.

Here again, the situation appears to be very different from the one described in [16, Theorem 1.4] for the cubic Szegő equation. Indeed equation (1.2) admits many stationary waves in $H^{1/2}_+$ that are given by Blaschke products (see the end of Section 3 for a precise definition). Moreover, it also admits a wider variety of traveling waves, whereas those of Theorem 6 all deduce from one another by an invariance argument (see Proposition 3.5).

The proof of Theorem 6 mainly consists in translating (1.3) into an algebraic relation between operators thanks to the Lax pair. The key of the proof consists in describing the spectrum of a perturbation of the operator $D = \frac{1}{i}\partial_x$ by a bounded Toeplitz operator over L^2_+. It is worth noticing that this kind of operators precisely arises in the study of the Lax pair associated to the Benjamin-Ono equation (see *e.g.* [66]). See also the pioneering work of Amick and Toland on the solitary waves for the Benjamin-Ono equation [1, 2], also dealing with a non-local operator such as Π.

In the present paper, we would also like to address the question of the nonlinear stability of the traveling wave solutions. Relatively to some norm $\|\cdot\|_X$, we will say that v_0 is X*-orbitally stable* if for any $\varepsilon > 0$, there exists $\eta > 0$ such that if $\|u_0 - v_0\|_X \leq \eta$, then, for all $t \in \mathbb{R}$,

$$\inf_{(\theta,\alpha)\in\mathbb{T}^2} \|u(t) - e^{-i\theta} v_0(ze^{-i\alpha})\|_X \leq \varepsilon,$$

where $t \mapsto u(t)$ refers to the solution of (1.1) starting from u_0 at time $t = 0$.

Classically, we have a global Gagliardo-Nirenberg inequality on $H^{1/2}_+$:

Proposition 1.1. *For any* $u \in H^{1/2}_+$,

$$E(u) \leq \frac{1}{2} Q(u)^2 (Q(u) + M(u)). \tag{1.4}$$

Equality in (1.4) *holds if and only if* $u(z) = \frac{\lambda}{1-pz}$ *for some* $\lambda, p \in \mathbb{C}$ *with* $|p| < 1$.

This enables us to prove the following stability result :

Corollary 1.2 (Stability of the ground states)**.** *For* $\lambda, p \in \mathbb{C}$ *with* $|p| < 1$*, the traveling wave* $z \mapsto \frac{\lambda}{1-pz}$ *is* $H^{1/2}_+$*-orbitally stable.*

Apart from this family of ground states, we are also going to study the stability of the second branch of the traveling waves of Theorem 6. Using the Lax pair, we know (see Corollary 2.2) that the set of functions of the form

$$z \mapsto b + \frac{cz}{1 - pz}, \quad b, c, p \in \mathbb{C},\ c \neq 0,\ c - bp \neq 0,\ |p| < 1,$$

is left invariant by the flow of (1.1), and on this set (which is called $\mathcal{V}(3)$), (1.1) reduces to a system of coupled ODEs on b, c and p, which have been made explicit in [62] and is recalled in Appendix B. Designing an appropriate perturbation of the "translated ground states" inside $\mathcal{V}(3)$, and studying the leading order of the new equations of motion, will yield the following result, which also makes use of the invariance argument mentioned above :

Proposition 1.3. *For* $\lambda, p \in \mathbb{C}$ *with* $|p| < 1$*, for* $N \in \mathbb{N}$*, the traveling waves* $z \mapsto -\lambda\frac{1+|p|^2}{1-|p|^2} + \frac{\lambda}{1-pz^N}$ *are* not $H^{1/2}_+$*-orbitally stable.*

The question of the stability of traveling waves of the form $\frac{\lambda}{1-pz^N}$, with $N \geq 2$, is left open. In the case of the cubic Szegő equation, a negative answer is given in [18] for all traveling waves of degree greater than 2, but the proof heavily relies on the use of action-angle coordinates which are still missing in our case.

Remark 11 (A remark on the traveling waves on $\mathbb{R}$). As equation (1.1) can be posed on $\mathbb{R}$ as well, the same question of finding traveling waves holds. Adapting rigourously the argument of O. Pocovnicu [54] leads to the same result as for the cubic Szegő equation on the line : the only solitons are given by the profiles

$$v_0(z) = \frac{\alpha}{z-p}, \quad \forall z \in \{\zeta \in \mathbb{C} \mid \operatorname{Im} \zeta > 0\},$$

where $\alpha, p \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} p < 0$. Moreover, these solitons are orbitally stable. This remark strengthens an observation that was already made in [62] : the cubic and quadratic Szegő equation look very similar on the line, unlike what happens on the torus. That is why it would be also very interesting to study the stability of the above solitons under a perturbation of the quadratic Szegő equation itself, as was done in [55] for a perturbation of the cubic Szegő equation by a Toeplitz potential.

This paper is organized as follows. In Section 2, we give the definition of the operators arising in the study of equation (1.1) and recall some results about them. In Section 3, we give the proof of the classification theorem. Section 4 is devoted to the questions about the stability of the traveling waves that are exhibited in Theorem 6.

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2 Notations and preliminaries

2.1 Hankel and Toeplitz operators

In this paragraph, we recall some notations, the Lax pair result from [62], and some of its consequences.

For $u \in BMO_+(\mathbb{T})$, we denote the Hankel operator of symbol u by $H_u : h \mapsto \Pi(u\bar{h})$. This defines a bounded $\mathbb{C}$-antilinear operator on L^2_+. Since $\forall h_1, h_2 \in L^2_+$, we have $(H_u(h_1)|h_2) = (H_u(h_2)|h_1)$, we see that H_u^2 is $\mathbb{C}$-linear, positive and self-adjoint. Similarly, for $b \in L^\infty(\mathbb{T})$, the Toeplitz operator of symbol b is given by $T_b : h \mapsto \Pi(bh)$. We also have $T_b \in \mathcal{L}(L^2_+)$, but T_b is $\mathbb{C}$-linear. The adjoint of T_b is $(T_b)^* = T_{\bar{b}}$. A fundamental example of a Toeplitz operator is the shift on the right, which we call $S := T_{e^{ix}}$. Combining these definitions, we can define the shifted Hankel operator of symbol $u \in BMO_+$: for $h \in L^2_+$, we set

$$K_u(h) := S^* H_u(h) = H_u S(h) = H_{S^*u}(h).$$

Now we can state the algebraic identities associated to the Lax pair for (1.1).

Theorem 7. *Let $u \in H^s_+$ for some $s > 1/2$. Set $X(u) := 2\Pi(|u|^2) + u^2$. Then we have*

$$K_{X(u)} = A_u K_u + K_u A_u$$
$$H_{X(u)} = A_u H_u + H_u A_u - (u|\cdot)u,$$

where $A_u := T_u + T_{\bar{u}}$ is a bounded self-adjoint operator on L^2_+.

Proof. The proof simply follows from the arguments of [62], but we recall it for the convenience of the reader. Taking $h \in L^2_+$, we see that

$$\Pi(\Pi(|u|^2)\bar{h}) = \Pi(|u|^2\bar{h}) = \Pi(u\overline{\Pi(uh)}) = \Pi(\bar{u}\Pi(u\bar{h})),$$

and on the other hand, decomposing $\Pi(u^2\bar{h}) = \Pi(u\Pi(u\bar{h})) + \Pi(u(I-\Pi)(u\bar{h}))$, we observe that

$$(I-\Pi)(u\bar{h}) = \overline{\Pi(\bar{u}h)} - (u|h),$$

so $\Pi(u^2\bar{h}) = T_u H_u(h) + H_u T_{\bar{u}}(h) - (u|h)u$. Now, writing $H_{X(u)}(h) = 2J\Pi(\Pi(|u|^2)\bar{h}) + \bar{J}\Pi(u^2\bar{h})$ leads to the second identity.

Besides, for $h \in L^2_+$,

$$\begin{aligned}(S^*A_u - A_uS^*)h &= \Pi(\bar{z}\Pi((u+\bar{u})h) - (u+\bar{u})\Pi(\bar{z}h)) \\ &= \Pi(\bar{z}uh) - \Pi(u\Pi(\bar{z}h)) - \Pi(\bar{z}(I-\Pi)(\bar{u}h)) \\ &= (h|1)S^*u,\end{aligned}$$

because $\bar{z}h = \Pi(\bar{z}h) + \bar{z}(h|1)$, and $\bar{z}(I-\Pi)(\bar{u}h) \perp L^2_+$. Then

$$\begin{aligned}K_{X(u)}(h) = S^*H_{X(u)}(h) &= A_uK_u(h) + K_uA_u(h) - (u|h)S^*u + [S^*, A_u]H_u(h) \\ &= A_uK_u(h) + K_uA_u(h),\end{aligned}$$

since $(H_u(h)|1) = (H_u(1)|h) = (u|h)$. □

A consequence of this theorem is the existence of finite-dimensional submanifolds of the energy space, of arbitrary complex dimension, which are stable by the flow of (1.1), as it is also true for the cubic Szegő equation (see [18]).

Definition. Let $N \in \mathbb{N}$. We denote by $\mathcal{V}(2N)$ the set of functions $u \in H^{1/2}_+(\mathbb{T})$ such that $\operatorname{rk} H_u = \operatorname{rk} K_u = N$, and by $\mathcal{V}(2N+1)$ the set of those such that $\operatorname{rk} H_u = N+1$ and $\operatorname{rk} K_u = N$.

It is easy to see that $\bigcup_{d\geq 0}\mathcal{V}(d)$ is simply the set of symbols u such that H_u (and K_u) is a finite-rank operator. From a theorem by Kronecker, the $\mathcal{V}(d)$'s can be described explicitely :

Proposition 2.1 ([16, 20]). *Let $d \in \mathbb{N}$. A function $u(z) \in H^{1/2}_+$ belongs to $\mathcal{V}(d)$ if and only if*

$$u(z) = \frac{A(z)}{B(z)}, \quad \forall z \in \mathbb{D},$$

where A and B are two polynomials such that $A \wedge B = 1$, $B(0) = 1$, B has no root in $\overline{\mathbb{D}}$, and in addition

- *$\deg A \leq N-1$ and $\deg B = N$ if $d = 2N$;*
- *$\deg A = N$ and $\deg B \leq N$ if $d = 2N+1$.*

In particular, each $\mathcal{V}(d)$ is composed of rational, hence smooth functions. Now we can state the announced consequence of Theorem 7 :

Corollary 2.2. *For each $d \in \mathbb{N}$, $\mathcal{V}(d)$ is preserved by the flow of* (1.1).

Proof. From [62] we already know that $\mathcal{V}(2N) \cup \mathcal{V}(2N+1)$ is preserved by the flow of (1.1). Thus it suffices to show that $\mathcal{V}(2N)$ is stable. Let $u_0 \in \mathcal{V}(2N)$ and $t \mapsto u(t)$ be the corresponding solution. Since u_0 belongs to $C^\infty_+(\mathbb{T})$, so does $u(t)$ for all time $t \in \mathbb{R}$, so we can apply Theorem 7 and compute, for a given $h \in L^2_+$,

$$i\frac{d}{dt}H_u(h) = H_{2J\Pi(|u|^2)+\bar{J}u^2}(h) = A_{\bar{J}u}H_u(h) + H_uA_{\bar{J}u}(h) - \bar{J}(u|h)u,$$

or in other words, using the $\mathbb{C}$-antilinearity of H_u, and the fact that $(u|h) = (H_u(h)|1)$,

$$\frac{d}{dt}H_u = \left(-iA_{\bar{J}u} + \frac{i}{2}\bar{J}(\cdot|1)u\right)H_u + H_u\left(iA_{\bar{J}u} - \frac{i}{2}J(\cdot|u)1\right).$$

We give a name to the operator on the left, say $Y_u := -iA_{\bar{J}u} + \frac{i}{2}\bar{J}(\cdot|1)u$, and note that since $A_{\bar{J}u}$ is self-adjoint, we have in fact

$$\frac{d}{dt}H_u = Y_uH_u + H_uY_u^*. \tag{2.1}$$

This identity is close to a Lax pair, but $Y_u^* \neq -Y_u$. It is still enough to deduce the stability of $\mathcal{V}(2N)$. Indeed, let $V(t)$ be the (global) solution to the following linear Cauchy problem on $\mathcal{L}(L^2_+)$:

$$\begin{cases} V'(t) = -Y_{u(t)}^*V(t), \\ V(0) = I. \end{cases}$$

Since $V \in GL(L^2_+)$ at time $t = 0$, this remains true for all time. Besides, compute

$$\frac{d}{dt}(V^*H_uV) = (-V^*Y_u)H_uV + V^*\left(\frac{d}{dt}H_u\right)V + V^*H_u(-Y_u^*V) = 0$$

by (2.1), hence $V^*(t)H_{u(t)}V(t) = H_{u_0}$ for all $t \in \mathbb{R}$. In particular, $\operatorname{Ran} H_{u(t)} = (V^*(t))^{-1}\operatorname{Ran} H_{u_0}$, so both spaces have the same dimension. This proves that $\forall t \in \mathbb{R}$, $u(t) \in \mathcal{V}(2N)$. □

Remark 12. As a weaker version of a Lax pair, an identity such as (2.1) also holds for H_u^2. It can be interpreted saying that H_u^2 remains equivalent to $H_{u_0}^2$ as a *quadratic form*, whereas K_u^2 remains equivalent to $K_{u_0}^2$ as an *operator* on L^2_+ (see [62]).

2.2 The spectral theory of K_u

For the sake of completeness, we recall below some properties of K_u which will be useful in the course of the proof of Theorem 6. For a more general picture, we refer to [21, Section 3].

Let $u \in H^{1/2}_+$. Then H_u^2, K_u^2 are positive compact self-adjoint operators [16] which satisfy the relation $H_u^2 = K_u^2 + (\cdot|u)u$. For $\rho, \sigma \geq 0$, we denote by

$$E_u(\rho) := \ker(H_u^2 - \rho^2 I), \quad F_u(\sigma) := \ker(K_u^2 - \sigma^2 I).$$

Proposition 2.3 ([21]). *Let $s > 0$ such that $\{0\} \subsetneq E_u(s) \cup F_u(s)$. Then one of the following holds true :*

(i) $\dim E_u(s) = 1 + \dim F_u(s)$, $u \not\perp E_u(s)$ *and* $F_u(s) = E_u(s) \cap u^\perp$ *(s is H-dominant) ;*

(ii) $\dim F_u(s) = 1 + \dim E_u(s)$, $u \not\perp F_u(s)$ *and* $E_u(s) = F_u(s) \cap u^\perp$ *(s is K-dominant).*

Denote by Σ_u^K the set of K-dominant eigenvalues of K_u^2 (plus 0). For $\sigma \in \Sigma_u^K$, let u_σ be the projection of u onto $F_u(\sigma)$. By the above proposition, we have $u_\sigma \neq 0$ for $\sigma > 0$, and

$$u = \sum_{\sigma \in \Sigma_u^K} u_\sigma. \tag{2.2}$$

We will also need a simple lemma :

Lemma 2.4. *Let $\sigma \in \Sigma_u^K \setminus \{0\}$. Then $E_u(\sigma) \subseteq F_u(\sigma) \cap 1^\perp$.*

Proof. Suppose $h \in E_u(\sigma)$. Then $H_u(h) \in E_u(\sigma)$, so $H_u(h) \perp u$ by the preceding proposition. Thus $0 = (H_u(h)|u) = (1|H_u^2(h)) = \sigma^2(h|1)$. □

3 Proof of the main theorem

As usual, we derive from (1.3) an equation on v_0 : indeed, if v satisfies both (1.1) and (1.3), then

$$i\partial_t v = \omega v + cDv,$$

and $J(v) = e^{-i\omega t}J^0$, where we set $J^0 := J(v_0) = (H^3_{v_0}(1)|1)$. As usual, $D := -i\partial_x = z\partial_z$. Note that $J^0 \neq 0$, since we assume that u is not a steady solution. Thus, we find an equation for the initial data v_0 :

$$\omega v_0 + cDv_0 = 2J^0\Pi(|v_0|^2) + \overline{J^0}v_0^2. \tag{3.1}$$

Our goal is thus to solve this differential equation. We begin with the necessary conditions and assume that there exists $v_0 \in H^{1/2}_+$ such that (3.1) holds.

3.1 The case of standing waves

Let us first examine the case when $c = 0$. To simplify our analysis, we write $v_0 = \frac{\omega J^0}{|J^0|^2}u$ so that equation (3.1) translates into

$$u = 2\Pi(|u|^2) + u^2. \tag{3.2}$$

In order to use Theorem 7, we first need to show that the operator A_u can be properly defined. Observe that since $|u|^2$ takes real values, we have

$$|u|^2 = \Pi(|u|^2) + \overline{\Pi(|u|^2)} - Q = \mathrm{Re}(u(1-u)) - Q.$$

Writing $|u|^2 + \mathrm{Re}(u^2) = |u|^2 + (\mathrm{Re}\, u)^2 - (\mathrm{Im}\, u)^2 = 2(\mathrm{Re}\, u)^2$, we infer that $\mathrm{Re}\, u = r_\pm$ almost everywhere on $\mathbb{T}$, where $r_\pm$ stands for the roots of the polynomial $2X^2 - X + Q$. In particular, $\mathrm{Re}\, u \in L^\infty(\mathbb{T})$, which proves that $A_u = T_u + T_{\bar{u}} = 2T_{\mathrm{Re}\, u}$ defines a self-adjoint bounded operator on L^2_+.

The Lax pair of Theorem 7 then applies, and yields that

$$K_u = A_uK_u + K_uA_u.$$

This shows that A_u and K_u^2 commute. Now suppose that K_u^2 is not zero. As K_u^2 is a compact operator (it is even trace class), there exists $V \subseteq L^2_+$ an eigenspace of K_u^2 of finite dimension $d \geq 1$. A_u stabilizes V, and $A_u|_V$ is self-adjoint. By the theory of Hermitian operators in finite dimension, $A_u|_V$ then admits a non-zero eigenvector, *i.e.* there exists $\varphi \in L^2_+ \setminus \{0\}$ and $\lambda \in \mathbb{R}$ such that $A_u\varphi = \lambda\varphi$. Hence

$$2\Pi(\varphi\,\mathrm{Re}\, u) = \lambda\varphi,$$

which means that $(2\,\mathrm{Re}\, u - \lambda)\varphi \perp L^2_+$. Multiplying by $\bar{\varphi}$ yields that $(2\,\mathrm{Re}\, u - \lambda)|\varphi|^2$ only has negative Fourier modes, but as this last function takes real values, we must have $(2\,\mathrm{Re}\, u - \lambda)|\varphi|^2 \equiv 0$, or equivalently $2\,\mathrm{Re}\, u \equiv \lambda$ on $\{\varphi \neq 0\}$. However, a classical result on the Hardy space L^2_+ (see [56, Theorem 17.18]) ensures that since φ is not identically zero, $\{\varphi = 0\}$ has zero measure in $\mathbb{T}$. So $2\,\mathrm{Re}\, u \equiv \lambda$ almost everywhere. Let us write $u = \frac{\lambda}{2} + i\psi$ with ψ a real function. As $\psi = \frac{1}{i}(u - \frac{\lambda}{2}) \in L^2_+$, which means that ψ is constant, and so is u. This contradicts the assumption that $K_u^2 \neq 0$. Hence $K_u^2 = 0$, and $K_u = 0$ (because $\ker K_u = \ker K_u^2$), and finally $S^*u = 0$, which means that u (therefore v_0) is constant.

Conversely, the set of constant functions corresponds to the manifold $\mathcal{V}(1) = \{u \in H^{1/2}_+ \mid K_u = 0\}$ which is preserved by the flow. On this manifold, (1.1) becomes the following ODE : $if' = 3|f|^4 f$. Its solutions satisfy $|f|^2 = \mathrm{cst}$, and so turn out to be standing waves.

Remark 13. However, it appears from the previous analysis that equation (3.2) admits many non-trivial solutions on $BMO_+(\mathbb{T})$, and we can also classify them. Indeed, pick some real number $0 < r_- < \frac{1}{6}$, and set $r_+ := \frac{1}{2} - r_-$. Define

$$\theta := \frac{r_-}{r_+ - r_-} = \frac{2r_-}{1 - 4r_-} \in (0, 1),$$

and let $\mathcal{B}_+ \subseteq \mathbb{T}$ be *any* Borel set of measure θ. Let $\mathcal{B}_- := \mathbb{T} \setminus \mathcal{B}_+$. Lastly, set $f : \mathbb{T} \to \mathbb{R}$, with $f = r_+ \mathbb{1}_{\mathcal{B}_+} + r_- \mathbb{1}_{\mathcal{B}_-}$, and introduce

$$u(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} f(e^{ix}) dx, \quad \forall z \in \mathbb{D}.$$

With these definitions, we notice that $f \in L^2(\mathbb{T})$, and thanks to the Poisson kernel, we see that $\operatorname{Re} u$ equals r_+ (resp. r_-) on $\mathcal{B}_+$ (resp. $\mathcal{B}_-$). It also appears that u is holomorphic on $\mathbb{D}$, and expanding $(1 - ze^{-ix})^{-1}$ as a power series, it can be checked that $u = 2\Pi(f) - (f|1)$, so $u \in \Pi(L^\infty) = BMO_+(\mathbb{T})$.

As above, we have $2(\operatorname{Re} u)^2 - \operatorname{Re} u + 2r_- r_+ = 0$, which implies that

$$|u|^2 = \operatorname{Re}(u(1 - u)) - 2r_- r_+.$$

Applying the Szegő projector, we get $\Pi(|u|^2) = \frac{1}{2}u(1 - u) + \frac{1}{2}(1|u)(1 - (1|u)) - 2r_- r_+$. Furthermore, $(u|1) = (f|1) = r_+\theta + r_-(1 - \theta)r_- = 2r_-$, so we compute

$$\frac{1}{2}(1|u)(1 - (1|u)) - 2r_- r_+ = r_-(1 - 2r_- - 2r_+) = 0.$$

This proves that u solves (3.2). Note that we have

$$u = \frac{1}{1 + 2\theta} \cdot \Pi(\mathbb{1}_{\mathcal{B}_+}).$$

3.2 The general case

Traveling waves are rational functions. From now on, we assume that $c \neq 0$. Observe that from (3.1), it implies that $v_0 \in C^\infty_+(\mathbb{T})$. If we make the ansatz $v_0 = \frac{cJ^0}{|J^0|^2} u$, then equation (3.1) reduces to the following equation on the profile u :

$$\varpi u + Du = 2\Pi(|u|^2) + u^2, \tag{3.3}$$

where we have set $\varpi := \omega / c$.

Writing $K_{Du} = DK_u + K_u D + K_u$ and $H_{Du} = DH_u + H_u D$, the Lax pair identities from Theorem 7 read

$$(\varpi + 1)K_u = (A_u - D)K_u + K_u(A_u - D), \tag{3.4}$$
$$\varpi H_u = (A_u - D)H_u + H_u(A_u - D) - (u|\cdot)u. \tag{3.5}$$

Now, since D is a selfadjoint operator with compact resolvent and A_u is a selfadjoint bounded operator, $D - A_u$ is also a selfadjoint operator with compact resolvent, so we can find an eigenbasis $\{\varepsilon_j\}_{j \in \mathbb{N}}$ of L^2_+ and real eigenvalues λ_j going to $+\infty$, such that $(D - A_u)\varepsilon_j = \lambda_j \varepsilon_j$, $\forall j \in \mathbb{N}$. Plugging this into (3.4), we find

$$(-\varpi - 1 - \lambda_j)K_u(\varepsilon_j) = (D - A_u)K_u(\varepsilon_j),$$

so $-\varpi - 1 - \lambda_j$ is also an eigenvalue of $D - A_u$, unless $K_u(\varepsilon_j) = 0$. Since $((D - A_u)h|h) \geq -M\|h\|^2$ for some $M > 0$, we deduce that $K_u(\varepsilon_j) = 0$ for j large enough, hence K_u has finite rank. This proves that u is a rational function by Proposition 2.1.

We are now going to deduce further informations on u thanks to equation (3.3) itself. Indeed expand $u(z)$ as a linear combination of

$$z^k,\ 0 \leq k \leq m_0, \quad \text{and} \quad \frac{1}{(1-p_\ell z)^k},\ 1 \leq k \leq m_\ell,$$

where $\mathcal{P} = \{p_\ell \mid 1 \leq \ell \leq N\}$ is some finite set of points of $\mathbb{D}\backslash\{0\}$, and the m_ℓ's are nonnegative integers. Now $\Pi(|u|^2) = H_u(u) \in \operatorname{Im} H_u$, so it is a combination of the same fractions (see [16, appendix 4] for a complete description of $\operatorname{Im} H_v$ when v is a rational function). As $D = z\partial_z$, Du is a linear combination of z^k, $0 \leq k \leq m_0$, and of $(1-p_\ell z)^{-k}$, $1 \leq k \leq m_\ell + 1$, $p_\ell \in \mathcal{P}$. However, the u^2 term in (3.3) generates terms like z^{2m_0} and $(1-p_\ell z)^{-2m_\ell}$, and they cannot be compensated by other terms unless $m_0 = 0$, and $m_\ell = 1$ for all ℓ. In other words, u must have simple poles, and we write

$$u = \beta + \sum_{\ell=1}^{N} \frac{\alpha_\ell}{1-p_\ell z},$$

for some $\beta, \alpha_1, \ldots, \alpha_N \in \mathbb{C}$.

Let us compute

$$\begin{aligned}
\Pi(|u|^2) &= \sum_{\ell=1}^{N} \left(\bar{\beta} + \sum_{\kappa=1}^{N} \frac{\alpha_\ell \overline{\alpha_\kappa}}{1-p_\ell \overline{p_\kappa}} \right) \frac{1}{1-p_\ell z} + |\beta|^2 + \sum_{\ell=1}^{N} \overline{\alpha_\ell}\beta, \\
u^2 &= \sum_{\ell=1}^{N} \frac{\alpha_\ell^2}{(1-p_\ell z)^2} + \sum_{\ell=1}^{N} \left(2\beta\alpha_\ell + \sum_{\kappa\neq\ell} 2p_\ell \frac{\alpha_\ell\alpha_\kappa}{p_\ell - p_\kappa} \right) \frac{1}{1-p_\ell z} + \beta^2, \\
Du &= \sum_{\ell=1}^{N} \frac{\alpha_\ell}{(1-p_\ell z)^2} - \sum_{\ell=1}^{N} \frac{\alpha_\ell}{1-p_\ell z}.
\end{aligned}$$

Considering the multiples of $\frac{1}{(1-p_\ell z)^2}$ which only appear when computing Du and u^2, we get that $\alpha_\ell = 1$, for all $1 \leq \ell \leq N$ (assuming without loss of generality that $\alpha_\ell \neq 0$).

Reduction to the case $u \in \mathcal{V}(2N)$. From the above formulae on u, $\Pi(|u|^2)$ and u^2, we also get an equation on β which reads

$$\varpi\beta = 2|\beta|^2 + 2N\beta + \beta^2.$$

This equation only has two solutions : either $\beta = 0$ or, dividing by β, $2\bar{\beta} + \beta = \varpi - 2N$. This shows that $\beta = \varpi - 2N - 2\operatorname{Re}(\beta) \in \mathbb{R}$, so we must have

$$\beta = \frac{1}{3}(\varpi - 2N). \tag{3.6}$$

Now, introduce $\tilde{u} = u - \beta$. Then

$$\begin{aligned}
2\Pi(|\tilde{u}|^2) + \tilde{u}^2 &= 2\Pi(|u|^2) + 2|\beta|^2 - 2\beta\Pi(\bar{u}) - 2\bar{\beta}u + u^2 - 2u\beta + \beta^2 \\
&= 2\Pi(|u|^2) + u^2 - 4\beta u + 3\beta^2 - 2\beta(1|u) \\
&= (\varpi - 4\beta)u + Du + \beta^2 - 2\beta N
\end{aligned}$$

$$= (\varpi - 4\beta)\tilde{u} + D\tilde{u}.$$

Consequently, up to a modification of ϖ, it suffices to treat the case $\beta = 0$, so from now on, we consider that u belongs to $\mathcal{V}(2N)$ and is of the form

$$u = \sum_{\ell=1}^{N} \frac{1}{1 - p_\ell z}. \tag{3.7}$$

In particular, we have

$$(u|1) = N. \tag{3.8}$$

To conclude, it suffices to describe the possible choices of $\mathcal{P} = \{p_\ell\}$. Note that the p_ℓ's solve the following system of equations :

$$\frac{\varpi - 1}{2} = \sum_{\kappa=1}^{N} \frac{1}{1 - p_\ell \overline{p_\kappa}} + \sum_{\substack{\kappa=1\\ \kappa\neq\ell}}^{N} \frac{p_\ell}{p_\ell - p_\kappa}, \quad \forall \ell = 1, \dots, N. \tag{3.9}$$

Spectral analysis of $A_u - D$. To solve the above system, we are going to perform a spectral analysis of the operator $A_u - D$, taking advantage from its relation with the self-adjoint finite-rank operator K_u^2, in the spirit of [16, Section 9]. Denote by Σ the (finite) set of the K-dominant eigenvalues of K_u^2. In this whole section, we fix $\sigma \in \Sigma$. To simplify the notations, we set $F := F_u(\sigma)$ and $E := E_u(\sigma)$ (see paragraph 2.2). Since $0 \notin \Sigma$ now that $u \in \mathcal{V}(2N)$, both F and E are finite-dimensional subspaces of $\operatorname{Ran} H_u = \operatorname{Ran} K_u$. We also define u_σ to be the (non-zero) orthogonal projection of u onto F. Our goal is to prove the following statement :

Proposition 3.1. *We have*

$$(A_u - D)u_\sigma = \frac{\varpi + n_\sigma}{2} u_\sigma,$$

where $n_\sigma := \dim F$.

Proof. The proof of this proposition decomposes into several steps.

First step. u_σ is an eigenvector of $A_u - D$.

From (3.4), we see that $(A_u - D)K_u^2 = K_u^2(A_u - D)$. This shows that $(A_u - D)(F) \subseteq F$. Now, as $E = F \cap u^\perp$, then by (3.5), $\varpi H_u = (A_u - D)H_u + H_u(A_u - D)$ on E.

If $h \in E$, then $H_u(h) \in E$, so we apply this identity to $H_u(h)$. We get

$$\begin{aligned}\sigma^2 \varpi h &= \sigma^2 (A_u - D)(h) + H_u(A_u - D)H_u(h)\\ &= \sigma^2(A_u - D)(h) + H_u(-H_u(A_u - D) + \varpi H_u)(h),\end{aligned}$$

so $H_u^2(A_u - D)(h) = \sigma^2(A_u - D)(h)$, which proves that $A_u - D$ also leaves E invariant. As $A_u - D$ is self-adjoint, it thus preserves $F \cap E^\perp = \mathbb{C}u_\sigma$, which means that there exists $\lambda \in \mathbb{R}$ such that

$$(A_u - D)u_\sigma = \lambda u_\sigma.$$

Second step. $2\lambda - \varpi$ is a positive integer.

Assume by contradiction that it is not true. Then by (3.4), $K_u u_\sigma$ satisfies $(A_u - D)K_u u_\sigma = (\varpi + 1 - \lambda)K_u u_\sigma$. As $\varpi + 1 - \lambda \neq \lambda$ by assumption, we find that $K_u u_\sigma \perp u_\sigma$, and since $K_u u_\sigma \in F$, $K_u u_\sigma \perp u$, hence $K_u u_\sigma \in E$. Applying identity (3.5), we get

$$(A_u - D)H_u K_u u_\sigma = (\lambda - 1)H_u K_u u_\sigma.$$

Note that $H_uK_uu_\sigma \neq 0$, for K_u (resp. H_u) is one-to-one on F (resp. E).

As $H_uK_uu_\sigma \in E$ and also in F, we can restart this argument, and prove by induction that

$$(A_u - D)(H_uK_u)^j u_\sigma = (\lambda - j)(H_uK_u)^j u_\sigma, \quad \forall j \in \mathbb{N}.$$

This of course cannot happen, otherwise $A_u - D$ would have infinitely many distinct eigenvalues on F which is a finite dimensional subspace of L^2_+.

From now on, we write $\lambda = \frac{1}{2}(\varpi + n)$. It remains to show that $n = \dim F$.

Third step. The action of S^* on E.

Our purpose now is to prove that if $e \in E\backslash\{0\}$ satisfies $(A_u - D)e = \mu e$, then $S^*e \in F\backslash\{0\}$ and satisfies $(A_u - D)S^*e = (\mu+1)S^*e$.

We thus have to compute the commutator $[A_u - D, S^*]$. Since for $h \in L^2_+$,

$$[D, S](h) = z\partial_z(zh) - z^2\partial_z h = zh = S(h),$$

we get $[S^*, D] = ([D, S])^* = S^*$. Besides, we know from the proof of Theorem 7 that $S^*A_u - A_uS^* = (\cdot|1)S^*u$. But $E \subseteq 1^\perp$ by Lemma 2.4. This proves that A_u and S^* commute on E.

Finally, taking $e \in E$ as above, we get

$$(A_u - D)S^*e = S^*(A_u - D)e + [A_u - D, S^*]e = \mu S^*e + [S^*, D]e = (\mu + 1)S^*e.$$

To see that $S^*e \in F\backslash\{0\}$, it suffices to notice that $\sigma^2 S^*e = S^*H_u^2 e = K_uH_ue$, and to conclude thanks to the injectivity of H_u and K_u as in the second step.

Fourth step. The eigenvalue λ of $A_u - D|_F$ is simple.

Otherwise, there would be an eigenvector $e \in F$ associated to λ such that $e \perp u_\sigma$. This would mean that $e \in E$, so by the previous point, S^*e would be an eigenvector of $A_u - D|_F$ associated to $\lambda + 1$, hence orthogonal to u_σ. Therefore, for all $j \geq 0$, $(S^*)^j e$ would be a (non-zero) eigenvector of $A_u - D|_F$ associated to $\lambda + j$, and this contradicts the fact that $\dim F < \infty$.

Similarly, $A_u - D|_F$ has no eigenvalue $\mu > \lambda$, otherwise, iterating the third step, $\mu + j$ would be an eigenvalue of $A_u - D|_F$ for all $j \in \mathbb{N}$.

Now we are able to finish the proof. Let μ be the smallest eigenvalue of $A_u - D$ on F and $\tilde{e}$ a corresponding eigenvector. Then $\nu := \lambda - \mu$ should be a nonnegative integer, or then the $(S^*)^j\tilde{e}$, $j \geq 0$, would give rise to an infinite sequence of orthogonal eigenvectors. We also have $(S^*)^\nu \tilde{e} \in \mathbb{C}u_\sigma$, thanks to the previous remark. Moreover, since S^* is injective on E (recall that $E \subseteq 1^\perp$), all the eigenvalues $\mu, \mu+1, \ldots, \mu+\nu-1$ are simple as well on F. Thus $\dim F = \nu + 1$.

It is now straightforward to see that $\nu = n - 1$, with $n = 2\lambda - \varpi$ as above. On the one hand, $K_u(u_\sigma)$ is an eigenvector of $A_u - D|_F$ associated to $\varpi + 1 - \lambda$, which means that $\nu = \lambda - \mu \geq \lambda - (\varpi + 1 - \lambda) = n - 1$ by minimality of μ. On the other hand, $K_u(\tilde{e})$ is an eigenvector associated to $\varpi + 1 - \mu$, so $\nu = \lambda - \mu \geq (\varpi + 1 - \mu) - \mu = \varpi + 1 - 2(\lambda - \nu)$ by maximality of λ, and hence $n - 1 \geq \nu$. This establishes the yielded formula :

$$(A_u - D)u_\sigma = \frac{\varpi + \dim F}{2} u_\sigma,$$

in addition to the fact that $\varpi + 1 - \lambda = \frac{1}{2}(\varpi + 2 - \dim F)$ is the smallest eigenvalue of $A_u - D|_F$. □

We point out a by-product of the last part of the proof :

Corollary 3.2. *There exists a complex number $\zeta_\sigma \in \mathbb{C} \setminus \{0\}$ such that we have*

$$K_u(u_\sigma) = \zeta_\sigma z^{n_\sigma - 1} u_\sigma, \tag{3.10}$$

where $n_\sigma = \dim F$ as above.

Proof. Indeed, with the terminology of the proof above, $K_u(u_\sigma)$ is non-zero and colinear to $\tilde{e}$. So $(S^*)^{n_\sigma-1}K_u(u_\sigma) = \zeta_\sigma u_\sigma$ for some $\zeta_\sigma \neq 0$. Now if $j < n_\sigma - 1$, then $(S^*)^j K_u(u_\sigma) \in E$, and on E, we have $SS^* = I - (\cdot|1)1 = I$. Therefore, $S^{n_\sigma-1}(S^*)^{n_\sigma-1}K_u(u_\sigma) = K_u(u_\sigma)$, and (3.10) is proved. □

We summarize the results of this paragraph on Figure 4.1.

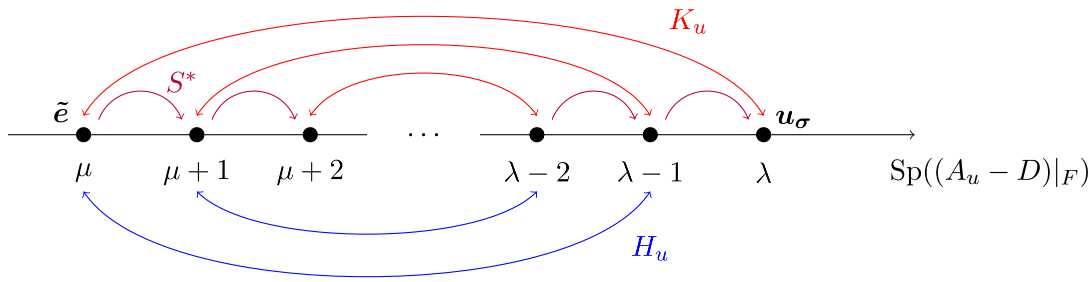

FIGURE 4.1: The action of H_u, K_u and S^* on F.

K_u^2 has at most one positive eigenvalue. Combining the informations of Proposition 3.1 on each K-dominant eigenspace of K_u^2, we claim that K_u cannot have more than one singular value. Let $\Sigma = \{\sigma_m \mid 1 \le m \le |\Sigma|\}$ be the set of the K-dominant eigenvalues of K_u^2, and $n_m = \dim(F_u(\sigma_m))$ for each m. As above, we introduce the orthogonal projection of u onto the eigenspace $F_u(\sigma_m)$ and call it u_m. The norm $\|\cdot\|$ without further precision denotes the L^2 norm, and Q refers to $\|u\|^2$.

We first prove a simple but crucial lemma.

Lemma 3.3. *We have, for each* $1 \le m \le |\Sigma|$,

$$(\varpi + n_m - 2N) \cdot (u_m|1) = 2\|u_m\|^2. \tag{3.11}$$

In particular, $(u_m|1)$ *is a positive real number.*

Proof. We know that u_m is a non-zero eigenvector of the operator $A_u - D$. Let λ be the associated eigenvalue. Since $(A_u - D)1 = \Pi(u + \bar{u})$, and by (3.8), $\Pi(\bar{u}) = (1|u) = N$, we can compute

$$\lambda(u_m|1) = ((A_u - D)u_m|1) = (u_m|(A_u - D)1) = \|u_m\|^2 + N(u_m|1),$$

so that $(u_m|1)(\lambda - N) = \|u_m\|^2$. But by Proposition 3.1, we have $\lambda = \frac{1}{2}(\varpi + n_m)$, which gives formula (3.11).

As $\|u_m\|^2 > 0$, it remains to show that $\varpi + n_m - 2N$ is a positive number. Taking the scalar product of equation (3.3) with 1 leads to the following identity :

$$\varpi N = 2Q + N^2. \tag{3.12}$$

However, the Cauchy-Schwarz inequality yields $N^2 = |(u|1)|^2 \le Q$, so $\varpi N \ge 3N^2$, which means that $\varpi \ge 3N$. A fortiori $\varpi + n_m - 2N \ge \varpi - 2N > 0$, and the proof of the lemma is complete. □

We are now able to prove our claim :

Lemma 3.4. *Necessarily,* Σ *is a singleton* $\{\sigma\}$*, and* $\dim F_u(\sigma) = N$*.*

Proof. We are going to sum identity (3.11) over m. This gives, using (2.2) and the fact that $u_m \perp u_{m'}$ for $m \neq m'$,

$$(\varpi - 2N) \cdot (u|1) + \sum_{m=1}^{|\Sigma|} n_m (u_m|1) = 2\|u\|^2.$$

As we know from (3.12) that $2Q = \varpi N - N^2 = (\varpi - N) \cdot (u|1)$, we get

$$\sum_{m=1}^{|\Sigma|} n_m (u_m|1) = N \cdot (u|1) = N \sum_{m=1}^{|\Sigma|} (u_m|1),$$

or equivalently

$$\sum_{m=1}^{|\Sigma|} (N - n_m) \cdot (u_m|1) = 0. \tag{3.13}$$

But $\sum_m n_m \leq \operatorname{rk} K_u^2 = N$, so $n_m \leq N$. Together with the preceding lemma, this shows that the sum in (3.13) only involves nonnegative terms. Each of them must then be zero, *i.e.* we must have $n_m = N$ for all $1 \leq m \leq |\Sigma|$. Hence $N \geq \sum_m n_m = N|\Sigma|$, so finally $|\Sigma| = 1$. □

Determination of u. Because of (2.2), Lemma 3.4 implies that u is an eigenvector of K_u^2, *i.e.* $K_u^2(u) = \sigma^2 u$ for some $\sigma > 0$. Hence $H_u^2(u) = (\sigma^2 + Q)u$, so u is also an eigenvector of H_u^2. Now, coming back to Proposition 3.1, this means that $u_\sigma = u$, and $\dim F = N$, hence

$$(A_u - D)u = \frac{\varpi + N}{2} u.$$

However, $(A_u - D)u = \Pi(|u|^2) + u^2 - Du = \varpi u - \Pi(|u|^2)$, by equation (3.3). Therefore,

$$H_u(u) = \varpi u - (A_u - D)u = \frac{\varpi - N}{2} u.$$

On the other hand, by Corollary 3.2, we also have, for some $\zeta \neq 0$,

$$K_u(u) = \zeta z^{N-1} u.$$

Combining the two informations, we can write $\zeta z^{N-1} u = K_u(u) = S^* H_u(u) = \frac{\varpi - N}{2} S^* u$. Applying S again, we get the existence of some $\alpha \neq 0$ such that $SS^* u = \alpha z^N u$. Besides, $SS^* u = u - (u|1) = u - N$. This finishes to prove that

$$u(z) = \frac{N}{1 - \alpha z^N}.$$

In other words, going back to formula (3.7), this shows that $\mathcal{P}$ in (3.9) has to be the set of the Nth roots of some complex number α such that $|\alpha| < 1$.

Remark 14. In terms of the inverse spectral transform of [21, Section 5], we have proved that H_u has one dominant singular value of multiplicity 1, and that K_u has one dominant singular value σ of multiplicity N. Corollary 3.2 says that the angle associated to σ is $e^{i\psi} z^{N-1}$ for some $\psi \in \mathbb{T}$.

The reverse statement. Now that we have found what form a traveling wave should have a priori, we need to show that functions of this form are traveling waves for (1.1) indeed. From a straightforward computation based on (3.9), we can easily find the traveling waves of $\mathcal{V}(2)$ and $\mathcal{V}(3)$: the function $u(z) = \frac{1}{1-pz}$, for $|p| < 1$, satisfies

$$\varpi u + Du = 2\Pi(|u|^2) + u^2, \quad \text{with } \varpi \in \mathbb{R},$$

and no other function of $\mathcal{V}(2)$ does. Besides, $\varpi = \frac{3-|p|^2}{1-|p|^2}$ because of (3.9). Then, if $\lambda \in \mathbb{C}$, set $v := \lambda u$. We have

$$J(v) = |\lambda|^2 \lambda J(u) = \frac{|\lambda|^2 \lambda}{(1-|p|^2)^2},$$

which leads to

$$2J(v)\Pi(|v|^2) + \overline{J(v)} v^2 = \frac{|\lambda|^4 \lambda}{(1-|p|^2)^2}(2\Pi(|u|^2) + u^2) = \frac{|\lambda|^4 (3-|p|^2)}{(1-|p|^2)^3} v + \frac{|\lambda|^4}{(1-|p|^2)^2} Dv.$$

Thus v gives rise to a traveling wave solution of (1.1) with

$$\omega = |\lambda|^4 \frac{3-|p|^2}{(1-|p|^2)^3}, \quad c = |\lambda|^4 \frac{1}{(1-|p|^2)^2}.$$

Similarly, in $\mathcal{V}(3)$, the only solution of $\varpi u + Du = 2\Pi(|u|^2) + u^2$ has the form $u = \beta + \frac{1}{1-pz}$, and by (3.6) and (3.9), we must have

$$\begin{cases} \varpi - 4\beta = \dfrac{3-|p|^2}{1-|p|^2}, \\ \varpi - 3\beta = 2, \end{cases}$$

or equivalently $\beta = -\frac{1+|p|^2}{1-|p|^2}$ and $\varpi = -\frac{1+5|p|^2}{1-|p|^2}$. Then $v := \lambda u$ for $\lambda \in \mathbb{C}$ is such that

$$J(v) = |\lambda|^2 \lambda J(u) = -\frac{|\lambda|^2 \lambda \cdot |p|^4 (3+5|p|^2)}{(1-|p|^2)^3},$$

as show the formulae of Appendix B for instance. As above, we finally get that v is the initial state of a traveling wave with

$$\omega = |\lambda|^4 \frac{|p|^4(3+5|p|^2)(1+5|p|^2)}{(1-|p|^2)^4}, \quad c = -|\lambda|^4 \frac{|p|^4(3+5|p|^2)}{(1-|p|^2)^3}.$$

The general case of $\mathcal{V}(2N)$ and $\mathcal{V}(2N+1)$ relies on an invariance argument discovered in [68] in the case of the cubic Szegő equation, and which also applies here.

Definition. An *inner function* is a function $f \in L^2_+$ such that for almost every $x \in \mathbb{T}$,

$$|f(e^{ix})| = 1.$$

Among such functions, some are rational functions : they are called *Blaschke products*, and are of the form

$$z \mapsto \prod_{\ell=1}^{L} \frac{z - \overline{p_\ell}}{1 - p_\ell z}, \quad p_\ell \in \mathbb{D}, \forall 1 \leq \ell \leq L.$$

The invariance result is the following :

Proposition 3.5 (Invariance by composition with an inner function)**.** *Let $f \in L^2_+$ be an inner function, and $t \mapsto u(t,z)$ a solution of* (1.1) *starting from $u(0,z) = u_0(z) \in BMO_+(\mathbb{T})$. Then*

$$t \mapsto u(t, zf(z))$$

is the solution of (1.1) *in BMO_+ starting from initial data $u_0(zf(z))$.*

Let us first show how this proposition allows us to conclude. Assume $v(t,z)$ is a traveling wave with $v(0,z) = v_0(z)$. Equation (1.3) is equivalent to

$$\widehat{v(t)}(k) = \widehat{v_0}(k) e^{-i(\omega + c\cdot k)t}, \quad \forall k \in \mathbb{Z},$$

where $\widehat{\cdot}$ is the Fourier transform. Now, if $N \geq 1$ and $f(z) = z^{N-1}$, the invariance proposition states that $v(t, z^N)$ is the solution starting from $v_0(z^N)$. Writing $w(t,z) := v(t, z^N)$, we get

$$\widehat{w(t)}(k) = \widehat{v_0}(k) e^{-i(\omega + \frac{c}{N}\cdot k)t}, \quad \forall k \in \mathbb{Z} \text{ s.t. } N|k.$$

Thus, the set of traveling waves for equation (1.1) is stable by composition with z^N (the pulsation does not change and the velocity is divided by N). This ends the proof of Theorem 6.

To prove Proposition 3.5, we introduce an operator on $L^2(\mathbb{T})$. If f is an inner function, we set

$$\mathcal{T}_f : \begin{cases} L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ \sum_{k=-\infty}^{+\infty} u_k e^{ikx} & \longmapsto & \sum_{k=-\infty}^{+\infty} u_k [e^{ix} f(e^{ix})]^k. \end{cases}$$

The operator $\mathcal{T}_f$ satisfies some useful algebraic properties that we sum up in the next lemma.

Lemma 3.6. *(i) $\mathcal{T}_f$ is an isometry of $L^2(\mathbb{T})$.*

(ii) $\mathcal{T}_f \circ \Pi = \Pi \circ \mathcal{T}_f$.

(iii) $\mathcal{T}_f$ induces a $*$-endomorphism of the C^*-algebra $L^\infty(\mathbb{T})$.

Proof. For $k \in \mathbb{Z}$, let us introduce $e_k^f(x) = e^{ikx} f(e^{ix})^k$ for $x \in \mathbb{T}$. To see that $\mathcal{T}_f$ is an isometry, it suffices to notice that $\{e_k^f\}_{k\in\mathbb{Z}}$ is a orthonormal family of L^2. If $p \geq q$, we have

$$(e_p^f | e_q^f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e_p^f(x) \overline{e_q^f(x)} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(p-q)x} f(e^{ix})^{p-q} dx = \delta_{pq},$$

for f has nonnegative frequencies only. Hence, for any $u = \sum_{k=-\infty}^{+\infty} u_k e^{ikx} \in L^2(\mathbb{T})$, $\|\mathcal{T}_f u\|_{L^2}^2 = \sum_{k=-\infty}^{+\infty} |u_k|^2 = \|u\|_{L^2}^2$. As $\mathcal{T}_f$ is linear, this proves *(i)*.

Property *(ii)* comes from the fact that $\Pi(e_k^f) = e_k^f$ if $k \geq 0$, and 0 otherwise.

Finally, if $v \in L^\infty$, then v can be identified to a function $\tilde{v}$ such that $v = \tilde{v}$ almost everywhere and $|\tilde{v}(x)| \leq \|v\|_{L^\infty}$ for *all* $x \in \mathbb{T}$. Now, if $x \in \mathbb{T}$, $\mathcal{T}_f v(e^{ix}) = \mathcal{T}_f \tilde{v}(e^{ix}) = \tilde{v}(e^{iy})$, where $y \in \mathbb{T}$ is such that $e^{iy} = e^{ix} f(e^{ix})$. This proves that $|\mathcal{T}_f \tilde{v}(e^{ix})| \leq \|v\|_{L^\infty}$, so the restiction of $\mathcal{T}_f$ to L^∞ gives rise to a continuous endomorphism of L^∞.

As we obviously have $\mathcal{T}_f 1 = 1$, it remains to show that $\mathcal{T}_f(uv) = (\mathcal{T}_f u)(\mathcal{T}_f v)$ and $\mathcal{T}_f \bar{u} = \overline{\mathcal{T}_f u}$ whenever $u, v \in L^\infty$. But this is immediate once we have interpreted $\mathcal{T}_f$ as a composition operator. □

Remark 15. The converse of Lemma 3.6 is true : any operator $\mathcal{G} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ that satisfies properties *(i)–(iii)* is of the form $\mathcal{T}_f$, where $f \in L^2_+$ is an inner function. It suffices to set $f(e^{ix}) := e^{-ix} \mathcal{G}(e^{ix})$ for $x \in \mathbb{T}$.

Proof of Proposition 3.5. Firstly, we note that since $u_0 = \Pi(b)$ for some $b \in L^\infty(\mathbb{T})$, then $\mathcal{T}_f u_0 = \mathcal{T}_f \Pi(b) = \Pi(\mathcal{T}_f b) \in BMO_+$. Indeed, $\mathcal{T}_f b \in L^\infty$ by the previous lemma.

If u is the solution starting from u_0, then we get

$$\begin{aligned} i\partial_t(\mathcal{T}_f u) &= \mathcal{T}_f(i\partial_t u) \\ &= 2J(u)\mathcal{T}_f\Pi(|u|^2) + \overline{J(u)}\mathcal{T}_f(u^2) \\ &= 2J(u)\Pi(|\mathcal{T}_f u|^2) + \overline{J(u)}(\mathcal{T}_f u)^2. \end{aligned}$$

Furthermore, $J(u) = (u^2|u) = (\mathcal{T}_f(u^2)|\mathcal{T}_f u) = J(\mathcal{T}_f u)$, because of property *(i)* of Lemma 3.6. So $\mathcal{T}_f u$ solves equation (1.1) in BMO_+, and $\mathcal{T}_f(u)(t=0) = \mathcal{T}_f u_0$. □

Remark 16. The mapping $\mathcal{T}_f$ also preserves the class of the standing waves in BMO_+, for if $\operatorname{Re} u$ only takes two different values on $\mathbb{T}$, then so does $\operatorname{Re}(\mathcal{T}_f u)$.

4 Stability and instability of the traveling waves

Hereafter we consider the problem of the stability of the solutions that Theorem 6 classifies.

4.1 The ground state

We begin with the "ground state", and show its stability via the functional inequality (1.4) as in [5, 16].

Proof of Proposition 1.1. Let $u \in H^{1/2}_+$, and write $u(z) = \sum_{k\geq 0} u_k z^k$. Then by Parserval's theorem and twice the Cauchy-Schwarz inequality,

$$\begin{aligned} E(u) = \frac{1}{2}|J(u)|^2 &= \frac{1}{2}\left|\sum_{k,\ell\geq 0} u_k u_\ell \overline{u_{k+\ell}}\right|^2 \\ &\leq \frac{1}{2}\left(\sum_{k\geq 0}|u_k|^2\right)\left(\sum_{k\geq 0}\left|\sum_{\ell\geq 0} u_\ell\overline{u_{k+\ell}}\right|^2\right) \\ &\leq \frac{1}{2}Q(u)\cdot\left(\sum_{\ell\geq 0}|u_\ell|^2\right)\left(\sum_{k,\ell\geq 0}|u_{k+\ell}|^2\right) \\ &= \frac{1}{2}Q(u)^2\left(\sum_{m\geq 0}(1+m)|u_m|^2\right) = \frac{1}{2}Q(u)^2(Q(u)+M(u)). \end{aligned}$$

If in addition $2E = Q^2(Q+M)$, then the case of equality in Cauchy-Schwarz allows us to say that necessarily, $\forall k \geq 0$, there exists $\gamma_k \in \mathbb{C}$ such that

$$u_\ell = \gamma_k u_{\ell+k}, \quad \forall \ell \geq 0.$$

In particular, $u_\ell = \gamma_1 u_{\ell+1}$, so $\{u_\ell\}$ must be a geometric sequence. There exists $\lambda, p \in \mathbb{C}$ such that

$$u_\ell = \lambda p^\ell, \quad \forall \ell \geq 0.$$

As $\sum_\ell |u_\ell|^2 < \infty$, we further have $|p| < 1$. Hence $u(z) = \frac{\lambda}{1-pz}$. Conversely, if $\{u_\ell\}$ is a geometric sequence, both inequalities in the previous computation become equalities. □

Now it is standard to prove the $H^{1/2}$-stability of the traveling waves $z \mapsto \frac{\lambda}{1-pz}$ (which are even stationary waves in the case when $p = 0$).

Proof of Corollary 1.2. Let $\lambda, p \in \mathbb{C}$ with $|p| < 1$, and $v_0(z) = \frac{\lambda}{1-pz}$. Simple computations show that

$$Q(v_0) = \frac{|\lambda|^2}{1-|p|^2}, \quad M(v_0) = \frac{|\lambda|^2|p|^2}{(1-|p|^2)^2}, \quad E(v_0) = \frac{1}{2}\frac{|\lambda|^6}{(1-|p|^2)^4}.$$

Then by Proposition 1.1, it appears that

$$\{e^{-i\theta}v_0(ze^{-i\alpha}) \mid (\theta,\alpha) \in \mathbb{T}^2\} = \{u \in H^{1/2}_+ \mid Q(u) = Q(v_0),\ M(u) = M(v_0),\ E(u) = E(v_0)\}.$$

In the sequel, we denote by $\mathcal{T}$ this two- (or maybe one-) dimensional torus, and we show that it is stable in the energy space.

Suppose that it is not. Then there exists a sequence $\{u_0^n\}$ of initial data in $H^{1/2}_+$ such that $d(u_0^n, \mathcal{T}) \to 0$ (where the distance is taken in $H^{1/2}_+$), and a sequence of times $\{t^n\}$ such that the solution $u^n(t)$ of (1.1) starting from u_0^n at time $t = 0$ satisfies

$$d(u^n(t^n), \mathcal{T}) \geq \varepsilon_0 > 0, \tag{4.1}$$

for some $\varepsilon_0 > 0$.

Because of the conservation laws, we know that $\{u^n(t^n)\}$ is bounded in $H^{1/2}_+$. Up to extracting a subsequence, there is an element f of the energy space such that $u^n(t^n) \rightharpoonup f$. By Rellich's theorem, we know that $u^n(t^n) \to f$ strongly in any L^p, $p < \infty$, so

$$\begin{aligned} Q(v_0) &= \lim_{n\to\infty} Q(u_0^n) = \lim_{n\to\infty} Q(u^n(t^n)) = Q(f), \\ E(v_0) &= \lim_{n\to\infty} E(u_0^n) = \lim_{n\to\infty} E(u^n(t^n)) = E(f). \end{aligned}$$

By the weak convergence, we also have $M(f) \leq \liminf_{n\to\infty} M(u^n(t^n)) = \liminf_{n\to\infty} M(u_0^n) = M(v_0)$. Hence

$$E(f) \leq \frac{1}{2}Q(f)^2(Q(f) + M(f)) \leq \frac{1}{2}Q(v_0)^2(Q(v_0) + M(v_0)) = E(v_0) = E(f),$$

so all the inequalities are in fact equalities. Thus $M(f) = M(v_0)$, so $u^n(t^n) \to f$ strongly in $H^{1/2}$ and $f \in \mathcal{T}$. This is a contradiction with (4.1). □

4.2 The second branch

Now it remains to consider the case of the second branch of the traveling waves in Theorem 6. We first focus on the simplest one, of degree 1 in z, namely

$$z \mapsto -\lambda\frac{1+|p|^2}{1-|p|^2} + \frac{\lambda}{1-pz} = -\lambda\frac{2|p|^2}{1-|p|^2} + \frac{\lambda pz}{1-pz},$$

where $\lambda, p \in \mathbb{C}\setminus\{0\}$, and $|p| < 1$. Observe that thanks to time rescaling, and to the invariance of equation (1.1) under phase rotation and phase translation, it is enough to study the stability of the following traveling waves :

$$v_r : z \in \mathbb{D} \longmapsto -\frac{2r}{1-r} + \frac{z\sqrt{r}}{1-z\sqrt{r}}, \quad r \in (0,1).$$

Lemma 4.1. *Let $0 < r < 1$ be fixed. Then v_r is not $H^{1/2}_+$-orbitally stable.*

Before starting the proof of this lemma, let us recall how equation (1.1) looks like on $\mathcal{V}(3)$. The exact formulae can be found in Appendix B. Let u be a solution of (1.1) of the form

$$u(z) = b + \frac{cz}{1-pz}, \quad \forall z \in \mathbb{D},$$

with $b, c, p \in \mathbb{C}$, $c \neq 0$, $c - bp \neq 0$, $|p| < 1$. Then Q, M and J can be computed explicitely in terms of b, c, p :

$$Q = |b|^2 + \frac{|c|^2}{1-|p|^2},$$
$$M = \frac{|c|^2}{(1-|p|^2)^2},$$
$$J = |b|^2 b + \frac{2b|c|^2}{1-|p|^2} + \frac{|c|^2 c\bar{p}}{(1-|p|^2)^2}.$$

Since $|c| = \sqrt{M}\cdot(1-|p|^2)$ and $|b|^2 = Q - |c|\sqrt{M}$, we can rewrite J in a simpler way as

$$J = (Q + |c|\sqrt{M})b + Mc\bar{p}.$$

Introduce the doubled energy $\mathcal{E} := 2E = |J|^2$. From the expression of J, we find

$$\mathcal{E} = (Q+|c|\sqrt{M})^2(Q-|c|\sqrt{M}) + M|c|^2(M-|c|\sqrt{M}) + 2M(Q+|c|\sqrt{M})\operatorname{Re}(b\bar{c}p).$$

Writing for simplicity $x := |c|\sqrt{M}$, and defining ψ by $\psi = \arg(b\bar{c}p)$ whenever $bp \neq 0$, and $\psi = 0$ when $b = 0$ or $p = 0$, we get

$$\mathcal{E} = (Q+x)^2(Q-x) + x^2(M-x) + 2x(Q+x)\sqrt{(Q-x)(M-x)}\cdot\cos\psi. \tag{4.2}$$

As far as the evolution of x is concerned, recall that

$$\begin{aligned} i\frac{dc}{dt} &= 2bc\bar{J} + 2\bar{b}cJ + \frac{2Jp|c|^2}{1-|p|^2} \\ &= \left[4\operatorname{Re}(\bar{b}J) + 2M\frac{|p|^2|c|^2}{1-|p|^2}\right]c + 2|c|^2(Q+|c|\sqrt{M})\frac{bp}{1-|p|^2}, \end{aligned}$$

hence

$$\frac{d|c|}{dt} = \frac{1}{|c|}\operatorname{Re}\left(\bar{c}\frac{dc}{dt}\right) = 2|c|(Q+|c|\sqrt{M})\frac{\operatorname{Im}(b\bar{c}p)}{1-|p|^2},$$

or equivalently,

$$\frac{dx}{dt} = 2x(Q+x)\sqrt{(Q-x)(M-x)}\cdot\sin\psi.$$

Using (4.2), and writing $(\sin\psi)^2 = 1-(\cos\psi)^2$, we can finally express the evolution of x in terms of conservation laws only :

$$\left(\frac{dx}{dt}\right)^2 = 4x^2(Q+x)^2(Q-x)(M-x) - \left[(Q+x)^2(Q-x) + x^2(M-x) - \mathcal{E}\right]^2. \tag{4.3}$$

Proof of Lemma 4.1. Thanks to the above formulae, we can easily compute

$$Q_r := Q(v_r) = \frac{r}{(1-r)^2}(3r+1), \qquad M_r := M(v_r) = \frac{r}{(1-r)^2},$$
$$J_r := J(v_r) = -\frac{r^2}{(1-r)^3}(5r+3), \qquad \mathcal{E}_r = |J_r|^2 = \frac{r^4}{(1-r)^6}(5r+3)^2.$$

Besides, since v_r gives rise to traveling wave solution, $|c|$ (hence x) will be constant, so $\forall t \in \mathbb{R}$, $x(t) = x_r := \sqrt{r}\sqrt{M} = \frac{r}{1-r}$. For the same reason, $\psi(t) = \psi_r := \pi$. Equation (4.3) then gives

$$0 = 4x_r^2(Q_r + x_r)^2(Q_r - x_r)(M_r - x_r) - \left[(Q_r + x_r)^2(Q_r - x_r) + x_r^2(M_r - x_r) - \mathcal{E}_r\right]^2. \quad (4.4)$$

Now we are going to perturb equation (4.3) around x_r and $\mathcal{E}_r$ without changing Q_r nor M_r. Let $\gamma > 0$ be a small enough parameter (in a sense that we will further make clear), and set

$$u_0^\gamma(z) = -e^{i\gamma}\frac{2r}{1-r} + \frac{z\sqrt{r}}{1-z\sqrt{r}}, \quad \forall z \in \mathbb{D}.$$

We have $\|u_0^\gamma - v_r\|_{H^{1/2}} = \frac{2r}{1-r}|1-e^{i\gamma}| = \mathcal{O}_r(\gamma)$, where the notation $\mathcal{O}_r$ means that the constant in the $\mathcal{O}$ only depends on r. In addition, $Q(u_0^\gamma) = Q_r$, $M(u_0^\gamma) = M_r$. Computing $\mathcal{E}(u_0^\gamma)$ thanks to (4.2), we see that only the angle ψ is modified at time 0, and changes from π to $\pi+\gamma$. If γ is sufficiently small, we have $\cos(\pi+\gamma) > -1$, and thus we can write $\mathcal{E}(u_0^\gamma) = \mathcal{E}_r + \delta\mathcal{E}$, where $\delta\mathcal{E} > 0$ and $\delta\mathcal{E} = \mathcal{O}_r(\gamma)$.

Denote by u (with an implicit dependence on γ) the solution of (1.1) starting from u_0^γ. Write $x = |c|\sqrt{M}$ as above, and decompose $x = x_r + y$, where y is initially zero, and is meant to remain small. Equation (4.3) yields

$$\left(\frac{dy}{dt}\right)^2 = 4(x_r+y)^2(Q_r+x_r+y)^2(Q_r-x_r-y)(M_r-x_r-y)$$
$$- \left[(Q_r+x_r+y)^2(Q_r-x_r-y) + (x_r+y)^2(M_r-x_r-y) - \mathcal{E}_r - \delta\mathcal{E}\right]^2.$$

We must develop this expression. The leading order term vanishes because of (4.4). Handling the $\delta\mathcal{E}$-terms with care, we can write

$$\left(\frac{dy}{dt}\right)^2 = \delta\mathcal{E}\cdot\left[-2\mathcal{E}_r + 2(Q_r+x_r)^2(Q_r-x_r) + 2x_r^2(M_r-x_r) + \mathcal{O}_r(y) - \delta\mathcal{E}\right]$$
$$+ y\cdot\left[\frac{d}{dy}F(y)|_{y=0}\right]$$
$$+ \mathcal{O}_r(y^2),$$

where

$$F(y) = 4(x_r+y)^2(Q_r+x_r+y)^2(Q_r-x_r-y)(M_r-x_r-y)$$
$$+ 2\mathcal{E}_r\cdot\left[(Q_r+x_r+y)^2(Q_r-x_r-y) + (x_r+y)^2(M_r-x_r-y)\right]$$
$$- \left[(Q_r+x_r+y)^2(Q_r-x_r-y) + (x_r+y)^2(M_r-x_r-y)\right]^2.$$

Here, the important point is that all the $\mathcal{O}$ do not depend on γ nor on $\delta\mathcal{E}$. After a tedious calculation, we get

$$\left(\frac{dy}{dt}\right)^2 = \delta\mathcal{E}\cdot\left[\frac{16r^4(1+r)}{(1-r)^5} + \mathcal{O}_r(y) - \delta\mathcal{E}\right] + \left[-\frac{64r^7(1+r)^2}{(1-r)^9} + \mathcal{O}_r(y)\right]\cdot y. \quad (4.5)$$

Now, assume that v_r is $H^{1/2}$-orbitally stable. Take $\varepsilon > 0$ to be adjusted, and the corresponding $\eta > 0$. Choose $\gamma_* > 0$ small so that $\forall \gamma \in (0, \gamma_*)$, we have $\|u_0^\gamma - v_r\|_{H^{1/2}} \leq \eta$, and consider u the corresponding solution. With the above notations for u, observe that for any $(\theta, \alpha) \in \mathbb{T}^2$, and $t \in \mathbb{R}$,

$$\begin{aligned}\|u(t,z) - e^{-i\theta} v_r(ze^{-i\alpha})\|_{H^{1/2}} &\geq \left|\left(u(t,z) - e^{-i\theta} v_r(ze^{-i\alpha}) \middle| z\right)\right| \\ &= |c(t) - e^{-i(\theta+\alpha)}\sqrt{r}| \\ &\geq \left||c(t)| - \sqrt{r}\right| = \frac{1}{\sqrt{M_r}} |y(t)|\end{aligned}$$

so $\forall t \in \mathbb{R}$,

$$|y(t)| \leq \sqrt{M_r} \cdot \inf_{(\theta,\alpha)\in\mathbb{T}^2} \|u(t) - e^{-i\theta} v_r(ze^{-i\alpha})\|_{H^{1/2}} \leq \sqrt{M_r} \cdot \varepsilon$$

because of the stability. Going back to (4.5), taking ε small enough (in a way that only depends on r), and shrinking γ_* if needed to lessen $\delta\mathcal{E}$, we get

$$\begin{aligned}\frac{16r^4(1+r)}{(1-r)^5} + \mathcal{O}_r(y) - \delta\mathcal{E} &\geq a_r, \\ -\frac{64r^7(1+r)^2}{(1-r)^9} + \mathcal{O}_r(y) &\leq -b_r,\end{aligned}$$

for all times $t \in \mathbb{R}$, where $a_r, b_r > 0$ are some constants only depending on r.

At time $t = 0$, we have $y = 0$, so

$$\left(\frac{dy}{dt}|_{t=0}\right)^2 \geq a_r \delta\mathcal{E} > 0.$$

We suppose that $dy/dt < 0$ at time $t = 0$ (it suffices to argue on negative times if the converse is true). Then $y < 0$ in some maximal time interval $(0, T)$. Thus (4.5) yields

$$\left(\frac{dy}{dt}\right)^2 \geq a_r \delta\mathcal{E} + b_r |y| \geq a_r \delta\mathcal{E} \quad \text{on } (0, T),$$

and since y is C^∞, we have

$$\frac{dy}{dt} < -\sqrt{a_r \delta\mathcal{E}} \quad \text{on } (0, T).$$

This yields $T = +\infty$ and $\forall t \in \mathbb{R}_+$, $y(t) \leq -t\sqrt{a_r \delta\mathcal{E}}$, which cannot happen. This ends the proof of the instability of v_r. $\square$

Let us now prove the instability of the traveling waves of higher degree.

Proof of Proposition 1.3. In Lemma 4.1, we have proved the existence of some $\varepsilon_0 > 0$, and constructed initial data u_0^γ arbitrary close to v_r in $H^{1/2}_+$, such that the corresponding solution u satisfies $\|u(T,z) - e^{-i\theta} v_r(ze^{-i\alpha})\|_{H^{1/2}} \geq \varepsilon_0 > 0$ for some $T \in \mathbb{R}$, and all $(\theta, \alpha) \in \mathbb{T}^2$. If now $N \geq 2$, Proposition 3.5 precisely states that the solution of (1.1) starting from $u_0^\gamma(z^N)$ is $u(t, z^N)$. On the one hand, we have

$$\|u_0^\gamma(z^N) - v_r(z^N)\|_{H^{1/2}} = \|u_0^\gamma(z) - v_r(z)\|_{H^{1/2}} \ll 1,$$

since u_0^γ only differs from v_r by its constant term. On the other hand, for all $(\theta, \alpha) \in \mathbb{T}^2$,

$$\|u(T, z^N) - e^{-i\theta} v_r\left((ze^{-i\alpha})^N\right)\|_{H^{1/2}} \geq \|u(T,z) - e^{-i\theta} v_r(ze^{-iN\alpha})\|_{H^{1/2}} \geq \varepsilon_0,$$

where we used the elementary fact that for any $w \in H^{1/2}_+$,

$$\|w(z^N)\|^2_{H^{1/2}} = \sum_{k\geq 0}(1+kN)|\hat{w}(k)|^2 \geq \sum_{k\geq 0}(1+k)|\hat{w}(k)|^2 = \|w\|^2_{H^{1/2}}.$$

This concludes our proof. □

A About equilibrium points

The purpose of this section is to discuss the case of the steady solutions of (1.1). Notice first that if for some $u \in H^{1/2}_+$, $2J\Pi(|u|^2) + \bar{J}u^2 = 0$, then taking the scalar product with u, we find $3|J|^2 = 0$, hence $E = 0$. Hence all steady solutions are issued from null-energy functions. Describing the set of steady solutions, say, in $H^{1/2}_+$ amounts to giving a characterization of the set $\{J = 0\}$.

As an illustration of how tough this may be, we are now going to give an explicit description of the subset of the steady solutions that are also in $\mathcal{V}(3)$ thanks to the inverse transform of Gérard and Grellier [21].

Proposition A.1. *The subset of $\mathcal{V}(3)$ consisting in steady solutions of (1.1), called $\mathcal{V}_0(3)$, is a submanifold of real codimension 2 given by*

$$\mathcal{V}_0(3) = \left\{ \lambda e^{ia}\left(-\frac{2\sqrt{3}}{3}\sin\theta + \frac{\frac{(1+2\cos 2\theta)^2}{3\sqrt{9+2\cos 2\theta - 2\cos 4\theta}} z e^{ib}}{1 - \frac{4(2+\cos 2\theta)\sin\theta}{\sqrt{3}\sqrt{9+2\cos 2\theta - 2\cos 4\theta}} z e^{ib}} \right) \;\middle|\; \lambda \in \mathbb{R},\ a, b \in \mathbb{T}, \theta \in [0, \tfrac{\pi}{3}) \right\}.$$

Proof. Let us fix u a function of $\mathcal{V}(3)$ such that $J(u) = 0$. We suppose that u is not identically zero, and rescale it so that $Q(u) = 1$. Denote by σ^2 the unique non-zero eigenvalue of K_u^2. We have $\sigma^2 = \operatorname{Tr} K_u^2 = M$. Moreover, it is an elementary fact (see [18]) that eigenvalues of H_u^2 and K_u^2 are interlaced. As $\operatorname{rk} H_u^2 = 2$, we denote by ρ_1^2, ρ_2^2 its eigenvalues, in such a way that

$$\rho_1^2 \geq \sigma^2 \geq \rho_2^2 \geq 0. \tag{A.1}$$

In addition, we have $\rho_1^2 + \rho_2^2 = \operatorname{Tr} H_u^2 = Q + M$, hence

$$\rho_1^2 - \sigma^2 + \rho_2^2 = Q = 1. \tag{A.2}$$

The case when $\rho_1 = \rho_2$ can be treated first, because it corresponds to a Blaschke product of degree 1. In that case, u should be of the form

$$u(z) = e^{ia}\frac{z - \bar{p}}{1 - pz}, \quad \forall z \in \mathbb{D},$$

for some $p \in \mathbb{D}$ and $a \in \mathbb{T}$. However, for such a function, $J(u) = \int_{\mathbb{T}} |u|^2 u = (u|1) = -e^{ia}\bar{p}$. Therefore, $p = 0$ and $u(z) = e^{ia}z$. From now on, we assume that $\rho_1 > \rho_2$ in (A.1).

Let us now briefly recall some notations. We denote by $E_u(\rho_j)$ the eigenspace of H_u^2 associated to the eigenvalue ρ_j^2, $j \in \{1, 2\}$, and we define u_j to be the orthogonal projection of u onto $E_u(\rho_j)$. Since $u = H_u(1) \in \operatorname{Ran} H_u = \operatorname{Ran} H_u^2$, we have $u = u_1 + u_2$ with $u_1 \perp u_2$. We can also characterize the action of H_u on u_j, noting that $\dim E_u(\rho_j^2) = 1$: there exists $\varphi_1, \varphi_2 \in \mathbb{T}$ such that

$$\begin{cases} H_u(u_1) = \rho_1 e^{i\varphi_1} u_1, \\ H_u(u_2) = \rho_2 e^{i\varphi_2} u_2. \end{cases}$$

Last of all, it is possible to compute the L^2 norm of the u_j's in terms of ρ_1^2, σ^2, ρ_2^2 (see *e.g.* [68]) :

$$\|u_1\|^2 = \rho_1^2 \frac{\rho_1^2 - \sigma^2}{\rho_1^2 - \rho_2^2}, \quad \|u_2\|^2 = \rho_2^2 \frac{\sigma^2 - \rho_2^2}{\rho_1^2 - \rho_2^2}.$$

Using the decomposition $u = u_1 + u_2$, we get

$$0 = J(u) = (u|H_u(u)) = \rho_1 e^{-i\varphi_1}\|u_1\|^2 + \rho_2 e^{-i\varphi_2}\|u_2\|^2.$$

Up to a rotation of u, we can assume that $\varphi_1 = 0$. As $\|u_2\|^2 \neq 0$, $e^{i\varphi_2}$ must then be real, so necessarily, $\varphi_2 = \pi$. Now using the expression of $\|u_1\|^2$, $\|u_2\|^2$, and multiplying by $\rho_1^2 - \rho_2^2$, we get

$$\rho_1^3(\rho_1^2 - \sigma^2) - \rho_2^3(\sigma^2 - \rho_2^2) = 0.$$

By (A.2), $\rho_1^2 - \sigma^2 = 1 - \rho_2^2$ and $\sigma^2 - \rho_2^2 = \rho_1^2 - 1$. Hence $\rho_1^3 + \rho_2^3 = \rho_1^2\rho_2^2(\rho_1 + \rho_2)$, or rather

$$\rho_1^2 - \rho_1\rho_2 + \rho_2^2 = \rho_1^2\rho_2^2.$$

This equation means that $x := \frac{1}{\rho_1}$ and $y := \frac{1}{\rho_2}$ belong to the ellipse of equation $x^2 - xy + y^2 = 1$. A standard reduction procedure indicates that all the points of this ellipse can be written as $(x, y) = \frac{2}{\sqrt{3}}(\cos(\theta + \frac{\pi}{6}), \cos(\theta - \frac{\pi}{6}))$, for some $\theta \in \mathbb{R}$. However, we must have $y > x$ and $x, y > 0$, which imposes $\theta \in (0, \frac{\pi}{3})$.

The matrix involved in the inverse spectral formula reads

$$\mathscr{C}(z) := \begin{pmatrix} \dfrac{\rho_1 - \sigma e^{i\psi}z}{\rho_1^2 - \sigma^2} & \dfrac{1}{\rho_1} \\ \dfrac{\rho_2 + \sigma e^{i\psi}z}{\sigma^2 - \rho_2^2} & -\dfrac{1}{\rho_2} \end{pmatrix}, \quad z \in \mathbb{D}.$$

A translation in z enables to reduce ourselves to the case when the angle associated to $\sigma^2/2$, called ψ, is zero. Then we can reconstruct

$$u(z) = \left\langle \mathscr{C}(z)^{-1}\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle_{\mathbb{C}^2} = \frac{1-\rho_1\rho_2}{\rho_1-\rho_2} - \frac{z}{\frac{\sigma(\rho_1-\rho_2)^2}{1-\rho_1\rho_2} + (1+\rho_1\rho_2)(\rho_1-\rho_2)z},$$

and $\sigma = \sqrt{\rho_1^2 + \rho_2^2 - 1}$, which leads to the claim, by means of a few trigonometric identities. □

Example. The function

$$u(z) = -\frac{\sqrt{3}}{3} + \frac{\frac{4}{3\sqrt{11}}z}{1 - \frac{5}{\sqrt{33}}z}$$

corresponds to an equilibrium point of (1.1).

B Formulae for solutions in $\mathcal{V}(3)$

We summarize below some useful formulae from [62, Section 4] for solutions of (1.1) of the form

$$u(z) = b + \frac{cz}{1 - pz}, \quad \forall z \in \mathbb{D},$$

where $b, c, p \in \mathbb{C}$, and $|p| < 1$.

We have

$$
Q = |b|^2 + \frac{|c|^2}{1-|p|^2},
$$

$$
M = \frac{|c|^2}{(1-|p|^2)^2},
$$

$$
J = |b|^2 b + \frac{2b|c|^2}{1-|p|^2} + \frac{|c|^2 c\bar{p}}{(1-|p|^2)^2}.
$$

As far as the evolution of u is concerned, (1.1) translates into

$$
\begin{cases}
i\dot{p} = c\bar{J}, \\
i\dot{c} = 2bc\bar{J} + 2\bar{b}cJ + \dfrac{2Jp|c|^2}{1-|p|^2}, \\
i\dot{b} = b^2\bar{J} + 2|b|^2 J + \dfrac{2J|c|^2}{1-|p|^2}.
\end{cases}
$$

Chapitre 5

About the quadratic Szegő hierarchy

Abstract

The purpose of this paper is to go further into the study of the quadratic Szegő equation, which is the following Hamiltonian PDE :

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2, \quad u(0,\cdot) = u_0,$$

where Π is the Szegő projector onto nonnegative modes, and $J = J(u)$ is the complex number given by $J = \int_{\mathbb{T}} |u|^2 u$. We exhibit an infinite set of new conservation laws $\{\ell_k\}$ which are in involution. These laws give us a better understanding of the "turbulent" behavior of certain rational solutions of the equation : we show that if the orbit of a rational solution is unbounded in some H^s, $s > \frac{1}{2}$, then one of the ℓ_k's must be zero. As a consequence, we characterize growing solutions which can be written as the sum of two solitons.

1 Introduction

The equation and its Hamiltonian structure. In this paper, we consider the following quadratic Szegő equation on the torus $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})$:

$$i\partial_t u = 2J\Pi(|u|^2) + \bar{J}u^2, \tag{1.1}$$

where $u : (t,x) \in \mathbb{R} \times \mathbb{T} \mapsto u(t) \in \mathbb{C}$, J is a complex number depending on u and given by $J(u) := \int_{\mathbb{T}} |u|^2 u$, and Π is the Szegő projector onto functions with only nonnegative modes :

$$\Pi\left(\sum_{k\in\mathbb{Z}} a_k e^{ikx}\right) = \sum_{k=0}^{+\infty} a_k e^{ikx}.$$

In particular, Π acts on $L^2(\mathbb{T})$ equipped with its usual inner product $(f|g) := \int_{\mathbb{T}} f\bar{g}$. We call $L^2_+(\mathbb{T})$ the closed subspace of L^2 which is made of square-integrable functions on $\mathbb{T}$ whose Fourier series is supported on nonnegative frequencies. Then Π induces the orthogonal projection from L^2 onto L^2_+. In the sequel, if G is a subspace of L^2, we denote by G_+ the subspace $G \cap L^2_+$ of L^2_+.

Equation (1.1) appears to be a Hamiltonian PDE : consider L^2_+ as the phase space, endowed with the standard symplectic structure given by $\omega(h_1, h_2) := \operatorname{Im}(h_1|h_2)$. The Hamiltonian associated to (1.1) is then the following functional :

$$\mathcal{H}(u) := \frac{1}{2}|J(u)|^2 = \left|\int_{\mathbb{T}} |u|^2 u\right|^2.$$

Indeed, if u is regular enough (say $u \in L^4_+$), and $h \in L^2_+$, we see that $\langle d\mathcal{H}(u), h\rangle = \operatorname{Re}(2J|u|^2 + \bar{J}u^2|h) = \omega(h|X_{\mathcal{H}}(u))$, where $X_{\mathcal{H}}(u) := -2iJ\Pi(|u|^2) - i\bar{J}u^2 \in L^2_+$ is called the symplectic gradient of $\mathcal{H}$. Equation (1.1) can be restated as

$$\dot{u} = X_{\mathcal{H}}(u), \tag{1.2}$$

where the dot stands for a time-derivative : in other words, (1.1) is the flow of the vector field $X_{\mathcal{H}}$.

If now $\mathcal{F}$ is some densely-defined differentiable functional on L^2_+, and if u is a smooth solution of (1.2), then

$$\frac{d}{dt}\mathcal{F}(u) = \langle d\mathcal{F}(u), \dot{u}\rangle = \omega(\dot{u}, X_{\mathcal{F}}(u)) = \omega(X_{\mathcal{H}}(u), X_{\mathcal{F}}(u)).$$

Defining the Poisson bracket $\{\mathcal{H}, \mathcal{F}\}$ to be the functional given by $\{\mathcal{H}, \mathcal{F}\} = \omega(X_{\mathcal{H}}, X_{\mathcal{F}})$, the evolution of $\mathcal{F}$ along flow lines of (1.2) is thus given by the equation $\dot{\mathcal{F}} = \{\mathcal{H}, \mathcal{F}\}$. In particular, $\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} = 0$, which means that the Hamiltonian $\mathcal{H}$ is conserved (at least for smooth solutions). Hence the factor J in (1.1) only evolves through its argument, explaining the terminology of "quadratic equation". Two other conservation laws arise from the invariances of $\mathcal{H}$: the mass Q and the momentum M defined by

$$Q(u) := \int_{\mathbb{T}} |u|^2,$$
$$M(u) := \int_{\mathbb{T}} \bar{u}Du, \quad D := -i\partial_x.$$

We have $\{\mathcal{H}, Q\} = \{\mathcal{H}, M\} = 0$. Moreover, as u only has nonnegative modes, these conservation laws control the $H^{1/2}$ regularity of u, namely $(Q + M)(u) \simeq \|u\|^2_{H^{1/2}}$.

Observe that replacing the variable $e^{ix} \in \mathbb{T}$ by $z \in \mathbb{D}$ in the Fourier series induces an isometry between L^2_+ and the Hardy space $\mathbb{H}^2(\mathbb{D})$, which is the set of holomorphic functions on the unit open disc $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ whose trace on the boundary $\partial\mathbb{D}$ lies in L^2. Therefore, we will often consider solutions of equation (1.1) as functions of $t \in \mathbb{R}$ and $z \in \mathbb{D}$.

Invariant manifolds. Equation (1.1) was first introduced in [62], following the seminal work of Gérard and Grellier on the cubic Szegő equation [16, 18, 20, 21] :

$$i\partial_t u = \Pi(|u|^2 u). \tag{1.3}$$

As in the case of (1.3), equation (1.1) can be considered as a toy model of a nonlinear non-dispersive equation, whose study is expected to give hints on physically more relevant equations, such as the conformal flow on $\mathbb{S}^3$ [4] or the cubic Lowest-Landau-Level (LLL) equation [15].

In [62, 64], using the formalism of equation (1.3), the author proved that there is a Lax pair structure associated to the quadratic equation (1.1), that we are now going to recall, as well as some of its consequences.

If $u \in H^{1/2}_+(\mathbb{T})$, we define the Hankel operator of symbol u by $H_u : L^2_+ \to L^2_+$, $h \mapsto \Pi(u\bar{h})$. It is a bounded $\mathbb{C}$-antilinear operator over L^2_+, and H_u^2 is trace class (hence compact), selfadjoint and positive. Consequently, we can write its spectrum as a decreasing sequence of nonnegative eigenvalues

$$\rho_1^2(u) \geq \rho_2^2(u) \geq \cdots \geq \rho_n^2(u) \geq \cdots \longrightarrow 0,$$

with possible repetitions according to their multiplicity. Analogous to the family of Hankel operators is the one of Toeplitz operators : given a symbol $b \in L^\infty(\mathbb{T})$, we define $T_b : L^2_+ \to L^2_+$, $h \mapsto \Pi(bh)$, which is $\mathbb{C}$-linear and bounded on L^2_+. Its adjoint is $(T_b)^* = T_{\bar{b}}$. A special Toeplitz operator is called the (right) shift $S := T_{e^{ix}}$. Thus we can define another operator which turns out to be of great importance in the study of (1.1) : for $u \in H^{1/2}_+$, the shifted Hankel operator is defined by $K_u := H_u S$. An easy computation shows that K_u also satisfies $K_u = S^* H_u = H_{S^* u}$. As a consequence,

$$K_u^2(h) = H_u^2(h) - (h|u)u, \quad \forall h \in L^2_+. \tag{1.4}$$

K_u^2 is compact as well, selfadjoint and positive, hence we can denote its eigenvalues by the decreasing sequence $\sigma_1^2(u) \geq \cdots \geq \sigma_n^2(u) \geq \cdots \to 0$. In fact, (1.4) leads to a more accurate interlacement property :

$$\rho_1^2(u) \geq \sigma_1^2(u) \geq \rho_2^2(u) \geq \sigma_2^2(u) \geq \cdots \geq \rho_n^2(u) \geq \sigma_n^2(u) \geq \cdots \longrightarrow 0. \tag{1.5}$$

The idea of a Lax pair is to look at the evolution of a solution $t \mapsto u(t)$ of (1.1) by associating to each $u(t)$ an operator $L_{u(t)}$ acting on some Hilbert space, and by computing the evolution of this operator rather than that of the function $u(t)$ itself. First of all, thanks to the conservation laws $\mathcal{H}$, Q and M, it can be shown that the flow of (1.1) is well defined on every $H^s_+(\mathbb{T})$ for $s > \frac{1}{2}$ [62]. We refer to solutions belonging to theses spaces as *smooth solutions*. The statement of the Lax pair theorem is then the following :

Theorem 8 ([62, 64])**.** *Let $t \mapsto u(t)$ be a smooth solution of the quadratic Szegő equation (1.1). Then the evolution of $H_{u(t)}$ and $K_{u(t)}$ is given by*

$$\frac{d}{dt}K_u = B_u K_u - K_u B_u,$$
$$\frac{d}{dt}H_u = B_u H_u - H_u B_u + i\bar{J}(u|\cdot)u,$$

where $B_u := -i(T_{\bar{J}u} + T_{J\bar{u}})$ is a bounded anti-selfadjoint operator over L^2_+.

Only the first identity concerning K_u is a rigorous Lax pair, but the second one turns out to give helpful informations about H_u as well [64]. In particular, we have the following corollary :

Corollary 1.1. *If $t \mapsto u(t)$ is a smooth solution of (1.1), then $\mathrm{rk}(K_u)$ and $\mathrm{rk}(H_u)$ are conserved. For any $j \geq 1$, $\sigma_j^2(u(t))$ is also conserved.*

This corollary is of particular interest when H_u has finite rank. In that case, since $K_u = H_u S$, we have $\mathrm{rk}(K_u) \leq \mathrm{rk}(H_u)$. Because $\mathrm{rk}(H_u) = \mathrm{rk}(H_u^2)$ and the same for K_u and K_u^2, we must have by (1.4) that $\mathrm{rk}(K_u) \in \{\mathrm{rk}(H_u), \mathrm{rk}(H_u) - 1\}$. Therefore, for $d \in \mathbb{N}$, we designate by $\mathcal{V}(d)$ the set of symbols $u \in H_+^{1/2}$ such that $\mathrm{rk}(H_u) + \mathrm{rk}(K_u) = d$.

It turns out that $\mathcal{V}(d)$ can be explicitely characterized (see [16]) : it is the set of rational functions of the variable z of the form

$$u(z) = \frac{A(z)}{B(z)},$$

where A and B are complex polynomials, such that $A \wedge B = 1$, $B(0) = 1$ and B has no root in the closed disc $\overline{\mathbb{D}}$, and such that

- (case $d = 2N$) the degree of B is exactly N and the degree of A is at most $N - 1$,
- (case $d = 2N + 1$) the degree of A is exactly N and the degree of B is at most N.

Since functions of $\mathcal{V}(d)$ obviously belong to C_+^∞, they give rise to smooth solutions of (1.1), and by the previous corollary, $\mathcal{V}(d)$ is left invariant by the flow of the quadratic Szegő equation.

Geometrically speaking, $\mathcal{V}(d)$ is a complex manifold of dimension d. Moreover, restricting the scalar product $(\cdot|\cdot)$ to the tangent space $T_u\mathcal{V}(d)$ for each $u \in \mathcal{V}(d)$ defines a Hermitian metric on $\mathcal{V}(d)$ whose imaginary part induces a symplectic structure on the $2d$-dimensional real manifold $\mathcal{V}(d)$. In other words, $\mathcal{V}(d)$ is a Kähler manifold.

Additional conservation laws. It is a natural question to ask whether the finite-dimensional ODE induced by (1.1) on $\mathcal{V}(d)$ is integrable or not, in the sense of the classical Hamiltonian mechanics. The celebrated Arnold-Jost-Liouville-Mineur theorem [3, 36, 46, 48] states that this problem first consists in finding d conservation laws (for a $2d$-dimensional real manifold) that are generically independent and in involution (*i.e.* such that $\{F, G\} = 0$ for any choice of F, G among these laws).

We denote by Ω_{gen} the set of $u \in H_+^{1/2}$ such that $u \in \mathcal{V}(d)$ for some $d \in \mathbb{N}$, and such that all the eigenvalues of H_u^2 and K_u^2 are simple (in that case, inequalities in (1.5) are strict inequalities). Thanks to the inverse spectral transform of [21], it appears that Ω_{gen} is a dense connected subset of $H_+^{1/2}$.

In the case of the cubic Szegő equation (1.3), the $\mathcal{V}(d)$'s are also invariant by the flow, and such conservation laws were first found in [16]. Relying on the fact that H_u and K_u satisfy an exact Lax pair, it can be proved [18] that both the ρ_j^2's and the σ_k^2's are generically independent conservation laws for solutions of (1.3), and they satisfy in addition

$$\{\rho_j^2, \rho_k^2\} = 0, \quad \{\sigma_j^2, \sigma_k^2\} = 0, \quad \{\rho_j^2, \sigma_k^2\} = 0,$$

on $\Omega_{\mathrm{gen}} \cap \mathcal{V}(d)$, for any choice of indices $j, k \geq 1$. (Note that on Ω_{gen}, the mappings $u \mapsto \rho_j^2(u)$ and $u \mapsto \sigma_k^2(u)$ are differentiable, so the Poisson brackets are well defined.)

In our case, Corollary 1.1 states that the σ_k^2's are conservation laws for (1.1). But the ρ_j^2's are no more conserved, that is why the Lax pair theorem only provides $\lfloor d/2 \rfloor$ conservation laws on $\mathcal{V}(d)$. The purpose of this paper is then to investigate and find the missing ones, to get the full quadratic Szegő hierarchy.

We can now state the main theorem of this paper. Let $u \in H_+^{1/2}$, and recall that

$$\sigma_1^2(u) \geq \sigma_2^2(u) \geq \cdots \geq \sigma_k^2(u) \geq \cdots$$

is the decreasing list of the eigenvalues of K_u^2. For $k \geq 1$, we set $F_u(\sigma_k(u)) := \ker(K_u^2 - \sigma_k^2(u)I)$, and we introduce $P^k := \mathbb{1}_{\{\sigma_k^2(u)\}}(K_u^2)$ in the sense of the functional calculus. In other terms, P^k is the orthogonal projection onto the subspace $F_u(\sigma_k)$ of L^2_+. We give a special name to the projections of u and $\Pi(|u|^2)$ on that space :

$$u_k^K := P^k(u),$$
$$w_k^K := P^k(\Pi(|u|^2)).$$

Finally, we set

$$\ell_k(u) := \frac{1}{\operatorname{Tr} P^k} \left[(2Q + \sigma_k^2)\|u_k^K\|^2_{L^2} - \|w_k^K\|^2_{L^2}\right],$$

where the trace factor $\operatorname{Tr} P^k$ equals the dimension $F_u(\sigma_k)$. By convention, we also call ℓ_∞ the quantity that we obtain by replacing σ_k^2 by $\sigma_\infty^2 := 0$ and $\operatorname{Tr} P^k$ by $\operatorname{Tr} P^\infty := 1$ in the above functional (thus considering the projection of u and $\Pi(|u|^2)$ onto the kernel of K_u^2).

The main result reads as follows :

Theorem 9. *We have the following identities on Ω_{gen} :*

$$\{\ell_j, \ell_k\} = 0, \quad \{\ell_j, \sigma_k^2\} = 0, \quad \{\sigma_j^2, \sigma_k^2\} = 0,$$

for any $1 \leq j, k \leq \infty$.

Furthermore, the ℓ_k's are conservation laws for the quadratic Szegő equation (1.1).

Let us comment on this result :

- The question of finding additional conservation laws was first raised in [16] for the cubic Szegő equation on $\mathbb{T}$, at a time when the Lax pair for K_u and the conservation laws σ_k^2 had not been discovered. These laws were found to be the $J_{2n}(u) := (H_u^{2n}(1)|1)$, $n \geq 1$. A similar inquiry turned out to be necessary in the study of related equations for which only one Lax pair is available, such as the cubic Szegő equation on $\mathbb{R}$ [53], or the cubic Szegő equation with a linear perturbative term on $\mathbb{T}$ [67, 68].

- We will prove in the beginning of Section 4 that the knowledge of the ℓ_k's and the σ_k^2's enables to reconstruct the a priori conservation laws M, Q and $\mathcal{H}$. We have for instance

$$Q^2 = \sum_{1 \leq k \leq \infty} \ell_k,$$
$$|J|^2 = \sum_{1 \leq k \leq \infty} (Q + \sigma_k^2)\ell_k,$$

  explaining why we need the trace factor $(\operatorname{Tr} P^k)^{-1}$ in the expression of ℓ_k, in the case of multiple singular values.

  However, the question of the generic independence of the ℓ_k's is left unanswered.

- The proof of Theorem 9 relies on generating series. For $u \in \Omega_{\text{gen}}$ (thus corresponding to a finite sequence of non-zero ℓ_k's), and an appropriate $x \in \mathbb{R}$, we will show that

$$\sum_{k=1}^{\infty} \frac{\ell_k}{1 - x\sigma_k^2} = \frac{x^2 \mathscr{J}^{(4)}(x)^2 - x|\mathscr{J}^{(3)}(x)|^2 - Q^2}{\mathscr{J}^{(0)}(x)},$$

where for $m \geq 0$,

$$\mathscr{J}^{(m)}(x) := ((I - xH_u^2)^{-1}(H_u^m(1))|1) = \sum_{j=0}^{+\infty} x^j J_{m+2j},$$

and $J_p := (H_u^p(1)|1)$ as above. Using the commutation relations between ρ_j^2 and σ_k^2 as well as the action-angle coordinates coming from the cubic Szegő equation [21], we will find that

$$\left\{ \sum_{k=1}^{\infty} \frac{\ell_k}{1 - x\sigma_k^2}, \sum_{k=1}^{\infty} \frac{\ell_k}{1 - y\sigma_k^2} \right\} = 0,$$

for all $x \neq y$.

Connection with the growth of Sobolev norms for rational solutions. An important question in the study of Hamiltonian PDEs is the question of the existence of "turbulent" trajectories : provided that M and Q are conserved, does there exist initial data $u_0 \in C_+^\infty$ giving rise to solutions of (1.1) such that

$$\limsup_{t \to +\infty} \|u(t)\|_{H^s} = +\infty$$

for some $s > \frac{1}{2}$?

A positive answer to this question is given in [62], where however it is shown that such a growth cannot happen faster than exponentially in time. An explicit computation tells us that this rate of growth is indeed achieved for solutions on $\mathcal{V}(3)$ satisfying the following condition :

$$|J|^2 = Q^3. \tag{1.6}$$

More precisely, solutions of the form

$$u(z) = b + \frac{cz}{1 - pz},$$

with $b, c, p \in \mathbb{C}$, $c \neq 0$, $b - cp \neq 0$ and $|p| < 1$, which also satisfy (1.6), are such that for any $s > 1/2$, there exists a constant $C_s > 0$ such that $\|u(t)\|_{H^s} \sim C_s e^{C_s|t|}$.

As in [68], it appears that the possible growth of Sobolev norms can be detected in terms of the new conservation laws ℓ_k.

Proposition 1.2. *Let v^n be some sequence in $\mathcal{V}(d)$ for some $d \in \mathbb{N}$. Assume that it is bounded in $H_+^{1/2}$ and that $\operatorname{Sp} K_{v^n}^2$ does not depend on n. Then the following statements are equivalent :*

(i) There exists $s_0 > \frac{1}{2}$ such that v^n is unbounded in $H_+^{s_0}$.

(ii) For every $s > \frac{1}{2}$, v^n is unbounded in H_+^s.

(iii) There exists a subsequence $\{n_k\}$ and $v_{\text{bad}} \in \mathcal{V}(d')$ (where $d' \leq d - 1$ if d is even, and $d' \leq d - 2$ if d is odd), such that

$$v^{n_k} \rightharpoonup v_{\text{bad}} \quad \text{in } H_+^{1/2}.$$

In the case of solutions of the quadratic Szegő equation (1.1), this proposition induces a necessary condition on the initial data for some growth of Sobolev norm to occur.

Corollary 1.3. *Assume that $u_0 \in \mathcal{V}(d)$ for some $d \in \mathbb{N}$, and assume that there exists $s_0 > \frac{1}{2}$ such that the corresponding solution $u(t)$ of (1.1) is unbounded in $H_+^{s_0}$. Then $u(t)$ is unbounded in every H^s, $s > \frac{1}{2}$. Furthermore, for some $k \geq 1$ such that σ_k^2 is the k-th non-zero eigenvalue of $K_{u_0}^2$, we must have*

$$\ell_k(u_0) = 0.$$

Remark 17. The proof of Proposition 1.2 relies on a connection between growth of Sobolev norms and loss of compactness, quantified by equipping $H_+^{1/2}$ with the weak topology and studying the cluster points of the strongly bounded sequence u^n. This idea can be illustrated by the following basic example. Pick some ℓ^2 sequence of positive numbers (a_k), and consider the periodic functions defined by

$$f_n(x) := \sum_{k=0}^{+\infty} a_k e^{i(kn)x}, \quad n \in \mathbb{N},\ x \in \mathbb{T}.$$

Then the sequence f_n is uniformly bounded in L^2, but the L^2-energy of f_n obviously moves toward high frequencies (or equivalently, the H^s norm of f_n, $s > 0$, is morally going to grow like n^s). This phenomenon can be described saying that the only weak cluster point of the sequence f_n in L^2 is a_0, and that $|a_0|^2 < \|f_n\|_{L^2}^2$. This energy loss through high-frequency energy transfer is precisely what is captured by Proposition 1.2.

This result enables to find the right counterpart of condition (1.6) for solutions in $\mathcal{V}(4)$:

Theorem 10. *A solution $t \mapsto u(t)$ of (1.1) in $\mathcal{V}(4)$ is unbounded in some H^s, $s > \frac{1}{2}$, if and only if $K_{u(t)}^2$ has two distinct eigenvalues of multiplicity 1, $\sigma_1^2 > \sigma_2^2$, and if $\ell_1(u) = 0$ or equivalently*

$$|J|^2 = Q^2(Q + \sigma_2^2). \tag{1.7}$$

In that case, for all $s > 1/2$, there exists constants $C_s, C_s' > 0$ such that we have

$$\frac{1}{C_s} e^{C_s'|t|} \leq \|u(t)\|_{H^s} \leq C_s e^{C_s'|t|}, \quad \text{as } t \to \pm\infty.$$

Example. A concrete example of a function of $\mathcal{V}(4)$ satisfying (1.7) is given by

$$v(z) := \frac{z}{(1 - pz)^2}, \quad \forall z \in \mathbb{D},$$

whenever $|p|^2 = 3\sqrt{2} - 4 \simeq 0,2426\ldots$

Remark 18. The interest of Theorem 10 is to display the case of an interaction between two solitons. Traveling waves for equation (1.1) are classified in [64], and this turbulent solution u appears to be the exact sum of two solitons. Indeed, a turbulent solution such as the one described above will be after some time T of the form

$$u(t, z) = \frac{\alpha}{1 - pz} + \frac{\beta}{1 - qz},$$

with $\alpha, \beta, p, q \in \mathbb{C}$, with $|p|, |q| < 1$ and $p \neq q$. One of the two poles approaches the unit circle $\partial\mathbb{D}$ exponentially fast, while the other remains inside a disc of radius $r < 1$.

Open questions. The picture we draw here remains far from being complete. First of all, now that we have d conservation laws in involution on $\mathcal{V}(d)$, we would like to apply the Arnold-Liouville theorem. For that purpose, we should give a description of the level sets of the ℓ_k's and the σ_k^2's on $\mathcal{V}(d)$, and find which ones are compact in $H_+^{1/2}$. Obviously, some are not, since we found solutions in $\mathcal{V}(3)$ and $\mathcal{V}(4)$ that leave every compact set of $H^{1/2}$.

Then, to solve explicitely the quadratic Szegő equation (1.1) on $\mathcal{V}(d)$, we should find angle coordinates in $\mathbb{T}^d$ (for compact level sets) or in $\mathbb{T}^{d'} \times \mathbb{R}^{d-d'}$ (in the general case), for some $d' < d$. Angle coordinates for the cubic Szegő equation are found in [18] in the framework of the torus, and in [53] for the real line. For the case of action-angle coordinates for other integrable PDEs, one can refer to [27, 37]. It is noteworthy that the angle associated to σ_k^2 in the case of the cubic Szegő coordinates does not evolve linearly in time through the flow of the quadratic equation (1.1) (see Lemma 3.7 below, where we compute its evolution).

The exact situation on $\mathcal{V}(4)$ is not completely understood either. Whereas on $\mathcal{V}(3)$, only ℓ_1 can cancel out, corresponding to (1.6), and $\ell_\infty(u) > 0$ for all $u \in \mathcal{V}(3)$, it is not even clear whether ℓ_2 can be zero on $\mathcal{V}(4)$. In any case, Theorem 10 is enough to say that solutions of (1.1) on $\mathcal{V}(4)$ such that $\ell_2 = 0$, if any, are bounded in every H^s topology.

A broadly open question naturally concerns the case of the $\mathcal{V}(d)$'s for $d \geq 5$. By the substitution principle that is stated in [64, Proposition 3.5], replacing z by z^N, $N \geq 2$, in turbulent solutions of $\mathcal{V}(3)$ and $\mathcal{V}(4)$ will allow us to give examples of exponentially growing solutions on each of the $\mathcal{V}(d)$'s, $d \geq 5$. However, can we completely classify such growing solutions ? Is it possible to find other types or rates of growth, such as a polynomial one, or an intermittent one (*i.e.* a solution satisfying both $\limsup \|u(t)\|_{H^s} = \infty$ and $\liminf \|u(t)\|_{H^s} < \infty$) ?

Going from rational solutions to general data in $H_+^{1/2}$ is our long-term objective. We would like to understand, as in [21] for the cubic Szegő equation or in [31] for the resonant NLS, which is the generic behaviour of solutions of (1.1) on that space. To this end, it seems unlikely that we can get around the construction of action-angle variables.

Plan of the paper. After some preliminaries in Section 2 about the spectral theory of H_u and K_u, we will see in Section 3 how to prove simply that the ℓ_k's are conserved along the evolution of (1.1), and we prove that the cancellation of at least one ℓ_k is a necessary condition for growth of Sobolev norms to occur. In Section 4, we analyse the case of $\mathcal{V}(4)$. Section 5 is finally devoted to the proof of the Poisson-commutation of the ℓ_k's.

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2 Preliminaries : spectral theory of H_u and K_u

For the sake of completeness, we recall in this section some of the results of [21], where the spectral theory of compact Hankel operators is studied in great detail.

We begin with a definition :

Definition (Finite Blaschke products)**.** A function $\Psi \in L^2_+$ is called a Blaschke product of degree $m \geq 0$ if there exists $\psi \in \mathbb{T}$ as well as m complex numbers $a_j \in \mathbb{D}$, $j \in [\![1, m]\!]$, such that

$$\Psi(z) = e^{i\psi} \prod_{j=1}^{m} \frac{z - \overline{a_j}}{1 - a_j z}, \quad \forall z \in \mathbb{D}.$$

ψ is called the *angle* of Ψ, and $D(z) = \prod_{j=1}^{m}(1 - a_j z)$ is called the *normalized denominator* of Ψ (*i.e.* with $D(0) = 1$).

Observe that a Blaschke product of degree m belongs to $\mathcal{V}(2m+1)$, but more importantly, if Ψ is a Blaschke product, then $|\Psi(e^{ix})|^2 = 1$ for all $x \in \mathbb{T}$. In particular, $\Psi \in L^\infty_+$.

Singular values. Now, fix $u \in H^{1/2}_+$. For $s \geq 0$, we introduce two subspaces of L^2_+ defined by

$$E_u(s) := \ker(H_u^2 - s^2 I),$$
$$F_u(s) := \ker(K_u^2 - s^2 I).$$

We denote by Ξ_u^H (resp. Ξ_u^K) the set of $s > 0$ such that $E_u(s)$ (resp. $F_u(s)$) is not $\{0\}$. It is the set of the square-roots of the non-zero eigenvalues of H_u^2 (resp. K_u^2). We call them the *singular values* associated to u. The link between Ξ_u^H and Ξ_u^K can be described more precisely :

Proposition 2.1 ([21, Lemma 3.1.1])**.** *Let $s \in \Xi_u^H \cup \Xi_u^K$. Then one of the following holds :*

(i) $\dim E_u(s) = \dim F_u(s) + 1$*,* $u \not\perp E_u(s)$*, and* $F_u(s) = E_u(s) \cap u^\perp$ *;*

(ii) $\dim F_u(s) = \dim E_u(s) + 1$*,* $u \not\perp F_u(s)$*, and* $E_u(s) = F_u(s) \cap u^\perp$*.*

In the first case, we say that s is H-dominant, and we write $s \in \Sigma_u^H$.
In the second case, we say that s is K-dominant, and we write $s \in \Sigma_u^K$.
It also appears that, writing $\Xi_u^H \cup \Xi_u^K$ as a decreasing sequence (with no repetition), H-dominant singular values are given by the odd terms, and K-dominant by the even ones.

Projections. Let $\{\rho_j\}_{j\geq 1}$ (resp. $\{\sigma_k\}_{k\geq 1}$) be the decreasing list of the eigenvalues of H_u^2 (resp. K_u^2) with possible repetitions. We define

u_j^H	the projection of u onto $E_u(\rho_j)$
u_k^K	the projection of u onto $F_u(\sigma_k)$
w_k^K	the projection of $\Pi(\lvert u\rvert^2)$ onto $F_u(\sigma_k)$

The notation u_j^H should be read as "the projection of u onto the eigenspace of H_u^2 corresponding to its j-th eigenvalue".

By Proposition 2.1, $u_j^H \neq 0$ if and only if $\rho_j \in \Sigma_u^H$, and $u_k^K \neq 0$ if and only if $\sigma_k \in \Sigma_u^K$. In particular, decomposing u onto the sum

$$L^2_+ = \overline{\bigoplus_{\sigma \in \Xi_u^K} \ker(K_u^2 - \sigma^2 I) \oplus \ker(K_u^2)},$$

we find

$$u = \sum_{1 \leq k < \infty} \frac{u_k^K}{\operatorname{Tr} P^k} + u_\infty^K = \sum_{\substack{1 \leq k < \infty \\ \sigma_k \in \Sigma_u^K}} \frac{u_k^K}{\operatorname{Tr} P^k} + u_\infty^K, \tag{2.1}$$

where u_∞^K stands for the projection of u onto $\ker K_u^2$. A similar formula holds for u_j^H, but the extra ∞ term is no more needed, since $u \perp \ker H_u^2$.

These decompositions of u appear to be very useful, for we can describe how H_u and K_u act on $E_u(s)$ and $F_u(s)$, $s > 0$. This is what is summed up in the next proposition :

Proposition 2.2 ([21, Proposition 3.5.1])**.** • *If $s \in \Sigma_u^H$, write $s = \rho_j$ for some $j \geq 1$. Let $m = \dim E_u(\rho_j) = \dim F_u(\rho_j) + 1$. Then there exists Ψ_j^H, a Blaschke product of degree $m-1$, such that*

$$\rho_j u_j^H = \Psi_j^H H_u\left(u_j^H\right).$$

In addition, if D is the normalized denominator of Ψ_j^H, then

$$E_u(\rho_j) = \left\{ \frac{f}{D} H_u\left(u_j^H\right) \,\middle|\, f \in \mathbb{C}_{m-1}[z] \right\},$$
$$F_u(\rho_j) = \left\{ \frac{g}{D} H_u\left(u_j^H\right) \,\middle|\, g \in \mathbb{C}_{m-2}[z] \right\},$$

and H_u (resp. K_u) acts on $E_u(\rho_j)$ (resp. $F_u(\rho_j)$) by reversing the order of the coefficients of the polynomial f (resp. g), conjugating them, and multiplying the result by $\rho_j e^{i\psi_j}$, where ψ_j is the angle of Ψ_j^H.

- *If $s \in \Sigma_u^K$, write $s = \sigma_k$ for some $k \geq 1$. Let $m' = \dim F_u(\sigma_k) = \dim E_u(\sigma_k) + 1$. Then there exists Ψ_k^K, a Blaschke product of degree $m'-1$, such that*

$$K_u\left(u_k^K\right) = \sigma_k \Psi_k^K u_k^K.$$

In addition, if D is the normalized denominator of Ψ_k^K, then

$$F_u(\sigma_k) = \left\{ \frac{f}{D} u_k^K \,\middle|\, f \in \mathbb{C}_{m'-1}[z] \right\},$$
$$E_u(\sigma_k) = \left\{ \frac{zg}{D} u_k^K \,\middle|\, g \in \mathbb{C}_{m'-2}[z] \right\},$$

and K_u (resp. H_u) acts on $F_u(\sigma_k)$ (resp. $E_u(\sigma_k)$) by reversing the order of the coefficients of the polynomial f (resp. g), conjugating them, and multiplying the result by $\sigma_k e^{i\psi_k}$, where ψ_k is the angle of Ψ_k^K.

We also recall a formula which enables to compute $\|u_k^K\|_{L^2}^2$ and $\|u_j^H\|_{L^2}^2$ in terms of the singular values. For $s \in \Sigma_u^H$, we call $\sigma(s)$ the biggest element of Σ_u^K which is smaller than s, if it exists, or 0 otherwise. With this notation, we have the following formulae :

Proposition 2.3 ([21, Proposition 3.2.1])**.** *Let $s \in \Sigma_u^H$. Assume that s is the j-th eigenvalue of H_u^2 and that $\sigma(s)$ is the k-th eigenvalue of K_u^2. We have*

$$\|u_j^H\|_{L^2}^2 = (s^2 - \sigma(s)^2) \prod_{s' \neq s} \frac{s^2 - \sigma(s')^2}{s^2 - s'^2},$$
$$\|u_k^K\|_{L^2}^2 = (s^2 - \sigma(s)^2) \prod_{s' \neq s} \frac{\sigma(s)^2 - s'^2}{\sigma(s)^2 - \sigma(s')^2},$$

where the products are taken over $s' \in \Sigma_u^H$.

The inverse spectral formula. In this paragraph, we state a weaker version of the main result in [21], restricted to functions in Ω_{gen}. In the previous propositions, we have associated to each $u \in H_+^{1/2}$ a set of H-dominant of K-dominant singular values, each of them being linked to some finite Blaschke product. Conversely, let $q \in \mathbb{N} \setminus \{0\}$ and $s_1 > s_2 > \cdots > s_{2q-1} > s_{2q} \geq 0$ some real numbers. Let also Ψ_n, $n \in [\![1, 2q]\!]$, be finite Blaschke products. We define a matrix $\mathscr{C}(z)$, where $z \in \mathbb{D}$ is a parameter, by its coefficients

$$c_{j,k} = \frac{s_{2j-1} - z s_{2k} \Psi_{2j-1}(z) \Psi_{2k}(z)}{s_{2j-1}^2 - s_{2k}^2}, \quad 1 \leq j,k \leq q.$$

Theorem 11 ([21, Theorem 1.0.3]). *For all $z \in \mathbb{D}$, $\mathscr{C}(z)$ is invertible, and if we set*

$$u(z) := \sum_{1\leq j,k\leq q} [\mathscr{C}(z)^{-1}]_{j,k}\Psi_{2k-1}(z),$$

then $u \in \mathcal{V}(2q)$ (or $u \in \mathcal{V}(2q-1)$ if $s_{2q} = 0$).

Furthermore, it is the unique function in $H^{1/2}_+$ such that the H-dominant and K-dominant singular values associated to u are given respectively by the s_{2j-1}'s, $j \in [\![1,q]\!]$, and by the s_{2k}'s, $k \in [\![1,q]\!]$, and such that the Blaschke products associated to these singular values are given respectively by Ψ_{2j-1}, $j \in [\![1,q]\!]$, and by Ψ_{2k}, $k \in [\![1,q]\!]$.

3 The additional conservation laws ℓ_k

In the sequel, we show how to prove simply that $\ell_k(u)$ is conserved along solutions of the quadratic Szegő equation (1.1). We then intend to prove Proposition 1.2 and its corollary : we give a necessary condition for growth of Sobolev norms to occur in the rational case. Let us mention that this condition will be an adaptation of the results of [68] in our context.

3.1 Evolution of u_k^K and w_k^K

Recall that if σ_k^2 is the k-th eigenvalue of K_u^2 in decreasing order (by convention, we set $\sigma_\infty = 0$), we have called $F_u(\sigma_k) := \ker(K_u^2 - \sigma_k^2 I)$, and we have defined u_k^K (resp. w_k^K) to be the orthogonal projection of u (resp. $H_u(u)$) onto $F_u(\sigma_k)$.

We first calculate the evolution of u_k^K and w_k^K.

Lemma 3.1. *Suppose that $t \mapsto u(t)$ is a smooth solution of (1.1). Then we have*

$$\dot{u}_k^K = B_u u_k^K - iJ w_k^K, \tag{3.1}$$

$$\dot{w}_k^K = B_u w_k^K + i\bar{J}(2Q + \sigma_k^2)u_k^K. \tag{3.2}$$

Proof. The proof of these identities relies on the Lax pair. First observe that in view of the expression of B_u and of equation (1.1), we have

$$\dot{u} = B_u(u) - iJH_u(u)$$

For the evolution of u_k^K, set $f := \mathbb{1}_{\{\sigma_k^2\}}$ and write

$$\begin{aligned}\frac{d}{dt}u_k^K = \frac{d}{dt}f(K_u^2)u &= [B_u, f(K_u^2)]u + f(K_u^2)(B_u u - iJH_u u)\\ &= B_u f(K_u^2)u - iJf(K_u^2)H_u u\\ &= B_u u_k^K - iJw_k^K,\end{aligned}$$

which corresponds to (3.1). As for w_k^K,

$$\begin{aligned}\frac{d}{dt}f(K_u^2)H_u u &= [B_u, f(K_u^2)]H_u u + f(K_u^2)([B_u, H_u]u + i\bar{J}Qu) + f(K_u^2)H_u(B_u u - iJH_u u)\\ &= B_u f(K_u^2)H_u u + i\bar{J}Qf(K_u^2)u + i\bar{J}f(K_u^2)H_u^2 u\\ &= B_u w_k^K + i\bar{J}Qu_k^K + i\bar{J}f(K_u^2)(K_u^2 u + Qu)\\ &= B_u w_k^K + i\bar{J}(2Q + \sigma_k^2)u_k^K,\end{aligned}$$

where we used the relation (1.4) between H_u^2 and K_u^2. Now (3.2) is proved. □

Proposition 3.2. *With the hypothesis of the preceding lemma, setting*

$$\ell_k(t) := \frac{1}{\operatorname{Tr} P^k}\left[(2Q+\sigma_k^2)\|u_k^K(t)\|_{L^2}^2 - \|w_k^K(t)\|_{L^2}^2\right],$$

we have $\frac{d}{dt}\ell_k = 0$.

Proof. As $\operatorname{Tr} P^k$, σ_k^2 and Q are constant, it suffices to compute the time derivative of $\|u_k^K(t)\|_{L^2}^2$ and $\|w_k^K(t)\|_{L^2}^2$. On the one hand, by (3.1),

$$\frac{d}{dt}\|u_k^K\|_{L^2}^2 = 2\operatorname{Re}(\dot{u}_k^K|u_k^K) = 2\operatorname{Im}(J(w_k^K|u_k^K)), \tag{3.3}$$

since B_u is anti-selfadjoint. On the other hand, by (3.2),

$$\frac{d}{dt}\|w_k^K\|_{L^2}^2 = 2\operatorname{Re}(\dot{w}_k^K|w_k^k) = -2(2Q+\sigma_k^2)\operatorname{Im}(\bar{J}(u_k^K|w_k^K)).$$

Thus $\frac{d}{dt}\|w_k^K\|_{L^2}^2 = (2Q+\sigma_k^2)\frac{d}{dt}\|u_k^K\|_{L^2}^2$, which yields the conservation of ℓ_k. □

Hereafter, we give another expression of ℓ_k by means of the spectral theory of H_u and K_u (see Section 2). Fix some $u \in H_+^{1/2}$.

Lemma 3.3. *Let σ_k^2 be a non-zero eigenvalue of K_u^2.*

(i) Suppose $\sigma_k \in \Sigma_u^K$. Then $u_k^K \neq 0$ and w_k^K is colinear to u_k^K. Consequently,

$$\ell_k(u) = \frac{\|u_k^K\|_{L^2}^2}{\operatorname{Tr} P^k}\left((2Q+\sigma_k^2) - |\xi_k|^2\right),$$

where

$$\xi_k := \left(\Pi(|u|^2)\,\middle|\,\frac{u_k^K}{\|u_k^K\|_{L^2}^2}\right). \tag{3.4}$$

(ii) Suppose $\sigma_k \in \Sigma_u^H$. Then $u_k^K = 0$ and $w_k^K \neq 0$, hence

$$\ell_k(u) < 0.$$

Proof. Let us first examine the case when $\sigma_k \in \Sigma_u^K$. Then $u_k^K \neq 0$ by Proposition 2.1. Now, if $h \in F_u(\sigma_k)$ and $h \perp u_k^K$, it means that $(h|u) = 0$ and $h \in E_u(\sigma_k)$, *i.e.* $H_u^2 h = \sigma_k^2 h$. We have thereby

$$(w_k^K|h) = (\Pi(|u|^2)|h) = (H_u u|h) = (H_u h|u) = 0,$$

since $H_u h \in E_u(\sigma_k)$, so $H_u h \perp u$. This proves that w_k^K and u_k^K are colinear, and the formula with ξ_k immediately follows, since

$$w_k^K = \left(w_k^K\,\middle|\,\frac{u_k^K}{\|u_k^K\|_{L^2}}\right)\frac{u_k^K}{\|u_k^K\|_{L^2}}.$$

In the case when $\sigma_k \in \Sigma_u^H$ and σ_k is also the j-th eigenvalue of H_u^2, by Proposition 2.1 again, we have $u_k^K = 0$. Let us turn to w_k^K. First, setting $f = \mathbb{1}_{\{\sigma_k^2\}}$, we observe that $f(H_u^2)(\Pi(|u|^2)) = H_u f(H_u^2)(u) = H_u u_j^H \neq 0$, because $u_j^H \neq 0$ and H_u is one-to-one on $E_u(\sigma_k)$. Now observe that $H_u u_j^H \notin \mathbb{C}u_j^H$, otherwise we would have $\dim E_u(\sigma_k) = 1$ by Proposition 2.2, and thus $\dim F_u(\sigma_k) = 0$, which contradicts the assumption that σ_k^2 is an eigenvalue of K_u^2. Therefore,

$$w_k^K = f(K_u^2)f(H_u^2)\left(\Pi(|u|^2)\right) = f(K_u^2)H_u u_j^H \neq 0,$$

since $F_u(\sigma_k) = E_u(\sigma_k) \cap (u_j^H)^\perp$. The second part of the lemma is proved. □

Remark 19. Let us make a series of remarks on the case $k=\infty$. For $\sigma_\infty = 0$, it is also true that w_∞^K and u_∞^K are colinear. Indeed, if $h \in \ker K_u^2$ and $h \perp u$, then

$$0 = (K_u^2(h)|1) = (h|K_u^2(1)) = \big(h|H_u^2(1) + (1|u)u\big) = (h|H_u(u)),$$

so $h \perp H_u(u)$. In particular, if $u \perp \ker K_u^2$, then $H_u(u) \perp \ker K_u^2$ (and then $\ell_\infty = 0$).

When $u_\infty^K \neq 0$, then

$$\xi_\infty^K = \left(H_u^2(1)\middle|\frac{u_\infty^K}{\|u_\infty^K\|_{L^2}^2}\right) = \left(1\middle|\frac{K_u^2(u_\infty^K) + (u_\infty^K|u)u}{\|u_\infty^K\|_{L^2}^2}\right) = (1|u),$$

thus (even if $u \perp \ker K_u^2$),

$$\ell_\infty = \|u_\infty^K\|_{L^2}^2 (2Q - |(u|1)|^2). \tag{3.5}$$

It is worth noticing that identity (3.5) yields another proof of the fact that the submanifold $\{\operatorname{rk} H_u^2 = D\}$ of L_+^2 is stable by the flow of (1.1) (see [64, Corollary 2.2]). Indeed, it suffices to show that when $\operatorname{rk} K_u^2 = D' < +\infty$, $\operatorname{rk} H_u^2$ cannot pass from D' to $D'+1$ or conversely. The condition $\operatorname{rk} H_u^2 = \operatorname{rk} K_u^2 = D'$ for some $u \in H_+^{1/2}$ means that $\operatorname{Im} H_u^2 = \operatorname{Im} K_u^2$ (since the inclusion $\supseteq$ is always true). As $u = H_u(1) \in \operatorname{Im} H_u = \operatorname{Im} H_u^2$, we then have $u \in \operatorname{Im} K_u^2$. Hence $u_\infty^K = 0$. But since $\|u_\infty^K\|_{L^2}^2(2Q - |(u|1)|^2)$ is conserved by the flow, and as $2Q - |(u|1)|^2 \geq Q$ by Cauchy-Schwarz, we see that if $u_\infty^K = 0$ at time 0, then it must remain true for all times.

Now, if $u_\infty^K = 0$ for some $u \in H_+^{1/2}$, it means that $u \in (\ker K_u^2)^\perp = \operatorname{Im} K_u^2$, so writing $K_u^2 = H_u^2 - (\cdot|u)u$ shows that $\operatorname{Im} H_u^2 = \operatorname{Im} K_u^2$. Conversely, if $u_\infty^K \neq 0$ at time 0, it will never cancel.

3.2 About the dominance of eigenvalues of K_u^2

During the proof of Corollary 1.3, we will need to know how often u_k^K may be zero. Indeed, the eigenvalues of K_u^2 are conserved, but as the eigenvalues of H_u^2 have a non trivial evolution in time, it could perfectly happen that a K-dominant singular value associated to u transforms into a H-dominant one : such a phenomenon is called *crossing* in [68], and we follow this terminology. The purpose of this section is then to prove the following proposition.

Proposition 3.4. *Suppose that $t \mapsto u(t)$ is a solution of the quadratic Szegő equation (1.1) in $\mathcal{V}(d)$, and suppose that u is not constant in time. Then there exists a discrete set $\Lambda \subset \mathbb{R}$ such that when $t \notin \Lambda$, all the eigenvalues of K_u^2 are K-dominant.*

To put it in a different way, if $t \notin \Lambda$, then

$$\Xi_{u(t)}^K = \Sigma_{u(t)}^K,$$

and every H-dominant singular value associated to $u(t)$ is therefore of multiplicity 1. It means that crossing cannot happen outside a discrete set of times.

To prove this proposition, we start from a lemma which applies to all smooth solutions (not only the rational ones) :

Lemma 3.5. *Let $s > \frac{1}{2}$ and $u_0 \in H_+^s(\mathbb{T})$. Then the solution $t \mapsto u(t)$ of (1.1) such that $u(0) = u_0$ is real analytic in the variable $t \in \mathbb{R}$, taking values in the Hilbert space H^s.*

Proof. It is enough to prove the lemma on compact sets of $\mathbb{R}$, so we fix $T > 0$, and if $f : [-T,T] \to H^s$ is a continuous function, we denote by

$$\|f\|_T := \max_{t \in [-T,T]} \|f(t)\|_{H^s}.$$

Recall that for $s > \frac{1}{2}$, $H^s_+(\mathbb{T})$ is an algebra, and that $\Pi : H^s \to H^s_+$ is bounded and has norm 1. Therefore, the proof we are going to give only resorts to an ODE framework.

First of all, from the Cauchy-Lipschitz theorem, $t \mapsto u(t)$ is C^∞ on $\mathbb{R}$. It then suffices to prove that there exists constants $c_0, C > 0$ such that

$$\left\| \frac{\partial^n u}{\partial t^n} \right\|_T \leq c_0 C^n n!\,, \qquad \forall n \geq 0.$$

Write equation (1.1) in the following way :

$$\partial_t u = -i \left(\int_{\mathbb{T}} |u|^2 u \right) \Pi(|u|^2) - i\bar{J} \left(\int_{\mathbb{T}} |u|^2 \bar{u} \right) u^2 - i \left(\int_{\mathbb{T}} |u|^2 u \right) \Pi(|u|^2), \tag{3.6}$$

so that $\partial_t u$ appears to be a sum of three terms, each of them being a “product” of five copies of u. Now, it is clear that for $n \geq 0$, $\partial_t^n u$ will be a sum of c_n terms, each of which contains a “product” of d_n copies of u.

Let us find the induction relation between c_{n+1} and c_n, and between d_{n+1} and d_n. If we differentiate $\partial_t^n u$, the time-derivative is going to hit, one after another, each of the d_n factors u of the c_n terms of the sum, and for each of them, it will create three terms by (3.6). Thus

$$c_{n+1} = 3 d_n c_n.$$

As for d_{n+1}, the time-derivative will remove one the u factors and replace it by 5 others, so

$$d_{n+1} = d_n + 4.$$

Consequently, $d_n = 4n + d_0 = 4n + 1$, and $c_{n+1} = 3(4n+1)c_n \leq 12(n+1)c_n$, for all $n \geq 0$. By an easy induction, using $c_0 = 1$, we thus have

$$c_n \leq 12^n (n!).$$

Finally, we bound each of the d_n factors of the c_n terms by $\|u\|_T$, so we get

$$\left\| \frac{\partial^n u}{\partial t^n} \right\|_T \leq 12^n \|u\|_T^{4n+1} (n!),$$

which gives the result. □

Corollary 3.6. *Let $\sigma_k \in \Xi_u^K$. Then $t \mapsto [u(t)]_k^K$ is real analytic.*

Proof. It suffices to choose $\varepsilon > 0$ small enough so that

$$\left[\sqrt{\sigma_k^2 - \varepsilon}, \sqrt{\sigma_k^2 + \varepsilon} \right] \cap \Xi_{u(t)}^K = \{\sigma_k\}$$

for all $t \in \mathbb{R}$ (which is possible, since $\Xi_{u(t)}^K$ does not depend on t by the Lax pair). Then, denoting by $\mathcal{C}(\sigma_k^2, \varepsilon)$ the circle of center σ_k^2 and of radius ε in $\mathbb{C}$, we have

$$[u(t)]_k^K = \frac{1}{2i\pi} \int_{\mathcal{C}(\sigma_k^2, \varepsilon)} (zI - K_{u(t)}^2)^{-1}(u(t))dz$$

by the residue formula. This proves the corollary. □

Now we turn to the proof of the main proposition of this section :

Proof of Proposition 3.4. Assume first that there exists $\sigma_k \in \Xi_u^K$ such that, for some accumulating sequence of times $\{t_n\}$, σ_k is an H-dominant singular value associated to $u(t_n)$. Then by Proposition 2.1, we then have $[u(t_n)]_k^K = 0$ for all $n \in \mathbb{N}$. Therefore, by the real analyticity of this function, it imposes that

$$u_k^K \equiv 0 \quad \text{on } \mathbb{R},$$

which means that σ_k remains H-dominant for all times. Besides, by Lemma 3.3 and the conservation of ℓ_k, we know that, in that case, $-\|w_k^K\|_{L^2}^2$ is conserved and negative. This proves that $w_k^K \neq 0$ for all $t \in \mathbb{R}$.

Now, recall from (3.1) that the evolution of u_k^K is given by

$$\dot{u}_k^K = B_u u_k^K - iJw_k^K.$$

In our case, this means that $-iJw_k^K$ is identically zero. As $w_k^K \neq 0$, we must have

$$J \equiv 0,$$

and this is equivalent to u being a steady solution ($i.e.$ $\partial_t u = 0$).

We therefore have proved that if $t \mapsto u(t)$ is not constant-in-time, the following set $\{t \in \mathbb{R} \mid [u(t)]_k^K = 0\}$ is discrete in $\mathbb{R}$. But since our solution belongs to $\mathcal{V}(d)$, the set Ξ_u^K is finite, so the times for which one at least of the u_k^K's, $k \geq 1$, cancels out, lie in a finite union of discrete sets, so they form a discrete subset of $\mathbb{R}$. This proves the proposition. □

3.3 About the motion of the Blaschke products Ψ_k^K

Now we turn to another evolution law. Recall from Proposition 2.2 that if σ_k is a K-dominant singular value associated to u, then there exists Ψ_k^K, a Blaschke product of degree $m(\sigma_k) - 1$, where $m(\sigma_k)$ is the dimension of $F_u(\sigma_k)$, such that

$$K_u(u_k^K) = \sigma_k \Psi_k^K u_k^K. \tag{3.7}$$

The evolution equation for Ψ_k^K plays an important role and can be computed :

Lemma 3.7. *Choose $t_0 \in \mathbb{R} \setminus \Lambda$ (where Λ is given by Proposition 3.4), and let I be the connected component of $\mathbb{R} \setminus \Lambda$ containing t_0. Let $\sigma_k \in \Xi_u^K$. Then for all $t \in I$, $\Psi_k^K(t)$ is well defined by (3.7), and there exists a smooth function $\psi_{k,I} : I \to \mathbb{T}$ with $\psi_{k,I}(t_0) = 0$, such that*

$$\Psi_k^K(t) = e^{i\psi_{k,I}(t)}\Psi_k^K(t_0), \quad \forall t \in I.$$

Proof. The fact that Ψ_k^K is well-defined comes from the fact that for all $t \in I$, $\sigma_k \in \Sigma_{u(t)}^K$. Differentiate (3.7), using (3.1) and the Lax pair :

$$B_u K_u(u_k^K) + i\bar{J}K_u(w_k^K) = \sigma_k \dot{\Psi}_k^K u_k^K + \sigma_k \Psi_k^K B_u u_k^K - i\sigma_k J \Psi_k^K w_k^K.$$

By (3.7) again, and the fact that $w_k^K = \|u_k^K\|^{-2}(\Pi(|u|^2)|u_k^K)u_k^K$, we get

$$\dot{\Psi}_k^K u_k^K = (B_u \Psi_k^K - \Psi_k^K B_u)u_k^K + 2i\,\mathrm{Re}\left(J\frac{(\Pi(|u|^2)|u_k^K)}{\|u_k^K\|^2}\right)\Psi_k^K u_k^K. \tag{3.8}$$

Our goal is to show that $(B_u\Psi_k^K - \Psi_k^K B_u)u_k^K = 0$. This is obvious when $m(\sigma_k) = 1$ (because in that case Ψ_k^K is only a complex number), so we assume that $m(\sigma_k) \geq 2$. Since $u \in L_+^2$, it is clear that $T_{\bar{J}u}(\Psi_k^K u_k^K) = \bar{J}u\Psi_k^K u_k^K = \Psi_k^K T_{\bar{J}u}(u_k^K)$. So it is enough to show that $T_{\bar u}(\Psi_k^K u_k^K) - \Psi_k^K T_{\bar u}(u_k^K) = 0$, and then to multiply this identity by J. This cancellation follows from a direct computation :

$$\begin{aligned} T_{\bar u}(\Psi_k^K u_k^K) - \Psi_k^K T_{\bar u}(u_k^K) &= \Pi\left(\Psi_k^K (I-\Pi)(\bar u u_k^K)\right) \\ &= \Pi\left(\Psi_k^K \bar z \overline{\Pi(\bar z u \overline{u_k^K})}\right) \\ &= \Pi(\bar z \Psi_k^K \overline{K_u(u_k^K)}) \\ &= \sigma_k \,\Pi(\bar z |\Psi_k^K|^2 \overline{u_k^K}) \\ &= \sigma_k \,\Pi(\bar z \overline{u_k^K}) \\ &= 0, \end{aligned}$$

where we used the elementary fact that for $h \in L^2_+$, we have $(I-\Pi)(h) = \bar z \overline{\Pi(\bar z \bar h)}$.

Going back to (3.8), since $u_k^K \neq 0$, we find

$$\dot\Psi_k^K = 2i\,\mathrm{Re}\left(J \frac{(\Pi(|u|^2)|u_k^K)}{\|u_k^K\|^2}\right)\Psi_k^K = 2i\,\mathrm{Re}(J\xi_k)\Psi_k^K,$$

with the notation of (3.4). This gives the yielded result, with $\psi_{k,I}(t) = 2\,\mathrm{Re}\left(\int_{t_0}^t J(s)\xi_k(s)ds\right)$. □

From Lemma 3.7, we are going to deduce an important corollary : the zeros of the Blaschke product associated to some $\sigma_k \in \Sigma_u^K$ remain unchanged from one connected component of $\mathbb{R}\setminus\Lambda$ to another. As a consequence, the Blaschke products associated to K-dominant values can be defined for all times.

Corollary 3.8. *Fix $t_0 \in \mathbb{R}\setminus\Lambda$*[1]*. For each $\sigma_k \in \Xi_u^K$, the Blaschke product Ψ_k^K is well-defined by (3.7) for every $t \in \mathbb{R}\setminus\Lambda$, and there exists a continuous function $\psi_k : \mathbb{R}\to\mathbb{T}$ with $\psi_k(t_0) = 0$, such that*

$$\Psi_k^K(t) = e^{i\psi_k(t)}\Psi_k^K(t_0), \quad \forall t \in \mathbb{R}\setminus\Lambda.$$

Proof. Pick $\sigma_k \in \Xi_u^K$, and assume that there exists a time $\tilde t \in \mathbb{R}$ such that σ_k is an H-dominant singular value associated to $u(\tilde t)$. Pick also $\varepsilon > 0$ such that $[\tilde t - \varepsilon, \tilde t + \varepsilon] \cap \Lambda = \{\tilde t\}$.

Now, we know from Lemma 3.3 that $w_k^K(\tilde t) \neq 0$. On the other hand, it can be shown as in Corollary 3.6 that w_k^K is a real analytic function. Up to changing ε, we assume that $w_k^K(t) \neq 0$ if $|t - \tilde t| \le \varepsilon$, and for such t, we can define

$$\Psi^\sharp(t) := \frac{K_{u(t)}(w_k^K(t))}{\sigma_k w_k^K(t)}.$$

$\Psi^\sharp$ is a continuous function of t on the interval $[\tilde t-\varepsilon, \tilde t+\varepsilon]$ which takes values into rational functions of $z \in \mathbb{D}$.

Besides, recall that w_k^K is colinear to u_k^K when $\sigma_k \in \Sigma_{u(t)}^K$ (see Lemma 3.3). Thus, if $t \in [\tilde t-\varepsilon, \tilde t+\varepsilon]\setminus\{\tilde t\}$, then $\Psi^\sharp(t)$ coincides with $\Psi_k^K(t)$, *i.e.* $e^{i\psi_{k,I_1}(t)}\Psi_1$ on the left of $\tilde t$, and $e^{i\psi_{k,I_2}(t)}\Psi_2$ on the right (where $\psi_{k,I_j} : \mathbb{R}\to\mathbb{T}$ are smooth, and Ψ_j are some constant-in-time finite Blaschke products of identical degree, by Lemma 3.7). Therefore, $\Psi^\sharp$ enables to extend continuously each of the two functions, which imposes that $\Psi_1 = \Psi_2$ and that the ψ_{k,I_j} coincide with a function which is continuous in $\tilde t$. □

1. In the sequel, we will assume without loss of generality that $t_0 = 0$.

3.4 Weakly convergent sequences in $H^{1/2}_+$

Before coming back to equation (1.1), let us prove three useful preliminary results about weakly convergent sequences in $H^{1/2}_+(\mathbb{T})$.

Lemma 3.9. *Let $v_n \in H^{1/2}_+$ such that $\{v_n\}$ converges weakly to some v in $H^{1/2}_+$. Then, for any $h \in L^2_+$, $H_{v_n}(h) \to H_v(h)$ strongly in L^2_+.*

Proof. Replacing v_n by $v_n - v$, we can assume that $v = 0$. By Rellich's theorem, since $v_n \rightharpoonup 0$ in $H^{1/2}(\mathbb{T})$, we have $v_n \to 0$ in every $L^p(\mathbb{T})$, $p < \infty$. Thus, given $h \in L^4_+$,

$$\|H_{v_n}(h)\|_{L^2} \le \|v_n \bar{h}\|_{L^2} \le \|v_n\|_{L^4}\|h\|_{L^4} \longrightarrow 0$$

when $n \to +\infty$. Now set $\varepsilon > 0$. If $h \in L^2_+$, there exists $\tilde{h} \in L^4_+$ such that $\|h - \tilde{h}\|_{L^2} \le \varepsilon$. Furthermore, by the principle of uniform boundedness, there exists $C > 0$ such that $\|v_n\|_{H^{1/2}} \le C$ for all $n \ge 0$, hence $\|H_{v_n}\| \le C$. Then, for n large enough,

$$\|H_{v_n}(h)\|_{L^2} \le \|H_{v_n}(h - \tilde{h})\|_{L^2} + \|H_{v_n}(\tilde{h})\|_{L^2} \le (C+1)\varepsilon,$$

which proves the lemma. □

Lemma 3.10. *Let $d \in \mathbb{N}$. Suppose $v_n \in \mathcal{V}(d)$ and $v_n \rightharpoonup v$ in $H^{1/2}_+$. Then $v \in \mathcal{V}(d')$ for some $d' \le d$.*

Proof. This is in fact a completely general result on sequences of bounded operators $\mathcal{T}_n$ on some Hilbert space H, such that $\sup_n \|\mathcal{T}_n\| < +\infty$, and $\operatorname{rk} \mathcal{T}_n = k$. Assume that for all $h \in H$, $\mathcal{T}_n(h) \to \mathcal{T}(h)$ strongly. Then $\operatorname{rk} \mathcal{T} \le k$.

Indeed, for any choice of $k+1$ vectors $h_1, \dots, h_{k+1} \in H$, the Gram matrix

$$\begin{pmatrix} (\mathcal{T}_n(h_1)|\mathcal{T}_n(h_1)) & (\mathcal{T}_n(h_1)|\mathcal{T}_n(h_2)) & \cdots & (\mathcal{T}_n(h_1)|\mathcal{T}_n(h_{k+1})) \\ (\mathcal{T}_n(h_2)|\mathcal{T}_n(h_1)) & (\mathcal{T}_n(h_2)|\mathcal{T}_n(h_2)) & \cdots & (\mathcal{T}_n(h_2)|\mathcal{T}_n(h_{k+1})) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathcal{T}_n(h_{k+1})|\mathcal{T}_n(h_1)) & (\mathcal{T}_n(h_{k+1})|\mathcal{T}_n(h_2)) & \cdots & (\mathcal{T}_n(h_{k+1})|\mathcal{T}_n(h_{k+1})) \end{pmatrix}$$

has determinant 0, since $\mathcal{T}_n(h_1), \dots, \mathcal{T}_n(h_{k+1})$ are not linearly independent. Passing to the limit $n \to +\infty$ in this determinant shows that $\mathcal{T}(h_1), \dots, \mathcal{T}(h_{k+1})$ are not independant either, whatever the choice of h_j. So $\operatorname{rk} \mathcal{T} \le k$. Applying this general result both to H_{v_n} and K_{v_n} gives the result. □

We will also need a refinement of Lemma 3.10 in the case of sequences of functions such that the corresponding shifted Hankel operator has constant spectrum.

Lemma 3.11. *Let $v_n \in H^{1/2}_+$ such that $v_n \rightharpoonup v$ in $H^{1/2}_+$. Suppose that $\operatorname{Sp} K^2_{v_n}$ does not depend of $n \ge 1$. Then if $\sigma^2 \in \operatorname{Sp} K^2_v$ with multiplicity m, then there exists $N \in \mathbb{N}$ such that $\forall n \ge N$, $\sigma^2 \in \operatorname{Sp} K^2_{v_n}$ with multiplicity at least m.*

Proof. For $\sigma \in \Xi^K_v$, denote by P^σ the projection onto $\ker(K^2_v - \sigma^2 I)$. By the residue theorem, given $\sigma \in \Xi^K_v$ and $0 < \varepsilon < \sigma^2$ such that $\sigma^2 \pm \varepsilon \notin \operatorname{Sp} K^2_v$, we have

$$\frac{1}{2i\pi} \int_{\mathcal{C}(\sigma^2, \varepsilon)} (zI - K^2_v)^{-1} dz = \sum_{\substack{\tilde{\sigma}^2 \in \Xi^K_v \\ |\tilde{\sigma}^2 - \sigma^2| < \varepsilon}} P^{\tilde{\sigma}}, \tag{3.9}$$

where $\mathcal{C}(\sigma^2, \varepsilon)$ is the circle of center σ^2 and of radius ε.

If σ^2 is a non-zero eigenvalue of K_v^2, let $\{e_j \mid j = 1, \dots, m\}$ be an orthonormal basis of the corresponding eigenspace, which must be of finite dimension for K_v^2 is compact. Let $\varepsilon > 0$ be sufficiently small so that $\{z \in \mathbb{C} \mid |z - \sigma^2| \leq \varepsilon\}$ does not contain any other eigenvalue of K_v^2, and contains at most one eigenvalue $\tilde{\sigma}^2$ of $K_{v_n}^2$ for all $n \geq 1$. For each $1 \leq j \leq m$,

$$e_j^n := \frac{1}{2i\pi} \int_{\mathcal{C}(\sigma^2,\varepsilon)} (zI - K_{v_n}^2)^{-1}(e_j)dz$$

is well defined, and by Lemma 3.9 and formula (3.9), we have

$$e_j^n \longrightarrow \frac{1}{2i\pi} \int_{\mathcal{C}(\sigma^2,\varepsilon)} (zI - K_v^2)^{-1}(e_j) = e_j$$

as $n \to \infty$. Thus, if n is large enough, the e_j^n, $j = 1, \dots, m$ form a family of (non-zero) independant vectors that all belong to $\ker(K_{v_n}^2 - \tilde{\sigma}^2 I)$. As this is true for any $\varepsilon > 0$ small enough, we must have $\tilde{\sigma} = \sigma$. So $\sigma \in \Xi_{v_n}^K$, and the dimension of $\ker(K_{v_n}^2 - \sigma^2 I)$ is at least m when n is large enough. □

3.5 An equivalent condition for the growth of Sobolev norms in $\mathcal{V}(d)$

Let us now fix an integer $d \geq 2$, and let u be a solution of (1.1) in $\mathcal{V}(d)$. Write

$$u(t,z) = \frac{A(t,z)}{B(t,z)}, \quad \forall z \in \mathbb{D}, \, \forall t \in \mathbb{R},$$

where $A(t,\cdot)$ and $B(t,\cdot)$ are polynomials whose degree depends on $N := \lfloor \frac{d}{2} \rfloor$ in the following way : $\deg A \leq N-1$ and $\deg B = N$ when d is even, and $\deg A = N$ and $\deg B \leq N$ when d is odd. Moreover, A and B are relatively prime, with $B(t,0) = 1$ and B having no roots inside the closed unit disc of $\mathbb{C}$. With these notations, we have $\operatorname{rk} K_u = N$, and $\operatorname{rk} H_u = d - N$. Write $B(t,z) = \prod_{j=1}^N (1 - p_j(t)z)$, with $|p_j(t)| < 1$ for all $1 \leq j \leq N$, $t \in \mathbb{R}$.

Observe that, as a smooth solution of (1.1), by the conservation of M and Q, the function $t \mapsto u(t)$ remains bounded in $H^{1/2}$, so by the Banach-Alaoglu theorem, the following set

$$\mathcal{A}^\infty(u) = \left\{ v \in H_+^{\frac{1}{2}} \;\middle|\; \exists t_n \to \pm\infty \text{ s.t. } u(t_n) \overset{H^{\frac{1}{2}}}{\rightharpoonup} v \right\}$$

is non-empty [2].

We are ready to state our proposition in terms of solutions of (1.1) — but it can be formulated and proved as well in the general framework of Proposition 1.2 :

Proposition 3.12. *The following statements are equivalent :*

(i) u is bounded in $H_+^{s_0}$ for some $s_0 > \frac{1}{2}$.

(ii) u is bounded in every *H_+^s, $s > \frac{1}{2}$.*

(iii) $\mathcal{A}^\infty(u) \subseteq \mathcal{V}(d)$ when d is even, and $\mathcal{A}^\infty(u) \subseteq \mathcal{V}(d) \cup \mathcal{V}(d-1)$ when d is odd.

Proof. Start with an observation. Writing $A(t,z) = \sum_{j=0}^N a_j(t)z^j$, we have by Cauchy-Schwarz

$$\sum_{j=0}^N |a_j(t)| \leq \sqrt{N} \cdot \|A(t,\cdot)\|_{L^2} \leq \sqrt{N} \|B(t,\cdot)\|_{L^\infty} \|u(t)\|_{L^2} \leq 2^N \sqrt{N} \|u_0\|_{L^2},$$

2. Here, the letter $\mathcal{A}$ stands for the French word *adhérence*, which means "closure".

which proves that all the coefficients of A remain bounded uniformly in time. So if $\{t_n\}$ is a sequence of times with $t_n \to \pm\infty$, we can assume up to an extraction that, for each $z \in \mathbb{D}$,

$$u(t_n, z) \longrightarrow \frac{\sum_{j=0}^N a_j^\infty z^j}{\prod_{j=1}^N (1 - p_j^\infty z)},$$

where $a_j^\infty, p_j^\infty \in \mathbb{C}$ with $|p_j^\infty| \leq 1$. Besides, if $u(t_n, \cdot) \rightharpoonup v \in \mathcal{A}^\infty(u)$, then $\forall k \in \mathbb{N}$, $\hat{u}(t_n, k) \to \hat{v}(k)$, which implies that, for each $z \in \mathbb{D}$, we also have

$$u(t_n, z) = \sum_{k=0}^\infty \hat{u}(t_n, k) z^k \longrightarrow \sum_{k=0}^\infty \hat{v}(k) z^k = v(z).$$

Now, if *(iii)* is satisfied, then there must be some $\rho < 1$ such that $|p_j(t)| \leq \rho$ for all $t \in \mathbb{R}$ and $1 \leq j \leq N$, otherwise, choosing an appropriate sequence $\{t_n\}$, one of the p_j^∞ at least would be of modulus 1 (say $p_1^\infty = e^{i\theta}$). Hence considering v a cluster point of $\{u(t_n)\}$ for the weak $H^{1/2}$ topology, we would have

$$v(z) = \frac{\sum_{j=0}^N a_j^\infty z^j}{(1 - e^{i\theta} z) \prod_{j=2}^N (1 - p_j^\infty z)},$$

by the previous remark. But $v \in L^2_+$, so $1 - e^{i\theta} z$ would have to divide the numerator. After simplification, we would get $v \in \mathcal{V}(d - \ell)$ with $\ell \geq 2$, and $v \in \mathcal{A}^\infty(u)$, which contradicts *(iii)*. But once we have such a $\rho < 1$, it is possible to control the H^s_+ norm of u. Indeed, $\|A(t, \cdot)\|_{H^s} \leq (1 + N^2)^{s/2} \|A(t, \cdot)\|_{L^2} \leq C(N, s, u_0)$ for all time $t \in \mathbb{R}$. In addition,

$$\frac{1}{B(t, z)} = \prod_{j=1}^N \left(\sum_{k \geq 0} p_j^k z^k \right) = \sum_{k \geq 0} z^k \left(\sum_{\substack{(k_1, \ldots, k_N) \in \mathbb{N}^N \\ k_1 + \cdots + k_N = k}} p_1^{k_1} p_2^{k_2} \ldots p_N^{k_N} \right),$$

so the coefficient of z^k is controlled by $k^N \rho^k$. This proves that

$$\left\| \frac{1}{B(t, \cdot)} \right\|_{H^s}$$

is uniformly bounded for any $s > 1/2$. Hence *(ii)* is proved.

Let us now prove that *(i)* implies *(iii)*. If u is bounded in some H^{s_0}, $s_0 > \frac{1}{2}$, then its orbit belongs to a compact set of $H^{1/2}$, for the injection $H^s \hookrightarrow H^{1/2}$ is compact. Therefore, for each $v \in \mathcal{A}^\infty(u)$, there exists a sequence of times $\{t_n\}$ such that $u(t_n) \to v$ strongly in $H^{1/2}_+$. But by the min-max formula, we know the k-th eigenvalue of K_u^2 depends continuously on u with respect to the $H^{1/2}$ topology, and as it is a conservation law of (1.1), we get in particular that $N = \operatorname{rk} K^2_{u(t_n)} = \operatorname{rk} K_v^2$. Furthermore, by Lemma 3.10, we get $\operatorname{rk} H_v^2 \leq \liminf_{n \to +\infty} \operatorname{rk} H^2_{u(t_n)}$. Since $\operatorname{rk} H_v^2 \geq \operatorname{rk} K_v^2 = N$, we have $\operatorname{rk} H_v^2 = N$ if d is even, and $\operatorname{rk} H_v^2 \in \{N, N+1\}$ if d is odd. This finishes the proof. □

3.6 Proof of Corollary 1.3

Now we translate Proposition 3.12 into a blow-up criterion for solutions of (1.1) in $\mathcal{V}(d)$:

Proposition 3.13. *Let $t \mapsto u(t)$ be a solution of* (1.1) *in $\mathcal{V}(d)$. The following alternative holds :*

- *either the trajectory $\{u(t) \mid t \in \mathbb{R}\}$ is bounded in every H^s, $s > 1/2$.*
- *or there exists $\sigma_k \in \Xi_u^K$ and a sequence t_n going to $\pm\infty$ such that $u_k^K(t_n) \neq 0$ for all $n \geq 1$, and*

$$\begin{cases} u_k^K(t_n) \to 0, \\ w_k^K(t_n) \to 0 \end{cases} \quad \text{in } L_+^2.$$

Proof. Suppose that $t \mapsto u(t)$ is *not* bounded in some H^{s_0}, $s_0 > 1/2$. By continuity of the solution in H^{s_0} and by Proposition 3.4, we can find a sequence t_n such that for all $n \geq 1$, $t_n \in \mathbb{R} \setminus \Lambda$ and $\|u(t_n)\|_{H^{s_0}} \to +\infty$. By Proposition 1.2, it means that up to passing to a subsequence, we can assume that there exists $v \in H_+^{1/2}$ such that $\operatorname{rk} K_v < \operatorname{rk} K_{u(t)} = N$ and $u(t_n) \rightharpoonup v$ in $H^{1/2}$.

We set $u^n := u(t_n)$. By Rellich's theorem, we have $u^n \to v$ strongly in L_+^2. Let $\sigma_k \in \Xi_{u^n}^K$, and denote by π^n (resp. π^∞) the orthogonal projection onto $F_{u^n}(\sigma_k)$ (resp. $F_v(\sigma_k)$). With this notation, $(u^n)_k^K = \pi^n(u^n)$ and $v_k^K = \pi^\infty(v)$. Since $K_{u^n}(h) \to K_v(h)$ for any fixed $h \in L_+^2$, adapting formula (3.9), we also have $\pi^n(h) \to \pi^\infty(h)$. As $\|\pi^n\| \leq 1$, we thus get

$$\|(u^n)_k^K - v_k^K\|_{L^2} \leq \|\pi^n(u^n) - \pi^n(v)\|_{L^2} + \|(\pi^n - \pi^\infty)(v)\|_{L^2} \leq \|u^n - v\|_{L^2} + \|(\pi^n - \pi^\infty)(v)\|_{L^2},$$

so $(u^n)_k^K \to v_k^K$ strongly in L_+^2.

Now, since all the eingevalues of $K_{u^n}^2$ are K-dominant by the hypothesis on t_n, we can write

$$K_{u^n}^2\left((u^n)_k^K\right) = \sigma_k^2 (u^n)_k^K,$$
$$K_{u^n}\left((u^n)_k^K\right) = \sigma_k \Psi^n \cdot (u^n)_k^K,$$

where $\Psi^n := \Psi_k^K(t_n)$, and $(u^n)_k^K \neq 0$ for all $n \geq 1$. We would like to pass to the limit in these identities. Since $\|K_{u^n}\| \leq C$, we see that $\|K_{u^n}((u^n)_k^K) - K_v(v_k^K)\|_{L^2} \leq C\|(u^n)_k^K - v_k^K\|_{L^2} + \|K_{u^n}(v_k^K) - K_v(v_k^K)\|_{L^2}$, so $K_{u^n}((u^n)_k^K) \to K_v(v_k^K)$ strongly in L_+^2. The same holds replacing K_{u^n} by $K_{u^n}^2$ and K_v by K_v^2. Eventually, by Lemma 3.8, we have $\Psi_k^K(t_n) = e^{i\psi_k(t_n)}\Psi_k^K(0)$. So up to passing to a subsequence, $\Psi^n \to e^{i\psi^\infty}\Psi_k^K(0)$ for some $\psi^\infty \in \mathbb{T}$. Hence, taking n to ∞, we get

$$K_v^2(v_k^K) = \sigma_k^2 v_k^K,$$
$$K_v(v_k^K) = \sigma_k e^{i\psi^\infty}\Psi_k^K(0) v_k^K. \tag{3.10}$$

If now $v_k^K \neq 0$ for every $\sigma_k \in \Xi_{u^n}^K = \Sigma_{u^n}^K$, then the previous equality shows that σ_k also belongs to Σ_v^K, and more precisely, as the dimension of $F_v(\sigma_k)$ is given by the degree of the associated Blaschke product plus 1, we get from (3.10) that $\dim F_{u^n}(\sigma_k) = \dim F_v(\sigma_k)$. This proves that

$$\operatorname{rk} K_v \geq \sum_{\sigma \in \Sigma_{u^n}^K} \dim(F_v(\sigma)) = \sum_{\sigma \in \Sigma_{u^n}^K} \dim(F_{u^n}(\sigma)) = \operatorname{rk} K_{u^n} = N,$$

since $t_n \notin \Lambda$. This is a contradiction. Consequently, for some $\sigma_k \in \Xi_{u^n}^K$, we must have $[u(t_n)]_k^K \to 0$ in L_+^2.

Besides, for such σ_k's, we call $(w^n)_k^K$ the projection of $\Pi(|u^n|^2)$ onto $F_{u^n}(\sigma_k)$. We know from Lemma 3.3 that $(w^n)_k^K$ is colinear to $(u^n)_k^K$. Denote by y_k^K the projection of $\Pi(|v|^2)$ onto $F_v(\sigma_k)$. Since $\Pi(|u^n|^2) \to \Pi(|v|^2)$ strongly in L^2, we get that $(w^n)_k^K \to y_k^K$ strongly in L^2, by the same argument as above. Then, passing to the limit in the expression of $K_{u^n}((w^n)_k^K)$ and $K_{u^n}^2((w^n)_k^K)$, we get as before

$$K_v^2(y_k^K) = \sigma_k^2 y_k^K,$$
$$K_v(y_k^K) = \sigma_k e^{i\psi^\infty}\Psi_k^K(0)y_k^K.$$

Let us show that these equalities impose on y_k^K to be 0 for at least one k. Assume that $y_k^K \neq 0$. Together with v_k^K, it means that $\sigma_k \in \Xi_v^K \backslash \Sigma_v^K$, *i.e.* σ_k is H-dominant, and corresponds to the j-th eigenvalue of H_v^2 for some $j \geq 1$. Denote by m_k the dimension of $F_{u^n}(\sigma_k)$ and by n_j the dimension of $E_v(\sigma_k)$. By Proposition 2.2, since $y_k^K \in F_v(\sigma_k)$, there exists a non-zero polynomial $g \in \mathbb{C}_{n_j-2}[z]$ as well as a polynomial $D(z)$ such that

$$y_k^K = \frac{g(z)}{D(z)} H_v\left(v_j^H\right),$$

and there exists $\varphi \in \mathbb{T}$ such that

$$K_v\left(y_k^K\right) = \rho_j e^{-i\varphi}\frac{\tilde{g}(z)}{D(z)} H_v\left(v_j^H\right),$$

where $\tilde{g}$ is the polynomial of degree at most $n_j - 2$ obtained by reversing the order of the coefficients of g and conjugating them. Thus, combining all the informations we have,

$$\frac{K_v\left(y_k^K\right)}{y_k^K} = \rho_j e^{-i\varphi}\frac{\tilde{g}}{g} = \sigma e^{i\psi^\infty}\Psi_k^K(0).$$

Since Ψ_k^K is an irreducible rational function whose numerator and denominator are both of degree $m_k - 1$, it means that $n_j - 2 \geq \deg \tilde{g} = \deg g \geq m_k - 1$, hence

$$n_j - 1 = \dim F_v(\sigma_k) \geq \dim F_{u^n}(\sigma_k) = m_k.$$

But this cannot happen for all σ_k's for which $v_k^K = 0$, otherwise we would still have $\operatorname{rk} K_v \geq N$. Therefore, there exists $\sigma_k \in \Xi_u^K$ such that both v_k^K and y_k^K are zero.

Conversely, if $t \mapsto u(t)$ is bounded in some H^{s_0}, $s_0 > 1/2$, we have seen during the proof of Proposition 3.12 that for any $v \in \mathcal{A}^\infty(u)$, we have $\Xi_u^K = \Xi_v^K$. Pick some $\sigma_k \in \Xi_u^K$. Then either σ_k is K-dominant for v, and then $v_k^K \neq 0$, or σ_k is H-dominant for v, but then $[\Pi(|v|^2)]_k^K \neq 0$ by Lemma 3.3. So in both cases, denoting by t_n a sequence of times such that $u^n := u(t_n) \rightharpoonup v$ in $H^{1/2}$, and defining $w^n := \Pi(|u^n|^2)$ as above, we cannot have $(u^n)_k^K \to 0$ and $(w^n)_k^K \to 0$ at the same time. □

Remark 20. As a by-product of the proof of Proposition 3.13, it appears that whenever u is a solution of (1.1) in $\mathcal{V}(d)$, $v \in \mathcal{A}^\infty(u)$, and $\sigma \in \Xi_{u(t)}^K$ with multiplicity $m(\sigma)$,

- either $\sigma \in \Xi_v^K$ with multiplicity $m(\sigma)$,
- or $\sigma \notin \Xi_v^K$.

Indeed, if $\sigma \in \Xi_v^K$, then it is either H-dominant or K-dominant, so one at least of the vectors $\mathbb{1}_{\{\sigma^2\}}(K_v^2)\Pi(|v|^2)$ and $\mathbb{1}_{\{\sigma^2\}}(K_v^2)(v)$ is non-zero. Then, a Blaschke product argument as in proof above shows that σ has multiplicity at least $m(\sigma)$. Of course, it cannot be strictly bigger than $m(\sigma)$ (by Lemma 3.11).

Corollary 1.3 is now a mere consequence of Proposition 3.13. We restate it for the convenience of the reader :

Corollary 3.14 (Necessary condition for norm explosion)**.** *Let $t \mapsto u(t)$ be a solution of* (1.1) *in $\mathcal{V}(d)$, and suppose that it is not bounded in some H^s topology, $s > \frac{1}{2}$. Then there exists $\sigma_k \in \Xi_u^K$ such that $\ell_k = 0$,* i.e.

$$(2Q + \sigma_k^2)\|u_k^K(t)\|_{L^2}^2 - \|w_k^K(t)\|_{L^2}^2 = 0, \quad \forall t \in \mathbb{R}.$$

Proof. The quantity ℓ_k is conserved by Proposition 3.2, and if $t \mapsto u(t)$ is unbounded in some H^s, $s > \frac{1}{2}$, it tends to zero along a sequence of times by Proposition 3.13. Thus it is identically zero. □

Remark 21. Thanks to Corollary 3.14 together with Proposition 3.13, if one wants to prove that a rational solution has growing Sobolev norms, it suffices to study the evolution of $[u(t)]_k^K$ if $\ell_k = 0$. If it tends to zero along a sequence of times, so does automatically $[w(t)]_k^K$ along the same sequence, and the conditions of Proposition 3.13 are then fulfilled. The convergence to zero of both u_k^K and w_k^K is what makes this situation very different from of crossing (where only u_k^K goes to zero).

4 The particular case of $\mathcal{V}(4)$: 2-soliton turbulence

4.1 A priori analysis

We begin this section by proving identities that make a link between all the objects we have defined so far :

Lemma 4.1. *Let $u \in H_+^{1/2}$. Write $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq \ldots$ the decreasing sequence of the eigenvalues of K_u^2. Then*

$$M(u) = \sum_{1 \leq k < \infty} \sigma_k^2,$$
$$Q(u)^2 = \sum_{1 \leq k \leq \infty} \ell_k,$$
$$|J(u)|^2 = \sum_{1 \leq k \leq \infty} (Q + \sigma_k^2)\ell_k.$$

In addition, if $\sigma_k \in \Sigma_u^K$, we have set $\xi_k := \|u_k^K\|_{L^2}^{-2}(\Pi(|u|^2)|u_k^K)$. Then

$$\overline{J(u)} = \sum_{\substack{1 \leq k \leq \infty \\ \sigma_k \in \Sigma_u^K \cup \{0\}}} \frac{\xi_k}{\operatorname{Tr} P^k} \|u_k^K\|_{L^2}^2.$$

Remark 22. The above formulae must take "infinity" terms into account, with the convention already mentionned that $\sigma_\infty = 0$.

Proof of Lemma 4.1. The first identity is proved in [18], but we recall here how it is proved. In the canonical basis of L^2_+, the matrix of H_u reads

$$H_u = \begin{pmatrix} \widehat{u}(0) & \widehat{u}(1) & \widehat{u}(2) & \cdots \\ \widehat{u}(1) & \widehat{u}(2) & \widehat{u}(3) & \cdots \\ \widehat{u}(2) & \widehat{u}(3) & \widehat{u}(4) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\widehat{u}$ is the Fourier transform of u. So taking the $\mathbb{C}$-antilinearity of H_u into account, the trace norm of H_u^2 is given by

$$\operatorname{Tr} H_u^2 = \sum_{j\geq 0}\sum_{m\geq 0} |\widehat{u}(j+m)|^2 = \sum_{n\geq 0}(1+n)|\widehat{u}(n)|^2 = Q(u) + M(u).$$

Therefore, by (1.4) (*i.e.* $H_u^2 = K_u^2 + (\cdot|u)u$), we find $\operatorname{Tr} K_u^2 = \operatorname{Tr} H_u^2 - \operatorname{Tr}((\cdot|u)u) = (Q + M) - Q = M$, and the first formula follows, computing $\operatorname{Tr} K_u^2$ in an orthonormal basis of eigenvectors.

Secondly, note that $\sigma_k^2\|u_k^K\|_{L^2} = (K_u^2(u_k^K)|u_k^K) = (K_u^2(u)|u_k^K)$. Decomposing u and $\Pi(|u|^2)$ along all the eigenspaces of K_u^2, it yields that

$$\begin{aligned}\sum_{1\leq k\leq\infty} \ell_k &= \sum_{1\leq k\leq\infty} \frac{1}{\operatorname{Tr} P^k}\left[(2Q+\sigma_k^2)\|u_k^K\|_{L^2}^2 - \|w_k^K\|_{L^2}^2\right] \\ &= 2Q^2 + (K_u^2(u)|u) - \|H_u(u)\|_{L^2}^2 \\ &= 2Q^2 + (H_u^2(u) - Qu|u) - (H_u^2(u)|u) \\ &= Q^2,\end{aligned}$$

where we used the orthogonality of the eigenspaces of K_u^2 to sum the squared norms of u_k^K and w_k^K corresponding to distinct eigenvalues.

Then, using extensively (1.4) again,

$$\begin{aligned}\sum_{1\leq k\leq\infty} (Q+\sigma_k^2)\ell_k &= Q^3 + \sum_{1\leq k\leq\infty} \frac{1}{\operatorname{Tr} P^k}\left[(2Q+\sigma_k^2)\sigma_k^2\|u_k^K\|_{L^2}^2 - \sigma_k^2\|w_k^K\|_{L^2}^2\right] \\ &= Q^3 + 2Q(K_u^2(u)|u) + (K_u^4(u)|u) - (K_u^2(H_u(u))|H_u(u)) \\ &= Q^3 + Q(K_u^2(u)|u) + (H_u^2(K_u^2(u))|u) - \left[(H_u^3(u)|H_u(u)) - |J|^2\right] \\ &= Q^3 + Q(K_u^2(u)|u) - Q(H_u^2(u)|u) + |J|^2 \\ &= |J|^2.\end{aligned}$$

It remains to prove the alternative expression of $J(u)$. Since w_k^K is colinear to u_k^K with $w_k^K = \xi_k u_k^K$ (when this last projection is not zero), and because of the decomposition (2.1),

$$\overline{J(u)} = (\Pi(|u|^2)|u) = \sum_{\substack{1\leq k\leq\infty \\ \sigma_k\in\Sigma_u^K\cup\{0\}}} \frac{1}{\operatorname{Tr} P^k}(\Pi(|u|^2)|u_k^K) = \sum_{\substack{1\leq k\leq\infty \\ \sigma_k\in\Sigma_u^K\cup\{0\}}} \frac{1}{\operatorname{Tr} P^k}(w_k^K|u_k^K),$$

which gives the formula in Lemma 4.1. □

Let us now make a few considerations on $\mathcal{V}(4)$. On $\mathcal{V}(4)$, we have $\operatorname{rk} K_u^2 = 2$ and $u \perp \ker K_u^2$ (since $\operatorname{Ran} H_u^2 = \operatorname{Ran} K_u^2$).

Corollary 4.2. *Let $u \in \mathcal{V}(4) \setminus \{0\}$. There exists $k \in \{1, 2\}$ such that $\ell_k(u) = 0$ if and only if Ξ_u^K has two distinct elements $\sigma_1 > \sigma_2$, and*

$$|J|^2 = Q^2(Q + \sigma_k^2),$$

for one $k \in \{1, 2\}$.

Proof. If K_u^2 has a unique eigenvalue with multiplicity 2, *i.e.* $\sigma_1^2 = \sigma_2^2$, then $\ell_1 = \ell_2 = Q^2/2 \neq 0$ by Lemma 4.1. So for one of the ℓ_k to cancel out, K_u^2 must have two distinct eigenvalues $\sigma_1^2 > \sigma_2^2$. In that case, we have

$$\begin{cases} \ell_1 + \ell_2 = Q^2, \\ \ell_1(Q + \sigma_1^2) + \ell_2(Q + \sigma_2^2) = |J|^2. \end{cases}$$

This system can be solved, and we find

$$\begin{cases} \ell_1 = \dfrac{|J|^2 - Q^2(Q + \sigma_2^2)}{\sigma_1^2 - \sigma_2^2}, \\ \ell_2 = \dfrac{Q^2(Q + \sigma_1^2) - |J|^2}{\sigma_1^2 - \sigma_2^2}, \end{cases}$$

which proves the corollary. □

Remark 23. Suppose that for some solution $t \mapsto u(t)$ in $\mathcal{V}(4)$, $u(t_n)$ is not bounded in H^s for some $s > \frac{1}{2}$ and some sequence of times t_n. Then by Proposition 3.12, there exists $v \in \mathcal{V}(d)$, $d \leq 2$, such that $u(t_n) \rightharpoonup v$ in $H^{1/2}$ up to extraction. In fact, since $J(u(t_n)) = J(v)$ and $Q(u(t_n)) = Q(v)$ by Rellich's theorem, we cannot have $v \in \mathcal{V}(d)$ for $d \leq 1$, otherwise we would have $|J(u(t_n))|^2 = Q(u(t_n))^3$, which is not the case by the preceding corollary. Therefore,

$$v(z) = \frac{\alpha_\infty}{1 - p_\infty z},$$

with $\alpha_\infty, p_\infty \in \mathbb{C}$, $0 < |p_\infty| < 1$. It means that one of the two poles of $u(t_n)$ goes to $\mathbb{T}$, whereas the other stays away from $\mathbb{T}$ and from infinity.

4.2 Growing Sobolev norms in $\mathcal{V}(4)$

The purpose of this paragraph is to prove the first part of Theorem 10 : solutions in $\mathcal{V}(4)$ have growing Sobolev norms if and only $\ell_1 = 0$.

Throughout we fix $u_0 \in \mathcal{V}(4)$ such that $\Xi_{u_0}^K = \{\sigma_1 > \sigma_2\}$, and

$$(\ell_1(u_0), \ell_2(u_0)) \in \{(0, Q(u_0)^2), (Q(u_0)^2, 0)\}.$$

We denote by $u(t)$ the solution of (1.1) such that $u(0) = u_0$. Begin with an obvious consequence of the previous results :

Lemma 4.3. *For all $t \in \mathbb{R}$, σ_1 and σ_2 are K-dominant.*

Proof. By Proposition 3.3, if there was a phenomenon of crossing at some time t, we would have $\ell_1 < 0$ or $\ell_2 < 0$, which is excluded by our hypothesis. □

Thus we call $\rho_1^2(t)$, $\rho_2^2(t)$ the simple eigenvalues of $H^2_{u(t)}$, satisfying $\forall t \in \mathbb{R}$,

$$\rho_1(t) > \sigma_1 > \rho_2(t) > \sigma_2 > 0.$$

We can also define ξ_1 and ξ_2 as in (3.4) for all times, and we denote by $u_1 := [u(t)]_1^K$, $u_2 := [u(t)]_2^K$ with an implicit time-dependence. In particular, we have by Lemma 4.1 and Proposition 3.3 :

$$\bar{J} = \xi_1 \|u_1\|^2_{L^2} + \xi_2 \|u_2\|^2_{L^2}, \tag{4.1}$$

$$\ell_1 = \|u_1\|^2_{L^2}(2Q + \sigma_1^2 - |\xi_1|^2), \tag{4.2}$$

$$\ell_2 = \|u_2\|^2_{L^2}(2Q + \sigma_2^2 - |\xi_2|^2). \tag{4.3}$$

The main lemma of this paragraph is the following :

Lemma 4.4.
- *Suppose that $\ell_1(u_0) = 0$. Then $\|u_1\|^2_{L^2}$ goes exponentially fast to zero in both time directions.*
- *Suppose that $\ell_2(u_0) = 0$. Then there exists a constant $C > 0$ such that*

$$\|u_2\|^2_{L^2} \geq C > 0,$$

uniformly in time.

Proof. Let us denote by $x := \|u_1\|^2_{L^2}$. We have $\|u_2\|^2_{L^2} = Q - x$. Recall from (3.3) that $\dot{x} = 2x \operatorname{Im}(J\xi_1)$. Using (4.1), we then have

$$\dot{x} = 2x(Q - x) \operatorname{Im}(\xi_1 \overline{\xi_2}).$$

Moreover, (4.1) shows that $|J|^2 = |\xi_1|^2 x^2 + |\xi_2|^2 (Q-x)^2 + 2x(Q-x)\operatorname{Re}(\xi_1\overline{\xi_2})$. Therefore, we get

$$(\dot{x})^2 = 4x^2(Q-x)^2|\xi_1|^2|\xi_2|^2 - \left(|J|^2 - |\xi_1|^2x^2 - |\xi_2|^2(Q-x)^2\right)^2. \tag{4.4}$$

Suppose now that $\ell_1 = 0$. Corollary 4.2 says that then $|J|^2 = Q^2(Q + \sigma_2^2)$ and $\ell_2 = Q^2$. Then by (4.2) and (4.3), we have

$$|\xi_1|^2 = 2Q + \sigma_1^2$$
$$|\xi_2|^2(Q-x) = (2Q + \sigma_2^2)(Q-x) - Q^2$$

Coming back to (4.4), this gives

$$(\dot{x})^2 = 4x^2(Q-x)(2Q+\sigma_1^2)((2Q+\sigma_2^2)(Q-x) - Q^2) \\ - \left(Q^2(Q+\sigma_2^2) - (2Q+\sigma_1^2)x^2 - (Q-x)((2Q+\sigma_2^2)(Q-x) - Q^2)\right)^2.$$

Since $(2Q+\sigma_2^2)(Q-x) - Q^2 = (Q+\sigma_2^2)Q - x(2Q+\sigma_2^2)$, we find a simplification in the large squared parenthesis, and we get

$$\left(\frac{\dot{x}}{x}\right)^2 = 4(2Q+\sigma_1^2)(Q-x)((Q+\sigma_2^2)Q - x(2Q+\sigma_2^2)) - \left(Q(3Q+2\sigma_2^2) - x(4Q+\sigma_1^2+\sigma_2^2)\right)^2.$$

If we now develop the different terms crudely, we end up at

$$\left(\frac{\dot{x}}{x}\right)^2 = Q^2 P\left(\frac{\sigma_1^2 - \sigma_2^2}{Q} x\right), \tag{4.5}$$

where

$$P(X) = \left[4(Q+\sigma_2^2)(\sigma_1^2-\sigma_2^2)-Q^2\right] - 2X(3Q+2\sigma_2^2) - X^2. \tag{4.6}$$

We thus find an equation which is of the same type of the one on $\mathcal{V}(3)$ while analysing the case $|J|^2 = Q^3$ (see [62]). The analysis here goes the same. We see that $P(X) \to -\infty$ as $X \to \pm\infty$, and equation (4.5) implies that P also takes at least one nonnegative value on $(0,+\infty)$ (because $x(t) > 0$ for all $t \in \mathbb{R}$). So P is real-rooted, and its roots λ_1, λ_2 cannot be both non-positive. Furthermore, they satisfy

$$\lambda_1 + \lambda_2 = -2(3Q + 2\sigma_2^2) < 0.$$

This equation implies that λ_1 and λ_2 cannot be both non-negative either. Hence they must have different signs : one of them is strictly negative, and the other must be strictly positive. In particular,

$$P(0) = 4(Q+\sigma_2^2)(\sigma_1^2-\sigma_2^2) - Q^2 > 0.$$

Setting $y := (\sigma_1^2 - \sigma_2^2)Q^{-1}x$, we can then write equation (4.5) in the form :

$$(\dot{y})^2 = A^2y^2(y+a)(b-y)$$

for some constants A, a, $b > 0$.

This equation can be solved explicitely : there exists $t_0 \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, we have

$$y(t) = \frac{2ab}{(a-b)+(a+b)\cosh(\tau(t-t_0))}, \qquad \tau := A\sqrt{ab}.$$

We see on this formula that y (hence $x = \|u_1\|_{L^2}^2$) decreases exponentially fast in both time directions. The rate is given by

$$\tau = Q\sqrt{|\lambda_1\lambda_2|} = Q\sqrt{4(Q+\sigma_2^2)(\sigma_1^2-\sigma_2^2)-Q^2}.$$

It remains to treat the case when $\ell_2 = 0$. Taking the computation back from the beginning, we see that the equation on $x := \|u_2\|_{L^2}^2$ is given by (4.5), changing x into $-x$ and exchanging the indices 1 and 2. Thus $\dot{x}$ satisfies the same equation as (4.5), but the polynomial is now $\tilde{P}$ and can be deduced from (4.6) :

$$\tilde{P}(X) = \left[-4(Q+\sigma_1^2)(\sigma_1^2-\sigma_2^2)-Q^2\right] - 2X(3Q+2\sigma_1^2) - X^2.$$

Yet it can been seen directly now that $\tilde{P}(0) < 0$, so it imposes that $x = \|u_2\|_{L^2}^2$ remains bounded away from 0. □

At this point, Lemma 4.4 shows that when $\ell_1(u_0) = 0$ in $\mathcal{V}(4)$, the the corresponding solution satisfies

$$\forall s > \frac{1}{2}, \qquad \lim_{t\to\pm\infty} \|u(t)\|_{H^s} = +\infty.$$

Indeed, the existence of a sequence t_n such that $\|u(t_n)\|_{H^{\underline{s}}} \leq C < +\infty$ for some $\underline{s} > \frac{1}{2}$ would imply that u_1 would not go to zero along this sequence t_n, which would contradict the result of Lemma 4.4.

4.3 Determination of the rate of growth

To prove Theorem 10, it remains to show that the growth of Sobolev norms is exponential in time in the case when $\ell_1 = 0$. This can be seen through the inverse formula of Theorem 11, which will enable us to prove the following result :

Lemma 4.5. *Let $u_0 \in \mathcal{V}(4)$ such that $\ell_1(u_0) = 0$. Let $t \mapsto u(t)$ be the corresponding solution of (1.1). Then there exists a time $t_0 > 0$ such that for all $t \in \mathbb{R}$ with $|t| \geq t_0$, $u(t)$ has two distinct poles, one of which comes close to the unit circle $\partial\mathbb{D} \subset \mathbb{C}$ exponentially fast in time.*

Proof. As above, we denote the singular values associated to $u(t)$ by $\rho_1 > \sigma_1 > \rho_2 > \sigma_2$, where ρ_1 and ρ_2 depend on time. Under the hypothesis of the lemma, we have seen that $u_1^K := [u(t)]_1^K$ goes to zero exponentially fast. Together with the formula coming from Proposition 2.3 :

$$\|u_1^K\|_{L^2}^2 = \frac{(\rho_1^2 - \sigma_1^2)(\sigma_1^2 - \rho_2^2)}{(\sigma_1^2 - \sigma_2^2)},$$

it means that at least one of the ρ_j's shrinks exponentially fast to σ_1. Notice that we have seen during the proof of Lemma 4.1 that $Q = \operatorname{Tr} H_u^2 - \operatorname{Tr} K_u^2$, so in our case,

$$Q = \rho_1^2 - \sigma_1^2 + \rho_2^2 - \sigma_2^2. \tag{4.7}$$

In particular, the ρ_j's both converge exponentially fast to some limit.

Define the angles $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathbb{T}$ (depending on time) so that

$$\begin{aligned} H_u\left(u_1^H\right) &= \rho_1 e^{i\varphi_1} u_1^H, \quad K_u\left(u_1^K\right) = \sigma_1 e^{i\psi_1} u_1^K, \\ H_u\left(u_2^H\right) &= \rho_2 e^{i\varphi_2} u_2^H, \quad K_u\left(u_2^K\right) = \sigma_2 e^{i\psi_2} u_2^K. \end{aligned}$$

Adapting Theorem 11 to this context of simple singular values (involving Blaschke products of degree 0 only) shows that $u(t,z)$ is simply given by the sum of the coefficients of the inverse of the following matrix :

$$\mathscr{C}(z) := \begin{pmatrix} \dfrac{\rho_1 e^{i\varphi_1} - \sigma_1 e^{i\psi_1} z}{\rho_1^2 - \sigma_1^2} & \dfrac{\rho_1 e^{i\varphi_1} - \sigma_2 e^{i\psi_2} z}{\rho_1^2 - \sigma_2^2} \\ \dfrac{\rho_2 e^{i\varphi_2} - \sigma_1 e^{i\psi_1} z}{\rho_2^2 - \sigma_1^2} & \dfrac{\rho_2 e^{i\varphi_2} - \sigma_2 e^{i\psi_2} z}{\rho_2^2 - \sigma_2^2} \end{pmatrix}.$$

Since all the coefficients of $\mathscr{C}(z)$ are polynomials in z, computing $\mathscr{C}^{-1}(z)$ thanks to the cofactor matrix, we see that the poles of $u(t,\cdot)$ will be given by the inverse of the roots of $\det \mathscr{C}(z)$. As we are only interested by the modulus of these roots, we can change z into $ze^{-i\theta}$ and $\det \mathscr{C}(z)$ into $e^{-i\theta'} \det \mathscr{C}(z)$, for $\theta, \theta' \in \mathbb{T}$. So we only have to look for the zeros of

$$z \mapsto \frac{\rho_1 e^{i\varphi} - \sigma_1 e^{i\psi} z}{\rho_1^2 - \sigma_1^2} \frac{\rho_2 - \sigma_2 z}{\rho_2^2 - \sigma_2^2} - \frac{\rho_1 e^{i\varphi} - \sigma_2 z}{\rho_1^2 - \sigma_2^2} \frac{\rho_2 - \sigma_1 e^{i\psi} z}{\rho_2^2 - \sigma_1^2},$$

where we have set $\varphi := \varphi_1 - \varphi_2$ and $\psi := \psi_1 - \psi_2$. Multiplying this polynomial by $(\rho_1^2 - \sigma_1^2)(\rho_1^2 - \sigma_2^2)(\sigma_1^2 - \rho_2^2)(\rho_2^2 - \sigma_2^2)$ means that we only have to seek the roots of

$$\begin{aligned} P_t(z) := (\rho_1^2 - \sigma_2^2)(\sigma_1^2 - \rho_2^2)(\rho_1 e^{i\varphi} - \sigma_1 e^{i\psi} z)(\rho_2 - \sigma_2 z) \\ + (\rho_1^2 - \sigma_1^2)(\rho_2^2 - \sigma_2^2)(\rho_1 e^{i\varphi} - \sigma_2 z)(\rho_2 - \sigma_1 e^{i\psi} z). \end{aligned}$$

We now have to distinguish three cases :

First case : $\rho_1^2 \to \sigma_1^2$, but $\rho_2^2 \to \tau^2$, where $\sigma_2 < \tau < \sigma_1$. In that case, it appears that if we define

$$P_t^{\lim,1}(z) := \sigma_1 e^{i\psi}(\sigma_1^2 - \sigma_2^2)(\sigma_1^2 - \tau^2)(e^{i(\varphi-\psi)} - z)(\tau - \sigma_2 z),$$

then the coefficients of P_t and $P_t^{\lim,1}$ become exponentially close to each other. But so do their roots, because $P_t^{\lim,1}$ has distinct roots (one of them is of modulus 1 and the other one is of modulus $\tau/\sigma_2 > 1$), and the discriminant formulae are differentiable in the coefficients in this case. So one of the roots of P_t converges exponentially fast to the unit circle $\partial\mathbb{D}$ (and so does the corresponding pole of $u(z)$).

Second case : $\rho_2^2 \to \sigma_1^2$, but $\rho_1^2 \to (\tau')^2$, where $\tau' > \sigma_1$. This case goes as the preceding one, by considering the second term in P_t as the leading order.

Third case : $\rho_1^2 \to \sigma_1^2$ and $\rho_2^2 \to \sigma_1^2$. By the formula (4.7) for Q, it implies that $Q = \sigma_1^2 - \sigma_2^2$, and then we obtain $\rho_1^2 - \sigma_1^2 = \sigma_1^2 - \rho_2^2$. So the coefficients of $(\rho_1^2 - \sigma_1^2)^{-1} P_t(z)$ come exponentially close to those of

$$P_t^{\lim,3}(z) := \sigma_1 e^{i\psi}(\sigma_1^2 - \sigma_2^2)\left[(e^{i(\varphi-\psi)} - z)(\sigma_1 - \sigma_2 z) + (\sigma_1 e^{i\varphi} - \sigma_2 z)(e^{-i\psi} - z)\right].$$

In that case, we need something more to get to the conclusion, and this is precisely what preserves the asymptotic behaviour of $u(t)$ of just disclosing a simple crossing phenomenon, namely the fact that we also have $w_1^K := [\Pi(|u(t)|^2)]_1^K \to 0$.

We first compute w_1^K in terms of the variables ρ_j, φ_j and σ_k. Writing $u = u_1^H + u_2^H$, we have

$$w_1^K = \left(H_u(u) \middle| \frac{u_1^K}{\|u_1^K\|_{L^2}^2}\right) u_1^K = \left(\rho_1 e^{i\varphi_1} u_1^H + \rho_2 e^{i\varphi_2} u_2^H \middle| \frac{u_1^K}{\|u_1^K\|_{L^2}^2}\right) u_1^K.$$

But $(u_j^H | u_1^K)$ is easy to compute. Indeed,

$$\begin{aligned}\rho_j^2 (u_j^H | u_1^K) &= (H_u^2(u_j^H) | u_1^K) = (u_j^H | H_u^2(u_1^K)) \\ &= (u_j^H | K_u^2(u_1^K) + (u_1^K | u) u) = \sigma_1^2 (u_j^H | u_1^K) + \|u_j^H\|_{L^2}^2 \|u_1^K\|_{L^2}^2,\end{aligned}$$

so

$$(u_j^H | u_1^K) = \frac{\|u_j^H\|_{L^2}^2 \|u_1^K\|_{L^2}^2}{\rho_j^2 - \sigma_1^2}.$$

Going back to the expression of w_1^K, with the help of Proposition 2.3 again, we find

$$\begin{aligned}w_1^K &= \left(\rho_1 e^{i\varphi_1} \frac{\|u_1^H\|_{L^2}^2}{\rho_1^2 - \sigma_1^2} - \rho_2 e^{i\varphi_2} \frac{\|u_2^H\|_{L^2}^2}{\sigma_1^2 - \rho_2^2}\right) u_1^K \\ &= \left(\rho_1 e^{i\varphi_1} \frac{\rho_1^2 - \sigma_2^2}{\rho_1^2 - \rho_2^2} - \rho_2 e^{i\varphi_2} \frac{\rho_2^2 - \sigma_2^2}{\rho_1^2 - \rho_2^2}\right) u_1^K.\end{aligned}$$

Now, since $\ell_1(u) = (2Q + \sigma_1^2)\|u_1^K\|_{L^2}^2 - \|w_1^K\|_{L^2}^2 = 0$, it implies that for all times,

$$(2Q + \sigma_1^2) = \frac{\|w_1^K\|_{L^2}^2}{\|u_1^K\|_{L^2}^2},$$

hence

$$\begin{aligned}(2Q + \sigma_1^2)(\rho_1^2 - \rho_2^2)^2 &= \left|\rho_1 e^{i\varphi_1}(\rho_1^2 - \sigma_2^2) - \rho_2 e^{i\varphi_2}(\rho_2^2 - \sigma_2^2)\right|^2 \\ &= \left|\rho_1 e^{i\varphi_1}(\rho_1^2 - \rho_2^2) + (\rho_1 e^{i\varphi_1} - \rho_2 e^{i\varphi_2})(\rho_2^2 - \sigma_2^2)\right|^2.\end{aligned}$$

Therefore, $|\rho_1 e^{i\varphi_1} - \rho_2 e^{i\varphi_2}|^2$ has to go to zero exponentially fast. Developping this expression, we see that in particular, $\cos(\varphi_1 - \varphi_2) = \cos\varphi \to 0$ exponentially fast. So $\varphi \to 0$ in $\mathbb{T}$ at an exponential rate. So we can replace the polynomial $P_t^{\lim,3}$ above by

$$\tilde{P}_t^{\lim,3}(z) := 2\sigma_1 e^{i\psi}(\sigma_1^2 - \sigma_2^2)(\sigma_1 - \sigma_2 z)(e^{-i\psi} - z).$$

This finishes to show that one of the roots of P_t in $\mathbb{C}$ has to approach $\partial\mathbb{D}$ exponentially fast. □

Thanks to Lemma 4.5, we can come to the conclusion of the proof of the growth result on $\mathcal{V}(4)$.

Proof of Theorem 10. Let $t_0 \in \mathbb{R}$ as in Lemma 4.5, and write $u(t)$ as

$$u(t,z) = \frac{\alpha}{1-pz} + \frac{\beta}{1-qz},$$

where $1 - |p| \sim e^{-\kappa|t|}$ as $|t| \to +\infty$, and $|q| \leq q_{\max} < 1$. We also know that $|\alpha|$ and $|\beta|$ are bounded functions (for instance, by the proof of Proposition 3.12).

Observe that for some constant $C > 0$, we have

$$\frac{1}{C} \leq \left\| \frac{\alpha}{1-pz} \right\|_{H^{1/2}}^2 \leq C.$$

The right bound is immediate with the one on q and on u. The left bound comes from the fact that u cannot come arbitrarily close to $\mathcal{V}(2)$ in $H^{1/2}$: if there was a sequence of times t_n such that

$$\left\| u(t_n) - \frac{\beta(t_n)}{1-q(t_n)z} \right\|_{H^{1/2}} \longrightarrow 0,$$

then by compacity (since β is bounded and q is bounded away from $\partial\mathbb{D}$), $\beta(t_n)/(1-q(t_n)z)$ would converge along some subsequence to some $v_\infty \in \mathcal{V}(2)$ strongly in $H^{1/2}$. Then $\|u(t_n) - v_\infty\|_{H^{1/2}}$ would go to zero, but this cannot happen, since $K_{u(t_n)}^2$ has to distinct constant eigenvalues, whereas $K_{v_\infty}^2$ has only one. This fact can also be proved by invoking the stability of $\mathcal{V}(2)$ in $H_+^{1/2}$, which is established in [64, Section 4].

As a consequence,

$$\frac{1}{C}(1-|p|^2)^2 \leq |\alpha|^2 \leq C(1-|p|^2)^2.$$

Besides, the study of the power series $\sum x^j j^{2s}$ as $x \to 1^-$ shows that

$$\left\| \frac{1}{1-pz} \right\|_{H^s}^2 = \sum_{j=0}^{\infty} |p|^{2j}(1+j^2)^s \simeq C_s(1-|p|^2)^{-(1+2s)}$$

as $|p| \to 1$, for some constant $C_s > 0$. Therefore, if $s > \frac{1}{2}$,

$$\frac{1}{C'}\left(\frac{1}{1-|p|^2}\right)^{2s-1} \leq \left\| \frac{\alpha}{1-pz} \right\|_{H^s}^2 \simeq \|u\|_{H^s}^2 \leq C'\left(\frac{1}{1-|p|^2}\right)^{2s-1},$$

which concludes the proof. □

4.4 Example of an initial data in $\mathcal{V}(4)$ with $\ell_1 = 0$

We conclude this section by giving an example showing that the condition $\ell_1 = 0$ can indeed occur on $\mathcal{V}(4)$. This will finish the proof of the existence of unbounded orbits in H^s inside $\mathcal{V}(4)$.

Proposition 4.6. *Let $p \in \mathbb{C}$ with $0 < |p| < 1$, and fix*

$$u(z) := \frac{z}{(1-pz)^2}, \quad \forall z \in \mathbb{D}.$$

Then $u \in \mathcal{V}(4)$, K_u^2 has two distinct eigenvalues, and $\ell_2(u) \neq 0$. In addition, $\ell_1(u) = 0$ if and only if $|p|^2 = 3\sqrt{2} - 4$.

Proof. We first compute Q and J. Observe that

$$Q(u) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{2ix}}{(1-pe^{ix})^2 (e^{ix} - \bar{p})^2} dx = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{z}{(1-pz)^2 (z-\bar{p})^2} dz,$$

where $\mathcal{C}$ denotes the unit circle in $\mathbb{C}$. To calculate this contour integral, we use the residue formula, so we compute

$$\mathrm{Res}_{z=\bar{p}} \left[\frac{z}{(1-pz)^2 (z-\bar{p})^2} \right] = \frac{d}{dz}\bigg|_{z=\bar{p}} \left[\frac{z}{(1-pz)^2} \right].$$

Let $r := |p|^2$. Then we find $Q(u) = \frac{1+r}{(1-r)^3}$. Similarly,

$$J(u) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{z^2}{(1-pz)^4 (z-\bar{p})^2} dz = \frac{d}{dz}\bigg|_{z=\bar{p}} \left[\frac{z^2}{(1-pz)^4} \right] = 2Q(u) \frac{\bar{p}}{(1-|p|^2)^2}.$$

So the following expressions are established :

$$|J(u)|^2 = \frac{4r(1+r)^2}{(1-r)^{10}}, \qquad \frac{|J(u)|^2}{Q(u)^2} = \frac{4r}{(1-r)^4}.$$

It remains to find the expression of the eigenvalues of K_u^2. Since $K_u = H_{S^*u}$, we define

$$\tilde{u}(z) := \frac{1}{(1-pz)^2},$$

and we study $H_{\tilde{u}}^2$. We know from [16, Appendix 4] that the image of $H_{\tilde{u}}$ is generated by $e_1(z) := \frac{1}{1-pz}$ and by $e_2(z) := \frac{1}{(1-pz)^2}$. By the means of a partial fraction decomposition, we find

$$H_{\tilde{u}}(e_1) = \frac{|p|^2}{(1-|p|^2)^2} e_1 + \frac{1}{1-|p|^2} e_2,$$
$$H_{\tilde{u}}(e_2) = \frac{2|p|^2}{(1-|p|^2)^3} e_1 + \frac{1}{(1-|p|^2)^2} e_2.$$

We can compute the matrix of $H_{\tilde{u}}^2$ in the basis $((1-r)^{-1} e_1, e_2)$. It reads

$$\frac{1}{(1-r)^4} \begin{bmatrix} r(2+r) & 1+r \\ 2r(1+r) & 1+2r \end{bmatrix}.$$

We have $\operatorname{Tr} H_{\tilde{u}}^2 = (1-r)^{-4}(1+4r+r^2)$, and $\det H_{\tilde{u}}^2 = (1-r)^{-8}r[(2+r)(1+2r) - 2(1+r)^2] = (1-r)^{-8}r^2$. Thus the characteristic polynomial of $H_{\tilde{u}}^2$ equals

$$\chi(X) = \frac{1}{(1-r)^8}P((1-r)^4X),$$

where $P(X) = r^2 - (1+4r+r^2)X + X^2$. We deduce from P the eigenvalues of K_u^2 :

$$\frac{1+4r+r^2 \pm (1+r)\sqrt{1+6r+r^2}}{2(1-r)^4},$$

where the $+$ sign corresponds to σ_1^2 and the $-$ sign to σ_2^2. Note that $\sigma_1 > \sigma_2$ indeed.

Compute

$$Q + \sigma_j^2 = \frac{3+4r-r^2 \pm (1+r)\sqrt{1+6r+r^2}}{2(1-r)^4},$$

so $|J|^2 = Q^2(Q+\sigma_j^2)$ if and only if

$$8r = 3+4r-r^2 \pm (1+r)\sqrt{1+6r+r^2}. \tag{4.8}$$

This implies that $(3-4r-r^2)^2 = (1+r)^2(1+6r+r^2)$, and developping this expression, the terms in r^4 and r^3 cancel out. We end up with an equation of degree 2 on r :

$$r^2 + 8r - 2 = 0.$$

This equation ony has one positive solution, $r = 3\sqrt{2} - 4$. Going back to (4.8), we see that only the $-$ sign is consistent. Consequently, if $|p|^2 = 3\sqrt{2} - 4$, then $|J|^2 = Q^2(Q+\sigma_2^2)$ and therefore $\ell_1(u) = 0$, whereas $\ell_2(u) = 0$ never occurs for functions of the type $\frac{z}{(1-pz)^2}$. □

5 Computation of the Poisson brackets

In the last part of this paper, we intend to finish the proof of Theorem 9 by proving the Poisson-commutation of the conservation laws of the quadratic Szegő equation. Throughout this section, the notation $\|\cdot\|$ will always refer to the L^2 norm.

5.1 The generating series

Recall some notations : for $u \in H_+^{1/2}$, and $n \geq 1$, we set $J_n(u) = (H_u^n(1)|1)$. In particular, $J_2 = Q$ and $J_3 = J$ — in the sequel, we prefer these harmonized notations. We also define, for $x \in \mathbb{R}$ such that $\frac{1}{x} \notin \operatorname{Sp}(H_u^2)$, and $m \geq 0$,

$$\mathscr{J}^{(m)}(x) := \left((I - xH_u^2)^{-1}H_u^m(1)|1\right) = \sum_{j=0}^{+\infty} x^j J_{m+2j}.$$

The first result we establish is the following alternative form for the generating series :

Proposition 5.1. *Let $u \in H_+^{1/2}$, and denote by σ_k, ℓ_k, $k \geq 1$, the conservation laws associated to u as defined above. Then*

$$\sum_{1\leq k\leq\infty} \frac{\ell_k}{1 - x\sigma_k^2} = \mathscr{R}(x) := \frac{J_2^2 + x|\mathscr{J}^{(3)}(x)|^2 - x^2\mathscr{J}^{(4)}(x)^2}{\mathscr{J}^{(0)}(x)}. \tag{5.1}$$

Remark 24. We should observe that to some extent, Lemma 5.1 is a generalization of the formulae of Lemma 4.1, that we can recover here by developping $\mathscr{R}(x)$ as a power series.

We are going to express the right hand side of (5.1) in terms of the resolvant of K_u. As above, we set, for appropriate $x \in \mathbb{R}$,

$$\begin{aligned} \mathscr{K}^{(0)}(x) &:= ((I - xK_u^2)^{-1}(1)|1), \\ \mathscr{K}^{(1)}(x) &:= ((I - xK_u^2)^{-1}(u)|1), \\ \mathscr{K}^{(2)}(x) &:= ((I - xK_u^2)^{-1}(u)|u). \end{aligned}$$

Lemma 5.2. *For all $x \in \mathbb{R}$ such that it is defined, we have*

$$2 + 2xJ_2 - x^2\mathscr{R}(x) = \mathscr{K}^{(0)}(x) + 2x\,\mathrm{Re}(\overline{J_1}\mathscr{K}^{(1)}(x)) + (1 - x\mathscr{K}^{(2)}(x))\left(1 + x(2J_2 - |J_1|^2)\right).$$

Proof. The proof relies on identities discovered in [21, 68] that we recall here. Since $K_u^2 = H_u^2 - (\cdot|u)u$ (see (1.4)), we have, for $h \in L^2_+$,

$$\begin{aligned} (I - xK_u^2)^{-1}(h) - (I - xH_u^2)^{-1}(h) &= (I - xK_u^2)^{-1}\left[(I - xH_u^2) - (I - xK_u^2)\right](I - xH_u^2)^{-1}(h) \\ &= -x(h|(I - xH_u^2)^{-1}(u))(I - xK_u^2)^{-1}(u). \end{aligned}$$

Taking $h = u$ yields

$$(I - xH_u^2)^{-1}(u) = \mathscr{J}^{(0)}(x) \cdot (I - xK_u^2)^{-1}(u), \tag{5.2}$$

and taking $h = 1$ gives, once we have made the scalar product with 1,

$$((I - xK_u^2)^{-1}(1)|1) - ((I - xH_u^2)^{-1}(1)|1) = -x\overline{\mathscr{J}^{(1)}(x)}\mathscr{K}^{(1)}(x).$$

Since $\mathscr{J}^{(1)}(x) = \mathscr{J}^{(0)}(x)\mathscr{K}^{(1)}(x)$ by (5.2), this can also be written as

$$\mathscr{K}^{(0)}(x) = \mathscr{J}^{(0)}(x)(1 - x|\mathscr{K}^{(1)}(x)|^2) = \mathscr{J}^{(0)}(x) - x\frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(x)}. \tag{5.3}$$

Observe also that $\mathscr{J}^{(0)}(x) = 1 + x\mathscr{J}^{(2)}(x) = 1 + x\mathscr{J}^{(0)}(x)\mathscr{K}^{(2)}(x)$, hence

$$\frac{1}{\mathscr{J}^{(0)}(x)} = 1 - x\mathscr{K}^{(2)}(x). \tag{5.4}$$

Finally, we have $J_2 + x\mathscr{J}^{(4)}(x) = \mathscr{J}^{(2)}(x)$ and $J_1 + x\mathscr{J}^{(3)}(x) = \mathscr{J}^{(1)}(x)$.

Now we are ready to transform the expression of $\mathscr{R}(x)$, using (5.3), and (5.4) together with (5.2) :

$$\begin{aligned} x^2\mathscr{R}(x) &= \frac{-x^4\mathscr{J}^{(4)}(x)^2 + x^3|\mathscr{J}^{(3)}(x)|^2 + x^2J_2^2}{\mathscr{J}^{(0)}(x)} \\ &= -\frac{x^2\left[\mathscr{J}^{(2)}(x) - J_2\right]^2}{\mathscr{J}^{(0)}(x)} + \frac{x^3}{\mathscr{J}^{(0)}(x)}\left|\frac{\mathscr{J}^{(1)}(x) - J_1}{x}\right|^2 + \frac{x^2J_2^2}{\mathscr{J}^{(0)}(x)} \\ &= -\frac{x^2\mathscr{J}^{(2)}(x)^2}{\mathscr{J}^{(0)}(x)} + 2x^2J_2\mathscr{K}^{(2)}(x) - (\mathscr{K}^{(0)}(x) - \mathscr{J}^{(0)}(x)) - \frac{2x\,\mathrm{Re}(\overline{J_1}\mathscr{J}^{(1)}(x))}{\mathscr{J}^{(0)}(x)} + \frac{x|J_1|^2}{\mathscr{J}^{(0)}(x)} \\ &= 2 - \frac{1}{\mathscr{J}^{(0)}(x)} + 2x^2J_2\mathscr{K}^{(2)}(x) - \mathscr{K}^{(0)}(x) - 2x\,\mathrm{Re}(\overline{J_1}\mathscr{K}^{(1)}(x)) + \frac{x|J_1|^2}{\mathscr{J}^{(0)}(x)} \\ &= 2 + 2xJ_2 - (1 - x\mathscr{K}^{(2)}(x))\left(1 + x(2J_2 - |J_1|^2)\right) - \mathscr{K}^{(0)}(x) - 2x\,\mathrm{Re}(\overline{J_1}\mathscr{K}^{(1)}(x)), \end{aligned}$$

and the lemma is proved. □

Now we can turn to the

Proof of Proposition 5.1. Let us first restrict to a convenient framework : we assume that $u \in \Omega_{\mathrm{gen}}$, with the further condition that $\operatorname{rk} H_u = \operatorname{rk} K_u$ (*i.e.* $u \perp \ker K_u$ and $u \in \mathcal{V}(2N)$ for some $N \in \mathbb{N}$). Thus, u is a rational function, and denoting by $\{\rho_j\}$ (resp. $\{\sigma_k\}$) the singular values of H_u (resp. K_u), all of them are of multiplicity one. In particular, this imposes all the eigenvalues of K_u^2 to be K-dominant. Then the general result will follow by density of such functions in $H_+^{1/2}$.

We are going to study the poles of $\mathscr{R}(x)$. Recall that if $h \in L^2_+$ is given, h_k^K refers to the orthogonal projection of h onto $\ker(K_u^2 - \sigma_k^2 I)$. In particular, we have

$$u = \sum_{1\leq k<\infty} u_k^K,$$
$$1 = \sum_{1\leq k\leq\infty} 1_k^K,$$

all the sums being finite. As a consequence, we can write

$$\mathscr{K}^{(0)}(x) = \sum_{1\leq k\leq\infty} \frac{\|1_k^K\|^2}{1 - x\sigma_k^2},$$
$$\mathscr{K}^{(1)}(x) = \sum_{1\leq k<\infty} \frac{(u_k^K|1)}{1 - x\sigma_k^2},$$
$$\mathscr{K}^{(2)}(x) = \sum_{1\leq k<\infty} \frac{\|u_k^K\|^2}{1 - x\sigma_k^2}.$$

By Lemma 5.2 and the previous expressions, we then see that $\mathscr{R}(x)$ is a rational function of x, that it has simple poles at each $\frac{1}{\sigma_k^2}$, and that its limit as $x \to +\infty$ equals 0.

Besides, multiplying the equality in Lemma 5.2 by $(1 - x\sigma_k^2)$ and evaluating at $x = 1/\sigma_k^2$ gives the following formula for the poles of $\mathscr{R}(x)$:

$$\alpha_k := \|u_k^K\|^2(2J_2 - |J_1|^2 + \sigma_k^2) - 2\sigma_k^2 \operatorname{Re}\left(\overline{J_1}(u_k^K|1)\right) - \sigma_k^4\|1_k^K\|^2.$$

It remains to show that $\alpha_k = \ell_k$ for all $1 \leq k < \infty$. Using the fact that 1_k^K is colinear to u_k^K because of our assumption on the dimension of the eigenspaces of K_u^2, we have

$$\begin{aligned}\alpha_k &= \|u_k^K\|^2(2Q - |(u|1)|^2 + \sigma_k^2) - 2\sigma_k^2 \operatorname{Re}\left((1|u)(u_k^K|1)\right) - \sigma_k^4\|1_k^K\|^2\\ &= \|u_k^K\|^2\left(2Q + \sigma_k^2 - |(u|1)|^2 - 2\sigma_k^2 \operatorname{Re}\left((1|u)\frac{(u_k^K|1)}{\|u_k^K\|^2}\right) - \sigma_k^4\frac{|(u_k^K|1)|^2}{\|u_k^K\|^4}\right)\\ &= \|u_k^K\|^2\left(2Q + \sigma_k^2 - \left|(u|1) + \sigma_k^2\frac{(u_k^K|1)}{\|u_k^K\|^2}\right|^2\right).\end{aligned}$$

Because of formula $K_u^2 + (\cdot|u)u = H_u^2$, we have

$$(u|1) + \sigma_k^2\frac{(u_k^K|1)}{\|u_k^K\|^2} = (u|1) + \frac{\left(H_u^2(u_k^K) - (u_k^K|u)u|1\right)}{\|u_k^K\|^2} = \frac{(u_k^K|H_u^2(1))}{\|u_k^K\|^2} = \overline{\xi_k},$$

hence $\alpha_k = \|u_k^K\|^2((2Q + \sigma_k^2) - |\xi_k|^2) = \ell_k$ by Lemma 3.3. The proof of Proposition 5.1 is now complete. □

In the sequel, we prefer manipulating another generating function, coming from $\mathscr{R}(x)$, which involves functionals $\mathscr{J}^{(m)}$ that are of lower order. We thus define

$$\mathscr{F}(x) := 2J_2 - x\mathscr{R}(x).$$

Since $J_2^2 - x^2\mathscr{J}^{(4)}(x)^2 = (J_2 - x\mathscr{J}^{(4)}(x))(J_2 + x\mathscr{J}^{(4)}(x)) = (2J_2 - \mathscr{J}^{(2)}(x))\mathscr{J}^{(2)}(x)$, we get

$$\begin{aligned}\mathscr{F}(x) &= \frac{2J_2\mathscr{J}^{(0)}(x) - x^2|\mathscr{J}^{(3)}(x)|^2 - x(J_2^2 - x^2\mathscr{J}^{(4)}(x)^2)}{\mathscr{J}^{(0)}(x)}\\ &= \frac{2J_2(\mathscr{J}^{(0)}(x) - x\mathscr{J}^{(2)}(x)) - x^2|\mathscr{J}^{(3)}(x)|^2 + x\mathscr{J}^{(2)}(x)^2}{\mathscr{J}^{(0)}(x)}\\ &= \frac{2J_2 + x\mathscr{J}^{(2)}(x)^2 - x^2|\mathscr{J}^{(3)}(x)|^2}{\mathscr{J}^{(0)}(x)}.\end{aligned}$$

Since $\mathscr{R}(x)$ is invariant by rotation of u by $e^{i\theta}$, we have $\{J_2, \mathscr{R}(x)\} = 0$, hence

$$\{\mathscr{F}(x), \sigma_k^2\} = -x\{\mathscr{R}(x), \sigma_k^2\}, \qquad \{\mathscr{F}(x), \mathscr{F}(y)\} = xy\{\mathscr{R}(x), \mathscr{R}(y)\},$$

so from now on, we only study $\mathscr{F}(x)$.

5.2 A Lax pair for $\mathscr{F}(x)$

As in [16, 18], it is of high importance to study the evolution of the Hamiltonian system generated by $\mathscr{F}(x)$ (where $x \in \mathbb{R}$ is fixed). In particular, we are going to prove that the evolution given by $\dot{u} = X_{\mathscr{F}(x)}(u)$ also admits a Lax pair for K_u. As a consequence, the k-th eigenvalue of K_u^2 will be conserved by this flow, so on Ω_{gen} we will obtain the identity

$$\{\mathscr{F}(x), \sigma_k^2\} = 0, \quad \forall k \geq 1.$$

In view of (5.1), and of the fact that $\{\sigma_j^2, \sigma_k^2\} = 0$ for any $j, k \geq 1$, we will get that

$$\{\ell_j, \sigma_k^2\} = 0, \quad \forall j, k \geq 1.$$

We first introduce th following notations :

$$\begin{aligned}w^0(x) &:= (I - xH_u^2)^{-1}(1),\\ w^1(x) &:= (I - xH_u^2)^{-1}(u).\end{aligned}$$

Note that $w^1(x) = H_u(w^0(x))$, and that we also recover w^0 from w^1 thanks to the formula $1 + xH_u(w^1(x)) = w^0(x)$.

Theorem 12. *The Hamiltonian vector field associated to the functional $\mathscr{F}(x)$ (where $x \in \mathbb{R}$ is fixed) is given by*

$$\begin{aligned}X_{\mathscr{F}(x)}(u) = \frac{-i}{\mathscr{J}^{(0)}}\Big(&4u + x(4\mathscr{J}^{(2)} - 2\mathscr{F})w^0 H_u(w^0)\\ &-2x^2\overline{\mathscr{J}^{(3)}}(H_u(w^0))^2 - 2x^3\mathscr{J}^{(3)}(H_u(w^1))^2 - 4x^2\mathscr{J}^{(3)}H_u(w^1)\Big)\end{aligned} \quad (5.5)$$

In addition, for any solution to the evolution equation $\dot{u} = X_{\mathscr{F}(x)}(u)$, we have

$$\frac{d}{dt}K_u = [B_u^x, K_u],$$

where B_u^x is a skew-symmetric operator given by

$$B_u^x = -iA_u^x, \tag{5.6}$$

$$A_u^x := \frac{1}{\mathscr{J}^{(0)}}\left(2I + (2\mathscr{J}^{(2)} - \mathscr{F})\cdot(xT_{w^0}T_{\overline{w^0}} + x^2T_{w^1}T_{\overline{w^1}}) - 2x^2\left(\mathscr{J}^{(3)}\, T_{w^0}T_{\overline{w^1}} + \overline{\mathscr{J}^{(3)}}T_{w^1}T_{\overline{w^0}}\right)\right) \tag{5.7}$$

Notice that in (5.5), (5.7), as well as in the rest of this paragraph, we omit the x dependence of functionals and functions, in order to shorten our formulae.

We will make use of an elementary lemma which we recall here :

Lemma 5.3. *Let $h \in L^2_+$. Then $(I - \Pi)(\bar{z}h) = \bar{z}\overline{\Pi(\bar{h})}$.*

Proof of Theorem 12. Recall that

$$\mathscr{F} = \frac{2Q + x(\mathscr{J}^{(2)})^2 - x^2|\mathscr{J}^{(3)}|^2}{1 + x\mathscr{J}^{(2)}}.$$

First of all, we compute, for $h \in L^2_+$,

$$\begin{aligned} d_u\mathscr{J}^{(2)}\cdot h &= (x(I - xH_u^2)^{-1}(H_uH_h + H_hH_u)(I - xH_u^2)^{-1}(u)|u) + 2\,\mathrm{Re}((I - xH_u^2)^{-1}(u)|h) \\ &= 2x\,\mathrm{Re}(H_u(w^1)|H_h(w^1)) + 2\,\mathrm{Re}(w^1|h) \\ &= 2\,\mathrm{Re}(w^0w^1|h) = 2\,\mathrm{Re}(w^0H_u(w^0)|h). \end{aligned}$$

Similarly, as $\mathscr{J}^{(3)} = ((I - xH_u^2)^{-1}(u)|H_u(u))$,

$$\begin{aligned} d_u\mathscr{J}^{(3)}\cdot h &= (x(H_uH_h + H_hH_u)w^1|H_u(w^1)) + 2(w^1|H_u(h)) + (w^1|H_h(u)) \\ &= x(H_u^2(w^1)|H_h(w^1)) + x(h|(H_u(w^1))^2) + 2(h|H_u(w^1)) + (uw^1|h) \\ &= ((H_u(w^0))^2|h) + x(h|(H_u(w^1))^2) + 2(h|H_u(w^1)), \end{aligned}$$

where we used that $xH_u^2(w^1) = w^1 - u = H_u(w^0) - u$.

We are now ready to compute

$$\begin{aligned} d_u\mathscr{F}\cdot h = &-\frac{\mathscr{F}}{\mathscr{J}^{(0)}}\cdot 2x\,\mathrm{Re}(h|w^0H_u(w^0)) + \frac{1}{\mathscr{J}^{(0)}}\left(4\,\mathrm{Re}(h|u) + 2x\mathscr{J}^{(2)}\cdot 2\,\mathrm{Re}(h|w^0H_u(w^0))\right) \\ &- \frac{2x^2}{\mathscr{J}^{(0)}}\mathrm{Re}\left(\mathscr{J}^{(3)}(h|(H_u(w^0))^2) + x\overline{\mathscr{J}^{(3)}}(h|(H_u(w^1))^2) + 2\overline{\mathscr{J}^{(3)}}(h|H_u(w^1))\right). \end{aligned}$$

Hence,

$$\begin{aligned} d_u\mathscr{F}\cdot h = \frac{1}{\mathscr{J}^{(0)}}\mathrm{Re}\Big(h\Big|&4u + x(4\mathscr{J}^{(2)} - 2\mathscr{F})w^0H_u(w^0) \\ &-2x^2\overline{\mathscr{J}^{(3)}}(H_u(w^0))^2 - 2x^3\mathscr{J}^{(3)}(H_u(w^1))^2 - 4x^2\mathscr{J}^{(3)}H_u(w^1)\Big), \end{aligned}$$

which is equivalent to formula (5.5).

Now, assume that $\dot{u} = X_{\mathscr{F}}(u)$, and compute $i\frac{d}{dt}K_u(h)$, for $h \in L^2_+$. Step by step, we have, as in [16], and using Lemma 5.3 :

$$\begin{aligned} \Pi(\bar{z}w^0H_u(w^0)\bar{h}) &= \Pi(\bar{z}(1 + xH_u^2(w^0))H_u(w^0)\bar{h}) \\ &= \Pi(\bar{z}u\overline{w^0h}) + x\Pi(H_u^2(w^0)[\Pi + (I - \Pi)](\bar{z}H_u(w^0)\bar{h})) \\ &= T_{\overline{w^0}}K_u(h) + xH_u^2(w^0)\Pi(\bar{z}u\overline{w^0h}) + x\Pi(H_u^2(w^0)\bar{z}\overline{\Pi(\overline{H_u(w^0)}h)}) \\ &= (1 + xH_u^2(w^0))T_{\overline{w^0}}K_u(h) + x\Pi(\bar{z}u\overline{w^1}\overline{\Pi(\overline{w^1}h)}) \\ &= T_{w^0}T_{\overline{w^0}}K_u(h) + xK_uT_{w^1}T_{\overline{w^1}}(h), \end{aligned}$$

which we can symmetrize as

$$K_{w^0 H_u(w^0)} = \frac{1}{2}(T_{w^0}T_{\overline{w^0}}K_u + K_u T_{w^0}T_{\overline{w^0}}) + \frac{x}{2}(T_{w^1}T_{\overline{w^1}}K_u + K_u T_{w^1}T_{\overline{w^1}}).$$

Then,

$$\begin{aligned}\Pi(\bar{z}(H_u(w^0))^2\bar{h}) &= H_u(w^0)\Pi(\bar{z}H_u(w^0)\bar{h}) + \Pi(u\overline{w^0}(I-\Pi)(\bar{z}H_u(w^0)\bar{h}))\\ &= w^1\Pi(\overline{w^0}\Pi(\bar{z}u\bar{h})) + \Pi(\bar{z}uw^0\overline{\Pi(\overline{w^1}h)}),\end{aligned}$$

so

$$K_{(H_u(w^0))^2} = T_{w^1}T_{\overline{w^0}}K_u + K_u T_{w^0}T_{\overline{w^1}}.$$

Replacing w^0 by w^1 in the previous expression, we get

$$\begin{aligned}xK_{(H_u(w^1))^2} &= xT_{H_u(w^1)}T_{\overline{w^1}}K_u + xK_u T_{w^1}T_{\overline{H_u(w^1)}}\\ &= T_{w^0}T_{\overline{w^1}}K_u + K_u T_{w^1}T_{\overline{w^0}} - T_{\overline{w^1}}K_u - K_u T_{w^1}.\end{aligned}$$

The minus terms will exactly be compensated by

$$2\Pi(\bar{z}H_u(w^1)\bar{h}) = 2\Pi(\bar{z}u\overline{w^1 h}) = \Pi(\overline{w^1}\Pi(\bar{z}u\bar{h})) + \Pi(\bar{z}u\overline{\Pi(w^1 h)}),$$

or equivalently, $2K_{H_u(w^1)} = T_{\overline{w^1}}K_u + K_u T_{w^1}$. This completes the proof of (5.6)-(5.7). □

5.3 Commutation between the additional conservation laws

In this paragraph, we conclude the proof of Theorem 9 by proving that $\{\mathscr{F}(x), \mathscr{F}(y)\} = 0$ on Ω_{gen} when $x \neq y \in \mathbb{R}$ are fixed. Because of (5.1) and by the preceding commutation identities, it will be enough to show that $\{\ell_j, \ell_k\} = 0$ when $j, k \geq 1$.

Theorem 13. *For any $x \neq y \in \mathbb{R}$, we have $\{\mathscr{F}(x), \mathscr{F}(y)\} = 0$.*

To prove such a result, we will restrict again on the dense subset of $H^{1/2}_+$ which consists of symbols $u \in H^{1/2}_+$ such that both K_u and H_u have finite rank N for some $N \in \mathbb{N}$, and so that the singular values of H_u and K_u satisfy

$$\rho_1^2 > \sigma_1^2 > \rho_2^2 > \sigma_2^2 > \cdots > \rho_N^2 > \sigma_N^2 > 0.$$

Recall that, under this assumption of genericity, we can write

$$u = \sum_{j=1}^{N} u_j^H = \sum_{k=1}^{N} u_k^K,$$

where u_j^H (resp. u_k^K) is the projection of u onto the one-dimensional eigenspace of H_u^2 (resp. K_u^2) associated to ρ_j^2 (resp. σ_k^2). In that case, Proposition 2.2 also simplifies, and Blaschke products are just real numbers modulo 2π : there exists angles $(\varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_N) \in \mathbb{T}^{2N}$, such that $H_u(u_j^H) = \rho_j e^{i\varphi_j} u_j^H$ and $K_u(u_k^K) = \sigma_k e^{i\psi_k} u_k^K$, for $j \in [\![1, N]\!]$ and $k \in [\![1, N]\!]$. Moreover, on this open subset of generic sates of $\mathcal{V}(2N)$, the symplectic form reads $\omega = \sum_{j=1}^N d(\rho_j^2/2) \wedge d\varphi_j + \sum_{k=1}^N d(\sigma_k^2/2) \wedge d\psi_k$ (see [18, 21]).

We begin by proving a lemma inspired by the work of Haiyan Xu [68].

Lemma 5.4. *For any $x \neq y \in \mathbb{R}$, we have*

$$\left\{|\mathscr{J}^{(1)}(x)|^2, |\mathscr{J}^{(1)}(y)|^2\right\} = \frac{4\,\mathrm{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)})}{x-y}\left[x\mathscr{J}^{(0)}(x)^2 - y\mathscr{J}^{(0)}(y)^2 + x^2|\mathscr{J}^{(1)}(x)|^2 - y^2|\mathscr{J}^{(1)}(y)|^2\right].$$

Proof. Recall that we defined $\mathscr{K}^{(0)}(x) = ((I - xK_u^2)^{-1}(1)|1)$, and that it obeys

$$\mathscr{K}^{(0)}(x) = \mathscr{J}^{(0)}(x) - x\frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(x)}$$

by (5.3).

From the theory of the cubic Szegő equation, it is known that $\{\mathscr{J}^{(0)}(x), \mathscr{J}^{(0)}(y)\} = 0$ (see [16]) and $\{\mathscr{K}^{(0)}(x), \mathscr{K}^{(0)}(y)\} = 0$ (see [68]). In view of (5.3), this last identity gives

$$\begin{aligned} 0 &= \left\{\mathscr{J}^{(0)}(x) - x\frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(x)}, \mathscr{J}^{(0)}(y) - y\frac{|\mathscr{J}^{(1)}(y)|^2}{\mathscr{J}^{(0)}(y)}\right\} \\ &= -y\left\{\mathscr{J}^{(0)}(x), \frac{|\mathscr{J}^{(1)}(y)|^2}{\mathscr{J}^{(0)}(y)}\right\} + x\left\{\mathscr{J}^{(0)}(y), \frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(x)}\right\} + xy\left\{\frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(x)}, \frac{|\mathscr{J}^{(1)}(y)|^2}{\mathscr{J}^{(0)}(y)}\right\}. \end{aligned}$$

First of all, we have to compute $\{\mathscr{J}^{(0)}(x), |\mathscr{J}^{(1)}(y)|^2\}$. This is done in [68], but for the seek of completeness, we recall the argument. We have

$$\mathscr{J}^{(1)}(x) = \left(\sum_{j=1}^N \frac{u_j^H}{1 - x\rho_j^2}\middle| \sum_{j=1}^N \frac{(1|u_j^H)u_j^H}{\|u_j^H\|^2}\right) = \sum_{j=1}^N \frac{\|u_j^H\|^2 e^{-i\varphi_j}}{\rho_j(1 - x\rho_j^2)},$$

because $(u_j^H|1) = \rho_j^{-1}e^{-i\varphi_j}(H_u(u_j^H)|1)$, and $(H_u(u_j^H)|1) = (H_u(1)|u_j^H) = \|u_j^H\|^2$. Besides, we know ([21]) that $\mathscr{J}^{(0)}(x) = \prod_{j=1}^N \frac{1 - x\sigma_j^2}{1 - x\rho_j^2}$, we can compute directly from the expression of ω :

$$\begin{aligned} \{\mathscr{J}^{(0)}(x), \mathscr{J}^{(1)}(y)\} &= \sum_{j=1}^N \frac{2x\mathscr{J}^{(0)}(x)}{1 - x\rho_j^2} \cdot \left(-i\frac{\|u_j^H\|^2 e^{-\varphi_j}}{\rho_j(1 - y\rho_j^2)}\right) \\ &= -2ix\mathscr{J}^{(0)}(x)\sum_{j=1}^N \frac{\|u_j^H\|^2 e^{-i\varphi_j}}{\rho_j(x - y)}\left(\frac{x}{1 - x\rho_j^2} - \frac{y}{1 - y\rho_j^2}\right) \\ &= -\frac{2ix}{x - y}\mathscr{J}^{(0)}(x)(x\mathscr{J}^{(1)}(x) - y\mathscr{J}^{(1)}(y)). \end{aligned}$$

This yields

$$\{\mathscr{J}^{(0)}(x), |\mathscr{J}^{(1)}(y)|^2\} = 2\,\mathrm{Re}(\overline{\mathscr{J}^{(1)}(y)}\{\mathscr{J}^{(0)}(x), \mathscr{J}^{(1)}(y)\}) = \frac{4x^2\mathscr{J}^{(0)}(x)}{x - y}\,\mathrm{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)}). \tag{5.8}$$

Secondly, we write

$$\begin{aligned} \left\{\frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(x)}, \frac{|\mathscr{J}^{(1)}(y)|^2}{\mathscr{J}^{(0)}(y)}\right\} &= \left\{\frac{1}{\mathscr{J}^{(0)}(x)}, |\mathscr{J}^{(1)}(y)|^2\right\}\frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(y)} \\ &\quad - \left\{\frac{1}{\mathscr{J}^{(0)}(y)}, |\mathscr{J}^{(1)}(x)|^2\right\}\frac{|\mathscr{J}^{(1)}(y)|^2}{\mathscr{J}^{(0)}(x)} + \frac{\{|\mathscr{J}^{(1)}(x)|^2, |\mathscr{J}^{(1)}(y)|^2\}}{\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)}, \end{aligned}$$

and we have, using (5.8),

$$\left\{\frac{1}{\mathscr{J}^{(0)}(x)}, |\mathscr{J}^{(1)}(y)|^2\right\}\frac{|\mathscr{J}^{(1)}(x)|^2}{\mathscr{J}^{(0)}(y)} = -\frac{4x^2|\mathscr{J}^{(1)}(x)|^2}{(x - y)\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)}\,\mathrm{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)}).$$

Now we can go back to $0 = \{\mathscr{K}^{(0)}(x),\mathscr{K}^{(0)}(y)\}$, and get

$$0 = \left(-\frac{4x^2y\mathscr{J}^{(0)}(x)}{(x-y)\mathscr{J}^{(0)}(y)} + \frac{4xy^2\mathscr{J}^{(0)}(y)}{(x-y)\mathscr{J}^{(0)}(x)}\right)\mathrm{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)})$$
$$+\left(\frac{-4x^3y|\mathscr{J}^{(1)}(x)|^2+4xy^3|\mathscr{J}^{(1)}(y)|^2}{(x-y)\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)}\right)\mathrm{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)})+\frac{xy\{|\mathscr{J}^{(1)}(x)|^2,|\mathscr{J}^{(1)}(y)|^2\}}{\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)},$$

and this ends the proof of Lemma 5.4. □

Lemma 5.5. *For $x \neq y \in \mathbb{R}$, we have*

$$\{\mathscr{J}^{(3)}(x),\mathscr{J}^{(3)}(y)\} = -\frac{2i}{x-y}\left[x\mathscr{J}^{(3)}(x) - y\mathscr{J}^{(3)}(y)\right]^2$$
$$\{\mathscr{J}^{(3)}(x),\overline{\mathscr{J}^{(3)}(y)}\} = \frac{2i}{x-y}\left[\frac{\mathscr{J}^{(0)}(x)^2}{x} - \frac{\mathscr{J}^{(0)}(y)^2}{y} - \frac{1}{x} + \frac{1}{y}\right].$$

Proof. We expand $\mathscr{J}^{(3)}(x)$ thanks to the decomposition $u = \sum_j u_j^H$:

$$\mathscr{J}^{(3)}(x) = \sum_{j=1}^{N} \frac{\rho_j\|u_j^H\|^2 e^{-i\varphi_j}}{1-x\rho_j^2}. \tag{5.9}$$

and the expression of $\|u_j^H\|^2$ is given in Proposition 2.3 :

$$\|u_j^H\|^2 = \frac{\prod_{l=1}^N(\rho_j^2-\sigma_l^2)}{\prod_{l\neq j}(\rho_j^2-\rho_l^2)}.$$

Thus we compute

$$\frac{\partial\mathscr{J}^{(3)}(x)}{\partial(\rho_j^2/2)} = \frac{\|u_j^H\|^2e^{-i\varphi_j}}{\rho_j(1-x\rho_j^2)} + \frac{2x\rho_j\|u_j^H\|^2e^{-i\varphi_j}}{(1-x\rho_j^2)^2} + \frac{\rho_je^{-i\varphi_j}}{1-x\rho_j^2}\frac{\partial\|u_j^H\|^2}{\partial(\rho_j^2/2)} + 2\sum_{l\neq j}\frac{\rho_l\|u_l^H\|^2e^{-i\varphi_l}}{(\rho_l^2-\rho_j^2)(1-x\rho_l^2)}$$

We also have $\frac{\partial\mathscr{J}^{(3)}(y)}{\partial\varphi_j} = -i\frac{\rho_j\|u_j^H\|^2e^{-i\varphi_j}}{1-y\rho_j^2}$, hence, symmetrizing in x and y, we get

$$\frac{\partial\mathscr{J}^{(3)}(x)}{\partial(\rho_j^2/2)}\frac{\partial\mathscr{J}^{(3)}(y)}{\partial\varphi_j} - \frac{\partial\mathscr{J}^{(3)}(y)}{\partial(\rho_j^2/2)}\frac{\partial\mathscr{J}^{(3)}(x)}{\partial\varphi_j} =$$
$$-2i\rho_j^2\|u_j^H\|^4e^{-2i\varphi_j}\left[\frac{x}{(1-x\rho_j^2)^2(1-y\rho_j^2)} - \frac{y}{(1-x\rho_j^2)(1-y\rho_j^2)^2}\right]$$
$$-2i\sum_{l\neq j}\frac{\rho_j\rho_l\|u_j^H\|^2\|u_l^H\|^2e^{-i(\varphi_j+\varphi_l)}}{\rho_l^2-\rho_j^2}\left[\frac{1}{(1-x\rho_l^2)(1-y\rho_j^2)} - \frac{1}{(1-x\rho_j^2)(1-y\rho_l^2)}\right].$$

Now,

$$\frac{x}{(1-x\rho_j^2)^2(1-y\rho_j^2)} - \frac{y}{(1-x\rho_j^2)(1-y\rho_j^2)^2} = \frac{x-y}{(1-x\rho_j^2)^2(1-y\rho_j^2)^2},$$
$$\frac{1}{(1-x\rho_l^2)(1-y\rho_j^2)} - \frac{1}{(1-x\rho_j^2)(1-y\rho_l^2)} = \frac{(\rho_l^2-\rho_j^2)(x-y)}{(1-x\rho_j^2)(1-x\rho_l^2)(1-y\rho_j^2)(1-y\rho_l^2)},$$

which yields

$$\frac{\partial \mathscr{J}^{(3)}(x)}{\partial(\rho_j^2/2)}\frac{\partial \mathscr{J}^{(3)}(y)}{\partial\varphi_j} - \frac{\partial \mathscr{J}^{(3)}(y)}{\partial(\rho_j^2/2)}\frac{\partial \mathscr{J}^{(3)}(x)}{\partial\varphi_j}$$
$$= -2i(x-y)\sum_{l=1}^{N}\frac{\rho_j\rho_l\|u_j^H\|^2\|u_l^H\|^2e^{-i(\varphi_j+\varphi_l)}}{(1-x\rho_j^2)(1-x\rho_l^2)(1-y\rho_j^2)(1-y\rho_l^2)}.$$

Summing over j then gives

$$\begin{aligned}\{\mathscr{J}^{(3)}(x),\mathscr{J}^{(3)}(y)\} &= -2i(x-y)\left(\sum_{j=1}^{N}\frac{\rho_j\|u_j^H\|^2e^{-i\varphi_j}}{(1-x\rho_j^2)(1-y\rho_j^2)}\right)^2\\ &= -2i(x-y)\left((I-xH_u^2)^{-1}(I-yH_u^2)^{-1}(u)|H_u(u)\right)^2\\ &= -\frac{2i}{x-y}\left((I-xH_u^2)^{-1}[(I-yH_u^2)-(I-xH_u^2)](I-yH_u^2)^{-1}(u)|1\right)^2\\ &= -\frac{2i}{x-y}(\mathscr{J}^{(1)}(x)-\mathscr{J}^{(1)}(y))^2 = -\frac{2i}{x-y}(x\mathscr{J}^{(3)}(x)-y\mathscr{J}^{(3)}(y))^2.\end{aligned}$$

This is the first part of Lemma 5.5.

We turn to the second part. We will first compute $\{\mathscr{J}^{(1)}(x),\mathscr{J}^{(1)}(y)\}$, then we will deduce $\{\mathscr{J}^{(1)}(x),\overline{\mathscr{J}^{(1)}(y)}\}$ from Lemma 5.4 and finally get the expression of $\{\mathscr{J}^{(3)}(x),\overline{\mathscr{J}^{(3)}(y)}\}$. The same computation as above also provides a formula :

$$\begin{aligned}\{\mathscr{J}^{(1)}(x),\mathscr{J}^{(1)}(y)\} &= -2i(x-y)\left((I-xH_u^2)^{-1}(I-yH_u^2)^{-1}(u)|1\right)^2\\ &= -2i(x-y)\left((I-xH_u^2)^{-1}[(I-xH_u^2)+xH_u^2](I-yH_u^2)^{-1}(u)|1\right)^2\\ &= -2i(x-y)\left(\mathscr{J}^{(1)}(y)+\frac{x}{x-y}(\mathscr{J}^{(1)}(x)-\mathscr{J}^{(1)}(y))\right)^2\\ &= -\frac{2i}{x-y}(x\mathscr{J}^{(1)}(x)-y\mathscr{J}^{(1)}(y))^2.\end{aligned}$$

Now,

$$\begin{aligned}\{|\mathscr{J}^{(1)}(x)|^2,|\mathscr{J}^{(1)}(y)|^2\} = 2\,\mathrm{Re}\Big(&\overline{\mathscr{J}^{(1)}(x)\mathscr{J}^{(1)}(y)}\{\mathscr{J}^{(1)}(x),\mathscr{J}^{(1)}(y)\}\\ &+\overline{\mathscr{J}^{(1)}(x)}\,\mathscr{J}^{(1)}(y)\{\mathscr{J}^{(1)}(x),\overline{\mathscr{J}^{(1)}(y)}\}\Big),\end{aligned}$$

and

$$2\,\mathrm{Re}\left(\overline{\mathscr{J}^{(1)}(x)\mathscr{J}^{(1)}(y)}\{\mathscr{J}^{(1)}(x),\mathscr{J}^{(1)}(y)\}\right) = \frac{4\,\mathrm{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)})}{x-y}\left(x^2|\mathscr{J}^{(1)}(x)|^2-y^2|\mathscr{J}^{(1)}(y)|^2\right),$$

so by Lemma 5.4,

$$2\,\mathrm{Re}\left(\overline{\mathscr{J}^{(1)}(x)}\,\mathscr{J}^{(1)}(y)\{\mathscr{J}^{(1)}(x),\overline{\mathscr{J}^{(1)}(y)}\}\right) = \frac{4\,\mathrm{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)})}{x-y}\left[x\mathscr{J}^{(0)}(x)^2-y\mathscr{J}^{(0)}(y)^2\right]. \tag{5.10}$$

Denote by $f(x,y) := \{\mathscr{J}^{(1)}(x),\overline{\mathscr{J}^{(1)}(y)}\}$. As above, we compute, for $j\in[\![1,N]\!]$,

$$\frac{\partial \mathscr{J}^{(1)}(x)}{\partial(\rho_j^2/2)}\frac{\partial \overline{\mathscr{J}^{(1)}(y)}}{\partial\varphi_j} - \frac{\partial \overline{\mathscr{J}^{(1)}(y)}}{\partial(\rho_j^2/2)}\frac{\partial \mathscr{J}^{(1)}(x)}{\partial\varphi_j} =$$

$$\frac{-2i\|u_j^H\|^4}{\rho_j^4(1-x\rho_j^2)(1-y\rho_j^2)}+\frac{2ix\|u_j^H\|^4}{\rho_j^2(1-x\rho_j^2)^2(1-y\rho_j^2)}+\frac{2iy\|u_j^H\|^4}{\rho_j^2(1-x\rho_j^2)(1-y\rho_j^2)^2}+\frac{4i\|u_j^H\|^2\frac{\partial\|u_j^H\|^2}{\partial\rho_j^2}}{\rho_j^2(1-x\rho_j^2)(1-y\rho_j^2)}$$
$$+2i\sum_{l\neq j}\frac{\|u_j^H\|^2\|u_l^H\|^2}{\rho_j\rho_l(\rho_l^2-\rho_j^2)}\left(\frac{e^{i(\varphi_j-\varphi_l)}}{(1-x\rho_l^2)(1-y\rho_j^2)}+\frac{e^{-i(\varphi_j-\varphi_l)}}{(1-x\rho_j^2)(1-y\rho_l^2)}\right).$$

The crucial fact is the following : when we sum over j, the term of the last line (involving $\sum_{l\neq j}$) vanishes. All the remaining terms are purely imaginary, and we proved that $f(x,y)\in i\mathbb{R}$. We write $f(x,y)=ig(x,y)$. Therefore,

$$2\operatorname{Re}\left(\overline{\mathscr{J}^{(1)}(x)}\mathscr{J}^{(1)}(y)\{\mathscr{J}^{(1)}(x),\overline{\mathscr{J}^{(1)}(y)}\}\right)=2\operatorname{Im}(\mathscr{J}^{(1)}(x)\overline{\mathscr{J}^{(1)}(y)})\cdot g(x,y).$$

and by (5.10),

$$\{\mathscr{J}^{(1)}(x),\overline{\mathscr{J}^{(1)}(y)}\}=\frac{2i}{x-y}\left[x\mathscr{J}^{(0)}(x)^2-y\mathscr{J}^{(0)}(y)^2\right].$$

To conclude, observe that $\mathscr{J}^{(1)}(x)=J_1+x\mathscr{J}^{(3)}(x)$, with $J_1=\mathscr{J}^{(1)}(0)$. Hence

$$\begin{aligned}xy\{\mathscr{J}^{(3)}(x),\overline{\mathscr{J}^{(3)}(y)}\}&=\{\mathscr{J}^{(1)}(x),\overline{\mathscr{J}^{(1)}(y)}\}-\{\mathscr{J}^{(1)}(x),\overline{J_1}\}-\{J_1,\overline{\mathscr{J}^{(1)}(y)}\}+\{J_1,\overline{J_1}\}\\&=\frac{2i}{x-y}\left[x\mathscr{J}^{(0)}(x)^2-y\mathscr{J}^{(0)}(y)^2\right]-2i\mathscr{J}^{(0)}(x)^2-2i\mathscr{J}^{(0)}(y)^2+2i\\&=\frac{2i}{x-y}\left[y\mathscr{J}^{(0)}(x)^2-x\mathscr{J}^{(0)}(y)^2+x-y\right].\end{aligned}$$

Dividing by xy gives the claim and completes the proof. □

We are now ready to prove Theorem 13.

Proof of Theorem 13. Begin by noticing that, since $J_2,\mathscr{J}^{(0)},\mathscr{J}^{(2)}$ only depend on the actions $\rho_j^2/2$ and $\sigma_k^2/2$, all the brackets which don't involve $\mathscr{J}^{(3)}$ are zero.

We thus only need to compute

- $\{J_2,|\mathscr{J}^{(3)}(x)|^2\}\equiv 0$, since the functional $|\mathscr{J}^{(3)}(x)|^2$ is invariant under phase rotation of functions.
- Because of the product formula for $\mathscr{J}^{(0)}(x)$ and (5.9), we have

$$\begin{aligned}\left\{\frac{1}{\mathscr{J}^{(0)}(x)},|\mathscr{J}^{(3)}(y)|^2\right\}&=\sum_{j=1}^N\frac{-4x}{\mathscr{J}^{(0)}(x)(1-x\rho_j^2)}\operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)}\frac{\rho_j\|u_j^H\|^2e^{-i\varphi_j}}{1-y\rho_j^2}\right)\\&=\sum_{j=1}^N\frac{-4x}{\mathscr{J}^{(0)}(x)}\operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)}\frac{\|u_j^H\|^2e^{-i\varphi_j}}{\rho_j(x-y)}\left[\frac{1}{1-x\rho_j^2}-\frac{1}{1-y\rho_j^2}\right]\right)\\&=\frac{-4x}{(x-y)\mathscr{J}^{(0)}(x)}\operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)}[\mathscr{J}^{(1)}(x)-\mathscr{J}^{(1)}(y)]\right)\\&=\frac{-4x^2}{(x-y)\mathscr{J}^{(0)}(x)}\operatorname{Im}\left(\mathscr{J}^{(3)}(x)\overline{\mathscr{J}^{(3)}(y)}\right).\end{aligned}$$

- A similar trick gives

$$\left\{\mathscr{J}^{(2)}(x)^2, |\mathscr{J}^{(3)}(y)|^2\right\} = 2\mathscr{J}^{(2)}(x)\left\{\frac{\mathscr{J}^{(0)}(x)}{x}, |\mathscr{J}^{(3)}(y)|^2\right\}$$
$$= \frac{2\mathscr{J}^{(2)}(x)}{x}\sum_{j=1}^{N}\frac{4x\mathscr{J}^{(0)}(x)}{1-x\rho_j^2}\operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)}\frac{\rho_j\|u_j^H\|^2 e^{-i\varphi_j}}{1-y\rho_j^2}\right)$$
$$= \frac{8x\mathscr{J}^{(2)}(x)\mathscr{J}^{(0)}(x)}{x-y}\operatorname{Im}\left(\mathscr{J}^{(3)}(x)\overline{\mathscr{J}^{(3)}(y)}\right).$$

- Finally, Lemma 5.5 enables to calculate

$$2\operatorname{Re}\left(\overline{\mathscr{J}^{(3)}(x)}\ \overline{\mathscr{J}^{(3)}(y)}\{\mathscr{J}^{(3)}(x), \mathscr{J}^{(3)}(y)\}\right)$$
$$= \frac{4}{x-y}\operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(x)\mathscr{J}^{(3)}(y)}(x\mathscr{J}^{(3)}(x) - y\mathscr{J}^{(3)}(y))^2\right)$$
$$= \frac{4}{x-y}\left[x^2|\mathscr{J}^{(3)}(x)|^2 - y^2|\mathscr{J}^{(3)}(y)|^2\right]\operatorname{Im}\left(\mathscr{J}^{(3)}(x)\overline{\mathscr{J}^{(3)}(y)}\right),$$

and

$$2\operatorname{Re}\left(\overline{\mathscr{J}^{(3)}(x)}\ \mathscr{J}^{(3)}(y)\{\mathscr{J}^{(3)}(x), \overline{\mathscr{J}^{(3)}(y)}\}\right)$$
$$= \frac{4}{x-y}\left[\frac{\mathscr{J}^{(0)}(x)^2}{x} - \frac{\mathscr{J}^{(0)}(y)^2}{y} + \frac{1}{x} - \frac{1}{y}\right]\operatorname{Im}\left(\mathscr{J}^{(3)}(x)\overline{\mathscr{J}^{(3)}(y)}\right),$$

so that

$$\left\{|\mathscr{J}^{(3)}(x)|^2, |\mathscr{J}^{(3)}(y)|^2\right\} =$$
$$\frac{4}{x-y}\left[x^2|\mathscr{J}^{(3)}(x)|^2 - y^2|\mathscr{J}^{(3)}(y)|^2 + \frac{\mathscr{J}^{(0)}(x)^2}{x} - \frac{\mathscr{J}^{(0)}(y)^2}{y} + \frac{1}{x} - \frac{1}{y}\right]\operatorname{Im}\left(\mathscr{J}^{(3)}(x)\overline{\mathscr{J}^{(3)}(y)}\right).$$

At last, we can compute the main Poisson bracket, expanding it as a double product :

$$\{\mathscr{F}(x), \mathscr{F}(y)\} = -\frac{y^2(2J_2 + x\mathscr{J}^{(2)}(x)^2 - x^2|\mathscr{J}^{(3)}(x)|^2)}{\mathscr{J}^{(0)}(y)}\left\{\frac{1}{\mathscr{J}^{(0)}(x)}, |\mathscr{J}^{(3)}(y)|^2\right\}$$
$$-\frac{xy^2}{\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)}\{\mathscr{J}^{(2)}(x)^2, |\mathscr{J}^{(3)}(y)|^2\}$$
$$-\frac{x^2(2J_2 + y\mathscr{J}^{(2)}(y)^2 - y^2|\mathscr{J}^{(3)}(y)|^2)}{\mathscr{J}^{(0)}(x)}\left\{|\mathscr{J}^{(3)}(x)|^2, \frac{1}{\mathscr{J}^{(0)}(y)}\right\}$$
$$-\frac{x^2y}{\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)}\{|\mathscr{J}^{(3)}(x)|^2, \mathscr{J}^{(2)}(y)^2\}$$
$$+\frac{x^2y^2}{\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)}\{|\mathscr{J}^{(3)}(x)|^2, |\mathscr{J}^{(3)}(y)|^2\}.$$

□

Summing up, and taking obvious cancellations into account, we have

$$(x-y)\mathscr{J}^{(0)}(x)\mathscr{J}^{(0)}(y)\{\mathscr{F}(x), \mathscr{F}(y)\} =$$

$$4xy\,\mathrm{Im}\left(\mathscr{J}^{(3)}(x)\overline{\mathscr{J}^{(3)}(y)}\right)\Big[x^2y\,\mathscr{J}^{(2)}(x)^2 - 2xy\,\mathscr{J}^{(2)}(x)\,\mathscr{J}^{(0)}(x) - xy^2\,\mathscr{J}^{(2)}(y)^2$$
$$+2xy\,\mathscr{J}^{(2)}(y)^2\,\mathscr{J}^{(0)}(y) + y\,\mathscr{J}^{(0)}(x)^2 - x\,\mathscr{J}^{(0)}(y)^2 + x - y\Big].$$

Now remember that $x\,\mathscr{J}^{(2)}(x) = \mathscr{J}^{(0)}(x) - 1$. So

$$\begin{aligned}
x^2y\,\mathscr{J}^{(2)}(x)^2 &- 2xy\,\mathscr{J}^{(2)}(x)\,\mathscr{J}^{(0)}(x) + y\,\mathscr{J}^{(0)}(x)^2 - y \\
&= y\left[(\mathscr{J}^{(0)}(x) - 1)^2 - 2\,\mathscr{J}^{(0)}(x)(\mathscr{J}^{(0)}(x) - 1) + \mathscr{J}^{(0)}(x)^2 - 1\right] \\
&= 0,
\end{aligned}$$

and the same holds interverting x and y. This concludes the proof of Theorem 13.

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universite PARIS-SACLAY
École doctorale de mathématiques Hadamard (EDMH)

Titre : Instabilité et croissance des normes de Sobolev pour certaines EDP hamiltoniennes

Mots Clefs : EDP hamiltonienne, complète intégrabilité, transfert d'énergie vers les hautes fréquences, équation de Szegő, paire de Lax, ondes progressives

Résumé : Cette thèse est consacrée à l'étude de solutions globales et régulières de certaines EDP hamiltoniennes, du point de vue de la croissance de leurs normes de Sobolev. Un tel phénomène traduit une modification de la répartition de l'énergie dans l'espace des fréquences, appelée parfois « turbulence faible ». On étudie d'abord une équation d'évolution non-linéaire où intervient un laplacien fractionnaire, et l'on prouve des estimées a priori sur la vitesse de croissance des normes de Sobolev. On introduit ensuite une équation où de telles estimées sont optimales : une équation de Szegő, intégrable, avec une non-linéarité quadratique, et où certaines solutions régulières croissent à vitesse exponentielle tout en restant bornées dans l'espace d'énergie. On classifie les ondes progressives de cette équation de Szegő quadratique, et l'on met en évidence l'instabilité d'une partie d'entre elles. Enfin, on exhibe pour cette équation une hiérarchie de lois de conservation, qui permet d'étudier plus précisément les solutions rationnelles turbulentes.

Title : Instability and growth of Sobolev norms for certain Hamiltonian PDEs

Keys words : Hamiltonian PDE, complete integrability, high-frequency energy transfer, Szegő equation, Lax pair, traveling waves

Abstract : In this thesis we study global smooth solutions of certain Hamiltonian PDEs, in order to capture the possible growth of their Sobolev norms. Such a phenomenon is typical for what is sometimes called "weak turbulence" : a change in the distribution of energy between Fourier modes. We first study a nonlinear evolution equation involving a fractional Laplacian, and we prove a priori estimates on the growth of Sobolev norms. We then introduce an equation where these estimates turn out to be optimal : an integrable Szegő equation with a quadratic nonlinearity, which admits exponentially growing smooth solutions that remain bounded in the energy space. We classify the traveling wave solutions of this quadratic Szegő equation, and show that some of them are unstable. Eventually we find a hierarchy of conservation laws for this equation, which leads us into a deeper study of rational turbulent solutions.