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Resolution of some optimisation problems on graphs and combinatorial games

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Acknowledgments

I never thought of finding myself in this situation: thinking of all the people that crossed my path these last few years. It is now that I realize with nostalgia, and also kind of pride, all the choices I have made up until this moment. Everything that lead me here, but most importantly, everyone that helped me (directly or not) to get here.
And for all these people I am grateful that our paths crossed, sad that some grew apart, happy that some hang on. And I have now the demanding task to name each of these persons that make my life.

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Contents

Chapter 1

Introduction

"I never travel without my diary. One should always have something sensational to read in the train."

Oscar Wilde

Hereby I present my work of the last few years. My research project started with my intership of master's degree directed by Éric Duchêne and Aline Parreau from University of Lyon 1. During this intership I worked with Clément Charpentier (University Lyon 2) and Brice Effantin (University Lyon 1) on the graph coloring game. At the end, I obtained an ANR funding for a doctoral position. This project is called *Games and Graphs* and is managed by Éric Duchêne, it aims to study some game combinatorial problems by using graph theory tools.

Along these years I had the opportunity to work on different combinatorial problems.
One of the main problems I worked on is the coloring game. In collaboration with Clément Charpentier and Brice Effantin, we worked on the graph coloring game where two parties face each other with different goals. Given a fixed set of colors, Alice and Bob alternate turns to properly color vertices of a graph: Alice wins if at the end the graph is properly colored and Bob wins otherwise. We studied this game on edge-wise decomposable graphs and improved some known results on planar graphs.
As well, with Paul Dorbec (my co-advisor, from University of Bordeaux), Éric Sopena (University of Bordeaux) and Elżbieta Sidorowicz (from University of Zielona Gora) we studied a slighty more general problem: the graph marking game. Here Alice and Bob alternate turns to mark unmarked vertices. When a vertex is marked a score is given to it that depends, only, on the number of marked neighbors it has. Alice wants to minimize the maximum score obtained along the game and Bob wants to maximize it. We focused on the difference of strategies when we modify the graph: if Alice has a strategy ensuring a maximum score of s on G, what is the score she can ensure on $f(G)$, f being a graph operator? We bounded above and below the score of $f(G)$ for the operators of minors: vertex deletion, edge deletion and edge contraction. As well, we looked at the union of graphs and cartesian products.

In 2016, I obtained a doctoral mobility scholarship: I went to Turku for 5 months to work with Tero Laihonen and Ville Junnila on identifying and locating codes on graphs. An identifying code C is a subset of vertices of a graph G such that the set of neighbors of v in C is non-empty and different from the one of u, for any two vertices u, v. The codes with minimal cardinality are called optimal. This codes allow to locate faults on networks of processors. During my stay in Finland we focused on these codes (and two other variants) on circulant graphs which have the particularity of being regular and embeddable on infinite grids. These properties allowed us to conclude on the optimal codes on some particular families of circulant graphs, moreover, these bounds are reached for infinitely many such circulant graphs.

I also had the chance of working on heap games, in collaboration with Urban Larsson (Technion - Israel

Institute of Technology), Antoine Dailly (doctoral student at University of Lyon 1) and Éric Duchêne. We focused on *pure breaking* games. In the litterature *taking* games and *taking and breaking* games have been largely studied. In the first category subtraction games and allbut games are included (finite or not) and in the second category there are octal games and hexadecimal games among others. In these games two players, Alice and Bob, alternate turns to take some tokens from a heap of tokens and then divide it into multiple non-empty heaps (this division step is only allowed in *taking and breaking games*). *Pure breaking* games are such that no taking is allowed: given a list of integers $\{\ell_1, \ldots, \ell_k\}$, two players, Alice and Bob, alternate turns to divide a heap of tokens in $\ell_i + 1$, $1 \leq i \leq k$, non-empty heaps. We introduce these games and give the winning strategies for some particular lists of integers. As well, we explicit also a test to compute the best strategies for some other lists.

My work can be summed up as the study of two party combinatorial problems. In the first case Alice faces an uncollaborative partner, Bob, in the coloring and marking games. For the identifying codes, captors face faults on processors networks. In the last case two players face each other on the exact same playground: they have the same tools and play; the smartest wins.
In what follows I give some definitions to attack these problems separately. Then I describe my work starting from pure graph theory (identifying codes) to pure combinatorial games (pure breaking games) passing through combinatorial games on graphs (coloring and marking games). Keep in mind that graph theory is used all along since it is an important tool to study combinatorial games.

1.1 Definitions

In the following $\mathbb{N}$ denotes the set of positive integers; $[\![a, b]\!]$, denotes the set of integers between a and b both included; if $f : X \to Y$ is a function and $S \subset X$, then $f\,|_S \colon S \to Y$ is the function restricted to the set S, and $\mathrm{Im}(f)$ is the set of *images* of X by f.

1.1.1 Graph theory

Definition 1.1 (Graph) *A* graph *G is an ordered pair (V, E), the set V is the set of* vertices *and $E \subset V^2$ is the set of* edges.
The edges denote links between vertices: for $e \in E$, there are $u, v \in V$ such that $e = uv$.

Said like that graphs seem to be obscure objets, when in fact, the abstract of the definition makes it a powerful tool: they can model a lot of different situations. For instance, the friendship between people: vertices are persons and there is an edge between two people if they are friends; or bus lines: each stop is a vertex and there is an edge between two stops if there is a bus going through these stops.

For instance a very interesting and known problem is *the three house-services problem*.
The three house-services problem: assume you have three houses h_1, h_2 and h_3 and you want to link them to the main services provided by the city: gaz s_1, electricity s_2 and water s_3 but you dont want the links to cross.
If you allow the links to meet, we obtain a graph with set of vertices $\{h_1, h_2, h_3, s_1, s_2, s_3\}$ and set of edges $\{h_1s_1, h_1s_2, h_1s_3, h_2s_1, h_2s_2, h_2s_3, h_3s_1, h_3s_2, h_3s_3\}$: there is an edge between each service and each house. The graph obtained is shown in Figure 1.1.1. We will see later if there is a solution where the links cannot cross.

Definition 1.2 *Let G be a graph of vertex set V and edge set E.*

- **Neighbors:**
  Let $u, v \in V$. Vertices u and v are neighbors *if $uv \in E$ or $vu \in E$. As well we say u and v are* adjacent.

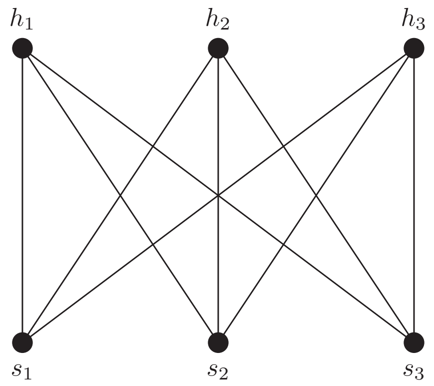


Figure 1.1.1: Example of the graph of the three house-services problem where edges can cross.

- **(Open) Neighborhood:**
  Let $v \in V$. The open neighborhood *of v, $N(v)$ is the set of neighbors of v.*

- **Closed Neighborhood:**
  Let $v \in V$, the closed neighborhood *of v, $N[v]$ is $N(v) \cup \{v\}$.*

- **Degree:**
  Let $v \in V$, the degree, *$d(v)$, of v is $|N(v)|$, the number of neighbors of v.*

- **Maximum degree:**
  The maximum degree of G, $\Delta(G)$ is $\max\{d(v) \mid v \in V\}$.

For instance in Figure 1.1.2, $N(0) = \{1, 3, 4, 7\}$ and $N(3) = \{0, 1, 2, 4\}$, and $\Delta(G) = 4$.

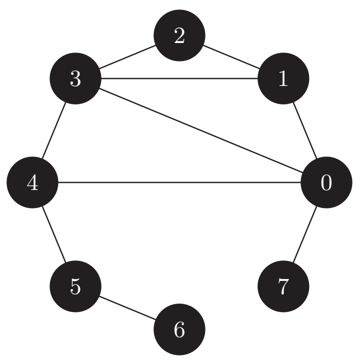


Figure 1.1.2: Example of a graph of maximum degree $\Delta(G) = 4$, obtained for the vertices 0 and 3.

Now, assume a graph G represents friendship between a group of persons and person u wants to get in touch with person v. If $v \in N[u]$ then u knows person v, but if not, u has to go through friends of friends to reach v... Or, assume the graph represents bus lines: how to go from point u to point v?

Definition 1.3 (Paths) *Let k be a positive integer and G a graph of vertex set $\{v_0, \ldots, v_{k-1}\}$. The graph G is a* path *if its edge set is $\{v_i v_{i+1} \mid 0 \leq i < k-1\}$.*
The length *of the path $v_0 \ldots v_{k-1}$ is $k-1$: the number of edges in the path.*

Definition 1.4 (Subgraph) *Let $G(V, E)$, $V' \subset V$ and $E' \subset E$ such that if $uv \in E'$, then u and v are in V'. The graph H of vertex set V' and edge set E' is a* subgraph *of G, denoted by $H \subset G$.*
Moreover, if for all $u, v \in V'$ such that if $uv \in E$ then uv is also in E', then H is an induced subgraph *of G.*

In a graph $G(V, E)$ there is a path between vertices u and v if there is a set of vertices in V, $v_0, \ldots, v_{k-1}$ such that $u = v_0$, $v = v_{k-1}$ and for all $i < k$, $v_i \in N(v_{i+1})$.
For instance in Figure 1.1.1 there is the path of length 5 as a subgraph: $s_1h_1s_2h_2s_3h_3$ and in Figure 1.1.2 there is a path of length 7 from 6 to 7.
We can talk about the distance between two vertices:

Definition 1.5 (Distance) *Let $G(V, E)$ be a graph and $u, v \in V$. The* distance *between u and v, $d(u, v)$, is the length of a shortest path between u and v. If there is no such path, then $d(u, v) = +\infty$.*

If G is such that for all $u, v \in V$, there is a path between u and v, G is said to be connected:

Definition 1.6 (Connected graph) *Let $G(V, E)$ be a graph. The graph G is* connected *if for all $u, v \in V$, $d(u, v) < +\infty$.*

Now, assume G is the path on $k+1$ vertices: $V = \{v_0, \ldots, v_k\}$ and we add the edge v_kv_0: the path closes.

Definition 1.7 (Cycles) *Let k be an integer and G a graph of vertex set $\{v_0, \ldots, v_{k-1}\}$. The graph G is a cycle if its edge set is $\{v_iv_{(i+1) \bmod k} \mid 0 \leq i \leq k-1\}$.*
The length of the cycle is k.

Remark that cycles have length at least 3.

Definition 1.8 (Class or Family of graphs) *A* class *or* family *of graphs is a collection of infinitely many graphs.*

Usually classes of graphs are defined by some property, like for instance *being connected* or *not having cycles of length 3*.

Definition 1.9 (Forests) *Let $G(V, E)$ be a graph. The graph G is a* forest *if G has no cycles.*
Moreover, if G is connected, it is called a tree.

Examples of trees are given in Figure 1.1.3. Here a vertex is chosen to be up and all their neighbors are drawn downward. The up most vertex is then called the *root*.

Definition 1.10 (Tree terminology) *Let $T(V, E)$ be a tree.*

- **Root:**
  Let r be a vertex of V. The vertex can be designated as a root *of T, which becomes* rooted.
- **Depth:**
  If T is rooted at r, the depth *of the tree is the length of the longest path starting at r. The* depth *of a vertex v is the length of the unique path between r and v.*
- **Fathers and Children:**
  If T is rooted at r, the neighbors of r are its children, *r is their* father. *For any other vertex v, its neighbor of lower depth is its* father *and the other neighbors are its* children.

Remark that normally *fathers* are called *parents* in the literature, here the use of *father* is justified in Section 3.4.

We have seen classes with forbidden subgraphs, another example are the empty graphs where edges are forbidden:

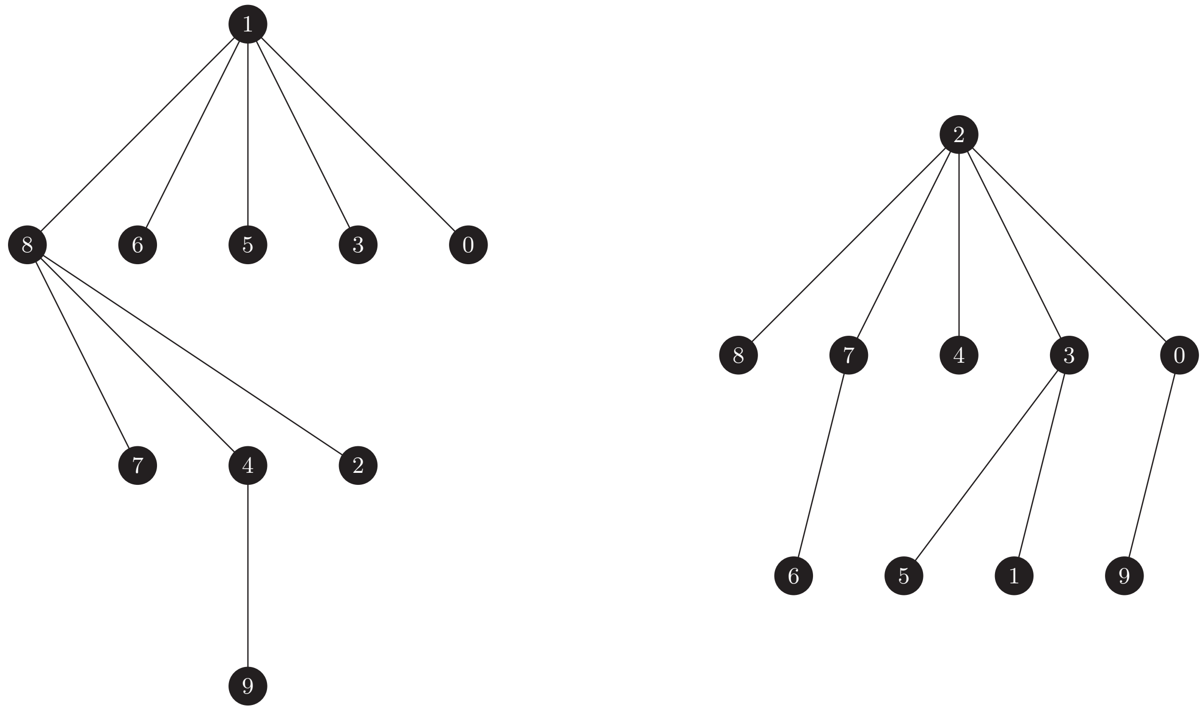


Figure 1.1.3: Example of two trees rooted at vertex 1 and 2 respectively.

Definition 1.11 (Empty graphs, stable sets) *Let G be a graph on n vertices.*

- *The graph G is the* empty graph *if for all $v_i, v_j \in V$, $v_iv_j \notin E$.*
- *A set of vertices $\{v_0, \dots, v_k\}$ of V is a* stable set *of G if for all i, j $v_iv_j \notin E$.*

Definition 1.12 (Complementary graph) *Let $G(V, E)$ be a graph. The* complementary *graph of G, $\overline{G}$ is the graph of vertex set V and edge set $\{v_iv_j \mid v_iv_j \notin E\}$.*

The complementary of an empty graph is called a *complete* graph.

Definition 1.13 (Complete graphs, cliques) *Let n, m be two positive integers.*

- *The* complete graph *on n vertices, K_n is the graph of vertex set $\{v_0, \dots, v_{n-1}\}$ and of edge set $\{v_iv_j \mid 0 \leq i, j \leq n-1, i \neq j\}$.*
  If in a graph G a set of vertices $\{v_0, \dots, v_k\}$ induces a complete graph we talk about a clique *of size $k+1$.*
- *The* complete bipartite graph *$K_{n,m}$ on $n+m$ vertices is the graph of vertex set $\{u_0, \dots u_{n-1}\} \cup \{v_0, \dots, v_{m-1}\}$ and edge set $\{u_iv_j \mid 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$.*

Examples of these graphs are shown in Figure 1.1.4.
In particular empty and complete graphs have something in common: all their vertices have the exact same degree. Graph with this particular feature are *regular*:

Definition 1.14 *Let $G(V, E)$ be a graph and r be a positive integer. The graph G is r-regular is for all $v \in V$, $d(v) = r$, meaning that all vertices have exactly r neighbors.*

Examples of regular graphs are given in Figure 1.1.5.
Other important class are *planar graphs*:

Definition 1.15 (Planar graphs) *A graph is planar if it can be embedded in the plane without edges crossing.*

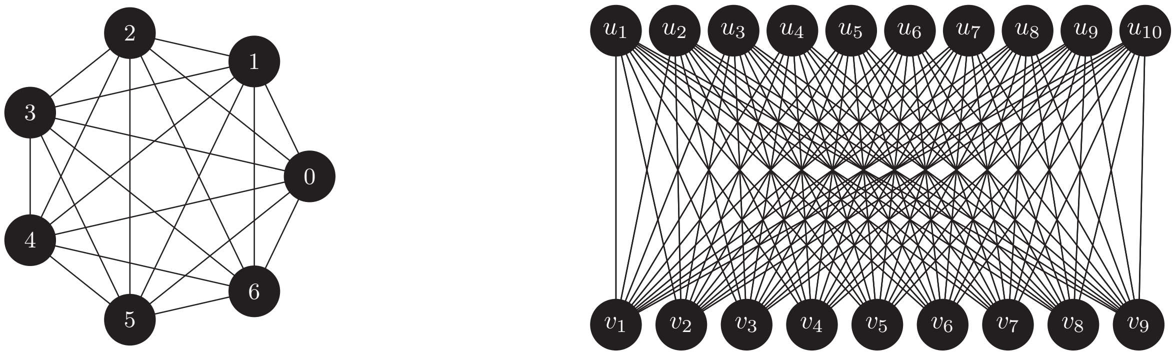


Figure 1.1.4: Example of a clique on 7 vertices and of a bipartite clique on $9 + 10$ vertices.

Figure 1.1.5: Example of regular graphs on 10 vertices of degree 4 and 6 respectively.

Take the three house-services problem again: is there a solution to link the services to the houses planary?

In fact planar graphs can also be defined by forbidding some particulars graphs. Here the restriction is not on subgraphs but with *minor graphs*.

Definition 1.16 (Minor graphs) *Let $G(V, E)$ be a graph.*

- *Let $v \in V$, the graph $G - \{v\}$ is the graph obtained from G by deletion of v and all the edges of v: $G - \{v\}(V \setminus \{v\}, E \setminus \{vu \mid u \in V\})$.*
- *Let $e \in E$, the graph $G \setminus \{e\}$ is the graph obtained from G by deletion of the edge e: $G \setminus \{e\}(V, E \setminus \{e\})$.*
- *Let $e \in E$, the graph $G/\{e\}$ is the graph obtained from G by the contraction of the edge e, meaning that if u and v are endpoints of e, these two vertices become one, w, that is neighbor to all the neighbors of u and all the neighbors of v.*

Graphs obtained from G by these operations are called minors *of G.*

Theorem 1.17 *[69] Planar graphs are exactly the $(K_{3,3}, K_5)$-minor-free graphs, meaning that for all graph P, P is planar if and only if minors of P do not contain $K_{3,3}$ nor K_5 as subgraphs.*

In particular, $K_{3,3}$ is not a planar graph and the three house-services problem does not have a feasible solution embeddable in the plane.

Minors are not *easy to see* as we need to modify the graph to obtain them (namely because of the edge contraction), an *easier-to-see* substructure tool is tree-decomposition.

Definition 1.18 (tree-decomposition) *Let $G(V, E)$ be a graph and* **a** *and* **d** *two integers.*

- *The graph G is* **a***-tree-decomposable if there are $E_1, \dots, E_a$ subsets of E such that for all i, j, $E_i \cap E_j = \varnothing$, $\cup_{1 \le i \le a} E_i = E$ and the graphs $G_i(V, E_i)$ are all trees. The minimum integer* **a** *such that G is* **a***-tree-decomposable is the* arboricity *of G.*
- *The graph G is $(\mathbf{a}, \mathbf{d})$-decomposable if the edges can be partitioned into* **a** *disjoint forests $E_1, \dots, E_a$ and a graph $D(V, E \setminus \{E_1 \cup \dots \cup E_a\})$ of maximum degree* **d***.*
  *If the graph of maximum degree is also a forest it is a $F(\mathbf{a}, \mathbf{d})$-*decomposition.

In Chapter 3 a generalization of these two last decompositions can be found.

Graphs are decomposable into disjoint forests. There are also other tools to decompose graphs into others graphs by doing some operations:

Definition 1.19 (Operations) *Let $G(V_G, E_G)$ and $H(V_H, E_H)$ be two graphs.*

- **Union:**
  The union U of G and H, denoted by $U = G \cup H$, is the graph of vertex set $V_G \cup V_H$ and of edge set $E_G \cup E_H$.
- **Cartesian product:**
  The cartesian product, C of G and H denoted by $C = G \Box H$ is the graph of vertex set $\{(u, v) \mid u \in V_G, v \in V_H\}$ and edge set $\{(u, v)(w, t) \mid u = w$ and $vt \in E_H$ or $v = t$ and $uw \in E_G\}$. In other words, for each vertex of G there is a copy of H and for each vertex of H there is a copy of G and for each copy the adjacencies are kept.
- **Join:**
  The join graph of G and H denoted by $J = G \vee H$ is $\overline{\overline{G} \cup \overline{H}}$, in other words, is an union of G and H with all the possible edges between G and H.

Remark the *complement* of a graph is also an operation on graphs. Examples of all of these operations can be found in Figures 1.1.6 to 1.1.8.

Figure 1.1.6: Example of a graph on 10 vertices and its complementary graph.

In what follows we consider also *directed* graphs or *digraphs* where edges have a given direction. This notion of direction is often shown with arrows on the edges.

Definition 1.20 (Directed graph) *Let $V = \{v_0, \dots, v_{n-1}\}$ be a set of vertices and $\overrightarrow{E} = \{\overrightarrow{v_i v_j} \mid 0 \le i, j \le n-1\}$ a set of* arcs*, or directed edges. Then $\overrightarrow{G}(V, \overrightarrow{E})$ is a* directed *graph.*

And the concept of neighborhood changes as follows:

Definition 1.21 (Neighborhood and degrees) *Let $\overrightarrow{G}(V, \overrightarrow{E})$ be a digraph and let $v \in V$.*

- **Inneighbors:**
  The inneighbors *of v, denoted by $N^-(v)$ are $\{u \mid \overrightarrow{uv} \in \overrightarrow{E}\}$. The* indegree *of v, denoted $d_-(v)$ is $|N^-(v)|$.*
  A vertex v such that $d_-(v) = 0$ is called a source.

Figure 1.1.7: Example of the union of the clique on 5 vertices and a cycle of length 4.

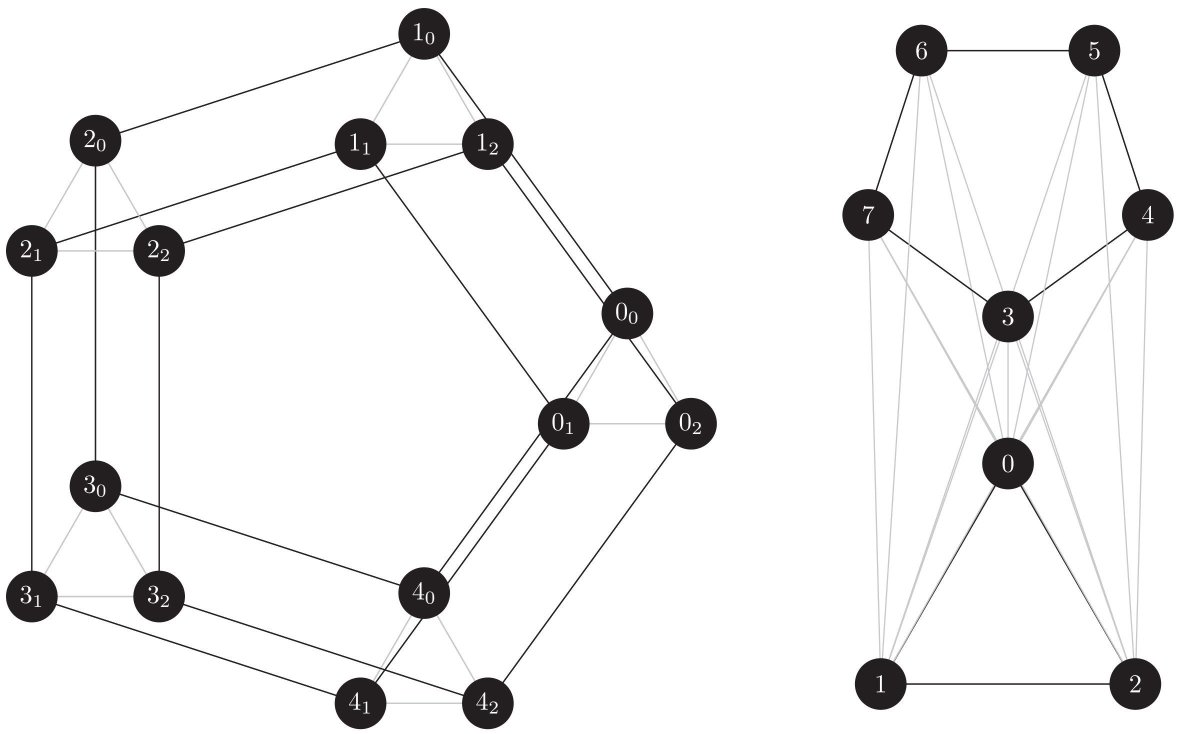


Figure 1.1.8: Example of a cartesian product of the cycle of length 5 and the cycle of length 3 (Left) and an example of a join between the same graphs (Right).

- **outneighbors:**
  The outneighbors *of v, denoted by $N^+(v)$ are $\{u \mid \overrightarrow{vu} \in \overrightarrow{E}\}$. The* outdegree *of v, denoted $d_+(v)$ is $|N^+(v)|$.*
  A vertex v such that $d_+(v) = 0$ is called a sink.

- **Maximum degree:**
  The maximum degree *of G, denoted by $\Delta(G)$ is* $\max\{d_+(v) + d_-(v) \mid v \in V\}$

As well there are directed paths and cycles:

Definition 1.22 (Directed paths and cycles) *Let $v_0, \dots, v_{k-2}$ and v_{k-1} be k vertices.*

- **Directed path:**
  The directed path of length k is the graph of vertex set $\{v_0, \dots, v_{k-1}\}$ and arc set $\{\overrightarrow{v_i v_{i+1}} \mid 0 \leq i < k-1\}$.

- **Directed cycle:**
  The directed cycle of length k is the graph of vertex set $\{v_0, \ldots, v_{k-1}\}$ and arc set $\{\overrightarrow{v_i v_{(i+1) \bmod k}} \mid 0 \leq i \leq k-1\}$.

Specific orientations on the edges define *tournaments*:

Definition 1.23 (Tournaments) *Let $V = \{v_0, \ldots, v_{n-1}\}$ be a set of n vertices. Let $\overrightarrow{E}$ be a set of arcs such that for $i, j \in [\![0, n-1]\!]$, $i \neq j$ we have: $\overrightarrow{v_i v_j} \in \overrightarrow{E}$ or $\overrightarrow{v_j v_i} \in \overrightarrow{E}$. Then $G(V, \overrightarrow{E})$ is a* tournament.
A transitive tournament *is such that for all v_i, v_j, v_k, if $\overrightarrow{v_i v_j} \in \overrightarrow{E}$ and $\overrightarrow{v_j v_k} \in \overrightarrow{E}$ then $\overrightarrow{v_i v_k} \in \overrightarrow{E}$.*

In other words, a tournament is a complete graph with an orientation. In Figure 1.1.9 we can see an example of these graphs. Remark that a transitive tournament on n vertices can be totally ordered by the edge relation: for all $i \in [\![0, n-1]\!]$, there is a unique vertex v_i such that $N^+(v_i) = i$. These last definitions

Figure 1.1.9: Examples of tournaments, the right one being transitive.

are necessary for the proof of Theorem 3.19.

Now, we present the basis of identifying vertices on graphs for Chapter 2.
The study of identifying codes on a graph uses a very specific vocabulary:

- *code*: a subset of the vertices,
- *codeword*: a vertex of a code,
- *non-codeword*: a vertex that is not in the code.

Definition 1.24 (Identifying set) *Let $G(V, E)$ be a graph and $C \subset V$ a code. For all $v \in V$, the* identifying set *or* Iset *of v is:*

$$I_{G,C}(v) = N[v] \cap C.$$

When no confusion is possible we skip the G, C subscripts.

Definition 1.25 (Identifying code) *Let $G(V, E)$ be a graph and $C \subset V$. The code C is an* identifying code *on G if:*

$$\forall u \neq v \in V, I_{G,C}(u) \neq I_{G,C}(v) \text{ and } I_{G,C}(u) \neq \varnothing$$

The identifying code of minimal cardinality on G is an optimal identifying code *of G and its cardinality is denoted by $\gamma^{ID}(G)$.*

Proposition 1.26 ([18]) *Let $G(V, E)$ be a graph. The graph G admits identifying codes if and only if for all $u \neq v \in V^2$, $N[u] \neq N[v]$.*

Hence, graph with twins dont have identifying codes.
Identifying codes can be used to identify faulty processors in multiprocessors systems. In Figure 1.1.10, if vertices are processors and the sensors in 0 and 2 are activated, then the faulty processor is 1, as it is the only one to have the set $\{0, 2\}$ as Iset.

This problem is hard:

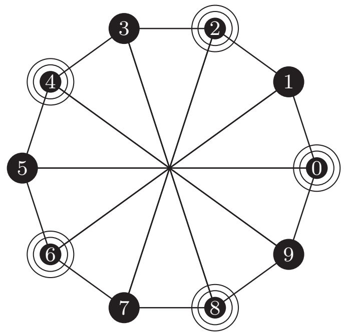


Figure 1.1.10: Example of an ID-code, codewords have halos.

Theorem 1.27 (IC-problem is NP-complete) *[22] Let $G(V, E)$ be a connected graph over n vertices and $k \leq n$ be a positive integer. Knowing if there is an ID-code of size at most k on G is a NP-complete problem.*

Reduction to a known NP-complete problem. This proof is done by reduction of the 3-SAT problem to the IC-problem. Let us recall the 3-SAT problem:

3-SAT PROBLEM: let ε be a collection of clauses over a set X of variables where each clause contains exactly three literals. Knowing if ε can be satisfied is NP-complete.

Here we are just giving the reduction to this problem, for more details we invite the readers to see the original paper.

Let $\varepsilon = \{C_1, \ldots, C_m\}$ be an instance of a 3-SAT over the set $X = \{x_1, \ldots, x_n\}$ where each clause contains exactly three literals taken in $U = \{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$. We construct a graph $G(V, E)$ such that ε can be satifsfied if and only if V contains an ID-code of size at most k for a certain k.
For each variable x_i of X, let G_{x_i} be the graph of vertex set $V_{x_i} = \{a_i, b_i, x_i, \overline{x_i}, c_i, d_i\}$ and edge set $E_{x_i} = \{a_ib_i, b_ix_i, b_i\overline{x_i}, x_ic_i, \overline{x_i}c_i, c_id_i\}$.
For each clause $C_j = \{u_{j,1}, u_{j,2}, u_{j,3}\}$, let G_{C_j} be the graph of vertex set $V_{C_j} = \{\alpha_j, \beta_j\}$ and edge set $E_{C_j} = \{\alpha_j\beta_j\}$.
Now, let $E'_{C_j} = \{\alpha_ju_{j,1}, \alpha_ju_{j,2}, \alpha_ju_{j,3}\}$ and G be the graph of vertex set $V = V_{x_1} \cup \cdots \cup V_{x_n} \cup V_{C_1} \cup \cdots \cup V_{C_m}$ and edge set $E = E_{x_1} \cup \cdots \cup E_{x_n} \cup E_{C_1} \cup \cdots \cup E_{C_m} \cup E'_{C_1} \cup \cdots \cup E'_{C_m}$ and let $k = 3n + m$. This construction is polynomial on the size of the 3-SAT, as $|V| = 6n + 2m$. If ε can be satisfied, then we can construct an ID-code C of size equal to k: for all $1 \leq i \leq n$, b_i, c_i and whichever of x_i or $\overline{x_i}$ is true, belong to C and for all $1 \leq j \leq m$, α_j belongs to C. This code is of size $3n+m = k$, and it is an ID-code of G.

As well, if C is an ID-code in G, then for all $1 \leq i \leq n$, $|C \cap \{x_i, \overline{x_i}\}| = 1$, C has size at least k and as α_j must be covered by a vertex other than β_j and itself, it has at least one of $u_{j,1}, u_{j,2}$ or $u_{j,3}$ in C. Hence ε is satisfied by setting x_i to true if $x_i \in C$ and to false otherwise. ■

In Chapter 2 identifying codes are studied over circulant graphs by using known results over infinite regular grids.

1.1.2 Games

Here we give basic definitions useful to understand the games studied in Chapters 3 and 4.
Combinatorial games as such were introduced in [10]:

Definition 1.28 (Combinatorial games) *A* combinatorial game *is a two-player game such that:*

- *players alternate turns,*

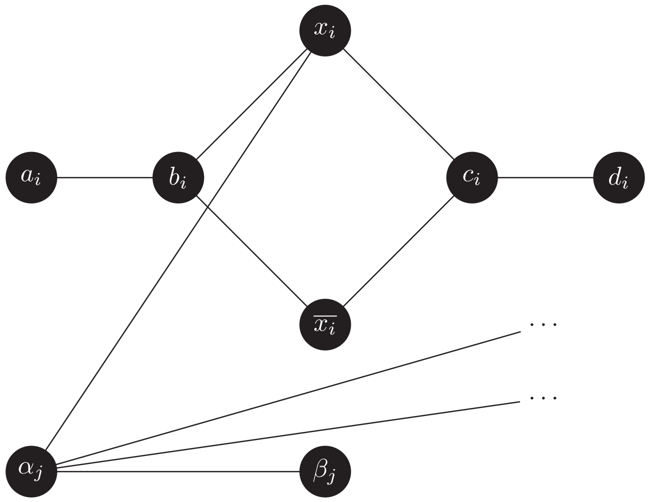


Figure 1.1.11: Example of a graph G_{x_i} and a graph G_{C_j}.

- *there is no chance in the game,*
- *there are no loops and the game is finite,*
- *it has perfect information: every player knows all the possible moves*
- *the last move determines the winner*

In normal *convention, the last player to play wins, in* misere *convention loses.*
A combinatorial game G is called a position *and the set available moves from a given* position *are called* options *of G.*

Here we consider only impartial games, meaning that both players always have the same moves, unlike partisan games where each player have its own moves.

In the game of CHESS, there are two players that alternate turns, there is no luck, the game is finite and has perfect information. But the last person to play is not necessarily the winner, as draws are possible. Moreover, in CHESS loops are also possible. For the game of GO it is similar: it has a lot of combinatorial characteristics, but the winner is determined by the score of the game and not the last person to play.

In what follows we mostly consider combinatorial games played on heaps of tokens in normal convention. A position of a game on t heaps with $n_1, \ldots, n_t$ tokens in each heap respectively is denoted by $(n_1, \ldots, n_k)$. Remark that their order is not important and hence the position $(n_1, \ldots, n_k)$ is the same as $(n_k, \ldots, n_1)$. From now on we assume them to be in increasing order.

An example of a combinatorial game on heaps is TOKENS:

TOKENS: The TOKENS game is played on t heaps of tokens. Players alternate turns to remove one or three tokens from a heap.

Assume now that Alice and Bob play TOKENS game on two heaps from the starting position $(1, 3)$.
If Alice starts by emptying one heap, Bob empties the other one and wins. Hence Alice starts by moving to the position $(1, 2)$. From this position Bob can either empty the first heap or remove one token from the second, in both cases, Alice does the other move and at his next turn he has only one non-empty heap with one token: he wins.
The **game tree** corresponds to a digraph where all the positions of the game are shown along with their options. The vertex set is the set of all positions (x, y) with $x \leq 1$ and $y \leq 3$ and the edge set is $\overrightarrow{(x, y), (u, v)}$

for all options (u, v) of the position (x, y). These games are without loops and finite, hence these graphs have a source (the starting position) and sinks (the final positions).

In Figure 1.1.12 we can see that from the starting position $(1, 3)$ (the source) there are 7 different positions $(1, 3)$, $(0, 3)$, $(1, 2)$, $(0, 2)$, $(1, 1)$, $(0, 1)$ and $(0, 0)$ (the sink).

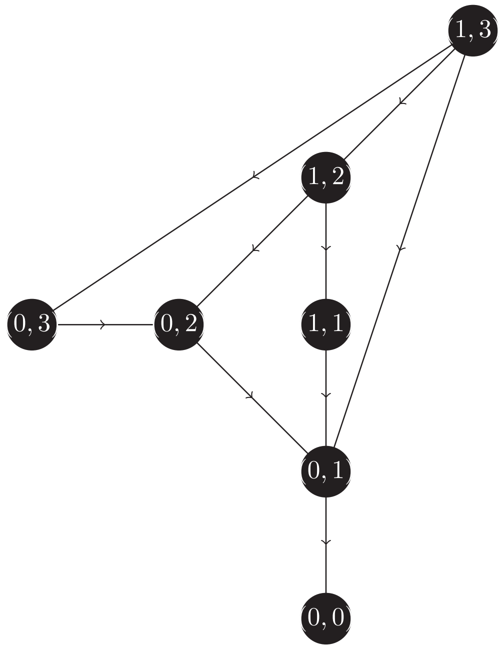


Figure 1.1.12: Game tree of TOKENS from the starting position $(3, 1)$.

As combinatorial games have no draws, an option of a game is either good or bad for the players. For instance from the last position $(0, 0)$ players have no moves, hence the player that has to play from this position loses. As well, the player starting at the position $(1, 0)$ only has one move and it allows him to win.
In the general case, if G is a position of a game and it is a final position, no player can win from this position: the *next* player loses. Let us call it a $\mathcal{P}$-position: the *previous* player wins. Now, if G is such that it has a position P among its options, the next player has a winning option: it plays to a $\mathcal{P}$-position and the player that comes after loses. Let us call this position an N-position: the *next* player wins.

Definition 1.29 (Outcomes) *Let G be a position of a game.*

- *If G is a final position or if all options of G are N-positions, then G is a $\mathcal{P}$-position.*
- *If G has an option that is a $\mathcal{P}$-position, G is an N-position.*

The outcome of G is denoted by $o(G)$.

From this and the game tree, we can deduce an algorithm to determine the outcomes of all positions of the game:
Hence in the TOKENS game from before: $(0, 0)$ is a $\mathcal{P}$-position and $(1, 0)$ is an $\mathcal{N}$-position. The application of Algorithm 1 is shown in Figure 1.1.13: the first player starts from a $\mathcal{P}$-position. If the players play optimally, the second player wins this game.

Assume now Alice and Bob play TOKENS on three heaps containing respectively $1, 3$ and 4 tokens. The options of this positions are either options of the game $(1, 3)$ leaving the third heap unchanged, or options

Algorithm 1 Outcomes of a game of tree $\overrightarrow{G}(V, \overrightarrow{E})$

while $V \neq \varnothing$ **do**
  $R = \varnothing$
  for all sink v **do**
    $R = R \cup \{v\}$
    $o(v) = \mathcal{P}$
    for all $u \in N^-(v)$ **do**
      $R = R \cup \{u\}$
      $o(u) = \mathcal{N}$
    end for
  end for
  $V = V \setminus R$
end while

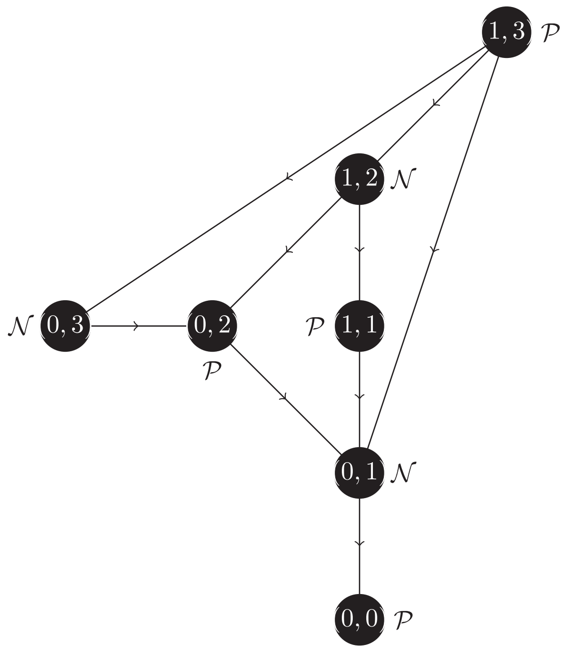


Figure 1.1.13: Outcomes for the game TOKENS from the starting position $(3, 1)$.

of (4) leaving $(1, 3)$ unchanged. The underlying operations is the sum of games: the game of TOKENS on n heaps is also n games of TOKENS on one heap. The players chose which game they play in and they make a move on that game. The set of options of a sum of two games is the cartesian product of the sets of options of each game.

Definition 1.30 (Sum of games) *Let G and H be two positions of two games of respective options $\{O_{G,0}, \ldots, O_{G,n-1}\}$ and $\{O_{H,0}, \ldots, O_{H,m-1}\}$. The* sum *of these games, $S = G+H$ is the position of options $\{(O_{G,0}, H), \ldots, (O_{G,n-1}, H), (G, O_{H,0}), \ldots, (G, O_{H,m-1})\}$, where each option (G_i, H_i) is then treated as a sum $G_i + H_i$. In other words, at each turn the players choose the game in which they play and choose an option of that game.*

Now assume Alice and Bob play a different game called NIM.

NIM: The NIM game is played on heaps of tokens. Players alternate turns to remove one or more tokens from a single non-empty heap.

If there is only one heap the game is quite simple: the first player takes all tokens from the heap and wins, as the second player is not able to play. In particular, the NIM game on one heap with n tokens is an $\mathcal{N}$-position if and only if $n \neq 0$. If they have two heaps say (n_1, n_2), the outcome of the game depends on $n_2 - n_1$. When $n_2 = n_1$ no matters what the first player does on one heap, the other player can always do the same on the other heap: when one heap is emptied, the second players empties the remaining one, winning the game. When $n_2 \neq n_1$ the first players removes tokens from the bigger heap to obtain two heaps of same size.

Assume, now, Alice and Bob want to spice things up: they play now a sum of the two games, they have four heaps of tokens, the first two heaps are for TOKENS and the two last heaps are for NIM. At each turn, each player choses a game between the two and plays on the corresponding heaps, they start on the position $(3, 1, 3, 2)$.
Remark all positions $(0, 0, x, y)$ have the same outcome as (x, y) in NIM, as well as positions $(x, y, 0, 0)$ in TOKENS. We can see the outcome of the game in Table 1.1.1.

			TOKENS						
			$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$
			(3,1)	(3,0)	(2,1)	(2,0)	(1,1)	(1,0)	(0,0)
	$\mathcal{N}$	(3,2)	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$
	$\mathcal{N}$	(3,1)	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
	$\mathcal{N}$	(3,0)	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
	$\mathcal{P}$	(2,2)	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$
NIM	$\mathcal{N}$	(2,1)	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
	$\mathcal{N}$	(2,0)	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
	$\mathcal{P}$	(1,1)	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$
	$\mathcal{N}$	(1,0)	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$
	$\mathcal{P}$	(0,0)	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$

Table 1.1.1: Outcome of the game MIXED TOKENS AND NIM from the starting position $(3, 1, 4, 1)$

The starting position $(3, 1, 3, 2)$ is a winning position for the first player. Positions $\mathcal{P} + \mathcal{P}$ seem to give $\mathcal{P}$-positions and positions $\mathcal{P} + \mathcal{N}$ seem to give $\mathcal{N}$-positions. The sum of two $\mathcal{N} + \mathcal{N}$ is more difficult to understand: sometimes it gives $\mathcal{P}$ (like for $(1, 0, 1, 0)$) and sometimes it gives $\mathcal{N}$ (for $(3, 1, 3, 0)$).
Here the game tree was not drawn for practical reasons: there are a lot of vertices and it will not be easy to read. These trees are good for small examples, to understand how the game works, but in practice this method takes a lot of space and time. Thankfully there are more tools to determine the outcome of sum of games.

Definition 1.31 (Equivalent classes) *Two positions G and G' are equivalent if for every position H, $o(G + H) = o(G' + H)$. This means that $G + H$ is a $\mathcal{P}$-position if and only if $G' + H$ is a $\mathcal{P}$-position.*

In particular, from Table 1.1.1 we remark that positions $(2, 0)$ in NIM and $(3, 0)$ in TOKENS are not equivalent since $o((2, 0)_{\text{NIM}} + (3, 0)_{\text{TOKENS}}) = \mathcal{N}$ and $o((3, 0)_{\text{TOKENS}} + (3, 0)_{\text{TOKENS}}) = \mathcal{P}$. For impartial game in normal convention there is a much more precise property:

Theorem 1.32 (Equivalent classes) *Two positions G and G' are equivalent if and only if $o(G + G') = \mathcal{P}$.*

In NIM two heaps are equivalent if and only if they are of same size: the $\mathcal{P}$-positions of NIM on two heaps are exactly (n, n), hence in the sum of two heaps, (n) and (m) we have a $\mathcal{P}$-position if and only if $n = m$. In particular, the heap of size n in NIM defines a unique equivalent class, denoted $\mathbf{n}$. There is a mapping between equivalent classes and positive integers.
For multiheap NIM games we can compute the class by using *Nim-sums*:

Definition 1.33 (Nim-sum) *Let $n_0, \ldots, n_{k-1}$ be non-negative integers. The* nim-sum *of these integers, $n_0 \oplus \cdots \oplus n_{k-1}$, is the bitwise XOR of these integers.*

Theorem 1.34 (Resolution of Nim) *Let $G = (n_0, \ldots, n_{k-1})$ be a game of* NIM *on k heaps. The game G is a $\mathcal{P}$-position if and only if $n_0 \oplus \cdots \oplus n_{k-1} = 0$.*
Moreover G is equivalent to the NIM *game $(n_0 \oplus \cdots \oplus n_{k-1})$ on one heap.*

Proof. First of all, the position $(0,\dots,0)$ has 0 as nim-sum and is a $\mathcal{P}$-position.
Now, let $(n_0,\dots,n_{k-1})$ be a position with 0 as nim-sum. In particular, for all i, $n_i = n_0 \oplus \dots \oplus n_{i-1} \oplus n_{i+1} \oplus \dots \oplus n_{k-1}$. All options of G are of the form $(n_0,\dots,n_{i-1},n'_i,n_{i+1},\dots,n_{k-1})$ for particulars i and $n'_i < n_i$. Assume one of this options has also 0 as nim-sum, say for i and $n'_i < n_i$, then $n'_i = n_0 \oplus \dots \oplus n_{i-1} \oplus n_{i+1} \oplus \dots \oplus n_{k-1}$ which is a contradiction since this nim-sum is equal to n_i and $n'_i < n_i$. Hence no option has 0 as nim-sum.
To finish, let $(n_0,\dots,n_{k-1})$ be a position with x as nim-sum, $x > 0$ and let $x_p 2^p + \dots + x_0 2^0$ be its binary decomposition with p such that $2^{p+1} > \max(n_0,\dots,n_{k-1},x) \geq 2^p$. As well, for $0 \leq i \leq k-1$, let $b_{i,p}2^p + \dots + b_{i,0}2^0$ be the binary decomposition of n_i. Let j be such that $x_j \neq 0$ and for all $i > j$, $x_i = 0$. There is some i such that $b_{i,j} \neq 0$, in particular, $n_i \oplus x < n_i$ since 2^j is removed and at most $2^p - 1$ is added. Let $n'_i = n_i \oplus x$, then $n_0 \oplus \dots \oplus n_{i-1} \oplus n'_i \oplus n_{i+1} \oplus \dots \oplus n_{k-1} = x \oplus n_i \oplus n'_i = 0$, meaning that the position $(n_0,\dots,n_{k-1})$ has an option with 0 as nim-sum.
In particular, positions with 0 as nim-sum are $\mathcal{P}$-positions.
Moreover, the game $(n_0,\dots,n_{k-1}) + (n_0 \oplus \dots \oplus n_{k-1})$ is a $\mathcal{P}$-position, hence the games $(n_0,\dots,n_{k-1})$ and $(n_0 \oplus \dots \oplus n_{k-1})$ are equivalent. ■

In fact, these classes are denoted *Grundy numbers* and we have:

Definition 1.35 (Grundy numbers) *The* Grundy number *of a position G of* NIM *on one heap of size n, denoted $\mathcal{G}(G)$, is* $\mathbf{n}$.

Moreover, these are the only possible classes, as Sprague and Grundy showed independently:

Theorem 1.36 (Sprague-Grundy theorem) *For every impartial game G there is a non-negative integer n such that G is equivalent to a* NIM *heap of size n.*

In particular, every impartial game has a *Grundy number*, and one of the main problems is to determine it. These numbers can be computed recursively from the final positions of the game:

Definition 1.37 (Minimal excluded value) *Let $S = \{n_1,\dots,n_k\}$ be a set of non-negative integers. The* minimal excluded value *of S, denoted by $mex(S)$ is the minimum non-negative integer m such that $m \notin S$.*

Proposition 1.38 *Let G be a position of a game of options $\{O_1,\dots O_t\}$.*

- *if G is a $\mathcal{P}$-position, then $\mathcal{G}(G) = \mathbf{0}$,*
- *otherwise, $\mathcal{G}(G) = mex(\{\mathcal{G}(O_1),\dots,\mathcal{G}(O_t)\})$.*

In other words, the final positions have Grundy value $\mathbf{0}$ and any other position has the minimum Grundy value that is not among the Grundy values of its options.
Moreover, it gives the Grundy numbers of sums:

Theorem 1.39 (Sum of games and Grundy numbers) *Let G and H be two impartial games such that $\mathcal{G}(G) = \mathbf{g}$ and $\mathcal{G}(H) = \mathbf{h}$. The Grundy number of $G + H$ is $\mathcal{G}(G+H) = \mathbf{g} \oplus \mathbf{h}$.*

This algorithm uses the game tree as a base whose sier is in genaral too large compared to the size of the game position. The goal is now to find a more practical way of computing Grundy numbers without computing the game tree. For instance, for NIM on k heaps, $n_0,\dots,n_{k-1}$, the game tree has $n_0 \times \dots \times n_{k-1}$ vertices and $n_0 + \dots + n_{k-1}$ edges, but computing the Grundy number needs only one operation: the nim-sum.

Here we study mainly *taking and breaking* games that are played on heaps where players can remove tokens or split heaps. The particularity of these games is that an instance of a game on k heaps, $(n_0,\dots,n_{k-1})$ is exactly the same as the sum of k games on one heap $(n_0) + \dots + (n_{k-1})$. Knowing the Grundy number of a heap of size n for all n gives all the information needed to compute the Grundy number of a multiheap game. Hence for these games on heaps we only need to compute the Grundy number of the heap of size n for all n to understand the behavior of the whole game.

Definition 1.40 (Grundy sequence) *Let* G *be a game played on a heap of tokens and denote by* $(n)_G$ *the heap of size* n*. The* Grundy sequence *of* G *is the non-negative sequence* $(\mathcal{G}((n)_G))_{n\in\mathbb{N}}$.

Remark that the sequence starts with the heap of size 1, as for the heap of size 0 the players have no options, the Grundy number is always **0**.

For the game TOKENS the Grundy sequence is **10101010101**... and for NIM, by definition, it is **123456**.... According to the games, these sequences can have different behaviors, among others there are periodic sequences, artithmetic-periodic sequences or a mix of the two:

Definition 1.41 (type of sequences) *Let* $S = (s_n)_{n\in\mathbb{N}}$ *be a sequence of integers.*

- **Periodicity**: *The sequence* A *is* periodic *if there is a positive integer* $\mathbf{p}$ *such that for all* $n \in \mathbb{N}$, $s_n = s_{n+p}$; $\mathbf{p}$ *is then called the* period.
  The sequence is ultimately periodic *if there are two positive integers* $\mathbf{e}$ *and* $\mathbf{p}$ *such that for all* $n \geq e$, $s_n = s_{n+p}$; $\mathbf{e}$ *is then called the* pre-period.

- **Arithmetic-Periodicity**: *The sequence* A *is* arithmetic-periodic *if there are two positive integers* $\mathbf{s}$ *and* $\mathbf{p}$ *such that for all* $n \in \mathbb{N}$, $s_{n+p} = s_n + \mathbf{s}$; $\mathbf{s}$ *is then called the* saltus *and* $\mathbf{p}$ *the* period.
  The sequence A *is* ultimately arithmetic-periodic *if there are three positive integers* $\mathbf{e}$, $\mathbf{p}$ *and* $\mathbf{s}$ *such that for all* $n \geq e$, $s_{n+p} = s_n + \mathbf{s}$; $\mathbf{e}$ *is then called the* pre-period.

- **Split arithmetic-periodic/periodic-regular or sapp-regular**: *The sequence* A *is* sapp-regular *if there is a partition of* $\mathbb{N}$ *into two sets* X, Y *such that* $A|_X = (s_n)_{n\in X}$ *is periodic and* $A|_Y = (s_n)_{n\in Y}$ *is arithmetic-periodic.*

To ease the notations we denote by:

$$(e_1, \dots, e_t)\ ((n_1, \dots, n_{m_1})^{k_1}(n_{m_1+1}, \dots, n_{m_2})^{k_2} \dots (n_{m_{r-1}}, \dots, n_{m_r})^{k_r})^k\ (+s)$$

the ultimately arithmetic-periodic sequence of preperiod $\mathbf{t}$ for $e_1, \dots, e_t$; of period $(\mathbf{m_1k_1} + \dots + \mathbf{m_rk_r})\mathbf{k}$ for $((n_1, \dots, n_{m_1})^{k_1} \dots (n_{m_{r-1}}, \dots, n_{m_r})^{k_r})^k$; and of saltus $\mathbf{s}$, where each subsequence $(n_{m_i}, \dots, n_{m_{i+1}})$ is repeated k_i times and the complete subsequence $(n_1, \dots, n_{m_1}, \dots, n_{m_{r-1}}, \dots, n_{m_r})$ is repeated k times. When there is no preperiod the first part is omitted.

There are other behaviors, like the *ruler-regularity*, that are not useful in the following. For more details we refer to [42].
In Chapter 4 details of Grundy sequences of some *taking and breaking* games are given.

Chapter 2

Identifying codes and variants

> "En essayant continuellement on finit par réussir. Donc : plus ça rate, plus on a de chance que ça marche."
>
> Jacques Rouxel - Les Shadoks

In this chapter I present the results obtained during a 5 months stay in Turku, Finland. This is a joint work with Ville Junnila and Tero Laihonen.
In this chapter we study identifying codes on graphs. We start with some definitions and examples. We introduce two variants of identifying codes and then we focus on our contributions. Recall that for $G(V, E)$ and $C \subset V$, C is called a *code*, items of the code are called *codewords* and for $v \in V$, the Iset of v is $I_{G,C}(v) = N[v] \cap C$.
Identifying codes on regular graphs have been largerly studied: optimal codes have been found for infinite grids (the square, the triangular and the king grid) and also on finite graphs like cycles and power of cycles. Here we focus on graphs linked to power of cycles: the circulant graphs. These graphs are also regular and some of them are embeddable on the infinite grids.

2.1 Introduction and some definitions

In 1984 Slater introduced locating-dominating codes that are now considered as a variant of identifying codes:

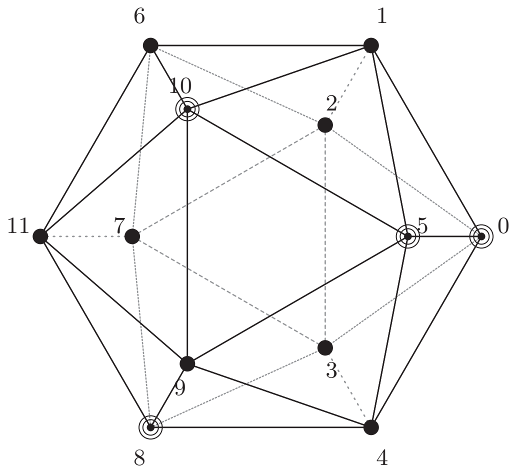
Definition 2.1 (Locating-dominating code) *[62] Let $G(V, E)$ be a graph and C a code of G. The code C is a* locating-dominating*, or LD- code if:*

$$\forall u, v \in V \setminus C, u \neq v, I_{C,G}(u) \neq I_{C,G}(v) \text{ and } I_{C,G}(u) \neq \varnothing.$$

A LD-code of G of minimal cardinality is an optimal *LD-code of G and its cardinality is denoted by $\gamma^{LD}(G)$.*

In Figure 2.1.1 the Isets of vertices $0, 5, 8, 10$ are not written as they are not needed to see if it is a *LD*-code. We can see that all the other Isets are pairwise different, hence the code $\{0, 5, 8, 10\}$ is a locating-dominating one.

Locating-dominating sets where introduced in 1984 by Slater in [62]. They are used to locate faulty processors. Indeed, if we assume the graph of Figure 2.1.1 is a network of processors and that the vertices $0, 5, 8$ and 10 have sensors on them, then if there is a faulty processor v, the sensors neighbors to that processor will be activated. For instance, if the sensors of $0, 5$ and 8 are activated, then we can identify the processor 3 as being the faulty one as its Iset is exactly $\{0, 5, 8\}$.



vertex	Iset
0	–
1	$\{0,5,10\}$
2	$\{0\}$
3	$\{0,8\}$
4	$\{0,5,8\}$
5	–
6	$\{10\}$
7	$\{8\}$
8	–
9	$\{5,8,10\}$
10	–
11	$\{8,10\}$

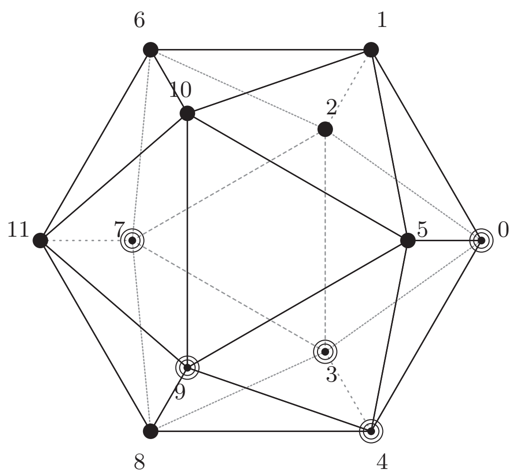
Figure 2.1.1: Example of a LD-code, $\{0,5,8,10\}$ on the dodecahedron.

In 1998,[54], Karpovsky, Chakrabarty and Levitin arose the question of identifying *all* the vertices, indeed, in locating-dominating codes the only vertices that are identified are the vertices that are not in the code, codewords are assumed to be non-faulty or to identify themselves right away.

Definition 2.2 (Identifying code) *Let $G(V,E)$ be a graph and $C \subset V$ be a code. The code C is an* identifying*, or ID- code if:*

$$\forall u,v \in V, I_{C,G}(u) \neq I_{C,G}(v) \text{ and } I_{C,G} \neq \varnothing.$$

An ID-code of G of minimal cardinality is an optimal *ID-code of G and its cardinality is denoted by $\gamma^{ID}(G)$.*



vertex	Iset
0	$\{0,3,4\}$
1	$\{0\}$
2	$\{0,3,7\}$
3	$\{0,3,4,7\}$
4	$\{0,3,4,9\}$
5	$\{0,4,9\}$
6	$\{7\}$
7	$\{3,7\}$
8	$\{3,4,7,9\}$
9	$\{4,9\}$
10	$\{9\}$
11	$\{7,9\}$

Figure 2.1.2: Example of an ID-code, $\{0,3,4,7,9\}$, on the dodecahedron.

Then again, if there is a faulty processor v, the active sensors on its neighborhood will help identify which vertex it is.

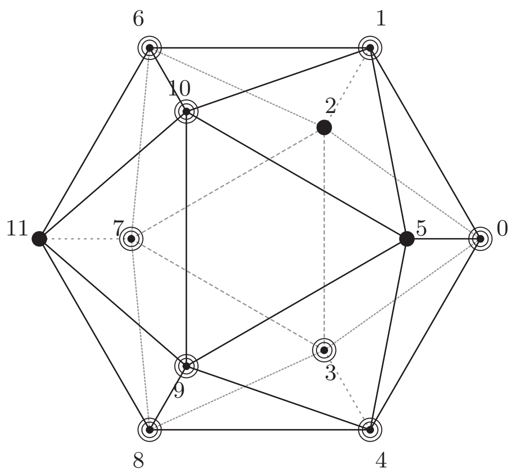
Now, assume there are two faulty processors, or more. None of these two codes help identify them. Assume for example the processors 5 and 6 are faulty. In the LD-case, the sensors activated are 0,4,9 and 7, which does not correspond to any Iset. The same hapepns for the ID-case.

To overcome this issue, Honkala and Laihonen defined a new type of codes in [48]: the *self-identifying* codes.

Definition 2.3 (Self-Identifying code) *Let $G(V, E)$ be a graph and $C \subset V$ a code. The code C is a* self-identifying*, or SID- code if:*

$$\forall u, v \in V, I_{C,G}(u) \not\subset I_{C,G}(v).$$

An SID-code of G of minimal cardinality is an optimal *SID-code of G and its cardinality is denoted by $\gamma^{SID}(G)$.*



vertex	Iset
0	$\{0, 1, 3, 4\}$
1	$\{0, 1, 6, 10\}$
2	$\{0, 1, 3, 6, 7\}$
3	$\{0, 3, 4, 7, 8\}$
4	$\{0, 3, 4, 8, 9\}$
5	$\{0, 1, 4, 9, 10\}$
6	$\{1, 6, 7, 10\}$
7	$\{3, 6, 7, 8\}$
8	$\{3, 4, 7, 8, 9\}$
9	$\{4, 8, 9, 10\}$
10	$\{1, 6, 9, 10\}$
11	$\{6, 7, 8, 9, 10\}$

Figure 2.1.3: Example of a SID-code, $\{0, 1, 3, 4, 6, 7, 8, 9, 10\}$, on the dodecahedron.

This new code has a very specific name *self*-identifying. This comes from the fact that, even if in the definition it seems necessary to compare all the Isets, in fact this is not needed.

Theorem 2.4 *[51] Let C be a code of G. Then the following statements are equivalent:*

- *The code C is self-identifying in G.*
- *For all $u \in C$ we have $I_{C,G}(u) \neq \emptyset$ and* $\bigcap_{c \in I_{C,G}(u)} N[c] = \{u\}$.

In particular this theorem gives a quick test for determining SID-codes.

Now, if in the graph and code of Figure 2.1.3 the sensors of vertices $\{0, 1, 3, 4\}$ are all activated, then it seems that $N[0] \cap N[1] \cap N[3] \cap N[4] = \{0, 1, 2, 3, 4, 5\} \cap \{0, 1, 2, 5, 6, 10\} \cap \{0, 2, 3, 4, 7, 8\} \cap \{0, 3, 4, 5, 8, 9\} = \{0\}$ is the faulty vertex.

Moreover, if two processors are faulty the code allows the user to identify at least one and know if there is another one to look for. If the sensors $0, 1, 6, 9, 10$ are activated, then $N[0] \cap ... \cap N[10] = \{\}$, which indicates that there is more than one faulty vertex. Now, let us do the intersections step by step:

vertex	cumulative intersections
0	$N[0] = \{0, 1, 2, 3, 4, 5\}$
1	$N[0] \cap N[1] = \{0, 1, 2\}$
6	$N[0] \cap N[1] \cap N[6] = \{1, 2\}$
9	$N[0] \cap N[1] \cap N[6] = \{\}$
10	$N[0] \cap N[1] \cap N[6] \cap N[10] = \{\}$

If we remove 9 from the list, we obtain $\{1\}$, and we identified a faulty vertex. In reverse order, we would have removed 0 and find $\{10\}$ as the faulty vertex.

From these definitions we can directly deduce that SID-codes are ID-codes, and that ID-codes are also LD-codes, in particular:

$$\gamma^{LD}(G) \leq \gamma^{ID}(G) \leq \gamma^{SID}(G)$$

In this chapter we focus on the ID-codes and then see briefly the modifications needed to treat the LD- and SID-codes.

2.2 Case of ID-codes

In the introductory paper, Karpovsky *et al.* gave lower and upper bounds for optimal ID-codes.

Proposition 2.5 *[54] Let G be a graph of order n. Then $\gamma^{ID}(G) \geq \lceil \log_2(n+1) \rceil$.*

Moreover, if the graph is regular, we have:

Proposition 2.6 *[54] Let G be a r-regular graph of order n. Then $\gamma^{ID}(G) \geq \frac{2n}{r+2}$.*

Proof. Let G be a r-regular graph of order n and C an identifying code on G.
As all vertices must have different Isets, there are at most $|C|$ vertices with Isets of cardinal 1. At best, all other vertices have in their Isets only two codewords. On one hand we count the minimum number of codewords seen by each vertex:

$$\sum_{v \in V} |I_{C,G}(v)| \geq 1 \times |C| + 2 \times (n - |C|),$$

on the other hand, in the sum each codeword appears $r+1$ times, as for each $c \in C$, $|N[c]| = r+1$, thus:

$$(r+1)|C| \geq 2n - |C| \;\Leftrightarrow\; |C| \geq \frac{2n}{r+2}.$$

■

This code has also been studied in infinite regular grids:

Definition 2.7 (Infinite square, triangular, hexagonal and king grids) *Here we define some infinite regular graphs.*

- ***Square grid**:*
The infinite square grid, *$\mathcal{S}$, is the graph of vertex set $\mathbb{Z}^2$ and such that for all $(i,j) \in \mathbb{Z}^2$, $N((i,j)) = \{(i-1,j),(i,j-1),(i,j+1),(i+1,j)\}$.*
- ***Triangular grid**:*
The infinite triangular grid, *$\mathcal{T}$, is the graph of vertex set $\mathbb{Z}^2$ and such that for all $(i,j) \in \mathbb{Z}^2$, $N((i,j)) = \{(i-1,j-1),(i-1,j),(i,j-1),(i,j+1),(i+1,j),(i+1,j+1)\}$.*
- ***King grid**: The* infinite king grid, *$\mathcal{K}$, is the graph of vertex set $\mathbb{Z}^2$ and such that for all $(i,j) \in \mathbb{Z}^2$, $N((i,j)) = \{(i-1,j-1),(i-1,j),(i-1,j+1),(i,j-1),(i,j+1),(i+1,j-1),(i+1,j),(i+1,j+1)\}$.*
- ***Hexagonal grid**: The* infinite hexagonal grid, *$\mathcal{H}$, is the graph of vertex set $\mathbb{Z}^2$ and such that for all $(i,j) \in \mathbb{Z}^2$, if $i+j \equiv 0 \bmod 2$ then $N((i,j)) = \{(i-1,j),(i+1,j),(i,j-1)\}$ and $N((i,j)) = \{(i-1,j),(i+1,j),(i,j+1)\}$ otherwise.*

For examples of these grids we refer to Figures 2.2.1 to 2.2.3. These graphs are infinite and we have not yet defined the optimal identifying codes in infinite graphs...

Definition 2.8 *[21] The* density *of an ID-code of an infinite grid over $\mathbb{Z}^2$, $\mathcal{G}$, is:*

$$D_{\mathcal{G}}(C) = \lim_{n \to \infty} \frac{|C|}{|Q_n|}$$

where $Q_n = [\![-n, n]\!]^2$.
An optimal *ID-code of an infinite grid over $\mathbb{Z}^2$, $\mathcal{G}$, is a code of minimum density and its density is denoted $\gamma^{ID}(\mathcal{G})$.*

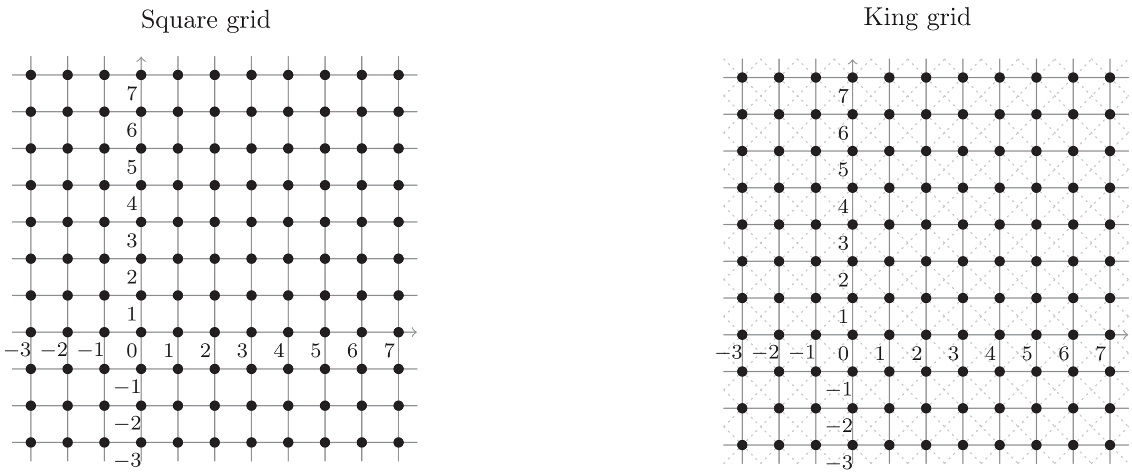


Figure 2.2.1: Example of square and king grids.

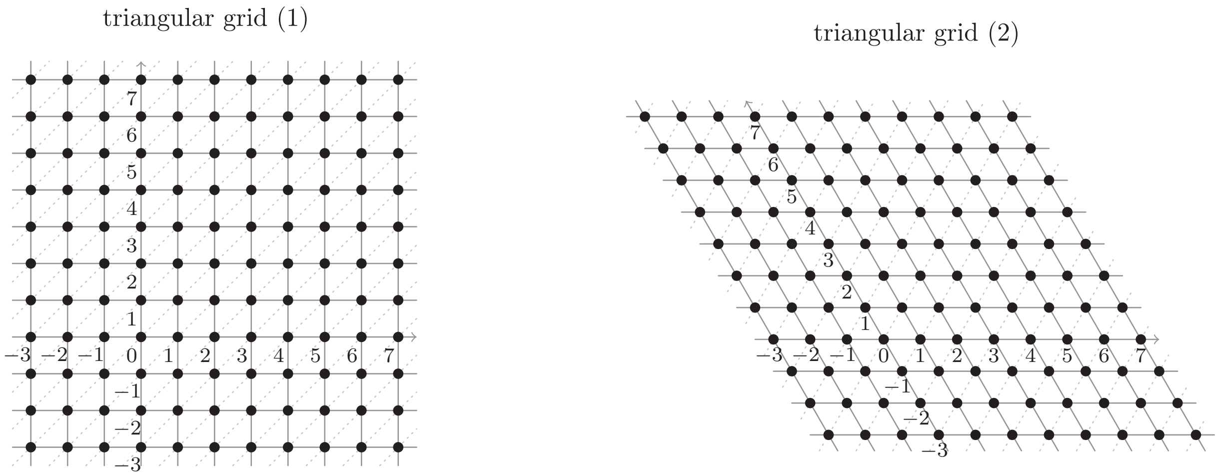


Figure 2.2.2: Example of triangular grids in the orthogonal plane and in the isometric plane

From Proposition 2.6:

Proposition 2.9 *The optimal ID-codes on the infinite grids $\mathcal{S}, \mathcal{T}, \mathcal{K}, \mathcal{H}$ are such that:*

$$\begin{array}{rcl} \gamma^{ID}(\mathcal{S}) & \geq & 1/3 \\ \gamma^{ID}(\mathcal{T}) & \geq & 1/4 \\ \gamma^{ID}(\mathcal{K}) & \geq & 1/5 \\ \gamma^{ID}(\mathcal{H}) & \geq & 2/5 \end{array}$$

In 2001 Cohen, Honkala, Lobstein and Zémor improved the lower bound for the ID-code on the square grid:

Proposition 2.10 *[22] Let C be an ID-code on the infinite square grid $\mathcal{S}$, then:*

$$D_{\mathcal{S}}(C) \geq \frac{15}{43}.$$

Moreover, the known codes with small densities in these grids are:

Proposition 2.11 *Known codes of smallest densities in $\mathcal{S}, \mathcal{T}, \mathcal{K}, \mathcal{H}$:*

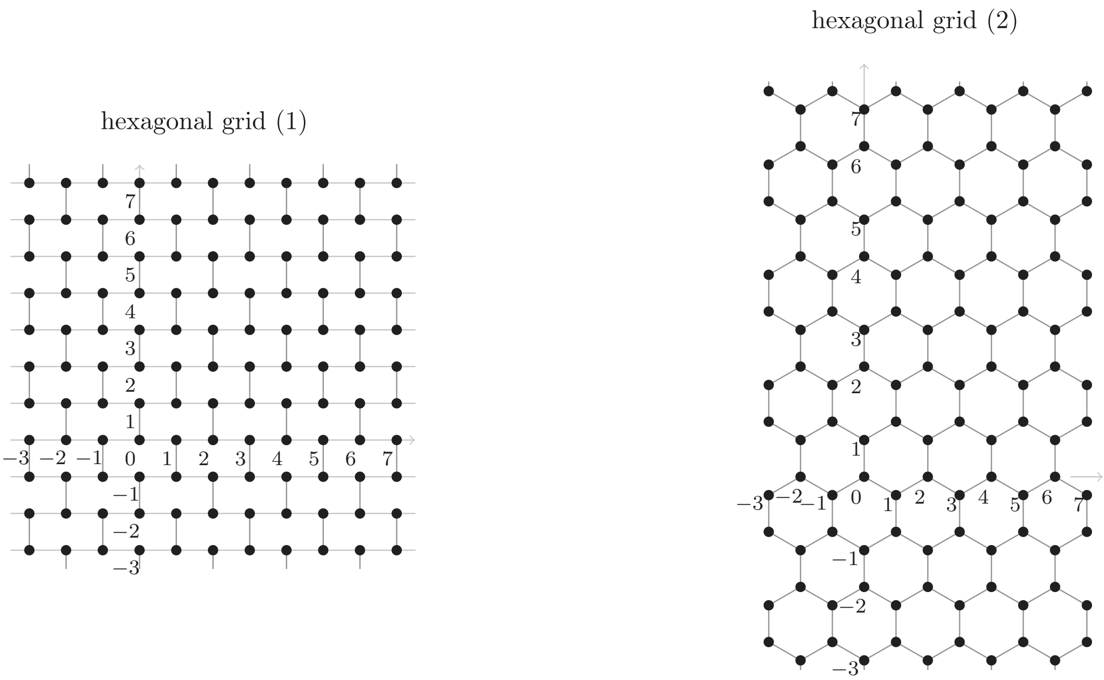


Figure 2.2.3: Example of hexagonal grids in the orthogonal plane and the hexagonal mesh

grid	*density*	*reference*
$\mathcal{S}$	*7/20*	*[8],[31]*
$\mathcal{T}$	*1/4*	*[22]*
$\mathcal{K}$	*2/9*	*[18],[21]*
$\mathcal{H}$	*3/7*	*[23]*

In fact, here we have an optimal code for the triangular grid with the expected density, $1/4$. In [8] Ben-Haim and Litsyn proved that ID-codes on the square grid have density at leat $7/20$. As well, Cohen, Honkala and Lobstein showed in [21] that ID-codes in the king grid have density at least $2/9$. In this table we have then three optimal values, and the only open question is

Open question 2.12 *Is* $3/7$ *the optimal density for* ID*-codes on the hexagonal grid* $\mathcal{H}$*?*

Other regular graphs that have already been studied are cycles. Recall that C_n denotes the cycle of length n.
From Proposition 2.6:

Proposition 2.13 *Let* n *be a positive integer,* $n \geq 3$ *and let* C *be an* ID*-code of* C_n*. Then*

$$|C_n| \geq \frac{n}{2}.$$

More precisely:

Theorem 2.14 *Let* n *be a positive integer.*

- *[11] If* $n \geq 4$ *is even:*

$$\gamma^{ID}(C_n) = \begin{cases} 3 & \text{if } n = 4 \\ \frac{n}{2} & \text{otherwise} \end{cases}$$

- *[40] If* $n \geq 5$ *is odd:*

$$\gamma^{ID}(C_n) = \begin{cases} 3 & \text{if } n = 5 \\ \frac{n+3}{2} & \text{otherwise} \end{cases}$$

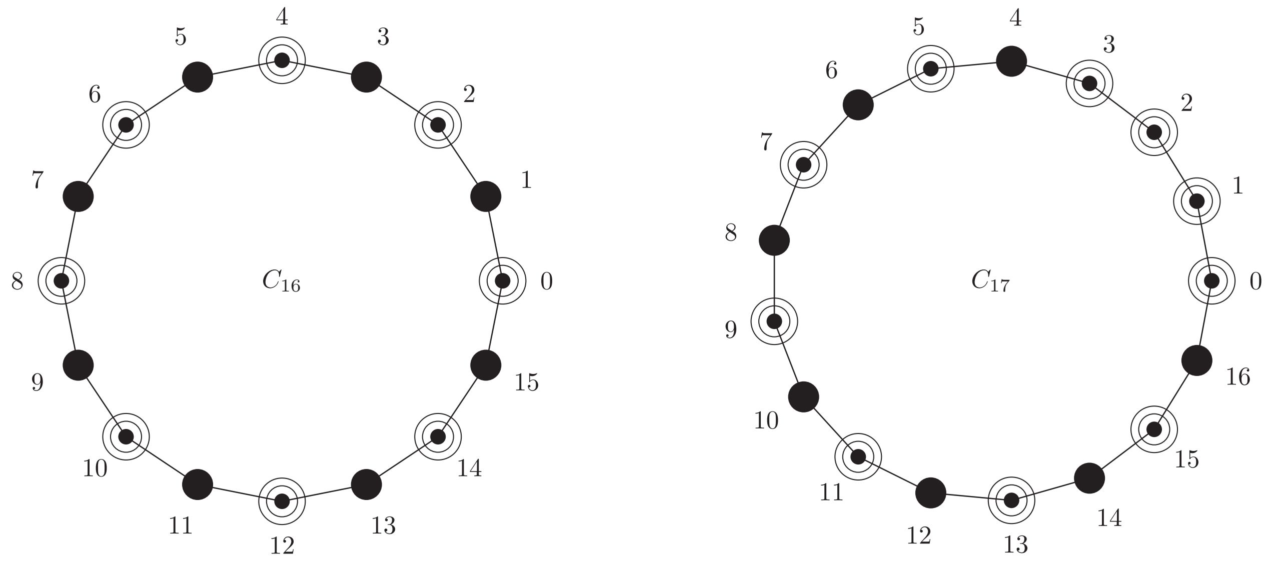


Figure 2.2.4: Examples of optimal ID-codes on an even cycle (left) and on an odd one (right). The codes are $\{0, 2, 4, 6, 8, 10, 12, 14\}$ in the first case and $\{0, 1, 2, 3, 5, 7, 9, 11, 13, 15\}$ in the second case.

Examples of optimal codes are shown in Figure 2.2.4.
As well as cycles, there has been also some interest on power of cycles.

Definition 2.15 *Let n be a positive integer. The cycle C_n to the p-th power, denoted by C_n^p, is the graph of vertex set $0, \ldots, n-1$ and such that $N(i) = \{(i-p) \bmod n, (i-p+1) \bmod n, \ldots, i-1, i+1, \ldots, (i+p) \bmod n\}$.*

In [11, 19, 40, 50, 63] and [74] is the complete study of ID-codes on powers of cycles. Here we are interested in a variant of these powers of cycles: the circulant graphs.

Definition 2.16 (Circulant Graph) *Let n be a positive integer and $d_1, \ldots, d_k$ be k integers such that for all i, $d_i \leq n/2$. The circulant graph $C_n(d_1, \ldots, d_k)$ is the graph of vertex set $0, \ldots, n-1$ and such that for all i:*

$$N(i) = \{i - d_k, i - d_{k-1}, \ldots, i - d_1, i + d_1, \ldots, i + d_k\}$$

where the computations are done modulo n.

Remark that the circulant graph $C_n(1, 2, \ldots, r)$ is exactly the r-th power of C_n, C_n^r. In Figure 2.2.5 we can see two examples of circulant graphs, where one of them is also a power of a cycle. These graphs are all regular, hence we can apply Proposition 2.6.
As the optimal ID-codes for power of cycles where already studied, in 2013, Ghebleh and Niepel considered circulant graphs that where not powers of graphs.

Theorem 2.17 *[37] Let n be a positive integer, $n \geq 15$, then:*

$$\gamma^{ID}(C_n(1,3)) \geq \left\lceil \frac{4n}{11} \right\rceil .$$

Moreover,

$$\gamma^{ID}(C_n(1,3)) \begin{cases} = & \lceil \frac{4n}{11} \rceil & n \not\equiv 8 \bmod 11 \\ \leq & \lceil \frac{4n}{11} \rceil + 1 & \textit{otherwise} \end{cases}$$

In the same article they explain why their methods are not easily applied to general circulant graphs of the form $C_n(1, d)$. They also suggested that for $n \equiv 8 \bmod 11$ the lower bound was not attainable.

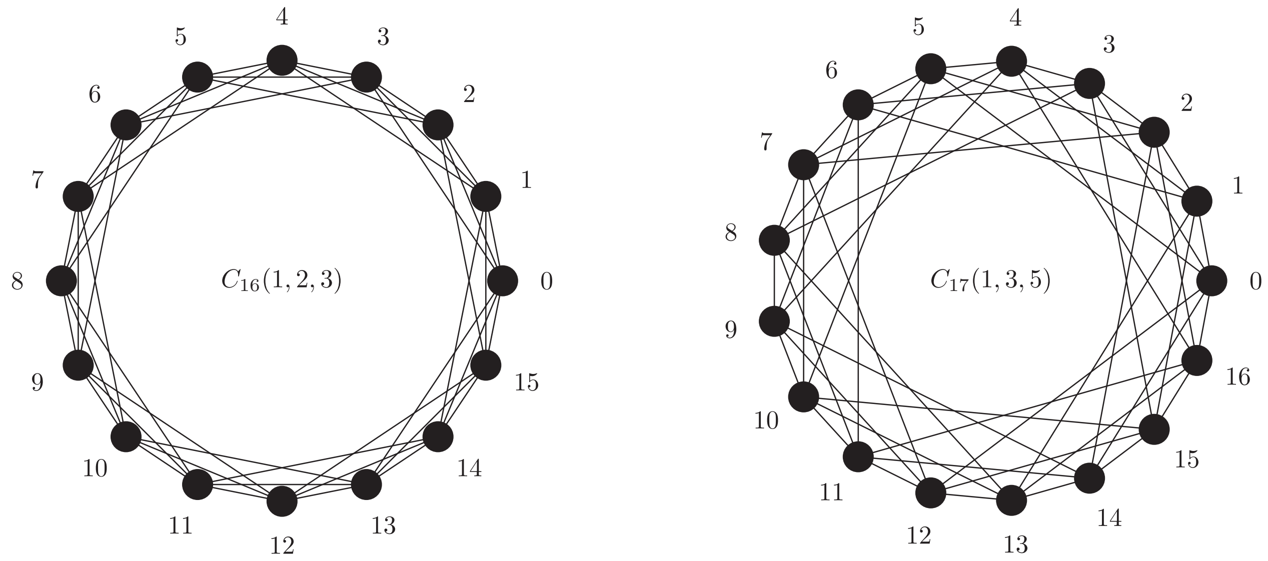


Figure 2.2.5: Example of circulant graphs: $C_{16}(1,2,3) = C_{16}^3$ and $C_{17}(1,3,5)$.

Sadly, $n \equiv 8 \bmod 11$ is not the only special case: there are no ID-codes on $C_{11q+2}(1,3)$ of cardinality $4q+1$ for $q \geq 5$, there is a problem in their codes. In 2017, in collaboration with V.Junnila and T.Laihonen, we correct this last part. Moreover we give the exact sizes of optimal ID-codes on circulant graphs $C_n(1,3)$ for $n \geq 11$. We introduce a new method to bound below the optimal ID-codes, which is also suitable for general circulant graphs as we see later.

Theorem 2.18 *[53] Let $n \geq 11$ and (a,b) the unique couple such that $n = 11a+b$ and $0 \leq b < 11$. Then:*

$$\gamma^{ID}(C_n(1,3)) = \begin{cases} \lceil \frac{4n}{11} \rceil + 1 & \text{if } b=2, a \geq 5 \text{ or } b=5, a\geq 3 \text{ or } b=8 \\ \lceil \frac{4n}{11} \rceil & \text{otherwise} \end{cases}$$

The proof of this theorem needs a strong tool introduced by Slater in 2002:

Definition 2.19 (Share) *[68] Let G be a graph and C a code on G such that for all $v \in V, I_{C,G}(v) \neq \varnothing$. The share of a codeword c, $\mathtt{s}_{C,G}(c)$ is:*

$$\mathtt{s}_{C,G}(c) = \sum_{v \in N[c]} \frac{1}{|I_{C,G}(v)|}.$$

In other terms, a *share* of a codeword corresponds to the fraction of identifying it does for its neighbors.

Proposition 2.20 *[68] Let G be a graph and C a code of G such that for all $v \in V, I_{C,G}(v) \neq \varnothing$. Then:*

$$\sum_{v \in C} \mathtt{s}_{C,G}(v) = n.$$

Proof. In the sum

$$\sum_{c \in C} \mathtt{s}(c) = \sum_{c \in C} \sum_{v \in N[c]} \frac{1}{|I_{C,G}(v)|}$$

each vertex $v \in V$ appears exactly $|I_{C,G}(v)|$ times (as it has that number of codewords in its Iset), hence:

$$\sum_{c \in C} \mathtt{s}(c) = \sum_{v \in V} |I_{C,G}(v)| \times \frac{1}{|I_{C,G}(v)|} = n.$$

■

In particular if all shares are upper bounded by a constant α we obtain directly: $\alpha|C| \geq n$, hence $|C| \geq n/\alpha$.

In the proof of Theorem 2.18 we take an ID-code and use a discharging method over the shares of each codeword to bound above the total shares. Here we present only the sketch of the proof, for the complete proof we refer to the Appendix A.

Sketch of the proof of Theorem 2.18. Let $n \geq 11$ and C be an ID-code of $C_n(1,3)$. Recall that the vertices are identified to the set $\mathbb{Z}_n$.

Let P be the pattern formed by 9 consecutive vertices $u, u+1, \ldots, u+8$ such that $u, u+1, u+3, u+4, u+5, u+6, u+7, u+8 \notin C$ and $u+2, u+3 \in C$. We denote by P' the reverse pattern of P, *i.e.*, $u+5, u+6 \in C$ and the rest are not in C. We say a codeword c is in a pattern P if c is one of the codewords $u+2, u+3$, or respectively of P' if c is one of the codewords $u+5$ or $u+6$.

We compute the shares in $C_n(1,3)$ with the code C and discharge the heavy ones to get a lower mean on the shares. The discharging method follows the rules:

1. If c is a codeword such that its surroundings are as in Figure 2.2.6(1), then 1/12 shares are shifted from c to $c+1$ by rule R1.1 and 1/24 shares are shifted to $c-1$ by rule R1.2.
2. If c is a codeword such that its surroundings are as in Figure 2.2.6(2), then 1/8 shares are shifted to $c+4$ from c by rule R2.1 and from $c+1$ by rule R2.2.
3. If c is a codeword such that its surroundings are as in Figure 2.2.6(3) then 1/24 shares are shifted from c to $c+1$ by rule R3.1, to $c+4$ by rule R3.2 and to $c+7$ by rule R3.3.
4. If c is a codeword such that its surroundings are as in Figure 2.2.6(4), then 1/8 shares are shifted to $c+11$ from c by rule R4.1, from $c+1$ by rule R4.2 and from $c+4$ by R4.3.
5. If c is a codeword such that its surroundings are as in Figure 2.2.6(5), then 1/8 shares are shifted from c to $c+3$ by rule R5.
6. If c is a codeword such that its surroundings are as in Figure 2.2.6(6), then 1/8 shares are shifted from c to $c+1$ by rule R6.
7. If c is a codeword such that its surroundings are as in Figure 2.2.6(7), then 1/12 shares are shifted from c to $c+1$ by the rule R7.

Of course, if the surroundings of a codeword are symmetric to one of these cases, then we shift the shares symmetrically. By shifting shares this way and denoting $\mathtt{s}_s(u)$ the share of codeword u after shifting, we have:

Claim 1 Let u be a codeword receiving shares by these rules. Then, if u is from a pattern P (or P') then $\mathtt{s}_s(u) \leq 11/4$, otherwise $\mathtt{s}_s(u) \leq 11/4 - 1/24$.

And:

Claim 2 Let u be a codeword not receiving shares by these rules. Then, if u is from a pattern P (or P') then $\mathtt{s}_s(u) \leq 11/4$, otherwise $\mathtt{s}_s(u) \leq 11/4 - 1/24$.

In particular for $n \equiv 2, 5, 8 \bmod 11$, if no codeword of C belongs to a pattern P or P':

- If $n = 11q_1 + 2$, $q_1 \geq 5$, then $|C| \geq 4q_1 + 2 = \lceil 4n/11 \rceil + 1$.
- If $n = 11q_2 + 5$, $q_2 \geq 3$, then $|C| \geq 4q_2 + 3 = \lceil 4n/11 \rceil + 1$.
- If $n = 11q_3 + 8$, $q_3 \geq 1$, then $|C| \geq 4q_3 + 4 = \lceil 4n/11 \rceil + 1$.

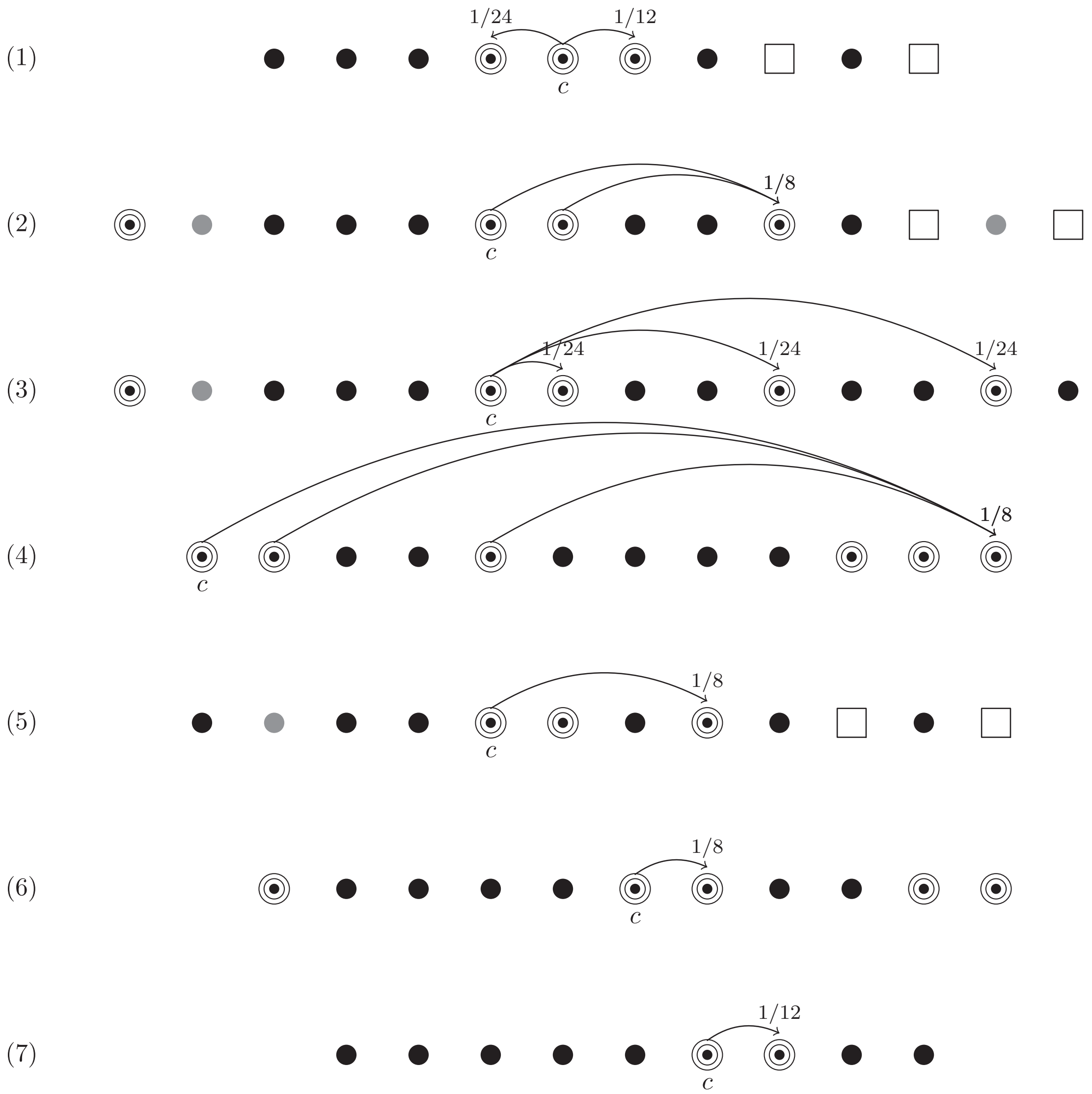


Figure 2.2.6: The rules of the discharging method illustrated. The haloed dots represent codewords, the ones without halo represent non-codewords, the gray ones can be either codewords or not. In figures (1), (2) and (5) at least one of the squares is a codeword. Note that the edges are omitted in the figure but from left to right vertices increase in value modulo n.

When $n = 11a + b$, $b \in \{2, 5, 8\}$, it is not possible to only have patterns P and P', hence there must be codewords not belonging to any pattern P or P'. This implies a drop of strictly more than 3/4 on the shares:

$$n = \sum_{c \in C} \mathtt{s}(c) < \frac{11}{4}|C| - \frac{3}{4}.$$

Hence:

- for $n = 11q_1 + 2$: $|C| > 4q_1 + 1$,
- for $n = 11q_2 + 5$: $|C| > 4q_2 + 2$,
- for $n = 11q_3 + 8$: $|C| > 4q_3 + 3$.

■

For a positive integer q, let $C_q = \{11i + j \mid 0 \le i \le q-1, j \in \{0,1,4,5\}\}$.

n	ID-code	$\|C\|$
11q	C_q	$4q$
11q+1	$C_q \cup \{11q\}$	$4q+1 = \lceil 4n/11 \rceil$
11q+2	$C_q \cup \{11q, 11q+1\}$	$4q+2 = \lceil 4n/11 \rceil + 1$
11q+3	$C_q \cup \{11q, 11q+1\}$	$4q+2 = \lceil 4n/11 \rceil$
11q+4	$C_q \cup \{11q, 11q+1\}$	$4q+2 = \lceil 4n/11 \rceil$
11q+5	$C_q \cup \{11q, 11q+1, 11q+2\}$	$4q+3 = \lceil 4n/11 \rceil + 1$
11q+6	$C_q \cup \{11q, 11q+1, 11q+2\}$	$4q+3 = \lceil 4n/11 \rceil$
11q+7	$C_q \cup \{11q, 11q+1, 11q+3\}$	$4q+3 = \lceil 4n/11 \rceil$
11q+8	$C_q \cup \{11q, 11q+1, 11q+2, 11q+3\}$	$4q+4 = \lceil 4n/11 \rceil + 1$
11q+9	$C_q \cup \{11q, 11q+1, 11q+2, 11q+3\}$	$4q+4 = \lceil 4n/11 \rceil$
11q+10	$C_q \cup \{11q, 11q+1, 11q+3, 11q+4\}$	$4q+4 = \lceil 4n/11 \rceil$

Table 2.2.1: Examples of optimal ID-codes on $C_n(1,3)$ for $n \ge 55$

Explicit codes attaining these bounds are given in Table 2.2.1 for $C_n(1,3)$, $n \ge 55$. And in Figure 2.2.7 there are examples of these codes on $C_n(1,3)$ graphs for $n \in [\![55, 65]\!]$.

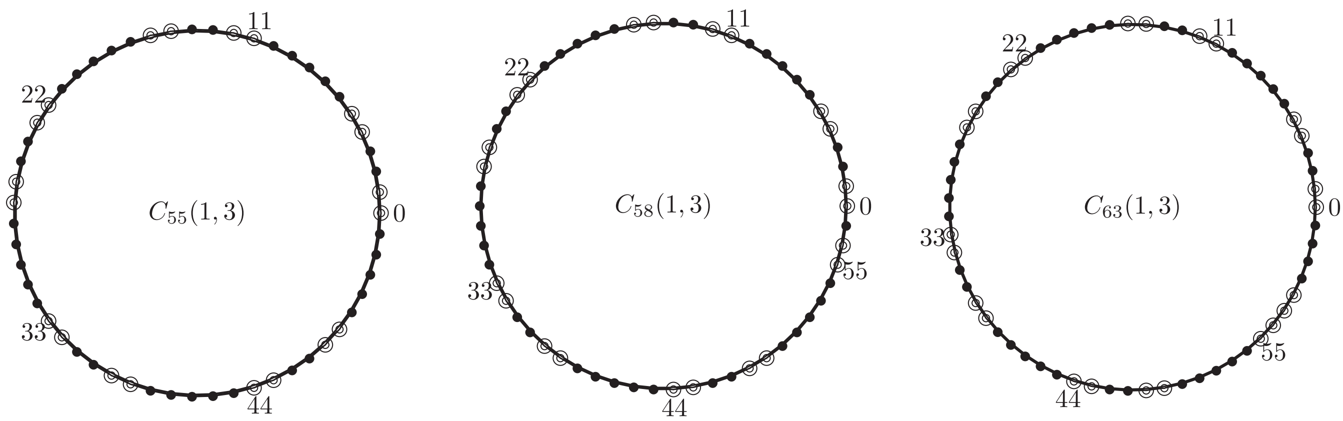


Figure 2.2.7: Example of optimal ID-codes on $C_n(1,3)$ for $n \in \{55, 58, 63\}$

In [52] we applied this same method to study circulant graphs of the form $C_n(1,d)$, $C_n(1,d,d+1)$ and $C_n(1,d-1,d,d+1)$.
First of all remark that these graphs are linked to the infinite grids:

Theorem 2.21 *[52] Let n, d and k be positive integers such that $d \ge 2$.*

- *If C is an identifying code in $C_n(1,d)$ with k codewords, then there exists an identifying code in the infinite square grid $\mathcal{S}$ with density k/n.*
- *If C is an identifying code in $C_n(1,d,d+1)$ with k codewords, then there exists an identifying code in the infinite triangular grid $\mathcal{T}$ with density k/n.*
- *If C is an identifying code in $C_n(1,d-1,d,d+1)$ with k codewords, then there exists an identifying code in the infinite king grid $\mathcal{K}$ with density k/n.*

Proof. Here we are only presenting the proof of the first result as the two others have very similar proofs. Let C be an identifying code in $C_n(1,d)$. We will use the following correspondence of the vertex $x = (x_1, x_2) \in \mathbb{Z}^2$ in the square grid with the vertex $x_1 + x_2 \cdot d$ in $C_n(1,d)$ where $x_1 + x_2 \cdot d$ is computed modulo n. Namely the closed neighbourhood of x is $N_{\mathcal{S}}[x] = \{(x_1,x_2), (x_1-1,x_2), (x_1+1,x_2)(x_1,x_2-1), (x_1,x_2+1)\}$ and the corresponding set in $C_n(1,d)$ is $\{x_1+x_2\cdot d, x_1-1+x_2\cdot d, x_1+1+x_2\cdot d, x_1+(x_2-1)\cdot d, x_1+(x_2+1)\cdot d\} = N_{C_n(1,d)}[x_1 + x_2 \cdot d]$ as shown in Figure 2.2.8. We define the following code in the square grid:

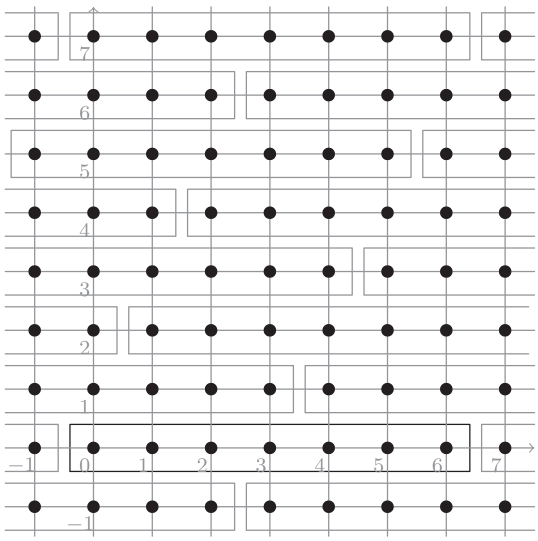


Figure 2.2.8: Example of $C_7(1,3)$ in the square grid where the rectangles represent copies of $C_7(1,3)$. Remark that the neighborhood of (x,y) in the grid corresponds to the neighborhood of $x+y\cdot d$ in the circulant graph.

$$C_{\mathcal{S}} = \{(x_1,x_2)\in\mathbb{Z}^2 \mid x_1+x_2\cdot d\in C\}.$$

In other words, a vertex (x_1,x_2) belongs to $C_{\mathcal{S}}$ if and only if the corresponding vertex $x_1+x_2\cdot d$ belongs to C. In what follows we show that $C_{\mathcal{S}}$ is an identifying code in $\mathcal{S}$.
First let us show that $C_{\mathcal{S}}$ is a dominating set in $\mathcal{S}$. Let $(x,y)\in\mathcal{S}$. Assume $I_{\mathcal{S}}((x,y))=\varnothing$, *i.e.*, $N[(x,y)]\cap C_{\mathcal{S}}=\varnothing$. This means that the vertices $(x-1,y),(x,y-1),(x,y),(x+1,y),(x,y+1)$ are not in $C_{\mathcal{S}}$, and by definition of $C_{\mathcal{S}}$, the vertices $(x-1+y\cdot d, x+(y-1)\cdot d, x+y\cdot d, x+1+y\cdot d, x+(y+1)\cdot d$ of $C_n(1,d)$ are not in C and, as $N[x+y\cdot d]=\{x+y\cdot d-d, x+y\cdot d-1, x+y\cdot, x+y\cdot d+1, x+y\cdot d+d\}$, $I(x+y\cdot d)=\varnothing$, hence C is not an ID-code in $C_n(1,d)$, a contradiction. Hence $C_{\mathcal{S}}$ is a dominating set in $\mathcal{S}$.
Now, assume there exist two distinct vertices $A=(x,y)\in\mathbb{Z}^2$ and $B=(z,t)\in\mathbb{Z}^2$ in the square grid such that $I_{\mathcal{S}}(A)=I_{\mathcal{S}}(B)$. Without loss of generality we can assume $t\geq y$. As $I_{\mathcal{S}}(A)\neq\varnothing$ and $I_{\mathcal{S}}(B)\neq\varnothing$, the vertices A and B are at distance at most 2 in $\mathcal{S}$, *i.e.* $B\in\{(x-2,y),(x-1,y),(x,y),(x+1,y),(x+2,y),(x-1,y+1),(x,y+1,(x+1,y+1),(x,y+2)\}$. Moreover, $I_{\mathcal{S}}(A)=I_{\mathcal{S}}(B)$ implies $I(x+y\cdot d)=I(z+t\cdot d)$, hence $x+y\cdot d\equiv z+t\cdot d \bmod n$ since C is an ID-code in $C_n(1,d)$. As B and A are at distance at most 2, we have: $z=x+a$, $t=y+b$ for a particular couple (a,b), $-2\leq a\leq 2$, $0\leq b\leq a$. This gives directly $a+b\cdot d\equiv 0 \bmod n$, *i.e.*$a=b=0$ or $a=0, b=2$ and $n=2d$. Assume $a=0,b=2$ and $n=2d$, in this case the only codeword in $I_{\mathcal{S}}(A)$ is $(x,y+1)$. As A and B correspond to the same vertex in C_n, $(z,t+1)\in I_{\mathcal{S}}(B)$, *i.e.* $I_{\mathcal{S}}(A)\neq I_{\mathcal{S}}(B)$, hence $C_{\mathcal{S}}$ is an ID-code in $\mathcal{S}$.
For the cases of $C_n(1,d,d+1)$ and $C_n(1,d-1,d,d+1)$ the proof is similar. ■

In particular, the lower bounds of ID-codes in these three grids give directly lower bounds of ID-codes for the graphs $C_n(1,d)$.

Corollary 2.22 *Let n and d be positive integers such that $d\geq 2$ and $d\leq n/2$. Then we have:*

$$\gamma^{ID}(C_n(1,d))\geq\left\lceil\frac{7n}{20}\right\rceil, \gamma^{ID}(C_n(1,d,d+1))\geq\left\lceil\frac{n}{4}\right\rceil \text{ and } \gamma^{ID}(C_n(1,d-1,d,d+1))\geq\left\lceil\frac{2n}{9}\right\rceil.$$

There are infinitely many values n,d such that the optimal ID-codes attain these lower bounds:

Theorem 2.23 *Let n and d be two positive integers such that $n\geq 2d$.*

1. ***For** $C_n(1,d)$**:***

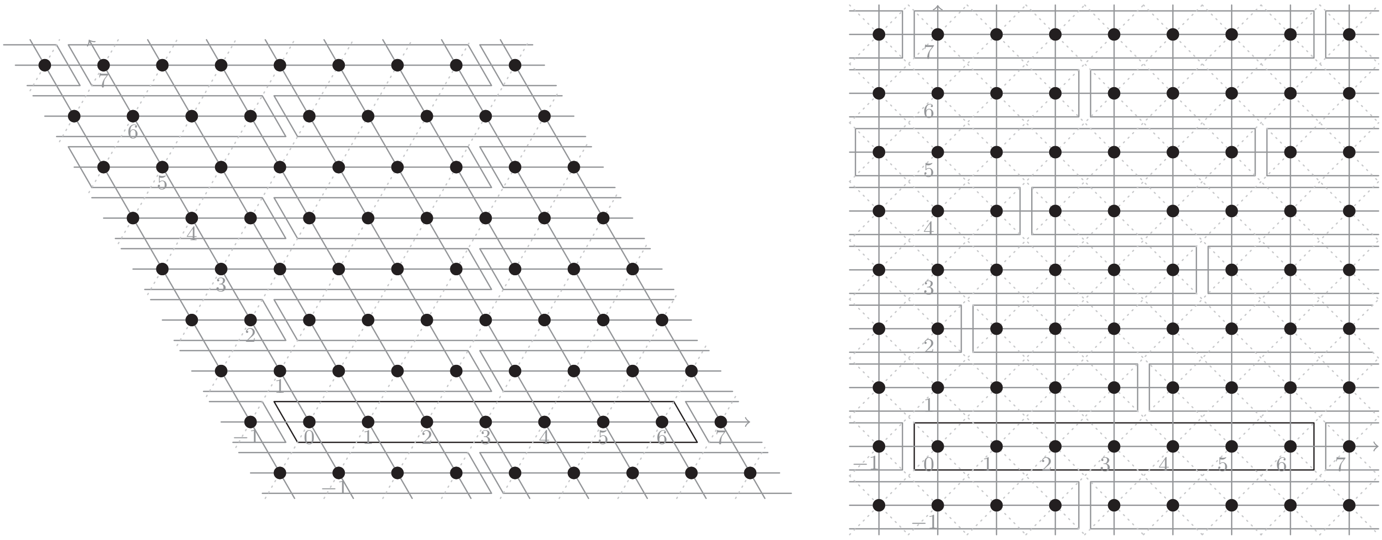


Figure 2.2.9: Example of $C_7(1,3)$ in the triangular and king grids. The rectangles represent copies of $C_7(1,3)$.

If $n \equiv 0 \bmod 40$ *and* $d \equiv 4 \bmod 40$, *then we have* $\gamma^{ID}(C_n(1,d)) = \frac{7n}{20}$.
If $n \equiv 0 \bmod 20$ *and* $d \equiv 6 \bmod 20$, *then we have* $\gamma^{ID}(C_n(1,d)) = \frac{7n}{20}$.

2. ***For*** $C_n(1,d-1,d)$***:***
There is a sequence $(n_i, d_i, C_i)_{i=1}^{\infty}$ *such that for all* i, C_i *is an ID-code on the graph* $C_{n_i}(1, d_i - 1, d_i)$,

$$\lim_{i\to\infty} n_i = \infty \text{ and } \lim_{i\to\infty} \frac{|C_i|}{n_i} = \frac{1}{4}.$$

3. ***For*** $C_n(1,d-1,d,d+1)$***:***
There is a sequence $(n_i, d_i, C_i)_{i=1}^{\infty}$ *such that for all* i, C_i *is an ID-code on the graph* $C_{n_i}(1, d_i - 1, d_i, d_i + 1)$,

$$\lim_{i\to\infty} n_i = \infty \text{ and } \lim_{i\to\infty} \frac{|C_i|}{n_i} = \frac{2}{9}.$$

Proof. Let n and d be two positive integers such that $n \geq 2d$.

1. For $C_n(1,d)$:
Let $n \equiv 0 \pmod{40}$ and $d \equiv 4 \pmod{40}$. Define

$$B_1 = \{0, 1, 2, 8, 10, 12, 16, 18, 22, 24, 26, 32, 33, 34\}$$

and

$$D_1 = \{u \in \mathbb{Z}_n \mid u \equiv b \pmod{40} \text{ for some } b \in B_1\}.$$

It is straightforward to verify that B_1 is an identifying code in $C_{40}(1,4)$. In what follows, we prove that D_1 is an identifying code in $C_n(1,d)$ by showing that all the identifying sets $I_{C_n(1,d),D_1}(x)$ are nonempty and unique. Observe first that by the construction of D_1 we obtain for all $x \in \mathbb{Z}_n$ that

$$I_{C_n(1,d),D_1}(x) \equiv I_{C_{40}(1,4),B_1}(x') \pmod{40},$$

where x' is an integer such that $x \equiv x' \pmod{40}$ and $0 \leq x' \leq 39$. Therefore, the identifying sets $I_{C_n(1,d),D_1}(x)$ are nonempty for all $x \in \mathbb{Z}_n$. Let x and y be distinct vertices of $\mathbb{Z}_n$. Assume first that $x \not\equiv y \pmod{40}$. Let then x' and y' be integers such that $x \equiv x' \pmod{40}$, $y \equiv y' \pmod{40}$, $0 \leq x' \leq 39$ and $0 \leq y' \leq 39$.

Therefore, by the previous observation, if $I_{C_n(1,d),D_1}(x) = I_{C_n(1,d),D_1}(y)$, then $I_{C_{40}(1,4),B_1}(x') = I_{C_{40}(1,4),B_1}(y')$ and we have a contradiction as B_1 is an identifying code in $C_{40}(1,4)$. Hence, we may assume that $x \equiv y \pmod{40}$. Let us then show that $N[C_n(1,d);x] \cap N[C_n(1,d);y] = \emptyset$. Suppose to the contrary that there exist $x, y \in \mathbb{Z}_n$ such that $x+j = y+j'$ for some $j, j' \in \{-d, -1, 0, 1, d\}$. Since $x \equiv y \pmod{40}$, we obtain that $j \equiv j' \pmod{40}$. This further implies that $j = j'$ and $x = y$ (a contradiction).

Therefore, as each vertex of $\mathbb{Z}_n$ is covered by a codeword of D_1, we have $I_{C_n(1,d),D_1}(x) \neq I_{C_n(1,d),D_1}(y)$. Thus, D_1 is an identifying code in $C_n(1,d)$.

Now, let $n \equiv 0 \pmod{20}$ and $d \equiv 6 \pmod{20}$. Define $B_2 = \{0, 2, 8, 9, 11, 12, 18\}$ and

$$D_2 = \{u \in \mathbb{Z}_n \mid u \equiv b \pmod{20} \text{ for some } b \in B_2\}.$$

It is straightforward to verify that B_2 is an identifying code in $C_{20}(1,6)$. Then, using similar arguments as in the case (i), we can prove that D_2 is an identifying code in $C_n(1,d)$.

2. For $C_n(1, d-1, d)$:
Let $d \geq 6$ be even and $n = 6d$. Denote $S = \{j \mid 0 \leq j \leq d, j \equiv 0 \pmod 2\}$. We define

$$C_d = \{v \in \mathbb{Z}_n \mid v \equiv b \pmod{2d} \text{ for some } b \in S\}.$$

The code C_d has cardinality $3(d/2+1)$. Thus $\lim_{d\to\infty} |C_d|/n = 1/4$.

We will show that C_d is identifying in $C_n(1, d-1, d)$. If $x \equiv s \pmod{2d}$ with $d \leq s \leq 2d-1$ and x is odd, then $\{x-d+1, x+d-1\} \subseteq I(x)$. Since $N[x-d+1] \cap N[x+d-1] = \{x\}$, it follows that $I(x) \neq I(y)$ for any $y \neq x$. If $x \equiv s \pmod{2d}$ where x is even and $d \leq s \leq 2d-1$ or $s = 0$, then $\{x-d, x+d\} \subseteq I(x)$. Since $N[x-d] \cap N[x+d] = \{x\}$, the $I(x)$ is distinguished from other $I(y)$'s. Suppose then that $x \equiv s \pmod{2d}$ with $1 \leq s \leq d-1$ and x is odd. Now $\{x-1, x+1\} \subseteq I(x)$ and again $I(x)$ is unique among I-sets. If $x \equiv s \pmod{2d}$ with $1 \leq s \leq d-1$ and x is even, then $I(x) = \{x\}$. It follows that C_d is identifying.

3. For $C_n(1, d-1, d, d+1)$:

Let $d \geq 15$, $d \equiv 3 \pmod 6$ and $n = 3d-9$. Notice that $n \equiv 0 \pmod 6$. We divide the vertices of the circulant graph into three sections denoted by $A_1 = \{0, 1, 2 \ldots, d-1\}$, $A_2 = \{d, d+1, \ldots, 2d-1\}$ and $A_3 = \{0, 1, \ldots, n-1\} \setminus (A_1 \cup A_2)$. We will first consider the code

$$C_d = \{v \mid v \in (A_1 \cup A_3), v \equiv 5 \pmod 6\} \cup \{v \mid v \in A_2, v \equiv 0, 4 \pmod 6\}.$$

Using this code we can construct (by adding later two more codewords) an identifying code in $C_n(1, d-1, d, d+1)$. The ratio $|C_d|/n$ tends to $2/9$ as d tends to infinitely. First we exclude some 'borderline' vertices from the three sections and denote $A_1' = A_1 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, d-1\}$, $A_2' = A_2 \setminus \{d, 2d-1\}$ and $A_3' = A_3 \setminus \{2d\}$. We consider the borderline vertices later. It is straightforward to check that the I-sets with regard to the code C_d are as follows for $x \in A_1' \cup A_2' \cup A_3'$:

$x \in A_1'$	$I(x)$	$d(c_1, c_2)$	$I(x)$ mod 6
$\equiv 0$ mod 6	$\{x-1, x+d+1\}$	$d+2$	$4, 5$
1	$\{x-d+1, x+d\}$	$d-8$	$4, 5$
2	$\{x-d, x+d-1, x+d+1\}$		$0, 4, 5$
3	$\{x-d-1, x+d\}$	$d-10$	$0, 5$
4	$\{x+1, x+d-1\}$	$d-2$	$0, 5$
5	$\{x\}$		

$x \in A'_2$	$I(x)$	$d(c_1, c_2)$	$I(x)$ mod 6
$\equiv 0$ mod 6	$\{x\}$		
1	$\{x-d+1, x-1, x+d+1\}$		$0, 5, 5$
2	$\{x-d, x+d\}$	$2d$	
3	$\{x-d-1, x+1, x+d-1\}$		$4, 5, 5$
4	$\{x\}$		
5	$\{x-1, x+1\}$	2	

$x \in A'_3$	$I(x)$	$d(c_1, c_2)$	$I(x)$ mod 6
$\equiv 0$ mod 6	$\{x-d+1, x-1\}$	$d-2$	$4, 5$
1	$\{x-d, x+d+1\}$	$d-10$	$4, 5$
2	$\{x-d-1, x-d+1, x+d\}$		$0, 4, 5$
3	$\{x-d, x+d-1\}$	$d-8$	$0, 5$
4	$\{x-d-1, x+1\}$	$d+2$	$0, 5$
5	$\{x\}$		

Let us compare these I-sets (that is, when $x \in A'_1 \cup A'_2 \cup A'_3$). Clearly, the I-sets of size one are distinguished. Consider then the I-sets of size two. In the tables above, one can found the distances $c_1 - c_2$ of the codewords in $I(x)$ with $c_1 > c_2$. If the distance is different, the I-sets cannot be the same. For those, which have the same distance, the $c_1 \pmod 6$ and $c_2 \pmod 6$ are different as shown in the table, and the I-sets again cannot be the same. Let us study the I-sets of size three then. According to the tables, the codewords in the I-sets are different modulo 6 unless $x \in A'_1$ where $x \equiv 2 \pmod 6$ and $y \in A'_3$ where $y \equiv 2 \pmod 6$. However, now $I(y)$ has distance 2 between its two largest codewords, but $I(x)$ has corresponding distance $d - 10$. Consequently, $I(x) \neq I(y)$.

For the rest of the vertices (i.e., the borderline vertices $x \notin A'_1 \cup A'_2 \cup A'_3$) we get the following I-sets: $I(0) = \{d+1, 2d-8, 3d-10\}$, $I(1) = \{d+1, 2d-8\}$, $I(2) = \{d+1, d+3, 2d-8, 2d-6\}$, $I(3) = \{d+3, 2d-6\}$, $I(4) = \{5, d+3, 2d-6\}$, $I(5) = \{5\}$, $I(6) = \{5, d+7, 2d-2\}$, $I(7) = \{d+7, 2d-2\}$, $I(8) = \{d+7, d+9, 2d-2\}$, $I(9) = \{d+9\}$, $I(d-1) = \{2d-2, 3d-10\}$, $I(d) = \{d+1, 3d-10\}$, $I(2d-1) = \{2d-2\}$ and $I(2d) = \{d+1\}$. It is straightforward to check (considering sizes of I-sets, codewords modulo 6 in I-sets and their distances) that we have exactly the following non-distinguished I-sets: $I(9) = I(d+9)$, $I(d-1) = I(d-2)$, $I(d+1) = I(2d)$ and $I(2d-2) = I(2d-1)$. We add two more codewords, namely, 0 and $2d$ to the code C_d to avoid these same I-sets. Denote $C'_d = C_d \cup \{0, 2d\}$. We should bear in mind that if $I_{C_d}(x) \neq I_{C_d}(y)$, then also $I_{C'_d}(x) \neq I_{C'_d}(y)$. Now we have (with respect to C'_d) that $2d \in I(9) \setminus I(d+9)$, $0 \in I(d-1) \setminus I(d-2)$, $2d \in I(2d-1) \setminus I(2d-2)$ and $0 \in I(d+1) \setminus I(2d)$. Therefore, C'_d is an identifying code and the proof is completed.

■

It is interesting to see that in all the cases these bounds are optimal, in one case we find infinity many values attaining it but in the other cases this bound is only approached. Recall that from Proposition 2.6, the optimal bound for an ID-code in $C_n(d_1, d_2, \ldots d_r)$ is of cardinal at least $\frac{n}{r+1}$.
The following result considers the situation when this bound can be achieved.

Theorem 2.24 *[52, 54] Let k be an integer such that $k \geq 2$ and $G(V, E)$ be a finite k-regular graph. Then:*

$$\gamma^{ID}(G) \geq \left\lceil \frac{2|V|}{k+2} \right\rceil$$

and an ID-code in G has cardinality $\frac{2|V|}{k+2}$ if and only if there exist exactly $|C|$ vertices $u \in V$ such that $|I_C(u)| = 1$ and for all other vertices $v \in V$ we have $|I_C(v)| = 2$.

Now for the lower bounds in $C_n(1, d, d+1)$:

Theorem 2.25 *Let n, r and $d_2, \ldots, d_r$ be integers such that $r \geq 3$ and $1 < d_2 < \cdots < d_r \leq n/2$. Then there does not exist any ID-code C in $C_n(1, d_2, \ldots, d_r)$ such that $|C| = n/(r+1)$.*

Proof. Let C be an ID-code in $C_n(1, d_2, \ldots, d_r)$ such that $|C| = n/(r+1)$. This is possible if and only if there are exactly $|C|$ vertices $x_1, \ldots, x_{|C|}$ such that $|I(x_i)| = 1$ and the rest of the vertices have identifying sets with exactly two vertices. If there exists a vertex of C, say u, such that $|I_C(u)| = 2$ then we have $I_C(u) = \{u, v\}$ and $I_C(v) = I_C(v)$ as all identifying sets have at most two codewords (a contradiction). Hence, the codewords of C are the vertices $x_1, \ldots, x_{|C|}$. Therefore, we have $I_C(x_1) = \{x_1\}$, implying that $x_1 + 1 \notin C$ and $|I_C(x_1 + 1)| = 2$. If $I_C(x_1 + 1) = \{x_1, x_1 + 1 \pm d_i\}$, for some i, then the vertex $v = x_1 \pm d_i$ contains $\{x_1, x_1 + 1 \pm d_i\}$ in its Iset (a contradiction). Hence, it has to be that $I_C(x_1 + 1) = \{x_1, x_1 + 2\}$. As $x_1 + 2 \in C$, $I_C(x_1 + 2) = \{x_1 + 2\}$, then, using similar arguments as above, we obtain that $x_1 + 3 \notin C$ and $I_C(x_1 + 3) = \{x_1 + 2, x_1 + 4\}$. Thus, by continuing this process, we obtain that every other vertex of $C_n(1, d_2, \ldots, d_r)$ is a codeword. Clearly, this leads to a contradiction with the chosen cardinality of C, thus we conclude that $\gamma^{ID}(C_n(1, d_2, \ldots, d_r)) > n/(r+1)$. ■

In particular the lower bound 1/4 of circulant graphs $C_n(1, d, d+1)$ is not attainable. This theorem does not helps answer the following question:

Open question 2.26 *Is there a triplet (n, d, C), $n \geq 2d$, $d \geq 3$, such that C is an ID-code of $C_n(1, d-1, d, d+1)$ of cardinality 2/9 ?*

In the following section we see similar results with LD- and SID-codes on the circulant graphs.

2.3 Differences with LD and SID

The same kind of results have been found for LD- and SID-codes. We start by showing the $C_n(1, 3)$ case and then focus on the graphs $C_n(1, d)$, $C_n(1, d, d+1)$ and $C_n(1, d-1, d, d+1)$.

In [37], Ghebleh and Niepel gave good bounds for optimal cardinalities of LD-codes on $C_n(1, 3)$ for most values of n:

Theorem 2.27 *[37] Let $n \geq 9$. Then*

$$\left\lceil \frac{n}{3} \right\rceil \leq \gamma^{LD}(C_n(1, 3)) \leq \left\lceil \frac{n}{3} \right\rceil + 1.$$

Moreover, if $n \equiv 0, 1, 4 \bmod 6$, $\gamma^{LD}(C_n(1, 3)) = \lceil n/3 \rceil$.

As well they conjectured:

Conjecture 2.28 *[37] Let $n \geq 13$ and $n \bmod 6 \in \{2, 3, 5\}$. The circulant graph $C_n(1, 3)$ does not admit a LD-code of size $\lceil n/3 \rceil$.*

In [53] we prove this conjecture for $n \geq 17$ and give the optimal codes.

Theorem 2.29 *[53] Let $n > 17$. Then*

$$\gamma^{LD}(C_n(1, 3)) = \begin{cases} \left\lceil \dfrac{n}{3} \right\rceil & \text{if } n \equiv 0, 1, 4 \bmod 6 \\ \\ \left\lceil \dfrac{n}{3} \right\rceil + 1 & \text{otherwise} \end{cases}$$

The proof of this last theorem is similar to Theorem 2.18 where we redistribute shares and remark some usual patterns that improve the lower bound in the cases $n \equiv 2, 3, 5 \bmod 6$. The proof is found in Appendix B.
As well, in [53] we studied the optimal SID-codes on the graphs $C_n(1, 3)$:

Theorem 2.30 *[52] The optimal cardinalities of SID-codes in $C_n(1,3)$ for $n > 11$ are:*

$$\gamma^{SID}((C_n(1,3)) = \begin{cases} 4k & \text{if } n = 7k \\ 4k+1 & \text{if } n = 7k+1 \\ 4k+2 & \text{if } n = 7k+2 \\ 4k+3 & \text{if } n \in \{7k+3, 7k+4\} \\ 4k+4 & \text{if } n \in \{7k+5, 7k+6\} \end{cases}$$

Proof. Let n be an integer such that $n > 11$. Observe first that we have the following characterization for self-identifying codes in $C_n(1,3)$:

- A code K in $C_n(1,3)$ is self-identifying if and only if $|I_K(c)| \geq 3$ for all $c \in K$ and $\{u-3, u+3\} \subseteq I_K(u)$ for all $u \in \mathbb{Z}_n \setminus K$.

Indeed, if K is a self-identifying code in $C_n(1,3)$, then the given conditions are met by the previous proposition. On the other hand, if K satisfies the conditions, then it is straightforward to verify that K is a self-identifying code by Theorem 2.4.
Let K be a self-identifying code in $C_n(1,3)$. In what follows, we study more closely what happens if there exists consecutive non-codewords in K:

- If there are four or more non-consecutive non-codewords, then the first one, say u, contradicts with the previous characterization as $u+3$ does not belong to K.

- If there are exactly three consecutive non-codewords, say $\{0,1,2\}$ (and thus $n-1$ and 3 are in the code), then $\{n-4, n-3, n-2, 4, 5, 6\}$ are all codewords (by the characterization). Let $P3$ be the pattern with 3 consecutive non-codewords followed by four consecutive codewords (see Figure 2.3.1).

- If there are exactly two consecutive non-codewords, say $\{0,1\}$, then $\{n-3, n-2, n-1, 2, 3, 4\}$ are in the code. Let $P2$ be the pattern with two consecutive non-codewords followed by three consecutive codewords as in Figure 2.3.1.

- Suppose then that there is only one consecutive codeword, say non-codeword 0 (and $n-1$ and 1 are in the code). If $2 \in K$, then we get the pattern $P1a$ with one non-codeword followed by two codewords. On the other hand, if $2 \notin K$, then we obtain (by the characterization) the pattern $P1b$ with five consecutive vertices with only the first and the third one being non-codewords.

Notice that the smallest density among the patterns is the one with three consecutive non-codewords followed by four codewords, i.e., the density of the codewords in the patter is $4/7$.
Due to the obtained patterns, we may conclude that there exists in the graph two consecutive codewords followed by a non-codeword. Without loss of generality, it can be assumed that $n-2, n-1 \in K$ and $0 \notin K$.
Furthermore, there exists a vertex x_1 such that the set $s_1 = \{0, 1, \ldots, x_1\}$ is one of the patterns $P3$, $P2$, $P1a$ or $P1b$. Hence $x_1 - 1$ and x_1 are codewords and we can do the same thing with the next non-codeword vertex x_2 (notice that x_2 may be different from $x_1 + 1$). Let x_3 be such that $s_2 = \{x_2, x_2+1, \ldots, x_3\}$ is one of the patterns. We can go on to the right and define all the sets $s_1, \ldots, s_r$ that correspond to the patterns. Note that the vertices that are not in these sets are all codewords. This partition the graph in patterns with maybe some codewords separating them. Notice also that the last pattern s_r do not intersect the first one s_1. For each of these sets s_i let d_i be its density and n_i the number of vertices. The density of K can then be estimated

$$d \geq \frac{1}{n}(\sum_{1 \leq i \leq r} d_i n_i + n - \sum_{1 \leq i \leq r} n_i) \geq \frac{1}{n}(\sum_{1 \leq i \leq r} \frac{4}{7} n_i + n - \sum_{1 \leq i \leq r} n_i) = \frac{4}{7}$$

This implies that the self-identifying code K has at least $\lceil 4n/7 \rceil$ codewords. The proof now divides into the following cases depending on the remainder of n when divided by 7:

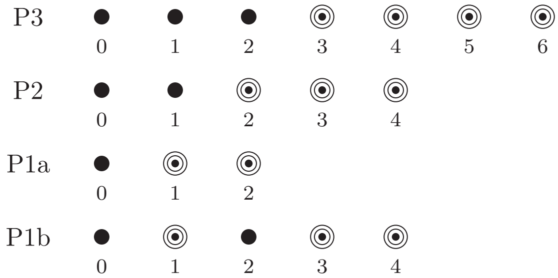


Figure 2.3.1: The patterns for $C_n(1,3)$. Codewords are haloed.

- If $n = 7k$, then the code has at least $\lceil \frac{4}{7}n \rceil$ codewords, that is, $4k$. The code $K_1 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k-1\}$ is self-identifying. Indeed, for every vertex $v \notin K$, we have $\{v-3, v+3\} \subseteq I(v)$. Furthermore, for every vertex $v \in K$, we have $|I_{K_1}(v)| \geq 3$. Thus, according to the characterization, the code K_1 is self-identifying in $C_n(1,3)$.
- If $n = 7k+1$, then the code has at least $\lceil \frac{4}{7}n \rceil$ codewords, that is, $4k+1$. By the same argument as for the case $n = 7k$, the code $K_2 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k-1\} \cup \{7k\}$ can be shown to be self-identifying.
- If $n = 7k+2$, then the code has at least $\lceil \frac{4}{7}n \rceil = 4k+2$ codewords. By the same argument as for the case $n = 7k$ the code $K_3 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k-1\} \cup \{7k, 7k+1\}$ works.
- If $n = 7k+4$ (notice that the more difficult case of $n = 7k+3$ will be dealt later), then the code has at least $4k+3$ codewords, the code $K_5 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k-1\} \cup \{7k-1, 7k, 7k+1\}$ works. Indeed, as above, it is straightforward to verify that $\{v-3, v+3\} \subseteq I_{K_5}(v)$ for all $v \notin K$ and $|I_{K_5}(c)| \geq 3$ for all $c \in K_5$. Thus, K_5 is self-identifying by the characterization.
- If $n = 7k+6$ (notice that the case $n = 7k+5$ is postponed), then the code has at least $4k+4$ codewords. As above, we can show that the code $K_7 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k\}$ is self-identifying in $C_n(1,3)$.
- Suppose $n = 7k+3$. We will first show that now a self-identifying code has at least $4k+3$ codewords. Every self-identifying code on $C_{7k+3}(1,3)$ needs at least $\lceil \frac{4}{7}n \rceil = 4k+2$ codewords. Assume that there is a self-identifying code K on $C_n(1,3)$ with $4k+2$ codewords. Recall that the density of codewords in the patterns is at least $3/5$ unless the pattern is $P3$. If there are at most $k-2$ patterns of $P3$, then $|K| \geq \frac{4}{7}(7(k-2)) + \frac{3}{5}(n - 7(k-2)) = 4k + \frac{11}{5} > 4k+2$. Consequently, there must be either k or $k-1$ patterns of $P3$. Suppose first that there are k of them. This implies that there are three vertices outside of them (not necessarily consecutive). Recall that if we have a pattern $P3$ starting from a vertex u, then the vertices $u-1$, $u-2$, $u-3$ and $u-4$ are all codewords. Therefore, as we have only three vertices outside of patterns $P3$, they all have to be codewords. Suppose then that there are $k-1$ patterns $P3$. Now there are 10 vertices not in these patterns. If a vertex u starts a pattern $P3$ such that $u-1$ is not part of a pattern $P3$ (indeed, such pattern has to exist), then $u-1$ is a codeword (as above) and does not belong to any pattern since none of the patterns other than $P3$ ends with four consecutive codewords. Therefore, we obtain that $7(k-1)$ vertices belongs to some pattern $P3$, one codeword does not belong to any pattern and the rest 9 of the vertices belong to patterns other than $P3$ (or not to any pattern). Thus, we obtain that $|K| \geq \frac{4}{7}(7(k-1)) + 1 + \frac{3}{5}(n - 7(k-1) - 1) = 4k + \frac{12}{5} > 4k+2$.

  Hence, there is no self-identifying code with $4k+2$ codewords and the size of the code is at least $4k+3$. By the same argument as above, we can show that the code $K_4 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k-1\} \cup \{7k, 7k+1, 7k+2\}$ works.
- If $n = 7k+5$, then we show next that the code has at least $4k+4$ codewords. It needs at least $4k+3$ codewords. Let us use the sets s_i of the patterns again. If there is at most $k-1$ patterns $P3$, then

$|K| \geq \frac{4}{7}(7(k-1)) + \frac{3}{5}(n - 7(k-1)) = 4k + \frac{16}{5} > 4k + 3$. Therefore, there must be k patterns of $P3$ and five vertices outside them (not necessarily consecutive). Suppose first that these five vertices are not consecutive. Then they all must be codewords since four consecutive vertices left to any pattern $P3$ are codewords. Suppose then that the five vertices are consecutive. This implies (with the same argument) that four of them has to be codewords. Thus, in both cases, at least four of the five vertices are codewords. Hence, we have $|K| \geq 4k + 4$. As above, it is straightforward to verify that $K_6 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k\}$ is an optimal self-identifying code with $4k + 4$ vertices.

■

Now, if we see the more general cases of $C_n(1,d)$, $C_n(1,d,d+1)$ and $C_n(1,d-1,d,d+1)$, we can do the same as for the ID-case: get lower bounds from the optimal LD- and SID-codes in the infinite grids.

These optimal densities give lower bounds for LD- and SID-codes on the circulant graphs $C_n(1,d)$, $C_n(1,d,d+1)$ and $C_n(1,d-1,d,d+1)$. Moreover, these are optimal lower bounds:

Theorem 2.31 *[52] Let n and d be positive integers such that $n \geq 2d$, $d \geq 4$.*

- ***For the LD-codes:***
  1. *If $n \equiv 0 \bmod 20$ and $d \equiv 5 \bmod 20$, then $\gamma^{LD}(C_n(1,d)) = 3n/10$.*
  2. *If $n \equiv 0 \bmod 57$ and $d \equiv 8 \bmod 57$, then $\gamma^{LD}(C_n(1,d-1,d)) = 13n/57$.*
  3. *If $n \equiv 0 \bmod 10$, $d \equiv 8 \bmod 10$ and $n \geq 4d+6$, then $\gamma^{LD}(C_n(1,d-1,d,d+1)) = n/5$.*
- ***For the SID-codes:***
  1. *If d and n are even and $n \geq 4d+1$, then $SID(C_n(1,d)) = n/2$.*
  2. *If $n \geq 4d+1$ and n is even, then $\gamma^{SID}(C_n(1,d-1,d)) = n/2$.*
  3. *If $d \equiv 1 \bmod 3$, $n \geq 4d+5$, $n \equiv 0 \bmod 3$, then $SID(C_n(1,d-1,d,d+1)) = n/3$.*

Proof.

- **For the LD-codes**:
  1. Let $n \equiv 0 \pmod{20}$ and $d \equiv 5 \pmod{20}$. Define $B_3 = \{0,4,7,11,14,17\}$ and
  $$D_3 = \{u \in \mathbb{Z}_n \mid u \equiv b \pmod{20} \text{ for some } b \in B_3\}.$$
  It is straightforward to verify that B_3 is a locating-dominating code in $C_{20}(1,5)$. Then, using similar arguments as in the case for ID-codes we have that D_3 is a locating-dominating code in $C_n(1,d)$.
  2. Let $d \equiv 8 \pmod{57}$, $d \geq 8$, $n \geq 2d$ and $n \equiv 0 \pmod{57}$. We denote
  $$B = \{0,2,4,6,15,18,27,29,31,33,43,45,47\}.$$
  Let further
  $$C = \{v \in \mathbb{Z}_n \mid v \equiv b \pmod{57} \text{ for some } b \in B\}.$$
  It is straightforward to check that B is a locating-dominating code in $C_{57}(1,d-1,d)$ for $d=8$. Next we will show that C is locating-dominating in $C_n(1,d-1,d)$. Let us first show that $I(x) = I(y)$ for $x \not\equiv y \pmod{57}$ and $x,y \notin C$. Denote $x' = x \pmod{57}$ and $y' = y \pmod{57}$ where $0 \leq x' \leq 56$ and $0 \leq y' \leq 56$. If $I(x) = I(y)$, then it follows that the codewords in $I(x)$ and in $I(y)$ would be equal modulo 57. However, that is not possible, since $I_B(x') \neq I_B(y')$ for distinct $x', y' \notin B$. Therefore, it suffices to consider $I(x) = I(y)$ for $x \equiv y \pmod{57}$, $x \neq y$ and $x, y \notin C$. Let $j \in \{-d, d+1, -1, 0, 1, d-1, d\}$ and $x + j \in I(x)$. Consequently, $x + j = y + j'$ for some $j' \in \{-d, d+1, -1, 0, 1, d-1, d\}$. Since $x \equiv y \pmod{57}$, we get $j = j'$ giving $x = y$. Hence C is locating-dominating.

3. Let $d \equiv 8 \pmod{10}$, $n \geq 4d+6$ and $n \equiv 0 \pmod{10}$. Next we will verify that the code

$$C' = \{v \in \mathbb{Z}_n \mid v \equiv 0, 4 \pmod{10}\}$$

is locating-dominating in $C_n(1, d-1, d, d+1)$. Since $d \equiv 8 \pmod{10}$, then we get the following I-sets depending on the value of non-codewords x modulo 10

$x \pmod{10}$	$I(x)$	$I(x) \pmod{10}$
1	$\{x-1, x-d+1, x+d+1\}$	$0, 4, 0$
2	$\{x-d, x+d\}$	$4, 0$
3	$\{x+1, x-d-1, x+d-1\}$	$4, 4, 0$
5	$\{x-1, x+d+1\}$	$4, 4$
6	$\{x+d\}$	4
7	$\{x-d+1, x+d-1\}$	$0, 4$
8	$\{x-d\}$	0
9	$\{x+1, x-d-1\}$	$0, 0.$

Let $x \neq y$. Clearly, $I(x) \neq I(y)$ for those x and y which have different sizes of the I-sets. Let us first consider the cases where the size of the I-sets equal one. If $x \equiv 6 \pmod{10}$ and $y \equiv 8 \pmod{10}$, then (see the table above) $c \in I(x)$ has $c \equiv 4 \pmod{10}$ and $c' \in I(y)$ has $c' \equiv 0 \pmod{10}$. Therefore, $I(x) \neq I(y)$. Obviously, the sets $I(x) \neq I(y)$ if $x \equiv y \equiv 6 \pmod{10}$ or if $x \equiv y \equiv 8 \pmod{10}$. Consider then the case of I-sets of size three. Let first $x \equiv 1 \pmod{10}$ and $y \equiv 3 \pmod{10}$. Now the set $I(x)$ has exactly one codeword c such that $c \equiv 4 \pmod{10}$ and the set $I(y)$ has exactly two such codewords. Therefore, $I(x) \neq I(y)$. Consider then the case $x \equiv y \equiv 1 \pmod{10}$. Now the only codeword which is 4 modulo 10 is $x-d+1$ in $I(x)$ and $y-d+1$ in $I(y)$. Consequently, if $I(x) = I(y)$, then $x-d+1 \equiv y-d+1 \pmod{n}$ giving $x = y$ (in $\mathbb{Z}_n$). The case if $x \equiv y \equiv 3 \pmod{10}$ goes similarly. Consider then the I-sets of size two. We start with the situation $I(x) = I(y)$ where $x \not\equiv y \pmod{10}$. If $x \equiv 5 \pmod{10}$ (resp. $x \equiv 9 \pmod{10}$), then in $I(x)$ both of the codewords are equal to 4 (resp. 0) modulo 10. If $x \equiv 2 \pmod{10}$ or $x \equiv 7 \pmod{10}$, then the $I(x)$ has exactly one codeword 0 modulo 10 and one 4 modulo 10. Therefore, it suffices to consider the case $x \equiv 2 \pmod{10}$ or $y \equiv 7 \pmod{10}$. Now $I(x) = \{x-d, x+d\}$ and $I(y) = \{y-d+1, y+d-1\}$. Due to the residue classes modulo 10, we must have $x-d \equiv y+d-1 \pmod{n}$ and $x+d \equiv y-d+1 \pmod{n}$. This implies that $2x \equiv 2y \pmod{n}$. If n is odd, we immediately have $x = y$ (in $\mathbb{Z}_n$). If n is even, we still have $x = y$ due to the fact that $n \geq 4d+6$.
The cases $x \equiv y \equiv 2 \pmod{10}$ and $x \equiv y \equiv 7 \pmod{10}$ go as above based on the residue classes modulo 10 of the codewords in $I(x)$ and $I(y)$. In the cases $x \equiv y \equiv 5, 9 \pmod{10}$ we use the fact that $n \geq 4d+6$. In summary $I(x) \neq I(y)$ for $x \neq y$ and we obtain the assertion.

- **For the SID-codes**:

1. We show that the code

$$C = \{v \in \mathbb{Z}_n \mid v \equiv 0 \pmod{2}\}$$

is self-identifying in the circulant graph $C_n(1, d)$. If $x \equiv 0 \pmod{2}$, then $I(x) = \{x-d, x, x+d\}$ and otherwise $I(x) = \{x-1, x+1\}$. Since $n \geq 4d+1$, we get that $N[x-d] \cap N[x-d] = \{x\}$ and $N[x-1] \cap N[x+1] = \{x\}$. Consequently, the condition for self-identification, namely, $\cap_{c \in I(x)} N[c] = \{x\}$, is satisfied. As $\frac{n}{2}$ is the lower bound, we showed that $\gamma^{SID}(C_n(1, d)) = \frac{n}{2}$.

2. Let $d \geq 4$, $n \geq 4d+1$ and n be even. The code

$$C = \{v \in \mathbb{Z}_n \mid v \equiv 0 \pmod{2}\}$$

is self-identifying in $C_n(1, d-1, d)$ as will be seen next. If d is even (resp. odd) and $x \equiv 0 \pmod{2}$, then $\{x-d, x+d\} \subseteq I(x)$ (resp. $\{x-d+1, x+d-1\} \subseteq I(x)$). Hence in both cases $\cap_{c \in I(x)} N[c] = \{x\}$. If d is even (resp. odd) and $x \equiv 1 \pmod{2}$, then $\{x-d+1, x+d-1\} \subseteq I(x)$ (resp. $\{x-d, x+d\} \subseteq I(x)$). Consequently, again $\cap_{c \in I(x)} N[c] = \{x\}$. Therefore, C is self-identifying. As $\frac{n}{2}$ is the lower bound, we showed that $\gamma^{SID}(C_n(1, d-1, d)) = \frac{n}{2}$.

3. Let
$$C = \{v \in \mathbb{Z}_n \mid v \equiv 0 \pmod 3\}.$$
We verify next that C is self-identifying in $C_n(1, d-1, d, d+1)$. If $x \equiv 0 \pmod 3$, we have $I(x) = \{x, x-d+1, x+d-1\}$ since $d \equiv 1 \pmod 3$. If $x \equiv 1 \pmod 3$ (resp. $x \equiv 2 \pmod 3$), then $I(x) = \{x-1, x-d, x+d+1\}$ (resp. $I(x) = \{x+1, x-d-1, x+d\}$). Now in each case, the intersection $\cap_{c \in I(x)} N[x] = \{x\}$ due to the fact that $n \geq 4d+5$. Hence C is self-identifying.

■

In Table 2.3.1 there is a summary of all the results obtained in this section.

	square grid $\mathcal{S}$	triangular grid $\mathcal{T}$	king grid $\mathcal{K}$
LD	3/10 [68]	13/57 [46]	1/5 [47]
ID	7/20 [8, 20]	1/4 [54]	2/9 [18, 22]
SID	1/2 [48]	1/2 [48]	1/3 [48]

Table 2.3.1: Optimal densities of ID, LD and SID-codes on the infinite square, triangular and king grids along with the corresponding references.

Here all the lower bounds are attained but it could be interesting to know if the lower bound 2/9 of ID-codes in the infinite king grid is a minimum, and of course, understand why the ID-codes behave differently in the triangular and the king grids.

Chapter 3

Graph marking and graph coloring games

> "S'il n'y a pas de solution c'est qu'il n'y a pas de problème."
>
> Jacques Rouxel - Les Shadoks

In this chapter two games are studied. First some definitions are given with the general context of both games, then the graph marking game is studied for graph operators and at the last section the edge coloring game is studied on graph decompositions.

3.1 Definitions and general context

In this chapter two two-players games on graphs are presented along with some of their variants. Historically, the first one to be studied was the vertex coloring game, then to simplify its study the vertex marking game was introduced. As their names suggest, the edge coloring and marking games have also been studied.

It all started in 1981 when Brams introduced a game in an attempt to find a non-computational proof for the 4-color theorem, [36]. Later on Bodlaender rediscovered that game and established some results for the tree class of graphs.

> COLORING GAME: The *coloring game* is played by two players, Alice and Bob on a graph G with a set C of colors. Players take turns to properly color an uncolored vertex v with a color of C. If, at the end, the graph is properly colored, then the first player (by convention is Alice) wins, otherwise, Bob wins.

For this game the parameter to consider is the number of colors. Indeed, if there are not enough colors, Alice can never win. Hence, for a given number of colors, the main question is: is there a strategy for Alice with k colors such that, no matter how Bob plays, at the end the graph is properly colored?

Definition 3.1 (game chromatic number) *The* game chromatic number $\chi_g(G)$ *is the minimum number of colors such that Alice has a winning strategy for the* COLORING GAME *on* G*, meaning that no matters how Bob plays, Alice can always win.*

Consider the first graph of Figure 3.1.1. Assume the players play with 4 colors: when Alice colors a vertex i with the color c, Bob colors the vertex $(i+3) \bmod 6$ with a new color. Each time Alice plays, she needs a new color, hence, after four turns there are no colors left and Bob wins. If they play with 5 colors, the last vertex to be played, say i, can always take the color of the vertex $(i+3) \bmod 6$. Hence $\chi_g(G) = 5$. Now consider the second graph, the vertices $1, 3, 4$ and 5 form a complete graph, hence they need at least 4 colors. With 4 colors, Alice's strategy is to play first vertices of degree 4 unless Bob colors the vertices 0 or 4, in which case she colors the other one (of $0, 4$) with the same color.

Figure 3.1.1: Example of graphs where Alice needs 5 (left) and 4 (right) colors to win the coloring game.

We have some trivial bounds, for the game chromatic number, depending on the maximum degree and the chromatic number.

Proposition 3.2 ([13]) *Given a graph G of maximum degree Δ and chromatic number $\chi(G)$:*

$$\chi(G) \leq \chi_g(G) \leq \Delta + 1$$

Proof. For the lower bound we have only to note that if Alice wins, the graph is properly colored, meaning that there are at least $\chi(G)$ colors.
If they play with $\Delta + 1$ colors, when a vertex is colored it has at most Δ colors in its neighborhood, hence there is always a color available. ∎

One way to simplify the study of the coloring game is by studying the *marking game.* It was first mentioned on a paper of Faigle *et al.* in 1993 [32], then properly introduced in a paper of Zhu in 1998 [75].

> MARKING GAME: The *marking game* is played on a graph G by two players Alice and Bob. They alternately take turns to mark unmarked vertices. At the beginning (no vertex is marked) the score of each vertex is 0. Each time a vertex is marked, its score changes to one plus the number of marked neighbors it has. At the end, the score of the graph is the maximum score obtained along the game.

In this case, Alice wants to minimize the score and Bob wants to maximize it.

Definition 3.3 (game coloring number) *The* game coloring number *of a graph G, noted $col_g(G)$ is the minimum score ensured by Alice, meaning that no matter how Bob plays, Alice has a strategy ensuring score at most $col_g(G)$.*

Please note that the *game chromatic number* is the parameter of the *coloring game* and the *game coloring number* is the one of the *marking game.*

Figure 3.1.2: Example of graphs where $col_g(G) = 4$.

Consider the first graph of Figure 3.1.2, Bob should play in his first two turns the vertices 1 and 7. This way the last vertex to be marked is of degree 3 and has all of its neighbors marked, hence the maximum score is at least 4. As the maximum degree is 3, the maximum score possible is 4, hence $col_g(G) = 4$.
Now consider the second graph of Figure 3.1.2. Alice starts by marking the vertices 1, 7 and 4 in that order. In the worst case Bob marks first two neighbors of 4 (different from 7) and the final score is 4 for the vertex 4, hence $col_g(G) \leq 4$. If Alice starts by marking 4, Bob marks 0 and Bob has the time to mark at least three neighbors of 1 or 7 before Alice marks both of them, ensuring a score of 4. It is the same if she starts by marking the vertex 7 but with the vertices 1 and 4. If she starts by marking another vertex one of the vertices 1, 4 or 7 has score at least 4 as Bob starts by marking 0 and 2. Hence $col_g(G) = 4$.

This game is useful for the study of the coloring game since the game coloring number is an upper bound for the game chromatic number, as Figure 3.1.3 suggests.

Proposition 3.4 ([75]) *Let G be a graph, then $\chi_g(G) \leq col_g(G)$.*

Proof. If Alice has a strategy with score k on the graph G for the vertex marking game, she can use this strategy: each time a vertex v is colored it has at most $k-1$ neighbors already colored, as they play with k colors there is always at least one color available to color it. ■

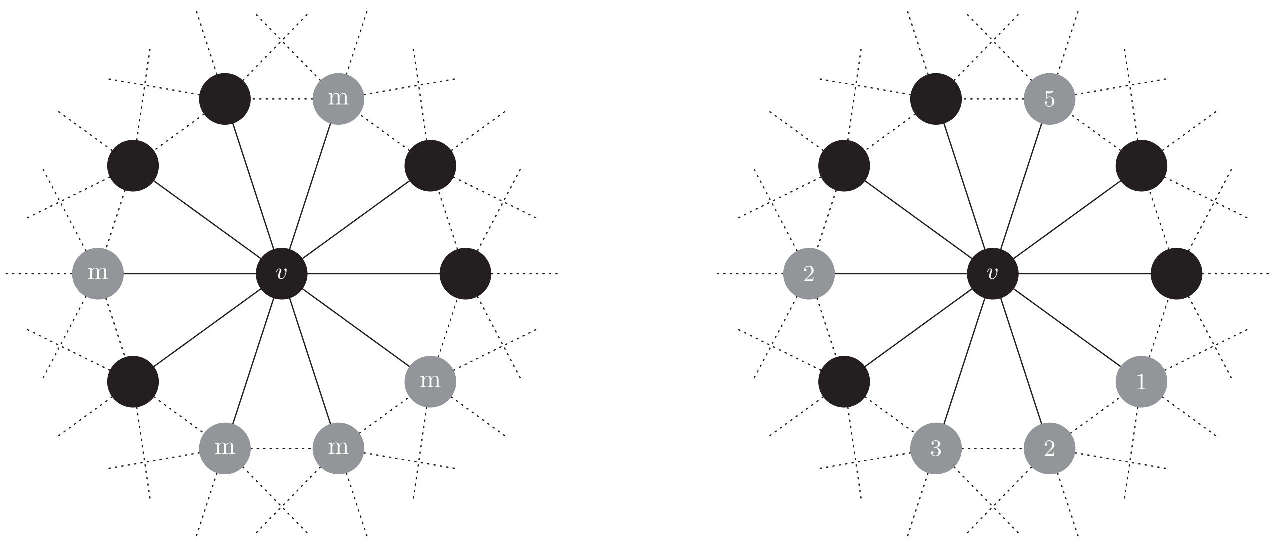


Figure 3.1.3: In one game we look at the number of neighbors, in the other, at the number of colors.

Another interesting remark about this new upper bound is that the gap between the two values can be as large as wanted. Indeed, for the bipartite clique $K_{n,n}$ we have: $\chi_g(K_{n,n}) = 3$ and $col_g(K_{n,n}) = n + 1$ as suggested in Figure 3.1.4.

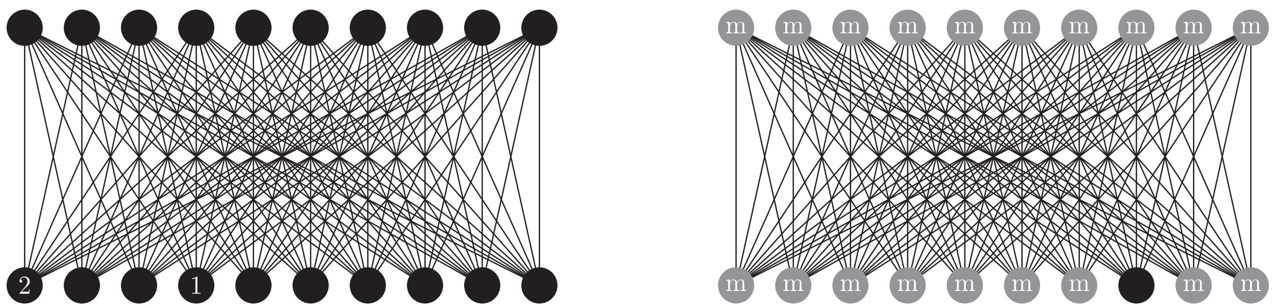


Figure 3.1.4: Gap as large as we want between col_g and χ_g

Indeed, for the coloring game, Bob uses his first turn to color with a second color a vertex on the same side as Alice (if Alice played the color 1, he plays the color 2 for instance): the vertices on the other side need a third color, hence $\chi_g(K_{n,n}) \geq 3$. In fact if they continue to play (with only 3 colors), on one side all the vertices will have the third color, and on the other side the players can always color with 1 or 2. Hence $\chi_g(K_{n,n}) = 3$. For the marking game, the last vertex to be marked has all of its neighbors marked, hence the final score is $n+1$.

Since 1983, the vertex marking game has been largely studied to give upper bounds for the vertex coloring game. Some of the most interesting results about the marking game where shown in [75]. Zhu showed that for spanning subgraphs of a graph G and for edge-partitions the game coloring number can be easily upper bounded:

Theorem 3.5 *[75] Let $G(V,E)$ be a graph.*

- *Let $G_1(V,E_1)$ and $G_2(V,E_2)$ such that E_1, E_2 is a partition of E, then:*

$$col_g(G) \leq col_g(G_1) + \Delta(G_2).$$

- *Let H be a spanning subgraph of G. Then*

$$col_g(H) \leq col_g(G).$$

These last results are helpful to bound the game coloring and game chromatic numbers by constructing the graphs step by step. Moreover, in this paper Zhu raises a very important, and still open, question:

Open question 3.6 *Assume $\chi_g(G) = k$. For $k' > k$, can Alice win with k' colors?*

Intuitively one would like it to be true, but no formal proof has been found yet. For the marking game, by definition, we obtain this monotonicity: Alice can always ensure a score at most k, $k > col_g(G)$ as she can always ensure a score of at most $col_g(G)$.

In 2003 Wu and Zhu improved the result about subgraphs:

Theorem 3.7 *[71] Let G be a graph and H be a subgraph of G (not necessarely spanning). Then $col_g(H) \leq col_g(G)$.*

And in fact, they also gave unfortunate counterexamples showing this cannot be the case for the coloring game. We are giving them as a proposition.

Proposition 3.8 *[71] There is a graph $G(V,E)$ and $e \in E$ such that removing the edge e gives: $\chi_g(G \setminus \{e\}) > \chi_g(G)$.*
As well, there is a graph $G(V,E)$ and $v \in V$ such that removing the vertex v gives: $\chi_g(G - \{v\}) > \chi_g(G)$.

Proof. Let $K_{n,n}$ be the complete bipartite graph on $2n$ vertices denoted $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ such that the vertices a_i form a stable set, as well as the b_i-ones. Let $M = \{a_0b_0, \ldots, a_{n-1}b_{n-1}\}$ be a perfect matching of $K_{n,n}$. Take $G = K_{n,n} \setminus (M \setminus \{a_0b_0\})$. We have then $\chi_g(G) = 3$ and $\chi_g(G \setminus \{a_0b_0\}) = n$.
For the first equality, assume they play with 3 colors. Alice starts by coloring a_0 with 1 and then Bob colors a_1 with 2: all the vertices $b_2, \ldots, b_{n-1}$ have to be colored with a third color (in particular Bob wins if they play with 2 colors). Alice then answers by coloring with 3 the vertex b_1: all the vertices a_i can be colored with 1 and 2 and all vertices b_i can be colored with 3.
Now, let us show that $\chi_g(G \setminus \{a_0b_0\}) = n$.
Bob's strategy is to always play the *unmatched* vertex of Alice's: when Alice plays a_i with color j, Bob colors b_i with color j. By doing this, after the i-th turn of Bob, each uncolored vertex is neighbor to i colors, hence, by the $n-1$-th turn of Bob, the remaining two vertices need a n-th color, as shown in Figure 3.1.5(a).

(a) $\chi_g(K_{n,n} \setminus M) = n$ (b) $\chi_g(G) = 3$ (c) $\chi_g(G) = 3$

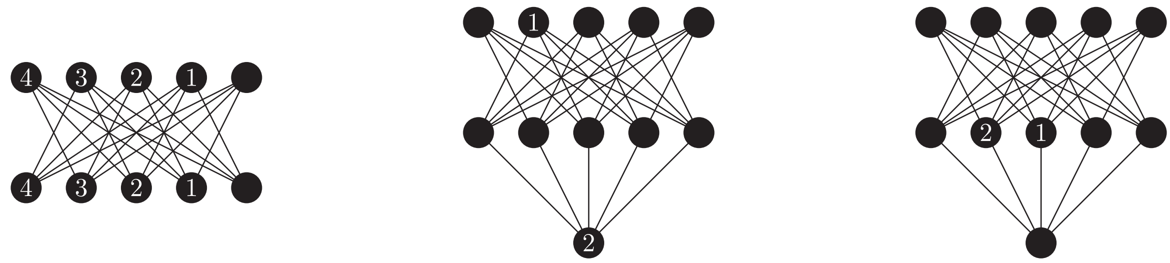


Figure 3.1.5: Examples of graphs for Proposition 3.8.

Now, let us prove the vertex case. Take the same graph $K_{n,n} \setminus M$ and this time add a vertex v neighbor of all the vertices $a_0, \dots, a_{n-1}$.
Bob's strategy is the following: if Alice starts by coloring v (or a vertex b_i respectively) with color j, then he colors a vertex b_i (v resp.) with another color. Then the vertices $a_0, \dots, a_{i-1}, a_{i+1}, \dots a_{n-1}$ are all neighbor with vertices colored with two colors, hence need a third one (see Figure 3.1.5(b)).
If Alice starts by coloring a vertex a_i, then Bob colors with a different color a_j, $j \neq i$, hence v needs a third color (see Figure 3.1.5(c)).
Playing this way, we obtain $\chi_g(G) \geq 3$. It is clear that Alice can win with three colors.
If we remove the vertex v, then $\chi_g(G) = n$ as shown before. ■

The marking game behaves much more nicely than the coloring game, while giving a good upper bound. Thus the interest in this new game that seems easier to get in charge of.

Instead of studying graph by graph, it is more practical to have results about classes of graphs. To do that, the following generalization is needed.

Definition 3.9 *Let $\mathcal{C}$ be a class of graphs. Then:*

$$\begin{array}{rcl} \chi_g(\mathcal{C}) & = & \sup\{\chi_g(G) \mid G \in \mathcal{C}\} \\ col_g(\mathcal{C}) & = & \sup\{col_g(G) \mid G \in \mathcal{C}\} \end{array}.$$

One of the classes the most studied is the class of trees. This class is interesting in the particular case of the marking and coloring games, because of the introduction of a powerful tool: *the activation strategy.* The first result about trees was given by Bodlaender in 1981, in his introductory paper.

Theorem 3.10 *[13] Let T be a tree, then $\chi_g(T) \leq 5$.*

Even though he gave examples of trees having game chromatic number 4, he raised the question of the existence of trees having it at 5. Faigle, Kern, Kierstead and Trotter answered this issue in 1993 by introducing the famous activation strategy, and proving a stronger result:

Theorem 3.11 *[32] Let T be a tree. Then $col_g(T) \leq 4$.*

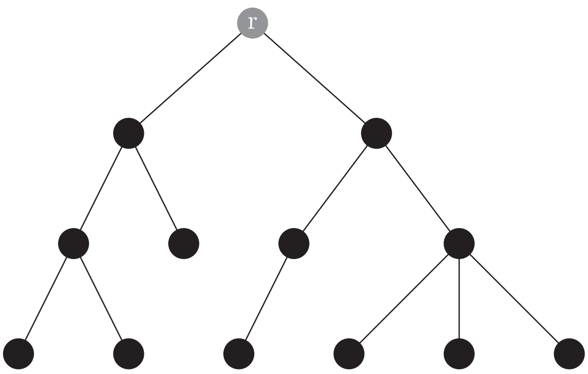
Proof by *the activation strategy.* We are giving a strategy for Alice such that, when playing the marking game on a tree T the final score is at most 4.
While reading this proof we recommend to look at Figure 3.1.6.

She starts by rooting the tree at a vertex r, meaning that she chooses a vertex r that we will call the *root.* She marks it. She will keep track of three sets of vertices: the marked ones M, the activated ones A and the unmarked and unactivated U. In particular, we will have $A \subset U$, and each time a vertex v of A is colored, A becomes $A \setminus \{v\}$ and M receives v, thus $A \cap M = \varnothing$. When deciding which vertex to mark Alice will give priority to the active vertices.

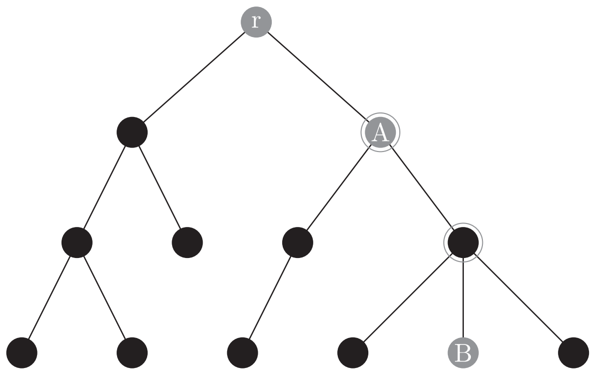
Each time Bob marks a vertex v, Alice looks at the unique path between v and r that we denote u_0, u_1,..., u_l, where $u_0 = v$ and $u_l = r$. Following this path from v to r she activates the vertices and she stops at the first vertex she cannot activate, say u_j, meaning that is was already either activated or marked. Then:

- if u_j was already activated, then she marks it,
- if u_j was already marked, then
  - if $j \neq 1$, she marks u_{j-1},
  - otherwise, she marks any vertex having a marked father.

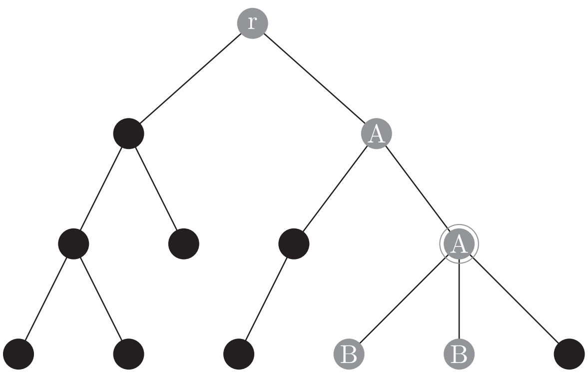
Alice starts by choosing a root r. She marks it.



Alice activates the path and marks the one before last.



When Alices encounters an already activated vertex, she marks it.



Or she marks a vertex having an already marked father.

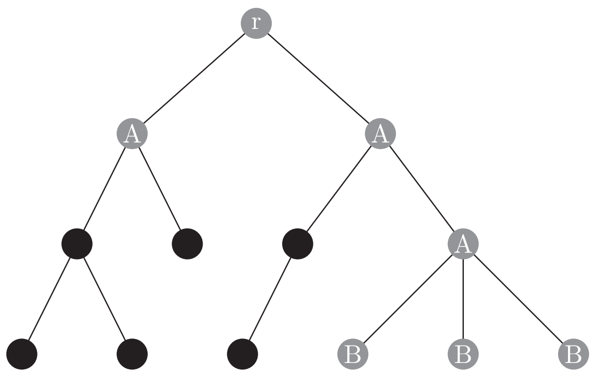


Figure 3.1.6: Steps of the activation stragegy. Marked vertices are drawn in gray and activated ones have a halo. For precision we marked the vertices Alice activates and marks at each step. The letters on each vertex if for the player that marks it.

By this strategy, each time Alice encounters an already activated vertex she marks it, hence when a vertex is marked, it has at most 2 children marked: the first made her activate it and the second mark it. As each vertex has exactly one father, when a vertex is marked it has at most 3 neighbors marked: the final score is at most 4.
In particular and by Proposition 3.4, we obtain that $\chi_g(G) \leq col_g(G) \leq 4$. ■

Remark that the chronology here is not respected: the marking game was not yet introduced when Faigle *et al.* did this the proof. In fact in their paper they talked about a colorblind version of the coloring game without properly introducing it.

In particular, if we mix this result with the fact that Bodlaender gave examples of trees with game chromatic number equal to 4 we obtain:

Theorem 3.12 *[13, 32] Let $\mathcal{T}$ be the class of trees. Then $\chi_g(\mathcal{T}) = 4$.*

The activation strategy has been modified and used to find new bounds in other classes of graphs. The main results are on classes of graphs liken to trees and planar graphs. Later on this chapter we modify it to find upper bounds for the edge coloring game on some particular classes of graphs.

Another class for which we have a tight result is the class of cactuses. A cactus is a graph such that any two cycles of the graph share at most one vertex.

Theorem 3.13 *[66] Let $\mathcal{C}$ be the class of cactuses. Then $\chi_g(\mathcal{C}) = col_g(\mathcal{C}) = 5$.*

But if we look at the wide class of planar graphs we can only give an upper and a lower bound.

Theorem 3.14 *[56, 71, 77] Let $\mathcal{P}$ be the class of planar graphs. Then $11 \leq col_g(\mathcal{P}) \leq 17$.*

In fact, this result comes from different papers: Kierstead *et al.* started by bounding below by 7 and above by 33. Since then, Zhu improved it to 30, then 19 and at the end by 17 (with Dinski in [26];[75];[77] respectively) and in between Kierstead also improved it from 19 to 18 [55]. In [71] Wu and Zhu improved the lower bound to 11. As well, in 2002, He, Hou, Lih, Shao, Wang and Zhu specified lower upper bounds in some particular cases that were later completed by Sekisgushi in 2014.

Theorem 3.15 *Let G be a planar graph of girth g. Then:*

$$\begin{array}{llll} \text{if } g \geq 4 & \text{then } col_g(G) \leq 13 & [64] \\ \text{if } g \geq 5 & \text{then } col_g(G) \leq 8 & [45] \\ \text{if } g \geq 7 & \text{then } col_g(G) \leq 6 & [45] \\ \text{if } g \geq 11 & \text{then } col_g(G) \leq 5 & [45] \end{array}$$

Moreover, the subclass of outerplanar graphs has also been study on its own:

Theorem 3.16 *[41] Let $\mathcal{O}$ be the class of outerplanar graphs. Then*

$$\chi_g(\mathcal{O}) \leq 7.$$

Let us look closer, the results from He *et al.* in [45] are proven using edge-decompositions of planar graphs. More precisely:

Theorem 3.17 *[45] Let G be a planar graph of girth g. Then G has an edge-partition into a forest T and a subgraph H such that:*

$$\begin{array}{ll} \Delta(H) \leq 4 & \text{if } g \geq 5 \\ \Delta(H) \leq 2 & \text{if } g \geq 7 \\ \Delta(H) \leq 1 & \text{if } g \geq 11 \end{array}.$$

Hence, by combining it with Theorems 3.5 and 3.11 we find the result.

Other classes of graphs that have been studied are k-trees, partial k-trees, the interval graphs and some very specific cartesian products (in particular the toroidal grids). Most of these results are shown with a modified activation strategy.

One very strong result using this method is found in [76]. In this paper, Zhu defines a new class of graphs, the *pseudo partial k-trees*, containing different known families of graphs (forests, interval graphs, outerplanar graphs, k-trees, chordal graphs. . .), and gives an upper bound that allows to find the known upper bounds for these known families. More precisely, he points out all the known (up to then) results on these classes were done by using kinds of *tree* structures and following the activation strategy. Hence, Zhu defines a more general and stronger strategy that leads to slighty better results on partial 2-trees.

Definition 3.18 *[76] Let $0 \leq a \leq b$ be two integers and $G(V, E)$ be a graph.*
*The graph G is a (a, b)-*pseudo partial k-tree *if there are two digraphs $\overrightarrow{G_1}(V, \overrightarrow{E_1})$ and $\overrightarrow{G_2}(V, \overrightarrow{E_2})$ such that:*

- *$E_1 \cap E_2 = \varnothing$, $E = E_1 \cup E_2$, where E_i is the set of edges obtained from $\overrightarrow{E_i}$ by omitting their orientation,*
- *$\overrightarrow{G_1}$ is acyclic and has a unique sink s with maximum outdegree k,*
- *$\overrightarrow{E_2}$ has maximum outdegree $\leq a$ and maximum degree $\leq b$,*
- *for all x, the set of outneighbors of x in $\overrightarrow{G_1}$, $N^+_{\overrightarrow{G_1}}(x)$ induces a transitive tournament in $\overrightarrow{G}^*$, the graph defined by $(V, \overrightarrow{E_1} \cup \overrightarrow{E_2})$.*

It is clear by the definition that $(0, 0)$-pseudo partial k-trees are partial k-trees. As well, forests are partial k-trees, interval graphs are chordal graphs and outerplanar are partial 2-trees. Thus all these graphs are particular (a, b)-pseudo partial k-trees.

In his paper, Zhu defines a strategy for Alice in the pseudo partial k-trees in which she keeps track of a set $T_a \subset V$ of activated vertices that always induces a tree in the graph.
To keep track of this set, he also defines two operations over T_a so that each time Bob plays, T_a is updated.

Theorem 3.19 *[76] Let G be a (a, b)-pseudo partial k-tree. Then $col_g(G) \leq 3k + 2a + b + 2$.*

Sketch of the proof by the *generalized activation strategy to pseudo partial k-trees.* Here we are just giving the strategy for Alice, the rest of the proof can be found in the original paper.
The graph G has an edge-decomposition as in Definition 3.18. In particular for $x \in V$, the outneighbors of x in $\overrightarrow{G}^*$ induce a transitive tournament, thus for each $0 \leq j \leq |N_{\overrightarrow{G_1}}(x)| - 1$ there is $v_j \in N_{\overrightarrow{G_1}}(x)$ having outdegree j, for each j we will talk about the j-th outneighbor of x. In particular if $j = 0$ the j-th outneighbor of x is denoted $f(x)$ (first outneighbor) and if $j = |N_{\overrightarrow{G_1}}(x)| - 1$ we will denote it by $l(x)$ (last outneighbor). Note that these two outneighbors exist if $x \neq s$ (where s is the unique sink of $\overrightarrow{G_1}$).

Let T be the spanning directed tree of $\overrightarrow{G_1}$ induced by the edges $xf(x)$ for $x \in V$. The set of activated vertices T_a will always contain s (the root of T which is also the sink of $\overrightarrow{G_1}$) and be an induced subtree of the graph T. Remark that here T_a can also contain marked vertices.

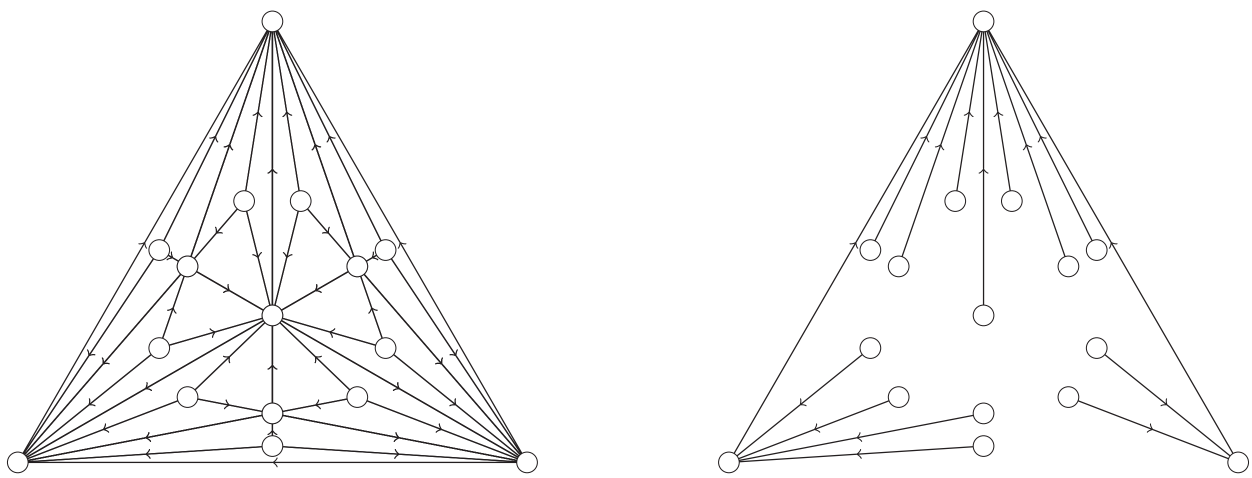


Figure 3.1.7: Example of k-tree with the orientation of $\overrightarrow{G_1}$ and the tree T of edges $xf(x)$.

We define two operations to update T_a. This operations are done over directed paths. Let $P = u_0 \dots u_k$ be a directed path of $\overrightarrow{G_1}$.

- **extension:** Let $P' = u_k \dots u_{k'}$ be the unique directed path in T connecting P to T_a, *i.e.* $u_{k'}$ is in T_a and $u_{k'-1}$ is not. Then the path $u_0 \dots u_{k'}$ is the *extension* of the path P, we denote it PP'.
  If u_k is already in T_a, then P is its own extension.
- **switch** If u_k is the j-th outneighbor of u_{k-1} and $u_k \neq l(u_{k-1})$ then the *switch* of P is the path $P' = u_0 \dots u_{k-1}u_{k'}$ where $u_{k'}$ is the $j+1$-th outneighbor of u_k.

We will say that a vertex v is free if it is unactivated and unmarked.

The strategy of Alice is the following: she starts by marking the root r. The activated set is now $T_a = \{r\}$. When Bob marks a vertex x, Alice chooses the vertex she plays by repeating the following procedure: let $P_1 = xf(x)$ and P_2 be the extension of P_1. Remark that as P_2 is a path from T, P_2 can be written as $xf(x)f^2(x)\dots f^{k-1}(x)$ if P_2 is of length k.

Start: Assume the path to consider is the path $P_{2t} = xf(x)u_2 \dots u_k$ then:

1. if u_k is a free vertex, Alice marks it;
2. if u_k is marked then:
   (a) if $u_k = l(u_{k-1})$:
      i. if u_{k-1} is free, Alice marks it,
      ii. otherwise Alice marks any free vertex having all of its outneighbors marked.
   (b) otherwise, let P_{2t+1} be the switch of P_{2t} and P_{2t+2} the extension of P_{2t+1}, and go back to **Start**.

If P_{2j} is the last path taken into account in this procedure, then Alice adds the vertices of the path and the vertex she marked to T_a. We add only the vertices, the edges of T_a are those induced by T, hence T_a is a subtree of T.
For an example of game using this strategy we refer to Figures 3.1.8 to 3.1.11. ■

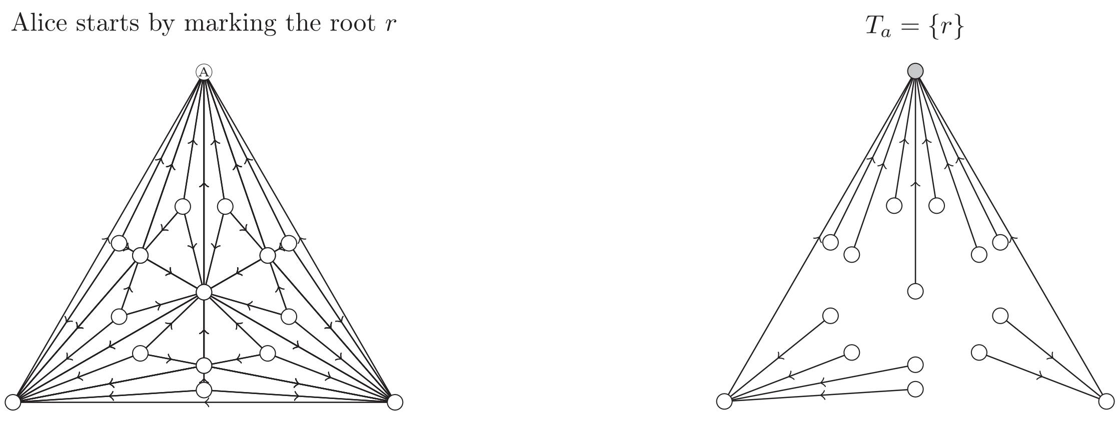


Figure 3.1.8: Example of generalized activation strategy in a 3-tree
Start of the game

In fact, if we look closer, this result does not give the best known bounds, but by doing this strategy in the particular case of forests, outerplanar graphs and interval graphs, Alice acheives the best bounds. This then a generalization of the activation strategies proposed in the corresponding papers.
In particular, this result improves the previous bound on k-trees that was quadratic on k, [26]. Moreover, by modifying a little this strategy for pseudo 2-trees Zhu also gives much better bounds, namely: if

Bob plays a vertex v | By 2.(b) Alice changes the path | By 2.$(a)i$. Alice marks u_{k-1}

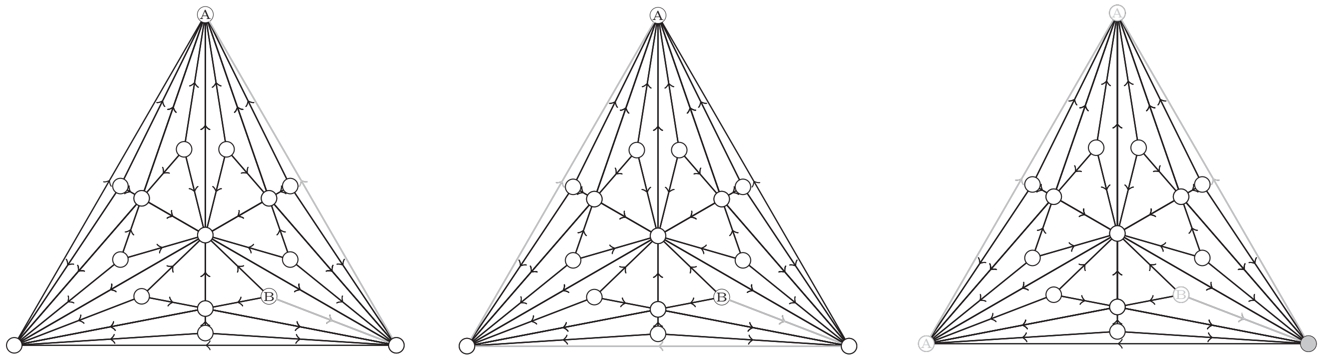


Figure 3.1.9: Example of generalized activation strategy in a 3-tree. Alices keeps the tree updated, after marking her vertex.

Bob marks another vertex, v | By 1. Alices marks u_k

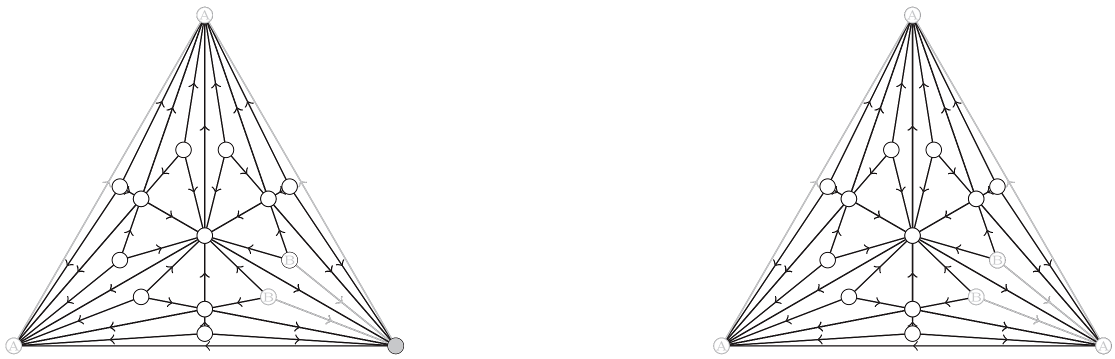


Figure 3.1.10: Example of generalized activation strategy in a 3-tree. Alice lands on an already activated vertex.

Bob marks a new vertex v | By three times $2(b)$ and then $2.(a)ii$. Alices marks a vertex having its outneighbors marked

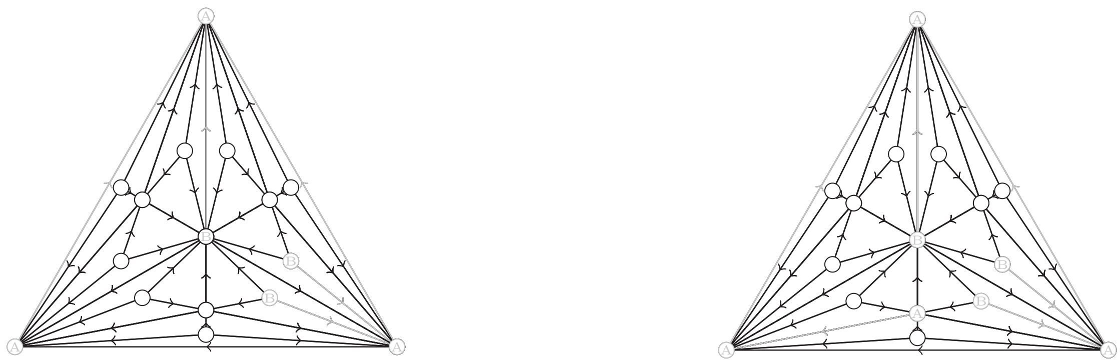


Figure 3.1.11: Example of generalized activation strategy in a 3-tree
the games goes on...

G is a (a, b)-pseudo 2-tree then $col_g(G) \leq a+b+8$ which depends only on the decomposition and not on k.

However, even if this result applies to all graphs, it is NP-complete to determine if a graph G is a k-tree, [4]. The advantage of the pseudo partial k-tree definition is that the graph $\overrightarrow{G_2}$ can be anything, hence it

is sufficient to find a subgraph of G that is a k-tree, for some k, and denote a, b the corresponding degrees of $\overrightarrow{G_2}$.
In [14] Bodlaender gives a linear time algorithm that takes an integer k and a graph G and gives either a tree decomposition of G if it is of treewidth k or returns that G is not of treewidth k. It should be feasible to find a polytime algorithm inspired by Bodlaender's to find a subgraph of treewidth k. The problem is then to find suitable values of a and b. Knowing if a graph G is a (a, b)-pseudo partial k-tree is not a solved problem... Even if this upper bound seems to be a good generalization, is not of much use without a good decomposition.

Remark that many classes of graphs have been studied, even though for now most of these results are only partial. We have seen that edge decompositions are also interesting when studying these problems, since Zhu's result about decompositions. As well, decompositions that lead to tree structures somehow give new strategies of activation. It becomes clear that the study of the possible decompositions could help understand better how the coloring game works, hence opening the door to a wider study.

In this chapter we study two different aspects of these games. In the first part we talk about the evolution of the *vertex*-marking game when modifying the graphs. The modifications we consider are the operators of minor graphs (deletion of vertices and edges and contraction of edges), the union of two graphs and the cartesian product. For each operator f we give upper and lower bounds for the game coloring number of $f(G)$ as functions of the game coloring number of G.
In the second part we focus on a variant of these games, when the players, instead of playing on the vertices, play on the edges: the *edge*-coloring game. For an introduction of this game, we refer to Section 3.3. More precisely, we define a new way of edge-decomposing the graphs that gives new bounds for some families of graphs (by the use of the activation strategy). Among others we improve known bounds on planar graphs.

3.2 Marking game and graph operations

This work was done in collaboration with Paul Dorbec, Éric Sopena and Elżbieta Sidorowicz.

We recall some notations that are widely used in what follows. We denote by $G - \{v\}$ the graph $G(V \setminus \{v\}, E)$, obtained from G by deleting the vertex v and by $G \setminus \{e\}$ the graph $G(V, E \setminus \{e\})$ obtained by deleting the edge e. We also note G/e the graph obtained by contraction of the edge e. In this last case, the two endpoints of e, say u and v are contracted into a unique vertex w.

3.2.1 Definitions and notations

Assume Alice and Bob play the marking game on a graph G. After her first turn, it is Bob's turn on a graph where there is a marked vertex v. We can consider this as a new instance of the game, where: there are already vertices marked (here just v) and Bob starts. In particular, if we study this game on graphs having already marked vertices and where each of the players can start we can see the progress of the whole game. We define then the *A-marking game* as the marking game where Alice starts and the *B-marking game* the one where Bob starts. Moreover, we denote by $G|M$ the graph where the vertices of the set M are already marked. As well, we define the *A-game coloring number* of $G|M$, $col_A(G \mid M)$, as the minimum score ensured by Alice on the A-marking game played on $G|M$ (the vertices of M have no score), and the *B-game coloring number* of $G|M$, $col_B(G \mid M)$, as the minimum score ensured by Alice on the B-marking game on $G|M$.

Assume they play the $X \in \{A, B\}$ marking game on a graph $G|M$. We say Alice has a strategy with score s if she has a strategy ensuring a score of at most s. As well, we say Bob has a strategy with score s if he has a strategy ensuring a score of at least s. Remark that if they both have a strategy with score s then $col_X(G) = s$.

We see first how the A- and the B-game coloring numbers interact and, then, using these results we focus on the operations.

Definition 3.20 *For an integer s and a graph G, we define $A_s(G) = \{v \in V_G \mid d(v) \geq s-1\}$ and $B_s(G) = V_G \setminus A_s(G)$.*

Lemma 3.21 *Let G be a graph, M a set of marked vertices of G and s an integer.*
For the A-marking game: if $|A_s \setminus M| > |B_s \setminus M|$ then Bob has a strategy with score s. For the B-marking game: if $|A_s \setminus M| \geq |B_s \setminus M|$ then Bob has a strategy with score s.

Proof. Let s be a positive integer.
If $|A_s \setminus M| \geq |B_s \setminus M|$ Bob marks in the first place vertices of B_s and when there are no more, the vertices of A_s. In the A-marking game, after each of Bob's turns there are at most the same number of vertices of A_s marked than vertices of B_s, hence, by this strategy, the last vertex of A_s to be marked has at least $s-1$ neighbors already marked, hence Bob has a strategy with score s if $|A_s \setminus M| > |B_s \setminus M|$.
In the B-marking game, if $|A_s \setminus M| = |B_s \setminus M|$ then the last vertex to be marked is a vertex of A_s hence it has all of its neighbors already marked: at least $s-1$, hence a score of at least s. ■

In particular, the largest integer s such that $|A_s| \leq |B_s|$ gives a lower bound for the A- and B-game coloring numbers.
Moreover, the A- and the B-game coloring numbers can differ by at most 1:

Lemma 3.22 *If Alice has a strategy in the B-marking game with score s on a graph $G|M$, then Alice has a strategy in the A-marking game with score s on the graph $G|M$.*
Moreover, if Alice has a strategy in the A-marking game with score s on a graph $G|M$, then Alice has a strategy in the B-marking game with score $s+1$ on the graph $G|M$.
In particular: $col_A(G|M) \leq col_B(G|M) \leq col_A(G|M)+1$.

Proof. Let's prove $col_A(G|M) \leq col_B(G|M)$ first.
Assume Alice has a strategy in the B-marking game with score s. Playing the A-marking game, Alice uses the same strategy. For her first move, she imagines Bob has already played on a vertex $x \in B_s \setminus M$ and she plays the vertex y she would have played in that case. We call this vertex the *phantom vertex* and we denote it by ϕ. For the following moves, she plays as if the phantom vertex was marked and she follows her strategy step by step. Each time Bob marks the phantom vertex, she imagines Bob plays another vertex $x' \in B_s \setminus (V_m \cup M)$, where V_m is the set of marked vertices during the game. The phantom vertex ϕ is now x'. She plays then the vertex y' she would have played in that case. Remark that as Alice has a strategy with score s for the B-marking game, then $|A_s \setminus M| < |B_s \setminus M|$ (by Lemma 3.21), which means that as long as there are unmarked vertices, we have $|A_s \setminus (V_m \cup \{\phi\} \cup M)| < |B_s \setminus (V_m \cup \{\phi\} \cup M)|$, hence when she needs to imagine Bob marked a vertex x she can always select it in B_s.
After $t+1$ moves on the B-marking game, there is one vertex marked that is not in the A-marking game after t moves. Hence, each time a vertex is marked in the A-marking game, it has at most as large number of neighbors marked than in the B-marking game, since Alice uses the same strategy. When the vertex ϕ is marked, it may have all of its neighbors marked, but as $d(\phi) < s-1$, it does not change the maximum score.
The maximum score is at most s.

Now, we prove $col_B(G \mid M) \leq col_A(G \mid M) + 1$.
Assume Alice has a strategy in the A-marking game with score s. Playing the B-marking game Alice uses the same strategy. Bob starts playing and Alice plays ignoring his move. If at some point she has to mark the vertex marked by Bob, Alice marks a vertex of A_s (if there is none, there are only vertices of B_s unmarked, hence no vertex with score bigger than s). At each step there is one more vertex marked in the A-marking game than in the B-marking game: when a vertex is marked it has at most s marked neighbors, instead of $s-1$. The final score is at most $s+1$. ■

This result shows that Alice has no gain in passing her first turn, as well as Bob his in the B-marking game. More precisely:

Lemma 3.23 *If Alice has a strategy on the graph $G|M$ with score s for the marking game, then if Bob passes a turn, she still has a strategy with score s.*
If Bob has a strategy for the marking game with score s on a graph $G|M$, then, if Alice passes a turn, he has still a strategy with score s.

Proof. Assume they play the X-marking game on a graph $G \mid M$, $X \in \{A, B\}$. As long as nobody passes a turn, nothing changes.
If Alice has a strategy with score s then by following her strategy all vertices will have score at most s. Everytime just before Bob plays, the game is equivalent to start over a B-marking game on the graph $G|(V_m \cup M)$, where V_m is the set of marked vertices. As the score before starting this new game is at most s, the score of the game is $\max(s, col_B(G|(V_m \cup M)))$, hence if Bob passes his turn, the score is at most $max(s, col_A(G|(V_m \cup M)))$, which is at most the same as before.
Hence if Bob passes a turn, Alice still has a strategy with score s.

Now, if Bob has a strategy with score s then if there is already a vertex with score s, if Alice passes a turn it changes nothing to the score. If there is no such vertex and Alice passes a turn: it is the same than starting over on $G|(V_m \cup M)$ the A-marking game and Alice passes her first turn which gives a score of at most $\max(s, col_A(G|(V_m \cup M))$ *i.e.* at least as large as $\max(s, col_B(G|(V_m \cup M)))$.
Hence if Alice passes a turn, Bob still has a strategy with score s. ■

Note that this result, for $M = \emptyset$, was already known, proved by Zhu in [75].
In particular this result shows that it is not advantageous to pass turns for either of them.
We assume from now on that nobody passes turns.

We saw earlier that for a given integer s the sets A_s and B_s give a bound for the coloring number. Indeed the vertices that are *dangerous* for Alice are those of A_s: if she wants to keep a score of at most s. In fact, we can show that these vertices can be the only vertices Alice plays.

Lemma 3.24 *Let G be a graph, M some marked vertices of G and $s \geq col_X(G|M), X \in \{A, B\}$. Then Alice has a strategy with score s by only playing on vertices of A_s as long as there are unmarked vertices in A_s.*

Proof. Let G be a graph and $M \subset V$, assume Alice has a strategy with score s on $G \mid M$. She changes her strategy as follows: she plays by her strategy if the vertex in her strategy is in A_s, otherwise she passes her turn. She does so as long as there are unmarked vertices in A_s. If there are none, she plays by her strategy.
Everytime a vertex of A_s is marked it has at most $s - 1$ neighbors marked, since she follows her strategy for these vertices. Hence the score is at most s. ■

In particular, from now on we can assume Alice only plays on vertices of A_s if she wants a score of at most s.

Now we understand, somehow, how the game takes place from start to finish and how the two game coloring numbers interact with each other. In the following the interactions with the graphs will be studied. But first, we define a new class of graphs that introduces smoothly the upcoming results.

3.2.2 Sunflower class of graphs

The sunflowers graphs are obtained by *joining* a clique and a stable set in a particular, regular way. More precisely:

Definition 3.25 *Let n, k be two integers, $n \geq k > 0$. The sunflower $SF_{n,k}$ is the graph where:*

- *we denote its vertices by $\{a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}\}$;*

- *and the edge set is* $\{a_i a_j \mid 0 \leq i, j < n, i \neq j\} \cup \{\, a_i b_j \mid 0 \leq i < n, j \in \{i, (i+1) \bmod n, \ldots, (i+k-1) \bmod n\}\}$.

We denote by A the set of vertices $\{a_0, \ldots, a_{n-1}\}$, we call them seed-vertices*; and by B the set of vertices $\{b_0, \ldots, b_{n-1}\}$ and we call them* petal-vertices.

Remark that the vertices of A form a clique and the vertices of B form a stable set. Examples of these graphs are shown in Figure 3.2.1.

As well, as we are considering vertex deletions, we can directly define the sunflowers without a petal.

Definition 3.26 *We denote by $SF^*_{n,k}$ the graph $SF_{n,k} - \{b_0\}$.*

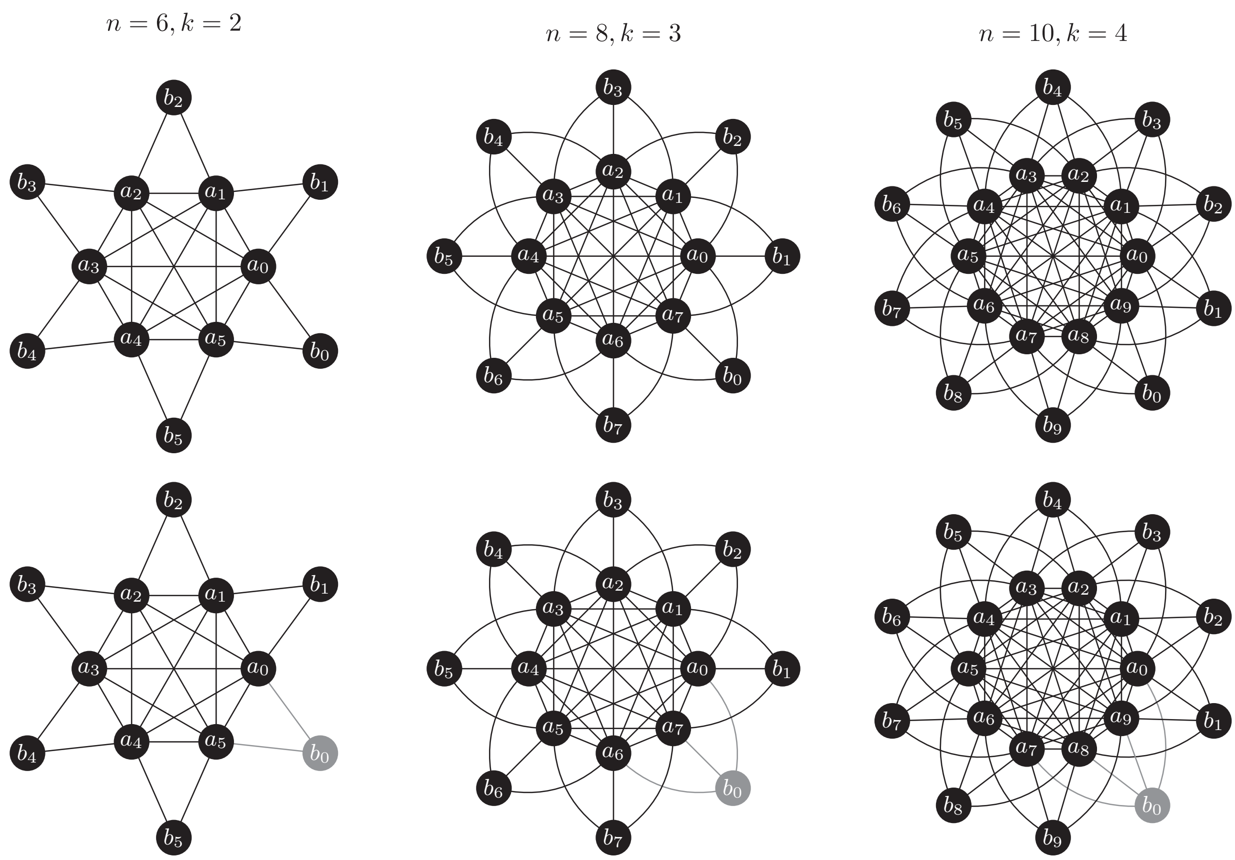


Figure 3.2.1: Examples of complete sunflowers $\mathrm{SF}_{n,k}$ and sunflowers missing a petal $\mathrm{SF}^*_{n,k}$

When (in the pictures) the vertices and/or edges are in gray, unless mentioned otherwise, is to mark their absence. Sometimes when the graph is too large the edges of the clique are not drawn.

By definition we have directly that the sets A_s and B_s, for $s = n+k$, defined in Definition 3.20 are exactly the sets A and B of the definition of sunflower graphs. In particular, both sets have the same number of vertices, hence by Lemma 3.21, Bob has a strategy with score $n+k$ for the B-marking game in the graph $\mathrm{SF}_{n,k}$. It is not always true for the A-marking game.

Theorem 3.27 *Let $n \geq k$ be two positive integers. Then:*

$$col_A(SF_{n,k}) = \begin{cases} n+k-1 & \text{if } n = k \\ n+k & \text{otherwise} \end{cases} \quad \text{and } col_B(SF_{n,k}) = n+k.$$

Proof. As mentioned before, we only need to prove the results for the A-marking game.
Let us start with the case $n = k$. Alice's strategy is to only play in A. No matter how Bob plays, when the last vertex of A is marked, say it is the vertex a_i, there is at most $n-1$ vertices in B that are marked, hence the vertex a_i has at most $2(n-1)$ neighbors marked, its score is then $2n-1 = n+k-1$. Bob ensures this score by playing only on the vertices of B.

Now, for the rest of the cases, we just have to give a strategy for Bob. Bob's strategy is the following: each time Alice plays a vertex a_i, he plays the vertex $b_{(i-1) \bmod n}$. If she plays elsewhere, he plays a vertex b_i having unmarked neighbors if possible. If they are playing the B-marking game, the last vertex to be marked is a vertex of A, hence the score is $n+k$. If they are playing the A-marking game, when the last vertex of A, say a_i, is marked there is maybe an unmarked vertex in B. As he has always followed his strategy, this unmarked vertex is exactly the one he would mark after Alice's turn, hence it is $b_{(i-1) \bmod n}$ which is not a neighbor of a_i since $k < n$, hence the score of a_i is $n+k$. ∎

In particular, if we recall the results of Lemma 3.22: $col_A(G) \leq col_B(G) \leq col_A(G)+1$ we remark that the lower bound is tight when $n = k$ and that the upper bound is tight in all the other cases.
Now we study the evolution of the X-game coloring number when applying *minor graph operators*, *i.e.*, how this number changes when we delete a vertex or an edge and when we contract an edge (the operators used to obtain minor graphs).

For the sunflower graphs, we only have to study two cases for the vertices and two for the edges: $v \in A$ or $v \in B$ for the vertices, and for the edges $e = a_i a_j$ for some $i \neq j$ or $e = a_i b_j$ for $j \leq i < (j+k) \bmod n$.

3.2.3 Vertex deletion

Let us start by the computation of the game coloring numbers on the graphs $\mathrm{SF}^*_{n,k}$.

Theorem 3.28 *Let n, k be two positive integers such that $n \leq k^2+3k-1$. Then $col_A(SF^*_{n,k}) = n+k-1$.*

Proof. Assume first that $n = k^2+3k-1$. We define $k+2$ A-zones and $k+2$ B-zones as follows (see Figure 3.2.2):

$$\begin{array}{lcll} Az_{-1} & = & \{a_1, \ldots, a_k\} & \\ Az_i & = & \{a_{(i+1)k+1}, \ldots, a_{(i+2)k}\} & \text{for } -1 < i < k \\ Az_k & = & \{a_{(k+1)k+1}, \ldots, a_{(k+2)k-1}\} & \\ & & & \\ Bz_{-1} & = & \{b_1, \ldots, b_k\} & \\ Bz_i & = & \{b_{(i+1)k+1}, \ldots, b_{(i+2)k}\} \cup \{a_{(n-i) \bmod n}\} & \text{for } -1 < i < k \\ Bz_k & = & \{b_{(k+1)k+1}, \ldots, b_{(k+2)k}\} & \end{array}$$

Remark that these zones do not form a partition on the vertices, since $b_{(k+2)k+i}$ for $1 \leq i \leq k-2$ are not included, but they for disjoint sets. Remark that every vertex of A is is one of the zones. Moreover we have $|Az_{-1}| = |Bz_{-1}|$ and for $i > -1$, $|Az_i| + 1 = |Bz_i|$. In addition, the vertices that can have a score of $n+k$ are exactly the vertices in the A-zones.
Alice's strategy is to play only in the A-zones and in decreasing order. More precisely, she starts by playing the vertex a_k. Then, each time Bob plays in a B-zone, say Bz_i, Alice plays the vertex a_j with maximum j in the zone Az_i. If Bob plays elsewhere, she plays the last a_i available with $i < n-k$.

By following this strategy, after each of Alice's turns, in every zone there is at least one more unmarked vertex in the zone Bz_i than in the zone Az_i for all i. If Bob plays on vertices of the A-zones, this difference increases, hence we can assume the last vertex to be marked in the zone Az_i is always $a_{(i+1)k+1}$: the only vertex of Az_i that is neighbor of all the vertices in Bz_i. The last vertex of the A-zones to be marked has always an unmarked vertex in the corresponding B-zone, hence its score is at most $n+k-1$. Before the last vertex of the A-zones is marked, all the vertices of A have a score $s < n+k-1$ since they have

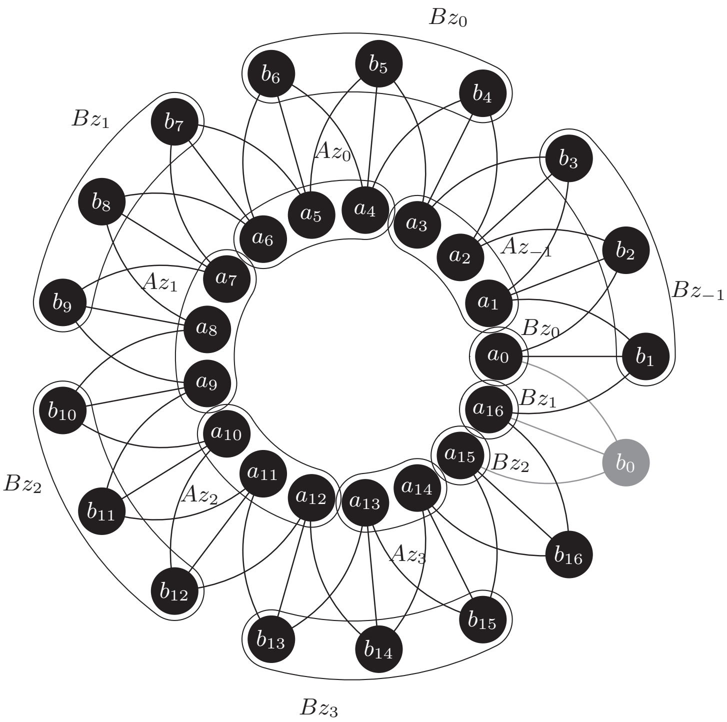


Figure 3.2.2: A- and B-zones for $k = 3$, $n = k^2 + 3k - 1$.

unmarked neighbors in A.

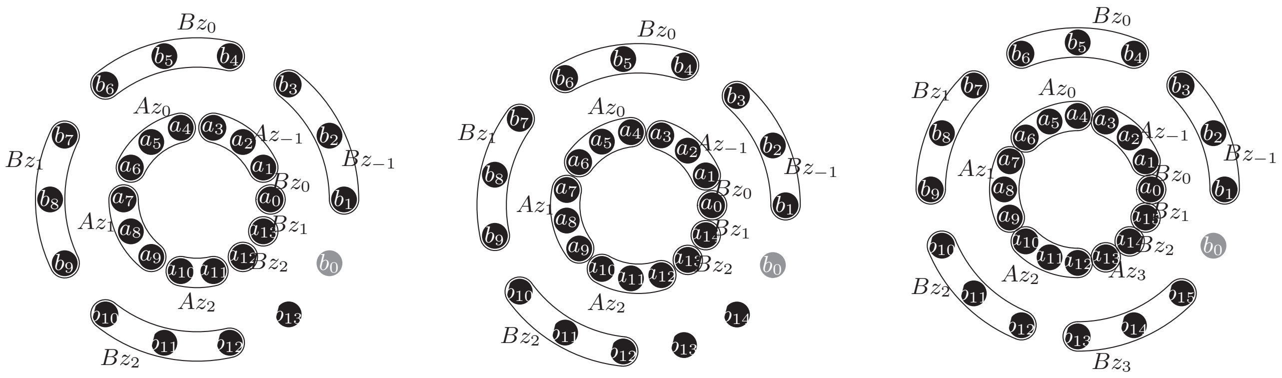


Figure 3.2.3: A- and B-zones for $n < k^2 + 3k - 1$, here $k = 3$ and $n \in \{14, 15, 16\}$. Here the edges are not drawn.

For the cases $n < k^2 + 3k - 1$ we take the most we can of each zone in increasing order, *i.e.*, the zones are the same for $-1 \leq i \leq j$ and $Az_j = \{a_{(k+1)k+1}, \ldots, a_{n-1}\}$ for some j and the same for the B-zones. This way there is always more vertices in Bz_i than in Az_i. The proof works the same way. For more details about the zones see Figure 3.2.3. ■

For the B-marking game we have the same kind of result, with the same kind of proof.

Theorem 3.29 *Let n, k be two positive integers such that $n \leq k^2 + k$. Then $col_B(SF^*_{n,k}) \leq n + k - 1$.*

Proof. It is the same kind of proof as above, so we are just giving the main lines for $n = k^2 + k$ and then show quickly how it works for $n < k^2 + k$.

Here we need a strategy for Alice ensuring a score of at most $n+k-1$ when $n=k^2+k$. Alice divides the graph in the following zones:

$$Az_i = \{a_{ik+1},\dots,a_{(i+1)k}\} \qquad \text{for } 0 \le i \le k-1$$

$$Bz_i = \{b_{ik+1},\dots,a_{(i+1)k}\} \cup \{a_{(n-i) \bmod n}\} \qquad \text{for } 0 \le i \le k-1$$

Here, each B-zone has exactly one more vertex than the corresponding A-zone, and the only dangerous vertices for Alice are exactly the vertices of the A-zones. We modify her strategy of above by removing her first move, since here Bob is the one that starts playing.

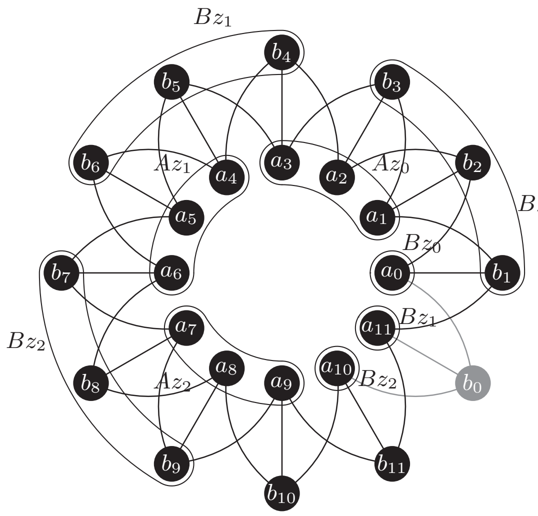


Figure 3.2.4: A- and B-zones for $k=3$ and $n=k^2+k$ of Theorem 3.29.

For $n<k^2+k$, we take the first complete zones and the most we can of the last one such that there is always one more vertex in the zone Bz_i than in the zone Az_i. And it is the same proof.
For a picture of the zones see Figure 3.2.4. ■

We will see later that in fact these two last theorems could be improved: when the final score is not $n+k-1$ then it is exactly $n+k$. There are some values for which we can give a strategy for Bob:

Theorem 3.30 *Let n,k be two integers such that $n \ge 2k^2+2$. Then $col_A(SF^*_{n,k}) = n+k$.*

Proof. We proceed as above: we take $n=2k^2+2$, we divide the graph in zones and give a strategy ensuring that score. Then we show how it changes for $n>2k^2+2$.
Let $n=2k^2+2$. The A- and B-zones are:

$$\begin{aligned} Bz_i &= \{b_{ik+1}\} && \text{for } i \in \{0,\dots,2k\} \\ Bz_{-1} &= B \setminus (Bz_0 \cup \dots \cup Bz_{2k}) && \\ \\ Az_0 &= \{a_{n-k+2},\dots,a_0,a_1\} && \\ Az_i &= \{a_{(i-1)k+2},\dots,a_{ik+1}\} && \text{for } i \in \{1,\dots 2k\} \end{aligned}$$

An example of these zones are given in Figure 3.2.5. Thus for every A-zone there is a B-zone, but not the converse. Moreover, Az_i is the exact neighborhoud of Bz_i.
Bob's strategy is to play first the vertices $\{a_{2k^2-k+3}, a_{2k^2-k+4},\dots,a_{n-1},a_0\}$. If in her first k turns Alice marks any of these vertices, Bob marks any vertex of any set Az_i. After $2k$ turns, (k for each player), all of these vertices are marked and there are k vertices in the A-zones that are also marked. After these $2k$

turns, Bob plays any vertex of Bz_{-1}. When this is no longer possible (all vertices of Bz_{-1} are marked) that means that at most $n-(2k-1)$ turns of Alice have passed and at most $n-(2k+1)-1$ turns of Bob. Thus, as there are $2k$ more vertices marked in A than in B, there is only one vertex of A that is unmarked and by this strategy, it is in one of the A-zones, say Az_j, the vertex $a_{j'}$. In the petals only the vertices b_{ik+1}, $i \in \{0,\dots,2k\}$ are unmarked, in particular, b_{jk+1} is unmarked and is the only remaining unmarked neighbor of $a_{j'}$. As it is Bob's turn, he can mark it and when Alice marks $a_{j'}$ its score is $n+k$. For $n = 2k^2+2$ Bob has a strategy ensuring a score of $n+k$.

Now, for $n > 2k^2+2$ we only need to redefine new zones and the same proof will hold. Remark that the vertices $a_{n-k+1},\dots,a_0$ do not need to be defined in a zone Az_i since they will be marked by Bob in the first k turns. Thus:

- the new B-zones are: $Bz_i = \{b_{ik+1}\}$ for $i \in \{1,\dots,\lceil (n-1)/k \rceil\}$,
- and the A-zones are: $Az_i = N(Bz_i)$

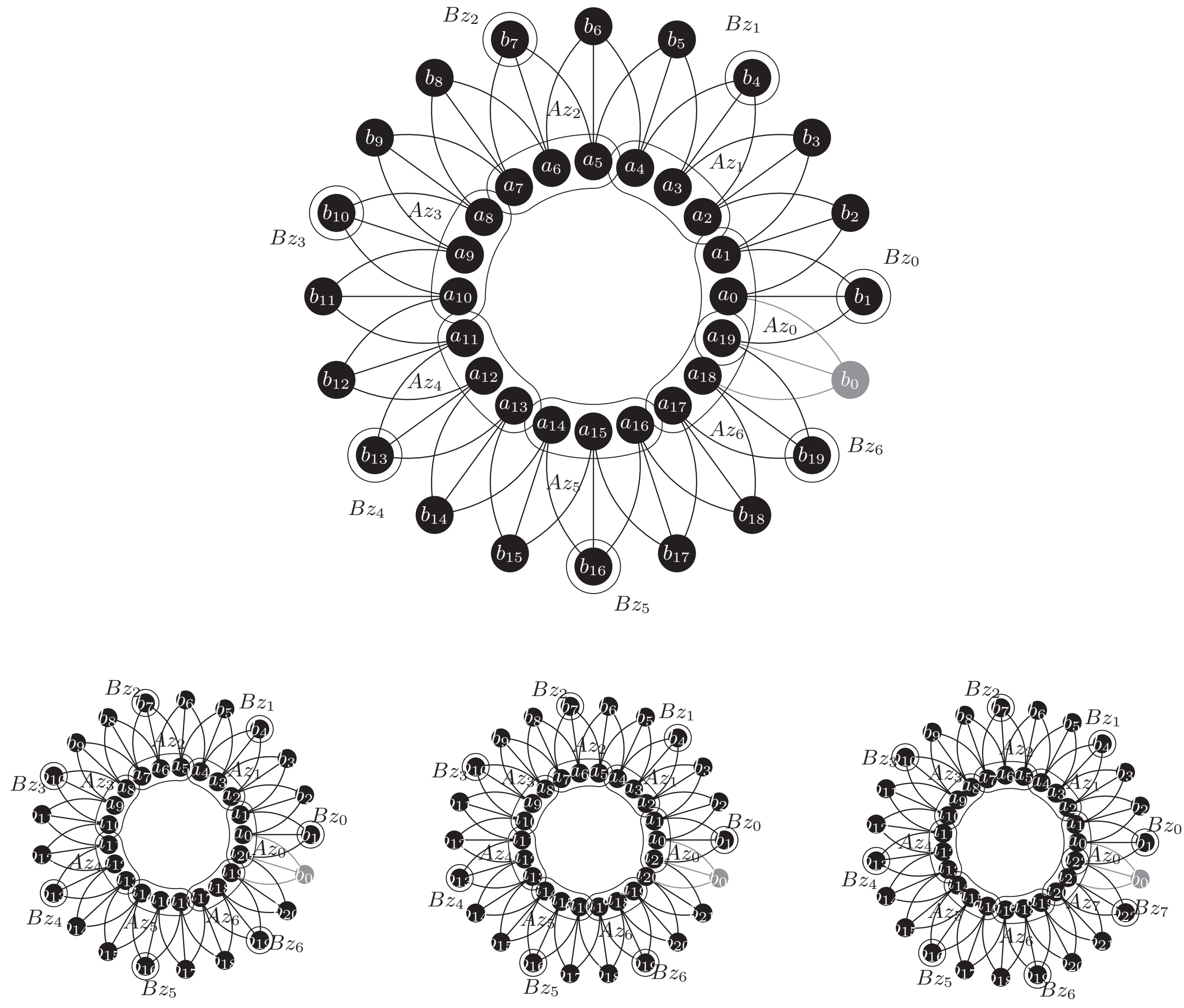


Figure 3.2.5: A- and B-zones for $n \geq 2k^2+2$ of Theorem 3.30. Here $k=3$ and $n \in \{20, 21, 22\}$, the edges of the clique are not drawn.

And Bz_{-1} is defined as the remaining vertices in B. This way Bob has more zones, his strategy changes only at the end: when there are no more vertices of Bz_{-1}, he marks vertices of B that have unmarked vertices if possible. By the same argument (the fact that the first $2k$ fully marked A-zones give $2k$ B-zones where Bob will not play), the last vertex to be marked in A has all of its neighbors marked. ■

As we know that $col_B(G) > col_A(G)$, we obtain the same result for the B-marking game for $n \geq 2k^2 + 2$, but in fact we can do better than that.

Theorem 3.31 *Let n, k be two positive integers, such that $n \geq 2k^2 - k + 2$. Then $col_B(SF^*_{n,k}) = n + k$.*

Proof. For $n = 2k^2 - k + 2$ the zones are:

$$\begin{aligned} Az_i &= \{a_{(i-1)k+2 \bmod n}, \dots, a_{ik+1}\} && \text{for } i \in \{0, \dots, 2k-1\} \\ Bz_i &= \{b_{ik+1}\} && \text{for } i \in \{0, \dots, 2k-1\} \\ Bz_{-1} &= B \setminus (Bz_0 \cup \dots \cup Bz_{2k-1}) \end{aligned}$$

Bob uses the same strategy as above and as he is not playing in the zones Bz_i, $i \geq 0$ until the end, he makes sure the last vertex Alice marks has all of its neighbors marked.
An example of these zones is given in Figure 3.2.6.

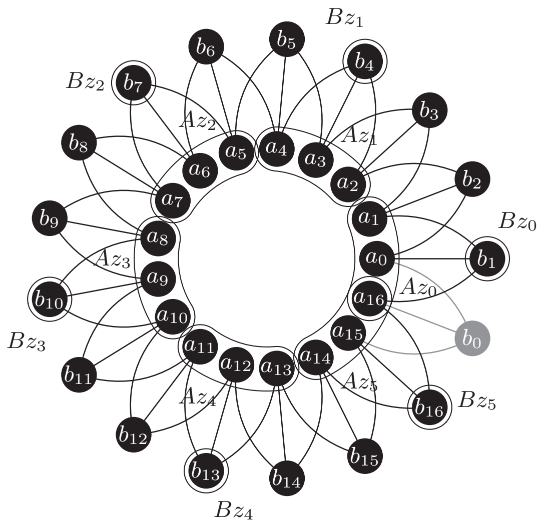


Figure 3.2.6: Example of A- and B-zones for $n = 17$ and $k = 3$ of Theorem 3.31.

For $n > 2k^2 - k + 2$, we define the exact same zones and the same strategy holds. ■

We know that for a graph $\mathrm{SF}^*_{n,k}$, the final score (if the players play smartly) is either $n + k$ or less. If for the A-marking game it is $n + k$, then for the B-one it is also $n + k$. And, conversely, if it is $< n + k$ for the B-one, it is also $< n + k$ for the A-marking game.
The previous theorems give two bounds (for each game) which separates the two possibilities ($n + k$ or not). Naturally, the following two questions are raised:

Open question 3.32 *for $k \geq 3$ and $k^2 + 3k - 1 < n < 2k^2 + 2$, what is the exact value of $col_A(SF^*_{n,k})$?*

Open question 3.33 *for $k \geq 2$ and $k^2 + k < n < 2k^2 - k + 2$, what is the exact value of $col_B(SF^*_{n,k})$?*

We have seen the deletion of a petal-vertex, and in particular, we do not know the exact values of col_A and col_B when we delete a petal in the general case. For the deletion of a seed-vertex we are only discussing three cases that are useful in the following.

Theorem 3.34 *Let n be a positive integer, $n > 2$. Then:*

$$\begin{array}{lcllcl} col_A(SF_{n,1} - \{a_0\}) & = & n & col_B(SF_{n,1} - \{a_0\}) & = & n \\ col_A(SF_{n,n-1} - \{a_0\}) & = & 2n-3 & col_B(SF_{n,n-1} - \{a_0\}) & = & 2n-2 \\ col_A(SF_{n,n} - \{a_0\}) & = & 2n-3 & col_B(SF_{n,n} - \{a_0\}) & = & 2n-2 \end{array}$$

Proof. We do the proof item by item.

- $\mathrm{SF}_{n,1} - \{a_0\}$:
  The maximum possible score is n. Alice's first turn is used to play on some vertex of A, say a_i. Bob's strategy is to play on vertices of B having unmarked neighbors, hence he is not playing b_0 nor b_i. He has the time to mark all the remaining vertices of B before Alice marks the last vertex of A, hence the score of this last vertex is n.
  For the B-marking game, it is the same strategy: he leaves b_0 for the end. When Alice marks the last vertex of A, all the vertices of B (except b_0) are marked, the score is then n.

- $\mathrm{SF}_{n,n-1} - \{a_0\}$:
  First, remark that $\Delta(\mathrm{SF}_{n,n-1} - \{a_0\}) = \Delta(\mathrm{SF}_{n,n-1}) - 1$, hence, as $col_A(\mathrm{SF}_{n,n-1}) = 2n-1 = \Delta(\mathrm{SF}_{n,k})$ we have that $col_A(\mathrm{SF}_{n,n-1} - \{a_0\}) \leq 2n-2$. Moreover, there is one more vertex in B than in A.
  Now for the A-marking game, assume Alice only plays on A. After each of Bob's turns, if there still are unmarked vertices on A then there are still at least two unmarked vertices on B. Hence, the last vertex of A to be marked has at least one unmarked neighbor. The score is thus $2n-3$. Bob ensures this score by playing the vertex b_{i-1} when Alice plays the vertex a_i.

  For the B-marking game we only need to give a strategy for Bob ensuring score at least $2n-2$. He starts by playing b_{n-1}, the only vertex that is not a neighbor of a_0. Now, all vertices in A have a neighbor marked. For the rest of the game, when Alice plays the vertex a_i, he plays the vertex b_{i-1}, the only vertex that is not a neighbor of a_i. By doing this, each time he marks a vertex, all the unmarked vertices in A have one more marked neighbor in B, hence the last vertex in A to be marked, say a_i, (after $n-1$ turns of Bob) has $n-1$ neighbors in B that are marked: its score is then $n-1+n-2+1 = 2n-2$. And the remaining vertex is b_{i-1} that is not a neighbor of a_i.

- $\mathrm{SF}_{n,n} - \{a_0\}$:
  Alice plays in order the vertices of A. For the A-marking game, the last vertex of A to be marked, say a_i, has $n-2$ marked vertices in a and at most $n-2$ neighbors marked in B. The score of a_i is thus at most $2n-3$. For the B-marking game, the only thing that changes is that B has played $n-1$ turns, hence at the end the score if at most $2n-2$. Bob ensures this score by only playing on vertices of B.

■

Here we have $col_X(\mathrm{SF}_{n,1}) = 1 + col_X(\mathrm{SF}_{n,1} - \{a_0\})$, a drop of exactly one. And $col_X(\mathrm{SF}_{n,k}) = 2 + col_X(\mathrm{SF}_{n,k} - \{a_0\})$ for $k \in \{n-1, n\}$, *id est*, a drop of 2.
We can raise the more general question:

Open question 3.35 *for which values of n and k do we have a drop of 1? of 2? for the A-game? for the B-game?*

We conjecture the drop is linear in k:

Conjecture 3.36 *For $X \in \{A, B\}$ and $n \in \mathbb{N}$, there is an integer $k_{c,X,n}$ such that:*

- *if $k \leq k_{c,X,n}$ then $col_X(SF_{n,k}) = 1 + col_X(SF_{n,k} - \{a_0\})$,*
- *otherwise: $col_X(SF_{n,k}) = 2 + col_X(SF_{n,k} - \{a_0\})$.*

Now, for non-particular graphs, we can see that the only possibilities are: a drop of 2, of 1 or not a drop at all. First, before comparing $col_X(G)$ to $col_X(G - \{v\})$ we are comparing $col_A(G)$ to $col_B(G - \{v\})$.

Lemma 3.37 *Let $G(V,E)$ be a graph, M a set of marked vertices and $v \in V \setminus M$, such that $|N(v) \cap M| < col_B(G - \{v\} \mid M)$. Then:*

$$col_A(G|M) \leq col_B(G - \{v\}|M) + 1$$

Proof. Assume Alice has a strategy for the B-marking game on $G - \{v\} \mid M$ with score s. Then, playing the A-marking game on $G \mid M$, she starts by marking v and then she follows her strategy. At the end, all the vertices have score at most s in $G - \{v\} \mid M$, hence at most $s + 1$ in $G \mid M$. ■

With this result we can now deduce:

Theorem 3.38 *Let G be a graph, M a set of marked vertices and $v \in V \setminus M$ with p $|N(v) \cap M| < col_B(G - \{v\} \mid M)$. We assume $G - \{v\}|M$ has at least one unmarked vertex.*

$$\begin{array}{ccccc} col_A(G|M) - 2 & \leq & col_A(G - \{v\}|M) & \leq & col_A(G|M) \\ col_B(G|M) - 2 & \leq & col_B(G - \{v\}|M) & \leq & col_B(G|M) \end{array}.$$

Both bounds are tight. Moreover the lower bound can be tight only for vertices of $A_{col_X(G)}$ for the X-marking game.

Proof. For the second inequality, Alice has a strategy with score s in $G \mid M$ that she uses also in $G - \{v\} \mid M$. Assume that, following this strategy she has to mark v after t turns. Let V_m be the set of marked vertices during this t turns: Alice has a strategy with score s on $G \mid (M \cup V_m)$ for the A-marking game. In particular, she also has a strategy with score s on $G \mid (M \cup V_m \cup \{v\})$ for the B-marking game and then, she has a strategy with score s on $G \mid (M \cup V_m \cup \{v\})$. Thus, she has a strategy with score s on $G - \{v\} \mid (M \cup V_m)$. If she does not have to mark v, then she just follows her strategy with score s in $G \mid M$.
By doing this, each unmarked vertex has at most the same number of marked neighbors than in $G \mid M$, which gives a score of at most s.
For the A-marking game, the graphs $\mathrm{SF}_{n,n}$ attain this bound when removing b_0 (see Theorem 3.28). For the B-marking game, the graphs $\mathrm{SF}_{n,k}$ attain this bound, for $n \geq 2k^2 - k + 2$, when removing b_0 (see Theorem 3.31).

For the first inequality, we use Lemma 3.22 and Lemma 3.37. The first one tells us $col_B(G - \{v\}) \leq col_A(G - \{v\}) + 1$ and the second one $col_A(G) \leq col_B(G - \{v\}) + 1$.
Thus $col_A(G) - 2 \leq col_A(G - \{v\})$ and the arguments go for the B-marking game.
For the A- and B-marking games the graphs $\mathrm{SF}_{n,n}$ attain the bound when removing a_0 (see Theorems 3.27 and 3.34).

Now, let us prove the tightness is achieved only for vertices of $A_{col_X(G)}$. Assume they are playing the X-game and let $v \notin A_{col_X(G)}$. Assume Alice has a strategy with score s in $G - \{v\}$. When they play in G she uses the same strategy. If at some point Bob plays v then it is like he passes his turn on $G - \{v\}$. Thus in $G - \{v\}$ the score will be at most s and counting v, at most $s + 1$. Hence $col_X(G) \leq col_X(G - \{v\}) + 1$.
■

In particular, if we recall Theorems 3.28 to 3.31 and 3.34 we can see that there are examples of the three different cases $(-2, -1, 0)$ inside the sunflower class of graphs. Moreover, we can even see that there are large families of graphs such that removing a vertex (any vertex) decreases the game coloring number.

Definition 3.39 *A graph $G(V,E)$ is* vertex-critical *if $\forall v \in V$, $col_A(G - \{v\}) < col_A(G)$.*
As well, a graph $G(V,E)$ is vertex-hyper-critical *if G is vertex-critical and $\forall v \in A_{col_A(G)}$, $col_A(G - \{v\}) = col_A(G) - 2$.*

Remark that the vertex-hyper-critical condition implies that for each vertex we have the maximum drop possible (depending on their degree). Moreover we know:

Corollary 3.40 (Theorems 3.28 and 3.34) *Let n, k be two positive integers.*

- *if $n \geq k^2 + 3k - 1$ then $SF_{n,k}$ is vertex-critical and*
- *if $k = n - 1$ then $SF_{n,k}$ is vertex-hyper-critical.*

We have seen the vertex deletion and stressed that not all the cases have been studied. However, we can bound below the maximum drop when removing a vertex and we exhibit infinite many graphs that have this maximum drop when removing any vertex.

3.2.4 Edge deletion

The study for the edge deletion is much simpler, so we will not get into the most precise details. On the contrary, we will just give some families such that edge deletions decrease the game coloring number.

Theorem 3.41 *Let n, k be two positive integers, such that $k \leq n \leq 3k$. Then for e any edge of $SF_{n,k}$ we have: $col_A(SF_{n,k} \setminus \{e\}) = col_A(SF_{n,k}) - 1$.*

Proof. Let e be an edge of $\mathrm{SF}_{n,k}$, by Theorem 3.5 we know that $col_A(\mathrm{SF}_{n,k}) \leq col_A(\mathrm{SF}_{n,k} \setminus \{e\}) + 1$. Let us show that this inequality is in fact tight.
As $col_A(\mathrm{SF}_{n,k}) = n+k$, we prove that $col_A(\mathrm{SF}_{n,k} \setminus \{e\}) = n+k-1$. As there are no edges between vertices of B, we can assume we remove an edge of the vertex a_0. Alice's strategy is the following: she starts by playing a_k. If Bob plays a_0 then she plays the minimal a_i such that $i \geq 2k$ (if $n \geq 2k$, otherwise she plays a_i with i maximal). If he plays a vertex of B, say b_j, she marks the minimal a_i such that $a_i \in N(b_j)$, $i \neq 0$. If he plays elsewhere or none of these possibilities is available, she plays the minimal a_i such that $i > 0$.
We define the A-zones:

$$\begin{array}{rcll} Az_i & = & \{a_{ik+1}, \ldots, a_{(i+1)k}\} & \text{for } i \in \{0, 1\} \\ Az_2 & = & \{a_{2k+1}, \ldots, a_{n-1}\} & \end{array}$$

Remark that a_0 is not in any of the A-zone.
By following this strategy, if Alice plays all the vertices of the A-zones and a_0 is still unmarked, then the maximum score is $n - 1 + k$ obtained either for a_0 or for the last vertex of A marked before it. If Bob marked at some point a_0, then the first vertex of each A-zone has been marked.
In this case, when there are $k - 1$ consecutive marked vertices in B then there are k consecutive vertices marked in A. Indeed, by following her strategy, for every k consecutive vertices in A there is at least one of them marked. Hence there are at most $k - 1$ consecutive unmarked vertices in A. By her strategy, she marks the minimum a_i neighbor of b_j marked by Bob, thus, these consecutive vertices, $b_j, \ldots, b_{j+k-1 \bmod n}$, make Alice mark the vertices $a_{j-k+1 \bmod n}, \ldots, a_j$ if unmarked, but there was already one marked, hence the vertices $a_{j-k+1}, \ldots, a_{j+1}$ are marked.
In fact, before Bob marks a_0, for every x consecutive vertices marked in B there are at least x consecutive vertices of A marked, thus when he marks a_0, there is no unmarked vertex of A with all of its neighbors in B already marked.
In particular, when Bob marks the k-th consecutive vertex in B, $b_j, \ldots, b_{j+k-1 \bmod n}$ their only common neighbor in A, a_j, is already marked. Thus the score is at most $n+k-1$ and as $col_A(\mathrm{SF}_{n,k} \setminus \{e\}) \geq n+k-1$, we have the equality. ∎

For the B-marking game we need to decrease the upper bound:

Theorem 3.42 *Let n, k be two positive integers such that $k \leq n \leq 2k$. Then for e any edge of $SF_{n,k}$ we have: $col_B(SF_{n,k} \setminus \{e\}) = col_B(SF_{n,k}) - 1$.*

Proof. We can assume the vertex a_0 is missing an edge. Alice's strategy is: if Bob plays a_0, she plays the minimal a_i such that $i \geq k$. If he plays a vertex of B, say b_j, she marks the minimal a_i such that $a_i \in N(b_j)$, $i \neq 0$. If he plays elsewhere or none of these possibilities is available, she plays the minimal a_i, $i > 0$.
The rest of the proof is exactly the same as for Theorem 3.41 but we define only two zones: $Az_0 = \{a_1, \ldots, a_k\}$ and $A_1 = \{a_{k+1}, \ldots, a_{n-1}\}$. ∎

These upper bounds may not be tight, but we know that for n big enough these results are no longer true:

Theorem 3.43 *Let n, k be two positive integers such that $n \geq 4k-1$, $k \geq 2$. Then $col_A(SF_{n,k}) = col_A(SF_{n,k} \setminus \{a_0b_0\})$.*

Proof. We recall that $col_A(\mathrm{SF}_{n,k}) = n + k$ as $n \neq k$. We give a strategy for Bob ensuring a score $n + k$ in the graph $\mathrm{SF}_{n,k} \setminus \{a_0b_0\}$.
We assume first that $n = 4k-1$ and we define the zones:

$$\begin{array}{rcll} Az_i & = & \{a_{ik}, \ldots, a_{(i+1)k-1}\} & \text{for } i \in \{0,1,2,3\} \\ Bz_{0i} & = & \{b_{k-1}, b_{2k-1}, b_{3k-1}, b_{4k-1}\} & \end{array}$$

Remark that Az_0 and Az_3 have a common vertex a_0.
Bob's strategy is to first plays a_0. If this is not possible (Alice played it at her first turn), then he marks a_1. After the second turn of Alice, there are three marked vertices in A. Then, for all of his other turns, he marks b_j for j minimal and $b_j \notin Bz_0$. He can play this way for at most $n-4$ turns. After these turns, when it is his turn (Alice has just played), the number of marked vertices in A is $(n-4)+1+1+1 = n-1$ where two 1's come from the first turns of Alice and Bob, the $(n-4)$ from the next $(n-4)$ turns and the last 1 comes from the turn Alice has just played. Hence there is still an unmarked vertex in A, say a_i, and it is Bob's turn. The only unmarked vertices in B are those of Bz_0. Remark that these vertices have no common neighbor: the vertex a_i has exactly one unmarked vertex, say b_j. Bob marks it and hence the score of a_i is $n+k$.
For $n > 4k-1$, we define the exact same zones (all vertices $a_i, b_i, i > 4k-1$ are just ignored) and his

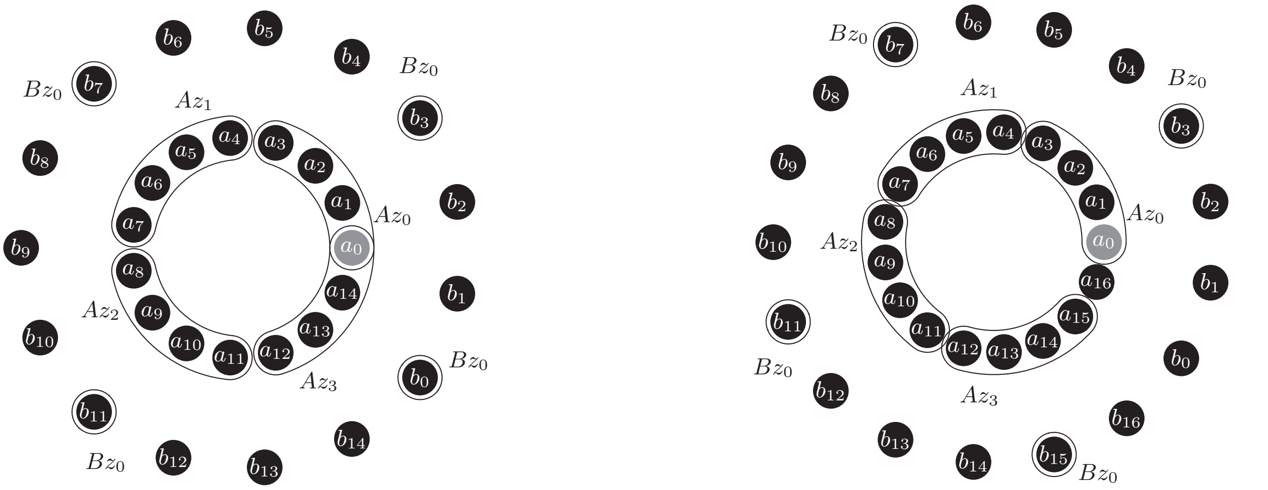


Figure 3.2.7: Example of zones for $k = 4$, $n = 4k-1$ and $n = 4k+1$. The edges are not drawn and the vertex a_0 is missing an edge.

strategy is the same: he doesnt play in Bz_0 as long as he can. This is possible for his first $n-4+1$ turns. If the remaining unmarked vertex of A, say a_i, is such that $i < 4k-1$, then Bob marks his unmarked neighbor, if $i \geq 4k-1$ then all its neighbors are already marked and the score is the same.
For some examples of these zones we refer to Figure 3.2.7. ■

And it is the same for the B-marking game with a lower upper bound:

Theorem 3.44 *Let n, k be two positive integers such that $n \geq 3k-1$. Then $col_B(SF_{n,k} \setminus \{a_0b_0\}) = col_B(SF_{n,k})$.*

Proof. We give a strategy for Bob ensuring a score of $n+k$ in $\mathrm{SF}_{n,k}\setminus\{a_0b_0\}$. Assume first that $n = 3k-1$. Bob divides the graph in zones:

$$\begin{array}{rcll} Az_i & = & \{a_{ik},\dots,a_{(i+1)k-1}\} & \text{for } i \in \{0,1,2\} \\ Bz_0 & = & \{b_{k-1}, b_{2k-1}, b_{3k-1}\} & \end{array}$$

Remark that a_0 is in Az_0 and Az_2.
Bob's strategy is to first play a_0. After Alice's first turn, his strategy is to play vertices of B that are not in Bz_0. By the same argument as in Theorem 3.43 the final score is $n+k$. ■

In fact, we know for $k \in \{1,2,3\}$, $n \geq 3k+1$ we can remove an edge and Bob has a strategy that gives a score $n+k$, but for $k > 3$ we do not know what happens.

Open question 3.45 *for k, n such that $3k+1 \leq n \leq 4k-2$, it is possible to remove an edge of $SF_{n,k}$ without changing the A-game marking number?*

Open question 3.46 *for k, n, $2k < n < 3k-2$, is it possible to remove an edge of $SF_{n,k}$ without changing the B-coloring number?*

These results call to mind the vertex-critical situations we introduced above. Let us see how the game coloring numbers evolve when deleting an edge.

Theorem 3.47 *Let G be a graph, M a set of marked vertices of G and e an edge of G. We assume G has at least one unmarked vertex.*

$$\begin{array}{rcccl} col_A(G|M)-1 & \leq & col_A(G\setminus\{e\}|M) & \leq & col_A(G|M) \\ col_B(G|M)-1 & \leq & col_B(G\setminus\{e\}|M) & \leq & col_B(G|M) \end{array}$$

And these bounds are tight.

Proof. Assume Alice has a strategy for the A-marking game on the graph $G\setminus\{e\} \mid M$ with score s. On the graph $G \mid M$ Alice uses her strategy: each time a vertex is marked it has at most $(s-1)+1$ neighbors marked. Hence at the end the score of $G \mid M$ is at most $s+1$. The same proof holds for the B-marking game.
The graphs $\mathrm{SF}_{n,n}$ attain this lower bound when removing the edge a_0a_1 (see Figure 3.2.8).
For these graphs we have: $col_A(\mathrm{SF}_{n,n}) = 2n-1$ and $col_B(\mathrm{SF}_{n,n}) = 2n$.

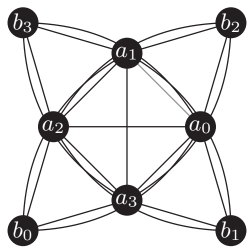


Figure 3.2.8: $\mathrm{SF}_{n,n}$ for $n = 4$

When an edge is removed the strategy for Alice is to mark last the two vertices of degree $2n-2$ (the two endpoints of e). By doing so, the best way to maximize the score is for Bob to play on the vertices of B, and the final score is: for the A-marking game $s = 2n-2$ and for the B-one $s = 2n-1$.

The second inequality is easily obtained: Alice plays by her strategy with score $col_X(G \mid M)$ and she ensures then a score of at most $col_X(G \mid M)$.
For the tightness we can take the graphs $\mathrm{SF}_{n,1}$, $n \geq 2$. We have $col_A(\mathrm{SF}_{n,1}) = n+1$ and when the edge a_0b_0 is removed we have $col_A(\mathrm{SF}_{n,1}\setminus\{a_0b_0\}) = n+1$. ■

In particular we obtain a stronger result than in Theorem 3.5:

Corollary 3.48 (Theorems 3.38 and 3.47) *Let H be a subgraph of a graph G with the set M of vertices marked such that $\forall v \in V(G)$, $|N(v) \cap M| < col_B(H|M_H)$, $M_H = M \cap V(H)$. . Then:*

$$\begin{array}{rcl} col_A(H|M_H) & \leq & col_A(G|M) \\ col_B(H|M_H) & \leq & col_B(G|M) \end{array}$$

Indeed, it is stronger since H is *any* subgraph, not necessarily induced nor spanning. For $M = \emptyset$ this result was already known [72], the fact that it works also for $M \neq \emptyset$ makes it stronger still.

Analogously to the vertex deletion results, we can introduce the *edge*-critical graphs:

Definition 3.49 *A graph $G(V, E)$ is* edge-critical *if $\forall e \in E$, $col_A(G \setminus \{e\}) = col_A(G) - 1$.*

We know already some edge-critical graphs.

Corollary 3.50 (Theorem 3.41) *Let n, k be two positive integers such that $k < n \leq 3k$. Then $SF_{n,k}$ is edge-critical.*

Remark that criticality has been only defined for the A-marking game. Indeed, since in the litterature only the A-marking game is studied we restrict only to this case. Obviously, a definition for the B-marking game with the same kind of results could be done. It could be interesting to know if the critical graphs are the same for both games.

3.2.5 Edge contraction

Even though we are using sunflower graphs for the tightness of the edge contraction results, we have not done much about the edge contraction on the sunflowers.
Indeed this operator is much more difficult to manipulate: removing an edge or a vertex leaves a graph very similar to the starting sunflower. Contracting an edge modifies the graph much more and describing precisely what remains it not the subject here. For now we just focus on the contraction of an edge in any graph.

Theorem 3.51 *Let G be a graph, M a set of marked vertices and $e = uv$ an edge of G.*

$$col_A(G|M) - 2 \quad \leq \quad col_A(G/e|M) \quad \leq \quad col_A(G|M) + 2$$

And these bounds are tight.

Proof. First, we prove the first inequality.
We remark that $G - \{u, v\} = G/e - \{w\}$ and by Lemma 3.37 we have
$col_A(G/e|M) \leq col_B(G - \{u, v\}|M) + 1$.
Then by Corollary 3.48: $col_B(G - \{u, v\}|M) + 1 \leq col_B(G|M) + 1$ and by Lemma 3.22 $col_B(G|M) + 1 \leq col_A(G|M) + 2$.

We have proved: $col_A(G/e|M) \leq col_A(G|M) + 2$.
For the other inequality, we prove $col_A(G|M) \leq col_A(G/e|M) + 2$. Assume Alice has a strategy for the A-marking game on $G/e|M$ with score s. She uses her strategy on $G|M$. If during the game she has to mark the vertex w on the graph $G/e|M$, she marks the vertex u in $G|M$. In the graph $G/e|M$ it is like she passed her turn (w has two corresponding vertices and Alice cannot mark them at the same time). The vertex v has at most $s-1$ neighbors marked before Alice marked u, then u is marked, hence at most s. At his next turn, Bob can mark another neighbor of v, hence its score is at most $s+2$. Alice marks v just after she marks u. Any other vertex has a score of at most s in G/e, in G it is at most s from G/e, plus 1 from v and plus 1 from the fact that Alice takes one of her turns to mark v. Hence the final score is at most $s+2$.

The tightness is reached for the family of graphs obtained as following: take $\mathrm{SF}^*_{n+1,n+1}$ (see Figure 3.2.9). The vertex contracted is one of A hence obtaining the graph $\mathrm{SF}_{n,n}$.

The graph $G = \mathrm{SF}^*_{n+1,n+1}$.

The graph $G/\{a_0a_1\} = \mathrm{SF}_{n,n}$.

Figure 3.2.9: tightness for the A-marking game: $col_A(G) = col_A(G/e) + 2$

In the Figure 3.2.9 $col_A(G) = 11$ and $col_A(G/e) = 9$.

The tightness of the other inequality is with a graph slightly different.
Let us take for G a subgraph of $\mathrm{SF}^*_{n+1,n+1}$ where we remove some edges in the following way: a_ib_i for $1 \leq i < n+1$ and the edges $a_{n+1}b_i$ for $i \geq \lfloor n/2 \rfloor$ and the edges a_0b_i for $i \leq \lfloor n/2 \rfloor$ and for $i = n$. In this case we have, if n is even $d(a_0) = d(a_{n+1}) + 1$ and if n is odd $d(a_0) = d(a_{n+1})$ (see Figure 3.2.10).

Graph G for $n = 6$

Graph G/e for $n = 6$

Figure 3.2.10: Tightness for $col_X(G/e) = 2 + col_X(G)$, for clarity we only drew the edges missing between A and B (dotted) except for a_0 and a_n in the first case, for which we show all of their edges.

The graph G/e is then the graph obtained from G by contracting the edge a_na_{n+1}, hence obtaining the graph $\mathrm{SF}_{n,n}$ without the edges a_ib_i for $0 \leq i \leq n$.
Let us prove that $col_A(G) = col_B(G) = 2n - 3$ if $n \geq 6$.
Indeed, Alice's strategy is to mark last a_n and before last a_0. By following this strategy, the last vertex she marks before marking a_{n+1} has at most all of its neighbors (except a_0 and a_n and maybe one in B) marked, hence a score of at most $1 + 2(n-2) = 2n - 3$. When Alice marks a_0 or a_n, they have at most $n - 1 + \lfloor n/2 \rfloor$ neighbors marked, hence a score of at most $\lfloor n/2 \rfloor + n$, and as $n \geq 6$, this implies the maximum score is at most $2n - 3$.

Bob can force this score by playing opposite to Alice, meaning that each time Alice plays a vertex a_i, he plays the vertex b_i that has no edge with a_i. If the last two vertices to be marked in A are not a_0 and a_n then the score of the last one is $2n$, hence Alice plays these two last. The last vertex to be marked, say a_i, before a_0 and a_n, has all of its neighbors in B marked but one: b_n. Hence its score is $2n-3$.

For the B-marking game it is the same strategy for Alice.

Now, for the graph G/e we only have to remark that a perfect matching is missing, hence by playing in B Bob ensures a score of $2n-1$ even when Alice starts: when she plays on a_i, he marks the matched one, b_i. ■

In particular, here, we have a lower and an upper bound that are both different from the starting game coloring number and these bounds are tight.
Moreover we can perceive the flexibility of sunflower graphs: even if their study for edge contraction is not developped, they verify the two bounds.

3.2.6 Union and cartesian product of graphs

Here we study separately the A- and B- marking games.

Theorem 3.52 *Let G and H be two graphs. Then:*

$$col_B(G \cup H) \quad = \quad \max(col_B(G), col_B(H))$$

Proof. For simplicity, we note $U = G \cup H$.
First of all, we remark that $G \subset U$ and $H \subset U$, thus $\max(col_B(G), col_B(H)) \leq col_B(U)$ by Corollary 3.50. Let us now prove $col_B(G \cup H) \leq \max(col_B(G), col_B(H))$. Without loss of generality we assume $col_B(G) \geq col_B(H)$. For this proof we are using the strategy of *following*: Alice plays always in the same graph as Bob. Assume Alice has a strategy for the B-marking game with score s on the graph G. As $col_B(G) \geq col_B(H)$, Alice has also a strategy for the B-marking game with score s on H. In this case, we need to prove Alice ensures a score s on the graph U for the B-marking game.
Bob starts playing. If he starts on G, then Alice plays in G by her strategy. If he starts on H, she plays on H by her strategy. She plays this way until playing on the same graph as *Bob* is no longer possible. The only case where she cannot play this way is when Bob marks the last vertex of G or H. In this case, she imagines a move of Bob in the other graph and plays accordingly (this is possible whenever there are still two vertices unmarked, if this is not the case no vertex has score $s+1$). By the imagination strategy, every time she marks a vertex, it has at most $s-1$ marked neighbors, hence all the vertices have score at most s.
We have just proved that if she has a strategy of score s for the B-marking game on G, then she has a strategy of score s for the B-marking game on $G \cup H$. Thus $col_B(G \cup H) = \max(col_B(G), col_B(H))$. ■

Now, for the marking game the expression of $col_A(G \cup H)$ is not so simple:

Theorem 3.53 *Let G and H be two graphs:*

$$col_A(G \cup H) \quad = \quad \min \left\{ \begin{array}{c} \max(col_A(G), col_B(H)) \\ \max(col_A(H), col_B(G)) \end{array} \right\}$$

Proof. Let U be the union graph. Without loss of generality, we can assume $col_A(G) \geq col_A(H)$. We can separate the proof in two cases. First of all, by Lemma 3.22 we have: $col_B(G) \geq col_A(H)$, hence we distinguish only the terms on the first maximum.

- if $col_A(G) = col_A(H)$ and $col_B(G) = col_B(H) = col_A(G) + 1$: $col_A(G \cup H) = col_A(G) + 1$.
  Assume Alice has a strategy for the A-marking game on the graph G with score s and one for the B-marking game on the graph H with score $s+1$. She starts by playing on G and each time Bob plays a vertex on $G \cup H$, Alice responds by her strategy of G with score s if Bob played on G or of H with score $s+1$ if he played on H.
  If she cannot respond by her strategy, it is because there is no unmarked vertex on that graph: she plays on the other graph. Notice that in this graph it is Bob's turn to play and Alice has a strategy with score s' ($s' = s$ or $s' = s+1$ depending on the graph), and it is as if Bob passes his turn. By

Lemma 3.23, Alice has a strategy with same score for the A-marking game. This way, she ensures s on G and $s+1$ on H, in total the score is at most $s+1$ on $G \cup H$.

In the other hand, if Bob has a strategy with score $s+1$ in H for the B-marking game, he has also a strategy with score $s+1$ in G for the B-marking game. His strategy is then to apply one of these strategies depending on where Alice played first: if Alice starts by playing on G, he applies his strategy on H, if she starts on H, he applies it on G. If she starts playing on G, he plays on H by his strategy. Each time she plays on H, he answers by his strategy and each time she plays on G, he plays randomly on G. Vice versa if she starts by playing on H. Either way, he obtains a score of $s+1$, hence proving that $col_A(G \cup H) = col_A(G) + 1 = col_B(G)$.

- Otherwise: $col_A(G \cup H) = col_A(G)$. In this case, without loss of generality, if $col_A(G) = col_A(H)$, we can assume $col_B(G) > col_B(H)$ (otherwise these values are all equal).

  By Lemma 3.22: $col_A(G) \leq col_A(G \cup H)$. As $col_B(H) \leq col_A(G)$ by the strategy used above, Alice can win with score $col_A(G)$ on $G \cup H$ by playing on G first and $col_A(G \cup H) = col_A(G)$ because Bob can decide to play by a strategy of score $col_A(G)$ on G.

If $col_A(H) \geq col_A(G)$ in the strategy we exchange H and G. In all the cases, we have the claimed equality. ■

It is interesting to see that, compared to the results we had before, here we have the exact value of the A- and the B-game coloring numbers, for any two graphs G and H. The only information we need is the values of the A- and B-game coloring numbers for each graph.

In 2009, Sia bounded above the coloring number of the cartesian product of two graphs.

Proposition 3.54 *[65] Let G and H be two graphs and denote by $\square$ the cartesian product of graphs. Then:*

$$col_g(G \square H) \leq col_g(\sqcup_{V(H)} G) + \Delta(H)$$

where $\sqcup_{V(H)} G$ is the union of each of the copies of G in the cartesian product.

Hence, we obtain almost directly:

Corollary 3.55 (Theorems 3.52 and 3.53 and [65]) *Let G and H be two graphs. Then:*

$$col_A(G \square H) \quad \leq \quad \min \left\{ \begin{array}{ccc} col_B(G) & + & \Delta(H) \\ col_B(H) & + & \Delta(G) \end{array} \right\}.$$

And this bound is tight.

Proof. Here we only need to give examples of tightness.
One class of graphs verifying this equality are the graphs $\mathrm{SF}_{n,k} \square K_l$ for $n \geq k \geq 1$, $l \geq 2$.
Indeed, let $G_{n,k,l}$ be such a graph for some n, k, l verifying the conditions. We give a strategy for Bob that ensures a score of $col_B(\mathrm{SF}_{n,k}) + l - 1 = n + k + l$.
Here we have l copies of $\mathrm{SF}_{n,k}$, hence for the copy i we are denoting the vertices $a_{0,i}, \ldots, a_{n-1,i}$ and $b_{0,i}, \ldots, b_{n-1,j}$. In particular, the set of A-vertices is $\{a_{j,i} | 0 \leq i \leq l-1, 0 \leq j \leq n-1\}$ and the set of B-vertices is the same for the $b_{j,i}$.
Bob's strategy is to play only on vertices $b_{j,i}$ under the conditions that each time he plays, there is $0 \leq j' \leq n-1$ such that $a_{j',i}$ is unmarked. In other words, it has an unmarked A-neighbor. This means that each time Alice marks the vertices $a_{0,i}, \ldots, a_{n-1,i}$ there is some integer $0 \leq j' \leq n-1$ such that $b_{j',i}$ is unmarked and has all of its A-neighbors marked. Thus, in the last copy of $\mathrm{SF}_{n,k}$ to be completed, say the copy i, Bob played at least one extra turn, ensuring that all $b_{j,i}$ are marked before all the $a_{j,i}$, hence the last $a_{j,i}$ to be marked has all of its A-neighbors marked $(n+l-1)$ and all of its B-neighbors marked

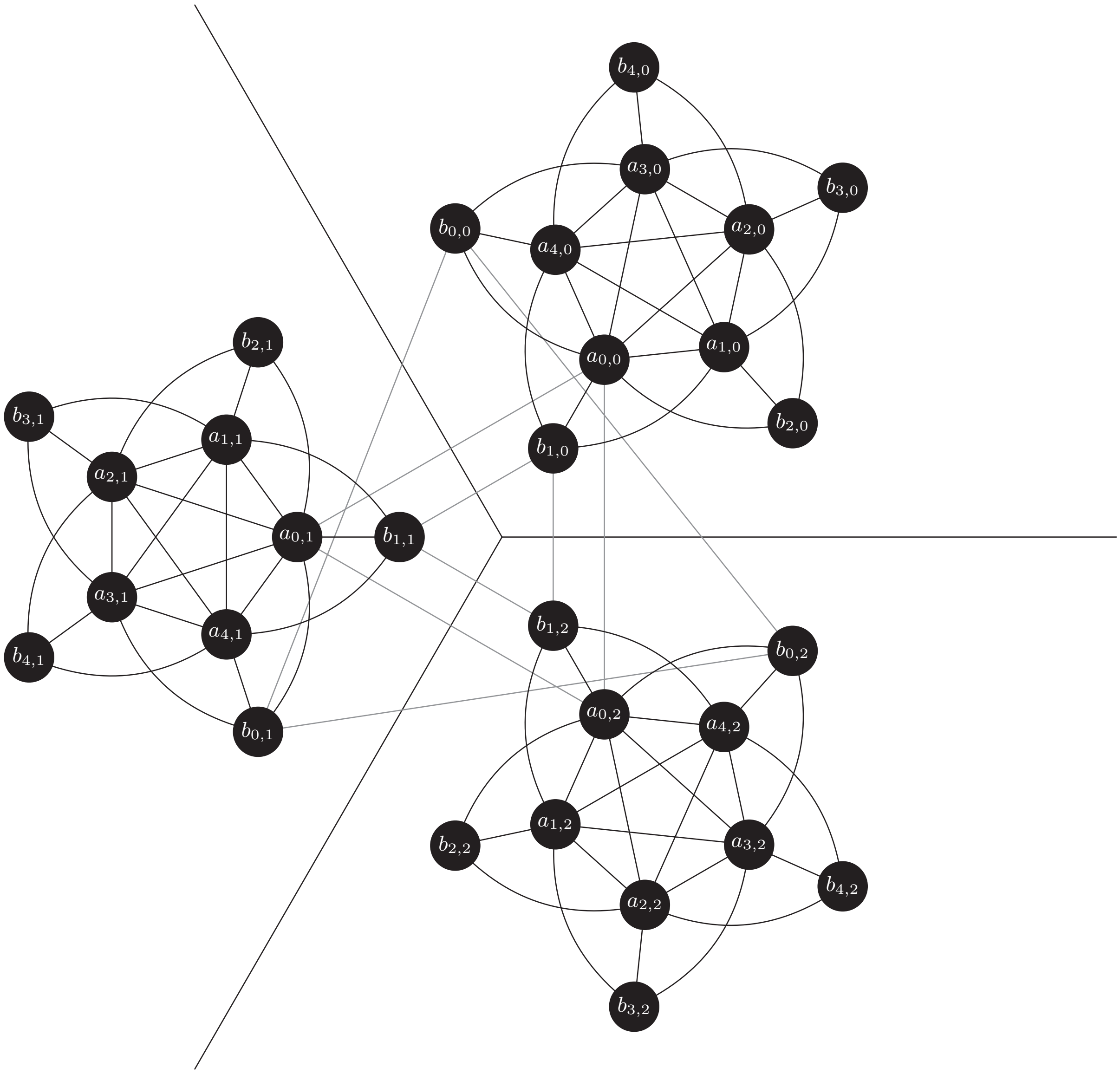


Figure 3.2.11: Example of a graph $G_{n,k,l} = \mathrm{SF}_{n,k} \square K_l$ for $n = 5, k = 3, l = 3$.
Here not all the graphs K_l are drawn. The black lines are there to emulate the correspondance between the graphs $\mathrm{SF}_{n,k}$.

(k), ensuring a score of $n + l - 1 + k + 1 = n + l + k$.
For an example of these graphs see Figure 3.2.11 ∎

We have seen good bounds for all the operators we considered along this section. It is interesting to see that the sunflower graphs allows to show the lower and upper bounds. But the study is not at all over. Some of the very interesting questions that were raised during this study are explained and developped in the following paragraph.

3.2.7 Open questions for the understanding of the marking game

Here we introduced a very practical tool to bound below the game coloring numbers: the sets A_s and B_s for s an integer. We have seen that just by looking at the difference of cardinalities of these sets we can estimate a good lower bound for their game parameters.

In fact, it would be interesting to characterize some graphs with some particularities regarding these sets. We suggest to look for the graphs such that:

Open question 3.56 *Which graphs are such that* $col_A(G) = \Delta(G) + 1$*?*

Open question 3.57 *Which graphs are such that* $|A_{col_A(G)}| = |B_{col_A(G)}|$*?*

Finding the answer to these questions does not seem easy, since there must be a lot of different families of graphs verifying either of these two conditions. Just finding necessary or sufficient conditions to verify these equalities could already open the way to more precise studies.

For the vertex deletion something interested about tightness is:

Open question 3.58 *Which are the graphs* G *having a vertex* v *such that* $col_A(G) = col_B(G - \{v\}) + 1$*?*

Open question 3.59 *Which are the graphs* G *such that* $\forall v \in V(G)$, $col_A(G - \{v\}) = col_A(G) - 1$*?*

Open question 3.60 *Which are the graphs* G *having a vertex* $v \in A_{col_A(G)}$ *such that* $col_A(G - \{v\}) = col_A(G) - 2$*?*

As well, and maybe an easier question to study, is about edge tightness:

Open question 3.61 *which are the graphs* G *and edges* e *such that* $col_A(G \setminus \{e\}) = col_A(G) - 1$*?*

Note that we defined the vertex-criticality and the edge-one. For the edge contraction we have not even given any results about the sunflowers graphs. A natural question that raises up for the edge contraction is about their existence:

Open question 3.62 *are there graphs* edge-contracting-criticals*?*

As well, we studied some binary operators: the union and the cartesian graphs. These two operators come often into play to defining some classes of graphs. For instance, the hamming spaces can be seen as the cartesian product between an edge and itself n times. As well, even if we have not studied the join operator, the cograph class could also be interesting.

Open question 3.63 *By recursion using the results on the cartesian product, is the A-game coloring number of hamming graphs computable?*

Open question 3.64 *Can we bound above and/or below the A-game coloring number of* $G \vee H$ *knowing the game coloring numbers of* G *and* H*?*

Open question 3.65 *Can we deduce the A-game coloring number for the cograph class of graphs?*

Of course, all of these questions are also pertinent for the B-marking game.

Now, if we look a little into the sunflower graphs we remark very nice properties: the vertices of A have all the same degree, as well for the vertices of B. Moreover, if we remove the vertices of B we obtain a regular graph. We obtain, most of the time (when $n \neq k$), just $n + k$ as their game coloring number. These particularities are not only found on this class of graphs, maybe there are other that give the same results about tightness. In particular, the way to connect the vertices between A and B was a choice we made.

Open question 3.66 *By connecting the vertices between the vertices of* A *and* B *in such a way that we obtain the same degree for all the vertices of* A*, the same degree for all the vertices of* B *and such that the graph induced by the vertices of* A *is also regular, can we obtain a graph verifying the same tightnesses as the sunflower graphs?*

Open question 3.67 *Is the regularity a necessary condition for any of these tightness results?*

Even if we are under the impression of openning a Pandora's box, our results help understand the relationship between graphs when studying the marking game as well as find new ways of determining the game coloring numbers of families of graphs that are constructed recursively.

3.3 Precise case of edge-games

Here we study the *edge*-marking game, an edge version of the marking game studied in the last section. In 1999, Lam, Shiu and Xu where the first to introduce this game:

EDGE COLORING GAME: The *edge coloring game* is played by two players, Alice and Bob with a set of colors C on a graph G. They alternate turns to properly color an uncolored edge. Normally Alice starts. If at the end all the graph is properly colored, then Alice wins. Otherwise, *i.e.*~there is an uncolored edge that cannot be properly colored, Bob wins. The *game chromatic index* of G, noted $\chi'_g(G)$, is the minimum number of colors such that Alice has a winning strategy, meaning that no matters how Bob plays, at the end the graph is properly colored.

As well, we can define the *edge marking game*. It was first introduced by Cai and Zhu in [16].

EDGE MARKING GAME: players alternate turns to mark edges. At the beginning all edges have score 0 and each time a player marks an edge e, its score is 1 plus the number of marked edges incident to e. The score of the graph is the maximum score obtained along the game.
The *game coloring index* of G, noted $col'_g(G)$ is the minimum number k such that no matters how Bob plays, Alice has a strategy ensuring a score of at most k.

It is clear that $\chi'_g(G) \leq col'_g(G)$.

In their paper, Lam *et al.* expose the trivial bounds of the game chromatic index:

Theorem 3.68 *[58] Let G be a graph. Then:* $\chi'(G) \leq \chi'_g(G) \leq 2\Delta - 1$.

Indeed, these are called trivial since it suffices to see that at the end of the game, if Alice won, the graph is properly edge-colored, hence there are more than $\chi'(G)$ colors. As well, if they play with $2\Delta - 1$ colors, when an edge e is colored it has at most $2\Delta - 2$ incident colored edges, hence there is always a color available for e.

They also adapted the activation strategy introduced by Kierstead and co. to study the game chromatic index of trees.

Theorem 3.69 *[58] Let T be a tree. Then* $\chi'_g(T) \leq \Delta(T) + 2$.

Proof. Here we are doing the proof using the marking game.
As always, Alice keeps track of the activated edges that are unmarked. Alice starts by marking an edge of T, that we are calling the root and denoting r. Each time Bob marks an edge e, Alice chooses her next edge as follows. There is a unique path starting from e and ending with r, say $e_0 \dots e_k$ with $e_0 = e$ and $e_k = r$. Starting at e_1 and following the path, Alice activates the edges. When this is not possible, say for an edge e_i, is because the edge is already activated or marked. She then does:

1. if e_i is activated but unmarked, Alice marks it,
2. if e_i is marked:
   (a) if $i \neq 1$ then Alice marks e_{i-1},
   (b) otherwise Alice marks an edge f such that in the path $f \dots r$ the second edge is already marked.

For more details see Figure 3.3.1.

By following this technic, when an edge is marked it has at most $\Delta - 1$ edges marked (on the side of the root) and at most two edges marked from the other side. Hence $col'_g(T) \leq \Delta(T)+2$, so $\chi'_g(G) \leq \Delta(T)+2$.
■

In particular, the activation strategy seems to be straight forward applicable to the case of the edges.

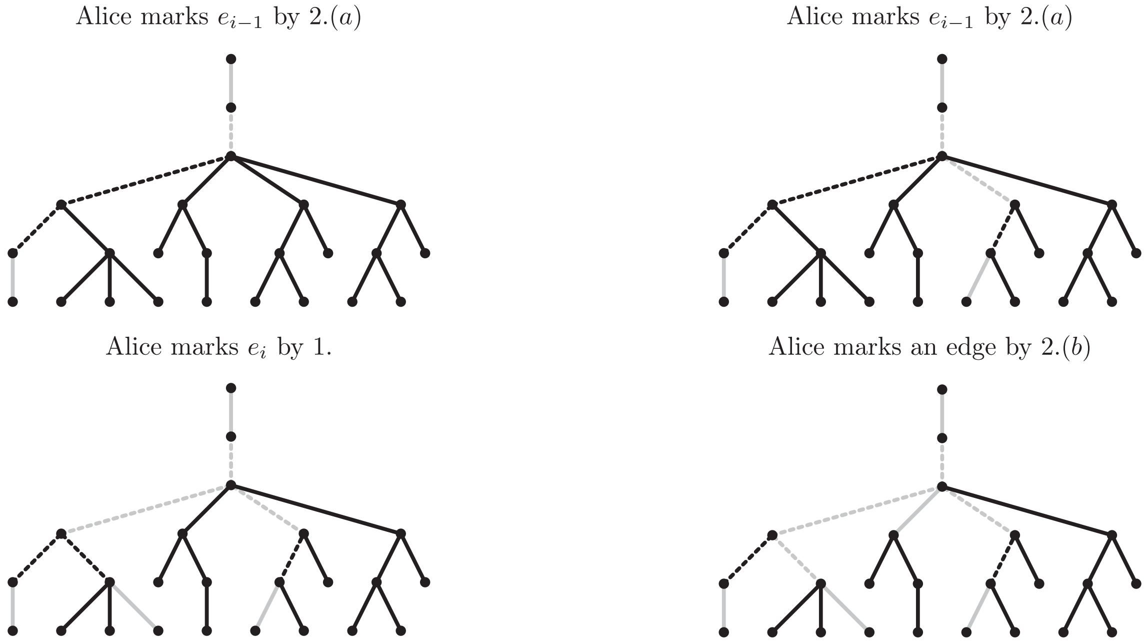


Figure 3.3.1: Example of activation strategy on edges. Here the black are unmarked, the gray edges are marked and the dotted edges are activated.
In this example, the last edge at depth 3 will have all $\Delta - 1$ edges marked on its side of the root.

Another result shown by Lam Shiu and Xu in their introductory paper is about the game chromatic index of wheels. A *wheel* of size n, denoted W_n, is the cycle of size n with a universal vertex in the center (*i.e.*~the central vertex sees all the vertices of the cycle). They give the exact value of $\chi'_g(W_n)$.

Theorem 3.70 *[58] Let $n \geq 3$ be an integer. Then:*

$$\chi'_g(W_n) = \begin{matrix} 5 & \text{for } n = 3 \\ n+1 & \text{otherwise} \end{matrix} .$$

This proof is done by showing that Alice can always color edges of the center with $n+1$ colors, since the others edges can always be colorable with $n+1$ colors (if $n \geq 4$).

They finish the paper by raising an interesting question in the hope of a generalized result to all graphs:

Open question 3.71 *Is there a constant $c \geq 2$ such that for all graph G: $\chi'_g(G) \leq \Delta + c$? If true, is $c = 2$ enough?*

In fact, in 2008 Beveridge, Bohman, Frieze and Pikhurko prove this conjecture not to be true.

Theorem 3.72 *[12] For large enough d, there is a graph G such that $\Delta(G) \leq d$ and $\chi'_g(G) \geq 1.008d$.*

They prove this by defining an integer $n = f(d)$ and giving a graph G of order n that satisfies the inequality. In particular, this inequality also holds for col'_g.

They wonder then about an upper bound linear on Δ:

Theorem 3.73 *[12] For $\mu > 0$, there is $\varepsilon > 0$ such that for all G, $\Delta(G) \geq (\mu + 1/2)|V|$, $\chi'_g(G) \leq (2 - \varepsilon)\Delta(G)$.*

Hence the open question is now about this upper bound:

Open question 3.74 *$\exists \varepsilon > 0$, $\forall G$, $\chi'_g(G) \leq (2 - \varepsilon)\Delta(G)$?*

If we go back to the proper study of the game, in 2001, Cai and Zhu studied the game chromatic index of k-degenerate graphs. This gave the first upper bounds for the classes of planar graphs, bounded arboricity graphs and some specific trees. Most importantly, they showed this result by using an improved activation strategy. We dont detail it here but we give the main results of that paper.

Theorem 3.75 *[16] Let G be a k-degenerate graph. Then $col'_g(G) \leq \Delta(G) + 3k - 1$.*

Corollary 3.76 *[16] Thus we obtain the following upper bounds:*

- *if G is planar: $col'_g(G) \leq \Delta(G) + 14$,*
- *if G is of arboricity a: $col'_g(G) \leq \Delta(G) + 6a - 4$,*
- *if G is a forest: $col'_g(G) \leq \Delta(G) + 2$,*
- *if G is a forest with $\Delta(G) = 3$: $col'_g(G) \leq 4$.*

Remark that we knew already about the forests. What we did not know is that in the specific case of forests of maximum degree 3 we had a better upper bound. In fact for almost all forests we have this improvement. Indeed, in 2005 Erdös, Faigle, Hochst and Kern gave the same results for $\Delta \geq 6$ and then, in 2006, Stephan Dominique Andres improved the upper bound for forests with maximum degree $\Delta \geq 5$. Erdös *et al.* gave a proof by "permitted" substructures. Each time Alice plays, the graph can be decomposed in permitted structures, and each time Bob plays there is at most one unpermitted structure that Alice can restore at her next turn. Andres generalizes this strategy to the case $\Delta = 5$ and he points out that this strategy, as he presents it, does not work for the case $\Delta = 4$.

Theorem 3.77 *[29] Let F be a forest with $\Delta(F) \geq 6$. Then: $col'_g(F) \leq \Delta(F) + 1$. Moreover, for $\Delta \geq 2$, there is a forest F with $\Delta(F) = \Delta$ such that $\chi'_g(F) = \Delta$.*

And Andres's result is the following:

Theorem 3.78 *[3] Let F be a forest with $\Delta(F) \geq 5$. Then: $col'_g(F) \leq \Delta(F) + 1$.*

The question is now, what happens for $\Delta = 4$?

Open question 3.79 *Is there a forest F with $\Delta = 4$ such that $col'_g(F) = \Delta + 2$?*

In 2008, Yang and Kierstead improved the bounds for planar graphs, and graphs of bounded arboricity. As well they gave bounds for outerplanar graphs. They did this by studying a variant of the marking game: for a, b to integers, at each turn they allow Alice to play a vertices and Bob to play b. They call this game the (a, b)-marking game and denote $(a, b)gcol(G)$ the (a, b)-coloring number of a graph G. Here we are just giving the results for $a = b = 1$. They also define a graph parameter: $\Delta^* = \min_{\overrightarrow{G} \in \mathcal{O}(G)} \Delta^+(\overrightarrow{G})$ where $\mathcal{O}(G)$ is the set of orientations of G and $\Delta^+(\overrightarrow{G})$ is the maximum outdegree of $\overrightarrow{G}$.

Theorem 3.80 *[57] Let G be a graph of maximum degree Δ. Then: $col'_g(G) \leq \Delta + 3\Delta^*(G) - 1$.*

The proof introduces a new strategy: the *limited harmonious strategy* in which Alice makes a sort of discharging method function of the edge marked by Bob. Each time Bob marks an edge e, she distributes shares to each of the outneighbors of e by following some rules (here we consider the orientation of G, $\overrightarrow{G}$ that gives the $\Delta^* = \Delta^+(\overrightarrow{G})$). She then takes the edge that, after some given number of distributions, received the least number of shares.
This result gives more specific upper bounds for some classes:

Corollary 3.81 *[57] We obtain then*

- *if G is planar: $col'_g(G) \leq \Delta + 8$,*
- *if G is outerplanar: $col'_g(G) \leq \Delta + 5$,*

- *if G is of arboricity a:* $col'_g(G) \leq \Delta + 3a - 1$,
- *and if* $\Delta^*(G) = -1$*:* $col'_g(G) \leq \Delta + 2$.

Independently of this results, Bartnicki and Grytczuk also obtained the same upper bound for graphs of arboricity a for the edge coloring game, this time using the activation stragegy.

If we recall what was done with the vertex version of these games, we see that there are tree decompositions that help the study of different classes of graphs. Their study is in fact simplifyed by the use of the activation strategy (on kind of tree structures) and the result about edge-decomposition shown by Zhu in [75]: Theorem 3.5.
In the next part we give a generalized activation strategy on very specific decompositions of graphs and improve some of the already known bounds.

3.4 Coloring game on F^+-decomposable graphs

This work was done in collaboration with Clément Charpentier and Brice Effantin and published in 2017 in the journal Discrete Applied Mathematics.

We defined a new way of decomposing graphs edge-wise.

Definition 3.82 *A graph G is $F^+(a, \{d_1, \ldots, d_k\}, d)$-decomposable if its edge-set can be partitioned into **a** forests of unbounded degree, **k** forests of maximum degrees at most $d_1, \ldots, d_k$ respectively and a graph of maximum degree **d**.*

An example of different decompositions can be found in Figure 3.4.1.
This is a generalisation of the (a, d)- and $F(a, d)$-decomposition where we decompose the graph in **a** forests and a graph of maximum degree **d** (this graph is a forest if it is a $F(a, d)$-decomposition). Indeed the decomposition (a, d) is the same as a decomposition $F^+(a, \{\}, d)$ and the $F(a, d)$ is a decomposition $F^+(a, \{d\}, 0)$ (as well as a $F^+(a, \{\}, d)$).

We give a strategy for Alice on a graph G with a given $F^+(a, \{d_1, \ldots, d_k\}, d)$-decomposition ($d_1 \geq \cdots \geq d_k$) and then see what is the score she can ensure by it.

3.4.1 Definitions and notations

As the graph decomposes in **a**+**k** forests we can give an orientation to the forests. For each $1 \leq i \leq k$, let A_i be the set of edges of the forest of bounded degree d_i, and let $A_- = A_1 \cup \cdots \cup A_k$. As well, let A_∞ be the edges of the **a** forests of unbounded degree and $A = A_\infty \cup A_-$. Let D be the remaining edges (those of the graph of maximum degree **d**).
We give an orientation to every forest such that each one of them has outdegree at most 1. This gives an orientation of A with maximum outdegree at most **a**+**k**.

Definition 3.83 *For every arc $\overrightarrow{uv}$ of $\overrightarrow{A}$ we define the following sets:*

- $F(\overrightarrow{uv}) = \{\overrightarrow{vx} \in \overrightarrow{A}\}$*, the set of* fathers *of* $\overrightarrow{uv}$*;*
- $S(\overrightarrow{uv}) = \{\overrightarrow{xu} \in \overrightarrow{A}\}$*, the set of* sons *of* $\overrightarrow{uv}$*;*
- $B(\overrightarrow{uv}) = \{\overrightarrow{xv} \in \overrightarrow{A}\}$*, the set of* brothers *of* $\overrightarrow{uv}$*;*
- $P(\overrightarrow{uv}) = \{\overrightarrow{ux} \in \overrightarrow{A}; u \neq v\}$*, the set of* partners *of* $\overrightarrow{uv}$.

*For each I of these sets, we denote by $I_\infty(\overrightarrow{uv}) = I(\overrightarrow{uv}) \cap A_\infty$, and by $I_-(\overrightarrow{uv}) = I(\overrightarrow{uv}) \cap A_-$.
We also define:*

A graph $F^+(1, \{1, 3, 1, 1\}, 2)$, one $F^+(1, \{1, 1, 1, 2, 1\}, 5)$ and one $F^+(1, \{2, 1, 1, 1\}, 2)$.

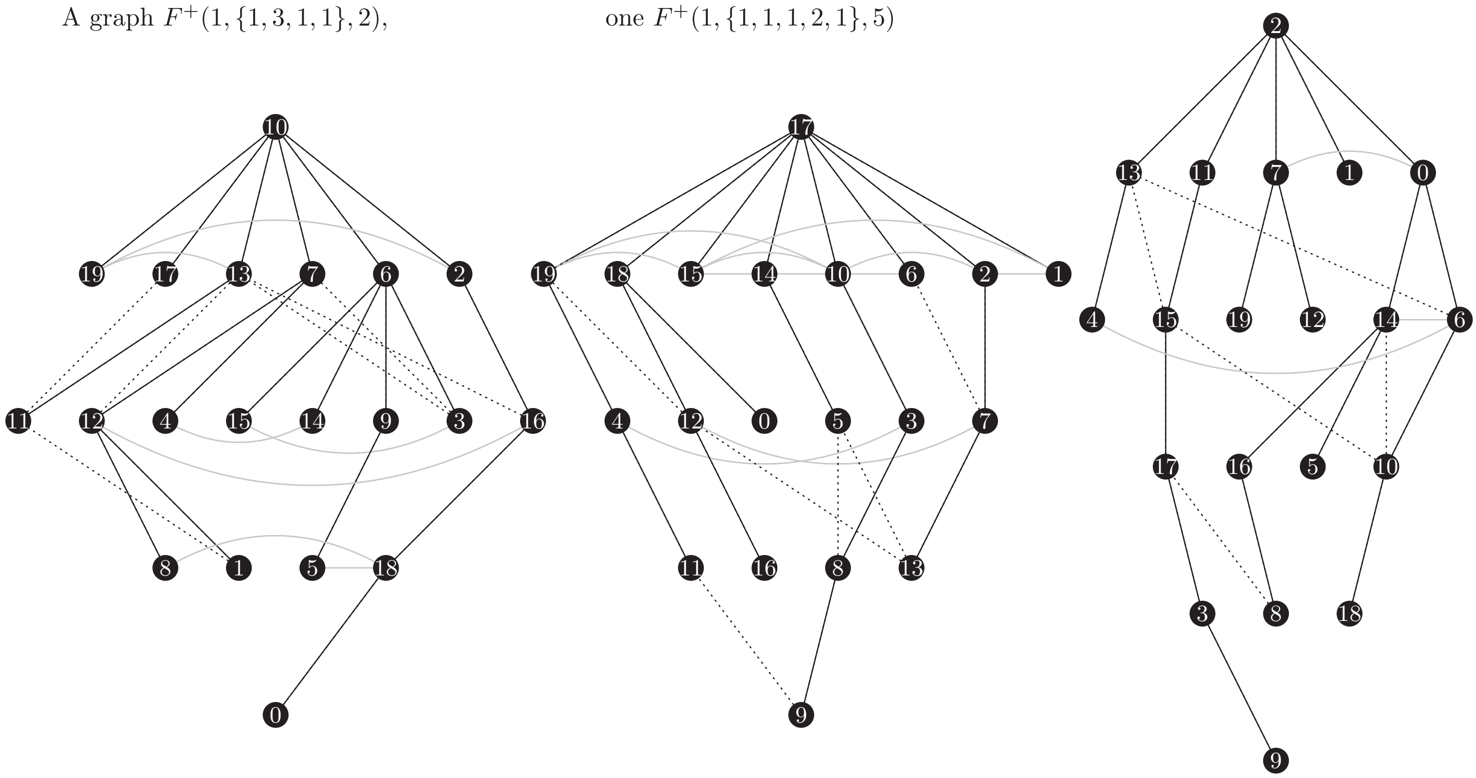


Figure 3.4.1: Example of decompositions: forests are marked with black edges (plain for unbounded ones, dotted for bounded ones) and the graph of bounded maximum degree is in gray. Each forest of bounded degree is a star graphs, for the first one they are centered in $\{17, 13, 7, 11\}$ respectively, the second in $\{19, 6, 12, 5, 11\}$ and the last in $\{13, 15, 14, 17\}$.

- $U(\overrightarrow{uv}) = \{vx \in F\}$, *the set of* uncles *of* $\overrightarrow{uv}$*;*
- $C(\overrightarrow{uv}) = \{ux \in D\}$, *the set of* cousins *of* $\overrightarrow{uv}$*.*

As well, for the edges of D we can define analogous parenthoods:

Definition 3.84 *For every edge $uv \in D$, we define the following sets:*

- $F(uv) = \{\overrightarrow{ux} \in \overrightarrow{A}\} \cup \{\overrightarrow{vx} \in \overrightarrow{A}\}$, *the set of* fathers *of uv;*
- $S(uv) = \{\overrightarrow{xu} \in \overrightarrow{A}\} \cup \{\overrightarrow{xv} \in \overrightarrow{A}\}$, *the set of* sons *of uv;*
- *and* $B(uv) = \{ux \in D, x \neq v\} \cup \{vx \in D, x \neq u\}$, *the set of* brothers *of uv.*

We remark directly that for each arc of $\overrightarrow{A}$ the sum of fathers, brothers and uncles is at most $\Delta - 1$. An illustration of these two definitions is given in Figure 3.4.2.

Now we can give a strategy for Alice depending on these decompositions and new definitions.

3.4.2 Alice's strategy

We give an activation strategy for Alice on graphs $F^+(a, \{d_1, \ldots, d_k\}, d)$-decomposables. Hence, all along the game, Alice keeps track of active edges/arcs. For each set E of edges we define by E_a and E_m its subset of activated and marked edges, respectively. The same notation holds for arcs.

Definition 3.85 *An arc or an edge is* neutral *if it is inactive and has no unmarked father or uncle.*

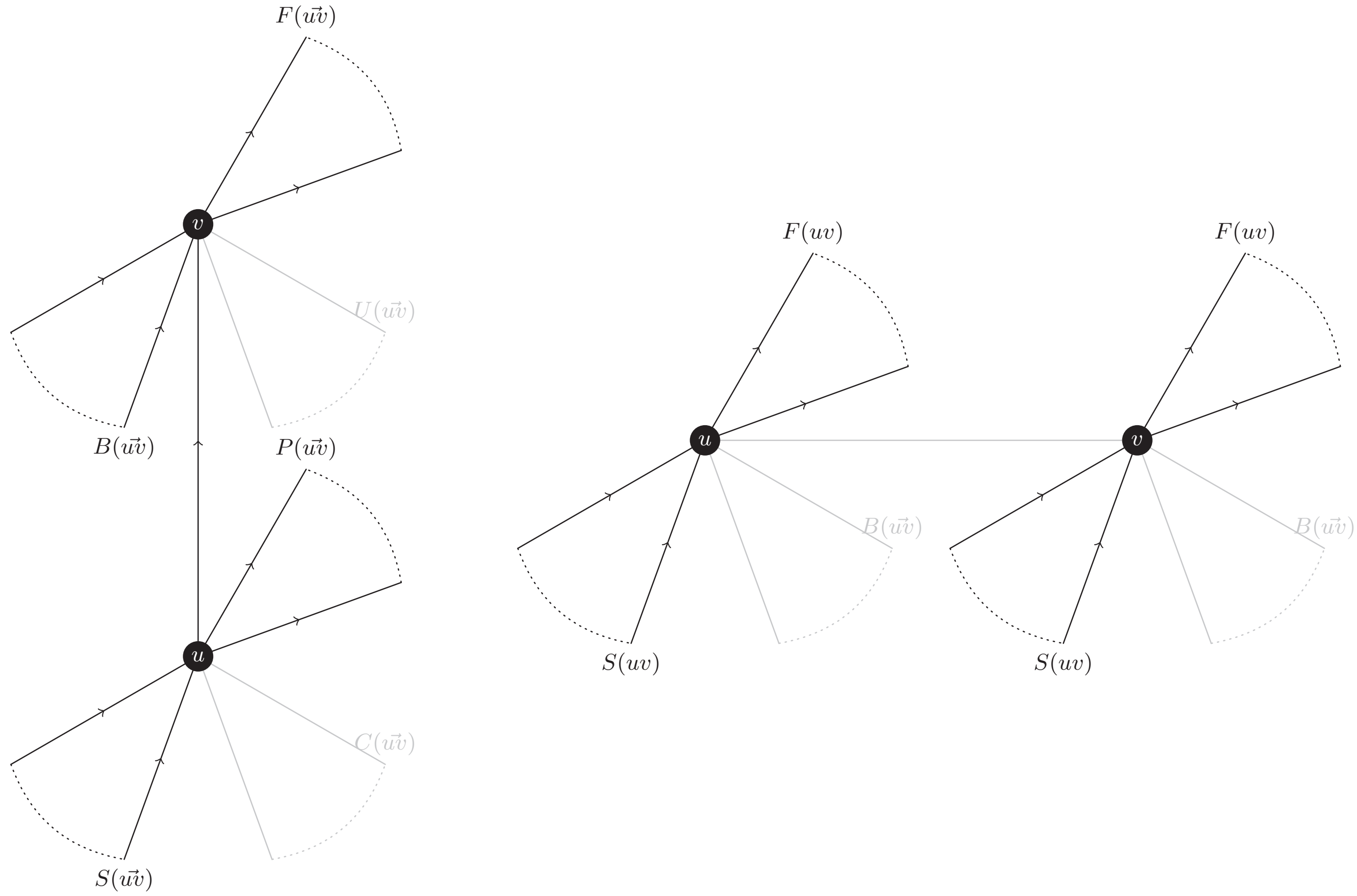


Figure 3.4.2: Illustration of the orientation and the notations.

When there is no neutral arc or edge in the graph, each arc or edge has at least one unmarked father or uncle, which implies there is at least one uncolored cycle such that, for each arc or edge, its successor is one of its fathers, and if it has none, its successor is one of its uncles. If all the arcs and edges of the graph are inactive, a *neutral move* consists for Alice to pick one of these cycles, to activate all its arcs and edges and to mark one of them. If there is no such cycle, she marks an edge or arc without a father.

Alice's strategy is as follows: she starts by doing a neutral move. Each time Bob marks an arc or an edge, say $e_1/\overrightarrow{e_1}$, Alice selects her next move by following the steps illustrated in Figure 3.4.3 and described below.
To simplify the notations, we are not drawing the arrows of arcs in the description of the strategy.

Start: Assume the edge/arc she is considering is e_i then:

1. if e_i is inactive (or if e_1 was inactive before Bob marked it), Alice activates it and:
   (a) if e_i has an unmarked father in $\overrightarrow{A}_\infty$, f, then $e_{i+1} = f$, and she goes back to **Start**;
   (b) if j is the smallest index for which e_i has an unmarked father on $\overrightarrow{A}_j$, f, then $e_{i+1} = f$ and she goes back to **Start**;
   (c) if j is the smallest index for which e_i has an unmarked brother on $\overrightarrow{A}_j$, then she marks it;
   (d) if e_i has an unmarked uncle, then Alice marks it;
2. if e_i is active:

(a) if it is marked, then Alice does a neutral move;

(b) otherwise, Alice marks it.

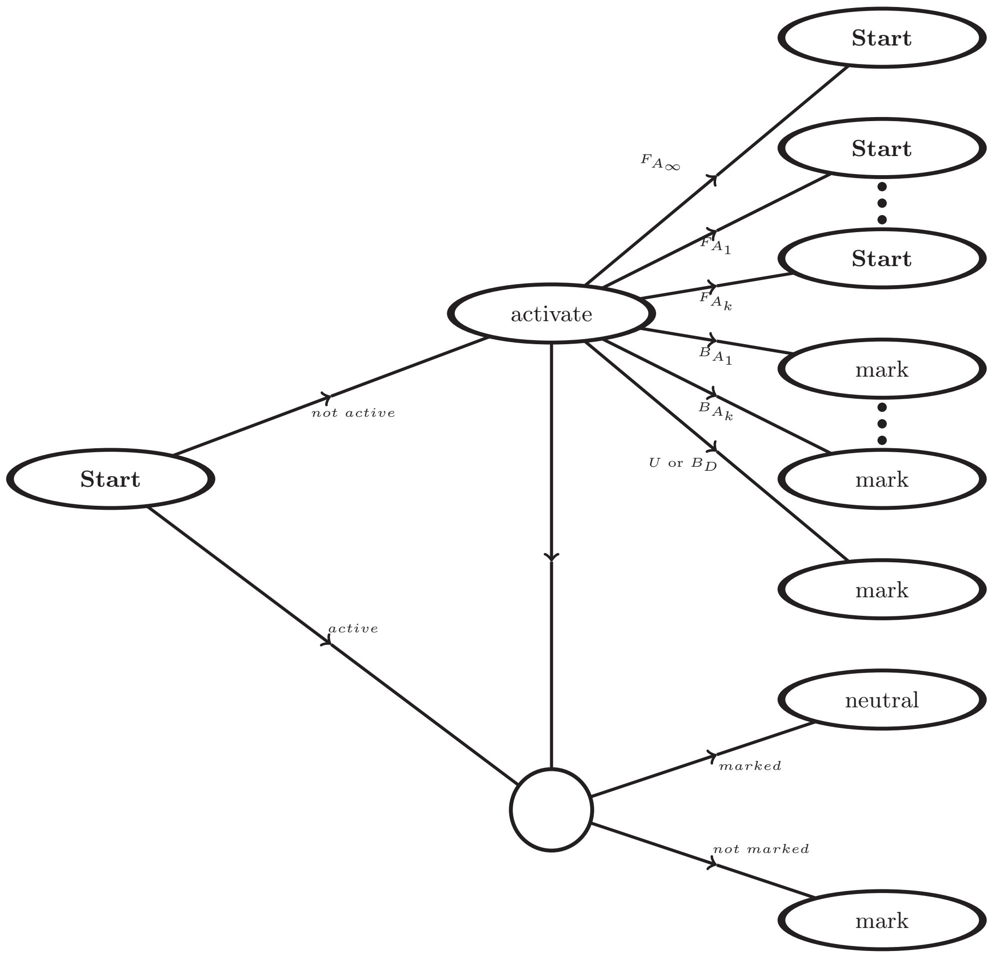


Figure 3.4.3: Sketch of the selection recursion.

In fact, with this strategy as it is, our results were not interesting, hence we decided to mix up the stages of selection of vertices by moving around the stages $2(b)$, $2(c)$ and $2(d)$. The $i-j-k$-strategy, for $i,j,k \in \{b,c,d\}$, is the strategy were those stages are ordered $2(i)$ first, $2(j)$ second and $2(k)$ third. We say $i > j$ if step $2(i)$ comes before $2(j)$, as well $\gamma_{i>j} = 1$ if $i > j$ and 0 otherwise. We introduce the maximum degree of the bounded forests as $S_k = \sum_{\ell=1}^{k} d_\ell$ and we define three values depending on the order of the three exchangeable steps:

- $\mu_1 = 2a + (1 - \gamma_{c>b})2k + \gamma_{d>b}d + \gamma_{c>b}2S_k$,
- $\mu_2 = 4a + (\gamma_{b>c} - \gamma_{c>b})2k + (\gamma_{d>b} + \gamma_{d>c})d + (A + \gamma_{c>b})2S_k - 2$,
- $\mu_3 = 2a + (\gamma_{b>d} - \gamma_{c>d})2k + \gamma_{c>d}2S_k + d$.

Lemma 3.86 *In every game and for a given strategy,*

1. *when an arc $\overrightarrow{e} \in \overrightarrow{A}_\infty$ is marked, if has at most $2a$ sons already marked;*
2. *when an arc $\overrightarrow{e} \in \overrightarrow{A}_-$ is marked, the number of its already marked sons is at most μ_1;*
3. *when an arc $\overrightarrow{e} \in \overrightarrow{A}_-$ is marked, the number of its sons and brothers already marked is at most μ_2;*

4. *when an edge $e = uv \in D$ is marked, the number of marked sons adjacent to u is at most μ_3, moreover the total number of already marked sons of e is at most $2\mu_3 - 1$.*

The proof of this lemma is a quite long case study. Interested readers can check the proof in Appendix C. As well, the proof of the following theorem will be given in the appendix.

Theorem 3.87 *For any $F^+(a, \{d_1, \ldots, d_k\}, d)$-decomposable graph G,*

$$col'_g(G) \leq \max \left\{ \begin{array}{c} \Delta + 3a + k + d - 1 \\ \min\{\Delta + \mu_1 + a + k + d - 1, \quad \mu_2 + 2a + 2k + 2d\} \\ \min\{\Delta + \mu_3 + a + k + d - 1, \quad 2\mu_3 + 2a + 2k + 2d - 2\} \end{array} \right\}$$

This theorem gives Table 3.4.1, depending on the strategy used.

Table 3.4.1: Upper bounds of col'_g for a $F^+(a, \{d_1, \ldots, d_k\}, d)$-decomposable graph.

Strategy	A_∞	A_-	
$b-c-d$	$\Delta + 3a + k + d - 1$	$\Delta + 3a + 3k + d - 1$	$6a + 4k + 2d + 2S_k - 2$
$b-d-c$		$\Delta + 3a + 3k + d - 1$	$6a + 4k + 3d + 2S_k - 2$
$c-b-d$		$\Delta + 3a + k + d + 2S_k - 1$	$6a + 2d + 4S_k - 2$
$c-d-b$		$\Delta + 3a + k + 2d + 2S_k - 1$	$6a + 3d + 4S_k - 2$
$d-b-c$		$\Delta + 3a + 3k + 2d - 1$	$6a + 4k + 4d + 2S_k - 2$
$d-c-b$		$\Delta + 3a + k + 2d + 2S_k - 1$	$6a + 4d + 4S_k - 2$

Strategy	D	
$b-c-d$	$\Delta + 3a + k + 2d + 2S_k - 1$	$6a + 2k + 4d + 4S_k - 2$
$b-d-c$	$\Delta + 3a + 3k + 2d - 1$	$6a + 6k + 4d - 2$
$c-b-d$	$\Delta + 3a + k + 2d + 2S_k - 1$	$6a + 2k + 4d + 4S_k - 2$
$c-d-b$	$\Delta + 3a - k + 2d + 2S_k - 1$	$6a - 2k + 4d + 4S_k - 2$
$d-b-c$	$\Delta + 3a + k + 2d - 1$	$6a + 2k + 4d - 2$
$d-c-b$	$\Delta + 3a + k + 2d - 1$	$6a + 2k + 4d - 2$

Now we only have to have a decomposition to compute the best upper bound, hence the best strategy for Alice. In the next section we exhibit the upper bounds for some particular decompositions.

3.4.3 Some particular graphs

Let us consider the particular case of (a, d)- and $F(a, d)$-decomposable graphs.

Corollary 3.88 (Theorem 3.87) *Let G be a graph:*

- *if G is (a, d)-decomposable:*

$$col'_g(G) \leq \begin{cases} \Delta + 3a + 2d - 1 & \text{if } \Delta \leq 3a + 2d - 1, \\ 6a + 4d - 2 & \text{if } 3a + 2d - 1 \leq \Delta \leq 3a + 2d - 1, \\ \Delta + 3a + d - 1 & \text{otherwise;} \end{cases}$$

- *if G is $F(a, 1)$-decomposable:*

$$col'_g(G) \leq \begin{cases} \Delta + 3a + 2 & \text{if } \Delta \leq 3a, \\ 6a + 2 & \text{if} 3a \leq \Delta \leq 3a + 2, \\ \Delta + 3a & \text{otherwise;} \end{cases}$$

- *if G is $F(a, d_1)$-decomposable and $d_1 > 1$:*

$$col'_g(G) \leq \begin{cases} \Delta + 3a + 2 & \text{if } \Delta \leq 3a + 2d_1, \\ 6a + 2d_1 + 2 & \text{if } 3a + 2d_1 \leq \Delta \leq 3a + 2d_1 + 2, \\ \Delta + 3a & \text{otherwise.} \end{cases}$$

Table 3.4.2: Upper bounds of col'_g for (a,d)-decomposable graphs.

Strategy	A_∞	A_-		D	
$b-c-d$	$\Delta+3a+d-1$	$\Delta+3a+d-1$	$6a+2d-2$	$\Delta+3a+2d-1$	$6a+4d-2$
$b-d-c$			$6a+3d-2$		
$c-b-d$			$6a+2d-2$		
$c-d-b$		$\Delta+3a+2d-1$	$6a+3d-2$		
$d-b-c$			$6a+4d-2$		
$d-c-b$					

Table 3.4.3: Upper bounds of col'_g for $F(a,d_1)$-decomposable graphs.

Strategy	A_∞	A_-		D	
$b-c-d$	$\Delta+3a$	$\Delta+3a+2$	$6a+2d_1+2$	$\Delta+3a+2d_1$	$6a+4d_1$
$b-d-c$				$\Delta+3a+2$	$6a+4$
$c-b-d$		$\Delta+3a+2d_1$	$6a+4d_1-2$	$\Delta+3a+2d_1$	$6a+4d_1$
$c-d-b$				$\Delta+3a+2d_1-2$	$6a+4d_1-4$
$d-b-c$		$\Delta+3a+2d_1$	$6a+2d_1+2$	$\Delta+3a$	$6a$
$d-c-b$		$\Delta+3a+2$	$6a+4d_1-2$		

It all comes from the Tables 3.4.2 and 3.4.3 obtained by replacing with the real values of $\mathbf{k}$, $\mathbf{S_k}$ and $\mathbf{d}$. Hence, if we look at the graph of Figure 3.4.4, we can look at it as a $F^+(1,\{4\},4)$-, $F^+(0,\{15,4\},4)$-, or $F^+(1,\{\},6)$-decomposable graph. Where the decompositions are done as follow: the plain edges are a forest of maximum degree 15, the dotted edges is a forest of maximum degree 4 and the gray edges is a graph of maximum degree 4; if we take the dotted and gray vertices at the same time we obtain a graph of maximum degree 6 (degree of the vertex 8). Hence, By applying the formulas with $\Delta = 15$ we obtain 26 for the two firsts decompositions and 28 for the last one: the knowledge of the maximum degree of each subgraph can give better bounds.
Remark that the difference between the trivial upper bound and the bounds given are 3 and 1 respectively. Fortunately, there are some graph decompositions that help estimate these upper bounds without doing much computations. For instance, for planar graphs there are known results of (a,d)- and $F(a,d)$-decompositions function of their girth.

Theorem 3.89 *Let P be a planar graph.*

1. *Then P is of arboricity at most 3. [61]*
2. *More precisely, P is $F(2,4)$-decomposable. [39]*
3. *If P has no 4-cycles then P is $(1,5)$-decomposable. [15]*
4. *If P is of girth g, the graph P has the decomposition type displayed in the following table: [60, 61, 70]*

$g \geq$	8	6	4
decomposition	$F(1,1)$	$F(1,2)$	$F(2,0)$

The best known upper bound for graphs of arboricity a was given by Bartnicki and Grytczuk in 2008, and it was a bound for the coloring game.

Theorem 3.90 *[7] Let G be a graph of arboricity a and maximum degree Δ. Then:*

$$\chi'_g(G) \leq \Delta + 3a - 1.$$

Remark that a graph of arboricity a is a graph $F^+(a,\{\},0)$ decomposable, hence by Table 3.4.1 we obtain the exact same result but for col'_g: it is a stronger result. Moreover, if we look at the decompositions of planar graphs given in Theorem 3.89 we obtain:

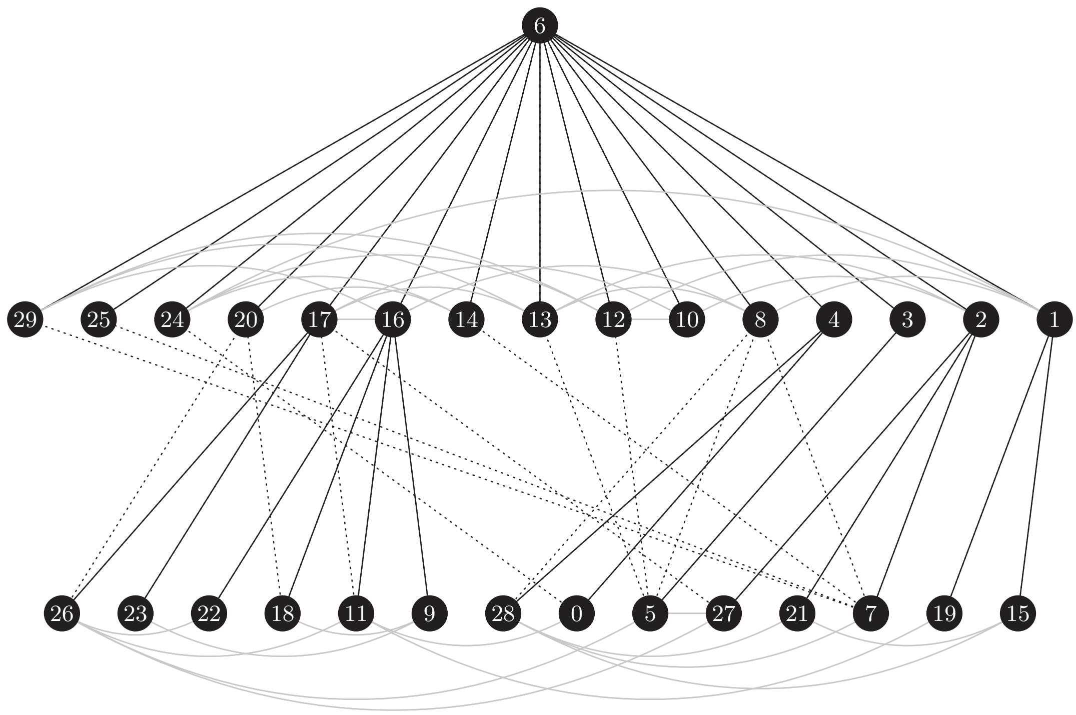


Figure 3.4.4: Example of a graph $F^+(1,\{4\},4)$-, $F^+(0,\{15,4\},4)$- and $F^+(1,\{\},6)$-decomposable

Corollary 3.91 *Let P be a planar graph.*

1. *Then P is of arboricity at most 3, hence:*

$$col'_g(G) \leq \Delta + 8.$$

2. *The graph P is $F(2,4)$-decomposable, hence:*

Δ	0	14	16	∞
$col'_g(P) \leq$	$\Delta + 8$	22	$\Delta + 6$	

3. *If P has no 4-cycles, it is $(1,5)$-decomposable, hence:*

Δ	0	12	17	∞
$col'_g(P) \leq$	$\Delta + 12$	24	$\Delta + 7$	

4. *If P is of girth g:*

   (a) *if $g \geq 4$, P is arboricity 2, hence:*

$$col'_g(P) \leq \Delta + 5,$$

   (b) *if $g \geq 6$, P if $F(1,2)$-decomposable, hence:*

Δ	0	7	9	∞
$col'_g(P) \leq$	$\Delta + 5$	12	$\Delta + 3$	

   (c) *if $g \geq 8$, P is $F(1,2)$-decomposable, hence:*

Δ	0	3	5	∞
$col'_g(P) \leq$	$\Delta + 5$	8	$\Delta + 3$	

a better bound is obtained for $\Delta < 4$ considering the graph as $(1,1)$-decomposable:

Δ	0 — 4	4 — 5	5 — ∞
$col'_g(P) \leq$	$\Delta+4$	8	$\Delta+3$

Details of the computations for $F(2,4)$-decomposable graphs..
Here $a = 2, k = 1, S_k = 4$ and $d = 0$, so, by Table 3.4.1 we obtain the table:

Strategy	A_∞	A_-		D	
$b-c-d$	$\Delta+6$	$\Delta+8$	22	$\Delta+14$	28
$b-d-c$				$\Delta+8$	16
$c-b-d$		$\Delta+14$	26	$\Delta+14$	28
$c-d-b$				$\Delta+12$	24
$d-b-c$		$\Delta+8$	22	$\Delta+6$	12
$d-c-b$		$\Delta+14$	26		

Depending on Δ it gives:

Strategy (Δ)	0 — 12	12 — 14	14 — 16	16 — 20	20 — 22	22 — ∞
$b-c-d$ and $c-b-d$	$\Delta+14$		28			$\Delta+6$
$b-d-c$ and $d-b-c$	$\Delta+8$		22	$\Delta+6$		
$c-d-b$ and $d-c-b$	$\Delta+14$	26			$\Delta+6$	
min: $*-b-*-c$	$\Delta+8$		22	$\Delta+6$		

■

In particular, according to the girth and the maximum degree of a planar graph the bound of Theorem 3.90 can be improved. For instance for planar graphs of girth $g \geq 8$ and $\Delta \geq 4$ we obtain an upper bound of $\Delta + 3$, instead of $\Delta + 5$.
Here, we have new ways to find suitable bounds for graphs depending on their decompositions. And, contrary to (a,b)-pseudo partial k-trees, these decompositions can be done in polytime. Indeed, by the BFS algorithm we can compute some forests and computing the maximum degree of the remaining graph can also be done in polytime. The problem here is to identify, for a graph G, the best decomposition in terms of strategy.

3.4.4 Conclusion

In this chapter we studied two games that are closely linked: first we focused on the marking game and the changes of strategy when modifying the graph, then we focused on the edge-coloring game on F^+-decomposable graphs and the best strategy for Alice as function of the decomposition.

In the first part we studied the minor-graph operators: the edge deletion, the vertex deletion and the edge contraction. For each of these operators we gave lower and upper bounds for the score obtained along the game as function of the score obtained on the initial graph. Each of these bounds are attained for infinitely many graphs and we introduced a class of graphs that give these bounds: sunflower graphs. Moreover, we studied also the strategy for the union of two graphs and the cartesian product of any graph and a complete graph. For these two last operations we give upper bounds and we show these bounds are attained for some graphs linked to the sunflower class of graphs.
This study helps understanding the behavior of the marking game even if a lot of questions remain open, namely, the characterisation of graphs with the same game coloring number for the A-marking game and the B-marking game.

In the second part, we studied the edge-coloring game on F^+-decomposable graphs. These new decomposition is a generalisation of the (a,d)- and $F(a,d)$-decompositions: we have forests with unknown maximum degree, forests with known maximum degree and a graph of maximum degree. We give a strategy for Alice for F^{+}-decomposable graphs and by doing this we improve bounds for the edge-chromatic index of planar graphs. Our strategy is an activation strategy that takes into account the number of forests with known maximum degree, the number of forests with unknown maximum degree and the maximum degree of the remaining graph. The bounds found as function of these three parameters is computable in constant time, the main difficulty is, for a given graph, to find the decomposition that gives the best bounds.

Chapter 4

Taking and Breaking games

"Là, vous faites sirop de vingt-et-un et vous dites : beau sirop, mi-sirop, siroté, gagne-sirop, sirop-grelot, passe-montagne, sirop au bon goût."

Perceval - Kaamelott

In this chapter we study combinatorial games played on heap of tokens. We focus on games where players can *take* tokens from a heap and/or can *break* a heap into multiple heaps.
In the litterature, *taking* and *taking and breaking* games have been largely studied. Namely, most of the known *taking* games have periodic or arithmetic-periodic Grundy sequences. When adding the *breaking* part to these games the study changes: some of them are conjectured to be ultimately periodic, some specific ones are arithmetic-periodic and some others have other behaviors as *sapp*-regularity or even *ruler-regularity*, for some no regularity has been found. For very specific classes of *taking and breaking* games there are tests for periodicity or arithmetic-periodicity, meaning that if the first values verify some conditions, then the sequence is periodic or arithmetic-periodic. *Taking and breaking* games are often studied one by one as no global result about their behavior exists.
These two big families of games are presented in the two following sections among with their important results.
In the third section a joint work with Éric Duchêne, Antoine Dailly and Urban Larsson is presented about *breaking* games. We introduce these games and we present some results about the behavior of their Grundy sequences.

4.1 Pure *taking* games

4.1.1 Subtraction games

In the game of Nim a particular constant is that the players can always take a whole heap. Subtraction games are such that this move is not possible: there is a maximum number of tokens that can be taken from a heap.

> SUBTRACTION $\text{S}(n_1, \ldots, n_\ell)$: Let $n_1, \ldots, n_\ell$ be a list of $\ell \in \mathbb{N}$ positive integers. The SUBTRACTION GAME $\text{S}(n_1, \ldots, n_\ell)$ is a game played on k heaps of tokens where the players alternate turns to remove $x \in \{n_1, \ldots, n_\ell\}$ tokens from a single heap.

For instance, take the game $S(1, 3)$. In this game a move consists on removing 1 or 3 tokens from a heap of tokens, in other words, this is exactly the same game as TOKENS, seen in Section 1.1.1.
Now, consider the game $S(1, 2, 3)$. From the starting positions $(1), (2)$ or (3) the first player can easily win

by taking all tokens from the heap. From (4), the options are $\{(1),(2),(3)\}$: this is a $\mathcal{P}$-position. As well, if they start with a heap of 5, 6 or 7 tokens, the first player can leave 4 tokens, hence these are $\mathcal{N}$-positions.

In fact, the particular case where $n_1, \ldots, n_\ell$ are consecutive integers with $n_1 = 1$ gives periodic Grundy sequences:

Theorem 4.1 *Let ℓ be a positive integer. The Grundy sequence of the game $S(1,2,\ldots,\ell)$ is $(\mathbf{1},\mathbf{2},\ldots,\ell,\mathbf{0})$ $(+\mathbf{0})$, which is periodic of period $\ell+1$.*

Proof. Let n be a non-negative integer and a, b such that $n = a(\ell+1)+b$ and $0 \leq b \leq \ell$. Assume n is the minimum integer such that $\mathcal{G}((n)) \neq b$. As for $m \in [\![0,\ell]\!]$ the options are exactly the same as in NIM, we can assume $n \geq \ell+1$.
The options from the heap of size n are $a(\ell+1)+b-\ell, a(\ell+1)+b-\ell+1, \ldots, a(\ell+1)+b-1$ which have Grundy number $\mathbf{b+1}, \ldots, \ell, \mathbf{0}, \ldots, \mathbf{b-1}$ respectively. In particular, there is no option of Grundy number $\mathbf{b}$ and for all $0 \leq i \leq b-1$ there is an option of Grundy value $\mathbf{i}$. A contradiction. Hence there is no minimum conterexample, meaning that the Grundy sequence is periodic of period $\ell+1$. ■

Hence, the important parameter in the game $S(1,\ldots,\ell)$ is the last integer in the list. For other games, like for instance $S(1,3,4)$ the Grundy sequence is quite different as the Grundy values are not consecutive. Indeed in Table 4.1.1 we give the options and Grundy numbers of the first heaps. In the first values there are multiples $\mathcal{P}$-positions and a period of length 7 seems to appear.

n	options	Grundy value
0	$\varnothing$	**0**
1	$\{0\}$	**1**
2	$\{1\}$	**0**
3	$\{0,2\}$	**1**
4	$\{0,1,3\}$	**2**
5	$\{1,2,4\}$	**3**
6	$\{2,3,5\}$	**2**
7	$\{3,4,6\}$	**0**
8	$\{4,5,7\}$	**1**
9	$\{5,6,8\}$	**0**
10	$\{6,7,9\}$	**1**
11	$\{7,8,10\}$	**2**
12	$\{8,9,11\}$	**3**
13	$\{9,10,12\}$	**2**
14	$\{10,11,13\}$	**0**
15	$\{11,12,14\}$	**1**

Table 4.1.1: First Grundy values for the game $S(1,3,4)$ on one heap.

In fact, all subtraction games have periodic Grundy sequences:

Theorem 4.2 *[1] The Grundy sequence of a subtraction game is always periodic.*

Proof. Let $L = \{\ell_1, \ldots, \ell_k\}$ be an increasing sequence of non-negative integers with $k \in \mathbb{N}$ and G_n the subtraction game $S(L)$ on one heap of size n.
Remark first that from the position G_n there are at most k options: $\{n-\ell_k, \ldots, n-\ell_1\}$, hence $\mathcal{G}(G_n) \leq \mathbf{k}$. As all Grundy values are bounded by $\mathbf{k}$, there are finitely many blocks of ℓ_k consecutive blocks. Hence there is a couple of integers (q,r) such that $\ell_k \leq q, r$ and $\mathcal{G}(G_{q-\ell_k-1}) = \mathcal{G}(G_{r-\ell_k-1})$, $\ldots$, $\mathcal{G}(G_{q-1}) = \mathcal{G}(G_{r-1})$. This gives directly that $\mathcal{G}(G_q) = \mathcal{G}(G_r)$.
Now, let m be a non-negative integer and assume that for all $m' < m$, $\mathcal{G}(G_{q+m'}) = \mathcal{G}(G_{r+m'})$. As for all $i \in \{\ell_1, \ldots, \ell_k\}$, $\mathcal{G}(G_{q+m-\ell_i}) = \mathcal{G}(G_{r+m-\ell_1})$, the grundy values of G_{q+m} and G_{r+m} are the same. Hence, for all $m \geq 0$, $\mathcal{G}(G_{q+m}) = \mathcal{G}(G_{r+m})$, meaning that the Grundy sequence is ultimately periodic of period $r-q$ and preperiod q. ■

Corollary 4.3 *Let $G = S(\ell_0, \dots, \ell_{k-1})$ be a subtraction game with $\ell_1 \leq \cdots \leq \ell_{k-1}$. Let e and p be positive integers such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ and $e \leq n < e+a$. The Grundy sequence of G is then periodic of period p and preperiod e.*

This gives a practical way of finding the Grundy sequence for any subtraction game. Indeed, we need to compute the first values until ℓ_{k-1} consecutive values are repeated and from this corollary, the periodicity comes straight forward.
An example using this corollary can be the computation of $S(2,4,7)$. The first 26 values are shown in Table 4.1.2

Table 4.1.2: First 26 values of the Grundy sequence of $S(2,4,7)$

heap of size	options	Grundy value	heap of size	options	Grundy value
25	18,21,23	**1**	12	5,8,10	**0**
24	17,20,22	**3**	11	4,7,9	**1**
23	16,19,21	**0**	10	3,6,8	**2**
22	15,18,20	**2**	9	2,5,7	**0**
21	14,17,19	**2**	8	1,4,6	**1**
20	13,16,18	**1**	7	0,3,5	**3**
19	12,15,17	**1**	6	2,4	**0**
18	11,14,16	**0**	5	1,3	**2**
17	10,13,15	**0**	4	0,2	**2**
16	9,12,14	**2**	3	1	**1**
15	8,11,13	**0**	2	0	**1**
14	7,10,12	**1**	1		**0**
13	6,9,11	**2**	0		**0**

Here, we remark that for $e = 0$, $p = 17$ and $\ell \leq n < \ell + 7$, we have $\mathcal{G}(n+p) = \mathcal{G}(n)$, hence this sequence is periodic of preperiod 0 and period 17, the sequence is $(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{0}, \mathbf{3}, \mathbf{1}, \mathbf{0}, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{2})$ $(+\mathbf{0})$.

Now we know how to compute in polynomial time the Grundy sequences of all subtraction games. Even though the behavior of all of these games is known, there is still an important question that remains open:

Open question 4.4 *What is the minimum period of the Grundy sequence of $S(\ell_1, \dots, \ell_k)$? is there a bound on the length of the preperiod?*

Having a good bound for this period as a function of $n_1, \dots, n_\ell$ could improve the computations of the Grundy sequence.

In the next section some infinite subtraction games are studied.

4.1.2 All-but games

Another way of looking to infinite subtraction games is to see what numbers of tokens are not takeable from the heaps:

> ALL-BUT $\mathrm{A}(n_1, \dots, n_\ell)$: Let $n_1, \dots, n_\ell$ be a list of $\ell \in \mathbb{N}$ positive integers. The ALL-BUT GAME $\mathrm{A}(n_1, \dots, n_\ell)$ is a game played on heaps of tokens where the players alternate turns to remove $x \in \mathbb{N} \setminus \{n_1, \dots, n_\ell\}$ tokens from a single heap.

For instance, consider the game $\mathrm{A}(1,3,4)$. The heaps of sizes 0 and 1 have no options, hence their value is **0**. The heap of size 2 has a unique option, 0, hence its value is **1**. The first 24 values are given in Table 4.1.3.
Here, the difference between the third and the last column is always 4. This Grundy sequence seems to be arithmetic-periodic with no preperiod.

Table 4.1.3: First 24 values of the Grundy sequence of $A(1, 3, 4)$.

heap of size	Grundy value	heap of size	Grundy value
23	**7**	11	**3**
22	**7**	10	**3**
21	**6**	9	**2**
20	**6**	8	**2**
19	**7**	7	**3**
18	**5**	6	**1**
17	**6**	5	**2**
16	**4**	4	**0**
15	**5**	3	**1**
14	**5**	2	**1**
13	**4**	1	**0**
12	**4**	0	**0**

In fact, as for subtraction games, there is also a result that reduces the computations as soon as the values repeat themselves:

Theorem 4.5 *[1] Let $G = \mathrm{A}(n_1, \dots, n_\ell)$ $n_1 \leq \cdots \leq n_\ell$. Assume there is s, p and e such that for all $e \leq n \leq e + 2n_\ell$, $\mathcal{G}(n+p) = \mathcal{G}(n) + s$. Then the Grundy sequence of G is arithmetic-periodic of preperiod e, period p and saltus s.*

This way, we are sure that the Grundy sequence of $\mathrm{A}(1, 3, 4)$ above is what it seems:

$$(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{2}, \mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{3}, \mathbf{3})\ (+\mathbf{4}).$$

In particular, once e and p have been identified, the only interesting values of the Grundy sequence are those for $0 \leq n \leq e + 2a + p$, all other values are given by arithmetic-periodicity.
In fact all ALL-BUT games are arithmetic-periodic:

Theorem 4.6 *[1] Let $\ell_1, \dots, \ell_\ell$ be a list of non-negative integers with $\ell_1 \leq \cdots \leq \ell_\ell$. Then the Grundy sequence of $\mathrm{A}(\ell_1, \dots, \ell_\ell)$ is arithmetic-periodic.*

The proof of Theorem 4.6 is similar to the one of Theorem 4.2 but quite more technical, interested readers are invited to read the 150^{th} page of [1].

Moreover, for some particular cases the study can be reduced to smaller lists of integers. For instance when the gap between the two greater values is big enough.

Theorem 4.7 *[1] Let $G = \mathrm{A}(\ell_0, \dots, \ell_{k-1})$ and $G' = \mathrm{A}(\ell_0, \dots, \ell_k)$ with $\ell_0 < \cdots < \ell_{k-1}$ and $2\ell_{k-1} < \ell_k$. Then G and G' have the exact same Grundy sequences.*

Of course this study does not give a bound on the size of the period nor the size of the preperiod. Moreover, here we only discussed finite subtraction games or finite all-but games, the cases where we remove an infinite set and keep also an infinite set are not so easy to compute as we see on some examples.

4.1.3 Non-finite games

Here the games SUBTRACT A POWER OF TWO and SUBTRACT A SQUARE are presented.

SUBTRACT A POWER OF TWO: is a game played on heaps of tokens where the players alternate turns to remove $x \in \{2^i \mid i \in \mathbb{N}\}$ tokens from a single heap.

Consider n be a positive integer and $a_0, \ldots, a_p$ its binary decomposition: $n = \sum_{0 \leq i \leq p} a_i 2^i$, with $a_p \neq 0$. A move from a heap of size n consists of removing 2^i, $i \leq p$ tokens from the heap. If there is only one i such that $a_i \neq 0$, one possible move is to the position (0), which is a $\mathcal{P}$-position, hence (n) is a $\mathcal{N}$-position. In the following table we give the first 16 Grundy numbers of its Grundy sequence.

Table 4.1.4: First 16 values of the Grundy sequence of $S(2^i, 0 \leq i)$.

heap of size	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Grundy value	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**

This is one example of an infinite subtraction game where the Grundy sequence is easy to understand: it is periodic of period three and the pattern repeated is $\mathbf{1, 2, 0}$.

Another example is SUBTRACT A SQUARE, introduced in [38] by S.W. Golomb.

SUBTRACT A SQUARE: is a game played on heaps of tokens where the players alternate turns to remove $x \in \{i^2 \mid i \in \mathbb{N}\}$ tokens from a single heap.

The first 50 Grundy numbers are shown in Table 4.1.5. No structure seems to appear on these values, and in fact no structure on this sequence has been found.

Table 4.1.5: First 100 values of the Grundy sequence of SUBTRACT A SQUARE.

heap	value	heap	value	heap	value	heap	value
99	**2**	74	**5**	49	**3**	24	**2**
98	**4**	73	**1**	48	**2**	23	**1**
97	**6**	72	**0**	47	**3**	22	**0**
96	**1**	71	**3**	46	**2**	21	**1**
95	**0**	70	**4**	45	**1**	20	**0**
94	**5**	69	**2**	44	**0**	19	**2**
93	**4**	68	**1**	43	**2**	18	**1**
92	**6**	67	**0**	42	**3**	17	**0**
91	**2**	66	**1**	41	**2**	16	**1**
90	**4**	65	**0**	40	**1**	15	**0**
89	**2**	64	**3**	39	**0**	14	**2**
88	**3**	63	**1**	38	**2**	13	**1**
87	**6**	62	**0**	37	**3**	12	**0**
86	**1**	61	**5**	36	**2**	11	**1**
85	**0**	60	**4**	35	**1**	10	**0**
84	**5**	59	**3**	34	**0**	9	**2**
83	**4**	58	**1**	33	**4**	8	**1**
82	**3**	57	**0**	32	**3**	7	**0**
81	**2**	56	**5**	31	**2**	6	**1**
80	**6**	55	**4**	30	**3**	5	**0**
79	**5**	54	**3**	29	**5**	4	**2**
78	**4**	53	**1**	28	**4**	3	**1**
77	**3**	52	**0**	27	**3**	2	**0**
76	**2**	51	**5**	26	**2**	1	**1**
75	**6**	50	**4**	25	**3**	0	**0**

In particular, infinite subtraction games do not behave as regularly as finite ones.

4.2 Some *taking and breaking* games

In this section *taking and breaking* games are studied, in these games players can take tokens from heaps and/or can split heaps into multiple ones. First some examples are shown and then octal and hexadecimal games are presented.

4.2.1 Some examples

1. **Kayles**
   Kayles game was invented by Dudeney and studied by Guy and Smith in [43].

   KAYLES: two players take turns to remove one or two consecutive tokens from a heap and then they can also split it into two heaps.

   In other words, players choose a heap and take tokens either from endpoints (leaving a smaller heap) or in the middle (leaving two smaller heaps). In Figure 4.2.1 we can see a game starting with 15 tokens.

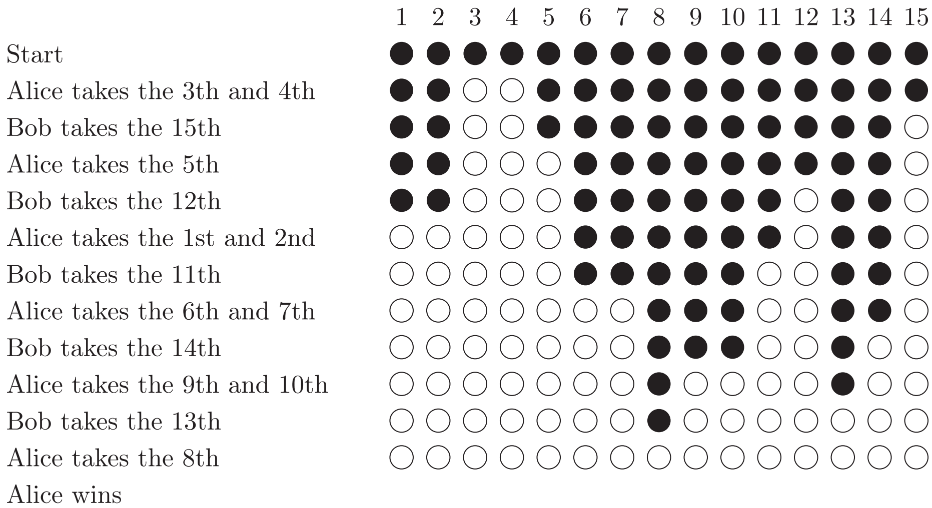


Figure 4.2.1: Example of game with 15 tokens initially.

In fact, here the players did not play the winning moves. Indeed, at her first turn, Alice should have taken the 8-th token hence leaving a sum of two games with the same size. As well, Bob could have done a winning move in his first turn by takng the 9-th token.
Remark that when they split a heap in two, the game on one heap becomes a game in two heaps or the sum of two games each one on one heap. Here the nim-sum over Grundy numbers come in handy to study the Grundy sequence of a single heap. In the general case, when they start with a heap with n tokens, $n > 2$, the first player can leave two heaps of equal size by removing 1 or 2 tokens from the middle. In particular, this says that the Grundy sequence of Kayles has only one 0, for the empty heap. The first 96 values of its Grundy sequence are shown in Table 4.2.1.

Table 4.2.1: First 96 values of the Grundy sequence of Kayles.

b	0	1	2	3	4	5	6	7	8	9	10	11
$12 \times 0 + b$	0	1	2	3	1	4	3	2	1	4	2	6
$12 \times 1 + b$	4	1	2	7	1	4	3	2	1	4	6	7
$12 \times 2 + b$	4	1	2	8	5	4	7	2	1	8	6	7
$12 \times 3 + b$	4	1	2	3	1	4	7	2	1	8	2	7
$12 \times 4 + b$	4	1	2	8	1	4	7	2	1	4	2	7
$12 \times 5 + b$	4	1	2	8	1	4	7	2	1	8	6	7
$12 \times 6 + b$	4	1	2	8	1	4	7	2	1	8	2	7
$12 \times 7 + b$	4	1	2	8	1	4	7	2	1	8	2	7

In fact, it has been proved that this sequence is periodic of preperiod 72 and period 12. In practice if you are playing this game, we recommend to have this table on sight, otherwise you will have to do a lot of computations if you want to play optimally.

2. **Dawson Kayles**

Consider the same game as before, but this time players are not able to take only one token.

DAWSON KAYLES: players alternate turns to take two tokens and split it if they want into two heaps.

By changing this rule, the sequence changes a lot: it is now of preperiod 52 and of period 34. The first 88 values are shown in Table 4.2.2.

Table 4.2.2: First 88 values of the Grundy sequence of Dawson Kayles.

b	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$17 \times 0 + b$	0	0	1	1	2	0	3	1	1	0	3	3	2	2	4	0	5
$17 \times 1 + b$	2	2	3	3	0	1	1	3	0	2	1	1	0	4	5	2	7
$17 \times 2 + b$	4	0	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5
$17 \times 3 + b$	5	2	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7
$17 \times 4 + b$	4	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5
$17 \times 5 + b$	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7

3. **Dawson Chess**

Consider now another version of the same game, introduced by Dawson in [25].

DAWSON CHESS: players alternate turns to either take an entire heap, if it has one, two or three tokens; or take two or three tokens from a pile. Moreover after removing tokens they can also break the remaining heap in two.

For instance, for a heap of size 10 the options are $7, 8, (1, 6), (2, 5), (3, 4)$. In Table 4.2.3 the Grundy values of heaps of sizes less than 34 are shown.

Table 4.2.3: First 34 Grundy values for Dawson Chess.

b	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$17 \times 0 + b$	0	1	1	2	0	3	1	1	0	3	3	2	2	4	0	5	2
$17 \times 1 + b$	2	3	3	0	1	1	3	0	2	1	1	0	4	5	2	7	4

If we compare Table 4.2.2 and Table 4.2.3 we remark there is a shift of one on the Grundy sequence, at least for these first values. In fact, this shift is kept all along. The Grundy sequence of Dawson Chess is periodic of preperiod 51 and period 34.

4.2.2 Octal and Hexadecimal games

We have seen some examples of games where the players can take and break heaps. If we remember the rules of the last game, Dawson Chess, the players can remove $1, 2$ or 3 tokens if they empty a heap, or they can remove 2 or 3 tokens from any heap, or they can take 3 tokens and divide the remaining heap in two. The rules start to be difficult to describe... In fact, the games we have seen earlier are easier to explain. In this section we see *octal* and *hexadecimal* games and some of their properties.

1. **Octal games**

OCTAL GAMES: Let $d_1, \ldots, d_t$ be t integers such that $\forall 1 \leq i \leq t$, $0 \leq d_i \leq 7$. The OCTAL GAME $\mathbf{0.d_1 \ldots d_k}$ is the game where for each $1 \leq i \leq t$:

$$d_i = e_{i,0} + 2e_{i,1} + 4e_{i,2}$$

with $e_{i,j} \in \{0, 1\}$ and:

- if $e_{i,0} = 1$: taking a heap of size i is permitted,
- if $e_{i,1} = 1$: taking i tokens from a heap of size $j > i$ is permitted, leaving a heap of size $j - i$,
- if $e_{i,2} = 1$: taking i tokens from a heap and split the remaining into two non-empty heaps is permitted.

The following table sums up the possible moves:

Table 4.2.4: Details of the possible moves in octal games

d_i	$(d_i)_2$	Removing i tokens...
0	000	is not possible,
1	001	is possible if it empties the heap,
2	010	is possible if it does not empty the heap,
3	011	is possible, no splitting is possible,
4	100	is possible as long as the remaining heap is cut in two,
5	101	is possible if the remaining heap is cut in two (if non-empty),
6	110	is possible if at least a token is left and the remaining may be split in two,
7	111	is possible, the remaining heap may be split in two.

For instance, the game **0.1234** is a game where removing 1 or 3 tokens is allowed if it empties the heap, removing 2 or 3 tokens is allowed if the heap is not emptied and removing 4 is allowed if the remaining heap is cut in two.
Now consider the game **0.137**: removing 1 is allowed if it empties the heap, removing 2 tokens is allowed if the remaining heap is not cut in two and removing 3 tokens is always allowed (emptying the heap, leaving a heap or cutting the heap). In other words, this is exactly Dawson Chess. As well, the game **0.07** is exactly Dawson Kayles, and Kayles is **0.77**.

In the octal games that we studied so far (the last three examples) the Grundy sequences tend to be periodic with some preperiod. In fact, to this day no octal game has been proven not to be periodic. Guy conjectured that all octal games are ultimately periodic. In practice, to see if an octal game is periodic we use a result similar to Corollary 4.3.

Theorem 4.8 (Octal periodicity test) *Let G be an octal game $\mathbf{0.d_1 \ldots d_k}$ of finite length k. If there exist $n_0 \geq 1$ and $p \geq 1$ such that:*

$$G(n + p) = G(n), \quad \forall n_0 \leq n < 2n_0 + p + k,$$

then G is ultimately periodic with period p and preperiod n_0.

The main difficulty is to find n_0 and p to deduce the regularity of the sequence. In fact, these values can be extremely big, as is the case for **0.454** which has a preperiod of 160949019 and a period of 60620715. For some games, the conjectured values n_0 and p have not been found yet, like **0.6**, **0.14**, **0.172** or even **0.007**, more details can be found in Table 4.2.5 taken from [33].

Remark that the game **0.07** is solved (Arithmetic-Periodic Grundy sequence) and the game **0.007** remains unknown.

Table 4.2.5: Details of computations of the games **0.6**, **0.14** and **0.007**

Games	maximum n	maximum $\mathcal{G}(i)$	number of $\mathcal{P}$-positions
0.6	2^{33}	363	14
0.14	2^{32}	85	172
0.172	2^{31}	387	10
0.007	$2^{2}8$	1689	37

Consider now the game **0.41**. In this game the players can take one token and split the remaining heap in two or take two tokens from a heap of size 2. The first 18 heaps are given in Table 4.2.6 along with their Grundy numbers.

Table 4.2.6: Grundy sequence of the game **0.41** up until the 119-th value.

b	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$17\times 0+b$	**0**	**0**	**1**	**1**	**0**	**2**	**1**	**3**	**0**	**1**	**1**	**3**	**2**	**2**	**3**	**4**	**1**
$17\times 1+b$	**5**	**3**	**2**	**2**	**3**	**1**	**1**	**0**	**3**	**1**	**2**	**0**	**1**	**1**	**4**	**4**	**2**
$17\times 2+b$	**6**	**4**	**1**	**1**	**0**	**2**	**1**	**3**	**0**	**1**	**1**	**3**	**2**	**2**	**3**	**4**	**4**
$17\times 3+b$	**5**	**7**	**2**	**2**	**3**	**1**	**1**	**0**	**3**	**1**	**2**	**0**	**1**	**1**	**4**	**4**	**3**
$17\times 4+b$	**6**	**4**	**1**	**1**	**0**	**2**	**1**	**3**	**0**	**1**	**1**	**3**	**2**	**2**	**3**	**4**	**4**
$17\times 5+b$	**5**	**7**	**2**	**2**	**3**	**1**	**1**	**0**	**3**	**1**	**2**	**0**	**1**	**1**	**4**	**4**	**3**
$17\times 6+b$	**6**	**4**	**1**	**1**	**0**	**2**	**1**	**3**	**0**	**1**	**1**	**3**	**2**	**2**	**3**	**4**	**4**
$17\times 7+b$	**5**	**7**	**2**	**2**	**3**	**1**	**1**	**0**	**3**	**1**	**2**	**0**	**1**	**1**	**4**	**4**	**3**
$17\times 8+b$	**6**	**4**	**1**	**1**	**0**	**2**	**1**	**3**	**0**	**1**	**1**	**3**	**2**	**2**	**3**	**4**	**4**

Here we remark that a period of 34 seems to rule the last values (starting at 34), hence the preperiod seems to be $n_0 = 34$, and the period also $p = 34$. Moreover, for all $n_0 \leq n \leq 2n_0 + p + 2 = 104$, we have $\mathcal{G}(n + p) = \mathcal{G}(n)$, hence, this sequence is periodic of preperiod n_0 and period p. Remark that here we need to compute all the values between 34 and 138 to be able to apply the theorem.

2. **Hexadecimal games**

A natural extension to octal games is when players are allowed to cut heap into three non-empty ones. These games can be coded with an hexadecimal code and are called *hexadecimal games*.

HEXADECIMAL GAMES: Let $d_1, \ldots, d_t$ be t integers such that for all $1 \leq i \leq t$ we have $0 \leq d_i \leq 15$. For clarity if $d_i > 9$, d_i is $10, 11, 12, 13, 14$ or 15, and we denote it by A, B, C, D, E of F respectively. The game $\mathbf{0.d_1 \ldots d_t}$ is then the game where for each $1 \leq i \leq t$:

$$d_i = e_0 + 2e_1 + 4e_2 + 8e_3$$

with $e_{i,j} \in \{0, 1\}$ and:

- if $e_{i,0} = 1$: taking a heap of size i is permitted,
- if $e_{i,1} = 1$: taking i tokens from a heap of size $j > i$ is permitted, leaving a heap of size $j - i$,
- if $e_{i,2} = 1$: taking i tokens from a heap and splitting it into two non-empty heaps is permitted,
- if $e_{i,3} = 1$: taking i tokens from a heap and splitting it into three non-empty heaps is permitted.

For instance, the game **0.F** is the game where players can remove one token and leave 0, 1, 2 or 3 non-empty heaps. This game exhibits a periodic Grundy sequence which is **(01)**. In fact, hexadecimal games have a behavior sometimes different from octal ones. For example, the game **0.89**. In Tables 4.2.7 and 4.2.8 the first heaps are shown with their options and Grundy values.

Table 4.2.7: Example of Hexadecimal game: **0.89**, first 11 options and Grundy values.

heap	options	Grundy value
0	−	**0**
1	−	**0**
2	0	**1**
3	−	**0**
4	$(1,1,1)$	**1**
5	$(1,1,2),(1,1,1)$	**2**
6	$(1,1,3),(1,2,2),(1,1,2)$	**2**
7	$(1,1,4),(1,2,3),(2,2,2),(1,1,3),(1,2,2)$	**2**
8	$(1,1,5),(1,2,4),(1,3,3),(2,2,3),(1,1,4),(1,2,3),(2,2,2)$	**3**
9	$(1,1,6),(1,2,5),(1,3,4),(2,2,4),(2,3,3),(1,1,5),(1,2,4),(1,3,3),(2,2,3)$	**4**
10	$(1,1,7),(1,2,6),(1,3,5),(1,4,4),(2,2,5),(2,3,4),(3,3,3),(1,1,6),(1,2,5),(1,3,4),(2,2,4),(2,3,3)$	**4**

Table 4.2.8: First 28 Grundy values of **0.89**.

b	0	1	2	3
$4\times 0+b$	**0**	**0**	**1**	**0**
$4\times 1+b$	**1**	**2**	**2**	**2**
$4\times 2+b$	**3**	**4**	**4**	**4**
$4\times 3+b$	**5**	**6**	**6**	**6**
$4\times 4+b$	**7**	**8**	**8**	**8**
$4\times 5+b$	**9**	**10**	**10**	**10**
$4\times 6+b$	**11**	**12**	**12**	**12**

This sequence is not periodic, but arithmetic-periodic, of preperiod 3, period 4 and saltus 2:
(**001**) (**0122**) (+**2**).
This is a usual behavior for hexadecimal games. Some results similar to Theorem 4.8 have been done for these games, for instance in [5], Austin gives a test to check arithmetic-periodicity when the saltus seems to be a power of 2, and in [49], Howse and Nowakowski give a similar test for arbitrary saltus. In both tests the number of values we have to check out is much larger than for the octal one: for octal games we checked 2 expected periods, and for hexadecimal games we need to check at least 7 expected periods. Moreover, hexadecimal games can also have other kind of Grundy sequences: the game **0.205200C** is sapp-regular and the game **0.20..48** (with an odd number of *0*s) is ruler-regular.
Anyhow, the difficulty when studying these games remains in founding the expected values for the period or understanding the behavior of the sequence when no period appears.

It goes without saying, most of the taking and breaking games can be described by using codes of the form $\mathbf{d_0.d_1}{\dots}\ \mathbf{d_k}$ for k finite or not, where the values of d_i's depend on the maximum number of breaks allowed. Higher bases are not of much interest here.

The *breaking* part of these games shows new behaviors for the Grundy sequences. Between octal games and hexadecimal games there is already a gap between a unique behavior and multiples ones when the only difference, rule-wise, is cutting twice the heaps. In the next section we focus on pure *breaking* games where players can cut heaps a given number of times and can never take tokens.

4.3 Pure *breaking* games

Taking games, when they are subtraction ones, are easy to study: there are all periodic and then the research for their period is what remains still open. When dealing with *taking and breaking* games, the behaviors of the Grundy sequence change and we can also see arithmetic-periodicity among other

types of sequences. The regularity of their sequences is not known, even if globally they show some king of periodicity. In both cases, for *taking* and *taking and breaking* games there are tests to show their regularities (periodicity and arithmetic-periodicity) by computing the first values of the sequence. The main problem is determine how far to take the computations as in some cases these are done for very big values without a final result.
Here we show *pure breaking games* where the players can only split the heaps, without taking tokens.

4.3.1 Introduction of *pure breaking games*

In [9], Grundy's game was introduced:

GRUNDY'S GAME: players alternate turns to chose a heap and split it into two non-empty heaps of different sizes.

An example of game is given in Figure 4.3.1.

Start with 15 tokens
Alice breaks in $(2, 13)$
Bob breaks in $(2, 5, 8)$
Alice breaks in $(2, 2, 3, 8)$
Bob breaks in $(2, 2, 3, 2, 6)$
Alice breaks in $(2, 2, 3, 2, 1, 5)$
Bob breaks in $(2, 2, 3, 2, 1, 2, 3)$
Alice breaks in $(2, 2, 3, 2, 1, 2, 2, 1)$
Bob breaks in $(2, 2, 1, 2, 2, 1, 2, 2, 1)$
Bob wins

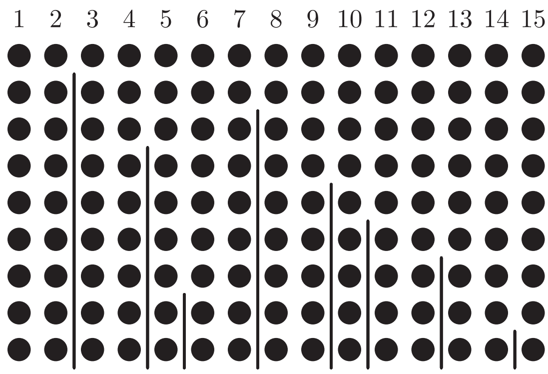


Figure 4.3.1: Example of GRUNDY'S game played by two players with an initial heap of 15 tokens

Here the players start with a heap of 15 tokens, but here Alice does a bad move, since $(2, 13)$ is an $\mathcal{N}$-position. For once we will not give the Grundy values of the first heaps, since, for now, it has been computed for the first 2^{35} heaps without a glimpse of regularity.

Remark that allowing to split heaps in equal-sized ones gives a very easy game to study: the Grundy sequence is periodic of period **(01)** and preperiod **1**.

As well, other *breaking* that has been studied in the litterature is COUPLES ARE FOREVER:

COUPLES ARE FOREVER: players alternate turns to split a heap of size $n > 3$ into two non-empty heaps.

As for GRUNDY'S game, the Grundy sequence of COUPLES ARE FOREVER does not show a regular behavior yet.

In this section we explore games where the players can split heaps in a given number of non-empty heaps and we call them PURE BREAKING GAMES. In this games no addition rule is given: the heaps resulting from the splitting can have same size, the only condition is that they are non-empty.

PURE BREAKING GAMES: Let $L = \{\ell_1, \ldots, \ell_k\}$ be a set of positive integers, called the *cut numbers*. The PURE BREAKING game PB(L) is the heap game such that the heap of size n has the following options:

$$\{\ (i_0, \ldots, i_\ell)\ \mid\ \ell \in L, \forall j, i_j > 0 \text{ and } i_0 + \cdots + i_\ell = n\ \}.$$

In other words, in PB(L), at each move the players chose an integer $\ell \in L$ and a heap of size $n > \ell$ and split the heap in $\ell + 1$ non empty heaps. Such a move is called an *ℓ-cut*. In practice, for the game

PB($\{\ell_1,\ldots,\ell_k\}$) we assume $\ell_1 < \cdots < \ell_k$.

Consider the game PB({2,3}). In Table 4.3.1 we can see the first heaps with their options and their Grundy value.

Table 4.3.1: First heaps of the game PB({2,3}).

heap	options	Grundy value
0	–	**0**
1	–	**0**
2	–	**0**
3	$(1,1,1)$	**1**
4	$(1,1,2),(1,1,1,1)$	**1**
5	$(1,1,3),(1,2,2),(1,1,1,2)$	**2**
6	$(1,1,4),(1,2,3),(2,2,2),(1,1,1,3),(1,1,2,2)$	**2**
7	$(1,1,5),(1,2,4),(1,3,3),(2,2,3),(1,1,1,4),(1,1,2,3),(1,2,2,2)$	**3**
8	$(1,1,6),(1,2,5),(1,3,4),(2,2,4),(2,3,3),(1,1,1,5),(1,1,2,4),(1,1,3,3),(1,2,2,3),(2,2,2,2)$	**3**

The Grundy sequence seems to be arithmetic-periodic of period 2 and saltus 1. We will see later that this is the expected behavior of pure breaking games.

4.3.2 Solving some particular families

In the last example, for the list $\{2,3\}$, the Grundy sequence seemed to be of period 2 and saltus 2. In fact, when 1 is not in the list, this is the normal behavior of pure breaking games.

Theorem 4.9 *Let $L = \{\ell_1,\ldots,\ell_k\}$ be a set of cut numbers such that $2 \leq \ell_1 < \cdots < \ell_k$. Then PB(L)'s Grundy sequence is* $(\mathbf{0})^{\ell_1}$ $(+\mathbf{1})$.

Proof. Let $L = \{\ell_1,\ldots,\ell_k\}$ such that $2 \leq \ell_1 < \cdots < \ell_k$. We prove this by contradiction, let n be a positive integer, then there is a unique couple (a,b) of non-negative integers such that $0 \leq b \leq \ell_1 - 1$ and $n = a\ell_1 + b + 1$. We prove that for every positive integer $\mathcal{G}(n) = \mathbf{a}$.
Assume that n is the smallest positive integer such that $\mathcal{G}(n) \neq \mathbf{a}$.
Assume $\mathcal{G}(n) > \mathbf{a}$. In particular, there is an option of the heap of size n with Grundy value $\mathbf{a}$, *i.e.*, there is an $m \in L$ and an m-cut of n, say $O_n = (a_0\ell_1 + b_0 + 1, \ldots, a_m\ell_1 + b_m + 1)$ such that:

- $\mathcal{G}(O_n) = \mathbf{a_0} \oplus \cdots \oplus \mathbf{a_m} = \mathbf{a}$, and
- $\sum_{i=0}^{m}(a_i\ell_1 + b_i + 1) = a\ell_1 + b + 1$.

As $b \leq \ell_1 - 1$, we have $a_0 + \cdots + a_m \leq a$, and since $a = a_0 \oplus \cdots \oplus a_m \leq a_0 + \cdots + a_m$, we know $a = a_0 + \cdots + a_m$. Directly, we have $1 + b = \sum_{i=0}^{m}(1 + b_i) = m + 1 + b_0 + \cdots + b_m$, which is a contradiction since $m \leq \ell_1$ and $b < \ell_1$. Thus, there is no option of the heap of size n with Grundy value $\mathbf{a}$, in particular $\mathcal{G}(n) < \mathbf{a}$.
We now prove that the heap of size n has options of Grundy values $\mathbf{i}$ for $i \in [\![0, a-1]\!]$. There are two cases:

1. if ℓ_1 is even, then for $i \in [\![0, a-1]\!]$, let $O_n = (i\ell_1 + b + 1, a - i, \ldots, a - i)$ be an ℓ_1-cut. This always exists since $\ell_1 \geq 2$. Moreover it is an option of n: $i\ell_1 + b + 1 + (a-i)\ell_1 = a\ell_1 + b + 1 = n$. Furhermore, we have $\mathcal{G}(O_n) = \mathcal{G}(i\ell_1 + b + 1) \oplus (\ell_1 \otimes \mathcal{G}(a - i))$. Since $\mathcal{G}(i\ell_1 + b + 1) = \mathbf{i}$, by minimality of n, and ℓ_1 is even, we have $\ell_1 \otimes \mathcal{G}(a - i) = \mathbf{0}$ and $\mathcal{G}(O_n) = \mathbf{i}$.

2. if ℓ_1 is odd, then for all $i \in [\![0, a-1]\!]$, we define an option O_n of n, obtained by an ℓ_1-cut, such that $\mathcal{G}(O_n) = \mathbf{i}$. We have two subcases:

(a) if $a-i$ is odd, let

$$\begin{array}{rcll} h_0 & = & i\ell_1+b+1 & \\ h_j & = & \frac{1}{2}(a-i-1)\ell_1+1 & \text{for } j=1,2 \\ h_j & = & 1 & \text{for } 3\leq j\leq \ell_1 \end{array}$$

this always exists since $\ell_1 \geq 3$ (if $\ell_1 = 3$ then there are only the first four heaps) and $(a-i-1)$ is even. Moreover, it is an option of n:

$$i\ell_1+b+1+2\left(\frac{1}{2}(a-i-1)\ell_1+1\right)+(\ell_1-2)=i\ell_1+b+1+(a-i-1)\ell_1+\ell_1=a\ell_1+b+1=n.$$

Furthermore, we have:

$$\mathcal{G}(O_n)=\mathcal{G}(i\ell_1+b+1)\oplus\left(2\otimes\mathcal{G}\left(\frac{1}{2}(a-i-1)\ell_1+1\right)\right)\oplus((\ell_1-2)\otimes\mathcal{G}(1))=\mathbf{i}$$

since $\mathcal{G}(i\ell_1+b+1)=\mathbf{i}$ by minimality of n and $\mathcal{G}(1)=\mathbf{0}$.

(b) If $a-i$ is even, let

$$\begin{array}{rcll} h_0 & = & i\ell_1+b+1 & \\ h_j & = & \frac{1}{2}((a-i-1)\ell_1+1) & \text{for } j=1,2 \\ h_j & = & 2 & \text{for } j=3 \\ h_j & = & 1 & \text{for } 4\leq j\leq \ell_1 \end{array}$$

This always exists since $\ell_1 \geq 3$, (if $\ell_1 = 3$, then there are only the first four heaps) and $(a-i-1)$ and ℓ_1 are odd so $(a-i-1)\ell_1+1$ is even. Moreover, it is an option of n:

$$i\ell_1+b+1+2\cdot\frac{1}{2}((a-i-1)\ell_1+1)+2+(\ell_1+3)=i\ell_1+b+1+(a-i-1)\ell_1+\ell_1=a\ell_1+b+1=n.$$

Furthermore, we have:

$$\mathcal{G}(O_n)=\mathcal{G}(i\ell_1+b+1)\oplus\left(2\otimes\mathcal{G}\left(\frac{1}{2}((a-i-1)\ell_1+1)\right)\right)\oplus\mathcal{G}(2)\oplus((\ell_1-3)\otimes\mathcal{G}(1))=\mathbf{i}$$

since $\mathcal{G}(i\ell_1+b+1)=\mathbf{i}$ by minimality of n and $\mathcal{G}(1)=\mathcal{G}(2)=\mathbf{0}$.

This proves that we have at least an option with Grundy value $\mathbf{i}$ for all $0 \leq i < a$, and thus that $\mathcal{G}(n) \geq \mathbf{a}$, a contradiction. Consequently, there is no counterexample to the sequence $(\mathbf{0})^{\ell_1}\ (+\mathbf{1})$. ∎

What is interesting here is that the number of cuts or the values of the cuts are of no importance at all, the only thing that we have to look at is the minimum value of cuts.
Allowing 1-cuts when the rest of the cuts are all odd gives always the same periodic Grundy sequence:

Theorem 4.10 *Let $L=\{1,\ell_2,\dots,\ell_k\}$ be a sequence of odd cut numbers. Then PB(L)'s Grundy sequence is* $(\mathbf{0},\mathbf{1})\ (+\mathbf{0})$.

Proof. We prove this by contradiction. Let n be the smallest positive integer for which the Grundy value of a heap of size n does not match with the sequence $(\mathbf{0},\mathbf{1})\ (+\mathbf{0})$.

First assume that n is even. We will prove that all the options of n have Grundy value $\mathbf{0}$. Let O_n be an option of n. Note that O_n exists since $n \geq 2$ and $1 \in L$. Since all the values of L are odd, O_n contains an even number of non empty heaps whose sum is even. Hence O_n contains an even number of odd-sized heaps. Since all the heaps in O_n are strictly smaller than n, their Grundy values satisfy the sequence $(\mathbf{0},\mathbf{1})$ $(+\mathbf{0})$, which implies that O_n contains an even number of heaps of Grundy value $\mathbf{1}$. Therefore, we have $\mathcal{G}(O_n) = \mathbf{0}$ and thus $\mathcal{G}(n) = \mathbf{1}$. Hence our counterexample n is necessarily odd.

We will show that n has no option of Grundy value 0. It is straightforward if n has no option. Otherwise, let O_n be an option of n. Since all the values of L are odd, O_n contains an even number of non empty heaps whose sum is odd. Hence O_n contains an odd number of odd-sized heaps and an odd number of even-sized heaps. Since all the heaps in O_n are strictly smaller than n, their Grundy values satisfy the sequence $(\mathbf{0},\mathbf{1})$ $(+\mathbf{0})$, which implies that O_n contains an odd number of heaps of Grundy value $\mathbf{1}$. Hence $\mathcal{G}(O_n) = \mathbf{1}$ and thus $\mathcal{G}(n) = \mathbf{0}$.

Consequently, there is no counterexample to the sequence $(\mathbf{0},\mathbf{1})$ $(+\mathbf{0})$. ■

Another easy case to study is when players can cut 1, 2 and 3 times.

Theorem 4.11 *Let $k \geq 3$ and $L = \{1, 2, 3, \ell_4, \dots, \ell_k\}$ be a sequence of cut numbers. The Grundy sequence of PB(L) is* $(\mathbf{0})$ $(+\mathbf{1})$.

Proof. We prove this result by contradiction. Let n be the smallest positive integer such that $\mathcal{G}(n) \neq \mathbf{n-1}$. Note that $n \geq 3$ since we have $\mathcal{G}(1) = \mathbf{0}$ and $\mathcal{G}(2) = \mathbf{1}$.

Suppose first that $\mathcal{G}(n) > \mathbf{n-1}$.
Then n has an option $O_n = (h_0, \dots, h_\ell)$ such that:

$$\sum_{i=0}^{\ell} h_i = n \quad \text{and} \quad \bigoplus_{i=0}^{\ell} \mathcal{G}(h_i) = \bigoplus_{i=0}^{\ell} (h_i - 1) = \mathbf{n-1}.$$

However, $\sum_{i=0}^{\ell}(h_i - 1) = n - \ell - 1$, and since $\ell \geq 1$ we have

$$\mathcal{G}(O_n) = \mathbf{n-1} > \sum_{i=0}^{\ell}(h_i - 1) \geq \bigoplus_{i=0}^{\ell}(\mathbf{h_i - 1}) = \mathcal{G}(O_n),$$

a contradiction.

Thus, there is no option of n with Grundy value $\mathbf{n-1}$, which implies $\mathcal{G}n < \mathbf{n-1}$.
We now prove that, from a heap of n counters, we can play to an option of Grundy value m for all $m < n-1$, which will lead to a contradiction.
If $m = n-2$, then let $O_n = (1, n-1)$ which is clearly an option of n with Grundy value $\mathbf{n-2}$ by minimality of n. Otherwise, let $m < n-2$. There are two cases:

1. If n is even, then there are two subcases:
   (a) If m is odd, $m \in [\![1, n-3]\!]$, let
   $$O_n = (m+1, \frac{n-1-m}{2}, \frac{n-1-m}{2})$$
   obtained by a 2-cut. It is an option of n and by minimality of n, $\mathcal{G}O_n = \mathcal{G}(m+1) = \mathbf{m}$.
   (b) If m is even, $m \in [\![0, n-4]\!]$, let:
   $$O_n = (m+1, 1, \frac{n-m-2}{2}, \frac{n-m-2}{2})$$
   obtained by a 3-cut. It is an option of n and by minimality of n, $\mathcal{G}(O_n) = \mathcal{G}(m+1) = \mathbf{m}$.
2. If n is odd, then there are two subcases:

(a) If m is odd, $m \in [\![1, n-4]\!]$, let:

$$O_n = (m+1, 1, \frac{n-m-2}{2}, \frac{n-m-2}{2})$$

obtained by a 3-cut. It is an option of n and by minimality of n, $\mathcal{G}(O_n) = \mathcal{G}(m+1) = \mathbf{m}$.

(b) If m is even, $m \in [\![0, n-3]\!]$, let:

$$O_n = (m+1, \frac{n-1-m}{2}, \frac{n-1-m}{2})$$

obtained by a 2-cut. It is an option of n and by minimality of n, $\mathcal{G}(O_n) = \mathcal{G}(m+1) = \mathbf{m}$.

Thus, for both cases, $\mathcal{G}(n) \geq \text{mex}(\{\mathbf{0}, ..., \mathbf{n-2}\}) = \mathbf{n-1}$, a contradiction.
Consequently, there is no counterexample to the sequence $(\mathbf{0})^{\mathbf{1}}$ $(+\mathbf{1})$. ■

Let us now make things a little harder. Consider now sequences where players can cut 1, 3 times and also an even number of times. Their Grundy sequences are just harder to describe, but they still have a nice behavior.

Theorem 4.12 *Let $k \geq 1$ and $L = \{1, 3, 2k\}$ be a sequence of cut numbers. The Grundy sequence of PB(L) is* $(\mathbf{0}, \mathbf{1})^{\mathbf{k}}$ $(+\mathbf{2})$.

Proof. We want to prove that for all $n = 2ka + b + 1 \geq 1$, $\mathcal{G}(n) = \mathbf{2a} + (\mathbf{b} \bmod \mathbf{2})$. We proceed by contradiction.
Let $n = 2ka + b + 1$, $0 \leq b < 2k$, be the smallest positive integer such that $\mathcal{G}(n) \neq \mathbf{2a} + (\mathbf{b} \bmod \mathbf{2})$. Note that $n \geq 3$ since we have $\mathcal{G}(1) = \mathbf{0}$ and $\mathcal{G}(2) = \mathbf{1}$.

Assume first that $\mathcal{G}(n) > \mathbf{2a} + (\mathbf{b} \bmod \mathbf{2})$.
Then n has an option $O_n = (2ka_0 + b_0 + 1, \ldots, 2ka_m + b_m + 1)$ with $m \in L$ such that $\mathcal{G}(O_n) = \mathbf{2a} + (\mathbf{b} \bmod \mathbf{2})$.

As O_n is an option of n with Grundy value $\mathbf{2a} + (\mathbf{b} \bmod \mathbf{2})$ and n is minimal, we have, on one hand:

$$\mathcal{G}(O_n) = \bigoplus_{i=0}^{m} (\mathbf{2a_i} + (\mathbf{b_i} \bmod \mathbf{2})) = 2\bigoplus_{i=0}^{m} \mathbf{a_i} + \bigoplus_{i=0}^{m} (\mathbf{b_i} \bmod \mathbf{2}) = \mathbf{2a} + (\mathbf{b} \bmod \mathbf{2}).$$

The second equality holds since 2 is a power of two and for all i, $(b_i \bmod 2) < 2$.
On the other hand we have:

$$n = \sum_{i=0}^{m} (2ka_i + b_i + 1) = 2k\sum_{i=0}^{m} a_i + \sum_{i=0}^{m} b_i + m + 1 = 2ka + b + 1.$$

Since a is the quotient of $n-1$ by $2k$, we have that $a_0 + \ldots + a_m \leq a$, and since $a_0 + \ldots + a_m \geq a_0 \oplus \ldots \oplus a_m$, we have $a = a_0 \oplus \ldots \oplus a_m = a_0 + \cdots + a_m$.
In particular $\sum_{i=0}^{m} b_i + m + 1 = b + 1$. Here we have two cases:

1. If $m = 2k$, then we have $b \geq m = 2k$, a contradiction.

2. If $m \in \{1, 3\}$, then we have:

$$b \bmod 2 = \bigoplus_{i=0}^{m} (b_i \bmod 2) = \left(\bigoplus_{i=0}^{m} b_i\right) \bmod 2 = \left(\sum_{i=0}^{m} b_i\right) \bmod 2 = \left(\sum_{i=0}^{m} b_i + m + 1\right) \bmod 2 = (b+1) \bmod 2$$

a contradiction.

Thus, there are no options of n with Grundy value $\mathbf{2a} + (\mathbf{b} \bmod \mathbf{2})$, which implies $\mathcal{G}(n) < \mathbf{2a} + (\mathbf{b} \bmod \mathbf{2})$. We now prove that, from a heap of n counters, we can play to an option of Grundy value $\mathbf{g}$ for any g in $[\![0, 2a + (b \bmod 2) - 1]\!]$, which will lead to a contradiction.
There are two cases:

1. If b is even, then $2a + (b \bmod 2) = 2a$ and from a heap of size n we can play to:
   (a) for all $x \in [\![0, a-1]\!]$, the options:
   $$O_n = (2kx + b + 1, a - x, \ldots, a - x)$$
   obtained by a $2k$-cut. By minimality of n, $\mathcal{G}(O_n) = \mathbf{2x}$. By doing this, we obtain the even Grundy values in $[\![0, 2a-2]\!]$.
   (b) if $b = 0$, for all $x \in [\![1, a-1]\!]$, the options:
   $$O_n = (2kx + b, 1, (a-x)k, (a-x)k)$$
   obtained by a 3-cut. By minimality of n, $\mathcal{G}(O_n) = \mathbf{2(x-1)} + (\mathbf{2k-1} \bmod \mathbf{2}) = \mathbf{2x-1}$ since $x \geq 1$. By doing this, we obtain the odd Grundy values in $[\![1, 2a-3]\!]$ and the value $2a-1$ is obtained by the option $O_n = (2ka, 1)$.
   (c) if $b > 0$, for all $x \in [\![0, a-1]\!]$, the options:
   $$O_n = (2kx + b, 1, (a-x)k, (a-x)k)$$
   obtained by a 3-cut. By minimality of n, $\mathcal{G}(O_n) = \mathbf{2x} + (\mathbf{b-1} \bmod \mathbf{2}) = \mathbf{2x+1}$ since b is even. By doing this, we obtain the odd Grundy values in $[\![1, 2a-1]\!]$.

   Putting the three previous cases altogether, this implies $\mathcal{G}(n) \geq \mathbf{2a}$, being a contradiction.

2. If b is odd, then $2a + (b \bmod 2) = 2a + 1$, and from a heap of size n we can play to:
   (a) for all $x \in [\![0, a-1]\!]$, the options:
   $$O_n = (2kx + b + 1, a - x, \ldots, a - x)$$
   obtained by a $2k$-cut. By minimality of n, $\mathcal{G}(O_n) = \mathbf{2x+1}$. By doing this, we obtain the odd Grundy values in $[\![1, 2a-1]\!]$.
   (b) for all $x \in [\![0, a-1]\!]$, the options:
   $$O_n = (2kx + b, 1, (a-x)k, (a-x)k)$$
   obtained by a 3-cut. By minimality of n, $\mathcal{G}(O_n) = \mathbf{2x}$. By doing this, we obtain the even Grundy values in $[\![0, 2a-2]\!]$ and the value $2a$ is obtained by the option $O_n = (2ka + b, 1)$.

   Altogether, this implies $\mathcal{G}(n) \geq \mathbf{2a+1}$, a contradiction.

Consequently, there is no counterexample to the sequence $(\mathbf{0}, \mathbf{1})^{\mathbf{k}}\ (+\mathbf{2})$.

■

This last case says that if $k = 1$, then the Grundy sequence of $\{1, 2, 3\}$ is $(\mathbf{0}, \mathbf{1})^{\mathbf{1}}\ (+\mathbf{2})$ which is exactly the same sequence as $(\mathbf{0})\ (+\mathbf{1})$, hence Theorem 4.12 contains the particular case $L = \{1, 2, 3\}$ but says nothing about cut lists of the form $L = \{1, 2, 3, \ell_4, \ldots, \ell_k\}$ for $k > 3$. As well, Theorem 4.12 says nothing about cut lists of the form $\{1, 3, 2k, \ell_4, \ldots, \ell_k\}$.
In Table 4.3.2 a summary is presented.
Here lies a global view of different classes of pure breaking games. All games seem, however, to be arithmetic-periodic or periodic. The remaining cases are not so natural to prove, and as, by computations, they seem arithmetic-periodic, we were inspired by Theorem 4.8 to look into arithmetic-periodicity tests. In the next section details of some other families are given and the arithmetic-periodic test is introduced.

Table 4.3.2: Summary of Grundy sequences of pure breaking Games

cut sequence	Grundy sequence	reduction
$L = \{\ell_1, \ldots, \ell_k\}$, $\ell_1 > 1$	$(\mathbf{0})^{\ell_1}$ $(+\mathbf{1})$	$\{\ell_1\}$
$L = \{1, \ell_2, \ldots, \ell_k\}$, ℓ_i odd	$(\mathbf{0}, \mathbf{1})$ $(+\mathbf{0})$	$\{1\}$
$L = \{1, 2, 3, \ell_4, \ldots, \ell_k\}$	$(\mathbf{0})$ $(+\mathbf{1})$	$\{1, 2, 3\}$
$L = \{1, 3, 2k\}$, $k \geq 1$	$(\mathbf{0}, \mathbf{1})^{\mathbf{k}}$ $(+\mathbf{2})$	–

4.3.3 Arithmetic-Periodicity test

Consider the games PB({1,4}) and PB({1,6}). The options for a heap of size 17 are shown in Table 4.3.3. As we can see, it becomes quickly very hard to keep track of all the options. In Tables 4.3.4 and 4.3.5 we present the first Grundy values for these two games.

Table 4.3.3: Options of a heap of size 17 for the games PB({1,4}) and PB({1,6})

game	17
1-cuts	(1, 16), (2, 15), (3, 14), (4, 13), (5, 12), (6, 11), (7, 10), (8, 9)
4-cuts	(1,1,1,1,13), (1,1,1,2,12), (1,1,1,3,11), (1,1,1,4,10), (1,1,1,5,9), (1,1,1,6,8), (1,1,1,7,7), (1,1,2,2,11), (1,1,2,3,10), (1,1,2,4,9), (1,1,2,5,8), (1,1,2,6,7), (1,1,3,3,9), (1,1,3,4,8), (1,1,3,5,7), (1,1,3,6,6), (1,1,4,4,7), (1,1,4,5,6), (1,1,5,5,5) (1,2,2,2,10), (1,2,2,3,9), (1,2,2,4,8), (1,2,2,5,7), (1,2,2,6,6), (1,2,3,3,8), (1,2,3,4,7), (1,2,3,5,6), (1,2,4,4,6), (1,2,4,5,5), (1,3,3,3,7), (1,3,3,4,6), (1,3,3,5,5), (1,3,4,4,5), (1,4,4,4,4), (2,2,2,2,9), (2,2,2,3,8), (2,2,2,4,7), (2,2,2,5,6), (2,2,3,3,7), (2,2,3,4,6), (2,2,3,5,5), (2,2,4,4,5), (2,3,3,3,6), (2,3,3,4,5), (2,3,4,4,4), (3,3,3,3,5), (3,3,3,4,4)
6-cuts	(1,1,1,1,1,1,11), (1,1,1,1,1,2,10), (1,1,1,1,1,3,9), (1,1,1,1,1,4,8), (1,1,1,1,1,5,7), (1,1,1,1,1,6,6), (1,1,1,1,2,2,9), (1,1,1,1,2,3,8), (1,1,1,1,2,4,7), (1,1,1,1,2,5,6), (1,1,1,1,3,3,7), (1,1,1,1,3,4,6), (1,1,1,1,3,5,5), (1,1,1,1,4,4,5), (1,1,1,2,2,2,8), (1,1,1,2,2,3,7), (1,1,1,2,2,4,6), (1,1,1,2,2,5,5), (1,1,1,2,3,3,6), (1,1,1,2,3,4,5), (1,1,1,2,4,4,4), (1,1,1,3,3,3,5), (1,1,1,3,3,4,4), (1,1,2,2,2,2,7), (1,1,2,2,2,3,6), (1,1,2,2,2,4,5), (1,1,2,2,3,3,5), (1,1,2,2,3,4,4), (1,1,2,3,3,3,4), (1,1,3,3,3,3,3), (1,2,2,2,2,2,6), (1,2,2,2,2,3,5), (1,2,2,2,2,4,4), (1,2,2,2,3,3,4), (1,2,2,3,3,3,3), (2,2,2,2,2,2,5), (2,2,2,2,2,3,4), (2,2,2,2,3,3,3)

Table 4.3.4: First 72 Grundy values of the game PB({1,4})

b	0	1	2	3	4	5	6	7	8	9	10	11
$12 \times 0 + b$	**0**	**1**	**0**	**1**	**2**	**3**	**2**	**3**	**1**	**4**	**5**	**4**
$12 \times 1 + b$	**3**	**2**	**3**	**2**	**4**	**5**	**4**	**5**	**6**	**7**	**6**	**7**
$12 \times 2 + b$	**8**	**9**	**8**	**9**	**10**	**11**	**10**	**11**	**9**	**12**	**13**	**12**
$12 \times 3 + b$	**11**	**10**	**11**	**10**	**12**	**13**	**12**	**13**	**14**	**15**	**14**	**15**
$12 \times 4 + b$	**16**	**17**	**16**	**17**	**18**	**19**	**18**	**19**	**17**	**20**	**21**	**20**
$12 \times 5 + b$	**19**	**18**	**19**	**18**	**20**	**21**	**20**	**21**	**22**	**23**	**22**	**23**

Table 4.3.5: First 108 Grundy values of the game PB({1,6})

b	0	1	2	3	4	5	6	7	8	9	10	11
$12 \times 0 + b$	**0**	**1**	**0**	**1**	**0**	**1**	**2**	**3**	**2**	**3**	**2**	**3**
$12 \times 1 + b$	**1**	**4**	**5**	**4**	**5**	**4**	**3**	**2**	**3**	**2**	**3**	**2**
$12 \times 2 + b$	**4**	**5**	**4**	**5**	**4**	**5**	**6**	**7**	**6**	**7**	**6**	**7**
$12 \times 3 + b$	**8**	**9**	**8**	**9**	**8**	**9**	**10**	**11**	**10**	**11**	**10**	**11**
$12 \times 4 + b$	**9**	**12**	**13**	**12**	**13**	**12**	**11**	**10**	**11**	**10**	**11**	**10**
$12 \times 5 + b$	**12**	**13**	**12**	**13**	**12**	**13**	**14**	**15**	**14**	**15**	**14**	**15**
$12 \times 6 + b$	**16**	**17**	**16**	**17**	**16**	**17**	**18**	**19**	**18**	**19**	**18**	**19**
$12 \times 7 + b$	**17**	**20**	**21**	**20**	**21**	**20**	**19**	**18**	**19**	**18**	**19**	**18**
$12 \times 8 + b$	**20**	**21**	**20**	**21**	**20**	**21**	**22**	**23**	**22**	**23**	**22**	**23**

These sequences look similar: the first one seems to have a period of 24 and a saltus of 8, and the second one seems to have a period of 36 and also a saltus of 8. Moreover, there are some common patterns like the repetition of **(0,1)**, **(2,3)**, **(4,5)** and **(6,7)**. Inspired by the octal periodicity test and the similar results for hexadecimal games, we eased up the *case by case* study by introducing an arithmetic-periodic test:

Definition 4.13 (Arithmetic-Periodic Test (*AP*-test).) *Let PB(L) be a pure breaking game. We say that PB(L) satisfies the AP-test if there exist a positive integer p and a power of two s such that:*

$AP1$ *for* $n \leq 3p$, $\mathcal{G}(n+p) = \mathcal{G}(n) + \mathbf{s}$,

$AP2$ $\mathrm{Im}(\mathcal{G}\ |_{[\![1,p]\!]}) = [\![\mathbf{0}, \mathbf{s}-\mathbf{1}]\!]$, *and*

$AP3$ *for all* $n \in [\![3p+1, 4p]\!]$ *and for all* $\mathbf{g} \in [\![\mathbf{0}, \mathbf{s}-\mathbf{1}]\!]$, *the heap of size n,* H_n *has an option* O_n *over* $(m+1)$ *non-empty heaps such that* $m \geq 2$, $m \in L$ *and* $\mathcal{G}(O_n) = \mathbf{g}$.

The first two conditions seem rather standard, compared to similar results on subtraction games, octal and hexadecimal games. The third condition, however, is unusual. We will see later that for some particular pure breaking games, the first two conditions imply the third one. Remark that here the period is a power of two: this is a strict condition, but, for pure breaking games the saltuses seem always to be powers of two.

Theorem 4.14 *Let* $L = \{\ell_1, \ldots, \ell_k\}$ *be a set of positive integers with* $\ell_k \geq 2$ *and such that PB(L) verifies the AP-test. Then for all* $n \geq 1$, $\mathcal{G}(n+p) = \mathcal{G}(n) + \mathbf{s}$.

In other words, if a pure breaking game verifies the *AP*-test, then it is arithmetic-periodic.
To prove this result we need first some lemmas:

Lemma 4.15 *Let* $L = \{\ell_1, \ldots, \ell_k\}$ *be a set of positive integers. In the game PB(L), if there exist two positive integers p and s, and* $n_0 \geq p$ *such that for all* $n \leq n_0$, $\mathcal{G}(n+p) = \mathcal{G}(n) + \mathbf{s}$ *then for all* $1 \leq n = ap+1+b \leq n_0 + p$ *with* $0 \leq b < p$, *we have*

$$\mathcal{G}(n) = \mathbf{as} + \mathcal{G}(1+b)$$

Proof. It is clear that for all $1 \leq n \leq p$, we have $n = ap+1+b$ with $a = 0$ and $0 \leq b < p$, and hence $\mathcal{G}(n) = \mathbf{as} + \mathcal{G}(1+b)$.
Let $n = ap+b+1 \leq n_0 + p$ be the smallest integer such that $\mathcal{G}(n) \neq \mathbf{as} + \mathcal{G}(1+b)$. From the previous remark, we know that $n > p$. The Grundy value of n is:

$$\mathcal{G}(n) = \mathcal{G}(n-p) + \mathbf{s} = \mathcal{G}((a-1)p+1+b) + \mathbf{s},$$

remark this equality holds since $n \leq n_0 + p$.
Since n is minimal and $(a-1)p+1+b < n$, we have $\mathcal{G}((a-1)p+1+b) = (\mathbf{a}-\mathbf{1})\mathbf{s} + \mathcal{G}(1+b)$, and thus

$$\mathcal{G}(n) = \mathbf{as} + \mathcal{G}(1+b),$$

which contradicts our initial hypothesis. ∎

As a direct consequence, if Lemma 4.15 is satisfied with the two additional constraints:

- s is a power of 2
- $\mathcal{G}(n) < s$ for all $1 \leq n \leq p$,

then any disjunctive sum $G = (a_0p+1+b_0, \ldots, a_mp+1+b_m)$ with $a_jp+1+b_j \leq n_0+p$ and $0 \leq b_j < p$ for all $0 \leq j \leq m$ satisfies

$$\mathcal{G}(G) = (\mathbf{a_0} \oplus \cdots \oplus \mathbf{a_m})\mathbf{s} + (\mathcal{G}(1+b_0) \oplus \cdots \oplus \mathcal{G}(1+b_m)) \tag{4.3.1}$$

The Theorem 4.14 is proved by induction, with a rather technical base case. Part of this base case is considered in the following lemma. Moreover, this lemma exposes why the condition $\ell_k \geq 2$ is necessary.

Lemma 4.16 *Let $L = \{\ell_1, \dots, \ell_k\}$ be a set of positive integers with $\ell_k \geq 2$ such that PB(L) verifies the test AP.*
Then for $i = 2, 3$, for all n in $[\![ip+1, (i+1)p]\!]$ and for all g in $[\![0, (i-1)s-1]\!]$, there is an option $O_n = (h_0, \dots, h_m), m \in L$ of n such that $m \geq 2$ and $\mathcal{G}(O_n) = \}$.

Proof. Let $L = \{\ell_1, \dots, \ell_k\}$ be such a set.

- We first consider the case $i = 3$. Let $n = 3p + 1 + b \in [\![3p+1, 4p]\!]$ and $g \in [\![0, 2s-1]\!]$.
  If $g \in [\![0, s-1]\!]$ then condition $AP3$ ensures such an option exists.
  Now, for $g \in [\![s, 2s-1]\!]$, by the conditions $AP1$ and $AP2$, Lemma 4.15 can be applied, implying that $\mathcal{G}(n) = \mathbf{3s} + \mathcal{G}(1+b)$ and hence that there is an option O_n of n such that $\mathcal{G}(O_n) = \mathbf{g}$. If $1 \notin L$, there is nothing to prove. Consequently, it suffices to prove that if $1 \in L$, and $O_n = (h_0, h_1)$ is an option of n obtained by a 1-cut, then $\mathcal{G}(O_n) \notin [\![s, 2s-1]\!]$. This result would indeed guarantee that all the options of n with Grundy value in $[\![s, 2s-1]\!]$ are obtained by m-cuts with $m \geq 2$.
  Assume $1 \in L$ and let $O_n = (h_0, h_1)$ be an option of n obtained by a 1-cut. There exist four unique non-negative integers a_0, b_0, a_1, b_1 such that $0 \leq b_0, b_1 < p$ and $O_n = (a_0 p + 1 + b_0, a_1 p + 1 + b_1)$. As O_n is an option of n we have:

  $$(a_0 + a_1)p + 1 + 1 + b_0 + b_1 = n = 3p + 1 + b$$

  which gives

  $$1 + b_0 + b_1 - b = (3 - a_0 - a_1)p.$$

  As $0 \leq a_0 + a_1 \leq 3$ and $b_0 + b_1 + 1 < 2p$, we have in one hand $0 \leq 1 + b_0 + b_1 - b < 2p$ and in the other hand that $1 + b_0 + b_1 - b \equiv 0 \pmod{p}$. Hence $1 + b_0 + b_1 - b \in \{0, p\}$. If it equals 0 then $a_0 + a_1 = 3$, otherwise $a_0 + a_1 = 2$. Without loss of generality the possible values for a_0, a_1 and $a_0 \oplus a_1$ are summarized in the following table:

a_0	a_1	$a_0 \oplus a_1$
0	2	2
	3	3
1	1	0
	2	3

  In particular, we remark that $a_0 \oplus a_1 \neq 1$. And, by Equation (4.3.1) we have: $\mathcal{G}(O_n) = (\mathbf{a_0} \oplus \mathbf{a_1})\mathbf{s} + \mathcal{G}(1+b_0) \oplus \mathcal{G}(1+b_1) \notin [\![s, 2s-1]\!]$ since s is a power of two and $\mathcal{G}(1+b_0), \mathcal{G}(1+b_1) < \mathbf{s}$.

- We now consider the case $i = 2$. Let $n \in [\![2p+1, 3p]\!]$ and $g \in [\![0, s-1]\!]$.
  Let $n' = n + p \in [\![3p+1, 4p]\!]$ and $g' = g + s \in [\![s, 2s-1]\!]$.
  By the first part of the proof, we know that there is an option $O_{n'} = (a_{0,n'}p + 1 + b_{0,n'}, \dots, a_{m,n'}p + 1 + b_{m,n'})$ of n' such that $m \geq 2$ and $\mathcal{G}(O_{n'}) = \mathbf{g'}$. Let $N = (a_{0,n'} \oplus \dots \oplus a_{m,n'})$, $S = a_{0,n'} + \dots + a_{m,n'}$ and $R = \mathcal{G}(1 + b_{0,n'}) \oplus \dots \oplus \mathcal{G}(1 + b_{m,n'})$. Remark that $N = 1$ and $\mathcal{G}(O_{n'}) = \mathbf{Ns} + \mathbf{R}$ since we can apply Equation (4.3.1) to $O_{n'}$ and $g' \in [\![s, 2s-1]\!]$. We define the following m-cut option O_n of n by:

  $$\begin{aligned} h_0 &= 1 + b_{0,n'} \\ h_j &= \tfrac{1}{2}(S-1)p + 1 + b_{j,n'} && \text{for } j = 1, 2 \\ h_j &= 1 + b_{j,n'} && \text{for } 3 \leq j \leq m \end{aligned}$$

  Remark that $S - 1 = S - N$ which is even and non-negative.
  Note that O_n is indeed an option of n since we have that $h_0 + \dots + h_m = (S-1)p + (1 + b_{0,n'} + \dots + 1 + b_{m,n'}) = n' - p = n$. By Equation (4.3.1), we have $\mathcal{G}(O_n) = \mathbf{R} = \mathbf{g'} - \mathbf{s} =_m athbfg$. Hence, O_n is indeed an option of n with $m \geq 2$ and $\mathcal{G}(O_n) = \mathbf{g}$.

■

We can now prove Theorem 4.14, meaning that if a pure breaking game verifies the AP-test, then its Grundy sequence is arithmetic periodic.

Proof of Theorem 4.14. Let us begin with some notations.

For all $1 \leq n \leq p$ we denote $\mathbf{r_n} = \mathcal{G}(n)$; thus for $0 \leq a < 4$ and $n = ap+b+1 \in [\![ap+1,(a+1)p]\!]$, and by Lemma 4.15 we have $\mathcal{G}(n) = \mathcal{G}(ap+b+1) = \mathbf{as} + \mathbf{r_{b+1}}$. Remark that for a family of non-negative integers $a_0, \ldots, a_m$, if $S = a_0 + \cdots + a_m$ and $N = a_0 \oplus \cdots \oplus a_m$ then $S \geq N$ and $S \equiv N \bmod 2$. In particular, $S - N$ is an even non-negative integer.

We will now prove by induction that for $n = ap+1+b \geq 1$, the following two properties hold:

(A) $\mathcal{G}(n) = \mathbf{as} + \mathbf{r_{1+b}}$ and

(B) for all $g \in [\![0,(a-1)s-1]\!]$, there is an option $O_n = (h_0, \ldots, h_m)$ of n such that $m \geq 2$ and $\mathcal{G}(O_n) = \mathbf{g}$.

Let $n = ap+1+b$ be the smallest positive integer such that either (A) or (B) is not verified. By Lemma 4.15, we know that (A) holds for all $n \leq 4p$. Moreover, by Lemma 4.16, we know that (B) holds for $a = 2,3$, and it is trivially true for $a \leq 1$. Thus $n > 4p$.

Let $n = ap+1+b > 4p$. We consider two cases:

1. Assume (A) is not verified. Thus either $\mathcal{G}(n) < \mathbf{as} + \mathbf{r_{1+b}}$ or $\mathcal{G}(n) > \mathbf{as} + \mathbf{r_{1+b}}$.

   (a) if $\mathcal{G}(n) < \mathbf{as} + \mathbf{r_{1+b}}$: by minimality of n, the heap of size $n' = n - 2p = a'p + 1 + b'$ verifies conditions (A) and (B). Let $O_{n'} = (a_{0,n'}p + 1 + b_{0,n'}, \ldots, a_{m,n'}p + 1 + b_{m,n'})$ be an option of n' with Grundy value g, for some $g < (a'-1)s$ and $m \geq 2$. Let $N = a_{0,n'} \oplus \cdots \oplus a_{m,n'}$, $S = a_{0,n'} + \cdots + a_{m,n'}$ and $R = \mathcal{G}(1+b_{0,n'}) \oplus \cdots \oplus \mathcal{G}(1+b_{m,n'})$. Let O_n be the following option:

$$\begin{aligned} h_0 &= Np + 1 + b_{0,n'} \\ h_j &= \tfrac{1}{2}(S - N + 2)p + 1 + b_{j,n'} && \text{for } j = 1,2 \\ h_j &= 1 + b_{m,n'} && \text{for } j > 2 \end{aligned}$$

   This is an option of n since $h_0 + \cdots + h_m = (2+S)p + 1 + b_{0,n'} + \cdots + 1 + b_{m,n'}$ and its Grundy value is $\mathcal{G}(O_n) = \mathbf{Ns} + \mathbf{R} = \mathbf{g}$ by Equation (4.3.1).
   Hence, the heap of size n has options to all Grundy values in $[\![0,(a'-1)s-1]\!]$, *i.e.* $\mathcal{G}(n) \geq (\mathbf{a'} - \mathbf{1})\mathbf{s}$.
   We now change O_n into O'_n as follows:

$$\begin{aligned} h'_0 &= (N+2)p + 1 + b_{0,n'} \\ h'_j &= \tfrac{1}{2}(S - N)p + 1 + b_{j,n'} && \text{for } j = 1,2 \\ h'_j &= 1 + b_{j,n'} && \text{for } j > 2 \end{aligned}$$

   This option is an option of n since $h'_0 + \cdots + h'_m = (2+S)p + 1 + b_{0,n'} + \cdots + 1 + b_{m,n'}$ and its Grundy value is $\mathcal{G}(O'_n) = (\mathbf{N}+\mathbf{2})\mathbf{s} + \mathbf{R} = \mathbf{g} + \mathbf{2s}$.
   Hence, the heap of size n has options of Grundy values in $[\![2s,(a-1)s-1]\!]$. If $a > 4$ then with the previous remark, the heap of size n has options to all Grundy values in $[\![0,(a-1)s-1]\!]$.
   Otherwise, if $a = 4$, then we take an option $O_{n'} = (a_{0,n'}p + 1 + b_{0,n'}, \ldots, a_{m,n'}p + 1 + b_{m,n'})$ of $n' = 3p+1+b = n-p$ with Grundy value $\mathbf{g}$ in $[\![0,s-1]\!]$ and $m \geq 2$, which exists by Lemma 4.16.
   We note $S = a_{0,n'} + \cdots + a_{m,n'}$, $N = a_{0,n'} \oplus \cdots \oplus a_{m,n'}$ and $R = \mathcal{G}(1+b_{0,n'}) \oplus \cdots \oplus \mathcal{G}(1+b_{m,n'})$.

We transform it into an option $O_n = (h_0, \dots, h_m)$ by:

$$\begin{array}{rcll} h_0 & = & (N+1)p + 1 + b_{0,n'} & \\ h_j & = & \frac{1}{2}(S-N)p + 1 + b_{j,n'} & \text{for } j = 1,2 \\ h_j & = & 1 + b_{j,n'} & \text{for } 3 \leq j \leq m \end{array}$$

it is an option of n since $h_0 + \dots + h_m = (S+1)p + 1 + b_{0,n'} + \dots + 1 + b_{m,n'} = n' + p = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(O_{n'}) + \mathbf{s} = \mathbf{g} + \mathbf{s}$.
Hence, even for $a = 4$, the heap of size n has options obtained by m-cuts, $m \geq 2$, to all Grundy values in $[\![0, (a-1)s]\!]$, hence the heap of size n verifies (B).

Now, let $n'' = n - (a-1)p = p + 1 + b$ and $g \in [\![0, s + r_{1+b} - 1]\!]$.
Let $O_{n''} = (a_{0,n''}p + 1 + b_{0,n''}, \dots, a_{m,n''}p + 1 + b_{m,n''})$ be an option of n'' such that $\mathcal{G}(O_{n''}) = \mathbf{g}$.
It exists since the heap of size n'' verifies (B) by minimality of n. Please remark that as $n'' \leq 2p$, if there is a j such that $a_{j,n''} \neq 0$, then it is unique, without loss of generality, assume that $a_{0,n''} \in \{0,1\}$ and for $j > 0$, $a_{j,n''} = 0$. Hence if $R = \mathcal{G}(1 + b_{0,n''}) \oplus \dots \oplus \mathcal{G}(1 + b_{m,n''})$ then $\mathcal{G}(O_{n''}) = \mathbf{a_{0,n''}s} + \mathbf{R}$ by Equation (4.3.1).
Let O_n be the following option:

$$\begin{array}{rcll} h_0 & = & (a_{0,n''} + a - 1)p + 1 + b_{0,n''} & \\ h_j & = & 1 + b_{j,n''} & \text{for } j > 0 \end{array}$$

This is an option of n since $h_0 + \dots + h_m = (a_{0,n''} + a - 1)p + 1 + b_{0,n''} + 1 + b_{1,n''} + \dots + 1 + b_{m,n''} = n'' + (a-1)p = n$. Its Grundy value is $\mathcal{G}(O_n) = (\mathbf{a_{0,n''}} + \mathbf{a} - \mathbf{1})\mathbf{s} + \mathbf{R} = \mathbf{g} + (\mathbf{a} - \mathbf{1})\mathbf{s}$. Hence, the heap of size n has options to all Grundy values in $[\![(a-1)s, as + r_{1+b} - 1]\!]$. With the previous remarks, the heap of size n has options to all Grundy values in $[\![0, as + r_{1+b} - 1]\!]$.
Altogether, this means $\mathcal{G}(n) \geq \mathbf{as} + \mathbf{r_{1+b}}$, a contradiction.

(b) Now, if $\mathcal{G}(n) > \mathbf{as} + \mathbf{r_{1+b}}$:
Let $O_n = (a_0 p + 1 + b_0, \dots, a_m p + 1 + b_m)$ be an option of n with Grundy value $\mathbf{as} + \mathbf{r_{1+b}}$. Let $N = a_0 \oplus \dots \oplus a_m$, $S = a_0 + \dots + a_m$ and $R = \mathcal{G}(1 + b_0) \oplus \dots \oplus \mathcal{G}(1 + b_m)$. Remark that by Equation (4.3.1) $a_0 \oplus \dots \oplus a_m = a$ and as $S \geq N$, $S = a$. Let $O_{n'}$ be the following option of $n' = n - 2p$:

$$\begin{array}{rcll} h'_0 & = & (a-2)p + 1 + b_0 & \\ h'_j & = & 1 + b_j & \text{for } j > 1 \end{array}$$

This is an option of n' since $h'_0 + \dots + h'_m = (a-2)p + 1 + b = n - 2p$ and its Grundy value is $\mathcal{G}(O_{n'}) = (\mathbf{a} - \mathbf{2})\mathbf{s} + \mathbf{R} = \mathbf{as} + \mathbf{r_{1+b}} - \mathbf{2s} = \mathcal{G}(n')$, a contradiction.

Hence, the heap of size n verifies (A).

2. Assume (B) is not verified:
By minimality of n, the heap of size $n' = n - 2p = a'p + 1 + b'$ verifies conditions (A) and (B). Let $O_{n'} = (h_{0,n'}, \dots, h_{m,n'}) = (a_{0,n'}p + 1 + b_{0,n'}, \dots, a_{m,n'}p + 1 + b_{m,n'})$ be an option of n' with Grundy value $\mathbf{g}$, for some $g < (a'-1)s$ and with $m \geq 2$. Let $N = a_{0,n'} \oplus \dots \oplus a_{m,n'}$, $S = a_{0,n'} + \dots + a_{m,n'}$ and $R = \mathcal{G}1 + b_{0,n'} \oplus \dots \oplus \mathcal{G}1 + b_{m,n'}$. Let O_n be the following option:

$$\begin{array}{rcll} h_0 & = & Np + 1 + b_{0,n'} & \\ h_j & = & \frac{1}{2}(S - N + 2) + 1 + b_{j,n'} & \text{for } j = 1,2 \\ h_j & = & 1 + b_{m,n'} & \text{for } j > 2 \end{array}$$

This is an option of n since $h_0 + \cdots + h_m = 2p + h_{0,n'} + \cdots + h_{m,n'}$ and its Grundy value is $\mathcal{G}(O_n) = \mathbf{Ns} + \mathbf{R} = \mathbf{g}$.
Hence, the heap of size n has options obtained by m-cuts with $m \geq 2$ to all Grundy values in $[\![0, (a'-1)s-1]\!]$.
We now change O_n into O'_n as follows:

$$\begin{aligned} h'_0 &= (N+2)p + 1 + b_{0,n'} \\ h'_j &= \tfrac{1}{2}(S-N) + 1 + b_{j,n'} \qquad \text{for } j = 1,2 \\ h'_j &= 1 + b_{j,n'} \qquad \text{for } j > 2 \end{aligned}$$

This option is an option of n since $h'_0 + \cdots + h'_m = 2p + h_{0,n'} + \cdots + h_{m,n'}$ and its Grundy value is $\mathcal{G}(O'_n) = (\mathbf{N}+\mathbf{2})\mathbf{s} + \mathbf{R} = \mathbf{g} + \mathbf{2s}$.
Hence, the heap of size n has options obtained by m-cuts, $m \geq 2$ to all Grundy values in $[\![2s, (a-1)s-1]\!]$. With the previous remark, this is true for all Grundy values in $[\![0, (a-1)s-1]\!]$. Hence the heap of size n verifies (B), a contradiction.

■

This result is strong as it ensures some sequences are arithmetic-periodic as long as they seem so for 4 periods. The third condition of the theorem seems complex to verify (as it needs to keep track of all options within the range of Grundy values, instead of only looking at the Grundy values). In fact, in some specific cases this last condition is unnecessary.

Theorem 4.17 *Let $L = \{\ell_1, \ldots, \ell_k\}$ be a sequence of positive integers, $k > 1$. If PB(L) verifies the conditions $AP1$ and $AP2$ if the AP-test and there are $m_1, m_2 \in L$ of different parities such that $2 \leq m_1, m_2 \leq 2p+1$, then PB(L) verifies the AP-test.*

Proof. It suffices to prove that L verifies the condition $AP3$ of the AP-test. Without loss of generality, we can consider that m_1 is even and m_2 is odd. We prove that for all $n \in [\![3p+1, 4p]\!]$ and $g \in [\![0, s-1]\!]$ there is an option $O_n = (h_0, \ldots, h_m)$ of n such that $m \geq 2$ and $\mathcal{G}(O_n) = \mathbf{g}$.
Let $n = 3p + 1 + b$ with $0 \leq b < p$ and $g \in [\![0, s-1]\!]$.
By $AP2$, there is $c \in [\![0, p-1]\!]$ such that $\mathcal{G}(1+c) = \mathbf{g}$. Let $n' = n - 1 - c = 3p + b - c$. We consider two cases:

- if n' is even: let (q_1, r_1) be the unique couple such that $0 \leq r_1 < m_1$ and $n' = m_1 q_1 + r_1$. In particular, r_1 is even, since m_1 and n' are also even. Moreover $q_1 > 0$ since $m_1 \leq 2p+1 \leq n'$. We define an option O_n of n by:

$$\begin{aligned} h_0 &= 1 + c \\ h_j &= q_1 + \tfrac{1}{2} r_1 \qquad \text{for } j = 1,2 \\ h_j &= q_1 \qquad \text{for } 3 \leq j \leq m_1 \end{aligned}$$

  It is indeed an option of n since $h_0 + \cdots + h_{m_1} = 1 + c + m_1 q_1 + r_1 = 1 + c + n' = n$ and in the expression $\mathcal{G}(h_0) \oplus \cdots \oplus \mathcal{G}(h_m)$, the terms $\mathcal{G}(h_1)$ and $\mathcal{G}(h_3)$ appear an even number of times, which gives directly $\mathcal{G}(O_n) = \mathcal{G}(1+c) = \mathbf{g}$.

- if n' is odd: let (q_2, r_2) be the unique couple such that $0 \leq r_2 < m_2$, $n' = m_2 q_2 + r_2$. Please remark that $q_2 > 0$ since $m_2 \leq 2p+1 \leq n'$. As n' and m_2 are odd, either q_2 is even and r_2 is odd or vice versa.

– if q_2 is even and r_2 is odd, we define the option O_n by:

$$\begin{aligned} h_0 &= 1+c \\ h_j &= \tfrac{3}{2}q_2 + \tfrac{1}{2}(r_2-1) && \text{for } j=1,2 \\ h_j &= 1 && \text{for } j=3 \\ h_j &= q_2 && \text{for } 4 \le j \le m_2 \end{aligned}$$

If $m_2 = 3$ then we only take the four first heaps.
The option O_n is an option of n since $h_0+\cdots+h_{m_2} = 1+c+3q_2+r_2-1+1+(m_2-1-2)q_2 = 1+c+m_2q_2+r_2 = 1+c+n' = n$. In the expression $\mathcal{G}(h_0)\oplus\cdots\oplus\mathcal{G}(h_{m_2})$ the terms $\mathcal{G}(h_1)$ and $\mathcal{G}(h_4)$ appear an even number of times and $\mathcal{G}(h_3) = \mathbf{0}$, hence $\mathcal{G}(O_n) = \mathbf{1}+\mathbf{c} = \mathbf{g}$.

– if q_2 is odd and r_2 is even, we define the option O_n by:

$$\begin{aligned} h_0 &= 1+c \\ h_j &= \tfrac{1}{2}(3q_2-1) + \tfrac{1}{2}r_2 && \text{for } j=1,2 \\ h_j &= 1 && \text{for } j=3 \\ h_j &= q_2 && \text{for } 4 \le j \le m_2 \end{aligned}$$

it is an option of n since $h_0+\cdots+h_{m_2} = 1+c+3q_2-1+r_2+1+(m_2-3)q_2 = 1+c+m_2q_2+r_2 = n$. In the expression $\mathcal{G}(h_0)\oplus\cdots\oplus\mathcal{G}(h_{m_2})$ the terms $\mathcal{G}(h_1)$ and $\mathcal{G}(h_4)$ appear an even number of times and $\mathcal{G}(h_3) = \mathbf{0}$, hence $\mathcal{G}(O_n) = \mathbf{g}$.

In every case, there is an option O_n of n obtained by an m-cut, $m \ge 2$, such that $\mathcal{G}(O_n) = \mathbf{g}$, *i.e.*, PB(L) verifies the condition $AP3$, which means PB(L) verifies the test AP.

■

This last result is, for instance, useful for the list $(1, 2k)$, $k \ge 3$.

Theorem 4.18 *Let $L = \{1, \ell\}$, $\ell > 2$, even. If PB(L) verifies the conditions $AP1$ and $AP2$ of the AP-test for some p with $\ell \le p$ and there are $x_1, x_2 \le p/2$ such that $\mathcal{G}(x_1) = \mathcal{G}(x_2) = 1$ and x_1 is odd and x_2 is even, then PB(L) verifies the AP-test.*

Proof. We prove that the game PB(L) verifies the condition $AP3$, *i.e.*, that for $n \in [\![3p+1, 4p]\!]$ and for $g \in [\![0, s-1]\!]$, there exists an option O_n of n such that $\mathcal{G}(O_n) = \mathbf{g}$. Since the condition $AP2$ is verified, this can be done by proving that for all $n \in [\![3p+1, 4p]\!]$ and for all $k \in [\![1, p]\!]$, there exists an option O_n of n such that $\mathcal{G}(O_n) = \mathcal{G}(k)$.
Let $n \in [\![3p+1, 4p]\!]$ and $k \in [\![1, p]\!]$. The proof is divided in four cases depending on the parities of k and n:

1. if $n = 2i$ is even:

   (a) if $k = 2j$ is even, then let $O_n = (h_0, \ldots, h_\ell)$ be the following option, obtained by an ℓ-cut:

$$\begin{aligned} h_0 &= 2j \\ h_j &= i-j+1-\tfrac{1}{2}\ell && \text{for } j=1,2 \\ h_j &= 1 && \text{for } 3 \le j \le \ell \end{aligned}$$

   This option exists since $i \ge (3p+1)/2$, $j \le p/2$ and $\ell \le p$, hence $i-j+1-\ell/2 > p/2 > 0$. Moreover it is an option of n since $2j+2i-2j+2-\ell+1\times(\ell-2) = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(k)$ since except $2j$, all the other values in O_n appear an even number of times.

(b) if $k = 2j + 1$ is odd, then let O_n be the following option, obtained by an ℓ-cut:

$$\begin{array}{rcll} h_0 &=& 2j+1 & \\ h_j &=& x_j & \text{for } j = 1,2 \\ h_j &=& \frac{1}{2}(2i-2j-\ell-x_1-x_2+3) & \text{for } j = 3,4 \\ h_j &=& 1 & \text{for } 5 \leq j \leq \ell \end{array}$$

This option exists since $i \geq (3p+1)/2$; $j, x_1, x_2 \leq p/2$ and $\ell \leq p$, hence $2i-2j-\ell-x_1-x_2+3 \geq 4$; and $2i - 2j - \ell - x_1 - x_2 + 3$ is even since $x_1 + x_2$ is odd.
Moreover, it is an option of n since $2j+1+x_1+x_2+\ell-4+(2i-2j-\ell-x_1-x_2+3) = 2i = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(k) \oplus \mathcal{G}(x_1) \oplus \mathcal{G}(x_2) = \mathcal{G}(k) \oplus \mathbf{1} \oplus \mathbf{1}$ since the other values in O_n each appear an even number of times.

2. if $n = 2i + 1$ is odd:

(a) if $k = 2j$ is even, then let O_n be the following option, obtained by an ℓ-cut:

$$\begin{array}{rcll} h_0 &=& 2j & \\ h_j &=& x_j & \text{for } j = 1,2 \\ h_j &=& \frac{1}{2}(2i-2j-\ell-x_1-x_2+5) & \text{for } j = 3,4 \\ h_j &=& 1 & \text{for } 5 \leq j \leq \ell \end{array}$$

This option exists since $i \geq (3p+1)/2$; $j, x_1, x_2 \leq p/2$ and $\ell \leq p$, hence $2i-2j-\ell-x_1-x_2+5 \geq 6$; and $2i - 2j - \ell - x_1 - x_2 + 5$ is even since $x_1 + x_2$ is odd.
Moreover, it is an option of n since $2j+x_1+x_2+\ell-4+(2i-2j-\ell-x_1-x_2+5) = 2i+1 = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(k) \oplus \mathcal{G}(x_1) \oplus \mathcal{G}(x_2) = \mathcal{G}(k)$ since the other values in O_n each appear an even number of times.

(b) if $k = 2j + 1$ is odd, then let O_n be the following option, obtained by an ℓ-cut

$$\begin{array}{rcll} h_0 &=& 2j+1 & \\ h_j &=& i-j+1-\frac{1}{2}\ell & \text{for } j = 1,2 \\ h_j &=& 1 & \text{for } 3 \leq j \leq \ell \end{array}$$

This option exists since $i \geq (3p+1)/2$ and $2j+1, \ell \leq p$, hence $i - j + 1 - \ell/2 > p/2 > 0$.
Moreover it is an option of n since $2j+1+1\times(\ell-2)+2(i-j+1)-\ell = 2i+1$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(2j+1) = \mathcal{G}(k)$ since all the other values in O_n appear an even number of times.

Hence, for all $k \in [\![1, p]\!]$, there exists an option of n with the same Grundy value. This implies that the condition $AP3$ is verified, and thus that the AP-test is verified for PB(L).

■

Moreover, we can go further than that for these games $L = \{1, \ell\}$, $\ell > 2$, ℓ even.

Corollary 4.19 *Let $L = \{1, \ell\}$ with $\ell > 2$, even. If PB(L) verifies the conditions $AP1$ and $AP2$ of the AP-test for some $p \geq 4\ell + 3$ then PB(L) verifies the condition $AP3$ of the AP-test.*

Proof. By Theorem 4.18, we only need to prove that there exists $x_1, x_2 < p/2$ such that $\mathcal{G}(x_1) = \mathcal{G}(x_2) = \mathbf{1}$ and x_1 is odd and x_2 is even.
Remark that $\mathcal{G}(2) = \mathbf{1}$ since the only option is $(1,1)$ which has Grundy value $\mathbf{0}$. Hence we can assume $x_2 = 2$.
We claim that we can choose $x_1 = 2\ell+1$. In order to do that, we prove that the beginning of the Grundy sequence of the game PB(L) is $(\mathbf{0},\mathbf{1})^{\ell/2}$ and the following ℓ values are different from $\mathbf{1}$ and $\mathbf{0}$, and the $2\ell+1$-th value is $\mathbf{1}$. Note that we trivially have $\mathcal{G}(1) = \mathbf{0}$ and $\mathcal{G}(2) = \mathbf{1}$.
Let $k \leq \ell$ be the smallest integer such that $\mathcal{G}(k) \neq ((\mathbf{k} \bmod \mathbf{2}) + \mathbf{1} \bmod \mathbf{2})$. The only possible options for k are obtained by 1-cuts. If k is odd, then all the options are of the form (i_0, i_1) with i_0 and i_1 of different parities, which have Grundy value $\mathbf{1}$ by minimality of k, a contradiction. If k is even, then all the options are of the form (i_0, i_1) with i_0 and i_1 of same parities, which have Grundy value $\mathbf{0}$ by minimality, a contradiction.
Now, let $k \in [\![\ell+1, 2\ell]\!]$. If k is odd, then k admits the 1-cut option $(k-\ell, \ell)$ of Grundy value $\mathbf{1}$ since ℓ is even, and the ℓ-cut option $(k-\ell, 1, \dots, 1)$ of Grundy value $\mathbf{0}$. If k is even, it admits the ℓ-cut option $(k-\ell, 1, \dots, 1)$ of Grundy value $\mathbf{1}$, and the 1-cut option $(k/2, k/2)$ of Grundy value 0. It thus implies that $\mathcal{G}(k) > \mathbf{1}$.

Finally, we prove $\mathcal{G}(2\ell+1) = \mathbf{1}$. We now set $k = 2\ell+1$.
From k, one can reach the value $\mathbf{0}$ by the option $(1, 2, \dots, 2)$ obtained by an ℓ-cut. All the 1-cuts (i_0, i_1) are such that without loss of generality $i_0 > \ell$ and $i_1 \leq \ell$, so $\mathcal{G}((i_0, i_1)) \neq \mathbf{1}$ since $\mathcal{G}(i_1) < \mathbf{2}$ and $\mathcal{G}(i_2) \geq \mathbf{2}$. Assume there is an ℓ-cut $O_k = (i_0, \dots, i_\ell)$ such that $\mathcal{G}(O_k) = \mathbf{1}$. If there is some j such that $i_j > \ell$, then it is unique and $\mathcal{G}(O_k) \geq \mathbf{2}$, hence, there is none: for all j, $i_j \leq \ell$. We necessarily have an odd number of i_j's, say $i_0, \dots, i_e$ with e even, such that $\mathcal{G}(i_j) = \mathbf{1}$ for $j \in [\![0, e]\!]$. And for $j > e$, $\mathcal{G}(i_j) = \mathbf{0}$. Hence there is an even number of odd i_j's and an odd number of even ones, this gives directly that $2\ell+1$ is even, which is a contradiction.
Therefore, $\mathcal{G}(2\ell+1) = \mathbf{1}$.
Moreover, $2\ell+1 < p/2$ since $4\ell+3 \leq p$, hence it suffices to take $x_1 = 2\ell+1$ and $x_2 = 2$ to meet the conditions of Theorem 4.18 and thus the condition $AP3$ of the AP-test. ■

Remark that the games PB({1,4}) and PB({1,6}) are indeed arithmetic-periodic of period 24 and 36 respectively and of saltus 8. Moreover, it seems these games obey a certain rule:

Conjecture 4.20 *Given $\ell \geq 2$, the game PB(1,2ℓ) is arithmetic-periodic of length* 12ℓ *and saltus* 8*.*

However, when we add other values to these cut sequences, the Grundy sequence changes and seems *easier*.

Conjecture 4.21 *Let K be a finite set of positive integers such that $2 \notin K$, $|K| \geq 2$ and K contains at least one even value. The game PB(L) with $L = \{1\} \cup K$ is arithmetic-periodic with period $(\mathbf{0},\mathbf{1})^\ell$ and saltus* 2*, where 2ℓ is the smallest even number of K.*

The case $L = \{1, 2\}$ remains the hardest to understand. If Table 4.3.6 suggest an arithmetic-periodic behavior when $|L| \geq 3$, we did not detect any regularity in the period. For example, when $|L| = 3$, the games $\{1,2,4\}$ and $\{1,2,6\}$ have identical Grundy sequence, whereas $\{1,2,5\}$ and $\{1,2,7\}$ are singular. Moreover, the game $\{1,2,8\}$ presents a preperiod.

4.3.4 Conclusion and perspectives

We have seen that pure breaking games tend to be at least ultimately arithmetic-periodic, the only case that has a non-empty preperiod being the cut sequences $\{1,2,8\}$ and $\{1,2,7,8\}$. The only case that is not clearly an arithmetic-periodic sequence seems to be for $\{1,2\}$ that shows a lot of regularity, except for some values that dive away from the main behavior. Computations carried out so far give not clue about a regular behavior. It would seem that this is the only sequence acting strangely.

Conjecture 4.22 *Every game PB(L) with $L \neq \{1,2\}$ has a Grundy sequence either ultimately periodic or ultimately arithmetic-periodic.*

Table 4.3.6: Some tested pure breaking games by using the AP-test and Theorem 4.14

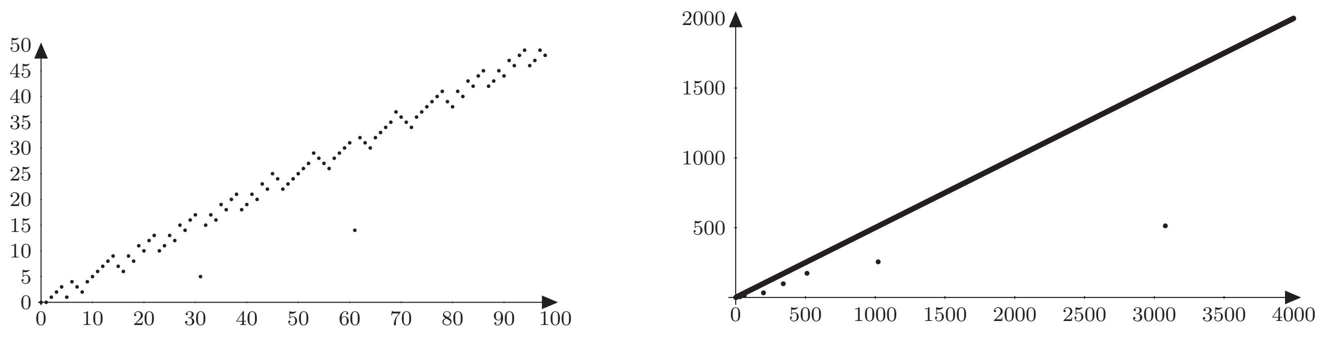
Cut sequence	Grundy sequence
$\{1,4\}$	$((\mathbf{0},\mathbf{1})^{\mathbf{2}}(\mathbf{2},\mathbf{3})^{\mathbf{2}},\mathbf{1},\mathbf{4},\mathbf{5},\mathbf{4},(\mathbf{3},\mathbf{2})^{\mathbf{2}}(\mathbf{4},\mathbf{5})^{\mathbf{2}}(\mathbf{6},\mathbf{7})^{\mathbf{2}})\ (+\mathbf{8})$
$\{1,6\}$	$((\mathbf{0},\mathbf{1})^{\mathbf{3}}(\mathbf{2},\mathbf{3})^{\mathbf{3}},\mathbf{1},\mathbf{4},(\mathbf{5},\mathbf{4})^{\mathbf{3}},(\mathbf{3},\mathbf{2})^{\mathbf{3}}(\mathbf{4},\mathbf{5})^{\mathbf{3}}(\mathbf{6},\mathbf{7})^{\mathbf{3}})\ (+\mathbf{8})$
$\{1,8\}$	$((\mathbf{0},\mathbf{1})^{\mathbf{4}}(\mathbf{2},\mathbf{3})^{\mathbf{4}},\mathbf{1},\mathbf{4},(\mathbf{5},\mathbf{4})^{\mathbf{4}},(\mathbf{3},\mathbf{2})^{\mathbf{4}}(\mathbf{4},\mathbf{5})^{\mathbf{4}}(\mathbf{6},\mathbf{7})^{\mathbf{4}})\ (+\mathbf{8})$
$\{1,10\}$	$((\mathbf{0},\mathbf{1})^{\mathbf{5}}(\mathbf{2},\mathbf{3})^{\mathbf{5}},\mathbf{1},\mathbf{4},(\mathbf{5},\mathbf{4})^{\mathbf{5}},(\mathbf{3},\mathbf{2})^{\mathbf{5}}(\mathbf{4},\mathbf{5})^{\mathbf{5}}(\mathbf{6},\mathbf{7})^{\mathbf{5}})\ (+\mathbf{8})$
$\{1,3\}\cup K$ with $K\subset\{3,5,6,7,8\}$, $K\neq\emptyset$	$(\mathbf{0},\mathbf{1})^{\mathbf{2}}\ (+\mathbf{2})$
$\{1,6\}\cup K$ with $K\subset\{3,5,7,8\}$, $K\neq\emptyset$	$(\mathbf{0},\mathbf{1})^{\mathbf{3}}\ (+\mathbf{2})$
$\{1,8\}\cup K$ with $K\subset\{3,5,7\}$, $K\neq\emptyset$	$(\mathbf{0},\mathbf{1})^{\mathbf{4}}\ (+\mathbf{2})$
$\{1,2,4\}\cup K$, $\{1,2,6\}\cup K'$ with $K\subset\{6,7,8\}$, $K'\in\{7,8\}$	$(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{54},\mathbf{3},\mathbf{2},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7})\ (+\mathbf{8})$
$\{1,2,5\}\cup K$ with $K\subset\{4,6,7,8\}$	$(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{4},\mathbf{3},\mathbf{6},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7})\ (+\mathbf{8})$
$\{1,2,7\}$	$(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8},\mathbf{9},\mathbf{7},\mathbf{6},\mathbf{9},\mathbf{8},\mathbf{11},\mathbf{10},\mathbf{12},\mathbf{13},\mathbf{10},\mathbf{11},\mathbf{13},\mathbf{12},\mathbf{15},\mathbf{14})\ (+\mathbf{16})$
$\{1,2,8\},\{1,2,7,8\}$	$(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{4})\ (\mathbf{3},\mathbf{2},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8},\mathbf{9},\mathbf{7},\mathbf{11},\mathbf{9},\mathbf{8})\ (+\mathbf{8})$

We present now a table summarizing the main results about pure breaking games along with two graphics showing the behavior of the $\{1,2\}$'s Grundy sequence. Even though this conjecture seems to treat all the

Table 4.3.7: The pure breaking games.

	Cut sequence	Grundy sequence
Solved	$\{\ell_1,\dots,\ell_k\}$, $\ell_1>1$	$(\mathbf{0})^{\ell_1}\ (+\mathbf{1})$
	$\{1,\ell_2,\dots,\ell_k\}$, ℓ_i odd	$(\mathbf{0},\mathbf{1})\ (+\mathbf{0})$
	$\{1,2,3,\ell_4,\dots,\ell_k\}$	$(\mathbf{0})^{\mathbf{1}}(+\mathbf{1})$
	$\{1,3,2k\}$, $k\geq 1$	$(\mathbf{0},\mathbf{1})^{\ell_1}\ (+\mathbf{2})$
Requires $AP1$ and $AP2$	$\{1,2\ell,2\ell'+1,\ell_1,\dots,\ell_k\}$	–
	$\{1,2\ell\}$, $\ell\geq 2$	–
Requires also $AP3$	$\{1,\ell_1,\dots,\ell_k\}$, ℓ_i even, $k\geq 1$	–

Figure 4.3.2: The Grundy sequence of PB({1,2}) for $n\leq 100$ and $n\leq 4000$



cases, there are still some relevant questions, especially when we look at the test:

Open question 4.23 *Do the conditions $AP1$ and $AP2$ of the AP-test immply the condition $AP3$ for any pure breaking game ?*

or the game PB({1,2}):

Open question 4.24 *What is the behavior of the Grundy sequence of PB({1,2})?*

In Figure 4.3.2 we see a kind of regular behavior where only few values are left apart. Remark that this sequence does not seem *saap*-regular as the left-apart values are not equal. As well the *ruler*-regularity does not seem to hold since the frequence of the left-apart values does not seem regular. This behavior looks like the one of the hexadecimal game **0.B33B**, for which no regularity has been found. The first 3000 Grundy numbers are shown in Figure 4.3.3.

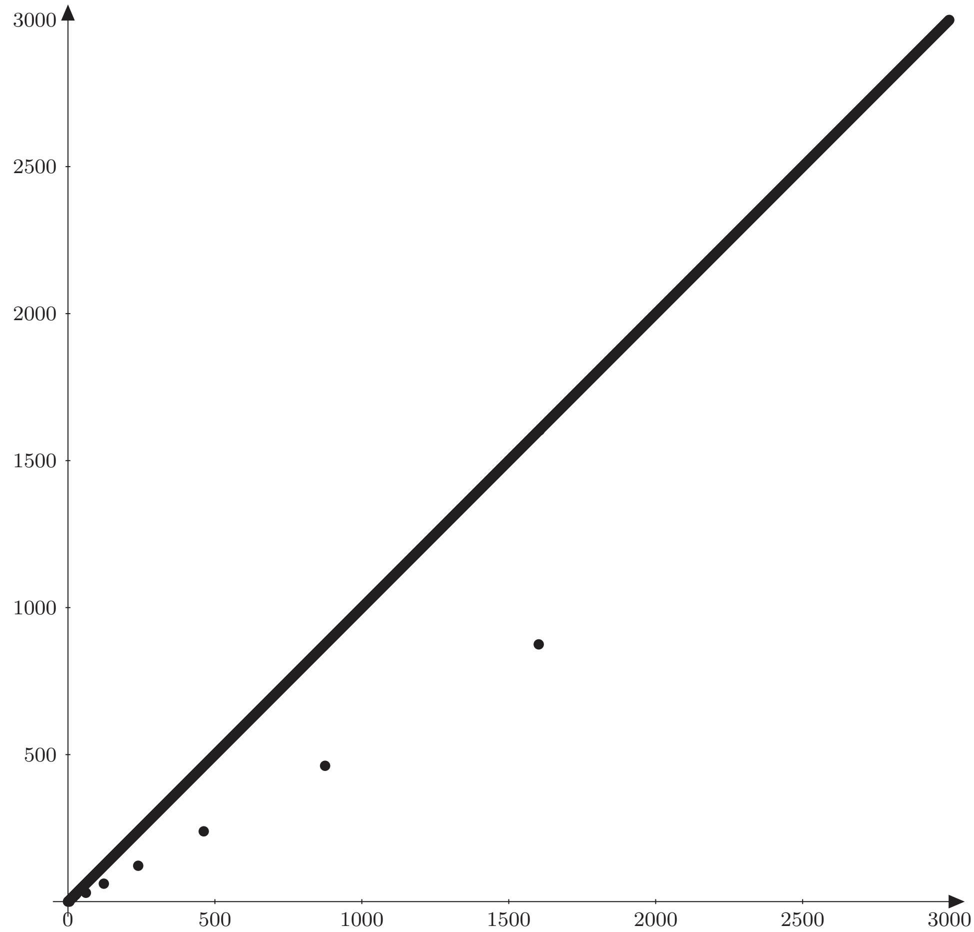


Figure 4.3.3: First 3000 Grundy numbers of the hexadecimal game **0.B33B**

Chapter 5

Conclusion

> I am my own man, I can wear my pants backwards.
>
> Sam - Atypical

Along these lines we saw different combinatorial problems that bring into play two parties.

First, in Chapter 2, identifying codes on circulant graphs were studied. This work was in collaboration with Ville Junnila and Tero Laihonen. Three main locating codes were presented: the locating-dominating, the identifying and the self-identifying codes. These codes counteract faulty vertices and are more or less suitable to identify either a faulty vertex or multiple ones. Moreover, self-identifying codes allow a quick identification of the faulty vertex. The study on circulant graphs was helped by the known optimal codes on infinite grids (the square, the king and the triangular): circulant graphs can be embedded into very particular grids ($C_n(1, d)$ into the square one, $C_n(1, d, d+1)$ into the triangular one and $C_n(1, d-1, d, d+1)$ into the king one). Once the best bounds were deduced from the grids, circulant graphs with these optimal bounds were given (for most of the cases) or sequences of circulant graphs coupled with locating codes approximating these bounds were given. All in all, one problem still open is for the identifying codes in the circulant graphs $C_n(1, d-1, d, d+1)$: is the optimal bound $2/9$ reached for some values of n and d? or is it impossible to obtain?

Second, the marking game was studied: Alice faces Bob, an uncollaborative partner. In the first place the graph operators were consired for the vertex game and in the second place the edge game was studied over a new, yet global, edge-partition of graphs.
In Section 3.2 my work with Paul Dorbec, Éric Sopena and Elżbieta Sidorowicz was displayed. We bounded above and below the game coloring number of $f(G)$ as a function of the game coloring number of G where f was either the deletion of a vertex, and edge or the contraction of an edge. As well, we did the same for union of graphs and the cartesian product of a graph G and a clique. These bounds were shown to be optimal thanks to the very flexible sunflower class of graphs. It could be interesting to study, using these bounds, families of graphs obtained by these operators, for instance hamming graphs.
In Section 3.3 my work with Clément Charpentier and Brice Effantin was explained. We defined a new edge-wise decomposition for graphs: the F^+-decomposition, which generalizes the (a, d)- and $F(a, d)$-decompositions. As well, we define a new activation strategy for the edge-marking game that uses the F^+ graphs and gives better bounds for specific decompositions. In particular better bounds for the edge coloring index of planar graphs and (a, d)- and $F(a, d)$-decomposable graphs. Our results are limited by the difficulty of finding good F^+-decompositions.
In global, there is a lot of work to do for the marking game, for vertices or edges, the results are very specific and a lot of open questions remain open.

Third, in Chapter 4, pure breaking games were shown. Alice still faces Bob, but here the main problem is to find the winning moves. This work was done with Éric Duchêne, Antoine Dailly and Urban Larsson.

First of all, taking games were presented: these games present very regular Grundy sequences and a test to compute them which depends only on the parameters of the game. As well, some taking and breaking games have regular sequences, but the more we breake the less the regularity: between octal games (breaking heaps once) and hexadecimal games (breaking heaps twice) the sequences go from ultimately periodic (conjectured) to sapp-regular or even ruler-regular. . . , in some cases no regularity is found. We focused on pure breaking games to try to understand this gap between *pure taking* and *taking and breaking* games. Our study showed that pure breaking games tend to have regular sequences: most of them seem to be arithmetic-periodic. We also created a test to compute the sequence with the first Grundy values. Even if all sequences seemed to be regular, our computations were limited and a lot of sequences, that we conjectured to be regular, have still unconfirmed behaviors. Moreover, the behavior of the sequence $\{1, 2\}$ remains a mistery.

All in all, my thesis focused on bilateral problems, where I had to prevent processors from faults, to play against an uncollaborative partner, or just try to find the best way to play against another clever player. In all the cases the main problem is to answer the optimal way: the less captors, less computations, the best decompositions of the graphs or the quicker way to determine regularity. Optimisation remains a main issue in the global subject of combinatorics and I had the chance to explore three problems that use different tools and have separate purposes.

Appendix A

Complete proof of Theorem 2.19

Here we prove Theorem 2.18 by steps: first we give a general upper bound for ID-codes in $C_n(1,3)$, then we refine these bounds with two lemmas and we conclude with the exact values for all $n \geq 11$.

Theorem A.1 *Let n be an integer such that $n \geq 11$. If $n \equiv 2,5,8 \pmod{11}$, then we have $\lceil 4n/11 \rceil \leq \gamma^{ID}(C_n(1,3)) \leq \lceil 4n/11 \rceil + 1$, and otherwise $\gamma^{ID}(C_n(1,3)) = \lceil 4n/11 \rceil$.*

Proof. Let n, q and r be integers such that $n = 11q + r$, $q \geq 0$ and $0 \leq r < 11$. Recall first that any identifying code in $C_n(1,3)$ has at least $\lceil 4n/11 \rceil$ codewords by [37]. For the constructions, we first define a code

$$C_q = \{11i + j \mid 0 \leq i \leq q-1 \text{ and } j \in \{0,1,4,5\}\}.$$

Let then A be the following set of vertices: $A = \{3,4,\ldots,11q-4\}$. In Table A.0.1, we have listed the identifying sets $I_{C_q}(u)$ and their reductions modulo 11 for all $u \in A$ depending on the remainder when u is divided by 11. Comparing the identifying sets $I_{C_q}(u) \pmod{11}$, we immediately observe that $I_{C_q}(u) \neq I_{C_q}(v)$ for all $u, v \in A$ and $u \not\equiv v \pmod{11}$. Moreover, if $u \equiv v \pmod{11}$ and $u \neq v$, then $I_{C_q}(u) \neq I_{C_q}(v)$ as $N[u] \cap N[v] = \emptyset$. This implies that C_q is an identifying set in $C_{11q}(1,3)$ since it is straightforward to verify that $I_{C_q}(u)$ are also non-empty and unique for all $u \in \{0,1,2,11q-3,11q-2,11q-1\}$. Similarly, it can be shown that the codes given in Table A.0.2 are identifying in $C_n(1,3)$. Observe that the cardinalities of the identifying codes are also given in the table. Therefore, as the cardinalities meet the ones given in the claim, the proof is concluded. ■

The general constructions given in the previous theorem can be improved for certain lengths n. These smaller identifying codes are given in Table A.0.3. It is straightforward to verify that these codes are indeed identifying. Observe also that the codes are optimal, i.e., attain the lower bound $\lceil 4n/11 \rceil$.

$u \in A \pmod{11}$	$I_{C_q}(u)$	$I_{C_q}(u) \pmod{11}$
0	$\{u, u+1\}$	$\{0,1\}$
1	$\{u-1, u, u+3\}$	$\{0,1,4\}$
2	$\{u-1, u+3\}$	$\{1,5\}$
3	$\{u-3, u+1\}$	$\{0,4\}$
4	$\{u-3, u, u+1\}$	$\{1,4,5\}$
5	$\{u-1, u\}$	$\{4,5\}$
6	$\{u-1\}$	$\{5\}$
7	$\{u-3\}$	$\{4\}$
8	$\{u-3, u+3\}$	$\{0,5\}$
9	$\{u+3\}$	$\{1\}$
10	$\{u+1\}$	$\{0\}$

Table A.0.1: Identifying sets $I_{C_q}(u)$ and their reductions modulo 11 for all $u \in A$

n	identifying code C	$\|C\|$
$11q$	C_q	$4q = \lceil 4n/11 \rceil$
$11q+1$	$C_q \cup \{11q\}$	$4q+1 = \lceil 4n/11 \rceil$
$11q+2$	$C_q \cup \{11q, 11q+1\}$	$4q+2 = \lceil 4n/11 \rceil + 1$
$11q+3$	$C_q \cup \{11q, 11q+1\}$	$4q+2 = \lceil 4n/11 \rceil$
$11q+4$	$C_q \cup \{11q, 11q+1\}$	$4q+2 = \lceil 4n/11 \rceil$
$11q+5$	$C_q \cup \{11q, 11q+1, 11q+2\}$	$4q+3 = \lceil 4n/11 \rceil + 1$
$11q+6$	$C_q \cup \{11q, 11q+1, 11q+2\}$	$4q+3 = \lceil 4n/11 \rceil$
$11q+7$	$C_q \cup \{11q, 11q+1, 11q+3\}$	$4q+3 = \lceil 4n/11 \rceil$
$11q+8$	$C_q \cup \{11q, 11q+1, 11q+2, 11q+3\}$	$4q+4 = \lceil 4n/11 \rceil + 1$
$11q+9$	$C_q \cup \{11q, 11q+1, 11q+2, 11q+3\}$	$4q+4 = \lceil 4n/11 \rceil$
$11q+10$	$C_q \cup \{11q, 11q+1, 11q+3, 11q+4\}$	$4q+4 = \lceil 4n/11 \rceil$

Table A.0.2: Identifying codes in $C_n(1,3)$ for $n = 11q + r$ and their cardinalities

n	identifying code C	$\|C\|$
13	$\{0,1,4,7,8\}$	$\lceil 4n/11 \rceil = 5$
16	$\{0,1,4,7,10,11\}$	$\lceil 4n/11 \rceil = 6$
24	$\{0,1,2,6,9,10,15,16,19\}$	$\lceil 4n/11 \rceil = 9$
27	$\{0,1,2,6,9,12,13,18,19,22\}$	$\lceil 4n/11 \rceil = 9$
35	$\{0,1,6,9,10,15,16,19,24,25,26,30,34\}$	$\lceil 4n/11 \rceil = 13$

Table A.0.3: Identifying codes in $C_n(1,3)$ for certain lengths n improving the general constructions

In what follows, we concentrate on improving the lower bound of $\gamma^{ID}(C_n(1,3))$ for $n \equiv 2,5,8 \pmod{11}$. For the rest of the section, assume first that C is an identifying code in the circulant graph $C_n(1,3)$. For the lower bound on $|C|$, we introduce a shifting scheme to even out the share among the codewords. The rules of the shifting scheme are illustrated in Figure A.0.1. In addition to the rules shown in the figure, we also have rules which are obtained by reflecting the figures over the line passing vertically through the codeword c. For example, corresponding to Figure A.0.1(1), we also have the symmetrical rules R1.1' and R1.2'. In what follows, we describe more carefully how share is shifted by the rules:

- Let c be a codeword such that its surroundings are as in Figure A.0.1(1). In other words, $\{c-1, c, c+1\} \subseteq C$, $\{c-4, c-3, c-2, c+2, c+4\} \cap C = \emptyset$ and at least one of $c+3$ and $c+5$ is a codeword. Now 1/12 units of share is shifted from c to $c+1$ by the ruleR1.1 and 1/24 units of share to $c-1$ by the ruleR1.2. Symmetrically, if c is a codeword such that its surroundings are as Figure A.0.1(1) when it is reflected over the line passing vertically through c, i.e., we have $\{c-1, c, c+1\} \subseteq C$, $\{c-2, c-4, c+2, c+3, c+4\} \cap C = \emptyset$ and at least one of $c-5$ and $c-3$ is a codeword, then 1/12 units of share is shifted from c to $c-1$ by the ruleR1.1' and 1/24 units of share to $c+1$ by the ruleR1.2'.

- If c is a codeword such that its surroundings are as in Figure A.0.1(2), then 3/24 units of share is shifted to $c+4$ from c by the ruleR2.1 and from $c+1$ by the ruleR2.2. In the symmetrical case, we have the analogous rulesR2.1' and R2.2'.

- If c is a codeword such that its surroundings are as in Figure A.0.1(3), then 1/24 units of share is shifted from c to $c+1$ by the ruleR3.1, to $c+4$ by the ruleR3.2 and to $c+7$ by the ruleR3.3. In the symmetrical case, we have the analogous rulesR3.1', R3.2' and R3.3'.

- If c is a codeword such that its surroundings are as in Figure A.0.1(4), then 3/24 units of share is shifted to $c+11$ from c by the rule4.1, from $c+1$ by the ruleR4.2 and from $c+4$ by the ruleR4.3. In the symmetrical case, we have the analogous rulesR4.1', R4.2' and R4.3'.

- If c is a codeword such that its surroundings are as in Figure A.0.1(5), then 3/24 units of share is shifted from c to $c+3$ by the ruleR5. In the symmetrical case, we have the analogous ruleR5'.

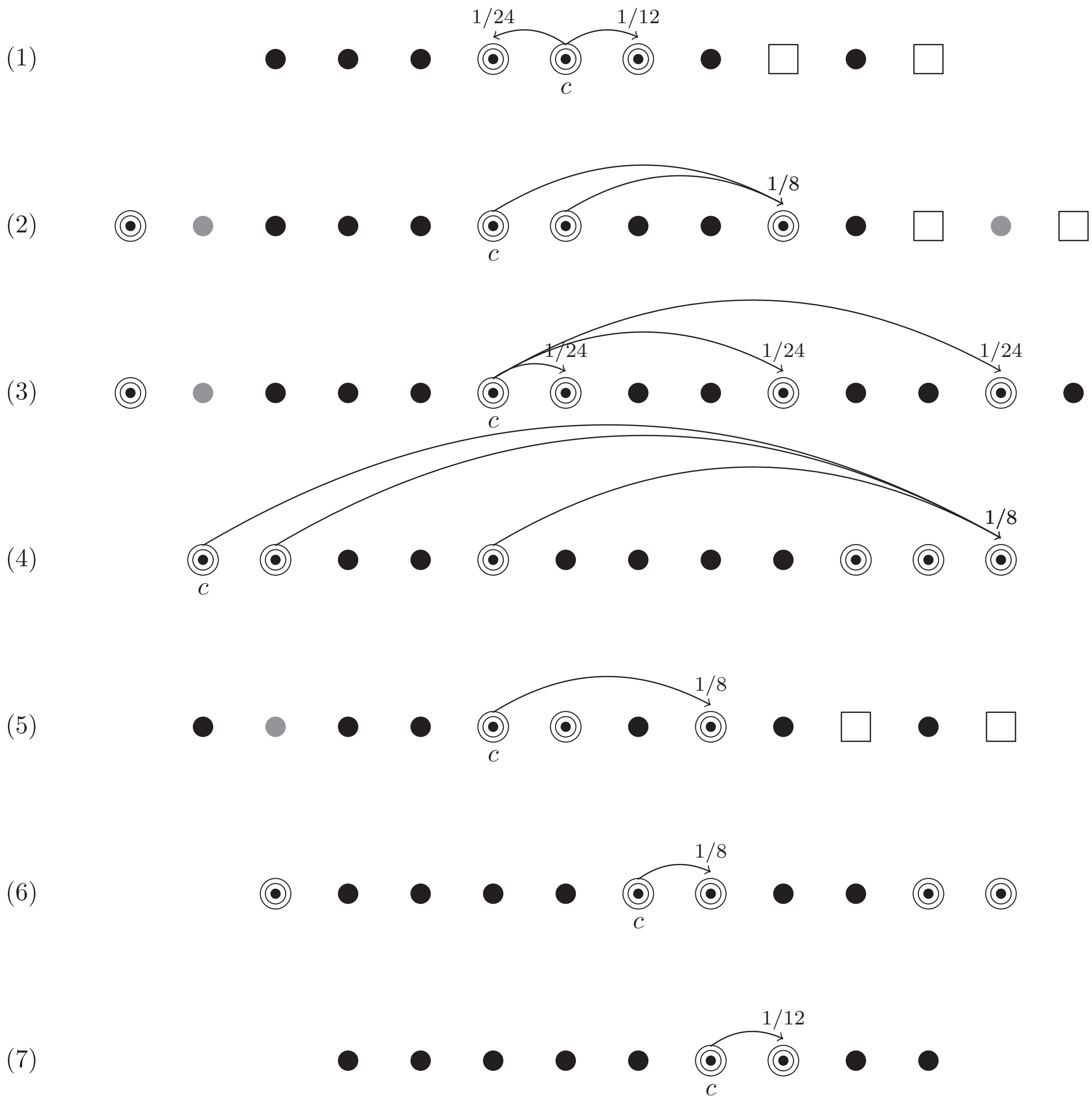


Figure A.0.1: The rules of the shifting scheme illustrated. The black dots represent codewords, the white dots represent non-codewords, and the grey dots can be either codewords or non-codewords. In the figures(a), (b) and (e), at least one of the vertices marked with a white square is a codeword. Notice that the edges of the circulant graph are omitted in the figure.

- If c is a codeword such that its surroundings are as in Figure A.0.1(6), then 3/24 units of share is shifted from c to $c+1$ by the ruleR6. In the symmetrical case, we have the analogous ruleR6'.
- If c is a codeword such that its surroundings are as in Figure A.0.1(7), then 1/12 units of share is shifted from c to $c+1$ by the ruleR7. In the symmetrical case, we have the analogous ruleR7'.

The modified share of a codeword $c \in C$, which is obtained after the shifting scheme has been applied, is denoted by $\mathtt{s}_s(c)$. The usage of the shifting scheme is illustrated in the following example.
Let u be a vertex in $C_n(1,3)$. We say that the consecutive vertices $u, u+1, \ldots, u+8$ form a *pattern* P (resp. P') if $\{u+2, u+3\} \subseteq C$ and $\{u, u+1, u+4, u+5, u+6, u+7, u+8\} \cap C = \emptyset$ (resp. $\{u+5, u+6\} \subseteq C$ and $\{u, u+1, u+2, u+3, u+4, u+7, u+8\} \cap C = \emptyset$). Furthermore, we say that a codeword $c \in \mathbb{Z}_n$ belongs to a pattern P (resp. P') if c is one of the codewords $u+2$ or $u+3$ (resp. $u+5$ or $u+6$) for some pattern P (resp. P'). Observe that all the codewords in the identifying code C_q belong to some pattern

P or P'. In what follows, we first show that after the shifting scheme has been applied the averaged share $\mathtt{s}_s(c) \leq 65/24 = 11/4 - 1/24$ for any $c \in C$ unless the codeword c belongs to some pattern P or P' when we have $\mathtt{s}_s(c) \leq 11/4$. Recall that in[37] Ghebleh and Niepel have shown using similar (albeit simpler) methods that on average the share of a codeword is at most $11/4$. Their method is based on a close study of connected components of codewords. Our refinement of the upper bound, which is based on recognizing the codewords achieving the upper bound of $11/4$ units of share, is essential to improving the lower bound for the lengths $n \equiv 2, 5, 8 \pmod{11}$ (as is shown later).

In what follows, we present two auxiliary lemmas for obtaining an upper bound on $\mathtt{s}_s(u)$; in the first one, we consider codewords receiving share according to some rule and, in the second one, we study codewords not receiving any share. In the following lemma, we begin by presenting an upper bound on $\mathtt{s}_s(u)$ when u is a codeword receiving share according to some rule.

Lemma A.2 *Let C be an identifying code in $C_n(1,3)$ and $u \in C$ be a codeword such that u receives share according to the previous rules. If u belongs to some pattern P or P', then we have $\mathtt{s}_s(u) \leq 11/4$, and otherwise $\mathtt{s}_s(u) \leq 65/24 = 11/4 - 1/24$.*

Proof. Let C be an identifying code in $C_n(1,3)$ and $u \in C$ be a codeword such that u receives share according to some rule. The proof now divides into different cases depending on which rule(s) are applied to u.

Suppose first that share is shifted to u according to the ruleR1.1. Observe first that $|I(u+1)| \geq 3$ and $|I(u+3)| \geq 2$ since $u+2$ or $u+4$ belongs to C. Therefore, we have $\mathtt{s}(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 \leq 13/6 = 11/4 - 7/12$. Furthermore, since $\{u-2, u-1\} \subseteq C$ and at least one of $u+2$ and $u+4$ is a codeword, we obtain that in addition to the ruleR1.1, u can receive share only according to the rulesR1.2', R4.1, R4.2 and R4.3. Therefore, $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 1/12 + 1/24 + 3 \cdot 3/24 \leq 8/3 = 11/4 - 2/24$ and we are done. If u receives share according to the symmetrical ruleR1.1', then we are again done since the reasoning is analogous to the considered case.

Suppose that u receives share according to the ruleR1.2; the case with the symmetrical ruleR1.2' is analogous. Now, as $u-1 \notin C$, $\{u+1, u+2\} \subseteq C$, and at least one of $u+4$ and $u+6$ is a codeword, it is straightforward to check that (in addition toR1.2) u can receive share only according to the ruleR1.1'. However, the case where u receives share according to the ruleR1.1' has already been considered above. Hence, we may assume that share is received only according to the ruleR1.2. Thus, as $|I(u+3)| \geq 3$, we obtain that $\mathtt{s}(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 8/3 = 11/4 - 1/12$. Therefore, $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 1/24 \leq 11/4 - 1/24$ and we are done.

Suppose that u receives share according to the rulesR2.1 and R2.2 (the case with the rulesR2.1' and R2.2' is analogous). Observe that since $u+2$ or $u+4$ is a codeword, at least one of the vertices $u+1$ and $u+3$ is adjacent to 3 codewords as otherwise $I(u+1) = I(u+3)$. Therefore, we obtain that $\mathtt{s}(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 13/6 = 11/4 - 7/12$. Furthermore, comparing the surroundings of u to the ones in other rules, it can be deduced that (besides the rulesR2.1 and R2.2) u can receive share only according to the rulesR2.1' and R2.2'. This implies that $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 4 \cdot 3/24 = 11/4 - 2/24$ and we are done.

Suppose that u receives share according to the ruleR3.1 (the case with R3.1' is analogous). Now it straightforward to check that u cannot receive share according to any other rule. Furthermore, we have $\mathtt{s}(u) = 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 8/3 = 11/4 - 2/24$. Therefore, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 1/24 = 11/4 - 1/24$ and we are done.

Suppose that u receives share according to the ruleR3.2 (the case with R3.2' is analogous). Now we have $\mathtt{s}(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 2/24$ as $|I(u-3)| = |I(u)| = 3$. Therefore, if share is not shifted to u by any other rule, then we are immediately done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 1/24 \leq 11/4 - 1/24$. Furthermore, it is straightforward to verify that in addition u can only receive share according to the ruleR3.3'. Then $u-4$, $u-3$, $u+3$, $u+6$ and $u+7$ are codewords and $s(u) \leq 1 + 1/2 + 3 \cdot 1/3 = 5/2 = 11/4 - 1/4$. Therefore, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 2 \cdot 1/24 \leq 11/4 - 4/24$ and we are done.

Suppose that u receives share according to the ruleR3.3 (the case with R3.3' is analogous). Observe first that if $u+2$ and $u+4$ are both non-codewords, then a contradiction follows as $I(u-1) = I(u+1) = \{u\}$. Hence, we may assume that $u+2$ or $u+4$ is a codeword. Therefore, one of the I-sets $I(u+1)$ and $I(u+3)$ contains at least 3 codewords. Thus, we have $s(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 \leq 11/4 - 2/24$. Observe that the

ruleR3.2' is the only other rule according to which u can receive; in particular, notice that share cannot be received by the rule3.3' since $u+2$ or $u+4$ is a codeword. Furthermore, the case where share is received according to the ruleR3.2' has already been considered above.
Suppose that u receives share according to the rulesR4.1, R4.2 and R4.3 (the case with R4.1', R4.2' and R4.3' is analogous). Observe first that $u+1$, $u+2$ or $u+4$ belongs to C since $I(u-3) \neq I(u+1)$. This implies that $s(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 \leq 13/6 = 11/4 - 7/12$. Furthermore, if u receives no share according to any other rule or receives share according to the rule1.1, then we are immediately done as in the case of the ruleR1.1. The only other possibility for u to receive share is according to the rulesR4.1', R4.2' and R4.3'. However, in this case, the vertices $u-2$, $u-1$, u, $u+1$ and $u+2$ are all codewords. This implies that $s(u) \leq 2 \cdot 1/2 + 1/3 + 2 \cdot 1/4 = 11/6$. Therefore, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 6 \cdot 3/24 \leq 49/20 = 11/4 - 1/6 = 11/4 - 4/24$ and we are done.
Suppose that u receives share according to the ruleR5 (the case with R5' is analogous). Now u cannot receive share according to any other rule. Furthermore, as $u+2$ or $u+4$ is a codeword, we obtain that $s(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 13/6 = 11/4 - 7/12$. Therefore, we are immediately done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 3/24 \leq 11/4 - 11/24$.
Suppose that u receives share according to the ruleR6 (the case with R6' is analogous). Now it straightforward to verify that u does not receive share according to any other rule. Furthermore, as $|I(u)| = 3$ and $|I(u+3)| \geq 3$, we immediately obtain that $s(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 13/6 = 11/4 - 14/24$. Hence, we are immediately done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 3/24 \leq 11/4 - 1/24$.
Suppose that u receives share according to the ruleR7 (the case with R7' is analogous). Again u cannot receive according to any other rule. Observe first that $u+3$ and $u+4$ are codewords since $I(u-1) \neq I(u)$ and $I(u-3) \neq I(u+1)$. Therefore, as $|I(u)| \geq 3$ and $|I(u+3)| \geq 3$, we immediately obtain that $s(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 8/3 = 11/4 - 2/24$. Thus, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 1/12 \leq 11/4$. However, now this is enough since u belongs to a pattern P'. Thus, in conclusion, the claim follows. ■

In the following lemma, we give an upper bound on $\mathtt{s}_s(u)$ when u is a codeword not receiving share according to any rule.

Lemma A.3 *Let C be an identifying code in $C_n(1,3)$ and $u \in C$ be a codeword such that u does not receive share according to any of the previous rules. If u belongs to some pattern P or P', then we have* $\mathtt{s}_s(u) \leq 11/4$, *and otherwise* $\mathtt{s}_s(u) \leq 65/24 = 11/4 - 1/24$.

Proof. Let C be an identifying code in $C_n(1,3)$ and $u \in C$ be a codeword such that u does not receive share according to the rules. Observe first that if $u+2$ is a codeword, then we are immediately done since at least two of the I-sets $I(u-1)$, $I(u+1)$ and $I(u+3)$ consists of at least three codewords implying $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 8/3 = 11/4 - 2/24$. The same argument also applies for $u-2 \in C$. Hence, we may assume that $u-2$ and $u+2$ do not belong to C. Now the proof divides into the following cases depending on the number of codewords in $I(u)$:

- Suppose first that $|I(u)| = 1$, i.e., $I(u) = \{u\}$. The previous observation taken into account, we now know that $u-3$, $u-2$, $u-1$, $u+1$, $u+2$ and $u+3$ are non-codewords. Therefore, as $I(u) \neq I(u-1)$ and $I(u) \neq I(u+1)$, we obtain that $u-4$ and $u+4$ belong to C. Furthermore, since $I(u-3) \neq I(u-1) = \{u-4, u\}$ and $I(u+3) \neq I(u+1) = \{u, u+4\}$, we have $|I(u-3)| \geq 3$ and $|I(u+3)| \geq 3$. Hence, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 2/24$ and we are done.

- Suppose then that $|I(u)| = 2$. Now we have a further split into the cases with $I(u) = \{u-3, u\}$ and $I(u) = \{u, u+1\}$ (the cases with $I(u) = \{u+3, u\}$ and $I(u) = \{u-1, u\}$ are analogous). Consider first the case with $I(u) = \{u-3, u\}$. If now $u+4 \in C$, then $|I(u-3)| \geq 3$ and $|I(u+3)| \geq 3$ since $I(u-3) \neq I(u)$ and $I(u+1) \neq I(u+3)$ and we are done as $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 2/24$. Hence, we may assume that $u+4 \notin C$. Therefore, as $I(u-1) \neq I(u+1) = \{u\}$, we have $u-4 \in C$. Furthermore, since $I(u+2) \neq \emptyset$, $I(u+1) \neq I(u+3)$ and $I(u+2) \neq I(u+4)$, we obtain respectively that $u+5$, $u+6$ and $u+7$ belong to C. Now $3/24$ units of share is shifted from u to $u+7$ according to the ruleR4.3. Thus, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 3/24 \leq 1 + 3 \cdot 1/2 + 1/3 - 3/24 = 11/4 - 1/24$.

  For the other case, suppose that $I(u) = \{u, u+1\}$. Observe first that $u+4$ belongs to C since $I(u) \neq I(u+1)$. It suffices to assume that $u-4 \notin C$ since otherwise $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 =$

$13/6 = 11/4 - 7/12$ and we are done. If now $u-5 \notin C$, then $u+5 \in C$ as $I(u+2) \neq I(u-2) = \{u\}$ and $1/12$ units of share is shifted from u to $u+1$ according to the ruleR7. Therefore, $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 1/12 \leq 1 + 3 \cdot 1/2 + 1/3 - 1/12 = 11/4$ and we are done since u belongs to a patternP'. Hence, we may assume that $u-5$ is a codeword. If $u+5$ is a codeword, then $3/24$ units of share is shifted from u to $u+1$ according to the ruleR6 and we are again done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 3/24 \leq 1 + 3 \cdot 1/2 + 1/3 - 3/24 = 11/4 - 1/24$. Hence, we may assume that $u+5 \notin C$. If at least one of $u+6$ and $u+8$ is a codeword, then $3/24$ units of share is shifted from u to $u+4$ according to the ruleR2.1 Thus, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 3/24 \leq 1 + 3 \cdot 1/2 + 1/3 - 3/24 = 11/4 - 1/24$ and we are done. Hence, we may assume that $u+6$ and $u+8$ do not belong to C. If $u+7 \in C$, then $1/24$ units of share is shifted from u to $u+1$, $u+4$ and $u+7$ according to the rulesR3.1, R3.2 and R3.3, respectively. Therefore, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 3 \cdot 1/24 \leq 1 + 3 \cdot 1/2 + 1/3 - 3/24 = 11/4 - 1/24$ and we are done. Hence, we may assume that $u+7$ is a non-codeword. Thus, since $I(u+6) \neq \emptyset$, $I(u+5) \neq I(u+7)$ and $I(u+6) \neq I(u+8)$, we obtain respectively that $u+9$, $u+10$ and $u+11$ belong to C. Now $3/24$ units of share is shifted from u to $u+11$ according to the ruleR4.1. Therefore, we are again done since $\mathtt{s}_s(u) \leq 11/4 - 1/24$. This concludes the proof of the current case.

- Suppose then that $|I(u)| = 3$. Observe first that if for some $v \in N(u)$ we have $|I(v)| \geq 3$, then we are immediately done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 2/24$. Now, for $|I(u)| = 3$, we have the following essentially different cases (others are analogous): $I(u) = \{u-3, u, u+3\}$, $I(u) = \{u-1, u, u+3\}$, $I(u) = \{u, u+1, u+3\}$ and $I(u) = \{u-1, u, u+1\}$. For future considerations, recall that the vertices $u-2$ and $u+2$ do not belong to C. Consider first the case with $I(u) = \{u-3, u, u+3\}$. By the previous observation, we may assume that $u-4$ and $u+4$ do not belong to C. However, this implies a contradiction since $I(u-1) = I(u+1) = \{u\}$.

  Consider then the case with $I(u) = \{u-1, u, u+3\}$. By the previous observation, we may assume that $u-4$, $u+4$ and $u+6$ are non-codewords. Thus, since $I(u-3) \neq I(u+1) = \{u\}$, $u-6$ is a codeword. If $u+5$ or $u+7$ is a codeword, then $3/24$ units of share is shifted from u to $u+3$ by the ruleR2.1. Therefore, we are done as $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 3/24 \leq 1 + 3 \cdot 1/2 + 1/3 - 3/24 = 11/4 - 1/24$. Hence, we may assume that $u+5$ and $u+7$ do not belong to C. Thus, since $I(u+5) \neq \emptyset$, $I(u+4) \neq I(u+6)$ and $I(u+5) \neq I(u+7)$, we obtain respectively that $u+8$, $u+9$ and $u+10$ belong to C. Now $3/24$ units of share is shifted from u to $u+10$ according to the ruleR4.2. Therefore, we are again done since $\mathtt{s}_s(u) \leq 11/4 - 1/24$.

  Suppose then that $I(u) = \{u, u+1, u+3\}$. By the previous observation, we may assume that $u+4$ and $u+6$ are non-codewords. If $u-4 \in C$, then we are immediately done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 13/6 = 11/4 - 7/12$. Hence, we may assume that $u+4 \notin C$. Now $u+5$ or $u+7$ belongs to C as otherwise $I(u+2) = I(u+4) = \{u+1, u+3\}$. Therefore, $3/24$ units of share is shifted from u to $u+3$ according to the ruleR5. Thus, we are done as $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 3/24 \leq 1 + 3 \cdot 1/2 + 1/3 - 3/24 = 11/4 - 1/24$.

  Finally, suppose that $I(u) = \{u-1, u, u+1\}$. Now at least one of $u-5$ and $u+5$ is a codeword since $I(u-2) \neq I(u+2)$. Without loss of generality, we may assume that $u+5 \in C$. Then $1/24$ and $1/12$ units of share is shifted from u to $u-1$ and $u+1$ according to the rulesR1.2 and R1.1, respectively. Therefore, we are done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 1/24 - 1/12 \leq 1 + 3 \cdot 1/2 + 1/3 - 3/24 = 11/4 - 1/24$.

- Suppose then that $|I(u)| = 4$. The proof now divides into the following essentially different cases: $I(u) = \{u-1, u, u+1, u+3\}$ and $I(u) = \{u-3, u, u+1, u+3\}$. In the former case, we may first assume that $u-4$ and $u+4$ are non-codewords by a similar argument as in the previous case. Then $1/24$ and $1/12$ units of share is shifted from u to $u-1$ and $u+1$ according to the rulesR1.2 and R1.1, respectively. Therefore, we are done since $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 1/24 - 1/12 \leq 1 + 3 \cdot 1/2 + 1/4 - 3/24 = 11/4 - 3/24$.

  Suppose now that $I(u) = \{u-3, u, u+1, u+3\}$. By the previous observations, we may assume that $u-4$, $u+4$ and $u+6$ are non-codewords. Now $u+5$ or $u+7$ belongs to C since $I(u+2) \neq I(u+4)$. Therefore, $3/24$ units of share is shifted from u to $u+3$ according to the ruleR5. Thus, we are done as $\mathtt{s}_s(u) \leq \mathtt{s}(u) - 3/24 \leq 1 + 3 \cdot 1/2 + 1/4 - 3/24 = 11/4 - 3/24$.

- Finally, suppose that $|I(u)| = 5$, i.e., $I(u) = \{u-3, u-1, u, u+1, u+3\}$. Now we are immediately done since $\mathtt{s}_s(u) \le \mathtt{s}(u) \le 1 + 3 \cdot 1/2 + 1/5 = 27/10 = 11/4 - 1/20 \le 11/4 - 1/24$. This concludes the proof of the claim.

■

In conclusion, the previous lemmas state that any codeword c not belonging to a pattern$P$ or P' has $\mathtt{s}_s(c) \le 11/4 - 1/24$. In the following lemma, we consider the case where C is an identifying code such that no codeword belongs to one of the patterns.

Lemma A.4 *Let C be an identifying code in $C_n(1,3)$ such that no codeword of C belongs to a pattern$P$ or P'. Then the following results hold:*

- *If $n = 11q_1 + 2$ with $q_1 \ge 5$, then $|C| \ge 4q_1 + 2 = \lceil 4n/11 \rceil + 1$.*
- *If $n = 11q_2 + 5$ with $q_2 \ge 3$, then $|C| \ge 4q_2 + 3 = \lceil 4n/11 \rceil + 1$.*
- *If $n = 11q_3 + 8$ with $q_3 \ge 1$, then $|C| \ge 4q_3 + 4 = \lceil 4n/11 \rceil + 1$.*

Proof. Let C be an identifying code in $C_n(1,3)$ such that no codeword of C belongs to a pattern$P$ or P'. Denote $n = 11q + r$, where q is a nonnegative integer and r is an integer such that $0 \le r < 11$. By Lemmas A.2 and A.3, we know that $\mathtt{s}_s(c) \le 65/24$ for all $c \in C$. Therefore, we obtain that

$$n = \sum_{c \in C} \mathtt{s}(c) = \sum_{c \in C} \mathtt{s}_s(c) \le \frac{65}{24}|C|.$$

This further implies that

$$|C| \ge \frac{24}{65}n = \frac{24}{65}(11q + r) = 4q + \frac{4q + 24r}{65}.$$

The rest of the proof now divides into the following cases:

- If $n = 11q_1 + 2$ with $q_1 \ge 5$, then $|C| \ge 4q_1 + (4q_1 + 24 \cdot 2)/65 \ge 4q_1 + 68/65$. Therefore, we have $|C| \ge \lceil 4q_1 + 68/65 \rceil = 4q_1 + 2 = \lceil 4n/11 \rceil + 1$.
- If $n = 11q_2 + 5$ with $q_2 \ge 3$, then $|C| \ge 4q_2 + (4q_2 + 24 \cdot 5)/65 \ge 4q_2 + 132/65$. Therefore, we have $|C| \ge \lceil 4q_2 + 132/65 \rceil = 4q_2 + 3 = \lceil 4n/11 \rceil + 1$.
- If $n = 11q_3 + 8$ with $q_3 \ge 1$, then $|C| \ge 4q_3 + (4q_3 + 24 \cdot 8)/65 \ge 4q_3 + 196/65$. Therefore, we have $|C| \ge \lceil 4q_3 + 196/65 \rceil = 4q_3 + 4 = \lceil 4n/11 \rceil + 1$.

■

In the following theorem, we improve the lower bound on $\gamma^{ID}(C_n(1,3))$ for lengths n such that n is large enough and $n \equiv 2, 5, 8 \pmod{11}$.

Theorem A.5 *Let n be a positive integer such that $n = 11q_1 + 2$ with $q_1 \ge 5$, $n = 11q_2 + 5$ with $q_2 \ge 3$, or $n = 11q_3 + 8$ with $q_3 \ge 1$. Now we have*

$$\gamma^{ID}(C_n(1,3)) \ge \left\lceil \frac{4n}{11} \right\rceil + 1.$$

Proof. Let C be an identifying code in $C_n(1,3)$. Recall that if no codeword of C belongs to a pattern$P$ or P', then the claim immediately follows by Lemma A.4. Hence, we may assume that there exist codewords of C belonging to a pattern$P$ or P'. Suppose first that all the codewords belong to a pattern$P$ or P'. This implies that the code is formed by consecutive repetitions of P and P'. Observe that consecutive patterns$P$ and P' form a segment of length 11 (with 4 codewords) similar to the identifying code C_q given in Theorem A.1. However, as now n is not divisible by 11, the identifying code C cannot entirely be formed by the segments of length 11. Thus, we obtain that all the codewords cannot belong to a pattern$P$

or P'. In other words, after a (finite) repetition of patterns$P$ and P', a codeword not belonging to the patterns has to appear. In what follows, we first show that the end of the repetition of the patterns$P$ and P' implies a drop of strictly more than 3/4 units of share in the sum $\sum_{c\in C} \mathtt{s}_s(c)$ compared to the average share of 11/4, i.e., $\sum_{c\in C} \mathtt{s}_s(c) < \frac{11}{4}|C| - \frac{3}{4}$. Based on this observation, we then show that the original lower bound of $\lceil 4n/11 \rceil$ can be improved by one.

Suppose first that the repetition of the patterns ends with a pattern$P$. More precisely, let $u-7$ and $u-6$ be codewords belonging to a pattern$P$, and assume that the next codeword to the right does not belong to a patternP'. Recall that due to the pattern P the vertices $u-9$, $u-8$, $u-5$, $u-4$, $u-3$, $u-2$ and $u-1$ are non-codewords. Now u and $u+1$ belong to C since $I(u-3) \neq I(u-5) = \{u-6\}$ and $I(u-2) \neq \emptyset$, respectively. By the assumption that u (and $u+1$) do no belong to a patternP', we can deduce that $u+2$ or $u+3$ is a codeword of C. These two cases are considered in the following:

(A1) Suppose first that $u+2 \in C$. If $u+3 \in C$, then no share is shifted to u according to any rule and we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 2\cdot 1/2 + 3\cdot 1/3 = 2 = 11/4 - 3/4$. Furthermore, by Lemmas A.2 and A.3, we have $\mathtt{s}_s(u+1) \leq 11/4 - 1/24$. Therefore, we are done since $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) < 2\cdot 11/4 - 3/4$. Hence, we may assume that $u+3$ does not belong to C. Now at least one of $u+4$ and $u+6$ is a codeword since $I(u-1) \neq I(u+3)$. Now we have $s(u) \leq 3\cdot 1/2 + 2\cdot 1/3 = 11/4 - 7/12$ and similarly $s(u+2) \leq 11/4 - 7/12$. Moreover, it is straightforward to verify that u and $u+2$ can receive share only according to the rulesR1.2 and R1.1, respectively. Therefore, we obtain that $\mathtt{s}_s(u) + \mathtt{s}_s(u+2) \leq (s(u) + 1/24) + (s(u+2) + 1/12) \leq 2\cdot 11/4 - 25/24 < 2\cdot 11/4 - 3/4$. This concludes the first case of the proof.

(A2) Suppose then that $u+2 \notin C$ and $u+3 \in C$. Observe first that $u+4$ or $u+7$ is a codeword since otherwise $I(u+2) = I(u+4) = \{u+1, u+3\}$ (a contradiction). Suppose first that $u+4$ is a codeword. Observe then that $|I(v)| \geq 3$ for all $v \in \{u, u+1, u+3, u+4\}$. Therefore, we have $s(v) \leq 1 + 1/2 + 3\cdot 1/3$ for all $v \in \{u, u+1, u+3, u+4\}$. It is straightforward to verify that u and $u+1$ do not receive share according to any rule. Moreover, either $u+3$ or $u+4$ can receive share according to the rulesR4.1', R4.2' and R4.3'. Furthermore, if this happens for one of the vertices, say v, then we have $\mathtt{s}_s(v) \leq 11/4 - 1/24$ by the previous lemmas and the other one does not receive share according to the rules. Thus, all the previous combined, we are done since $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) + \mathtt{s}_s(u+3) + \mathtt{s}_s(u+4) \leq 4\cdot 11/4 - 3\cdot 1/4 - 1/24 < 4\cdot 11/4 - 3\cdot 1/4$. Hence, we may assume that $u+4 \notin C$ and $u+7 \in C$.

Suppose that $u+6$ is a codeword. Then it is straightforward to verify that $u+3$ can receive share only according to the rulesR2.1' and R2.2'. Therefore, since $s(u+3) \leq 1/2 + 4\cdot 1/3 = 11/6$, we obtain that $\mathtt{s}_s(u+3) \leq \mathtt{s}(u+3) + 2\cdot 3/24 \leq 11/4 - 16/24$. Thus, we are done as $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) + \mathtt{s}_s(u+3) + \mathtt{s}_s(u+6) \leq 3(11/4 - 1/24) + 11/4 - 16/24 = 4\cdot 11/4 - 19/24 < 4\cdot 11/4 - 3/4$. Hence, we may assume that $u+6 \notin C$. Suppose then that $u+5$ or $u+9$ is a codeword; denote the codeword by v. Now we have $s(u+3) \leq 2\cdot 1/2 + 3\cdot 1/3$. Moreover, $u+3$ can receive only 3/24 units of share according to the ruleR5. Therefore, we obtain that $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) + \mathtt{s}_s(u+3) + \mathtt{s}_s(u+7) + \mathtt{s}_s(v) \leq 4(11/4 - 1/24) + 11/4 - 15/24 = 4\cdot 11/4 - 19/24 < 4\cdot 11/4 - 3/4$. Hence, we may assume that $u+5$ and $u+9$ are both non-codewords. Now $u+8$ is a codeword since $I(u+5) \neq \emptyset$. Furthermore, at least one of $u+10$ and $u+12$ is a codeword, say v, since $I(u+5) \neq I(u+9)$. Now we have $s(u+3) \leq 3\cdot 1/2 + 2\cdot 1/3 = 11/4 - 14/24$ and as above $u+3$ can receive only 3/34 units of share according to the ruleR5. Moreover, we have $s(w) \leq 1 + 2\cdot 1/2 + 2\cdot 1/3 = 11/4 - 2/24$ for any $w \in \{u+1, u+7, u+8\}$ and none of the codewords receive share according to any rule. Furthermore, we have $\mathtt{s}_s(u) \leq 11/4 - 1/24$ and $\mathtt{s}_s(v) \leq 11/4 - 1/24$ by Lemmas A.2 and A.3 since neither of the vertices u and v belongs to a pattern$P$ or P'. Thus, combining the previous observation, we obtain that $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) + \mathtt{s}_s(u+3) + \mathtt{s}_s(u+7) + \mathtt{s}_s(u+8) + \mathtt{s}_s(v) \leq 2(11/4 - 1/24) + 3(11/4 - 2/24) + (11/4 - 14/24 + 3/24) = 6\cdot 11/4 - 19/24 < 6\cdot 11/4 - 3/4$. This concludes the proof of the current case.

Suppose then that the repetition of the patterns ends with a patternP'. More precisely, let $u-4$ and $u-3$ be codewords belonging to a patternP', and assume that the next codeword to the right does not belong to a pattern$P$. Recall that due to the pattern P' the vertices $u-9$, $u-8$, $u-7$, $u-6$, $u-5$, $u-2$ and $u-1$ are non-codewords. Now u and $u+1$ belong to C since $I(u-3) \neq I(u-4) = \{u-4, u-3\}$ and $I(u-2) \neq I(u-6) = \{u-3\}$, respectively. By the assumption that u (and $u+1$) do no belong to a pattern$P$, we can deduce that one of the vertices $u+2$, $u+3$, $u+4$, $u+5$ and $u+6$ is a codeword of C. The proof now divides into the following five cases:

(B1) Suppose that $u+2 \in C$. Now we have $s(u) \leq 1/2 + 4 \cdot 1/3 = 11/4 - 22/24$. Furthermore, u can receive share only according to the rulesR4.1', R4.2' and R4.3'. Obviously, if u receives no share, then we are immediately done as $\mathtt{s}_s(u) \leq \mathtt{s}(u) \leq 11/4 - 22/24 \leq 11/4 - 3/4$. Hence, we may assume that share is shifted to u according to the rulesR4.1', R4.2' and R4.3'. This implies that $u+7$, $u+10$ and $u+11$ are codewords. Therefore, we have $s(u+1) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 14/24$ and $u+1$ cannot receive share according to any rule. Thus, we are done since $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) \leq (11/4 - 22/24 + 3 \cdot 3/24) + (11/4 - 14/24) = 2 \cdot 11/4 - 27/24 \leq 2 \cdot 11/4 - 3/4$.

(B2) Suppose that $u+2 \notin C$ and $u+3 \in C$. Now we have $s(u) \leq 3 \cdot 1/2 + 1/3 + 1/4 = 11/4 - 16/24$ and similarly $s(u+1) \leq 11/4 - 16/24$ (as $I(u+2) \neq I(u+4)$). Hence, as neither u nor $u+1$ receives share according to any rule, we obtain that $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) \leq \mathtt{s}(u) + \mathtt{s}(u+1) \leq 2(11/4 - 16/24) < 2 \cdot 11/4 - 3/4$. Thus, we are done.

(B3) Suppose that $u+2, u+3 \notin C$ and $u+4 \in C$. Now we have $s(u) \leq 2 \cdot 1/2 + 3 \cdot 1/3 = 2 = 11/4 - 3/4$. Furthermore, u does not receive share according to any rule. Therefore, as $u+1$ does not belong to any pattern P or P', we are done since $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) \leq (11/4 - 3/4) + (11/4 - 1/24) = 2 \cdot 11/4 - 19/24 < 2 \cdot 11/4 - 3/4$.

(B4) Suppose that $u+2, u+3, u+4 \notin C$ and $u+5 \in C$. Observe first that $u+7 \in C$ since $I(u+4) \neq I(u+2) = \{u+1, u+5\}$. Now we have $s(u) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 2/24$ and $s(u+1) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 14/24$. Furthermore, neither u nor $u+1$ receives share according to any rule. Moreover, at least one of $u+6$, $u+8$ and $u+11$, say v, is a codeword since $I(u+6) \neq I(u+8)$. Observe that if $v = u+6$ or $v = u+8$, then v does not belong to any pattern$P$ or P'. Assuming $u+6$ and $u+8$ do not belong to C, then $v = u+11$ does not belong to P or P'. Therefore, we are done as $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) + \mathtt{s}_s(u+5) + \mathtt{s}_s(u+7) + \mathtt{s}_s(v) \leq (11/4 - 2/24) + (11/4 - 14/24) + 3(11/4 - 1/24) = 5 \cdot 11/4 - 19/24 < 5 \cdot 11/4 - 3/4$.

(B5) Finally, suppose that $u+2, u+3, u+4, u+5 \notin C$ and $u+6 \in C$. Observe first that $u+7 \in C$ since $I(u+4) \neq I(u+2) = \{u+1\}$. This implies that $s(u) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 14/24$. It is also straightforward to verify that u can only receive 3/24 units of share according to the ruleR6'. Therefore, we have $\mathtt{s}_s(u) \leq \mathtt{s}(u) + 3/24 \leq 11/4 - 11/24$. Furthermore, since $I(u+6) \neq I(u+7)$, we know that at least one of $u+8$, $u+9$ and $u+10$ has to be a codeword. Suppose first that $u+8 \in C$. Now we have $s(u+6) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 14/24$ (as $I(u+5) \neq I(u+9)$), and $u+6$ can only receive 1/24 units of share according to the ruleR1.2. (In particular, notice that if share is shifted to $u+6$ according to the rulesR4.1', R4.2' or R4.3', then $I(u+5) = I(u+9)$ implying a contradiction.) Thus, we have $\mathtt{s}_s(u) + \mathtt{s}_s(u+6) \leq (11/4 - 11/24) + (11/4 - 14/24 + 1/24) \leq 2 \cdot 11/4 - 1 < 2 \cdot 11/4 - 3/4$. Hence, we may assume that $u+8 \notin C$. Suppose then that $u+9 \in C$. Now we have $s(u+7) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 14/24$, and $u+7$ cannot receive share according to any rule. Therefore, we are done as $\mathtt{s}_s(u) + \mathtt{s}_s(u+7) \leq (11/4 - 11/24) + (11/4 - 14/24) = 2 \cdot 11/4 - 25/24 < 2 \cdot 11/4 - 3/4$. Hence, we may assume that $u+9 \notin C$ and $u+10 \in C$.

Suppose first that $u+11 \in C$. Now we have $s(u+7) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 14/24$, and $u+7$ can receive share only according to the ruleR6 (3/24 units). Therefore, we are done since $\mathtt{s}_s(u) + \mathtt{s}_s(u+7) \leq (11/4 - 11/24) + (11/4 - 14/24 + 3/24) = 2 \cdot 11/4 - 22/24 < 2 \cdot 11/4 - 3/4$. Hence, we may assume that $u+11 \notin C$. Suppose then that $u+12 \in C$ or $u+14 \in C$. This implies that $s(u+10) \leq 3 \cdot 1/2 + 2 \cdot 1/3 = 11/4 - 14/24$. Furthermore, $u+10$ receives share according to the rulesR2.1 (3/24 units) and R2.2 (3/24 units), and it can possibly receive share also by the rulesR2.1' (3/24 units) and R2.2' (3/24 units). If no share is shifted to $u+10$ according to the rulesR2.1' and R2.2', then we are done since $\mathtt{s}_s(u) + \mathtt{s}_s(u+10) \leq (11/4 - 11/24) + (11/4 - 14/24 + 2 \cdot 3/24) = 2 \cdot 11/4 - 19/24 <$

$2 \cdot 11/4 - 3/4$. Hence, we may assume that $u+10$ receives share also according to the rulesR2.1' and R2.2'. This implies that $u+13$, $u+14$ and $u+19$ are codewords of C. Observe that the codewords $u+1$, $u+6$, $u+10$, $u+13$, $u+14$ and $u+19$ do not belong to any pattern$P$ or P'. In particular, $u+19$ does not belong to P or P' since $u+15 \notin C$. Thus, all the previous taken into account, we obtain that $\mathtt{s}_s(u)+\mathtt{s}_s(u+1)+\mathtt{s}_s(u+6)+\mathtt{s}_s(u+7)+\mathtt{s}_s(u+10)+\mathtt{s}_s(u+13)+\mathtt{s}_s(u+14)+\mathtt{s}_s(u+19) \leq (11/4 - 11/24) + (11/4 - 14/24 + 4 \cdot 3/24) + 6(11/4 - 1/24) = 8 \cdot 11/4 - 19/24 < 8 \cdot 11/4 - 3/4$. Hence, we may assume that $u+12 \notin C$ and $u+14 \notin C$.

Suppose that $u+13 \notin C$. Now $u+15$, $u+16$ and $u+17$ belong to C since $I(u+12) \neq \emptyset$, $I(u+13) \neq I(u+11) = \{u+10\}$ and $I(u+14) \neq I(u+12) = \{u+15\}$, respectively. Furthermore, at least one of the vertices $u+18$, $u+19$ and $u+21$, say v, is a codeword since $I(u+14) \neq I(u+18)$. Thus, if $v = u+18$, $v = u+19$, or $v = u+21$ and v does not belong to P or P', then we are done as $\mathtt{s}_s(u)+\mathtt{s}_s(u+1)+\mathtt{s}_s(u+6)+\mathtt{s}_s(u+7)+\mathtt{s}_s(u+10)+\mathtt{s}_s(u+15)+\mathtt{s}_s(u+16)+\mathtt{s}_s(u+17)+\mathtt{s}_s(v) \leq (11/4 - 11/24) + 8(11/4 - 1/24) = 9 \cdot 11/4 - 19/24 < 9 \cdot 11/4 - 3/4$ (none of the other codewords either belong to a pattern$P$ or P'). Hence, we may assume that v belongs to a pattern$P$ or P'. This implies that $u+18, u+19 \notin C$ and $v = u+21$. Now $u+20$ also belongs to the pattern$P$ and the codewords $u+15$, $u+16$ and $u+17$ form a case symmetrical to the case(B1). Hence, we may assume that $u+13 \in C$. Now $u+17 \in C$ because $I(u+14) \neq I(u+12)$. It is straightforward to verify that $u+13$ can now receive share only according to the rulesR3.3 (1/24 units) and R3.2' (1/24 units). If $u+15$ is a codeword, then $s(u+13) \leq 2 \cdot 1/2 + 3 \cdot 1/3 = 11/4 - 3/4$. Furthermore, if $u+16 \in C$, then we have $s(u+13) \leq 1 + 1/2 + 3 \cdot 1/3 = 11/4 - 6/24$. Thus, in both cases, we have $\mathtt{s}_s(u+13) \leq \mathtt{s}(u+13) + 2 \cdot 1/24 \leq 11/4 - 4/24$. Therefore, all the previous taken into account, we are done since $\mathtt{s}_s(u) + \mathtt{s}_s(u+1) + \mathtt{s}_s(u+6) + \mathtt{s}_s(u+7) + \mathtt{s}_s(u+10) + \mathtt{s}_s(u+13) + \mathtt{s}_s(u+17) \leq (11/4 - 11/24) + (11/4 - 4/24) + 5(11/4 - 1/24) = 7 \cdot 11/4 - 20/24 < 7 \cdot 11/4 - 3/4$ (none of the codewords belong to a pattern$P$ or P'). Hence, we may assume that $u+15$ and $u+16$ are non-codewords. Now $u+18$ and $u+19$ belong to C since $I(u+15) \neq \emptyset$ and $I(u+16) \neq I(u+14) = \{u+13, u+17\}$, respectively. Therefore, we have $\mathtt{s}_s(u)+\mathtt{s}_s(u+1)+\mathtt{s}_s(u+6)+\mathtt{s}_s(u+7)+\mathtt{s}_s(u+10)+\mathtt{s}_s(u+13)+\mathtt{s}_s(u+17)+\mathtt{s}_s(u+18)+\mathtt{s}_s(u+19) \leq (11/4 - 11/24) + 8(11/4 - 1/24) = 9 \cdot 11/4 - 19/24 < 7 \cdot 11/4 - 3/4$ (again none of the codewords belong to a pattern$P$ or P'). Thus, in conclusion, we achieve a drop of more than 3/4 units of share in the sum $\sum_{c \in C} \mathtt{s}_s(c)$ in all the cases compared to the average share of 11/4, i.e., $\sum_{c \in C} \mathtt{s}_s(c) < \frac{11}{4}|C| - \frac{3}{4}$.

In the previous detailed case analysis, we have achieved a drop of more than 3/4 units of share in the sum $\sum_{c \in C} \mathtt{s}_s(c)$. In what follows, we show how this implies the improved lower bound. The proof now splits into the following cases depending on the remainder when n is divided by 11:

- Suppose first that $n = 11q_1 + 2$ with $q_1 \geq 5$. By the previous considerations, we now have

$$n = \sum_{c \in C} \mathtt{s}(c) = \sum_{c \in C} \mathtt{s}_s(c) < \frac{11}{4}|C| - \frac{3}{4}.$$

  This implies that

$$|C| > \frac{4}{11}\left(n + \frac{3}{4}\right) = 4q_1 + 1.$$

  Thus, we have $|C| \geq 4q_1 + 2 = \lceil 4n/11 \rceil + 1$.

- Suppose then that $n = 11q_2 + 5$ with $q_2 \geq 3$. As in the previous case, we obtain that

$$|C| > \frac{4}{11}\left(n + \frac{3}{4}\right) = 4q_2 + 2 + \frac{1}{11}.$$

  Thus, we have $|C| \geq 4q_2 + 3 = \lceil 4n/11 \rceil + 1$.

- Finally, suppose then that $n = 11q_3 + 8$ with $q_3 \geq 1$. As in the previous case, we obtain that

$$|C| > \frac{4}{11}\left(n + \frac{3}{4}\right) = 4q_3 + 3 + \frac{2}{11}.$$

Thus, we have $|C| \geq 4q_3 + 4 = \lceil 4n/11 \rceil + 1$.

Thus, in conclusion, we have shown that $\gamma^{ID}(C_n(1,3)) \geq \lceil 4n/11 \rceil + 1$ for $n = 11q_1 + 2$ with $q_1 \geq 5$, $n = 11q_2 + 5$ with $q_2 \geq 3$, and $n = 11q_3 + 8$ with $q_3 \geq 8$. ■

Recall the general constructions of Theorem A.1, the constructions for the specific lengths in Table A.0.3 and the improved lower bound of Theorem A.5. Combining all these results, we know the exact values of $\gamma^{ID}(C_n(1,3))$ for all the lengths n except for $n = 46$. In the open case $n = 46$, we have $17 = \lceil 4n/11 \rceil \leq \gamma^{ID}(C_n(1,3)) \leq \lceil 4n/11 \rceil + 1 = 18$ by the general lower and upper bounds. Using an exhaustive computer search, it can be shown that there does not exist an identifying code in $C_{46}(1,3)$ with 17 codewords, i.e., $\gamma^{ID}(C_{46}(1,3)) = 18$. The method of the exhaustive search is briefly explained in the following remark.

Remark: Let C be a code in $C_{46}(1,3)$ with 17 codewords. Without loss of generality, we may assume that 8 of the codewords belong to $\{0, 1, \ldots, 22\}$ and the rest 9 codewords belong to $\{23, 24, \ldots, 45\}$. Observe that if C is an identifying code in $C_{46}(1,3)$, then the vertices in $\{3, 4, \ldots, 19\}$ have a unique identifying set among the codewords in $\{0, 1, \ldots, 22\}$ and the vertices in $\{26, 27, \ldots, 42\}$ have a unique identifying set among the codewords in $\{23, 24, \ldots, 45\}$. Using a computer search, we obtain that there exist 1919 codes $C_1 \subseteq \{0, 1, \ldots, 22\}$ with $|C_1| = 8$ such that $I_{C_1}(u)$, where $u \in \{3, 4, \ldots, 19\}$, are all non-empty and unique, and 23137 codes $C_2 \subseteq \{23, 24, \ldots, 45\}$ with $|C_2| = 9$ such that $I_{C_2}(u)$, where $u \in \{26, 27, \ldots, 42\}$, are all non-empty and unique. By an exhaustive search, we obtain that no union of such codes C_1 and C_2 is an identifying code in $C_{46}(1,3)$. Therefore, by the previous observation, there does not exists an identifying code in $C_{46}(1,3)$ with 17 codewords. Hence, we have $\gamma^{ID}(C_{46}(1,3)) = 18$.
The following theorem summarizes all these results and gives the exact values of $\gamma^{ID}(C_n(1,3))$ for all lengths $n \geq 11$. The exact values of $\gamma^{ID}(C_n(1,3))$ for the lengths n smaller than 11 have been determined in[37].

Theorem A.6 *Let n be an integer such that $n \geq 11$. Now we have the following results:*

- *Assume that $n \leq 37$. If $n \equiv 8 \pmod{11}$, then we have $\gamma^{ID}(C_n(1,3)) = \lceil 4n/11 \rceil + 1$, and otherwise $\gamma^{ID}(C_n(1,3)) = \lceil 4n/11 \rceil$.*
- *Assume that $n \geq 38$. If $n \equiv 2, 5, 8 \pmod{11}$, then we have $\gamma^{ID}(C_n(1,3)) = \lceil 4n/11 \rceil + 1$, and otherwise $\gamma^{ID}(C_n(1,3)) = \lceil 4n/11 \rceil$.*

Appendix B

Complete proof of Theorem 2.30

Here we prove that for $n \equiv 2, 3, 5 \mod 6$ the optimal codes on $C_n(1,3)$ are of cardinal $\lceil n/3 \rceil + 1$, *i.e.*, we show there cannot be of cardinal $\lceil n/3 \rceil$ and then we give LD-codes attaining with cardinal $\lceil n/3 \rceil + 1$.

Proposition B.1 *Let $n \geq 14$ and C be a locating-dominating code in $C_n(1,3)$. For all $c \in C$, we have either $\mathtt{s}(c) \leq 17/6$ or $\mathtt{s}(c) \in \{3, 37/12, 10/3\}$. Moreover, the following statements hold:*

- $\mathtt{s}(c) = 3$ *if and only if c belongs to a pattern$S1$ or $S3$ (defined below).*
- $\mathtt{s}(c) = 37/12$ *if and only if c belongs to a pattern$S4$ (defined below).*
- $\mathtt{s}(c) = 10/3$ *if and only if c belongs to a pattern$S6$ (defined below).*

Proof. Let c be a codeword in C. The proof now divides into three parts depending on whether $|I(c)| \geq 3$, $|I(c)| \geq 2$ or $|I(c)| = 1$.

- Suppose first that $|I(c)| \geq 3$. Observe that there exists at most one vertex u in $N[c]$ such that $|I(u)| = 1$, and the other vertices are covered by at least two codewords. Hence, we immeadiately obtain that $\mathtt{s}(c) \leq 1 + 3 \cdot 1/2 + 1/3 = 17/6$.
- Assume then that $|I(c)| = 2$. If all the vertices $v \in N[c]$ have $|I(v)| \geq 2$, then we get $\mathtt{s}(c) \leq 5/2 < 17/6$. Therefore, it is enough to consider the case where there is at least one vertex $v \in N[c]$ with $|I(v)| = 1$. There cannot be more than one such vertex. Indeed, such a vertex must be a non-codeword, and if there were two, say u and w, then $I(u) = I(w)$, which is not possible. Moreover, if there is one vertex $v \in N[v]$ such that $|I(v)| \geq 3$, we have $\mathtt{s}(c) \leq 17/6$. Therefore, $\mathtt{s}(u) = 3$ if and only if all the vertices in $N[c]$ have the size of the I-sets equal to 2 except one equal to 1. Next we analyze this case more carefully.
  - Let first $c - 3 \in I(c)$ (the case $c + 3$ goes analogously). If $I(c-1) = \{c\}$ (resp. $I(c+1) = \{c\}$), then $c+4$ (resp. $c-4$) belongs to C implying $|I(c+3)| \geq 3$ (resp. $|I(c-3)| \geq 3$). If $I(c+3) = \{c\}$, then $|I(c-3)| \geq 3$ (since $I(c-1) \neq I(c+1) = \{c\}$). In all cases, the share is at most 17/6.
  - Assume then that $c-1 \in I(c)$ (the case $c+1$ is analogous). If $I(c-3) = \{c\}$, then $|I(c+3)| \geq 3$. If $I(c+3) = \{c\}$, then $|I(c-1)| \geq 3$. In these cases $\mathtt{s}(c) \leq 17/6$. Therefore, we can assume that $I(c+1) = \{c\}$ and $|I(c-3)| = |I(c+3)| = 2$. Due to $c+3$, we must have $c+6 \in C$. Now either $c-4$ or $c-6$ belong to C (if both we are done). Moreover, we may assume that $c-4 \notin C$ as otherwise $|I(c-1)| \geq 3$ implying $\mathtt{s}(c) \leq 17/6$. Therefore, it is enough to consider the case $c-6 \in C$. Consequently, we have the pattern:

$$x * ooox\underline{x}oooo * x,$$

where c is denoted by the underlined codeword $\underline{x}$. Both of the unknowns cannot be non-codewords because then $I(c-2) = I(c+2)$ and $c-2, c+2 \notin C$. Moreover, we have $c-7 \in C$ since otherwise $I(c-2) = I(c-4)$. This leads to the following two patterns when $\mathtt{s}(c) = 3$:

$$\begin{array}{ll} S1 & xxxooooxx\underline{x}oooo*x \\ S3 & xxooooxx\underline{x}ooooxx \end{array} .$$

- Let then $|I(c)| = 1$. If there is no vertex $v \in N(c)$ such that $|I(v)| = 1$, then it is easy to check that $\mathtt{s}(c) \leq 17/6$ as at least one I-set has at least three codewords. Consequently, let us assume that such v exists (clearly only one such vertex is possible). Without loss of generality, we may assume that v is either $c-1$ or $c-3$.

  – Let us assume first that $v = c-1$. Due to $c+1$, we must have $c+4 \in C$. Moreover, since $I(c+1) \neq I(c+3)$, we get $c+6 \in C$. As $I(c+2) \neq \emptyset$ (resp. $c-2$), we have $c+5 \in C$ (reps. $c-5 \in C$). In order to have $I(c-1) \neq I(c-3)$ we must have $c-6 \in C$. In addition, $c-7 \in C$, since $I(c-2) \neq I(c-4)$. This leads to $\mathtt{s}(c) = 10/3$ and the only way to achieve this is by the pattern:
$$S6 \quad xxxoooo\underline{x}oooxxx \ .$$

  – Suppose then that $v = c-3$. In order to have $I(c-3) \neq I(c-1)$, we must have $c+2 \in C$. Also $c-5 \in C$ because $I(c-2)$ cannot be the empty set. Moreover, $c+4 \in C$ due to $I(c-1) \neq I(c+1)$. We also have $c+6 \in C$ to get $I(c+1) \neq I(c+3)$. Now $\mathtt{s}(c) = 37/12$ and it comes from the pattern:
$$S4 \quad oxoooo\underline{x}oxox*x \ .$$

■

Next we show that shifting the shares among codewords gives us the situation where the share of each vertex is (after the shifting) less than 17/6 or equal to 3. Moreover, the share is equal to 3 if and only if we have the case of pattern $S3$. The share of a vertex $v \in C$ after shifting is denoted by $\mathtt{s}_s(v)$. We do the shifting using the following three shifting rules and their symmetric counterparts (where the pattern is read from right to left):

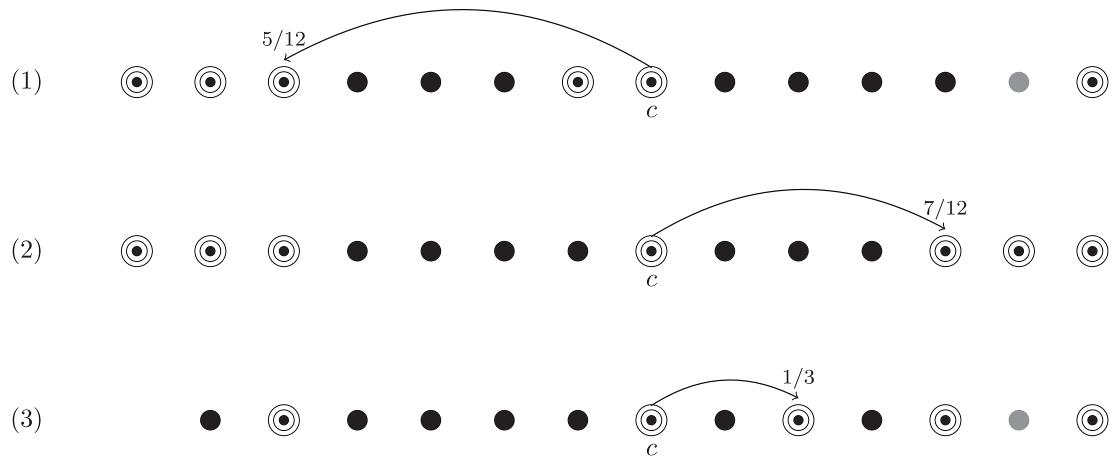


Figure B.0.1: The black nodes are non-codewords, halo nodes are codewords and gray nodes can be anything. The edges of the circulant graph are omitted in the figure.

- R1: The vertex c gives 5/12 units of share to $c-5$.
- R2: The vertex c gives 7/12 units of share to $c+4$.
- R3: The vertex c gives 1/3 units of share to $c+2$.

Notice that if a vertex receives shares by any of the rules, it cannot receive more share by another rule (including its symmetric counterpart).

Proposition B.2 *Let C be a locating-dominating code in $C_n(1,3)$ where $n \geq 14$. Then we have $\mathtt{s}_s(c) \leq 17/6$ for all $c \in C$ unless c belongs to a pattern $S3$ when $\mathtt{s}_s(c) = 3$.*

Proof. If in the code C there are only codewords with share at most 17/6, then there is nothing to do. Let us now consider the other cases:

- Let there be a codeword c with $\mathtt{s}(c) = 3$ in the pattern $S1$. By the rule R1, we shift 5/12 units of share to $c-5$. Notice that $\mathtt{s}(c-5) \leq 13/6$. Consequently, we have $\mathtt{s}_s(c) = 31/12 < 17/6$ and $\mathtt{s}_s(c-5) \leq 31/12$.
- If there is a codeword c with share $\mathtt{s}(c) = 37/12$ in the pattern $S4$. Using the rule R3 we shift 1/3 units of share to $c+2$. The share $\mathtt{s}(c+2) \leq 29/12$. Therefore, $\mathtt{s}_s(c) = 11/4 < 17/6$ and $\mathtt{s}_s(c+2) \leq 11/4$.
- Let there be a codeword c with share $\mathtt{s}(c) = 10/3$ in the pattern $S6$. Now R2 shifts 7/12 units of a share to $c+4$ with $\mathtt{s}(c+4) \leq 13/6$. Consequently, $\mathtt{s}_s(c) = 11/4$ and $\mathtt{s}_s(c+4) \leq 11/4$.

■

Before our main theorem on locating-dominating codes, let us give the following technical lemma.

Lemma B.3 *Let $n > 17$ be an integer such that $n \equiv 3 \pmod 6$ or $n \equiv 2 \pmod 3$, and let C be a locating-dominating code in $C_n(1,3)$. If there is no pattern $S3$, then $|C| > \lceil n/3 \rceil$.*

Proof. By Proposition B.2, after shifting the shares and knowing there is no $S3$ we have $\mathtt{s}_s(c) \leq 17/6$ for all $c \in C$. Hence,

$$n = \sum_{i \in C} \mathtt{s}(i) = \sum_{i \in C} \mathtt{s}_s(i) \leq \frac{17}{6}|C|.$$

The proof divides now into the following cases:

- if $n = 6k+3$, we have $|C| \geq 2k+1+\frac{2k+1}{17} > 2k+1$
- and if $n = 3k+2$, we have $|C| \geq k + \frac{k+12}{17}$ and as $n > 17$, $k > 5$, which gives $|C| > k+1$.

■

Theorem B.4 *Let $n > 17$. Then*

$$\gamma^{LD}(C_n(1,3)) = \begin{cases} \left\lceil \dfrac{n}{3} \right\rceil & \text{if } n \equiv 0,1,4 \pmod 6 \\ \left\lceil \dfrac{n}{3} \right\rceil + 1 & \text{otherwise} \end{cases}$$

Proof. Let $n > 17$. For $n \equiv 0,1,4 \pmod 6$ the result for locating-dominating codes is given in [37]. We need to prove that for $n \equiv 2,3,5 \pmod 6$ the bound $\lceil n/3 \rceil$ is not attainable. On the other hand, in [37] there are constructions of cardinality $\lceil \frac{n}{3} \rceil + 1$ given in these cases. We will write $n = 6k + r$ with $r \in \{2,3,5\}$. Notice that for $r = 2,5$, we can write n in the form $3l+2$, which implies $n \equiv 2 \pmod 6$ for $l \equiv 0 \pmod 2$ and $n \equiv 5 \pmod 6$ otherwise. By Lemma B.3 we know that if there is no $S3$ patterns, then the bound is not attainable. Assume then than there is a pattern $S3$. Note that these patterns $S3$ can overlap each other. We denote by $P6$ the pattern $xxoooo$. Therefore, we can divide overlapping $S3$-patterns into non-overlapping patterns $P6$.

Without loss of generality, we can assume that the vertices $0,1,2,3,4,5,6,7,8,9,10,11$ form two patterns $P6$ and $\mathtt{s}(7) = 3$, that is, $0,1,6,7,12,13 \in C$. As $n \not\equiv 0 \pmod 6$, we cannot have only patterns $P6$ in the graph. Therefore, we can assume that there are t consecutive patterns $P6$ starting in 0 (to the right) and that there is no pattern $P6$ on the vertices $n-6, n-5, n-4, n-3, n-2, n-1$.

We want to prove that the sum of all the shares of codewords is strictly less than $3|C|$ for $n = 6k+3$ and strictly less than $3|C|-1$ for $n = 6k+2$ and $n = 6k+5$. This will imply the lower bounds as we shall see.

- Let $n = 6k+3$. Since there is no pattern $P6$ on the vertices $n-6, n-5, n-4, n-3, n-2, n-1$, the vertex 1 does not have the surroundings of the pattern $S3$. Therefore, $\mathtt{s}_s(1) < 3$ by Proposition B.2.

  Hence, we have

$$n = \sum_{i\in C} \mathtt{s}(i) = \sum_{i \in C} \mathtt{s}_s(i) < 3|C| \quad \Leftrightarrow \quad 6k+3 < 3|C|$$
$$\Leftrightarrow \quad 2k+1 < |C|.$$

  Consequently, $|C| \geq 2k+2 = \lceil n/3 \rceil + 1$.

- Now, let $n \equiv 2, 5 \pmod 6$. It is easy to check that $\mathtt{s}(1) \leq 3$ and $\mathtt{s}(0) \leq 3$. We will try to find such vertices, say b of them, that their shares (after the shifting by the above rules) is less than $3b - 1 - \varepsilon$ for some $\varepsilon > 0$.

  - Let first $\mathtt{s}(1) = 3$. This implies that $n-5 \in C$ and $n-3, n-2, n-1$ are not codewords (due to patterns$S1$ and $S3$). If $\mathtt{s}(0) = 3$, then we get a pattern $P6$ on the vertices $n-6, n-5, n-4, n-3, n-2, n-1$, hence, as we assumed there was no such pattern, $\mathtt{s}(0) < 3$. Consequently, $n-4 \in C$. In addition, $n-6 \in C$ due to $I(n-1) \neq I(n-3)$. This implies that $\mathtt{s}(0) \leq 7/3$.

    The share of $n-4$ is then at most $13/6$, hence we have that the share of 0 and $n-4$ (before the shifting by the rules) drops from $2 \cdot 3$ by at least $-2/3 - 5/6 = -3/2$. Only the rule R1 applies here and it can give to $n-4$ the amount of $5/12$. Therefore, the total drop in shares is at least $-3/2 + 5/12 = -13/12$ (which is enough as we try to have drop of $-1-\varepsilon$). Hence,

$$\sum_{i\in C} \mathtt{s}(i) = \sum_{i\in C} \mathtt{s}_s(i) \leq 3|C| - 13/12.$$

  - Suppose then that $\mathtt{s}(1) < 3$. In what follows, we study vertices $n-1$, $n-2$, etc., and different variants of possible codewords among them in order to find the codewords whose shares drop enough (of course, excluding the cases $\mathtt{s}(1) = 3$).

    We start by consider separately the cases $n-1 \in C$ and $n-1 \notin C$.

    (i) Suppose first that $n-1 \in C$. We divide further the study into two cases $n-2 \in C$ and $n-2 \notin C$. Let $n-2 \in C$. Now $\mathtt{s}(1) \leq 2$ and $\mathtt{s}(0) \leq 2$. Therefore, the drop of the vertices 0 and 1 is -2 compared to $2 \cdot 3$ and since no rules gives these vertices any additional share, we are done. Assume then that $n-2 \notin C$. Now $\mathtt{s}(1) \leq 13/6$ and $\mathtt{s}(0) \leq 17/6$. Thus $\mathtt{s}(0) + \mathtt{s}(1) \leq 5$. If $\mathtt{s}(0) + \mathtt{s}(1) < 5$, then the drop is $-1-\varepsilon$ and no rules give extra share to them, so we are done. If $\mathtt{s}(0) + \mathtt{s}(1) = 5$, then we have $n-5 \in C$ and $n-4, n-3, n-2 \notin C$ and $\mathtt{s}(n-1) \leq 13/6$. The drop among the vertices 0,1 and $n-1$ is altogether $-11/6$. The rules R1 and R2 can give at most $7/12$ to the vertex $n-1$ (not both at the same time), and the codewords 0 and 1 do not receive share according to any rule. Therefore, the total drop is at least $-11/6 + 7/12 = -5/4$. Hence we have

$$\sum_{i\in C} \mathtt{s}_s(i) \leq 3|C| - \frac{5}{4}.$$

    (ii) Let then $n-1 \notin C$. Notice that in this case no rules give any additional share to vertices 0 and 1. In the following, we consider the cases depending on which of the vertices in $\{n-5, n-4, n-3, n-2\}$ are codewords. If there are three (or four) codewords in that set, it is easy to compute that $\mathtt{s}(0) + \mathtt{s}(1) \leq 29/6$. Hence the drop of the vertices 0 and 1 is at least $-7/6$. Recall that the rules give no extra here. Consequently,

$$\sum_{i\in C} \mathtt{s}_s(i) \leq 3|C| - 7/6.$$

    The remaining cases are listed below. Notice that since $\mathtt{s}(1) < 3$, the case where $n-5, n-4 \in C$ and $n-3, n-2 \notin C$ and also the case where $n-5 \in C$ and $n-4, n-3, n-2 \notin C$ can be excluded. Furthermore, the case $n-4 \in C$ and $n-5, n-3, n-2 \notin C$ is excluded because in that case $I(n-2) = I(2) = \{1\}$ and $2, n-2 \notin C$ which is impossible since C is locating-dominating. In the table below, we have all the other cases:

Case	*pattern*	$\mathtt{s}(1)$	$\mathtt{s}(0) \leq$	$\mathtt{s}(p) \leq$	*drop*
Case 1	$ooxxoxxooooxx$ 0 7	5/2	2		−3/2
Case 2	$xoxooxxooooxx$ p 0 7	8/3	17/6	2	−1/2−1
Case 3	$xooxoxxooooxx$ 0 7	8/3	13/6		−7/6
Case 4	$ooxooxxooooxx$ p 0 7	17/6	17/6	17/6	−1/3−1/6
Case 5	$oxxooxxooooxx$ p 0 7	17/6	13/6	8/3	−1−1/3
Case 6	$oooxoxxooooxx$ p 0 7	17/6	13/6	8/3	−1−1/3
Case 7	$oxoxoxxooooxx$ 0 7	17/6	2		−7/6

In all the cases except Case 4, we have a drop strictly smaller than -1 and the rules do not give any extra share to the vertex marked by p (the vertices 0 and 1 did not get any as mentioned earlier). To examine Case 4 more carefully, we study the vertices $n-9, n-8, n-7$ and $n-6$ and codewords among them. The cases where the codewords among these four vertices are as follows $\{oooo, xoxo, xooo, ooxo, oxxo, oxoo\}$ are forbidden in a locating-dominating code. Indeed, the first four combinations give $I(n-5) = \emptyset$ and the two last ones give $I(n-6) = I(n-4)$. All the other cases are studied in the following table. In Case 4.6 we have added one more codeword, namely, the p_0 (which necessarily must be a codeword).

Case	*pattern*	$\mathtt{s}(1)$	$\mathtt{s}(0)$	$\mathtt{s}(n-3)$	$\sum \mathtt{s}(p_i) \leq$	*drop*
Case 4.1	$xxxxooxooxxooooxx$ 0	17/6	8/3	23/12		−19/12
Case 4.2	$xxxoooxooxxooooxx$ 0	17/6	17/6	13/6		−7/6
Case 4.3	$xxoxooxooxxooooxx$ p 0	17/6	8/3	5/2	2	−2
Case 4.4	$xoxxooxooxxooooxx$ 0	17/6	8/3	23/12		−19/12
Case 4.5	$oxxxooxooxxooooxx$ 0	17/6	8/3	2		−3/2
Case 4.6	$x\,x\,x\,ooooxooxxooooxx$ $p_0 p_1 p_2$ 0	17/6	17/6	17/6	$2 * 8/3 + 17/6$	−1/2−5/6
Case 4.7	$xooxooxooxxooooxx$ p 0	17/6	8/3	5/2	5/2	−3/2
Case 4.8	$oxoxooxooxxooooxx$ p 0	17/6	8/3	8/3	7/3	−3/2
Case 4.9	$ooxxooxooxxooooxx$ 0	17/6	8/3	2		−3/2
Case 4.10	$oooxooxooxxooooxx$ p 0	17/6	8/3	8/3	8/3	−7/6

Notice that the vertices 0, 1 and $n-3$ cannot receive any extra share from the rules. In addition, the codewords marked by p also do not receive share by the rules. Now let us consider the special case Case 4.6. The vertices p_2 and p_1 do not receive share from the rules. If the vertex p_0 does not receive extra share, then the drop is enough. However, the vertex p_0 can get a share from the left by the rules R1 or R2. Suppose first that p_0 (the vertex $n-10$) receives 5/12 units of share by the rule R1 from the vertex $n-15$. But then the vertices $n-14$ and $n-15$ belong to C and thus $\mathtt{s}(p_2) \leq 8/3$, $\mathtt{s}(p_1) \leq 7/3$ and $\mathtt{s}(p_0) \leq 13/6$. Hence the new drop (taking into account the vertices 0, 1, $n-3$, p_0, p_1 and p_2) is at least $-7/3$. So even with the extra share the drop is enough $-7/3 + 5/12 = -23/12$. Assume then that p_0 gets extra share 7/12 by the rule R2 from $n-14$ (which belongs to C). Now $\mathtt{s}(p_2) \leq 8/3$, $\mathtt{s}(p_1) \leq 17/6$ and $\mathtt{s}(p_0) \leq 13/6$. Consequently, the drop of the six vertices is at least $-11/6$ and with the extra share $-11/6 + 7/12 = -5/4$, which is enough.

In all the cases studied above, we get that the drop of the share is strictly more than 1. Recall that we consider the cases $n \equiv 2, 5 \pmod 6$. We write n as $3k+2$, which implies $n \equiv 2 \pmod 6$ for $k \equiv 0$

(mod 2) and $n \equiv 5 \pmod 6$ otherwise. Therefore, we have, for some $\varepsilon > 0$:

$$\begin{aligned} n = \sum_{i \in C} \mathtt{s}_s(i) \leq 3|C| - 1 - \varepsilon \quad &\Leftrightarrow \quad 3k + 2 + 1 + \varepsilon \leq 3|C| \\ &\Leftrightarrow \quad k + 1 + \frac{\varepsilon}{3} \leq |C| \\ &\underset{|C| \in \mathbb{N}}{\Rightarrow} \quad k + 2 \leq |C| \end{aligned}$$

This implies that $|C| \geq \lceil n/3 \rceil + 1$.

■

Appendix C

Alice's strategy (details)

Here we give the details of Alice's strategy and we prove the announced bounds. Recall Alice's strategy:

Alice's strategy is as follows: she starts by doing a neutral move (*i.e.*an inactive edge having no marked father nor uncle) . Each time Bob marks an arc or an edge, say $e_1/\overrightarrow{e_1}$, Alice selects her next move by following the steps illustrated in Figure C.0.1 and described below.
To simplify the notations, we are not drawing the arrows of arcs in the description of the strategy.

Start: Assume the edge/arc she is considering is e_i then:

1. if e_i is inactive (or if e_1 was inactive before Bob marked it), Alice activates it and:
   (a) if e_i has an unmarked father in $\overrightarrow{A}_\infty$, f, then $e_{i+1} = f$, and she goes back to **Start**;
   (b) if j is the smallest index for which e_i has an unmarked father on $\overrightarrow{A}_j$, f, then $e_{i+1} = f$ and she goes back to **Start**;
   (c) if j is the smallest index for which e_i has an unmarked brother on $\overrightarrow{A}_j$, then she marks it;
   (d) if e_i has an unmarked uncle, then Alice marks it;
2. if e_i is active:
   (a) if it is marked, then Alice does a neutral move;
   (b) otherwise, Alice marks it.

Please remark that considering an edge for the first time means to activate it (and to mark it if it is a brother or an uncle of the previous considerated one). The second time an edge is considered, it is marked (if it is not).

Observation C.1 (by rules {1} and {2}) *In all strategies, each arc of $\vec{A}$ is considered at most twice.*

Notice that from the definitions we obtain directly:

Observation C.2 *For every arc $\vec{e}$ of $\vec{A}$, $|F(\vec{e})| + |B(\vec{e})| + |U(\vec{e})| \leq \Delta(G) - 1$. Moreover,*

- *if $\vec{e} \in \vec{A_\infty}$, then $|P_\infty(\vec{e})| \leq a - 1$, $|P_-(\vec{e})| \leq k$ and $|C(\vec{e})| \leq d$;*
- *if $\vec{e} \in \vec{A_-}$, then $|P_\infty(\vec{e})| \leq a$, $|P_-(\vec{e})| \leq k - 1$ and $|C(\vec{e})| \leq d$.*

In fact, with this strategy as it is, our results were not interesting, hence we decided to mix up the stages of selection of vertices by moving around the stages $2(b)$, $2(c)$ and $2(d)$. The $i - j - k$-strategy, for $i, j, k \in \{b, c, d\}$, is the strategy were those stages are ordered $2(i)$ first, $2(j)$ second and $2(k)$ third. We say $i > j$ if step $2(i)$ comes before $2(j)$, as well $\gamma_{i>j} = 1$ if $i > j$ and 0 otherwise. We introduce the maximum degree of the bounded forests as $S_k = \sum_{\ell=1}^{k} d_\ell$ and we define three values depending on the order of the three exchangeable steps:

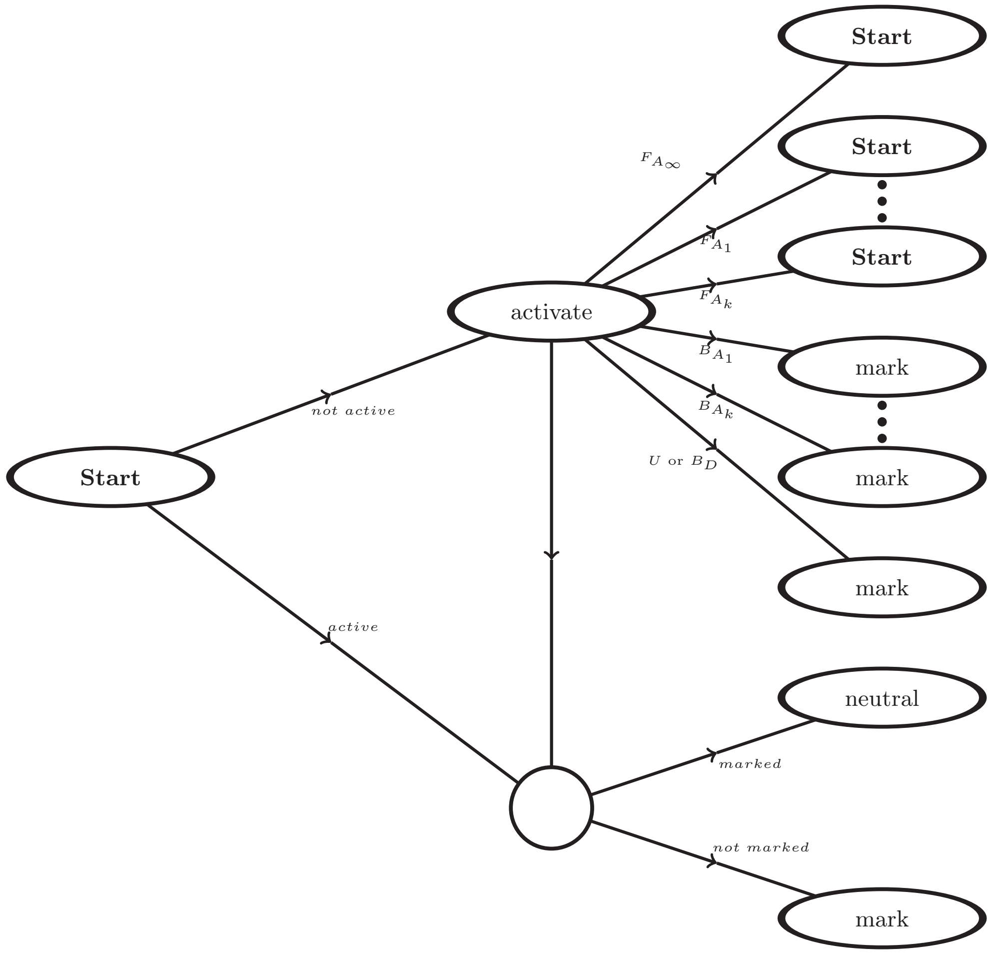


Figure C.0.1: Sketch of the selection recursion.

- $\mu_1 = 2a + (1 - \gamma_{c>b})2k + \gamma_{d>b}d + \gamma_{c>b}2S_k$,
- $\mu_2 = 4a + (\gamma_{b>c} - \gamma_{c>b})2k + (\gamma_{d>b} + \gamma_{d>c})d + (A + \gamma_{c>b})2S_k - 2$,
- $\mu_3 = 2a + (\gamma_{b>d} - \gamma_{c>d})2k + \gamma_{c>d}2S_k + d$.

Lemma C.3 *In every game and for a given strategy,*

1. *when an arc $\vec{e} \in \vec{A_\infty}$ is marked, it has at most $2a$ sons already marked.*
2. *when an arc $\vec{e} \in A_-$ is marked, the number of its already marked sons is at most μ_1.*
3. *when an arc $\vec{e} \in A_-$ is marked, the number of its sons and brothers already marked is at most μ_2.*
4. *when an edge $e = uv \in D$ is marked, the number of marked sons adjacent to u is at most μ_3. Moreover the total number of already marked sons of e is at most $2\mu_3 - 1$.*

Proof. We define $S_k = \sum_{l=1}^{k} d_l$.

1. Each time a son of $\vec{e} \in \vec{A_\infty}$ is activated, Alice considers an arc of $\{\vec{e}\} \cup P_\infty(\vec{e})$ (of size at most a by Observation C.2) by rule $2(a)$. By Observation C.1 an arc/edge is considered at most twice (and marked the second time). Thus $\vec{e}$ is marked when at most $2a$ of its sons are marked.
2. Take an unmarked arc $\vec{e} \in \vec{A_-}$. Each time a son of $\vec{e}$ is activated, one of the following may occur:

- By rule 2(a), Alice considers an arc of $P_\infty(\vec{e})$. By Observation C.1, this happens at most $2a$ times. This gives at most $2a$ marked sons by this rule.
- By rule 2(b), Alice considers an arc of $\{\vec{e}\} \cup P_-(\vec{e})$. The arc $\vec{e}$ is marked right after or already marked when this happens $2k$ times. This gives at most $2k$ marked sons by this rule.
- By rule 2(c), Alice marks an arc of $S_-(\vec{e})$. This can happen only if $c > b$, otherwise, by rule 2(b), $\vec{e}$ is already marked. It happens at most $\sum_{l=1}^{k}(d_l - 1) = S_k - k$ times. Note that each time Alice does it, it is because another arc of $S_-(\vec{e})$ was activated or marked just before. This gives at most $2\gamma_{c>b}(S_k - k)$ marked sons by this rule.
- By rule 2(d), Alice marks an edge of $C(\vec{e})$. As $\vec{e}$ is unmarked, this can happen only if $d > b$ and at most d times. This gives at most $\gamma_{d>b}d$ marked sons by this rule.

Thus $\vec{e}$ is marked when at most $(2a + 2k) + 2\gamma_{c>b}(S_k - k) + \gamma_{d>b}d = \mu_1$ sons are marked.

3. Take an arc $\vec{e} \in \vec{A_-}$, say $\vec{e} \in \vec{A_\infty}$ for some $1 \leq i \leq k$. As seen in the previous case, when μ_1 sons of $\vec{e}$ are marked, then if $\vec{e}$ is not, it is immediately marked by Alice.

   Each time $\vec{e}$ or a brother of $\vec{e}$ is marked, one of the following may occur:

   - Alice considers an arc of $F_\infty(\vec{e})$ by rule 2(a). This may only occur $2a$ times, which gives at most $2a$ brothers.
   - Alice considers an arc of $F_-(\vec{e})$ by rule 2(b). As $\vec{e}$ is unmarked, this may only occur if $b > c$ and only $2k$ times. This gives at most $\gamma_{b>c}2k$ brothers.
   - Alice marks an arc of $\{\vec{e}\} \cup B(\vec{e})$ by rule 2(c). If this occurs $S_k - k$ times, arc $\vec{e}$ is marked right after or already marked. As above, each time Alice does it, is because another arc of $\{\vec{e}\} \cup B(\vec{e})$ has been activated or marked. This gives in total $2(S_k - k) - 1$ brothers marked before $\vec{e}$.
   - Alice marks an edge of $U(\vec{e})$ by rule 2(d). This occurs only if $d > c$ and at most d times. This gives $\gamma_{d>c}d$ brothers.

   Be careful, if $|S_m(\vec{e})| = \mu_1$, then $\vec{e}$ is marked right after. Thus $\vec{e}$ is marked right after $|S_m(\vec{e})| \geq \mu_1$ or right after rule 2(c) happens $S_k - k$ times. This means that, when $\vec{e}$ is marked, it has at most:

   $$(\mu_1 - 1) + (2a) + (\gamma_{b>c}2k) + (2(S_k - k) - 1 - 1) + (\gamma_{d>c}d) + 1$$

   sons and brothers already marked.
   In total

   $$\mu_2 = 4a + (\gamma_{b>c} - \gamma_{c>b})2k + (1 + \gamma_{c>b})2S_k + (\gamma_{d>b} + \gamma_{d>c})d - 2$$

   sons and brothers already marked.

4. Take an edge $e = uv \in D$. Each time a son of e adjacent to u is marked, one of the following may occur:

   - Alice considers an arc of $F_\infty(\vec{e})$ adjacent to u by rule 2(a). This may only occur $2a$ times.
   - Alice considers an arc of $F_-(\vec{e})$ adjacent to u by rule 2(b). This may only occur if $b > d$ and only $2k$ times.
   - Alice marks an arc of $S_-(\vec{e})$ adjacent to u by rule 2(c). As e is unmarked, this may only occur if $c > d$ and at most $-k + \sum_{1 \leq l \leq k} d_l$ times. This makes in total at most $2(S_k - k)$ sons marked.
   - Alice marks an edge of $\{e\} \cup B(\vec{e})$ by rule 2(d). Arc $\vec{e}$ is marked if it occurs d times.

   Thus the number of sons of e adjacent to u marked before e is at most μ_3. This is also true for v, but e is marked right after one of these bounds is reached, so the total number of sons of e marked before is at most $2\mu_3 - 1$.

■

Theorem C.4 *For any $F^+(a, \{d_1, \ldots, d_k\}, d)$-decomposable graph \$G \$,*

$$col(G) \leq \max\{x_1, x_2, x_3\},$$

with

$$\begin{aligned} x_1 &= \Delta + 3a + k + d - 1 \\ x_2 &= \min\{\Delta + \mu_1 + a + k + d - 1, \mu_2 + 2a + 2k + 2d\} \\ x_3 &= \min\{\Delta + \mu_3 + a + k + d - 1, 2\mu_3 + 2a + 2k + 2d - 2\} \end{aligned}$$

Proof. Consider Alice uses one of our strategies. For any arc or edge of G, we give an upper bound on the number of adjacent arcs and edges already marked, using Observation C.2.

- First take an arc $\vec{e} \in \vec{A_\infty}$. It has at most $\Delta - 1$ fathers, uncles and brothers, at most $a + k - 1$ partners, d cousins, and when it is marked, it has at most $2a$ sons already marked by Lemma C.3.1. Thus it has at most $\Delta + 3a + k + d - 2$ neighbors marked before itself.
- Now take an arc $\vec{e} = uv \in \vec{A_-}$. At the endvertex v there is at most $\Delta - 1$ arcs and edges. Depending on the value of Δ it is more interesting to count the number of already marked arcs and edges instead of the general bound $\Delta - 1$. We study both cases.
  - It has at most $\Delta - 1$ fathers, brothers and uncles, at most $a + k - 1$ partners and at most d cousins. By Lemma C.3.2, we know that when $\vec{e}$ is marked, its number of sons already marked is bounded by μ_1. Altogether, $\vec{e}$ has at most $\Delta + \mu_1 + a + k + d - 2$ neighbors already marked.
  - Edge $\vec{e}$ has at most $a+k$ fathers, $a+k-1$ partners, d uncles and d cousins. Using Lemma C.3.3, we know $\vec{e}$ has at most μ_2 sons and brothers already marked, and so $\mu_2 + 2a + 2k + 2d - 1$ neighbors marked before itself.
- Finally, take an edge $e \in D$. As above, we study the general bound $\Delta - 1$ at one endvertex v and the number of already marked arcs and edges at both endvertices.
  - Edge $e = uv$ has at most $\Delta - 1$ adjacent edges adjacent to vertex v. It also has at most $a + k$ fathers and $d - 1$ brothers adjacent to u. When it is marked, by Lemma C.3.4, it has at most μ_3 sons indicent to u already marked. Altogether, in this case, e has at most $\Delta + \mu_3 + a + k + d - 2$ neighbors marked before itself.
  - Edge e has at most $2a + 2k$ fathers and at most $2d - 2$ brothers. When e is marked, its number of marked sons is bounded by $2\mu_3 - 1$ by Lemma C.3.4. Thus e has at most $2\mu_3 + 2a + 2k + 2d - 3$ neighbors marked before itself.

Therefore this gives

$$col(G) \leq 1 + \max \left\{ \begin{array}{ll} \Delta + 3a + k + d - 2, & (A_\infty) \\ \min\{\Delta + \mu_1 + a + k + d - 2, \mu_2 + 2a + 2k + 2d - 1\}, & (A_-) \\ \min\{\Delta + \mu_3 + a + k + d - 2, 2\mu_3 + 2a + 2k + 2d - 3\} & (D) \end{array} \right\}.$$

This completes the proof. ∎

We display in Table C.0.1 the details of Theorem C.4 for each of Alice's strategy.

Table C.0.1: Upper bounds of *col* for a F^+-decomposable graph (Theorem C.4)

Strategy	A_∞	A_-	
$b-c-d$	$\Delta+3a+k+d-1$	$\Delta+3a+3k+d-1$	$6a+4k+2d+2S_k-2$
$b-d-c$		$\Delta+3a+3k+d-1$	$6a+4k+3d+2S_k-2$
$c-b-d$		$\Delta+3a+k+d+2S_k-1$	$6a+2d+4S_k-2$
$c-d-b$		$\Delta+3a+k+2d+2S_k-1$	$6a+3d+4S_k-2$
$d-b-c$		$\Delta+3a+3k+2d-1$	$6a+4k+4d+2S_k-2$
$d-c-b$		$\Delta+3a+k+2d+2S_k-1$	$6a+4d+4S_k-2$

Strategy	D	
$b-c-d$	$\Delta+3a+k+2d+2S_k-1$	$6a+2k+4d+4S_k-2$
$b-d-c$	$\Delta+3a+3k+2d-1$	$6a+6k+4d-2$
$c-b-d$	$\Delta+3a+k+2d+2S_k-1$	$6a+2k+4d+4S_k-2$
$c-d-b$	$\Delta+3a-k+2d+2S_k-1$	$6a-2k+4d+4S_k-2$
$d-b-c$	$\Delta+3a+k+2d-1$	$6a+2k+4d-2$
$d-c-b$	$\Delta+3a+k+2d-1$	$6a+2k+4d-2$

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