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Fragilité financière par l'analyse des réseaux et l'approche comportementale

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Title : Financial fragility by network analysis and behavioral approach

Abstract :
This thesis studies financial fragility, *i.e.* the sensitivity of the financial system with respect to shocks. The main issue of financial fragility in the current context is the increased financial complexity. To address this problem, this study draws inspiration from two relatively recent streams of literature : economics of networks and behavioral economics. The main concepts in use are diffusion, cascade and bounded rationality. Chapter 1 studies how patterns of links, specifically, the length of transitive cycles affect the extent of financial contagion. Chapter 2 proposes a dynamic model in which bank runs arise as cascades of withdrawals. The aim is to better understand how bank runs occur. Chapter 3 studies bank runs in a dynamic and behavioral setting, with herding and heterogeneity of depositors.

Keywords : *Financial fragility, bounded rationality, systemic risk, behavioral economics, financial networks, computational economics.*

Titre : Fragilité financière par l'analyse des réseaux et l'approche comportementale

Résumé :
L'objectif ce cette thèse est d'étudier la fragilité financière, *c.a.d* la sensibilité du système financier par rapport aux perturbations. La difficulté principale concernant la fragilité financière dans le contexte actuel est la complexité croissante du système financier. Pour remédier à ce problème, cette thèse s'inspire des deux courants relativement récents de la recherche économique : l'analyse des réseaux et l'économie comportementale. Les principaux concepts mobilisés sont les mécanismes de diffusion, de cascade et la rationalité limitée. Chapitre 1 étudie les effets des structures locales des liens, spécifiquement la longueur des cycles transitifs, sur la magnitude de la contagion financière. Chapitre 2 propose un modèle dynamique des paniques bancaires, dans lequel les paniques émergement par un mécanisme de cascade des retraits. Le but est de mieux comprendre comment les paniques se forment. Chapitre 3 étudie les paniques bancaires dans un contexte à la fois dynamique et comportemental, avec la présence du mimétisme et l'hétérogénéité des déposants.

Mots clés : *Fragilité financière, risque systémique, rationalité limitée, économie comportementale, réseaux financiers, économie computationnelle.*

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Preface

This dissertation is the result of my PhD research at the University of Bordeaux, officially started in 2013. However, for various reasons[1], the topic changed and I begin this work in early 2015. Looking backward, the premises of this thesis kindled many years ago, when I enrolled in the economic program. As many young fellows, I started out with a futile motivation: to make the world a better place. Years later, after completing this thesis, the futile motivation remains and I still have no idea how to do it. The only way to find out is to move forward.

Doing a thesis is a journey and I would like to express my gratitude to people who have accompanied and helped me along the way.

First and foremost, I would like to thank my main supervisor, Emmanuelle Gabillon, for her steady support, for trusting and guiding me in unconventional terrains. Also, I would like to thank Nicolas Carayol for helpful advice and lessons, for being a fantastic coach even in difficult times.

Of course, I am indebted to my coauthors. Many thanks to Noemí Navarro for her patience and guidance for the first research paper, for her understanding when my health problems turned up. Many thanks to Emmanuelle Augeraud-Veron for her interests and excitement in our work. Although we just met recently, it is confirmed that her energy is unlimited and also has positive spillover effects. I have learned a lot from working with them.

During the time at the University of Bordeaux, I have been advised and guided by many admirable professors. Particularly, many thanks to Marc-Alexandre Sénégas, Murat Yildizoglu, Francesco Lissoni, Hervé Hocquard, Jean-Christophe Pereau, Emmanuel Petit, Pascale Roux, and Vincent Frigant for being excellent mentors and educators. I have greatly benefited from them in scientific research, teaching and personal growth.

It is obvious that I am grateful to all my colleagues, PhD-candidate fellows and friends at the University of Bordeaux. Particularly, thanks to Lucie for her material and mental support in time badly needed. Thanks to Robin for inspiring discussions in many remotely research-related subjects. Thanks to François and Viola for the longtime friendship. Thanks to Nicolas Mauhé, Selma, Johannes and Lionel for serious discussions and their helps. Special thanks to Nicolas Yol for countless discussions on famous economists and other delightful subjects, for being the most reliable person in both academia and everyday life. Many thanks to Fadoua, Jeanne, Romain, Samuel, Suneha, Lucile, Julien, Valentina, with whom I have shared many pleasant moments.

It would take a lot space if I have to thank my family and friends here, I will find another place to do it soon.

Last but not least, I am grateful to the members of the jury, who take time and effort to evaluate this dissertation. It's my honor to have them as thesis jury.

[1]one of the reasons is probably my laziness, as pointed out by some people. It might be true but hasn't been econometrically proven

“Only in the darkness, we can see the stars”

Thomas Carlyle

For you

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Introduction

This time is different

The global financial crisis of 2008 took us by surprise. As Bernanke (2013) pointed out "almost universally, economists failed to predict the nature, timing, or severity of the crisis". To illustrate the point, not long before the outbreak of the turbulence, Olivier Blanchard[2] declared "the state of macro is good".

However, the surprise element is not really a surprise. Crises are recurrent and they are hard to predict. To the point that they became "business as usual" as discussed by Mattick (2011). The markets (and eventually, economists) are said to have short memory. Events such as the 2000 Internet crisis or the 1998 LTCM crisis quickly faded away in history.

But "this time is different", as Krugman (2009) summarizes the sentiment shared by many economists. The most unexpected elements of the 2008 crisis were its severity and magnitude. Worldwide, 71 countries plunged into recession, according to OECD (2015). The total loss was unprecedented. The U.S. suffered heavy consequences until this day: $20 trillion included output losses, more than its entire annual GDP (Luttrell et al. (2013)). Full recovery is not yet reached 10 years afterward (Barnichon et al. (2018)). And after all, these consequences were not the worst scenario. An emergency rescuing package of $1 trillion was issued right after the peak of the crisis.

This event demonstrates the emergence of a new threat, inconceivable until recently: a potential systemic collapse of the world financial and economic system (*systemic risk*). All of it was started by a small shock in the U.S. housing market. How was this possible? This question is the central theme of this thesis.

Financial fragility

The term *financial fragility* is used to designate the phenomenon of "small shock, large crisis", as summarized by Gorton and Ordonez (2014). In other words, it reflects how the system as a whole (financial markets or macroeconomic state) is sensitive to small shocks. This idea has been formalized by various works appeared in the same period such as Diamond and Rajan (2001); Lagunoff and Schreft (2001); Allen and Gale (2004). The literature on systemic risk also started to grow, with the pioneer works of Allen and Gale (2000); Freixas et al. (2000); Eisenberg and Noe (2001). That was well before the crisis.

Then, as Krugman asked, "How Did Economists Get It Wrong?" To their defense, one possible argument is that the theories were in accordance with the circumstances of their time[3]. An essential ingredient is yet to come: *complexity*.

In the early 2000s, the global financial system has rapidly transformed. Two of the main reasons were the exponential advance of technology and heavy financial deregulation[4]. Financial complexity escalated in two directions: connectivity and opacity.

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[2]soon to be Chief Economist of the IMF by that time. The article is published one year later in Blanchard (2009)

[3]there are other arguments which involve ideological assertions that take a lot of space to explain, interested readers can see Krugman (2009, 2011)

[4]following the efforts of the Fed to pull the U.S. economy out of the 2000 Internet crash

First, financial linkages increased rapidly, as observed by Haldane (2013). Obviously, it became easier to make transactions across all distances. More than that, the Glass-Steagall act was repealed. Financial institutions were then allowed to combine commercial banking, securities intermediation and insurance operations. There were more transactions to make.

Second, complex financial contracts which emerged in the 1990s became widespread (Gorton (2008, 2009)). Along with financial deregulation and improved computational power, a new banking paradigm replaced the old one: originate-to-sell rather than originate-to-hold. In consequences, derivatives such as Assets-Backed-Securities and Collateralized-Debt-Obligation flooded the markets. These are financial contracts whose payoffs depend on the payoffs of the underlying products, such as loans or mortgages. Securitization then became even more sophisticated: combine derivatives to issue higher order derivatives. This trend reinforced the first mechanism, making more connections at the same time with more opacity.

Overall, the system became highly sensitive. Higher connectedness among financial institutions increases potential direct shock transmission and indirect exposure through asset prices. Higher opacity increases the risk of runs and panics, if any shock takes place. Furthermore, these mechanisms have self-amplifying effects. Combined with high leverage and massive short-term funding, this environment has produced one of the most severe economic downturns in history.

10 years after the crisis, economic literature on this subject has been growing rapidly. However, many questions remained unanswered, there are still a lot of work to be done.

The two approaches

To tackle new problems, this thesis employed and combined tools from two relatively recent streams of literature: economics of networks and behavioral economics.

Obviously, one might question the motivation to these approaches. An apparently straightforward argument is to evoke Kuhn's law: when anomalies accumulate and existing theories cannot explain these anomalies, there will be a shift in scientific paradigms. For economics, this argument is, at most, inaccurate. Empirically, a glance on economic literature does not indicate any paradigm shifting soon, or maybe at all. On the philosophical standpoint, economics differs from physics. Physics is about discoveries of permanent and universal truth, therefore physicists dream of the "Theory of everything". Unexplained anomalies are unacceptable, new paradigm should replace the old one. On the contrary, economics studies people, human organizations and societies. It is fair to say nothing of the above is permanent nor universal, rather continuously changing and diversified in many aspects. Indeed, the ideal economic science should be built upon the "many-model thinking" paradigm (Miller and Page (2009)). All economic theories are partial and more partial theories are better than one partial theory.

More specifically, there are two motivations for these new approaches. The first argument is that many respectable economists, such as Krugman (2009); Bernanke et al. (2010); Haldane (2013) among others, have advocated the application of these tools. The second motivation is that they probably make a point, as developed in the following paragraphs.

The application of network analysis is straightforward. It addresses two main assumptions of neoclassical economics which limits the scope of the analysis of financial fragility. The first assumption is that agents are assumed to freely interact with other agents. This is not the case in financial networks, where possible interactions are conditioned by the structure of financial linkages. For some specific problems such as loss spillover, financial contracts also determine the nature of interactions: debtors diffuse losses while creditors take losses (Elliott et al. (2014); Glasserman and Young (2015)). The second assumption is that there is a continuum of agents and they are homogeneous. This is also not the case in the financial system. Merges and bailouts give way to a few systemically important financial institutions (SIFIs). They clearly differ in size, connectivity and market power compared to others. Network analysis, especially computational network models (Nier et al. (2007); Gai and Kapadia (2010); Battiston et al. (2012a)), has shed new light on the link between financial complexity and financial fragility. Moreover, quantitative models are already in use to monitor systemic risk (DebtRank of Battiston et al. (2012b), Contagion Index of Cont et al. (2010)).

The behavioral approach is perhaps more controversial. It addresses the most central assumption of mainstream economics: perfect rationality. This assumption is the main workhorse of economic theories. It asserts that economic agents are infinitely intelligent and have stable preferences, they always act in such a way to maximize their utilities in a precise and invariable manner. Basically, it can be split into several usual implicit assumptions: agents know a lot of information, have infinite computational power to process the needed information and deliver the optimal solution, no matter how complex is the problem. However, in the context of financial complexity, Haldane (2012) argued that "humans follow simple rules" and "less may be more" by using a simple analogy. A dog can easily catch a frisbee without knowing or applying Newton's mechanics. All it takes are simple rules. Bernanke et al. (2010) pointed out another idea: "at certain times, decision makers simply cannot assign meaningful probabilities to alternative outcomes – indeed, cannot even think of all the possible outcomes – is known as Knightian uncertainty". In other words, there might not be enough information to compute rational expectation. In consequences, the point is that agents are rational, but not perfectly[5]. This is known as *bounded rationality*, a term coined by Simon (1972). Experimental economics is exploring the effects of biases and bounded rationality on bank runs and panics. One direct theoretical consequence of bounded rationality is non-linear dynamics and chaos. This subject is at the core of computational economics, which attracts increasing attention as summarized in Battiston et al. (2016).

It is worth stressing two important points. First, computational models are rigorous mathematical models, despite the apparent "lazy thinking" by letting the machines do the job, as one might suspect. These models address problems where it is impossible to derive analytical solutions, or such solutions require tremendous time and computational power. This is exactly what bounded rationality entails. Simplicity is elegant, but sometimes, "economists will have to learn to live with messiness", pointed out by Krugman (2009). This immediately leads to the second point as

[5] After all, the point is not to say that economists should stop doing models with rational expectation. The reason is twofold. First, there is only one way to be rational (the perfect way). Economists need a benchmark to compare models and to detect "anomalies". Second, in some situations such as absence of complexity, perfect rationality might work. It is imperative to leave the debates open with a lighten comment from Krugman (2011) "my problem is obvious: I'm an economist, and it seems that we need some kind of sociologist to solve our profession's problems."

discussed by Haldane (2012) “as you do not fight fire with fire, you do not fight complexity with complexity”, to remind that one should not treat complexity with over-complex models. Technical complexity is a double-edge knife, it should be employed only when necessary. Indeed, simplicity would be the ultimate complexity, where simple computational models can generate rich patterns that we actually observe, as the pioneer work of Brock and Hommes (1998).

Finally, Bernanke et al. (2010) offered a satisfactory summary: “both older and more recent ideas drawn from economic research have proved invaluable to policymakers attempting to diagnose and respond to the financial crisis”.

Methodology & Contributions

This thesis consists of 3 related chapters. The methodology and contributions of these chapters are summarized in what follows.

Chapter 1 studies how the presence of transitive cycles in the network affects the extent of financial contagion. In a regular network setting, where the same pattern of links repeats for each node, we allow an external shock to propagate losses through the system of linkages. The extent of contagion (contagiousness) of the network is measured by the limit of the losses when the initial shock is diffused into an infinitely large network. This measure indicates how a network may or may not facilitate shock diffusion in spite of other external factors. Our analysis highlights two main results. First, contagiousness decreases as the length of the minimal transitive cycle increases, keeping the degree of connectivity constant. Second, as density increases the extent of contagion can decrease or increase, because the addition of new links might decrease the length of the minimal transitive cycle. Our results provide new insights to better understand systemic risk and could be used to build complementary indicators for financial regulation.

Chapter 2 proposes a dynamic model in which bank runs arise as cascades of withdrawals. The aim is to better understand the patterns of how bank runs occur. With bounded rationality, agents employ a switching strategy that combines strategic complementarity and heuristics. When a fraction of random agents withdraw, under the right conditions, some depositors preemptively withdraw in response, increasing the probability that other depositors will run subsequently. The model is able to characterize two distinct patterns of runs. Immediate runs develop instantly following the shock with a stable trajectory. On the contrary, sudden runs occur “out of nowhere”, with massive withdrawals concentrate in a very short time window after a period of apparent inactivity. We provide analytical calculation of the tipping point, where the panic burst out.

Chapter 3 studies bank runs in a dynamic and behavioral setting. Current theoretical models mainly consider bank run as mis-coordination in simultaneous games. From another perspective, bank runs arise in this model as dynamic cascades of withdrawals, through strategic complementarity and herding. Within a network, agents can observe the actions of their neighbors. Agents make decisions based on (i) their types, (ii) their private signals and (iii) the observed actions of others. The model is able to characterize the frequency, speed and abruptness of bank runs. Particularly, there are two distinct patterns: sequential withdrawals build up progressively or massive withdrawals suddenly occur “out of nowhere”. Regarding the behavioral aspect, increase herding generates a tension between activation and speed, runs are more frequent but also slower to build up. By contrast, increase heterogeneity facilitates both activation and speed of runs.

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Chapter 1

Shock diffusion in large regular networks: the role of transitive cycles

Disclaimers[1] [2]

[1] This chapter is a joint work with Noemí Navarro

[2] We thank Stefano Battiston, Olivier Brandouy, Nicolas Carayol, Nicolas Castro Cienfuegos, Vincent Frigant, Emmanuelle Gabillon, Jaromír Kovárík, Ion Lapteacru, Francesco Lissoni, Barry Quinn, Christoph Siebenbrunner and participants at the Belgian Financial Research Forum (National Bank of Belgium), 33rd Annual meeting of the European Research Group on Money, Banking and Finance (CERDI), Network group workshop at CES (Paris 1 University), 7th Workshop on Networks in Economics and Finance (IMT Lucca), International Workshop on Financial System Architecture and Stability 2018 (Cass Business School) and and GREThA doctoral workshop (Bordeaux University) for useful comments. The usual disclaimer applies.

1 Introduction

Financial contagion is commonly regarded as the hallmark of the 2007-2008 financial crisis. Since the pioneering works by Allen and Gale (2000); Freixas et al. (2000), many studies have analyzed how the structure of financial networks affects the propagation of shocks[3]. The literature has uncovered the role played by certain characteristics of the network, focusing notably on density, which relates to the average number of neighbors or average degree in the network[4]. With different methodologies, this stream of literature shows that the effect of network density on shock diffusion is non-monotonic and depends on factors as the size of the shock, the presence of financial acceleration, level of integration, or the diversification of the system[5].

Nevertheless, little is known about the effect of other characteristics of the network with the exceptions of Craig et al. (2014) and Rogers and Veraart (2013) on individual centrality, or Allen et al. (2012) on clustering. We contribute to this literature by studying the role of transitive cycles in facilitating or restraining the propagation of a shock in financial networks. Our model shows that the length of transitive cycles is an important factor that shapes the relationship between network density and shock diffusion.

To lay out the intuitive foundation, consider two different structures of financial networks as depicted in Figure 1.1. We will provide formal definitions in the next section. An arrow from bank 1 to bank 2 indicates that bank 2 will take a loss if bank 1 fails. We call bank 1 an in-neighbor of bank 2 and bank 2 an out-neighbor of bank 1. In both networks represented below, each institution has two in-neighbors and two out-neighbors. Nevertheless, these two networks are not identical, or isomorphic, due to the different structure of *cycles* they each possess.

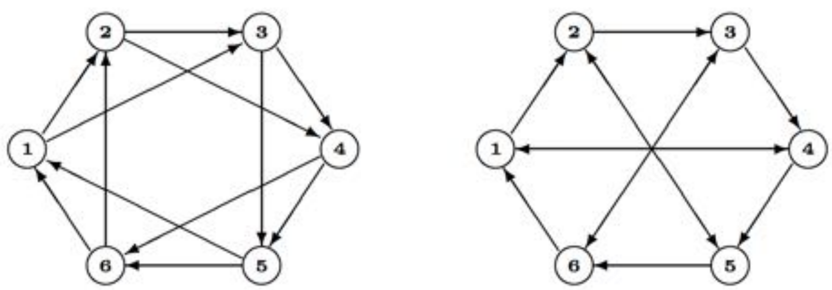

Figure 1.1: Same degree, different cycle length. The network (a) is on the left, (b) is on the right.

We observe cycles of different length for each structure. In network (a) 1 can affect 2, 2 can affect 3, and 1 can affect 3. We call this transitivity of loss-given-default among financial institutions a *transitive cycle*. In network (b) the transitive cycles always include at least four banks, while in

[3]see Summer (2013); Cabrales et al. (2015); Glasserman and Young (2016) for reviews of this stream of literature.

[4]Acharya (2009); Gai et al. (2011); Battiston et al. (2012a); Elliott et al. (2014); Acemoglu et al. (2015); Gofman (2017); Castiglionesi and Eboli (2018), among others.

[5]A higher density implies higher *individual* diversification but it does not necessarily mean more *systemic* diversity.

network (a) they only include three banks. Therefore, the length of the minimal transitive cycle is smaller in network (a) than in network (b).

We model the structure of financial liabilities as a directed network. When a bank defaults after taking a large external shock, it will impose losses on other banks to which it has liabilities. The losses-given-default in turn may cause these banks to fail. Thus, losses propagate into the network as a flow through a system of linkages. Inspired by Morris (2000), we assume that the population is infinite but each bank has a finite number of links, in our case with an identical pattern[6]. This type of structure is what we consider a large regular network. In this setting, we measure how a structure facilitates shock diffusion by computing the limit of the individual loss when the distance between a bank and the initial shock goes to infinity. A small value of this measure indicates that the structure itself is robust and can restrain the diffusion of the initial shock to a long distance. We therefore take this measure as an indicator of the contagiousness of the network.

In our setting, we show that the contagiousness of the network decreases as the length of the minimal transitive cycle increases, while keeping the number of links equal and constant for all nodes. Furthermore, increasing the connectivity of the network can have ambiguous effects on contagiousness. This ambiguity arises because when connectivity increases, additional links may or may not decrease the length of the minimal transitive cycle. On the one hand, when additional links do not change the length of minimal transitive cycle (long links are added), contagiousness decreases as connectivity increases. On the other hand, when additional links are made to banks at a closer distance than the length of minimal transitive cycle (short links are added), the length of the minimal transitive cycles decreases. In this case, contagiousness decreases as connectivity increases if and only if the length of the minimal transitive cycle is above a certain threshold. If the length of the minimal transitive cycle is lower than the threshold, increasing connectivity by adding links to banks that are relatively close will result in an increase in contagiousness.

To extend our analysis, we study the contagiousness of regular networks versus different structures having some related characteristics. First, we compare regular networks to tree networks with the same out-degree. The contagiousness of the tree networks always tends to zero as long as the out-degree of each node is greater than 1. We note that the contagiousness of regular network approaches the one of tree networks as the length of the minimal transitive cycles approaches infinity. We next use complete multipartite networks as a benchmark for comparison. Complete multipartite networks have the property of keeping the losses constant as the initial shock diffuses into the system. This constant loss is equal to the reduction in asset value of the direct neighbors of the first defaulted bank. Again we find a threshold for the length of the minimal transitive cycles, above which the contagiousness of regular structures is smaller than the one of the multipartite networks.

These results suggest some policy implications. First, many systemic-risk indicators have been developed, with several ones that take into account the structure of the financial system together with financial acceleration (for example, DebtRank by Battiston et al. (2012b), or Contagion Index by Cont et al. (2010)). Our measure, focusing solely on the structure of the network, could be

[6]The assumption of an infinite population allows us to draw more general conclusions about the effect of the length of the minimal transitive cycle. If each bank has assets and liabilities to a finite number of other banks, and the total number of banks is finite, a few values of length of minimal transitive cycle are compatible. By allowing the total number of banks to be large enough we also allow for the length of the minimal transitive cycle to go from 3 to infinity.

useful to build complementary indicators. Knowing which region has high potential for shock diffusion may help regulators to devise appropriate interventions in time of crisis. Furthermore, as the measure is derived without complex financial mechanisms, its application can be adapted to other type of financial interdependencies, such as networks of payments.

Second, the Basel Committee on Banking Supervision has compiled a set of global standards for financial institutions since 1982. One of the most important objectives is to improve the banking sector's ability to absorb shocks arising from financial and economic stress. In response to the 2007 global financial crisis, Basel III specifies extra recommendations for systemically important financial institutions (SIFI). Going one step further, the European Commission has decided to transpose some of the Basel III recommendations into laws that will be enforced starting in 2019 for the European Union. These recommendations focus mainly on variables at the individual level such as capital requirement, liquidity, and leverage ratio, with surcharge to SIFIs due to their potential important impact to the financial system. In what concerns the results presented in this paper, it would be useful to have complementary regulations on the structure itself of the financial linkages. Banks have to be more careful when choosing their diversification strategies, as increasing the level of diversification might facilitate the diffusion of potential shocks, especially when the length of the minimal transitive cycle decreases.

This paper is organized as follows. We introduce the setting in Section 2. The results are stated in Section 3. We provide a discussion of our results in Section 4 and conclude in Section 5.

2 The model

2.1 The financial interdependencies

In this section we introduce the basic notions and definitions that are needed in the subsequent analysis. More exhaustive definitions and measures can be found in Goyal (2012); Jackson (2010).

Let $N = \{1,2,...,n\}$ denote the set of financial institutions (or banks, for short). Each bank $i \in N$ holds a capital buffer $w_i \geq 0$, owns external assets for an amount of $a_i \geq 0$, and has liabilities to other banks $l_{ij} \geq 0$, where $j \in N$, $j \neq i$. The total interbank liability held by bank i is given by $L_i = \sum_j l_{ij}$. Bank i's total assets are therefore given by $a_i + \sum_k l_{ki}$ and banks i's total liabilities are given by $w_i + \sum_j l_{ij}$.

This interdependence can be represented by a directed graph over N where the set of links g is defined by $ij \in g$ for $i \in N$ and $j \in N$ if and only if $l_{ij} > 0$. To keep the model tractable, we have taken some regularity assumptions regarding the financial interdependence network.

Given a bank i, we define i's out-neighborhood to be the set of banks to whom i has a liability, i.e., $N_i^{out}(g) = \{j \in N \text{ such that } l_{ij} > 0\}$. The cardinality of i's out-neighborhood is called i's out-degree and denoted by k_i^{out}. Similarly, let i's in-neighborhood be the set of banks that have a liability with i, i.e., $N_i^{in}(g) = \{j \in N \text{ such that } l_{ji} > 0\}$. The cardinality of i's in-neighborhood is called i's in-degree and denoted by k_i^{in}.

A path in the network (N,g) is a set of consecutive links $\{i_1i_2,\ i_2i_3,\ ...,\ i_{r-1}i_r\} \subseteq g$ with $i_s \in N$

for all $s = 1,..,r$ and $i_s i_{s+1} \in g$ for all $s = 1,..,r-1$. The length of a path is the number of links in it. We say that j is connected to i if there is a path $\{i_1 i_2,\ i_2 i_3,\ ...,\ i_{r-1} i_r\} \subseteq g$, such that $i_1 = i$ and $i_r = j$. The distance between i and j in the network (N, g), denoted $d(i, j)$, is the number of links in the shortest path that connects i to j or vice versa (the path with smallest distance between two players is called a geodesic). A subset of nodes $S \subseteq N$ is connected in the network (N, g) if for every pair of nodes i and j in S either i is connected to j or j is connected to i. The network (N, g) is connected if N is connected in (N, g). We denote by $N_i^{out,\infty}$ the set of nodes that are connected to i in (N, g) and by $N_i^{in,\infty}$ the set of nodes to whom i is connected in (N, g).

A *transitive cycle* in the network is a path such that there exists *distinct* nodes $\{i_1, ..., 1_c\} \subseteq N$ satisfying that $\{i_1 i_2,\ i_2 i_3,\ ...,\ i_{c-1} i_c,\ i_1 i_c\} \subseteq g$. An intransitive cycle in the network is a path such that there exists *distinct* nodes $\{i_1, ..., 1_c\} \subseteq N$ satisfying that $\{i_1 i_2,\ i_2 i_3,\ ...,\ i_{c-1} i_c,\ i_c i_1\} \subseteq g$. Note that our cycles are "minimally" defined because in our definition the nodes in the cycle are distinct (a node cannot be visited several times). The length of a cycle is the number of links in the cycle, which by our definition of a cycle is also equal to the number of participants in the cycle. Figure 1.2 below shows a transitive and an intransitive cycle of length $c = 4$.

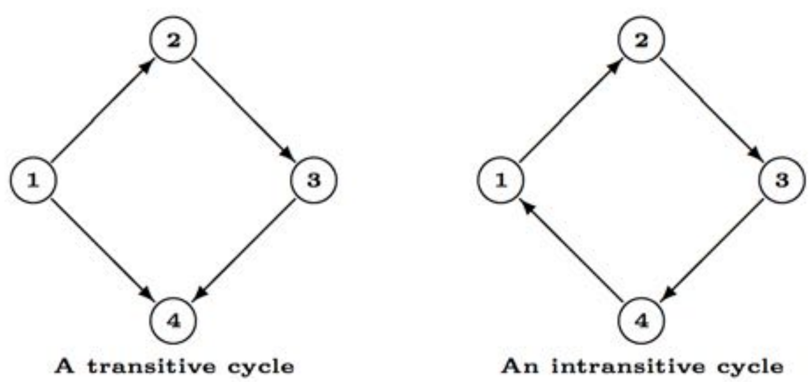

Figure 1.2: Cycles of length 4

To keep the model tractable, we make some regularity assumptions regarding the structure of the network. A financial network is homogeneous if all banks have the same and equal out-degree and in-degree, i.e. $k_i^{in} = k_i^{out} = k$ and it is transitive if (i) all cycles are transitive and (ii) for any two nodes i and j in N, if i is connected to j then j is not connected to i. For simplicity, we assume that all positive claims are of equal value, normalized to 1.

2.2 Bankruptcy and shock diffusion

Define x_i as the total loss in external and interbank assets that bank i receives in case of a shock.

We use the standard defaulting rules in the literature, as inEisenberg and Noe (2001). Creditors have priority over shareholders and interbank liabilities are of equal priority. When a bank receives a shock, the losses on its external and interbank assets are reflected in capital loss. When its capital is depleted, the bank defaults. The condition of default of bank i is given by $x_i \geq w_i$. Then, the total loss-given-default that bank i impose on its creditors is

$$LGD^i = x_i - w_i \geq 0$$

A bankruptcy event is organized as follows: the defaulted bank liquidates all of its remaining assets and the liquidation proceeds are shared among creditors proportionally according to bank i's relative liabilities. We assume that for all assets, liquidating value is identical to book value, so that defaulted banks do not generate additional losses. Then, sharing liquidation proceeds is equivalent to share loss-given-default proportionally among creditors. Let's consider an example, depicted in Figure 1.3.

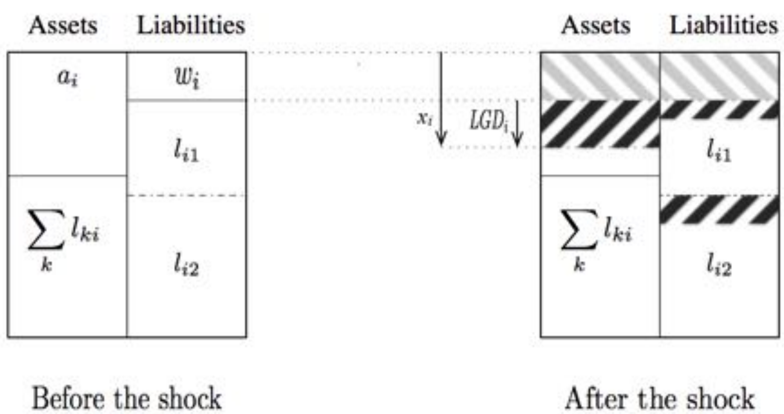

Figure 1.3: The shock and LGD

When bank i defaults from the external shock x_i, its liquidation proceeds are $a_i + \sum_k l_{ki} - x_i$. The loss-given-default that bank j suffers from the default of bank i is the difference between nominal liability and proportional repayment made by bank i to bank j.

$$\begin{aligned} LGD^i_j &= l_{ij} - (a_i + \sum_k l_{ki} - x_i)\frac{l_{ij}}{L_i} \\ &= \frac{l_{ij}}{L_i}\left[L_i - (a_i + \sum_k l_{ki})\right] + x_i \frac{l_{ij}}{L_i} \\ &= \frac{l_{ij}}{L_i}(-w_i) + \frac{l_{ij}}{L_i}x_i = LGD^i \frac{l_{ij}}{L_i} \end{aligned}$$

Thus, the shock is distributed proportionally according to relative liabilities. If the network is transitive, the shock diffuses in waves that do not come back to nodes who have been already affected by it.

3 Results

3.1 Limiting behavior of the shock

In order to compute the limit of losses in homogeneous, transitive networks as the number of banks gets large (when $n \to \infty$), we define regular networks of degree k and minimal transitive cycles of length c as follows.

Definition 1. We say that a homogeneous, transitive network is a regular network with degree k and minimal transitive cycle of length $c \geq 3$ if (i) all nodes have in-degree and out-degree equal to k and (ii) starting from any bank $b \in N$ we can relabel the banks in a way such that for any $i \in N_b^{out}$

$$N_i^{out} = \{i+1, i+c-1, i+c, i+c+1, ..., i+c+k-3\}$$

and

$$N_i^{in} = \{i-1, i-c+1, i-c, i-c-1, ..., i-c-k+3\}.$$

Figure 1.4 shows parts of (infinite) regular networks of degree $k = 2$ and minimal transitive cycle of length $c = 3$, $c = 4$, and $c = 5$, respectively. Each of the patterns shown below is assumed to be repeated infinitely because $n \to \infty$.

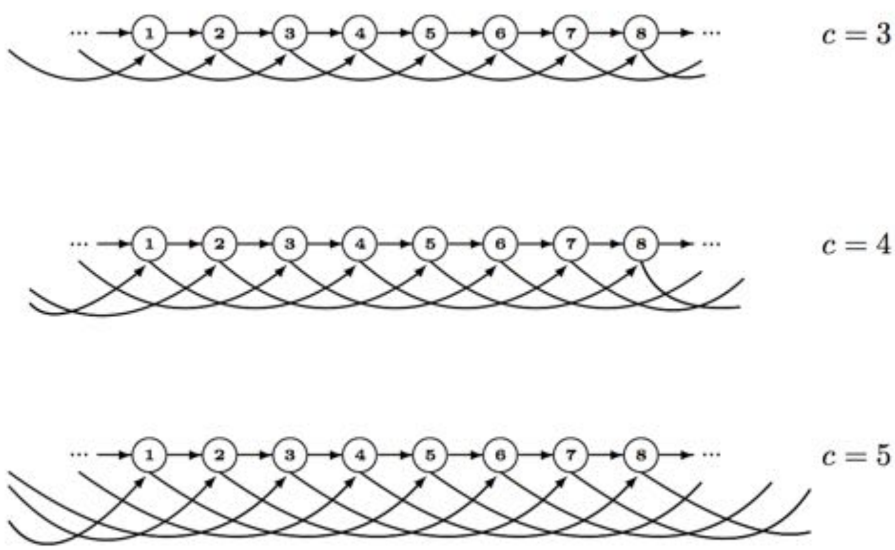

Figure 1.4: Regular networks of degree 2

The term *minimal* transitive cycle of length c is used because a regular network, as defined previously, has many transitive cycles if $k > 2$. For example, if $k = c = 3$ and labeling the nodes as in the examples shown in Figure 1.4, we have that $\{12, 23, 13\} \subseteq g$ (transitive cycle of minimal length 3). Nevertheless, $\{12, 23, 34, 14\} \subseteq g$ is also a transitive cycle, but of length greater than 3.

We have the following result regarding the limit behavior of a single shock.

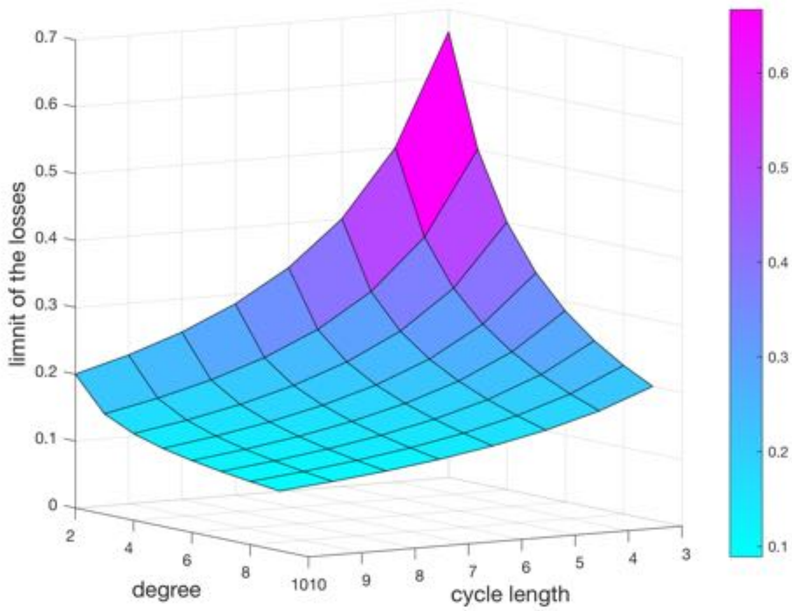

Figure 1.5: The limiting value of $\frac{x_i}{x_j}$ as $d(i,j)$ goes to infinity and j receives the unique initial, external shock, for $k = 2,...,9$ and $c = 3,...,10$

Theorem 1. *Let w_i be equal to 0 for all $i \in N$ and assume one single external initial shock: there is one unique $j \in N$ such that (i) $x_j > 0$, and (ii) if $x_i > 0$ for $i \neq j$ then $i \in N_j^{out,\infty}$ and $x_i = \sum_{m \in N_i^{in}(g)} \frac{1}{k} x_m$. If the interdependency network of liabilities is a regular network of degree $k \geq 2$ and minimal cycle length $c \geq 3$ then for $i \in N_j^{out,\infty}$*

$$x_i \rightarrow \frac{2k}{2k + (k-1)(k+2c-6)} x_j \quad as \quad d(i,j) \rightarrow \infty$$

The proof is in the Appendix and it is built considering a natural relabeling/ordering of the nodes from their position/distance with respect to the node suffering the initial shock $j \in N$. We can then consider x_i for $i \in N_j^{out,\infty} = \{2,3,4,....\}$ as an infinite sequence in $\Re_+$. This sequence is convergent in $\Re_+$ and its limit depends on x_j, k, and c as stated in Theorem 1. Figure 1.5 shows a numerical example of the behavior of the limit $\frac{x_i}{x_j}$ as c and k vary.

Theorem 1 shows that the losses received by banks that are connected to the node receiving the initial shock $j \in N$ do not go to zero even if banks are located infinitely far from j (as far as k and c are finite). A large value for the limit of the sequence x_i indicates that the structure itself facilitates the propagation of the losses without further consideration of other factors. Therefore we can consider the limit value of the losses as a measure of the contagiousness of the network.

3.2 Comparative statics

We discuss now how the limiting value of $\frac{x_i}{x_j}$, where j is the bank with the external, initial shock and $i \in N_j^{out,\infty}$ changes as k and/or c vary.

We observe from Theorem 1 that the limit of $\frac{x_i}{x_j}$ decreases with higher values of k or higher values of c (recall that $c \geq 3$). Therefore, according to Theorem 1, we can make two statements regarding the contagiousness of the network. First, increasing the length of minimal transitive cycles, while keeping the degree of connectivity constant, will make the network more robust, in the sense that it will dissipate a larger fraction of the shock during the diffusion process. Figure 1.4 provides an example of networks with degree equal to 2 but different lengths of minimal transitive cycles. Secondly, increasing the degree of connectivity, *while keeping the length of the minimal transitive cycle constant*, will also reduce the contagiousness of the network. Both of these effects can be observed in figure 1.5, as we move down along either one of the axis from any point.

Increasing the degree of connectivity might nevertheless decrease the length of the minimal transitive cycle. An example can be found in Figure 1.6 below. Starting from a regular network with $k = 2$ and $c = 4$, increasing the degree to $k = 3$ can be done in two different ways, such that the network remains regular as previously defined. First, we could add the link $i, i+2$ to the initial network, which would decrease the length of the minimal transitive cycle to 3. Secondly, we could also add the link $i, i+4$ to the initial network, which would keep the length of the minimal transitive cycle equal to 4. In general, to obtain a regular network of degree $k+1$ by adding one link per node to a regular network of degree k and minimal transitive cycle length c, there are two possible results. If we add the link $i, i+c-2$ for each $i \geq 1$ to the initial network (new *short* links) the length of the minimal transitive cycle decreases to $c-1$. If we add the link $i, i+k+c-2$ for each $i \geq 1$ to the initial network (new *long* links) the length of the minimal transitive cycle stays equal to c.

With regard to the addition of short links, we have the following proposition for the limit of losses, when the degree increases by one unit while the length of minimal transitive cycle decreases by one unit.

Proposition 1. *Let* $\bar{x}(k,c) = \frac{2k}{2k+(k-1)(k+2c-6)}$. *We have that* $\bar{x}(k+1, c-1) < \bar{x}(k,c)$ *if and only if* $c > 3 + \frac{k(k-1)}{2}$.

The proof of Proposition 1 is straightforward and therefore omitted. Proposition 1 states that there is a threshold for the length of the minimal transitive cycle such that the addition of a short link to each node reduces the contagiousness of the network.

Summing up, if the length of the minimal transitive cycle c is large enough, the contagiousness of the network is reduced if we consider an increase in degree, regardless of the type of additional links. When the length of the minimal transitive cycle is low, increasing the degree of connectivity has ambiguous results. If short links are added, the network becomes more contagious, while if long links are added then the network is less contagious.

This result allows us to identify another factor that contributes to the non-monotonic relationship between density and systemic risk. Proposition 1 shows that increasing density may decrease or increase the extent of contagion depending on how the length of transitive cycles in the network changes as density varies.

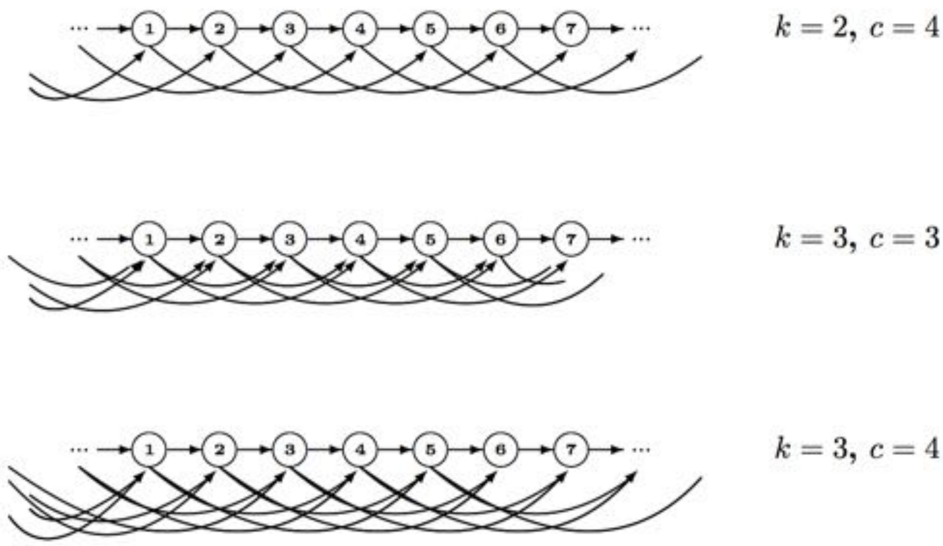

Figure 1.6: Increasing the degree of a regular network might decrease the length of the minimal transitive cycles

4 Discussion

In this section, we extend our analysis of contagiousness and compare the regular networks with other families of networks that share some characteristics: the tree and the complete multipartite network. The families of networks that serve as benchmarks are all connected, transitive networks. This analysis will provide more insights to better understand the effect of the length of transitive cycles on the contagiousness of the network. Let us define the following two types of networks.

First, a connected, transitive network is considered to be a *tree of out-degree k* if (i) all nodes have out-degree equal to k and in-degree equal to 1, and (ii) for any two nodes i and j in N, if i is connected to j there is a *unique* path from j to i. Secondly, a connected, transitive, *homogeneous* network of degree k is a *complete multipartite network of degree k* if for any node $b \in N$ we can find (i) a set S_b of $k-1$ nodes such that for all $i \in S_b$ it holds that $N_i^{out} = N_b^{out}$, and (ii) a sequence of sets $\left\{S_b^t\right\}_{t=2,3,4,...}$ such that for all t and $i \in S_b^t$ it holds that $N_i^{out} = S_b^{t+1}$. Figure 1.7 shows an example of a tree of out-degree 3, a complete multipartite network of degree 3, and a regular network of degree 3 and minimal transitive cycle length equal to 3.

It is easy to see that in the case of the tree of out-degree equal to k the shock received by banks that are far from the source approaches zero when $w_i = 0$ for all $i \in N$. Recall that in a tree there will be a unique path connecting any $i \in N_j^{out,\infty}$ to j (the bank receiving the unique external shock). For any $i \in N_j^{out,\infty}$, each node in the path connecting i to j diffuses $\frac{1}{k}$ of the shock received because $w_i = 0$ for all $i \in N$. Hence, $x_i = \frac{1}{k^{d(i,j)}} x_j$, where, recall, $d(i,j)$ is the distance from i to j (in this case the length of the unique path connecting them). As $d(i,j)$ tends to infinity for $i \in N_j^{out,\infty}$, we

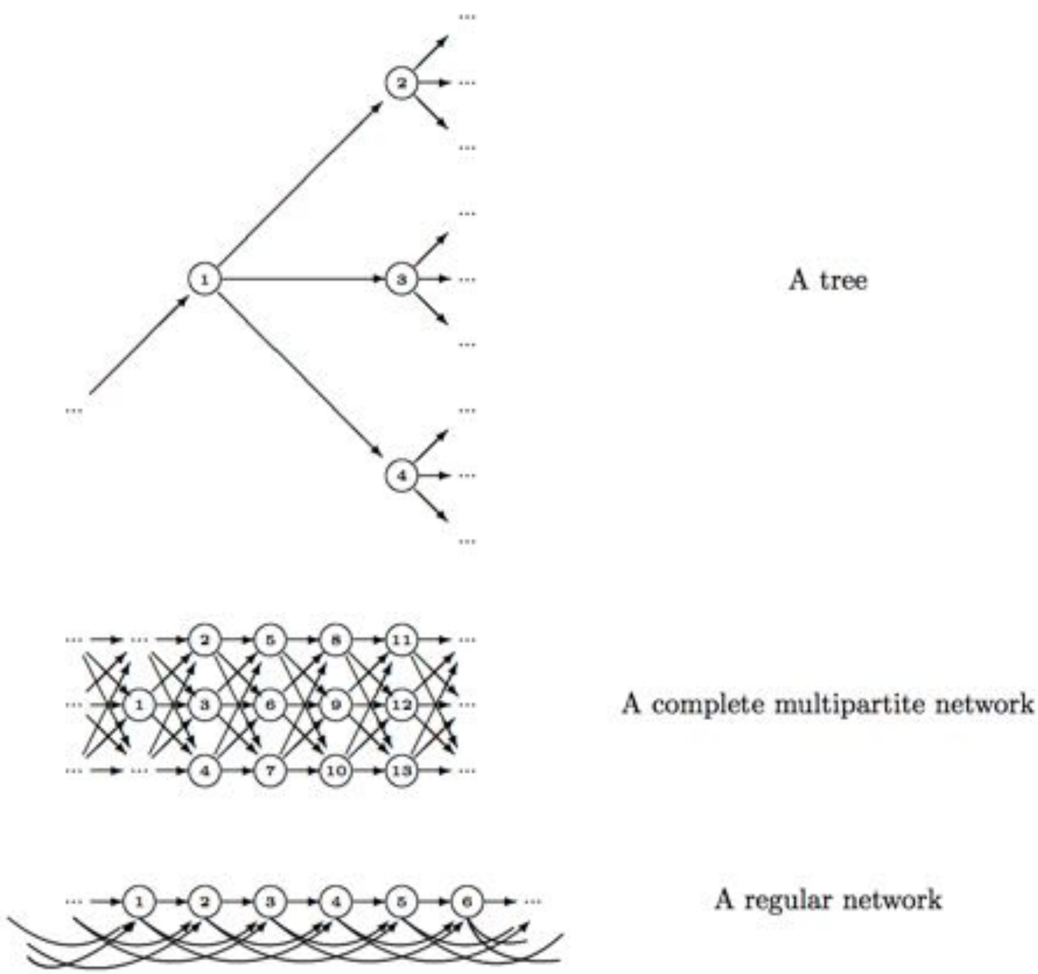

Figure 1.7: Networks with out-degree equal to 3

see that x_i tends to zero.

The case of the complete multipartite network of degree k is also easy to compute. The node receiving the initial external shock, j, diffuses $\frac{1}{k}x_j$ to each $i \in N_j^{out}$. Each $i \in N_j^{out}$ diffuses $\frac{1}{k^2}x_j$ to each $h \in N_i^{out}$. By definition of the complete multipartite network, each $h \in N_i^{out}$ is connected to all $i \in N_j^{out}$, hence receiving $x_h = \sum_{i \in N_j^{out}} \frac{1}{k^2}x_j = \frac{1}{k}x_j$. The shock received and transmitted by $i \in N_j^{out}$ is always equal to $\frac{1}{k}x_j$ and hence, as $d(i,j)$ tends to infinity for $i \in N_j^{out}$, x_i stays equal to $\frac{1}{k}x_j$.

These two types of networks, the tree and the complete multipartite one, illustrate well the role that in and out degrees have in the contagiousness properties of financial networks. If $k = 1$ both the tree and the complete multipartite network are equal to the infinite line $\{12, 23, 34, 45, 56, 67, ...\}$ (up to a relabelling of the nodes) and the shock received and transmitted by any $i \in N_j^{out,\infty}$ (j being the bank receiving the unique external, initial shock) is constant and equal to x_j. When $k \geq 2$

the tree and the complete multipartite network have a different shape which results in a different diffusion of the shock. In the tree, the shock received and transmitted by any $i \in N_j^{out,\infty}$ is decreasing exponentially until it reaches zero because the out-degree being greater than the in-degree helps spread the shock, making it smaller as it travels further through the network. In the complete multipartite network the in-degree and the out-degree are equal. This creates the possibility of connecting banks in $N_j^{out,\infty}$ to j through many different paths.[7] This multiplicity of paths prevents the shock to decrease to zero as it gets further away from j because there is *accumulation without amplification* through the multiple paths connecting the nodes.

This distinct behavior of shock diffusion in these two networks can also be related to the neighborhood growth in Morris (2000). In the tree, the bank receiving the initial external shock has k out-neighbors. Each of these k out-neighbors have k distinct out-neighbors, the initial external shock has an effect over k^2 *new* nodes after two iterations of the set of out-neighborhood. We note that after l iterations of the set of out-neighborhood k^l nodes are *newly* added. In the complete multipartite network, the bank receiving the initial external shock also has k out-neighbors, but each of these k out-neighbors have the same k out-neighbors. After l iterations of the set of out-neighborhood we still find k new banks being affected by the initial external shock in the multipartite network. Morris (2000) shows that in social coordination games (coordination games played on a network) new behaviors are *potentially* more contagious in networks where there is slow neighborhood growth, which means that the number of new out-neighbors at each iteration of the set of out-neighborhood does not grow exponentially. The diffusion behavior of the shock is consistent with this view. The tree is less contagious because the shock goes to zero as we get far from the initial shock in our analysis and the neighborhood growth is exponential. The complete multipartite network is very contagious because the shock does not go to zero as we get far from the initial shock in our analysis and the neighborhood growth is constant.

What happens in the case of the regular network? It is also true that the neighborhood growth is constant given the regularity of the network: after the node $j+c-1$ is reached, there are always $k+c-3$ new out-neighbors added at each iteration step. It might be tempting to assume that the regular network is less contagious than the complete multipartite network by looking at the neighborhood growth, as $c \geq 3$. We have the following proposition comparing the two limiting values of the shock as we get far from the bank receiving the initial shock.

Proposition 2. *Recall that* $\bar{x}(k,c) = \frac{2k}{2k+(k-1)(k+2c-6)}$. *We have that* $\bar{x}(k,c) < \frac{1}{k}$ *if and only if* $c > 3 + \frac{k}{2}$.

The proof of Proposition 2 is straightforward and therefore omitted. Proposition 2 states that there is a threshold for the length of the minimal transitive cycle such that a regular network can be less contagious than a complete multipartite network. For an illustration, Figure 1.8 shows that with the same degree of 3, the regular network with $c = 5$ is less contagious than the multipartite network, while the regular network with $c = 3$ is more contagious.

In particular, if $c = 3$ the regular network will be more contagious than the complete multipartite network for any value of $k > 1$. As c approaches infinity the shape of the regular network

[7] This multiplicity of paths does not imply the existence of cycles in the network because links are directed.

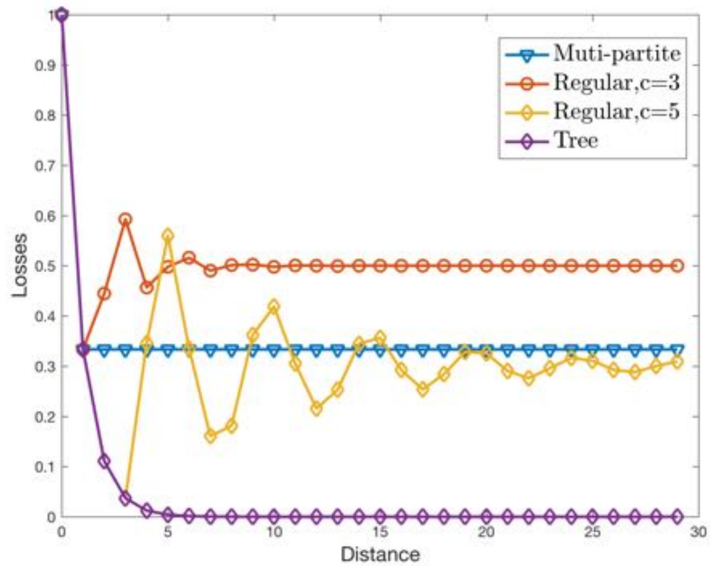

Figure 1.8: The value of $\frac{x_i}{x_j}$ as a function of the distance $d(i, j)$ to bank j receiving the unique initial, external shock, for networks of degree equal to 3

approaches the one of the tree. We also note that the value of the threshold increases with the degree of the network. If the network gets denser (in the sense of higher in and out degree) the minimal transitive cycle length has to be greater too so that the regular network is less contagious than the complete multipartite network of the same degree. This result demonstrates another important role of minimal transitive cycles. Networks with very similar patterns and characteristics can have different behaviors regarding shock diffusion, depending on the value of the length of minimal transitive cycles.

5 Concluding comments

Our analysis provides new insights on shock diffusion in financial networks, by focusing on the role of minimal transitive cycles. Using large regular networks, where all nodes have equal in-degree and out-degree and with the same pattern of links repeated infinitely, we allow an initial shock to diffuse as a flow into the system. The contagiousness of a network is measured by the limit of the losses of banks that are located infinitely far to the first defaulted bank. This measure captures how a pattern of links may or may not facilitate the propagation of losses.

This analysis allows variations of the length of the minimal transitive cycle as far as the number of financial institutions tends to infinity. We find that contagiousness is decreasing in the length of the minimal transitive cycle. Increasing the degree has ambiguous effects, depending on whether the length of minimal transitive cycle decreases or not after the addition of new links. Finally, similar network structures can have different level of contagiousness when the length of minimal transitive cycles is above or below a certain threshold.

The results contribute to the literature by showing that beside density, transitive cycles have important effects on the extent of contagion, independently of financial factors. The results might be useful to build better indicators for systemic risks. Further work would include applying numerical methods to compute how the extent of contagion in more realistic financial networks depends on the length of transitive or intransitive cycles.

Appendix

Proof of Theorem 1

Let us fix $i = 1$ to be the institution receiving the unique external shock. Given the transitivity nature of our network, only nodes in $N_1^{out,\infty}$ can potentially receive a shock from their in-neighbors. Given the regularity of our network, we can now label the nodes following the natural order defined by the network. Formally, the labeling satisfies that (i) $N_1^{out,\infty} = \{2,3,4,5,....\}$, and (2) for every i and j in $N_1^{out,\infty}$: $i < j$ if and only if $j \in N_i^{out,\infty}$. The regularity of the network and the transitivity requirements guarantee that the labeling makes sense. The examples shown in Figure 1.4 are an illustration of such a natural labeling of the nodes.

We make use of the following Lemma.

Lemma. Let (N,g) be a regular network of degree k and minimal cycle length c. Assume $w_i = 0$ for all $i \in N$. We fix $i = 1$ as the label for the node that receives the unique external shock. Starting from $i = 1$ we consider a labeling of nodes as explained above. Recall that x_i denotes total loss in assets that bank i receives in case of a shock (coming from the external asset or from interbank assets). We have that if $c = 3$ then

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = x_1,$$

while if $c \geq 4$ then

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + \frac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} = x_1.$$

Proof of Lemma. We consider first the case when $c = 3$. Recall that $w_i = 0$ for all $i \in N$. Hence node k receives a fraction $\frac{1}{k}x_j$ from each $j \in N_k^{in}$. By definition of the network and the labeling of the nodes the only nodes $j \in N_k^{in}$ such that $x_j > 0$ are the ones in the set $\{1,...,k-1\}$. Hence,

$$x_k = \frac{1}{k}\sum_{j=1}^{k-1} x_j.$$

Substituting x_k we obtain

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = \frac{2}{k}x_1 + \frac{3}{k}x_2 + ... + \frac{k-1}{k}x_{k-2} + x_{k-1}.$$

We proceed to substitute x_{k-1}. Following the same argument as before,

$$x_{k-1} = \frac{1}{k}\sum_{j=1}^{k-2} x_j.$$

Substituting x_{k-1} we obtain

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = \frac{3}{k}x_1 + \frac{4}{k}x_2 + ... + \frac{k-1}{k}x_{k-3} + x_{k-2}.$$

Applying the argument recursively, we arrive to

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = \frac{k-1}{k}x_1 + x_2.$$

Given that $x_2 = \frac{1}{k}x_1$ we obtain, by substituting x_2, that

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = x_1.$$

We consider now the case when $c \geq 4$. We apply a similar argument as before. Recall that $w_i = 0$ for all $i \in N$. Hence node $k+c-3$ receives a fraction $\frac{1}{k}x_j$ from each $j \in N^{in}_{k+c-3}$. We note that, by definition of the network and the labelling of the nodes, the only nodes $j \in N^{in}_{k+c-3}$ such that $x_j > 0$ are the ones in the set $\{1,...,k-2,k+c-4\}$. Hence,

$$x_{k+c-3} = \frac{1}{k}\sum_{j=1}^{k-2} x_j + \frac{1}{k}x_{k+c-4}.$$

Substituting x_{k+c-3} we obtain

$$\begin{gathered}\tfrac{1}{k}x_1 + \tfrac{2}{k}x_2 + ... + \tfrac{k-1}{k}x_{k-1} + \tfrac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} = \\ \tfrac{2}{k}x_1 + \tfrac{3}{k}x_2 + ... + \tfrac{k-1}{k}x_{k-2} + \tfrac{k-1}{k}x_{k-1} + \tfrac{k-1}{k}\left(x_k + ... + x_{k+c-5}\right) + x_{k+c-4}.\end{gathered}$$

We proceed to substitute x_{k+c-4}. Following the same argument as before,

$$x_{k+c-4} = \frac{1}{k}\sum_{j=1}^{k-3} x_j + \frac{1}{k}x_{k+c-5}.$$

Substituting x_{k+c-4} we obtain

$$\begin{gathered}\tfrac{1}{k}x_1 + \tfrac{2}{k}x_2 + ... + \tfrac{k-1}{k}x_{k-1} + \tfrac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} = \\ \tfrac{3}{k}x_1 + \tfrac{4}{k}x_2 + ... \tfrac{k-2}{k}x_{k-3} + \tfrac{k-1}{k}\left(x_{k-2} + x_{k-1} + x_k + ... + x_{k+c-6}\right) + x_{k+c-5}.\end{gathered}$$

Applying the argument recursively for $j \geq c$, noting that $c \in N^{out}_1$ but $i \notin N^{out}_1$ for $2 < i \leq c-1$, we arrive to

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k = \frac{k-1}{k}\left(x_1 + ... + x_{c-2}\right) + x_{c-1}.$$

Given that $x_i = \frac{1}{k}x_{i-1}$ for $1 < i \leq c-1$ we obtain, by substituting recursively, that

$$\frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + \frac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} = x_1.$$

This completes the proof of the Lemma. □

We proceed now to prove the statement of Theorem 1. Recall that we have labeled the nodes such that (i) $i = 1$ is the node receiving the unique external shock, (ii) $N_1^{out,\infty} = \{2,3,4,5,....\}$, and (iii) for every i and j in $N_1^{out,\infty}$: $i < j$ if and only if $j \in N_i^{out,\infty}$.

We can rewrite the sequence $x_1, x_2, x_3, ...$ in matrix form as

$$x^{[i+1]} = Ax^{[i]}$$

with $x^{[i+1]} = \begin{pmatrix} x_{i+1} \\ x_{i+2} \\ \vdots \\ x_{i+k+c-3} \end{pmatrix}$, $x^{[i]} = \begin{pmatrix} x_i \\ x_i \\ \vdots \\ x_{i+k+c-4} \end{pmatrix}$ and

$$A_{(k+c-3 \times k+c-3)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} & 0 & \cdots & 0 & \frac{1}{k} \end{pmatrix}.$$

In the last row of matrix A we find the first $k-1$ elements and the last element to be equal to $\frac{1}{k}$ (so k elements are equal to $\frac{1}{k}$) and the rest of elements to be equal to 0. It is easy to see that

$$x^{[n]} = A^n x^{[1]} \tag{1.1}$$

with $x^{[1]} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+c-3} \end{pmatrix}$.

Given that A is a row stochastic matrix we have that 1 is a simple eigenvalue of A and that the spectral radius of A is equal to 1. We also know that A is irreducible and primitive.[8] Hence, by

[8] A nonnegative $n \times n$ matrix A is irreducible if and only if the graph $G(A)$, defined to be the directed graph on nodes $1, 2, ..., n$ in which there is a directed edge leading from i to j if and only if $a_{ij} > 0$, is strongly connected (see Meyer 2000, p. 671). A nonnegative $n \times n$ matrix A is primitive if it is irreducible and at least one diagonal element is positive, i.e., the trace of the matrix is positive (see Meyer 2000, p. 678). Furthermore, we also know that A^{k+2c-6} is a positive matrix.

equation (8.3.10) in Meyer (2000b), we have that

$$\underset{n\to\infty}{Lim}\, A^n = \frac{r.l^T}{l^T.r} \tag{1.2}$$

where r and l are, respectively, the right and left eigenvectors corresponding to the eigenvalue 1, l^T is the transpose of l (l is written as a column vector, so l^T is a row vector).

Given that A is row stochastic, the right eigenvector is equal to the vector of ones. To compute the left eigenvector we solve

$$(l_1,...,l_{k+c-3})\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} & 0 & \cdots & 0 & \frac{1}{k} \end{pmatrix} = (l_1,...,l_{k+c-3}),$$

and obtain

$$l_i = \tfrac{i}{k}, \text{ for } 1 \le i \le k-1$$
$$l_i = \tfrac{k-1}{k}, \text{ for } k-1 < i \le k+c-4$$
$$l_{k+c-3} = 1.$$

Substituting in (1.2) to compute the limit of A^n we obtain

$$\underset{n\to\infty}{Lim}\, A^n = \frac{1}{\sum_{i=1}^{k+c-3} l_i}\begin{pmatrix} l_1 & l_2 & \cdots & l_{k+c-3} \\ l_1 & l_2 & \cdots & l_{k+c-3} \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & l_{k+c-3} \end{pmatrix}$$

Hence,

$$\underset{n\to\infty}{Lim}\,(x_n) = \frac{1}{\sum_{i=1}^{k+c-3} l_i}\sum_{i=1}^{k+c-3} l_i x_i$$

Note that

$$\sum_{i=1}^{k+c-3} l_i x_i = \begin{cases} \frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + x_k, & \text{if } c = 3 \\ \frac{1}{k}x_1 + \frac{2}{k}x_2 + ... + \frac{k-1}{k}x_{k-1} + \frac{k-1}{k}\left(x_k + ... + x_{k+c-4}\right) + x_{k+c-3} & \text{if } c \ge 4. \end{cases}$$

Hence, by Lemma,

$$\underset{n\to\infty}{Lim}\,(x_n) = \frac{1}{\sum_{i=1}^{k+c-3} l_i}x_1 = \frac{2k}{2k+(k-1)(k+2c-6)}x_1.$$

This completes the proof of Theorem 1. □

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Chapter 2

The dynamics of bank runs by a simple cascade model

Disclaimers[1]

[1] This chapter is a joint work with Emmanuelle Augeraud-Veron

1 Introduction

Bank runs and panics are at the heart of financial crises. Bernanke (2013) and Gorton (2008) stressed that the global crisis started in 2007 was similar to a large-scale bank run, akin to the Panic of 1907. “The forces that hit financial markets in the U.S. in the summer of 2007 seemed like a force of nature [...] something beyond human control” (Gorton (2008)). This quote showed a widely shared sentiment among economists: the crisis took us by surprise. The question that how a small shock in subprime mortgages can suddenly trigger a generalized panic remains troubling. This paper aims to shed some light on this question.

Theoretical literature on bank runs, and panics more generally, is built upon the coordination game framework of Diamond and Dybvig (1983). The main driving force is strategic complementarity: in the panic equilibrium, all depositors withdraw because they expect others would also withdraw and the bank will fail.

While this elegant framework provides many insights to understand bank runs, there is one shortfall. Symmetric and simultaneous actions make it difficult to study the dynamics of bank runs. By design, the panic state is achieved instantly in equilibrium. However, sequentiality of actions is an important feature, as Brunnermeier (2001) pointed out that withdrawals are made sequentially in reality. Existing literature has paid little attention to some important questions: how withdrawals are made over time? How fast a bank defaults from a run? When and how depositors synchronize their actions? Better understanding of these matters might be useful to devise interventions in time of crisis.

This paper proposes a dynamic model of bank runs to address these issues. In a finite horizon, depositors can withdraw at any point in time. Depositors have private information on total withdrawal with some errors. As Bernanke (2013) observed, in time of crisis, agents often have to face Knightian uncertainty. Therefore, we assume that agents do not know the distribution of private information. With bounded rationality, agents make decisions by following a switching strategy that combines strategic complementarity and heuristics. A depositor withdraws when her perceived total withdrawal reaches a precautionary threshold, which is determined by liquidity of the bank. Bank runs in this model are purely panic-driven. When a fraction of random agents withdraw, under the right conditions, signals get worse and trigger preemptive withdrawals from some other depositors. The additional fraction of withdrawals in turn increases the probability to withdraw for remaining depositors. By this feedback mechanism, bank runs arise as dynamic cascades of sequential withdrawals.

The model generates two stylized patterns of bank runs. Immediate runs take place when withdrawals build up following a stable increasing pattern. On the contrary, after a period of apparent inactivity, sudden runs occur “out of nowhere” without any visible sign. One important result of the paper is the explicit computation of the tipping point, where the panic burst out.

The second pattern of runs is interesting because it might explain the phenomenon commonly referred as “the calm before the storm”. Sometimes, panics do not manifest immediately following a shock. Only tiny changes build up over time, then a generalized panic suddenly breaks out as if there is an unexpected shift at the aggregate level. The idea can be illustrated with a popular game called Jenga. A wooden tower is constructed from removable rectangular blocks beforehand, then each player takes turn to remove one block at a time, until the tower collapse. In the early stage, when each block is removed, the fragility of the tower increase by an imperceptible margin. From a moderate distance, it would be impossible to tell whether the tower has some blocks removed. At some critical point, where enough blocks are removed, the tower becomes visibly unstable. Removing one more block would make the tower collapse.

Our paper contributes to the literature in two directions. First, the model is able to replicate and characterize the patterns of bank runs that can be observed empirically. It provides a possible explanation on why massive withdrawals suddenly occur, as if depositors synchronize their actions, even if they don't have to. To our limited knowledge, this issue has not been addressed in existing theoretical models. Second, this paper offers a novel approach to study bank runs and panics more generally, with unconstrained sequence of actions. Bank runs arise as path-dependent cascades rather than being instantly achieved. It is worth noting that the model is simplistic in some aspects. It is an attempt toward building a quantitative model that might be useful to detect and identify panic patterns for policy interventions.

With respect to existing literature, this paper is linked to both empirical and theoretical researches on bank run. Recent empirical works showed that the sequentiality of withdrawals is important and has influences on the outcomes (Schotter and Yorulmazer (2009)). Furthermore, there are evidences that decision of depositors are affected by the past withdrawals (Garratt and Keister (2009); Kiss et al. (2012)). Among a few exceptions, Gu (2011) proposed a bank run model with sequential actions. However, in her model, only one agent can take action at a time. The majority of existing theoretical models are built upon simultaneous coordination game and do not take into account these features. This is the main reason that our model introduces continuous, unconstrained sequence of actions and partial observability of past withdrawals. Although conceptually different, the model presented here share some features with theoretical work on global games (see Carlsson and Van Damme (1993); Morris and Shin (2001)). The framework of global game has been applied to model panic events such as panic-based bank run (Goldstein and Pauzner (2005)) and currency attack (Morris and Shin (1998)). These models use a setting of coordination game with structural uncertainty, where payoffs are random variables and agents receive signals on the payoffs. The common mechanism with our model is the switching strategy: agents choose an action by default and switch to the other action if they observe a signal above a critical threshold. However, the central assumption of these models is that the structure of information is common knowledge, such that agents can infer the distribution of signals of others to apply iterative deletion of dominated strategy. In our model, agents have bounded rationality and do not know the complete structure of information. The threshold comes from individual perception of risk, rather than strategic deduction of beliefs. Our assumption retains rational behaviors while simplifies the strategic aspect of the problem, to focus more on the dynamic aspect of panics.

On the technical side, this paper is inspired by models of collective behavior (Granovetter (1978)) and dynamic diffusion (Bass (1969)). The pioneer work of Granovetter (1978) introduced a global interaction "all-to-all" mechanism: agents use simple rules to adopt an action based on the "popularity" of that action on the aggregate scale. Each individual action can affect the decision of all other agents. Under the right conditions, the same action is adopted at the individual level over time. The collective behavior then emerges as if there is a shift on the aggregate scale. In a different setting, Bass (1969) modeled the speed of diffusion of a new product when a fraction of agents try to imitate others. The diffusion process is path dependence: new adoptions depend on the fraction of past adoptions made by others. Our model makes use of these features to model the dynamics of bank runs as a cascade mechanism.

The remainder of the paper is organized as follows. Section 2 presents the model, then establishes the dynamics of bank runs and characterizes these dynamics. Section 3 discusses some real-world examples and policy implications. Section 4 concludes the paper.

2 Model

2.1 Setting

The economy. A continuum of agents are depositors to a common bank. The time horizon T is finite with n periods. The time step is denoted by $\Delta t = \frac{T}{n}$. If $\Delta t = 1$, then time is indexed by $t \in \{1, 2, \ldots, n\}$ and $n = T$.

Depositors. By default, each agent has one unit of deposit in the bank. Depositors decide to keep their deposit in the bank (wait) or to withdraw (run) in each period. To simplify the analysis, each depositor can only withdraw all of her deposit at once. Depositors who withdrew cannot put their deposit back in the bank and become inactive. Let $r_t \in [0,1]$ denote the fraction of agents who withdraw at date t and $R_t = \sum_{k=0}^{t} r_k$ with $R_t \in [0,1]$ denote the total fraction of withdrawals up to the end of date t.

In each period, depositors have private information on the total fraction of withdrawal : $\tilde{R}_{it} = R_{t-1} + \tilde{z}_{it}$. The distribution of private information represents diversity in opinion and capacity to process aggregate information. For technical simplicity, $\tilde{z}_{it}$ are i.d.d and uniformly distributed in the interval $[-\varepsilon, \varepsilon]$. In the remaining of the paper, $\tilde{R}_{it}$ are labeled as signals and $\tilde{z}_{it}$ are labeled as noises.

Bank & deposit contract. A fraction of deposits $L \in (0,1)$ is kept as liquidity reserve to meet liquidity demand at short term. The remaining fraction of deposits is invested in illiquid assets. The investment yields positive profit with certainty at T. The bank has better investment opportunity and offers demandable debt-deposit contracts to depositors. Deposit contracts have maturity T. The time horizon T represents the maturity mismatch between short-term demandable deposits and the long-term investment.

The model assumes an implicit deposit contract and agents behave as if they are offered the contract described here. For each unit of deposits, there is a positive return at maturity, the payoff is $C > 1$ at $t = T$. At any interim period, depositors can choose to forgo the interest at maturity to get back their deposit, the short-term payoff is normalized to 1. Thus, L equals the fraction of the population to which the bank can pay back at short term without going bankrupt. If the total withdrawal is greater than the available liquidity at any period $t^* < T$, the bank defaults and liquidates the long-term investment. Depositors are paid in a first come first served basis. Each late runner get a payoff $c < 1$ until the liquidation proceeds are depleted. If $0 < c < 1 < R$, the structure of the payoffs is sufficient to generate strategic complementarity. The payoffs *per se* have little influence on the results of the model.

Timeline. At $t = 0$, a fraction r_0 of random agents withdraws. At $t > 0$, active agents choose to wait or withdraw in each period. The bank fails if at any time $t^* < T$, total fraction of withdrawals exceeds liquidity reserve *i.e.* $R_t > L$. Otherwise, the bank survives. The parameter r_0 reflects a "random" liquidity shock, when the bank already committed to the investment.

Decision-making. Since the distribution of noises is not assumed to be common knowledge, each particular depositor can not deduce the distribution of signals of other agents using her own signal. Payoff maximization will depend on individual-specific additional priors. The solution of the dynamic optimization problem would be subjected to a large confidence interval and require tremendous amount of computational power.

Subjected to this large amount of uncertainty, agents follow a switching strategy to approximate payoff maximization. Let a_{it} the action of agent i in period t. If $a_{i,t-k} \neq$ withdraw, $\forall k = 1, 2, \dots, t$ then:

$$a_{it}(\tilde{R}_{it}, \tau_i) = \begin{cases} \text{withdraw} & \text{if } \tilde{R}_{it} > \tau_i \\ \text{wait} & \text{otherwise} \end{cases}$$

where $\tilde{R}_{it}$ is taken as the perceived expected total withdrawal and τ_i is the precautionary threshold.

In a broad sense, the threshold reflects individual perception on the fragility of the bank, as if agents "discount" the true liquidity level. Let θ_i be the discounting factor of agent i. The precautionary threshold is obtained by a convex transformation $f_{\theta_i} : L \to \tau_i$ such that 3 conditions must be satisfied:

1. $\tau_i < L$:
2. if $L \to 0$, then $\tau_i \to 0$
3. if $L \to 1$, then $\tau_i \to 1$

These 3 conditions make sure that individual behaviors are rational: (1) agents always withdraw before the perceived total withdrawal reaches the true liquidity reserve; (2) when the bank has little liquidity, agent has low incentive to wait, as any small fraction of withdrawals could make the bank fail; (3) vice-versa, agents have high incentive to wait when the bank has high liquidity reserve.

Specifically, the precautionary threshold is given by:

$$\tau_i = L^{\theta_i}$$

with $L \in (0, 1)$ and $\theta > 1$.

The discounting factor reflects characteristics of depositors that influence their willingness to wait. To simplify mathematical operations, let assume that depositors share a unique discounting factor, such that $\theta_i = \theta$. This unique value can be regarded as the mean of the distribution of individual values. Therefore, the switching threshold has a unique value:

$$\tau_i = \tau = L^{\theta}$$

A higher value of θ implies a larger precautionary gap, such that agents are more sensitive to withdraw. Figure 2.1 illustrates an example of precautionary threshold. Empirically, the parameter θ can be linked to bank-client relationship, for example. Iyer and Puri (2012) found that depositors with better relationship with the bank delay their decisions to withdraw in a bank run.

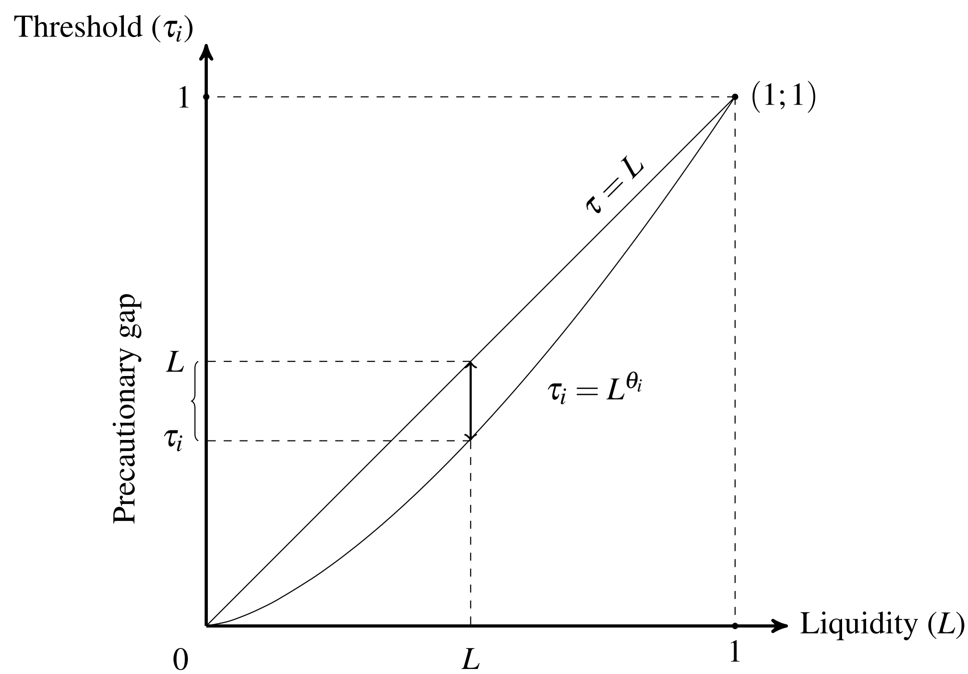

Figure 2.1: Precautionary threshold, $\theta > 1$. Higher value of θ make the curve bend downward, resulting in a larger precautionary gap.

The decision-making process is built upon both strategic complementarity and heuristics. Strategic complementarity reflects rationality in individual decisions. As withdrawals drain liquidity, the bank becomes more sensitive to failure. Depositors withdraw preemptively as if their expected payoffs are decreasing with the perceived total withdrawal. However, given the large amount of uncertainty, the threshold can not be strategically determined. This model assumes that agents have bounded rationality and thresholds are determined by characteristics of depositors, as suggested by empirical evidences. This assumption does not rule out the case that thresholds can have a common

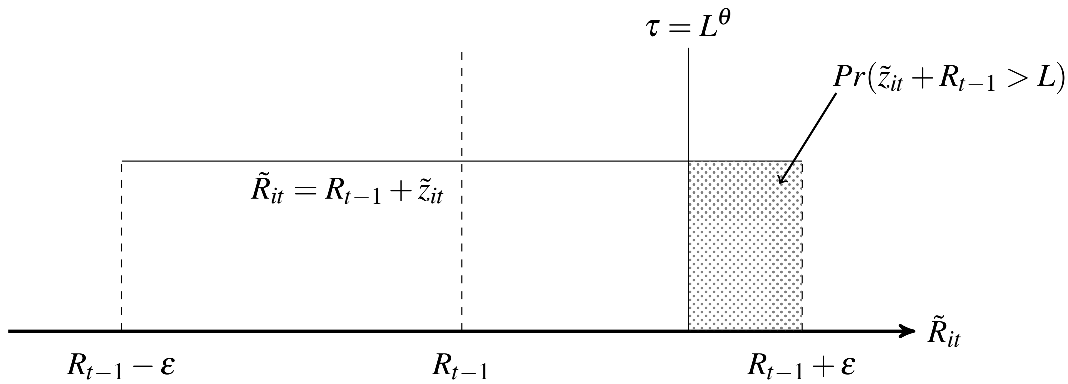

Figure 2.2: Probability to withdraw. When total withdrawal increases, the distribution shifts to the right, making the probability larger.

value, which coincides with the unique threshold determined by strategic deduction of beliefs as in global games (interested readers can see Morris and Shin (2001) for more details).

Intuitively, the switching strategy can be understood as a set of simple rules. First, agent avoid running too late, when the bank already failed. Secondly, agents also avoid running too early, when liquidity reserve is sufficiently high compared to the perceived total withdrawal. Running early is to deny the highest payoff when the chance of bankruptcy in the next period is low. Therefore, agents wait as long as possible, in an attempt to get the highest payoff. Only when the perceived total withdrawal becomes large enough, agents will run to avoid losses. There are experimental evidences that depositors use cutoff thresholds to make withdrawals (Garratt and Keister (2009)).

Interactions. In this model, the dynamics are driven by a feedback mechanism. The aggregate information R_t acts a global interaction device. Whenever a fraction of depositors with high signals withdraw early, these withdrawals make R_t slightly larger, thus increases the chance to have a larger signal $\tilde{R}_{i,t+1}$ to all other agents. This stochastic feedback mechanism can generate a cascade of actions, when a relatively small fraction of withdrawals has positive probability to induce a larger fraction of depositors to run. Over time, depositors might be caught up in a generalized panic and forced to run.

2.2 Dynamics of withdrawals

Given the switching strategy, the probability that an agent withdraws in period t is $Pr\left(\tilde{R}_{it} > \tau\right)$. Since $\tilde{R}_{it} = R_{t-1} + \tilde{z}_{it}$, signals are uniformly distributed in $[R_{t-1} - \varepsilon, R_{t-1} + \varepsilon]$. Figure 2.2 illustrates the individual probability to withdraw.

As a thought experiment, the analysis begins with an extreme case, in which $\varepsilon = 0$. The distribution of signals collapses into a vertical line. When the noises disappear, agents have the same information and they always know the true value of total withdrawal. Moreover, agents also have the same threshold. If the shock is larger than the unique threshold, then everybody withdraws.

Otherwise, nobody withdraws at all. The outcomes are two symmetric equilibria: all agents run if $r_0 > \tau$ and all agents wait if $r_0 < \tau$. It is important to notice that the economy reaches equilibrium state immediately in this case, there is no sequential withdrawals.

In what follows, we assume that $\varepsilon > 0$. By the law of large numbers, the fraction of withdrawal in period t is the individual probability to withdraw times the fraction of remaining (waiting) agents :

$$r_t = Pr\,(\tilde{z}_{it} + R_{t-1} > \tau)\,(1 - R_{t-1}) \tag{2.1}$$

Given that signals are uniformly distributed, the probability to withdraw is zero if τ is greater than the highest possible signal $R_{t-1} + \varepsilon$ at any period t. Hence, Lemma 1 characterizes the no-withdrawal condition.

Lemma 1. *Given an initial shock r_0, no additional withdrawal is made if the following condition holds:*

$$L^\theta \geq r_0 + \varepsilon$$

Proof. From equation (2.1): $Pr\,(\tilde{z}_{i1} + r_0 > \tau) = 0$ when $r_0 + \varepsilon \leq L^\theta$, then $r_1 = 0$ and $R_1 = r_0$. By forward induction: $\forall t > 0, r_t = 0$ and $R_t = r_0$. □

Lemma 1 states that if the discounted liquidity reserve is higher than the maximum signal, no depositor will withdraw at all. In other words, following the initial withdrawals, if the most pessimistic depositor (with the highest signal) does not think that the bank will fail in the next period, then no one withdraws. Given that no depositor withdraws in the past period, the same argument applies recursively and no withdrawal is ever made.

Otherwise, given the condition $L^\theta < r_0 + \varepsilon$, there is always a positive fraction of withdrawal every period. The dynamics of sequential withdrawals are described by the following system:

$$\begin{cases} r_{t+1} &= \frac{1}{2\varepsilon}(\varepsilon - L^\theta + R_t)(1 - R_t) \\ R_t &= \sum\limits_{j=0}^{t} r_j \end{cases} \tag{2.2}$$

with initial condition $1 > r_0 > L^\theta - \varepsilon$.

The system (2.2) is similar to a discrete logistic map. The sequence $(r_0, r_1, \ldots, r_t)$ may exhibit chaotic behaviors and has no closed-form solution. To simplify the analysis, an approximation in continuous time is used for the remaining of the paper.

For any time step $\Delta t < 1$, the fraction of withdrawal *per time step* is given by

$$r_{t+\Delta t} = \frac{1}{2\varepsilon}(\varepsilon - L^\theta + R_t)(1 - R_t)\Delta t = r_t \Delta t$$

By definition, $r_{t+\Delta t}$ is the change in the cumulative fraction of withdrawal: $r_{t+\Delta t} = R_{t+\Delta t} - R_t$. The change in cumulative fraction of withdrawal *per time step* is given by

$$\frac{R_{t+\Delta t} - R_t}{\Delta t} = r_t$$

For a finite time horizon T, time steps become smaller when the number of periods increases. Small time periods allow for an approximation of the system (2.2) in continuous time.

$$\underset{\Delta t \to 0}{Lim}\ (\frac{R_{t+\Delta t} - R_t}{\Delta t}) = \frac{dR}{dt} = r_t$$

Therefore, the accumulation of sequential withdrawals is given by the following differential equation (law of motion):

$$R'(t) = \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + R(t))(1 - R(t)) \tag{2.3}$$

with initial condition $1 > r_0 > L^{\theta} - \varepsilon$. As the RHS of equation (2.3) is a quadratic concave function of $R(t)$ which becomes zero at $L^{\theta} - \varepsilon$ and 1, $R(t)$ is always non-decreasing.

The solution of equation (2.3) will provide a complete description of the dynamics of sequential withdrawals.

Proposition 3. *(Dynamics of withdrawals) The total withdrawal at time t is given by*

$$R(t) = \frac{\left(r_0 + \varepsilon - L^{\theta}\right) e^{ht} - \left(\varepsilon - L^{\theta}\right)(1 - r_0)}{\left(r_0 + \varepsilon - L^{\theta}\right) e^{ht} + 1 - r_0} \tag{2.4}$$

where $h = \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + 1)$*, with initial condition* $r_0 > L^{\theta} - \varepsilon$*.*

Proposition (3) states that for a range of parameters, the cumulative total fraction of withdrawals can be described by a generalized logistic function. From equation (2.4), when the no-withdrawal condition does not hold, the economy is set into motion toward the steady-state $R(t) = 1$ by a cascade mechanism: any positive fraction of withdrawals will induce more withdrawals.

Most importantly, Proposition 1 shows how withdrawals are made over time. There are two stylized dynamics of runs depicted in Figure 2.3. For some cases, runs are apparent immediately after the shock and follow a stable increasing trajectory. On the contrary, after a period of inactivity, sudden runs occur “out of nowhere” without any visible sign. At first, the shock may not seem to trigger any visible fraction of withdrawals, then a massive withdrawal takes place. In what follows, we refer to sudden run as “tipping”.

Immediate runs occur when the shock is relatively high compared to liquidity reserve. On the contrary, sudden runs require a certain degree of balance between liquidity reserve and shock. Tipping only occur when parameters break the no-withdrawal condition by a small margin. Intuitively, when $r_0 = \varepsilon - L^{\theta}$, there is no withdrawal and $R(t)$ is a flat line. With a small perturbation, infinitesimal fractions of withdrawals take place, each time raising the probability to withdraw by a small margin. As the term $(1 - R(t))$ remains close to 1, the cumulative process builds up with increasing speed. When the panic becomes visible, its growth rate is already high. This high growth rate produces an apparent jump in total withdrawal, compared to the previous periods. This is the “surprise effect”: following imperceptible changes, the cascade bursts out in a very short time window, making a bank run seemingly occurs out of nowhere.

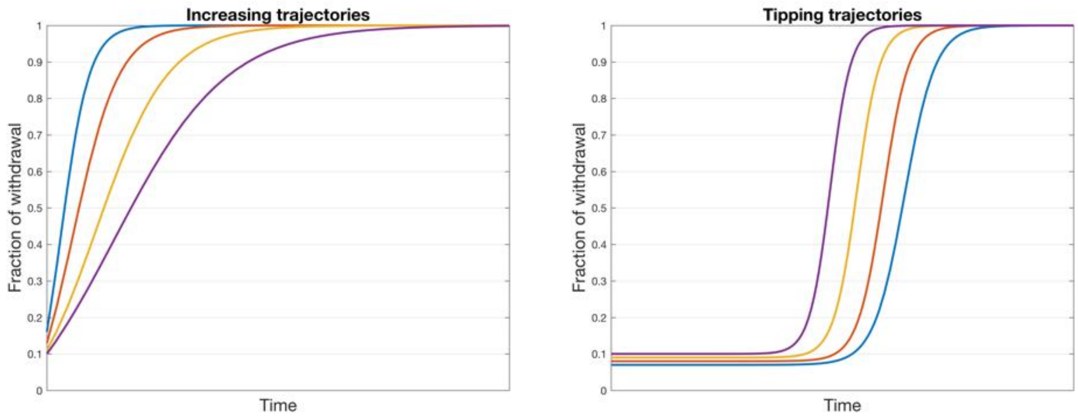

Figure 2.3: Stylized dynamics of bank runs.

2.3 Defaulting patterns

Given the evolution of cumulative withdrawal and the liquidity reserve, it is possible to determine the hitting time t^* when the liquidity reserve is completely exhausted. Visually, it is the moment that the curve $R(t)$hits the horizontal line L. Proposition 4 gives existence conditions and characterizes the hitting time.

Proposition 4. *(Hitting time) If the conditions $r_0 < L < (r_0 + \varepsilon)^{\frac{1}{\theta}}$ are satisfied, the hitting time is well defined and given by*

$$t^* = \frac{2\varepsilon}{\varepsilon - L^\theta + 1} \ln \left[\frac{\left(\varepsilon - L^\theta + L\right)(1 - r_0)}{\left(r_0 + \varepsilon - L^\theta\right)(1 - L)} \right] \tag{2.5}$$

In this model, the bank only defaults within the time horizon T. Therefore, the hitting time is also the defaulting time if $0 < t^* < T$.

Following equation (2.5), the condition $L > r_0$ ensures that t^* will only take strictly positive values. The lower bound of the defaulting time is 0, when $L = r_0$. By design, this condition is consistent with the definition of default: if the initial shock already depletes the liquidity reserve, the bank fails immediately. There is no upper bound for t^*. If L approaches the limit $(r_0 + \varepsilon)^{\frac{1}{\theta}}$, the hitting time tends to infinity. This result agrees with Lemma 1: when liquidity is too high, no additional withdrawal is made, thus the bank never defaults. Proposition 4 establishes that if the conditions $r_0 < L < (r_0 + \varepsilon)^{\frac{1}{\theta}}$ holds, continuous withdrawals are made following the dynamics described in Proposition 3 and the exact time that total withdrawal reaches the liquidity level is explicitly given. The following corollaries analyze the survival duration of the bank with respect to the parameters, when the conditions in Proposition 4 hold.

Corollary 1. *The hitting time t^* is strictly increasing with L and strictly decreasing with r_0.*

If a bank run occurs, more liquidity helps to satisfy more withdrawals, thus increasing the time needed to deplete liquidity reserve. On the contrary, higher shocks imply higher initial conditions for the dynamic process $R(t)$, making cumulative withdrawal reach any predetermined level sooner with the same speed.

Corollary 2. *The hitting time t^* is decreasing with ε, for small values of ε such that $t^*(\varepsilon) < \lambda(\varepsilon)$, with $\lambda(\varepsilon) = \frac{2\varepsilon^2(L-r_0)}{(r_0+\varepsilon-L^\theta)(L+\varepsilon-L^\theta)(1-L^\theta)}$.*

The dependency of t^* with respect to ε is more ambiguous, technical details are provided in the Appendix. Intuitively, this result follows the mechanism depicted in Figure 2.2. When ε is small initially, an increase in ε significantly boost the probability to withdraw in the early stage, leading to a higher early growth rate of cumulative withdrawal. Everything being equal, liquidity reserve is depleted sooner when withdrawals accumulate faster.

The next step is to find out how much liquidity the bank must hold in order to survive the time horizon T. From Proposition 4, it is possible to determine the liquidity level that will be depleted at a given time. However, solving the equation (2.5) for L involves a Lambert W function, which has no analytical solution. Thus, it is necessary to use an implicit function.

Definition 2. Define the set I such that for any $L \in I$ then $r_0 < L < (r_0+\varepsilon)^{\frac{1}{\theta}}$. Let the function of hitting time $\varphi : I \to \mathbb{R}_+$ be defined by the equation (2.5): $t^* = \varphi(L)$. Because $\varphi(L)$ is strictly increasing in L, the inverse function of hitting time is uniquely defined by $\varphi^{-1} : \mathbb{R}_+ \to I$

$$L = \varphi^{-1}(t^*)$$

In other words, for a specific value t^*, the function $\varphi^{-1}(t^*)$ gives the value of L that satisfies the equation (2.5) such that $R(t^*) = L$. In what follows, we use the notation $L^*(t)$ to indicate the liquidity level that will be depleted at time t.

Because $\varphi(L)$ is strictly increasing in L, thus $L^*(t)$ is also strictly increasing in t. In other words, to survive longer in a bank run, the bank must hold more liquidity. Using this monotonicity of inverse function, it is possible to characterize the defaulting patterns and occurrence of bank runs.

Proposition 5. *(Defaulting patterns) For a given triplet (L, r_0, ε), with any values of t and T that satisfy $0 < t < T$, the following conditions hold:*

$$r_0 < L^*(t) < L^*(T) < (r_0+\varepsilon)^{\frac{1}{\theta}}$$

such that a bank run occurs when $r_0 < L < (r_0+\varepsilon)^{\frac{1}{\theta}}$ and the bank always defaults from a bank run when $r_0 < L < L^*(T)$.

The proof is straightforward and therefore omitted. Proposition 5 establishes four possible outcomes for the bank when hit by a "random" liquidity shock. The results are depicted in Figure 2.4. When liquidity reserve is lower than or equal to the shock, the bank defaults immediately.

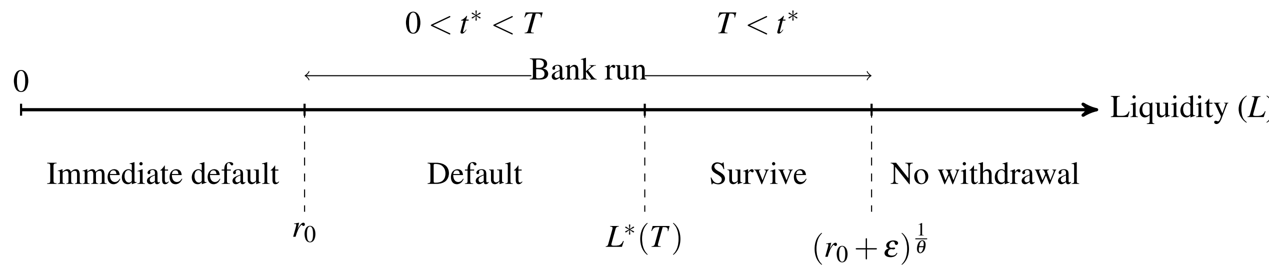

Figure 2.4: Defaulting patterns.

By contrast, when liquidity is above the no-withdrawal threshold, the bank never defaults and the economy reaches a steady state immediately. When liquidity is moderate compared to the shock, a bank run occurs dynamically. The bank fails if the liquidity reserve is lower than a critical value $L^*(T)$, because withdrawals will deplete the liquidity before the maturity. The relative distances between the critical values are important because they determine the probability of bank run and the probability of default.

2.4 Comparative statics

The next step is to study how the regions in Figure 2.4 vary with the main parameters. Even without an explicit expression of $L^*(t)$, it is possible to derive properties of this function to advance the analysis.

Corollary 3. *Longer maturities (T) strictly increase the probability of default from a bank run*

The proof is straightforward: since the function $\varphi^{-1}(\cdot)$ is strictly increasing, for any $T' > T$, it is true that $L^*(T') > L^*(T)$. While the conditions for run do not change, when T increases, we have $L^*(T) \to (r_0 + \varepsilon)^{\frac{1}{\theta}}$ such that the survival region shrinks. When the maturity increases, if a run is triggered, it has more time to reach the predetermined liquidity level. If T is large enough, the chance to survive a run is almost zero. Recall the assumption that the bank cannot obtain additional liquidity in the time horizon T. In practice, banks can sell liquid assets or borrow at short term to face withdrawals, but these measures are costly and may deteriorate fundamentals of the bank, increasing the speed of withdrawals. Thus, reducing the illiquid time horizon could be an effective measure against panic runs. This result may explain why financial institutions without access to low-cost liquidity (such as Fed funds) increasingly turn to short-term funding. However, relying too much on short-term funding can lead to inefficient outcomes, such as the "maturity rat race" pointed out by Brunnermeier and Oehmke (2013).

Corollary 4. *Larger shocks* (r_0) *strictly increase the probability of default. However, the probability of bank run increases with* r_0 *if and only if* $r_0 < \theta^{\frac{\theta}{1-\theta}} - \varepsilon$.

Using the implicit function theorem, it is possible to show that $\frac{\partial L^*}{\partial r_0} > 0$, therefore the default region expands with higher r_0. However, increasing r_0 will also shift the other critical values forward. When the condition $r_0 < \theta^{\frac{\theta}{1-\theta}} - \varepsilon$ holds, we have $\frac{\partial}{\partial r_0}(r_0+\varepsilon)^{\frac{1}{\theta}} > 1$ such that the bank-run region expands with higher r_0. Otherwise, the critical value $(r_0+\varepsilon)^{\frac{1}{\theta}}$ advances slower than the first boundary, making the bank-run region shrink.

The probability of bank run increases simply because the no-withdrawal condition is tightening: for larger shocks, it requires more liquidity to keep even the most pessimistic depositor from withdrawing. Everything being equal, a larger shock provides a higher initial condition for the dynamic process $R(t)$, as if the trajectory of $R(t)$ shifts to the left with a higher starting point. For small shocks, this mechanism will increase the probability of default because total withdrawal can reach a higher level of liquidity for the same time horizon. However, above a certain threshold, large shocks will more likely make the bank default immediately rather than trigger a bank run.

Corollary 5. *Larger magnitudes of noises* (ε) *increase both the probability of default and the probability of bank run, for small values of* ε.

Similar to the precedent result, it is trivial that $\frac{\partial}{\partial r_0}(r_0+\varepsilon)^{\frac{1}{\theta}} > 0$. The probability of bank run increases with ε because depositors have more "extreme" signals, such that withdrawals can be triggered even when liquidity is high. For small values of ε, recall that $t^*(\varepsilon)$ is a decreasing function, then it is possible to show that $\frac{\partial L^*}{\partial \varepsilon} > 0$. Intuitively, higher dispersion of private signals increases the chance to draw high signals, especially in the early stage, thus making the probability to withdraw larger. This early boost of withdrawals makes the bank more likely default.

2.5 Abruptness & tipping point

The previous results provide some insights on the factors that facilitate the occurrence of bank runs. Given that the bank-run conditions hold, one interesting question is how the run would occur. Figure 2.3 showed that there are two patterns: immediate runs and sudden tipping. This section studies the abruptness of these dynamics.

The first element is the steepness of the fast ascending phase of the trajectory. A steeper curve implies that the run is more abrupt: a large fraction of withdrawals is concentrated in a small time window. Given that cumulative withdrawal follows a generalized logistic function described by equation (2.4) , the term e^{ht} mainly determines the speed of growth of the trajectory $R(t)$. This leads to the following proposition.

Proposition 6. *(Abruptness) The abruptness of runs is decreasing with liquidity* (L) *and magnitude of noises* (ε)*, given that the bank-run conditions hold.*

Precisely, the higher value of the exponent $h = \frac{(\varepsilon - L^{\theta}+1)}{2\varepsilon}$, the trajectory will be steeper in its fast ascending phase. It is straightforward that h is decreasing with both L and ε. The first element is obvious: a higher liquidity level makes a higher threshold, such that depositors withdraw less in every time step. Therefore, the cumulative withdrawal takes more time to build up.

The second result may seem counter-intuitive. The explanation resides in the law of motion, equation (2.3) . Given that the bank-run conditions hold, a lower magnitude of noises decreases the probability to withdraw in the early stage. This effect is more apparent in the beginning, when the probability to withdraw is minimal. Therefore, the early growth rate of the trajectory is significantly slower. This low growth rate in turn makes the probability to withdraw increase very slowly. However, this effect preserves the fraction of waiting depositors. When cumulative withdrawal is large enough to make an apparent shift in the probability to withdraw, the fraction of waiting depositors is still large. Therefore, the multiplicative effect of these factors produces a large fraction of withdrawals in a small time window. On the contrary, large magnitude of noises generates high early growth rate. When the probability to withdraw reaches a high level, the fraction of waiting depositors is already small. The speed of growth is more stable because the two terms in the law of motion balance each other over time.

Given that bank runs can be abrupt, there is little to do for an immediate run. However, sudden runs are interesting for two reasons. First, these runs are triggered by small shocks that are hard to detect and could be ignored until it is already too late. Second, as the early growth rate of the panic is very low, it requires less costly intervention to avoid bankruptcy, if interventions are made on time. Further analysis of tipping trajectories requires some definitions.

Definition 3. Define the curvature κ of the graph $R(t)$ at a specific point $(t, R(t))$ as the measure of sensitivity for the slope of the tangent line at the point $(t, R(t))$. Specifically, the curvature of a curve $(t, R(t))$ is given by

$$\kappa(t) = \frac{R''(t)}{\left(1 + R'(t)^2\right)^{\frac{3}{2}}}$$

Higher curvature at a point implies that the tangent line "turns" faster around that point. Define the tipping time $(\hat{t})$ as the moment where the curvature of the graph $R(t)$ is maximum. The level of total withdrawal $\hat{R} = R(\hat{t})$ is labeled as tipping point.

Proposition 7. *(Tipping point) The tipping time is given by*

$$\widehat{t} = \frac{2\varepsilon}{m+1} \ln \left[\frac{\left(\widehat{R} + m\right)(1 - r_0)}{(r_0 + m)\left(1 - \widehat{R}\right)} \right]$$

and the tipping point is given by

$$\hat{R} = \frac{1-m}{2} - \sqrt{r}$$

with $m = \varepsilon - L^\theta$ and r is the only positive solution of

$$12r^3 - 5(m+1)^2 r^2 + \left(\frac{(m+1)^4}{4} - \frac{12}{a^2} \right) r + (m+1)^2 \left(\frac{1}{a^2} + \frac{(m+1)^4}{16} \right) = 0$$

Explicit computation is given in the Appendix. The prominent feature of tipping trajectories is the low growth rate in the early stage. Proposition 7 gives explicit computation of the tipping point, where a trajectory turns into a sudden tipping. The tipping time marks the end of the early stage of the crisis, beyond which the panic will burst out. Theoretically, it is possible that the tipping time is close to zero or even negative. This simply indicates that under some conditions, such as a noticeably large shock, the "acceleration" of the panic is already very high at the beginning. Otherwise, the tipping point is positively defined and the trajectory $R(t)$ is very sensitive around this point. Any small perturbation in the neighborhood of the tipping point could substantially change the course of the trajectory. It is worth noting that the cascade process can have a "mixed" trajectory, meaning that the tipping point is positively defined and the first segment of the curve is not flat.

3 Discussions

From an empirical perspective, we will discuss some real-world examples to illustrate how small shocks can trigger large crises following the two patterns shown in the model.

The first pattern is immediate run, where a panic take off instantly after the shock. One example is the Russian-LTCM crisis in 1998 (Allen and Gale (2009)). LTCM was one of the largest investment firms of Wall Street. Its success was spectacular by the end of 1997. However, in August 1998, their fortunes changed when Russia unexpectedly defaulted on its government debts and devalued its currency. Being highly leveraged, LTCM lost half of its value in one month. Despite the small scale of the loss in total value, this shock triggered an instant panic in financial markets. The cascade pattern was recognizable: investors "run" away from risky assets, make asset prices depreciate and induce more runs (sales). If LTCM was liquidated, the resulted drop in asset prices could make the system collapse. The Federal Reserve Bank of New York organized a bailout with major banks to avoid the potential systemic crisis.

The second pattern is tipping, where a bank run suddenly occurs after a period of apparent calm following the shock. One example is the run on Lehman Brothers in repurchase agreement (repo) markets in September 2008 (Copeland et al. (2014)) .Three months prior to the bankruptcy, Lehman Brothers reported unprecedented losses. However, there was no panic. The repo division of Lehman Brothers was able to secure uninterrupted funding by tri-party repos with unchanged borrowing conditions for weeks. Repo remained as one of the main sources of funding for Lehman Brothers. Then it suddenly collapsed : in 5 days, investors pulled out massively, took away about 40% of short-term funding. This synchronized massive withdrawal pushed Lehman Brothers to declare bankruptcy subsequently.

From a policy perspective, the model showed that there is an optimal time window for interventions. If one admits the cascade argument, then panics are path dependent. This implies that it is possible to dissolve or at least dampen some large crises with relatively small effort, if interventions are made at the right moment. In this model, we have identified the tipping point, above which the panic will burst out. It would be either too late or very costly to react if this point is reached. It is obvious that this simple model cannot provide the exact mathematical description of the underlying mechanism, it could only generate the stylized dynamics. However, it puts forward the importance of building a descriptive model that able to identify panic-sensitive patterns to minimize the required efforts when interventions are inevitable.

4 Conclusions

This paper has studied the dynamics of bank runs in a model that allows unrestricted continuous actions. Panic bank runs arise as cascades of withdrawals by partial observability and strategic complementarity.

There are two distinct patterns of runs. For immediate runs, noticeable withdrawals take place right after the shock and follow stable increasing trajectories. On the contrary, for sudden runs, massive withdrawals burst out in a very short time window without visible signs. The paper is able to characterize how fast and how frequent bank runs occurs. Furthermore, we provide explicit computation of the critical point where the panic burst out, defined as tipping point. These results might be useful to devise interventions in time of crisis.

The model is simple and has several limitations. First, in time of crisis, individual decisions are likely to be correlated. To make the model more realistic, one can allow depositors to observe and learn from the actions of others. Second, the bank does not react to runs. One can introduce short-term borrowing with liquidity cost or fire sales of assets to have a richer set of dynamics. These ideas serve as directions for future research.

Appendix

Proof of Proposition 1

Solve the following differential equation by integration

$$\frac{dR}{dt} = \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + R(t))(1 - R(t))$$

Define $\alpha = \frac{1}{2\varepsilon}$ and $m = \varepsilon - L^{\theta}$, we obtain

$$\frac{dR}{(m + R(t))(1 - R(t))} = \alpha dt$$

The fraction on the LHS can be expressed as a sum:

$$\frac{1}{(m + R(t))(1 - R(t)} = \frac{A}{m + R(t)} + \frac{B}{1 - R(t)}$$

It is easy to show that $A = B = \frac{1}{m+1}$. Define $k = \frac{1}{m+1}$ and integrate both side. The LHS is given by

$$\int \frac{k}{m+R(t)}dR + \int \frac{k}{1-R(t)}dR = k.ln(m+R(t)) - k.ln(1-R(t)) + c_1 + c_2$$

where c_1, c_2 are integration constants. It is straightforward for the RHS:

$$\int \alpha dt = \alpha t + c_3$$

Taking exponential of both sides and rearranging terms yield

$$\left(\frac{m+R(t)}{1-R(t)}\right)^k = e^{\alpha t}.e^{(c_3-c_1-c_2)}$$

To solve for $R(t)$, take both sides to the power of $\frac{1}{k}$, define $C = e^{\frac{c_3-c_1-c_2}{k}}$, we obtain the result after araging terms:

$$R(t) = \frac{Ce^{ht} - m}{Ce^{ht} + 1}$$

with $h = \frac{\alpha}{k} = \alpha(m+1) = \frac{\varepsilon - L^\theta + 1}{2\varepsilon}$.
Plug the initial condition $R(0) = r_0$ into the equation yields: $C = \frac{r_0+m}{1-r_0}$

Proof of Proposition 2

To find the exact time when liquidity reserve is depleted, we solve the following equation for t:

$$L = \frac{C.e^{ht} - m}{C.e^{ht} + 1}$$

with $m = \varepsilon - L^\theta, h = \frac{1}{2\varepsilon}(m+1), C = \frac{r_0+m}{1-r_0}$.
Arranging terms yields

$$e^{ht} = \frac{1}{C} \cdot \frac{m+L}{1-L}$$

Taking natural logarithm of both sides and araging terms yield the final result

$$t^* = \frac{2\varepsilon}{\varepsilon - L^\theta + 1} \ln \left[\frac{\left(\varepsilon - L^\theta + L\right)(1-r_0)}{\left(r_0 + \varepsilon - L^\theta\right)(1-L)} \right]$$

Proof of Corollary 1-2

Let us remind that the condition $(r_0 > L^\theta - \varepsilon)$ holds.

If $r_0 > L$ then we have $\frac{(\varepsilon - L^\theta + L)(1-r_0)}{(r_0+\varepsilon-L^\theta)(1-L)} < 1$, that is $t^* < 0$. It implies that $R(t^*) = L$ only holds for $r_0 < L$.

From the equation of t^*, it is trivial that t^* is strictly decreasing with respect to r_0.

Let denote $t^*(L) = \frac{2\varepsilon}{\varepsilon - L^\theta + 1} \ln\left[\frac{(\varepsilon - L^\theta + L)(1-r_0)}{(r_0+\varepsilon-L^\theta)(1-L)}\right]$

We have

$$\begin{aligned}\frac{dt^*}{dL} &= \frac{2\varepsilon\theta L^{\theta-1}}{\varepsilon - L^\theta + 1}\ln\left[\frac{\left(\varepsilon - L^\theta + L\right)(1-r_0)}{\left(r_0+\varepsilon-L^\theta\right)(1-L)}\right] \\ &\quad + \frac{2\varepsilon}{\varepsilon - L^\theta + 1}\left(\frac{1}{L+\varepsilon-L^\theta} + \frac{L}{1-L} + \theta L^{\theta-1}\left(\frac{L-r_0}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)}\right)\right)\end{aligned}$$

None of these terms can be negative for any $L \in \left(r_0, (r_0+\varepsilon)^{\frac{1}{\theta}}\right)$ with $0 < r_0 < 1$ and $\theta > 1$, therefore $\frac{dt^*}{dL} > 0$.

The dependency of t^* with respect to ε is more ambiguous. It is worth denoting t^* as $t^*(\varepsilon)$

$$\frac{dt^*}{d\varepsilon} = \frac{2\left(1-L^\theta\right)}{\left(\varepsilon - L^\theta + 1\right)^2}\ln\left[\frac{\left(\varepsilon - L^\theta + L\right)(1-r_0)}{\left(r_0+\varepsilon-L^\theta\right)(1-L)}\right] - \frac{2\varepsilon}{\varepsilon - L^\theta + 1}\frac{L-r_0}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)}$$

Devide both side by t^* yields

$$\left(\frac{1}{t^*}\right)\frac{dt^*}{d\varepsilon} = \frac{1}{\varepsilon\left(\varepsilon - L^\theta + 1\right)}\left[\left(1-L^\theta\right) - \frac{2\varepsilon^2(L-r_0)}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)}\left(\frac{1}{t^*}\right)\right]$$

The first term is always positive, the second term is negative if and only if

$$1 - L^\theta < \frac{2\varepsilon^2(L-r_0)}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)}\left(\frac{1}{t^*}\right)$$

Define

$$\tau(\varepsilon) = \frac{2\varepsilon^2(L-r_0)}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)\left(1-L^\theta\right)}$$

If $t^*(\varepsilon) < \tau(\varepsilon)$, then $\frac{dt^*}{d\varepsilon} < 0$. It is straighforward that for small ε, we have $t(\varepsilon) < \tau(\varepsilon)$, thus $t(\varepsilon)$ is a decreasing function of ε.

To obtain a general result, we need to consider

$$\tau'(\varepsilon) = 2\frac{L-r_0}{1-L^\theta}\left[\left(-L^\theta + \frac{L+r_0}{2}\right)\varepsilon + \left(L-L^\theta\right)\left(r_0-L^\theta\right)\right]$$

If $L^{\theta}-\varepsilon<r_0<2L^{\theta}-L$ and $\ln\left(\frac{1-r_0}{1-L}\right)<\frac{L-r_0}{1-L^{\theta}}$, then $\tau(\varepsilon)$ is a decreasing function of ε and its graph is above the graph of $t(\varepsilon)$. Then $t(\varepsilon)$ is an increasing function of ε.

If $L^{\theta}-\varepsilon<r_0<L^{\theta}$ and $\ln\left(\frac{1-r_0}{1-L}\right)>\frac{L-r_0}{1-L^{\theta}}$, it implies that there exists a unique ε^* such that $t^*(\varepsilon)$ is a decreasing function of ε if $\varepsilon<\varepsilon^*$ and $t(\varepsilon)$ is an increasing function with ε if $\varepsilon>\varepsilon^*$.

If $2L^{\theta}-L<r_0<L^{\theta}$ and $\ln\left(\frac{1-r_0}{1-L}\right)<\frac{L-r_0}{1-L^{\theta}}$, then $\tau(\varepsilon)$ has a minimum value and crosses twice the graph of $t(\varepsilon)$. It means that for small values of ε, then $t(\varepsilon)$ is a decreasing function of ε, then for greater values it is an increasing function of ε, then a decreasing function of ε when ε increases.

Proof of Corollary 4-5

Compute the derivative of $(r_0+\varepsilon)^{\frac{1}{\theta}}$ with respect to r_0 yields:

$$\frac{\partial}{\partial r_0}(r_0+\varepsilon)^{\frac{1}{\theta}}=\frac{1}{\theta}(r_0+\varepsilon)^{\frac{1}{\theta}-1}$$

The bank run zone expands with r_0 if the derivative is greater than 1

$$\frac{1}{\theta}(r_0+\varepsilon)^{\frac{1}{\theta}-1}>1$$

$$r_0+\varepsilon<\theta^{\frac{\theta}{1-\theta}}$$

$$r_0<\theta^{\frac{\theta}{1-\theta}}-\varepsilon$$

The sign changed because $\theta>1$.

Next, we procced to prove that $\frac{\partial L^*}{\partial r_0}>0$. Denote the function of hitting time in equation (2.5) as :

$$F(L^*,r_0)=t^*$$

Consider the relation:

$$\phi(L^*,r_0)=F(L^*,r_0)-t^*$$

The following conditions are satisfied:

1. $\phi(L^*,r_0)=F(L^*,r_0)-t^*=0$
2. $\frac{\partial\phi(L^*,r_0)}{\partial L^*}=\frac{\partial F}{\partial L^*}>0$

Thus, there exist an implicit function $L^*(r_0)$ such that:

$$F(L^*(r_0),r_0)=t^*$$

Using implicit differentiation with respect to r_0:

$$\frac{\partial F}{\partial r_0}+\frac{\partial F}{\partial L^*}.\frac{\partial L^*}{\partial r_0}=0$$
$$\frac{\partial L^*}{\partial r_0}=-\frac{\frac{\partial F}{\partial r_0}}{\frac{\partial F}{\partial L^*}}$$

We have $\frac{\partial F}{\partial r_0}<0$ and $\frac{\partial F}{\partial L^*}>0$, therefore $\frac{\partial L^*}{\partial r_0}>0$. The proof for ε is similar and therefore omitted.

Proof of Proposition 5

The curvature of a curve $(t,R(t))$ is defined as

$$\kappa(t)=\frac{R''(t)}{\left(1+R'(t)^2\right)^{\frac{3}{2}}}$$

Denote $R'(t)=f(R)=\frac{1}{2\varepsilon}(1-R)(m+R)$, with $m=\varepsilon-L^{\theta}$. We can express the curvature as a function of R:

$$\widetilde{\kappa}(R)=\frac{f(R)f'(R)}{(1+f(R)^2)^{3/2}}$$

It is worth seeing that $\widetilde{\kappa}(R)$ is anti-symmetric according to $\frac{m-1}{2}$ (that is $\widetilde{\kappa}(R)=-\widetilde{\kappa}(1-m-R)$). We can therefore consider $\rho(R)=\widetilde{\kappa}(R+\frac{1-m}{2})$

$$\rho'(R)=\frac{\left(f'\left(R+\frac{1-m}{2}\right)^2-2f\left(R+\frac{1-m}{2}\right)^2f'\left(R+\frac{1-m}{2}\right)^2-2f\left(R+\frac{1-m}{2}\right)-2f\left(R+\frac{1-m}{2}\right)^3\right)}{\left(1+f(R+\frac{1-m}{2})^2\right)^{5/2}}$$

The maximum of the curvature is reached at $\widehat{R}=\frac{1-m}{2}-\sqrt{r}$, where r is a positive solution of

$$12r^3-5(m+1)^2r^2+\left(\frac{(m+1)^4}{4}-\frac{12}{a^2}\right)r+(m+1)^2\left(\frac{1}{a^2}+\frac{(m+1)^4}{16}\right)=0$$

Explicit computation gives

$$r=-\frac{18}{a}(A+\frac{(w^2+27)}{A}-\frac{5}{2}w+i\sqrt{3}(A-\frac{(w^2+27)}{A})$$

where

$$\begin{aligned}A &= \left(-w^3+\frac{81}{4}w+\frac{9}{4}\sqrt{-24w^4-351w^2-3888}\right)^{1/3}\\ w &= a(m+1)^2\end{aligned}$$

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Chapter 3

Dynamic bank runs: from individual to collective behavior

Disclaimers[1] [2]

[1] Job Market Paper

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1 Introduction

Bank runs have been and remain an important threat to financial stability. Lindgren et al. (1996) document that 133 countries have experienced severe banking crises between 1980 and 1996. Large financial institutions such as Bear Stearns, Northern Rock, IndyMac Bank and Wachovia were subjected to massive runs in 2007-2008.

Theoretical literature on bank runs is mainly built upon the coordination game of Diamond and Dybvig (1983). In this common framework, two notable features are simultaneous moves and symmetric equilibria. In the panic equilibrium, all depositors withdraw simultaneously as they expect others would also withdraw and the bank would fail. Bank runs arise from strategic complementarity.

While this framework provides many insights to understand bank runs, there are two shortfalls. First, the absence of sequential actions makes it difficult to study the dynamics of bank runs. Reducing the set of possible actions of depositors to one single decision might overlook relevant issues. Second, beyond strategic complementarity, empirical studies show that behavioral factors also play an important role. Typically, decisions of depositors are influenced by actions that they observe[3], their social network[4] and their heterogeneity[5].

The strategic dimension of bank runs has been extensively studied by the literature. However, the dynamic and behavioral aspects have received little attention, leaving several questions unanswered: how does the panic start out? How fast and frequent does the economy reaches the panic equilibrium? How do behavioral factors affect bank runs?

This paper develops a dynamic and behavioral model of bank runs to address these questions. Within a network, depositors can observe the actions of their direct neighbors. The network is localized, such that each agent can observe a small subset of other agents. The time horizon is finite with multiple periods. Every period, active depositors choose to withdraw or wait based on (i) their types, (ii) their private signals on total withdrawal and (iii) the observed actions of others. Agents make decision by following a switching strategy that combines rationality and heuristics. This decision-making mechanism is inspired by the hybrid system of thinking proposed by Kahneman (2003) and is supported by empirical evidences. This setting allows for a wide range of features that affect the dynamics of bank runs: continuous sequential withdrawals, heterogeneity of depositors, herding and partial observability.

Bank runs in this model are purely panic-based. Initially, a fraction of random agents withdraw. If this fraction is large enough, the bank becomes vulnerable to bankruptcy, signals get worse and increase the probability to withdraw for remaining agents. Furthermore, agents are more likely to withdraw when they observe their neighbors doing so. Thus, bank runs can emerge as cascades of withdrawals through these two feedback mechanisms.

There are two sets of results. Regarding the dynamics, there are two distinct patterns of cascades: slow runs in which withdrawals build up progressively or sudden run in which massive withdrawals occur abruptly “out of nowhere”. Regarding the behavioral dimension, this model

[3]Schotter and Yorulmazer (2009); Kiss et al. (2012)

[4]Kelly and O Grada (2000); Iyer and Puri (2012); Atmaca et al. (2017)

[5]Iyer et al. (2016)

considers two main factors: degree of herding and diversity of depositors. Diversity is defined in a broad sense, with respect to information and types. Increase herding has non-monotonic effects on bank failure, because runs are more frequent but slower to reach the bankruptcy level. In contrast, increase diversity makes runs more frequent and faster, such that the probability of failure is strictly increasing.

This paper contributes to existing literature in two directions. First, the model is able to characterize not only the frequency, but also the speed and abruptness of bank runs. While recent theoretical models successfully eliminate the multiplicity of equilibria to determine the probability of bank run, the issues that how and why bank runs can occur in different manners are not addressed, to my limited knowledge. Furthermore, the abruptness of bank runs is often assumed, because all depositors take action simultaneously. This model offers a possible explanation on why massive withdrawals occurs as if depositors synchronize their actions, even if they don't have to. Second, the model shows how behavioral factors affect the trajectory of bank runs, making potential bridges between recent empirical findings and theoretical research. It is an attempt to build a descriptive framework on how bank runs occur to complement existing theoretical models. The results have several policy implications and could serve to build a quantitative model that would be able to monitor and detect panic-sensitive patterns.

This paper is related to three streams of literature. First, it takes inspirations from the empirical literature on bank runs. Several empirical features are integrated into the framework: sequential withdrawals (Schotter and Yorulmazer (2009); Kiss et al. (2014)); diversity of depositors' characteristics (Iyer and Puri (2012); Iyer et al. (2016)); herding through social networks (Kelly and O Grada (2000); Iyer and Puri (2012)) and partial observability (Garratt and Keister (2009); Kiss et al. (2012)). Furthermore, the model might provide background to discuss recent experimental findings, for examples, the indeterminacy zone in Arifovic et al. (2013) and the group-size effect in Arifovic et al. (2018).

Although conceptually different, this paper shares some features with recent developments in theoretical literature on bank runs. Goldstein and Pauzner (2005) employed a switching strategy in a context of global game (see Carlsson and Van Damme (1993); Morris and Shin (2001)) to eliminate the multiplicity of equilibria. In their model, the cutoff threshold is uniquely derived by symmetric deduction of beliefs. By contrast, thresholds in the present model are determined by agents' characteristics and reflect their diversity. Gu (2011) proposed a herding model that allows for dynamic withdrawals. Depositors withdraw if their expected payoffs are below a cutoff threshold that is determined by a perfect Bayesian equilibrium. However, there is an exogenous sequence, such that only one agent can take action every period and actions are observable to all other agents. Her model focuses more on herding rather than the dynamics of bank runs *per se*.

On the technical side, this paper takes inspiration from the literature on cascades of actions and collective behavior. Schelling (1971) proposed a pioneer model of local interaction, in which agents are influenced by their direct neighbors. By contrast, Granovetter (1978) proposed an "all-to-all" interaction model, in which every agent is equally influenced by all other agents. In a different direction, Banerjee (1992) studied herding as cascades of information by sequential actions on a line. Using random graph theory, Watts (2002) introduced an explicit network dimension to cascade models. This literature is large and growing, see Watts and Dodds (2009); Miller and Page (2004) for comprehensive reviews, among others. The present paper combines various features of cascade models to integrate dynamics and behavioral factors into the analysis of bank runs.

The remainder of the paper is organized as follows. Section 2 presents the general model and explains the main mechanism. Section 3 derive analytical results from a simple version of the model. Section 4 studies the complete model using numerical simulations. Section 5 discusses policy implications and concludes.

2 General Model

The economy. N agents are depositors to a common bank. Depositors are linked together in an undirected network, in which they can observe the actions of their direct neighbors.

The time horizon T is finite with multiple periods to allow for quasi-continuous actions (*e.g.* online banking). Periods are indexed by $t = 0, 1, 2, 3 \ldots, T$.

Depositors. By default, each agent has a unit of deposit in the bank. Agents choose to keep their deposit in the bank (wait) or to withdraw (run). Each depositor can only withdraw all of her deposit at once. Depositors who withdrew cannot put their deposit back in the bank and become inactive, their actions remain observable. Let $r_t \in [0,1]$ denote the fraction of agents who withdraw at date t and $R_t = \sum_{k=0}^{t} r_k$ denote the total fraction of withdrawals up to the end of date t.

In each period, depositors have private information on the total withdrawal: $\tilde{R}_{it} = R_{t-1} + \tilde{z}_{it}$. The distribution of private information represents diversity in opinion and capacity to process aggregate information. The random variable $\tilde{z}_{it}$ are i.d.d, have mean zero and the standard deviation will be specified in relevant sections. In the remaining of the paper, $\tilde{R}_{it}$ are labeled as signals and $\tilde{z}_{it}$ are labeled as noises.

Finally, depositors differ in their types, which is defined by θ_i. Types are private information and normally distributed. Types determine the sensitivity to make withdrawals. Agents with higher values of θ_i are more likely to withdraw.

Bank & deposit contract. A fraction of deposits $L \in (0,1)$ is kept as liquidity reserve. The remaining fraction of deposits is invested in illiquid assets. The investment yields positive profit with certainty at T. The bank always has better investment opportunity and offers debt-deposit contracts to depositors. Deposit contracts have maturity T. The time horizon T reflects the maturity mismatch between short-term demandable deposits and the long-term investment.

The model assumes an implicit deposit contract and agents behave as if they are offered the contract described here. For each unit of deposit, there is a positive return at maturity, the payoff is $C > 1$ at $t = T$. At any interim period, depositors can choose to forgo the interest at maturity to get back their deposit, the short-term payoff is normalized to 1. Thus, L equals the fraction of the population to which the bank can pay back at short term without going bankrupt. If the total withdrawal is greater than the available liquidity at any period $t^* < T$, the bank defaults and liquidates the long-term investment. Depositors are paid in a first come first served basis. Each late runner get a payoff $c < 1$ until the liquidation proceeds are depleted. If $0 < c < 1 < R$, the structure of the payoffs is sufficient to generate strategic complementarity. The payoffs *per se* have little influence on the model.[6]

Timeline. At $t = 0$, a fraction r_0 of random agents withdraws. At $t > 0$, active agents choose to wait or withdraw in each period. The bank fails if at any time $t^* < T$, total fraction of withdrawals exceeds liquidity reserve *i.e.* $R_t > L$. Otherwise, the bank survives. The parameter r_0 reflects a "random" liquidity shock, when the bank already committed to the investment.

Decision-making. Since the distributions of noises and types are not assumed to be common knowledge, rational expectations will depend on individual-specific additional priors. The solution of the dynamic optimization problem would require tremendous amount of computational power and would be subjected to a large confidence interval.

Subjected to this large amount of uncertainty, agents follow a switching strategy to approximate payoff maximization. Let a_{it} the action of agent i in period t. If $a_{i,t-k} \neq$ withdraw, $\forall k = 1, 2, \ldots, t$ then:

$$a_{it}(s_{it}, \tau_i) = \begin{cases} \text{withdraw} & \text{if } s_{it} > \tau_i \\ \text{wait} & \text{otherwise} \end{cases}$$

where s_{it} is taken as the perceived expected total withdrawal and τ_i the precautionary threshold.

The perceived total withdrawal of agent i in period t is given by:

$$s_{it} = (1-h).\tilde{R}_{it} + h.n_{i,t-1} \tag{3.1}$$

with $h \in [0, 1]$ is the herding factor, $n_{i,t-1}$ the fraction of neighbors who withdrew up to period $t-1$ and $\tilde{R}_{it} = R_{t-1} + \tilde{z}_{it}$ the private signal on total withdrawal. To some extent, the perceived total withdrawal is a weighted average of two channels of information: private signal and information embedded in the observed actions of neighbors.

The precautionary threshold τ_i reflects individual perception on the fragility of the bank, as if agents "discount" the true liquidity level. Let θ_i be the type of agent i, her precautionary threshold is obtained by a convex transformation $f_{\theta_i} : L \to \tau_i$ such that 3 conditions must be satisfied:

1. $\tau_i < L$:
2. if $L \to 0$, then $\tau_i \to 0$
3. if $L \to 1$, then $\tau_i \to 1$

These 3 conditions make sure that individual incentives are rational : (1) agents always withdraw before the perceived total withdrawal reaches the level of liquidity reserve; (2) when the bank has no liquidity, agent has low incentive to wait, as any small withdrawal could make the bank fail; (3) vice-versa, agents have high incentive to wait when the bank has high liquidity.

[6] Optimal contracts have short-term payoff greater than 1, but the payoff structure remains similar.

Specifically, the precautionary threshold is given by:

$$\tau_i = L^{\theta_i} \tag{3.2}$$

with $L \in (0,1)$ and $\theta_i > 1, \forall i$. The distribution of θ_i represents individual characteristics of agents that influence their perception on the bank. Notice that a higher value of θ_i implies a lower threshold, hence higher sensitivity to run. With respect to theoretical literature, θ_i can be regarded as the inverse of degree of patience in Azrieli and Peck (2012). Empirically, Iyer and Puri (2012); Iyer et al. (2016) documented that individual characteristics and relationships with the bank have significant influences on the decision to withdraw. Figure 3.1 illustrates the features of precautionary thresholds.

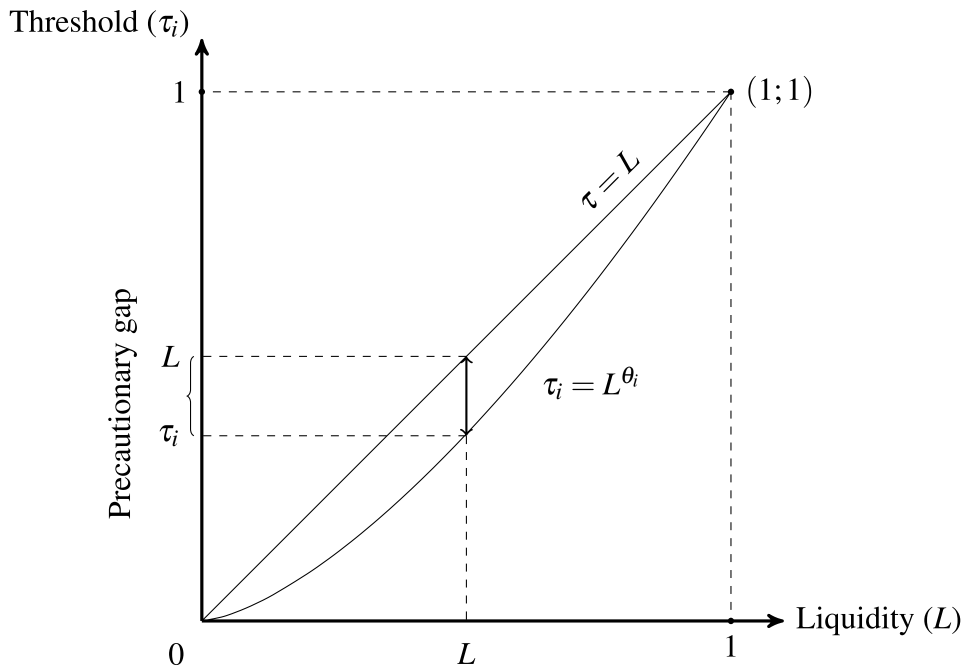


Figure 3.1: A precautionary threshold, $\theta_i > 1$. Higher value of θ_i make the curve bend downward, resulting in a larger precautionary gap.

In this setting, the decision function has two dimensions, similar to the hybrid system of thinking as in Kahneman (2003). First, strategic complementarity reflects rationality in individual decisions. As withdrawals drain liquidity, the bank becomes more sensitive to failure. Depositors withdraw preemptively as if their expected payoffs are decreasing with the perceived total withdrawal. However, given the structural uncertainty and heterogeneity, cutoff thresholds can not be computed as in global games. This model assumes that individual thresholds are linked to individual characteristics of depositors, as suggested by empirical evidences. This assumption does not rule out the case that thresholds can have a common value, that coincides with the unique threshold determined by strategic deduction of beliefs as in global games (see Carlsson and Van Damme (1993); Morris and Shin (2001) for more details).

Second, individual decisions can be directly influenced by the observed actions of others. In a broad sense, let define this behavior as herding[7]. Herding is the phenomenon in which agents follow what they observe rather than taking decision independently based on their private information. One can rationalize herding as in Banerjee (1992): agents extract additional information from the observed actions of others using quantitative belief-updating mechanism. There are other explanations for herding, such as mechanisms characterized by "animal spirits" as in Akerlof and Shiller (2010). In this model, herding is broadly defined and encompasses both views.

Intuitively, the switching strategy can be understood as a set of simple rules. First, agent avoid running too late, when the bank already failed. Secondly, agents also avoid running too early, when liquidity reserve is sufficiently high compared to the perceived total withdrawal. Running early is to deny the highest payoff when the chance of bankruptcy in the next period is low. Therefore, agents wait as long as possible, in an attempt to get the highest payoff. Only when the perceived total withdrawal becomes large enough, agents will run to avoid losses. There are experimental evidences that depositors use cutoff thresholds to make withdrawals (Garratt and Keister (2009)).

Interactions. In this model, the dynamics are driven by two feedback loops, as depicted in Figure 3.2. The public information R_t acts a global interaction device. Whenever an agent with high θ_i withdraws early, this action makes R_t slightly larger, thus increases the chance to have a larger signal $\tilde{R}_{i,t+1}$ to all other agents. Meanwhile, the agent who withdrew also increases the desire to run of her neighbors due to the observability of her action, this is the local mechanism.

The global interaction has unlimited range and ignores the network structure, but the effect of each individual withdrawal is small. Furthermore, the global effect is stochastic because a higher value of R_t increases the chance but does not ensure a larger signals for remaining agents. On the contrary, the local interaction has very limited range, but each withdrawal has significant impact on the set of neighbors. In addition, the local effect is deterministic and results in a direct increase in decision score of connected agents. These feedback mechanisms can generate a cascade of decisions, when relatively small numbers of withdrawals have positive probability to induce a larger fraction of agents to run. Over time, waiting agents might be caught up in a generalized panic and forced to run.

[7] see Baddeley (2010) for an extensive review on how individual behavior is influenced by actions of others

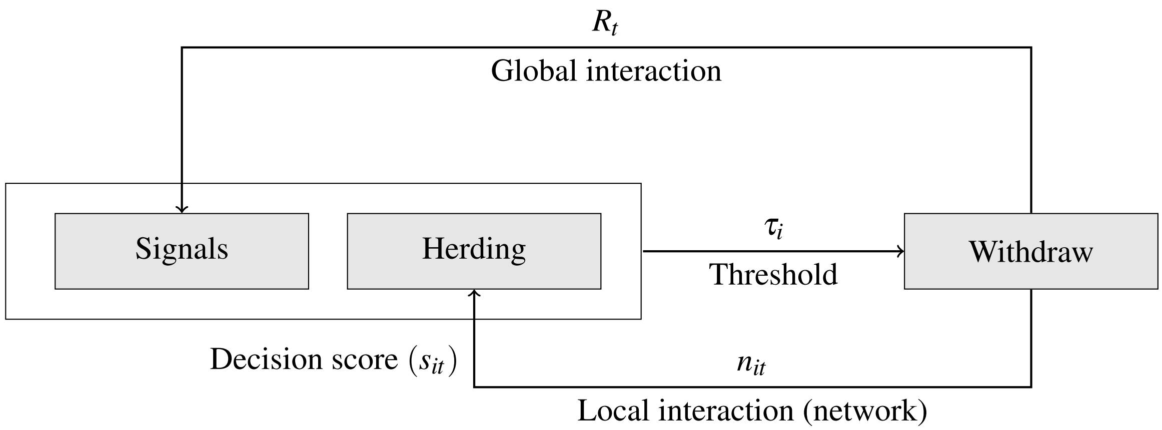


Figure 3.2: Feedback mechanisms

3 Mean-field analysis

This section studies a simplified version of the model to derive some insights, before getting into higher order of complexity. The following temporary assumptions are used:

1. There is no herding i.e. $h = 0$. In this case, the network has no effect.
2. Agents have their types equal to the mean: $\theta_i = \bar{\theta}$ such that $\tau_i = \tau$. Furthermore, let $\bar{\theta} = 1$.
3. The noises are uniformly distributed around 0: $z_{it} \sim \mathscr{U}[-\varepsilon, \varepsilon]$.

The first hypothesis allows us to focus on the strategic complementarity feature. The second and third hypotheses are made to simplify mathematical expressions, in order to derive explicit analytical results. These hypotheses do not alter the qualitative results and they will be removed in the next section. For the moment, the parameter space of the model is reduced to 3 dimensions: magnitude of noises, liquidity reserve and shock.

Given private signals and the switching strategy, the probability that an agent withdraws in period t is given by $Pr\left(\tilde{R}_{it} > \tau\right) = Pr\left(R_{t-1} + \tilde{z}_{it} > L\right)$. By the third temporary assumption, signals are uniformly distributed $\tilde{R}_{it} \in [R_{t-1} - \varepsilon, R_{t-1} + \varepsilon]$. Figure 3.3 illustrates the individual probability to withdraw.

As a thought experiment, our analysis begins with an extreme case, in which $\varepsilon = 0$. The distribution of signals collapses into a vertical line. When the noises disappear, agents have perfect information on total withdrawal. Moreover, agents also have the same threshold as they are perfectly homogenous in type. If the shock is larger than the common threshold, then everybody withdraws. Otherwise, nobody will withdraw at all. The resulting outcomes are two symmetric equilibria: all agents run if $r_0 > \tau$ and all agents wait if $r_0 \leq \tau$. It is important to notice that the economy reaches a steady state immediately at $t = 1$ in this case.

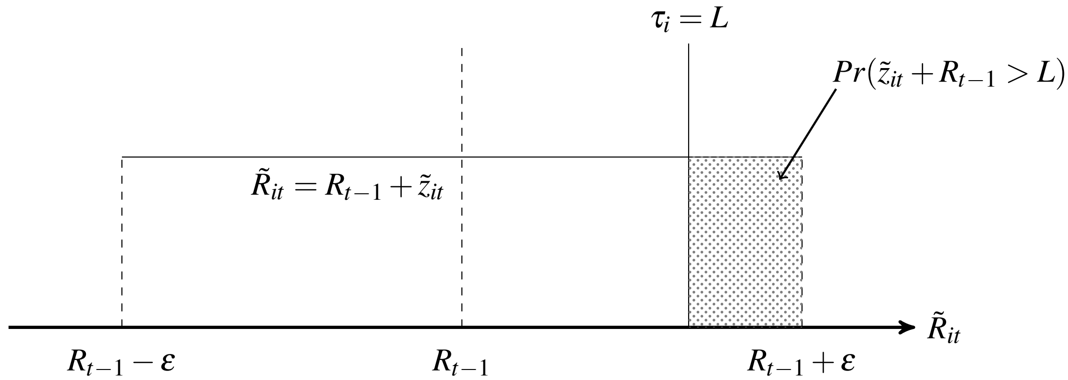


Figure 3.3: Probability to withdraw. When total withdrawal increases, the distribution shifts to the right, making the probability larger.

Then, we proceed to the case where $\varepsilon > 0$. By the law of large numbers, the fraction of withdrawal in period t is the probability to withdraw times the fraction of remaining agents :

$$r_t = Pr\left(\tilde{z}_{it} + R_{t-1} > L\right)\left(1 - R_{t-1}\right) \tag{3.3}$$

From equation (3.3), it is straightforward that if $\varepsilon + r_0 \leq L$, then $r_t = 0, \forall t$ because the probability to withdraw is always zero. In other words, following the initial withdrawals, if the most pessimistic depositor (with the highest signal) does not think that the bank will fail in the next period, then no one withdraws. Given that no depositor withdraws in the past period, the same argument applies recursively and no withdrawal is ever made.

Otherwise, if $L < \varepsilon + r_0$ then $r_t > 0$ and R_t is non-decreasing. Over time, if R_t increases, the probability to withdraw becomes larger as the distribution of signals shifts to the right, while the fraction of remaining agents decreases. Conditional on $\varepsilon > L - r_0 > 0$, the dynamics of sequential withdrawals are described by the system of equations:

$$\begin{cases} r_t & = \frac{1}{2\varepsilon}(\varepsilon - L + R_{t-1})(1 - R_{t-1}) \\ R_{t-1} & = \sum_{j=0}^{t-1} r_j \end{cases}$$

with initial condition $r_0 \in [0, 1]$.

The mean-field dynamics are depicted in Figure 3.4. L_1^* is a critical value for L, below which the bank immediately fails in period 1. L_1^* is obtained using the binding condition of $r_0 + r_1 = L$. The value of L_1^* indicates that if liquidity is too low compared to the shock, the fraction of withdrawal in period 1 will be large enough to entail bankruptcy. L_N^* is another critical value for L, above which no additional withdrawal is made after the initial shock. L_N^* is obtained by the condition $r_1 = 0$. The value of L_N^* indicates that if liquidity is very high compared to the shock, the risk of bankruptcy is zero.

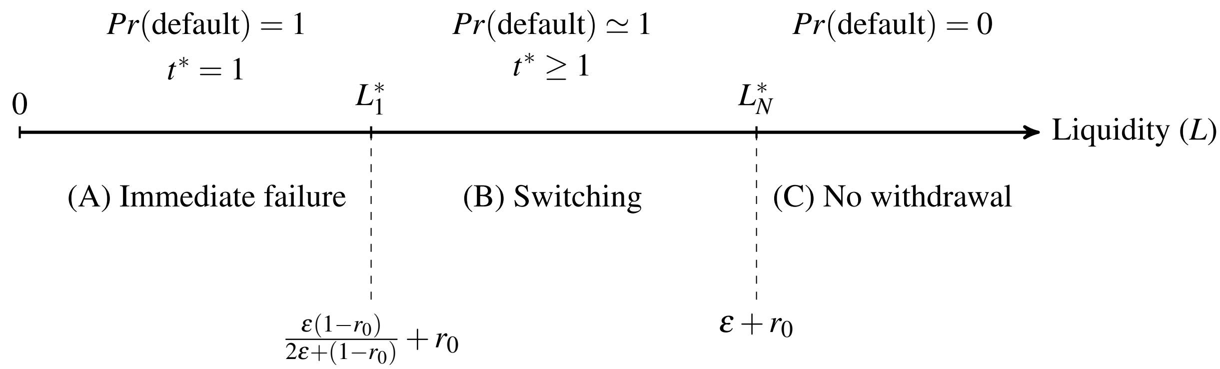


Figure 3.4: Mean-field dynamics of bank runs

For intermediate values of L, the switching zone emerges, in which sequential withdrawals build up over time, putting the economy in motion toward bankruptcy. There exists an ordered sequence $L_1^* \leq L_2^* \leq \ldots \leq L_T^* \leq L_N^*$ such that if $L_t^* \leq L < L_{t+1}^*$ with $(t = 1, .., T-1)$, the bank survives t periods and defaults in the next period. Thus, the probability of default is 1 in such cases. Otherwise, if L falls between L_T^* and L_N^*, the bank survives T periods despite continuous withdrawals. In this model, we only consider defaults within the time horizon T. Given that the illiquid investment will be realized at date T, the bank will always have enough liquidity if it survives T periods. However, L_T^* converges to L_N^* as T increases, so $Pr(L_T^* \leq L < L_N^*)$ tends to zero. If T is reasonably large, L_N^* can be used as an approximation for L_T^*, such that $Pr(\text{default}) \to 1$ in the switching zone. Any critical value L_t^* can be obtained by using the binding condition $r_0 + r_1 + \ldots r_t = L$. As $r_t \geq 0$, it is straightforward that the sequence L_t^* is non-decreasing. The analytical solution to the sequence (r_t) is not the focus of the paper, interested readers can see Chiarella (2012) for example.

Finally, varying the value of ε will affect the relative positions of the critical values $L_1^*, L_2^*, \ldots, L_N^*$. It is straightforward that $0 < \frac{\partial L_1^*}{\partial \varepsilon} < \frac{\partial L_N^*}{\partial \varepsilon} = 1$. In Figure 3.4, when ε increases, both critical values move forward but L_N^* advances faster than L_1^*. Thus, the immediate-failure zone and the switching zone expand, both bank runs and bank failure are more frequent. Vice versa, when ε decreases, the boundaries move backward and closer to each other. The extreme case is when $\varepsilon = 0$, all the boundaries collapse into one unique critical value, as in the thought experiment. The switching zone disappears along with sequential withdrawals in this case.

Although obtained from a simple setting, these analytical results provide two main insights. First, liquidity reserve and shock are important factors that determine not only the probability but also the dynamics of runs, particularly for intermediate liquidity compared to shock. Second, perfect symmetry across agents virtually eliminates sequential moves.

4 Numerical analysis

This section studies the complete model using numerical experiments, with three additional dimensions: herding, diversity and network. The aim of this section is to consolidate previous findings and investigate how the main parameters affect the dynamics of bank runs.

4.1 Calibration

There are $N = 1000$ agents. The time horizon is $T = 30$, according to 30 calendar days specified in Basel III. The distribution of types is given by $\theta_i \sim \mathcal{N}(\mu_\theta = 2, \sigma_\theta = 0.25)$. Noises are normally distributed around zero $\tilde{z}_{it} \sim \mathcal{N}(\mu_z = 0, \sigma_z = \gamma R_{t-1})$, with $\gamma = 0.25$, as if agents perceive available information on total withdrawal with reasonable variation in opinions. Furthermore, noises are endogenized with respect to the total withdrawal, as if quality of information declines when the crisis builds up. On the technical side, endogenized noises avoid the implicit bias that fixed noises impose on the economy, when high (low) noises bring up (down) the perceived total withdrawal on average[8]. The network is a two-dimension lattice, every agent has four direct neighbors: north, south, east, west. The network is localized in the sense that every agent only has a small set of neighbors that are close together. Incremental steps are 0.01 for liquidity and shock, 0.1 for herding. Each vector of parameters is used 100 times.

The results are studied through three generated variables: probability of default, survival time and tip measure. Probability of default is the percentage of failures. Survival time indicates how many periods the bank survives before going bankrupt. Tip measure is a metric for the abruptness of runs. If t_{max} is the period in which daily withdrawals reach the maximum level, tip measure is defined as the ratio between the maximum daily withdrawal and the average of daily withdrawals up to t_{max}. For example, a tip measure of 5 indicates that in one particular day, daily withdrawal is maximum and equal 5 times the averaged withdrawal up to that day included.

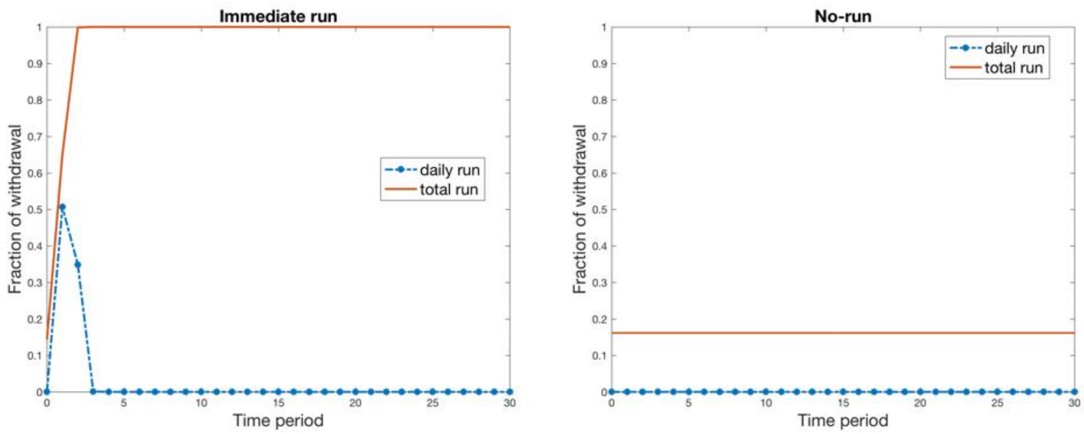
4.2 Stylized dynamics

Numerical experiments show four stylized trajectories of bank runs, illustrated in Figure 3.5.

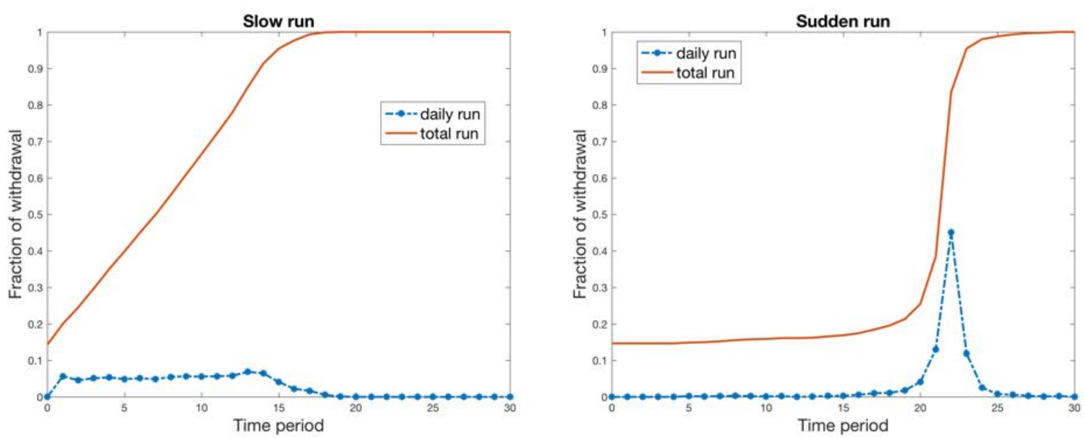
1. All depositors withdraw almost simultaneously (immediate run)
2. Insignificant additional withdrawal (no-run)
3. Stable sequential withdrawals over time (slow run)
4. Massive run is triggered abruptly (sudden run)

These trajectories can be put into two groups: immediate convergence and state-switching. First, immediate convergence occurs when the economy precipitates toward a steady state, in which

[8] It is possible that some agents might get a signal higher than 1, when total withdrawal is very large. Theses signals can be treated as 1 to avoid erratic behaviors. Similarly, negative signals might also be drawn, they are treated as 0. These implementations do not alter the model in any significant way.



(a) Immediate convergence. Tip measure is zero for both cases.



(b) State-switching. Tip measure is 1.2 for the slow run and 11.5 for the sudden run.

Figure 3.5: Stylized dynamics of bank runs. Initial shock is 0.15 for all cases.

all depositors either withdraw or wait, depicted in Figure 3.5(a). The intuition follows from the properties of thresholds. When liquidity is low, agents are very sensitive to any small amount of noises or fraction of withdrew neighbors. It takes only one or a few periods for all agents to withdraw. The reasoning is similar for high liquidity. To some extent, these instantaneous outcomes are related to symmetric equilibria in existing literature. Agents synchronize their action as if they move simultaneously.

Second, state-switching occurs when withdrawals are made sequentially. There are two distinct state-switching patterns: slow transition or sudden tip, depicted in Figure 3.5(b). Slow runs happen when withdrawals build up smoothly over time. In this case, daily withdrawals are visibly positive and remain stable in a large time window. By contrast, sudden runs occurs in an abrupt manner. Daily withdrawals stay close to zero and suddenly jump to the maximum level. Switching pattern is driven by the interplay between local and global mechanisms. Local interactions favor smooth transition while global interactions facilitate sudden tip. More explanations will be provided in subsequent sections.

To distinguish these scenarios, survival time and tip measure are used. Survival time is close to zero for immediate run, while maximum for no-run. By convention, tip measure is zero for these trajectories. By contrast, switching trajectories have intermediate survival times with positive tip measures. Slow run has very low tip measure, the more sudden the run is, the higher its tip measure.

To summary, the stylized dynamics complement findings of the mean-field analysis, showing how dynamic bank runs occur in different ways. In some cases, they span over a large time window. In other cases, they occur suddenly without any visible sign. The following sections investigate the effects of the main parameters on the occurrence and characteristics of bank runs.

4.3 Liquidity vs. shock

The baseline model is studied in this section to focus on the interplay between liquidity reserve (L) and initial shock (r_0). Herding is fixed at $h = 0.5$. The main findings are presented in Figure 3.6.

First, by survival time, panel 3.6(a) shows three distinct regions. Closest to the 45 degrees line, the deep blue region corresponds to the immediate failure zone, in which runs occurs instantly, such that survival time is close to 0. Conversely, the deep red region corresponds to the no-run zone, in which the bank always survives to the last period. This region is occupied mostly by no-run trajectories. Finally, the region with mixing colors from light blue to light red corresponds to the switching zone, in which survival time follows a positive and rather smooth evolution with relative liquidity. These 3 regions confirm the previous findings, showing that dynamic bank runs occurs when liquidity level is moderate compared to shock.

Second, pannel 3.6(b) identifies a very abrupt transition of bank failure, similar to a phase transition. The outcomes are very stable for a wide range of parameters, then abruptly switches to another state. Furthermore, a closer look at the frontier shows a very narrow subregion, characterized by $Pr(\text{default}) \in (0,1)$. This subregion can be defined as the indeterminacy corridor, where switching trajectories occur randomly. The bank fails when switching occurs, otherwise no-run trajectories take place and the bank survives.

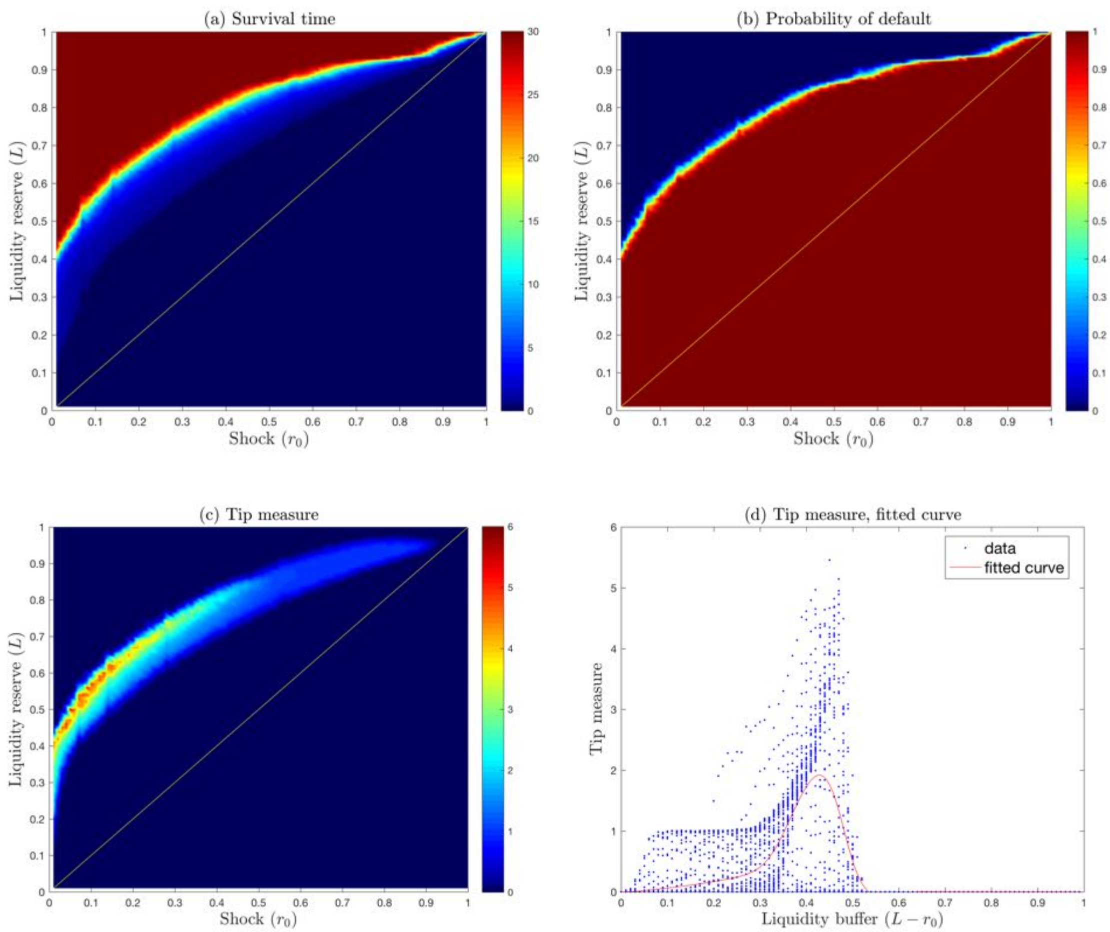


Figure 3.6: The dynamics of bank runs, baseline model $h = 0.5$. Higher temperature color represents higher value for the variable of interest.

The indeterminacy corridor might be explained by stochastic elements in the model. There are 3 random factors: thresholds, signals and relative positions of runners. In general, the law of large numbers stabilizes the outcomes, which are determined by whether liquidity or shock dominates the other. But when the two opposing forces are in balance, the system becomes sensitive to small events. The course of a trajectory can be altered by successive events that boost the number of withdrawals, such as: agents with low thresholds draw high signals in an ordering manner, runners are clustered together within an optimal distance that increases contact with waiting agents and facilitates local activation. To some extent, this finding is similar to the indeterminacy zone documented in Arifovic et al. (2013), in which the experimental economy randomly switches between no-run and run equilibria for the same parameters. While their setting is different from this model, one common feature of the two findings is the path dependence property of bank runs trajectories.

Third, pannel 3.6(c) show the colored region in which tip measure is positive. This region is an approximation for the switching zone, allowing the distinction between sudden runs and slow runs. Sudden run occurs for low liquidity levels, because the average threshold is low. A moderate jump in total withdrawal is enough to make a large fraction of depositors withdraw in the subsequent period. For high liquidity level, the argument is simply reversed. Tip measure is highest around the transition frontier, particularly within the indeterminacy corridor because of random switching.

Finally, the fitted curve in pannel 3.6(d) shows that the abruptness of runs is non-monotonic with respect to liquidity buffers $(L - r_0)$. Switching trajectories become closer to immediate run for higher shocks or lower liquidity levels, vice-versa for no-run. Combined with the phase transition frontier, this result implies that banks need to hold liquidity above a critical level to avoid panic runs. Otherwise, the surplus of liquidity makes runs delayed but more abrupt.

Overall, these results are aligned with previous findings and provide a general view on the dynamics of bank runs. Clearly low liquidity leads to immediate run, while clearly high liquidity leads to no run. For moderate liquidity, there are two distinct patterns: progressive withdrawals or sudden massive run.

4.4 Herding

This section investigates the effects of herding on the dynamics and the occurrence of bank runs. When the herding factor (h) decreases, individual decision is inclined toward private signals that are related to the available liquidity of the bank. On the contrary, when h increases, agents will progressively disregard their own information and put more weight on the observed actions of their neighbors. Variation of the herding factor will alter the micro behaviors and have impact on the aggregate dynamics. The main results are presented in Figure 3.7, the stacked heat maps show regions where tip measure is positive.

First, there is a very clear trend regarding the colors: runs become less abrupt as herding increases. This can be explained by the range of feedback mechanisms. When herding is strong, local interactions build up the panic similarly to the spread of contagious diseases. Depositors are more likely to be infected through direct contact with withdrew neighbors. Large clusters of runners develop progressively and infect depositors who reside close to the boundaries of those clusters. In consequence, the panic takes time to spread out. By contrast, for weak herding, global mechanism

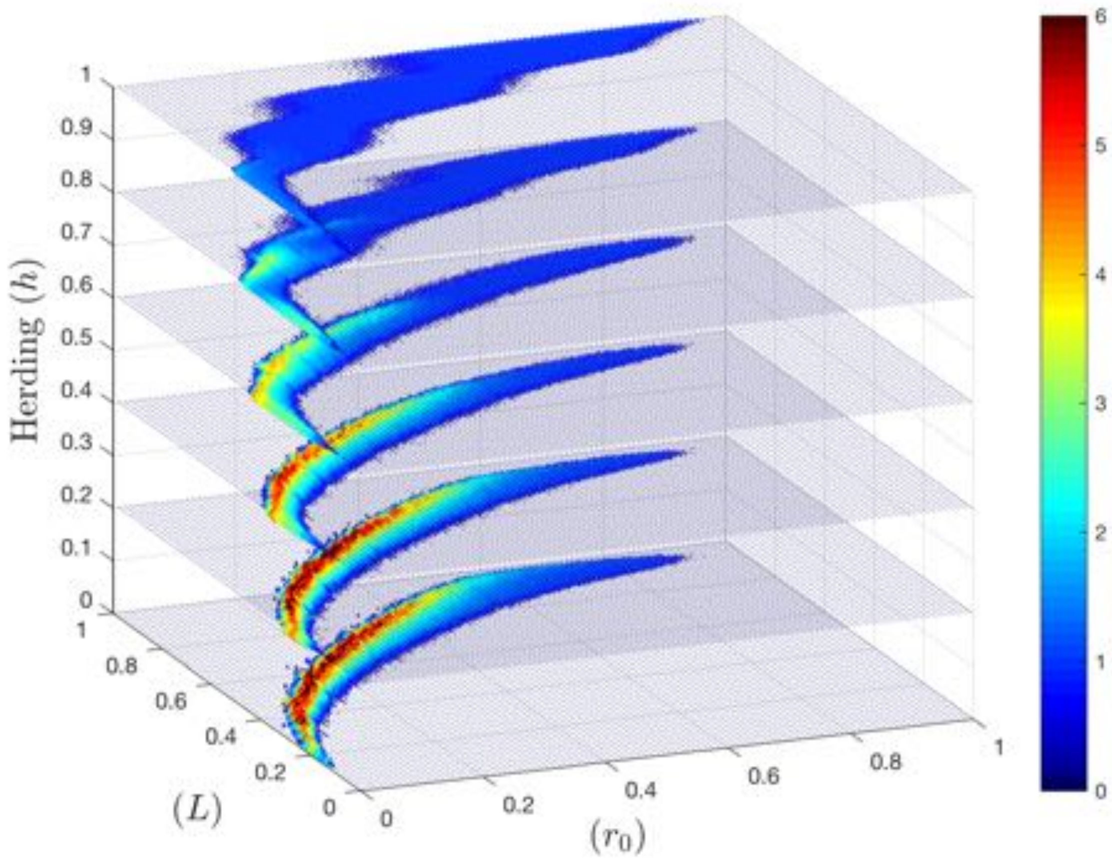


Figure 3.7: Effect of herding on tip measure. High temperature colors represent high values of tip measure.

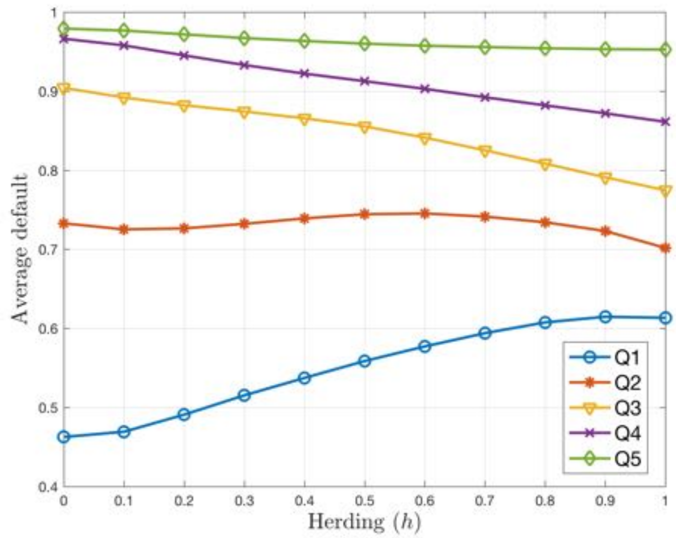


Figure 3.8: Effects of herding on averaged probability of default. The shock (r_0) is decomposed into 5 quintiles. Each line represents the averaged probability of default for one specific quintile of shock.

make signals diffuse "over the air" and bypass the localized network structure. Total withdrawal accumulates slowly but it eventually reaches a critical mass, such that the chance to receive signals higher than the average threshold is significant. Then, global interaction creates a generalized panic instantly.

Second, runs are more frequent with strong herding. From bottom to top layers, the no-run region shrinks (upper left triangle). Furthermore, the switching region expands on the left (low r_0) and becomes flatten on the right (high r_0). This change in aggregate behavior shows that the system does not respond monotonically to different shock regimes, as herding increases.

To further investigate this effect, Figure 3.8 decomposes the distribution of shocks into 5 quintiles. There are 2 distinct macro behaviors with regard to shocks of different sizes. For the first quintile of shocks *i.e.* $r_0 \in (0.01, 0.2)$, the average probability of default increases with h. On the contrary, for the third and fourth quintiles, the tendency is clearly reversed. This observation shows that the bank becomes more fragile to small shocks and more robust to large shocks as herding increases.

These results can be explained as follows. Except for extreme values of h, the liquidity shock has two initial impacts. First, agents surrounded by initial runners are likely to withdraw by local interactions. Second, if the shock is not too small, it will generate enough noises that make other random agents withdraw. The combined impact sets the initial condition for the cascading process.

For small shock, the initial withdrawals generate low signals on average. When herding is weak, small shocks can hardly make random agents withdraw by the global mechanism. However, when herding increases, small number of initial runners can trigger their neighbors to withdraw with certainty. The global mechanism is weak in the beginning, but the local mechanism fills in to build up cumulative withdrawal. In consequence, particularly for small shocks, increasing herding favors the occurrence of runs, making the bank more fragile.

For large shocks, the initial withdrawals generate high signals on average. If liquidity is low, the bank defaults immediately, only high liquidity levels are relevant to the discussion. When herding is weak, random agents have high probability to withdraw regardless of their distance. However, when herding is strong, only agents located around initial runners will withdraw. Moreover, for large shocks, initial runners are more likely to form large clusters, their total contact with the remaining agents is further reduced. In consequence, increasing herding will reduce the early impact of large shocks. Combined with the loss in speed due to herding, the reduction in early activation makes large shocks less dangerous for banks, conditional on having high liquidity.

Overall, herding generates a tension between activation and speed, such that increase herding make runs more frequent but less abrupt. Strong herding favors early activation, but the cascade is slow because local interaction has limited range. By contrast, weak herding facilitates tipping. It is more difficult for runs to start out, however, at some critical level, global interaction causes a panic instantly. Depending on other parameters, these two opposing effects of herding may result in a higher or lower probability of default.

4.5 Diversity: noises & thresholds

As shown in the mean-field analysis, to have switching dynamics, it is necessary that agents have different inputs to their decision-making process. To some extent, the differences across agents can be defined as diversity, with respect to information and type. In this model, γ captures the variations of private signals and σ_θ captures the differences in types that results in variations of thresholds. This section will show how these two parameters affect dynamic bank runs.

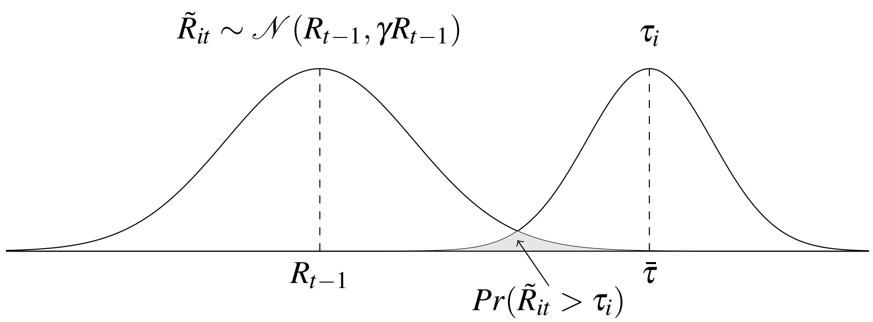


Figure 3.9: Signals vs. thresholds. When diversity increases, the distributions have fatter tails.

Before getting into the results, it is important to show how the distributions of signals and thresholds shape the trajectories of runs. The effect of γ is stronger with low values of h, as more weight is put on private signals. To derive some insights, let consider the extreme case in which $h = 0$, depicted in Figure 3.9. The probability to withdraw is the joint probability to draw high signals and low thresholds. The distribution of thresholds is fixed while the distribution of signals is endogenous and evolves over time. When total withdrawal increases, the distribution of signals shifts to the right and get larger tails, proportional to γ. The probability to withdraw determines the magnitude of the shift and how fat the tails become in the next period. This mechanism is similar to the mean-field case while allowing us to visualize the effect of diversity.

First, when diversity is low, the two distributions concentrate around their means, the probability to withdraw is likely to be very small or even zero in early periods. On the contrary, more diversity makes fatter tails, such that the probability to withdraw is likely to be high. Hence, other things being equal, high levels of diversity will trigger more activation.

Second, as the distribution of signals evolves, diversity plays an important role in the abruptness of runs. The key element is the expansion of the joint area. With low diversity, early fractions of withdrawals will be very small, as the two distributions have very thin tails. However, if total withdrawal keeps building up, the two distributions come progressively closer together. The overlapped area suddenly gets very large around the means. This is the surprise effect: the jump in probability to withdraw produces a massive fraction of run compared to previous periods. Therefore, runs will be delayed and abrupt if they occur in low diversity environment. On the contrary, high diversity will make the joint area grow faster in a more stable way from the beginning.

Figure 3.10 shows the discussed results. Accordingly, increases in either γ or σ_θ (left panels) result in a higher average default rate. Furthermore, averaged tip measures decreases with respect to both parameters (right panels). The average tip measure decreases because runs are faster, by contrast with the effect of herding. It may seem counterintuitive, but abruptness requires slow early accumulation of withdrawals. High diversity accelerates the speed of accumulation, raising the initial slope of switching trajectories, making runs more apparent but less abrupt. Thus, increase diversity makes runs occur more frequently and faster, strictly increasing the probability of default.

The idea that more diversity increases vulnerability may appear counterintuitive. It is helpful to think in probabilistic dispersion. More diversity implies that there is a high chance that some depositors are extremely "impatient" compared to others. There is also a high chance that some depositors have very bad opinions about the robustness of the bank, compared to the average opinion. Together, these factors are more likely to trigger early withdrawals. The same process can apply sequentially, making the panic cascade over the population. On the contrary, when depositors are symmetric, they behave like one representative agent. There is strength in unity, because it is hard to make some depositors withdraw before others. When necessary conditions are met, depositors are likely to withdraw simultaneously. Although using a different setting, the financial cascade model of Iori et al. (2006) has a similar finding, in which heterogeneity among agents increases the fragility of the system.

The result on heterogeneity of this model might provide alternative explanation to a recent finding in Arifovic et al. (2018). In their experiment, subjects play a repeated version of the simultaneous game and have many decisions to make over time. They found that larger groups of

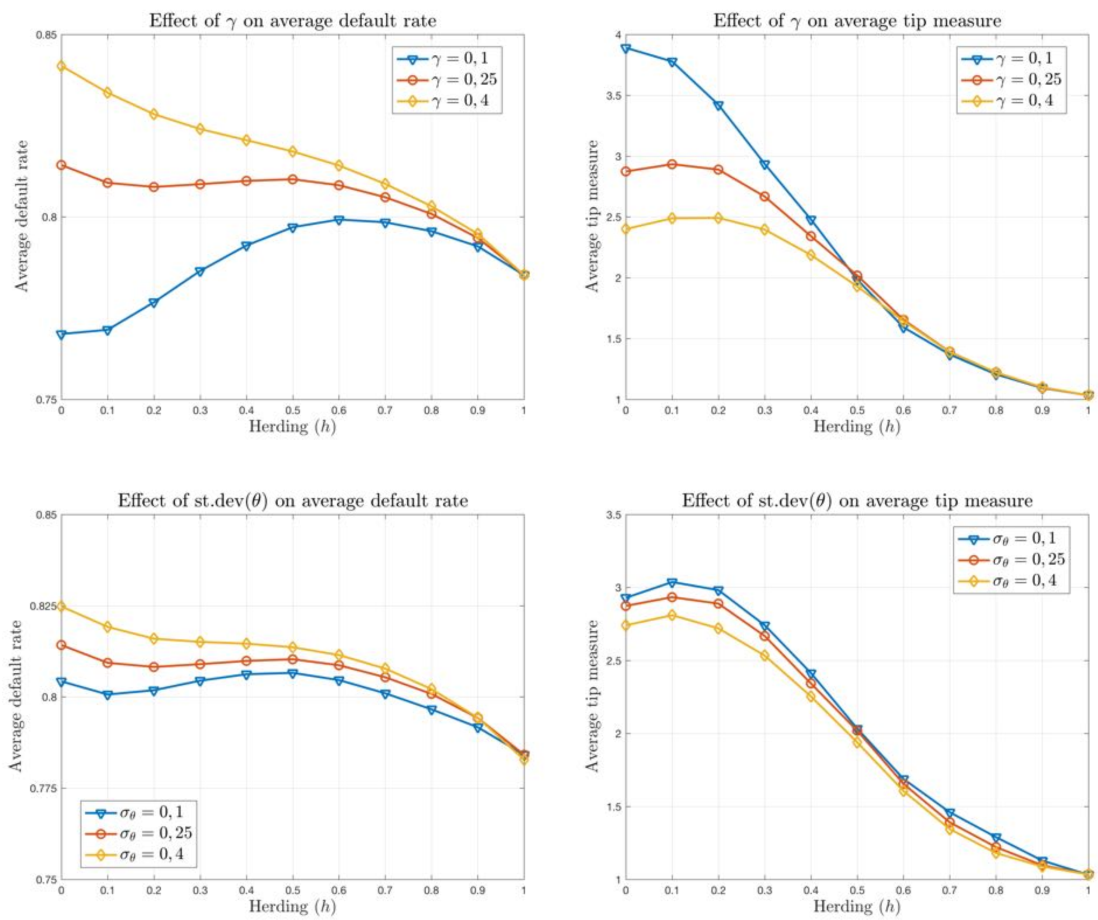


Figure 3.10: Effects of diversity. Increase diversity will result in a higher average default and a lower average tip measure. The base line model is the setting with $h = 0.5, \gamma = 0.25, \sigma_\theta = 0.25$.

subjects generate more bank runs compared to smaller groups, conditional on moderate liquidity (high early payoffs). To some extent, larger groups might imply larger intervals for the sensitivity to withdraw. For example, there is a higher chance to have subjects with extreme loss aversion. There are two consequences of high dispersion of sensitivity in large groups. First, the *perceived* riskiness of the "wait" option might increase for any depositor, when they believe that some other depositors (even a small number) will withdraw with certainty. Second, when subjects make many decisions over time and they know the total withdrawals in the past, there is a potential cascade process similar to this model. As discussed previously, distribution of sensitivity with fatter tails facilitate activation of cascades. When depositors with extreme sensitivity withdraw in an early round, other highly sensitive depositors will follow in subsequent rounds. Over time, the average depositors are caught up in the cascade, when they learn that a large fraction of depositors withdrew in the last round.

4.6 Herding vs. diversity

To complete the picture, this section investigates the interaction between diversity and herding. At a first glance, diversity and herding have the same qualitative effects: they both facilitate activation and reduce abruptness of runs. However, there is a main difference: the speed of runs is increased with high diversity while reduced with high herding. Runs are less abrupt in both cases by design of the tipping measure.

Figure 3.10 shows the interplay between diversity and herding. It appears that strong herding reduce the impacts of diversity, such that all the curves converge when h increases. By design, it is obvious that the effects of noises are washed out for high values of h. But why it is also the case for thresholds? The answer comes from the network, precisely the small number of direct neighbors. The idea can be illustrated with an extreme case in which $h = 1$. When local interaction is the only active force, each withdrawing neighbor has very large impact on individual decision. The variance in thresholds among waiting agents makes little difference when these agents are surrounded by many withdrew neighbors. Unless the variance of thresholds is unusually large, such that the population is polarized: one group has very low thresholds and the other group has very high thresholds. In such case, the latter group will form clusters of resistance, like islands in a sea of runners.

Furthermore, different levels of diversity can counter or enhance the effects of herding on bank failure. Low diversity implies weak global mechanism, leaving the two opposing effects of herding compete with each other. This combination generates non-monotonic effects when herding increases (lowest curves in the left panels). By contrast, high diversity brings in additional boost for activation and speed. This boost is maximal when herding is low, but it fades out quickly when herding increases. Thus, in high diversity environment, the effect of herding is quasi-monotonic rather than non-monotonic as in low diversity environment.

4.7 On network topology

As discussed in previous sections, the network plays a significant role in channeling herding behavior. The network acts as a platform on which herding operates. The topology of the network and the parameter h go hand in hand to determine the extent of local interactions. However, this paper focuses more on herding. There are many different network topologies with a wide range of characteristics, a complete network analysis is beyond the scope of this paper. Some comparative statics for different topologies can be found in the appendix. The results remain qualitatively robust to changes in network topology. This section provides some reflections on the network dimension of the model.

First, observations are essential for herding. A reasonable assumption is that monitoring the actions of many agents at the same time is costly. Furthermore, it is also difficult to observe agents that are far way. These limits are not applied only to spatial dimension, but also to social and sectorial dimension, if depositors are firms or financial institutions. Therefore, to capture the effect of herding, the network should be localized. Furthermore, this type of topology is also consistent with empirical findings (Iyer and Puri (2012); Atmaca et al. (2017)), in which only close neighbors or family members have significant effects on depositors' decision.

Second, the effect of the topology become dominant only when herding is strong. Then, the decision-making process approach naïve learning, where agents strongly incline to imitate their neighbors. However, actions are costly in bank run, unlike other social phenomena such as fashion or opinion diffusion. Thus, very strong herding appears to be unrealistic for this model. The effects of the network will be more influential when herding is endogenized, such that agents determine their tendency to herd base on the structure of links and their local position. This will be a future extension of the model.

5 Discussion and conclusion

This paper has studied bank runs in a dynamic and behavioral setting. Panic bank runs arise as cascades of decisions by both strategic complementarity and herding.

The model is able to characterize the frequency, speed and abruptness of runs. Depending on the parameters, there are three distinct patterns: immediate run, no-run and switching. For low, or high liquidity reserve respectively, all depositors either withdraw immediately, or don't withdraw at all. For moderate liquidity, there are two patterns of bank runs: slow runs with sequential withdrawals build up over time or sudden runs with massive withdrawals occur abruptly. Increase herding generates a tension between activation and speed, such that runs are more frequent but also slower to spread out. Because of this tension, the overall effect of herding on bank failure is non-monotonic. By contrast, increase diversity facilitates both activation and speed, such that the probability of bank failure is strictly increasing.

From a policy perspective, banks are currently required to hold at least 1% of certain liabilities, mainly customer deposits, as liquidity reserve. The liquidity requirement has a maintenance window of 2 weeks in the US and 6 weeks in the EU. This paper suggests several implications for banks and policies makers.

First, it is important to actively monitor day-by-day withdrawals and adjust dynamically liquidity reserve, in addition to the minimum reserve requirements. The reason is that sequential withdrawals are path-dependent, a small change in early withdrawals can have large impact on the trajectory of runs. Therefore, it might be useful to develop a quantitative framework that is able to detect run-sensitive patterns of withdrawals, to devise interventions that dissolve or at least dampen potential panics in the early stage.
Second, there might exist a trade off between stability and uncertainty. Increase liquidity holding and restrain herd behavior can reduce the probability of runs, but also make potential runs more unpredictable. Furthermore, banks may face double layers of uncertainty on whether a run might occur (random switching), but also when it could occur (sudden tip) if it is on the way. Any measure aiming at increasing the robustness of banks should also be followed by efforts to reduce uncertainty in forecasting sudden large withdrawals. One direct measure is to supervise factors that affect the behavior of depositors such as social networks, tendency to herd and degree of heterogeneity. This paper showed that these specific factors play an important role in generating bank runs.

It is worth stressing that the present model is simple in some aspects. It serves as a first step toward building a dynamic framework for panic events. There are several limitations that can be addressed.
The first limit is the exogenous nature of important elements, such as the herding factor and the static information network. Furthermore, depositors stick with simple decision rules and do not change their behaviors during the crisis. To make the model more realistic, one can use a sophisticated decision-making by allowing several features: endogenous herding with respect to the network and withdrawals; memory with weighted averaged signals; dynamic evolution of network; exchange of information among connected agents. These features will allow the model to focus more on the individual aspect, such as learning and social interactions.
The second limit is the absence of reactions from the bank. In practice, banks do not sit still and wait for their liquidity reserves to be depleted. One can introduce an interbank market such that banks can borrow liquidity to face withdrawals. This feature can be used to study the systemic aspect, such as market freeze and contagious bank runs. When a bank experiences withdrawals, depositors might panic and run preemptively from connected banks, creating a generalized banking panic. Furthermore, when banks fear that depositors might herd, they will be reluctant to lend to others, creating a feedback cycle.

Overall, integrating dynamics and behavioral factors to models with strategic complementarity is an important step for future research, at both micro and aggregate level.

Appendix

A. Network topologies

The network used in the model is a two-dimension grid. This section introduces 4 additional structures: random network, small-world, circle, regular-8. The random network and small-world both have average degree of 4, similar to the grid network used in the model. The average degree

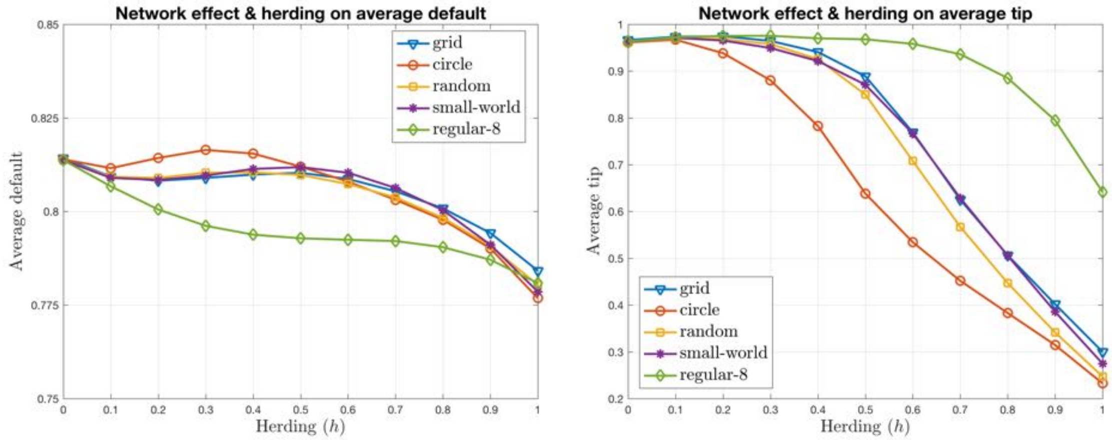


Figure 3.11: Macro behaviors of networks

of the circle is 2 and 8 for the regular-8.

Circle is a regular network with degree 2, in which every agent has one neighbor each side: left and right. All agents form a closed ring. Regular-8 is similar to the circle, but each agent is connected to 4 other agents on each side, instead of 1. Small-world starts out as a regular network with degree 4. Then, every link has a small probability to be rewired, such that some shortcuts are made inside the ring. Random network starts out as a regular network with degree 4. Every link has a probability 1 to be rewired, such that all links are random.

Figure 10 compares the macro behavior for all networks, including the grid network as the benchmark. Networks with the same average degree (grid, random, small-world) behave very similarly, regardless the structure. Higher average degree (regular-8) appears to decrease fragility but increase abruptness, vice-versa for low average degree. These observations suggest that the average number of neighbors might play an important role, otherwise the pattern of links has little influence in this model.

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