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Linéarisation de structures algébriques à l'aide d'opérades et de foncteurs polynomiaux:

Les équivalences quadratiques et la formule de Baker-Campbell-Hausdorff pour les variétés 2-nilpotentes

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"L'interdépendance de toutes choses nous rappelle que rien, ici-bas, n'a d'existence absolue en tant qu'entité fixe et isolée",
Christophe André

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Titre de la thèse

Linéarisation de structures algébriques à l'aide d'opérades et de foncteurs polynomiaux:
Les équivalences quadratiques et la formule de Baker-Campbell-Hausdorff pour les variétés 2-nilpotentes

Résumé de la thèse

Le travail de thèse contribue à établir des liens entre structures algébriques non-linéaires, décrites par des théories algébriques, et des structures algébriques linéaires, encodées par des algèbres sur une opérade linéaire. Pour les théories algébriques dont les modèles forment une catégorie semi-abélienne (ce qui inclut la plupart des structures intéressantes), un tel lien a été exhibé récemment par M. Hartl, au niveau des objets gradués associés à une nouvelle notion de suite centrale descendante des modèles d'une théorie donnée : il s'avère qu'ils ont une structure naturelle d'algèbre graduée sur une certaine opérade de groupes abéliens associée à la théorie.

Le sujet de thèse s'inscrit dans le projet d'étendre ce lien au niveau global, c'est-à-dire d'établir des correspondances du type Mal'cev et Lazard dans le cas des groupes, à savoir entre les modèles nilpotents suffisamment radicables et les algèbres nilpotentes sur l'opérade linéaire correspondante (après tensorisation avec un sous-anneau des rationnels approprié). Ces correspondances jouent un rôle fondamental en théorie des groupes et commencent à faire leurs preuves en théorie des loops grâce au développement plus récent d'une théorie de Lie non-associative; on peut s'attendre à ce qu'il en soit de même dans un contexte plus général. Il est important de noter qu'aussi bien dans les correspondances classiques de Mal'cev et Lazard que dans leurs généralisations à des variétés multiples de loops (Moufang, Bruck, Bol etc.), le passage des algèbres (de Lie, de Mal'cev etc.) appropriées aux objets non-linéaires (groupes, voire loops) qui leur correspondent, est donné par une formule de Baker-Campbell-Hausdorff appropriée, déduite d'une étude de fonctions exponentielles et logarithmes.
Dans la thèse, une nouvelle approche est développée pour construire une correspondance (en fait, une équivalence de catégories) du type Lazard entre une variété (dite aussi catégorie algébrique) 2-nilpotente 2-radicable (dans un sens approprié) $\mathcal{C}$ donnée et les algèbres sur une opérade symétrique unitaire linéaire et 2-nilpotente $AbOp(\mathcal{C})$ dépendant de la variété, vivant dans la catégorie monoïdale des $\mathbb{Z}[\frac{1}{2}]$-modules à gauche. L'anneau de fraction $\mathbb{Z}[\frac{1}{2}]$ apparaît car notre définition de 2-divisibilité d'objets de $\mathcal{C}$ se traduit par la condition de 2-divisibilité classique sur le premier terme de l'opérade. L'équivalence de type Lazard se construit grâce à la théorie des foncteurs polynomiaux (plus précisément quadratiques) et à la notion d'extension linéaire de catégories. L'idée principale est de chercher une équivalence quadratique (i.e un foncteur quadratique qui est une équivalence de catégories) entre une variété semi-abélienne 2-nilpotente 2-radicable donnée $\mathcal{C}$ et la catégorie des algèbres sur $AbOp(\mathcal{C})$, que nous appellerons le foncteur de Lazard.
La nouveauté principale de cette approche est de ne pas construire ce foncteur explicitement sur tous les objets et les morphismes, en utilisant une formule de BCH établie au préalable; mais au contraire de construire l'"ADN" du foncteur de Lazard, c'est-à-dire un ensemble de données minimales le caractérisant étudié dans ce travail de thèse, et d'en *déduire* une formule de type BCH dans notre contexte. Cette démarche devrait pouvoir se généraliser et ainsi fournir une approche nouvelle et intéressante même de la formule BCH classique.

Title of the thesis

Linearization of algebraic structures with operads and polynomial functors:
Quadratic equivalences and the Baker-Campbell-Hausdorff formula for 2-step nilpotent varieties

Abstract

The aim of this work consists of establishing the foundations and first steps of a research project which aims at a new understanding and generalization of the classical Baker-Campbell-Hausdorff formula with a conceptual approach, and its main application in group theory: refining a result of Mal'cev adapting the classical Lie correspondence to abstract groups, Lazard proved that the category of n-divisible n-step nilpotent groups is equivalent with the category of n-step nilpotent Lie algebras over the coefficient ring $\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{n}]$. Generalizations to other algebraic structures than groups were obtained in the literature first for several varieties of loops (in particular Moufang, Bruck and Bol loops), and finally for all loops in recent work of Mostovoy, Pérez-Izquierdo and Shestakov. They invoke other types of algebras replacing Lie algebras in the respective context, namely Mal'cev algebras related with Moufang loops, Lie triple systems related with Bruck loops, Bol algebras with Bol algebras and finally Sabinin algebras with arbitrary loops. In each case, the associated type of algebras can be viewed as a linearization of the non-linear structure given by a given type of loops.

This situation motivates a research program initiated by M. Hartl, namely of exhibiting suitable linearizations of all non-linear algebraic structures satisfying suitable conditions, namely all semi-abelian varieties (of universal algebras, in the sense of universal algebra or of Lawvere). In fact, Hartl associated with any semi-abelian category $\mathcal{C}$ a multi-right exact (and hence multi-linear) functor operad on its abelian core. In the special case where $\mathcal{C}$ is a variety, this functor operad is even multi-colimit preserving and by specialization is equivalent with an operad in abelian groups; the algebra type encoded by this operad provides a linearization of the given variety. Indeed, for each of the above-mentioned varieties of loops this algebra type coincides (over rational coefficients) with the one exhibited in the literature. These constructions and results are based on a new commutator theory in semi-abelian categories which itself relies on a calculus of functors in the framework of semi-abelian categories, both developed by Hartl in partial collaboration with B. Loiseau and T. Van der Linden. Now the project mentioned at the beginning constitutes the next major goal in this emerging general theory of linearization of algebraic structures: to generalize the Lazard equivalence and Baker-Campbell-Hausdorff formula to the context of semi-abelian varieties, and to deduce a way of explicitly computing the operad $AbOp(\mathcal{C})$ from a given presentation of the variety $\mathcal{C}$ (more precisely, the operad obtained from $AbOp(\mathcal{C})$ by tensoring its term of arity n with $\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{n}]$). In the classical example of groups this would amount to deducing the structure of the Lie operad directly from the usual group axioms.

In this thesis, we provide the starting point of this new theoric approach for the case $n = 2$. In contrast with all existing *local* approaches to the subject (defining the desired equivalence object-by-object), in the classical framework of groups or loops, the approach investigated in this thesis for the first time is of an essentially *global* nature; in fact, it is not based on the use of an exponential function, but exclusively relies on the theory of polynomial functors. More precisely, we first study the *DNA*-like condensed data encoding such quadratic functors. The latter data should allow to exhibit a 2-*truncated logarithm functor* from a given 2-step nilpotent variety $\mathcal{C}$ satisfying a certain 2-divisibility condition to 2-step nilpotent algebras over the operad $AbOp(\mathcal{C})$, that is an equivalence of categories. Then the latter may be termed Lazard correspondence of degree 2 and provides an explicit 2-truncated Baker-Campbell-Hausdorff formula, that is a formula expressing *all non-linear operations* in the variety $\mathcal{C}$ by the linear operations of algebras over the operad $AbOp(\mathcal{C})$.

Contents

Chapter 0

Introduction

The Baker-Campbell-Hausdorff formula (BCH formula for short) has a long history and has applications in a wide variety of problems. The classical, associative version (for groups) was more recently extended to various non-associative contexts (that is, for numerous varieties of loops). In the associative case, one of the old references about the BCH series had been provided in 1906 by F. Hausdorff in [15]. Some forty years later, Mal'cev gave a bijective correspondence (called now the Mal'cev correspondence) between torsion-free radicable nilpotent groups and nilpotent Lie algebras over the rational field $\mathbb{Q}$ in [29]. In 1954, M. Lazard then improved this result by establishing a bijective correspondence between n-step nilpotent n-radicable groups and n-step nilpotent Lie algebras over the subring $\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{n}]$ of $\mathbb{Q}$. The idea of these correspondences consists in making a (nilpotent or complete) Lie algebra into a group by using the BCH formula. Explicitely, this group structure is given by the element $H(X,Y) = log(exp(X).exp(Y))$ of the rational non-commutative power series ring in two variables X and Y, expressed as an infinite sum of nested commutators of X and Y of increasing weight. Thus

$$H(X,Y) = X + Y + \frac{1}{2}\,[X,\,Y] + \frac{1}{12}\,[X,\,[X,\,Y]] - \frac{1}{12}\,[Y,\,[X,\,Y]] - \frac{1}{24}\,[Y,\,[X,\,[X,\,Y]]] + \ldots$$

Specializing the variables X and Y to any elements of an n-step nilpotent Lie algebra $\mathcal{G}$ over the subring $\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{n}]$ of $\mathbb{Q}$ then defines an n-step nilpotent and n-radicable group structure on $\mathcal{G}$. The above explicit expression of the first few terms of the BCH formula is probably due to E. Dynkin in [8]. The infinite BCH series is well-known to arise in the classical equivalence between simply-connected Lie groups and Lie algebras; it expresses the multiplication of the Lie group in terms of the linear structure of its Lie algebra.

In the nonassociative case, generalizations to other algebraic structures than groups were obtained in the literature first for several varieties of loops (in particular Moufang, Bruck and Bol loops). According to Lev L. Sabinin, Mal'cev has made a pioneering work of a great importance in this area by providing the first generalization of Lie theory to a non-associative context. In fact, Mal'cev established a bijective correspondence between simply-connected Monfang loops and Mal'cev algebras (which was called Moufang-Lie algebras by Mal'cev). The latter correspondence was further studied by Kuzmin in [23] and by Kerdman in [20]. Then Sabinin proved that local Bruck loops and local symmetric spaces are essentially the same (see [38]), while it is known that the latter spaces are classified by Lie triple systems (see [25] and [27]). In an independent work, Kikkawa introduced in [21] homogeneous Lie loops (some generalized version of Lie groups), and in particular symmetric Lie loops (a slightly generalized version of Bruck loops) which are also classified by Lie triple systems (see also [22] for a short history). Next the correspondence between Bol algebras and simply-connected Bol loops was well-studied in [30], [33] and [37]. Finally, J. Mostovoy, J.M. Izquierdo and P. Shestakov in [32] gave an equivalence between the category of nilpotent Sabinin algebras over the real numbers and the category of simply-connected nilpotent loops, using a nonassociative BCH formula which they exhibit in this context.

In the present work, we introduce a functor theoretic approach in order to exhibit a BCH type formula in a much more general context than for varieties of groups and loops, namely for semi-abelian varieties all of whose objects are n-step nilpotent and n-radicable. In fact, in this thesis we content ourselves of studying only the first, but already highly non-trivial case where $n = 2$. However, the methods developed here are designed to serve as a *model* for a future treatment of higher values of n once the necessary theory of polynomial functors will have been developed.

More precisely, given a 2-step nilpotent 2-radicable variety $\mathcal{C}$, we establish a Lazard type correspondence in a more general context (equivalence of categories in fact) between $\mathcal{C}$ and the category of algebras over the operad in abelian groups $AbOp(\mathcal{C})$ associated with $\mathcal{C}$ (by specialization of a more general construction of Manfred Hartl for arbitrary semi-abelian categories). Taking $\mathcal{C}$ to be the variety of 2-step nilpotent 2-radicable groups our equivalence recovers the classical Lazard correspondence in nilpotency class 2. However, the classical methods based on a thorough study of the exponential and other functions are not available at this level of generality, so we develop a new approach based on the use of *functors* instead of *functions* which proceeds in four steps: first of all, we find minimal algebraic data, which we call *DNA*, characterizing quadratic functors with domain an appropriate pointed category and with values in algebras over a given linear operad, which generalizes the work of M. Hartl and C. Vespa for functors taking values in abelian groups in [12]. Secondly, we give a criterion for a quadratic functor between categories of regular projective objects in 2-step nilpotent categories, and from there between entire 2-step nilpotent varieties, to be an equivalence of categories. This criterion is based on Baues's notion of linear extension of categories and Hartl's commutator calculus for functors. Thirdly, we construct a specific DNA giving rise to a functor on a given 2-step nilpotent 2-radicable variety with values in algebras over $AbOp(\mathcal{C})$, which we prove to be an equivalence by using the criterion obtained in the second step. Finally, analyzing this equivalence in detail provides a way to recover not only the 2-step nilpotent group structure in $\mathcal{C}$ but also any operation of arbitrary arity, in terms of a BCH formula for all operations in $\mathcal{C}$.

In summary, the method in this paper is based not on any kind of exponential function as all classical theory on the subject, but on universal algebra and the construction of a logarithm functor via its DNA, by using the theory of quadratic functors.

We now give a detailed account of the content of each chapter of this thesis.

Chapter 1. In this chapter, we give the necessary background of the thesis. First we recall the notion of varieties (in the sense of Lawvere). It provides a convenient formal setting to describe algebraic structures consisting of a given family of operations of any numbers of variables (called arities) on sets which satisfy a given family of equational axioms (or relations). Then we give generalities about polynomial functors that are functors defined by the vanishing of their cross-effect of a certain degree. Moreover we recall the notion of (linear) operads and algebras over such operads. A linear operad may be seen as a way to describe a collection of modules of abstract operations with (potentially) several entries and one output, endowed with multilinear composition operations satisfying certain relations. Then an algebra over such an operad is a module endowed with multilinear structure maps, which morally realize the abstract operations of the operad as concrete multilinear multiplication operations on the given module. In addition, we recall the notion of commutators relative to a functor introduced by M. Hartl, which play a fundamental role in our work. They are a generalization of commutators in semi-abelian categories defined by cross-effects of the identity functor as introduced by Hartl and Loiseau. This tool allows us to establish interesting links between polynomial functors and nilpotent objets (i.e. objects in a given semi-abelian category such that its commutator of a certain order is trivial).

Chapter 2. In this part, we first recall the main results of the paper [12] in which M. Hartl and C. Vespa provide the minimal algebraic data (or DNA) characterizing quadratic functors with domain an appropriate category $\mathcal{C}$ and values in the category of abelian groups Ab. More precisely, their DNA are *quadratic $\mathcal{C}$-modules* that are diagrams of abelian groups homormophisms of the form

$$M = \Big(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e \xrightarrow{H} M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e\Big) \tag{0.0.1}$$

satisfying certain conditions, where E is a fixed object in $\mathcal{C}$. In addition, M. Hartl and C. Vespa also give a functorial construction of quadratic functors (with domain $\mathcal{C}$ and values in abelian groups) from quadratic $\mathcal{C}$-modules, namely: if M is a quadratic $\mathcal{C}$-module as above, they construct the quadratic functor $-\otimes M : \mathcal{C} \to Ab$, called the *quadratic tensor product* (associated with M) determined by the data which constitute M, as being the pushout of two natural transformations. It is proved that the quadratic tensor product preserves filtered colimits and suitable coequalizers (more precisely coequalizers of reflexive graphs if $\mathcal{C}$ is a semi-abelian category).
Let R be a (unitary) ring and let $\mathcal{P}$ be a unitary symmetric operad in the category of abelian groups endowed with its standard monoidal structure given by the tensor product. In the present work, we generalize the above results by providing DNA's that characterize first quadratic functors with domain $\mathcal{C}$ and values in (right) R-modules, and then those with the same domain and values in $\mathcal{P}$-algebras. In fact, we show that quadratic functors taking values in (right) R-modules are entirely characterized by quadratic $\mathcal{C}$-modules as in (0.0.1) endowed with a structure of R-modules, i.e. each component is a R-module and the maps preserve R-module structures. Next the first main result of this thesis provide DNA's of quadratic functors with values in $\mathcal{P}$-algebras, namely:

Theorem 0.0.1. *Quadratic functors from an arbitrary pointed algebraic theory $\mathcal{C}$ to $\mathcal{P}$-algebras are functorially equivalent to quadratic $\mathcal{C}$-modules over $\mathcal{P}$. Also, quadratic functors from any semi-abelian variety $\mathcal{C}$ to $\mathcal{P}$-algebras preserving filtered colimits and coequalizers of reflexive graphs are functorially equivalent to quadratic $\mathcal{C}$-modules.*

Here a quadratic $\mathcal{C}$-module over $\mathcal{P}$ is a pair $M^{\mathcal{P}} = (M,\ \phi^M : M^2 \to M)$ where M is a quadratic $\mathcal{C}$-module enriched with a structure of right $\mathcal{P}(1)$-module, M^2 is another such object depending on M and $\phi : M^2 \to M$ is a morphism of these kinds of objects, see definitions 1.4.3 and 1.4.6 for details. In fact, the morphism $\phi^M : M^2 \to M$ between quadratic $\mathcal{C}$-modules recovers binary structure linear maps of the quadratic tensor product $-\otimes M : \mathcal{C} \to Ab$ so as to make it take values in the category of $\mathcal{P}$-algebras.

Chapter 3. In this chapter, we first recall the notion of linear extensions of categories introduced by H.-J. Baues in 5.1 of [4]. Then we provide the five lemma in this context, already given by Baues in 5.5 of [4], but whose assumption is slightly weakened here in this thesis. Then it permits us to establish a criterion for a quadratic functor between 2-step nilpotent categories to induce an equivalence between suitable subcategories, or even to be an equivalence between the entire categories if they are varieties. For this we need some technical results using abstract tools such as commutators relative to functors in semi-abelian categories established in the forthcoming paper [10], and recalled in the first chapter in the thesis. This criterion is the second main result of this thesis, whose explicit form is as follows:

Theorem 0.0.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-step nilpotent varieties. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a reduced (i.e. sending the nul object of $\mathcal{C}$ to the nul object of $\mathcal{D}$) quadratic functor. Assume that the following conditions are satisfied.*

(1) F preserves filtered colimits, binary coproducts and coequalizers of reflexive graphs.

(2) F sends the free object E of rank 1 in $\mathcal{C}$ to the free object E' of rank 1 in $\mathcal{D}$.

(3) F commutes up to an isomorphism with the abelianization functors of $\mathcal{C}$ and $\mathcal{D}$ (i.e. a certain natural triangle commutes).

(4) The functor $Ab(F)\colon Ab(\mathcal{C}) \to Ab(\mathcal{D})$ given by the restriction of F (well-defined thanks to condition (3)) is an equivalence of categories.

(5) F preserves the class of monomorphism constitued to binary commutators of free object of finite rank in $\mathcal{C}$.

Then F is an equivalence of categories.

Here note that this kind of functor necessarly preserves free objects of finite rank (by conditions (1) and (2) of the latter theorem). Moreover an explicit construction of a weak inverse of the equivalence F, in the above statement, has been given in the thesis (see Lemma 3.5.12 for details).

Next we show that quadratic equivalences with domain $\mathcal{C}$ and values in $\mathcal{P}$-algebras can be equivalently seen as a certain type of quadratic $\mathcal{C}$-modules over $\mathcal{P}$ in which all the terms (in particular $\mathcal{P}$ itself) are explicitly determined by the condition of being an equivalence, except an appropriate action of the monoid $\mathcal{C}(E,E)$ on M_e and the morphism $H : T_{11}cr_2U_E(E,\, E) \otimes_\Lambda M_e \to M_{ee}$ in the structure of M in $M^{\mathcal{P}}$ (see (0.0.1)). This says that taking a quadratic equivalence with domain $\mathcal{C}$ and values in $\mathcal{P}$-algebras amounts to giving an appropriate explicit expression for H.

Chapter 4. In this part, we establish the desired Lazard type correspondence between a 2-radicable 2-step nilpotent variety $\mathcal{C}$ and the category of algebras over $AbOp(\mathcal{C})$ (see section 3.1 for details) depending on $\mathcal{C}$. Let E be a distinguished object of rank 1 in $\mathcal{C}$, let $\mathcal{F}_{AbOp(\mathcal{C})}$ be the canonical free $AbOp(\mathcal{C})$-algebra of rank 1 and let us denote by $Alg - AbOp(\mathcal{C})$ the category of $AbOp(\mathcal{C})$-algebras. The first step towards establishing our Lazard correpondence is to find an appropriate action of the monoid $\mathcal{C}(E,E)$ on $\mathcal{F}_{AbOp(\mathcal{C})}$ such that the adjoint morphism of monoids

$$L_{E,E} : \mathcal{C}(E,E) \to Alg - AbOp(\mathcal{C})\big(\mathcal{F}_{AbOp(\mathcal{C})}, \mathcal{F}_{AbOp(\mathcal{C})}\big)\,,$$

which will become the effect of the functor L on $\mathcal{C}(E,E)$) is an isomorphism. Then we exhibit the remaining structure of an appropriate quadratic $\mathcal{C}$-module over the operad $AbOp(\mathcal{C})$ providing a quadratic functor $L : \mathcal{C} \to Alg - AbOp(\mathcal{C})$, called the *Lazard functor*, which satisfies the above-mentioned necessary conditions for being an equivalence, in particular whose evaluation to the endomorphisms of E in $\mathcal{C}$ is given by the above isomorphism of monoids. Next we prove that the Lazard functor preserves finite coproducts so that its restriction to the full subcategory $\langle E\rangle$ of free objects of finite rank of $\mathcal{C}$ takes values in the category $\langle \mathcal{F}_{AbOp(\mathcal{C})}\rangle$ of free $AbOp(\mathcal{C})$-algebras of finite rank. Then we show that its restriction functor is an equivalence of categories between the algebraic theories $\langle E\rangle$ and $\langle \mathcal{F}_{AbOp(\mathcal{C})}\rangle$ by applying the criterion given in the third chapter.

Chapter 5. In this chapter, we prove that the equivalence of categories between the varieties $Alg - AbOp(\mathcal{C})$ and $\mathcal{C}$, induced by the equivalence between their underlying theories established in the previous chapter, provide an explicit Baker-Campbell-Hausdorff formula recovering any (non linear) operation of the variety $\mathcal{C}$ from structure linear maps of $AbOp(\mathcal{C})$-algebras. This actually is the third main result of this thesis, whose explicit form is as follows:

Theorem 0.0.3. *Let $\mathcal{C}$ be a 2-step nilpotent semi-abelian variety, then there is a Lazard equivalence*

$$L^* : Alg - AbOp(\mathcal{C}) \to \mathcal{C}$$

given by $|L^*(A)| = |A|$ and the following Baker-Campbell-Hausdorff formula : an n-ary operation θ of the variety $\mathcal{C}$ acts on $|L^*(A)|$ by

$$\begin{aligned}\theta(a_1, \ldots, a_n) = \sum_{p=1}^{n} \Big(\lambda_1(a_p \otimes \overline{\theta_p(e)}) + \frac{1}{2}\, \lambda_2(a_p \otimes a_p \otimes H(\theta_p)) \Big) \\ + \frac{1}{2} \sum_{1 \leqslant p < q \leqslant n} \lambda_2\big(a_p \otimes a_q \otimes \gamma_{1,1;2}(\overline{\theta_q(e)} \otimes \overline{\theta_p(e)} \otimes [e_1,\ e_2]_M)\big) \\ + \sum_{1 \leqslant p < q \leqslant n} \lambda_2\big(a_q \otimes a_p \otimes (\theta_{pq}(e_1, e_2) -_M (\theta_p(e_1) + \theta_q(e_2)))\big)\end{aligned}$$

for $a_1, \ldots, a_n \in A$.

Here $\lambda_k : A^{\otimes k} \otimes \mathcal{P}(k) \to A$ are the multiplication maps in the structure of algebra A over $AbOp(\mathcal{C})$, for $k = 1, 2$. The unary operations θ_p and binary ones θ_{pq} in $\mathcal{C}$ are defined by

$$\theta_l(a) = \theta(0, \ldots, 0, a, 0, \ldots, 0) \quad \text{and} \quad \theta_{pq}(a_1, a_2) = \theta(0, \ldots, 0, a_1, 0, \ldots, 0, a_2, 0, \ldots, 0)$$

where a is placed in the l-th place and a_1, a_2 are respectively placed in the p-th and q-th places, for $1 \leqslant p \leqslant n$ and $1 \leqslant p < q \leqslant n$. Moreover, for any unary operation $\mathcal{V}$ of $\mathcal{C}$, we have

$$H(\mathcal{V}) = \mathcal{V}_{E+E}(e_1 +_M e_2) -_M \big(\mathcal{V}_{E+E}(e_1) +_M \mathcal{V}_{E+E}(e_2)\big)$$

where $i_k : E \to E + E$ is the injection of the k-th summand, $e_k = i_k(e)$ and $k = 1, 2$. The element $[a, b]_M$ of $|A|$ is the commutator of a and b for the group structure $+$ which is given by $[a, b]_M = (a +_M b) -_M (b +_M a)$.
Now if θ is a binary operation for which 0 is a both-sided unit, θ_p is the identity for $p = 1, 2$, whence

$$\theta(a, b) = a + b + \frac{1}{2}\, \lambda_2\big(b \otimes a \otimes [e_1, e_2]_M\big) + \lambda_2\big(a \otimes b \otimes (\theta(e_1, e_2) -_M (e_1 +_M e_2))\big)$$

In particular, $a +_M b = a + b + \frac{1}{2}\lambda_2(b \otimes a \otimes [e_1, e_2]_M)$, which in case $\mathcal{C}$ is the variety of 2-radicable 2-step nilpotent groups becomes the classical 2-truncated Baker-Campbell-Hausdorff formula. Moreover, if θ is a binary bireduced operation (that is $\theta_p = 0$ for $p = 1, 2$), then

$$\theta(a, b) = \lambda_2\big(b \otimes a \otimes \theta(e_1, e_2)\big)$$

In particular, $[a, b]_M = \lambda_2(b \otimes a \otimes [e_1, e_2]_M)$. Thus when $\mathcal{C}$ is the variety of 2-radicable 2-step nilpotent groups, then $[a, b]_M$ equals the Lie commutator of a and b, in accordance with the BCH formula for commutators.

To summarize, we have given this concrete formula by using abstract concepts of a global nature such as quadratic functors and linear extensions of categories. Let us have a look at the perspectives resulting from a possible generalization of the methods introduced in this thesis. Denote by Gr_n and Lie_n respectively the theories of radicable n-step nilpotent groups, respectively n-step nilpotent Lie algebras over $\mathbb{Q}$. In the classical case, one observes that the classical Mal'cev equivalence preserves the underlying sets and hence also free objects, in particular those of finite rank. This means that it induces an isomorphism of algebraic theories $L_n : Gr_n \to Lie_n$ (L_n for logarithm and also Lazard).

It follows from general polynomial functor theory that an isomorphism L_n as above is polynomial of degree n, as well as its composite with the natural forgetful functor from Lie_n to rational vector spaces $Vect$. Then generalizing the methods developed in this thesis should provide a new, exponential-free approach to the Lazard equivalence and BCH formula for groups, which may shed some new light on the combinatorics of the coefficients of the BCH formula, by presenting it as kind of a fusion of the combinatorics of free Lie algebras and the one of non-linear pseudo-Mackey-functors introduced in [13] as a DNA of polynomial functors from groups to abelian groups. If this approach works out one may carry out a similar program for loops since the necessary ingredients from polynomial theory have also been provided in [13]. On the long term, one may then hope to obtain similar results for arbitrary nilpotent semi-abelian varieties based on the theory of polynomial functors after finding their corresponding DNA. In fact, there is unpublished work of Xantcha in this direction (see [40]); however, it remains to be investigated whether it indeed provides the desired general results in a satisfactory form.

Chapter 1

Background

In this part, we give the necessary background before tackling the main subject of the thesis. Throughout the thesis, $\mathcal{C}$ denotes a pointed category (i.e. having a null object denoted by 0) with small Hom-sets, but with a possibly large set of objects with respect to a fixed universe. Thus our categories of functors with domain $\mathcal{C}$ may have large Hom-sets, according to the conventions in Mac Lane's book [28]. We also suppose that $\mathcal{C}$ is a pointed category with finite coproducts denoted by $+$. For the case where $\mathcal{C}$ (or any category) has finite products, we denote by $\times$ the product.

Notation 1.0.1. Troughout this thesis, we use the following conventions:

- We denote by Set the category of sets, Gr the category of groups and Ab the category of abelian groups. If $\mathcal{D}$ and $\mathcal{E}$ are categories, then $\mathcal{E}^{\mathcal{D}}$ denotes the category of functors with domain $\mathcal{D}$ and values in $\mathcal{E}$;

- We denote by $U : Ab \to Set$ the canonical forgetful functor assigning each abelian group to its underlying set;

- If $\mathcal{D}$ is any pointed category, $Func_*(\mathcal{C}, \mathcal{D})$ denotes the category of reduced functors (i.e. $F(0) = 0$) with domain $\mathcal{C}$ and values in $\mathcal{D}$;

- Let $\mathcal{D}$ and $\mathcal{E}$ be two categories and let $F, F' : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{D} \to \mathcal{E}$ be functors. We denote by $G \cdot F : \mathcal{C} \to \mathcal{E}$ the composite functor. If $\alpha : F \Rightarrow F'$ and $\beta : G \Rightarrow G'$ are two natural transformations, then we denote by $G_* \cdot \alpha : G \cdot F \Rightarrow G \cdot F'$ the image of α by G and by $F^* \cdot \beta : G \cdot F \Rightarrow G' \cdot F$ the restriction of β to image objects of F.

- For $n \in \mathbb{N}^*$ and n objects $X_1, \dots X_n$ in $\mathcal{C}$, we denote by $i_k^n : X_k \rightarrowtail X_1 + \ldots + X_n$ the injection of the k-th summand and $r_k^n : X_1 + \ldots + X_n \to X_k$ its corresponding retraction, i.e. the unique morphism such that $r_k^n \circ i_k^n = id$ and $r_k^n \circ i_l^n = 0$, for $l \neq k$.

- Let X be an object in $\mathcal{C}$, we write as usual $\nabla_X^n : X^{+n} \to X$ the unique morphism such that, for $k = 1, \dots, n$, $\nabla_X^n \circ i_k^n = id$.

- If moreover $\mathcal{C}$ has finite products, and $X_1, \dots X_n$ are n objects in $\mathcal{C}$, we denote by $\pi_k^n : X_1 \times \ldots \times X_n \to X_k$ the projection onto the k-th summand, for $k = 1, \dots, n$. In addition, we denote their corresponding injectons $\iota_k^n : X_k \rightarrowtail X_1 \times \ldots \times X_n$ such that $\pi_k^n \circ \iota_k^n = id$ and $\pi_k^n \circ \iota_l^n = 0$, for $l \neq k$.

- If moreover $\mathcal{C}$ has finite products, we denote by $\Delta^n : \mathcal{C} \to \mathcal{C}^{\times n}$ the diagonal functor and by $\Delta_X^n : X \to X^{\times n}$ the unique morphism such that $\pi_k^n \circ \Delta_X^n = id$, for $k = 1, \dots, n$.

- For two objects X_1 and X_2 in $\mathcal{C}$, we denote by $\tau_{X_1,X_2}^2 : X_1 + X_2 \to X_2 + X_1$ the canonical switch. If $X_1 = X_2 = X$, then we write $\tau_{X_1,X_2}^2 = \tau_X^2$. If moreover $\mathcal{C}$ has finite products, we denote by $T^2_{X_1,X_2} : X_1 \times X_2 \to X_2 \times X_1$ the canonical switch. If $X_1 = X_2 = X$, we write $T^2_{X_1,X_2} = T^2_X : X \times X \to X \times X$.

- If f_1, f_2 are two morphisms in $\mathcal{C}$ with domain respectively X_1 and X_2 taking values in the same codomain Y, then $(f_1, f_2) : X_1 + X_2 \to Y$ denotes the unique morphism given by the universal property of the coproduct $X_1 + X_2$.

- If moreover $\mathcal{C}$ has finite products, and g_1, g_2 are two morphisms in $\mathcal{C}$ with the same domain X and values respectively in Y_1 and Y_2, then we write $(g_1,\ g_2)^t : X \to Y_1 \times Y_2$ the unique morphism given by the universal property of the product $Y_1 \times Y_2$.

- If moreover $\mathcal{C}$ is finitely complete and cocomplete, we recall that a regular morphism in $\mathcal{C}$ is a coequalizer of some parallel pair of morphisms. In addition we say that, for X and Y objects in $\mathcal{C}$, a morphism $f : X \to Y$ in $\mathcal{C}$ has a *regular epi-mono factorization* if we have $f = i \circ e$, where $i : I \to Y$ is a monomorphism, $e : X \to I$ is a regular epimorphism and I is an object in $\mathcal{C}$. The object I is usually called the *image* (unique up to an isomorphism) of f and it is denoted by $Im(f)$. It is a general fact that any morphism in a regular category (hence in a semi-abelian category) has an epi-mono factorization. It is said that $e : X \to I$ and $i : I \to Y$ are respectively the *coimage* and the *image* of f (unique up to an isomorphism).

- If moreover $\mathcal{C}$ has kernels and cokernels, for any morphism $f : X \to Y$ in $\mathcal{C}$, we denote respectively by $ker(f) : Ker(f) \rightarrowtail X$ and by $coker(f) : Y \to Coker(f)$ the kernel and the cokernel of f.

1.1 Varieties

We here recall and discuss the definition of a variety and a pointed algebraic theory used in this paper.
Roughly speaking, a *variety* (or an algebraic category) in the sense of classical universal algebra is a collection of sets X endowed with a familly of operations $X^{\times n} \to X$, for some $n \in \mathbb{N}$ (for the case $n = 0$, it is the same as taking a constant in X) and a set of equational relations. This definition is a part of classical universal algebra and it is more detailed in the definition 3.2.1 of [5], in terms of logic syntax and all of whose axioms are universally quantified equations. As an example, the category Gr is in particular a variety: given a group G, it may be considered as a set containing a constant $0 \in G$ (i.e. a 0-ary operation), a unary operation $- : G \to G$ and a binary operation $+ : G \times G \to G$ satisfying the usual axioms

$$(x + y) + z = x + (y + z), \quad x + 0 = 0 + x, \quad x + (-x) = 0 = (-x) + x$$

where $x, y, z \in G$. One should observe that these axioms are now presented in a very elementary form: just equalities between algebraic composites, without any existential quantifier, implication symbol, conjunction, disjunction or negation. Then there is another definition of a variety in the sense of categorical universal algebra (which is equivalent to the one in the sense of classical universal algebra). In fact, it is first given precisely by an algebraic theory, or simply theory, whose definition is the following:

Definition 1.1.1. A *pointed (algebraic) theory* is a pointed category $\mathcal{T}$ with a given object E in $\mathcal{C}$ such that any object of $\mathcal{T}$ is a finite sum of copies of E, i.e. E^{+n} for some $n \in \mathbb{N}$ with a specific choice of injections $E \rightarrowtail E^{+n}$ (and the convention $E^{+0} = 0$). We denote by $\langle E \rangle_{\mathcal{C}}$, or simply $\langle E \rangle$, the theory generated by E.

Note that this definition of an algebraic theory is dual to the classical one as being a category encoding algebraic operations. Now a variety may be seen as a category of models (associated with a theory), in the sense of categorical universal algebra, defined as follows:

Definition 1.1.2. A *model* of a theory $\mathcal{T}$ in our sense is a contravariant functor from $\mathcal{T}$ to *Set* transforming coproducts into products. A *variety* (in the sense of categorical universal algebra) is the category of models of some theory $\mathcal{T}$.

The advantage of this definition is that here $\mathcal{T}$ identifies with a full subcategory of its category of models, namely the category of free models of $\mathcal{T}$ of finite rank. This allows us to define certain quadratic functors, from data just depending on $\mathcal{T}$, to be naturally defined on the whole category of models of $\mathcal{T}$.
Now we are interested in certain properties of any variety and preservation properties of the forgetful functor assigning each object of a given variety its underlying set. Let $\mathcal{C}$ be variety (in the sense of 1.1.2) and let $V : \mathcal{C} \to Set$ denote the canonical forgetful functor. Then we provide the theorem 3.5.4 of [5] below:

Proposition 1.1.3. *The variety $\mathcal{C}$ is regular and exact.*

Here we recall that a category is *regular* when it has finite limits, every kernel pair has a coequalizer and the pull-back of a regular epimorphism along any morphism is a regular epimorphism (see the definition in A.5.1 of [6]). In addition a category is *exact* when it is a regular category and every equivalence relation is a kernel pair relation (see the definition in A.5.11 of [6]). Then we give the following proposition already given in 3.5.2 of [6]:

Proposition 1.1.4. *The forgetful functor $V : \mathcal{C} \to Set$ preserves and reflects coequalizers of equivalence relation.*

Corollary 1.1.5. *If moreover $\mathcal{C}$ is a Mal'cev variety (i.e. it has finite limits and every reflexive relation in $\mathcal{C}$ is an equivalent relation), then the forgetful functor $V : \mathcal{C} \to Set$ preserves and reflects coequalizers of reflexive graphs.*

Proof. First we observe that in $\mathcal{C}$ every equivalence relation is in particular a reflexive graph. We assume that $d_0, d_1 : R \to X$ is an equivalence relation of an object X in $\mathcal{C}$ (*relation* means that the morphism $(d_0, d_1)^t : R \to X \times X$ is a monomorphism). In particular, this (equivalence) relation is reflexive, i.e. there is a morphism $s : R \to X$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[d, "\Delta_X^2"] \\ R \arrow[r, "(d_0,d_1)^t"'] & X \times X \end{tikzcd}$$

Hence it is clear that the morphism s is a common section of d_0 and d_1. This implies that $d_0, d_1 : R \to X$ is a reflexive graph in $\mathcal{C}$.
Then we prove that a coequalizer of a reflexive graph is also a coequalizer of some equivalence relation. Let $\delta_0, \delta_1 : T \to X$ be a reflexive graph with common section $\sigma : X \to T$, and $q : X \to Q$ be its coequalizer. Hence we have the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[d, "\Delta_X^2"] \\ T \arrow[r, "(\delta_0,\delta_1)^t"'] & X \times X \end{tikzcd} \tag{1.1.1}$$

As the morphism $(\delta_0, \delta_1)^t : T \to X \times X$ is not a monomorphism (i.e. a relation on X) in general, we consider its regular epi-mono factorization as follows:

$$\begin{tikzcd} T \arrow[rr, "(\delta_0,\delta_1)^t"] \arrow[dr, "p"', two heads] & & X \times X \\ & R \arrow[ur, "r"', tail] & \end{tikzcd}$$

where p and r are respectively the coimage and the image of $(\delta_0, \delta_1)^t$. Thus $r : R \rightarrowtail X \times X$ is a relation on X (because it is a monomorphism) and it is reflexive because we get

$$r \circ \big(p \circ \sigma\big) = r \circ p \circ \sigma = (\delta_0, \delta_1)^t \circ \sigma = \Delta_X^2$$

As the variety $\mathcal{C}$ is supposed to be Mal'cev by assumption, $r : R \rightarrowtail X \times X$ in an equivalence relation. In addition, $q : X \to Q$ is also the coequalizer of r because $p : T \to R$ is a (regular) epimorphism. Finally, the forgetful functor $V : \mathcal{C} \to Set$ preserves and reflects coequalizers of reflexive graphs. $\square$

Next we recall the proposition 3.4.2 of [5] as follows:

Proposition 1.1.6. *The variety $\mathcal{C}$ has filtered colimits and these are computed pointwise. In particular, the forgetful functor $V : \mathcal{C} \to Set$ preserves and reflects filtered colimits.*

In this thesis, we mostly use the semi-abelian context.

Remark 1.1.7. Any semi-abelian category (see the definition in 5.1.1 of [6] or in [17]) is Mal'cev by 5.1.2 of [6]. If the variety $\mathcal{C}$ is semi-abelian, then $\mathcal{C}$ has filtered colimits, and the forgetful functor $V : \mathcal{C} \to Set$ preserves and reflects filtered colimits and coequalizers of reflexive graphs. It is a direct consequence of 1.1.5 and 1.1.6.

1.2 Generalities about polynomial functors

Let $\mathcal{D}$ be a semi-abelian category. Here we mainly use the second cross-effect of a reduced functor $F : \mathcal{C} \to \mathcal{D}$ with domain $\mathcal{C}$ and values in $\mathcal{D}$ defined as follows:

$$F(X|Y) = cr_2F(X,\, Y) = Ker\Big(\widehat{r_2^F} = (F(r_{\hat{1}}^2),\ F(r_{\hat{2}}^2))^t : F(X+Y) \to F(X) \times F(Y)\Big)\,,$$

More generally, we have the following definition (also see 1.2 of [12]):

Definition 1.2.1. The *n-th cross-effect of F*, denoted by $cr_nF : \mathcal{C}^{\times n} \to \mathcal{D}$, is the multireduced multifunctor such that, for $X_1, \ldots, X_n$ objects in $\mathcal{C}$, $cr_nF(X_1, \ldots, X_n)$ also denoted by $F(X_1|\ldots|X_n)$ is the kernel of the following natural homomorphism

$$\widehat{r_n^F} = \prod_{k=1}^{n} F(r_{\hat{k}}^n) : F(X_1 + \ldots + X_n) \to \prod_{k=1}^{n} F(X_1 + \ldots + \hat{X}_k + \ldots X_n) \qquad (1.2.1)$$

where, for $k = 1, \ldots, n$, $r_{\hat{k}}^n : X_1 + \ldots + X_n \to X_1 + \ldots + \hat{X}_k + \ldots + X_n$ is the morphism whose restriction to X_i is its canonical injection if $i \neq k$ and is the zero morphism otherwise, see 1.3 of [12].

Remark 1.2.2. From 1.2.1, it defines a functor $cr_n : Func_*\big(\mathcal{C},\, \mathcal{D}\big) \to Func_*\big(\mathcal{C}^{\times n},\, \mathcal{D}\big)$, for $n \in \mathbb{N}^*$.

There is also an inductive definition of the n-th cross-effect of a functor given in 1.2 of [12].

Notation 1.2.3. We denote by $\iota_n^F : cr_nF(X_1, \ldots, X_n) \rightarrowtail F(X_1 + \ldots + X_n)$ the kernel of $\widehat{r_n^F}$.

Now we give a property (also given in 2.25 of [14]) of the comparison morphism $\widehat{r_2^F}$ (see (1.2.1)), as follows:

Lemma 1.2.4. *Let X and Y be objects in $\mathcal{C}$. Then the comparison morphism $\widehat{r_2^F} : F(X+Y) \to F(X) \times F(Y)$ (defined in (1.2.1)) is a regular epimorphism.*

Proof. Since the functor $F : \mathcal{C} \to \mathcal{D}$ is reduced, we have the following equalities:

$$\widehat{r_2^{Id_\mathcal{D}}} = \widehat{r_2^F} \circ \big(F(i_1^2), F(i_2^2)\big) : F(X) + F(Y) \to F(X) \times F(Y)$$

where $\big(F(i_1^2), F(i_2^2)\big) : F(X)+F(Y) \to F(X+Y)$ is the morphism given by the universal property of the coproduct $F(X)+F(Y)$. Since the comparison morphism $\widehat{r_2^{Id_\mathcal{D}}} : F(X)+F(Y) \to F(X) \times F(Y)$ is a regular epimorphism (by protomodularity of the category $\mathcal{D}$), so is the comparison morphism $\widehat{r_2^F} : F(X+Y) \to F(X) \times F(Y)$. □

Intuitively the cross-effect of a functor can be seen as a (categorical) version of the "derivatives" of a functor. This point of view is supported by the notation 2.21 and Lemma 2.22 of [14]. Now it permits to define the notion of polynomial functors, already given in 1.6 of [12]:

Definition 1.2.5. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *polynomial of degree* $\leqslant n$ whenever its $(n+1)$-th cross-effect is trivial, i.e. $cr_{n+1}F = 0$. We denote $Func_{\leqslant n}(\mathcal{C},\, \mathcal{D})$ the full subcategory of $Func_*(\mathcal{C},\, \mathcal{D})$ constituted by polynomial functors of degree $\leqslant n$. In particular, $Lin(\mathcal{C},\, \mathcal{D})$ and $Quad(\mathcal{C},\, \mathcal{D})$ are respectively the full subcategories of $Func_*(\mathcal{C},\, \mathcal{D})$ formed by linear and quadratic functors.

The following proposition says that the composition of a polynomial functor taking values in an abelian category with another polynomial functor with abelian source and target is polynomial. It has appeared in the theorem 1.9 of [34], the proposition 1.9 of [19] and the proposition 2.20 of [13].

Proposition 1.2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $\mathcal{C} \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{B}$ be functors. If F is polynomial of degree $\leqslant n$ and G is polynomial of degree $\leqslant m$, then the composite functor $G \cdot F : \mathcal{C} \to \mathcal{B}$ is polynomial of degree $\leqslant n\, m$.*

Then we define an important natural transformation, already given in 1.7 of [12], as follows:

Definition 1.2.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor. Then the natural transformation $S_n^F : cr_nF \cdot \Delta^n \Rightarrow F$ is such that, for X object in $\mathcal{C}$,

$$(S_n^F)_X = F(\nabla_X^n) \circ \iota_n^F : cr_nF(X, \dots, X) \to F(X)$$

where $cr_nF : \mathcal{C}^{\times n} \to \mathcal{D}$ is the n-th cross-effect of F defined in 1.2.1.

Notation 1.2.8. Here we assume that $\mathcal{C}$ is semi-abelian. For the special case where F is the identity functor $Id_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ (the identity functor of $\mathcal{C}$), the morphism $(S_n^{Id_\mathcal{C}})_X$ (given in 1.2.7) is also written $c_n^X : cr_nId_\mathcal{C}(X, \dots, X) \to X$.

Then it is possible to construct the "polynomialization" of a reduced functor. For this, M. Hartl and C. Vespa introduce the notion of n-Taylorization of a reduced functor in 1.9 of [12].

Definition 1.2.9. The *n-Taylorization functor* $T_n : Func_*(\mathcal{C},\, \mathcal{D}) \to Func_{\leqslant n}(\mathcal{C},\, \mathcal{D})$ is such that, for a (reduced) functor $F : \mathcal{C} \to \mathcal{D}$, $T_n(F) = Coker\big(S_{n+1}^F : cr_{n+1}F \cdot \Delta^{n+1} \Rightarrow F\big)$. The proposition 1.10 in [12] says that T_n is left adjoint to the inclusion functor.

Notation 1.2.10. We denote by $t_n^F : F \Rightarrow T_nF$ the unit of the adjunction that is the cokernel of $S_{n+1}^F : cr_{n+1}F \cdot \Delta^{n+1} \Rightarrow F$, defined in 1.2.7.

The universal property of t_n^F gives:

Proposition 1.2.11. *Any natural transformation with source F and target a polynomial functor of degree $\leqslant n$ factorizes uniquely through $t_n^F : F \Rightarrow T_n F$.*

Let us denote by $BiFunc_{*,*}(\mathcal{C}^{\times 2}, \mathcal{D})$ the category of bireduced bifunctors (i.e that are trivial whenever one of their place is the zero object). The notion of polynomial functors has been extended for bifunctors as bipolynomial bifunctors, appeared in 1.11 of [12]:

Definition 1.2.12. A bireduced bifunctor $B : \mathcal{C}^{\times 2} \to \mathcal{D}$ is said *bipolynomial of bidegree* $\leqslant (n, m)$ whenever, for an object X in $\mathcal{C}$, the reduced functors $B(-, X) : \mathcal{C} \to \mathcal{D}$ and $B(X, -) : \mathcal{C} \to \mathcal{D}$ are respectively polynomial of degree $\leqslant n$ and $\leqslant m$. If $n = m = 1$, we say that B is *bilinear*. We consider $BiFunc_{\leqslant(n,m)}(\mathcal{C}^{\times 2}, \mathcal{D})$ its corresponding category.

Remark 1.2.13. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor. By 1.2 of [12], F is a quadratic functor if, and only if, its second cross-effect $cr_2 F = F(-|-) : \mathcal{C}^{\times 2} \to \mathcal{D}$ defined above is bilinear.

Now we recall the bilinearization bifunctor $T_{11} : BiFunc_{*,*}(\mathcal{C}^{\times 2}, \mathcal{D}) \to BiFunc_{\leqslant(1,1)}(\mathcal{C}^{\times 2}, \mathcal{D})$ defined in 1.13 of [12]. It is the left adjoint of the inclusion functor by 1.14 of [12]. For a bireduced bifunctor $B : \mathcal{C}^{\times 2} \to \mathcal{D}$, we denote by $t_{11}^B : B \Rightarrow T_{11}B$ the unit of the adjunction. The universal property of t_{11}^B gives:

Proposition 1.2.14. *Any natural transformation (between bifunctors) with source B and target a bilinear bifunctor factorizes uniquely through $t_{11} : B \Rightarrow T_{11}B$.*

Notation 1.2.15. We write $t_{11}^B(a)$, or simply $t_{11}(a)$, the equivalence class of an element $a \in B(X,\, Y)$ in $T_{11}B(X,\, Y)$ where X and Y are objects in $\mathcal{C}$.

1.3 Commutators relative to functors in semi-abelian categories

Let $\mathcal{D}$ be a semi-abelian category and let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor. For this, we give the notion of commutators relative to a functor, introduced by M. Hartl in the forthcoming paper [10]. Here we shall recall the definition of commutators and nilpotent objects in semi-abelian categories.

Definition 1.3.1. Consider $F : \mathcal{C} \to \mathcal{D}$ a reduced functor and A an object in $\mathcal{D}$. Let X be an object in $\mathcal{C}$ and let $(x_i : X_i \rightarrowtail X)_{1 \leqslant i \leqslant n}$ be n subobjects of X. Then

- the *n-weighted commutator $[X_1, \ldots, X_n]_F$ relative to F* is the image of the morphism $(S_n^F)_X \circ F(x_1|\ldots|x_n) : F(X_1|\ldots|X_n) \to F(X)$, where $S_n^F : cr_n F \cdot \Delta^n \Rightarrow F$ is the natural transformation given in 1.2.7 and $F(x_1|\ldots|x_n) : F(X_1|\ldots|X_n) \to F(X|\ldots|X)$ is the restriction of the morphism $F(x_1 + \ldots + x_n) : F(X_1 + \ldots + X_n) \to F(X^{+n})$ to $F(X_1|\ldots|X_n)$.

- the object A is said *n-step nilpotent* whenever the $(n+1)$-weighted commutator $[A, \ldots, A]_{Id_{\mathcal{D}}}$ is trivial. We denote by $Nil_n(\mathcal{D})$ the full subcategory of $\mathcal{D}$ formed by n-step nilpotent objects in $\mathcal{D}$.

- for the case $n = 1$, we say that A is an *abelian object* of $\mathcal{D}$. We denote by *Ab*$(\mathcal{D})$ the full subcategory formed by abelian objects in $\mathcal{D}$, usually called the abelian core of $\mathcal{D}$.

We now consider the notion of central subobject in semi-abelian categories using the commutators in the sense of 1.3.1 as follows:

Definition 1.3.2. Let X be an object in $\mathcal{D}$ and $z : Z \rightarrowtail X$ be a subobject of X. We say that $z : Z \rightarrowtail X$, or simply Z, is a *central subobject* of X if $[X,\, Z]_{Id_{\mathcal{D}}} = 0$.

Notation 1.3.3. Let X be an object in $\mathcal{C}$. Here

- we write $\gamma_n^F(X) = [X, \ldots, X]_F$ for the n-weighted commutator of X relative to F;
- we denote by $e_{X,n}^F : F(X|\ldots|X) \twoheadrightarrow \gamma_n^F(X)$ and $i_{X,n}^F : \gamma_n^F(X) \rightarrowtail F(X)$ respectively the coimage and the image of $(S_n^F)_X : F(X|\ldots|X) \to F(X)$ (see 1.2.7);
- we denote by $I_n : Nil_n(\mathcal{D}) \to \mathcal{D}$ and $I = I_1 : Ab(\mathcal{D}) \to \mathcal{D}$ the inclusion functors.

It provides a functor associating to any object in $\mathcal{C}$ its n-weighted commutator of X relative to a functor in $\mathcal{D}$, as follows:

Definition 1.3.4. We define the functor $\gamma_n^F : \mathcal{C} \to \mathcal{D}$ such that $\gamma_n^F(X)$ is the n-weighted commutator of X relative to F (given in 1.3.3), and $\gamma_n^F(f) : \gamma_n^F(X) \to \gamma_n^F(Y)$ is the unique factorization of $F(f) \circ i_{X,n}^F$ through $i_{Y,n}^F$ (which exists by naturality of S_n^F). It also satisfies

$$\gamma_n^F(f) \circ e_{X,n}^F = e_{Y,n}^F \circ cr_n F(f, \ldots, f)$$

where $X \in \mathcal{C}$ and $f : X \to Y$ is any morphism in $\mathcal{C}$.

Notation 1.3.5. We introduce the following notations:

- If $F = Id_{\mathcal{D}}$, we write $\gamma_n^{Id_{\mathcal{D}}} = \gamma_n^{\mathcal{D}}$.
- For an object X in $\mathcal{D}$, we denote respectively by $e_X = e_{X,2}^{Id_{\mathcal{D}}} : X \to \gamma_2^{\mathcal{D}}(X)$ and $i_X = i_{X,2}^{Id_{\mathcal{D}}} : \gamma_2^{\mathcal{D}}(X) \rightarrowtail X$ the coimage and the image of the morphism $c_2^X : Id_{\mathcal{D}}(X|X) \to X$.

Remark 1.3.6. Consider an object X in $\mathcal{C}$ and a subobject $z : Z \rightarrowtail X$ of X. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor. Then the 1-weighted commutator $\gamma_1^F(Z) = [Z]_F$ relative to F is the image of the morphism $F(z) : F(Z) \to F(X)$ (because $cr_1 F(X) = F(X)$ since F is reduced, see 1.2.1). It gives the deviation of $F(z)$ to be a monomorphism.

Notation 1.3.7. Consider an object X in $\mathcal{C}$ and a subobject $z : Z \rightarrowtail X$ of X. We denote respectively by $e_Z^F = e_{Z,1}^F : F(Z) \to [Z]_F$ and $i_Z^F = i_{X,1}^F : [Z]_F \rightarrowtail F(X)$ the coimage and the image of $F(z) : F(Z) \to F(X)$.

Remark 1.3.8. Let $F : \mathcal{C} \to \mathcal{D}$ be a polynomial functor of degree $\leqslant n$ (see 1.2.5). Then, for any object X in $\mathcal{C}$ and $k \geq n+1$, $\gamma_k^F(X) = [X, \ldots, X]_F = 0$ by 1.3.1 because the k-th cross-effect of F is trivial implying that the natural transformation $S_k^F : cr_k F \cdot \Delta^k \Rightarrow F$ is trivial.

The next proposition provides the properties of commutators relative to functors, that appear in the forthcoming paper [10]:

Proposition 1.3.9. *Let $\mathcal{E}$ be a semi-abelian category and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be two reduced functors. If the functor $G : \mathcal{D} \to \mathcal{E}$ preserves regular epimorphisms, then we have*

$$[\,[X_{1,1}, \ldots, X_{1,n_1}]_F, \ldots, [X_{m,1}, \ldots, X_{m,n_m}]_F\,]_G \subset [X_{1,1}, \ldots, X_{m,n_m}]_{G \cdot F}$$

where, for $k \in \{1, \ldots m\}$ and $j \in \{1, \ldots, n_k\}$, $X_{k,j}$ is a subobject of an object X in $\mathcal{C}$.

Now we are naturally led in this thesis to consider semi-abelian categories whose objects are nilpotent defined in an appropriate sense (see 1.3.1 for details). It is given by the notion of nilpotent category whose definition is the following:

Definition 1.3.10. A category $\mathcal{D}$ is called *n-step nilpotent* when it is a semi-abelian category whose identity functor $Id_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is polynomial of degree $\leqslant n$ in the sense of 1.2.5.

As an example, the full subcategory $Nil_n(\mathcal{D})$ of $\mathcal{D}$ (see 1.3.1) formed by n-step nilpotent objects in $\mathcal{D}$ is an n-step nilpotent category. The case $n = 2$ has an immediate consequence:

Proposition 1.3.11. *Let $\mathcal{D}$ be a 2-step nilpotent category and let X be an object in $\mathcal{D}$. Then the 2-weighted commutator $[X, X]_{Id_{\mathcal{D}}}$ is a central subobject of X.*

Proof. By 1.3.10, the identity functor $Id_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ of $\mathcal{D}$ is quadratic. Hence it is a direct consequence of 1.3.12. □

Let X be an object in $\mathcal{D}$. We give a condition for the 1-weighted commutator $[[X, X]_{Id_{\mathcal{C}}}]_F$ relative to F to be a central subobject of $F(X)$, as follows:

Proposition 1.3.12. *We here suppose that $\mathcal{C}$ is a semi-abelian category. Consider a (reduced) functor $F : \mathcal{C} \to \mathcal{D}$ and an object X in $\mathcal{C}$. If F is a quadratic functor and preserves regular epimorphisms, then the 1-weighted commutator $[[X, X]_{Id_{\mathcal{C}}}]_F$ relative F is a central subobject of $F(X)$.*

Proof. As F is a reduced functor, $F(Y) = [Y]_F$ is the 1-weighted commutator of Y relative to F by 1.3.1. By 1.3.9, we have

$$[\,[[Y,\,Y]_{Id_{\mathcal{C}}}]_F,\; F(Y)\,]_{Id_{\mathcal{D}}} = [\,[[Y,\,Y]_{Id_{\mathcal{C}}}]_F,\; [Y]_F\,]_{Id_{\mathcal{D}}} \subset [\,[Y,\,Y]_{Id_{\mathcal{C}}},\; Y\,]_F \subset [Y,\,Y,\,Y]_F = 0\,, \text{by 1.3.8}$$

because F is a quadratic functor which preserves regular epimorphisms. □

1.4 Polynomial functors and nilpotent objects

In this part, $\mathcal{D}$ denotes a semi-abelian category. We here study links between polynomial functors and nilpotent objects. The next proposition says that polynomial functors of degree $\leqslant n$, $n \in \mathbb{N}^*$, with values in a semi-abelian category takes in fact values in n-step nilpotent objects:

Proposition 1.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced polynomial functor of degree n, then F takes values in $Nil_n(\mathcal{D})$.*

Proof. Let X be an object in $\mathcal{C}$. The result is a direct consequence of 1.3.9. The last result says in particular that the commutator $\gamma_{n+1}^{Id_{\mathcal{D}}}(F(X))$ is a subobject of the commutator $\gamma_{n+1}^{F}(X)$ relative to the functor F of weight $n+1$. As $F : \mathcal{C} \to \mathcal{D}$ is polynomial of degree $\leqslant n$, $\gamma_{n+1}^{F}(X)$ is trivial (see 1.3.8). Hence $\gamma_{n+1}^{Id_{\mathcal{D}}}(F(X))$ is trivial as well. Consequently $F(X)$ is an n-step nilpotent object in $\mathcal{D}$. □

It follows from 1.4.1 that the second cross-effect of the identity functor of a given 2-step nilpotent category takes values in the abelian core, as follows:

Lemma 1.4.2. *If $\mathcal{D}$ is a 2-step nilpotent category, then the second cross-effect of the identity functor $Id_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ seen as a bifunctor from $\mathcal{D} \times \mathcal{D}$ to $\mathcal{D}$ takes in fact values in the abelian core $Ab(\mathcal{D})$ of $\mathcal{D}$.*

Proof. By 1.2.13, the bifunctor $Id_{\mathcal{D}}(-|-) : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ is bilinear because the identity functor of $\mathcal{D}$ is quadratic by 1.3.10 (since the category $\mathcal{D}$ is 2-step nilpotent). Hence it is a direct consequence of 1.4.8 that the (bilinear) bifunctor $Id_{\mathcal{D}}(-|-)$ takes values in $Ab(\mathcal{D})$. □

Remark 1.4.3. For $n \in \mathbb{N}^*$, the functor $T_n Id_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ (defined in 1.2.9) is a polynomial functor of degree n. By 1.4.1, it takes values in $Nil_n(\mathcal{D})$.

Notation 1.4.4. We consider the following notations:

- We denote by $Nil_n : \mathcal{D} \to Nil_n(\mathcal{D})$ the functor $T_n Id_{\mathcal{D}} : \mathcal{D} \to Nil_n(\mathcal{D})$ defined in 1.2.9.

- We write $nil_n = t_n^{Id_\mathcal{D}} : Id_\mathcal{D} \Rightarrow I_n.Nil_n$ for the cokernel of the natural transformation $S_{n+1}^{Id_\mathcal{D}} : cr_{n+1}Id_\mathcal{D} \cdot \Delta^{n+1} \Rightarrow Id_\mathcal{D}$ given in 1.2.7, where $I_n : Nil_n(\mathcal{D}) \to \mathcal{D}$ is the inclusion functor (see 1.3.1).

- For the case $n = 1$, we denote by $Nil_1(\mathcal{D})$, respectively $Ab^\mathcal{D} : \mathcal{D} \to Ab(\mathcal{D})$, the category $Ab(\mathcal{D})$, respectively the functor $Nil_1 : \mathcal{D} \to Nil_1(\mathcal{D})$. The category $Ab(\mathcal{D})$ is called the *abelian core* of $\mathcal{D}$, and the functor $Ab^\mathcal{D} : \mathcal{D} \to Ab(\mathcal{D})$ is called the *abelianization functor* (of $\mathcal{D}$).

- Moreover we write $ab = nil_1 : Id_\mathcal{D} \Rightarrow I.Ab^\mathcal{D}$ for the cokernel of $S_2^{Id_\mathcal{D}} : cr_2 Id_\mathcal{D} \cdot \Delta^{n+1} \Rightarrow Id_\mathcal{D}$, where $I : Ab(\mathcal{C}) \to \mathcal{C}$ is the inclusion functor (see 1.3.1).

Remark 1.4.5. For an n-step nilpotent object X in $\mathcal{D}$, we consider that $X = Nil_n(X)$ and $(nil_n)_X = id : X \to Nil_n(X) = X$, i.e. the functor $Nil_n : \mathcal{D} \to Nil_n(\mathcal{D})$ (given in 1.4.4) restricted to $Nil_n(\mathcal{D})$ is the identity functor of $Nil_n(\mathcal{D})$.

Proposition 1.4.6. *For any $n \in \mathbb{N}^*$, the functor $Nil_n : \mathcal{D} \to Nil_n(\mathcal{D})$ is a reflection of the inclusion functor $I_n : Nil_n(\mathcal{D}) \to \mathcal{D}$. The unit of this pair of adjoint functors is the cokernel $nil_n : Id_\mathcal{D} \Rightarrow I_n.Nil_n$ of the natural transformation $S_{n+1}^{Id_\mathcal{D}} : cr_{n+1}Id_\mathcal{D} \cdot \Delta^{n+1} \Rightarrow Id_\mathcal{D}$.*

Proof. Let X be an object in $\mathcal{D}$. We first observe that $Nil_n(X) = T_n Id_\mathcal{D}(X)$ is an n-step nilpotent object of $\mathcal{D}$. For this we have

$$\big(S_{n+1}^{Id_\mathcal{D}}\circ\big)_{Nil_n(X)} \circ Id_\mathcal{D}\big((nil_n)_X|(nil_n)_X\big) = (nil_n)_X \circ \big(S_{n+1}^{Id_\mathcal{D}}\big)_X = 0 \tag{1.4.1}$$

by naturality of $\big(S_{n+1}^{Id_\mathcal{D}}\big)_X : cr_{n+1}Id_\mathcal{D} \cdot \Delta^{n+1}(X) \to X$ in X and because $(nil_n)_X : X \to Nil_n(X)$ is the cokernel of $\big(S_{n+1}^{Id_\mathcal{D}}\big)_X$. As the identity functor $Id_\mathcal{D} : \mathcal{D} \to \mathcal{D}$ clearly preserves regular epimorphisms and $(nil_n)_X : X \to Nil_n(X)$ is a regular epimorphism, the morphism $Id_\mathcal{D}\big((nil_n)_X|(nil_n)_X\big) : Id_\mathcal{D}\big(X|X\big) \to Id_\mathcal{D}\big(Nil_n(X)|Nil_n(X)\big)$ is a (regular) epimorphism. It follows from (1.4.1) that the morphism

$$\big(S_{n+1}^{Id_\mathcal{D}}\big)_{Nil_n(X)} : Id_\mathcal{D}\big(Nil_n(X)|Nil_n(X)\big) \to Nil_n(X)$$

is trivial. By 1.3.1, $Nil_n(X)$ is an n-step nilpotent object in $\mathcal{D}$. Then the universal property of the unit $(nil_n)_X : X \to Nil_n(X)$ of the pair of adjoint functors is a direct consequence of the naturality of $\big(S_{n+1}^{Id_\mathcal{D}}\big)_X$ in X, and of the universal property of the cokernel $(nil_n)_X : X \to Nil_n(X)$ of $\big(S_{n+1}^{Id_\mathcal{D}}\big)_X$. □

Remark 1.4.7. For an object X in $\mathcal{D}$, we observe that $Nil_n(X)$ is the quotient of X by the $(n+1)$-weighted commutator $\gamma_{n+1}^{Id_\mathcal{D}}(X)$ (defined in 1.3.1).

Then, for $n \geq 1$, the next proposition says that polynomial functors of degree $\leqslant n$ with semi-abelian source and target and preserving coequalizers of reflexive graphs can be entirely described by restricting them to n-step nilpotent objects of the source category.

Proposition 1.4.8. *We here assume that $\mathcal{C}$ is a semi-abelian category. Let $F : \mathcal{C} \to D$ be a polynomial functor of degree $\leqslant n$ preserving coequalizers of reflexive graphs. Then the functors F and $F.Nil_n$ with domain $\mathcal{C}$ and values in $\mathcal{D}$ are isomorphic to each other. More precisely, the natural transformation $F^*.nil_n : F \Rightarrow F.Nil_n$ is an isomorphism.*

Proof. Let X be an object in $\mathcal{C}$. Then we have the following short exact sequence:

$$0 \longrightarrow \gamma_{n+1}^{Id_\mathcal{C}}(X) \xrightarrow{i_{n,X}^{Id_\mathcal{D}}} X \xrightarrow{(nil_n)_X} Nil_n(X) \longrightarrow 0 \tag{1.4.2}$$

because $(nil_n)_X : X \to Nil_n(X)$ is the cokernel of $(S_{n+1}^{Id_{\mathcal{C}}})_X : cr_{n+1}Id_{\mathcal{C}} \cdot \Delta^{n+1}(X) \to X$ and $\gamma_{n+1}^{Id_{\mathcal{C}}}(X) = Im\big((S_{n+1}^{Id_{\mathcal{C}}})_X\big)$ by 1.3.1. By 2.31 of [14], we get the following right exact sequence:

$$F\big(\gamma_{n+1}^{Id_{\mathcal{C}}}(X)|X\big) \rtimes F\big(\gamma_{n+1}^{Id_{\mathcal{C}}}(X)\big) \xrightarrow{\left\langle \begin{array}{c} S_2^F \circ Id_{\mathcal{C}}(i_{n+1,X}^{Id_{\mathcal{C}}}|id) \\ F(i_{n+1,X}^{Id_{\mathcal{C}}}) \end{array} \right\rangle} F(X) \xrightarrow{F\big((nil_n)_X\big)} F\big(Nil_n(X)\big) \longrightarrow 0$$

By 1.3.1 and by 1.3.9, we have

$$Im\big(S_2^F \circ Id_{\mathcal{C}}(i_{n+1,X}^{Id_{\mathcal{C}}}|id)\big) = \big[\gamma_{n+1}^{Id_{\mathcal{C}}}(X), X\big]_F = \big[[X, \ldots, X]_{Id_{\mathcal{C}}}, X\big]_F \subset [X, \ldots, X]_F = \gamma_{n+2}^F(X)$$

because the functor $F : \mathcal{C} \to \mathcal{D}$ preserves coequalizers of reflexive pairs (hence regular epimorphisms). Similarly, we get

$$Im\big(\gamma_{n+1}^{Id_{\mathcal{C}}}(X)\big) = \big[\gamma_{n+1}^{Id_{\mathcal{C}}}(X)\big]_F = \big[[X, \ldots, X]_{Id_{\mathcal{C}}}\big]_F \subset [X, \ldots, X]_F = \gamma_{n+1}^F(X)$$

As the functor $F : \mathcal{C} \to \mathcal{D}$ is polynomial of degree $\leqslant n$, we have

$$\gamma_{n+2}^F(X) = 0 = \gamma_{n+1}^F(X)$$

by 1.3.8. It implies that $F((nil_n)_X) : F(X) \to F(Nil_n(X))$ is an isomorphism, as desired. $\square$

1.5 Effective actions on morphism sets in semi-abelian categories

We note that actions on morphism sets have been already treated by Bourn in a more general context of unital categories (see the definition in 1.2.5 of [6]). This work has been studied in the paper [7]. It also appears in the book [6] of Bourn and Borceux, in which moreover they consider these actions in strongly unital categories (see the definition in 1.8.3 of [6]), which are in particular unital categories by 1.8.4 of [6].

In this part, we first recall the necessary notions and the main result relative to actions on morphism sets in [6]. In the case of semi-abelian categories, which are strongly unital, we give an alternative definition of actions on morphism sets, which is equivalent to those of Bourn ; these are effective.

Now we recall that a *unital* category $\mathcal{E}$ is a pointed category having finite limits such that, for X and Y objects in $\mathcal{E}$, the pair (ι_1^2, ι_2^2) is strongly epimorphic:

$$X \xrightarrow{\iota_1^2} X \times Y \xleftarrow{\iota_2^2} Y\,,$$

see the notations given in 1.0.1. Then we provide the notion of centrality in unital categories, already defined in 1.3.12 of [6]. Before this, we give the following definition:

Definition 1.5.1. Let $\mathcal{E}$ be a unital category. Two morphisms $f : X \to Z$ and $g : Y \to Z$ with the same codomain *cooperate* when there exists a factorization $\varphi_{f,g} : X \times Y \to Z$ such that the following diagram

$$\begin{tikzcd} X \arrow[r, "\iota_1^2"] \arrow[dr, "f"'] & X \times Y \arrow[d, "\varphi_{f,g}"] & Y \arrow[l, "\iota_2^2"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd} \tag{1.5.1}$$

commutes. The morphism $\varphi_{f,g}$ is called the *cooperator* of f and g; it is necessarily unique because the pair $(\iota_1^2; \iota_2^2)$ is epimorphic. To simplify the notation, we may write $\varphi = \varphi_{f,g}$ for the cooperator of f and g.

Remark 1.5.2. Let $\mathcal{E}$ be a unital category. Consider two morphisms $f : X \to Z$ and $g : Y \to Z$ in $\mathcal{E}$ which cooperate with cooperator $\phi_{f,g} : X \times Y \to Z$ as in 1.5.1. If moreover $\mathcal{E}$ has binary coproducts, then f and g cooperate if, and only if, the cooperator $\phi_{f,g}$ makes the following triangle commute:

$$\begin{tikzcd}
X + Y \arrow[rr, "r_2^{\widehat{Id_{\mathcal{E}}}}"] \arrow[drr, "(f{,}g)"'] & & X \times Y \arrow[d, "\varphi_{f,g}"] \\
& & Z
\end{tikzcd}$$

commutes.

Then we are able to define central morphisms in unital categories already given in 1.3.12 of [6], as follows:

Definition 1.5.3. Let $\mathcal{E}$ be a unital category. A morphism $f : X \to Y$ is *central* when it cooperates with the identity of Y. We write φ_f the cooperator of f and id_Y. We denote by $Z(X, Y)$ the set of central morphisms from X to Y, and by $Z(\mathcal{E})$ the class of central morphisms in $\mathcal{E}$.

Let $\mathcal{E}$ be a unital category. For X and Y objects in $\mathcal{E}$, we have a map $+ : \mathcal{E}(X, Y) \times Z(X, Y) \to \mathcal{E}(X, Y)$ defined by

$$f + g = \varphi_g \circ (f, id_X)^t \tag{1.5.2}$$

where $f \in \mathcal{E}(X, Y), g \in Z(X, Y)$ and $\varphi_g : Y \times X \to Y$ is the cooperator of g and id_Y (see 1.5.3). It gives the following proposition present in 1.3.22 of [6]:

Proposition 1.5.4. *Let $\mathcal{E}$ be a unital category. For all objects $X, Y \in \mathcal{E}$, the set $Z(X, Y)$ of central morphisms from X to Y is a commutative monoid which acts on $\mathcal{E}(X, Y)$. The monoid operation and the monoid action are both given by the addition in* (1.5.2).

In any unital category $\mathcal{E}$, each central morphism from X to Y doesn't need to have an inverse in the commutative monoid $Z(X, Y)$, for $X, Y \in \mathcal{E}$. We shall consider those morphisms having such an inverse, as follows:

Definition 1.5.5. Let $\mathcal{E}$ be a unital category. We say that a morphism $f : X \to Y$ is *symmetrizable* if f is a central morphism having an inverse in the commutative monoid $Z(X, Y)$. We denote by $\Sigma(X, Y)$ the subset of $Z(X, Y)$ formed by symmetrizable morphisms form X to Y, and by $\Sigma(\mathcal{E})$ the class of symmetrizable morphisms in $\mathcal{E}$.

Remark 1.5.6. Taking the notations of 1.5.5, the set $\Sigma(X, Y)$ of symmetrizable morphisms is clearly an abelian group.

There are some unital categories in which all central morphisms are symmetrizable. This fact holds in the general context of strongly unital categories. We briefly recall that a category is *strongly unital* (see 1.8.3 of [6]) when it is pointed, has finite limits and satifies the property: every split right punctual relation is undiscrete (see 1.1.1 of [6]). Moreover 1.5.4 also holds in strongly unital categories because any such category is unital by 1.8.4 of [6].

Remark 1.5.7. Let $\mathcal{E}$ be a strongly unital category. For $X, Y \in \mathcal{E}$, $Z(X, Y) = \Sigma(X, Y)$, hence the set $Z(X, Y)$ is an abelian group, see 1.8.19 of [6].

Note that any semi-abelian category is strongly unital because it is in particular a finitely complete protomodular category and any such category is strongly unital by 3.1.18 of [6]. Now we define certain actions on morphism sets in semi-abelian categories induced by central subobjects in the sense of 1.3.2, and we compare them with actions given in the context of (strong) unital categories as above.

From now on, we assume that $\mathcal{E}$ is a semi-abelian category (or merely a pointed finitely complete regular protomodular category having binary coproducts). The next proposition says that a central subobject of an object in $\mathcal{E}$ in the sense of 1.3.2 is exactly a monomorphism and a central morphism as in 1.5.3.

Proposition 1.5.8. *Let $z : Z \rightarrowtail Y$ be a monomorphism in $\mathcal{E}$. Then $z : Z \rightarrowtail Y$ is a central subobject of Y in the sense of 1.3.2 if, and only if, it is a central morphism from Z to Y in $\mathcal{E}$ as in 1.5.3.*

Proof. First we assume that $z : Z \rightarrowtail Y$ is a central subobject of Y in $\mathcal{E}$ in the sense of 1.3.2. It means that $[Y, Z]_{Id_{\mathcal{E}}} = 0$, that is equivalent to say that the morphism $c_2^Y \circ Id_{\mathcal{E}}(id|z) : Id_{\mathcal{E}}(Y|Z) \to Y$ is trivial. Hence there is a unique factorization $\varphi_z : Y \times Z \to Y$ of $(id, z) : Y + Z \to Y$ though the comparison morphism $\widehat{r_2^{Id_{\mathcal{E}}}} : Y + Z \to Y \times Z$, i.e. we get

$$\varphi_z \circ \widehat{r_2^{Id_{\mathcal{E}}}} = (id, z) \tag{1.5.3}$$

because $(id, z) \circ \iota_2^{Id_{\mathcal{E}}} = c_2^Y \circ Id_{\mathcal{E}}(id|z) = 0$ and the comparison morphism $\widehat{r_2^{Id_{\mathcal{E}}}} : Y + Z \to Y \times Z$ is the cokernel of its kernel $\iota_2^{Id_{\mathcal{E}}} : Id_{\mathcal{E}}(Y|Z) \rightarrowtail Y + Z$. By 1.5.2 and 1.5.1, it says that z and the identity of Y cooperate with cooperator φ_z. By 1.5.3, we deduce that z is a central morphism from Z to Y.
Next we suppose that $z : Z \rightarrowtail Y$ is a central morphism in $\mathcal{E}$. By 1.5.3 and 1.5.2, there is a unique morphism $\varphi_z : Y \times Z \to Y$ such that the relation (1.5.3) holds. Hence we get the equalities as follows:

$$\begin{aligned} c_2^Y \circ Id_{\mathcal{E}}(id|z) &= (id, z) \circ \iota_2^{Id_{\mathcal{E}}} \\ &= \varphi_z \circ \widehat{r_2^{Id_{\mathcal{E}}}} \circ \iota_2^{Id_{\mathcal{E}}} \\ &= 0 \end{aligned}$$

because $\iota_2^{Id_{\mathcal{E}}} : Id_{\mathcal{E}}(Y|Z) \rightarrowtail Y + Z$ is the kernel of the comparison morphism $\widehat{r_2^{Id_{\mathcal{E}}}} : Y + Z \to Y \times Z$ (see 1.2.3). By 1.3.1, we have

$$[Y, Z]_{Id_{\mathcal{E}}} = Im\big(c_2^Y \circ Id_{\mathcal{E}}(id|z)\big) = 0$$

implying that $z : Z \rightarrowtail Y$ is a central subobject in the sense of 1.3.2. □

Now we assume that $z : Z \to Y$ is a central subobject of Y in $\mathcal{E}$ (i.e. a central morphism from X to Y in $\mathcal{E}$).

Notation 1.5.9. Let X and Y be two objects in $\mathcal{E}$. We write $Z_0(X, Y)$ for the set of morphisms of the form $z \circ \alpha$, where $\alpha \in \mathcal{E}(X, Z)$.

Then we show that, for $X, Y \in \mathcal{E}$, each morphism belonging to $Z_0(X, Y)$ is a central morphism in $\mathcal{E}$.

Proposition 1.5.10. *Let X and Y be two objects in $\mathcal{E}$. Then $Z_0(X, Y) \subset Z(X, Y)$. Moreover for $\alpha \in \mathcal{E}(X, Z)$ the cooperator $\varphi_{z\circ\alpha} : Y \times X \to Y$ of $z \circ \alpha$ and id_Y is given by*

$$\varphi_{z\circ\alpha} = \varphi_z \circ (id_Y \times \alpha) \tag{1.5.4}$$

Proof. It is a direct consequence of 1.3.20 and 1.3.6 of [6]. □

Remark 1.5.11. Let X and Y be two objects in $\mathcal{E}$. The set $Z_0(X, Y)$ is clearly stable under the additive law of the abelian group $Z(X, Y)$ (by 1.5.7 because $\mathcal{E}$ is strongly unital since it is a semi-abelian category by 1.5.8). Hence it has a canonical abelian group structure.

Now we are interested in an action on sets of morphisms with codomain Z (a central subobject of Y in $\mathcal{E}$) from(1.5.3); in fact, for $X, Y \in \mathcal{E}$, we define the map $\bullet : \mathcal{E}(X,Y) \times \mathcal{E}(X,Z) \to \mathcal{E}(X,Y)$ by

$$f \bullet \alpha = \varphi_z \circ (f, \alpha)^t \tag{1.5.5}$$

where $f \in \mathcal{E}(X,Y)$ and $\alpha \in \mathcal{E}(X,Z)$. We prove that it is an action of $\mathcal{E}(X,Z)$ on $\mathcal{E}(X,Y)$ which coincides with the restriction of the action given in (1.5.2) to $\mathcal{E}(X,Y) \times Z_0(X,Y)$.

Lemma 1.5.12. *Let X and Y be two objects in $\mathcal{E}$. For $f \in \mathcal{E}(X,Y)$ and $\alpha \in \mathcal{E}(X,Z)$, we have*

$$f \bullet \alpha = f + \alpha$$

Proof. It suffices to prove that $\varphi_{z\circ\alpha} = \varphi_z \circ (id_Y \times \alpha)$. For this, we have the equalities as follows:

$$\begin{aligned} f + \alpha &= \varphi_{z\circ\alpha} \circ (f, id)^t \\ &= \varphi_{z\circ\alpha} \circ (f \times id) \circ \Delta_X^2 \\ &= \varphi_z \circ (id \times \alpha) \circ (f \times id) \circ \Delta_X^2 \text{, by (1.5.4)} \\ &= \varphi_z \circ (f \times \alpha) \circ \Delta_X^2 \\ &= \varphi_z \circ (f, \alpha)^t \\ &= f \bullet \alpha , \end{aligned}$$

as desired. □

Remark 1.5.13. We recall that $z : Z \rightarrowtail Y$ is a central subobject of Y in $\mathcal{E}$. Let X and Y be objects in $\mathcal{E}$. Then the abelian group $\mathcal{E}(X,Z)$ acts on $\mathcal{E}(X,Y)$ as

$$f + \alpha = \varphi_z \circ (f, \alpha)^t \tag{1.5.6}$$

where $f \in \mathcal{E}(X,Y)$ and $\alpha \in \mathcal{E}(X,Z)$. Moreover this action coincides with the one given in (1.5.2) by 1.5.12.

Then we prove that each abelian object in $\mathcal{E}$ has an internal binary operation in the sense of Definition A.1.1 of [6].

Proposition 1.5.14. *Let A be an abelian object in $\mathcal{E}$. Then A has an internal binary operation $m_A : A \times A \to A$ in $Ab(\mathcal{E})$, that is the unique factorization of $\nabla_A^2 : A + A \to A$ through the comparison morphism $\widehat{r_2^{Id_\mathcal{E}}} : A + A \to A \times A$.*

Proof. The morphism $m_A : A \times A \to A$ exists because $\nabla_A^2 \circ \iota_2^{Id_\mathcal{E}} = c_2^A = 0$ by 1.3.1 (since A is an abelian object in $\mathcal{E}$). □

Remark 1.5.15. Since Z is a central subobject of Y (hence an abelian object in $\mathcal{E}$), there is an internal binary operation $m_Z : Z \times Z \to Z$ by 1.5.14. It determines an abelian group structure on $\mathcal{E}(X,Z)$ as follows:

$$f + g = m_Z \circ (f, g)^t$$

where $f, g \in \mathcal{E}(X,Z)$. It is in fact the restriction of the action $+ : \mathcal{E}(X,Y) \times \mathcal{E}(X,Z) \to \mathcal{E}(X,Y)$ (see 1.5.13) to the set $\mathcal{E}(X,Z) \times \mathcal{E}(X,Z)$.

The next proposition gives an isomorphism of abelian groups:

Proposition 1.5.16. *Consider an object X and an abelian object A both in $\mathcal{E}$. Then the map $(ab_X)^* : \mathcal{E}(X^{ab},\ A) \to \mathcal{E}(X,\ A)$ is an isomorphism of abelian groups.*

Proof. It suffices to observe that the morphism $c_2^X : Id_{\mathcal{E}}(X|X) \to X$ (given in (1.2.8)) is natural in X, i.e. $f \circ c_2^X = c_2^A \circ Id_{\mathcal{E}}(f|f)$, for $f \in \mathcal{E}(X, Z)$. As A is an abelian object in $\mathcal{E}$, we have $c_2^A = 0$ by 1.3.1. Hence f factorizes uniquely through the cokernel $ab_X : X \to X^{ab}$ of c_2^X. ☐

Notation 1.5.17. Consider an object X and an abelian object A in $\mathcal{E}$, and $f \in \mathcal{E}(X,\ A)$, then we write $f^{ab} \in \mathcal{C}(X^{ab},\ A)$ for the unique factorization of f through $ab_X : X \to X^{ab}$.

Now we provide a criterion for a subobject of an object in $\mathcal{E}$ to be central, already given by D. Bourn and M. Gran. The proof has been adapted for our own context.

Proposition 1.5.18. *Let $e : Y \to Q$ be a regular epimorphism in $\mathcal{E}$. Consider the following diagram:*

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d, "k"'] & K \arrow[d, "ker(e)"] \\
R \arrow[r, "d_1"] \arrow[d, "d_0"] & Y \arrow[d, "e"] \\
Y \arrow[r, "e"] \arrow[u, bend left, "s"] & Q
\end{tikzcd} \tag{1.5.7}$$

where the bottom rectangle is the kernel pair of e, $s : Y \to R$ is the canonical common section of d_0 and d_1, and $k : K \rightarrowtail R$ is the unique morphism such that $k \circ d_0 = 0$ and $k \circ d_1 = ker(e)$ by the universal property of the kernel pair of e.
Then K is a central subobject of Y (in the sense of 1.3.2) if, and only if, there is a morphism $\sigma : R \to K$ in $\mathcal{E}$ such that $\sigma \circ k = id$, $\sigma \circ s = 0$ and $(d_0, \sigma)^t : R \to Y \times K$ is an isomorphism. In other terms, there is a morphism $\sigma : R \to K$ in $\mathcal{E}$ such that the span

$$\begin{tikzcd}
K \arrow[r, "k"'] & R \arrow[l, dashed, bend right, "\sigma"'] \arrow[r, "d_0"'] & Y \arrow[l, bend right, "s"']
\end{tikzcd}$$

in $\mathcal{E}$ is a split punctual and undiscrete relation (see the definitions in 1.11 *of [6]).*

Proof. First we assume that K is a central subobject of Y, i.e. $[Y, K]_{Id_{\mathcal{E}}} = 0$. As $s : Y \rightarrowtail R$ is a subobject of Y, it implies that $(s, k) \circ \iota_2^{Id_{\mathcal{E}}} = c_2^Y \circ Id_{\mathcal{E}}(s|k) = 0$ because $Im(c_2^R \circ Id_{\mathcal{E}}(s|k)) = [Y, K]_{Id_{\mathcal{E}}} = 0$ by 1.3.1. As the comparison morphism $\widehat{r_2^{Id_{\mathcal{E}}}} : Y + K \to Y \times K$ is the cokernel of its kernel $\iota_2^{Id_{\mathcal{E}}} : Id_{\mathcal{E}}(Y|K) \rightarrowtail Y + K$ (since it is a regular epimorphism), there is a unique $\varphi_{(s,k)} : Y \times K \to R$ such that

$$\varphi_{(s,k)} \circ \widehat{r_2^{Id_{\mathcal{E}}}} = (s, k) \tag{1.5.8}$$

Note that $k : K \rightarrowtail R$ is the kernel of $d_0 : R \to Y$ by a categorical argument because the bottom rectangle of diagram (1.5.7) is a pull-back. Then we consider the following morphism of split short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\iota_2^2"] \arrow[d, equal] & Y \times K \arrow[r, "\pi_1^2"] \arrow[d, "\varphi_{(s,k)}"] & Y \arrow[r] \arrow[d, equal] \arrow[l, bend right, "\iota_1^2"'] & 0 \\
0 \arrow[r] & K \arrow[r, "k"] & R \arrow[r, "d_0"] & Y \arrow[r] \arrow[l, bend right, "s"'] & 0
\end{tikzcd}$$

By applying the split five lemma (by protomodularity of $\mathcal{E}$) to the above diagram, it proves the $\varphi_{(s,k)} : Y \times Z \to R$ is an isomorphism. Next we set the morphism $\sigma = \pi_2^2 \circ \varphi_{(s,k)}^{-1} : R \to K$. We first have

$$\begin{aligned}
\sigma \circ k &= \sigma \circ (s,k) \circ i_1^2 \\
&= \sigma \circ \varphi_{(s,k)} \circ \widehat{r_2^{Id\mathcal{E}}} \circ i_2^2 \text{, by } (1.5.8) \\
&= \pi_2^2 \circ \varphi_{(s,k)}^{-1} \circ \varphi_{(s,k)} \circ \widehat{r_2^{Id\mathcal{E}}} \circ i_2^2 \\
&= \pi_2^2 \circ \widehat{r_2^{Id\mathcal{E}}} \circ i_2^2 \\
&= \pi_2^2 \circ \iota_2^2 \\
&= id
\end{aligned}$$

Then we get

$$\begin{aligned}
\sigma \circ s &= \sigma \circ (s,k) \circ i_1^2 \\
&= \pi_2^2 \circ \widehat{r_2^{Id\mathcal{E}}} \circ i_1^2 \\
&= \pi_2^2 \circ \iota_1^2 \\
&= 0
\end{aligned}$$

Hence we obtain

$$\sigma \circ k = id \quad \text{and} \quad \sigma \circ s = 0 \tag{1.5.9}$$

It remains to prove that the morphism $(d_0, \sigma)^t : R \to Y \times K$ is an isomorphism. For this, it suffices to show that $(d_0, \sigma)^t \circ \varphi_{(s,k)} = id$. First we have the equalities as follows:

$$\begin{aligned}
\pi_1^2 \circ (d_0, \sigma)^t \circ \varphi_{(s,k)} \circ \widehat{r_2^{Id\mathcal{E}}} &= d_0 \circ \varphi_{(s,k)} \circ \widehat{r_2^{Id\mathcal{E}}} \\
&= d_0 \circ (s,k) \text{, by } (1.5.8) \\
&= (d_0 \circ s, d_0 \circ k) \\
&= (id, 0) \\
&= r_1^2 \\
&= \pi_1^2 \circ \widehat{r_2^{Id\mathcal{E}}}
\end{aligned}$$

Hence we obtain

$$\pi_1^2 \circ (d_0, \sigma)^t \circ \varphi_{(s,k)} = \pi_1^2 \tag{1.5.10}$$

because the comparison morphism $\widehat{r_2^{Id\mathcal{E}}} : Y + K \to Y \times K$ is an epimorphism. Next we get

$$\begin{aligned}
\pi_2^2 \circ (d_0, \sigma)^t \circ \varphi_{(s,k)} \circ \widehat{r_2^{Id\mathcal{E}}} &= \sigma \circ \varphi_{(s,k)} \circ \widehat{r_2^{Id\mathcal{E}}} \\
&= \sigma \circ (s,k) \text{, by } (1.5.8) \\
&= (\sigma \circ s, \sigma \circ k) \\
&= (0, id) \text{, by } (1.5.9) \\
&= r_2^2 \\
&= \pi_2^2 \circ \widehat{r_2^{Id\mathcal{E}}}
\end{aligned}$$

Thus we have

$$\pi_2^2 \circ (d_0, \sigma)^t \circ \varphi_{(s,k)} = \pi_2^2 \tag{1.5.11}$$

By (1.5.10) and (1.5.11), it implies that

$$(d_0, \sigma)^t \circ \varphi_{(s,k)} = id \tag{1.5.12}$$

by uniqueness in the universal property of the product $Y \times K$. Consequently, the morphism $(d_0, \sigma)^t : R \to Y \times K$ is an isomorphism.
Now we assume that there is a morphism $\sigma : R \to K$ such that $\sigma \circ k = id$, $\sigma \circ s = 0$ and $(d_0, \sigma)^t : R \to Y \times K$ is an isomorphism. We aim at proving that $ker(e) : K \rightarrowtail Y$ is a central subobject of Y. By using similar calculations as above, we get

$$(d_0, \sigma)^t \circ (s, k) = \widehat{r_2^{Id_{\mathcal{E}}}} \tag{1.5.13}$$

We set the morphism $\varphi_{s,k} = ((d_0, \sigma)^t)^{-1}$. Hence we have

$$c_2^Y \circ Id_{\mathcal{E}}(id|ker(e)) = (id, ker(e)) \circ \iota_2^{Id_{\mathcal{E}}} = d_1 \circ (s, k) \circ \iota_2^{Id_{\mathcal{E}}} = d_1 \circ \varphi_{(s,k)} \circ \widehat{r_2^{Id_{\mathcal{E}}}} \circ \iota_2^{Id_{\mathcal{E}}} = 0$$

implying that $[Y, K]_{Id_{\mathcal{E}}} = 0$. Thus it proves that $ker(e) : K \rightarrowtail Y$ is a central subobject of Y. □

We recall that $z : Z \rightarrowtail Y$ is a central subobject of Y in $\mathcal{E}$. We denote by $q = coker(z) : Z \rightarrowtail Y) : Y \to Coker(z)$ the cokernel of z. Note that the monomorphism z is normal by using Lemma 4.2 of [36].

Remark 1.5.19. By Proposition 3.2.20 and Lemma 4.2.6 both in [6], the monomorphism $z : Z \rightarrowtail Y$ is the kernel of q.

Then we determine the orbits of actions on morphism sets as in (1.5.6).

Proposition 1.5.20. *Let X and Y be two objects in $\mathcal{E}$. Consider $f, g : X \to Y$ two morphisms in $\mathcal{E}$. Then we have $q \circ f = q \circ g$ if, and only if, there is a morphism $d : X \to Z$ such that*

$$g = f + d$$

whose action is defined in (1.5.6).

Proof. Consider the following diagram:

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "k"'] & Z \arrow[d, "z"] \\ R \arrow[r, "d_1"] \arrow[d, "d_0"] & Y \arrow[d, "q"] \\ Y \arrow[u, bend left, "s"] \arrow[r, "q"] & Coker(z) \end{tikzcd} \tag{1.5.14}$$

where the bottom rectangle of the above diagram is the kernel pair of q. It is a similar diagram as in (1.5.7) in the statement of 1.5.18 (replacing respectively e, $ker(e)$ and K with q, z and Z). By 1.5.18, there is a morphism $\sigma : R \to Z$ such that $\sigma \circ k = id$, $\sigma \circ s = 0$ and $(d_0, \sigma)^t : R \to Y \times Z$ is an isomorphism (because $z : Z \rightarrowtail Y$ is a central subobject of Y). We set the morphism $\varphi_{(s,k)} = ((d_0, \sigma)^t)^{-1} : Y \times K \to R$

We first assume that $q \circ f = q \circ g$. By the universal property of the kernel pair of q, there is a unique morphism $\alpha : X \to R$ such that

$$f = d_0 \circ \alpha \quad \text{and} \quad g = d_1 \circ \alpha \tag{1.5.15}$$

We set the morphism $d = \sigma \circ \alpha : X \to Z$. We have

$$\varphi_z = d_1 \circ \varphi_{(s,k)} \tag{1.5.16}$$

by (**??**). Then we get the equalities as follows:

$$\begin{aligned} f + d &= \varphi_z \circ (f, d)^t \\ &= d_1 \circ \varphi_{(s,k)} \circ (f, d)^t \text{, by (1.5.17)} \\ &= d_1 \circ \varphi_{(s,k)} \circ (d_0 \circ \alpha, \sigma \circ \alpha)^t \text{, by (1.5.17)} \\ &= d_1 \circ \varphi_{(s,k)} \circ (d_0, \sigma)^t \circ \alpha \\ &= d_1 \circ \alpha \text{, by (1.5.12)} \\ &= g \,, \end{aligned}$$

as desired. Now we assume that there is a morphism $d : X \to Z$ such that $g = f + d$. We have the following equalities:

$$q \circ g = q \circ (f + d) = q \circ \varphi_z \circ (f, d)^t = q \circ \varphi_z \circ (f \times d) \circ \Delta_X^2$$

Then we get

$$\begin{aligned} q \circ \varphi_z \circ (f \times d) \circ \widehat{r_2^{Id\varepsilon}} &= q \circ \varphi_z \circ \widehat{r_2^{Id\varepsilon}} \circ (f + d) \text{, by naturality} \\ &= q \circ (id, z) \circ (f + d) \text{, by (1.5.4)} \\ &= q \circ (f, z \circ d) \\ &= (q \circ f, 0) \\ &= q \circ f \circ r_1^2 \\ &= q \circ f \circ \pi_1^2 \circ \widehat{r_2^{Id\varepsilon}} \end{aligned}$$

Hence we obtain

$$q \circ \varphi_z \circ (f \times d) = q \circ f \circ \pi_1^2 \tag{1.5.17}$$

Thus we have

$$\begin{aligned} q \circ g &= q \circ \varphi_z \circ (f \times d) \circ \Delta_X^2 \\ &= q \circ f \circ \pi_1^2 \circ \Delta_X^2 \text{, by (1.5.17)} \\ &= q \circ f \,, \end{aligned}$$

as desired. $\square$

Now we give an important property of actions on morphism sets as in 1.5.5.

Proposition 1.5.21. *Let X be an object in $\mathcal{E}$. Then the action of the abelian group $\mathcal{E}(X,Z)$ on $\mathcal{E}(X,Y)$ given in 1.5.5 is simple, i.e. if $f : X \to Y$ and $d : X \to Z$ are two morphisms such that*

$$f + d = f$$

then it implies that $d = 0$.

Proof. Let $f : X \to Y$ and $d : X \to Z$ be two morphisms such that $f = f + d$. We have the following relations:

$$\begin{aligned} f = f + d &\Longleftrightarrow f = \varphi_z \circ (f,d)^t \text{, by (1.5.6)} \\ &\Longleftrightarrow f = d_1 \circ \varphi_{(s,k)} \circ (f,d)^t \text{, by (1.5.17)} \end{aligned}$$

First we observe that $d_1 \circ (s \circ f) = f = d_1 \circ \varphi_{(s,k)} \circ (f,d)^t$, i.e.

$$d_1 \circ (s \circ f) = d_1 \circ \varphi_{(s,k)} \circ (f,d)^t \tag{1.5.18}$$

Then we have

$$d_0 \circ \varphi_{(s,k)} \circ (f,d)^t = \pi_2^2 \circ (f,d)^t \text{, by (1.5.10)} = f = d_0 \circ (s \circ f)$$

Hence we obtain

$$d_0 \circ \varphi_{(s,k)} \circ (f,d)^t = d_0 \circ (s \circ f) \tag{1.5.19}$$

The relations (1.5.18) and (1.5.19) imply that

$$s \circ f = \varphi_{(s,k)} \circ (f,d)^t \tag{1.5.20}$$

by uniqueness in the universal property of the kernel pair of q (see (1.5.14)). Now we have

$$s \circ f = (s,k) \circ i_1^2 \circ f = \varphi_{(s,k)} \circ \widehat{r_2^{Id_{\mathcal{E}}}} \circ i_1^2 \circ f \text{, by (1.5.4)} = \varphi_{(s,k)} \circ \iota_1^2 \circ f$$

Then we have the relations as follows:

$$\begin{aligned} s \circ f = \varphi_{(s,k)} \circ \circ (f,d)^t &\Longleftrightarrow \varphi_{(s,k)} \circ \iota_1^2 \circ f = \varphi_{(s,k)} \circ \circ (f,d)^t \text{, by (1.5.4)} \\ &\Longleftrightarrow \iota_1^2 \circ f = (f,d)^t \end{aligned}$$

Thus we obtain

$$d = \pi_2^2 \circ (f,d)^t = \pi_2^2 \circ \iota_1^2 \circ f = 0$$

□

Then we are led to consider a certain type of actions on morphism sets (induced by specific central subobjects) in 2-step nilpotent categories as follows:

Corollary 1.5.22. *Let $\mathcal{E}$ be a 2-step nilpotent category (see 1.3.10). Then, for $X, Y \in \mathcal{E}$, the abelian group $\mathcal{E}(X, [Y,Y]_{Id_{\mathcal{E}}})$ simply acts on $\mathcal{E}(X,Y)$ as*

$$f + \alpha = \varphi_{i_Y} \circ (f,\alpha)^t \tag{1.5.21}$$

where $f \in \mathcal{E}(X,Y)$, $\alpha \in \mathcal{E}(X, [Y,Y]_{Id_{\mathcal{E}}})$, $i_Y : [Y,Y]_{Id_{\mathcal{E}}} \rightarrowtail Y$ is the image of $c_2^Y : Id_{\mathcal{E}}(Y|Y) \to Y$ (see 1.3.5) and $\varphi_{i_Y} : Y \times [Y,Y]_{Id_{\mathcal{E}}} \to Y$ is the cooperator given in (1.5.3) (replacing z with i_Y).
Moreover, for $f, g : X \to Y$ two morphisms in $\mathcal{E}$, then we have $ab_Y \circ f = ab_Y \circ g$ if, and only if, there is a morphism $d : X \to [Y,Y]_{Id_{\mathcal{E}}}$ such that

$$g = f + d$$

Proof. In this case, $[Y,Y]_{Id_{\mathcal{E}}}$ is a central subobject of Y by 1.3.11. Then it is a direct consequence of 1.5.13, 1.5.21 and 1.5.20. $\square$

Notation 1.5.23. In (1.5.21), we often write $\varphi = \varphi_{i_Y}$.

Proposition 1.5.24. *Let $\mathcal{E}$ be a 2-step nilpotent category and let $X, X', Y, Y' \in \mathcal{E}$. Then for $f \in \mathcal{E}(Y,Y')$, $g \in \mathcal{E}(X,Y)$, $h \in \mathcal{E}(X',X)$ and $\alpha \in \mathcal{E}(X,[Y,Y]_{Id_{\mathcal{E}}})$ we have*

$$\begin{cases} f \circ (g+\alpha) = f \circ g + \gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha \\ (g+\alpha) \circ h = g \circ h + \alpha \circ h \end{cases}$$

where the action $+ : \mathcal{E}(X,Y) \times \mathcal{E}(X,[Y,Y]_{Id_{\mathcal{E}}}) \to \mathcal{E}(X,Y)$ is defined in (1.5.21).

Proof. For this we have the following equalities:

$$\begin{aligned}
f \circ \varphi_{i_Y} \circ (g \times \alpha) \circ \widehat{r_2^{Id_{\mathcal{E}}}} &= f \circ \varphi_{i_Y} \circ \widehat{r_2^{Id_{\mathcal{E}}}} \circ (g+\alpha)\,, \text{by naturality} \\
&= f \circ (id, i_Y) \circ (g+\alpha) \\
&= (f \circ g, f \circ i_Y \circ \alpha) \\
&= (f \circ g, i_{Y'} \circ \gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha) \\
&= (id, i_{Y'}) \circ \big((f \circ g) + (\gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha)\big) \\
&= \varphi_{i_{Y'}} \circ \widehat{r_2^{Id_{\mathcal{E}}}} \circ \big((f \circ g) + (\gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha)\big) \\
&= \varphi_{i_{Y'}} \circ \big((f \circ g) \times (\gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha)\big) \circ \widehat{r_2^{Id_{\mathcal{E}}}}
\end{aligned}$$

Hence we obtain

$$f \circ \varphi_{i_Y} \circ (g \times \alpha) = \varphi_{i_{Y'}} \circ \big((f \circ g) \times (\gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha)\big) \tag{1.5.22}$$

Then we get

$$\begin{aligned}
f \circ (g+\alpha) &= f \circ \varphi_{i_Y} \circ (g,\alpha)^t \\
&= f \circ \varphi_{i_Y} \circ (g \times \alpha) \circ \Delta_X^2 \\
&= \varphi_{i_{Y'}} \circ \big((f \circ g) \times (\gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha)\big) \circ \Delta_X^2 \\
&= \varphi_{i_{Y'}} \circ \big(f \circ g,\ \gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha\big)^t \\
&= f \circ g + \gamma_2^{Id_{\mathcal{E}}}(f) \circ \alpha\,,
\end{aligned}$$

as desired. Next we consider the equalities as follows:

$$(g+\alpha) \circ h = \varphi_{i_Y} \circ (g,\alpha)^t \circ h = \varphi_{i_Y} \circ (g \circ h, \alpha \circ h)^t = g \circ h + \alpha \circ h\,,$$

as desired. $\square$

1.6 Linear operads and algebras over such an operad

In this part, we recall the notion of (algebraic) operads wich can be found in chapter 5 of [26], and in chapter 1 of [9]. Intuitively, the notion of operad consists of a collection of objects $\mathcal{P}(r)$ (indexed in $\mathbb{N}$) in some monoidal category which collects (formal) operations with r variables. It is formally defined by a structure given by such a collections of objects $\mathcal{P}(r)$ together with composition product that model the composition of operations. In our context, we consider (right) linear operads in the whole thesis, namely the objects $\mathcal{P}(r)$ are abelian groups (or modules over a commutative ring) together with structure maps which are linear.

Definition 1.6.1. We consider the following definitions:

- We say that $\mathcal{P}$ is a right (resp. left) *linear operad* when it is an operad in the monoidal category of modules over a commutative ring $\Bbbk$, i.e. it consists of a sequence of $\Bbbk$-modules $\mathcal{P}(r)$, $r \in \mathbb{N}$, together with structure linear maps

$$\gamma_{k_1,\dots,k_n;n} : \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) \otimes \mathcal{P}(n) \to \mathcal{P}(k_1 + \dots + k_n) \tag{1.6.1}$$

$$\Big(\text{resp. } \gamma_{n;k_1,\dots,k_n} : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \dots + k_n)\Big) \tag{1.6.2}$$

  defined for all $k_1, \dots, k_n, n \in \mathbb{N}$, satisfying associativity relations expressed in the diagram of Figure 1.3 of [9]. We often omit the term *righ* and *left* for an operad. For $r \in \mathbb{N}$, we call $\mathcal{P}(r)$ the *k-th term* of $\mathcal{P}$ and the integer r its *arity*. An element $p \in \mathcal{P}(r)$ is called an *operation of arity* r.

- A (linear) operad $\mathcal{P}$ is *symmetric* if, for $r \in \mathbb{N}$, the $\Bbbk$-module $\mathcal{P}(r)$ has an additional (left) $\mathfrak{S}_r$-module structure and the structure linear maps in (1.6.1) verify the equivariance relations expressed in the diagram of Figure 1.1 of [9].

- A (linear) operad $\mathcal{P}$ is *unitary* when there is a unit morphism $\eta : \Bbbk \to \mathcal{P}(1)$ satisfying the unit relation given in the diagram of Figure 1.2 of [9]. In fact, it is equivalent to have an element $1_{\mathcal{P}} \in \mathcal{P}(1)$, called the *unit* of $\mathcal{P}$, such that certain appropriate axioms hold.

- A (linear) operad $\mathcal{P}$ is *reduced* if $\mathcal{P}(0)$ is the zero object.

- If $\mathcal{Q}$ is another such operad, a morphism $\phi : \mathcal{P} \to \mathcal{Q}$ of (linear) operads is a sequence of $\Bbbk$-module homomorphisms $\phi_r : \mathcal{P}(r) \to \mathcal{Q}(r)$, $r \in \mathbb{N}$, which commutes with the operad structure.

From now on, we consider $\Bbbk$ a commutative ring. One can see a unitary operad as a *monad*, i.e. a monoid in the category of endofunctors of the category of $\Bbbk$-modules endowed with its standard (strict) monoidal structure given by the composition of functors. This is a point of view already given in 5.2.1 of [26] by seing the operad $\mathcal{P}$ as an endofunctor of the category of $\Bbbk$-modules defined on objects by

$$\mathcal{P}(M) = \bigoplus_{n \in \mathbb{N}} M^{\otimes n} \otimes_{\Bbbk} \mathcal{P}(n) \tag{1.6.3}$$

for the case where $\mathcal{P}$ is not symmetric, or by

$$\mathcal{P}(M) = \bigoplus_{n \in \mathbb{N}} M^{\otimes n} \otimes_{\mathfrak{S}_n} \mathcal{P}(n) \tag{1.6.4}$$

for the case where $\mathcal{P}$ is symmetric. For $n \in \mathbb{N}$, the summand $M^{\otimes n} \otimes_{\mathfrak{S}_n} \mathcal{P}(n)$ is the quotient of $M^{\otimes n} \otimes_{\Bbbk} \mathcal{P}(n)$ by equivariance relations involving operations of arity n in the operad $\mathcal{P}$.

Remark 1.6.2. Now we recall that a linear operad $\mathcal{P}$ can be seen as an endofunctor defined on objects in (1.6.3) and (1.6.4) according to the structure of $\mathcal{P}$. In any case, it provides a left adjoint functor to the forgetful functor from the category of (nonsymmetric or symmetric) linear operads to the category of $\Bbbk$-modules. Moreover setting $M = \Bbbk$ in (1.6.4) resp. (1.6.3) determines the free (symmetric resp. nonsymmetric) $\mathcal{P}$-algebra of rank 1, denoted by $\mathcal{F}_{\mathcal{P}}$.

Now we provide a fundamental example of (left) linear operad given as follows:

Example 1.6.3. Let A be a $\Bbbk$-module. We write $Hom_{\Bbbk}(B, A)$ the set of $\Bbbk$-module homomorphisms from B to A, for any $\Bbbk$-module B. Then we consider the (left) *endomorphism operad* End_A associated with A that is a unitary symmetric linear operad given by a collection of hom-sets

$$End_A(r) = Hom_{\Bbbk}(A^{\otimes r}, A)$$

for $r \in \mathbb{N}$, together with the structure linear maps

$$\begin{array}{rcccl} \gamma_{n;k_1,\ldots,k_n} & : & End_A(n) \otimes End_A(k_1) \otimes \ldots \otimes End_A(k_n) & \longrightarrow & End_A(k_1 + \ldots + k_n) \\ & & f \otimes f_1 \otimes \ldots \otimes f_n & \longmapsto & f \circ \big(f_1 \otimes \ldots \otimes f_n\big) \,, \end{array}$$

for all $k_1, \ldots, k_n, n \in \mathbb{N}$, and the unit of this operad is the identity $id_A \in End_A(1) = Hom_{\Bbbk}(A, A)$. Moreover, for $r \in \mathbb{N}$, the abelian group $End_A(r)$ is canonically endowed with a $\mathfrak{S}_r$-module structure whose action is given by the permutations of the inputs $A^{\otimes r}$, and the operad End_A clearly satisfies the equivariance relations.

In the whole thesis, we are led to consider right linear operads and a certain type of them given as follows:

Definition 1.6.4. A (linear) operad $\mathcal{P}$ (endowed with any monoidal structure) is *n-step nilpotent* when any k-th term of $\mathcal{P}$ is trivial, for $k > n$.

We observe that this definition can be extended to operads in any pointed monoidal category. Now we can construct canonically a nilpotent operad from a given linear operad.

Definition 1.6.5. Let $\mathcal{P}$ be a linear operad. We define $Nil_n(\mathcal{P})$ the (linear) n-step nilpotent operad associated with $\mathcal{P}$ such that the collection of abelian groups $\{Nil_n(\mathcal{P})(k)\}_{k\in\mathbb{N}}$ is given by

$$Nil_n(\mathcal{P})(k) = \begin{cases} \mathcal{P}(k)\,, & \text{if } 1 \leqslant k \leqslant n \\ 0\,, & \text{otherwise} \end{cases}$$

endowed with the structure linear maps of $\mathcal{P}$ but truncated to order n.

Next we define algebras compatible with the structure of linear operad as follows:

Definition 1.6.6. Let $\mathcal{P}$ be a linear operad. A *$\mathcal{P}$-algebra* A is a $\Bbbk$-module endowed with a morphism $\phi_A : \mathcal{P} \to End_A$ of linear operads. Equivalently speaking, it consists of a $\Bbbk$-module A together with structure linear maps

$$\lambda^A_r : A^{\otimes r} \otimes \mathcal{P}(r) \to A\,,$$

for $r \in \mathbb{N}$, satisfying appropriate associativity relations. If moreover the operad $\mathcal{P}$ is supposed to be unitary (resp. symmetric), then the structure linear maps should verify the unitary (resp. symmetric) relations.

Notation 1.6.7. Let $\mathcal{P}$ be a linear operad. Then we denote by $Alg - \mathcal{P}$ the category of $\mathcal{P}$-algebras.

Remark 1.6.8. Let $\mathcal{P}$ be a linear operad. Then the category $Alg - \mathcal{P}$ can be seen as a semi-abelian variety associated with an algebraic theory containing an abelian group whose presentation is taken in such a way that associative relations and multilinearity relation for operations hold. If moreover $\mathcal{P}$ is unitary and symmetric, unitary and equivariance relations must hold.

Let $\mathcal{P}$ be a linear operad. Any $\mathcal{P}$-algebra can be clearly considered as an abelian group or a set. Hence it gives rise to two forgetful functors as follows:

Notation 1.6.9. We denote by $W : Alg - \mathcal{P} \to Ab$, respectively $V : Alg - \mathcal{P} \to Set$, the forgetful functor from the category of $\mathcal{P}$-algebras to the category of abelian groups, respectively the category of sets.

The next proposition says that the forgetful functor $W : Alg - \mathcal{P} \to Ab$ has the following preservation properties:

Proposition 1.6.10. *The forgetful functor $W : Alg - \mathcal{P} \to Ab$ preserves and reflects filtered colimits and coequalizers of reflexive graphs.*

Proof. For this, we consider the following commutative diagram:

$$\begin{tikzcd}
 & Set \\
Alg - \mathcal{P} \arrow[ur, "V"] \arrow[r, "W"'] & Ab \arrow[u, "U"']
\end{tikzcd}$$

As the category $Alg - \mathcal{P}$ is a semi-abelian variety (see 1.6.8), it follows that the forgetful functor $W : Alg - \mathcal{P} \to Set$ preserves and reflects filtered colimits and coequalizers of reflexive graphs by 1.1.7. It is the same for the forgetful functor $U : Ab \to Set$ because Ab is an abelian category (hence in particular a semi-abelian category). Hence it is straighforward that the (forgetful) functor $W : Alg - \mathcal{P} \to Ab$ preserves and reflects filtered colimits and coequalizers of reflexive pairs. $\square$

The proof of the proposition 1.6.10 can be slightly extented replacing the forgetful functor $W : Alg - \mathcal{P} \to Ab$ with a certain type of functors, as follows:

Proposition 1.6.11. *Let $\mathcal{C}$ be any category having filtered colimits and coequalizers of reflexive graphs, and let $F : \mathcal{C} \to Alg - \mathcal{P}$ be a functor. If the composite functors $W.F : \mathcal{C} \to Ab$ preserves filtered colimits and coequalizers of reflexive graphs, then F preserves these colimits.*

Proof. For this we consider the following commutative diagram:

$$\begin{tikzcd}
 & Ab \\
\mathcal{C} \arrow[ur, "W.F"] \arrow[r, "F"'] & Alg - \mathcal{P} \arrow[u, "W"']
\end{tikzcd}$$

By assumption, the composite functors $W.F$ preserves filtered colimits and coequalizers of reflexive graphs. As the forgetful functor $W : Alg - \mathcal{P} \to Ab$ reflects these colimits by 1.6.10, it follows that the functor F preserves also filtered colimits and coequalizers of reflexive graphs. $\square$

1.7 Nilpotent algebras over a linear operad

We assume that $\mathcal{P}$ is a linear operad as in 1.6.1. In this part, we determine nilpotent $\mathcal{P}$-algebras. First, for $n \geq 2$, we shall compute on objects the n-th cross-effect of the identity functor of the category $Alg - \mathcal{P}$ on objects. Then we provide commutators of any $\mathcal{P}$-algebra. Next we verify that taking a 2-step nilpotent $\mathcal{P}$-algebra is the same as taking an algebra over a certain 2-step nilpotent operad depending on $\mathcal{P}$. Let $A_1, \ldots, A_n$ n be $\mathcal{P}$-algebras and $i_k : A_k \rightarrowtail A_1 + \ldots + A_n$ be the injection of the k-th summand, for $1 \leqslant k \leqslant n$. To simplify, we write $S = A_1 + \ldots + A_n$.

Proposition 1.7.1. *The n-th cross-effect $Id_{Alg-\mathcal{P}}(A_1|\ldots|A_n)$ is generated as a subgroup of S by elements of the following form*

$$\lambda_k^S\Big(i_1(a_{1,1})\otimes\ldots\otimes i_1(a_{1,k_1})\otimes\ldots\otimes i_n(a_{n,1})\otimes\ldots\otimes i_n(a_{n,k_n})\otimes p\Big) \tag{1.7.1}$$

where k is a natural number such that $k\geq n$, $a_{j,l}\in A_j$ (for $1\leqslant j\leqslant n$ and $1\leqslant l\leqslant k_j$), $p\in\mathcal{P}(k)$ and $k_1,\ldots,k_n$ are natural numbers such that $k_1,\ldots,k_n\in\{1,\ldots k-n+1\}$ and $k_1+\ldots+k_n=k$.

Proof. Denote by P_n the assertion for a given $n\geqslant 2$. We prove this result by induction. First we prove that P_2 is true. Let $i_1:A_1\to A_1+A_2$ and $i_2:A_2\to A_1+A_2$ be respectively the injections of the first and the second summand, and $r_1:A_1+A_2\to A_1$, $r_2:A_1+A_2\to A_2$ their corresponding retractions. We first observe that A_1+A_2 may be seen as a (right) $\mathcal{P}(1)$-module generated by elements of the following form:

$$\lambda_k^{A_1+A_2}\big(i_1(a_{1,1})\otimes\ldots\otimes i_1(a_{1,k_1})\otimes i_2(a_{2,1})\otimes\ldots\otimes i_2(a_{2,k_2})\otimes p\big) \tag{1.7.2}$$

where $a_{1,i}\in A_1$ (for $1\leqslant i\leqslant k_1$), $a_{2,j}\in A_2$ (for $1\leqslant j\leqslant k_2$), $p\in\mathcal{P}(k)$ and k_1, k_2 are natural numbers such that $k_1+k_2=k$. By convention, the case $k_1=k$ (resp. $k_2=k$) and $k_2=0$ (resp. $k_1=0$) corresponds to the fact that there are only elements in A_1 (resp. A_2) in $\lambda_k^{A_1+A_2}$ above. Then there is a retraction $\rho_2^{Id}:A_1+A_2\to Id_{Alg-\mathcal{P}}(A_1|A_2)$ (as only a right $\mathcal{P}(1)$-module homomorphism) of the inclusion map $\iota_2^{Id}:Id_{Alg-\mathcal{P}}(A_1|A_2)\to A_1+A_2$ which is the kernel of the comparison morphism $\widehat{r_2^{Id_{Alg-\mathcal{P}}}}=(r_1,\ r_2)^t:A_1+A_2\to A_1\times A_2$, given by:

$$\forall x\in A_1+A_2\,,\qquad \rho_2^{Id}(x)=x-(i_1\circ r_1)(x)-(i_2\circ r_2)(x)$$

As $\rho_2^{Id}:A_1+A_2\to Id_{Alg-\mathcal{P}}(A_1|A_2)$ is surjective, the elements of $\rho_2^{Id}(A_1+A_2)$ generate $Id_{Alg-\mathcal{P}}(A_1|A_2)$ as a sub-$\mathcal{P}(1)$-module of A_1+A_2. It suffices to evaluate the morphism ρ_2^{Id} on the generators of A_1+A_2 given in (1.7.2). Hence we find

$$\rho_2^{Id}\Big(\lambda_k^{A_1+A_2}\big(i_1(a_1)\otimes\ldots\otimes i_1(a_k)\otimes p\big)\Big)=\rho_2^{Id}\Big(\lambda_k^{A_1+A_2}\big(i_2(b_1)\otimes\ldots\otimes i_2(b_k)\otimes p\big)\Big)=0$$

where $a_i\in A_1$ and $b_i\in A_2$, for $1\leqslant i\leqslant k$, and $p\in\mathcal{P}(k)$. However we have

$$\begin{aligned}&\rho_2^{Id}\Big(\lambda_k^{A_1+A_2}\Big(i_1(a_{1,1})\otimes\ldots\otimes i_1(a_{1,k_1})\otimes i_2(a_{2,1})\otimes\ldots\otimes i_2(a_{2,k_2})\otimes p\Big)\Big)\\ &=\lambda_k^{A_1+A_2}\Big(i_1(a_{1,1})\otimes\ldots\otimes i_1(a_{1,k_1})\otimes i_2(a_{2,1})\otimes\ldots\otimes i_2(a_{2,k_2})\otimes p\Big)\end{aligned}$$

if we assume that $k\geq 2$ and k_1, k_2 are natural numbers such that $1\leqslant k_1,k_2\leqslant k-1$ such that $k_1+k_2=k$. It proves that the property P_2 is verified.
Now, proceeding by induction, we assume that P_{n-1} is true , for a given $n\geqslant 3$. We aim at proving that the property P_n is also true. For this, we first consider the coproduct S to be $(A_1+A_2)+A_3+\ldots+A_n$ the coproduct of $n-1$ $\mathcal{P}$-algebras whose $\widetilde{i_1^{n-1}}=(i_1^n,\ i_2^n):A_1+A_2\to(A_1+A_2)+A_3+\ldots+A_n$ is the injection of the first summand and, for $2\leqslant p\leqslant n-1$, $\widetilde{i_p^{n-1}}=i_{p+1}^n:A_{p+1}\to(A_1+A_2)+A_3+\ldots+A_n$ the injection of the $(p+1)$-th summand of S (seen as the injection of the p-th summand of $(A_1+A_2)+A_3+\ldots+A_n$). Then we use the inductive definition of the cross-effect of the identity functor $Id_{Alg-\mathcal{P}}:Alg-\mathcal{P}\to Alg-\mathcal{P}$ given in 2.20 of [14] as follows:

$$Id_{Alg-\mathcal{P}}(A_1|\ldots|A_n)=Id_{Alg-\mathcal{P}}(-|A_3|\ldots|A_n)(A_1|A_2)$$

To simplify, we write $G=Id_{Alg-\mathcal{P}}(-|A_3|\ldots|A_n)$ which is a functor with domain and range $Alg-\mathcal{P}$. There is a retraction $\rho_2^G:G(A_1+A_2)\to G(A_1|A_2)$ (as a $\mathcal{P}(1)$-module homomorphism) of the kernel

$\iota_2^G : G(A_1|A_2) \to G(A_1 + A_2)$ of the comparison morphism $\widehat{r_2^G} = (G(r_1),\ G(r_2))^t : G(A_1 + A_2) \to G(A_1) \times G(A_2)$; this retraction is given by

$$\forall x \in G(A_1 + A_2)\,, \quad \rho_2^G(x) = x - G(i_1 \circ r_1)(x) - G(i_2 \circ r_2)(x)$$

As $\rho_2^G : G(A_1+A_2) \to G(A_1|A_2)$ is surjective, the elements of $\rho_2^G\big(G(A_1+A_2)\big)$ generate $G(A_1|A_2) = Id_{Alg-\mathcal{P}}(A_1|\ldots|A_n)$ as a $\mathcal{P}(1)$-module. Then it suffices to evaluate ρ_2^G on the generators of $G(A_1+A_2)$ given by induction, namely $G(A_1+A_2) = Id_{Alg-\mathcal{P}}(A_1+A_2|\ldots|A_n)$ is generated (as a $\mathcal{P}(1)$-module) by elements of the following form:

$$x_k^S = \lambda_k^S\Big(\bigotimes_{i=1}^{k_1} \widetilde{i_1^{n-1}}(s_i) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} \widetilde{i_l^{n-1}}(a_{\alpha,\beta}) \otimes p\Big)$$

where k is a natural number such that $k \geq n-1$, $s_i \in A_1 + A_2$ (for $1 \leqslant i \leqslant k_1$), $a_{j,l} \in A_{j+1}$ (for $2 \leqslant j \leqslant n-1$ and $1 \leqslant l \leqslant k_j$), $p \in \mathcal{P}(k)$ and $k_1, \ldots, k_{n-1}$ are natural numbers such that $k_1, \ldots, k_{n-1} \in \{1, \ldots k-n+2\}$ and $k_1 + \ldots + k_{n-1} = k$. We know that, for $1 \leqslant i \leqslant k_1$, we have

$$s_i = \big(i_1 \circ r_1\big)(s_i) + \big(i_2 \circ r_2\big)(s_i) + \rho_2^{Id}(s_i)$$

where $\rho_2^{Id} : A_1 + A_2 \to Id_{Alg-\mathcal{P}}(A_1|A_2)$ is the map given above. By the previous argument, we know that, for $1 \leqslant i \leqslant k_1$, $\rho_2^{Id}(s_i) \in Id_{Alg-\mathcal{P}}(A_1|A_2)$. Then we have

$$\begin{aligned}
x_k^S &= \lambda_k^S\Big(\bigotimes_{i=1}^{k_1} \widetilde{i_1^{n-1}}\Big((i_1 \circ r_1)(s_i) + (i_2 \circ r_2)(s_i) + \rho_2^{Id}(s_i)\Big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p\Big) \\
&= \lambda_k^S\Big(\bigotimes_{i=1}^{k_1} \Big(i_1^n\big(r_1(s_i)\big) + i_2^n\big(r_2(s_i)\big) + \widetilde{i_1^{n-1}}\big(\rho_2^{Id}(s_i)\big)\Big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p\Big) \\
&= \lambda_k^S\Big(\bigotimes_{i=1}^{k_1} i_1^n\big(r_1(s_i)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p\Big) \\
&\qquad + \lambda_k^S\Big(\bigotimes_{i=1}^{k_1} i_2^n\big(r_2(s_i)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p\Big) \\
&\qquad + \lambda_k^S\Big(\bigotimes_{i=1}^{k_1} \widetilde{i_1^{n-1}}\big(\rho_2^{Id}(s_i)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p\Big) \ (\clubsuit) \\
&\qquad + \text{ sum of mixed terms } (\spadesuit)
\end{aligned}$$

It is not necessary to make the sum explicit but it is interesting to see in which form are the mixed terms:

$$\left\{\begin{array}{c}
\lambda_k^S\Big(\bigotimes_{i=1}^{k_1^1} i_1^n\big(r_1(s_i)\big) \otimes \bigotimes_{j=1}^{k_1^2} i_2^n\big(r_2(s_j)\big) \otimes \bigotimes_{l=1}^{k_1^3} \widetilde{i_1^{n-1}}\big(\rho_2^{Id}(s_l)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p_1\Big) \\
\lambda_k^S\Big(\bigotimes_{i=1}^{k_1^1} i_1^n\big(r_1(s_i)\big) \otimes \bigotimes_{j=1}^{k_1^2} \widetilde{i_1^{n-1}}\big(\rho_2^{Id}(s_j)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p_2\Big) \\
\lambda_k^S\Big(\bigotimes_{i=1}^{k_1^1} i_2^n\big(r_2(s_i)\big) \otimes \bigotimes_{j=1}^{k_1^2} \widetilde{i_1^{n-1}}\big(\rho_2^{Id}(s_j)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p_3\Big) \\
\lambda_k^S\Big(\bigotimes_{i=1}^{k_1^1} i_1^n\big(r_1(s_i)\big) \otimes \bigotimes_{j=1}^{k_1^2} i_2^n\big(r_2(s_i)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} i_{l+1}^n(a_{\alpha,\beta}) \otimes p_4\Big)
\end{array}\right.$$

where $k_1^l \in \mathbb{N}^*$ and $p_i \in \mathcal{P}(k)$, for $i = 1, \ldots, 4$ and $l : 1, \ldots, 3$. Moreover we know that $\rho_2^{Id}(s_i) \in Id_{Alg-\mathcal{P}}(A_1|A_2)$ so that it can be expressed as a sum of the elements as in (1.7.3) (for $n = 2$). Hence (♣) and (♠) are sums of elements having the form of those in (1.7.3). In addition, we have

$$\rho_2^G(x_k^S) = \lambda_k^S\Big(\bigotimes_{i=1}^{k_1} \widetilde{i_1^{n-1}}\big(\rho_2^{Id}(s_i)\big) \otimes \bigotimes_{\alpha=2}^{n-1}\bigotimes_{\beta=1}^{k_\alpha} \widetilde{i_l^{n-1}}(a_{\alpha,\beta}) \otimes p\Big) \;+\; (\spadesuit)$$

Consequently, the property P_n is true. This proves the result. □

A direct consequence of 1.7.1 is to find a generating set of commutators (in the sense of 1.3.1) in the category of $\mathcal{P}$-algebras as an abelian group.

Proposition 1.7.2. *For a $\mathcal{P}$-algebra A, the n-weighted commutator $\gamma_n^{Id_{Alg-\mathcal{P}}}(A) = [A, \ldots, A]_{Id_{Alg-\mathcal{P}}}$ is generated as a subgroup of A by elements of the following form*

$$\lambda_k^A\Big(a_1 \otimes \ldots \otimes a_k \otimes p\Big) \tag{1.7.3}$$

where k is a natural number such that $k \geq n$, $a_i \in A$ for $i = 1, \ldots, k$.

Proof. Let $i_l^n : A \rightarrowtail A^{+n}$ be the injection of the l-th summand, for $1 \leqslant l \leqslant n$. By 1.7.1, the n-th cross-effect $Id_{Alg-\mathcal{P}}(A| \ldots |A)$ is the $\mathcal{P}$-algebra generated by elements of the following form:

$$\lambda_k^S\Big(i_1^n(a_{1,1}) \otimes \ldots \otimes i_1^n(a_{1,k_1}) \otimes \ldots \otimes i_n^n(a_{n,1}) \otimes \ldots \otimes i_n^n(a_{n,k_n}) \otimes p\Big)$$

where $S = A^{+n}$, k is a natural number such that $k \geq n$, $a_{j,l} \in A$ (for $1 \leqslant j \leqslant n$ and $1 \leqslant l \leqslant k_j$), $p \in \mathcal{P}(k)$ and $k_1, \ldots, k_n$ are natural numbers such that $k_1, \ldots, k_n \in \{1, \ldots k-n+1\}$ and $k_1 + \ldots + k_n = k$. Hence we get

$$\begin{aligned}
&c_n^A\Big(\lambda_k^S\big(i_1^n(a_{1,1}) \otimes \ldots \otimes i_1^n(a_{1,k_1}) \otimes \ldots \otimes i_n^n(a_{n,1}) \otimes \ldots \otimes i_n^n(a_{n,k_n}) \otimes p\big)\Big) \\
&= (\nabla_A^n \circ \iota_2^{Id_{Alg-\mathcal{P}}})\Big(\lambda_k^S\big(i_1^n(a_{1,1}) \otimes \ldots \otimes i_1^n(a_{1,k_1}) \otimes \ldots \otimes i_n^n(a_{n,1}) \otimes \ldots \otimes i_n^n(a_{n,k_n}) \otimes p\big)\Big) \\
&= \nabla_A^n\Big(\lambda_k^S\big(i_1^n(a_{1,1}) \otimes \ldots \otimes i_1^n(a_{1,k_1}) \otimes \ldots \otimes i_n^n(a_{n,1}) \otimes \ldots \otimes i_n^n(a_{n,k_n}) \otimes p\big)\Big) \\
&= \lambda_k^A\big(a_{1,1} \otimes \ldots \otimes a_{1,k_1} \otimes \ldots \otimes a_{n,1} \otimes \ldots \otimes a_{n,k_n} \otimes p\big)
\end{aligned}$$

because $\nabla_A^n : A^{+n} \to A$ is the unique homomorphism of $\mathcal{P}$-algebras such that, for $1 \leqslant l \leqslant n$, $\nabla_A^n \circ i_l^n = id$. Since by 1.3.1 the n-weighted commutator $\gamma_n^{Id_{Alg-\mathcal{P}}}(A)$ of A is the image of $c_n^A : Id_{Alg-\mathcal{P}}(A| \ldots |A) \to A$, it concludes the proof. □

Then it is now possible to characterize nilpotent $\mathcal{P}$-algebras as follows:

Corollary 1.7.3. *Let A be a $\mathcal{P}$-algebra. Then A is an n-step nilpotent $\mathcal{P}$-algebra if, and only if, its structure linear maps $\lambda_k^A : A^{\otimes k} \otimes \mathcal{P}(k) \to A$ are trivial for $k > n$.*

Proof. It is a direct consequence of 1.7.2. □

Corollary 1.7.4. *Let A be a $\mathcal{P}$-algebra. If $\mathcal{P}$ is a 2-step nilpotent operad (see 1.6.4), then the 2-weighted commutator $[A, A]_{Id_{Alg-\mathcal{P}}}$, defined in 1.3.1, is*

$$[A, A]_{Id_{Alg-\mathcal{P}}} = Im\big(\lambda_2^A : A^{\otimes 2} \otimes \mathcal{P}(2) \to A\big)$$

where λ_2^A is a structure linear map of A.

Proof. It also is a direct consequence of 1.7.3. □

Corollary 1.7.5. *The abelian core $Ab(Alg-\mathcal{P})$ is exactly the category of (right) $\mathcal{P}(1)$-modules.*

Proof. Let A be an abelian object in $Alg-\mathcal{P}$. It means that $\gamma_2^{Id_{Alg-\mathcal{P}}}(A)$ is trivial by 1.3.1. By 1.7.3, it implies that the structure linear maps $\lambda_k^A : A \otimes \mathcal{P}(k) \to A$ are trivial, for $k \geqslant 2$. □

Now it is possible to determine an explicit expression of the abelianization functor $Ab^{Alg-\mathcal{P}}$: $Alg-\mathcal{P} \to Ab(Alg-\mathcal{P}) = Mod_{\mathcal{P}(1)}$ on objects and on morphisms.

Notation 1.7.6. We consider the following notations, as follows:

- For a $\mathcal{P}$-algebra A, we denote by A^2 the ideal of A consists of elements of the form $\lambda_2^A(a_1 \otimes a_2 \otimes p_2)$ where $a_1, a_2 \in A$, $p_2 \in \mathcal{P}(2)$ and $\lambda_2^A : A^{\otimes 2} \otimes \mathcal{P}(2) \to A$ is the structure linear map of A encoding binary bilinear operations in A parametrized by $\mathcal{P}(2)$. Moreover we denote by $\overline{A}$ the quotient of A by A^2.

- For $a \in A$, we write $\overline{a}$ the equivalence class of A in $\overline{A}$.

Corollary 1.7.7. *The abelianization functor $Ab^{Alg-\mathcal{P}} : Alg-\mathcal{P} \to Ab(Alg-\mathcal{P}) = Mod_{\mathcal{P}(1)}$ is such that*

- *On objects, for a $\mathcal{P}$-algebra A, $Ab^{Alg-\mathcal{P}}(A) = A^{ab} = \overline{A}$ the quotient of A by the ideal A^2 (see 1.7.6).*

- *On morphisms, for a morphism $f : A \to B$ of $\mathcal{P}$-algebras, $Ab^{Alg-\mathcal{P}}(f) : A^{ab} = \overline{A} \to B^{ab} = \overline{B}$ is the unique canonical factorization.*

The next result now says that taking an n-step nilpotent $\mathcal{P}$-algebra amounts to picking an algebra over the n-step nilpotent operad $Nil_n(\mathcal{P})$.

Proposition 1.7.8. *Let $\mathcal{P}$ be a (linear) operad as in 1.7.3. We have the following isomorphism of categories*

$$Nil_n(Alg-\mathcal{P}) \cong Alg - Nil_n(\mathcal{P})$$

where $Nil_n(\mathcal{P})$ is the n-step nilpotent linear operad defined in 1.6.5.

Proof. It is an immediate consequence of 1.7.3. □

1.8 Binary coproducts of 2-step nilpotent algebras over a linear operad

Let $\mathcal{P}$ be a linear symmetric unitary operad as in 1.6.1 supposed here 2-step nilpotent (see 1.6.4). In this part, we give an explicit expression of binary coproducts in the category of $\mathcal{P}$-algebras. We first give an explicit expression of the free $\mathcal{P}$-algebra of rank 1, denoted by $\mathcal{F}_{\mathcal{P}}$. As $\mathcal{P}$ is a 2-step nilpotent operad, the free $\mathcal{P}$-algebra of rank 1 has the following expression:

$$\bigoplus_{n=1}^{+\infty} \mathbb{Z}^{\otimes n} \otimes_{\mathfrak{S}_n} \mathcal{P}(n) = \big(\mathbb{Z} \otimes \mathcal{P}(1)\big) \oplus \big(\mathbb{Z}^{\otimes 2} \otimes_{\mathfrak{S}_2} \mathcal{P}(2)\big) \cong \mathcal{P}(1) \oplus \mathcal{P}(2)_{\mathfrak{S}_2} \tag{1.8.1}$$

where $\mathcal{P}(2)_{\mathfrak{S}_2}$ is the set of the coinvariants by the action of the symmetric group $\mathfrak{S}_2$ on $\mathcal{P}(2)$ present in the structure of the operad $\mathcal{P}$.

Notation 1.8.1. We denote by $q : \mathcal{P}(2) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ the canonical quotient map. If $p_2 \in \mathcal{P}(2)$, we write $q(p_2) = \overline{p_2}$ to denote the equivalence class of p_2 in the set of coinvariants $\mathcal{P}(2)_{\mathfrak{S}_2}$.

Remark 1.8.2. By (1.8.1), we here consider $\mathcal{F}_\mathcal{P}$ to be $\mathcal{P}(1) \oplus \mathcal{P}(2)_{\mathfrak{S}_2}$ together with its structure linear maps given by:

- $\lambda_1^{\mathcal{F}_\mathcal{P}} : \mathcal{F}_\mathcal{P} \otimes \mathcal{P}(1) \to \mathcal{F}_\mathcal{P}$ is given by

$$\lambda_1^{\mathcal{F}_\mathcal{P}}\big((p_1, \overline{p_2}) \otimes p_1'\big) = \big(\gamma_{1;1}(p_1 \otimes p_1'),\ \overline{\gamma_{2;1}(p_2 \otimes p_1')}\big) \tag{1.8.2}$$

  where $p_1, p_1' \in \mathcal{P}(1)$, $p_2 \in \mathcal{P}(2)$.

- $\lambda_2^{\mathcal{F}_\mathcal{P}} : \mathcal{F}_\mathcal{P}^{\otimes 2} \otimes \mathcal{P}(2) \to \mathcal{F}_\mathcal{P}$ is defined by

$$\lambda_2^{\mathcal{F}_\mathcal{P}}\big((p_1^1, \overline{p_2^1}) \otimes (p_1^2, \overline{p_2^2}) \otimes p_2\big) = \big(0,\ \overline{\gamma_{1,1;2}(p_1^1 \otimes p_1^2 \otimes p_2)}\big) \tag{1.8.3}$$

  where $p_1^k \in \mathcal{P}(1)$, $p_2^k, p_2 \in \mathcal{P}(2)$ and $k = 1, 2$.

Notation 1.8.3. We set $0_{\mathcal{F}_\mathcal{P}} = (0, \overline{0})$.

Let A be a $\mathcal{P}$-algebra and let $ev_{(id,\overline{0})} : Alg-\mathcal{P}(\mathcal{F}_\mathcal{P}, A) \to A$ be the canonical isomorphism that assigns each morphism with source $\mathcal{F}_\mathcal{P}$ and target A to its evaluation to the basis element $(id, \overline{0})$ of $\mathcal{F}_\mathcal{P}$. We observe that its inverse is given by:

$$ev_{(id,\overline{0})}^{-1}(a)(p_1, \overline{p_2}) = \lambda_1^A\big(a \otimes p_1\big) + \lambda_2^A\big(a \otimes a \otimes p_2\big) \tag{1.8.4}$$

where $p_1 \in \mathcal{P}(1)$, $p_2 \in \mathcal{P}(2)$ and $a \in A$. Now we observe that the ideal $(\mathcal{F}_\mathcal{P})^2$ of $\mathcal{F}_\mathcal{P}$ consists of elements of the form $(0, \overline{p_2})$ where $p_2 \in \mathcal{P}(2)$. Consequently, $(\mathcal{F}_\mathcal{P})^2$ is isomorphic to $\mathcal{P}(2)_{\mathfrak{S}_2}$ in the abelian category $Mod_{\mathcal{P}(1)}$. Let $i_2 : \mathcal{P}(2)_{\mathfrak{S}_2} \rightarrowtail \mathcal{F}_\mathcal{P}$ and $\pi_1 : \mathcal{F}_\mathcal{P} \to \mathcal{P}(1)$ be respectively the injection of the second summand and the projection onto the first one. Then we have clearly the following canonical short exact sequence in $Mod_{\mathcal{P}(1)}$:

$$0 \to \mathcal{P}(2)_{\mathfrak{S}_2} \overset{i_2}{\rightarrowtail} \mathcal{F}_\mathcal{P} \overset{\pi_1}{\twoheadrightarrow} \mathcal{P}(1) \to 0$$

This implies that there is an isomorphism of $\mathcal{P}(1)$-modules as follows:

$$Ab^{Alg-\mathcal{P}}(\mathcal{F}_\mathcal{P}) = \overline{\mathcal{F}_\mathcal{P}} \cong \mathcal{P}(1) \tag{1.8.5}$$

Remark 1.8.4. It is straightforward to see that the (right) $\mathcal{P}(1)$-module $\overline{\mathcal{F}_\mathcal{P}}$ consists of elements of the form $\overline{(p_1, \overline{0})}$, with $p_1 \in \mathcal{P}(1)$.

Then we give an explicit expression of binay coproducts in the category of $\mathcal{P}$-algebras (that is here specific for the case where $\mathcal{P}$ is a 2-step nilpotent operad) as follows:

Proposition 1.8.5. *Let A and B be two $\mathcal{P}$-algebras. Then the coproduct $A + B$ is the abelian group $A \times B \times \big(\overline{A} \otimes \overline{B} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\big)$ together with its structure linear maps defined below:*

- $\lambda_1^{A+B} : (A+B) \otimes \mathcal{P}(1) \to A + B$ *is given by*

$$\lambda_1^{A+B}\big((a,\ b,\ \overline{a'} \otimes \overline{b'} \otimes p_2) \otimes p_1\big) \;=\; \Big(\lambda_1^A(a \otimes p_1),\ \lambda_1^B(b \otimes p_1),\ \overline{a'} \otimes \overline{b'} \otimes \gamma_{2;1}(p_2 \otimes p_1)\Big)$$

  where $a, a' \in A$, $b, b' \in B$, $p_1 \in \mathcal{P}(1)$ and $p_2 \in \mathcal{P}(2)$.

- $\lambda_2^{A+B} : (A+B)^{\otimes 2} \otimes \mathcal{P}(2) \to A + B$ *is defined by*

$$\begin{aligned}&\lambda_2^{A+B}\big((a_1 \otimes b_1 \otimes u_1) \otimes (a_2, b_2, u_2) \otimes p_2\big)\\ &= \Big(\lambda_2^A(a_1 \otimes a_2 \otimes p_2),\ \lambda_2^B(b_1 \otimes b_2 \otimes p_2),\ \overline{a_1} \otimes \overline{b_2} \otimes p_2 \ +\ \overline{a_2} \otimes \overline{b_1} \otimes (p_2.t)\Big)\end{aligned}$$

  where $a_1, a_2 \in A$, $b_1, b_2 \in B$, $u_1, u_2 \in \overline{A} \otimes \overline{B} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)$ and $p_2 \in \mathcal{P}(2)$.

and the injections of the first and the second summands are given by $i_A : A \to A+B$, $a \mapsto (a,0,0)$ *and* $i_B : B \to A+B$, $b \mapsto (0,b,0)$. *Here, for a* $\mathcal{P}$*-algebra* A, $\overline{A} = Ab^{Alg-\mathcal{P}}(A)$ *is the quotient of* A *by the ideal* A^2 *(see 1.7.6 and 1.7.7)*

Proof. It is straightforward to check that $A+B$ defined above together with its structure linear maps is a $\mathcal{P}$-algebra. Then we need to prove that $A+B$ verifies the universal property of the coproduct. Let $f : A \to C$ and $g : B \to C$ be two morphisms in $Alg - \mathcal{P}$. Then we define the morphism $h : A+B \to C$ by

$$h(a,\ b,\ \overline{a'} \otimes \overline{b'} \otimes p_2) \ = \ f(a) + g(b) + \lambda_2^C(f(a') \otimes g(b') \otimes p_2) \tag{1.8.6}$$

where $a, a' \in A$, $b, b' \in B$ and $p_2 \in \mathcal{P}(2)$. It is easy to check that $h \circ i_A = f$ and $h \circ i_B = g$. Now we prove that $h : A+B \to C$ is a morphism in $Alg - \mathcal{P}$. For this, we consider the two following diagrams.

- We verify that the following diagram commutes:

$$\begin{tikzcd}
(A+B)\otimes\mathcal{P}(1) \arrow[r, "h\otimes id"] \arrow[d, "\lambda_1^{A+B}"'] & C\otimes\mathcal{P}(1) \arrow[d, "\lambda_1^C"] \\
A+B \arrow[r, "h"] & C
\end{tikzcd}$$

  We have

$$\begin{aligned}
&h \circ \lambda_1^{A+B}\big((a,b,\overline{a'}\otimes\overline{b'}\otimes p_2)\otimes p_1\big)\\
&= h\big(\lambda_1^A(a\otimes p_1), \lambda_1^B(b\otimes p_1), \overline{a'}\otimes\overline{b'}\otimes\gamma_{2;1}(p_2\otimes p_1)\big)\\
&= f(\lambda_1^A(a\otimes p_1)) + g(\lambda_1^B(b\otimes p_1)) + \lambda_1^C\big(f(a')\otimes g(b')\otimes\gamma_{2;1}(p_2\otimes p_1)\big)\\
&= \lambda_1^C(f(a)\otimes p_1) + \lambda_1^C(g(b)\otimes p_1) + \lambda_1^C\big(\lambda_2^C(f(a')\otimes g(b')\otimes p_2)\otimes p_1\big)\\
&= \lambda_1^C\Big(\big(f(a)+g(b)+\lambda_2^C(f(a')\otimes g(b')\otimes p_2)\big)\big)\otimes p_1\Big)\\
&= \lambda_1^C \circ (h\otimes id)\big((a,b,\overline{a'}\otimes\overline{b'}\otimes p_2)\otimes p_1\big)
\end{aligned}$$

  where $a, a' \in A$, $b, b' \in B$, $p_1 \in \mathcal{P}(1)$ and $p_2 \in \mathcal{P}(2)$.

- We prove that the following diagram commutes:

$$\begin{tikzcd}
(A+B)^{\otimes 2}\otimes\mathcal{P}(2) \arrow[r, "h^{\otimes 2}\otimes id"] \arrow[d, "\lambda_2^{A+B}"'] & C^{\otimes 2}\otimes\mathcal{P}(2) \arrow[d, "\lambda_2^C"] \\
A+B \arrow[r, "h"] & C
\end{tikzcd}$$

We have

$$\begin{aligned}
&h \circ \lambda_2^C((a_1,b_1,\overline{a_1'}\otimes\overline{b_1'}\otimes p_2^1)\otimes(a_2,b_2,\overline{a_2'}\otimes\overline{b_2'}\otimes p_2^2)\otimes p_2)\\
&= h\big(\lambda_2^A(a_1\otimes a_2\otimes p_2), \lambda_2^B(b_1\otimes b_2\otimes p_2), \overline{a_1}\otimes\overline{b_2}\otimes p_2 + \overline{a_2}\otimes\overline{b_1}\otimes(p_2.t)\big)\big)\\
&= f(\lambda_2^A(a_1\otimes a_2\otimes p_2)) + g(\lambda_2^B(b_1\otimes b_2\otimes p_2)) + \lambda_2^C(f(a_1)\otimes g(b_2)\otimes p_2) + \lambda_2^C(f(a_2)\otimes g(b_1)\otimes(p_2.t))\\
&= \lambda_2^C(f(a_1)\otimes f(a_2)\otimes p_2) + \lambda_2^C(g(b_1)\otimes g(b_2)\otimes p_2) + \lambda_2^C(f(a_1)\otimes g(b_2)\otimes p_2) + \lambda_2^C(g(b_1)\otimes f(a_2)\otimes p_2)
\end{aligned}$$

Moreover we obtain

$$\lambda_2^C \circ (h^{\otimes 2} \otimes id)\Big((a_1, b_1, \overline{a_1'} \otimes \overline{b_1'} \otimes p_2^1)) \otimes (a_2, b_2, \overline{a_2'} \otimes \overline{b_2'} \otimes p_2^2) \otimes p_2\Big)$$

$$= \lambda_2^C\Big(\big(f(a_1) + g(b_1) + \lambda_2^C(f(a_1') \otimes g(b_1') \otimes p_2^1)\big) \otimes \big(f(a_2) + g(b_2) + \lambda_2^C(f(a_2') \otimes g(b_2'))\big) \otimes p_2^2\Big)$$

$$= \lambda_2^C(f(a_1) \otimes f(a_2) \otimes p_2) + \lambda_2^C(g(b_1) \otimes g(b_2) \otimes p_2) + \lambda_2^C(f(a_1) \otimes g(b_2) \otimes p_2) + \lambda_2^C(g(b_1) \otimes f(a_2) \otimes p_2)$$

because the other terms of the sum disappear as they generate ternary linear operations that are trivial, where $a_1, a_2, a_1', a_2' \in A$, $b_1, b_2, b_1', b_2' \in B$ and $p_2^1, p_2^2, p_2 \in \mathcal{P}(2)$. □

Let $r_A : A + B \to A$ and $r_B : A + B \to B$ be respectively the two retractions onto the first and the second summand. Then we give the second cross-effect of the identity functor of $\mathcal{P}$-algebras as follows:

Corollary 1.8.6. *Let A and B be two $\mathcal{P}$-algebras, then we have*

$$Id_{Alg-\mathcal{P}}(A \mid B) = \overline{A} \otimes \overline{B} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)$$

and the kernel $\iota_2^{Id} : Id_{Alg-\mathcal{P}}(A \mid B) \to A + B$ of the comparison morphism $r_2^{\widehat{Id_{Alg-\mathcal{P}}}} = (r_A,\ r_B)^t : A + B \to A \times B$ is given by $\iota_2^{Id}(u) = (0,\ 0,\ u)$, where $u \in Id_{Alg-\mathcal{P}}(A \mid B)$.

Remark 1.8.7. Let A be a $\mathcal{P}$-algebra. We see that the (right) $\mathcal{P}(1)$-module $Id_{Alg-\mathcal{P}}(A|A)$ is endowed with the involution $T \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} t$ where $T : (\overline{A})^{\otimes 2} \to (\overline{A})^{\otimes 2}$ is the canonical switch.

Notation 1.8.8. For a $\mathcal{P}$-algebra A, we denote by $Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2}$ the coinvariants set and by $\pi : Id_{Alg-\mathcal{P}}(A|A) \to Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2}$ the canonical quotient map.

By 1.8.5 (or by 1.8.4), we get the following isomorphism of $\mathcal{P}(1)$-modules:

$$Id_{Alg-\mathcal{P}}(\mathcal{F}_{\mathcal{P}}|\mathcal{F}_{\mathcal{P}}) = \overline{\mathcal{F}_{\mathcal{P}}} \otimes \overline{\mathcal{F}_{\mathcal{P}}} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \cong \mathcal{P}(1) \otimes \mathcal{P}(1) \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \cong \mathcal{P}(2)$$

because $\mathcal{P}(2)$ is a $(\mathcal{P}(1) \otimes \mathcal{P}(1))$-$\mathcal{P}(1)$-bimodule.

Remark 1.8.9. It permits us to consider that the binary coproduct $\mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}}$ has the following expression

$$\mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}} = \mathcal{F}_{\mathcal{P}} \times \mathcal{F}_{\mathcal{P}} \times \mathcal{P}(2)$$

endowed with the following structure linear maps:

- $\lambda_1^{\mathcal{F}_{\mathcal{P}}^{+2}} : (\mathcal{F}_{\mathcal{P}}^{+2}) \otimes \mathcal{P}(1) \to \mathcal{F}_{\mathcal{P}}^{+2}$ is given by

$$\lambda_1^{\mathcal{F}_{\mathcal{P}}^{+2}}\Big((p_1^1, \overline{p_2^1}),\ (p_1^2, \overline{p_2^2}),\ p_2) \otimes p_1\Big)$$

$$= \Big(\big(\gamma_{1;1}(p_1^1 \otimes p_1), \overline{\gamma_{2;1}(p_2^1 \otimes p_1)}\big),\ \big(\gamma_{1;1}(p_1^2 \otimes p_1), \overline{\gamma_{2;1}(p_2^2 \otimes p_1)}\big),\ \gamma_{2;1}(p_2 \otimes p_1))\Big)$$

  where $p_1, p_1^k \in \mathcal{P}(1)$, $p_2^k, p_2 \in \mathcal{P}(2)$ with $k = 1, 2$.

- $\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}} : (\mathcal{F}_{\mathcal{P}}^{+2})^{\otimes 2} \otimes \mathcal{P}(2) \to \mathcal{F}_{\mathcal{P}}^{+2}$ is defined by

$$\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}\Big(\big((p_1^{1,1}, \overline{p_2^{1,1}}), (p_1^{1,2}, \overline{p_2^{1,2}}), u_1\big) \otimes \big((p_1^{2,1}, \overline{p_2^{2,1}}), (p_1^{2,2}, \overline{p_2^{2,2}}), u_2\big) \otimes p_2\Big)$$

$$= \Big(\lambda_2^{\mathcal{F}_{\mathcal{P}}}\big((p_1^{1,1}, \overline{p_2^{1,1}}) \otimes (p_1^{2,1}, \overline{p_2^{2,1}}) \otimes p_2\big),\ \lambda_2^{\mathcal{F}_{\mathcal{P}}}\big((p_1^{1,2}, \overline{p_2^{1,2}}) \otimes (p_1^{2,2}, \overline{p_2^{2,2}}) \otimes p_2\big),$$

$$\gamma_{1,1;2}\big(p_1^{1,1} \otimes p_1^{2,2} \otimes p_2\big)\ +\ \gamma_{1,1;2}\big(p_1^{2,1} \otimes p_1^{1,2} \otimes (p_2.t)\big)\Big)$$

  where $p_1^{i,j} \in \mathcal{P}(1)$ and $p_2^{i,j}, p_2, u_1, u_2 \in \mathcal{P}(2)$, for $i, j = 1, 2$.

and the injections of the first and the second summands are given by $i_1 : \mathcal{F}_{\mathcal{P}} \to \mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}}$, $(p_1, \overline{p_2}) \mapsto \big((p_1, \overline{p_2}), 0, 0\big)$ and $i_2 : \mathcal{F}_{\mathcal{P}} \to \mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}}$, $(p_1, \overline{p_2}) \mapsto \big(0, (p_1, \overline{p_2}), 0\big)$.

Then it is possible to know the generating set of $\overline{\mathcal{F}_{\mathcal{P}}^{+2}}$ as a (right) $\mathcal{P}(1)$-module.

Remark 1.8.10. The (right) $\mathcal{P}(1)$-module $\overline{\mathcal{F}_{\mathcal{P}}^{+2}}$ consists of elements of the form $\overline{\big((p_1^1, \overline{0}), (p_1^2, \overline{0}), 0\big)}$, where $p_1^1, p_1^2 \in \mathcal{P}(1)$. This is due to the fact that we have

$$\begin{cases} \lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}\Big(\big((id, \overline{0}), 0_{\mathcal{F}_{\mathcal{P}}}, 0\big) \otimes \big((id, \overline{0}), 0_{\mathcal{F}_{\mathcal{P}}}, 0\big) \otimes p_2\Big) = \big((0, \overline{p_2}), 0_{\mathcal{F}_{\mathcal{P}}}, 0\big) \\ \lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}\Big(\big(0_{\mathcal{F}_{\mathcal{P}}}, (id, \overline{0}), 0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}}, (id, \overline{0}), 0\big) \otimes p_2\Big) = \big(0_{\mathcal{F}_{\mathcal{P}}}, (0, \overline{p_2}), 0\big) \\ \lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}\Big(\big((id, \overline{0}), 0_{\mathcal{F}_{\mathcal{P}}}, 0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}}, (id, \overline{0}), 0\big) \otimes p_2\Big) = \big(0_{\mathcal{F}_{\mathcal{P}}}, 0_{\mathcal{F}_{\mathcal{P}}}, p_2\big) \end{cases}$$

for $p_2 \in \mathcal{P}(2)$.

Chapter 2

Quadratic functors

In this chapter, we are interested in studying quadratic functors. We first give the appropriate context for those taking values in *Ab*. Then we provide minimal algebraic data (or also called DNA) characterizing quadratic functors taking values in (right) modules, and those with values in algebras over a linear symmetric unitary operad.

Assumption: we recall that $\mathcal{C}$ denotes a pointed category (whose zero object is denoted by 0) having finite coproducts (whose coproduct is denoted by $+$), and E is a fixed object in $\mathcal{C}$.

For a set S, let $\mathbb{Z}[S]$ denote the free abelian group with basis S. Let $X \in \mathcal{C}$, then we consider the pointed set $\mathcal{C}(E,\ X)$ with basepoint the zero map $0_{EX} : E \to X$, and we can define a subfunctor $\mathbb{Z}[0]$ of $\mathbb{Z}[\mathcal{C}(E,\ -)] : \mathcal{C} \to Ab$ such that, for $X \in \mathcal{C}$, $\mathbb{Z}[0](X) = \mathbb{Z}|\{0_{EX}\}] \subseteq \mathbb{Z}[\mathcal{C}(E,\ X)]$. This allows us to give the following definition:

Definition 2.0.1. The universal functor $U_E : \mathcal{C} \to Ab$ relative to E is the quotient of $\mathbb{Z}[\mathcal{C}(E,\ -)] : \mathcal{C} \to Ab$ by the subfunctor $\mathbb{Z}[0] : \mathcal{C} \to Ab$.

Moreover there is a retraction $\rho_2 : U_E(E+E) \to U_E(E|E)$ of the kernel $\iota_2 : U_E(E|E) \to U_E(E+E)$ of the comparison morphism $\widehat{r_2^{U_E}}$ (see (1.2.1)) defined by

$$\rho_2(\xi) = \xi - i_1^2 \circ r_1^2 \circ \xi - i_2^2 \circ r_2^2 \circ \xi\,, \tag{2.0.1}$$

for $\xi \in \mathcal{C}(E,\ E^{+2})$. The above definition is given in 1.1 of [12].

Notation 2.0.2. The abelian groups $U_E(E)$ and $T_1U_E(E)$ are rings denoted respectively by Λ and $\overline{\Lambda}$ where T_1 is the linearization functor defined in 1.2.9. To keep notation simple we write also f for the equivalence class in $U_E(X)$ of an element f of $\mathcal{C}(E,\ X)$ and $t_1(f)$ for the equivalence class of f in $T_1U_E(X)$.

For an object X in $\mathcal{C}$, we observe that $U_E(X)$ has clearly a left Λ-module structure whose action of Λ is given by the precomposition of elements in the monoid $\mathcal{C}(E, E)$. It also provides a left Λ-module structure on $T_1U_E(X)$. More precisely, we get

Remark 2.0.3. For an object X in $\mathcal{C}$, the abelian group $T_1U_E(X)$ is a left $\overline{\Lambda}$-module. This is a direct consequence of 3.8 of [12] because $T_1U_E : \mathcal{C} \to Mod_{\Lambda}$ is a linear functor by 1.2.9.

Notation 2.0.4. Let $\mathcal{D}$ be any variety. If $\mathcal{C}$ is supposed to be a semi-abelian variety (or merely a Mal'cev variety), then we denote by $QUAD(\mathcal{C}, \mathcal{D})$ the full-subcategory of $Quad(\mathcal{C}, \mathcal{D})$ constituted with quadratic functors from $\mathcal{C}$ to $\mathcal{D}$ preserving filtered colimits and coequalizers of reflexive graphs.

In chapter 3 and 4 of the thesis, we are led to study quadratic functors between (2-step nilpotent) semi-abelian varieties.

2.1 Quadratic functors with values in abelian groups

In this part, we mainly recall definitions and results of [12]. The main result of [12] provide minimal algebraic data characterizing quadratic functors taking values in abelian groups.

Definition 2.1.1. A *quadratic $\mathcal{C}$-module (relative to E)* is a diagram of homomorphisms of abelian groups

$$M = \Big(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e \xrightarrow{H_M} M_{ee} \xrightarrow{T_M} M_{ee} \xrightarrow{P_M} M_e\Big)\ ,$$

where

- M_e is a left Λ-module;
- M_{ee} is a symmetric $(\overline{\Lambda} \otimes \overline{\Lambda})$-module with involution $T_M : M_{ee} \to M_{ee}$;
- $P_M : M_{ee} \to M_e$ is a homomorphism of Λ-modules with respect to the diagonal action of Λ on M_e, i.e. for $\alpha \in \mathcal{C}(E,E)$ and $m \in M_{ee}$,

  $$P_M(t_1(\alpha) \otimes t_1(\alpha).m) \ = \ \alpha P_M(m)$$

  and satisfies $P_M \circ T_M = P_M$.
- $H_M : T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e \to M_{ee}$ is a homomorphism of symmetric $\overline{\Lambda} \otimes \overline{\Lambda}$-modules such that, for $\xi \in \mathcal{C}(E, E^{+2})$, $m \in M_{ee}$ and $a \in M_e$,

  $$(\nabla_E^2 \circ \xi)a = (r_1^2 \circ \xi)a + (r_2^2 \circ \xi)a + (P_M \circ H_M)\big(t_{11}(\rho_2^{U_E}(\xi)) \otimes a\big) \qquad (QM1)$$

  and

  $$H_M\big(t_{11}(\rho_2^{U_E})(\xi) \otimes P_M(m)\big) \ = \ t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\Big(m + T_M(m)\Big) \qquad (QM2)$$

  where $\rho_2 = \rho_2^{U_E} : U_E(E+E) \to U_E(E|E)$ is the retraction of the inclusion $\iota_2 = \iota_2^{U_E} : U_E(E|E) \to U_E(E+E)$ defined in 2.0.1.

A morphism between quadratic $\mathcal{C}$-modules is a pair of homomorphisms of abelian groups $(\phi_e,\ \phi_{ee}) : M \to N$ such that $\phi_e : M_e \to N_e$ and $\phi_{ee} : M_{ee} \to N_{ee}$ are respectively homomorphisms of left Λ-modules and $\overline{\Lambda} \otimes \overline{\Lambda}$-modules which make an obvious diagram commute. We denote by $QMod_{\mathcal{C}}$ the corresponding category.

We shall know the structure of $Coker(P_M)$, where $P_M : M_{ee} \to M_e$ is the morphism involved in a quadratic $\mathcal{C}$-module as in 2.1.1. This is given by the following remark:

Remark 2.1.2. Let M be a quadratic $\mathcal{C}$-module, then $Coker(P_M)$ is a left $\overline{\Lambda}$-module by 5.2 of [12].

For an object X in $\mathcal{C}$, we now define the map $\phi_1' : T_1U_E(X)^{\otimes 2} \otimes_{\Lambda \otimes \Lambda} T_{11}cr_2U_E(E,E) \to T_2U_E(X)$ as being the following composite map:

$$\begin{tikzcd}
T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \arrow[r, "\phi_1'"] \arrow[d, "id^{\otimes 2}\otimes\overline{cr_2(t_2)}"', "\cong"] & T_2U_E(X) \\
T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} cr_2(T_2U_E)(E,E) \arrow[r, "u'_{cr_2(T_2U_E)}"'] & cr_2(T_2U_E)(X,X) \arrow[u, "S_2^{T_2U_E}"']
\end{tikzcd} \qquad (2.1.1)$$

Here

- the natural transformation $\overline{cr_2(t_2)} : T_{11}(cr_2 U_E) \Rightarrow cr_2(T_2 U_E)$ between these bifunctors is the unique factorization of $cr_2(t_2) : cr_2 U_E \Rightarrow cr_2(T_2 U_E)$ through $t_{11} : cr_2 U_E \Rightarrow T_{11}(cr_2 U_E)$, see (2.4.1) in [12]. It is a natural isomorphism by 2.5 of [12].

- the natural transformation $u_{cr_2(T_2 U_E)} : T_1 U_E \otimes T_1 U_E \otimes T_{11} cr_2(U_E)(E, E) \Rightarrow cr_2(T_2 U_E)$ between bifunctors from $\mathcal{C} \times \mathcal{C}$ to Ab is defined in 3.21 of [12] (replacing B with the bilinear bifunctor $cr_2(T_2 U_E)$).

Then, we recall the construction of a quadratic functor with values in Ab corresponding to an arbitrary quadratic $\mathcal{C}$-module. This is given by taking the push-out of two natural transformations, see 6.2 and 6.4 of [12], called the quadratic tensor product whose definition is given below:

Definition 2.1.3. Let M be a quadratic $\mathcal{C}$-module and X be an object in $\mathcal{C}$. The *quadratic tensor product* $X \otimes M \in Ab$ is defined by the following push-out diagram of homomorphisms of abelian groups:

$$\begin{tikzcd}
\big(T_1 U_E(X)^{\otimes 2} \otimes_{\Lambda \otimes \Lambda} T_{11} cr_2(U_E)(E, E) \otimes_\Lambda M_e\big)_{\mathfrak{S}_2} \oplus (U_E(X) \otimes M_{ee}) \arrow[r, "\overline{\phi_X} = (\overline{\phi_1' \otimes id}, t_2 \otimes P_M)"] \arrow[d, "\overline{\psi_X} = (\overline{\psi_1'}, \pi(\delta \otimes id))"'] & T_2 U_E(X) \otimes_\Lambda M_e \arrow[d, "\widehat{\psi_X^M}"] \\
\big(T_1 U_E(X)^{\otimes 2} \otimes_{\Lambda \otimes \Lambda} M_{ee}\big)_{\mathfrak{S}_2} \arrow[r, "\widehat{\phi_X^M}"'] & X \otimes M
\end{tikzcd}$$

where $\psi_1' = id \otimes id \otimes H$, $\delta : U_E(X) \to T_1 U_E(X) \otimes T_1 U_E(X), f \mapsto t_1(f) \otimes t_1(f)$, π is the cokernel of $\widehat{T} \otimes id - id \otimes T$ with $\widehat{T} : T_1 U_E(X) \otimes T_1 U_E(X) \to T_1 U_E(X) \otimes T_1 U_E(X)$ being the canonical switch.

In the sequel, we shall give the explicit expression for the morphism ϕ_1' involved in the definition of the quadratic tensor product (see 2.1.3):

Lemma 2.1.4. *The abelian group homomorphism ϕ_1' has the following explicit expression:*

$$\phi_1'\Big(t_1(f_1) \otimes t_1(f_2) \otimes_{\Lambda \otimes \Lambda} t_{11}(\rho_2(\xi))\Big) = t_2(\nabla_X^2 \circ (f_1 + f_2) \circ \xi) - t_2(f_1 \circ r_1^2 \circ \xi) - t_2(f_2 \circ r_2^2 \circ \xi)$$

where $f_1, f_2 \in \mathcal{C}(E,\ X)$ and $\xi \in \mathcal{C}(E,\ E^{+2})$.

Proof. Let $f_1, f_2 \in \mathcal{C}(E,\, X)$ and $\xi \in \mathcal{C}(E,\, E^{+2})$. We have

$$
\begin{aligned}
&\phi_1'\Big(t_1(f_1)\otimes t_1(f_2)\otimes_{\Lambda\otimes\Lambda} t_{11}(\rho_2(\xi))\Big)\\
&= S_2^{T_2U_E}\circ u'_{cr_2(T_2U_E)}\circ\big(id\otimes id\otimes_{\Lambda\otimes\Lambda}\overline{cr_2(t_2)}\big)\big(t_1(f_1)\otimes t_1(f_2)\otimes t_{11}(\rho_2(\xi))\big)\\
&= T_2U_E(\nabla_X^2)\circ\iota_2^{T_2U_E}\circ u'_{cr_2(T_2U_E)}\big(t_1(f_1)\otimes t_1(f_2)\otimes cr_2(t_2)(\rho_2(\xi))\big)\\
&= T_2U_E(\nabla_X^2)\circ\iota_2^{T_2U_E}\circ cr_2(T_2U_E)(f_1,f_2)\big(cr_2(t_2)(\rho_2(\xi))\big)\\
&= T_2U_E(\nabla_X^2)\circ T_2U_E(f_1+f_2)\circ\iota_2^{T_2U_E}\circ cr_2(t_2)(\rho_2(\xi))\\
&= T_2U_E(\nabla_X^2)\circ T_2U_E(f_1+f_2)\circ t_2(\iota_2\circ\rho_2(\xi))\\
&= T_2U_E(\nabla_X^2)\circ T_2U_E(f_1+f_2)\circ t_2(\xi) - T_2U_E(\nabla_X^2)\circ T_2U_E(f_1+f_2)\circ t_2(i_1^2\circ r_1^2\circ\xi)\\
&\qquad - T_2U_E(\nabla_X^2)\circ T_2U_E(f_1+f_2)\circ t_2(i_2^2\circ r_2^2\circ\xi)\\
&= t_2(\nabla_X^2\circ(f_1+f_2)\circ\xi) - t_2(\nabla_X^2\circ(f_1+f_2)\circ i_1^2\circ r_1^2\circ\xi) - t_2(\nabla_X^2\circ(f_1+f_2)\circ i_2^2\circ r_2^2\circ\xi)\\
&= t_2(\nabla_X^2\circ(f_1+f_2)\circ\xi) - t_2(\nabla_X^2\circ i_1^2\circ f_1\circ r_1^2\circ\xi) - t_2(\nabla_X^2\circ i_2^2\circ f_2\circ r_2^2\circ\xi)\\
&= t_2(\nabla_X^2\circ(f_1+f_2)\circ\xi) - t_2(f_1\circ r_1^2\circ\xi) - t_2(f_2\circ r_2^2\circ\xi)
\end{aligned}
$$

as desired. $\square$

Corollary 2.1.5. *Let $f_1, f_2 \in \mathcal{C}(E,\, X)$, $h \in \mathcal{C}(E,\, Id_{\mathcal{C}}(E|E))$ and $a \in M_e$. Then we have*

$$\overline{\phi_X}\Big(\overline{t_1(f_1)\otimes t_1(f_2)\otimes_{\Lambda\otimes\Lambda} t_{11}(\rho_2(\iota_2^{Id_{\mathcal{C}}}\circ h))\otimes a},\ 0\Big) = t_2\big(c_2^X\circ Id_{\mathcal{C}}(f_1|f_2)\circ h\big)\otimes a$$

Proof. It is a direct consequence of 2.1.4 replacing ξ with $\iota_2^{Id_{\mathcal{C}}}\circ h$ and of the relations $r_k^2\circ\iota_2^{Id_{\mathcal{C}}} = 0$, for $k = 1, 2$ (because $\iota_2^{Id_{\mathcal{C}}} : Id_{\mathcal{C}}(E|E) \rightarrowtail E+E$ is the kernel of the comparison morphism $\widehat{r_2^{Id_{\mathcal{C}}}} : E+E \to E\times E$, see 1.2.1). Moreover we observe that we have

$$\nabla_X^2\circ(f_1+f_2)\circ\iota_2^{Id_{\mathcal{C}}}\circ h = \nabla_X^2\circ\iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(f_1|f_2)\circ h = c_2^X\circ Id_{\mathcal{C}}(f_1|f_2)\circ h\,,$$

see 1.2.8. $\square$

The diagram of 2.1.3 is clearly functorial. Let M be a quadratic $\mathcal{C}$-module, then the Proposition 6.5 of [12] says that $-\otimes M : \mathcal{C}\to Ab$ is a quadratic functor. It allows to define the following functor:

Definition 2.1.6. The functor $\mathbb{T}_2 : QMod_{\mathcal{C}} \to Quad(\mathcal{C},\, Ab)$ is given as follows:

1. *On objects*, for a quadratic $\mathcal{C}$-module M, $\mathbb{T}_2(M) = -\otimes M : \mathcal{C}\to Ab$ such that, for all $X\in\mathcal{C}$, $(-\otimes M)(X) = X\otimes M$ is the corresponding quadratic tensor product given in 2.1.3.

2. *On morphisms*, let $\phi = (\phi_e, \phi_{ee}) : M\to N$ be a morphism of quadratic $\mathcal{C}$-modules. Then $\mathbb{T}_2(\phi) : \mathbb{T}_2(M) \Rightarrow \mathbb{T}_2(N)$ is a natural transformation such that, for all $X\in\mathcal{C}$, $\mathbb{T}_2(\phi)_X : X\otimes M\to X\otimes N$ is the unique morphism given by the universal property of the pushout in 2.1.3 satisfying

$$\begin{cases} \mathbb{T}_2(\phi)_X\circ\widehat{\psi_X^M} = \widehat{\psi_Y^M}\circ\big(t_2(f)\otimes_\Lambda id\big) \\ \mathbb{T}_2(\phi)_X\circ\widehat{\phi_X^M} = \widehat{\phi_Y^M}\circ\big(t_1(f_1)\otimes t_1(f_2)\otimes_{\Lambda\otimes\Lambda} id\big)_{\mathfrak{S}_2} \end{cases} \tag{2.1.2}$$

where $f, f_1, f_2 \in \mathcal{C}(X,Y)$.

Moreover a quadratic functor with domain $\mathcal{C}$ and values in Ab gives rise to a quadratic $\mathcal{C}$-module, see 5.16 of [12]. This defines a functor as follows:

Definition 2.1.7. The functor $\mathbb{S}_2 : Quad(\mathcal{C},\ Ab) \to QMod_{\mathcal{C}}$ is defined as follows:

1. *On objects*, with a quadratic functor $F : \mathcal{C} \to Ab$, we associate a corresponding quadratic $\mathcal{C}$-module $\mathbb{S}_2(F)$ as follows:

$$\mathbb{S}_2(F) \;=\; \Big(T_{11}cr_2(U_E)(E,E)\otimes_\Lambda F(E) \xrightarrow{H_E^F} F(E|E) \xrightarrow{T_E^F} F(E|E) \xrightarrow{(S_2^F)_E} F(E)\Big).$$

   Here we have

   - $H^F : T_{11}cr_2(U_E)\cdot\Delta^2 \otimes_\Lambda F(E) \Rightarrow cr_2F$ the natural transformation given by the following diagram

$$\begin{tikzcd}
T_{11}cr_2(U_E)(X,X)\otimes_\Lambda F(E) \arrow[r, "H_X^F"] & cr_2F(X,X) \\
cr_2(U_E)(X,X)\otimes_\Lambda F(E) \arrow[u, "t_{11}^{cr_2(U_E)}\otimes_\Lambda id"] \arrow[ur, "cr_2(u'_F)_{X,X}"'] &
\end{tikzcd}$$

     where $X \in \mathcal{C}$ and $u'_F : U_E \otimes_\Lambda F(E) \Rightarrow F$ is a natural transformation given by $(u'_F)_X : U_E(X)\otimes_\Lambda F(E) \to F(X)$, $f \otimes_\Lambda x \mapsto F(f)(x)$, for $f \in U_E(X)$ and $x \in F(E)$;
   - $T_X^F : cr_2F(X,X) \to cr_2F(X,X)$ the restriction of the involution $F(\tau_X^2) : F(X+X) \to F(X+X)$ to $cr_2F(X,X)$.
   - $S_2^F : cr_2F \cdot \Delta^2 \Rightarrow F$ the natural transformation given in 1.8 of [12] and defined by $(S_2^F)_X = F(\nabla_X)\circ \iota_2^F$, for an object X in $\mathcal{C}$.

2. *On morphisms*, let $\alpha : F \Rightarrow G$ be a natural transformation between quadratic functors, then $\mathbb{S}_2(\alpha) = (\alpha_E,\ cr_2(\alpha)_{E,E}) : \mathbb{S}_2(F) \to \mathbb{S}_2(G)$.

Notation 2.1.8. Let $F : \mathcal{C} \to Ab$ be a quadratic functor. Then we set $M^F = \mathbb{S}_2(F)$ its corresponding quadratic $\mathcal{C}$-module.

Then the following result says that the two functors $\mathbb{S}_2 : Quad(\mathcal{C},\ Ab) \to QMod_{\mathcal{C}}$ and $\mathbb{T}_2 : QMod_{\mathcal{C}} \to Quad(\mathcal{C},\ Ab)$ are both additive.

Proposition 2.1.9. *The functors $\mathbb{S}_2 : Quad(\mathcal{C},\ Ab) \to QMod_{\mathcal{C}}$ and $\mathbb{T}_2 : QMod_{\mathcal{C}} \to Quad(\mathcal{C},\ Ab)$ are additive.*

Proof. The proof is given in two steps.

1. First we prove that the functor $\mathbb{S}_2 : Quad(\mathcal{C},\ Ab) \to QMod_{\mathcal{C}}$ is additive. Let $\alpha,\ \beta : F \Rightarrow G$ be two natural transformations. Then we have

$$\mathbb{S}_2(\alpha+\beta) = \big(\alpha_E + \beta_E,\ cr_2(\alpha+\beta)_{E,E}\big)$$

   We verify that, for two objects X and Y in $\mathcal{C}$, we get

$$cr_2(\alpha+\beta)_{X,Y} = cr_2(\alpha)_{X,Y} + cr_2(\beta)_{X,Y}$$

We have the following equalities as follows:

$$
\begin{aligned}
\iota_2^G \circ cr_2(\alpha+\beta)_{X,Y} &= (\alpha_{X+Y} + \beta_{X+Y}) \circ \iota_2^F \\
&= (\alpha_{X+Y} \circ \iota_2^F + \beta_{X+Y} \circ \iota_2^F) \\
&= \big(\iota_2^G \circ cr_2(\alpha)_{X,Y} + \iota_2^G \circ cr_2(\beta)_{X,Y}\big) \\
&= \iota_2^G \circ \big(cr_2(\alpha)_{X,Y} + cr_2(\beta)_{X,Y}\big)
\end{aligned}
$$

As $\iota_2^G : G(X|Y) \rightarrowtail G(X+Y)$ is a monomorphism, we get

$$cr_2(\alpha+\beta)_{X,Y} = cr_2(\alpha)_{X,Y} + cr_2(\beta)_{X,Y}$$

Hence we have

$$
\begin{aligned}
\mathbb{S}_2(\alpha+\beta) &= \big(\alpha_E + \beta_E,\ cr_2(\alpha+\beta)_{E,E}\big) \\
&= \big(\alpha_E + \beta_E,\ cr_2(\alpha)_{E,E} + cr_2(\beta)_{E,E}\big) \\
&= \big(\alpha_E,\ cr_2(\alpha)_{E,E}\big) + \big(\beta_E,\ cr_2(\beta)_{E,E}\big) \\
&= \mathbb{S}_2(\alpha) + \mathbb{S}_2(\beta)
\end{aligned}
$$

as desired.

2. Then we prove that the functor $\mathbb{T}_2 : QMod_{\mathcal{C}} \to Quad(\mathcal{C},\ Ab)$ is additive. Let X be an object in $\mathcal{C}$. For this it suffices to observe that the functors $T_2U_E(X) \otimes -$ and $\big(T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} -\big)_{\mathfrak{S}_2}$ with domain $\mathcal{C}$ and values in Ab are additive.

□

Now we recall the theorem 7.1 of [12] which says that quadratic functors taking values in Ab can be characterized by quadratic $\mathcal{C}$-modules:

Theorem 2.1.10. *Let $\mathcal{C}$ be a pointed category with finite coproducts.*

- *If $\mathcal{C}$ is a small category, the functors*

$$\mathbb{S}_2 : Quad(\mathcal{C},\ Ab) \rightleftarrows QMod_{\mathcal{C}} : \mathbb{T}_2$$

form a pair of adjoint functors.

- *If $\mathcal{C} = \langle E \rangle$, the functors $\mathbb{S}_2$ and $\mathbb{T}_2$ are equivalences of categories inverse to each other.*

- *If $\mathcal{C}$ has sums and if E is a small regular-projective generator object of $\mathcal{C}$, then the functors*

$$\mathbb{S}_2' : QUAD_E(\mathcal{C}, Ab) \rightleftarrows QMod_{\mathcal{C}} : \mathbb{T}_2'$$

are equivalences of categories inverse to each other, where $\mathbb{T}_2'$ is given by $\mathbb{T}_2$ which actually takes values in $QUAD_E(\mathcal{C}, Ab)$ (by 6.24 of [12]), and where $\mathbb{S}_2'$ is the restriction of $\mathbb{S}_2$.

Here $QUAD_E(\mathcal{C}, Ab)$ denotes the full subcategory of $Quad(\mathcal{C}, Ab)$ formed by (reduced) quadratic functors from $\mathcal{C}$ to Ab preserving filtered colimits and E-saturated coequalizers (see the definition in 6.21 of [12]); from the proposition 6.23 of [12], E-saturated coequalizers can be replaced with E-saturated E-free coequalizers, and with coequalizers of reflexive graphs if $\mathcal{C}$ is Mal'cev and Barr exact (as all semi-abelian categories).

Remark 2.1.11. The third point of the statement in 2.1.10 can be replaced with the following one: if $\mathcal{C}$ is a semi-abelian variety and if E denotes the free object of rank 1 in $\mathcal{C}$, then the functors

$$\mathbb{S}_2' : QUAD(\mathcal{C}, Ab) \rightleftarrows QMod_{\mathcal{C}} : \mathbb{T}_2'$$

are equivalences of categories inverse to each other, where $\mathbb{T}_2'$ is given by $\mathbb{T}_2$ which actually takes values in $QUAD(\mathcal{C}, Ab)$ (see the definition of this category in 2.0.4), and where $\mathbb{S}_2'$ is the restriction of $\mathbb{S}_2$.

Notation 2.1.12. We denote respectively by $\eta : Id \Rightarrow \mathbb{S}_2 \cdot \mathbb{T}_2$ and $\varepsilon : \mathbb{T}_2 \cdot \mathbb{S}_2 \Rightarrow Id$ the unit and the counit of the adjunction 2.1.10.

Remark 2.1.13. The unit $\eta : Id \Rightarrow \mathbb{S}_2 \cdot \mathbb{T}_2$ is a natural equivalence by 7.10 of [12]; but not the counit $\varepsilon : \mathbb{T}_2 \cdot \mathbb{S}_2 \Rightarrow Id$ in general.

2.2 Quadratic functors with values in right modules

In this part, we take R a ring, $F : \mathcal{C} \to Mod_R$ a quadratic functor and Mod_R denotes the category of right R-modules. The functor $F : \mathcal{C} \to Mod_R$ is the same as considering a pair $(F,\ \lambda_1^F)$ where F is seen as taking values in abelian groups and $\lambda_1^F : F \otimes R \Rightarrow F$ is the right action of R on F. As F is a quadratic functor taking also values in Ab, it allows us to apply the functor $\mathbb{S}_2$ to F (see 2.1.7) representing a part of its minimal algebraic data given by the following quadratic $\mathcal{C}$-module:

$$\mathbb{S}_2(F) \ = \ \left(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda F(E) \xrightarrow{H_E^F} F(E|E) \xrightarrow{T_E^F} F(E|E) \xrightarrow{(S_2^F)_E} F(E)\right)$$

Moreover there is a homomorphism of rings $\alpha : R^{op} \to End(F), r^{op} \mapsto \alpha^{r^{op}}$ that is the right action of R on F, more precisely, for $r \in R$, $\alpha^{r^{op}} : F \Rightarrow F$ is the natural transformation defined by

$$\alpha_X^{r^{op}} : F(X) \to F(X),\ x \mapsto (\lambda_1^F)_X(x \otimes r)$$

where X is an object in $\mathcal{C}$ and $(\lambda_1^F)_X : F(X) \otimes R \to F(X)$ represents the right action of R on $F(X)$. By restriction of $\alpha_{X+X}^{r^{op}}$ to $F(E|E)$, we obtain a right action of R on $F(X|X)$ denoted by $cr_2(\alpha^{r^{op}})_{E,E}$ making $F(E|E)$ into a right R-module. Then, for any $r \in R$, we have the following commutative diagram by applying $\mathbb{S}_2$ to the natural transformation $\alpha^{r^{op}} : F \Rightarrow F$:

$$\begin{tikzcd}
T_{11}cr_2(U_E)(E,\ E) \otimes F(E) \arrow[r, "H_E^F"] & F(E|E) \arrow[r, "T_E^F"] & F(E|E) \arrow[r, "(S_2^F)_E"] & F(E) \\
T_{11}cr_2(U_E)(E,\ E) \otimes F(E) \arrow[r, "H_E^F"] \arrow[u, "id\otimes_\Lambda \alpha_E^{r^{op}}"] & F(E|E) \arrow[r, "T_E^F"'] \arrow[u, "cr_2(\alpha^{r^{op}})_{E,E}"] & F(E|E) \arrow[r, "(S_2^F)_E"'] \arrow[u, "cr_2(\alpha^{r^{op}})_{E,E}"] & F(E) \arrow[u, "\alpha_E^{r^{op}}"]
\end{tikzcd}$$

This commutative diagram expresses the fact that H_E^F, T_E^F and $(S_2^F)_E$ are homomorphisms of right R-modules.

2.2.1 Quadratic $\mathcal{C}$-modules over a ring R

We define the notion of quadratic $\mathcal{C}$-modules enriched with a right R-module structure as follows:

Definition 2.2.1. A *quadratic $\mathcal{C}$-module over R* is a quadratic $\mathcal{C}$-module as follows:

$$M \ = \ \Big(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e \xrightarrow{H} M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e\Big)$$

as in 2.1.1 such that

- M_e and M_{ee} are right R-modules; moreover the action of Λ (resp. $\overline{\Lambda} \otimes \overline{\Lambda}$) on M_e (resp. M_{ee}) commutes with the action of R on M_e (resp. M_{ee}).
- P, H and T are homomorphisms of right R-modules.

We denote by $QMod_{\mathcal{C}}^R$ the corresponding category.

Let $\mathcal{A}$ be a preadditive category. We denote by $Mod_R(\mathcal{A})$ the category of right R-modules whose objects are pairs $(A,\ \phi^A)$ where A is an object in $\mathcal{A}$ and $\phi^A : R^{op} \to End(A)$, $r^{op} \mapsto \phi^A_{r^{op}}$ is a homomorphism of rings. A morphism $f : (A,\ \phi^A) \to (B,\ \phi^B)$ in $Mod_R(\mathcal{A})$ is a morphism $f : A \to B$ in $\mathcal{A}$ preserving the right R-modules structure in the following sense: for $r^{op} \in R^{op}$, we have

$$f \circ \phi^A_{r^{op}} = \phi^B_{r^{op}} \circ f$$

Remark 2.2.2. We remark that $QMod_{\mathcal{C}}^R$ is isomorphic to the category $Mod_R(QMod_{\mathcal{C}})$. It makes sense because $QMod_{\mathcal{C}}$ is clearly a preadditive category. Similarly we also observe that the category $Quad(\mathcal{C},\ Mod_R)$ is isomorphic to $Mod_R\big(Quad(\mathcal{C},\ Ab)\big)$.

2.2.2 The functors $\mathbb{S}_2^R$ and $\mathbb{T}_2^R$

In this part, we define two functors so as to settle a similar theorem as in 2.1.10 for quadratic functors with domain $\mathcal{C}$ and values in Mod_R. First we check that a quadratic $\mathcal{C}$-module over R provides a quadratic functor taking values in Mod_R.

Proposition 2.2.3. *Let M be a quadratic $\mathcal{C}$-module over R, then the quadratic functor $\mathbb{T}_2(M) = - \otimes M : \mathcal{C} \to Ab$, defined in 2.1.6, lifts into a functor from $\mathcal{C}$ to Mod_R.*

Proof. It remains to recover the right action of R on $- \otimes M$. We write $\beta = (\beta_e,\ \beta_{ee}) : R^{op} \to End(M)$, $r^{op} \mapsto \beta^{r^{op}} = (\beta_e^{r^{op}},\ \beta_{ee}^{r^{op}})$ denoting the right R-module structure for M. For each $r \in R$, $\alpha^{r^{op}} = \mathbb{T}_2(\beta^{r^{op}}) : - \otimes M \Rightarrow - \otimes M$ is the natural transformation given by applying $\mathbb{T}_2$ to the morphism $\beta^{r^{op}} : M \to M$ of quadratic $\mathcal{C}$-modules. The uniqueness in the universal property of the push-out defined in 2.1.3 says that $\alpha : R^{op} \to End(- \otimes M)$, $r^{op} \mapsto \alpha^{r^{op}}$ is a homomorphism of rings. Finally $\mathbb{T}_2(M) = - \otimes M$ is a quadratic functor taking values in Mod_R. □

Now it is convenient to define two functors in order to summarize the above arguments.

Definition 2.2.4. We define two functors as follows:

1. The functor $\mathbb{S}_2^R : Quad(\mathcal{C},\ Mod_R) \to QMod_{\mathcal{C}}^R$ is defined by:
   - *On objects*, let $F : \mathcal{C} \to Mod_R$ be a quadratic functor, $\mathbb{S}_2^R(F)$ is the quadratic $\mathcal{C}$-module

     $$\mathbb{S}_2^R(F) \ = \ \Big(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda F(E) \xrightarrow{H_E^F} F(E|E) \xrightarrow{T_E^F} F(E|E) \xrightarrow{(S_2^F)_E} F(E)\Big)$$

     equipped with the (right) action of R on $\mathbb{S}_2^R(F)$ given by

     $$R^{op} \to End(\mathbb{S}_2(F))\,, \quad r^{op} \mapsto (\alpha_E^{r^{op}}, cr_2(\alpha^{r^{op}})_{E,E})$$

     Here $\alpha : R^{op} \to End(F)$ is the homomorphism of rings corresponding to the action of R on the quadratic functor F and $\mathbb{S}_2 : Quad(\mathcal{C},\ Ab) \to QMod_{\mathcal{C}}$ is the functor defined in 2.1.7;

- *On morphisms*, $\mathbb{S}_2^R(\beta) = \mathbb{S}_2(\beta)$, for $\beta : F \Rightarrow G$ a natural transformation in $Quad(\mathcal{C},\ Mod_R)$.

2. The functor $\mathbb{T}_2^R : QMod_{\mathcal{C}}^R \to Quad(\mathcal{C},\ Mod_R)$ is defined by:

   - *On objects*, let M be a quadratic $\mathcal{C}$-module over R as in 2.2.1, $\mathbb{T}_2^R(M^R) = (-\otimes M,\ \lambda_1^M)$, where $\lambda_1^M : (-\otimes M)\otimes R \Rightarrow -\otimes M$ is the natural transformation representing the right action of R on $-\otimes M$; more precisely, for all $X \in \mathcal{C}$, we have

   $$(\lambda_1^M)_X : (X\otimes M)\otimes R \to X\otimes M,\ x\otimes r \mapsto \alpha_X^{r^{op}}(x)$$

   where $\alpha^{r^{op}} = \mathbb{T}_2(\beta^{r^{op}}) : -\otimes M \Rightarrow -\otimes M$ is the natural transformation given by applying $\mathbb{T}_2 : QMod_{\mathcal{C}} \to Quad(\mathcal{C},\ Ab)$ (see 2.1.6) to $\beta^{r^{op}} : M \to M$, and $\beta : R^{op} \to End(QMod_{\mathcal{C}})$, $r^{op} \mapsto \beta^{r^{op}}$ is the homomorphism of rings associated with the (right) action of R on M.

   - *On morphisms*, for $\phi = (\phi_e,\ \phi_{ee}) : M \to N$ a morphism of quadratic $\mathcal{C}$-modules over R, $\mathbb{T}_2^R(\phi) = \mathbb{T}_2(\phi)$.

Notation 2.2.5. We give a similar notation as in 2.1.8. Let $F : \mathcal{C} \to Mod_R$ be a quadratic functor. Then we also set $M_R^F = \mathbb{S}_2^R(F)$ its corresponding quadratic $\mathcal{C}$-module over R.

Remark 2.2.6. If we assume that $\mathcal{C}$ is a semi-abelian variety and if E denotes the free object of rank 1 in $\mathcal{C}$, then the functor $\mathbb{T}_2^R$ takes in fact values in $QUAD(\mathcal{C}, Mod_R)$. This is due to the fact that, for a quadratic $\mathcal{C}$-module M over R, the composite functors $W \cdot \mathbb{T}_2^R(M) = W \cdot (-\otimes M) : \mathcal{C} \to Ab$ preserves filtered colimits and coequalizers of reflexive graphs by 2.1.11, where $W : Mod_R \to Ab$ is the forgetful functor. By 1.6.11, the (quadratic) functor $\mathbb{T}_2^R(M) = -\otimes M : \mathcal{C} \to Mod_R$ preserves filtered colimits and coequalizers of reflexive graphs.

2.2.3 The adjunction between $\mathbb{S}_2^R$ and $\mathbb{T}_2^R$

The two functors $\mathbb{S}_2^R$ and $\mathbb{T}_2^R$ defined in 2.2.4 give rise to the following theorem:

Theorem 2.2.7. *Let $\mathcal{C}$ be a pointed category with finite coproducts.*

- *If $\mathcal{C}$ is a small category, the functors*

$$\mathbb{S}_2^R : Quad(\mathcal{C},\ Mod_R) \rightleftarrows QMod_{\mathcal{C}}^R : \mathbb{T}_2^R$$

form a pair of adjoint functors extending $\mathbb{S}_2$ and $\mathbb{T}_2$.

- *If $\mathcal{C} = \langle E\rangle$, the functors $\mathbb{S}_2^R$ and $\mathbb{T}_2^R$ are equivalences of categories inverse to each other.*

- *If $\mathcal{C}$ is a semi-abelian variety and if E denotes the free object of rank 1 in $\mathcal{C}$, then the functors*

$$(\mathbb{S}_2^R)' : QUAD(\mathcal{C}, Mod_R) \rightleftarrows QMod_{\mathcal{C}}^R : (\mathbb{T}_2^R)'$$

are equivalences of categories inverse to each other, where $(\mathbb{T}_2^R)'$ is given by $\mathbb{T}_2^R$ which actually takes values in $QUAD(\mathcal{C}, Mod_R)$ (by 2.2.6), and where $(\mathbb{S}_2^R)'$ is the restriction of $\mathbb{S}_2^R$.

Before tackling the proof of this theorem, we need a technical lemma providing a pair of adjoint additive functors between categories of modules in pre-additive categories from such a pair between preadditive categories.

Lemma 2.2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two preadditive categories. Suppose that there is a pair of adjoint additive functors*

$$F : \mathcal{A} \leftrightarrows \mathcal{B} : G$$

Then it fits into another pair of adjoint additive functors

$$F_R : Mod_R(\mathcal{A}) \leftrightarrows Mod_R(\mathcal{B}) : G_R$$

where F_R is the functor defined by

1. *On objects, let $(A,\ \phi^A)$ be an object in $Mod_R(\mathcal{A})$, $F_R((A,\ \phi^A)) = (F(A),\ \phi^{F(A)})$, and $\phi^{F(A)} : R^{op} \to End_{\mathcal{B}}(F(A))$ is the homomorphism of rings given by:*

$$\forall r\ \in R, \quad \phi^{F(A)}_{r^{op}} = F(\phi^A_{r^{op}})$$

2. *On morphisms, for any $f : (A, \phi^A) \to (B, \phi^B)$ morphism in $Mod_R(\mathcal{A})$, we set*

$$F_R(f) = F(f) : (F(A), \phi^{F(A)}) \to (F(B), \phi^{F(B)})$$

In addition, G_R is defined in the same way.

Proof. By 2.1.9, we know that $\mathbb{S}_2 : Quad(\mathcal{C},\ Ab) \to QMod_{\mathcal{C}}$ and $\mathbb{T}_2 : QMod_{\mathcal{C}} \to Quad(\mathcal{C},\ Ab)$ are additive functors. Let $\eta : Id_{\mathcal{A}} \Rightarrow G \cdot F$ be the unit of the adjunction.

- Given an object (A, ϕ^A) in $Mod_R(\mathcal{A})$ and set $\phi^{(G\cdot F)(A)} : R^{op} \to End_{\mathcal{A}}((G \cdot F)(A))$ the homomorphism of rings given by:

$$\forall r\ \in R, \quad \phi^{(G\cdot F)(A)}_{r^{op}} = (G \cdot F)(\phi^A_{r^{op}})$$

  By naturality of η, we have

$$\phi^{(G\cdot F)(A)}_{r^{op}} \circ \eta_A = (G \cdot F)(\phi^A_{r^{op}}) \circ \eta_A = \eta \circ \phi^A_{r^{op}}$$

  This proves that $\eta_A : (A, \phi^A) \to ((G \cdot F)(A), \phi^{(G\cdot F)(A)}) = G_R \cdot F_R((A, \phi^A))$ is a morphism in $Mod_R(\mathcal{A})$ and that $\eta : Id_{Mod_R(\mathcal{A})} \Rightarrow G_R \cdot F_R$ is a natural transformation from $Id_{Mod_R(\mathcal{A})}$ to $G_R \cdot F_R$.

- It suffices to prove that the universal property of $\eta : Id_{Mod_R(\mathcal{A})} \Rightarrow G_R \cdot F_R$ is satisfied in $Mod_R(\mathcal{A})$. Let (A, ϕ^A) be an right R-module in $\mathcal{A}$ and $f : (A, \phi^A) \to G_R((B, \psi^B)) = (G(B), \psi^{G(B)})$ be a morphism in $Mod_R(\mathcal{B})$. As the universal property of $\eta : Id_{\mathcal{A}} \Rightarrow G \cdot F$ works in $\mathcal{A}$, there exists a unique $\overline{f} : F(A) \to B$ morphism in $\mathcal{B}$ such that

$$f = G(\overline{f}) \circ \eta_A$$

  Then we prove that $\overline{f} : F_R((A, \phi^A)) = (F(A), \phi^{F(A)}) \to (B, \psi^B))$ is a morphism in $Mod_R(\mathcal{B})$. Let $r \in R$, then we have

$$\begin{aligned}
G(\overline{f} \circ \phi^{F(A)}_{r^{op}}) \circ \eta_A &= G(\overline{f}) \circ G(\phi^{F(A)}_{r^{op}})\eta_A \\
&= G(\overline{f}) \circ (G.F)(\phi^A_{r^{op}}) \circ \eta_A \\
&= G(\overline{f}) \circ \eta_A \circ \phi^A_{r^{op}} \\
&= f \circ \phi^A_{r^{op}} \\
&= \psi^{G(B)}_{r^{op}} \circ f \\
&= G(\psi^B_{r^{op}}) \circ f \\
&= G(\psi^B_{r^{op}}) \circ G(\overline{f}) \circ \eta_A \\
&= G(\psi_{r^{op}} \circ \overline{f}) \circ \eta_A
\end{aligned}$$

By uniqueness in the universal property of η,

$$\psi^B_{r^{op}} \circ \overline{f} = \phi^{F(A)}_{r^{op}}$$

Consequently $\overline{f} : F_R((A, \phi^A)) \to (B, \psi^B)$ is a morphism in $Mod_R(\mathcal{B})$. This proves the result.

$\square$

Proof. of Theorem 2.2.7. We consider the following commutative diagram:

$$\begin{tikzcd}
Mod - R(Quad(\mathcal{C},\ Ab)) \arrow[r, shift left, "(\mathbb{S}_2)_R"] \arrow[d, "\cong"'] & Mod - R(QMod_{\mathcal{C}}) \arrow[l, shift left, "(\mathbb{T}_2)_R"] \arrow[d, "\cong"] \\
Quad(\mathcal{C},\ Mod_R) \arrow[r, shift left, "\mathbb{S}_2^R"] & QMod^R_{\mathcal{C}} \arrow[l, shift left, "\mathbb{T}_2^R"]
\end{tikzcd}$$

The left and right isomorphisms of categories comes from 2.2.2. By 2.2.8, $(\mathbb{S}_2)_R$ and $(\mathbb{T}_2)_R$ form a pair of adjoint functors because $\mathbb{S}_2$ and $\mathbb{T}_2$ is a pair of adjoint additive functors, see 2.1.10. This implies that $\mathbb{S}_2^R$ and $\mathbb{T}_2^R$ form also an adjunction pair.
The unit, respectively the counit of the adjunction pair 2.2.7 is exactly the unit $\eta : Id \Rightarrow \mathbb{S}_2 \cdot \mathbb{T}_2$, respectively the counit $\varepsilon : \mathbb{T}_2 \cdot \mathbb{S}_2 \Rightarrow Id$ of the pair of adjoint functors 2.1.10 (see the notations given in 2.1.12); they both preserve the (right) R-module structure (by naturality of η, respectively ε) if restricted to the category $QMod_R$, respectively to the category $Quad(\mathcal{C}, Mod_R)$. Then we can consider the unit η (respectively the counit ε) as a natural transformation from the identity functor of $QMod^R_{\mathcal{C}}$ (respectively the composite functors $\mathbb{T}_2^R \cdot \mathbb{S}_2^R$), to the composite functors $\mathbb{S}_2^R \cdot \mathbb{T}_2^R$ (respectively the identity functor of $Quad(\mathcal{C}, Mod_R)$).
As $\eta : Id \Rightarrow \mathbb{S}_2^R \cdot \mathbb{T}_2^R$ is a natural equivalence by 2.1.13, it suffices to prove that $\varepsilon : \mathbb{T}_2^R \cdot \mathbb{S}_2^R \Rightarrow Id$ is a natural equivalence for the second and third points in the statement.
If we assume that $\mathcal{C} = \langle E \rangle$, then ε is a natural equivalence by the second point of 2.1.10 implying that the functors $\mathbb{S}_2^R$ and $\mathbb{T}_2^R$ form a pair of adjoint equivalences.
Now we suppose that $\mathcal{C}$ is a semi-abelian variety and E is the free object of rank 1 in $\mathcal{C}$. For a quadratic functor $F : \mathcal{C} \to Mod_R$ preserving filtered colimits and coequalizers of reflexive graphs, the counit ε_F : $\mathbb{T}_2^R \cdot \mathbb{S}_2^R(F) = -\otimes \mathbb{S}_2(F) \Rightarrow F$ (evaluated to F) is a natural transformation between quadratic functors preserving filtered colimits and coequalizers of reflexive graphs which is a natural isomorphism if restricted to the full subcategory $\langle E \rangle$ of $\mathcal{C}$ (by the second point in the above statement). Hence it is a natural isomorphism by 6.25 of [12]. Thus the functors $(\mathbb{S}_2^R)'$ and $(\mathbb{T}_2^R)'$ in the statement form a pair of adjoint equivalences. $\square$

Notation 2.2.9. We denote respectively by $\eta : Id \Rightarrow \mathbb{S}_2^R \cdot \mathbb{T}_2^R$ and $\varepsilon : \mathbb{T}_2^R \cdot \mathbb{S}_2^R \Rightarrow Id$ the unit and the counit of the adjunction pair 2.2.7.

2.3 The linearization of the quadratic tensor product

Let M be a quadratic $\mathcal{C}$-module. Here we give an explicit expression of the linearization of the functor $\mathbb{T}_2(M) = - \otimes M : \mathcal{C} \to Ab$ defined in 2.1.6. However we shall give two results before.

Proposition 2.3.1. *Let $\mathcal{A}$, $\mathcal{B}$ be two abelian categories, $G : \mathcal{C} \to \mathcal{A}$ be a reduced functor and $L : \mathcal{A} \to \mathcal{B}$ be a functor preserving right exact sequences. Then we have*

$$T_n(L \cdot G) \cong L \cdot T_n G$$

where $T_n : Func_*(\mathcal{C},\,\mathcal{A}) \to Func_{\leqslant n}(\mathcal{C},\,\mathcal{A})$ is the n-Taylorization functor defined in 1.9 *of [12]. More precisely, the unique factorization $\overline{L_* \cdot t^G} : T_n(L\cdot G) \Rightarrow L\cdot T_nG$ of $L_*\cdot t_n^G : L\cdot G \Rightarrow L\cdot T_nG$ through $t_n^{L\cdot G} : L\cdot G \Rightarrow T_n(L\cdot G)$ is a natural isomorphism.*

Proof. We observe that the following diagram commutes by naturality of $t_n^{G\otimes_A M} : L\cdot G \Rightarrow T_n(L\cdot G)$:

$$\begin{tikzcd}
L\cdot\big(cr_{n+1}G\cdot\Delta^{n+1}\big) \arrow[r, "L_*\cdot S^G_{n+1}"] \arrow[d, "t_n"'] & L\cdot G \arrow[r, "L_*\cdot t_n^G"] \arrow[d, "t_n^{L\cdot G}"'] & L\cdot T_nG \arrow[r] & 0 \\
T_n\Big(L\cdot\big(cr_{n+1}G\cdot\Delta^{n+1}\big)\Big) \arrow[r, "T_n(L_*\cdot S^G_{n+1})"'] & T_n(L\cdot G) & &
\end{tikzcd}$$

As the functor $L\cdot T_nG : \mathcal{C}\to Ab$ is polynomial of degree $\leqslant n$ (by Theorem 1.9 in [34] or Proposition 1.6 in [19] because it is a polynomial functor of degree $\leqslant n$ postcomposed by a linear functor with abelian source and target), the universal property of $t_n^{L\cdot G} : L\cdot G \Rightarrow T_n(L\cdot G)$ (see 1.10 of [12]) says that there is a unique natural transformation $\overline{L_*\cdot t_n^G} : T_n(L\cdot G)\Rightarrow L\cdot T_nG$ such that

$$(\overline{L_*\cdot t_n^G})\circ t_n^{L\cdot G} = L_*\cdot t_n^G \tag{2.3.1}$$

Moreover the n-Taylorisation of the functor $L\cdot\big(cr_{n+1}G\cdot\Delta^{n+1}\big) : \mathcal{C}\to Ab$ is trivial by 2.19 of [13], i.e

$$T_n\Big(L\cdot\big(cr_{n+1}G\cdot\Delta^{n+1}\big)\Big) = 0$$

since $L_*\cdot cr_{n+1}G : \mathcal{C}^{\times(n+1)}\to Ab$ is a multireduced multifunctor (the latter fact is also expressed by saying that $L\cdot\big(cr_{n+1}G\cdot\Delta^{n+1}\big) : \mathcal{C}\to Ab$ is cohomogenous of degree $\leqslant n$). Hence we have

$$t_n^{L\cdot G}\circ\big(L_*\cdot S^G_{n+1}\big) = 0$$

By 1.9 of [12], $t_n^G : G\Rightarrow T_nG$ is the cokernel of $S^G_{n+1} : cr_{n+1}G\cdot\Delta^{n+1}\Rightarrow G$. As the functor $L:\mathcal{A}\to\mathcal{B}$ preserves colimits (right exact sequences in particular), the top sequence of the above diagram is right exact. Consequently, there is a unique natural transformation $\overline{t_n^{L\cdot G}} : L\cdot T_nG\Rightarrow T_n(L\cdot G)$ such that

$$\overline{t_n^{L\cdot G}}\circ\big(L_*\cdot t_n^G\big) = t_n^{L\cdot G} \tag{2.3.2}$$

The two composites of the maps $\overline{t_n^{L\cdot G}}\circ\big(\overline{L_*\cdot t_n^G}\big)$ and $\big(\overline{L_*\cdot t_n^G}\big)\circ\overline{t_n^{L\cdot G}}$ respectively with the epimorphisms $t_n^{L\cdot G}$ and $L_*\cdot t_n^G$, and the equations (2.3.1) and (2.3.2) show that $\overline{t_n^{L\cdot G}}$ and $\overline{L_*\cdot t_n^G}$ are inverse to each other. □

Proposition 2.3.2. *Let $\mathcal{D}$ be a semi-abelian category, $F:\mathcal{C}\to\mathcal{D}$ be a reduced functor, and n, m be two natural integers with $1\leqslant n<m$. Then we have the following natural isomorphism*

$$T_n(T_mF)\cong T_nF$$

More precisely, the natural transformation $T_n^*\cdot t_m^F : T_nF\Rightarrow T_n(T_mF)$ is an isomorphism.

Proof. As $t_m^F : F\Rightarrow T_mF$ is the cokernel of the natural transformation $S_m^F : cr_mF\cdot\Delta^m\Rightarrow F$ in $Func_*(\mathcal{C},\,Ab)$ and the n-Taylorization functor $T_n : Func_*(\mathcal{C},\,Ab)\to Func_{\leqslant n}(\mathcal{C},\,Ab)$, given in 1.9 of [12], preserves colimits (because T_n is a left adjoint functor by 1.10 of [12]), we obtain the following right exact sequence:

$$T_n(cr_mF\cdot\Delta^m) \overset{(T_n)_*\cdot S_m^F}{\Longrightarrow} T_nF \overset{(T_n)_*\cdot t_m^F}{\Longrightarrow} T_n(T_mF)\Longrightarrow 0$$

Applying 2.19 of [13], we deduce that $T_n(cr_mF\cdot\Delta^m)=0$ because the m-th cross-effet $cr_mF : \mathcal{C}^{\times m}\to Ab$ of F (see 1.2 of [12]) is a multireduced multifunctor and $1\leqslant n<m$. This proves that $(T_n)_*\cdot S_m^F=0$, so that $(T_n)_*\cdot t_m^F : T_nF\Rightarrow T_n(T_mF)$ is a natural isomorphism. Consequently, the functors $T_n(T_mF)$ and T_nF are isomorphic in $Func_{\leqslant n}(\mathcal{C},\,Ab)$. □

Proposition 2.3.3. *Let M be a quadratic $\mathcal{C}$-module and X be an object in $\mathcal{C}$, then we have*

$$T_1(-\otimes M)(X) \cong T_1U_E(X) \otimes Coker(P)\,,$$

which is natural in X. In particular, we obtain the following isomorphism of left $\overline{\Lambda}$-modules:

$$T_1(-\otimes M)(E) \cong Coker(P)$$

Proof. Let M be a quadratic $\mathcal{C}$-module. By 2.1.3, we recall that the quadratic functor $-\otimes M : \mathcal{C} \to Ab$ is the pushout in $Func(\mathcal{C},\ Ab)$ given below:

$$\begin{tikzcd}
\Big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e\Big)_{\mathfrak{S}_2} \oplus (U_E \otimes M_{ee}) \arrow[r, "\overline{\phi}=(\overline{\phi_1'\otimes id},t_2\otimes P)"] \arrow[d, "\overline{\psi}=(\overline{\psi_1},\pi(\delta\otimes id))"'] & T_2U_E \otimes_\Lambda M_e \arrow[d, "\widehat{\psi^M}"] \\
\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} M_{ee}\big)_{\mathfrak{S}_2} \arrow[r, "\widehat{\phi^M}"'] & -\otimes M
\end{tikzcd}$$

Let $i_2 : U_E \otimes M_{ee} \Rightarrow \big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e\big)_{\mathfrak{S}_2} \oplus (U_E \otimes M_{ee})$ be the injection of the second summand. As the functor $T_1 : Func_*(\mathcal{C}, Ab) \to Lin(\mathcal{C}, Ab)$ is left adjoint to the inclusion functor, then T_1 preserves colimits in $Func_*(\mathcal{C}, Ab)$. This leads to the following pushout in $Lin(\mathcal{C}, Ab)$:

$$\begin{tikzcd}
T_1\Big(\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e\big)_{\mathfrak{S}_2} \oplus (U_E \otimes M_{ee})\Big) \arrow[r, "T_1(\overline{\phi})"] \arrow[d, "T_1(\overline{\psi})"'] & T_1(T_2U_E \otimes_\Lambda M_e) \arrow[d, "T_1(\widehat{\psi}^M)"] \\
T_1\Big(\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} M_{ee}\big)_{\mathfrak{S}_2}\Big) \arrow[r, "T_1(\widehat{\phi^M})"'] & T_1(-\otimes M)
\end{tikzcd}$$

We know that $T_1\Big(\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} M_{ee}\big)_{\mathfrak{S}_2}\Big) : \mathcal{C} \to Ab$ is trivial by 2.4 of [12] or 2.19 of [13] because $\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} M_{ee}\big)_{\mathfrak{S}_2} : \mathcal{C} \to Ab$ is a diagonalizable functor (also called cohomogenous of degree $\leqslant 1$), see the definifion at the beginning of section 2.2 in [12]. The same argument also works for the diagonalizable functor $\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e\big)_{\mathfrak{S}_2} : \mathcal{C} \to Ab$, so that its linearization is trivial. Finally, we obtain the following right exact sequence in $Lin(\mathcal{C}, Ab)$:

$$T_1\Big((T_1U_E^{\otimes 2}\otimes_{\Lambda\otimes\Lambda}T_{11}cr_2(U_E)(E,E)\otimes_\Lambda M_e)_{\mathfrak{S}_2}\oplus(U_E\otimes M_{ee})\Big) \overset{T_1(\overline{\phi})}{\Longrightarrow} T_1(T_2U_E\otimes_\Lambda M_e) \overset{T_1(\widehat{\psi}^M)}{\Longrightarrow} T_1(-\otimes M) \Longrightarrow 0$$

As the functor $T_1 : Func_*(\mathcal{C}, Ab) \to Lin(\mathcal{C}, Ab)$ preserves colimits in $Func_*(\mathcal{C},\ Ab)$, we obtain the following isomorphisms in $Lin(\mathcal{C}, Ab)$:

$$\begin{aligned}
&T_1\Big(\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e\big)_{\mathfrak{S}_2} \oplus (U_E \otimes M_{ee})\Big) \\
&\cong T_1\Big(\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e\big)_{\mathfrak{S}_2}\Big) \oplus T_1(U_E \otimes M_{ee})) \\
&\cong T_1(U_E \otimes M_{ee}) \\
&\cong T_1U_E \otimes M_{ee}\,,\ \text{by 2.3.1.}
\end{aligned}$$

The third isomorphism above holds because the functor $\big(T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e\big)_{\mathfrak{S}_2} :$ $\mathcal{C} \to Ab$ is a diagonalizable functor so that its linearization is trivial by 2.4 of [12]. Evaluating the above isomorphisms on X, we have the following isomorphism in Ab:

$$B(X)\ \cong\ T_1U_E(X) \otimes M_{ee}$$

where $B(X) = T_1\Big((T_1U_E^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e)_{\mathfrak{S}_2} \oplus (U_E \otimes M_{ee})\Big)(X)$ whose injection of the first summand can be chosen to be the zero morphism, and the injection of the second summand can be taken to be $T_1(i_2) : T_1(U_E \otimes M_e)(X) \rightarrowtail B(X)$. In fact, the morphism $T_1(i_2)$ is an isomorphism thanks to the isomorphisms above. Moreover we get the following isomorphisms by using 2.3.1 and 2.3.2:

$$T_1\big(T_2U_E \otimes_\Lambda M_e\big)(X) \cong T_1(T_2U_E)(X) \otimes M_e \;\cong\; T_1U_E(X) \otimes M_e$$

Then we have the following diagram in Ab:

$$\begin{tikzcd}
B(X) \arrow[r, "T_1(\overline{\phi})_X"] \arrow[d, "\alpha_X"', "\cong"] & T_1\big(T_2U_E \otimes M_e\big)(X) \arrow[r, "T_1(\widehat{\psi^M})_X"] \arrow[d, "\cong"', "\beta_X"] & T_1(- \otimes M)(X) \arrow[r] \arrow[d, dashed, "\cong"', "\gamma_X"] & 0 \\
T_1U_E(X) \otimes M_{ee} \arrow[r, "id\otimes P"] & T_1U_E(X) \otimes M_e \arrow[r, "id\otimes coker(P)"] & T_1U_E(X) \otimes Coker(P) \arrow[r] & 0
\end{tikzcd}$$

where $B(X)$ is defined just above and $\alpha_X : B(X) \to T_1U_E(X) \otimes M_{ee}$, $\beta_X : T_1\big(T_2U_E \otimes M_e\big) \to T_1U_E(X) \otimes M_e$ are the unique morphisms in Ab such that

$$\alpha_X \circ T_1(i_2)_X \circ \big(t_1^{U_E\otimes M_{ee}}\big)_X = t_1 \otimes id \quad \text{and} \quad \beta_X \circ \big(t_1^{T_2U_E\otimes M_e}\big)_X \circ \big((t_2)_X \otimes id\big) = t_1 \otimes id \tag{2.3.3}$$

We prove that the left-hand square commutes, as follows:

$$\begin{aligned}
\beta_X \circ T_1(\overline{\phi})_X \circ T_1(i_2)_X \circ \big(t_1^{U_E\otimes M_{ee}}\big)_X &= \beta_X \circ T_1(\overline{\phi} \circ i_2)_X \circ \big(t_1^{U_E\otimes M_{ee}}\big)_X \\
&= \beta_X \circ T_1(t_2 \otimes P)_X \circ \big(t_1^{U_E\otimes M_{ee}}\big)_X \\
&= \beta_X \circ \big(t_1^{T_2U_E\otimes M_e}\big)_X \circ \big((t_2)_X \otimes P\big) \\
&= \beta_X \circ \big(t_1^{T_2U_E\otimes M_e}\big)_X \circ \big((t_2)_X \otimes id\big) \circ \big(id \otimes P\big) \\
&= \big((t_1)_X \otimes id\big) \circ \big(id \otimes P\big) \\
&= \big(id \otimes P\big) \circ \big((t_1)_X \otimes id\big) \\
&= \big(id \otimes P\big) \circ \alpha_X \circ T_1(i_2)_X \circ \big(t_1^{U_E)\otimes M_{ee}}\big)_X
\end{aligned}$$

As $\big(t_1^{U_E\otimes M_{ee}}\big)_X : U_E(X) \otimes M_{ee} \to T_1\big(U_E \otimes M_{ee}\big)(X)$ is an epimorphism, we have

$$\beta_X \circ T_1(\overline{\phi})_X \circ T_1(i_2)_X = \big(id \otimes P\big) \circ \alpha_X \circ T_1(i_2)_X$$

As $T_1(i_2)_X : T_1(U_E \otimes M_e)(X) \rightarrowtail B(X)$ is an epimorphism, we get

$$\beta_X \circ T_1(\overline{\phi})_X = \big(id \otimes P\big) \circ \alpha_X$$

as desired. Then a categorical argument provides a unique isomorphism $\gamma_X : T_1(- \otimes M)(X) \to T_1U_E(X) \otimes Coker(P)$ (natural in X) which makes the right-hand square of the above diagram commutes, i.e. such that

$$\gamma_X \circ T_1(\widehat{\psi^M})_X = \big(id \otimes_\Lambda coker(P)\big) \circ \beta_X\,,$$

that is equivalent to

$$\gamma_X \circ (t_1^{-\otimes M})_X \circ \widehat{\psi_X^M} = \gamma_X \circ T_1(\widehat{\psi^M})_X \circ \big(t_1^{T_2U_E\otimes_\Lambda M_e}\big)_X = (t_1)_X \otimes_\Lambda coker(P) \tag{2.3.4}$$

by (2.3.3) and because $\big(t_1^{T_2U_E\otimes_\Lambda M_e}\big)_X$ is a regular epimorphism (see 1.2.10). Moreover $Coker(P)$ is a left $\overline{\Lambda}$-module by 2.1.2, and it gives us the following isomorphism:

$$T_1(- \otimes M)(E) \cong T_1U_E(E) \otimes Coker(P) \cong \overline{\Lambda} \otimes Coker(P) \cong Coker(P)\,,$$

as desired. □

Notation 2.3.4. We denote by $\overline{\gamma} : T_1(-\otimes M)(E) \to Coker(P)$ the isomorphism obtained by precomposing $\gamma_E : T_1(-\otimes M)(E) \to \overline{\Lambda} \otimes Coker(P)$ with the evaluation isomorphism from $\overline{\Lambda} \otimes Coker(P)$ onto $Coker(P)$. If $x \in Coker(P)$, we write $\overline{x} = \overline{\gamma}^{-1}(x)$ to simplify notations.

2.4 Quadratic functors with values in algebras over a linear symmetric operad $\mathcal{P}$

Here we give the assume the following important hypothesis:

Assumption: from now on, we assume here that $\mathcal{C}$ is a semi-abelian variety and E is the free object of rank 1 in $\mathcal{C}$.

Notation 2.4.1. Let $\mathcal{P}$ be a linear symmetric unitary operad in the category of abelian groups endowed with its standard monoidal structure given by the tensor product. The unit of $\mathcal{P}$ is denoted by $1_{\mathcal{P}} \in \mathcal{P}(1)$.

In this part, we intend to make the same work as before for quadratic functors with domain $\mathcal{C}$ and values in $\mathcal{P}$-algebras.

Notation 2.4.2. For a $\mathcal{P}$-algebra A, we denote by $\lambda_k^A : A^{\otimes k} \otimes \mathcal{P}(k) \to A$, for $k \in \mathbb{N}^*$, the structure linear maps of A. Moreover $Alg-\mathcal{P}$ denotes the category of $\mathcal{P}$-algebras.

2.4.1 Aim and main arguments

We aim at finding DNA describing quadratic functors with domain $\mathcal{C}$ and values in $\mathcal{P}$-algebras.

Assumption: we suppose that $F : \mathcal{C} \to Alg-\mathcal{P}$ is a (reduced) quadratic functor in this section.

First we observe that F may be considered as taking values in $Mod_{\mathcal{P}(1)}$, so we know that a part of its DNA is given by the following quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ (see 2.2.7):

$$\mathbb{S}_2^{\mathcal{P}(1)}(F) = \left(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda F(E) \xrightarrow{H_E^F} F(E|E) \xrightarrow{T_E^F} F(E|E) \xrightarrow{(S_2^F)_E} F(E) \right)$$

where $\mathbb{S}_2^{\mathcal{P}(1)} : Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)}) \to QMod_{\mathcal{C}}^{\mathcal{P}(1)}$ is the functor defined in 2.2.4. It says that unary operations in the $\mathcal{P}$-algebra structure for F are entirely described by the notion of quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$, see 2.2.1.
Then we need three main steps to describe multilinear operations in the $\mathcal{P}$-algebra structure for F by using the notion of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$. The first step is to remark that the linear operad $\mathcal{P}$ can be supposed to be 2-step nilpotent, i.e. the abelian groups $\mathcal{P}(k)$ are trivial, for $k > 2$ (see 1.4.1 and 1.7.8 for details). Hence we observe that the quadratic functor $F : \mathcal{C} \to Alg-\mathcal{P}$ can be interpreted as a triple $(F, \lambda_1^F, \lambda_2^F)$, where F is seen as taking values in abelian groups, endowed with the structure natural transformations $\lambda_1^F : F \otimes \mathcal{P}(1) \Rightarrow F$ and $\lambda_2^F : F^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow F$ encoding respectively unary and binary operations.
The second step is to describe binary operations $\lambda_2^F : F^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow F$ encoded by $\mathcal{P}(2)$ as a certain morphism between quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$. For this, we compress the $\mathcal{P}$-algebras structure for F into just one natural transformation $\overline{\lambda_2^F} : T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)) \Rightarrow F$ which is the unique factorization of $\lambda_2^F : F^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow F$ through a certain natural transformation. Here T_2 is the

quadratization functor defined in 1.2.9 and $S = (\mathcal{P}(1) \otimes \mathcal{P}(1)) \wr \mathfrak{S}_2$ is the wreath product recalled in 2.4.9. Then it provides the morphism $\mathbb{S}_2^{\mathcal{P}(1)}(\overline{\lambda_2^F}) : \mathbb{S}_2^{\mathcal{P}(1)}(T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2))) \to \mathbb{S}_2^{\mathcal{P}(1)}(F)$ of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$ by applying the functor $\mathbb{S}_2^{\mathcal{P}(1)}$ to the natural transformaton $\overline{\lambda_2^F}$.
The third step is to prove the existence of a natural isomorphism $\phi^F : T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)) \Rightarrow T_1 F^{\otimes 2} \otimes_S \mathcal{P}(2)$ between quadratic functors with values in $Mod_{\mathcal{P}(1)}$. It leads to the isomorphism $\mathbb{S}_2^{\mathcal{P}(1)}(\phi^F)$ from $\mathbb{S}_2^{\mathcal{P}(1)}(T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)))$ onto $\mathbb{S}_2^{\mathcal{P}(1)}(T_1 F^{\otimes 2} \otimes_S \mathcal{P}(2))$ in the category $QMod_{\mathcal{C}}^{\mathcal{P}(1)}$. The main interest of this result is that $\mathbb{S}_2^{\mathcal{P}(1)}(T_1 F^{\otimes 2} \otimes_S \mathcal{P}(2))$ is a more understandable quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ than $\mathbb{S}_2^{\mathcal{P}(1)}(T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)))$.
Finally it gives the morphism $\mathbb{S}_2^{\mathcal{P}(1)}(\overline{\lambda_2^F} \circ (\phi^F)^{-1}) : \mathbb{S}_2^{\mathcal{P}(1)}(T_1 F^{\otimes 2} \otimes_S \mathcal{P}(2)) \to \mathbb{S}_2^{\mathcal{P}(1)}(F)$ of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$. This leads us to define quadratic $\mathcal{C}$-modules over $\mathcal{P}$ as pairs of the following form:

$$M^{\mathcal{P}} = (M,\ \phi^M : M^2 \to M)$$

where M is any quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$, M^2 is another such objects depending on M and ϕ^M is a morphism between these kinds of object, see 2.4.23 for details. The aim of this section is to prove that minimal algebraic data describing quadratic functors with domain $\mathcal{C}$ and values in $\mathcal{P}$-algebras are quadratic $\mathcal{C}$-modules over $\mathcal{P}$, see the theorem 2.4.37.

2.4.2 Assumption on the linear operad $\mathcal{P}$

In this part, we observe that quadratic functors with domain $\mathcal{C}$ and values in algebras over a linear operad can be considered as taking values in algebras over a 2-step nilpotent linear operad (i.e. the linear n-ary operations of the operad are trivial for $n > 2$).
As $F : \mathcal{C} \to Alg - \mathcal{P}$ is a (reduced) quadratic functor, it takes values in the full subcategory $Nil_2(Alg - \mathcal{P})$ of $Alg - \mathcal{P}$ constituted with 2-step nilpotent $\mathcal{P}$-algebras. By 1.7.8, there is an isomorphism of categories between $Nil_2(Alg - \mathcal{P})$ and $Alg - Nil_2(\mathcal{P})$, the category of $Nil_2(\mathcal{P})$-algebras. Here we recall that $Nil_2(\mathcal{P})$ is the 2-step nilpotent linear (unitary and symmetric) operad associated with $\mathcal{P}$, see 1.6.5 for details.
Consequently, taking a quadratic functor with domain $\mathcal{C}$ and values in $Alg - \mathcal{P}$ is equivalent to take a quadratic functor with domain $\mathcal{C}$ and values in $Alg - Nil_2(\mathcal{P})$. This explains why we don't need to consider the multi-linear maps $(\lambda_n^F)_X : F(X)^{\otimes n} \otimes \mathcal{P}(n) \to F(X)$ for $n > 2$ present in the $\mathcal{P}$-algebra structure for $F(X)$ where X an object in $\mathcal{C}$.

Assumption: from now on, the linear unitary symmetric operad $\mathcal{P}$ supposed to be 2-*step nilpotent*. In this case, $\mathcal{P} = Nil_2(\mathcal{P})$.

2.4.3 The structure bilinear maps for F encoded by $\mathcal{P}(2)$

Let X be an object in $\mathcal{C}$. In this part, we prove that the natural homomorphism $(\lambda_2^F)_X : F(X)^{\otimes 2} \otimes \mathcal{P}(2) \to F(X)$ is $\mathcal{P}(1) \otimes \mathcal{P}(1)$-bilinear and that it is also a homomorphism of right $\mathcal{P}(1)$-modules thanks to the axioms of $\mathcal{P}$-algebras for $F(X)$. First we give some notations:

Notation 2.4.3. Let B be a ring and A be a subring of B. Take M and N be respectively right and left B-modules (hence A-modules). As the tensor product $\otimes_B : M \times N \to M \otimes_B N$ is clearly A-bilinear, there is a unique homomorphism of abelian groups $q_A^B : M \otimes_A N \to M \otimes_B N$ such that the following diagram

$$\begin{tikzcd} M \times N \arrow[r, "\otimes_A"] \arrow[dr, "\otimes_B"'] & M \otimes_A N \arrow[d, "q_A^B"] \\ & M \otimes_B N \end{tikzcd}$$

commutes. We observe that $q_A^B : M \otimes_A N \to M \otimes_B N$ is B-bilinear and is an epimorphism. It is a straightforward exercice to prove that it satisfies the following universal property, namely: any B-bilinear map with domain $M \otimes_A N$ and values in an abelian group factorizes uniquely through $q_A^B : M \otimes_A N \to M \otimes_B N$ (it is a direct consequence of the universal property of the tensor product $\otimes_B$ and the fact that $\otimes_A$ is an epimorphism).

Then we denote by $\gamma_{1;1} : \mathcal{P}(1) \otimes \mathcal{P}(1) \to \mathcal{P}(1)$, $\gamma_{2;1} : \mathcal{P}(2) \otimes \mathcal{P}(1) \to \mathcal{P}(2)$ and $\gamma_{1,1;2} : (\mathcal{P}(1) \otimes \mathcal{P}(1)) \otimes \mathcal{P}(2) \to \mathcal{P}(2)$ the structure linear maps of the operad $\mathcal{P}$. By the axioms of the linear operad $\mathcal{P}$, $\gamma_{1;1}$ conferes a ring structure on $\mathcal{P}(1)$. Moreover the homomorphisms $\gamma_{2;1}$ and $\gamma_{1,1;2}$ confers $\mathcal{P}(2)$ a $(\mathcal{P}(1) \otimes \mathcal{P}(1))$-$\mathcal{P}(1)$-bimodule structure. Now we observe that $F(X)$ is a right $\mathcal{P}(1)$-module, with action of $\mathcal{P}(1)$ given by $(\lambda_1^F)_X : F(X) \otimes \mathcal{P}(1) \to F(X)$. This is due to the following commutative diagram

$$\begin{tikzcd}
F(X) \otimes (\mathcal{P}(1) \otimes \mathcal{P}(1)) \arrow[rr, "\cong"] \arrow[d, "id \otimes \gamma_{1;1}"'] & & (F(X) \otimes \mathcal{P}(1)) \otimes \mathcal{P}(1) \arrow[d, "(\lambda_1^F)_X \otimes id"] \\
F(X) \otimes \mathcal{P}(1) \arrow[dr, "(\lambda_1^F)_X"'] & & F(X) \otimes \mathcal{P}(1) \arrow[dl, "(\lambda_1^F)_X"] \\
& F(X) &
\end{tikzcd}$$

Then one of the axioms of the $\mathcal{P}$-algebra for $F(X)$ is given by the following commutative diagram:

$$\begin{tikzcd}
F(X)^{\otimes 2} \otimes (\mathcal{P}(2) \otimes \mathcal{P}(1)) \arrow[rr, "\cong"] \arrow[d, "id^{\otimes 2} \otimes \gamma_{2;1}"'] & & (F(X)^{\otimes 2} \otimes \mathcal{P}(2)) \otimes \mathcal{P}(1) \arrow[d, "(\lambda_2^F)_X \otimes id"] \\
F(X)^{\otimes 2} \otimes \mathcal{P}(2) \arrow[dr, "(\lambda_2^F)_X"'] & & F(X) \otimes \mathcal{P}(1) \arrow[dl, "(\lambda_1^F)_X"] \\
& F(X) &
\end{tikzcd}$$

It says that $(\lambda_2^F)_X : F(X)^{\otimes 2} \otimes \mathcal{P}(2) \to F(X)$ is a homomorphism of right $\mathcal{P}(1)$-modules. Hence the natural transformation $\lambda_2^F : F^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow F$ is a morphism in the category $Func_*(\mathcal{C}, Mod_{\mathcal{P}(1)})$. Then another axiom is given by the following commutative diagram:

$$\begin{tikzcd}
F(X)^{\otimes 2} \otimes (\mathcal{P}(1) \otimes \mathcal{P}(1)) \otimes \mathcal{P}(2) \arrow[rr, "\cong"] \arrow[d, "id^{\otimes 2} \otimes \gamma_{1,1;2}"'] & & \big((F(X) \otimes \mathcal{P}(1)) \otimes (F(X) \otimes \mathcal{P}(1))\big) \otimes \mathcal{P}(2) \arrow[d, "(\lambda_1^F)_X^{\otimes 2} \otimes id"] \\
F(X)^{\otimes 2} \otimes \mathcal{P}(2) \arrow[dr, "(\lambda_2^F)_X"'] & & F(X)^{\otimes 2} \otimes \mathcal{P}(2) \arrow[dl, "(\lambda_2^F)_X"] \\
& F(X) &
\end{tikzcd}$$

It says that $(\lambda_2^F)_X : F(X)^{\otimes 2} \otimes \mathcal{P}(2) \to F(X)$ is $\mathcal{P}(1) \otimes \mathcal{P}(1)$-bilinear. By 2.4.3, there is a unique homomorphism of abelian groups $(\widehat{\lambda_2}^F)_X : F(X)^{\otimes 2} \otimes_{\mathcal{P}(1) \otimes \mathcal{P}(1)} \mathcal{P}(2) \to F(X)$ such that

$$(\widehat{\lambda_2}^F)_X \circ q_{\mathbb{Z}}^{\mathcal{P}(1) \otimes \mathcal{P}(1)} = (\lambda_2^F)_X \tag{2.4.1}$$

As $(\lambda_2^F)_X$ is a homomorphism of right $\mathcal{P}(1)$-modules and $q_{\mathbb{Z}}^{\mathcal{P}(1)\otimes\mathcal{P}(1)}$ is an epimorphism, $(\widehat{\lambda_2}^F)_X$ also is a homomorphism of right $\mathcal{P}(1)$-modules. As the construction is functorial, $\widehat{\lambda_2}^F$ is a natural transformation living in $Func_*(\mathcal{C},\ Mod_{\mathcal{P}(1)})$.

We here point out that the above arguments in this subsection also hold for any $\mathcal{P}$-algebra. Then we here provide a classical way to compress the axioms of $\mathcal{P}$-algebra for A in term of a unique morphism. Consider a $\mathcal{P}$-algebra A with its structure linear maps $\lambda_1^A : A\otimes\mathcal{P}(1)\to A$ and $\lambda_2^A : A^{\otimes 2}\otimes\mathcal{P}(2)\to A$. By 1.8.6, we recall that, for two $\mathcal{P}$-algebras A and B, the second cross-effect of the identity functor of $Alg-\mathcal{P}$ is defined on objects by

$$Id_{Alg-\mathcal{P}}(A|B) = \overline{A}\otimes\overline{B}\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2)$$

where $\overline{A}$ (resp. $\overline{B}$) is the quotient of A (resp. B) by the ideal A^2 (resp. B^2), see the notations given in 1.7.6. Now since the operad $\mathcal{P}$ is supposed to be 2-step nilpotent, we obtain

$$\lambda_2^A\circ\big(\lambda_2^A\otimes id\otimes id\big) = 0 = \lambda_2^A\circ\big(id\otimes\lambda_2^A\otimes id\big)$$

by an associativity relation for $\mathcal{P}$-algebras. Hence there is a unique morphism $\widetilde{\lambda_2^A} : (\overline{A})^{\otimes 2}\otimes\mathcal{P}(2)\to A$ such that

$$\widetilde{\lambda_2^A}\big(\overline{a_1}\otimes\overline{a_2}\otimes p\big) = \lambda_2^A\big(a_1\otimes a_2\otimes p\big)$$

where $a_1, a_2\in A$ and $p\in\mathcal{P}(2)$. It is a (right) $\mathcal{P}(1)$-module homomorphism because so is λ_2^A. Similarly, we know that the map λ_2^A is a (right) $\mathcal{P}(1)$-modules homomorphism. In addition, there is a unique abelian groups homomorphims $\widehat{\lambda_2^A} : (\overline{A})^{\otimes 2}\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2)\to A$ such that

$$\widehat{\lambda_2^A}\circ q_{\mathbb{Z}}^{\mathcal{P}(1)\otimes\mathcal{P}(1)} = \widetilde{\lambda_2^A}$$

by 2.4.3. It is a (right) $\mathcal{P}(1)$-module homomorphism because so is $\widetilde{\lambda_2^A}$. The equivariance axiom says that

$$\lambda_2^A\big(a_1\otimes a_2\otimes p\big) = \lambda_2^A\big(T(a_1\otimes a_2)\otimes(p.t)\big)$$

where $T : A^{\otimes 2}\to A^{\otimes 2}$, $x\otimes y\mapsto y\otimes x$ is the canonical switch and $t:\mathcal{P}(2)\to\mathcal{P}(2)$ is the (right) action of $\mathfrak{S}_2$ on $\mathcal{P}(2)$ in the operad structure of $\mathcal{P}$. As $q_{\mathbb{Z}}^{\mathcal{P}(1)\otimes\mathcal{P}(1)} : A^{\otimes 2}\otimes\mathcal{P}(2)\to A^{\otimes 2}\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2)$ is natural and is surjective, we also have

$$\widehat{\lambda_2^A}\big(\overline{a_1}\otimes\overline{a_2}\otimes p\big) = \widehat{\lambda_2^A}\big(\overline{T}(\overline{a_1}\otimes\overline{a_2})\otimes(p.t)\big) \tag{2.4.2}$$

where $\overline{T} : (\overline{A})^{\otimes 2}\to(\overline{A})^{\otimes 2}$ also is the canonical switch.

Notation 2.4.4. We denote by

$$Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2} = \big((\overline{A})^{\otimes 2}\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2)\big)_{\mathfrak{S}_2}$$

the set of coinvariants of $Id_{Alg-\mathcal{P}}(A|A)$ and by $\pi : Id_{Alg-\mathcal{P}}(A|A)\to Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2}$ is the canonical quotient map which clearly satisfies

$$\pi = \pi\circ\big(\overline{T}\otimes t\big)$$

where $\overline{T} : A^{\otimes 2}\to A^{\otimes 2}$ is the canonical switch.

The relation (2.4.2) implies that there is a unique abelian group homomorphism $\overline{\lambda_2^A}$: $Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2} \to A$ such that

$$\overline{\lambda_2^A} \circ \pi = \widehat{\lambda_2^A}$$

It also is a (right) $\mathcal{P}(1)$-module homomorphism because so is $\widehat{\lambda_2^A}$. By 1.7.4, we recall that $\gamma_2^{Id_{Alg-\mathcal{P}}}(A) = [A,A]_{Id_{Alg-\mathcal{P}}} = Im\big(\lambda_2^A : A^{\otimes 2} \otimes \mathcal{P}(2) \to A\big)$.

Notation 2.4.5. We denote by $\overline{\overline{\lambda_2^A}}$: $Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2} \to [A,A]_{Id_{Alg-\mathcal{P}}}$ the restriction map of $\overline{\lambda_2^A}$: $Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2} \to A$ onto its image. It is clearly a surjective map.

Remark 2.4.6. In summary, for $a_1, a_2 \in A$ and $p \in \mathcal{P}(2)$, we have

$$\overline{\overline{\lambda_2^A}}(\overline{\overline{a_1} \otimes \overline{a_2} \otimes p}) = \lambda_2^A(a_1 \otimes a_2 \otimes p)$$

We now are able to give another description of the image $[A,A]_{Id_{Alg-\mathcal{P}}} = Im(\lambda_2^A)$ (see 1.7.4) whenever A is a free $\mathcal{P}$-algebra of finite rank.

Proposition 2.4.7. *If A is a free $\mathcal{P}$-algebra of finite rank, then the surjective map*

$$\overline{\overline{\lambda_2^A}} : Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2} \to [A,A]_{Id_{Alg-\mathcal{P}}}$$

given in 2.4.5 is an isomorphism of $\mathcal{P}(1)$-modules.

Proof. First we recall that $\overline{\overline{\lambda_2^A}} : Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2} \to Im(\lambda_2^A)$ is defined by

$$\overline{\overline{\lambda_2^A}}\big(\overline{\overline{a_1} \otimes \overline{a_2} \otimes p_2}\big) = \lambda_2^A(a_1 \otimes a_2 \otimes p_2) \tag{2.4.3}$$

where $a_1, a_2 \in A$ and $p_2 \in \mathcal{P}(2)$. Then we observe that $\overline{\overline{\lambda_2^A}} : Id_{Alg-\mathcal{P}}(A|A)_{\mathfrak{S}_2} \to [A,A]_{Alg-\mathcal{P}}$ is natural in A so that it gives rise to a natural transformation

$$\overline{\overline{\lambda_2}} : I \cdot \big((Ab^{Alg-\mathcal{P}})^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \Longrightarrow \gamma_2^{Id_{Alg-\mathcal{P}}}$$

in the category of $\mathcal{P}$-algebras, where $I : Mod_{\mathcal{P}(1)} = Ab(Alg-\mathcal{P}) \to Alg-\mathcal{P}$ is the inlcusion functor. We observe that $Im(\lambda_2)$ can be seen as a subfunctor of the identity functor $Id_{Alg-\mathcal{P}} : Alg-\mathcal{P} \to Alg-\mathcal{P}$. As $\mathcal{P}$ is a 2-step nilpotent operad, the category of $\mathcal{P}$-algebras is 2-step nilpotent so that the functor $Id_{Alg-\mathcal{P}} : Alg-\mathcal{P} \to Alg-\mathcal{P}$ is quadratic by 1.3.10. It implies that the functor $Im(\lambda_2) : Alg-\mathcal{P} \to Alg-\mathcal{P}$ also is quadratic. Then the functor

$$I \cdot \big((Ab^{Alg-\mathcal{P}})^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\big)_{\mathfrak{S}_2} : Alg-\mathcal{P} \to Alg-\mathcal{P}$$

is quadratic by 1.2.6. Hence the natural transformation $\overline{\overline{\lambda_2}}$ with quadratic source and target restricted to $\langle\mathcal{F}_{\mathcal{P}}\rangle$ (the full subcategory of free $\mathcal{P}$-algebras of finite rank of $Alg-\mathcal{P}$) is an isomorphism if, and only if, $\overline{\overline{\lambda_2^A}}$ is an isomorphism of $\mathcal{P}(1)$-modules, for $A = \mathcal{F}_{\mathcal{P}}$ and $A = \mathcal{F}_{\mathcal{P}}^{+2}$, by 1.17 of [12].
We first prove that, for $A = \mathcal{F}_{\mathcal{P}}$, $\overline{\overline{\lambda_2^A}}$ is an isomorphism. We observe that $Im(\lambda_2^{\mathcal{F}_{\mathcal{P}}})$ consists of elements of the form $(0, \overline{p_2})$, for $p_2 \in \mathcal{P}(2)$ (see 1.8.3). For this we defined the $\mathcal{P}(1)$-module homomorphism $r_{\mathcal{F}_{\mathcal{P}}} : Im(\lambda_2^{\mathcal{F}_{\mathcal{P}}}) \to \big((\overline{\mathcal{F}_{\mathcal{P}}})^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\big)_{\mathfrak{S}_2}$ by

$$r_{\mathcal{F}_{\mathcal{P}}}(0, \overline{p_2}) = \overline{\overline{(id, \overline{0})} \otimes \overline{(id, \overline{0})} \otimes p_2} \tag{2.4.4}$$

where $p_2 \in \mathcal{P}(2)$. Then we have the equalities as follows:

$$\begin{aligned}
r_{\mathcal{F}_\mathcal{P}} \circ \overline{\overline{\lambda_2^{\mathcal{F}_\mathcal{P}}}}\big(\overline{\overline{(p_1,\bar{0})},\ \overline{(p_1',\bar{0})} \otimes p_2}\big) &= r_{\mathcal{F}_\mathcal{P}}\big(\lambda_2^{\mathcal{F}_\mathcal{P}}\big((p_1,\bar{0}) \otimes (p_1',\bar{0}) \otimes p_2\big)\big) \text{, by (1.8.2)} \\
&= r_{\mathcal{F}_\mathcal{P}}\big(0,\ \overline{\gamma_{1,1;2}(p_1 \otimes p_1' \otimes p_2)}\big) \\
&= \overline{\overline{(id,\bar{0})} \otimes \overline{(id,\bar{0})} \otimes \gamma_{1,1;2}(p_1 \otimes p_1' \otimes p_2)} \\
&= \overline{\overline{(p_1,\bar{0})} \otimes \overline{(p_1',\bar{0})} \otimes p_2}
\end{aligned}$$

where $p_1, p_1' \in \mathcal{P}(1)$ and $p_2 \in \mathcal{P}(2)$. Thus we obtain $r_{\mathcal{F}_\mathcal{P}} \circ \overline{\overline{\lambda_2^{\mathcal{F}_\mathcal{P}}}} = id$ implying that $\overline{\overline{\lambda_2^{\mathcal{F}_\mathcal{P}}}} : \big((\overline{\mathcal{F}_\mathcal{P}})^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \to Im(\lambda_2^{\mathcal{F}_\mathcal{P}})$ is a monomorphism. Since $\overline{\overline{\lambda_2^{\mathcal{F}_\mathcal{P}}}}$ is a surjective map, it is an isomorphism.

Next we check that, for $A = \mathcal{F}_\mathcal{P}^{+2}$, $\overline{\overline{\lambda_2^A}}$ is an isomorphism. By 1.8.10, we recall that the (right) $\mathcal{P}(1)$-module $\overline{\mathcal{F}_\mathcal{P}^{+2}}$ consists of elements of the form $\overline{\big((p_1^1,\bar{0}),(p_1^2,\bar{0}),0\big)}$, where $p_1^1, p_1^2 \in \mathcal{P}(1)$. Then we define the $\mathcal{P}(1)$-module homomorphism $r_{\mathcal{F}_\mathcal{P}^{+2}} : Im(\lambda_2^{\mathcal{F}_\mathcal{P}^{+2}}) \to \big((\overline{\mathcal{F}_\mathcal{P}^{+2}})^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\big)_{\mathfrak{S}_2}$ defined by

$$\begin{aligned}
&r_{\mathcal{F}_\mathcal{P}^{+2}}\big((0,\overline{p_2^1}),\ (0,\overline{p_2^2}),\ p_2\big) \\
&= \overline{\big((id,\bar{0}),0_{\mathcal{F}_\mathcal{P}},0\big) \otimes \big((id,\bar{0}),0_{\mathcal{F}_\mathcal{P}},0\big) \otimes p_2^1} + \overline{\big(0_{\mathcal{F}_\mathcal{P}},(id,\bar{0}),0\big) \otimes \big(0_{\mathcal{F}_\mathcal{P}},(id,\bar{0}),0\big) \otimes p_2^2} \\
&\quad + \overline{\big((id,\bar{0}),0_{\mathcal{F}_\mathcal{P}},0\big) \otimes \big(0_{\mathcal{F}_\mathcal{P}},(id,\bar{0}),0\big) \otimes p_2}
\end{aligned}$$

where $p_2^1, p_2^2, p_2 \in \mathcal{P}(2)$. Let $p_1^{i,j} \in \mathcal{P}(1)$ with $i, j = 1, 2$. Then we have the following equalities:

$$
\begin{aligned}
& r_{\mathcal{F}_{\mathcal{P}}^{+2}} \circ \overline{\overline{\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}}} \Big(\overline{((p_1^{1,1},\overline{0}),(p_1^{1,2},\overline{0}),0) \otimes ((p_1^{2,1},\overline{0}),(p_1^{2,2},\overline{0}),0) \otimes p_2} \Big) \\
&= r_{\mathcal{F}_{\mathcal{P}}^{+2}} \Big(\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}} \Big(((p_1^{1,1},\overline{0}),(p_1^{1,2},\overline{0}),0) \otimes ((p_1^{2,1},\overline{0}),(p_1^{2,2},\overline{0}),0) \otimes p_2 \Big) \Big) \\
&= r_{\mathcal{F}_{\mathcal{P}}^{+2}} \Big(\lambda_2^{\mathcal{F}_{\mathcal{P}}} \big((p_1^{1,1},\overline{0}) \otimes (p_1^{2,1},\overline{0}) \otimes p_2 \big),\ \lambda_2^{\mathcal{F}_{\mathcal{P}}} \big((p_1^{1,2},\overline{0}) \otimes (p_1^{2,2},\overline{0}) \otimes p_2 \big), \\
&\qquad \gamma_{1,1;2}\big((p_1^{1,1},\overline{p_2^{1,1}}) \otimes (p_1^{2,2},\overline{p_2^{2,2}}) \otimes p_2 \big) \ + \ \gamma_{1,1;2}\big((p_1^{2,1},\overline{p_2^{2,1}}) \otimes (p_1^{1,2},\overline{p_2^{1,2}}) \otimes (p_2.t) \big) \Big) \text{, by 1.8.9} \\
&= r_{\mathcal{F}_{\mathcal{P}}^{+2}} \Big(\big(0, \overline{\gamma_{1,1;2}(p_1^{1,1} \otimes p_1^{2,1} \otimes p_2)}\big),\ \big(0, \overline{\gamma_{1,1;2}(p_1^{1,2} \otimes p_1^{2,2} \otimes p_2)}\big), \\
&\qquad \gamma_{1,1;2}((p_1^{1,1} \otimes p_1^{2,2} \otimes p_2) \ + \ \gamma_{1,1;2}(p_1^{2,1} \otimes p_1^{1,2} \otimes (p_2.t) \big) \Big) \text{, by (1.8.3)} \\
&= \overline{\big((id,\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big((id,\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \gamma_{1,1;2}(p_1^{1,1} \otimes p_1^{2,1} \otimes p_2)} \\
&\qquad + \overline{\big(0_{\mathcal{F}_{\mathcal{P}}},(id,\overline{0}),0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(id,\overline{0}),0\big) \otimes \gamma_{1,1;2}(p_1^{1,2} \otimes p_1^{2,2} \otimes p_2)} \\
&\qquad + \overline{\big((id,\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(id,\overline{0}),0\big) \otimes \big(\gamma_{1,1;2}((p_1^{1,1} \otimes p_1^{2,2} \otimes p_2) \ + \ \gamma_{1,1;2}(p_1^{2,1} \otimes p_1^{1,2} \otimes (p_2.t))\big)} \\
&= \overline{\big((p_1^{1,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big((p_1^{2,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes p_2} \ + \ \overline{\big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{1,2},\overline{0}),0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{2,2},\overline{0}),0\big) \otimes p_2} \\
&\qquad + \overline{\big((id,\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(id,\overline{0}),0\big) \otimes \gamma_{1,1;2}((p_1^{1,1} \otimes p_1^{2,2} \otimes p_2)} \\
&\qquad + \overline{\big((id,\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(id,\overline{0}),0\big) \otimes \gamma_{1,1;2}(p_1^{2,1} \otimes p_1^{1,2} \otimes (p_2.t))} \\
&= \overline{\big((p_1^{1,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big((p_1^{2,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes p_2} \ + \ \overline{\big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{1,2},\overline{0}),0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{2,2},\overline{0}),0\big) \otimes p_2} \\
&\qquad + \overline{\big((p_1^{1,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{2,2},\overline{0}),0\big) \otimes p_2} \ + \ \overline{\big((p_1^{2,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{1,2},\overline{0}),0\big) \otimes (p_2.t)}
\end{aligned}
$$

Moreover we get

$$
\overline{\big((p_1^{2,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{1,2},\overline{0}),0\big) \otimes (p_2.t)} = \overline{\big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{1,2},\overline{0}),0\big) \otimes \big((p_1^{2,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes p_2}\,,
$$

see 2.4.4. Then we obtain

$$
\begin{aligned}
& r_{\mathcal{F}_{\mathcal{P}}^{+2}} \circ \overline{\overline{\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}}} \Big(\overline{((p_1^{1,1},\overline{0}),(p_1^{1,2},\overline{0}),0) \otimes ((p_1^{2,1},\overline{0}),(p_1^{2,2},\overline{0}),0) \otimes p_2} \Big) \\
&= \overline{\big((p_1^{1,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big((p_1^{2,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes p_2} \ + \ \overline{\big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{1,2},\overline{0}),0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{2,2},\overline{0}),0\big) \otimes p_2} \\
&\qquad + \overline{\big((p_1^{1,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes \big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{2,2},\overline{0}),0\big) \otimes p_2} \ + \ \overline{\big(0_{\mathcal{F}_{\mathcal{P}}},(p_1^{1,2},\overline{0}),0\big) \otimes \big((p_1^{2,1},\overline{0}),0_{\mathcal{F}_{\mathcal{P}}},0\big) \otimes p_2} \\
&= \overline{\big((p_1^{1,1},\overline{0}),(p_1^{1,2},\overline{0}),0\big) \otimes \big((p_1^{2,1},\overline{0}),(p_1^{2,2},\overline{0}),0\big) \otimes p_2}
\end{aligned}
$$

Thus we have

$$
r_{\mathcal{F}_{\mathcal{P}}^{+2}} \circ \overline{\overline{\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}}} = id
$$

implying that the map $\overline{\overline{\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}}}$ is a monomorphism. Since $\overline{\overline{\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}}}$ is surjective, it is an isomorphism. Hence $\overline{\overline{\lambda_2^{A}}}$ is an isomorphism of $\mathcal{P}(1)$-modules, for any free $\mathcal{P}$-algebra A of finite rank. $\square$

In summary, a $\mathcal{P}$-algebra A can be equivalently seen as a right $\mathcal{P}(1)$-module endowed with a $\mathcal{P}(1)$-module homomorphism $\overline{\lambda_2^A} : \big((\overline{A})^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \to A$. In the next subsection, we give a different way to describe the equivariance axiom by taking the cokernel of a certain natural transformation. Then we use the (polynomial) functors calculus providing a functorial way to describe the structure linear natural transformations $\lambda_1^F : F^{\otimes 2} \Rightarrow F$, $\lambda_2^F : F^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow F$ and their relations into a unique natural transformation.

2.4.4 The equivariance axiom

Here we show that the equivariance axiom for F (as it takes values in $\mathcal{P}$-algebras) allows to factorize the natural transformation $\widehat{\lambda_2}^F : F^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \Rightarrow F$ through a certain cokernel.

Notation 2.4.8. For a (reduced) functor $G : \mathcal{C} \to Ab$ and an object X in $\mathcal{C}$, we denote by $\widehat{T_X^G} : G(X)^{\otimes 2} \to G(X)^{\otimes 2}$, $x \otimes y \mapsto y \otimes x$ the canonical switch which is clearly natural in X.

As $\mathcal{P}$ is a symmetric 2-step nilpotent operad, there is just one diagram left as follows:

$$\begin{tikzcd} F(X)^{\otimes 2} \otimes \mathcal{P}(2) \arrow[rr, "id\otimes t"] \arrow[d, "\widehat{T_X^F}\otimes id"'] & & F(X)^{\otimes 2} \otimes \mathcal{P}(2) \arrow[d, "(\lambda_2^F)_X"] \\ F(X)^{\otimes 2} \otimes \mathcal{P}(2) \arrow[rr, "(\lambda_2^F)_X"] & & F(X) \end{tikzcd}$$

where $\widehat{T_X^F} : F(X)^{\otimes 2} \to F(X)^{\otimes 2}$, $x \otimes y \mapsto y \otimes x$ is the canonical switch and t the right action of $\mathfrak{S}_2$ on $\mathcal{P}(2)$. This above diagram equivalently says that we have

$$(\lambda_2^F)_X \circ (\widehat{T_X^F} \otimes id - id \otimes t) = 0$$

Then we have the following equations:

$$\begin{aligned} &(\lambda_2^F)_X \circ \big(\widehat{T_X^F} \otimes id - id \otimes t\big) = 0 \\ &\Leftrightarrow (\widehat{\lambda_2}^F)_X \circ q_{\mathbb{Z}}^{\mathcal{P}(1)\otimes\mathcal{P}(1)} \circ \big(\widehat{T_X^F} \otimes id - id \otimes t\big) = 0 \\ &\Leftrightarrow (\widehat{\lambda_2}^F)_X \circ \big(\widehat{T_X^F} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} id - id \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} t\big) \circ q_{\mathbb{Z}}^{\mathcal{P}(1)\otimes\mathcal{P}(1)} = 0 \\ &\Leftrightarrow (\widehat{\lambda_2}^F)_X \circ \big(\widehat{T_X^F} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} id - id \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} t\big) = 0\,, \text{ because } q_{\mathbb{Z}}^{\mathcal{P}(1)\otimes\mathcal{P}(1)} \text{ is a regular epimorphism.} \end{aligned}$$

Finally we obtain

$$(\widehat{\lambda_2}^F)_X \circ \big(\widehat{T_X^F} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} id - id \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} t\big) = 0 \tag{2.4.5}$$

Notation 2.4.9. Here we denote respectively $R = \mathcal{P}(1)$, and $S = (R \otimes R) \wr \mathfrak{S}_2$ the wreath product defined by

$$(R \otimes R) \wr \mathfrak{S}_2 = (R \otimes R) \oplus (R \otimes R).t$$

whose multiplication is given by

$$\big(r_1 \otimes r_2 + (s_1 \otimes s_2).t\big)\big(r'_1 \otimes r'_2 + (s'_1 \otimes s'_2).t\big) = \big(r_1r'_1 \otimes r_2r'_2 + s_1s'_2 \otimes s_2s'_1\big) + \big(r_1s'_1 \otimes r_2s'_2 + s_1r'_2 \otimes s_2r'_1\big).t$$

where $r_i, r'_i, s_i, s'_i \in R$ and t denotes the generator of $\mathfrak{S}_2$. It is defined in 3.24 of [12].

Remarks 2.4.10. We have the following three observations:

- $R \otimes R$ is a subring of S.

- Any left (resp. right) S-module is a left (resp. right) $(R \otimes R)$-module, and any S-bilinear map is $(R \otimes R)$-bilinear.

- A right symmetric $(R \otimes R)$-module M with involution T has a right S-module structure and it is given by
$$m.\big(r_1 \otimes r_2 + (s_1 \otimes s_2).t\big) = m.(r_1 \otimes r_2) + T(m).(s_2 \otimes s_1) \tag{2.4.6}$$
where $r_i, r_i', s_i, s_i' \in R$.

Example 2.4.11. As an example, we observe that

- $F(X)^{\otimes 2}$ has a right S-module structure given by
$$x \otimes y.\big((r_1 \otimes r_2) + (s_1 \otimes s_2).t\big) = \lambda_1^F(x \otimes r_1) \otimes \lambda_1^F(y \otimes r_2) + \lambda_1^F(y \otimes s_2) \otimes \lambda_1^F(x \otimes s_1)$$
where $x, y \in F(X)$. This structure commutes with the right $\mathcal{P}(1)$-module structure on $\mathcal{P}(2)$ so that $\mathcal{P}(2)$ actually is an S-$\mathcal{P}(1)$-bimodule;

- $\mathcal{P}(2)$ has a left S-module structure given by
$$\big((r_1 \otimes r_2) + (s_1 \otimes s_2).t\big).p = \gamma_{1,1;2}(r_1 \otimes r_2 \otimes p) + \gamma_{1,1;2}(s_1 \otimes s_2 \otimes p.t)$$
where $r_k,\ s_k \in \mathcal{P}(1)$ and $p \in \mathcal{P}(2)$.

Notation 2.4.12. For X an object in $\mathcal{C}$, we write $q_X^F = q_{R\otimes R}^S : F(X)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to F(X)^{\otimes 2} \otimes_S \mathcal{P}(2)$ (see 2.4.3) which is natural in X.

Then we prove that $q_X^F : F(X)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to F(X)^{\otimes 2} \otimes_S \mathcal{P}(2)$ is the cokernel of $\widehat{T_X^F} \otimes_{R\otimes R} id - id \otimes_{R\otimes R} t$.

Proposition 2.4.13. *The natural transformation $q^F : F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \Rightarrow F^{\otimes 2} \otimes_S \mathcal{P}(2)$ is the cokernel of $\widehat{T^F} \otimes_{R\otimes R} id - id \otimes_{R\otimes R} t : F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \Rightarrow F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$ in $Func_*(\mathcal{C},\ Mod_{\mathcal{P}(1)})$.*

Proof. Let X be an object in $\mathcal{C}$. First we remark that $q_X^F : F(X)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to F(X)^{\otimes 2} \otimes_S \mathcal{P}(2)$ is an epimorphism and that is a homomorphism of right $\mathcal{P}(1)$-modules. Then we check that q_X^F is the cokernel of $\widehat{T_X^F} \otimes_{R\otimes R} id - id \otimes_{R\otimes R} t$. Let $\phi : F(X)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to C$ be a homomorphism of right $\mathcal{P}(1)$-modules such that
$$\phi \circ \big(\widehat{T_X^F} \otimes_{R\otimes R} id\big) = \phi \circ \big(id \otimes_{R\otimes R} t\big)$$
It suffices to prove that $\phi : F(X)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to C$ is S-bilinear as follows:
$$\begin{aligned}&\phi\Big(x \otimes y.\big((r_1 \otimes r_2) + (s_1 \otimes s_2).t\big) \otimes_{R\otimes R}\ p\Big)\\ &= \phi\big(\lambda_1^F(x \otimes r_1) \otimes \lambda_1^F(y \otimes r_2) \otimes_{R\otimes R} p\big) + \phi\big(\lambda_1^F(y \otimes s_2) \otimes \lambda_1^F(x \otimes s_1) \otimes_{R\otimes R} p\big)\end{aligned}$$
However we get
$$\begin{aligned}\phi\big(\lambda_1^F(y \otimes s_2) \otimes \lambda_1^F(x \otimes s_1) \otimes_{R\otimes R} p\big) &= \phi \circ \big(\widehat{T_X^F} \otimes_{R\otimes R} id\big)\big(\lambda_1^F(x \otimes s_1) \otimes \lambda_1^F(y \otimes s_2) \otimes_{R\otimes R} p\big)\\ &= \phi \circ \big(id \otimes_{R\otimes R} t\big)\big(\lambda_1^F(x \otimes s_1) \otimes \lambda_1^F(y \otimes s_2) \otimes_{R\otimes R} p\big)\\ &= \phi\big(\lambda_1^F(x \otimes s_1) \otimes \lambda_1^F(y \otimes s_2) \otimes_{R\otimes R} p.t\big)\\ &= \phi\big(x \otimes y \otimes_{R\otimes R} \gamma_{1,1;2}(s_1 \otimes s_2 \otimes p.t)\big)\end{aligned}$$

Hence we have

$$\begin{aligned}
&\phi\Big(x \otimes y.\big((r_1 \otimes r_2) + (s_1 \otimes s_2).t\big) \otimes_{R\otimes R} \ p\Big) \\
&= \phi\big(x \otimes y \otimes_{R\otimes R} \ \gamma_{1,1;2}(r_1 \otimes r_2 \otimes p)\big) + \phi\big(x \otimes y \otimes_{R\otimes R} \ \gamma_{1,1;2}(s_1 \otimes s_2 \otimes p.t)\big) \\
&= \phi\Big(x \otimes y \otimes_{R\otimes R} \ \big(\gamma_{1,1;2}(r_1 \otimes r_2 \otimes p) + \gamma_{1,1;2}(s_1 \otimes s_2 \otimes p.t)\big)\Big) \\
&= \phi\Big(x \otimes y \otimes_{R\otimes R} \ \big((r_1 \otimes r_2) + (s_1 \otimes s_2).t\big).p\Big)
\end{aligned}$$

This proves that $\phi : F(X)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to C$ is S-bilinear. By 2.4.3, there is a unique homomorphism of abelian groups $\overline{\phi_X} : F(X)^{\otimes 2} \otimes_S \mathcal{P}(2) \to C$ such that

$$\phi = \overline{\phi_X} \circ q_X^F$$

Moreover $\overline{\phi_X}$ is a homomorphism of right $\mathcal{P}(1)$-modules because so are ϕ_X and q_X^F. It proves the result. $\square$

To summarize, we have the natural transformation $\widehat{\lambda_2}^F : F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \Rightarrow F$ verifying

$$\widehat{\lambda_2}^F \circ \big(\widehat{T^F} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} id - id \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} t\big) = 0$$

By 2.4.13 and (2.4.5), there is a unique natural transformation $\widetilde{\lambda_2}^F : F^{\otimes 2} \otimes_S \mathcal{P}(2) \Rightarrow F$ factorizing $\widehat{\lambda_2}^F$ through $q^F : F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \Rightarrow F^{\otimes 2} \otimes_S \mathcal{P}(2)$, i.e.

$$\widehat{\lambda_2}^F = \widetilde{\lambda_2}^F \circ q^F \tag{2.4.7}$$

However we observe that we can not apply the functor $\mathbb{S}_2^{\mathcal{P}(1)} : Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)}) \to Qmod_{\mathcal{C}}^{\mathcal{P}(1)}$ (see 2.2.4) to the natural transformation $\widetilde{\lambda_2}^F$ because $F^{\otimes 2} \otimes_S \mathcal{P}(2) : \mathcal{C} \to Mod_{\mathcal{P}(1)}$ is not a quadratic functor in general but polynomial of degree 4. That is the reason why we use the universal property of the unit of the adjunction given in 1.10 of [12], which makes sense because F is a quadratic functor. More precisely, there is a unique natural transformation $\overline{\lambda_2}^F : T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)) \Rightarrow F$ such that

$$\widetilde{\lambda_2}^F = \overline{\lambda_2}^F \circ t_2^{F^{\otimes 2}\otimes_S \mathcal{P}(2)} \tag{2.4.8}$$

The proposition 1.10 of [12] also says that $T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)) : \mathcal{C} \to Ab$ is a quadratic functor taking values in right $\mathcal{P}(1)$-modules. Finally we have expressed the $\mathcal{P}$-algebra structure for F in terms of a natural transformation $\overline{\lambda_2}^F : T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)) \Rightarrow F$ between quadratic functors with domain $\mathcal{C}$ and values in right $\mathcal{P}(1)$-modules. By applying $\mathbb{S}_2^{\mathcal{P}(1)}$ to this natural transformation, it provides the morphism $\mathbb{S}_2^{\mathcal{P}(1)}(\overline{\lambda_2^F}) : \mathbb{S}_2^{\mathcal{P}(1)}(T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2))) \to \mathbb{S}_2^{\mathcal{P}(1)}(F)$ between quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$. Finally, it interprets the binary bilinear operations involved in the $\mathcal{P}$-algebra structure for the functor $F : \mathcal{C} \to Alg - \mathcal{P}$ in terms of a morphism between quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$.

2.4.5 Isomorphism between two quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$

Here we prove that the quadratic $\mathcal{C}$-module $\mathbb{S}_2^{\mathcal{P}(1)}(T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2)))$ over $\mathcal{P}(1)$ is isomorphic to an another such object more understandable. On the one hand, we show that the quadratic functors $T_2(F^{\otimes 2} \otimes_S \mathcal{P}(2))$ and $T_1 F^{\otimes 2} \otimes_S \mathcal{P}(2)$ with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$ are isomorphic to each other. On the second hand, we give the explicit expression of each component and morphism involved in the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ corresponding to the quadratic functor $T_1 F^{\otimes 2} \otimes_S \mathcal{P}(2)$.

First we provide a technical result which gives the bilinearization of a specific (bireduced) diagonalizable bifunctor (see 2.2 of [12]) depending on the functor F. By the universal property of the bilinearization of the bifunctor $F \otimes F : \mathcal{C}^{\times 2} \to Ab$ (see 1.14 of [12]), there is a unique morphism between bilinear bifunctors $\overline{t_1^F \otimes t_1^F} : T_{11}\big(F \otimes F\big) \Rightarrow T_1F \otimes T_1F$ factorizing $t_1^F \otimes t_1^F : F \otimes F \Rightarrow T_1F \otimes T_1F$ through $t_{11}^{F\otimes F} : F \otimes F \Rightarrow T_{11}\big(F \otimes F\big)$. The following proposition says that this morphism is a natural isomorphism.

Proposition 2.4.14. *The natural transformation* $\overline{t_1^F \otimes t_1^F} : T_{11}\big(F \otimes F\big) \Rightarrow T_1F \otimes T_1F$ *is an isomorphism between bilinear bifunctors.*

Proof. It is an immediate consequence of 2.3.1 and of right-exactness of the tensor product. $\square$

There is a more general setting than 2.4.14 given in Example 1.15 of [12]. Now we give the following natural isomorphism between quadratic functors:

Proposition 2.4.15. *The quadratic functors* $T_2\big(F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)\big)$ *and* $T_1F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$ *with domain* $\mathcal{C}$ *and values in* $Mod_{\mathcal{P}(1)}$ *are isomorphic to each other in* $Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)})$.

Proof. By 2.3.1, 2.7 of [12] and 2.4.14, we have the following natural isomorphisms between quadratic functors with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$:

$$T_2\big(F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)\big) \cong T_2\big(\big(F \otimes F\big) \cdot \Delta^2\big) \otimes_{R\otimes R} \mathcal{P}(2) \cong T_{11}\big(F \otimes F\big) \cdot \Delta^2 \otimes_{R\otimes R} \mathcal{P}(2) \cong T_1F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$$

By 1.2.11, there is a unique factorization $\overline{(t_1^F)^{\otimes 2} \otimes id} : T_2\big(F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)\big) \Rightarrow T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)$ (that exists because its target object is quadratic) of $(t_1^F)^{\otimes 2} \otimes id : F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \Rightarrow T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)$ through $t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)} : F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \Rightarrow T_2\big(F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)\big)$, i.e.

$$\overline{(t_1^F)^{\otimes 2} \otimes id} \circ t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)} = (t_1^F)^{\otimes 2} \otimes id \tag{2.4.9}$$

We prove that $\overline{(t_1^F)^{\otimes 2} \otimes id}$ is a natural isomorphism. We have the following equalities:

$$\begin{aligned}
&\big((\overline{t_1 \otimes t_1})_{X,X} \otimes_{R\otimes R} id\big) \circ \big((\overline{\Delta_2^* \cdot t_{11}^{F\otimes F}})_X \otimes_{R\otimes R} id\big) \circ \big((\overline{t_2^{F^{\otimes 2}} \otimes id})_X \otimes_{R\otimes R} id\big) \circ \big(t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)}\big)_X \\
&= \big((\overline{t_1 \otimes t_1})_{X,X} \otimes_{R\otimes R} id\big) \circ \big((\overline{\Delta_2^* \cdot t_{11}^{F\otimes F}})_X \otimes_{R\otimes R} id\big) \circ \big((t_2^{F^{\otimes 2}})_X \otimes_{R\otimes R} id\big)\text{, by 2.3.1} \\
&= \big((\overline{t_1 \otimes t_1})_{X,X} \otimes_{R\otimes R} id\big) \circ \big((t_{11}^{F\otimes F})_{X,X} \otimes_{R\otimes R} id\big)\text{, by 2.7 of [12]} \\
&= (t_1^F)_X^{\otimes 2} \otimes_{R\otimes R} id\text{, by 2.4.14} \\
&= \big(\overline{(t_1^F)_X^{\otimes 2} \otimes id}\big) \circ \big(t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)}\big)_X\text{, by (2.4.9)}S
\end{aligned}$$

Hence we get

$$\big(\overline{(t_1^F)_X^{\otimes 2} \otimes id}\big) = \big((\overline{t_1 \otimes t_1})_{X,X} \otimes_{R\otimes R} id\big) \circ \big((\overline{\Delta_2^* \cdot t_{11}^{F\otimes F}})_X \otimes_{R\otimes R} id\big) \circ \big((\overline{t_2^{F^{\otimes 2}} \otimes id})_X \otimes_{R\otimes R} id\big)$$

because $\big(t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)}\big)_X$ is a (regular) epimorphism. Thus $\overline{(t_1^F)_X^{\otimes 2} \otimes id}$ is an isomorphism as a composite of isomorphisms.
By using 2.4.13, we have the following right exact sequence in $Func_*(\mathcal{C}, Mod_{\mathcal{P}(1)})$ as follows:

$$F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \overset{\widehat{T^F}\otimes id - id\otimes t}{\Longrightarrow} F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \overset{q^F}{\Longrightarrow} F^{\otimes 2} \otimes_S \mathcal{P}(2) \Longrightarrow 0$$

The quadratization functor $T_2 : Func_*(\mathcal{C}, Mod_{\mathcal{P}(1)}) \to Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)})$ is the left adjoint functor of the inclusion functor by 1.10 of [12]. Hence T_2 is a right exact functor. Then we have the following right exact sequence in $Quad(\mathcal{C}, Mod_{\mathcal{P}(1)})$:

$$T_2\big(F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)\big) \overset{T_2\left(\widehat{T^F}\otimes id - id\otimes t\right)}{\Longrightarrow} T_2\big(F^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)\big) \overset{T_2(q^F)}{\Longrightarrow} T_2\big(F^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \Longrightarrow 0$$

Let X be any object in $\mathcal{C}$. Then we consider the following diagram:

$$\begin{tikzcd}
T_2\big(F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)\big)(X) \arrow[r, "T_2(\widehat{T^F}\otimes id - id\otimes t)_X"] \arrow[d, "\overline{(t_1^F)^{\otimes 2}_X\otimes id}", "\cong"'] & T_2\big(F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)\big)(X) \arrow[r, "T_2(q^F)_X"] \arrow[d, "\overline{(t_1^F)^{\otimes 2}_X\otimes id}", "\cong"'] & T_2\big(F^{\otimes 2}\otimes_S\mathcal{P}(2)\big)(X) \arrow[d, dashed, "\overline{\phi^F_X}", "\cong"'] \\
T_1F(X)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2) \arrow[r, "\widehat{T_X^{T_1F}}\otimes id - id\otimes t"] & T_1F(X)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2) \arrow[r, "q_X^{T_1F}"] & T_1F(X)^{\otimes 2}\otimes_S\mathcal{P}(2)
\end{tikzcd}$$

Now we prove that the left-hand rectangle is commutative:

$$\begin{aligned}
&\big(\overline{(t_1^F)^{\otimes 2}_X\otimes id}\big)\circ T_2(\widehat{T^F}\otimes id - id\otimes t)_X\circ (t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)})_X\\
&= \big(\overline{(t_1^F)^{\otimes 2}_X\otimes id}\big)\circ (t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)})_X\circ(\widehat{T_X^F}\otimes id - id\otimes t)\\
&= \big((t_1^F)^{\otimes 2}_X\otimes id\big)\circ(\widehat{T_X^F}\otimes id - id\otimes t)\text{, by (2.4.9)}\\
&= (\widehat{T_X^{T_1F}}\otimes id - id\otimes t)\circ\big((t_1^F)^{\otimes 2}_X\otimes id\big)\\
&= (\widehat{T_X^{T_1F}}\otimes id - id\otimes t)\circ\big(\overline{(t_1^F)^{\otimes 2}_X\otimes id}\big)\circ (t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)})_X\text{, by 2.4.9}
\end{aligned}$$

Hence we obtain

$$\big(\overline{(t_1^F)^{\otimes 2}_X\otimes id}\big)\circ T_2(\widehat{T^F}\otimes id - id\otimes t)_X = (\widehat{T_X^{T_1F}}\otimes id - id\otimes t)\circ\big(\overline{(t_1^F)^{\otimes 2}_X\otimes id}\big)$$

because $(t_2^{F^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)})_X$ is a (regular) epimorphism. As $\big(\overline{(t_1^F)^{\otimes 2}_X\otimes id}\big)$ is an isomorphism (see 2.4.15 and (2.4.9)), a category argument provides an unique isomorphism $\overline{\phi^F_X} : T_2\big(F^{\otimes 2}\otimes_S\mathcal{P}(2)\big)(X)\to T_1F(X)^{\otimes 2}\otimes_S\mathcal{P}(2)$ such that

$$\overline{\phi^F_X}\circ T_2(q)_X = q_X^{T_1F}\circ\big(\overline{(t_1^F)^{\otimes 2}_X\otimes id}\big) \tag{2.4.10}$$

We remark that the morphism $\overline{\phi^F_X}$ is natural in X. Then it defines a natural isomorphism $\overline{\phi^F}$: $T_2(F^{\otimes 2}\otimes_S\mathcal{P}(2))\Rightarrow T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)$ in $Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)})$. □

Remark 2.4.16. By applying the functor $\mathbb{S}_2^{\mathcal{P}(1)} : Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)})\to QMod_{\mathcal{C}}^{\mathcal{P}(1)}$ to the natural isomorphism $\overline{\phi^F} : T_2(F^{\otimes 2}\otimes_S\mathcal{P}(2))\Rightarrow T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)$, we get an isomorphism $\mathbb{S}_2^{\mathcal{P}(1)}(\overline{\phi^F}) = (\overline{\phi^F_E},\ cr_2(\overline{\phi^F})_{E,E}) : \mathbb{S}_2^{\mathcal{P}(1)}(T_2(F^{\otimes 2}\otimes_S\mathcal{P}(2)))\to\mathbb{S}_2^{\mathcal{P}(1)}(T_1F^{\otimes 2}\otimes_S\mathcal{P}(2))$ in $QMod_{\mathcal{C}}^{\mathcal{P}(1)}$.

Before giving the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ associated with the quadratic functor $T_1F\otimes_S\mathcal{P}(2) : \mathcal{C}\to Mod_{\mathcal{P}(1)}$, we give the following proposition:

Proposition 2.4.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, $G : \mathcal{C}\to\mathcal{A}$ be a reduced functor and $L : \mathcal{A}\to\mathcal{B}$ be an additive functor. Then the n-th cross-effect of the composite functor $L\cdot G : \mathcal{C}\to\mathcal{B}$ is*

$$cr_n(L\cdot G)(X_1,\ldots,X_n) = L\big(cr_nG(X_1,\ldots,X_n)\big)$$

where $X_1,\ldots,X_n\in\mathcal{C}$. Moreover the kernel $\iota_n^{L\cdot G} : cr_n(L\cdot G)(X_1,\ldots,X_n)\rightarrowtail (L\cdot G)(X_1+\ldots+X_n)$ of the comparison morphism $\widehat{r_n^{L\cdot G}}$ (see (1.2.1)) is given by

$$\iota_n^{L\cdot G} = L(\iota_n^G)$$

with $\iota_n^G : cr_nG(X_1,\ldots,X_n)\rightarrowtail G(X_1+\ldots+X_n)$ being the kernel of the comparaisaon morphism $\widehat{r_n^G}$.

Proof. It is an immediate consequence of the inductive definition of the n-th cross-effect of G (see 1.2 of [12]) and of the fact that L preserves finite products (because it is an additive functor between abelian categories). $\square$

Proposition 2.4.18. *The quadratic $\mathcal{C}$-module $\mathbb{S}_2^{\mathcal{P}(1)}\big(T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)\big)$ over $\mathcal{P}(1)$ is as follows:*

$$T_{11}cr_2(U_E)(E,E) \otimes_\Lambda \big(T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \xrightarrow{\widehat{H_E^F}} T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \xrightarrow{q_E^{T_1F}} T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)$$

Here

- *The involution involved is the morphism $\widehat{T_E^{T_1F}} \otimes_{R\otimes R} t : T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$, where $\widehat{T_E^{T_1F}} : T_1F(E)^{\otimes 2} \to T_1F(E)^{\otimes 2}, x \otimes y \mapsto y \otimes x$ is the canonical switch;*
- *the morphism $q_E^{T_1F}$, respectively the map $\widehat{H_E^F}$, is the cokernel of $\widehat{T_E^{T_1F}} \otimes_{R\otimes R} id - id \otimes_{R\otimes R} t$ (see 2.4.13), respectively the homomorphism of $(\overline{\Lambda} \otimes \overline{\Lambda})$-$\mathcal{P}(1)$-bimodules defined as follows:*

$$\widehat{H_E^F}\big(t_{11}(\rho_2(\xi)) \otimes (x \otimes y \otimes p)\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\Big(x \otimes y \otimes p + y \otimes x \otimes (p.t)\Big)$$

where $x, y \in T_1F(E)$, $\xi \in \mathcal{C}(E, E+E)$ and $p \in \mathcal{P}(2)$.

Proof. Let X and Y be two objects in $\mathcal{C}$. We denote respectively by $i_1^2 : X \rightarrowtail X+Y$ and $i_2^2 : Y \rightarrowtail X+Y$ the injections of the first and the second summand. Moreover consider $x, x' \in T_1F(X)$, $y, y' \in T_1F(Y)$ and $p \in \mathcal{P}(2)$. There are several steps to prove this proposition:

1. *Computation of $cr_2\big(T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)\big)$.* First we observe that the quadratic functor $T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)$ is the postcomposite of the additive functor $- \otimes_S \mathcal{P}(2) : Mod_S \to Mod_{\mathcal{P}(1)}$ with the (quadratic) functor $\big(T_1F \otimes T_1F\big) \cdot \Delta^2 = T_1F^{\otimes 2} : \mathcal{C} \to Mod_S$. Moreover the second cross-effect of the functor $\big(T_1F \otimes T_1F\big) \cdot \Delta^2 = T_1F^{\otimes 2} : \mathcal{C} \to Mod_S$ is

$$cr_2\big(T_1F^{\otimes 2}\big)(X,Y) = \big(T_1F(X) \otimes T_1F(Y)\big) \oplus \big(T_1F(Y) \otimes T_1F(X)\big) \tag{2.4.11}$$

by 2.6 of [12] because the bifunctor $T_1F \otimes T_1F : \mathcal{C}^{\times 2} \to Mod_S$ is bilinear. In addition, the kernel $\iota_2^{T_1F^{\otimes 2}} : cr_2\big(T_1F^{\otimes 2}\big)(X,Y) \rightarrowtail T_1F(X+Y)^{\otimes 2}$ of the comparison morphism $\widehat{r_2^{T_1F^{\otimes 2}}}$ (see (1.2.1)) is given by

$$\begin{aligned} \iota_2^{T_1F^{\otimes 2}} &= \Big(\big(T_1F \otimes T_1F\big)(i_1^2,\, i_2^2),\ \big(T_1F \otimes T_1F\big)(i_2^2,\, i_1^2)\Big) \\ &= \big(T_1F(i_1^2) \otimes T_1F(i_2^2),\ T_1F(i_2^2) \otimes T_1F(i_1^2)\big) \end{aligned}$$

We remark that $cr_2\big(T_1F^{\otimes 2}\big)(X,Y)$ is a right $(R \otimes R)$-module with involution T defined by

$$T\big(a \otimes b,\ c \otimes d\big) = (d \otimes c,\ b \otimes a)$$

Hence $cr_2\big(T_1F^{\otimes 2}\big)(X,Y)$ has a canonical right S-module structure by 2.4.10. By 2.4.17, the second cross-effect of the functor $T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)$ is given as follows:

$$cr_2\big(T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)\big)(X,Y) = cr_2\big(T_1F^{\otimes 2}\big)(X,Y) \otimes_S \mathcal{P}(2)\,, \tag{2.4.12}$$

and the kernel $\iota_2^{T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)} : cr_2\big(T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)\big)(X,Y) \rightarrowtail T_1F(X+Y)^{\otimes 2} \otimes_S \mathcal{P}(2)$ of the comparison morphism (see (1.2.1)) is given by

$$\iota_2^{T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)} = \iota_2^{T_1F^{\otimes 2}} \otimes_S id\,.$$

Now let us define the morphism $\phi : T_1F(X) \otimes T_1F(Y) \otimes_{R\otimes R} \mathcal{P}(2) \to cr_2\big(T_1F^{\otimes 2} \otimes_S \mathcal{P}(2)\big)(X,Y)$ by

$$\phi(x \otimes y \otimes p) = (x \otimes y,\ 0) \otimes_S p\,,$$

Then we prove that ϕ is an isomorphism in $Mod_{\mathcal{P}(1)}$. For this, it suffices to find its inverse. Let $\varphi : cr_2\big(T_1F^{\otimes 2}\big)(X,Y) \otimes_{R\otimes R} \mathcal{P}(2) \to T_1F(X) \otimes T_1F(Y) \otimes_{R\otimes R} \mathcal{P}(2)$ be defined by

$$\varphi\big((x \otimes y,\ y' \otimes x') \otimes p\big) = x \otimes y \otimes p\ +\ x' \otimes y' \otimes (p.t)$$

where $x, x' \in T_1F(X)$, $y, y' \in T_1F(Y)$ and $p \in \mathcal{P}(2)$. We verify that φ is S-bilinear. For this, we get

$$\begin{aligned}
&\varphi\Big((x \otimes y,\ y' \otimes x').\big(r_1 \otimes r_2 + (s_1 \otimes s_2).t\big) \otimes p\Big) \\
&= \lambda_1^F(x \otimes r_1) \otimes \lambda_1^F(y \otimes r_2) \otimes p + \lambda_1^F(x' \otimes r_2) \otimes \lambda_1^F(y' \otimes r_1) \otimes (p.t) \\
&\qquad + \lambda_1^F(x' \otimes s_2) \otimes \lambda_1^F(y' \otimes s_1) \otimes p + \lambda_1^F(x \otimes s_1) \otimes \lambda_1^F(y \otimes s_2) \otimes (p.t)\,, \text{by (2.4.6)} \\
&= x \otimes y \otimes \gamma_{1,1;2}(r_1 \otimes r_2 \otimes p) + x \otimes y \otimes \gamma_{1,1;2}(s_1 \otimes s_2 \otimes (p.t)) \\
&\qquad + x' \otimes y' \otimes \gamma_{1,1;2}(r_2 \otimes r_1 \otimes (p.t)) + x' \otimes y' \otimes \gamma_{1,1;2}(s_2 \otimes s_1 \otimes p)
\end{aligned}$$

By the equivariance axiom of the operad $\mathcal{P}$, we have

$$\gamma_{1,1;2}(r_2 \otimes r_1 \otimes (p.t)) + \gamma_{1,1;2}(s_2 \otimes s_1 \otimes p) = \Big(\gamma_{1,1;2}(r_1 \otimes r_2 \otimes p) + \gamma_{1,1;2}(s_1 \otimes s_2 \otimes (p.t))\Big).t$$

Hence we get

$$\begin{aligned}
&\varphi\Big((x \otimes y,\ y' \otimes x').\big(r_1 \otimes r_2 + (s_1 \otimes s_2).t\big) \otimes p\Big) \\
&= x \otimes y \otimes \big(\gamma_{1,1;2}(r_1 \otimes r_2 \otimes p) + \gamma_{1,1;2}(s_1 \otimes s_2 \otimes (p.t))\big) \\
&\qquad x' \otimes y' \otimes \big(\gamma_{1,1;2}(r_1 \otimes r_2 \otimes p) + \gamma_{1,1;2}(s_1 \otimes s_2 \otimes (p.t))\big).t \\
&= x \otimes y \otimes \big(r_1 \otimes r_2 + (s_1 \otimes s_2).t\big).p \\
&\qquad x' \otimes y' \otimes \Big(\big(r_1 \otimes r_2 + (s_1 \otimes s_2).t\big).p.\Big).t\,, \text{by 2.4.11} \\
&= \varphi\Big((x \otimes y,\ y' \otimes x') \otimes \big(r_1 \otimes r_2 + (s_1 \otimes s_2).t\big).p\Big)
\end{aligned}$$

where $r_1, r_2, s_1, s_2 \in R$. By 2.4.3, there is a morphism $\psi : cr_2\big(T_1F^{\otimes 2}\big)(X,Y) \otimes_S \mathcal{P}(2) \to T_1F(X) \otimes T_1F(Y) \otimes_{R\otimes R} \mathcal{P}(2)$ such that $\psi \circ q^S_{R\otimes R} = \varphi$. It is clear that it is a (right) $\mathcal{P}(1)$-module homomorphism. Let us prove that ϕ and ψ are inverse to each other.

- On the one hand, we have $\phi \circ \psi = id$ because we get

$$\begin{aligned}
(\phi \circ \psi)\big((x \otimes y,\ y' \otimes x') \otimes_S p\big) &= \phi\big(x \otimes y \otimes p + x' \otimes y' \otimes (p.t)\big) \\
&= \big(x \otimes y,\ 0\big) \otimes_S p + \big(0,\ y' \otimes x'\big) \otimes_S p \\
&= (x \otimes y,\ y' \otimes x') \otimes_S p
\end{aligned}$$

  as desired.
- On the other hand, we get $\psi \circ \phi = id$ as we have

$$(\psi \circ \phi)(x \otimes y \otimes p) = \psi((x \otimes y,\ 0) \otimes_S p) = x \otimes y \otimes p$$

  as desired.

Now let us define $k : T_1F(X) \otimes T_1F(Y) \otimes_{R\otimes R} \mathcal{P}(2) \to T_1F(X+Y)^{\otimes 2} \otimes_S \mathcal{P}(2)$ to be the composite morphism $k = \iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)} \circ \psi$, which is clearly a kernel of the comparison morphism $r_2^{T_1\widehat{F^{\otimes 2}\otimes_S}\mathcal{P}(2)}$ (see (1.2.1)). We have the following expression of k:

$$\begin{aligned}
k(x\otimes y\otimes p) &= (\iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)} \circ \psi)(x\otimes y\otimes p)\\
&= \big(\iota_2^{T_1F^{\otimes 2}} \otimes_S id\big)\big((x\otimes y,\, 0)\otimes p\big)\\
&= \big(\big(T_1F\otimes T_1F\big)(i_1^2, i_2^2),\ \big(T_1F\otimes T_1F\big)(i_2^2, i_1^2)\big)(x\otimes y,\, 0)\otimes_S p\\
&= \big(T_1F(i_1^2)\otimes T_1F(i_2^2),\ T_1F(i_2^2)\otimes T_1F(i_1^2)\big)(x\otimes y,\, 0)\otimes_S p\\
&= T_1F(i_1^2)(x)\otimes T_1F(i_2^2)(y)\otimes_S p\\
&= q_{X+Y}^F \circ \big(T_1F(i_1^2)\otimes T_1F(i_2^2)\otimes_{R\otimes R} id\big)(x\otimes y\otimes p)
\end{aligned}$$

Moreover we deduce that we have the following isomorphism of (right) $\mathcal{P}(1)$-modules:

$$cr_2\big(T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)\big)(X,Y) \cong T_1F(X)\otimes T_1F(Y)\otimes_{R\otimes R}\mathcal{P}(2)$$

From now on, we consider that the second cross-effect of the functor $B_S^F\cdot\Delta^2 : \mathcal{C} \to Mod_{\mathcal{P}(1)}$ is

$$cr_2\big(T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)\big)(X,Y) = T_1F(X)\otimes T_1F(Y)\otimes_{R\otimes R}\mathcal{P}(2) \tag{2.4.13}$$

setting here $(\iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{X,Y} = q_{X+Y}^{T_1F}\circ\big((T_1F(i_1^2)\otimes T_1F(i_2^2))\otimes_{R\otimes R} id\big)$, for $X, Y\in\mathcal{C}$.

2. *Computation of* $(S_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_E : cr_2\big(T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)\big)(E,E) \to T_1F(E)^{\otimes 2}\otimes_S\mathcal{P}(2)$. We prove that $(S_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_E = q_E^{T_1F}$. We have the following equalities:

$$\begin{aligned}
(S_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_E &= \big(T_1F(\nabla_E^2)^{\otimes 2}\otimes_S id\big)\circ(\iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E,E}\,, \quad \text{by definition}\\
&= \big(T_1F(\nabla_E^2)\otimes T_1F(\nabla_E^2)\otimes_S id\big)\circ q_{E+E}^{T_1F}\circ\big(T_1F(i_1^2)\otimes T_1F(i_2^2)\otimes_{R\otimes R} id\big)\\
&= q_E^{T_1F}\circ\big(T_1F(\nabla_E^2)\otimes T_1F(\nabla_E^2)\otimes_{R\otimes R} id\big)\circ\big(T_1F(i_1^2)\otimes T_1F(i_2^2)\otimes_{R\otimes R} id\big)\,,\\
&\qquad \text{by naturality of } q_E^F \text{ in } E\\
&= q_E^{T_1F}
\end{aligned}$$

3. *Computation of* $H_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)} : T_{11}cr_2U_E(E,E)\otimes_\Lambda\big(T_1F(E)^{\otimes 2}\otimes_S\mathcal{P}(2)\big) \to T_1F(E)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)$. We prove that $H_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)} = \widehat{H_E^F}$ where $\widehat{H_E^F}$ is given in the statement of 2.4.18. We recall that the morphism $H_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}$ is defined in 5.15 of [12] as follows:

$$\begin{tikzcd}
T_{11}cr_2(U_E)(E,E)\otimes_\Lambda(T_1F(E)^{\otimes 2}\otimes_S\mathcal{P}(2)) \arrow[r, "H_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}"] & T_1F(E)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)\\
cr_2(U_E)(E,E)\otimes_\Lambda(T_1F(E)^{\otimes 2}\otimes\mathcal{P}(2)) \arrow[u, "(t_{11}^{cr_2(U_E)})_{E,E}\otimes_\Lambda id"] \arrow[ur, "cr_2(u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E,E}"'] &
\end{tikzcd}$$

where, for X object in $\mathcal{C}$, $(u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_X : U_E(X)\otimes_\Lambda(T_1F(E)^{\otimes 2}\otimes_S\mathcal{P}(2)) \to T_1F(X)^{\otimes 2}\otimes_S\mathcal{P}(2)$ is the morphism defined by

$$(u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_X\big(f\otimes_\Lambda(x\otimes y\otimes_S p)\big) = T_1F(f)(x)\otimes T_1F(f)(y)\otimes_S p$$

where $f \in \mathcal{C}(E, X)$, $x, y \in T_1F(E)$ and $p \in \mathcal{P}(2)$. Moreover we recall that $cr_2(u'_{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E,E}$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd}
cr_2(U_E)(E,E) \otimes_\Lambda \big(T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \arrow[r, "\iota_2 \otimes_\Lambda id"] \arrow[d, "cr_2(u'_{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E,E}"'] & U_E(E^{+2}) \otimes_\Lambda \big(T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \arrow[d, "(u'_{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E^{+2}}"] \\
T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \arrow[r, "\iota_2^{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)}"] & (T_1F(E^{+2})^{\otimes 2} \otimes_S \mathcal{P}(2))
\end{tikzcd}$$

with $\iota_2 : U_E(E|E) \rightarrowtail U_E(E^{+2})$ being the kernel of the comparison morphism $\widehat{r_2^{U_E}} : U_E(E^{+2}) \to U_E(E)^{\times 2}$ (see 1.3 of [12]). We recall that ι_2 has a retraction $\rho_2 : U_E(E^{+2}) \to U_E(E|E)$ (as the functor U_E takes values in the abelian category Mod_Λ). Hence $\rho_2(U_E(E^{+2}))$ generates $U_E(E|E)$ (as a left Λ-module). Let $x, y \in T_1F(E)$, $\xi \in \mathcal{C}(E,\, E^{+2})$ and $p \in \mathcal{P}(2)$. It suffices to prove that the above diagram commutes if $cr_2(u'_{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E,E}$ has the following explicit expression:

$$\begin{aligned}
&cr_2(u'_{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E,E}\big(\rho_2(\xi) \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&= t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(x \otimes y \otimes_{R\otimes R} p + y \otimes x \otimes_{R\otimes R} (p.t)\big)\\
&= T_1F(r_1^2 \circ \xi)(x) \otimes T_1F(r_2^2 \circ \xi)(y) \otimes_{R\otimes R} p + T_1F(r_1^2 \circ \xi)(y) \otimes T_1F(r_2^2 \circ \xi)(x) \otimes_{R\otimes R} (p.t)
\end{aligned}$$

by uniqueness of $cr_2(u'_{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E,E}$. On the one hand, we have

$$\begin{aligned}
&(\iota_2^{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E,E} \circ cr_2(u'_{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)})_{E,E}\big(\rho_2(\xi) \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&= q_{E^{+2}}^{T_1F} \circ \big(T_1F(i_1^2) \otimes T_1F(i_2^2) \otimes_{R\otimes R} id\big)\Big(T_1F(r_1^2 \circ \xi)(x) \otimes T_1F(r_2^2 \circ \xi)(y) \otimes_{R\otimes R} p\\
&\qquad + T_1F(r_1^2 \circ \xi)(y) \otimes T_1F(r_2^2 \circ \xi)(x) \otimes_{R\otimes R} (p.t)\Big)\\
&= q_{E^{+2}}^{T_1F}\Big(T_1F(i_1^2 \circ r_1^2 \circ \xi)(x) \otimes T_1F(i_2^2 \circ r_2^2 \circ \xi)(y) \otimes_{R\otimes R} p\\
&\qquad + T_1F(i_1^2 \circ r_1^2 \circ \xi)(y) \otimes T_1F(i_2^2 \circ r_2^2 \circ \xi)(x) \otimes_{R\otimes R} (p.t)\Big)\\
&= T_1F(i_1^2 \circ r_1^2 \circ \xi)(x) \otimes T_1F(i_2^2 \circ r_2^2 \circ \xi)(y) \otimes_S p\\
&\qquad + T_1F(i_1^2 \circ r_1^2 \circ \xi)(y) \otimes T_1F(i_2^2 \circ r_2^2 \circ \xi)(x) \otimes_S (p.t)\\
&= T_1F(i_1^2 \circ r_1^2 \circ \xi)(x) \otimes T_1F(i_2^2 \circ r_2^2 \circ \xi)(y) \otimes_S p\\
&\qquad + T_1F(i_2^2 \circ r_2^2 \circ \xi)(x) \otimes T_1F(i_1^2 \circ r_1^2 \circ \xi)(y) \otimes_S p
\end{aligned}$$

On the other hand, we obtain

$$
\begin{aligned}
&(u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E^{+2}} \circ (\iota_2 \otimes_\Lambda id)\big(\rho_2(\xi) \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&= (u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E^{+2}} \circ \big((\iota_2 \circ \rho_2)(\xi) \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&= (u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E^{+2}}\big(\xi \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&\qquad - (u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E^{+2}}\big((i_1^2 \circ r_1^2 \circ \xi) \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&\qquad - (u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E^{+2}}\big((i_2^2 \circ r_2^2 \circ \xi) \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&= T_1F(\xi)(x) \otimes T_1F(\xi)(y) \otimes_S p - T_1F(i_1^2 \circ r_1^2\xi)(x) \otimes T_1F(i_1^2 \circ r_1^2 \circ \xi)(y) \otimes_S p\\
&\qquad - T_1F(i_2^2 \circ r_2^2 \circ \xi)(x) \otimes T_1F(i_2^2 \circ r_2^2 \circ \xi)(y) \otimes_S p
\end{aligned}
$$

As $T_1F : \mathcal{C} \to Ab$ is a linear functor, we have the following relation:

$$T_1F(\xi) \;=\; T_1F(i_1^2 \circ r_1^2 \circ \xi) + T_1F(i_2^2 \circ r_2^2 \circ \xi)\,,$$

by 2.14 of [12]. Hence it follows that we have

$$
\begin{aligned}
&(u'_{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)})_{E^{+2}} \circ (\iota_2 \otimes_\Lambda id)\big(\rho_2(\xi) \otimes_\Lambda (x \otimes y \otimes_S p)\big)\\
&= T_1F(i_1^2 \circ r_1^2 \circ \xi)(x) \otimes T_1F(i_2^2 \circ r_2^2 \circ \xi)(y) \otimes_S p\\
&\qquad + T_1F(i_2^2 \circ r_2^2 \circ \xi)(x) \otimes T_1F(i_1^2 \circ r_1^2 \circ \xi)(y) \otimes_S p
\end{aligned}
$$

as desired. Finally we get $H_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)} = \widehat{H_E^F}$.

4. *Computation of the involution* $T_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)} : T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2) \to T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)$. By the definition of $T_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}$, it is the unique morphism such that the following diagram commutes:

$$
\begin{tikzcd}
T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \arrow[r, "\iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}"] \arrow[d, "T_E^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}"'] & T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2) \arrow[d, "T_1F(\tau_E^2)^{\otimes 2}\otimes_S\mathcal{P}(2)"] \\
T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \arrow[r, "\iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}"] & T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)
\end{tikzcd}
$$

where the morphism $\tau_E^2 : E^{+2} \to E^{+2}$ is the canonical switch. We get the following equalities:

$$
\begin{aligned}
&\big(T_1F(\tau_E^2)^{\otimes 2} \otimes_S id\big) \circ \iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}(x \otimes y \otimes p)\\
&= \big(T_1F(\tau_E^2)^{\otimes 2} \otimes_S id\big) \circ q_{E^{+2}}^{T_1F} \circ \big(T_1F(i_1^2) \circ T_1(i_2^2) \otimes_{R\otimes R} id\big)(x \otimes y \otimes p)\\
&= T_1F(\tau_E^2 \circ i_1^2)(x) \otimes T_1F(\tau_E^2 \circ i_2^2)(y) \otimes_S p\\
&= T_1F(i_2^2)(x) \otimes T_1F(i_1^2)(y) \otimes_S p\\
&= T_1F(i_1^2)(y) \otimes T_1F(i_2^2)(x) \otimes_S (p.t)\\
&= q_{E^{+2}}^{T_1F} \circ \big(T_1F(i_1^2) \otimes T_1F(i_2^2) \otimes_{R\otimes R} id\big) \circ (\widehat{T_E^{T_1F}} \otimes_{R\otimes R} t)(x \otimes y \otimes p)\\
&= \iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)} \circ (\widehat{T_E^{T_1F}} \otimes_{R\otimes R} t)(x \otimes y \otimes p)
\end{aligned}
$$

where $x, y \in T_1F(E)$ and $p \in \mathcal{P}(2)$. This proves that

$$T_E^{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)} = \widehat{T_E^{T_1F}} \otimes_{R\otimes R} t$$

by uniqueness of $T_E^{T_1F^{\otimes 2}\otimes_S \mathcal{P}(2)}$.

This proves the result. □

In summary, we get the morphism $\mathbb{S}_2^{\mathcal{P}(1)}(\overline{\lambda_2^F} \circ (\overline{\phi}^F)^{-1}) : \mathbb{S}_2^{\mathcal{P}(1)}(T_1F^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)) \to \mathbb{S}_2^{\mathcal{P}(1)}(F)$ in the category $QMod_{\mathcal{C}}^{\mathcal{P}(1)}$. It describes the binary bilinear operations encoded by $\mathcal{P}(2)$ (and their relations) involved in the $\mathcal{P}$-algebra structure for the functor F.

2.4.6 Quadratic $\mathcal{C}$-modules over $\mathcal{P}$

Here we give the definition of quadratic $\mathcal{C}$-modules over a symmetric unitary linear operad. We recall that R and S are rings respectively equal to $\mathcal{P}(1)$ and $(R \otimes R) \wr \mathfrak{S}_2$ (see 2.4.9 or 3.24 of [12]). First we define a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ from a given such object as follows:

Definition 2.4.19. Let M be a quadratic $\mathcal{C}$-module over the ring $\mathcal{P}(1)$. We define M^2 the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ depending on M by

$$M^2 = \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1(- \otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)\big)$$

Explicitly the quadratic $\mathcal{C}$-module M^2 has the following form by 2.2.4:

$$M^2 = \left(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda M_e^2 \xrightarrow{\widehat{H^M}} M_{ee}^2 \xrightarrow{\widehat{T^M}\otimes_{R\otimes R} t} M_{ee}^2 \xrightarrow{q^M} M_e^2\right)$$

where

- M_e^2 is the left Λ-module $T_1(- \otimes M)(E)^{\otimes 2} \otimes_S \mathcal{P}(2)$;
- M_{ee}^2 is the symmetric $(\overline{\Lambda} \otimes \overline{\Lambda})$-module $T_1(- \otimes M)(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$ with involution $\widehat{T^M} \otimes_{R\otimes R} t$, where $\widehat{T^M} : T_1(- \otimes M)(E)^{\otimes 2} \to T_1(- \otimes M)(E)^{\otimes 2}$, $x \otimes y \mapsto y \otimes x$ is the canonical switch;
- the map q_M, respectively $\widehat{H^M}$, is the cokernel of $\widehat{T^M} \otimes_{R\otimes R} id - id \otimes_{R\otimes R} t$, respectively the homomorphism of symmetric $(\overline{\Lambda} \otimes \overline{\Lambda})$-modules defined by:

  $$\widehat{H^M}\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (x \otimes y \otimes_S p)\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\Big(x \otimes y \otimes_{R\otimes R} p + y \otimes x \otimes_{R\otimes R} (p.t)\Big)$$

  where $x, y \in T_1(- \otimes M)(E)$, $\xi \in \mathcal{C}(E,\ E^{+2})$ and $p \in \mathcal{P}(2)$.

Let M be a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$. Then we give an expression of the other such object M^2 (depending on M) up to isomorphism as follows:

Proposition 2.4.20. *Let M be a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$. Then up to isomorphism the quadratic $\mathcal{C}$-module $\overline{M^2}$ has the following form:*

$$\overline{M^2} = \left(T_{11}cr_2(U_E)(E,E) \otimes_\Lambda \overline{M_e^2} \xrightarrow{\widehat{H_M}} \overline{M_{ee}^2} \xrightarrow{\widehat{T_M}\otimes_{R\otimes R} t} \overline{M_{ee}^2} \xrightarrow{q_M} \overline{M_e^2}\right)$$

where

- $\overline{M^2_e}$ *is the left* Λ*-module* $Coker(P_M)^{\otimes 2} \otimes_S \mathcal{P}(2)$ *(induced by the left* Λ*-module structure of* $Coker(P_M)$*, see 2.1.2) where* $P_M : M_{ee} \to M_e$ *is the morphism involved in the structure of quadratic* $\mathcal{C}$*-module over* $\mathcal{P}(1)$ *for* M *(see 2.1.1);*

- $\overline{M^2_{ee}}$ *is the symmetric* $(\overline{\Lambda}\otimes\overline{\Lambda})$*-module* $Coker(P_M)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)$ *(as* $Coker(P_M)$ *is a left* $\overline{\Lambda}$*-module by 2.1.2) with involution* $\widehat{T_M} \otimes_{R\otimes R} t$*, where* $\widehat{T_M} : Coker(P)^{\otimes 2} \to Coker(P_M)^{\otimes 2}$*,* $x \otimes y \mapsto y \otimes x$ *is the canonical switch;*

- *the map* q_M*, respectively* $\widehat{H_M}$*, is the cokernel of* $\widehat{T_M} \otimes_{R\otimes R} id - id \otimes_{R\otimes R} t$*, respectively the homomorphism of symmetric* $(\overline{\Lambda}\otimes\overline{\Lambda})$*-modules defined by:*

$$\widehat{H_M}\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (x \otimes y \otimes_S p)\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\Big(x \otimes y \otimes_{R\otimes R} p + y \otimes x \otimes_{R\otimes R} (p.t)\Big)$$

where $x, y \in Coker(P_M)$*,* $\xi \in \mathcal{C}(E,\ E^{+2})$ *and* $p \in \mathcal{P}(2)$*.*

Proof. First we have an explicit expression of the quadratic $\mathcal{C}$-module M^2 (over $\mathcal{P}(1)$ by 2.4.19. Then it suffices to observe that the isomorphism $\overline{\gamma} : T_1(- \otimes M)(E) \to Coker(P_M)$ of Λ-$\mathcal{P}(1)$-bimodules given in 2.3.4 implies that $\overline{M^2}$ is a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$, and that $\big((\overline{\gamma})^{\otimes 2} \otimes_S id,\ (\overline{\gamma})^{\otimes 2} \otimes_{R\otimes R} id\big) : M^2 \to \overline{M^2}$ is an isomorphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$. □

Notation 2.4.21. Let $f = (f_e, f_{ee}) : M \to N$ be a morphism in $QMod_{\mathcal{C}}^{\mathcal{P}(1)}$. We set $f^2 = (f_e^2, f_{ee}^2)$ where

- $f_e^2 = T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E^{\otimes 2} \otimes_S id : M_e^2 \to N_e^2$

- $f_{ee}^2 = T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E^{\otimes 2} \otimes_{R\otimes R} id : M_{ee}^2 \to N_{ee}^2$.

by keeping the notations in 2.4.19, where $\mathbb{T}_2^{\mathcal{P}(1)} : Func_*(\mathcal{C}, Mod_{\mathcal{P}(1)}) \to Quad(\mathcal{C}, Mod_{\mathcal{P}(1)})$ is the functor defined in 2.2.4.

Proposition 2.4.22. *Let* M *and* N *be two quadratic* $\mathcal{C}$*-modules over* $\mathcal{P}(1)$*, and let* $f = (f_e, f_{ee}) : M \to N$ *between these objects. Then the pair of morphisms* $f^2 = (f_e^2, f_{ee}^2) : M^2 \to N^2$ *is a morphism between quadratic* $\mathcal{C}$*-modules over* $\mathcal{P}(1)$*.*

Proof. We recall that $t_1^M : E \otimes M \Rightarrow T_1\big(-\otimes M\big)(E)$ is the cokernel of the morphism $S_2^{-\otimes M} : cr_2(-\otimes M)(E, E) \to E \otimes M$; it is a (regular) epimorphism. Let $x, y \in E \otimes M$, $p \in \mathcal{P}(2)$ and $\xi \in \mathcal{C}(E,\ E^{+2})$. We prove that we have

- $\underline{f_{ee}^2 \circ \widehat{H^M} = \widehat{H^N} \circ (id \otimes_\Lambda f_e^2)}$. For this, we get

$$\widehat{H^N} \circ (id \otimes_\Lambda f_e^2)\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda \big(t_1^M(x) \otimes t_1^M(y) \otimes_S p\big)\big)$$

$$= \widehat{H^N}\Big(t_{11}(\rho_2(\xi)) \otimes_\Lambda \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(x)) \otimes T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(y)) \otimes_S p\big)\Big)$$

$$= \widehat{H^N}\Big(t_{11}(\rho_2(\xi)) \otimes_\Lambda \big(t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big) \otimes t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(y)\big) \otimes_S p\big)\Big)$$

$$= t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\Big(t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big) \otimes t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(y)\big) \otimes_{R\otimes R} p$$

$$+\ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(y)\big) \otimes t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big) \otimes_{R\otimes R} p.t\Big)$$

$$= \big(T_1(-\otimes N)(r_1^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big)\big) \otimes \big(T_1(-\otimes N)(r_2^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(y)\big)\big) \otimes_{R\otimes R} p$$

$$+\ \big(T_1(-\otimes N)(r_1^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(y)\big)\big) \otimes \big(T_1(-\otimes N)(r_2^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big)\big) \otimes_{R\otimes R} p.t$$

Then we observe that, for $k = 1, 2$, we have

$$\begin{aligned} T_1(-\otimes N)(r_k^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big) &= t_1^N\Big(\big((r_k^2 \circ \xi) \otimes N\big) \circ \mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\Big) \\ &= t_1^N\Big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E \circ \big((r_k^2 \circ \xi) \otimes M\big)(x)\Big) \\ &= T_1\Big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\Big)_E \circ t_1^M\big(\big((r_k^2 \circ \xi) \otimes M\big)(x)\big) \\ &= T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_k^2 \circ \xi)(t_1^M(x)) \end{aligned}$$

Hence we get

$$T_1(-\otimes N)(r_k^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big) = T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_k^2 \circ \xi)(t_1^M(x)) \quad (2.4.14)$$

Thus we have the equalities as follows:

$$\begin{aligned} &\widehat{H^N} \circ (id \otimes_\Lambda f_e^2)\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda \big(t_1^M(x) \otimes t_1^M(y) \otimes_S p\big)\big) \\ &= \big(T_1(-\otimes N)(r_1^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big)\big) \otimes \big(T_1(-\otimes N)(r_2^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(y)\big)\big) \otimes_{R\otimes R} p \\ &\quad + \ \big(T_1(-\otimes N)(r_1^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(y)\big)\big) \otimes \big(T_1(-\otimes N)(r_2^2 \circ \xi) \circ t_1^N\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)_E(x)\big)\big) \otimes_{R\otimes R} p.t \\ &= \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_1^2 \circ \xi)(t_1^M(x))\big) \otimes \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_2^2 \circ \xi)(t_1^M(Y))\big) \otimes p \\ &= \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_1^2 \circ \xi)(t_1^M(x))\big) \otimes \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_2^2 \circ \xi)(t_1^M(y))\big) \otimes p \\ &\quad + T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_1^2 \circ \xi)(t_1^M(y))\big) \otimes \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E \circ T_1(-\otimes M)(r_2^2 \circ \xi)(t_1^M(x))\big) \otimes p.t \\ &= \Big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)\big)_E^{\otimes 2} \otimes_{R\otimes R} id\Big) \circ \Big(T_1(-\otimes M)(r_1^2 \circ \xi)(t_1^M(x)) \otimes T_1(-\otimes M)(r_2^2 \circ \xi)(t_1^M(y)) \otimes p \\ &\quad + T_1(-\otimes M)(r_1^2 \circ \xi)(t_1^M(y)) \otimes T_1(-\otimes M)(r_2^2 \circ \xi)(t_1^M(x)) \otimes p.t\Big) \\ &= f_{ee}^2\Big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(t_1^M(x) \otimes t_1^M(y) \circ p + t_1^M(y) \otimes t_1^M(x) \otimes p.t\big)\Big) \\ &= f_{ee}^2 \circ \widehat{H^M}\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda \big(t_1^M(x) \otimes t_1^M(y) \otimes_S p\big)\big)\,, \end{aligned}$$

as desired.

- $\underline{f_{ee}^2 \circ \big(\widehat{T^M} \otimes_{R\otimes R} t\big) = \big(\widehat{T^N} \otimes_{R\otimes R} t\big) \circ f_{ee}^2}$. For this, we get

$$\begin{aligned} &f_{ee}^2 \circ \big(\widehat{T^M} \otimes_{R\otimes R} t\big)(t_1^M(x) \otimes t_1^M(y) \otimes_{R\otimes R} p) \\ &= f_{ee}^2\big(t_1^M(y) \otimes t_1^M(x) \otimes_{R\otimes R} (p.t)\big) \\ &= T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(y)) \otimes T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(x)) \otimes p.t \\ &= \big(\widehat{T^N} \otimes_{R\otimes R} t\big)\big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(x))\big) \otimes T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(y)) \otimes p\big) \\ &= \big(\widehat{T^N} \otimes_{R\otimes R} t\big) \circ \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E^{\otimes 2} \otimes_{R\otimes R} id\big)(t_1^M(x) \otimes t_1^M(y) \otimes p) \\ &= \big(\widehat{T^N} \otimes_{R\otimes R} t\big) \circ f_{ee}^2(t_1^M(x) \otimes t_1^M(y) \otimes p) \end{aligned}$$

- $\underline{q^N \circ f^2_{ee} = f^2_e \circ q^M}$. For this, we have the following equalities

$$\begin{aligned}
&q^N \circ f^2_{ee}(t_1^M(x) \otimes t_1^M(y) \otimes_{R\otimes R} p) \\
&= q^N \circ \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E^{\otimes 2} \otimes_{R\otimes R} id\big)(t_1^M(x) \otimes t_1^M(y) \otimes_{R\otimes R} p) \\
&= T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(x)) \otimes T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E(t_1^M(y)) \otimes_S id \\
&= \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E^{\otimes 2} \otimes_S id\big)(t_1^M(x) \otimes t_1^M(y) \otimes_S p) \\
&= \big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(f)\big)_E^{\otimes 2} \otimes_S id\big) \circ q^M(t_1^M(x) \otimes t_1^M(y) \otimes_{R\otimes R} p)\,,
\end{aligned}$$

as desired.

□

Then we give the definition of a quadratic $\mathcal{C}$-module over $\mathcal{P}$ by relying on the previous arguments. It just consists in interpreting them entirely in terms of quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ as follows:

Definition 2.4.23. A *quadratic $\mathcal{C}$-module over $\mathcal{P}$*, denoted by $M^{\mathcal{P}}$, is a pair $(M,\ \phi^M)$ where M is a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ (see 2.2.1), M^2 is the other such object defined in 2.4.19 and $\phi^M = (\phi^M_e,\ \phi^M_{ee}) : M^2 \to M$ is a morphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$.
A morphism $f : M^{\mathcal{P}} \to N^{\mathcal{P}}$ between such objects is a morphism $f : M \to N$ between the two underlying quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$ making the following diagram commute in $QMod_{\mathcal{C}}^{\mathcal{P}(1)}$:

$$\begin{tikzcd}
M^2 \arrow[r, "\phi^M"] \arrow[d, "f^2"'] & M \arrow[d, "f"] \\
N^2 \arrow[r, "\phi^N"'] & N
\end{tikzcd}$$

where $f^2 : M^2 \to N^2$ is defined in 2.4.21. We denote by $QMod_{\mathcal{C}}^{\mathcal{P}}$ the corresponding category.

2.4.7 The functors $\mathbb{S}_2^{\mathcal{P}}$ and $\mathbb{T}_2^{\mathcal{P}}$

Let $M^{\mathcal{P}} = (M,\ \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$ (see 2.4.23). First we prove that, for an object X in $\mathcal{C}$, the abelian group $X \otimes M$, given in 2.1.3, is endowed with a $\mathcal{P}$-algebra structure implying that the quadratic functor $- \otimes M : \mathcal{C} \to Ab$ takes in fact values in $Alg - \mathcal{P}$. Next we define two functors $\mathbb{S}_2^{\mathcal{P}} : Quad(\mathcal{C},\ Alg - \mathcal{P}) \to QMod_{\mathcal{C}}^{\mathcal{P}}$ and $\mathbb{T}_2^{\mathcal{P}} : QMod_{\mathcal{C}}^{\mathcal{P}} \to Quad(\mathcal{C},\ Alg - \mathcal{P})$, see 2.4.27 for details.
Now we first determine up to isomorphism the quadratic functor taking values in $Mod_{\mathcal{P}(1)}$ corresponding to M^2, the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ defined in 2.4.19.

Proposition 2.4.24. *Let M be a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ and M^2 be the other such object defined in 2.4.19. Then the following natural transformation between quadratic functors with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$*

$$\varepsilon_{T_1(-\otimes M)^{\otimes 2}\otimes_S \mathcal{P}(2)} :\ \mathbb{T}_2^{\mathcal{P}(1)}(M^2) = \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1(- \otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \Longrightarrow T_1(- \otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2) \tag{2.4.15}$$

is an isomorphism, where $\varepsilon : \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)} \Rightarrow Id$ is the counit of the adjoint pair of functors given in 2.2.7.

Proof. First we recall that the (linear) functor $T_1(-\otimes M)$ and $T_1U_E \otimes_\Lambda \mathcal{P}(1)$ whith domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$ are isomorphic to each other by 2.3.3. By 6.24 of [12] and by 1.6.11, the functor $T_1U_E : \mathcal{C} \to Mod_\Lambda$ preserves filtered colimits and coequalizers of reflexive pairs. As the functor $-\otimes_S \mathcal{P}(2) : Mod_S \to Mod_{\mathcal{P}(1)}$ prerserves colimits, it follows that the quadratic functor $T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2) : \mathcal{C} \to Mod_{\mathcal{P}(1)}$ preserves also filtered colimits and coequalizers of reflexive pairs. Moreover the following quadratic functor

$$\mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)\big) : \mathcal{C} \to Mod_{\mathcal{P}(1)}$$

preserves also filtered colimits and coequalizers of reflexive pairs by 6.24 of [12] and by 1.6.11. Hence the natural transformation $\varepsilon_{T_1(-\otimes M)^{\otimes 2}\otimes_S \mathcal{P}(1)}$ is a natural isomorphism by 6.25 of [12] because it is a natural transformation whose source and target are functors which preserve filtered colimits and coequalizers of reflexive graphs, and it is a natural isomorphism if restricted to the full subcategory $\langle E\rangle$ of $\mathcal{C}$ by 2.2.7. $\square$

Now we verify that, for any object X in $\mathcal{C}$, $X\otimes M : \mathcal{C} \to Ab$ is a $\mathcal{P}$-algebra. We know that there is a morphism $\phi^M = (\phi^M_e,\ \phi^M_{ee}) : M^2 \to M$ of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$ involved in the definition of $M^{\mathcal{P}} = (M,\ \phi^M)$, see 2.4.23. Applying the functor $\mathbb{T}_2^{\mathcal{P}(1)} : QMod_{\mathcal{C}}^{\mathcal{P}(1)} \to Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)})$ to this morphism, we get its corresponding natural transformation

$$\mathbb{T}_2^{\mathcal{P}(1)}(\phi^M) : \mathbb{T}_2^{\mathcal{P}(1)}(M^2) = -\otimes M^2 \Rightarrow \mathbb{T}_2^{\mathcal{P}(1)}(M) = -\otimes M$$

between quadratic functors with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$. Then we define the natural transformation $\lambda_2^M : (-\otimes M)^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow -\otimes M$ to be the following composite of natural transformations:

$$\lambda_2^M = \mathbb{T}_2^{\mathcal{P}(1)}(\phi^M) \circ \varepsilon^{-1}_{T_1(-\otimes M)^{\otimes 2}\otimes_S \mathcal{P}(2)} \circ ((t_1^M)^{\otimes 2} \otimes_S id) \circ q^M \circ q_{\mathbb{Z}}^{R\otimes R} \tag{2.4.16}$$

where we recall that

- $\varepsilon_{T_1(-\otimes M)^{\otimes 2}\otimes_S \mathcal{P}(2)} : \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \Rightarrow T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)$ is the counit of the adjunction pair of functors 2.2.7; it is a natural isomorphism by 2.4.24 (see also (2.4.15));
- $q^M : T_1(-\otimes M)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \Rightarrow T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)$ is the cokernel of $\widehat{T^M} \otimes_{R\otimes R} id - id \otimes_{R\otimes R} t$ (see 2.4.13 replacing F with $-\otimes M$);
- $t_1^M : -\otimes M \Rightarrow T_1(-\otimes M)$ is the cokernel of $S_2^{-\otimes M} : cr_2(-\otimes M) \cdot \Delta^2 \Rightarrow -\otimes M$ (see 1.9 of [12]);
- $q_{\mathbb{Z}}^{R\otimes R} : (-\otimes M)^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow (-\otimes M)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$ is the natural transformation given in 2.4.9.

We also recall that we have the natural isomorphism $\overline{\phi^{-\otimes M}} : T_2\big((-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \Rightarrow T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)$ between quadratic functors with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$, and the natural transformation $\overline{\lambda_2^M} : T_2((-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)) \Rightarrow -\otimes M$ given respectively in (2.4.10) and (2.4.8) (replacing F with $-\otimes M$). The following proposition says that there is another expression of the natural transformation $\mathbb{T}_2^{\mathcal{P}(1)}(\phi^M) : T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2) \Rightarrow -\otimes M$ as follows:

Proposition 2.4.25. *The natural transformation $\mathbb{T}_2^{\mathcal{P}(1)}(\phi^M) : T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2) \Rightarrow -\otimes M$ is equal to the following composite:*

$$\mathbb{T}_2(\phi^M) \circ \varepsilon^{-1}_{T_1(-\otimes M)^{\otimes 2}\otimes_S \mathcal{P}(2)} = \overline{\lambda_2^M} \circ (\overline{\phi^{-\otimes M}})^{-1}$$

Proof. Let X be an object in $\mathcal{C}$. We have the following equalities:

$$
\begin{aligned}
&(\overline{\lambda_2^M})_X \circ (\overline{\phi_X^{-\otimes M}})^{-1} \circ \big((t_1^M)_X^{\otimes 2} \otimes_S id\big) \circ q^{-\otimes M} \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= (\overline{\lambda_2^M})_X \circ (\overline{\phi_X^{-\otimes M}})^{-1} \circ q_X^M \circ \big((t_1^M)_X^{\otimes 2} \otimes_{R\otimes R} id\big) \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= (\overline{\lambda_2^M})_X \circ T_2(q^M)_X \circ \big(t_2^{(-\otimes M)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)}\big)_X \circ q_{\mathbb{Z}}^{R\otimes R}\,, \text{by } (2.4.10) \\
&= (\overline{\lambda_2^M})_X \circ \big(t_2^{(-\otimes M)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)}\big)_X \circ q_X^M \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= (\widetilde{\lambda_2^M})_X \circ q_X^M \circ q_{\mathbb{Z}}^{R\otimes R}, \text{ by } (2.4.1) \\
&= (\widehat{\lambda_2^M})_X \circ q_{\mathbb{Z}}^{R\otimes R}, \text{ by } (2.4.7) \\
&= (\lambda_2^M)_X \\
&= \mathbb{T}_2^{\mathcal{P}(1)}(\phi^M) \circ \varepsilon^{-1}_{T_1(-\otimes M)^{\otimes 2}\otimes_S\mathcal{P}(2)} \circ \big((t_1^M)_X^{\otimes 2} \otimes_S id\big) \circ q_X^M \circ q_{\mathbb{Z}}^{R\otimes R}\,, \text{by definition of } \lambda_2^M
\end{aligned}
$$

As $\big((t_1^M)_X^{\otimes 2} \otimes_S id\big) \circ q^{-\otimes M} \circ q_{\mathbb{Z}}^{R\otimes R}$ is a natural epimorphism, we get

$$\mathbb{T}_2(\phi^M) \circ \varepsilon^{-1}_{T_1(-\otimes M)^{\otimes 2}\otimes_S\mathcal{P}(2)} = \overline{\lambda_2^M} \circ (\overline{\phi^{-\otimes M}})^{-1}\,,$$

as desired. □

Proposition 2.4.26. *Let X be an object in $\mathcal{C}$ and $M^{\mathcal{P}} = (M,\ \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$ (see 2.4.23). Then $\big(X \otimes M,\ (\lambda_1^M)_X,\ (\lambda_2^M)_X\big)$ is a $\mathcal{P}$-algebra, where $\lambda_1^M : (-\otimes M) \otimes \mathcal{P}(1) \Rightarrow -\otimes M$ and $\lambda_2^M : (-\otimes M)^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow -\otimes M$ are the natural transformations respectively given in 2.2.4 and* (2.4.16).

Proof. Let X be an object in $\mathcal{C}$ and $M^{\mathcal{P}} = (M,\ \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$. By 2.4.23, we know that the object M, involved in the definition of $M^{\mathcal{P}}$, is a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$. By applying the functor $\mathbb{T}_2^{\mathcal{P}(1)} : QMod_{\mathcal{C}}^{\mathcal{P}(1)} \to Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)})$, we obtain its corresponding quadratic functor with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$:

$$\mathbb{T}_2^{\mathcal{P}(1)}(M) = \big(-\otimes M,\ \lambda_1^M\big)\,, \text{by } 2.2.4$$

Hence we know that $X \otimes M$ is a right $\mathcal{P}(1)$-module whose action of $\mathcal{P}(1)$ on $X \otimes M$ is given by $(\lambda_1^M)_X$. It can be interpreted by the following commutative diagram:

$$
\begin{tikzcd}
(X\otimes M)\otimes\big(\mathcal{P}(1)\otimes\mathcal{P}(1)\big) \arrow[rr, "\cong"] \arrow[d, "id\otimes\gamma_{1;1}"'] & & \big((X\otimes M)\otimes\mathcal{P}(1)\big)\otimes\mathcal{P}(1) \arrow[d, "(\lambda_1^M)_X\otimes id"] \\
(X\otimes M)\otimes\mathcal{P}(1) \arrow[dr, "(\lambda_1^M)_X"'] & & (X\otimes M)\otimes\mathcal{P}(1) \arrow[dl, "(\lambda_1^M)_X"] \\
& X\otimes M &
\end{tikzcd}
$$

As $\mathbb{T}_2^{\mathcal{P}(1)}(\phi)_X : T_1(-\otimes M)(X)^{\otimes 2} \otimes_S \mathcal{P}(2) \to X \otimes M$ and $(t_1^M)_X^{\otimes 2} \otimes_S id : (X\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2) \to T_1(-\otimes M)(X)^{\otimes 2} \otimes_S \mathcal{P}(2)$ are S-bilinear, they are both $(R\otimes R)$-bilinear by 2.4.10. Hence the composite morphism $(\lambda_2^M)_X : (X\otimes M)^{\otimes 2}\otimes \mathcal{P}(2) \to X\otimes M$, given in 2.4.16, is also $(R\otimes R)$-bilinear. It is equivalent to say that the following diagram commutes:

$$\begin{tikzcd}
(X\otimes M)^{\otimes 2}\otimes \big(\mathcal{P}(1)\otimes\mathcal{P}(1)\otimes\mathcal{P}(2)\big) \arrow[rr, "\cong"] \arrow[d, "id\otimes\gamma_{1,1;2}"'] & & \Big(\big((X\otimes M)\otimes\mathcal{P}(1)\big)\otimes\big((X\otimes M)\otimes\mathcal{P}(1)\big)\Big)\otimes\mathcal{P}(2) \arrow[d, "(\lambda_1^M)_X^{\otimes 2}\otimes id"] \\
(X\otimes M)^{\otimes 2}\otimes\mathcal{P}(2) \arrow[dr, "(\lambda_2^M)_X"'] & & (X\otimes M)^{\otimes 2}\otimes\mathcal{P}(2) \arrow[dl, "(\lambda_2^M)_X"] \\
& X\otimes M &
\end{tikzcd}$$

Moreover we observe that $(\lambda_2^M)_X : (X\otimes M)^{\otimes 2}\otimes\mathcal{P}(2)\to X\otimes M$ is a homomorphism of right $\mathcal{P}(1)$-modules as it is a composite of right $\mathcal{P}(1)$-module homomorphisms. It is equivalent to say that the following diagram commutes:

$$\begin{tikzcd}
(X\otimes M)^{\otimes 2}\otimes\big(\mathcal{P}(2)\otimes\mathcal{P}(1)\big) \arrow[rr, "\cong"] \arrow[d, "id\otimes\gamma_{2;1}"'] & & \big((X\otimes M)^{\otimes 2}\otimes\mathcal{P}(2)\big)\otimes\mathcal{P}(1) \arrow[d, "(\lambda_2^M)_X\otimes id"] \\
(X\otimes M)^{\otimes 2}\otimes\mathcal{P}(2) \arrow[dr, "(\lambda_2^M)_X"'] & & (X\otimes M)\otimes\mathcal{P}(1) \arrow[dl, "(\lambda_1^M)_X"] \\
& X\otimes M &
\end{tikzcd}$$

Then it remains to check that $\big(X\otimes M,\ (\lambda_1^M)_X,\ (\lambda_2^M)_X\big)$ satisfies the equivariance axiom. It holds because we have the following relation:

$$(\lambda_2^M)_X\circ(\widehat{T^M}\otimes id) = (\lambda_2^M)_X\circ(id\otimes t)$$

by 2.4.13 and (2.4.16). Finally, $\big(X\otimes M,\ (\lambda_1^M)_X,\ (\lambda_2^M)_X\big)$ is a $\mathcal{P}$-algebra. $\square$

We recall that the functors $\mathbb{S}_2^{\mathcal{P}(1)} : Quad(\mathcal{C}, Mod_{\mathcal{P}(1)}) \to QMod_{\mathcal{C}}^{\mathcal{P}(1)}$ and $\mathbb{T}_2^{\mathcal{P}(1)} : QMod_{\mathcal{C}}^{\mathcal{P}(1)} \to Quad(\mathcal{C}, Mod_{\mathcal{P}(1)})$ both defined in 2.2.4 form a pair of adjoint functors by 2.2.7. We now give two functors which summarize the previous arguments.

Definition 2.4.27. We define the functors $\mathbb{S}_2^{\mathcal{P}} : Quad(\mathcal{C},\ Alg-\mathcal{P}) \to QMod_{\mathcal{C}}^{\mathcal{P}}$ and $\mathbb{T}_2^{\mathcal{P}} : QMod_{\mathcal{C}}^{\mathcal{P}} \to Quad(\mathcal{C},\ Alg-\mathcal{P})$ as follows:

1. The functor $\mathbb{S}_2^{\mathcal{P}} : Quad(\mathcal{C},\ Alg-\mathcal{P}) \to QMod_{\mathcal{C}}^{\mathcal{P}}$ is such that

   - *On objects*, let $F : \mathcal{C}\to Alg-\mathcal{P}$ be a quadratic functor, then $\mathbb{S}_2^{\mathcal{P}}(F)$ is the pair

     $$\Big(M^F,\ \phi^{M^F} : (M^F)^2 \to M^F\Big)$$

     Here
     - $M^F = \mathbb{S}_2^{\mathcal{P}(1)}(F)$ (see 2.2.5) is the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ corresponding to F seen as a functor with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$;
     - $(M^F)^2 = \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(M^F)\big)^{\otimes 2}\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2)\big)$ is the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ associated with the quadratic functor

$$T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(M^F)\big)^{\otimes 2}\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2) = T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}\cdot\mathbb{S}_2^{\mathcal{P}(1)}(F)\big)^{\otimes 2}\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2) : \mathcal{C}\to Mod_{\mathcal{P}(1)}\,;$$

– $\phi^{M^F} = \mathbb{S}_2^{\mathcal{P}(1)}\Big(\overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big(T_1(\varepsilon_F)^{\otimes 2} \otimes_S id\big)\Big) : (M^F)^2 \to M^F$;

where $\overline{\lambda_2^F} : T_2\big(F^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \Rightarrow F$ and $\overline{\phi^F} : T_2\big(F^{\otimes 2} \otimes_S \mathcal{P}(2)\big) \Rightarrow T_1 F^{\otimes 2} \otimes_S \mathcal{P}(2)$ are respectively the natural transformations given in 2.4.8 and (2.4.10), and $\varepsilon_F : \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}(F) \Rightarrow F$ is the counit of the adjunction 2.2.7 evaluated to F.

- *On morphisms*, let $\alpha : F \Rightarrow G$ be a natural transformation in $Quad(\mathcal{C},\ Alg - \mathcal{P})$, then $\mathbb{S}_2^{\mathcal{P}}(\alpha) = (\alpha_E,\ cr_2(\alpha)_{E,E})$.

2. The functor $\mathbb{T}_2^{\mathcal{P}} : QMod_{\mathcal{C}}^{\mathcal{P}} \to Quad(\mathcal{C},\ Alg - \mathcal{P})$ is such that

   - *On objects*, let $M^{\mathcal{P}} = (M,\ \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$ as in 2.4.23, $\mathbb{T}_2^{\mathcal{P}}(M^{\mathcal{P}}) = \big(-\otimes M,\ \lambda_1^M,\ \lambda_2^M\big)$, or simply $\mathbb{T}_2^{\mathcal{P}}(M^{\mathcal{P}}) = -\otimes M$, where
     – M is the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ (see 2.2.1) involved in the definition of $M^{\mathcal{P}}$.
     – $\big(-\otimes M,\ \lambda_1^M\big) = \mathbb{T}_2^{\mathcal{P}(1)}(M)$ where $\mathbb{T}_2^{\mathcal{P}(1)} : QMod_{\mathcal{C}}^{\mathcal{P}(1)} \to Quad(\mathcal{C},\ Mod_{\mathcal{P}(1)})$ is the functor defined in 2.4.27.
     – $\lambda_2^M : (-\otimes M)^{\otimes 2} \otimes \mathcal{P}(2) \Rightarrow -\otimes M$ is the natural transformation defined in (2.4.16).
   - *On morphims*, let $f = (f_e, f_{ee}) : M \to N$ be a morphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}$, then $\mathbb{T}_2^{\mathcal{P}}(f) = \mathbb{T}_2^{\mathcal{P}(1)}(f) = \mathbb{T}_2(f)$ is the unique natural transformation given by the universal property of the push-out 2.1.3.

Remark 2.4.28. If we assume that $\mathcal{C}$ is a semi-abelian variety and if E denotes the free object of rank 1 in $\mathcal{C}$, then the functor $\mathbb{T}_2^{\mathcal{P}}$ takes in fact values in $QUAD(\mathcal{C}, Alg - \mathcal{P})$. This is due to the fact that, for a quadratic $\mathcal{C}$-module $M^{\mathcal{P}} = (M, \phi^M)$ over $\mathcal{P}$, the composite functors $W \cdot \mathbb{T}_2^{\mathcal{P}}(M) = W \cdot (-\otimes M) : \mathcal{C} \to Ab$ preserves filtered colimits and coequalizers of reflexive graphs by 2.1.11, where $W : Alg - \mathcal{P} \to Ab$ is the forgetful functor. By 1.6.11, the (quadratic) functor $\mathbb{T}_2^{\mathcal{P}}(M^{\mathcal{P}}) : \mathcal{C} \to Alg - \mathcal{P}$ preserves filtered colimits and coequalizers of reflexive graphs.

2.4.8 The DNA of a quadratic functor from $\mathcal{C}$ to *Alg*-$\mathcal{P}$ is a quadratic $\mathcal{C}$-module over $\mathcal{P}$

We prove that the minimal algebraic data (called DNA) which characterize quadratic functors with domain $\mathcal{C}$ taking values in $Alg - \mathcal{P}$ are quadratic $\mathcal{C}$-modules over $\mathcal{P}$. Before giving the main theorem of this section, we recall that $\eta : Id \Rightarrow \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}$ and $\varepsilon : \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)} \Rightarrow Id$ are respectively the unit and the counit of the adjunction 2.2.7 (that is the same as in the adjunction pair 2.1.10), see 2.2.9. Let M be a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$. Then $\eta_M : M \to \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)$ is clearly a morphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$, hence so is the pair of morphisms

$$(\eta_M)^2 = \mathbb{S}_2^{\mathcal{P}(1)}\Big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(\eta_M)\big)^{\otimes 2} \otimes_S id\Big) : M^2 \to \big(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\big)^2 \tag{2.4.17}$$

by 2.4.22 (replacing f with η_M) where $T_1 : Func_*(\mathcal{C}, Mod_{\mathcal{P}(1)}) \to Lin(\mathcal{C}, Mod_{\mathcal{P}(1)})$ is the linearization functor (see 1.2.9), and $\big(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\big)^2$ is the quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ as follows:

$$\begin{aligned}\big(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\big)^2 &= \mathbb{S}_2^{\mathcal{P}(1)}\Big(T_1\big(-\otimes \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\big)^{\otimes 2} \otimes_S \mathcal{P}(2)\Big) \\ &= \mathbb{S}_2^{\mathcal{P}(1)}\Big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\big)^{\otimes 2} \otimes_S \mathcal{P}(2)\Big)\end{aligned}$$

Notation 2.4.29. Let $M = (M, \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$. We set the following composite morphisms of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$:

$$\phi^{\mathbb{T}_2 \cdot \mathbb{S}_2(M)} = \eta_M \circ \phi^M \circ \mathbb{S}_2^{\mathcal{P}(1)}\Big(T_1\big(\varepsilon_{-\otimes M}\big)^{\otimes 2} \otimes_S id\Big) : \big(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\big)^2 \to \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)$$

where $\varepsilon_{-\otimes M} : \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}(-\otimes M) = \mathbb{T}_2^{\mathcal{P}}(1) \cdot \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M) \Rightarrow -\otimes M = \mathbb{T}_2^{\mathcal{P}(1)}(M)$ is the counit of the adjunction 2.2.7 evaluated to the quadratic functor $-\otimes M : \mathcal{C} \to Mod_{\mathcal{P}(1)}$.

Remark 2.4.30. Let $M = (M, \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$. We observe that the pair

$$\left(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M), \phi^{\mathbb{T}_2 \cdot \mathbb{S}_2(M)}\right)$$

is a quadratic $\mathcal{C}$-module over $\mathcal{P}$ (see the definition given in 2.4.23).

Lemma 2.4.31. *Let $M^{\mathcal{P}} = (M, \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$, then the following diagram in $QMod_{\mathcal{C}}^{\mathcal{P}(1)}$*

$$\begin{tikzcd}
M^2 \arrow[r, "\phi^M"] \arrow[d, "(\eta_M)^2"'] & M \arrow[d, "\eta_M"] \\
\left(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\right)^2 \arrow[r, "\phi^{\mathbb{T}_2 \cdot \mathbb{S}_2(M)}"'] & \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)
\end{tikzcd}$$

commutes.

Proof. We have the equalities as follows:

$$\begin{aligned}
\phi^{\mathbb{T}_2 \cdot \mathbb{S}_2(M)} \circ (\eta_M)^2 &= \eta_M \circ \phi^M \circ \mathbb{S}_2^{\mathcal{P}(1)}\Big(\big(T_1(\varepsilon_{-\otimes M}\big)^{\otimes 2} \otimes_S id\big)\Big) \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(\eta_M)\big)^{\otimes 2} \otimes_S id\big) \\
&= \eta_M \circ \phi^M \circ \Big(T_1\big(\varepsilon_{-\otimes M} \circ \mathbb{T}_2^{\mathcal{P}(1)}(\eta_M)\big)^{\otimes 2} \otimes_S id\big)\Big) \\
&= \eta_M \circ \phi^M
\end{aligned}$$

because $\varepsilon_{-\otimes M} \circ \mathbb{T}_2^{\mathcal{P}(1)}(\eta_M) = id$. □

Corollary 2.4.32. *Let $M^{\mathcal{P}} = (M, \phi^M)$ be a quadratic $\mathcal{C}$-module over $\mathcal{P}$. Then the unit $\eta_M : M \to \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)$ of the pair of adjoint functors 2.2.7 is a morphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}$ from $M^{\mathcal{P}} = (M, \phi^M)$ to $\big(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M), \phi^{\mathbb{T}_2 \cdot \mathbb{S}_2(M)}\big)$.*

Proof. It is a direct consequence of 2.4.31. □

Remark 2.4.33. We observe that the unit $\eta : Id \Rightarrow \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}$ of the pair of adjoint functors 2.2.7 can be considered as a natural transformation from the identity functor of $QMod_{\mathcal{C}}^{\mathcal{P}}$ to the composite functors $\mathbb{S}_2^{\mathcal{P}} \cdot \mathbb{T}_2^{\mathcal{P}}$. It is denoted by $\eta^{\mathcal{P}} : Id \Rightarrow \mathbb{S}_2^{\mathcal{P}} \cdot \mathbb{T}_2^{\mathcal{P}}$ in this case.

Now we prove that, for a quadratic functor $F : \mathcal{C} \to Alg - \mathcal{P}$ and an object X in $\mathcal{C}$, the counit $(\varepsilon_F)_X : \big(\mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}(F)\big)(X) = X \otimes M^F \to F(X)$ is a homomorphism of $\mathcal{P}$-algebras. We known that it is a homomorphism of (right) $\mathcal{P}(1)$-modules by 2.2.7. Then we have the following proposition:

Proposition 2.4.34. *Let $F : \mathcal{C} \to Alg - \mathcal{P}$ be a quadratic functor and let X be an object in $\mathcal{C}$. Then the counit $(\varepsilon_F)_X : \big(\mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}(F)\big)(X) \to F(X)$ satisfies*

$$(\lambda_2^F)_X \circ (\varepsilon_F)_X = (\varepsilon_F)_X \circ (\lambda_2^{M^F})_X$$

where $(\lambda_2^F)_X : F(X)^{\otimes 2} \otimes \mathcal{P}(2) \to F(X)$, respectively $(\lambda_2^{M^F})_X : (X \otimes M^F)^{\otimes 2} \otimes \mathcal{P}(2) \to X \otimes M^F$ (defined in 2.4.16) is the structure linear map involved in the $\mathcal{P}$-algebra structure of $F(X)$, respectively $X \otimes M^F$.

Proof. First we recall that $R = \mathcal{P}(1)$. Then we get the following equalities:

$$\begin{aligned}
&\varepsilon_F \circ \lambda_2^{M^F} \\
&= \varepsilon_F \circ \mathbb{T}_2^{\mathcal{P}(1)}(\phi^{M^F}) \circ \varepsilon^{-1}_{T_1(-\otimes M^F)^{\otimes 2}\otimes_S \mathcal{P}(2)} \circ \big((t_1^{M^F})^{\otimes} \otimes_S id\big) \circ q^{M^F} \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= \varepsilon_F \circ \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}\Big(\overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big(T_1(\varepsilon_F)^{\otimes 2} \otimes_S id\big)\Big) \circ \varepsilon^{-1}_{T_1(-\otimes M^F)^{\otimes 2}\otimes_S \mathcal{P}(2)} \circ \big((t_1^{M^F})^{\otimes} \otimes_S id\big) \circ q^{M^F} \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= \overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big(T_1(\varepsilon_F)^{\otimes 2} \otimes_S id\big) \circ \varepsilon_{T_1(-\otimes M^F)^{\otimes 2}\otimes_S \mathcal{P}(2)} \circ \varepsilon^{-1}_{T_1(-\otimes M^F)^{\otimes 2}\otimes_S \mathcal{P}(2)} \circ \big((t_1^{M^F})^{\otimes} \otimes_S id\big) \circ q^{M^F} \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= \overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big(T_1(\varepsilon_F)^{\otimes 2} \otimes_S id\big) \circ \big((t_1^{M^F})^{\otimes} \otimes_S id\big) \circ q^{M^F} \circ q_{\mathbb{Z}}^{R\otimes R}
\end{aligned}$$

By naturality of $t_1^{M^F}$ in M^F, we have $T_1(\varepsilon_F) \circ t_1^{M^F} = t_1^F \circ \varepsilon_F$. Hence we obtain

$$\begin{aligned}
&\varepsilon_F \circ \lambda_2^{M^F} \\
&= \overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big(T_1(\varepsilon_F)^{\otimes 2} \otimes_S id\big) \circ \big((t_1^{M^F})^{\otimes} \otimes_S id\big) \circ q^{M^F} \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= \overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big((t_1^{F})^{\otimes} \otimes_S id\big) \circ \big((\varepsilon_F)^{\otimes 2} \circ_S id\big) \circ q^{M^F} \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= \overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big((t_1^{F})^{\otimes} \otimes_S id\big) \circ q^F \circ \big((\varepsilon_F)^{\otimes 2} \circ_{R\otimes R} id\big) \circ q_{\mathbb{Z}}^{R\otimes R} \\
&= \overline{\lambda_2^F} \circ (\overline{\phi^F})^{-1} \circ \big((t_1^{F})^{\otimes} \otimes_S id\big) \circ q^F \circ q_{\mathbb{Z}}^{R\otimes R} \circ \big((\varepsilon_F)^{\otimes 2} \circ id\big) \\
&= \lambda_2^F \circ \big((\varepsilon_F)^{\otimes 2} \circ id\big)\,,
\end{aligned}$$

as desired. □

Corollary 2.4.35. *Let $F : \mathcal{C} \to Alg - \mathcal{P}$ be a quadratic functor and let X be an object in $\mathcal{C}$. Then the morphism $(\varepsilon_F)_X : \big(\mathbb{T}_2^{\mathcal{P}(1)}.\mathbb{S}_2^{\mathcal{P}(1)}(F)\big)(X) \to F(X)$ is a homomorphism of $\mathcal{P}$-algebras.*

Remark 2.4.36. Let $F : \mathcal{C} \to Alg - \mathcal{P}$ be a quadratic functor. By 2.4.35, we observe that the counit $\varepsilon : \mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)} \Rightarrow Id$ can be seen as a natural transformation from the composite functors $\mathbb{T}_2^{\mathcal{P}} \cdot \mathbb{S}_2^{\mathcal{P}}$ to the identity functor of $Quad(\mathcal{C}, Alg - \mathcal{P})$. It is denoted by $\varepsilon^{\mathcal{P}} : \mathbb{T}_2^{\mathcal{P}} \cdot \mathbb{S}_2^{\mathcal{P}} \Rightarrow Id$ in this case.

The main result of this section generalizes the theorems 7.1 of [12] and 2.2.7. It says that quadratic functors with domain $\mathcal{C}$ and values in $Alg - \mathcal{P}$ are entirely characterized by quadratic $\mathcal{C}$-modules over $\mathcal{P}$ (see 2.4.23) which constitute their DNA.

Theorem 2.4.37. *Let $\mathcal{P}$ be an operad as in 2.4.1. Then*

- *the functors*
$$\mathbb{S}_2^{\mathcal{P}} : Quad(\langle E\rangle, Alg - \mathcal{P}) \rightleftarrows QMod_{\mathcal{C}}^{\mathcal{P}} : \mathbb{T}_2^{\mathcal{P}}$$
are equivalences of categories, inverse to each other.

- *if $\mathcal{C}$ is a semi-abelian variety, then the functors*
$$(\mathbb{S}_2^{\mathcal{P}})' : QUAD(\mathcal{C}, Alg - \mathcal{P}) \rightleftarrows QMod_{\mathcal{C}}^{\mathcal{P}} : (\mathbb{T}_2^{\mathcal{P}})'$$
also are equivalences of categories, inverse to each other. Here the functor $(\mathbb{T}_2^{\mathcal{P}})'$ is given by $\mathbb{T}_2^{\mathcal{P}}$ (defined in 2.4.27) which actually takes values in $QUAD(\mathcal{C}, Alg - \mathcal{P})$ (by 2.4.28), and where the functor $(\mathbb{S}_2^{\mathcal{P}})'$ is the restriction of $\mathbb{S}_2^{\mathcal{P}}$ (given in 2.4.27).

Proof. First we assume that $\mathcal{C}$ is a semi-abelian variety. In the whole proof, X, $M^{\mathcal{P}}$ and $G : \mathcal{C} \to Alg - \mathcal{P}$ will denote respectively an indeterminate object of $\mathcal{C}$, a quadratic $\mathcal{C}$-module over $\mathcal{P}$ (see 2.4.23) and a quadratic functor. Here we prove that the natural transformation $\eta^{\mathcal{P}} : Id \Rightarrow \mathbb{S}_2^{\mathcal{P}} \cdot \mathbb{T}_2^{\mathcal{P}}$ (given in (2.4.33)) is the unit of the pair of adjoint functors in the statement. For this we check that $\eta_M = \eta_{M^{\mathcal{P}}} : M^{\mathcal{P}} \to \mathbb{S}_2^{\mathcal{P}} \cdot \mathbb{T}_2^{\mathcal{P}}(M^{\mathcal{P}})$ satisfies the universal property of the unit of the adjunction in the statement.

Let $\alpha = (\alpha_e,\, \alpha_{ee}) : M^{\mathcal{P}} \to \mathbb{S}_2^{\mathcal{P}}(G)$ be any morphism in $QMod_{\mathcal{C}}^{\mathcal{P}}$. We check that there is a unique natural transformation $\beta : \mathbb{T}_2^{\mathcal{P}}(M^{\mathcal{P}}) \Rightarrow G$ in $Quad(\mathcal{C},\, Alg - \mathcal{P})$ such that $\alpha = \mathbb{S}_2^{\mathcal{P}}(\beta) \circ \eta_M$. First we remark that $\alpha : M \to \mathbb{S}_2^{\mathcal{P}(1)}(G)$ is also a morphism in $QMod_{\mathcal{C}}^{\mathcal{P}(1)}$. Hence there is a unique natural transformation $\beta : \mathbb{T}_2^{\mathcal{P}(1)}(M) \Rightarrow G$ in $Quad(\mathcal{C},\, Mod_{\mathcal{P}(1)})$ such that

$$\alpha = \mathbb{S}_2^{\mathcal{P}(1)}(\beta) \circ \eta_M \tag{2.4.18}$$

by the universal property of the unit $\eta_M : M \to \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)$ of the adjunction 2.2.7. Next we have

$$\begin{aligned}
\varepsilon_G \circ \mathbb{T}_2^{\mathcal{P}(1)}(\alpha) &= \varepsilon_G \circ \big(\mathbb{T}_2^{\mathcal{P}(1)} \cdot \mathbb{S}_2^{\mathcal{P}(1)}\big)(\beta) \circ \mathbb{T}_2^{\mathcal{P}(1)}(\eta_M)\\
&= \beta \circ \varepsilon_{-\otimes M} \circ \mathbb{T}_2^{\mathcal{P}(1)}(\eta_M)\\
&= \beta
\end{aligned}$$

Thus we get

$$\varepsilon_G \circ \mathbb{T}_2^{\mathcal{P}(1)}(\alpha) = \beta \tag{2.4.19}$$

Then we verify that $\alpha = \mathbb{S}_2^{\mathcal{P}}(\beta) \circ \eta_{M^{\mathcal{P}}}$. It remains to prove that the following diagram commutes:

$$\begin{tikzcd}
 & M^2 \arrow[rr, "\phi^M"] \arrow[d, "(\eta_M)^2"] \arrow[ddl, "\alpha^2"'] & & M \arrow[d, "\eta_M"] \arrow[ddl, "\alpha"'] \\
 & \big(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M)\big)^2 \arrow[rr, "\phi^{\mathbb{S}_2 \cdot \mathbb{T}_2(M)}"'] \arrow[dl, "\big(\mathbb{S}_2^{\mathcal{P}(1)}(\beta)\big)^2"] & & \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M) \arrow[dl, "\mathbb{S}_2^{\mathcal{P}(1)}(\beta)"] \\
\big(\mathbb{S}_2^{\mathcal{P}(1)}(G)\big)^2 \arrow[rr, "\phi^{M^G}"'] & & \mathbb{S}_2^{\mathcal{P}(1)}(G) &
\end{tikzcd}$$

Here we recall that $\phi^{M^G} = \mathbb{S}_2^{\mathcal{P}(1)}\big(\overline{\lambda_2^G} \circ (\phi^G)^{-1} \circ \big(T_1(\varepsilon_G)^{\otimes 2} \otimes_S id\big)\big) = \phi^{M^F} \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1(\varepsilon_G)^{\otimes 2} \otimes_S id\big)$, see the definition in 2.4.27. The top rectangle and the right-hand triangle of the above diagram commute by 2.4.31 and (2.4.18). Moreover the diagonal rectangle commutes because $\alpha : M^{\mathcal{P}} \to \mathbb{S}_2^{\mathcal{P}}(G)$ is a morphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}$. Then we prove that the left-hand triangle commutes. We have the equalities as follows:

$$\begin{aligned}
\big(\mathbb{S}_2^{\mathcal{P}(1)}(\beta)\big)^2 \circ (\eta_M)^2 &= \mathbb{S}_2^{\mathcal{P}(1)}\big(\big(T_1(\beta)^{\otimes 2} \otimes_S id\big)\big) \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(\big(T_1(\mathbb{T}_2^{\mathcal{P}(1)}(\eta_M)\big)^{\otimes 2} \otimes_S id\big)\big)\\
&= \mathbb{S}_2^{\mathcal{P}(1)}\Big(T_1\big(\beta \circ \mathbb{T}_2^{\mathcal{P}(1)}(\eta_M)\big)^{\otimes 2} \otimes_S id\Big)\\
&= \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1\big(\mathbb{T}_2^{\mathcal{P}(1)}(\alpha)\big)^{\otimes 2} \otimes_S id\big)\\
&= \alpha^2\,,
\end{aligned}$$

as desired. As $(\eta_M)^2$ is an isomorphism (hence an epimorphism), the bottom rectangle of the above diagram commutes, i.e.

$$\phi^{M^G} \circ \big(\mathbb{S}_2^{\mathcal{P}(1)}(\beta)\big)^2 = \mathbb{S}_2^{\mathcal{P}(1)}(\beta) \circ \phi^{\mathbb{S}_2.\mathbb{T}_2(M)} \tag{2.4.20}$$

implying that $\mathbb{S}_2^{\mathcal{P}(1)}(\beta) : \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M) \to \mathbb{S}_2^{\mathcal{P}(1)}(G)$ is in fact a morphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}$.
Next we prove that, for all $X \in \mathcal{C}$, $\beta_X : \mathbb{T}_2^{\mathcal{P}(1)}(M)(X) = X \otimes M \to G(X)$ is a homomorphism of $\mathcal{P}$-algebras. By 2.4.18 and 2.2.7, we know that β_X is a $\mathcal{P}(1)$-module homomorphism. It remains to prove that

$$\beta_X \circ (\lambda_2^M)_X = (\lambda_2^G)_X \circ \big((\beta_X)^{\otimes 2} \otimes_S id\big)$$

For this we consider the following equalities:

$$\begin{aligned}
&\mathbb{S}_2^{\mathcal{P}(1)}\Big(\beta \circ \mathbb{T}_2^{\mathcal{P}(1)}(\phi^M)\Big) \circ \eta_{M^2} \\
&= \mathbb{S}_2^{\mathcal{P}(1)}(\beta) \circ \big(\mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}\big)(\phi^M) \circ \eta_{M^2} \\
&= \mathbb{S}_2^{\mathcal{P}(1)}(\beta) \circ \eta_M \circ \phi^M \\
&= \mathbb{S}_2^{\mathcal{P}(1)}(\beta) \circ \eta_M \circ \phi^M \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1\big(\varepsilon_{-\otimes M}\big)^{\otimes 2} \otimes_S id\big) \circ (\eta_M)^2 \\
&= \mathbb{S}_2^{\mathcal{P}(1)}(\beta) \circ \phi^{\mathbb{S}_2 \cdot \mathbb{T}_2(M)} \circ (\eta_M)^2 \\
&= \phi^{M^G} \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1(\beta)^{\otimes 2} \otimes_S id\big) \circ (\eta_M)^2 \\
&= \mathbb{S}_2^{\mathcal{P}(1)}\Big(\overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\varepsilon_G)^{\otimes 2} \otimes_S id\big)\Big) \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1(\beta)^{\otimes 2} \otimes_S id\big) \circ (\eta_M)^2 \\
&= \mathbb{S}_2^{\mathcal{P}(1)}\Big(\overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\varepsilon_G \circ \beta)^{\otimes 2} \otimes_S id\big)\Big) \circ (\eta_M)^2 \\
&= \mathbb{S}_2^{\mathcal{P}(1)}\big(\overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1\big(\beta \circ \varepsilon_{-\otimes M}\big)^{\otimes 2}\big) \otimes_S id\big) \circ (\eta_M)^2 \\
&= \mathbb{S}_2^{\mathcal{P}(1)}\big(\overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\beta)^{\otimes 2} \otimes_S id\big)\big) \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(T_1\big(\varepsilon_{-\otimes M}\big)^{\otimes 2} \otimes_S id\big) \circ (\eta_M)^2 \\
&= \mathbb{S}_2^{\mathcal{P}(1)}\big(\overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\beta)^{\otimes 2} \otimes_S id\big)\big) \\
&= \mathbb{S}_2^{\mathcal{P}(1)}\big(\overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\beta)^{\otimes 2} \otimes_S id\big)\big) \circ \mathbb{S}_2^{\mathcal{P}(1)}\big(\varepsilon_{T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)}\big) \circ \eta_{M^2} \\
&= \mathbb{S}_2^{\mathcal{P}(1)}\big(\overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\beta)^{\otimes 2} \otimes_S id\big) \circ \varepsilon_{T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)}\big) \circ \eta_{M^2}
\end{aligned}$$

By the uniqueness in the universal property of the unit $\eta_{M^2} : M^2 \to \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}(M^2)$, we obtain

$$\beta \circ \mathbb{T}_2^{\mathcal{P}(1)}(\phi^M) = \overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ T_1(\beta)^{\otimes 2} \otimes_S id\big) \circ \varepsilon_{T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)} \tag{2.4.21}$$

Then we get the equalities as follows:

$$\begin{aligned}
\beta \circ \lambda_2^M &= \beta \circ \mathbb{T}_2^{\mathcal{P}(1)}(\phi^M) \circ \varepsilon^{-1}_{T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)} \circ \big((t_1^M)^{\otimes 2} \otimes_S id\big) \circ q^M \circ q_{\mathbb{Z}}^{R \otimes R} \\
&= \overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\beta)^{\otimes 2} \otimes_S id\big) \circ \varepsilon_{T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)} \circ \varepsilon^{-1}_{T_1(-\otimes M)^{\otimes 2} \otimes_S \mathcal{P}(2)} \circ \big((t_1^M)^{\otimes 2} \otimes_S id\big) \circ q^M \circ q_{\mathbb{Z}}^{R \otimes R} \\
&= \overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big(T_1(\beta)^{\otimes 2} \otimes_S id\big) \circ \big((t_1^M)^{\otimes 2} \otimes_S id\big) \circ q^M \circ q_{\mathbb{Z}}^{R \otimes R} \\
&= \overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big((t_1^G)^{\otimes 2} \otimes_S id\big) \circ \big(\beta^{\otimes 2} \otimes_S id\big) \circ q^M \circ q_{\mathbb{Z}}^{R \otimes R} \\
&= \overline{\lambda_2^G} \circ (\overline{\phi^G})^{-1} \circ \big((t_1^G)^{\otimes 2} \otimes_S id\big) \circ q^G \circ q_{\mathbb{Z}}^{R \otimes R} \circ \big(\beta^{\otimes 2} \otimes id\big) \\
&= \lambda_2^G \circ \big(\beta^{\otimes 2} \otimes id\big)\,,
\end{aligned}$$

as desired.
As $\eta : Id \Rightarrow \mathbb{S}_2^{\mathcal{P}(1)} \cdot \mathbb{T}_2^{\mathcal{P}(1)}$ is a natural equivalence by 2.1.13, so is $\eta^{\mathcal{P}} : Id \Rightarrow \mathbb{S}_2^{\mathcal{P}} \cdot \mathbb{T}_2^{\mathcal{P}}$ (see the notation given in 2.4.33). Hence it suffices to prove that the counit $\varepsilon^{\mathcal{P}} : \mathbb{T}_2^{\mathcal{P}} \cdot \mathbb{S}_2^{\mathcal{P}} \Rightarrow Id$ (see 2.4.36) is a natural equivalence for the second and third points in the statement.
If we assume that $\mathcal{C} = \langle E \rangle$, then ε is a natural equivalence by the second point of 2.1.10 implying that $\varepsilon^{\mathcal{P}}$ is also a natural equivalence. Hence the functors $\mathbb{S}_2^{\mathcal{P}}$ and $\mathbb{T}_2^{\mathcal{P}}$ form a pair of adjoint equivalences.
Now we suppose that $\mathcal{C}$ is a semi-abelian variety and E is the free object of rank 1 in $\mathcal{C}$. For a quadratic functor $F : \mathcal{C} \to Alg - \mathcal{P}$ preserving filtered colimits and coequalizers of reflexive graphs, the counit $\varepsilon_F^{\mathcal{P}} : \mathbb{T}_2^{\mathcal{P}} \cdot \mathbb{S}_2^{\mathcal{P}}(F) = - \otimes \mathbb{S}_2(F) \to F$ (evaluated to F) is a natural transformation between quadratic functors preserving filtered colimits and coequalizers of reflexive graphs which is a natural isomorphism if restricted to the full subcategory $\langle E \rangle$ of $\mathcal{C}$ (by the above argument). Hence it is a natural isomorphism by 6.25 of [12]. Thus the functors $(\mathbb{S}_2^{\mathcal{P}})'$ and $(\mathbb{T}_2^{\mathcal{P}})'$ in the statement form a pair of adjoint equivalences. $\square$

This concludes the proof of 1.4.42, which gives the "DNA" of quadratic functors with domain $\mathcal{C}$ and values in $\mathcal{P}$-algebras.

Chapter 3

Quadratic equivalences

Here we assume that $\mathcal{C}$ is a 2-step nilpotent category. Moreover we recall that $\mathcal{P}$ is a linear symmetric unitary operad in the category of abelian groups endowed with its standard monoidal structure given by the tensor product. The unit of $\mathcal{P}$ is denoted by $1_{\mathcal{P}} \in \mathcal{P}(1)$, and $Alg - \mathcal{P}$ denotes the category of $\mathcal{P}$-algebras.
In this chapter, we first give a criterion for certain quadratic functors to be quadratic equivalences by using the notion of linear extension of categories. Then we characterize quadratic $\mathcal{C}$-modules over $\mathcal{P}$ which correspond to quadratic equivalences with values in $\mathcal{P}$-algebras.

Notation 3.0.1. We denote by $\mathcal{C}^{op}$ the dual category of $\mathcal{C}$ whose objects are the same as those in $\mathcal{C}$ and, for X and Y objects in $\mathcal{C}$, morphisms are of the form $f^{op} : Y \to X$, where $f : X \to Y$ is a morphism in $\mathcal{C}$. In addition, we consider $Op^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{op}$ the contravariant functor which is the identity on objects and reverses direction of any morphism in $\mathcal{C}$.

If $G : \mathcal{C} \to \mathcal{D}$ is any functor, then $G^{OP} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is the unique functor factorizing $Op^{\mathcal{D}} \cdot G$ through $Op^{\mathcal{C}}$. Then we recall that $Ab(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ formed by abelian objects (see 1.3.1) and that $Ab^{\mathcal{C}} : \mathcal{C} \to Ab(\mathcal{C})$ is the abelianization functor defined in 1.4.4. As $[X,\, X]_{Id_{\mathcal{C}}}$ is a normal subobject of X, we have the following short exact sequence in $\mathcal{C}$:

$$0 \longrightarrow [X,\, X]_{Id_{\mathcal{C}}} \xrightarrow{i_X} X \xrightarrow{ab_X} X^{ab} \longrightarrow 0 \tag{3.0.1}$$

where $i_X : [X,\, X]_{Id_{\mathcal{C}}} \rightarrowtail X$ is the image of c_2^X and we denote $e_X : Id_{\mathcal{C}}(X|X) \twoheadrightarrow [X, X]_{Id_{\mathcal{C}}}$ its coimage (see the notations given in 1.3.5).

3.1 2-step nilpotent categories as linear extensions of abelian categories

In this part, we shall use the notion of linear extensions of categories given in 5.1 of [4]. There are countless examples of this setting in algebra as well as in homotopy theory. As an example, it is used to characterize the category of the Moore spaces $M(A, 2)$, for A an abelian group, as a non trivial cohomology class of the second cohomology of Ab in coefficients on the Ab-bimodule $Ext(-,\, \Gamma) : Ab^{op} \times Ab \to Ab$, where $\Gamma : Ab \to Ab$ is the J.H.C Whitehead's quadratic functor [39], see [3]. Here we recall the basic definition:

Definition 3.1.1. Let $\mathcal{B}$ be a category and let $D : \mathcal{B}^{op} \times \mathcal{B} \to Ab$ be a bifunctor. We say that

$$D \xrightarrow{+} \mathcal{A} \xrightarrow{p} \mathcal{B}$$

is a *linear extension of the category* $\mathcal{B}$ if

1. $\mathcal{A}$ is a category with the same objects as $\mathcal{B}$, and p is a full functor which is the identity on objects;

2. for each $f : A \to B$ in $\mathcal{B}$, the abelian group $D(A,\, B)$ acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\mathcal{A}$. We write $f_0+\alpha$ for the action of $\alpha \in D(A,\, B)$ on $f_0 \in p^{-1}(f)$. Any $f_0 \in p^{-1}(f)$ is called a lift of f;

3. the action satisfies the linear distributive law:

$$(f_0 + \alpha) \circ (g_0 + \beta) = f_0 \circ g_0 + \big(D(id_C, f)(\beta) + D(g, id_B)(\alpha)\big)$$

where $f : A \to B$ and $g : C \to A$ are morphisms in $\mathcal{B}$ and $f_0 : A \to B$ (resp. $g_0 : C \to A$) are respecitvely lifts of f (resp. g).

The fundamental algebraic example studied and exploited in many contexts is the following; it provides the model of our generalization to arbitrary 2-step nilpotent categories below.

Example 3.1.2. Let us denote by $\langle\mathbb{Z}\rangle_{Nil_2(Gr)}$ the full subcategory of the category of groups Gr whose objects are free 2-step nilpotent groups of finite rank. We denote by $\langle\mathbb{Z}\rangle_{Ab}$ the full subcategory of the category of abelian groups Ab formed by free abelian groups of finite rank, and by $Ab : Gr \to Ab$ the abelianization functor. Here we take the restriction of the abelianization functor to $\langle\mathbb{Z}\rangle_{Nil_2(Gr)}$ taking values in $\langle\mathbb{Z}\rangle_{Ab}$ (since the functor Ab preserves coproducts), also denoted by $Ab : \langle\mathbb{Z}\rangle_{Nil_2(Gr)} \to \langle\mathbb{Z}\rangle_{Ab}$. Then we consider the category $Im(Ab)$ that has the same objects as $\langle\mathbb{Z}\rangle_{Nil_2(Gr)}$ and, for F and H two free 2-step nilpotent groups of finite rank, $Im(Ab)(F,\, H)$ is the set of morphisms $Ab(f) = f^{ab} : F^{ab} \to H^{ab}$ where $f : F \to H$ is a morphism in $\langle\mathbb{Z}\rangle_{Nil_2(Gr)}$. We define the functor $Ab' : \langle\mathbb{Z}\rangle_{Nil_2(Gr)} \to Im(Ab)$ that is the identity on objects and the abelianization functor on morphisms. Moreover we consider the functor $\tilde{\gamma_2} : Im(Ab) \to Ab$ given by

- *On objects*, let F be an object in $\langle\mathbb{Z}\rangle_{Nil_2(Gr)}$, then $\tilde{\gamma_2}(F) = \gamma_2(F) = [F,\, F] \in Ab$. We recall that $[F,\, F]$ is here the classical binary commutator in Gr.

- *On morphisms*, let F and H be two free 2-step nilpotent groups of finite rank and $g : F^{ab} \to H^{ab}$ be a morphism in $Im(Ab)$, then we set $\tilde{\gamma_2}(g) = \gamma_2(f_0)$ where $f_0 : F \to H$ is any lift of $g \circ ab_F$ through ab_H, i.e. such that

$$g \circ ab_F = ab_H \circ f$$

where $ab_F : F \to F^{ab}$ is the quotient map. Such morphisms exist because F is projective (as any free group). Let $f_1,\, f_2 : F \to H$ be two such lifts, then their "difference" takes values in $\gamma_2(H) = [H,\, H]$, i. e.

$$\forall x \in F, \quad (f_1\, f_2^{-1})(x) = f_1(x) f_2(x)^{-1} \in [H,\, H] = \gamma_2(H)$$

because, for $x \in F$, $ab_H(f_1(x)) = ab_H(f_2(x))$ and $Ker(ab_H) = [H, H]$. Moreoever we observe that $f_1.f_2^{-1} : F \to \gamma_2(H)$ is a group homomorphism because $\gamma_2(H)$ is central in H (as H is a 2-step nilpotent group). Then we deduce that f_1 and f_2 are equal if we both restrict them on $\gamma_2(F) = [F,\, F]$, i.e. $\gamma_2(f_1) = \gamma_2(f_2)$ because

$$(f_1\, f_2^{-1})([F,\, F]) = [(f_1\, f_2^{-1})(F),\, (f_1\, f_2^{-1})(F)] \subset [\gamma_2(H),\, \gamma_2(H)] = 0$$

This is due to the fact that $\gamma_2(H)$ is central in H (hence it is an abelian group) because H is a 2-step nilpotent group. As a consequence, the functor $\tilde{\gamma_2} : Im(Ab) \to Ab$ is well-defined on morphisms.

We consider F a free 2-step nilpotent group of finite rank. As we know, there is a natural extension of $Ab(F) = F^{ab}$ by the binary commutator $\gamma_2(F) = [F,\ F]$ (which is an abelian group because it is a central subgroup in F) as follows:

$$0 \longrightarrow [F,\ F] \xrightarrow{i_F} F \xrightarrow{ab_F} F^{ab} \longrightarrow 0 \tag{3.1.1}$$

where $i_F : [F,\ F] \to F$ is the inclusion map and $ab_F : F \to F^{ab}$ is the quotient map. Then such "concrete" extensions can be put into a global structure by considering the following linear extension of the category $Im(Ab)$:

$$Hom\big(-,\ \tilde{\gamma_2}\big) \xrightarrow{+} \langle \mathbb{Z} \rangle_{Nil_2(Gr)} \xrightarrow{Ab'} Im(Ab) \tag{3.1.2}$$

Here the bifunctor $Hom\big(-,\ \tilde{\gamma_2}\big) : Im(Ab)^{op} \times Im(Ab) \to Ab$ is defined on objects by

$$Hom\big(-,\ \tilde{\gamma_2}\big)(F, H) = Hom(F,\ \tilde{\gamma_2}(H)) = Hom(F,\ \gamma_2(H)),$$

for F and H objects in $\langle \mathbb{Z} \rangle_{Nil_2(Gr)}$. Note that, for any free 2-step nilpotent group of finite rank F, the abelian group $\gamma_2(F) = [F, F]$ may be seen as the second exterior algebra $\wedge^2 F^{ab}$ of F^{ab}. For this we observe that $[-, -] : F \times F \to F$ factorizes through the surjection $ab_F \times ab_F : F \times F \to F^{ab} \times F^{ab}$ and clearly maps into the central subobject $[F, F]$ of F. We denote by $\overline{[-, -]} : F^{ab} \times F^{ab} \to [F, F]$ its (unique) factorization which is bilinear since F is 2-step nilpotent. By the universal property of the tensor product, there is a unique abelian group homomorphism $\phi_F : F^{ab} \otimes F^{ab} \to [F, F]$ that factorizes $\overline{[-, -]} : F^{ab} \times F^{ab} \to [F, F]$ through $\otimes : F^{ab} \times F^{ab} \to F^{ab} \otimes F^{ab}$. Hence we have

$$\phi_F(\overline{g} \otimes \overline{g'}) = [g, g']$$

where $g, g' \in F$. If $g = g'$, then $\phi_F(\overline{g} \otimes \overline{g})$ is trivial. Hence there is a unique abelian group homomorphism $\overline{\phi_F} : \wedge^2 F^{ab} \to [F, F]$ by the universal property of the second exterior algebra of F^{ab}. It is clearly a surjection (already true if F is any 2-step nilpotent group). Now thanks to Witt's theorem, the abelian group homomorphism $\overline{\phi_F} : \wedge^2 F^{ab} \to [F, F]$ is an isomorphism whenever F is a free 2-step nilpotent group of finite rank. Thus we get back the classical central extension for 2-step nilpotent groups as follows:

$$Hom\big(-,\ \wedge^2\big) \xrightarrow{+} \langle \mathbb{Z} \rangle_{Nil_2(Gr)} \xrightarrow{Ab'} Im(Ab)$$

Now we generalize this example to any 2-step nilpotent category.

Definition 3.1.3. We define the category $Im(Ab^{\mathcal{C}})$ such that it has the same objects as $\mathcal{C}$ and, for X and Y objects in $\mathcal{C}$, $Im(Ab^{\mathcal{C}})(X,\ Y)$ is the set of morphisms $f^{ab} : X^{ab} \to Y^{ab}$ where $f : X \to Y$ is a morphism in $\mathcal{C}$. If we take the restriction of the above linear extension to any full subcategory $\mathcal{C}'$ whose objects are regular projective, then $Im(Ab^{\mathcal{C}'})(X,\ Y) = Ab(\mathcal{C}')(X^{ab},\ Y^{ab})$ for X and Y objects in $\mathcal{C}'$.

Notation 3.1.4. We consider the functor $(Ab^{\mathcal{C}})' : \mathcal{C} \to Im(Ab^{\mathcal{C}})$ that is the identity on objects and the abelianization functor $Ab^{\mathcal{C}} : \mathcal{C} \to Ab(\mathcal{C})$ on morphisms.

Let X and Y be two objects in $\mathcal{C}$. We set $D(X,\ Y) = \mathcal{C}(X^{ab},\ [Y,\ Y]_{Id_{\mathcal{C}}})$, and it is an abelian group because $[Y,\ Y]_{Id_{\mathcal{C}}}$ is an abelian object in $\mathcal{C}$ (since it is a central subobject of Y). Since $(ab_X)^* : \mathcal{C}(X^{ab}, [Y, Y]_{Id_{\mathcal{C}}}) \to \mathcal{C}(X, [Y, Y]_{Id_{\mathcal{C}}})$ is an isomorphism of abelian groups, the abelian group $D(X,\ Y)$ simply acts on $\mathcal{C}(X,\ Y)$ by 1.5.22 as follows:

$$f +_D \alpha = f + (ab_X)^*(\alpha) = \varphi_{i_Y} \circ (f,\ \alpha \circ ab_X)^t, \tag{3.1.3}$$

for $f \in \mathcal{C}(X, Y)$ and $\alpha \in D(X, Y)$, where $\varphi_{i_Y} : Y \times [Y,\ Y]_{Id_{\mathcal{C}}} \to Y$ is the unique factorization of $(id,\ i_Y) : Y + [Y,\ Y]_{Id_{\mathcal{C}}} \to Y$ through $\widehat{r_2^{Id_{\mathcal{C}}}} : Y + [Y,\ Y]_{Id_{\mathcal{C}}} \to Y \times [Y,\ Y]_{Id_{\mathcal{C}}}$ (see 1.5.4).

Remark 3.1.5. Let $f, g : X \to Y$ be two morphisms in $\mathcal{C}$. Then, $ab_Y \circ f = ab_Y \circ g$ if, and only if, there is $\alpha \in D(X, Y)$ such that

$$g = f +_D \alpha$$

It is a direct consequence of 1.5.22.

Remark 3.1.6. Let $X, X', Y, Y' \in \mathcal{C}$. For $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(X', X)$, $h \in \mathcal{C}(Y, Y')$ and $\alpha \in D(X, Y)$, we have

$$\begin{cases} (f +_D \alpha) \circ g = f \circ g +_D \alpha \circ g^{ab} \\ h \circ (f +_D \alpha) = h \circ f +_D \gamma_2^{\mathcal{C}}(h) \circ \alpha \end{cases}$$

It is a direct consequence of 1.5.22.

Definition 3.1.7. We define the functor $\tilde{\gamma_2}^{\mathcal{C}} : Im(Ab^{\mathcal{C}}) \to Ab(\mathcal{C})$ of the following way:

- *On objects*, let X be an object in $\mathcal{C}$, then we set $\tilde{\gamma_2}^{\mathcal{C}}(X) = \gamma_2^{\mathcal{C}}(X) = [X,\ X]_{Id_{\mathcal{C}}}$.
- *On morphisms*, let $f : X \to Y$ be a morphism in $\mathcal{C}$, then $\tilde{\gamma_2}^{\mathcal{C}}(f^{ab}) = \gamma_2^{\mathcal{C}}(f)$. We shall verify that the functor $\tilde{\gamma_2}^{\mathcal{C}}$ is well-defined on morphisms. We consider $\tilde{f} : X \to Y$ another morphism such that $f^{ab} \circ ab_X = ab_Y \circ \tilde{f}$. It follows that $ab_Y \circ f = ab_Y \circ \tilde{f}$. By 1.5.22, there is a morphism $\alpha : X^{ab} \to [Y,\ Y]_{Id_{\mathcal{C}}}$ such that

  $$f = \tilde{f} +_D \alpha$$

  By precomposing with the image $i_X : \gamma_2^{\mathcal{C}}(X) \rightarrowtail X$ of $c_2^X : Id_{\mathcal{D}}(X \mid X) \to X$, we get

  $$f \circ i_X = (\tilde{f} +_D \alpha) \circ i_X \quad \Leftrightarrow \quad f \circ i_X = \tilde{f} \circ i_X +_D \alpha \circ (i_X)^{ab} \quad \Leftrightarrow \quad i_Y \circ \gamma_2^{\mathcal{C}}(f) = i_Y \circ \gamma_2^{\mathcal{C}}(\tilde{f})$$

  because $ab_X \circ i_X \circ e_X = ab_X \circ i_x \circ e_X = 0$ implying that $ab_X \circ i_X = 0$ (since $e_X : Id_{\mathcal{C}}(X|X) \to [X, X]_{Id_{\mathcal{C}}}$ is an epimorphism). As $i_Y : [Y, Y]_{Id_{\mathcal{C}}} \to Y$ is a monomorphism, we have $\gamma_2^{\mathcal{C}}(f) = \gamma_2^{\mathcal{C}}(\tilde{f})$ as desired.

Then we define the following bifunctor:

Definition 3.1.8. The bifunctor $D : Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}}) \to Ab$ is defined such that:

- *On objects*, let X and Y be objects in $\mathcal{C}$, then $D(X,\ Y) = \mathcal{C}(X^{ab},\ [Y,\ Y]_{Id_{\mathcal{C}}})$.
- *On morphisms*, let $f \in D(X, X')$ and $g \in D(Y, Y')$, we set $D((f^{ab})^{op},\ g^{ab}) = \tilde{\gamma_2}^{\mathcal{C}}(g^{ab})_* \circ (f^{ab})^* = \gamma_2^{\mathcal{C}}(g)_* \circ (f^{ab})^* : D(X', Y) \to D(X, Y')$.

Now we are able to see any 2-step nilpotent category as a linear extension of categories, as follows:

Proposition 3.1.9. *Let $\mathcal{C}$ be a 2-step nilpotent category. Then we have the following linear extension of categories:*

$$D \xrightarrow{+} \mathcal{C} \xrightarrow{(Ab^{\mathcal{C}})'} Im(Ab^{\mathcal{C}})$$

where $Im(Ab^{\mathcal{C}})$ is the category given in 3.1.3, and $(Ab^{\mathcal{C}})'$ and $D : Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}}) \to Ab$ are respectively the functor and the bifunctor defined in 3.1.4 and 3.1.8.

Proof. The first point of 3.1.1 is clearly satisfied by construction of the category $Im(Ab^{\mathcal{C}})$ and of the functor $(Ab^{\mathcal{C}})' : \mathcal{C} \to Im(Ab^{\mathcal{C}})$. Now let $g \in Im(Ab^{\mathcal{C}})(X, Y)$. Then there is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $g = f^{ab}$. Then putting

$$f +_D \alpha = f + (ab_X)^*(\alpha) \quad \text{(see (3.1.3))}$$

defines a simple action of $D(X,Y)$ on the set $\mathcal{C}(X,Y)$ since $(ab_X)^* : \mathcal{C}(X^{ab},[Y,Y]_{Id_\mathcal{C}}) \to \mathcal{C}(X,[Y,Y]_{Id_\mathcal{C}})$ is an isomorphism of abelian groups by 1.5.16; the orbits of this action is the set $\big((Ab^\mathcal{C})'\big)^{-1}(f)$ by 1.5.22. Hence the second point of 3.1.1 holds. Then it remains to check that the third point is verified, i.e. the *linear distributive law*. Let $f \in Im(Ab^\mathcal{C})(Y,Y')$ and $g \in Im(Ab^\mathcal{C})(X,Y)$. Consider $f_0 : Y \to Y'$ and $g_0 : X \to Y$ respectively two lifts of f and g. Let $\alpha \in D(Y,Y')$ and $\beta \in D(X,Y)$. First we get

$$(g_0 +_D \beta)^{ab} \circ ab_X = ab_Y \circ (g_0 +_D \beta) = ab_Y \circ g_0 = g \circ ab_X$$

by 1.5.22. Hence we obtain $(g_0 +_D \beta)^{ab} = g$ because $ab_X : X \to X^{ab}$ is an epimorphism. Then we have

$$\begin{aligned}(f_0 +_D \alpha) \circ (g_0 +_D \beta) &= f_0 \circ (g_0 +_D \beta) \ +_D \ \alpha \circ (g_0 +_D \beta)^{ab}\,, \text{by } 3.1.6\\ &= f_0 \circ g_0 +_D \gamma_2^\mathcal{C}(f_0) \circ \beta +_D \alpha \circ g\,, \text{by } 3.1.6\\ &= f_0 \circ g_0 +_D \widetilde{\gamma_2}^\mathcal{C}(f) \circ \beta +_D \alpha \circ g\,, \text{see } 3.1.7\\ &= f_0 \circ g_0 +_D \widetilde{\gamma_2}^\mathcal{C}(f)_*(\beta) +_D g^*(\alpha)\\ &= f_0 \circ g_0 +_D \big(D(id,f)(\beta) + D(g,id)(\alpha)\big)\,, \text{see } 3.1.8\end{aligned}$$

□

3.2 The five lemma for linear extensions of categories

First taking a linear extension of category as in 3.1.1, the functor $p : \mathcal{A} \to \mathcal{B}$ has the following property already proved in [1] by H.J Baues:

Proposition 3.2.1. *Given a linear extension of categories as in 3.1.1, then the functor $p : \mathcal{A} \to \mathcal{B}$ reflects isomorphisms, equivalently speaking it satisfies the sufficiency condition in the sense of H.J. Baues, see* 1.3 *of [1].*

Proof. This is a straightforward application of 2.12 of [1] because the mixed term (see 2.7 of [1]) of the action $D : \mathcal{B}^{op} \times \mathcal{B} \to Ab$ on $p : \mathcal{A} \to \mathcal{B}$ is trivial. □

Then we give the five lemma in this context of linear extension of categories which has been provided in 5.5 of [4] for the first time. However we provide a slightly generalized assumption of this lemma and a more detailed proof as follows:

Lemma 3.2.2. *Consider the following morphism of linear extensions of categories:*

$$\begin{tikzcd} D' \arrow[r, "+"] & \mathcal{A}' \arrow[r, "p'"] & \mathcal{B}' \\ D \arrow[r, "+"] & \mathcal{A} \arrow[u, "F"] \arrow[r, "p"] & \mathcal{B} \arrow[u, "G"'] \end{tikzcd}$$

that is, F and G are functors as indicated such that the right-hand square commutes, and $\phi : D \Rightarrow (G^{OP} \times G)^* \cdot D'$ is a natural isomorphism between bifunctors with domain $\mathcal{B}^{op} \times \mathcal{B}$ and values in Ab such that

$$F(f + \alpha) = F(f) + \phi_{X,Y}(\alpha)$$

where X and Y are objects in $\mathcal{A}$, $f \in \mathcal{A}(X,\ Y)$, $\alpha \in D(X,\ Y)$. Suppose that $G : \mathcal{B} \to \mathcal{B}'$ is an equivalence of categories. Then F is an equivalence of categories.

Proof. First we prove that $F : \mathcal{A} \to \mathcal{B}$ is essentially surjective, i.e. for each object B in $\mathcal{A}'$, there exists $A \in \mathcal{A}$ such that $F(A) \cong B$ in $\mathcal{A}'$. As $G : \mathcal{B} \to \mathcal{B}'$ is an equivalence of categories (in particular essentially surjective), there exists $A \in \mathcal{B}$ and an isomorphism $\varphi : G(A) \cong p'(B) = B$ in $\mathcal{B}'$. Moreover the right square of the above diagram commutes, then we have

$$G(A) = G \cdot p(A) = p' \cdot F(A)$$

As $p' : \mathcal{A}' \to \mathcal{B}'$ is full, there exists $\tilde{\varphi} : F(A) \to B$ morphism in $\mathcal{A}'$ such that $\varphi = p'(\tilde{\varphi})$. By 3.2.1, $\tilde{\varphi} : F(A) \to B$ is an isomorphism in $\mathcal{A}'$. Then it suffices to prove that F is full and faithful. Let X and Y be two objects in $\mathcal{A}$, we have

- *F is full.* Let $g \in \mathcal{A}'(F(X),\, F(Y))$. We consider the diagram below:

$$\begin{tikzcd}
\mathcal{A}(X,\, Y) \arrow[r, "F"] \arrow[d, "p"'] & \mathcal{A}'(F(X),\, F(Y)) \arrow[d, "p'"] \\
\mathcal{B}(X,\, Y) \arrow[r, "\cong", "G"'] & \mathcal{B}'(F(X),\, F(Y)) = \mathcal{B}'(G(X),\, G(Y))
\end{tikzcd}$$

  As p, p' are surjective on morphisms and G is a bijection on morphisms, there is $f \in \mathcal{A}(X,\, Y)$ such that
$$p'(g) = G \cdot p'(f) = p' \cdot F(f) = p'(F(f))$$
  Then there exists $\beta \in D'(p'(F(X)),\, p'(F(Y)))$ such that $g = F(f) + \beta$ by 3.1.1. As $\phi_{X,Y}$ is surjective, we have
$$\exists \alpha \in D(p(X),\, p(Y)), \quad \beta = \phi_{X,Y}(\alpha)$$
  Consequently, we have
$$g = F(f) + \beta = F(f) + \phi_{X,Y}(\alpha) = F(f + \alpha)\,,$$
  It proves that F is full.

- *F is faithful.* Let $f, g \in \mathcal{A}(X,\, Y)$ such that
$$F(f) = F(g)$$
  By applying p', we obtain $G \cdot p(f) = G \cdot p(g)$, which is equivalent to $p(f) = p(g)$ because G is faithful. Then there exits $\alpha \in D(p(X),\, p(Y))$ such that $g = f + \alpha$. By applying the functor F, we have
$$F(g) = F(f + \alpha) = F(f) + \phi_{X,Y}(\alpha) = F(g) + \phi_{X,Y}(\alpha)$$
  By 1.5.21, $\phi_{X,Y}(\alpha) = 0$ implying that $\alpha = 0$ because $\phi_{X,Y}$ is injective. Finally $g = f$. Hence F is faithful.

□

3.3 Existence of a morphism

Let $\mathcal{C}$ be a semi-abelian category. We need to recall the "join" between two subobjects, defined in 2.7 of [14]. For subobjects
$$L \overset{l}{\rightarrowtail} X \overset{m}{\leftarrowtail} M$$

of an object X in $\mathcal{C}$ (merely in homological categories), we write

$$L \vee M = Im\big((l,\, m) : L + M \to X\big)\,,$$

see [6]. Let $\mathcal{D}$ be a semi-abelian category, $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor and $\delta : A \to G$ be a morphism. We start to recall the existence and uniqueness of a certain morphism, already determined in [14]. Here we denote respectively by $r_G : A + G \to G$, $i_G : G \to A + G$ and $i_A : A \to A + G$ the retraction onto the second summand, the injections of the second and first summand. First we have the split epimorphism $F(r_G) : F(A + G) \to F(G)$ whose section is $F(i_G) : F(G) \to F(A + G)$ and its kernel is denoted by $ker(F(r_G)) : Ker(F(r_G)) \to F(A+G)$. As F is a reduced functor, there is a unique morphism $s : F(A) \to Ker(F(r_G))$ factorizing $F(i_A) : F(A) \to F(A+G)$ through $ker(F(r_G))$. In addition, there is a unique morphism $k : F(A|G) \to Ker(F(r_G))$ such that $ker(F(r_G)) \circ k = \iota_2^F$ where $\iota_2^F : F(A|G) \to F(A + G)$ is the kernel of the regular epimorphism $\widehat{r_2^F} = (F(r_A), F(r_G))^t : F(A + G) \to F(A) \times F(G)$. Setting $p = F(r_A) \circ ker(F(r_G)) : Ker(F(r_G)) \to F(A)$, it is a split epimorphism whose section is s. The nine lemma applied to the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & 0 \arrow[u] \arrow[r] & F(G) \arrow[u] \arrow[r, two heads] & F(G) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & F(A|G) \arrow[u, two heads] \arrow[r, tail, "\iota_2^F"] & F(A+G) \arrow[u, two heads, "F(r_G)"'] \arrow[r, two heads, "\widehat{r_2^F}"] & F(A) \times F(G) \arrow[u, two heads, "\pi_G"'] \arrow[r] & 0 \\
0 \arrow[r] & F(A|G) \arrow[u, equal] \arrow[r, tail, "k"] & Ker(F(r_G)) \arrow[u, "ker(F(r_G))"'] \arrow[r, "p"] & F(A) \arrow[u, "\iota_A"'] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

ensures that the bottom sequence of the diagram is split short exact, where $\pi_G : F(A)\times F(G) \to F(G)$ and $\iota_A : F(A) \to F(A) \times F(G)$ are respectively the projection onto the second summand and the injection of the first summand. By 3.10 of [11] or 3.1 of [14], there is a strict action core (see 3.5 of [11]), or simply an action (see 3.1 of [14]), $\psi : Id_{\mathcal{D}}(F(A|G)\,|\,F(A)) \to F(A|G)$ which is the restriction of the regular epimorphism $(k,\ s) : F(A|G) + F(A) \to Ker(F(r_G))$ (by protomodularity of $\mathcal{D}$) to $Id_{\mathcal{D}}(F(A|G) \mid F(A))$, such that

$$\iota_2^F \circ \psi = (\iota_2^F,\ F(i_A)) \circ \iota_2^{Id_{\mathcal{D}}}$$

Then we prove that the following diagram commutes:

$$\begin{tikzcd}
Id_{\mathcal{D}}(F(A|G) \mid F(A)) \arrow[r, "\psi"] \arrow[d, "Id_{\mathcal{D}}\left(S_2^F \circ F(\delta|id) \,\middle|\, F(\delta)\right)"'] & F(A|G) \arrow[d, "S_2^F \circ F(\delta|id)"] \\
Id_{\mathcal{D}}(F(G) \mid F(G)) \arrow[r, "S_2^{Id} = c_2^{F(G)}"'] & F(G)
\end{tikzcd}$$

We have

$$\begin{aligned}
S_2^{Id} \circ Id_{\mathcal{D}}\big(S_2^F \circ F(\delta|id) \mid F(\delta)\big) &= \big(S_2^F \circ F(\delta|id),\ F(\delta)\big) \circ \iota_2^{Id} \\
&= \big(F((\delta, id)) \circ \iota_2^F,\ F(\delta)\big) \circ \iota_2^{Id} \\
&= \big(F((\delta, id)) \circ \iota_2^F,\ F((\delta, id)) \circ F(i_A)\big) \circ \iota_2^{Id} \\
&= F((\delta, id)) \circ (\iota_2^F,\ F(i_A)) \circ \iota_2^{Id} \\
&= F((\delta, id)) \circ \iota_2^F \circ \psi \\
&= S_2^F \circ F(\delta|id) \circ \psi
\end{aligned}$$

By 4.4 of [11], there exists a unique morphism $h = \left\langle \begin{array}{c} (S_2^F)_G \circ F(\delta|id) \\ F(\delta) \end{array} \right\rangle : F(A|G) \rtimes F(A) \to F(G)$ such that $h \circ k = (S_2^F)_G \circ F(\delta|id)$ and $h \circ s = F(\delta)$.

3.4 Commutators and the effect of functors on exact sequences

In this part, we provide short exact sequences by applying reduced functors preserving coequalizers of reflexive graphs to right short exact sequences. The next proposition gives a useful short exact sequences as follows:

Proposition 3.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor preserving coequalizers of reflexive graphs. Consider the following right exact sequence in $\mathcal{C}$:*

$$A \xrightarrow{\delta} G \xrightarrow{q} Q \longrightarrow 0$$

Then it gives rise to the following short exact sequence:

$$0 \longrightarrow [A,\ G]_F \vee [A]_F \longrightarrow F(G) \xrightarrow{F(q)} F(Q) \longrightarrow 0$$

Proof. As F preserves coequalizers of reflexive graphs, we have the following exact sequence by 2.31 of [14]:

$$F(A|G) \rtimes F(A) \xrightarrow{h} F(G) \xrightarrow{F(q)} F(Q) \longrightarrow 0$$

Then it suffices to determine the image of h so as to have our desired short exact sequence because $im(h) : Im(h) \rightarrowtail F(G)$ is the kernel of $F(q)$. We remark that the images of $(S_2^F)_G \circ F(\delta|id)$ and $F(\delta)$ are respectively $[A,\ G]_F$ and $[A]_F$ by definition of these commutators. Hence we have

$$\begin{aligned}
Im(h) &= Im(h \circ (k,\ s)) \\
&= Im\Big(\big((S_2^F)_G \circ F(\delta|id),\ F(\delta)\big)\Big) \\
&= Im\left((S_2^F)_G \circ F(\delta|id)\right) \vee Im\left(F(\delta)\right) \\
&= [A,\ G]_F \vee [A]_F
\end{aligned}$$

as desired. □

Corollary 3.4.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor preserving coequalizers of reflexive graphs and Y be an object in $\mathcal{C}$. If, moreover, we suppose that F is quadratic, then we have the following short exact sequence*

$$0 \longrightarrow [[Y,\ Y]_{Id_{\mathcal{C}}}]_F \xrightarrow{i^F_{[Y,\ Y]_{Id_{\mathcal{C}}}}} F(Y) \xrightarrow{F(ab_Y)} F(Y^{ab}) \longrightarrow 0$$

where we recall that $i^F_{[Y,\,Y]_{Id_\mathcal{C}}} : [[Y,\,Y]_{Id_\mathcal{C}}]_F \to F(Y)$ *is the image of the morphism* $F(i_Y) : F\big([Y,\,Y]_{Id_\mathcal{C}}\big) \to F(Y)$.

Proof. We just apply 3.4.1 to the following short exact sequence:

$$0 \longrightarrow [Y,\,Y]_{Id_\mathcal{C}} \xrightarrow{i_Y} Y \xrightarrow{ab_Y} Y^{ab} \longrightarrow 0$$

If, moreover, F is a quadratic functor, then

$$[[Y,\,Y]_{Id_\mathcal{C}},\,Y]_F \subset [Y,\,Y,\,Y]_F = 0\,,$$

by 1.3.8 and by 1.3.9 because F preserves regular epimorphisms (since it preserves coequalizers of reflexive graphs by 2.31 of [14]). $\square$

3.5 Criteria for quadratic equivalences

In this part, we find criteria for certain quadratic functors between 2-step nilpotent categories respectively varieties to be quadratic equivalences, at least when restricted to suitable subcategories. We start by stating the precise results.

Theorem 3.5.1. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two 2-step nilpotent categories with distinguished full subcategories* $\mathcal{C}'$ *respectively* $\mathcal{D}'$ *all of whose objects are regular projective. Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a reduced quadratic functor which preserves coequalizers of reflexive graphs and carries* $\mathcal{C}'$ *into* $\mathcal{D}'$*. Denote by* $F'\colon \mathcal{C}' \to \mathcal{D}'$, $Ab^{\mathcal{C}'}\colon \mathcal{C}' \to Ab(\mathcal{C})$ *and* $Ab^{\mathcal{D}'}\colon \mathcal{D}' \to Ab(\mathcal{D})$ *the functors given by the corresponding restrictions of* F, $Ab^{\mathcal{C}}$ *and* $Ab^{\mathcal{D}}$*, respectively.*

Also let $Ab(\mathcal{C}')$ *(resp.* $Ab(\mathcal{D}')$*) be the full subcategories of* $Ab(\mathcal{C})$ *(resp.* $Ab(\mathcal{D})$*) whose objects are isomorphic to the abelianization of some object in* $\mathcal{C}'$ *(resp.* $\mathcal{D}'$*).*

Then the functor $F'\colon \mathcal{C}' \to \mathcal{D}'$ *given by restriction of* F *is an equivalence provided the following three conditions hold.*

1. *There is a natural isomorphism* $\sigma\colon Ab^{\mathcal{D}} \cdot F \Rightarrow F \cdot Ab^{\mathcal{C}}$ *of functors from* $\mathcal{C}$ *to* $\mathcal{D}$ *such that the triangle*

$$\begin{tikzcd} & F(X) \arrow[dl, "F(ab_X)"'] \arrow[dr, "ab_{F(X)}"] & \\ F(X^{ab}) & & F(X)^{ab} \arrow[ll, "\sigma_X", "\cong"'] \end{tikzcd} \tag{3.5.1}$$

   commutes for all objects X *in* $\mathcal{C}$.

2. *The functor* $Ab(F)\colon Ab(\mathcal{C}) \to Ab(\mathcal{D})$ *given by restriction of* F *(which is defined thanks to condition 1.) is full and faithful, and its restriction* $Ab(F')\colon Ab(\mathcal{C}') \to Ab(\mathcal{D}')$ *is essentially surjective. Here* $Ab(\mathcal{C}')$ *and* $Ab(\mathcal{D}')$ *denote respectively the full subcategories of* $Ab(\mathcal{C})$ *and* $Ab(\mathcal{D})$ *whose objects are isomorphic to abelianizations of objects in* $\mathcal{C}'$ *and* $\mathcal{D}'$*, respectively.*

3. *For every object* Y *in* $\mathcal{C}'$*, the morphism* $F(i_Y)\colon F(\gamma_2^{\mathcal{C}}(Y)) \to F(Y)$ *is a monomorphism.*

This result can be considerably strengthened for 2-nilpotent varieties, as follows.

Theorem 3.5.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two 2-step nilpotent varieties. Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a reduced quadratic functor. Suppose that* F *satisfies the following properties.*

1. *F preserves binary coproducts, filtered colimits and coequalizers of reflexive graphs.*

2. *F carries a given free object E of rank* 1 *in* $\mathcal{C}$ *to a free object of rank* 1 *in* $\mathcal{D}$.

3. *Up to isomorphism F commutes with the abelienization functors of* $\mathcal{C}$ *and* $\mathcal{D}$, *as in condition 1. of Theorem 3.5.1.*

4. *The functor* $Ab(F)\colon Ab(\mathcal{C}) \to Ab(\mathcal{D})$ *given by restriction of F (which is defined thanks to condition 3.) is an equivalence.*

5. *For* $n \geq 1$ *and* $Y = E^{+n}$, *the morphism* $F(i_Y)\colon F(\gamma_2^{\mathcal{C}}(Y)) \to F(Y)$ *is a monomorphism.*

Then F is an equivalence, with a weak inverse F^{-1} *described in Lemma 3.5.12 below.*

Remark 3.5.3. Condition 4. may be replaced with the condition that the map

$$F_{E^{ab},E^{ab}}\colon \mathcal{C}(E^{ab}, E^{ab}) \to \mathcal{C}(F(E^{ab}), F(E^{ab}))$$

is bijective. In fact, as both F and abelianization functors preserve binary sums and F commutes with the abelianization functors (up to isomorphism), it follows that $Ab(F)$ preserves direct coproducts and thus is additive. Hence if $F_{E^{ab},E^{ab}}$ is bijective it is a ring isomorphism and hence the functor

$$\widetilde{Ab(F)}\colon \langle E^{ab}\rangle \to \langle F(E^{ab})\rangle$$

is full and faithfull since morphisms in these categories can be described by matrices with coefficients in the rings $\mathcal{C}(E^{ab}, E^{ab})$ respectively $\mathcal{C}(F(E^{ab}), F(E^{ab}))$, and composition of morphisms corresponds to matrix multiplication. Now Lemma 3.5.12 applied to $Ab(F)$ instead of F shows that $Ab(F)$ is an equivalence (note that $Ab(F)$ preserves coequalizers of reflexive graphs and filtered colimits since F does and commutes with abelianization functors).

Now let $\mathcal{C}$ and $\mathcal{D}$ be 2-step nilpotent categories. Recall the bifunctors $D : Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}}) \to Ab$ and $D' : Im(Ab^{\mathcal{D}})^{op} \times Im(Ab^{\mathcal{D}}) \to Ab$ the bifunctors defined in 3.1.8. For all $X, Y \in \mathcal{C}$, we know by 1.5.22 that the abelian group $D(X,Y)$ simply acts on the set $\mathcal{C}(X,Y)$ as follows:

$$f +_D \alpha = \varphi_{i_Y} \circ (f, \alpha \circ ab_X)^t$$

where $f \in \mathcal{C}(X,Y)$, $\alpha \in D(X,Y)$ and $\varphi_{i_Y} : Y \times [Y,Y]_{Id_{\mathcal{C}}} \to Y$ is the cooperator of $i_Y : [Y,Y]_{Id_{\mathcal{C}}} \rightarrowtail Y$ and the identity of Y (see (1.5.4)). Similarly, for $A, B \in \mathcal{D}$, the abelian group $D'(A,B)$ acts on the set $\mathcal{D}(A,B)$.

Remark 3.5.4. Let X and Y be two objects in $\mathcal{C}$, and let $F : \mathcal{C} \to \mathcal{D}$ be a reduced quadratic functor preserving coequalizers of reflexive graphs (hence regular epimorphisms). Since $[F(Y), F(Y)]_{Id_{\mathcal{D}}}$ is a central subobject of $F(Y)$ by 1.3.11, we observe that the abelian group $D(F(X), F(Y))$ acts on the set $\mathcal{D}(F(X), F(Y))$ by (1.5.6), as follows:

$$g +_{D'} \beta = \varphi_{i_{F(Y)}} \circ (g, \beta \circ ab_{F(Y)})^t \tag{3.5.2}$$

where $g \in \mathcal{D}(F(X), F(Y))$, $\beta \in D'(F(X), F(Y))$ and $\varphi_{i_{F(Y)}} : F(Y) \times [F(Y), F(Y)]_{Id_{\mathcal{D}}} \to F(Y)$ is the cooperator of $i_{F(Y)}$ and the identity of Y (see (1.5.4)).

Now we define a functor $\tilde{\gamma_1}^F : Im(Ab^{\mathcal{C}}) \to Ab(\mathcal{D})$ as follows:

- *On objects*, let X be an object in $\mathcal{C}$, then $\tilde{\gamma_1}^F(X) = [[X,\ X]_{Id_{\mathcal{C}}}]_F$.

- *On morphisms*, let $f : X \to Y$ be a morphism in $\mathcal{C}$, then $\tilde{\gamma_1}^F(f^{ab})$ is the unique morphism such that the two rectangles of the following diagram commute:

$$\begin{tikzcd}
F(\gamma_2^{\mathcal{C}}(X)) \arrow[rr, "F(\gamma_2^{\mathcal{C}}(f))=F(\tilde{\gamma_2}^{\mathcal{C}}(f^{ab}))"] \arrow[d] & & F(\gamma_2^{\mathcal{C}}(Y)) \arrow[d] \\
{[\gamma_2^{\mathcal{C}}(X)]_F} \arrow[rr, dashed, "\tilde{\gamma_1}^F(f^{ab})"] \arrow[d] & & {[\gamma_2^{\mathcal{C}}(Y)]_F} \arrow[d] \\
F(X) \arrow[rr, "F(f)"] & & F(Y)
\end{tikzcd}$$

where the functor $\tilde{\gamma_2}^{\mathcal{C}} : Im(Ab^{\mathcal{C}}) \to Ab(\mathcal{C})$ is defined in subsection 2.1. Is is clear that the functor $\tilde{\gamma_1}^F : Im(Ab^{\mathcal{C}}) \to Ab(\mathcal{D})$ is well-defined on morphisms.

Next we define a specific bifunctor $D^F : Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}}) \to Ab$ depending on F which kind of "interpolates" between D and D', as follows:

Definition 3.5.5. We define the bifunctor $D^F : Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}}) \to Ab$ as follows:

- *On objects*, let X and Y be two objects in $\mathcal{C}$, then $D^F(X, Y) = \mathcal{D}\big(F(X^{ab}), [[Y, Y]_{Id_{\mathcal{C}}}]_F\big)$ which is an abelian group because $[[Y, Y]_{Id_{\mathcal{C}}}]_F$ is an abelian object since it is a central subobject of $F(Y)$ by 1.3.12.

- *On morphisms*, let $f \in \mathcal{C}(X', X)$ and $g \in \mathcal{C}(Y, Y')$, then $D^F((f^{ab})^{op}, g^{ab}) = F(f^{ab})^* \circ \tilde{\gamma_1}^F(g^{ab})_* : D^F(X, Y) \to D^F(X', Y')$.

Remark 3.5.6. It is a consequence of 1.5.13 that the abelian group $D^F(X, Y)$ acts on $\mathcal{D}(F(X), F(Y))$ as follows:

$$g +_{D^F} \alpha = \varphi' \circ \big(g,\ \alpha \circ F(ab_X)\big)^t$$

where $g \in \mathcal{D}(F(X),\ F(Y))$, $\alpha \in D^F(X,\ Y)$ and $\varphi' : F(Y) \times [[Y,\ Y]_{Id_{\mathcal{C}}}]_F \to F(Y)$ is the unique factorization of $(id,\ i^F_Y) : F(Y) + [[Y,\ Y]_{Id_{\mathcal{C}}}]_F \to F(Y)$ through the comparison morphism $\widehat{r_2^{Id_{\mathcal{D}}}} : F(Y) + [[Y,\ Y]_{Id_{\mathcal{C}}}]_F \to F(Y) \times [[Y,\ Y]_{Id_{\mathcal{C}}}]_F$. It is due to the fact that $[[Y, Y]_{Id_{\mathcal{C}}}]_F$ is a central subobject of $F(Y)$ by 1.3.12.

Then we provide a natural transformation between the bifunctors D and D^F both with domain $Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}})$ and values in *Set*. For all $X, Y \in \mathcal{C}$, we first define the map

$$\begin{array}{cccc} \phi^F_{X,\,Y} & : & D(X,\ Y) & \longrightarrow & D^F(X,\ Y) \\ & & \alpha & \longmapsto & e^F_{[Y,Y]_{Id_{\mathcal{C}}}} \circ F(\alpha)\ . \end{array} \tag{3.5.3}$$

where $e^F_{[Y,Y]_{Id_{\mathcal{C}}}} : F([Y,Y]_{Id_{\mathcal{C}}}) \twoheadrightarrow [[Y,Y]_{Id_{\mathcal{C}}}]_F$ is the coimage of $F(i_Y) : F([Y,\ Y]_{Id_{\mathcal{C}}}) \to F(Y)$.

Proposition 3.5.7. *The collection of maps $\phi^F_{X,Y}$ for $X, Y \in \mathcal{C}$ defines a natural transformation $\phi^F : D \Rightarrow D^F$ between bifunctors with domain $Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}})$ and values in Set.*

Remark 3.5.8. Without further hypothesis on F the map $\phi^F_{X,Y}$ is not a homomorphism; a sufficient condition would be to require that F preserves binary products of abelian objects. This in fact is a consequence of the hypothesis of Theorem 3.5.1, see its proof below.

Next we give the following proposition:

Proposition 3.5.9. *If F is a quadratic reduced functor preserving regular epimorphisms, then for all $X, Y \in \mathcal{C}$ the map $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ carries the action $+_D$ of $D(X, Y)$ on $\mathcal{C}(X, Y)$ to the action $+_{D^F}$ of $D^F(X, Y)$ on $\mathcal{D}(F(X), F(Y))$ along $\phi^F_{X,Y} : V\big(D(X,Y)\big) \to V\big(D^F(X,Y)\big)$; more precisely, for $f \in \mathcal{C}(X, Y)$ and $\alpha \in D(X, Y)$, we have*

$$F(f +_D \alpha) = F(f) +_{D^F} \phi^F_{X,\,Y}(\alpha)$$

where

- *$V : Ab \to Set$ is the canonical forgetful functor;*
- *D and D^F are the bifunctors with domain $Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}})$ and values in Ab respectively defined in 3.1.8 and 3.5.5, and $+_D$ and $+_{D^F}$ are the actions given in (3.1.3) and (3.5.6);*
- *for all $X, Y \in \mathcal{C}$, $\phi^F_{X,Y} : D(X,Y) \to D^F(X,Y)$ is the map defined in (3.5.3).*

Proof. By (3.5.6), we recall that for $f \in \mathcal{C}(X,Y)$ and $\alpha \in D(X,Y)$ we have

$$f +_D \alpha = \varphi_{i_Y} \circ (f, \alpha \circ ab_X)^t \quad \text{(see (3.1.3)}$$

where $\varphi_{i_Y} : Y \times [Y,Y]_{Id_{\mathcal{C}}} \to Y$ is the cooperator of i_Y and the identity of Y given in (1.5.4). Now we consider the following diagram:

$$\begin{tikzcd}
F(Y + [Y,\, Y]_{Id_{\mathcal{C}}}) \arrow[r, "F(\widehat{r_2^{Id_{\mathcal{C}}}})"] \arrow[dr, "\widehat{r_2^F}"] \arrow[rr, bend left, "F((id,\, i_Y))"] & F(Y \times [Y,\, Y]_{Id_{\mathcal{C}}}) \arrow[r, "F(\varphi_{i_Y})"] \arrow[d, "(F(\pi_1),\, F(\pi_2))^t"] & F(Y) \arrow[dd, equal] \\
F(Y) + F([Y,\, Y]_{Id_{\mathcal{C}}}) \arrow[u, "(F(i_1),\, F(i_2))"] \arrow[r, "\widehat{r_2^{Id_{\mathcal{D}}}}"] \arrow[d, "id + e^F_Y"'] & F(Y) \times F([Y,\, Y]_{Id_{\mathcal{C}}}) \arrow[d, "id \times e^F_Y"] & \\
F(Y) + [[Y,\, Y]_{Id_{\mathcal{C}}}]_F \arrow[r, "\widehat{r_2^{Id_{\mathcal{D}}}}"] \arrow[rr, bend right, "(id,\, i^F_Y)"'] & F(Y) \times [[Y,\, Y]_{Id_{\mathcal{C}}}]_F \arrow[r, "\varphi'"] & F(Y)
\end{tikzcd} \tag{3.5.4}$$

However it does not allows us to conclude that the right-hand square commutes. This happens whenever the composite morphism $F(\widehat{r_2^{Id_{\mathcal{C}}}}) \circ (F(i_1^2),\, F(i_2^2)) : F(Y) + F([Y,\, Y]_{Id_{\mathcal{C}}}) \to F(Y \times [Y,\, Y]_{Id_{\mathcal{C}}})$ is a regular epimorphism (which is false in general). For this, we search the deviation of the morphism $(F(i_1^2),\, F(i_2^2)) : F(Y) + F([Y,\, Y]_{Id_{\mathcal{C}}}) \to F(Y + [Y,\, Y]_{Id_{\mathcal{C}}})$ to be a regular epimorphism. We take the pull-back of the morphism $\widehat{r_2^{Id_{\mathcal{C}}}} : F(Y) + F([Y,\, Y]_{Id_{\mathcal{C}}}) \to F(Y) \times F([Y,\, Y]_{Id_{\mathcal{C}}})$ along $\widehat{r_2^F}$: $F(Y + [Y,\, Y]_{Id_{\mathcal{C}}}) \to F(Y) \times F([Y,\, Y]_{Id_{\mathcal{C}}})$ as follows:

$$\begin{tikzcd}
0 \arrow[r] & F(Y \mid [Y,\, Y]_{Id_{\mathcal{C}}}) \arrow[r, "k"] & P \arrow[r, "p"] \arrow[d, "q"] & F(Y) + F([Y,\, Y]_{Id_{\mathcal{C}}}) \arrow[r] \arrow[l, bend right, "s"'] \arrow[d, "\widehat{r_2^{Id_{\mathcal{C}}}}"] & 0 \\
& & F(Y + [Y,\, Y]_{Id_{\mathcal{C}}}) \arrow[r, "\widehat{r_2^F}"] & F(Y) \times F([Y,\, Y]_{Id_{\mathcal{C}}}) &
\end{tikzcd}$$

By regularity of the category $\mathcal{D}$, the morphisms p and q are regular epimorphisms as pull-back of regular epimorphisms. Next, $k : F(Y \mid [Y,\, Y]_{Id_\mathcal{C}}) \to P$ is the unique morphism such that $k \circ p = 0$ and $k \circ q = \iota_2^F$ by the universal property of the pull-back. A categorical argument says that k is the kernel of $p : P \to F(Y) + F([Y,\, Y]_{Id_\mathcal{C}})$ (it works in any finite complete category). Finally $s : F(Y) + F([Y,\, Y]_{Id_\mathcal{C}}) \to P$ is the unique morphism such that $p \circ s = id$ and $q \circ s = (F(i_1^2),\, F(i_2^2))$ by the universal property of the pull-back. Hence we deduce that the top sequence of the above diagram is short split exact. By protomodularity of the category $\mathcal{D}$ (as any semi-abelian category), the morphism $(s,\, k) : \big(F(Y) + F([Y,\, Y]_{Id_\mathcal{C}})\big) + F(Y \mid [Y,\, Y]_{Id_\mathcal{C}}) \to P$ is a regular epimorphism. So the morphism $\big((F(i_1^2), F(i_2^2)), \iota_2^F\big) = q \circ (k,\, s) : \big(F(Y) + F([Y, Y]_{Id_\mathcal{C}})\big) + F(Y \mid [Y, Y]_{Id_\mathcal{C}}) \to F(Y + [Y, Y]_{Id_\mathcal{C}})$ is also a regular epimoprhism as a composite of two regular epimorphisms. Now we consider the following diagram:

$$\begin{tikzcd}
F(Y + [Y,\, Y]_{Id_\mathcal{C}}) \arrow[r, "F(\widehat{r_2^{Id_\mathcal{C}}})"] \arrow[dr, "\widehat{r_2^F}"] \arrow[rr, bend left, "F((id,\, i_Y))"] & F(Y \times [Y,\, Y]_{Id_\mathcal{C}}) \arrow[r, "F(\varphi_{i_Y})"] \arrow[d, "(F(\pi_1),\, F(\pi_2))^t"] & F(Y) \arrow[dd, equal] \\
\big(F(Y) + F([Y,\, Y]_{Id_\mathcal{C}})\big) + F(Y \mid [Y,\, Y]_{Id_\mathcal{C}}) \arrow[u, "\big((F(i_1),\, F(i_2)),\, \iota_2^F\big)"] \arrow[r, "(\widehat{r_2^{Id_\mathcal{D}}},\, 0)"'] \arrow[d, "\big(id + e_Y^F,\, 0\big)"'] & F(Y) \times F([Y,\, Y]_{Id_\mathcal{C}}) \arrow[d, "id \times e_Y^F"] & \\
F(Y) + [[Y,\, Y]_{Id_\mathcal{C}}]_F \arrow[r, "\widehat{r_2^{Id_\mathcal{D}}}"] \arrow[rr, bend right, "(id,\, i_Y^F)"'] & F(Y) \times [[Y,\, Y]_{Id_\mathcal{C}}]_F \arrow[r, "\varphi'"] & F(Y)
\end{tikzcd}$$

Note that $F(\widehat{r_2^{Id_\mathcal{C}}}) : F(Y + [Y, Y]_{Id_\mathcal{C}}) \to F(Y \times [Y, Y]_{Id_\mathcal{C}})$ is a regular epimorphism because F preserves regular epimorphisms. The outside and left-hand rectangles commute and $F(\widehat{r_2^{Id_\mathcal{C}}}) \circ (F(i_1^2),\, F(i_2^2)) : F(Y) + F([Y,\, Y]_{Id_\mathcal{C}}) \to F(Y \times [Y,\, Y]_{Id_\mathcal{C}})$ is a regular epimorphism as a composite of two regular epimorphisms. Hence the right-hand rectangle commutes. Finally, the result of the assumption comes from the following commutative diagram:

$$
\begin{tikzcd}
F(X) \arrow[r, "F((f,\,\alpha\circ ab_X)^t)"] \arrow[rr, bend left, "F(f+_D\alpha)"] \arrow[dr, bend right=20, "\left(F(f),\,\phi^F_{X,Y}(\alpha)\circ F(ab_X)\right)^t" description] \arrow[drr, bend right=60, "F(f)+_{D^F}\phi^F_{X,Y}(\alpha)"'] & F(Y\times[Y,\,Y]_{Id_{\mathcal{C}}}) \arrow[r, "F(\varphi_{i_Y})"] \arrow[d, "(F(\pi_1),\,F(\pi_2))^t"] & F(Y) \arrow[dd, equal] \\
 & F(Y)\times F([Y,\,Y]_{Id_{\mathcal{C}}}) \arrow[d, "id\times e^F_Y"] & \\
 & F(Y)\times[[Y,\,Y]_{Id_{\mathcal{C}}}]_F \arrow[r, "\varphi'"] & F(Y)
\end{tikzcd}
$$

$\square$

Now we assume that the functor $F:\mathcal{C}\to\mathcal{D}$ commutes with the abelianization functors, i.e. there is a natural isomorphism $\sigma : Ab^{\mathcal{D}}\cdot L \Rightarrow L\cdot Ab^{\mathcal{C}}$ such that for all $X\in\mathcal{C}$ the triangle (3.5.1) commutes. Then we consider the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & {[[Y,\,Y]_{Id_{\mathcal{C}}}]_F} \arrow[r, "i^F_Y"] & F(Y) \arrow[r, "F(ab_Y)"] & F(Y^{ab}) \arrow[r] & 0 \\
0 \arrow[r] & {[F(Y),\,F(Y)]_{Id_{\mathcal{D}}}} \arrow[r, "i_{F(Y)}"] \arrow[u, dashed, "\widehat{\sigma_Y}", "\cong"'] & F(Y) \arrow[r, "ab_{F(Y)}"] \arrow[u, equal] & F(Y)^{ab} \arrow[r] \arrow[u, "\cong", "\sigma_Y"'] & 0
\end{tikzcd}
$$

The top sequence is short exact by 3.4.2 because $F:\mathcal{C}\to\mathcal{D}$ is a quadratic functor preserving regular epimorphisms. The bottom one is also short exact by definition of $ab_{F(Y)}$ as a cokernel of the image of $c_2^{F(Y)} : Id_{\mathcal{D}}(F(Y)|F(Y))\to F(Y)$. Since the right-hand square commutes, there is a unique morphism $\widehat{\sigma_Y} : [F(Y),\,F(Y)]_{Id_{\mathcal{D}}}\to[[Y,\,Y]_{Id_{\mathcal{C}}}]_F$ such that

$$
i^F_{[Y,Y]_{Id_{\mathcal{C}}}}\circ\widehat{\sigma_Y} = i_{F(Y)} \tag{3.5.5}
$$

Moreover it is an isomorphism by the five lemma. Hence it induces a natural isomorphism

$$
(\widehat{\sigma}^{-1})_*\circ(\sigma)^* : D^F \Longrightarrow (G^{OP}\times G)^*\cdot D'
$$

between bifunctors with domain $Im(Ab^{\mathcal{C}})^{op}\times Im(Ab^{\mathcal{C}})$ and values in Ab. For all $X,Y\in\mathcal{C}$, we define the map

$$
\begin{array}{ccccc}
\phi_{X,Y} & : & D(X,\,Y) & \longrightarrow & D'(F(X),\,F(Y)) \\
 & & \alpha & \longmapsto & \widehat{\sigma_Y}^{-1}\circ e^F_{[Y,Y]_{Id_{\mathcal{C}}}}\circ F(\alpha)\circ\sigma_X\ .
\end{array} \tag{3.5.6}
$$

We observe that for $\alpha\in D(X,Y)$ we get

$$
\phi_{X,Y}(\alpha) = (\widehat{\sigma_Y})^{-1}\circ\phi^F_{X,Y}\circ\sigma_X\,, \tag{3.5.7}
$$

see (3.5.3). Then we have a condition for a quadratic reduced functor $F:\mathcal{C}\to\mathcal{D}$ preserving coequalizers of reflexive graphs to carry the action $+_D$ of $D(X,\,Y)$ on $\mathcal{C}(X,\,Y)$ to the action $+_{D'}$ of $D'(F(X),\,F(Y))$ on $\mathcal{D}(F(X),\,F(Y))$.

Corollary 3.5.10. *Let X and Y be two objects in $\mathcal{C}$. We assume that $F : \mathcal{C} \to \mathcal{D}$ is a quadratic reduced functor preserving coequalizers of reflexive graphs. If, moreover, there is a natural isomorphism $\sigma : Ab^{\mathcal{D}} \cdot F \Rightarrow F \cdot Ab^{\mathcal{C}}$ on $\mathcal{C}$ such the triangle 3.5.1 commutes, then the map $F : \mathcal{C}(X,\, Y) \to \mathcal{D}(F(X),\, F(Y))$ carries the action $+_D$ of $D(X,\, Y)$ on $\mathcal{C}(X,\, Y)$ to the action $+_{D'}$ of $D'(F(X),\, F(Y))$ on $\mathcal{D}(F(X),\, F(Y))$ along $\phi_{X,Y} : D(X,Y) \to D'(F(X), F(Y))$; more precisely for $f \in \mathcal{C}(X,\, Y)$ and $\alpha \in D(X,\, Y)$, we have*

$$F(f +_D \alpha) = F(f) +_{D'} \phi_{X,\,Y}(\alpha)$$

where

- *D is the bifunctor with domain $Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}})$ and values in Ab defined in 3.1.8 and D' is the bifunctor with domain $Im(Ab^{\mathcal{D}})^{op} \times Im(Ab^{\mathcal{D}})$ and values in Ab defined in 3.5.5;*
- *$+_D$ and $+_{D'}$ are the actions given in* (3.1.3) *and* (3.5.4)*;*
- *for all $X, Y \in \mathcal{C}$, $\phi_{X,Y}$ is the map defined in* (3.5.6).

Proof. Let $f \in \mathcal{C}(X,Y)$ and $\alpha \in D(X,Y)$. By 3.5.9, we have

$$F(f +_D \alpha) = F(f) +_{D^F} \phi^F_{X,Y}(\alpha) = \varphi' \circ \big(F(f), \phi^F_{X,Y}(\alpha) \circ F(ab_X)\big)^t$$

It remains to prove that

$$F(f) +_{D^F} \phi^F_{X,Y}(\alpha) = F(f) +_{D'} \phi_{X,\,Y}(\alpha)$$

First we get

$$\begin{aligned}
\varphi' \circ \widehat{r_2^{Id_\mathcal{D}}} &= (id, i^F_{[Y,Y]_{Id_\mathcal{C}}}) \\
&= \big(id, i_{F(Y)} \circ (\widehat{\sigma_Y})^{-1}\big)\,, \text{by } (3.5.7) \\
&= (id, i_{F(Y)}) \circ \big(id + (\widehat{\sigma_Y})^{-1}\big) \\
&= \varphi_{i_{F(Y)}} \circ \circ \widehat{r_2^{Id_\mathcal{D}}} \circ \big(id + (\widehat{\sigma_Y})^{-1}\big)\,, \text{by } (1.5.4) \\
&= \varphi_{i_{F(Y)}} \circ \big(id \times (\widehat{\sigma_Y})^{-1}\big) \circ \widehat{r_2^{Id_\mathcal{D}}}\,, \text{by naturality}
\end{aligned}$$

Hence we obtain

$$\varphi' = \varphi_{i_{F(Y)}} \circ \big(id \times (\widehat{\sigma_Y})^{-1}\big) \tag{3.5.8}$$

because the comparison morphism $\widehat{r_2^{Id_\mathcal{C}}} : F(Y) + [[Y,Y]_{Id_\mathcal{C}}]_F \to F(F) \times [[Y,Y]_{Id_\mathcal{C}}]_F$ is a (regular) epimorphism. Then we get the equalities as follows:

$$\begin{aligned}
F(f) +_{D^F} \phi^F_{X,Y}(\alpha) &= \varphi' \circ \big(F(f), \phi^F_{X,Y}(\alpha) \circ F(ab_X)\big)^t \\
&= \varphi_{i_{F(Y)}} \circ \big(id \times (\widehat{\sigma_Y})^{-1}\big) \circ \big(F(f), \phi^F_{X,Y}(\alpha) \circ F(ab_X)\big)^t\,, \text{by } (3.5.8) \\
&= \varphi_{i_{F(Y)}} \circ \big(F(f), (\widehat{\sigma_Y})^{-1} \circ \phi^F_{X,Y}(\alpha) \circ F(ab_X)\big)^t \\
&= \varphi_{i_{F(Y)}} \circ \big(F(f), (\widehat{\sigma_Y})^{-1} \circ \phi^F_{X,Y}(\alpha) \circ \sigma_X \circ ab_{F(X)}\big)^t\,, \text{by } (3.5.1) \\
&= \varphi_{i_{F(Y)}} \circ \big(F(f), \phi_{X,Y}(\alpha) \circ ab_{F(X)}\big)^t\,, \text{by } (3.5.7) \\
&= F(f) +_{D'} \phi_{X,\,Y}(\alpha)\,, \text{by } (3.5.4)
\end{aligned}$$

as desired. $\square$

Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced quadratic functor preserving coequalizers of reflexive graphs (hence regular epimorphisms), which commutes with the abelianization functors as in condition 1. of Theorem 3.5.1. We now define a functor depending on F as follows:

Definition 3.5.11. We define the functor $Im(Ab(F)) : Im(Ab^{\mathcal{C}}) \to Im(Ab^{\mathcal{D}})$ such that:

- *On objects*, let X be an object in $\mathcal{C}$, $Im(Ab(F))(X) = F(X)$,
- *On morphisms*, let $f : X \to Y$ be a morphism in $\mathcal{C}$, we set $Im(Ab(F))(f) = F(f)^{ab}$. We prove that it is well-defined on morphisms. For this, consider morphism $g : X \to Y$ in $\mathcal{C}$ such that

$$f^{ab} = g^{ab}\,, \quad \text{i.e.} \quad ab_Y \circ f = ab_Y \circ g$$

By 3.1.5, there exists $d \in D(X,Y)$ such that

$$g = f +_D d$$

By 1.5.22, we get

$$F(g) = F(f) +_{D'} \phi_{X,Y}(d)$$

Hence we have

$$\begin{aligned} ab_{F(Y)} \circ F(g) &= ab_{F(Y)} \circ \big(F(f) +_{D'} \phi_{X,Y}(d)\big) \\ &= ab_{F(Y)} \circ F(f)\,, \text{by 3.1.5} \end{aligned}$$

Thus we get $F(g)^{ab} \circ ab_{F(X)} = F(f)^{ab} \circ ab_{F(X)}$ implying that $F(g)^{ab} = F(f)^{ab}$, because $ab_{F(X)} : F(X) \to F(X)^{ab}$ is a (regular) epimorphism.

Then we observe that we have the following diagram of linear extensions of categories:

$$\begin{tikzcd} D' \arrow[r, "+"] & \mathcal{D} \arrow[r, "(Ab^{\mathcal{D}})'"] & Im(Ab^{\mathcal{D}}) \\ D \arrow[r, "+"] & \mathcal{C} \arrow[u, "F"] \arrow[r, "(Ab^{\mathcal{C}})'"] & Im(Ab^{\mathcal{C}}) \arrow[u, "Im(Ab(F))"'] \end{tikzcd} \tag{3.5.9}$$

where the two linear extensions of categories are given in 3.1.9, the bifunctor D (resp. D') with domain $Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}})$ (resp. $Im(Ab^{\mathcal{D}})^{op} \times Im(Ab^{\mathcal{D}})$) and values in Ab is defined in 3.1.8, and the functor $Im(Ab(F)) : Im(Ab^{\mathcal{C}}) \to Im(Ab^{\mathcal{D}})$ is defined in 3.5.11.

Now we are ready to prove the equivalence criteria stated at the beginning of this section.

Proof of Theorem 3.5.1. Suppose that the hypothesis 1. and 2. hold. Consider the following diagram.

$$\begin{tikzcd} D'_{\mathcal{D}'} \arrow[r, "+"] & \mathcal{D}' \arrow[r, "(Ab^{\mathcal{D}'})'"] & Im(Ab^{\mathcal{D}'}) \arrow[r, "J_{\mathcal{D}'}"] & Ab(\mathcal{D}') \\ D_{\mathcal{C}'} \arrow[r, "+"] & \mathcal{C}' \arrow[u, "F'"] \arrow[r, "(Ab^{\mathcal{C}'})'"] & Im(Ab^{\mathcal{C}'}) \arrow[u, "G"] \arrow[r, "J_{\mathcal{C}'}"] & Ab(\mathcal{C}') \arrow[u, "Ab(F')"] \end{tikzcd}$$

where $G = Im(Ab(F'))$ is the functor defined in 3.5.11, the bifunctor $D_{\mathcal{C}'}$ is given by restriction of D to $Im(Ab^{\mathcal{C}'})^{op} \times Im(Ab^{\mathcal{C}'})$, and similarly for $D'_{\mathcal{D}'}$; in fact, omitting the right-hand square the lines are sub-linear extensions of the ones in diagram (3.5.9). Next, G is defined by $G(X) = F(X)$ and

$G(f^{ab}) = F(f)^{ab} = \sigma_Y^{-1} \circ F(f^{ab}) \circ \sigma_X$ for X, Y in $\mathcal{C}'$ and $f \in \mathcal{C}(X,Y)$; the second identity ensures that $G(f^{ab})$ does indeed only depend on f^{ab}. Thus the left-hand square commutes. Moreover, the functor $J_{\mathcal{C}'}$ defined by $J_{\mathcal{C}'}(X) = X^{ab}$ and $J_{\mathcal{C}'}(f^{ab}) = f^{ab}$ is an equivalence since it is essentially surjective, full (by regular projectivity of the objects of $\mathcal{C}'$) and faithful by definition of the category $Im(Ab^{\mathcal{C}'})$; the same holds for $J_{\mathcal{D}'}$. As the right-hand square commutes up to the isomorphism σ and as $Ab(F')$ is an equivalence by condition 2. so is G.

Now let X, Y be objects of $\mathcal{C}'$ and consider the following decomposition of the map $\phi_{X,Y}\colon D(X,Y) \to D'(F(X), F(Y))$:

$$\mathcal{C}(X^{ab}, \gamma_2^{\mathcal{C}}(Y)) \xrightarrow{F_{X^{ab},\gamma_2^{\mathcal{C}}(Y)}} \mathcal{D}(F(X^{ab}), F(\gamma_2^{\mathcal{C}}(Y))) \xrightarrow{(\sigma_X)^*(e_Y^F)_*} \mathcal{D}(F(X)^{ab}, \gamma_2^{\mathcal{D}}(F(Y))).$$

Note that the functor $Ab(F)$ is additive since being an equivalence it preserves binary coproducts. Hence the map $F_{X^{ab},\gamma_2^{\mathcal{C}}(Y)}$ is an isomorphism of abelian groups since X^{ab} and $\gamma_2^{\mathcal{C}}(Y)$ are abelian objects. Moreover, $(\sigma_X)^*$ is an isomorphism of abelian groups since σ is an isomorphism in $\mathcal{D}$; the same with $(e_Y^F)_*$ since being a regular epimorphism e_Y^F is an isomorphism iff it is monic. But this is equivalent with $F(i_Y) = i^F_{\gamma_2^{\mathcal{C}}(Y)} \circ e_Y^F$ being monic. Thus $\phi_{X,Y}$ is an isomorphism of abelian groups.

Together with 3.5.10 we conclude that the five-lemma for linear extension 3.2.2 applies to the above diagram of linear extensions (omitting the right-hand square) and shows that F' is an equivalence. $\square$

The proof of Theorem 3.5.2 now heavily relies on the following general lemma.

Lemma 3.5.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-step nilpotent semi-abelian categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite coproducts, coequalizers of reflexive graphs and filtered colimits. Moreover, suppose that F carries a given free object E of rank 1 in $\mathcal{C}$ to a free object of rank 1 in $\mathcal{D}$, and that its restriction to $\langle E\rangle$ is full and faithful. Then F is an equivalence, and a weak inverse F^{-1} of F is given as follows: consider that the objects of $\mathcal{C}$ and $\mathcal{D}$ are sets endowed with operations satisfying given equational axioms, and let X be an object of $\mathcal{D}$. Then $F^{-1}(X)$ has the same underlying set $|X|$ as X, and for an n-ary operation θ of the variety $\mathcal{C}$ and an element $\underline{x} = (x_1, \ldots, x_n) \in |X|^n$ the element $\theta(x_1, \ldots, x_n)$ is given as follows: let e′ be a basis element of $F(E)$ and $\hat{\theta} \in \mathcal{C}(E, E^{+n})$ be the morphism sending e′ to $\theta(i_1^n(e'), \ldots, i_n^n(e'))$. Furthermore, let $s\colon F(E)^{+n} \to F(E^{+n})$ be the isomorphism such that $s \circ i_k^n = F(i_k^n)$ for $k = 1, \ldots, n$, and $\hat{\underline{x}} \in \mathcal{D}(F(E)^{+n}, X)$ be such that $|\hat{\underline{x}} \circ i_k^n|(e') = x_k$ for $k = 1, \ldots, n$. Then $\theta(x_1, \ldots, x_n) = |\hat{\underline{x}} \circ s^{-1} \circ F(\hat{\theta})|(e')$.*

Proof. We proceed in several steps.

Step 1: *passing to the language of models of an algebraic theory.* Consider the theory $\langle E\rangle$ in $\mathcal{C}$. By hypothesis, the injections $F(i_k^n)\colon F(E) \to F(E^{+n})$, $k = 1, \ldots, n$, make $F(E^{+n})$ into a coproduct of n copies of $F(E)$; in particular, $F(E^{+n})$ is a free object of rank n. So let the theory $\langle F(E)\rangle$ in $\mathcal{D}$ be given by taking $F(E)^{+n} = F(E^{+n}) \times \{n\}$ with injections $(pr_1)^{-1} \circ F(i_k^n)$ (the products with the sets $\{n\}$ render the objects $F(E)^{+n}$ formally distinct). Now by hypothesis the restriction $\tilde{F}\colon \langle E\rangle \to \langle F(E)\rangle$ of F is an isomorphism of theories. We thus obtain a diagram of functors, cf. section 1.1:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[d, "\rho_{\mathcal{C}}"', "\sim"] & & \mathcal{D} \arrow[d, "\rho_{\mathcal{D}}"', "\sim"] \\ Model(\langle E\rangle) \arrow[rr, shift right, leftarrow, "\tilde{F}^*"'] \arrow[rr, shift right=2, "(\tilde{F}^{-1})^*"'] & & Model(\langle F(E)\rangle) \end{tikzcd}$$

Both $\rho_{\mathcal{C}}$ and $\rho_{\mathcal{D}}$ are equivalences as $\mathcal{C}$ and $\mathcal{D}$ are varieties, and the functors $\tilde{F}^*, (\tilde{F}^{-1})^*$ are mutually inverse to each other.

Step 2: *Reduction to the construction of a certain natural isomorphism* Γ. We contend to construct a natural isomorphism $\Gamma\colon (\tilde{F}^{-1})^* \cdot \rho_{\mathcal{C}} \to \rho_{\mathcal{D}} \cdot F$ as then we can deduce isomorphisms

$$\rho_{\mathcal{C}}^{-1} \cdot \tilde{F}^* \cdot \rho_{\mathcal{D}} \cdot F \;\cong\; \rho_{\mathcal{C}}^{-1} \cdot \tilde{F}^* \cdot (\tilde{F}^{-1})^* \cdot \rho_{\mathcal{C}} \;\cong\; Id_{\mathcal{C}}.$$

Since $G = \rho_{\mathcal{C}}^{-1} \cdot \tilde{F}^* \cdot \rho_{\mathcal{D}}$ is an equivalence since its three factors are it follows that F is an equivalence with weak inverse G. Using the construction of $\rho_{\mathcal{C}}^{-1}$ in section 1.1 it then is easily seen that G is the functor F^{-1} described in the assertion.

Step 3: *Construction of* Γ *on* $\langle E\rangle$. Abbreviate $F_1 = (\tilde{F}^{-1})^* \cdot \rho_{\mathcal{C}}$ and $F_2 = \rho_{\mathcal{D}} \cdot F$. Let $m, n \geq 0$. Then

$$\begin{aligned} F_1(E^{+m})(F(E^{+n})) &= \mathcal{C}(E^{+n}, E^{+m}) \\ F_2(E^{+m})(F(E^{+n})) &= \mathcal{D}(F(E^{+n}), F(E^{+m})) \end{aligned}$$

Hence we may define a map

$$(\Gamma_{E^{+m}})_{E^{+n}} = F_{E^{+n},E^{+m}}\colon F_1(E^{+m})(F(E^{+n})) \to F_2(E^{+m})(F(E^{+n}))$$

which is a bijection since F is full and faithful on $\langle E\rangle$. Now it is immediate to check that for fixed m the collection $(\Gamma_{E^{+m}})_{E^{+n}}$, $n \geq 0$, is a natural transformation and hence natural isomorphism $\Gamma_{E^{+m}}\colon F_1(E^{+m}) \to F_2(E^{+m})$, and again that the collection of maps $\Gamma_{E^{+m}}$, $m \geq 0$, is a natural transformation and hence a natural isomorphism Γ_1 between the restrictions of F_1 and F_2 to $\langle E\rangle$. It is then clear that Γ_1 extends to a natural isomorphism Γ_2 between the restrictions of F_1 and F_2 to $Free(\mathcal{C})$, the full subcategory of all finite sums E with itself. More precisely, for a set S let $L(S) = \coprod_{s\in S} E$ be a chosen coproduct with injections $i_t\colon E \to \coprod_{s\in S}$ for all $t \in S$. Then the objects of $Free(\mathcal{C})$ are of the form $L(S)$ for S a finite set.

Step 4: *Extension of* Γ *to all free objects.* First note that the above construction is a way of describing a left-adjoint $L\colon Set \to \mathcal{C}$ of the forgetful functor $X \mapsto |X|$ where for a map $f\colon S \to T$ in Set the morphism $L(f)\colon L(S) \to L(T)$ is defined by $L(f) \circ i_s = i_{f(s)}$ for $s \in S$. Now let $\langle\!\langle E\rangle\!\rangle_E$ be the full subcategory of $\mathcal{C}$ whose objects are of the form $L(S)$ for any set S; it is equivalent with the full subcategory of all free objects. Let S be a set. Denote by $Fin(S)$ the filtered category of finite subsets of S and their mutual inclusions, and for $T \in |Fin(S)|$ let $j_{T,S}\colon T \hookrightarrow S$. Note that the morphisms $L(j_{T,S})\colon L(T) \to L(S)$ with T varying over the objects of $Fin(S)$ form a colimit cone of the diagram given by applying L to $Fin(S)$. As F and also the equivalences $\tilde{F}^*, (\tilde{F}^{-1})^*, \rho_{\mathcal{C}}$ and $\rho_{\mathcal{D}}$ preserve filtered colimits so do F_1 and F_2, hence for $k = 1, 2$ the morphisms $F_k \circ L(j_{T,S})\colon F_k \circ L(T) \to F_k \circ L(S)$, $T \in |Fin(S)|$, again form a colimit cone. It follows that there exists a unique morphism $\Gamma_{L(S)}\colon F_1(L(S)) \to F_2(L(S))$ such that $\Gamma_{L(S)} \circ F_1(L(j_{T,S})) = F_2(L(j_{T,S})) \circ \Gamma_{L(T)}$ for all $T \in |Fin(S)|$. To prove that the collection of maps $\Gamma_{L(S)}$, S a set, is a natural transformation let S, S' be two sets and $f \in \mathcal{C}(S, S')$. Note that in order to show that $\Gamma_{L(S')} \circ F_1(f) = F_2(f) \circ \Gamma_{L(S)}$ it suffices to show that this identity holds after precomposing with $F_1(L(j_{T,S}))$ for all $T \in |Fin(S)|$. Fix such a set T. The key fact is that E and hence $L(T)$ are small objects, so that there exists a finite subset T' of S' such that $f \circ j_{T,S}$ factors through the monomorphism $L(j_{T',S'})$, so that there exists $f_T \in \mathcal{C}(L(T), L(T'))$

such that $f \circ j_{T,S} = j_{T',S'} \circ f_T$. Hence

$$\begin{aligned}
F_2(f) \circ \Gamma_{L(S)} \circ F_1(L(j_{T,S})) &= F_2(f) \circ F_2(L(j_{T,S})) \circ \Gamma_{L(T)} \\
&= F_2(f \circ L(j_{T,S})) \circ \Gamma_{L(T)} \\
&= F_2(L(j_{T',S'}) \circ f_T) \circ \Gamma_{L(T)} \\
&= F_2(L(j_{T',S'})) \circ F_2(f_T) \circ \Gamma_{L(T)} \\
&= F_2(L(j_{T',S'})) \circ \Gamma_{L(T')} \circ F_1(f_T) \\
&= \Gamma_{L(S')} \circ F_1(L(j_{T',S'})) \circ F_1(f_T) \\
&= \Gamma_{L(S')} \circ F_1(f \circ L(j_{T,S})) \\
&= \Gamma_{L(S')} \circ F_1(f) \circ F_1(L(j_{T,S})),
\end{aligned}$$

as desired.

Step 5: *Extension of* Γ *to all objects.* It is clear that it suffices to present any object X of $\mathcal{C}$ as a coequalizer of a *natural* reflexive graph in $\langle\!\langle E \rangle\!\rangle_E$. The natural way to do this is the following. For any object Y of $\mathcal{C}$ let $p_Y : L(|Y|) \to Y$ be given by $p_Y \circ i_y(e') = y$ for $y \in |Y|$. Now let X be an object of $\mathcal{C}$ and $R \underset{d_1}{\overset{d_0}{\underset{\longleftarrow s_0 \longrightarrow}{\longrightarrow}}} L(|X|)$ be the kernel pair of p_X. Let U be the disjoint union of R and $|X|$ and for $k = 0, 1$ let $d'_k : L(U) \to L(|X|)$ be defined by $d'_k \circ i_u(e') = d_k(u)$ if $u \in R$ and $d'_k \circ i_u(e') = i_u(e')$ if $u \in |X|$. Defining also $s'_0 : L(|X|) \to L(U)$ by $s'_0 \circ i_x = i_x$ for $x \in |X|$ we obtain a natural reflexive graph $L(U) \underset{d'_1}{\overset{d'_0}{\underset{\longleftarrow s'_0 \longrightarrow}{\longrightarrow}}} L(|X|)$ whose coequalizer is p_X. $\square$

Proof of Theorem 3.5.2: Apply Theorem 3.5.1 with $\mathcal{C}' = \langle E \rangle$ and $\mathcal{D}' = \langle F(E) \rangle$. $\square$

3.6 Quadratic $\mathcal{C}$-modules over $\mathcal{P}$ associated with quadratic equivalences

In this section, we characterize quadratic $\mathcal{C}$-modules over $\mathcal{P}$ (see 2.4.23) corresponding to quadratic equivalences taking values in $\mathcal{P}$-algebras. Let $\mathcal{C}$ be a semi-abelian category and $F : \mathcal{C} \to Alg - \mathcal{P}$ be a quadratic equivalence. As F is bijective on objects, there is an object E in $\mathcal{C}$ such that $F(E)$ is isomorphic to the free algebra of rank 1 in $Alg - \mathcal{P}$, denoted by $\mathcal{F}_{\mathcal{P}}$. To simplify calculations, we can suppose that $F(E) = \mathcal{F}_{\mathcal{P}}$.
We know that the functor $F_{|\langle E \rangle} : \langle E \rangle \to Alg - \mathcal{P}$ restricted to the algebraic theory $\langle E \rangle$ generated by E (as in 1.1.1) preserves finite coproducts because F is an equivalence of categories. Then the quadratic equivalence $F_{|\langle E \rangle}$, or simply F, takes values in the category of free $\mathcal{P}$-algebras of finite rank, denoted by $\langle \mathcal{F}_{\mathcal{P}} \rangle_{Alg-\mathcal{P}}$. We now give the second cross-effect $F(E|E)$ evaluated twice on E as below:

Proposition 3.6.1. *Let $F : \mathcal{C} \to Alg - \mathcal{P}$ be a quadratic equivalence, then there is a natural isomorphism of $(\overline{\Lambda} \otimes \overline{\Lambda})$-$\mathcal{P}(1)$-bimodules $F(E|E) \cong \mathcal{P}(2)$.*

Proof. As F is a reduced functor in particular, we know that the right-hand square commutes in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & F(E\mid E) \arrow[r, tail, "\iota_2^F"] & F(E+E) \arrow[r, "\widehat{r_2^F}"] & F(E)\times F(E) \arrow[r] & 0 \\
0 \arrow[r] & Id_{Alg-\mathcal{P}}(F(E)\mid F(E)) \arrow[u, dashed, "\alpha", "\cong"'] \arrow[r, "\iota_2^{Id_{Alg-\mathcal{P}}}"] & F(E)+F(E) \arrow[u, "(F(i_1^2),\,F(i_2^2))"', "\cong"] \arrow[r, "\widehat{r_2^{Id_{Alg-\mathcal{P}}}}"] & F(E)\times F(E) \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

The morphism $(F(i_1^2),\ F(i_2^2)) : F(E) + F(E) \to F(E + E)$ is an isomorphism because F preserves finite coproducts. Then there is a unique morphism $\alpha : Id_{Alg-\mathcal{P}}(F(E)|F(E)) \to F(E|E)$ such that the left-hand square of the above diagram commutes, i.e.

$$\iota_2^F \circ \alpha = (F(i_1^2),\ F(i_2^2)) \circ \iota_2^{Id_{Alg-\mathcal{P}}} \tag{3.6.1}$$

by using a categorical argument. Then, by a straightforward categorical argument, α is an isomorphism in $Alg - \mathcal{P}$. By 1.8.6 and (1.8.5), we deduce that

$$F(E\mid E) \cong Id_{Alg-\mathcal{P}}(F(E)\mid F(E)) = Id_{Alg-\mathcal{P}}(\mathcal{F}_{\mathcal{P}}\mid \mathcal{F}_{\mathcal{P}})$$

By 1.8.6 and (1.8.5), we get

$$Id_{Alg-\mathcal{P}}(\mathcal{F}_{\mathcal{P}}\mid \mathcal{F}_{\mathcal{P}}) = \overline{\mathcal{F}_{\mathcal{P}}} \otimes \overline{\mathcal{F}_{\mathcal{P}}} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) = \mathcal{P}(1)\otimes\mathcal{P}(1)\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\,,$$

and there is a canonical isomorphism $ev_2 : Id_{Alg-\mathcal{P}}(\mathcal{F}_{\mathcal{P}}|\mathcal{F}_{\mathcal{P}}) \to \mathcal{P}(2)$ defined by:

$$ev_2\Big(\overline{(p_1^1,\ \overline{p_2^1})} \otimes \overline{(p_1^2,\ \overline{p_2^2})} \otimes p_2\Big) = \gamma_{1,1;2}(p_1^1 \otimes p_1^2 \otimes p_2) \tag{3.6.2}$$

where $p_2^k, p_2 \in \mathcal{P}(2)$ and $p_1^k \in \mathcal{P}(1)$, for $k = 1, 2$. $\square$

Since $F(E)$ is a left Λ-module by 3.2 of [12], so is $\mathcal{F}_{\mathcal{P}}$ whose action is given by

$$f.(p_1,\ \overline{p_2}) = F(f)(p_1,\ \overline{p_2})\,,$$

for $p_1 \in \mathcal{P}(1)$ and $p_2 \in \mathcal{P}(2)$. Moreover, 3.17 of [12] says that $F(E|E)$ is a left $(\overline{\Lambda}\otimes\overline{\Lambda})$-module (because the second cross-effect of F is a bilinear bifunctor 1.2.13 since the functor F is quadratic), hence so is $\mathcal{P}(2)$ whose action is given by

$$t_1(f)\otimes t_1(g).p_2 = ev_2 \circ \alpha^{-1} \circ F(f|g) \circ \alpha \circ ev_2^{-1}(p_2) \tag{3.6.3}$$

where $f, g \in \mathcal{C}(E,\ E)$, $\alpha : Id_{Alg-\mathcal{P}}(F(E)|F(E)) \to F(E|E)$ is the isomorphism defined in (3.6.1) and $ev_2 : Id_{Alg-\mathcal{P}}(\mathcal{F}_{\mathcal{P}}|\mathcal{F}_{\mathcal{P}}) \to \mathcal{P}(2)$ is the canonical isomorphism defined in 3.6.2.

Proposition 3.6.2. *Let $f, g \in \mathcal{C}(E,\ E)$ and $p_2 \in \mathcal{P}(2)$. Then we have*

$$t_1(f)\otimes t_1(g).(p_2.t) = t_1(g)\otimes t_1(f).p_2$$

Proof. First we check that $\alpha\circ ev_2^{-1}(p_2.t) = T_E\circ\alpha\circ ev_2^{-1}(p)$, where $\alpha : Id_{Alg-\mathcal{P}}(F(E)|F(E)) \to F(E|E)$ is the isomorphism in the proof of 3.6.1 and $ev_2 : Id_{Alg-\mathcal{P}}(F(E)|F(E)) \to \mathcal{P}(2)$ is the evaluation isomorphism given in (3.6.2). We have the following relations:

$$\begin{aligned}
\iota_2^F \circ \alpha \circ ev_2^{-1}(p_2.t) &= \big(F(i_1^2),\ F(i_2^2)\big) \circ \iota_2^{Id}\big(\overline{(1_{\mathcal{P}},\overline{0})} \otimes \overline{(1_{\mathcal{P}},\overline{0})} \otimes (p_2.t)\big)\\
&= \big(F(i_1^2),\ F(i_2^2)\big)\big(0,\ 0,\ \overline{(1_{\mathcal{P}},\overline{0})} \otimes \overline{(1_{\mathcal{P}},\overline{0})} \otimes (p_2.t)\big)\\
&= (\lambda_2^F)_{E^{+2}}\big(F(i_1^2)(1_{\mathcal{P}},\overline{0}) \otimes F(i_2^2)(1_{\mathcal{P}},\overline{0}) \otimes (p_2.t)\big)\ \text{, by 1.8.5}\\
&= (\lambda_2^F)_{E^{+2}}\big(F(i_2^2)(1_{\mathcal{P}},\overline{0}) \otimes F(i_1^2)(1_{\mathcal{P}},\overline{0}) \otimes p_2\big)\ \text{, by the axioms of algebra over } \mathcal{P}\\
&= (\lambda_2^F)_{E^{+2}}\big(F(\tau_E^2\circ i_1^2)(1_{\mathcal{P}},\overline{0}) \otimes F(\tau_E^2\circ i_2^2)(1_{\mathcal{P}},\overline{0}) \otimes p_2\big)\\
&= F(\tau_E^2)\Big((\lambda_2^F)_{E^{+2}}(F(i_1^2)(1_{\mathcal{P}},\overline{0}) \otimes F(i_2^2)(1_{\mathcal{P}},\overline{0}) \otimes p_2)\Big)
\end{aligned}$$

because $F(\tau_E^2) : F(E+E) \to F(E+E)$ is a morphism in the category $Alg - \mathcal{P}$. Then we have

$$\begin{aligned}
\iota_2^F \circ \alpha \circ ev_2^{-1}(p_2.t) &= F(\tau_E^2) \circ (F(i_1^2),\ F(i_2^2)) \circ \iota_2^{Id} \circ ev_2^{-1}(p_2) \\
&= F(\tau_E^2) \circ \iota_2^F \circ \alpha \circ ev_2^{-1}(p_2) \\
&= \iota_2^F \circ T_E \circ \alpha \circ ev_2^{-1}(p_2)
\end{aligned}$$

As $\iota_2^F : F(E|E) \to F(E+E)$ is a monomorphism, we obtain

$$\alpha \circ ev_2^{-1}(p_2.t) = T_E \circ \alpha \circ ev_2^{-1}(p_2)$$

Hence we have

$$\begin{aligned}
t_1(f) \otimes t_1(g).(p_2.t) &= ev_2 \circ \alpha^{-1} \circ F(f|g) \circ \alpha \circ ev_2^{-1}(p_2.t) \\
&= ev_2 \circ \alpha^{-1} F(f|g) \circ T_E \circ \alpha \circ ev_2^{-1}(p_2) \\
&= ev_2 \circ \alpha^{-1} F(g|f) \circ \alpha \circ ev_2^{-1}(p_2)\, , \text{because } F(f|g) \circ T_E = F(g|f) \\
&= t_1(g) \otimes t_1(f).p_2
\end{aligned}$$

as desired. Now it is easy to see that the left action of Λ on $\mathcal{P}(2)_{\mathfrak{S}_2}$ is well-defined. $\square$

Remark 3.6.3. The abelian group $\mathcal{P}(2)_{\mathfrak{S}_2}$ is a left Λ-module whose action is given by:

$$f.\overline{p_2} = \overline{t_1(f) \otimes t_1(f).p_2}$$

where $f \in \mathcal{C}(E,\ E)$ and $p_2 \in \mathcal{P}(2)$. It is immediate that the left Λ action on $\mathcal{P}(2)$ is well-defined by 3.6.2.

Notation 3.6.4. We define the map $\dot{q} : \mathcal{P}(2) \to \mathcal{F}_{\mathcal{P}}$ by $\dot{q} = i_2 \circ q$ where $q : \mathcal{P}(2) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ is the canonical quotient map and $i_2 : \mathcal{P}(2)_{\mathfrak{S}_2} \rightarrowtail \mathcal{F}_{\mathcal{P}}$ is the inclusion of the second summand.

Now we give the linearization of the functor F evaluated on E:

Proposition 3.6.5. *There is an isomorphism of $\overline{\Lambda}$-$\mathcal{P}(1)$-bimodules between the linearization of F evaluated to E and $\mathcal{P}(1)$.*

Proof. We consider the following diagram:

$$\begin{tikzcd}
F(E|E) \arrow[r, "(S_2^F)_E"] & F(E) \arrow[r, "(t_1^F)_E"] \arrow[d, equal] & T_1F(E) \arrow[r] \arrow[dd, dashed, "\cong"', "\beta"] & 0 \\
Id_{Alg-\mathcal{P}}(F(E)|F(E)) \arrow[u, "\alpha", "\cong"'] \arrow[r, "c_2^{F(E)}"] & F(E) \arrow[d, equal] & & \\
\mathcal{P}(2) \arrow[u, "ev_2^{-1}", "\cong"'] \arrow[r, "\dot{q}"] & \mathcal{F}_{\mathcal{P}} \arrow[r, "\pi_1"] & \mathcal{P}(1) \arrow[r] & 0
\end{tikzcd}$$

where $\pi_1 : \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(1)$ is the projection onto the first summand and ev_2 is the isomorphism given in (3.6.2). We recall that $\alpha : Id_{Alg-\mathcal{P}}(F(E)|F(E)) \to F(E|E)$ is the isomorphism such that $\iota_2^F \circ \alpha = (F(i_1^2),\ F(i_2^2)) \circ \iota_2^{Id}$ given in the proof of 3.6.1.

- The top left-hand rectangle commutes because we have

$$(S_2^F)_E \circ \alpha = F(\nabla_E^2) \circ \iota_2^F \circ \alpha = F(\nabla_E^2) \circ (F(i_1^2),\ F(i_2^2)) \circ \iota_2^{Id} = \nabla_{F(E)}^2 \circ \iota_2^{Id} = c_2^{F(E)}$$

- The left bottom one commutes because we have

$$\begin{aligned} c_2^{F(E)} \circ ev_2^{-1}(p) &= \nabla_{F(E)}^2 \circ \iota_2^{Id}(\overline{(1_{\mathcal{P}},\ \overline{0})} \otimes \overline{(1_{\mathcal{P}},\ \overline{0})} \otimes p_2) \\ &= \nabla_{F(E)}^2\big(0,\ 0,\ \overline{(1_{\mathcal{P}},\ \overline{0})} \otimes \overline{(1_{\mathcal{P}},\ \overline{0})} \otimes p_2\big) \text{ , by 1.8.6} \\ &= \lambda_2^{\mathcal{F}_{\mathcal{P}}}\big((1_{\mathcal{P}},\ \overline{0}) \otimes (1_{\mathcal{P}},\ \overline{0}) \otimes p_2\big) \text{ , by 1.8.5} \\ &= \big(0,\ \overline{\gamma_{1,1;2}(1_{\mathcal{P}} \otimes 1_{\mathcal{P}} \otimes p_2)}\big) \\ &= (0,\ \overline{p_2}) \\ &= \dot{q}(p_2) \end{aligned}$$

where $p_2 \in \mathcal{P}(2)$.

As $(t_1^F)_E : F(E) \to T_1F(E)$ and $\pi_1 : \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(1)$ are respectively the cokernels of $(S_2^F)_E : F(E|E) \to F(E)$ and $\dot{q} : \mathcal{P}(2) \to \mathcal{F}_{\mathcal{P}}$, there is a unique isomorphism (of abelian groups) $\beta : T_1F(E) \to \mathcal{P}(1)$ such that $\beta \circ (t_1^F)_E = \pi_1$. As each (not dotted) morphism in the diagram is a homomorphism of Λ-$\mathcal{P}(1)$-bimodules, so is the isomorphism $\beta : T_1F(E) \to \mathcal{P}(1)$. □

Notation 3.6.6. If $(p_1,\ \overline{p_2})$ is an element of $F(E) = \mathcal{F}_{\mathcal{P}}$, then we set $\overline{(p_1,\ \overline{p_2})}$ the equivalence class of $(p,\ \overline{q})$ in $T_1F(E)$.

The next result gives a specific quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$. We shall use it to determine the one corresponding to the quadratic equivalence $F : \mathcal{C} \to Alg - \mathcal{P}$.

Proposition 3.6.7. *The diagram of homomorphisms of right $\mathcal{P}(1)$-modules*

$$M_2^{\mathcal{P}} = \left(T_{11}(cr_2U_E)(E,E) \otimes_{\Lambda} \mathcal{P}(2)_{\mathfrak{S}_2} \xrightarrow{\hat{H}} \mathcal{P}(2) \xrightarrow{.t} \mathcal{P}(2) \xrightarrow{q} \mathcal{P}(2)_{\mathfrak{S}_2} \right)$$

where

- *$.t : \mathcal{P}(2) \to \mathcal{P}(2)$ is the (right) action of $t = (1,2)$ on $\mathcal{P}(2)$ involved in the structure of operads;*
- *$\hat{H} : T_{11}(cr_2U_E)(E,E) \otimes_{\Lambda} \mathcal{P}(2)_{\mathfrak{S}_2} \to \mathcal{P}(2)$ is defined by*

$$\hat{H}\big(t_{11}(\rho_2(\xi)) \otimes_{\Lambda} \overline{p_2}\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).(p_2 + p_2.t)$$

where $p_2 \in \mathcal{P}(2)$ and $\xi \in \mathcal{C}(E,\ E^{+2})$;

is a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$.

Proof. Let $\xi \in \mathcal{C}(E,\ E^{+2})$, $f, g \in \mathcal{C}(E,\ E)$, $p_1 \in \mathcal{P}(1)$ and $p_2 \in \mathcal{P}(2)$. First we know that $\mathcal{P}(2)_{\mathfrak{S}_2}$ and $\mathcal{P}(2)$ are respectively left Λ-module and $(\overline{\Lambda} \otimes \overline{\Lambda})$-module by (3.6.3) and 3.6.3. We verify that $q : \mathcal{P}(2) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ is a homomorphism of Λ-modules with respect to the diagonal action of Λ on $\mathcal{P}(2)$ as follows:

$$q(t_1(f) \otimes t_1(f).p_2) = \overline{t_1(f) \otimes t_1(f).p_2} = f.\overline{p_2} = f.q(p_2)$$

by using 3.6.3. By definition, $\hat{H} : T_{11}(cr_2U_E)(E,E) \otimes_{\Lambda} \mathcal{P}(2)_{\mathfrak{S}_2} \to \mathcal{P}(2)$ clearly satisfies $(QM2)$ in 2.1.1:

$$\hat{H}(t_{11}\big(\rho_2(\xi)) \otimes_{\Lambda} q(p_2)\big)) = \hat{H}\big(t_{11}(\rho_2(\xi)) \otimes_{\Lambda} \overline{p_2}\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).(p_2 + p_2.t)$$

In addition, $\hat{H}$ also satisfies $(QM1)$ as follows:

$$\begin{aligned}
(\nabla_E^2\circ\xi).\overline{p_2} &= \overline{t_1(\nabla_E^2\circ\xi)\otimes t_1(\nabla_E^2\circ\xi).p_2}\\
&= \overline{t_1(r_1^2\circ\xi)\otimes t_1(\nabla_E^2\circ\xi).p_2}+\overline{t_1(r_2^2\circ\xi)\otimes t_1(\nabla_E^2\circ\xi).p_2}\text{ ,by 2.14 of [12]}\\
&= \overline{t_1(r_1^2\circ\xi)\otimes t_1(r_1^2\circ\xi).p_2}+\overline{t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).p_2}\\
&\quad+\overline{t_1(r_2^2\circ\xi)\otimes t_1(r_1^2\circ\xi).p_2}+\overline{t_1(r_2^2\circ\xi)\otimes t_1(r_2^2\circ\xi).p_2}\\
&= (r_1^2\circ\xi).\overline{p_2}+(r_2^2\circ\xi).\overline{p_2}+\overline{t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).p_2}+\overline{t_1(r_2^2\circ\xi)\otimes t_1(r_1^2\circ\xi).p_2}\text{ , by 3.6.3}\\
&= (r_1^2\circ\xi).\overline{p_2}+(r_2^2\circ\xi).\overline{p_2}+\overline{t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).p_2}+\overline{t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).(p_2.t)}\\
&= (r_1^2\circ\xi).\overline{p'}+(r_2^2\circ\xi).\overline{p_2}+\big(q\circ\hat{H}\big)(t_{11}(\rho_2(\xi))\otimes_\Lambda\overline{p_2})
\end{aligned}$$

as desired. As $\mathcal{P}(2)$ is a right $\mathcal{P}(1)$-module, $\mathcal{P}(2)_{\mathfrak{S}_2}$ is also a right $\mathcal{P}(1)$-module defined by

$$\overline{p_2}.p_1=\overline{\gamma_{2;1}(p_2\otimes p_1)}$$

and the quotient map $q:\mathcal{P}(2)\to\mathcal{P}(2)_{\mathfrak{S}_2}$ is a homomorphism of right $\mathcal{P}(1)$-modules because we have

$$q(\gamma_{2;1}(p_2\otimes p_1))=\overline{\gamma_{2;1}(p_2\otimes p_1)}=\overline{p_2}.p_1=q(p_2).p_1$$

To prove that $\hat{H}:T_{11}(cr_2U_E)(E,E)\otimes_\Lambda\mathcal{P}(2)_{\mathfrak{S}_2}\to\mathcal{P}(2)$ is a homomorphism of right $\mathcal{P}(1)$-modules, it is sufficient to prove that

$$\gamma_{2;1}(t_1(f)\otimes t_1(g).p_2\otimes p_1)=t_1(f)\otimes t_1(g).\gamma_{2;1}(p_2\otimes p_1)$$

For this, we use relation (3.6.3), and the fact that $\alpha:Id_{Alg-\mathcal{P}}(F(E)|F(E))\to F(E|E)$ (defined in (3.6.1)) is a homomorphism of right $\mathcal{P}(1)$-modules (as it is the restriction of the homomorphism of right $\mathcal{P}(1)$-modules $(F(i_1^2),F(i_2^2)):F(E)+F(E)\to F(E+E)$ to $Id_{Alg-\mathcal{P}}(F(E)|F(E))$). $\square$

Before giving the quadratic $\mathcal{C}$-module corresponding to the quadratic equivalence $F:\mathcal{C}\to Alg\text{-}\mathcal{P}$, we give the following lemma:

Proposition 3.6.8. *The objects $\mathbb{S}_2^{\mathcal{P}(1)}(T_1F^{\otimes 2}\otimes_S\mathcal{P}(2))$, given in 2.4.18, and $M_2^{\mathcal{P}}$, given in 3.6.7, are isomorphic in $Mod_{\mathcal{C}}^{\mathcal{P}(1)}$.*

Proof. We set $R=\mathcal{P}(1)$ and $S=(R\otimes R)\wr\mathfrak{S}_2$ (as in 2.4.9) and we define the canonical isomorphism of $(\mathcal{P}(1)\otimes\mathcal{P}(1))$-$\mathcal{P}(1)$-bimodules $ev_{\mathfrak{S}_2}:(\mathcal{P}(1)\otimes\mathcal{P}(1))\otimes_S\mathcal{P}(2)\to\mathcal{P}(2)_{\mathfrak{S}_2}$ by:

$$ev_{\mathfrak{S}_2}((p_1\otimes p_1')\otimes_S p_2)=\overline{\gamma_{1,1;2}(p_1\otimes p_1'\otimes p_2)}\,,$$

for $p_1,p_1'\in\mathcal{P}(1)$ and $p_2\in\mathcal{P}(2)$. We prove that $(ev_{\mathfrak{S}_2}\circ(\beta^{\otimes 2}\otimes_S id),ev_2\circ(\beta^{\otimes 2}\otimes_{R\otimes R}id)):\mathbb{S}_2^{\mathcal{P}(1)}(T_1F^{\otimes 2}\otimes_S\mathcal{P}(2))\to M_2^{\mathcal{P}}$ is an isomorphism of quadratic $\mathcal{C}$-modules over $\mathcal{P}(1)$.

1. *Computation of $q_E^F:T_1F(E)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2)\to T_1F(E)^{\otimes 2}\otimes_S\mathcal{P}(2)$.* We check that the following diagram commutes:

$$\begin{tikzcd}
T_1F(E)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2) \arrow[r, "q_E^F"] \arrow[d, "\beta^{\otimes 2}\otimes_{R\otimes R}id"', "\cong"] & T_1F(E)^{\otimes 2}\otimes_S\mathcal{P}(2) \arrow[d, "\beta^{\otimes 2}\otimes_S id", "\cong"'] \\
(\mathcal{P}(1)\otimes\mathcal{P}(1))\otimes_{R\otimes R}\mathcal{P}(2) \arrow[d, "ev_2"', "\cong"] & (\mathcal{P}(1)\otimes\mathcal{P}(1))\otimes_S\mathcal{P}(2) \arrow[d, "ev_{\mathfrak{S}_2}", "\cong"'] \\
\mathcal{P}(2) \arrow[r, "q"] & \mathcal{P}(2)_{\mathfrak{S}_2}
\end{tikzcd}$$

Let $(p_1^k,\, \overline{p_2^k}) \in F(E)$, $p_2 \in \mathcal{P}(2)$ and $k = 1, 2$. First we have

$$\begin{aligned} q \circ ev_2 \circ (\beta^{\otimes 2} \otimes_{R\otimes R} id)\big(\overline{(p_1^1,\, \overline{p_2^1})} \otimes \overline{(p_1^2,\, \overline{p_2^2})} \otimes p_2\big) &= q \circ ev_2(p_1^1 \otimes p_1^2 \otimes p_2) \\ &= q(\gamma_{1,1;2}(p_1^1 \otimes p_1^2 \otimes p_2)) \\ &= \overline{\gamma_{1,1;2}(p_1^1 \otimes p_1^2 \otimes p_2)} \end{aligned}$$

Then we have

$$\begin{aligned} ev_{\mathfrak{S}_2} \circ (\beta^{\otimes 2} \otimes_S id) \circ q_E^F\big(\overline{(p_1^1,\, \overline{p_2^1})} \otimes \overline{(p_1^2,\, \overline{p_2^2})} \otimes_{R\otimes R} p_2\big) &= ev_{\mathfrak{S}_2}(p_1^1 \otimes p_1^2 \otimes_S p_2) \\ &= \overline{\gamma_{1,1;2}(p_1^1 \otimes p_1^2 \otimes p_2)} \end{aligned}$$

This proves that the above diagram commutes.

2. *Computation of the involution* $\widehat{T_E^F} \otimes_{R\otimes R} t : T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \to T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$. We verify that the following diagram commutes:

$$\begin{tikzcd} T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \arrow[rr, "\widehat{T_E^F} \otimes_{R\otimes R} t"] \arrow[d, "\beta^{\otimes 2} \otimes_{R\otimes R} id"', "\cong"] & & T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \arrow[d, "\cong"', "\beta^{\otimes 2} \otimes_{R\otimes R} id"] \\ (\mathcal{P}(1) \otimes \mathcal{P}(1)) \otimes_{R\otimes R} \mathcal{P}(2) \arrow[d, "ev_2"', "\cong"] & & (\mathcal{P}(1) \otimes \mathcal{P}(1)) \otimes_{R\otimes R} \mathcal{P}(2) \arrow[d, "\cong"', "ev_2"] \\ \mathcal{P}(2) \arrow[rr, ".t"] & & \mathcal{P}(2) \end{tikzcd}$$

We have

$$\begin{aligned} & ev_2 \circ (\beta^{\otimes 2} \otimes_{R\otimes R} id) \circ (\widehat{T_E^F} \otimes_{R\otimes R} t)\big(\overline{(p_1^1,\, \overline{p_2^1})} \otimes \overline{(p_1^2,\, \overline{p_2^2})} \otimes_{R\otimes R} p_2\big) \\ &= ev_2 \circ (\beta^{\otimes 2} \otimes_{R\otimes R} id)\big(\overline{(p_1^2,\, \overline{p_2^2})} \otimes \overline{(p_1^1,\, \overline{p_2^1})} \otimes_{R\otimes R} (p_2.t)\big) \\ &= ev_2\big(p_1^2 \otimes p_1^1 \otimes (p_2.t)\big) = \gamma_{1,1;2}(p_1^2 \otimes p_1^1 \otimes (p_2.t)) = \gamma_{1,1;2}(p_1^1 \otimes p_1^2 \otimes p_2).t \\ &= ev_2\big(p_1^1 \otimes p_1^2 \otimes p_2\big).t \\ &= (.t) \circ ev_2 \circ (\beta^{\otimes 2} \otimes_{R\otimes R} id)\big(\overline{(p_1^1,\, \overline{p_2^1})} \otimes \overline{(p_1^2,\, \overline{p_2^2})} \otimes_{R\otimes R} p_2\big) \end{aligned}$$

where $.t : \mathcal{P}(2) \to \mathcal{P}(2)$ denotes the action of $t = (1, 2) \in \mathfrak{S}_2$ on $\mathcal{P}(2)$ (involved in the structure of the operad $\mathcal{P}$).

3. *Computation of* $\widehat{H}_E^F : T_{11}cr_2(U_E)(E,\, E) \otimes (T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)) \to T_1F(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2)$. We

prove that the following diagram commutes:

$$\begin{tikzcd}
T_{11}(cr_2U_E)(E,\ E)\otimes_\Lambda (T_1F(E)^{\otimes 2}\otimes_S \mathcal{P}(2)) \arrow[r, "\widehat{H}^F_E"] \arrow[d, "id\otimes_\Lambda(\beta^{\otimes 2}\otimes_S id)"', "\cong"] & T_1F(E)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2) \arrow[d, "\cong"', "\beta^{\otimes 2}\otimes_{R\otimes R} id"] \\
T_{11}(cr_2U_E)(E,\ E)\otimes_\Lambda ((\mathcal{P}(1)\otimes\mathcal{P}(1))\otimes_{(\mathcal{P}(1)\otimes\mathcal{P}(1))\wr\mathfrak{S}_2}\mathcal{P}(2)) \arrow[d, "id\otimes_\Lambda ev_{\mathfrak{S}_2}"', "\cong"] & (\mathcal{P}(1)\otimes\mathcal{P}(1))\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2) \arrow[d, "\cong"', "ev_2"] \\
T_{11}(cr_2U_E)(E,\ E)\otimes_\Lambda \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[r, "\widehat{H}"] & \mathcal{P}(2)
\end{tikzcd}$$

Let $p_2\in\mathcal{P}(2)$ and $\xi\in\mathcal{C}(E,\ E+E)$. First we have

$$t_{11}(\rho_2(\xi))\otimes\overline{p_2} = (id\otimes_\Lambda ev_{\mathfrak{S}_2})\circ\big(id\otimes_\Lambda(\beta^{\otimes 2}\otimes_S id)\big)\big(t_{11}(\rho_2(\xi))\otimes_\Lambda\overline{(1_\mathcal{P},\overline{0})}\otimes\overline{(1_\mathcal{P},\overline{0})}\otimes_S p_2\big)$$

where $ev_1:\overline{\mathcal{F}_\mathcal{P}}\to\mathcal{P}(1)$ denotes the isomorphism in (1.8.5). Then we also have

$$\begin{aligned}
&ev_2\circ(\beta^{\otimes 2}\otimes_{R\otimes R} id)\circ\widehat{H}^F_E\big(t_{11}(\rho_2(\xi))\otimes_\Lambda\overline{(1_\mathcal{P},\overline{0})}\otimes\overline{(1_\mathcal{P},\overline{0})}\otimes_S p_2\big)\\
&= ev_2\circ(\beta^{\otimes 2}\otimes_{R\otimes R} id)\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).\big((1_\mathcal{P},\overline{0})\otimes(1_\mathcal{P},\overline{0})\otimes_{R\otimes R} p_2\\
&\qquad\qquad + (1_\mathcal{P},\overline{0})\otimes(1_\mathcal{P},\overline{0})\otimes_{R\otimes R}(p_2.t)\big)\Big)\\
&= t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).\,ev_2\circ(\beta^{\otimes 2}\otimes_{R\otimes R} id)\Big((1_\mathcal{P},\overline{0})\otimes(1_\mathcal{P},\overline{0})\otimes_{R\otimes R} p_2\\
&\qquad\qquad + (1_\mathcal{P},\overline{0})\otimes(1_\mathcal{P},\overline{0})\otimes_{R\otimes R}(p_2.t)\big)\Big)\\
&= t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).\,ev_2\Big(1_\mathcal{P}\otimes 1_\mathcal{P}\otimes_{R\otimes R} p_2 + 1_\mathcal{P}\otimes 1_\mathcal{P}\otimes_{R\otimes R}(p_2.t))\Big)\\
&= t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).(p_2+p_2.t)\\
&= \widehat{H}\big(t_{11}(\rho_2(\xi))\otimes\overline{p_2}\big)
\end{aligned}$$

This proves that the above diagram commutes.

□

Now we give the quadratic $\mathcal{C}$-module over $\mathcal{P}$ corresponding to the quadratic equivalence $F:\mathcal{C}\to Alg-\mathcal{P}$ as follows:

Proposition 3.6.9. *The quadratic $\mathcal{C}$-module over $\mathcal{P}$ corresponding to the equivalence $F:\mathcal{C}\to Alg-\mathcal{P}$ is the following commutative diagram of homomorphisms of right $\mathcal{P}(1)$-modules up to an isomorphism:*

$$\begin{tikzcd}
T_{11}cr_2(U_E)(E,\ E)\otimes_\Lambda\mathcal{F}_\mathcal{P} \arrow[r, "\widehat{H}^F"] & \mathcal{P}(2) \arrow[r, "\dot{q}"] \arrow[d, equal] & \mathcal{F}_\mathcal{P} \\
T_{11}cr_2(U_E)(E,\ E)\otimes_\Lambda\mathcal{P}(2)_{\mathfrak{S}_2} \arrow[r, "\widehat{H}"'] \arrow[u, "id\otimes_\Lambda i_2"] & \mathcal{P}(2) \arrow[r, "q"'] & \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[u, "i_2"']
\end{tikzcd}$$

Here

- $\widehat{H}^F : T_{11}cr_2(U_E)(E,\ E) \otimes_\Lambda \mathcal{F}_\mathcal{P} \to \mathcal{P}(2)$ *is a homomorphism of* $(\overline{\Lambda} \otimes \overline{\Lambda})$*-modules satisfying the following relations*

$$\begin{cases} (\nabla_E^2 \circ \xi).(p_1, \overline{p_2}) = (r_1^2 \circ \xi).(p_1, \overline{p_2}) + (r_2^2 \circ \xi).(p_1, \overline{p_2}) + \big(\dot{q} \circ \widehat{H}^F\big)\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (p_1, \overline{p_2})\big) \\ \widehat{H}^F\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (0, \overline{p_2})\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(p_2 + p_2.t\big) \end{cases}$$

where $\xi\ \in \mathcal{C}(E,\ E^{+2})$, $p_1\ \in \mathcal{P}(1)$ *and* $p_2\ \in \mathcal{P}(2)$*;*

- *the bottom diagram is the quadratic* $\mathcal{C}$*-module over* $\mathcal{P}(1)$ *given in 3.6.7.*

Proof. By applying the functor $\mathbb{S}_2^{\mathcal{P}} : Quad(\mathcal{C},\ Alg - \mathcal{P}) \to Mod_{\mathcal{C}}^{\mathcal{P}}$, defined in 2.4.27, to the quadratic functor $F : \mathcal{C} \to Alg - \mathcal{P}$, we know that its corresponding quadratic $\mathcal{C}$-module over $\mathcal{P}$ is given by the following diagram:

$$\begin{tikzcd} T_{11}cr_2(U_E)(E,\ E) \otimes F(E) \arrow[r, "H_E^F"] & F(E|E) \arrow[r, "(S_2^F)_E"] & F(E) \\ T_{11}cr_2(U_E)(E,\ E) \otimes (T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2)) \arrow[r, "\widehat{H}_E^F"'] \arrow[u, "id \otimes_\Lambda \psi_E^F"] & T_1F(E)^{\otimes 2} \otimes_{R \otimes R} \mathcal{P}(2) \arrow[r, "q_E^F"'] \arrow[u, "cr_2(\psi^F)_{E,E}"] & T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2) \arrow[u, "\psi_E^F"'] \end{tikzcd}$$

where $R = \mathcal{P}(1)$ and $S = (R \otimes R) \wr \mathfrak{S}_2$ (see 2.4.9). By 3.6.8, the quadratic $\mathcal{C}$-module $\mathbb{S}_2(T_1F^{\otimes 2} \otimes_S \mathcal{P}(2))$ over $\mathcal{P}(1)$ is isomorphic to $M_2^{\mathcal{P}}$ given in 3.6.7. Let us denote by M^F the top quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ of the diagram in the assumption, then we prove that $(id,\ ev_2 \circ \alpha^{-1}) : \mathbb{S}_2(F) \to M^F$ where $\alpha : Id_{Alg-\mathcal{P}}(F(E)|F(E)) \to F(E|E)$ is the isomorphism given in (3.6.1), such that $i_2^F \circ \alpha = (F(i_1^2),\ F(i_2^2)) \circ \iota_2^{Id}$.

1. *Computation of* ψ_E^F. We consider the diagram below:

$$\begin{tikzcd} T_1F(E)^{\otimes 2} \otimes_S \mathcal{P}(2) \arrow[rr, "\psi_E^F"] \arrow[d, "\beta^{\otimes 2} \otimes_S id"', "\cong"] & & F(E) \arrow[dd, equal] \\ (\mathcal{P}(1) \otimes \mathcal{P}(1)) \otimes_{(\mathcal{P}(1) \otimes \mathcal{P}(1)) \wr \mathfrak{S}_2} \mathcal{P}(2) \arrow[d, "ev_{\mathfrak{S}_2}"', "\cong"] & & \\ \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[rr, "i_2"] & & \mathcal{F}_\mathcal{P} \end{tikzcd}$$

Let $p_2 \in \mathcal{P}(2)$. We have the following equalities:

$$\begin{aligned} \psi_E^F \circ (\beta^{\otimes 2} \otimes_S id)^{-1} \circ ev_{\mathfrak{S}_2}^{-1}(\overline{p_2}) &= \psi_E^F(\overline{(1_\mathcal{P}, \overline{0})} \otimes \overline{(1_\mathcal{P}, \overline{0})} \otimes_S p_2) \\ &= \lambda_2^{\mathcal{F}_\mathcal{P}}\big((1_\mathcal{P}, \overline{0}) \otimes (1_\mathcal{P}, \overline{0}) \otimes p_2\big) \\ &= \big(0, \overline{\gamma_{1,1;2}(1_\mathcal{P} \otimes 1_\mathcal{P} \otimes p_2)}\big) \\ &= (0, \overline{p_2}) \\ &= i_2(\overline{p_2}) \end{aligned}$$

This proves that the above diagram commutes.

2. *Computation of $cr_2(\psi^F)_{E,E}$.* We verify that the following diagram commutes:

$$\begin{tikzcd}
T_1F(E)^{\otimes 2}\otimes_{R\otimes R}\mathcal{P}(2) \arrow[rr, "cr_2(\psi^F)_{E,E}"] \arrow[d, "\beta^{\otimes 2}\otimes_{R\otimes R} id"', "\cong"] & & F(E|E) \arrow[d, "\cong"', "\alpha^{-1}"] \\
(\mathcal{P}(1)\otimes\mathcal{P}(1))\otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)}\mathcal{P}(2) \arrow[d, "ev_2"', "\cong"] & & Id_{Alg-\mathcal{P}}(F(E)|F(E)) \arrow[d, "\cong"', "ev_2"] \\
\mathcal{P}(2) \arrow[rr, equal] & & \mathcal{P}(2)
\end{tikzcd}$$

Let $p_2\in\mathcal{P}(2)$, then we have

$$\begin{aligned}
&\iota_2^F\circ cr_2(\psi^F)_{E,E}\big(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2\big)\\
&=\psi^F_{E+2}\circ\iota_2^{T_1F^{\otimes 2}\otimes_S\mathcal{P}(2)}\big(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2\big)\\
&=\psi^F_{E+2}\big(\overline{F(i_1^2)(1_{\mathcal{P}},\overline{0})}\otimes\overline{F(i_2^2)(1_{\mathcal{P}},\overline{0})}\otimes_S p_2\big)\\
&=\lambda_2^F\big(F(i_1^2)(1_{\mathcal{P}},\overline{0})\otimes F(i_2^2)(1_{\mathcal{P}},\overline{0})\otimes p_2\big)\\
&=(F(i_1^2),\ F(i_2^2))\big(0,\ 0,\ \overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2\big)\text{, by 1.8.5}\\
&=(F(i_1^2),\ F(i_2^2))\circ\iota_2^{Id}(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2)\text{, by 1.8.6}\\
&=\iota_2^F\circ\alpha\big(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2\big)\text{, by definition of }\alpha
\end{aligned}$$

As ι_2^F is a monomorphism, we have the following relation:

$$cr_2(\psi^F)_{E,E}(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2)=\alpha(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2)$$

Therefore we have

$$\begin{aligned}
&ev_2\circ\alpha^{-1}\circ cr_2(\psi^F)_{E,E}\circ(\beta^{\otimes 2}\otimes_{R\otimes R}id)^{-1}\circ ev^{-1}(p_2)\\
&=ev_2\circ\alpha^{-1}\circ cr_2(\psi^F)_{E,E}\big(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2\big)\\
&=ev_2\big(\overline{(1_{\mathcal{P}},\overline{0})}\otimes\overline{(1_{\mathcal{P}},\overline{0})}\otimes_{R\otimes R}p_2\big)\\
&=p_2
\end{aligned}$$

This proves that we have

$$ev_2\circ\alpha^{-1}\circ cr_2(\psi^F)_{E,E}=ev_2\circ(\beta^{\otimes 2}\otimes_{R\otimes R}id)$$

as desired.

3. *Definition of $\widehat{H}^F$.* We define $\widehat{H}^F: T_{11}(cr_2U_E)(E,E)\otimes_\Lambda\mathcal{F}_{\mathcal{P}}\to\mathcal{P}(2)$ by the following composite

of morphisms:

$$\begin{tikzcd}
T_{11}(cr_2U_E)(E,E)\otimes_\Lambda \mathcal{F}_{\mathcal{P}} \arrow[rr, "\widehat{H}^F"] \arrow[dd, equal] & & \mathcal{P}(2) \\
& & Id_{Alg-\mathcal{P}}(F(E)|F(E)) \arrow[u, "\cong"', "ev_2"] \\
T_{11}(cr_2U_E)(E,E)\otimes_\Lambda F(E) \arrow[rr, "\widehat{H}^F_E"] & & F(E|E) \arrow[u, "\cong"', "\alpha^{-1}"]
\end{tikzcd}$$

By using appropriated isomorphisms, it is easy to check that

$$\widehat{H}^F\big(t_{11}(\rho_2(\xi))\otimes(0,\,\overline{p_2})\big)=t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).(p_2+p_2.t)$$

where $\xi\in\mathcal{C}(E,\,E+E)$ and $p_2\in\mathcal{P}(2)$. This proves that the left-hand diagram in the statement commutes. As $H^F_E:T_{11}(cr_2U_E)(E,E)\otimes_\Lambda F(E)\to F(E|E)$ verifies $(QM1)$ in 2.1.1, the morphism $\widehat{H}^F:T_{11}(cr_2U_E)(E,E)\otimes_\Lambda\mathcal{F}_{\mathcal{P}}\to\mathcal{P}(2)$ satisfies the following relation:

$$(\nabla^2_E\circ\xi).(p_1,\,\overline{p_2})=(r_1^2\circ\xi).(p_1,\,\overline{p_2})+(r_2^2\circ\xi).(p_1,\,\overline{p_2})+\big(\dot{q}\circ\widehat{H}^F\big)\big(t_{11}(\rho_2)(\xi)\otimes_\Lambda(p_1,\,\overline{p_2})\big)$$

where $\xi\in\mathcal{C}(E,\,E+E)$ and $(p_1,\,\overline{p_2})\in\mathcal{F}_{\mathcal{P}}$.

$\square$

We point out that $\widehat{H}^F:T_{11}(cr_2U_E)(E,E)\otimes_\Lambda\mathcal{F}_{\mathcal{P}}\to\mathcal{P}(2)$, given in 3.6.9, has no explicit expression comparing with the other maps in the diagram of 3.6.9. Then taking a quadratic equivalence $F:\langle E\rangle_{\mathcal{C}}\to Alg-\mathcal{P}$ with domain an algebraic theory $\langle E\rangle$ generated by E and values in $Alg-\mathcal{P}$ amounts to taking an appropriate explicit expression of $\widehat{H}^F:T_{11}(cr_2U_E)(E,E)\otimes_\Lambda\mathcal{F}_{\mathcal{P}}\to\mathcal{P}(2)$.

Chapter 4

Lazard correspondence for 2-step nilpotent varieties

In this chapter, we aim at finding the Lazard correspondence between any 2-radicable 2-step nilpotent variety and the category of algebras over a 2-step nilpotent linear symmetric unitary operad depending on the variety. In the final chapter this equivalence of categories will then provide the BCH formula for arbitrary operations in the variety.

Notation 4.0.1. Recall the following notations:

- We denote by E the free object of rank 1 in $\mathcal{C}$, and $\langle E\rangle$ the algebraic theory generated by E (as in 1.1.1) representing the full subcategory formed by free objects of finite rank in $\mathcal{C}$.

- We denote by $ev_e : \mathcal{C}(E,\, X) \to |X|,\, f \mapsto f(e)$ the canonical bijection. Given $x \in |X|$, we write $\hat{x} = ev_e^{-1}(x)$.

Then the object E^{ab} is the distinguished free object of rank 1 in the abelian core $Ab(\mathcal{C})$ whose basis element is $\overline{e} = ab_E(e)$.

Notation 4.0.2. We also consider the following notations:

- We denote by $\langle E^{ab}\rangle$ the theory (as in 1.1.1) generated by E^{ab} representing the full subcategory of free abelian objects of finite rank in the abelian core $Ab(\mathcal{C})$.

- We consider $ev_{\overline{e}} : \mathcal{C}(E^{ab},\, A) \to |A|,\, g \mapsto g(\overline{e})$ the canonical bijection, where A is an abelian object in $\mathcal{C}$. Given $a \in |A|$, we write $\tilde{a} = ev_{\overline{e}}^{-1}(a)$. Let X be any object in $\mathcal{C}$ and $x \in |X|$, we write $\overline{x} = ab_X(x)$.

- If $f : X \to Y$ is any morphism in $\mathcal{C}$, we denote $\overline{f} = |f^{ab}|$ and we clearly have $\overline{f}(\overline{x}) = \overline{f(x)}$.

Now we point out the following important property of the variety $\mathcal{C}$ (in fact of any variety).

Remark 4.0.3. The free object E of rank 1 of $\mathcal{C}$ is a small regular-projective generator.

We recall that *small* means that the functor $\mathcal{C}(E,-) : \mathcal{C} \to Set_*$ preserves filtered colimits, *regular-projective* means that E is projective with respect to the class of all regular epimorphisms, and *generator* means that any object X in $\mathcal{C}$ is a colimit of copies of E, or equivalently, admits a regular epimorphism $+_{i\in I}E \to X$, where I is a set. The property in 4.0.3 permits us to use the next proposition about certain natural transformations (holding here for merely Mal'cev and Barr exact categories) that has been already given in 6.25 of [12] as follows:

Proposition 4.0.4. *Let $\mathcal{D}$ be any category. Let $\varphi : F \Rightarrow G$ be any natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$ preserving both filtered colimits and coequalizers of reflexive pairs. Then φ is an isomorphism if, and only if, $\varphi_{E^{+n}}$ is an isomorphism for all $n \geq 1$. Similarly let $\psi : B \Rightarrow D$ be any natural transformation between bifunctors $B, D : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ preserving both filtered colimits and coequalizers of reflexive pairs. Then ψ is an isomorphism if, and only if, $\psi_{E^{+n}, E^{+m}}$ is an isomorphism for all $n, m \geq 1$.*

4.1 2-step nilpotent varieties

Assumption: Throughout this section we assume that $\mathcal{C}$ is a 2-step nilpotent variety (see 1.3.10).

We here establish certain specific properties of these varieties which are needed to construct the quadratic $\mathcal{C}$-module which gives rise to the Lazard correspondence. They could basically be deduced from the theory of square ringoids in [4] using the framework of linear extensions of categories which in section 2 was made available in our context; however, as several important formulas in that paper are wrong and many proofs not explicitely developed we give a (mostly) independent treatment here.

In particular, we show that there exists a (non-unique) 2-step nilpotent group structure among the operations in $\mathcal{C}$, denoted by $+$. Thus for any object X in $\mathcal{C}$ its underlying set $|X|$ has a natural 2-step nilpotent group structure. If X is an abelian object, this structure is abelian and coincides with the natural one. Moreover, we study the compatibility between the induced group structure on morphism sets between free objects and the composition operation.

Assumption: In this subsection, we consider X an object in $\langle E \rangle$, the full subcategory of free objects of finite rank in $\mathcal{C}$, and Y, Z any objects in $\mathcal{C}$.

First we recall the definition of a cogroup:

Definition 4.1.1. Let $\mathcal{D}$ be a pointed category having finite coproducts (also denoted by $+$). A cogroup in $\mathcal{D}$ is a triplet (Z, μ, v) such that Z is an object in $\mathcal{D}$, and $\mu : Z \to Z + Z$ and $v : Z \to Z$ are morphisms in $\mathcal{D}$ satisfying the following properties:

- the *counity property*: $(0 + id) \circ \mu = (id + 0) \circ \mu = id$, where $0 : Z \to 0$ is the zero morphism;
- the *coassociativity property*: $(\mu + id) \circ \mu = (id + \mu) \circ \mu$;
- the *coinverse property*: $(v + id) \circ \mu = 0 = (id + v) \circ \mu$, where here $0 : Z \to Z$ is the zero morphism.

As X is a regular-projective object in $\mathcal{C}$ (because it is a finite coproduct of copies of E that is a regular-projective object), there is a morphism $\mu_X : X \to X + X$ in $\mathcal{C}$ such that

$$\widehat{r_2^{Id_{\mathcal{C}}}} \circ \mu_X = \Delta_X^2 \tag{4.1.1}$$

because the comparison morphism $\widehat{r_2^{Id_{\mathcal{C}}}} : X + X \to X \times X$ (see (1.2.1)) is a regular epimorphism, where $\Delta_X^2 : X \rightarrowtail X \times X$ is the morphism given in 1.0.1. Then we verify that the morphism $\mu_X : X \to X + X$ satisfies the *counit property* as in 4.1.1, as follows:

$$\begin{aligned} r_X \circ (0 + id) \circ \mu_X &= \pi_2^2 \circ \widehat{r_2^{Id_{\mathcal{C}}}} \circ (0 + id) \circ \mu_X \text{, by (1.2.1)} \\ &= \pi_2^2 \circ (0 \times id) \circ \widehat{r_2^{Id_{\mathcal{C}}}} \circ \mu_X \text{, by naturality of } \widehat{r_2^{Id_{\mathcal{C}}}} \\ &= id \circ \pi_2^2 \circ \Delta_X^2 \text{, by (4.1.1)} \\ &= id \end{aligned}$$

where $r_X : 0+X \to X$ is the canonical isomorphism (retraction) and $\pi_1^2 : 0\times X \to X$ is the projection onto the second summand (that is also an isomorphism). Similarly we also have $r_X \circ (id+0) \circ \mu_X = id$, as desired.
Hence we use the Lemma 6.4 of [4] saying that a morphism with domain an object X and with target object $X + X$, that satisfies the *counit property* as in 4.1.1, provides a structure of cogroup on X. Let X_1, X_2, X_3 be objects in $\mathcal{C}$. For this, we point out that the authors only use the injectivity of the comparison morphism

$$\widehat{r_3}^{Id_{\mathcal{C}}} : X_1 + X_2 + X_3 \to (X_1 + X_2) \times (X_1 + X_3) \times (X_2 + X_3)\,, \text{see } (1.2.1)$$

for the *coassociativity property* as in 4.1.1. By 1.2.1, the kernel of the comparison morphism $\widehat{r_3}$ is $Id_{\mathcal{C}}(X_1|X_2|X_3)$, the third cross-effect of the identity functor of $\mathcal{C}$. As the category $\mathcal{C}$ is supposed to be 2-step nilpotent (see 1.3.10), it follows that

$$Id_{\mathcal{C}}(X_1|X_2|X_3) = 0$$

Hence it permits us to use this lemma. By 6.4 of [4], there is a morphism $v_X : X \to X$ in $\langle E\rangle$ such that (X, μ_X, v_X) is a cogroup in $\mathcal{C}$ with μ_X being its comultiplication. By 6.6 of [4], it yields a group structure (written additively) on the set $\mathcal{C}(X,\, Y)$ given by

$$\forall f, g \in \mathcal{C}(X,\, Y)\,, \quad f + g = (f,\, g) \circ \mu_X \tag{4.1.2}$$

whose neutral element is the zero morphism $0 : X \to Y$ and, for each morphism $f \in \mathcal{C}(X, Y)$, the inverse of f, denoted by f^{-1}, is given by $f^{-1} = f \circ v_X$. In the case where $Y = X + X$, $f = i_1^2$ and $g = i_2^2$, we get

$$\mu_X = i_1^2 + i_2^2 \tag{4.1.3}$$

Notation 4.1.2. We write $2_X = id + id$, or simply 2 for the case $X = E$.

The upshot of these considerations is the following result.

Proposition 4.1.3. *Every object X of $\langle E\rangle$ admits a (non unique) cogroup structure (X, μ_X, ν_X) which we choose once and for all. Then the representable functor $\mathcal{C}(X, -) : \mathcal{C} \to Set$ takes its values in Gr. In particular, $\mathcal{C}$ is a variety of ω-groups where a group law on any object Y (depending on the choice of μ_E) is defined by $x + y = |(\hat{x}, \hat{y}) \circ \mu_E|(e)$ for $x, y \in |Y|$.*

The following proposition says that the group structure given in (4.1.2) is left distributive:

Proposition 4.1.4. *Let Y' be an object in $\mathcal{C}$, $f_1, f_2 \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Y')$. Then we have*

$$g \circ \big(f_1 + f_2\big) = g \circ f_1 + g \circ f_2$$

Proof. We have the equalities as follows:

$$g \circ \big(f_1 + f_2\big) = g \circ (f_1, f_2) \circ \mu_X = (g \circ f_1, g \circ f_2) \circ \mu_X = g \circ f_1 + g \circ f_2\,,$$

as desired. □

Now for an abelian object Z of $\mathcal{C}$ the internal binary operation $m_Z : Z \times Z \to Z$ on Z (see 1.5.14) provides an abelian group structure on $\mathcal{C}(X,\, Z)$ given by

$$\forall f, g \in \mathcal{C}(X,\, Z), \quad f \bullet g = m_Z \circ (f,\, g)^t \tag{4.1.4}$$

whose neutral element is the zero morphism $0 : X \to Z$. Then we observe that $\mathcal{C}(X,\, Z)$ has two group structures. However we have the following proposition:

Proposition 4.1.5. *Let Z be an abelian object of $\mathcal{C}$. Then the two group structures on $\mathcal{C}(X,\ Z)$ given in* (4.1.2) *and* (4.1.4) *coincide.*

Proof. Let $f, g \in \mathcal{C}(X, Z)$. Then we have

$$\begin{aligned} f+g &= (f,g)\circ\mu_X\,, \text{by (4.1.2)}\\ &= \nabla_Z^2\circ(f+g)\circ\mu_X\\ &= m_Z\circ\widehat{r_2^{Id_{\mathcal{C}}}}\circ(f+g)\circ\mu_X\,, \text{by 1.5.14}\end{aligned}$$

where $\widehat{r_2^{Id_{\mathcal{C}}}} : Z+Z \to Z\times Z$ is the comparison morphism given in (1.2.1). Then we have

$$\begin{aligned} f+g &= m_Z\circ(f\times g)\circ\widehat{r_2^{Id_{\mathcal{C}}}}\circ\mu_X\,, \text{by naturality of } \widehat{r_2^{Id_{\mathcal{C}}}}\\ &= m_Z\circ(f\times g)\circ\Delta_X^2\,, \text{by (4.1.1)}\\ &= m_Z\circ(f,g)^t\\ &= f\bullet g\,, \text{by 4.1.4}\end{aligned}$$

as desired. □

Now we show that the abelian group $\mathcal{C}(X, Z)$ has an additional (right) module structure. For this, we need the following remark:

Remark 4.1.6. Let Z be an abelian object in $\mathcal{C}$. We remark that the abelian group $\mathcal{C}(X^{ab}, Z)$ is a (right) $\mathcal{C}(X^{ab}, X^{ab})$-module whose action is given by the precomposition of elements in the endomorphism ring of X^{ab}. Hence it provides a (right) $\mathcal{C}(X^{ab}, X^{ab})$-module structure on $\mathcal{C}(X, Z)$, as follows:

$$f.\alpha = (ab_X)^*\Big(((ab_X)^*)^{-1}(f)\circ\alpha\Big)$$

for $\alpha \in \mathcal{C}(X^{ab}, X^{ab})$ and $f \in \mathcal{C}(X, Z)$.

The linear functors $Id_{\mathcal{C}}(-|Y)$ and $Id_{\mathcal{C}}(Y|-) : \mathcal{C} \to Ab(\mathcal{C})$ (cf. Definition 1.2.5 and Proposition 1.4.2) are "additive" on $\langle E\rangle$, as follows:

Proposition 4.1.7. *Consider an object X' in $\langle E\rangle$ and an object Y' in $\mathcal{C}$. Let $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(X', Y')$. Then the morphism $Id_{\mathcal{C}}(f|g) : Id_{\mathcal{C}}(X|X') \to Id_{\mathcal{C}}(Y|Y')$ is linear in f and g in the sense of [4], i.e. we have*

$$\begin{cases} Id_{\mathcal{C}}(f_1+f_2\mid g) = Id_{\mathcal{C}}(f_1|g) + Id_{\mathcal{C}}(f_2|g)\\ Id_{\mathcal{C}}(f\mid g_1+g_2) = Id_{\mathcal{C}}(f|g_1) + Id_{\mathcal{C}}(f|g_2)\end{cases}$$

where $f_1, f_2 \in \mathcal{C}(X, Y)$ and $g_1, g_2 \in \mathcal{C}(X', Y')$.

Proof. Let $f_1, f_2 \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(X', Y')$. Then we have

$$\begin{aligned} Id_{\mathcal{C}}\big(f_1+f_2\mid g\big) &= Id_{\mathcal{C}}\big((f_1,f_2)\circ\mu_X\mid g\big)\,, \text{by (4.1.2)}\\ &= Id_{\mathcal{C}}\big((f_1,f_2)\mid g\big)\circ Id_{\mathcal{C}}\big(\mu_X\mid g\big)\end{aligned}$$

As for any object Z in $\mathcal{C}$, the functor $Id_{\mathcal{C}}(-|Z) : \mathcal{C} \to \mathcal{C}$ is linear in the sense of 1.2.5. By 3.6 of [12], we have

$$Id_{\mathcal{C}}(id|id) = Id_{\mathcal{C}}\big(i_1^2\circ r_1^2|\ id\big) + Id_{\mathcal{C}}\big(i_2^2\circ r_2^2\mid id\big)$$

Hence we have

$$Id_{\mathcal{C}}\big(f_1 + f_2 \mid g\big) = Id_{\mathcal{C}}\big((f_1, f_2) \mid g\big) \circ \Big(Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 | \, id\big) + Id_{\mathcal{C}}\big(i_2^2 \circ r_2^2 \mid id\big)\Big) \circ Id_{\mathcal{C}}\big(\mu_X \mid g\big)$$
$$= Id_{\mathcal{C}}\big((f_1, f_2) \circ i_1^2 \circ r_1^2 \circ \mu_X \mid g\big) + Id_{\mathcal{C}}\big((f_1, f_2) \circ i_2^2 \circ r_2^2 \circ \mu_X \mid g\big)$$

By (4.1.1), we get $r_k^2 \circ \mu_X = id$, for $k = 1, 2$. Then it implies that we have

$$Id_{\mathcal{C}}\big(f_1 + f_2 \mid g\big) = Id_{\mathcal{C}}\big(f_1 \mid g\big) + Id_{\mathcal{C}}\big(f_2 \mid g\big)\,,$$

as desired. Similarly $Id_{\mathcal{C}}(f|g)$ is linear in g. $\square$

As the representable functor $\mathcal{C}(X, -) : \mathcal{C} \to Gr$ is exact and preserves finite products, the second cross-effect of $\mathcal{C}(X, -) : \mathcal{C} \to Gr$ is given by

$$cr_2\big(\mathcal{C}(X, -)\big)(Y, Z) = \mathcal{C}\big(X,\ Id_{\mathcal{C}}(Y|Z)\big) \tag{4.1.5}$$

where $(\iota_2^{Id_{\mathcal{C}}})_* : \mathcal{C}\big(X, Id_{\mathcal{C}}(Y|Z)\big) \rightarrowtail \mathcal{C}\big(X, Y + Z\big)$ is the kernel of the comparison morphism $\widehat{r_2^{\mathcal{C}(X,-)}}$ (see (1.2.1)). Note that $Id_{\mathcal{C}}(Y|Z)$ is abelian by Lemma 1.4.2, hence the bifunctor $cr_2\big(\mathcal{C}(X, -)\big)$: $\mathcal{C}^{\times 2} \to Gr$ takes in fact values in Ab by 4.1.4.

Notation 4.1.8. The second cross-effect of the representable functor $\mathcal{C}(X, -) : \mathcal{C} \to Gr$ is denoted by $\mathcal{C}(X, -|-) : \mathcal{C} \times \mathcal{C} \to Ab$. More precisely, we have

- *On objects*, for two objects Y and Z in $\mathcal{C}$, $\mathcal{C}(X, Y|Z) = \mathcal{C}\big(X, Id_{\mathcal{C}}(Y|Z)\big)$.
- *On morphisms*, let $f : Y \to Y'$ and $g : Z \to Z'$ be two morphisms in $\mathcal{C}$, then $\mathcal{C}(X, f|g) = Id_{\mathcal{C}}(f|g)_*$.

Notation 4.1.9. The abelian group $\mathcal{C}(X, Y|Z) = \mathcal{C}\big(X,\ Id_{\mathcal{C}}(Y|Z)\big)$ is equipped with the involution $T_{X,Y,Z}$ where we write $T_{X,Y,Z} = (T_{Y,Z})_* : \mathcal{C}(X, Y|Z) \to \mathcal{C}(X, Y|Z)$, with $T_{Y,Z} : Id_{\mathcal{C}}(Y|Z) \to Id_{\mathcal{C}}(Z|Y)$ being the restriction of the canonical switch $\tau^2_{Y,Z} : Y + Z \to Z + Y$ to $Id_{\mathcal{C}}(Y|Z)$. It clearly satisfies $T_{X,Y,Z} \circ T_{X,Z,Y} = id$. In the case where $X = Y = Z = E$, we write $T_{X,Y,Z} = T$.

Notation 4.1.10. We denote by $\big(\mathcal{C}\big(X,\ Id_{\mathcal{C}}(Y|Y)\big)\big)_{\mathfrak{S}_2}$ the abelian group of coinvariants and by π : $\mathcal{C}\big(X,\ Id_{\mathcal{C}}(Y|Y)\big) \to \big(\mathcal{C}\big(X,\ Id_{\mathcal{C}}(Y|Y)\big)\big)_{\mathfrak{S}_2}$ the canonical quotient map.

Since the representable functor $\mathcal{C}(X, -) : \mathcal{C} \to Gr$ preserves finite products and the bifunctor $Id_{\mathcal{C}}(-|-) : \mathcal{C}^{\times 2} \to \mathcal{C}$ is bilinear (by 1.3.10 since $\mathcal{C}$ is supposed to be 2-step nilpotent), we have the following remark:

Remark 4.1.11. The second cross-effect of the representable functor $\mathcal{C}(X, -) : \mathcal{C} \to Gr$ is a bilinear bifunctor (see 1.2.12). It implies that the functor $\mathcal{C}(X, -) : \mathcal{C} \to Gr$ is quadratic by 1.2.13. Hence the representable functor $\mathcal{C}(X, -) : \mathcal{C} \to Gr$ takes values in $Nil_2(Gr)$, i.e. the full subcategory of Gr formed by 2-step nilpotent groups by 1.4.1.

By 3.21 of [12], we get a natural transformation $u'_{\mathcal{C}(E, Id_{\mathcal{C}}(-|-))} : T_1U_E \otimes T_1U_E \otimes_{\Lambda \otimes \Lambda} \mathcal{C}(E, Id_{\mathcal{C}}(E|E)) \Rightarrow \mathcal{C}(E, Id_{\mathcal{C}}(-|-))$ between bifunctors defined by

$$(u'_{\mathcal{C}(E, Id_{\mathcal{C}}(-|-))})_{X,Y}\big(t_1(f_1) \otimes t_1(f_2) \otimes h\big) = Id_{\mathcal{C}}(f_1|f_2) \circ h \tag{4.1.6}$$

By 4.1.11 and 3.22 of [12], $u'_{\mathcal{C}(E, Id_{\mathcal{C}}(-|-))}$ restricted to $\langle E \rangle \times \langle E \rangle$ is an isomorphism. In addition, for all X_1 and X_2 objects in $\mathcal{C}$, $Id_{\mathcal{C}}(X_1|X_2)$ is an abelian object in $\mathcal{C}$ implying that $(ab_E)^* : \mathcal{C}(E^{ab}, Id_{\mathcal{C}}(E|E)) \to \mathcal{C}(E, Id_{\mathcal{C}}(E|E))$ is an isomorphism by 1.5.16. Hence we get the natural isomorphism $u'_{\mathcal{C}(E^{ab}, Id_{\mathcal{C}}(-|-))} : T_1U_E \otimes T_1U_E \otimes_{\Lambda \otimes \Lambda} \mathcal{C}(E^{ab}, Id_{\mathcal{C}}(E|E)) \Rightarrow \mathcal{C}(E^{ab}, Id_{\mathcal{C}}(-|-))$ between bifunctors defined by

$$(u_{\mathcal{C}(E^{ab}, Id_{\mathcal{C}}(-|-))'})_{X,Y} = ((ab_E)^*)^{-1} \circ (u'_{\mathcal{C}(E, Id_{\mathcal{C}}(-|-))})_{X,Y} \circ \big(id \otimes id \otimes_{\Lambda \otimes \Lambda} (ab_E)^*\big) \tag{4.1.7}$$

where X and Y objects in $\mathcal{C}$.

Notation 4.1.12. For an object X in $\mathcal{C}$, we denote by

$$\left(\overline{u'_{\mathcal{C}(E,Id_{\mathcal{C}}(-|-))}}\right)_X : \left(T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)\right)_{\mathfrak{S}_2} \to \mathcal{C}(E, Id_{\mathcal{C}}(X|X))_{\mathfrak{S}_2}$$

the canonical factorization between the sets of coinvariants.

Remark 4.1.13. The map $\left(\overline{u'_{\mathcal{C}(E,Id_{\mathcal{C}}(-|-))}}\right)_X$, given in 4.1.12, is a $\mathcal{P}(1)$-module homomorphism which is an isomorphism by the five lemma applied to an appropriated diagram.

Now we recall that $i_Y : [Y,Y]_{Id_{\mathcal{C}}} \rightarrowtail Y$ is the image of the morphism $c_2^Y = \nabla_Y^2 \circ \iota_2^{Id} : Id_{\mathcal{C}}(Y|Y) \to Y$ (see 1.2.8) and $ab_Y : Y \to Y^{ab}$ is its cokernel. Then we need the following technical lemma:

Lemma 4.1.14. *The morphisms $Id_{\mathcal{C}}(ab_Y|id) : Id_{\mathcal{C}}(Y|Z) \to Id_{\mathcal{C}}(Y^{ab}|Z)$ and $Id_{\mathcal{C}}(id|ab_Z) : Id_{\mathcal{C}}(Y|Z) \to Id_{\mathcal{C}}(Y|Z^{ab})$ are isomorphisms.*

Proof. By 2.26 of [14], the functor $Id_{\mathcal{C}}(-|Z) : \mathcal{C} \to \mathcal{C}$ preserves coequalizers of reflexive graphs. Hence the claim follows from Proposition 1.4.8 □

Then the following proposition says that the morphism $(c_2^Y)_* : \mathcal{C}(X, Id_{\mathcal{C}}(Y|Y)) \to \mathcal{C}(X,Y)$ maps to the center of the group $\mathcal{C}(X,Y)$.

Proposition 4.1.15. *Let Y be an object in $\mathcal{C}$, $f \in \mathcal{C}(X,Y)$ and $\xi \in \mathcal{C}(X,Y|Y)$. Then we have*

$$f + c_2^Y \circ \xi = c_2^Y \circ \xi + f$$

Proof. First we observe that we get

$$Id_{\mathcal{C}}(id|ab_Y) \circ Id_{\mathcal{C}}(f|c_2^Y \circ \xi) = Id_{\mathcal{C}}(f|ab_Y \circ c_2^Y \circ \xi) = 0$$

implying that we have $Id_{\mathcal{C}}(f|c_2^Y \circ \xi) = 0$ because $ab_Y : Y \to Y^{ab}$ is the cokernel of $c_2^Y : Id_{\mathcal{C}}(Y|Y) \to Y$ and $Id_{\mathcal{C}}(Y|ab_Y) : Id_{\mathcal{C}}(Y|Y) \to Id_{\mathcal{C}}(Y|Y^{ab})$ is an isomorphism by 4.1.14. Hence we have

$$(f, c_2^Y \circ \xi) \circ \iota_2^{Id_{\mathcal{C}}} = c_2^Y \circ Id_{\mathcal{C}}(f|c_2^Y \circ \xi) = 0$$

As the comparison morphism $\widehat{r_2^{Id_{\mathcal{C}}}}$ is the cokernel of $\iota_2^{Id_{\mathcal{C}}} : Id_{\mathcal{C}}(X|X) \rightarrowtail X + X$ (see 1.2.3 and 1.2.1), there is a unique factorization $\phi_1 : X \times X \to Y$ of $(f, c_2^Y \circ \xi) : X + X \to Y$ though $\widehat{r_2^{Id_{\mathcal{C}}}} : X + X \to X \times X$, i.e.

$$(f, c_2^Y \circ \xi) = \phi_1 \circ \widehat{r_2^{Id_{\mathcal{C}}}} \tag{4.1.8}$$

Similarly there is a unique $\phi_2 : X \times X \to Y$ such that

$$(c_2^Y \circ \xi, f) = \phi_2 \circ \widehat{r_2^{Id_{\mathcal{C}}}} \tag{4.1.9}$$

Next we have the equalities as follows:

$$\begin{aligned} f + c_2^Y \circ \xi &= (f, c_2^Y \circ \xi) \circ \mu_X \text{ , by (4.1.2)} \\ &= \phi_1 \circ \widehat{r_2^{Id_{\mathcal{C}}}} \circ \mu_X \\ &= \phi_1 \circ \Delta_X^2 \text{ , by (4.1.1)} \\ &= \phi_1 \circ T_X^2 \circ \Delta_X^2 \text{ , because } T_X^2 \circ \Delta_X^2 = \Delta_X^2 \end{aligned}$$

where $T_X^2 : X \times X \to X \times X$ is the canonical switch and $\Delta_X^2 : X \rightarrowtail X \times X$ is the diagonal morphism (see 1.0.1). However we have

$$\begin{aligned}\phi_1 \circ T_X^2 \circ \widehat{r_2^{Id_\mathcal{C}}} &= \phi_1 \circ \widehat{r_2^{Id_\mathcal{C}}} \circ \tau_X^2 \text{, by naturality of } \widehat{r_2^{Id_\mathcal{C}}} \\ &= (f, c_2^Y \circ \xi) \circ \tau_X^2 \text{, by (4.1.8)} \\ &= (c_2^Y \circ \xi, f) \\ &= \phi_2 \circ \widehat{r_2^{Id_\mathcal{C}}} \text{, by (4.1)}\end{aligned}$$

where $\tau_X^2 : X + X \to X + X$ is the canonical switch given in 1.0.1. As $\widehat{r_2^{Id_\mathcal{C}}} : X + X \to X \times X$ is an epimorphism, we have

$$\phi_1 \circ T_X^2 = \phi_2 \tag{4.1.10}$$

Hence we get

$$\begin{aligned}f + c_2^Y \circ \xi &= \phi_1 \circ T_X^2 \circ \Delta_X^2 \\ &= \phi_2 \circ \Delta_X^2 \text{, by (4.1.10)} \\ &= (c_2^Y \circ \xi, f) \circ \widehat{r_2^{Id_\mathcal{C}}} \circ \Delta_X^2 \text{, by (4.1)} \\ &= (c_2^Y \circ \xi, f) \circ \mu_X \text{, by (4.1.1)} \\ &= c_2^Y \circ \xi + f \text{, by 4.1.2}\end{aligned}$$

□

Remark 4.1.16. From 4.1.15, we deduce that, for each morphism $\xi \in \mathcal{C}(X, Id_\mathcal{C}(Y|Z))$, the morphism $\iota_2^{Id_\mathcal{C}} \circ \xi \in \mathcal{C}(X, Y + Z)$ belongs to the center of the group $\mathcal{C}(X, Y + Z)$. Because we have

$$\iota_2^{Id_\mathcal{C}} \circ \xi = (i_1^2, i_2^2) \circ \iota_2^{Id_\mathcal{C}} \circ \xi = \nabla_{Y+Z}^2 \circ (i_1^2 + i_2^2) \circ \iota_2^{Id_\mathcal{C}} \circ \xi = \nabla_{Y+Z}^2 \circ \iota_2^{Id_\mathcal{C}} \circ Id_\mathcal{C}(i_1^2|i_2^2) \circ \xi = c_2^{Y+Z} \circ Id_\mathcal{C}(i_1^2|i_2^2) \circ \xi$$

The set $\mathcal{C}(X, [Y, Y]_{Id_\mathcal{C}})$ may be seen as a subgroup of the group $\mathcal{C}(X, Y)$ whose inclusion map is the injection map $(i_Y)_* : \mathcal{C}(X, [Y, Y]_{Id_\mathcal{C}}) \to \mathcal{C}(X, Y)$.

Corollary 4.1.17. *The subgroup $\mathcal{C}(X, [Y, Y]_{Id_\mathcal{C}})$ is central in the group $\mathcal{C}(X, Y)$.*

Proof. Let $h \in \mathcal{C}(X, [Y, Y]_{Id_\mathcal{C}})$. As X is a regular-projective object in $\mathcal{C}$ (as a finite coproduct of copies of the regular-projective object E in $\mathcal{C}$) and $e_Y : Id_\mathcal{C}(Y|Y) \to [Y, Y]_{Id_\mathcal{C}}$ is a regular epimorphism (because it is the coimage of the morphism c_2^Y), there is a morphism $\hat{h} \in \mathcal{C}(X, Id_\mathcal{C}(Y|Y))$ such that $h = e_X \circ \hat{h}$. Hence we get

$$(i_Y)_*(h) = i_Y \circ h = i_Y \circ e_Y \circ \hat{h} = c_2^Y \circ \hat{h}$$

Then it is a direct consequence of 4.1.15. □

Moreover there is a (set-theoretic) retraction $r_2 : \mathcal{C}(X, Y + Z) \to \mathcal{C}(X, Y|Z)$ of the kernel $\iota_2^{\mathcal{C}(X,-)} = (\iota_2^{Id_\mathcal{C}})_* : \mathcal{C}(X, Y|Z) \to \mathcal{C}(X, Y + Z)$ of the comparison morphism $\widehat{r_2^{\mathcal{C}(X,-)}}$ such that, for $\xi \in \mathcal{C}(X, Y + Z)$, $r_2(\xi) \in \mathcal{C}(X, Y|Z)$ is the unique map satisfying

$$\iota_2^{Id_\mathcal{C}} \circ r_2(\xi) = \xi - \left(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi\right) = \xi - i_2^2 \circ r_2^2 \circ \xi - i_1^2 \circ r_1^2 \circ \xi \tag{4.1.11}$$

Then the map $H_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y|Y)$ (already given in 2.11 of [4]) is defined such that, for $\alpha \in \mathcal{C}(X,Y)$, $H(\alpha) \in \mathcal{C}(X,Y|Y)$ is the unique morphism satisfying

$$\iota_2^{Id_{\mathcal{C}}} \circ H_{X,Y}(\alpha) = (i_1^2 + i_2^2) \circ \alpha - \big(i_1^2 \circ \alpha + i_2^2 \circ \alpha\big) = \iota_2^{Id_{\mathcal{C}}} \circ r_2\big((i_1^2 + i_2^2) \circ \alpha\big) \,, \tag{4.1.12}$$

implying that we get

$$H_{X,Y}(\alpha) = r_2\big((i_1^2 + i_2^2) \circ \alpha\big)$$

where here Y is supposed to be in $\langle E \rangle$.

Notation 4.1.18. For $X = Y$, we write $H_{X,Y} = H_X : \mathcal{C}(X,X) \to \mathcal{C}(X,X|X)$.

In the special case where $X = Y = E$, we consider the map $H : \mathcal{C}(E,E) \to \mathcal{C}(E^{ab}, E|E) = \mathcal{C}(E^{ab}, Id_{\mathcal{C}}(E|E))$ such that, for $\alpha \in \mathcal{C}(E,E)$, we have

$$H(\alpha) = H_E(\alpha)^{ab} \tag{4.1.13}$$

where $H_E(\alpha)^{ab} \in \mathcal{C}(E^{ab}, Id_{\mathcal{C}}(E|E))$ is the unique factorization of $H_E(\alpha) \in \mathcal{C}(E, Id_{\mathcal{C}}(E|E))$ through $ab_E \in \mathcal{C}(E, E^{ab})$ (which exists because the target object of H_E is an abelian object in $\mathcal{C}$, see 1.5.16 and 1.5.17).

Remark 4.1.19. We have the following observations:

1. First we have $\iota_2^{Id_{\mathcal{C}}} \circ H_X(2_X) = [i_2^2,\, i_1^2] = i_2^2 + i_1^2 - i_2^2 - i_1^2$. This is due to the following equalities:

$$\begin{aligned}
\iota_2^{Id_{\mathcal{C}}} \circ H_X(2_X) &= (i_1^2 + i_2^2) \circ 2_X - i_2^2 \circ 2_X - i_1^2 \circ 2_X \,, \text{by (4.1.12)}\\
&= i_1^2 + i_2^2 + i_1^2 + i_2^2 - i_2^2 - i_2^2 - i_1^2 - i_1^2\\
&= i_1^2 + \big(i_2^2 + i_1^2 - i_2^2 - i_1^2\big) - i_1^2\\
&= i_1^2 + [i_2^2,\, i_1^2] - i_1^2
\end{aligned}$$

   Then we observe that the morphism $[i_2^2,\, i_1^2]$ belongs to the kernel of the comparison morphism $\widehat{r_2^{\mathcal{C}(X,-)}}$, see (4.1.5). By 4.1.16, $[i_2^2,\, i_1^2]$ is in the center of the group $\mathcal{C}(X, X+X)$. Hence we have

$$\iota_2^{Id_{\mathcal{C}}} \circ H_X(2_X) = i_1^2 + [i_2^2,\, i_1^2] - i_1^2 = [i_2^2,\, i_1^2]$$

2. Next it is straightforward to see that $T_X \circ H_X(2_X) = -id$, where $T_X : Id_{\mathcal{C}}(X|X) \to Id_{\mathcal{C}}(X|X)$ is the restriction of the canonical switch $\tau_X^2 : X + X \to X + X$ to $Id_{\mathcal{C}}(X|X)$.

Now we determine the deviation of the group structure given in 4.1.2 to be commutative:

Proposition 4.1.20. *Let $f, g \in \mathcal{C}(X,Y)$. Then we have*

$$g + f = f + g + c_2^Y \circ Id_{\mathcal{C}}(f|g) \circ H_X(2_X)$$

Proof. We have the following equalities:

$$\begin{aligned}
g + f &= (g,f) \circ \mu_X\\
&= (f,g) \circ \tau_X^2 \circ (i_1^2 + i_2^2) \,, \text{by 4.1.3}\\
&= (f,g) \circ \big(\tau_X^2 \circ i_1^2 + \tau_X^2 \circ i_2^2\big) \,, \text{by 4.1.4}\\
&= (f,g) \circ (i_2^2 + i_1^2)\\
&= (f,g) \circ \big(i_1^2 + i_2^2 + \iota_2^{Id_{\mathcal{C}}} \circ H_X(2_X)\big) \,, \text{by 4.1.19 and 4.1.16}\\
&= (f,g) \circ i_1^2 + (f,g) \circ i_2^2 + (f,g) \circ \iota_2^{Id_{\mathcal{C}}} \circ H_X(2_X) \,, \text{by 4.1.4}\\
&= f + g + (f,g) \circ \iota_2^{Id_{\mathcal{C}}} \circ H_X(2_X)
\end{aligned}$$

In addition, we have

$$(f,g)\circ \iota_2^{Id_{\mathcal{C}}} = \nabla_Y^2 \circ (f+g)\circ \iota_2^{Id_{\mathcal{C}}} = \nabla_Y^2 \circ \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(f|g) = c_2^Y \circ Id_{\mathcal{C}}(f|g)\,, \text{by } 1.2.8$$

Hence we get

$$g+f = f+g+(f,g)\circ \iota_2^{Id_{\mathcal{C}}} \circ H_X(2_X) = f+g+c_2^Y \circ Id_{\mathcal{C}}(f|g)\circ H_X(2_X)$$

□

The map $H_{XY} : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y|Y)$ defined in (4.1.12) is not a homomorphism of groups in general, where Y is here supposed to be in $\langle E\rangle$. The next proposition gives the deviation of the map $H_{X,Y}$ to be a homomorphism of groups.

Proposition 4.1.21. *We consider an object X' in $\langle E\rangle$. Let $f,g \in \mathcal{C}(X,X')$. Then we have*

$$H_{X,X'}(f+g) = H_{X,X'}(f) + H_{X,X'}(g) + Id_{\mathcal{C}}(g|f)\circ H_X(2_X)$$

Proof. We have the following equalities:

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ H_{X,X'}(f+g)\\
&= (i_1^2+i_2^2)\circ(f+g) - i_2^2\circ(f+g) - i_1^2\circ(f+g)\,, \text{by } (4.1.12)\\
&= (i_1^2+i_2^2)\circ f + (i_1^2+i_2^2)\circ g - i_2^2\circ g - i_2^2\circ f - i_1^2\circ g - i_1^2 \circ f\,, \text{by } (4.1.4)\\
&= (i_1^2+i_2^2)\circ f + (i_1^2+i_2^2)\circ g - i_2^2\circ g - \big(i_1^2\circ g + i_2^2\circ f\big) - i_1^2\circ f
\end{aligned}$$

Moreover we have

$$\begin{aligned}
i_1^2\circ g + i_2^2\circ f &= i_2^2\circ f + i_1^2\circ g + c_2^{X'+X'}\circ Id_{\mathcal{C}}(i_2^2\circ f|i_1^2\circ g)\circ H_X(2_X)\,, \text{by } (4.1.20)\\
&= i_2^2\circ f + i_1^2\circ g + c_2^{X'+X'}\circ T_{X'+X'}\circ Id_{\mathcal{C}}(i_2^2\circ f|i_1^2\circ g)\circ H_X(2_X)\\
&= i_2^2\circ f + i_1^2\circ g + c_2^{X'+X'}\circ Id_{\mathcal{C}}(i_1^2\circ g|i_2^2\circ f)\circ T_X\circ H_X(2_X)\\
&= i_2^2\circ f + i_1^2\circ g - c_2^{X'+X'}\circ Id_{\mathcal{C}}(i_1^2\circ g|i_2^2\circ f)\circ H_X(2_X)\,, \text{by } 4.1.19\\
&= i_2^2\circ f + i_1^2\circ g - \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(g|f)\circ H_X(2_X)
\end{aligned}$$

where, for any object Z in $\mathcal{C}$, $T_Z : Id_{\mathcal{C}}(Z|Z) \to Id_{\mathcal{C}}(Z|Z)$ is the restriction of the canonical switch $\tau_Z^2 : Z+Z \to Z+Z$ to $Id_{\mathcal{C}}(Z|Z)$. Hence we get

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ H_{X,X'}(f+g)\\
&= (i_1^2+i_2^2)\circ f + (i_1^2+i_2^2)\circ g - i_2^2\circ g - \big(i_1^2\circ g + i_2^2\circ f\big) - i_1^2\circ f\\
&= (i_1^2+i_2^2)\circ f + (i_1^2+i_2^2)\circ g - i_2^2\circ g - \big(i_2^2\circ f + i_1^2\circ g - \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(g|f)\circ H_X(2_X)\big) - i_1^2\circ f\\
&= (i_1^2+i_2^2)\circ f + (i_1^2+i_2^2)\circ g - i_2^2\circ g - i_1^2\circ g - i_2^2\circ f - i_1^2\circ f + \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(g|f)\circ H_X(2_X)\,, \text{by } (4.1.16)\\
&= (i_1^2+i_2^2)\circ f + \iota_2^{Id_{\mathcal{C}}}\circ H_{X,X'}(g) - i_2^2\circ f - i_1^2\circ f + \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(g|f)\circ H_X(2_X)\,, \text{by } (4.1.12)\\
&= (i_1^2+i_2^2)\circ f - i_2^2\circ f - i_1^2\circ f + \iota_2^{Id_{\mathcal{C}}}\circ H_{X,X'}(g) + \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(g|f)\circ H_X(2_X)\,, \text{by } (4.1.16)\\
&= \iota_2^{Id_{\mathcal{C}}}\circ H_{X,X'}(f) + \iota_2^{Id_{\mathcal{C}}}\circ H_{X,X'}(g) + \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(g|f)\circ H_X(2_X)\,, \text{by } (4.1.12)\\
&= \iota_2^{Id_{\mathcal{C}}}\circ \Big(H_{X,X'}(f) + H_{X,X'}(g) + Id_{\mathcal{C}}(g|f)\circ H_X(2_X)\Big)\,, \text{by } (4.1.4)
\end{aligned}$$

As $\iota_2^{Id_{\mathcal{C}}} : Id_{\mathcal{C}}(X'|X') \rightarrowtail X'+X'$ is a monomorphism, it concludes the proof. □

In addition we provide the deviation of the group structure given in (4.1.2) to be right distributive:

Proposition 4.1.22. *Consider an object X' in $\langle E\rangle$. Let $g_1, g_2 \in \mathcal{C}(X,Y)$ and $f \in \mathcal{C}(X',X)$. Then we have*

$$\big(g_1+g_2\big)\circ f = g_1\circ f + g_2\circ f + c_2^Y \circ Id_{\mathcal{C}}(g_1|g_2)\circ H_{XX'}(f)$$

Proof. We have the following equalities:

$$\begin{aligned}
\big(g_1+g_2\big)\circ f &= (g_1,g_2)\circ \mu_X\circ f\,, \text{by (4.1.2)}\\
&= (g_1,g_2)\circ (i_1^2+i_2^2)\circ f\,, \text{by (4.1.3)}\\
&= (g_1,g_2)\circ \big(i_1\circ f + i_2\circ f + \iota_2^{Id_{\mathcal{C}}}\circ H_{XX'}(f)\big)\,, \text{by (4.1.12) and 4.1.16}\\
&= (g_1,g_2)\circ i_1\circ f + (g_1,g_2)\circ i_2\circ f + (g_1,g_2)\circ \iota_2^{Id_{\mathcal{C}}}\circ H_{XX'}(f)\,, \text{by 4.1.4}
\end{aligned}$$

Moreover we have

$$\begin{aligned}
(g_1,g_2)\circ \iota_2^{Id_{\mathcal{C}}}\circ H_{XX'}(f) &= \nabla_Y^2\circ (g_1+g_2)\circ \iota_2^{Id_{\mathcal{C}}}\circ H_{XX'}(f)\\
&= \nabla_Y^2\circ \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(g_1|g_2)\circ H_{XX'}(f)\\
&= c_2^Y\circ Id_{\mathcal{C}}(g_1|g_2)\circ H_{XX'}(f)\,, \text{by 1.2.8}
\end{aligned}$$

Hence we get

$$\begin{aligned}
\big(g_1+g_2\big)\circ f &= (g_1,g_2)\circ i_1\circ f + (g_1,g_2)\circ i_2\circ f + (g_1,g_2)\circ \iota_2^{Id_{\mathcal{C}}}\circ H_{XX'}(f)\\
&= g_1\circ f + g_2\circ f + c_2^Y\circ Id_{\mathcal{C}}(g_1|g_2)\circ H_{XX'}(f)
\end{aligned}$$

□

There is another expression of the involution $T_{X,Y,Y}: \mathcal{C}(X,Y|Y)\to\mathcal{C}(X,Y|Y)$ given in the next proposition.

Proposition 4.1.23. *Consider an object X' in $\langle E\rangle$. We have $T_{X,X',X'} = H_{X,X'}\circ (c_2^{X'})_* - id$.*

Proof. Let $\xi\in\mathcal{C}(X,X'|X')$. Then we have

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}}\circ H_{X,X'}\circ (c_2^{X'})_*(\xi)\\
&= \iota_2^{Id_{\mathcal{C}}}\circ H_{X,X'}(c_2^{X'}\circ\xi)\\
&= (i_1^2+i_2^2)\circ c_2^{X'}\circ\xi - i_1^2\circ c_2^{X'}\circ\xi - i_2^2\circ c_2^{X'}\circ\xi\,, \text{by (4.1.12)}\\
&= c_2^{X'+X'}\circ Id_{\mathcal{C}}\big(i_1^2+i_2^2|i_1^2+i_2^2\big)\circ\xi - c_2^{X'+X'}\circ Id_{\mathcal{C}}\big(i_1^2|i_1^2\big)\circ\xi - c_2^{X'+X'}\circ Id_{\mathcal{C}}\big(i_2^2|i_2^2\big)\circ\xi
\end{aligned}$$

By 4.2.8, we get

$$\begin{aligned}
Id_{\mathcal{C}}\big(i_1^2+i_2^2|i_1^2+i_2^2\big) &= Id_{\mathcal{C}}\big(i_1^2|i_1^2+i_2^2\big) + Id_{\mathcal{C}}\big(i_2^2|i_1^2+i_2^2\big)\\
&= Id_{\mathcal{C}}\big(i_1^2|i_1^2\big) + Id_{\mathcal{C}}\big(i_1^2|i_2^2\big) + Id_{\mathcal{C}}\big(i_2^2|i_1^2\big) + Id_{\mathcal{C}}\big(i_2^2|i_2^2\big)
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ H_{X,X'} \circ (c_2^{X'})_*(\xi) \\
&= c_2^{X'+X'} \circ Id_{\mathcal{C}}\big(i_1^2|i_2^2\big) \circ \xi + c_2^{X'+X'} \circ Id_{\mathcal{C}}\big(i_2^2|i_1^2\big) \circ \xi \\
&= c_2^{X'+X'} \circ Id_{\mathcal{C}}\big(i_1^2|i_2^2\big) \circ \xi + c_2^{X'+X'} \circ T_{X'+X'} \circ Id_{\mathcal{C}}\big(i_2^2|i_1^2\big) \circ \xi \\
&= c_2^{X'+X'} \circ Id_{\mathcal{C}}\big(i_1^2|i_2^2\big) \circ \xi + c_2^{X'+X'} \circ Id_{\mathcal{C}}\big(i_1^2|i_2^2\big) \circ T_{X'} \circ \xi \\
&= \iota_2^{Id_{\mathcal{C}}} \circ \xi + \iota_2^{Id_{\mathcal{C}}} \circ T_{X'} \circ \xi \\
&= \iota_2^{Id_{\mathcal{C}}} \circ \big(\xi + T_{X,X',X'}(\xi)\big)
\end{aligned}
$$

As $\iota_2^{Id_{\mathcal{C}}} : Id_{\mathcal{C}}(X'|X') \rightarrowtail X' + X'$ is a monomorphism, it concludes the proof. □

Then the next proposition says that the full subcategory $\langle E\rangle$ of $\mathcal{C}$ formed by free objects of finite rank in $\mathcal{C}$ has a *square ringoid* structure, as introduced in definition 3.1 of [4].

Proposition 4.1.24. *The full subcategory $\langle E\rangle$ of $\mathcal{C}$ is a square ringoid, when endowed with the multifunctor $\mathcal{C}(-,-|-) : \langle E\rangle^{op} \times \langle E\rangle \times \langle E\rangle \to Gr$ and, for X, Y and Z objects in $\langle E\rangle$, the following diagram of maps*

$$\Big(\mathcal{C}(X,Y) \overset{H_{X,Y}}{\longrightarrow} \mathcal{C}(X,Y|Y) \overset{P_{X,Y}}{\longrightarrow} \mathcal{C}(X,Y)\Big), \tag{4.1.14}$$

and with $T_{X,Y,Z} : \mathcal{C}(X,Y|Z) \to \mathcal{C}(X,Z|Y)$ being the bijection given in 4.1.9 and $P_{X,Y} = (c_2^Y)_*$, where $c_2^Y : Id_{\mathcal{C}}(Y|Y) \to Y$ is the morphism given in 1.2.8. It means that the maps $H_{X,Y}$, $(c_2^Y)_*$ and $T_{X,Y,Z}$ satisfy the following properties:

- *$P_{X,Y} = (c_2^Y)_* : \mathcal{C}(X, Id_{\mathcal{C}}(Y|Y)) \to \mathcal{C}(X,Y)$ is a homomorphism which is natural in both variables, i.e. for $f_1 \in \mathcal{C}(X,X')$ and $f_2 \in \mathcal{C}(Y,Y')$, we have*

$$(f_1)^* \circ (c_2^Y)_* = (c_2^Y)_* \circ (f_1)^* \quad \text{and} \quad (c_2^{Y'})_* \circ Id_{\mathcal{C}}(f_2|f_2)_* = (f_2)_* \circ (c_2^Y)_* \tag{4.1.15}$$

- *Let X', Y and Y' be objects in $\langle E\rangle$. For $f_1 \in \mathcal{C}(X,X')$, $f_2 \in \mathcal{C}(Y,Y')$, $\xi_1 \in \mathcal{C}(X,X|X')$ and $\xi_2 \in \mathcal{C}(Y,Y'|Y')$, the maps*

$$Id_{\mathcal{C}}\big(f_1 \mid c_2^{Y'} \circ \xi_2\big)_*, Id_{\mathcal{C}}\big(c_2^{X'} \circ \xi_1 \mid f_2\big)_* : \mathcal{C}(Z,X|Y) \to \mathcal{C}(Z,X'|Y')$$

are trivial, i.e.

$$Id_{\mathcal{C}}\big(f_1 \mid c_2^{X'} \circ \xi_1\big)_* = Id_{\mathcal{C}}\big(c_2^{Y'} \circ \xi_2 \mid f_2\big)_* = 0 \tag{4.1.16}$$

- *Let $f, g \in \mathcal{C}(X,Y)$, we have*

$$H_{X,Y}(f+g) = H_{X,Y}(f) + H_{X,Y}(g) + Id_{\mathcal{C}}(g|f) \circ H_X(2_X) \tag{4.1.17}$$

- *For $f_1 \in \mathcal{C}(X,Y)$ and $f_2 \in \mathcal{C}(Y,Z)$, we have*

$$H_{X,Z}(f_2 \circ f_1) = Id_{\mathcal{C}}(f_2|f_2) \circ H_{X,Y}(f_1) + H_{Y,Z}(f_2) \circ f_1 \tag{4.1.18}$$

- *For $\alpha \in \mathcal{C}(X,Y)$, we get*

$$T_{X,Y,Y}\big(H_{X,Y}(\alpha)\big) = H_{X,Y}(\alpha) + H_Y(2_Y) \circ \alpha - Id_{\mathcal{C}}(\alpha|\alpha) \circ H_X(2_X) \tag{4.1.19}$$

- *Let Z be an object in $\langle E\rangle$ (or merely in $\mathcal{C}$). For $g_1, g_2 \in \mathcal{C}(Y,Z)$ and $f \in \mathcal{C}(X,Y)$, we get "the quadratic left distributivity law":*

$$(g_1+g_2)\circ f = g_1\circ f + g_2\circ f + c_2^Z \circ Id_{\mathcal{C}}(g_1|g_2)\circ H_{X,Y}(f) \tag{4.1.20}$$

- *Let Z be an object in $\langle E\rangle$ (or merely in $\mathcal{C}$). For $g \in \mathcal{C}(Y,Z)$ and $f_1, f_2 \in \mathcal{C}(X,Y)$, we have "the linear right distributivity law":*

$$g\circ(f_1+f_2) = g\circ f_1 + g\circ f_2 \tag{4.1.21}$$

- *The bijection $T_{X,Y,Z} : \mathcal{C}(X,Y|Z)\to\mathcal{C}(X,Z|Y)$ satisfies*

$$T_{X,Y,Y} = H_{X,Y}\circ P_{X,Y} - id = H_{X,Y}\circ (c_2^Y)_* - id \tag{4.1.22}$$

Proof. The first property is verified because the map $(c_2^Y)_* : \mathcal{C}(X, Id_{\mathcal{C}}(Y|Y)) \to \mathcal{C}(X,Y)$ clearly satisfies (4.1.15). The property (4.1.16) is satisfied because we have

$$Id_{\mathcal{C}}\big(id\,|\,ab_{Y'}\big)\circ Id_{\mathcal{C}}\big(f_1\,|\,c_2^{Y'}\circ\xi_2\big) = Id_{\mathcal{C}}\big(f_1\,|\,ab_{Y'}\circ c_2^{Y'}\circ\xi_2\big) = 0$$

and $Id_{\mathcal{C}}\big(id\,|\,ab_{Y'}\big) : Id_{\mathcal{C}}\big(X'|\,Y'\big)\to Id_{\mathcal{C}}\big(X'\,|\,(Y')^{ab}\big)$ is an isomorphism by 4.1.14. Then the properties (4.1.17), (4.1.20), (4.1.21), (4.1.20) and (4.1.22) are respectively given by 4.1.21, 4.1.22, 4.1.4 and 4.1.23. Hence it remains to prove the properties (4.1.18) and (4.1.19).

- First we prove that the property (4.1.18) holds. Let $f_1 \in \mathcal{C}(X,Y)$ and $f_2 \in \mathcal{C}(Y,Z)$. Then we have

$$\begin{aligned}
&\iota_2^{Idc}\circ H_{XZ}(f_2\circ f_1)\\
&= (i_1^2+i_2^2)\circ(f_2\circ f_1) - i_2^2\circ(f_2\circ f_1) - i_1^2\circ(f_2\circ f_1)\,, \text{by (4.1.12)}\\
&= \Big(i_1^2\circ f_2 + i_2^2\circ f_2 + \iota_2^{Idc}\circ H_{YZ}(f_2)\Big)\circ f_1 - i_2^2\circ f_2\circ f_1 - i_1^2\circ f_2\circ f_1\,, \text{by (4.1.12) and (4.1.16)}
\end{aligned}$$

In addition we get

$$\Big(i_1^2\circ f_2 + i_2^2\circ f_2 + \iota_2^{Idc}\circ H_{YZ}(f_2)\Big)\circ f_1 = \big(i_1^2\circ f_2 + i_2^2\circ f_2\big)\circ f_1 + \iota_2^{Idc}\circ H_{YZ}(f_2)\circ f_1$$

by 4.1.22 and (4.1.17) because we have

$$\iota_2^{Idc}\circ H_{YZ}(f_2)\circ f_1 = c_2^{Z+Z}\circ Id_{\mathcal{C}}(i_1^2|i_2^2)\circ H_{YZ}(f_2)\circ f_1$$

Hence we have

$$\begin{aligned}
&\iota_2^{Idc}\circ H_{XZ}(f_2\circ f_1)\\
&= i_1^2\circ f_2\circ f_1 + i_2^2\circ f_2\circ f_1 + c_2^{Z+Z}\circ Id_{\mathcal{C}}(i_1^2\circ f_2|i_2^2\circ f_2)\circ H_{XY}(f_1) + \iota_2^{Idc}\circ H_{YZ}(f_2)\circ f_1\,, \text{by 4.1.22}\\
&\qquad - i_2^2\circ f_2\circ f_1 - i_1^2\circ f_2\circ f_1\\
&= c_2^{Z+Z}\circ Id_{\mathcal{C}}(i_1^2\circ f_2|i_2^2\circ f_2)\circ H_{XY}(f_1) + \iota_2^{Idc}\circ H_{YZ}(f_2)\circ f_1\,, \text{by (4.1.15) and (4.1.16)}\\
&= c_2^{Z+Z}\circ Id_{\mathcal{C}}(i_1^2|i_2^2)\circ Id_{\mathcal{C}}(f_2|f_2)\circ H_{XY}(f_1) + \iota_2^{Idc}\circ H_{YZ}(f_2)\circ f_1\\
&= \iota_2^{Idc}\circ Id_{\mathcal{C}}(f_2|f_2)\circ H_{XY}(f_1) + \iota_2^{Idc}\circ H_{YZ}(f_2)\circ f_1\\
&= \iota_2^{Idc}\circ\Big(Id_{\mathcal{C}}(f_2|f_2)\circ H_{XY}(f_1) + H_{YZ}(f_2)\circ f_1\Big)\,, \text{by 4.1.4}
\end{aligned}$$

As $\iota_2^{Idc} : Id_{\mathcal{C}}(Z|Z) \rightarrowtail Z+Z$ is a monomorphism, it gives the desired relation.

- Then we prove that the property (4.1.19) holds. For $\alpha \in \mathcal{C}(X,Y)$, we have the following equalities:

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ T_{X,Y,Y}\big(H_{XY}(\alpha)\big)\\
&= \iota_2^{Id_{\mathcal{C}}} \circ T_{Y,Y} \circ H_{XY}(\alpha)\,, \text{see 4.1.9}\\
&= \tau_Y^2 \circ \Big((i_1^2 + i_2^2)\circ\alpha - i_2^2\circ\alpha - i_1^2\circ\alpha\Big)\\
&= (i_2^2 + i_1^2)\circ\alpha - i_1^2\circ\alpha - i_2^2\circ\alpha\,, \text{by (4.1.4)}\\
&= \big(i_1^2 + i_2^2 + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\big)\circ\alpha - i_1^2\circ\alpha - i_2^2\circ\alpha\\
&= (i_1^2 + i_2^2)\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\circ\alpha - i_1^2\circ\alpha - i_2^2\circ\alpha\,, \text{by (4.1.22) and 4.1.16}\\
&= (i_1^2 + i_2^2)\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\circ\alpha - \big(i_2^2\circ\alpha + i_1^2\circ\alpha\big)
\end{aligned}$$

By (4.1.20), we have

$$\begin{aligned}
i_2^2\circ\alpha + i_1^2\circ\alpha &= i_1^2\circ\alpha + i_2^2\circ\alpha + c_2^{Y+Y}\circ Id_{\mathcal{C}}(i_1^2\circ\alpha|i_2^2\circ\alpha)\circ H_X(2_X)\\
&= i_1^2\circ\alpha + i_2^2\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(\alpha|\alpha)\circ H_X(2_X)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ T_{X,Y,Y}\big(H_{XY}(\alpha)\big)\\
&= (i_1^2 + i_2^2)\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\circ\alpha - \big(i_2^2\circ\alpha + i_1^2\circ\alpha\big)\\
&= (i_1^2 + i_2^2)\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\circ\alpha\\
&\qquad - \Big(i_1^2\circ\alpha + i_2^2\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(\alpha|\alpha)\circ H_X(2_X)\Big)\\
&= (i_1^2 + i_2^2)\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\circ\alpha - i_2^2\circ\alpha - i_1^2\circ\alpha - \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(\alpha|\alpha)\circ H_X(2_X)\,, \text{by (4.1.16)}\\
&= (i_1^2 + i_2^2)\circ\alpha - i_2^2\circ\alpha - i_1^2\circ\alpha + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\circ\alpha - \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(\alpha|\alpha)\circ H_X(2_X)\,, \text{by (4.1.16)}\\
&= \iota_2^{Id_{\mathcal{C}}}\circ H_{XY}(\alpha) + \iota_2^{Id_{\mathcal{C}}}\circ H_Y(2_Y)\circ\alpha - \iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(\alpha|\alpha)\circ H_X(2_X)\,, \text{by (4.1.12)}\\
&= \iota_2^{Id_{\mathcal{C}}}\Big(H_{XY}(\alpha) + H_Y(2_Y)\circ\alpha - Id_{\mathcal{C}}(\alpha|\alpha)\circ H_X(2_X)\Big)\,, \text{by (4.1.4)}
\end{aligned}$$

As $\iota_2^{Id_{\mathcal{C}}} : Id_{\mathcal{C}}(Y|Y) \rightarrowtail Y+Y$ is a monomorphism, we obtain the desired relation.

□

Remark 4.1.25. We have the following observations:

1. Let $f \in \mathcal{C}(Y,Z)$. Using (4.1.20) with $g_1 = 0$ and $g_2 = -f$, we get
$$(-f)\circ g = -(f\circ g) + c_2^Z\circ Id_{\mathcal{C}}(f|f)\circ H_{Y,Z}(g) \tag{4.1.23}$$

2. The morphisms in $\mathcal{C}$ involved in the relations (4.1.17), (4.1.18), (4.1.19) and (4.1.22) have an abelian object as a target object. By 1.5.17, these relations remain the same if we replace each morphism with its unique factorization through the abelianization morphism (see 1.5.16 and 1.5.17). For exemple, (4.1.17) is equivalent to the following relation:
$$H_{X,Y}(f+g)^{ab} = H_{X,Y}(f)^{ab} + H_{X,Y}(g)^{ab} + Id_{\mathcal{C}}(f|g)\circ H_X(2_X)^{ab}\,,$$
for $f,g \in \mathcal{C}(X,Y)$.

Now we give the deviation of the retraction $r_2 : \mathcal{C}(X, Y+Y) \to \mathcal{C}(X, Y|Y)$ to preserve the (right) action of the monoid $\mathcal{C}(X,X)$, as follows:

Proposition 4.1.26. *Let X be an object in $\langle E\rangle$ and Y be an object in $\mathcal{C}$. Then, for $\xi \in \mathcal{C}(Y, Z+Z)$ and $\alpha \in \mathcal{C}(X,Y)$, we have*

$$r_2(\xi \circ \alpha) = r_2(\xi) \circ \alpha + Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ H_{X,Y}(\alpha)$$

Proof. We have the following equalities:

$$\begin{aligned} \iota_2^{Id_{\mathcal{C}}} \circ r_2(\xi) \circ \alpha &= \Big(\xi - \big(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi\big)\Big) \circ \alpha \\ &= \xi \circ \alpha + \Big(- \big(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi\big)\Big) \circ \alpha \\ &\qquad - c_2^{E^{+2}} \circ Id_{\mathcal{C}}(\xi | i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi) \circ H_{X,Y}(\alpha) \text{ by } (4.1.20) \end{aligned}$$

By (4.1.23), we get

$$\begin{aligned} &\Big(- \big(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi\big)\Big) \circ \alpha \\ &= -\big(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi\big) \circ \alpha \\ &\quad + c_2^{E^{+2}} \circ Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi \big| i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi\big) \circ H_{X,Y}(\alpha) \end{aligned}$$

Denoting $i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi$ by h, we have

$$Id_{\mathcal{C}}(\xi|h) = Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | h) + Id_{\mathcal{C}}(i_2^2 \circ r_2^2 \circ \xi | h) = Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 | h) = Id_{\mathcal{C}}(h|h)$$

by 3.6 of [12] because the functor $Id_{\mathcal{C}}(-|E^{+2}) : \mathcal{C} \to \mathcal{C}$ is linear. Hence we have

$$\begin{aligned} \iota_2^{Id_{\mathcal{C}}} \circ r_2(\xi) \circ \alpha &= \xi \circ \alpha - \big(i_1^2 \circ r_1^2 \circ \xi + i_2^2 \circ r_2^2 \circ \xi\big) \circ \alpha \\ &= \xi \circ \alpha - \Big(i_1^2 \circ r_1^2 \circ \xi \circ \alpha + i_2^2 \circ r_2^2 \circ \xi \circ \alpha \\ &\qquad + c_2^{E^{+2}} \circ Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | i_2^2 \circ r_2^2 \circ \xi) \circ H_{X,Y}(\alpha)\Big) \end{aligned}$$

However we get

$$\begin{aligned} c_2^{E^{+2}} \circ Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | i_2^2 \circ r_2^2 \circ \xi) \circ H_{X,Y}(\alpha) &= (i_1^2, i_2^2) \circ \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ H_{X,Y}(\alpha) \\ &= \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ H_{X,Y}(\alpha) \end{aligned}$$

It implies that

$$\begin{aligned} \iota_2^{Id_{\mathcal{C}}} \circ r_2(\xi) \circ \alpha &= \xi \circ \alpha - \big(i_1^2 \circ r_1^2 \circ \xi \circ \alpha + i_2^2 \circ r_2^2 \circ \xi \circ \alpha\big) - \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | i_2^2 \circ r_2^2 \circ \xi) \circ H_{X,Y}(\alpha) \\ &= \iota_2^{Id_{\mathcal{C}}} \circ r_2(\xi \circ \alpha) - \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ H_{X,Y}(\alpha) \\ &= \iota_2^{Id_{\mathcal{C}}} \circ \Big(r_2(\xi \circ \alpha) - Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ H_{X,Y}(\alpha)\Big) \text{ by } (4.1.20) \end{aligned}$$

As $\iota_2^{Id_{\mathcal{C}}} : \mathcal{C}(X, Y|Y) \rightarrowtail \mathcal{C}(X, Y+Y)$ is a monomorphism, it concludes the proof. $\square$

In addition we give the deviation of the retraction $r_2 : \mathcal{C}(X, Y+Y) \to \mathcal{C}(X, Y|Y)$ to be a homomorphism of groups, as follows:

Proposition 4.1.27. *Let X be an object in $\langle E\rangle$ and Y be an object in $\mathcal{C}$. Then, for $f, g \in \mathcal{C}(X, Y+Y)$, we have*

$$r_2(f+g) = r_2(f) + r_2(g) + Id_{\mathcal{C}}\big(r_1^2 \circ g | r_2^2 \circ f\big) \circ H_X(2_X)$$

Proof. We have the following equalities:

$$\begin{aligned}
\iota_2^{Id_{\mathcal{C}}} \circ r_2(f+g) &= (f+g) - i_2^2 \circ r_2^2 \circ (f+g) - i_1^2 \circ r_1^2 \circ (f+g) \\
&= f + g - i_2^2 \circ r_2^2 \circ g - i_2^2 \circ r_2^2 \circ f - i_1^2 \circ r_1^2 \circ g - i_1^2 \circ r_1^2 \circ f \text{ , by (4.1.21)} \\
&= f + g - i_2^2 \circ r_2^2 \circ g - \big(i_1^2 \circ r_1^2 \circ g + i_2^2 \circ r_2^2 \circ f\big) - i_1^2 \circ r_1^2 \circ f
\end{aligned}$$

By (4.1.18), we get

$$\begin{aligned}
&i_1^2 \circ r_1^2 \circ g + i_2^2 \circ r_2^2 \circ f \\
&= i_2^2 \circ r_2^2 \circ f + i_1^2 \circ r_1^2 \circ g + c_2^{Y+Y} \circ Id_{\mathcal{C}}(i_2^2 \circ r_2^2 \circ f | i_1^2 \circ r_1^2 \circ g) \circ H_X(2_X) \\
&= i_2^2 \circ r_2^2 \circ f + i_1^2 \circ r_1^2 \circ g + c_2^{Y+Y} \circ T_{Y+Y} \circ Id_{\mathcal{C}}(i_2^2 \circ r_2^2 \circ f | i_1^2 \circ r_1^2 \circ g) \circ H_X(2_X) \\
&= i_2^2 \circ r_2^2 \circ f + i_1^2 \circ r_1^2 \circ g + c_2^{Y+Y} \circ Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ g | i_2^2 \circ r_2^2 \circ f) \circ T_{X,X} \circ H_X(2_X) \\
&= i_2^2 \circ r_2^2 \circ f + i_1^2 \circ r_1^2 \circ g - c_2^{Y+Y} \circ Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ g | i_2^2 \circ r_2^2 \circ f) \circ H_X(2_X) \text{ , by 4.1.19} \\
&= i_2^2 \circ r_2^2 \circ f + i_1^2 \circ r_1^2 \circ g - \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ g | r_2^2 \circ f) \circ H_X(2_X)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ r_2(f+g) \\
&= f + g - i_2^2 \circ r_2^2 \circ g - \big(i_1^2 \circ r_1^2 \circ g + i_2^2 \circ r_2^2 \circ f\big) - i_1^2 \circ r_1^2 \circ f \\
&= f + g - i_2^2 \circ r_2^2 \circ g - \Big(i_2^2 \circ r_2^2 \circ f + i_1^2 \circ r_1^2 \circ g - \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ g | r_2^2 \circ f) \circ H_X(2_X)\Big) - i_1^2 \circ r_1^2 \circ f \\
&= f + g - i_2^2 \circ r_2^2 \circ g - i_1^2 \circ r_1^2 \circ g - i_2^2 \circ r_2^2 \circ f - i_1^2 \circ r_1^2 \circ f \\
&\qquad + \iota_2^{Id_{\mathcal{C}}} \circ \circ Id_{\mathcal{C}}(r_1^2 \circ g | r_2^2 \circ f) \circ H_X(2_X) \text{ , by 4.1.16} \\
&= f + \iota_2^{Id_{\mathcal{C}}} \circ r_2(g) - i_2^2 \circ r_2^2 \circ f - i_1^2 \circ r_1^2 \circ f + \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ g | r_2^2 \circ f) \circ H_X(2_X) \text{ , by (4.1.11)} \\
&= f - i_2^2 \circ r_2^2 \circ f - i_1^2 \circ r_1^2 \circ f + \iota_2^{Id_{\mathcal{C}}} \circ r_2(g) + \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ g | r_2^2 \circ f) \circ H_X(2_X) \text{ , by 4.1.16} \\
&= \iota_2^{Id_{\mathcal{C}}} \circ r_2(f) + \iota_2^{Id_{\mathcal{C}}} \circ r_2(g) + \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(r_1^2 \circ g | r_2^2 \circ f) \circ H_X(2_X) \text{ , by (4.1.11)} \\
&= \iota_2^{Id_{\mathcal{C}}} \circ \Big(r_2(f) + r_2(g) + Id_{\mathcal{C}}(r_1^2 \circ g | r_2^2 \circ f) \circ H_X(2_X)\Big) \text{ , by (4.1.21)}
\end{aligned}$$

As $\iota_2^{Id_{\mathcal{C}}} : Id_{\mathcal{C}}(Y|Y) \rightarrowtail Y + Y$ is a monomorphism, it concludes the proof. □

Remark 4.1.28. From 4.1.27 and 4.1.7, we deduce that the map $r_2 : \mathcal{C}(X, Y+Y) \to \mathcal{C}(X, Y|Y)$ is a quadratic map in the sense of [4].

Then we give the following relation:

Proposition 4.1.29. *Consider X an object in $\langle E\rangle$, and Y and Z two objects in $\mathcal{C}$. Then, for $f_1, f_2 \in \mathcal{C}(Y, Z+Z)$ and $\xi \in \mathcal{C}(X, Y|Y)$, we have*

$$r_2\big(c_2^{Z+Z} \circ Id_{\mathcal{C}}(f_1|f_2) \circ \xi\big) = Id_{\mathcal{C}}(r_1^2 \circ f_1 | r_2^2 \circ f_2) \circ \xi + Id_{\mathcal{C}}(r_1^2 \circ f_2 | r_2^2 \circ f_1) \circ T_{X,Y,Y}(\xi)$$

Proof. We get the following equalities:

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ r_2\big(c_2^{Z+Z} \circ Id_{\mathcal{C}}(f_1|f_2) \circ \xi\big)\\
&= c_2^{Z+Z} \circ Id_{\mathcal{C}}(f_1|f_2) \circ \xi - i_2^2 \circ r_2^2 \circ c_2^{Z+Z} \circ Id_{\mathcal{C}}(f_1|f_2) \circ \xi - i_1^2 \circ r_1^2 \circ c_2^{Z+Z} \circ Id_{\mathcal{C}}(f_1|f_2) \circ \xi\,, \text{by (4.1.11)}\\
&= c_2^{Z+Z} \circ Id_{\mathcal{C}}(f_1|f_2) \circ \xi - c_2^{Z+Z} \circ Id_{\mathcal{C}}\big(i_2^2 \circ r_2^2 \circ f_1 | i_2^2 \circ r_2^2 \circ f_2\big) \circ \xi\\
&\qquad - c_2^{Z+Z} \circ Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 \circ f_1 | i_1^2 \circ r_1^2 \circ f_2\big) \circ \xi\,, \text{by naturality of } c_2^{Z+Z}
\end{aligned}$$

As the bifunctor $Id_{\mathcal{C}}(-|-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear, we have

$$\begin{aligned}
Id_{\mathcal{C}}(id|id) &= Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 | id\big) + Id_{\mathcal{C}}\big(i_2^2 \circ r_2^2 | id\big)\\
&= Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 | i_1^2 \circ r_1^2\big) + Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 | i_2^2 \circ r_2^2\big) + Id_{\mathcal{C}}\big(i_2^2 \circ r_2^2 | i_1^2 \circ r_1^2\big) + Id_{\mathcal{C}}\big(i_2^2 \circ r_2^2 | i_2^2 \circ r_2^2\big)
\end{aligned}$$

by 3.6 of [12]. Hence we have

$$\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}} \circ r_2\big(c_2^{Z+Z} \circ Id_{\mathcal{C}}(f_1|f_2) \circ \xi\big)\\
&= c_2^{Z+Z} \circ Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 \circ f_1 | i_2^2 \circ r_2^2 \circ f_2\big) \circ \xi + c_2^{Z+Z} \circ Id_{\mathcal{C}}\big(i_2^2 \circ r_2^2 \circ f_1 | i_1^2 \circ r_1^2 \circ f_2\big) \circ \xi\\
&= Id_{\mathcal{C}}\big(r_1^2 \circ f_1 | r_2^2 \circ f_2\big) \circ \xi + c_2^{Z+Z} \circ T_{Z+Z} \circ Id_{\mathcal{C}}\big(i_2^2 \circ r_2^2 \circ f_1 | i_1^2 \circ r_1^2 \circ f_2\big) \circ \xi\\
&= Id_{\mathcal{C}}\big(r_1^2 \circ f_1 | r_2^2 \circ f_2\big) \circ \xi + c_2^{Z+Z} \circ Id_{\mathcal{C}}\big(i_1^2 \circ r_1^2 \circ f_2 | i_2^2 \circ r_2^2 \circ f_1\big) \circ T_X \circ \xi\\
&= \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}\big(r_1^2 \circ f_1 | r_2^2 \circ f_2\big) \circ \xi + \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}\big(r_1^2 \circ f_2 | r_2^2 \circ f_1\big) \circ T_{X,Y,Y}(\xi)\\
&= \iota_2^{Id_{\mathcal{C}}} \circ \Big(Id_{\mathcal{C}}\big(r_1^2 \circ f_1 | r_2^2 \circ f_2\big) \circ \xi + Id_{\mathcal{C}}\big(r_1^2 \circ f_2 | r_2^2 \circ f_1\big) \circ T_{X,Y,Y}(\xi)\Big)\,, \text{by (4.1.21)}
\end{aligned}$$

As $\iota_2^{Id_{\mathcal{C}}} : Id_{\mathcal{C}}(Z|Z) \rightarrowtail Z + Z$ is a monomorphism, it concludes the proof. □

Remark 4.1.30. As in 4.1.25, the relations in 4.1.26, 4.1.27 and 4.1.29 also hold if we replace each morphism with its unique factorization through the abelianization morphism by 1.5.16 because it has an abelian object as a target object. For exemple, the relation in 4.1.26 is equivalent to the following one:

$$r_2(\xi \circ \alpha)^{ab} = r_2(\xi)^{ab} \circ \alpha^{ab} + Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ H_{X,Y}(\alpha)^{ab}\,,$$

for $\xi \in \mathcal{C}(Y, Z + Z)$ and $\alpha \in \mathcal{C}(X, Y)$.

4.2 Definition of the operad $AbOp(\mathcal{C})$

First we recall that the set of morphisms with abelian source and target has a natural abelian group structure and we consider the following notation:

Notation 4.2.1. For A and B abelian objects in $\mathcal{C}$, we denote by $Ab(\mathcal{C})(A, B) = \mathcal{C}(A, B)$ the indicated morphism set endowed with its natural abelian group structure.

From now on, we suppose that we have the 2-*divisibility condition* as follows:

Assumption: $id_{E^{ab}} + id_{E^{ab}}$ is invertible in the endomorphism ring $\mathcal{C}(E^{ab},\ E^{ab})$.

Hence it permits us to consider that, for an abelian object Z in $\mathcal{C}$, the abelian group $\mathcal{C}(E^{ab}, Z)$ (see 4.1.4) is a left $\mathbb{Z}[\frac{1}{2}]$-module. Then the 2-*divisibility condition* implies the next proposition:

Proposition 4.2.2. *Let X be an object in $\langle E \rangle$. If we assume that the 2-divisibility condition given above holds, then we have the following isomorphism of Λ-$\mathcal{P}(1)$-bimodules*

$$\mathcal{C}\big(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}}\big) \cong \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)_{\mathfrak{S}_2}$$

Proof. We recall that the morphism $i_X : [X, X]_{Id_{\mathcal{C}}} \rightarrowtail X$ and $e_X : Id_{\mathcal{C}}(X|X) \to [X, X]_{Id_{\mathcal{C}}}$ are respectively the image and the coimage of $c_2^X : Id_{\mathcal{C}}(X|X) \to X$ given in 1.3.5. We have the following equalities:

$$c_2^X \circ T_{X,X} = \nabla_X^2 \circ \iota_2^{Id_{\mathcal{C}}} \circ T_{X,X} = \nabla_X^2 \circ \tau_X^2 \circ \iota_2^{Id_{\mathcal{C}}} = \nabla_X^2 \circ \iota_2^{Id_{\mathcal{C}}} = c_2^E$$

As $c_2^X = i_X \circ e_X$ and $i_X : [X,\, X]_{Id_{\mathcal{C}}} \rightarrowtail X$ is a monomorphism, we get $e_X = e_X \circ T_{X,X}$ implying that we have

$$(e_X)_* = (e_X)_* \circ (T_{X,X})_* = (e_X)_* \circ T_{E^{ab},X,X}$$

Hence there is a unique morphism $\overline{(e_X)_*} : \big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)_{\mathfrak{S}_2} \to \mathcal{C}(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}})$ such that

$$(e_X)_* = \overline{(e_X)_*} \circ \pi \tag{4.2.1}$$

where $\pi : \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big) \to \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)_{\mathfrak{S}_2}$ is the cokernel of the morphism $T_{E^{ab},X,X} - id : \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big) \to \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)$. We observe that $(e_X)_* : \mathcal{C}(E^{ab},\, Id_{\mathcal{C}}(X|X)) \to \mathcal{C}(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}})$ is a surjective homomorphism of Λ-$\mathcal{P}(1)$-bimodules because E^{ab} is a regular-projective object in the abelian core $Ab(\mathcal{C})$ and $e_X : Id_{\mathcal{C}}(X|X) \to [X, X]_{Id_{\mathcal{C}}}$ is a surjective homomorphism of Λ-$\mathcal{P}(1)$-bimodules. As $(e_X)_* : \mathcal{C}(E^{ab},\, Id_{\mathcal{C}}(X|X)) \to \mathcal{C}(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}})$ and $\pi : \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big) \to \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)_{\mathfrak{S}_2}$ are also surjective homomorphisms of Λ-$\mathcal{P}(1)$-bimodules, so is $\overline{(e_X)_*} : \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)_{\mathfrak{S}_2} \to \mathcal{C}(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}})$ by regularity of the category $\mathcal{C}$. Then we define a set map $\theta_X : \mathcal{C}(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}}) \to \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)_{\mathfrak{S}_2}$ by:

$$\theta_X(f) = \frac{1}{2}\,\overline{H_X(i_X \circ f \circ ab_E)^{ab}} \tag{4.2.2}$$

where $f \in \mathcal{C}(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}})$ and the set map $H_X : \mathcal{C}(X, X) \to \mathcal{C}\big(E, Id_{\mathcal{C}}(X|X)\big)$ is given in (4.1.12) (also see 4.1.18). As $\overline{(e_X)_*} : \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)_{\mathfrak{S}_2} \to \mathcal{C}(E^{ab},\, [X,\, X]_{Id_{\mathcal{C}}})$ is surjective, it suffices to prove that $\theta_X \circ \overline{(e_X)*} = id$. Let $h \in \mathcal{C}\big(E^{ab},\, Id_{\mathcal{C}}(X|X)\big)$, then we get

$$\begin{aligned}
\theta_X \circ \overline{(e_X)_*}(\overline{h}) &= \theta_X \circ \overline{(e_X)_*} \circ \pi(h) \\
&= \theta_X(e_X \circ h) \\
&= \frac{1}{2}\,\overline{H_X(i_X \circ e_X \circ h \circ ab_E)^{ab}} \\
&= \frac{1}{2}\,\overline{H_X(c_2^X \circ h \circ ab_E)^{ab}} \\
&= \frac{1}{2}\,\overline{h} + \frac{1}{2}\,\overline{T_{E^{ab},X,X}(h)}\,, \text{by (4.1.22) and 4.1.25} \\
&= \frac{1}{2}\,\overline{h} + \frac{1}{2}\,\overline{h}\,, \text{because } \overline{h} = \overline{T_{E^{ab},X,X}(h)} \\
&= \overline{h}
\end{aligned}$$

as desired. □

Remark 4.2.3. By 4.2.2, we deduce that we have the following isomorphism of $\mathcal{P}(1)$-modules

$$|Id_{\mathcal{C}}(X|X)|_{\mathfrak{S}_2} \cong |[X,X]_{Id_{\mathcal{C}}}|\,,$$

for an object X in $\mathcal{C}$ (see the notations given in 4.0.1). In the category of $\mathcal{P}$-algebras with $\mathcal{P}$ here being an operad as in 2.4.1 supposed to be 2-step nilpotent (see 1.6.4), the above isomorphism is exactly the one given in 2.4.7.

Then we define a linear 2-step nilpotent operad depending on the variety $\mathcal{C}$, already constructed by M. Hartl.

Definition 4.2.4. The 2-step nilpotent symmetric unitary (right) operad $AbOp(\mathcal{C})$ actually is an operad in the monoidal category of $\mathbb{Z}[\frac{1}{2}]$-modules as

$$AbOp(\mathcal{C})(1) = \mathcal{C}(E^{ab},\ E^{ab}) \quad \text{and} \quad AbOp(\mathcal{C})(2) = \mathcal{C}(E^{ab},\ Id_{\mathcal{C}}(E|E)) \tag{4.2.3}$$

and the second term of $AbOp(\mathcal{C})$ is endowed with the involution $T = (T_E)_*$, where $T_E : Id_{\mathcal{C}}(E|E) \to Id_{\mathcal{C}}(E|E)$ is given in 4.1.9. Then abbreviating $\mathcal{P} = AbOp(\mathcal{C})$ the only non-trivial ones among the composition operations

$$\gamma_{k_1,\ldots,k_m;m} : \mathcal{P}(k_1) \otimes \ldots \otimes \mathcal{P}(k_m) \otimes \mathcal{P}(m) \to \mathcal{P}(k_1 + \ldots + k_m)$$

are given as follows:

$$\gamma_{1;1}(a \otimes b) = a \circ b\,, \quad \gamma_{2;1}(\mu \otimes a) = \mu \circ a\,, \quad \gamma_{1,1;2}(a \otimes b \otimes \mu) = Id_{\mathcal{C}}(a'|b') \circ \mu \tag{4.2.4}$$

where $a, b \in \mathcal{P}(1)$, $\mu \in \mathcal{P}(2)$ and $a', b' \in \mathcal{C}(E,\ E)$ are respectively a (non-unique) factorization of $a \circ ab_E$ and $b \circ ab_E$ through ab_E (which exist because E is a regular-projective object). The structure linear map $\gamma_{1,1;2} : \mathcal{P}(1) \otimes \mathcal{P}(1) \otimes \mathcal{P}(2) \to \mathcal{P}(2)$ is well-defined because $Id_{\mathcal{C}}(ab_E|ab_E) : Id_{\mathcal{C}}(E|E) \to Id_{\mathcal{C}}(E^{ab}|E^{ab})$ is an isomorphism.

Now we recall the rings $\Lambda = U_E(E)$ and $\overline{\Lambda} = T_1U_E(E)$ where $U_E : \mathcal{C} \to Ab$ is a reduced standard projective functor associated with E, defined in 2.0.1. Then we remark that $\mathcal{P}(1)$ has a left Λ-module structure given by

$$\alpha.a = \alpha^{ab} \circ a = \gamma_{1;1}(\alpha^{ab} \otimes a)\,, \tag{4.2.5}$$

for $a \in \mathcal{P}(1)$ and $\alpha \in \mathcal{C}(E, E)$. As $\mathcal{C}$ is a 2-step nilpotent category, the bifunctor $Id_{\mathcal{C}}(-|-) : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is bilinear making $\mathcal{P}(2)$ into a left $\overline{\Lambda} \otimes \overline{\Lambda}$-module (see 3.17 and 3.26 of [12]) given by

$$t_1(\alpha) \otimes t_1(\beta).b = Id_{\mathcal{C}}(\alpha|\beta) \circ b = \gamma_{1,1;2}(\alpha^{ab} \otimes \beta^{ab} \otimes b) \tag{4.2.6}$$

where $\alpha, \beta \in \mathcal{C}(E, E)$ and $b \in \mathcal{P}(2)$. Now we recall that $T_E : Id_{\mathcal{C}}(E|E) \to Id_{\mathcal{C}}(E|E)$ be the involution of $Id_{\mathcal{C}}(E|E)$ obtained by taking the restriction of the canonical switch $\tau_E^2 : E + E \to E + E$ to $Id_{\mathcal{C}}(E|E)$.

Notation 4.2.5. We denote by $\mathcal{P}(2)_{\mathfrak{S}_2}$ the coinvariants of $\mathcal{P}(2)$ (i.e here the quotient of $\mathcal{P}(2)$ by the image of $T - id$) and we consider $q : \mathcal{P}(2) \twoheadrightarrow \mathcal{P}(2)_{\mathfrak{S}_2}$ the canonical quotient map, where $T = (T_E)_* : \mathcal{P}(2) \to \mathcal{P}(2)$.

Then we observe that the linear unitary (whose unity is here equal to $id \in \mathcal{P}(1) = \mathcal{C}(E^{ab},\ E^{ab})$) operad $AbOp(\mathcal{C})$ is symmetric because we have the following equality:

$$\gamma_{1,1;2}\big(a \otimes b \otimes T(\mu)\big) = Id_{\mathcal{C}}(a'|b') \circ T(\mu) = Id_{\mathcal{C}}(a'|b') \circ T_E \circ \mu = T_E \circ Id_{\mathcal{C}}(b'|a') \circ \mu = T\big(\gamma_{1,1;2}(b \otimes a \otimes \mu)\big)$$

by taking the same notations as in (4.2.4). Moreover we recall that the free $\mathcal{P}$-algebra of rank 1 is $\mathcal{F}_{\mathcal{P}} = \mathcal{P}(1) \oplus \mathcal{P}(2)_{\mathfrak{S}_2}$, see the beginning of section 2.4. Now we give specific abelian objects in the category of $\mathcal{P}$-algebras obtained from abelian objects in $\mathcal{C}$, as follows:

Proposition 4.2.6. *Let Z be an abelian object in $\mathcal{C}$. The sets $\mathcal{C}(E, Z)$ and $\mathcal{C}(E^{ab}, Z)$ are abelian objects in the category of $\mathcal{P}$-algebras, i.e. right $\mathcal{P}(1)$-modules by 1.7.5.*

Proof. It is a direct consequence of (4.1.4) (providing an abelian group structure), 4.1.6 (giving a right $\mathcal{P}(1)$-module structure) and 1.7.5 (saying that abelian objects in $Alg - \mathcal{P}$ are (right) $\mathcal{P}(1)$-modules). $\square$

Remark 4.2.7. Hence 4.2.6 says that the representable functors $\mathcal{C}(E, -)$ and $\mathcal{C}(E^{ab}, -)$ with domain $\mathcal{C}$ taking values in Gr (see (4.1.2)) restricted to the abelian core $Ab(\mathcal{C})$ preserve abelian objects. Hence their restrictions to the abelian core $Ab(\mathcal{C})$ (see 1.3.1) are functors between abelian categories.

Proposition 4.2.8. *The representable functors $\mathcal{C}(E, -)$ and $\mathcal{C}(E^{ab}, -)$ restricted to the abelian core $Ab(\mathcal{C})$ and values in $Mod_{\mathcal{P}(1)}$ are linear in the sense of 1.2.5.*

Proof. The restriction of the representable functors $\mathcal{C}(E, -)$ and $\mathcal{C}(E^{ab}, -)$ to the abelian core $Ab(\mathcal{C})$ take values in the abelian category $Mod_{\mathcal{P}(1)}$ and preserve finite coproducts. Hence their comparison morphism $\widehat{r_2^{\mathcal{C}(E,-)}}$ and $\widehat{r_2^{\mathcal{C}(E^{ab},-)}}$ (see 1.2.1) are isomorphisms (as coproducts and products coincide in the abelian category $Ab(\mathcal{C})$) implying that their second cross-effet (see 1.2.1) are trivial. $\square$

We define the natural transformation $\widehat{t_1} : U_E \Rightarrow \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big)$ between functors $\mathcal{C} \to Ab$ such that, for X an object in $\mathcal{C}$ and $\alpha \in \mathcal{C}(E, X)$, $\widehat{t_1}(\alpha) = t_1(\alpha^{ab})$. As the functor $\mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big) : \mathcal{C} \to Ab$ is linear by 1.2.6 (because it is a linear functor postcomposed by a linear functor with abelian source and target), there is a unique morphism $\overline{t_1} : T_1 U_E \Rightarrow \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big)$ such that

$$\overline{t_1} \circ t_1 = \widehat{t_1} \tag{4.2.7}$$

by 1.2.11.

Proposition 4.2.9. *The natural transformation $\overline{t_1} : T_1 U_E \Rightarrow \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big)$ is an isomorphism.*

Proof. Consider the following factorization of $\overline{t_1}$:

$$T_1 U_E \xrightarrow{(T_1 U_E)_*.ab} (T_1 U_E) \circ Ab^{\mathcal{C}} \xrightarrow{\overline{t_1}} \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big)$$

The first factor is an isomorphism by 1.4.8 since $T_1 U_E$ preserves coequalizers of reflexive graphs by 6.24 of [12]. The second factor also is an isomorphism since for an abelian object A the map $\mathcal{C}\big(E^{ab}, A\big) \to T_1 U_E(A)$, $f \mapsto t_1(f \circ ab_E)$, is an inverse of $(\overline{t_1})_A$. $\square$

We recall the rings $\Lambda = U_E(E)$ and $\overline{\Lambda} = T_1 U_E(E)$, see 2.0.2.

Corollary 4.2.10. *The ring homomorphism $(\overline{t_1})_E : \overline{\Lambda} \to \mathcal{P}(1)$ is an isomorphism.*

We now consider the natural transformation $u'_{\mathcal{C}(E^{ab}, Ab^{\mathcal{C}})} : T_1 U_E \otimes_\Lambda \mathcal{P}(1) \Rightarrow \mathcal{C}\big(E^{ab}, Ab^{\mathcal{C}}\big)$ such that, for any object X in $\mathcal{C}$, $\alpha \in \mathcal{C}(E, X)$ and $f \in \mathcal{P}(1)$, we have

$$\big(u'_{\mathcal{C}(E^{ab}, Ab^{\mathcal{C}})}\big)_X \big(t_1(\alpha) \otimes f\big) = (\alpha^{ab})_*(f) = \alpha^{ab} \circ f\,, \tag{4.2.8}$$

see 3.5 of [12]. Combining 4.2.9 and 4.2.10 we obtain

Proposition 4.2.11. *The natural transformation*

$$u'_{\mathcal{C}(E^{ab}, Ab^{\mathcal{C}})} : T_1 U_E \otimes_\Lambda \mathcal{P}(1) \Rightarrow \mathcal{C}\big(E^{ab}, Ab^{\mathcal{C}}\big)$$

(defined in (4.2.8)) is an isomorphism living in the category of functors from $\mathcal{C}$ to the category of $\overline{\Lambda}$-$\mathcal{P}(1)$-bimodules.

4.3 The graded algebra over $AbOp(\mathcal{C})$

As a first "approximation" of the intended Lazard correspondence functor, we here consider the *graded algebra* over the linear symmetric unitary operad $\mathcal{P} = AbOp(\mathcal{C})$ (see (4.2.3)) associated with any object in $\mathcal{C}$, by taking the associated graded of its lower central series. Then we prove that this defines a quadratic functor with domain $\mathcal{C}$ and values in $Alg-\mathcal{P}$. Finally we exhibit its corresponding quadratic $\mathcal{C}$-module over $\mathcal{P}$. In the following sections we will modify the latter by a suitable "twist" (or "perturbation") in order to construct the quadratic $\mathcal{C}$-module over $\mathcal{P}$ whose associated functor, in contrast with the associated graded functor, is an equivalence, thus establishing the desired Lazard correspondence.

For X an object in $\mathcal{C}$ we associate its graded $\mathcal{P}$-algebra as follows:

$$\bigoplus_{n\geq 1} \gamma_n^{Id_\mathcal{C}}(X)/\gamma_{n+1}^{Id_\mathcal{C}}(X) = X^{ab} \oplus [X,\, X]_{Id_\mathcal{C}}$$

in the category $\mathcal{C}$, because $\gamma_n^{Id_\mathcal{C}}(X) = 0$ for $n \geqslant 3$ (as $\mathcal{C}$ is a 2-step nilpotent category) and $X^{ab} = X/\gamma_2^{Id_\mathcal{C}}(X) = \gamma_1^{Id_\mathcal{C}}(X)/\gamma_2^{Id_\mathcal{C}}(X)$. Hence we have the following abelian groups isomorphism:

$$|X^{ab}| \oplus |[X,\, X]_{Id_\mathcal{C}}| \cong \mathcal{C}(E^{ab},\, X^{ab}) \oplus \mathcal{C}(E^{ab},\, [X,\, X]_{Id_\mathcal{C}}) = Grad(X)$$

Note that $|[X,\, X]_{Id_\mathcal{C}}|$ is an abelian group because $[X,\, X]_{Id_\mathcal{C}}$ is a central subobject of X in $\mathcal{C}$ (as $\mathcal{C}$ is a 2-step nilpotent category). Then the structure linear maps of $Grad(X)$ are given by:

- the map $\lambda_1^{Grad} : Grad(X) \otimes \mathcal{P}(1) \to Grad(X)$ is such that

$$\lambda_1^{Grad}((f,h)\otimes a) = \big(f\circ a,\; h\circ a\big)$$

- the map $\lambda_2^{Grad} : Grad(X)^{\otimes 2} \otimes \mathcal{P}(2) \to Grad(X)$ is such that

$$\lambda_2^{Grad}\big((f_1,h_1)\otimes(f_2,h_2)\otimes b\big) = \big(0,\; e_X \circ Id_\mathcal{C}(ab_X|ab_X)^{-1} \circ Id_\mathcal{C}(f_1|f_2) \circ Id_\mathcal{C}(ab_E|ab_E)\circ b\big)$$

  Here we recall that the morphism $Id_\mathcal{C}(ab_X|ab_X) : Id_\mathcal{C}(X|X) \to Id_\mathcal{C}(X^{ab}|X^{ab})$ is an isomorphism by 4.1.14.

This gives rise to the functor $Grad : \mathcal{C} \to Alg - \mathcal{P}$ that is defined on morphisms by $Grad(f) = (f^{ab},\, \gamma_2^{Id_\mathcal{C}}(f))$, for any morphism f in $\mathcal{C}$.

4.3.1 The second cross-effect of the functor $Grad$ from $\mathcal{C}$ to $AbOp(\mathcal{C})$-algebras

Before determining the second cross-effect of $Grad : \mathcal{C} \to Alg-\mathcal{P}$, we need the following proposition:

Proposition 4.3.1. *Let X and Y be two objects in $\mathcal{C}$, then the second cross-effect of $\gamma_2^{Id_\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is given by*

$$\gamma_2^{Id_\mathcal{C}}(X|Y) = Id_\mathcal{C}(X|Y)$$

and the kernel $\iota_2^{\gamma_2^{Id_\mathcal{C}}} : \gamma_2^{Id_\mathcal{C}}(X|Y) \rightarrowtail \gamma_2^{Id_\mathcal{C}}(X+Y)$ of the comparison morphism $\widehat{r_2^{\gamma_2^{Id_\mathcal{C}}}} : \gamma_2^{Id_\mathcal{C}}(X+Y) \to \gamma_2^{Id_\mathcal{C}}(X)\times\gamma_2^{Id_\mathcal{C}}(Y)$ is the unique factorization of $\iota_2^{Id_\mathcal{C}} : Id_\mathcal{C}(X|Y) \rightarrowtail X+Y$ through $i_{X+Y} : \gamma_2^{Id_\mathcal{C}}(X+Y) \rightarrowtail X+Y$, i.e. $\iota_2^{Id_\mathcal{C}} = i_{X+Y} \circ \iota_2^{\gamma_2^{Id_\mathcal{C}}}$.

Proof. Let X and Y be two objects in $\mathcal{C}$. Moreover we denote by $i_X^2 : X \rightarrowtail X+Y$, $i_Y^2 : Y \rightarrowtail X+Y$ respectively the injections of the first and the second summands, and by $r_X^2 : X+Y \to X$, $r_Y^2 : X+Y \to Y$ their respective retractions. We consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Id_{\mathcal{C}}(X|Y) \arrow[r, tail, "\iota_2^{Id_{\mathcal{C}}}"] & X+Y \arrow[r, two heads, "\widehat{r_2^{Id_{\mathcal{C}}}}"] & X \times Y \arrow[r] & 0 \\
0 \arrow[r] & Id_{\mathcal{C}}(X|Y) \arrow[u, equal] \arrow[r, tail, dashed, "\overline{k}"] & \gamma_2^{Id_{\mathcal{C}}}(X+Y) \arrow[u, tail, "i_{X+Y}"] \arrow[r, two heads, "\widehat{r_2^{\gamma_2^{Id_{\mathcal{C}}}}}"] & \gamma_2^{Id_{\mathcal{C}}}(X) \times \gamma_2^{Id_{\mathcal{C}}}(Y) \arrow[u, tail, "i_X \times i_Y"] \arrow[r] & 0 \\
0 \arrow[r] & Id_{\mathcal{C}}(X|Y) \oplus Id_{\mathcal{C}}(Y|X) \arrow[u, two heads, "\left(id,\, T_{YX}\right)"] \arrow[r, tail, "k"] & Id_{\mathcal{C}}(X+Y|X+Y) \arrow[u, two heads, "e_{X+Y}"] \arrow[r, two heads, "r_2^{\widehat{Id_{\mathcal{C}}(-|-)\cdot\Delta^2}}"] & Id_{\mathcal{C}}(X|X) \times Id_{\mathcal{C}}(Y|Y) \arrow[u, two heads, "e_X \times e_Y"] \arrow[r] & 0
\end{tikzcd}$$

where $k = \big(Id_{\mathcal{C}}(i_X^2|i_Y^2),\ Id_{\mathcal{C}}(i_Y^2|i_X^2)\big)$ is the kernel of $r_2^{\widehat{Id_{\mathcal{C}}(-|-)\cdot\Delta^2}}$ (see the Lemma 1.20 of [12]), $T_{YX} : Id_{\mathcal{C}}(Y|X) \to Id_{\mathcal{C}}(X|Y)$ is the restrinction of the canonical switch $\tau_{YX}^2 : Y+X \to X+Y$ on $Id_{\mathcal{C}}(Y|X)$. The two right-hand rectangles commute by naturality of the comparison morphism $\widehat{r_2^{\gamma_2^{Id_{\mathcal{C}}}}} : \gamma_2^{Id_{\mathcal{C}}}(X+Y) \to \gamma_2^{Id_{\mathcal{C}}}(X) \times \gamma_2^{Id_{\mathcal{C}}}(Y)$ in $\gamma_2^{Id_{\mathcal{C}}}$. Then it remains to prove that the outside left-hand rectangle commutes. It commutes because we have

$$c_2^{X+Y} \circ Id_{\mathcal{C}}(i_X^2|i_Y^2) = (i_X^2,\ i_Y^2) \circ \iota_2^{Id_{\mathcal{C}}} = \iota_2^{Id_{\mathcal{C}}}$$

and

$$c_2^{X+Y} \circ Id_{\mathcal{C}}(i_Y^2|i_X^2) = (i_Y^2,\ i_X^2) \circ \iota_2^{Id_{\mathcal{C}}} = (i_X^2,\ i_Y^2) \circ \tau_{YX}^2 \circ \iota_2^{Id_D} = \iota_2^{Id_{\mathcal{C}}} \circ T_{YX}$$

We observe that $(id,\ T_{YX}) = \nabla^2 \circ (id \oplus T_{YX})$ is a regular epimorphism as a composite of two regular epimorphisms. Hence we deduce that $Id_{\mathcal{C}}(X|Y)$ is the image of the morphism $\iota_2^{Id_{\mathcal{C}}} \circ (id,\ T_{YX})$. By the universal property of the image, there is a unique morphism $\overline{k} : Id_{\mathcal{C}}(X|Y) \rightarrowtail \gamma_2^{Id_{\mathcal{C}}}(X+Y)$ such that

$$\iota_2^{Id_{\mathcal{C}}} = i_{X+Y} \circ \overline{k} \tag{4.3.1}$$

Moreover we get

$$e_{X+Y} \circ k = \overline{k} \circ \big(id, T_{YX}\big) \tag{4.3.2}$$

because we have the relation (4.3.1), $c_2^{X+Y} \circ k = \iota_2^{Id_{\mathcal{C}}} \circ (id,\ T_{Y,X})$, and $i_{X+Y} : \gamma_2^{Id_{\mathcal{C}}}(X+Y)$ is a monomorphism. It remains to prove that the morphism $\overline{k} : Id_{\mathcal{C}}(X|Y) \rightarrowtail \gamma_2^{Id_{\mathcal{C}}}(X+Y)$ satisfies the universal property of the kernel of the comparison morphism $\widehat{r_2^{\gamma_2^{Id_{\mathcal{C}}}}}$. Let $f : Z \to \gamma_2^{Id_{\mathcal{C}}}(X+Y)$ any morphism in $\mathcal{C}$ such that $\widehat{r_2^{\gamma_2^{Id_{\mathcal{C}}}}} \circ f = 0$. By postcomposing with $i_X \times i_Y$, we obtain

$$0 = (i_X \times i_Y) \circ \widehat{r_2^{\gamma_2^{Id_{\mathcal{C}}}}} \circ f = \widehat{r_2^{Id_{\mathcal{C}}}} \circ i_{X+Y} \circ f$$

By definition of $\iota_2^{Id_{\mathcal{C}}}$, there is a unique morphism $\tilde{f} : Z \to Id_{\mathcal{C}}(X|Y)$ such that $\iota_2^{Id_{\mathcal{C}}} \circ \tilde{f} = i_{X+Y} \circ f$. By (4.3.2), we get

$$i_{X+Y} \circ \overline{k} \circ \tilde{f} = \iota_2^{Id_{\mathcal{C}}} \circ \tilde{f} = i_{X+Y} \circ f$$

As $i_{X+Y} : \gamma_2^{Id_{\mathcal{C}}}(X+Y) \rightarrowtail X+Y$ is a monomorphism, we have $\overline{k} \circ \tilde{f} = f$. It finishes the proof. $\square$

Corollary 4.3.2. *Let X and Y be two objects in $\mathcal{C}$, then the second cross-effect of Grad is given by:*

$$Grad(X|Y) = I\Big(\mathcal{C}\big(E^{ab},\ Id_{\mathcal{C}}(X|Y)\big)\Big)$$

and the kernel $\iota_2^{Grad} : Grad(X|Y) \rightarrowtail Grad(X+Y)$ of the comparison morphism $\widehat{r_2^{Grad}} : Grad(X+Y) \to Grad(X) \times Grad(Y)$ is defined by:

$$\iota_2^{Grad}(\alpha) = (0,\ \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ \alpha)$$

where $\alpha \in \mathcal{C}(E^{ab},\ Id_{\mathcal{C}}(X|Y))$, and $I : Mod_{\mathcal{P}(1)} = Ab(Alg - \mathcal{P}) \to Alg - \mathcal{P}$ is the inclusion functor (see the notation given in 1.3.3 for $\mathcal{C} = Alg - \mathcal{P}$).

Proof. Let $(f,\ h) \in Grad(X+Y)$ such that $\widehat{r_2^{Grad}}(f,\ h) = 0$, i. e.

$$\begin{cases} \big((r_X^2)^{ab} \circ f,\ \gamma_2^{Id_{\mathcal{C}}}(r_X^2) \circ h\big) = 0 \\ \big((r_Y^2)^{ab} \circ f,\ \gamma_2^{Id_{\mathcal{C}}}(r_Y^2) \circ h\big) = 0 \end{cases} \iff \begin{cases} \widehat{r_2^{Ab^{\mathcal{C}}}} \circ f = 0 \\ \widehat{r_2^{\gamma_2^{Id_{\mathcal{C}}}}} \circ h = 0 \end{cases} \iff \begin{cases} f = 0 \\ \widehat{r_2^{\gamma_2^{Id_{\mathcal{C}}}}} \circ h = 0 \end{cases}$$

because $Ab^{\mathcal{C}} : \mathcal{C} \to Ab(\mathcal{C})$ is a linear functor which is equivalent to say that $\widehat{r_2^{Ab^{\mathcal{C}}}} : (X+Y)^{ab} \to X^{ab} \times Y^{ab}$ is an isomorphism by 1.3 of [12]. By 4.3.1, there is a unique morphism $\tilde{h} : E^{ab} \to Id_{\mathcal{C}}(X|Y)$ such that $h = \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ \tilde{h}$. Then we have

$$(f,\ h) = (0,\ h) = (0,\ \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ \tilde{h}) = \iota_2^{Grad}(\tilde{h})$$

as desired. □

Corollary 4.3.3. *The functor $Grad : \mathcal{C} \to Alg - \mathcal{P}$ is quadratic.*

Proof. The bifunctor $Grad(-|-) : \mathcal{C} \times \mathcal{C} \to Alg - \mathcal{P}$ (whose expression is given in 4.3.2) is bilinear because the functor $\mathcal{C}(E^{ab},\ -) : \mathcal{C} \to Ab$ restricted to $Ab(\mathcal{C})$ is linear and $Id_{\mathcal{C}}(-|-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bilinear bifunctor (because $\mathcal{C}$ is a 2-step nilpotent category). Hence it proves that the functor $Grad : \mathcal{C} \to Alg - \mathcal{P}$ is quadratic by Remark 1.2.13. □

4.3.2 The linearization of the functor *Grad*

We now determine the linearization of the functor $Grad : \mathcal{C} \to Alg - \mathcal{P}$.

Proposition 4.3.4. *Let X be an object in $\mathcal{C}$. The linearization of $Grad : \mathcal{C} \to Alg - \mathcal{P}$ is given by*

$$T_1(Grad)(X) = I\Big(\mathcal{C}(E^{ab},\ X^{ab})\Big)$$

where $t_1^{Grad} : Grad(X) \to T_1(Grad)(X)$ is the projection onto the first summand, and $I : Mod_{\mathcal{P}(1)} = Ab(Alg - \mathcal{P}) \to Alg - \mathcal{P}$ is the inclusion functor (see the notation given in 1.3.3 for $\mathcal{C} = Alg - \mathcal{P}$).

Proof. Let X be an object in $\mathcal{C}$. We give another expression of the morphism $(S_2^{Grad})_X : Grad(X|X) \to Grad(X)$ as follows:

$$(S_2^{Grad})_X(\alpha) = Grad(\nabla_X^2) \circ \iota_2^{Grad}(\alpha) = Grad(\nabla_X^2)(0, \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ \alpha) = (0, \gamma_2^{Id_{\mathcal{C}}}(\nabla_X^2) \circ \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ \alpha) = (0, (S_2^{\gamma_2^{Id_{\mathcal{C}}}})_X \circ \alpha)$$

where $\alpha \in \mathcal{C}(E^{ab},\ Id_{\mathcal{C}}(X|X))$. Let $g : E^{ab} \to [X,\ X]_{Id_{\mathcal{C}}}$ be any morphism in $Ab(\mathcal{C})$. As E^{ab} is a regular-projective object in $Ab(\mathcal{C})$, there is a (non-unique) morphism $\tilde{g} : E^{ab} \to Id_{\mathcal{C}}(X|X)$ such that

$$g = e_X \circ \tilde{g}$$

Then we have

$$
\begin{aligned}
i_X \circ (S_2^{\gamma_2^{Id_{\mathcal{C}}}})_X \circ \tilde{g} &= i_X \circ \gamma_2^{Id_{\mathcal{C}}}(\nabla_X^2) \circ \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ \tilde{g} \\
&= \nabla_X^2 \circ i_{X+X} \circ \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ \tilde{g} \\
&= \nabla_X^2 \circ \iota_2^{Id_{\mathcal{C}}} \circ \tilde{g}\,, \quad \text{by definition of } \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \\
&= c_2^X \circ \tilde{g} \\
&= i_X \circ e_X \circ \tilde{g} \\
&= i_X \circ g
\end{aligned}
$$

Hence we have $g = (S_2^{\gamma_2^{Id_{\mathcal{C}}}})_X \circ \tilde{g}$ because $i_X : [X,\, X]_{Id_{\mathcal{C}}} \rightarrowtail X$ is a monomorphism. It proves that the canonical injection of the second summand of $Grad(X)$ is the image of the morphism $(S_2^{Grad})_X : Grad(X|X) \to Grad(X)$. Finally we deduce that the projection of $Grad(X)$ onto the first summand is the cokernel of $(S_2^{Grad})_X : Grad(X|X) \to Grad(X)$, denoted by $(t_1^{Grad})_X : Grad(X) \to T_1(Grad)(X) = \mathcal{C}(E^{ab},\, X^{ab})$. $\square$

4.3.3 The quadratic $\mathcal{C}$-module over $AbOp(\mathcal{C})$ associated with the functor $Grad$

In this part, we give the quadratic $\mathcal{C}$-module over $\mathcal{P} = AbOp(\mathcal{C})$ (see (4.2.3)) associated with the quadratic functor $Grad : \mathcal{C} \to Alg - \mathcal{P}$.

Remark 4.3.5. We first observe that the 2-*divisibility condition* ensures that $Grad(E)$ is isomorphic to $\mathcal{F}_{\mathcal{P}}$ the free $\mathcal{P}$-algebra of rank 1 because we get

$$Grad(E) = \mathcal{C}\big(E^{ab}, E^{ab}\big) \oplus \mathcal{C}\big(E^{ab}, [E,E]_{Id_{\mathcal{C}}}\big) = \mathcal{P}(1) \oplus \mathcal{C}\big(E^{ab}, [E,E]_{Id_{\mathcal{C}}}\big) \cong \mathcal{P}(1) \oplus \mathcal{P}(2)_{\mathfrak{S}_2} = \mathcal{F}_{\mathcal{P}}$$

by 4.2.2 (replacing X with E) because the 2-*divisibility condition* holds.

As $Grad : \mathcal{C} \to Alg - \mathcal{P}$ is a quadratic functor by 4.3.3, it makes sense to consider $\mathbb{S}_2^{\mathcal{P}}(Grad)$ its corresponding quadratic $\mathcal{C}$-module over $\mathcal{P}$ where $\mathbb{S}_2^{\mathcal{P}} : Quad(\mathcal{C},\, Alg - \mathcal{P}) \to QMod_{\mathcal{C}}^{\mathcal{P}}$ is the functor defined in 2.4.27.

Remark 4.3.6. Recall that $\theta_E : \mathcal{C}(E^{ab},\, [E,\, E]_{Id_{\mathcal{C}}}) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ is the isomorphism of (right) $\mathcal{P}(1)$-modules defined in (4.2.2) and that $\overline{e_E} : \mathcal{P}(2)_{\mathfrak{S}_2} \to \mathcal{C}(E^{ab},\, [E,\, E]_{Id_{\mathcal{C}}})$ is its inverse given in 4.2.1, then we have another left Λ-module structure on $\mathcal{F}_{\mathcal{P}}$ given by

$$
\begin{aligned}
\alpha.(f,\, \overline{h}) &= (id \oplus \theta_E) \circ Grad(\alpha) \circ (id \oplus \overline{e_E})(f,\, \overline{h}) \\
&= \big(\alpha^{ab} \circ f,\, \overline{Id_{\mathcal{C}}(\alpha|\alpha) \circ h}\big) \\
&= \big(\gamma_{1;1}(\alpha^{ab} \otimes f),\, \overline{\gamma_{1,1;2}(\alpha^{ab} \otimes \alpha^{ab} \otimes h)}\big)\,,
\end{aligned}
$$

for $\alpha \in \mathcal{C}(E,\, E)$, $f \in \mathcal{P}(1)$, $h \in \mathcal{P}(2)$ and $\overline{h}$ denotes the equivalence class of h in $\mathcal{P}(2)_{\mathfrak{S}_2}$. The structure linear maps of the linear symmetric unitary operad $\mathcal{P}$ are defined in (4.2.4).

The next result provides another quadratic $\mathcal{C}$-module over $\mathcal{P}$ isomorphic to $\mathbb{S}_2^{\mathcal{P}}(Grad)$ in $QMod_{\mathcal{C}}^{\mathcal{P}}$.

Proposition 4.3.7. *If the 2-divisibility condition holds, the quadratic $\mathcal{C}$-module over $\mathcal{P}$ corresponding to Grad is isomorphic to the following one in the category $QMod_{\mathcal{C}}^{\mathcal{P}}$:*

$$\begin{tikzcd}
T_{11}cr_2(U_E)(E,\ E) \otimes_\Lambda \mathcal{F}_{\mathcal{P}} \arrow[r, "H^{Grad}"] & \mathcal{P}(2) \arrow[r, "\dot{q}"] & \mathcal{F}_{\mathcal{P}} \\
T_{11}cr_2(U_E)(E,\ E) \otimes_\Lambda \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[r, "\widehat{H}"'] \arrow[u, "id\otimes_\Lambda i_2"] & \mathcal{P}(2) \arrow[r, "q"'] \arrow[u, equal] & \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[u, "i_2"']
\end{tikzcd}$$

Here

- *we define the map $H^{Grad} : T_{11}cr_2(U_E)(E,\ E) \otimes \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(2)$ by*

$$H^{Grad}\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f, \overline{h})\big) = \widehat{H}\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda \overline{h}\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(h + T(h)\big)$$

  where $\xi \in \mathcal{C}(E,\ E^{+2})$, $f \in \mathcal{P}(1)$ and $h \in \mathcal{P}(2)$.

- *$i_2 : \mathcal{P}(2)_{\mathfrak{S}_2} \rightarrowtail \mathcal{F}_{\mathcal{P}}$ is the canonical injection of the second summand and $\dot{q} : \mathcal{P}(2) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ is the map defined by $\dot{q} = i_2 \circ q$.*

Proof. By 2.4.27, the quadratic $\mathcal{C}$-module over $\mathcal{P}$ corresponding to $Grad$ is given by:

$$\begin{tikzcd}
T_{11}cr_2(U_E)(E,\ E) \otimes_\Lambda Grad(E) \arrow[r, "H_E^{Grad}"] & Grad(E|E) \arrow[r, "(S_2^{Grad})_E"] & Grad(E) \\
T_{11}cr_2(U_E)(E,\ E) \otimes (T_1Grad(E)^{\otimes 2} \otimes_S \mathcal{P}(2)) \arrow[r, "\widehat{H_E}^{Grad}"'] \arrow[u, "id\otimes_\Lambda\left((\overline{\lambda_2^{Grad}})_E\circ(\overline{\phi_E^{Grad}})^{-1}\right)"] & T_1Grad(E)^{\otimes 2} \otimes_{R\otimes R} \mathcal{P}(2) \arrow[r, "q_E^{Grad}"'] \arrow[u, "cr_2\left(\overline{\lambda_2^{Grad}}\circ(\overline{\phi^{Grad}})^{-1}\right)_{E,E}"] & T_1Grad(E)^{\otimes 2} \otimes_S \mathcal{P}(2) \arrow[u, "(\overline{\lambda_2^{Grad}})_E\circ(\overline{\phi_E^{Grad}})^{-1}"]
\end{tikzcd}$$

By 4.3.4 and 4.3.2, we know that $T_1Grad(E) = \mathcal{P}(1)$ and $Grad(E|E) = \mathcal{P}(2)$. The calculations are the same than those in the proof of 3.6.9. It is just necessary to focus on the morphism H^{Grad}. Let $\xi \in \mathcal{C}(E,\ E^{+2})$ and $(f,\ \overline{h}) \in \mathcal{F}_{\mathcal{P}}$. We define the morphism $H^{Grad} : T_{11}cr_2(U_E)(E,\ E) \otimes_\Lambda \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(2)$ by

$$H^{Grad} = H_E^{Grad} \circ \big(id \otimes (id \oplus \overline{e_E})\big) \tag{4.3.3}$$

where $\overline{e_E} : \mathcal{P}(2)_{\mathfrak{S}_2} \to \mathcal{C}(E^{ab},\ [E,\ E]_{Id_{\mathcal{C}}})$ is the isomorphism given in (4.2.1). Then we have explicitly

$$\begin{aligned}
H^{Grad}\big(t_{11}(\rho_2(\xi)) \otimes (f,\ \overline{h})\big) &= H_E^{Grad}\big(t_{11}(\rho_2(\xi)) \otimes (f,\ \overline{e_E}(\overline{h}))\big)\,, (4.3.3) \\
&= H_E^{Grad}\big(t_{11}(\rho_2(\xi)) \otimes (f,\ e_E \circ h)\big)\,, \text{by (4.2.1)} \\
&= cr_2(u'_{Grad})_{E,E}\big(\rho_2(\xi) \otimes (f,\ e_E \circ h)\big)\,, \text{by 2.1.7}
\end{aligned}$$

We recall that $u'_{Grad} : U_E \otimes_\Lambda Grad(E) \Rightarrow Grad$ is the natural transformation given by

$$(u'_{Grad})_X(\alpha \otimes (g,\ b)) = Grad(\alpha)(g,\ b)$$

where X is an object in $\mathcal{C}$, $\alpha \in \mathcal{C}(E,\, X)$ and $(g,\, b) \in Grad(E)$ (see 2.1.1). Now we have

$$\begin{aligned}
&\iota_2^{Grad} \circ H^{Grad}\big(t_{11}(\rho_2(\xi)) \otimes (f,\, \overline{h})\big)\\
&= \iota_2^{Grad} \circ cr_2(u'_{Grad})_{E,E}\big(\rho_2(\xi) \otimes (f,\, e_E \circ h)\big)\\
&= (u'_{Grad})_{E^{+2}} \circ (\iota_2 \otimes id)\big(\rho_2(\xi) \otimes (f,\, e_E \circ h)\big)\\
&= (u'_{Grad})_{E^{+2}}\big((\iota_2 \circ \rho_2(\xi)) \otimes (f,\, e_E \circ h)\big)\\
&= (u'_{Grad})_{E^{+2}}\big(\xi \otimes (f,\, e_E \circ h)\big) - (u'_{Grad})_{E^{+2}}\big((i_1^2 \circ r_1^2 \circ \xi) \otimes (f,\, e_E \circ h)\big)\\
&\qquad - (u'_{Grad})_{E^{+2}}\big((i_2^2 \circ r_2^2 \circ \xi) \otimes (f,\, e_E \circ h)\big)\\
&= Grad(\xi)(f,\, e_E \circ h) - Grad(i_1^2 \circ r_1^2 \circ \xi)(f,\, e_E \circ h) - Grad(i_2^2 \circ r_2^2 \circ \xi)(f,\, e_E \circ h)\\
&= \big(\xi^{ab} \circ f,\, \gamma_2^{Id_{\mathcal{C}}}(\xi) \circ e_E \circ h\big) - \big((i_1^2 \circ r_1^2 \circ \xi)^{ab} \circ f,\, \gamma_2^{Id_{\mathcal{C}}}(i_1^2 \circ r_1^2 \circ \xi) \circ h\big)\\
&\qquad - \big((i_2^2 \circ r_2^2 \circ \xi)^{ab} \circ f,\, \gamma_2^{Id_{\mathcal{C}}}(i_2^2 \circ r_2^2 \circ \xi) \circ h\big)
\end{aligned}$$

As the functor $Ab^{\mathcal{C}} : \mathcal{C} \to Ab(\mathcal{C})$ is linear, we get

$$\xi^{ab} = \big(i_1^2 \circ r_1^2 \circ \xi\big)^{ab} + \big(i_2^2 \circ r_2^2 \circ \xi\big)^{ab}$$

by 3.6 of [12]. Hence we have

$$\begin{aligned}
&\iota_2^{Grad} \circ H^{Grad}\big(t_{11}(\rho_2(\xi)) \otimes (f,\, \overline{h})\big)\\
&= \Big(0,\, \gamma_2^{Id_{\mathcal{C}}}(\xi) \circ e_E \circ h - \gamma_2^{Id_{\mathcal{C}}}(i_1^2 \circ r_1^2 \circ \xi) \circ e_E \circ h - \gamma_2^{Id_{\mathcal{C}}}(i_2^2 \circ r_2^2 \circ \xi) \circ e_E \circ h\Big)\\
&= \Big(0,\, e_{E^{+2}} \circ Id_{\mathcal{C}}(\xi|\xi) \circ h - e_{E^{+2}} \circ Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | i_1^2 \circ r_1^2 \circ \xi) \circ h\\
&\qquad - e_{E^{+2}} \circ Id_{\mathcal{C}}(i_2^2 \circ r_2^2 \circ \xi | i_2^2 \circ r_2^2 \circ \xi) \circ h\Big)\\
&= \Big(0,\, e_{E^{+2}} \circ Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | i_2^2 \circ r_2^2 \circ \xi) \circ h + Id_{\mathcal{C}}(i_2^2 \circ r_2^2 \circ \xi | i_1^2 \circ r_1^2 \circ \xi) \circ h\Big)\text{, by 3.6 of [12]}
\end{aligned}$$

As the bifunctor $Id_{\mathcal{C}}(-|-) : \mathcal{C}^{\times 2} \to \mathcal{C}$ is bilinear, it implies that

$$\begin{aligned}
Id_{\mathcal{C}}(\xi|\xi) &= Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | i_1^2 \circ r_1^2 \circ \xi) + Id_{\mathcal{C}}(i_2^2 \circ r_2^2 \circ \xi | i_2^2 \circ r_2^2 \circ \xi)\\
&\qquad + Id_{\mathcal{C}}(i_1^2 \circ r_1^2 \circ \xi | i_2^2 \circ r_2^2 \circ \xi) + Id_{\mathcal{C}}(i_2^2 \circ r_2^2 \circ \xi | i_1^2 \circ r_1^2 \circ \xi)
\end{aligned}$$

by 3.6 of [12]. Then we get

$$\begin{aligned}
&\iota_2^{Grad} \circ H^{Grad}\big(t_{11}(\rho_2(\xi)) \otimes (f,\, \overline{h})\big)\\
&= \Big(0,\, e_{E^{+2}} \circ \big(Id_{\mathcal{C}}(i_1^2|i_2^2),\, Id_{\mathcal{C}}(i_1^2|i_2^2)\big) \circ \big(Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \oplus Id_{\mathcal{C}}(r_2^2 \circ \xi | r_1^2 \circ \xi)\big) \circ h\Big)\\
&= \Big(0,\, \iota_2^{\gamma_2^{Id_{\mathcal{C}}}} \circ (id,\, T_E) \circ \big(Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \oplus Id_{\mathcal{C}}(r_2^2 \circ \xi | r_1^2 \circ \xi)\big) \circ h\Big)\text{, by (4.3.2)}\\
&= \iota_2^{Grad}\Big((id,\, T_E) \circ (Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \oplus Id_{\mathcal{C}}(r_2^2 \circ \xi | r_1^2 \circ \xi)) \circ h\Big)\text{, by 4.3.2}\\
&= \iota_2^{Grad}\Big(Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ h + T_E \circ Id_{\mathcal{C}}(r_2^2 \circ \xi | r_1^2 \circ \xi) \circ h\Big)\\
&= \iota_2^{Grad}\Big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).h + Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ T_E \circ h\Big)\text{, by 3.17 of [12]}\\
&= \iota_2^{Grad}\Big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).h + t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).T(h)\Big)
\end{aligned}$$

As $\iota_2^{Grad} : Grad(E|E) \rightarrowtail Grad(E + E)$ is a monomorphism, it proves the result. $\square$

We also denote $Grad : \mathcal{C} \to Alg - \mathcal{P}$ the quadratic functor corresponding to the quadratic $\mathcal{C}$-module over $\mathcal{P}$ given in 4.3.7. Note that the functor $Grad : \mathcal{C} \to Alg - \mathcal{P}$ is not an equivalence of categories because the map $Grad : \mathcal{C}(E,\, E) \to Alg - \mathcal{P}(\mathcal{F}_{\mathcal{P}},\, \mathcal{F}_{\mathcal{P}})$ is not a bijection. In order to see this first note that we have the canonical isomorphism $ev_{(id,\overline{0})} : Alg - \mathcal{P}(\mathcal{F}_{\mathcal{P}},\, \mathcal{F}_{\mathcal{P}}) \to \mathcal{F}_{\mathcal{P}}\, , f \mapsto f(id, \overline{0})$. Then, for $\alpha \in \mathcal{C}(E,\, E)$, we have

$$ev_{(id,\overline{0})} \circ Grad(\alpha) = \alpha.(id,\, \overline{0}) = (\alpha^{ab},\, \overline{0})$$

where $\alpha.(id,\, \overline{0})$ is given in 4.3.6. Hence we deduce that the map $Grad : \mathcal{C}(E,\, E) \to Alg - \mathcal{P}(\mathcal{F}_{\mathcal{P}},\, \mathcal{F}_{\mathcal{P}})$ is not a bijection.
This observation leads us to modify the left Λ-module structure for $\mathcal{F}_{\mathcal{P}}$ and the expression of the morphism $H^{Grad} : T_{11}cr_2(U_E)(E,\, E) \otimes_{\Lambda} \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(2)$ (present in 4.3.7) by 3.6.9. It is the first step to determine the Lazard equivalence with domain $\mathcal{C}$ and values in $Alg - \mathcal{P}$. Before tackling this, we need to provide a version of the five lemma for quadratic maps relative a (normal) subgroup.

4.4 The quadratic five lemma

In this part, we first give the definition of quadratic maps, already given in 2 of [4] and in [18] (as weakly quadratic maps), and those relative to a subgroup introduced by M. Hartl. Then a five lemma is provided for quadratic maps relative to a subgroup. It recovers the classical five lemma in the category of groups by considering the homomorphisms of groups (equivalently saying linear maps) as a particular case of quadratic maps.
Let $f : G \to H$ be some function between arbitrary groups. We shall however, write the group law of H additively since in many applications H is abelian, and in those in [4], where H is genuinely nonabelian, it is written additively anyway to match the conventions in homotopy theory which originally motivated these developments. Define the deviation function, or the cross-effect of f, to be the map:

$$d_f : G \times G \to H \quad \text{by} \quad d(a,\, b) = f(a + b) - \big(f(a) + f(b)\big) \tag{4.4.1}$$

Furthermore, let I_f and D_f denote respectively the subgroup of H generated by $Im(f)$ and $Im(d_f)$.

Definition 4.4.1. We say that f as above is

1. *linear* if $d_f = 0$, i.e. f is a group homomorphism;
2. *quadratic* if d_f is bilinear and D_f is central in I_f, or more explicitly, $\forall a, b, c \in G$, $[d_f(a; b), f(c)] = 0$.

We also need the relative version of quadratic maps as follows. Let A be a subgroup of G, then we say that f as above is quadratic relative A if f is quadratic and $d_f(A \times G) = d_f(G \times A) = 0$. Note that f is quadratic if, and only if, it is quadratic relative the trivial subgroup $\{0\}$. Now we establish the five lemma for quadratic maps relative a subgroup, here called *the quadratic five lemma*:

Lemma 4.4.2. *Given a commutative diagram where the two horizontable short sequences are exact in the category of groups:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail] \arrow[d, "f_1"] & G \arrow[r, "q_1"] \arrow[d, "f_2"] & U \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B \arrow[r, tail] & H \arrow[r, "q_2"] & V \arrow[r] & 0
\end{tikzcd}$$

If f_2 is a quadratic map relative A, f_1 is a reduced (i.e. $f_1(0) = f_2(0) = 0$) bijection map and f_3 is a bijection, then f_2 is a bijection.

Proof. This proof is mainly the same as the classical five lemma, except we need the following property $d_{f_2}(A \times G) = d_{f_2}(G \times A) = 0$ which holds because f is a quadratic map relative A.

1. First we prove that f_2 is surjective. Let h be any element in H. As $q_2(h) \in V$ and f_3 is surjective, there is $u \in U$ such that $q_2(h) = f_3(u)$. Moreover there exists $g \in G$ such that $u = q_1(g)$ because q_1 is also surjective. Hence we have
$$q_2(h) = f_3(u) = (f_3 \circ q_1)(g) = (q_2 \circ f_2)(g)\,.$$
It implies that $h - f_2(g) \in B$. As f_1 is surjective, there is $a \in A$ such that $h - f_2(g) = f_1(a) = f_2(a)$ (because f_1 is the restriction of f_2 to A). Then the map f_2 is quadratic relative A implying that we get $f_2(a+g) = f_2(a) + f_2(g)$ because $d_{f_2}(A \times G) = 0$. Hence we have
$$h = f_2(a) + f_2(g) = f_2(a+g)$$
Finally f_2 is surjective.

2. Then we prove that f_2 is injective. Let g and g' be two elements in G such that $f_2(g) = f_2(g')$. By postcomposing with q_2, we have
$$\begin{aligned}(q_2 \circ f_2)(g) = (q_2 \circ f_2)(g') &\iff (f_3 \circ q_1)(g) = (f_3 \circ q_1)(g') \\ &\iff q_1(g) = q_1(g') \quad \text{because } f_3 \text{ is injective} \\ &\iff g - g' \in A\end{aligned}$$
As f_2 is a quadratic map relative A, we get $f_2(g) = f_2((g-g')+g') = f_2(g-g') + f_2(g')$. Hence we have $f_2(g-g') = 0$ because $f_2(g) = f_2(g')$. But $g - g' \in A$ and f_1 is the restriction of f_2 to A, so we have $f_1(g-g') = 0$ implying that we get $g = g'$ because f_1 is injective. Finally f_2 is injective.

$\square$

4.5 Construction of the Lazard functor

Now we search for a quadratic equivalence with domain $\mathcal{C}$ and values in $Alg - \mathcal{P}$. We call this the Lazard correspondence for the 2-radicable 2-step nilpotent variety $\mathcal{C}$. For this we first modify the left Λ-module structure for $\mathcal{F}_{\mathcal{P}}$, the free $\mathcal{P}$-algebra of rank 1, given in 4.3.6 in such a way that we obtain a bijection $L_{E,E} : \mathcal{C}(E,\, E) \to Alg - \mathcal{P}(\mathcal{F}_{\mathcal{P}},\, \mathcal{F}_{\mathcal{P}})$. Then we find an appropriate quadratic $\mathcal{C}$-module over $\mathcal{P}$ whose associated quadratic functor $L : \mathcal{C} \to Alg - AbOp(\mathcal{C})$ will be proved to be an equivalence in the next section.

4.5.1 The quadratic functor $L : \mathcal{C} \to Alg - AbOp(\mathcal{C})$

Here we find the quadratic functor $L : \mathcal{C} \to Alg - AbOp(\mathcal{C})$ by picking a particular quadratic $\mathcal{C}$-module over $\mathcal{P} = AbOp(\mathcal{C})$ (see (4.2.3)) with a suitable Λ-module structure on $\mathcal{F}_{\mathcal{P}}$. As the quadratic functor L has to be an equivalence of categories, we know that its corresponding quadratic $\mathcal{C}$-module over $\mathcal{P}$ is of the following form (by 3.6.9):

$$\begin{tikzcd} T_{11}cr_2(U_E)(E,\, E) \otimes_\Lambda \mathcal{F}_{\mathcal{P}} \arrow[r, "H^L"] & \mathcal{P}(2) \arrow[r, "\dot{q}"] \arrow[d, equal] & \mathcal{F}_{\mathcal{P}} \\ T_{11}cr_2(U_E)(E,\, E) \otimes_\Lambda \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[u, "id\otimes_\Lambda i_2"] \arrow[r, "\widehat{H}"'] & \mathcal{P}(2) \arrow[r, "q"'] & \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[u, "i_2"'] \end{tikzcd} \tag{4.5.1}$$

where $i_2 : \mathcal{P}(2)_{\mathfrak{S}_2} \rightarrowtail \mathcal{F}_{\mathcal{P}}$ is the canonical inclusion of the second summand, $q : \mathcal{P}(2) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ is the cokernel of $T - id$ (see 4.2.5), $\dot{q} : \mathcal{P}(2) \to \mathcal{F}_{\mathcal{P}}$ is the composite $i_2 \circ q$, $H^L : T_{11}cr_2(U_E)(E,\, E) \otimes_\Lambda \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(2)$ is a morphism satisfying the relations $(QM1)$ and $(QM2)$ (the relation $(QM2)$ holds if, and ony if, the left-hand square commutes) in 2.1.1 and $\widehat{H} : T_{11}cr_2(U_E)(E,\, E) \otimes_\Lambda \mathcal{P}(2)_{\mathfrak{S}_2} \to \mathcal{P}(2)$ is the morphism defined by

$$\widehat{H}\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda \overline{h}\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(h + T(h)\big)\,, \tag{4.5.2}$$

for $\xi \in \mathcal{C}(E,\, E^{+2})$ and $h \in \mathcal{P}(2)$.

Notation 4.5.1. The top and the bottom rows of the diagram (4.5.1) are respectively denoted by M^L and by $(M^L)^2$.

On the one hand, we give another structure of left Λ-module for $\mathcal{F}_{\mathcal{P}} = \mathcal{P}(1) \oplus \mathcal{P}(2)_{\mathfrak{S}_2}$, the free $\mathcal{P}$-algebra of rank 1. For this we first define the map

$$\phi^L_{E,E} : \mathcal{C}(E,\, E) \to \mathcal{F}_{\mathcal{P}} \quad \text{by} \quad \phi^L_{E,E}(\alpha) = \big(\alpha^{ab},\, \frac{1}{2}\,\overline{H(\alpha)}\big) = \alpha.(id,\, \overline{0}) + \frac{1}{2}\,\dot{q}(H(\alpha)) \tag{4.5.3}$$

where $\alpha \in \mathcal{C}(E,\, E)$ and $\alpha.(id,\, \overline{0})$ is given in 4.3.6.

Proposition 4.5.2. *The map $\phi^L_{E,E} : \mathcal{C}(E,\, E) \to \mathcal{F}_{\mathcal{P}}$ is quadratic relative $\mathcal{C}\big(E,\, [E,\, E]_{Id_{\mathcal{C}}}\big)$.*

Proof. First we prove that the deviation of $\phi^L_{E,E}$ to be a homomorphism of groups (see (4.4.1) for $f = \phi^L_{E,E}$) is bilinear. Let $f_1, f_2 \in \mathcal{C}(E, E)$, then we have

$$\begin{aligned}
d_{\phi^L_{E,E}}(f_1,\, f_2) &= \phi^L_{E,E}(f_1 + f_2) - \phi^L_{E,E}(f_1) - \phi^L_{E,E}(f_2)\\
&= \Big((f_1 + f_2)^{ab},\, \frac{1}{2}\,\overline{H(f_1 + f_2)}\Big) - \Big(f_1^{ab},\, \frac{1}{2}\,\overline{H(f_1)}\Big) - \Big(f_2^{ab},\, \frac{1}{2}\,\overline{H(f_2)}\Big)\\
&= \Big(0,\, \frac{1}{2}\,\overline{\big(H(f_1 + f_2) - H(f_1) - H(f_2)\big)}\Big)\\
&= \Big(0,\, \frac{1}{2}\,\overline{Id_{\mathcal{C}}(f_2|f_1) \circ H(2)}\Big)\,, \text{by (4.1.17)}\\
&= \Big(0,\, \frac{1}{2}\,\overline{\gamma_{1,1;2}(f_2^{ab} \otimes f_1^{ab} \otimes H(2))}\Big)\,, (4.2.4)
\end{aligned}$$

We deduce that $d_{\phi^L_{E,E}} : \mathcal{C}(E,\, E) \times \mathcal{C}(E,\, E) \to \mathcal{C}(E,\, E)$ is bilinear. Next we verify that $d_{\phi^L_{E,E}}\big(\mathcal{C}(E,\, [E,\, E]_{Id_{\mathcal{C}}}) \times \mathcal{C}(E,\, E)\big) = d_{\phi^L_{E,E}}\big(\mathcal{C}(E,\, E) \times \mathcal{C}(E,\, [E,\, E]_{Id_{\mathcal{C}}})\big) = 0$. Let $f \in \mathcal{C}(E,\, [E,\, E]_{Id_{\mathcal{C}}})$. If we replace f_2 (or f_1) with $(i_E)_*(f) = i_E \circ f$, then we clearly have observe that

$$H(f_1 + f) = H(f_1) + H(f)$$

because $ab_E : E \to E^{ab}$ is the cokernel of $i_E : [E,\, E]_{Id_{\mathcal{C}}} \rightarrowtail E$ implying that $(i_E \circ f)^{ab} = 0$. □

Proposition 4.5.3. *The map $\phi^L_{E,E} : \mathcal{C}(E,\, E) \to \mathcal{F}_{\mathcal{P}}$ is a bijection.*

Proof. First we prove that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}(E, [E,\, E]_{Id_{\mathcal{C}}}) \arrow[r, tail, "(i_E)_*"] \arrow[d, "\cong"', "\theta_E \circ ((ab_E)^*)^{-1}"] & \mathcal{C}(E, E) \arrow[r, "(ab_E)_*"] \arrow[d, "\phi^L_{E,E}"] & \mathcal{C}(E, E^{ab}) \arrow[r] \arrow[d, "\cong"', "((ab_E)^*)^{-1}"] & 0\\
0 \arrow[r] & \mathcal{P}(2)_{\mathfrak{S}_2} \arrow[r, tail, "i_2"] & \mathcal{F}_{\mathcal{P}} \arrow[r, "\pi_1"] & \mathcal{P}(1) \arrow[r] & 0
\end{tikzcd}$$

where $\theta_E : \mathcal{C}(E^{ab},\ [E,\ E]_{Id_\mathcal{C}}) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ is the isomorphism given in 4.2.2, i_2 is the injection of the second summand and π_1 is the projection onto the first summand. The top sequence of the above diagram is exact because we apply the representable functor $\mathcal{C}(E,\ -) : \mathcal{C} \to Gr$ to the following short exact sequence

$$0 \longrightarrow [E,\ E]_{Id_\mathcal{C}} \xrightarrow{i_E} E \xrightarrow{ab_E} E^{ab} \longrightarrow 0$$

For $f \in \mathcal{C}(E,\ [E,\ E]_{Id_\mathcal{C}})$, then we check that the right-hand square is commutative:

$$\begin{aligned} i_2 \circ \theta_E \circ ((ab_E)^*)^{-1}(f) &= (i_2 \circ \theta_E)(f^{ab}) \\ &= i_2\Big(\frac{1}{2}\,\overline{H(i_E \circ f^{ab} \circ ab_E)}\Big) \\ &= \Big(0,\ \frac{1}{2}\,\overline{H(i_E \circ f)}\Big) \\ &= \Big((i_E \circ f)^{ab},\ \frac{1}{2}\,\overline{H(i_E \circ f)}\Big) \\ &= \phi^L_{E,E}(i_E \circ f) \\ &= \phi^L_{E,E} \circ (i_E)_*(f) \end{aligned}$$

Let $g \in \mathcal{C}(E,\ E)$, then we prove that the left-hand square commutes:

$$((ab_E)^*)^{-1} \circ (ab_E)_*(f) = f^{ab} = \pi_1\Big(f^{ab},\ \frac{1}{2}\,\overline{H(f)}\Big) = \pi_1 \circ \phi^L_{E,E}(f)$$

As $\phi^L_{E,E} : \mathcal{C}(E,\ E) \to \mathcal{F}_\mathcal{P}$ is a quadratic map relative the subgroup group $\mathcal{C}\big(E,\ [E,\ E]_{Id_\mathcal{C}}\big)$ by 4.5.2, it is a bijection by 4.4.2. □

Let us recall the canonical isomorphism $ev_{(id,\bar{0})} : Alg - \mathcal{P}(\mathcal{F}_\mathcal{P},\ \mathcal{F}_\mathcal{P}) \to \mathcal{F}_\mathcal{P}$, then we define the composite map $L_{E,E} = ev^{-1}_{(id,\bar{0})} \circ \phi^L_{E,E} : \mathcal{C}(E,\ E) \to Alg - \mathcal{P}(\mathcal{F}_\mathcal{P},\ \mathcal{F}_\mathcal{P})$ that is a bijection (as a composite of two bijections). It will correspond to the image of endomorphisms of E in $\mathcal{C}$ by the Lazard equivalence L (see 3.2, 6.12 and 6.16 of [12]), and it has the following explicit expression:

Lemma 4.5.4. *Let $\alpha \in \mathcal{C}(E,\ E)$, $f \in \mathcal{P}(1)$ and $h \in \mathcal{P}(2)$. Then we have*

$$L_{E,E}(\alpha)(f,\ \overline{h}) = \Big(\alpha^{ab} \circ f,\ \overline{\gamma_{1,1;2}(\alpha^{ab} \otimes \alpha^{ab} \otimes h)} + \frac{1}{2}\,\overline{\gamma_{2;1}(H(\alpha) \otimes f)}\Big)$$

Proof. We consider the equalities as follows:

$$\begin{aligned} & L_{E,E}(\alpha)(f,\ \overline{h}) \\ &= ev^{-1}_{(id,\bar{0})} \circ \phi^L_{E,E}(\alpha)(f,\ \overline{h}) \\ &= ev^{-1}_{(id,\bar{0})}\Big(\alpha^{ab},\ \frac{1}{2}\,\overline{H(\alpha)}\Big)(f,\ \overline{h}),\ \text{by 4.5.3} \\ &= \lambda_1^{\mathcal{F}_\mathcal{P}}\Big(\Big(\alpha^{ab},\ \frac{1}{2}\,\overline{H(\alpha)}\Big) \otimes f\Big) + \lambda_2^{\mathcal{F}_\mathcal{P}}\Big(\Big(\alpha^{ab},\ \frac{1}{2}\,\overline{H(\alpha)}\Big) \otimes \Big(\alpha^{ab},\ \frac{1}{2}\,\overline{H(\alpha)}\Big) \otimes h\Big),\ \text{by (1.8.4)} \\ &= \Big(\alpha^{ab} \circ f,\ \frac{1}{2}\,\overline{\gamma_{2;1}(H(\alpha) \otimes f)}\Big) + \Big(0,\ \overline{\gamma_{1,1;2}(\alpha^{ab} \otimes \alpha^{ab} \otimes h)}\Big),\ \text{by (1.8.2) and (1.8.3)} \\ &= \Big(\alpha^{ab} \circ f,\ \overline{\gamma_{1,1;2}(\alpha^{ab} \otimes \alpha^{ab} \otimes h)} + \frac{1}{2}\,\overline{\gamma_{2;1}(H(\alpha) \otimes f)}\Big), \end{aligned}$$

as desired. □

The bijection $L_{E,E} : \mathcal{C}(E,E) \to Alg - \mathcal{P}(L(E), L(E))$ gives rise to the following left Λ-module structure on $\mathcal{F}_{\mathcal{P}}$:

Proposition 4.5.5. *The map $L_{E,E} : \mathcal{C}(E,E) \to Alg - \mathcal{P}(L(E), L(E))$ gives rise to a left Λ-module structure on $\mathcal{F}_{\mathcal{P}}$ as follows:*

$$\begin{aligned}
\alpha *_L (f,\, \overline{h}) &= L_{E,E}(\alpha)(f,\, \overline{h}) \\
&= \Big(\alpha^{ab} \circ f,\, \overline{\gamma_{1,1;2}(\alpha^{ab} \otimes \alpha^{ab} \otimes h)} + \frac{1}{2}\, \overline{\gamma_{2;1}(H(\alpha) \otimes f)}\Big)\, , \textit{by 4.5.4} \\
&= \alpha.(f,\, \overline{h}) + \frac{1}{2}\, \dot{q}\big(\gamma_{2;1}(H(\alpha) \otimes f)\big)
\end{aligned}$$

where $\alpha \in \mathcal{C}(E,\, E)$, $f \in \mathcal{P}(1)$, $h \in \mathcal{P}(2)$ and $\alpha.(f,\, \overline{h})$ is given in 4.3.6.

Proof. Let $\alpha_1, \alpha_2 \in \mathcal{C}(E,\, E)$ and $(f,\, \overline{h}) \in \mathcal{F}_{\mathcal{P}}$, then it suffices to prove that we have

$$(\alpha_2 \circ \alpha_1) *_L (f,\, \overline{h}) = \alpha_2 *_L \big(\alpha_1 *_L (f,\, \overline{h})\big)$$

We get

$$\begin{aligned}
&(\alpha_2 \circ \alpha_1) *_L (f,\, \overline{h}) \\
&= (\alpha_2 \circ \alpha_1).(f,\, \overline{h}) + \frac{1}{2}\, \dot{q}\big(\gamma_{2;1}(H(\alpha_2 \circ \alpha_1) \otimes f)\big) \\
&= \alpha_2.\big(\alpha_1.(f,\, \overline{h})\big) + \frac{1}{2}\, \dot{q}\Big(\gamma_{2;1}\big((H(\alpha_2) \circ \alpha_1^{ab}) \otimes f\big) + \gamma_{2;1}\big((Id_{\mathcal{C}}(\alpha_2|\alpha_2) \circ H(\alpha_1)) \otimes f\big)\Big)\, , \text{by (4.1.18)} \\
&= \alpha_2.\big(\alpha_1.(f,\, \overline{h})\big) + \frac{1}{2}\, \dot{q}\Big(\gamma_{2;1}\big(\gamma_{2;1}(H(\alpha_2) \otimes \alpha_1^{ab}) \otimes f\big) + \gamma_{2;1}\big(\gamma_{1,1;2}(\alpha_2^{ab} \otimes \alpha_2^{ab} \otimes H(\alpha_1)) \otimes f\big)\Big)\, , \text{by (4.2.4)} \\
&= \alpha_2.\big(\alpha_1.(f,\, \overline{h})\big) + \alpha_2.(0,\, \frac{1}{2}\, \overline{\gamma_{2;1}(H(\alpha_1) \otimes f)}\big) + \frac{1}{2}\, \dot{q}\Big(\gamma_{2;1}\big(H(\alpha_2) \otimes (\alpha_1^{ab} \circ f)\big)\Big) \\
&= \alpha_2.\Big(\alpha_1.(f,\, \overline{h}) + \frac{1}{2}\, \dot{q}\big(\gamma_{2;1}(H(\alpha_1) \otimes f)\big)\Big) + \frac{1}{2}\, \dot{q}\Big(\gamma_{2;1}\big(H(\alpha_2) \otimes (\alpha_1^{ab} \circ f)\big)\Big) \\
&= \alpha_2.\big(\alpha_1 *_L (f,\, \overline{h})\big) + \frac{1}{2}\, \dot{q}\Big(\gamma_{2;1}\big(H(\alpha_2) \otimes (\alpha_1^{ab} \circ f)\big)\Big) \\
&= \alpha_2 *_L \big(\alpha_1 *_L (f,\, \overline{h})\big)
\end{aligned}$$

as desired. □

Assumption: we now consider that $\mathcal{F}_{\mathcal{P}}$ is equipped with the left Λ-module structure given in 4.5.5.

On the other hand, we are looking for the expression of the map $H^L : T_{11}cr_2(U_E)(E,\, E) \otimes_\Lambda \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(2)$. For this we define the (right) $\mathcal{P}(1)$-module homomorphism $(v_L)_{X,Y} : U_E(X+Y) \otimes \mathcal{F}_{\mathcal{P}} \to T_1U_E(X) \otimes T_1U_E(Y) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)$ by

$$(v_L)_{X,Y}\big(\xi \otimes (f, \overline{h})\big) = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi) \otimes \big(h + T(h) - \frac{1}{2}\, H(2) \circ f\big) + (u'_{\mathcal{C}(E^{ab}, Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}(r_2(\xi)^{ab} \circ f)$$

where X and Y are objects in $\langle E \rangle$, and $u'_{\mathcal{C}(E^{ab}, Id_{\mathcal{C}}(-|-))} : T_1U_E \otimes T_1U_E \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2) \to \mathcal{C}(E^{ab},\, Id_{\mathcal{C}}(X|Y))$ is the natural transformation between bifunctors given in 4.1.7 which is an isomorphism on $\langle E \rangle \times \langle E \rangle$.

Proposition 4.5.6. *Let X and Y be two objects in $\langle E\rangle$. Then the (right) $\mathcal{P}(1)$-module homomorphism $(v_L)_{X,Y}: U_E(X+Y)\otimes\mathcal{F}_{\mathcal{P}} \to T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ is Λ-bilinear, i.e.*

$$(v_L)_{X,Y}\big(\xi\otimes\alpha *_L (f,\overline{h})\big) = (v_L)_{X,Y}\big((\xi\circ\alpha)\otimes(f,\overline{h})\big)$$

where $\xi\in\mathcal{C}(E,X+Y)$, $\alpha\in\mathcal{C}(E,E)$ and $(f,\overline{h})\in\mathcal{F}_{\mathcal{P}}$.

Proof. Let $\xi\in\mathcal{C}(E,X+Y)$, $\alpha\in\mathcal{C}(E,E)$ and $(f,\overline{h})\in\mathcal{F}_{\mathcal{P}}$. Then we have the following equalities:

$$\begin{aligned}
&(v_L)_{X,Y}\big(\xi\otimes\alpha *_L (f,\overline{h})\big)\\
&= (v_L)_{X,Y}\Big(\xi\otimes\big(\alpha^{ab}\circ f, \overline{\gamma_{1,1;2}(\alpha^{ab}\otimes\alpha^{ab}\otimes h)} + \frac{1}{2}\,\overline{\gamma_{2;1}(H(\alpha)\otimes f)}\big)\Big)\\
&= t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\Big(\gamma_{1,1;2}(\alpha^{ab}\otimes\alpha^{ab}\otimes h)\\
&\qquad + \frac{1}{2}\,H(\alpha)\circ f + T\big(\gamma_{1,1;2}(\alpha^{ab}\otimes\alpha^{ab}\otimes h) + \frac{1}{2}\,H(\alpha)\circ f\big)\\
&\qquad - \frac{1}{2}\,H(2)\circ\alpha^{ab}\circ f\Big) + (u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi)^{ab}\circ\alpha^{ab}\circ f\big)\\
&= t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\big(t_1(\alpha)\otimes t_1(\alpha).(h+T(h)\big)\\
&\qquad + t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\frac{1}{2}\,\Big(H(\alpha)\circ f + T(H(\alpha))\circ f\\
&\qquad - H(2)\circ\alpha^{ab}\circ f\Big) + (u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi)^{ab}\circ\alpha^{ab}\circ f\big)\\
&= t_1(r_1^2\circ\xi\circ\alpha)\otimes t_1(r_2^2\circ\xi\circ\alpha)\otimes\big(h+T(h)\big)\\
&\qquad + t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\frac{1}{2}\,\Big(H(\alpha)\circ f + T(H(\alpha))\circ f\\
&\qquad - H(2)\circ\alpha^{ab}\circ f\Big) + (u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi)^{ab}\circ\alpha^{ab}\circ f\big)
\end{aligned}$$

Now we consider the following term:

$$\begin{aligned}
&(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})_{X,Y}\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\frac{1}{2}\,\Big(H(\alpha)\circ f + T(H(\alpha))\circ f\\
&\qquad - H(2)\circ\alpha^{ab}\circ f\Big) + (u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi)^{ab}\circ\alpha^{ab}\circ f\big)\Big)\\
&= \frac{1}{2}\,Id_{\mathcal{C}}(r_1^2\circ\xi|r_2\circ\xi)\circ H(\alpha)\circ f + \frac{1}{2}\,Id_{\mathcal{C}}(r_1^2\circ\xi|r_2\circ\xi)\circ T(H(\alpha))\circ f\\
&\qquad - \frac{1}{2}\,Id_{\mathcal{C}}(r_1^2\circ\xi|r_2\circ\xi)\circ H(2)\circ\alpha^{ab}\circ f\Big) + r_2(\xi)^{ab}\circ\alpha^{ab}\circ f
\end{aligned}$$

By 4.1.26, we have

$$r_2(\xi)\circ\alpha^{ab} = r_2(\xi\circ\alpha)^{ab} - Id_{\mathcal{C}}(r_1^2\circ\xi|r_2^2\circ\xi)\circ H(\alpha)$$

Hence we get

$$
\begin{aligned}
&(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})_{X,Y}\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\frac{1}{2}\Big(H(\alpha)\circ f+T(H(\alpha))\circ f\\
&\qquad -H(2)\circ\alpha^{ab}\circ f\Big)+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi)^{ab}\circ\alpha^{ab}\circ f\big)\Big)\\
&=-\frac{1}{2}\,Id_{\mathcal{C}}(r_1^2\circ\xi|r_2\circ\xi)\circ H(\alpha)\circ f+\frac{1}{2}\,Id_{\mathcal{C}}(r_1^2\circ\xi|r_2\circ\xi)\circ T(H(\alpha))\circ f\\
&\qquad -\frac{1}{2}\,Id_{\mathcal{C}}(r_1^2\circ\xi|r_2\circ\xi)\circ H(2)\circ\alpha^{ab}\circ f\Big)+r_2(\xi\circ\alpha)^{ab}\circ f
\end{aligned}
$$

By (4.1.19), we have

$$T(H(\alpha))=H(\alpha)+H(2)\circ\alpha^{ab}-Id_{\mathcal{C}}(\alpha|\alpha)\circ H(2)$$

Then it gives the following equalities:

$$
\begin{aligned}
&(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})_{X,Y}\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\frac{1}{2}\Big(H(\alpha)\circ f+T(H(\alpha))\circ f\\
&\qquad -H(2)\circ\alpha^{ab}\circ f\Big)+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi)^{ab}\circ\alpha^{ab}\circ f\big)\Big)\\
&=-\frac{1}{2}\,Id_{\mathcal{C}}(r_1^2\circ\xi\circ\alpha|r_2^2\circ\xi\circ\alpha).H(2)\circ f+r_2(\xi\circ\alpha)^{ab}\circ f\\
&=(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})_{X,Y}\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\big(-\frac{1}{2}\,Id_{\mathcal{C}}(\alpha|\alpha)\circ H(2)\circ f\big)\\
&\qquad +(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi\circ\alpha)^{ab}\circ f\big)\Big)
\end{aligned}
$$

implying that we have

$$
\begin{aligned}
&t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\frac{1}{2}\Big(H(\alpha)\circ f+T(H(\alpha))\circ f-H(2)\circ\alpha^{ab}\circ f\Big)+(\overline{u'_{cr_2U_E}})^{-1}_{X,Y}\big(r_2(\xi)^{ab}\circ\alpha^{ab}\circ f\\
&=t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\big(-\frac{1}{2}\,Id_{\mathcal{C}}(\alpha|\alpha)\circ H(2)\circ f\big)+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi\circ\alpha)^{ab}\circ f\big)
\end{aligned}
$$

because $(\overline{u'_{cr_2U_E}})_{X,Y}:T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)\to\mathcal{C}(E^{ab},\ Id_{\mathcal{C}}(X|Y))$ is an isomorphism, see 4.1.11 and 4.1.7. Hence we have

$$
\begin{aligned}
&(v_L)_{X,Y}\big(\xi\otimes\alpha *_L(f,\overline{h})\big)\\
&=t_1(r_1^2\circ\xi\circ\alpha)\otimes t_1(r_2^2\circ\xi\circ\alpha)\otimes\big(h+T(h)\big)\\
&\qquad +t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\big(-\frac{1}{2}\,Id_{\mathcal{C}}(\alpha|\alpha)\circ H(2)\circ f\big)+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi\circ\alpha)^{ab}\circ f\big)\\
&=t_1(r_1^2\circ\xi\circ\alpha)\otimes t_1(r_2^2\circ\xi\circ\alpha)\otimes\big(h+T(h)-\frac{1}{2}\,H(2)\circ f\big)+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X,Y}\big(r_2(\xi\circ\alpha)^{ab}\circ f\big)\\
&=(v_L)_{X,Y}\big((\xi\circ\alpha)\otimes(f,\overline{h})\big)\,,
\end{aligned}
$$

as desired. □

Then it follows that the (right) $\mathcal{P}(1)$-module homomorphism $(v_L)_{X,Y}:U_E(X+Y)\otimes\mathcal{F}_{\mathcal{P}}\to T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ factorizes through the quotient map $q^{\Lambda}_{\mathbb{Z}}:U_E(X+Y)\otimes\mathcal{F}_{\mathcal{P}}\to U_E(X+Y)\otimes_{\Lambda}\mathcal{F}_{\mathcal{P}}$ by 2.4.3. We denote by $(\overline{v_L})_{X,Y}:U_E(X+Y)\otimes_{\Lambda}\mathcal{F}_{\mathcal{P}}\to T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ its factorization.

Proposition 4.5.7. *For two objects X and Y in $\langle E\rangle$, the $\mathcal{P}(1)$-module homomorphism $(v_L)_{X,Y}$: $U_E(X+Y)\otimes\mathcal{F}_{\mathcal{P}}\to T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ is natural in X and Y.*

Proof. Let $\alpha\in\mathcal{C}(X_1,X_2)$, $\beta\in\mathcal{C}(Y_1,Y_2)$, $\xi\in\mathcal{C}(X,X_1+Y_1)$ and $(f,\overline{h})\in\mathcal{F}_{\mathcal{P}}$. Then we have

$$
\begin{aligned}
&(v_L)_{X_2,Y_2}\big(((\alpha+\beta)\circ\xi\big)\otimes(f,\overline{h})\big)\\
&=t_1\big(r_1^2\circ(\alpha+\beta)\circ\xi\big)\otimes t_1\big(r_2^2\circ(\alpha+\beta)\circ\xi\big)\otimes\big(h+T(h)-\frac{1}{2}\,H(2)\circ f\big)\\
&\qquad+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X_2,Y_2}\big(r_2\big((\alpha+\beta)\circ\xi\big)^{ab}\circ f\big)\\
&=t_1\big(\alpha\circ r_1^2\circ\xi\big)\otimes t_1\big(\beta\circ r_2^2\circ\xi\big)\otimes\big(h+T(h)-\frac{1}{2}\,H(2)\circ f\big)\\
&\qquad+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X_2,Y_2}\big(r_2\big((\alpha+\beta)\circ\xi\big)^{ab}\circ f\big)\\
&=t_1(\alpha)\otimes t_1(\beta).\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi)\otimes\big(h+T(h)-\frac{1}{2}\,H(2)\circ f\big)\Big)\\
&\qquad+(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X_2,Y_2}\big(r_2\big((\alpha+\beta)\circ\xi\big)^{ab}\circ f\big)
\end{aligned}
$$

However we have

$$
\begin{aligned}
&\iota_2^{Id_{\mathcal{C}}}\circ r_2\big((\alpha+\beta)\circ\xi\big)\\
&=(\alpha+\beta)\circ\xi-\Big(i_1^2\circ r_1^2\circ\big((\alpha+\beta)\circ\xi\big)+i_1^2\circ r_1^2\circ\big((\alpha+\beta)\circ\xi\big)\Big)\,,\text{by (4.1.11)}\\
&=(\alpha+\beta)\circ\xi-\Big((\alpha+\beta)\circ i_1^2\circ r_1^2\circ\xi+(\alpha+\beta)\circ i_2^2\circ r_2^2\circ\xi\Big)\\
&=(\alpha+\beta)\circ\xi-(\alpha+\beta)\circ\big(i_1^2\circ r_1^2\circ\xi+i_2^2\circ r_2^2\circ\xi\big)\,,\text{by (4.1.21)}\\
&=(\alpha+\beta)\circ\Big(\xi-\big(i_1^2\circ r_1^2\circ\xi+i_2^2\circ r_2^2\circ\xi\big)\Big)\,,\text{by (4.1.21)}\\
&=(\alpha+\beta)\circ\iota_2^{Id_{\mathcal{C}}}\circ r_2(\xi)\,,\text{by (4.1.11)}\\
&=\iota_2^{Id_{\mathcal{C}}}\circ Id_{\mathcal{C}}(\alpha|\beta)\circ r_2(\xi)
\end{aligned}
$$

As $\iota_2^{Id_{\mathcal{C}}}:Id_{\mathcal{C}}(X,X_2|Y_2)\rightarrowtail\mathcal{C}(X,X_2+Y_2)$ is a monomorphism, we get

$$r_2\big((\alpha+\beta)\circ\xi\big)=Id_{\mathcal{C}}(\alpha|\beta)\circ r_2(\xi)=t_1(\alpha)\otimes t_1(\beta).r_2(\xi)$$

Hence we have

$$(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X_2,Y_2}\big(r_2\big((\alpha+\beta)\circ\xi\big)^{ab}\circ f\big)=\big(t_1(\alpha)\otimes t_1(\beta)\otimes_{\Lambda\otimes\Lambda}id\big)\circ(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X_1,Y_1}\big(r_2(\xi)^{ab}\circ f\big)$$

by naturality of the natural transformation $(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})_{X_1,Y_1}:T_1U_E(X_1)\otimes T_1U_E(Y_1)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)\to$

$\mathcal{C}(X, Id_{\mathcal{C}}(X_1|Y_1))$ in X_1 and Y_1. Finally we get

$$\begin{aligned}
&(v_L)_{X_2,Y_2}\big(((\alpha+\beta)\circ\xi)\otimes(f,\overline{h})\big)\\
&= t_1\big(\alpha\circ r_1^2\circ\xi\big)\otimes t_1\big(\beta\circ r_2^2\circ\xi\big)\otimes\big(h+T(h)-\frac{1}{2}\,H(2)\circ f\big)\\
&\qquad + (u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X_2,Y_2}\big(r_2\big((\alpha+\beta)\circ\xi\big)^{ab}\circ f\big)\\
&= \big(t_1(\alpha)\otimes t_1(\beta)\otimes id\big)\Big(t_1\big(r_1^2\circ\xi\big)\otimes t_1\big(r_2^2\circ\xi\big)\otimes\big(h+T(h)-\frac{1}{2}\,H(2)\circ f\big)\\
&\qquad + (u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))})^{-1}_{X_1,Y_1}\big(r_2(\xi)^{ab}\circ f\big)\\
&= \big(t_1(\alpha)\otimes t_1(\beta)\otimes_{\Lambda\otimes\Lambda} id\big)\circ(v_L)_{X_1,Y_1}\big(\xi\otimes(f,\overline{h})\big)
\end{aligned}$$

as desired. □

Consequently the (right) $\mathcal{P}(1)$-module homomorphism $(\overline{v_L})_{X,Y} : U_E(X+Y)\otimes_\Lambda \mathcal{F}_{\mathcal{P}} \to T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ is also natural in X and Y. Then we define the map $(w_L)_{X,Y} : cr_2U_E(X,Y)\otimes_\Lambda \mathcal{F}_{\mathcal{P}} \to T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ by

$$(w_L)_{X,Y} = (\overline{v_L})_{X,Y}\circ\big(\iota_2\otimes_\Lambda id\big)$$

Remark 4.5.8. Let X and Y be two objects in $\langle E\rangle$. If $\xi\in\mathcal{C}(E,X+Y)$ and $(f,\overline{h})\in\mathcal{F}_{\mathcal{P}}$, then we have the following relation:

$$(w_L)_{X,Y}\big(\rho_2(\xi)\otimes(f,\overline{h})\big) = (\overline{v_L})_{X,Y}\big(\xi\otimes(f,\overline{h})\big)\,, \tag{4.5.4}$$

because we have

$$\begin{aligned}
(w_L)_{X,Y}\big(\rho_2(\xi)\otimes(f,\overline{h})\big) &= (\overline{v_L})_{X,Y}\circ\big(\iota_2\otimes_\Lambda id\big)\big(\rho_2(\xi)\otimes(f,\overline{h})\big)\\
&= (\overline{v_L})_{X,Y}\circ\big((\iota_2\circ\rho_2)(\xi)\otimes_\Lambda(f,\overline{h})\big)\\
&= (\overline{v_L})_{X,Y}\circ\big(\xi\otimes_\Lambda(f,\overline{h})\big)\\
&\qquad - (\overline{v_L})_{X,Y}\circ\big((i_1^2\circ r_1^2\circ\xi)\otimes_\Lambda(f,\overline{h})\big)\\
&\qquad - (\overline{v_L})_{X,Y}\circ\big((i_2^2\circ r_2^2\circ\xi)\otimes_\Lambda(f,\overline{h})\big)\\
&= (\overline{v_L})_{X,Y}\circ\big(\xi\otimes_\Lambda(f,\overline{h})\big)
\end{aligned}$$

since it is straightforward to check that $r_2(i_k^2\circ r_k^2\circ\xi)=0$, for $k=1,2$. Moreover the map $(w_L)_{X,Y} : cr_2U_E(X,Y)\otimes_\Lambda\mathcal{F}_{\mathcal{P}} \Rightarrow T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ is natural in X and Y because so is the map $(\overline{v_L})_{X,Y} : U_E(X+Y)\otimes_\Lambda\mathcal{F}_{\mathcal{P}} \to T_1U_E(X)\otimes T_1U_E(Y)\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ by 4.5.7.

In summary, we get a natural transformation $w_L : cr_2U_E\otimes_\Lambda\mathcal{F}_{\mathcal{P}} \Rightarrow T_1U_E\otimes T_1U_E\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ between bifunctors whose target is a bilinear bifunctor. By 1.2.14, there is a unique factorization $\overline{w_L} : T_{11}cr_2U_E\otimes_\Lambda\mathcal{F}_{\mathcal{P}} \Rightarrow T_1U_E\otimes T_1U_E\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ of $w_L : cr_2U_E\otimes_\Lambda\mathcal{F}_{\mathcal{P}} \Rightarrow T_1U_E\otimes T_1U_E\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)$ through $t_{11}\otimes_\Lambda id : cr_2U_E\otimes_\Lambda\mathcal{F}_{\mathcal{P}} \Rightarrow T_{11}cr_2U_E\otimes_\Lambda\mathcal{F}_{\mathcal{P}}$, i.e.

$$\overline{w_L}\circ\big(t_{11}\otimes_\Lambda id\big) = w_L \tag{4.5.5}$$

Let $ev_{\mathcal{P}(2)} : \overline{\Lambda}\otimes\overline{\Lambda}\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)\to\mathcal{P}(2)$ be the canonical isomorphism, where $T_1U_E(E)=\overline{\Lambda}$ (see 2.0.2). Then we choose a specific expression of the morphism $H^L : T_{11}cr_2(U_E)(E,\,E)\otimes\mathcal{F}_{\mathcal{P}}\to\mathcal{P}(2)$ such that the diagram (4.5.1) is a quadratic $\mathcal{C}$-module over $\mathcal{P}$.

Lemma 4.5.9. *If we define the morphism* $H^L : T_{11}cr_2(U_E)(E,\, E) \otimes_\Lambda \mathcal{F}_\mathcal{P} \to \mathcal{P}(2)$ *by*

$$H^L\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\, \overline{h})\big) = ev_{\mathcal{P}(2)} \circ (\overline{w_L})_{E,E}\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\, \overline{h})\big)$$
$$= t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\Big(h + T(h) - \frac{1}{2}\,\gamma_{2;1}(H(2) \otimes f)\Big) + \gamma_{2;1}(r_2(\xi)^{ab} \otimes f)\,,$$

for $\xi \in \mathcal{C}(E,\, E^{+2})$ *and* $(f,\, \overline{h}) \in \mathcal{F}_\mathcal{P}$*, then the diagram of homomorphisms of abelian groups* (4.5.1) *is a quadratic* $\mathcal{C}$*-module over* $\mathcal{P}$*.*

Proof. First we know that the bottom row in (4.5.1) is a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ by 3.6.7. Then it suffices to verify that the top one is a quadratic $\mathcal{C}$-module over $\mathcal{P}(1)$ and that the diagram (4.5.1) commutes. We recall that $\mathcal{F}_\mathcal{P}$ and $\mathcal{P}(2)$ are respectively a left Λ-module and a left $\overline{\Lambda} \otimes \overline{\Lambda}$-module (see (4.5.5) and (4.2.6)). The map $\dot{q} : \mathcal{P}(2) \to \mathcal{P}(2)_{\mathfrak{S}_2}$ is a homomorphism of $\Lambda - \mathcal{P}(1)$-bimodules, in fact:

$$\begin{cases} \alpha *_L \dot{q}(h) = \alpha *_L (0,\, \overline{h}) = \big(0,\, \overline{\gamma_{1,1;2}(\alpha^{ab} \otimes \alpha^{ab} \otimes h)}\big) = \dot{q}\big(t_1(\alpha) \otimes t_1(\alpha).h\big) \\ \lambda_1^{\mathcal{F}_\mathcal{P}}\big(\dot{q}(h) \otimes f\big) = \lambda_1^{\mathcal{F}_\mathcal{P}}\big((0,\, \overline{h}) \otimes f\big) = \big(0,\, \overline{\gamma_{2;1}(h \otimes f)}\big) = \dot{q}\big(\gamma_{2;1}(h \otimes f)\big) \end{cases}$$

where $\alpha \in \mathcal{C}(E,\, E)$, $f \in \mathcal{P}(1)$ and $h \in \mathcal{P}(2)$. Moreover the map $\dot{q}$ clearly satisfies $\dot{q} = \dot{q} \circ T$ because $\dot{q} \circ T = i_2 \circ q \circ T = i_2 \circ q = \dot{q}$. Now it remains to prove the relations $(QM1)$ and $(QM2)$ in 2.1.1. First we prove that $(QM2)$ is verified. We remark that $(QM2)$ holds whenever the right-hand square of the diagram (4.5.1) commutes. We have

$$\begin{aligned} \big(H^L \circ (id \otimes i_2)\big)\big(t_{11}(\rho_2(\xi)) \otimes \overline{h}\big) &= H^L\big(t_{11}(\rho_2(\xi)) \otimes (0, \overline{h})\big) \\ &= t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(h + T(h)\big)\,, \text{by definition of } H^L \\ &= \widehat{H}(t_{11}(\rho_2(\xi)) \otimes_\Lambda \overline{h})\,, \text{by (4.5.2)} \end{aligned}$$

Next we verify that $(QM1)$ holds. By 4.5.5, we have

$$\begin{aligned} (\nabla_E^2 \circ \xi) *_L (f,\, \overline{h}) &= \Big((\nabla_E^2 \circ \xi)^{ab} \circ f,\, \overline{\gamma_{1,1;2}((\nabla_E^2 \circ \xi)^{ab} \otimes (\nabla_E^2 \circ \xi)^{ab} \otimes h)} + \frac{1}{2}\,\overline{\gamma_{2;1}(H(\nabla_E^2 \circ \xi) \otimes f)}\Big) \\ &= \Big((\nabla_E^2 \circ \xi)^{ab} \circ f,\, \overline{Id_\mathcal{C}(\nabla_E^2 \circ \xi | \nabla_E^2 \circ \xi) \circ h} + \frac{1}{2}\,\overline{\gamma_{2;1}(H(\nabla_E^2 \circ \xi) \otimes f)}\Big) \end{aligned}$$

As $Id_\mathcal{C}(-|-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bilinear bifunctor, we have the following equalities by 3.7 of [12]:

$$\begin{aligned} Id_\mathcal{C}(\nabla_E^2 \circ \xi | \nabla_E^2 \circ \xi) &= Id_\mathcal{C}(r_1^2 \circ \xi | \nabla_E^2 \circ \xi) + Id_\mathcal{C}(r_2^2 \circ \xi | \nabla_E^2) \\ &= Id_\mathcal{C}(r_1^2 \circ \xi | r_1^2 \circ \xi) + Id_\mathcal{C}(r_1^2 \circ \xi | r_2^2 \circ \xi) + Id_\mathcal{C}(r_2^2 \circ \xi | r_1^2 \circ \xi) + Id_\mathcal{C}(r_2^2 \circ \xi | r_2^2 \circ \xi) \end{aligned}$$

Since the abelianization functor $Ab^\mathcal{C} : \mathcal{C} \to Ab(\mathcal{C})$ is linear, we have $(\nabla_E^2 \circ \xi)^{ab} = (r_1^2 \circ \xi)^{ab} + (r_2^2 \circ \xi)^{ab}$ by 3.6 of [12]. Moreover we have the following equalities:

$$\begin{aligned} &H(\nabla_E^2 \circ \xi) \\ &= H\big((r_1^2 \circ \xi) + (r_2^2 \circ \xi) + (c_2^E \circ r_2(\xi))\big)\,, \text{by postcomposing with } \nabla_E^2 \text{ to the equation (4.1.11)} \\ &= H\big((r_1^2 \circ \xi) + (r_2^2 \circ \xi)\big) + H(c_2^E \circ r_2(\xi))\,, \text{by (4.1.17) and (4.1.16)} \\ &= H(r_1^2 \circ \xi) + H(r_2^2 \circ \xi) + Id_\mathcal{C}\big(r_2^2 \circ \xi | r_1^2 \circ \xi\big) \circ H(2) + H(c_2^E \circ r_2(\xi))\,, \text{by (4.1.17)} \\ &= H(r_1^2 \circ \xi) + H(r_2^2 \circ \xi) + \gamma_{1,1;2}\big((r_2^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes H(2)\big) + H(c_2^E \circ r_2(\xi))\,, \text{by (4.2.4)} \\ &= H(r_1^2 \circ \xi) + H(r_2^2 \circ \xi) + \gamma_{1,1;2}\big((r_2^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes H(2)\big) + H(c_2^E \circ r_2(\xi)^{ab} \circ ab_E) \\ &= H(r_1^2 \circ \xi) + H(r_2^2 \circ \xi) + \gamma_{1,1;2}\big((r_2^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes H(2)\big) + r_2(\xi)^{ab} + T(r_2(\xi)^{ab})\,, \text{by (4.1.22)} \end{aligned}$$

Finally we obtain

$$
\begin{aligned}
&(\nabla_E^2 \circ \xi) *_L (f,\, \overline{h}) \\
&= \Big((r_1^2 \circ \xi)^{ab},\, \overline{\gamma_{1,1;2}((r_1^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes h)} + \frac{1}{2}\, \overline{\gamma_{2;1}(H(r_1^2 \circ \xi) \otimes f)} \Big) \\
&\quad + \Big((r_2^2 \circ \xi)^{ab},\, \overline{\gamma_{1,1;2}((r_2^2 \circ \xi)^{ab} \otimes (r_2^2 \circ \xi)^{ab} \otimes h)} + \frac{1}{2}\, \overline{\gamma_{2;1}(H(r_1^2 \circ \xi) \otimes f)} \Big) \\
&\quad + \Big(0,\, \overline{\gamma_{1,1;2}((r_1^2 \circ \xi)^{ab} \otimes (r_2^2 \circ \xi)^{ab} \otimes h)} + \overline{\gamma_{1,1;2}((r_2^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes h)} \Big) \\
&\quad + \Big(0,\, \frac{1}{2}\, \overline{\gamma_{1,1;2}((r_2^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes H(2)) \circ f} + \frac{1}{2}\, \overline{r_2(\xi)^{ab} \circ f} + \frac{1}{2}\, \overline{T(r_2(\xi)^{ab}) \circ f} \Big) \\
&= (r_1^2 \circ \xi) *_L (f,\, \overline{h}) + (r_2^2 \circ \xi) *_L (f,\, \overline{h}) \\
&\quad + \Big(0,\, \overline{Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ h} + \overline{Id_{\mathcal{C}}(r_1^2 \circ \xi | r_2^2 \circ \xi) \circ T(h)} \Big) \\
&\quad + \Big(0,\, \frac{1}{2}\, \overline{\gamma_{1,1;2}((r_2^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes \gamma_{2;1}(H(2) \otimes f))} + \frac{1}{2}\, \overline{r_2(\xi)^{ab} \circ f} + \frac{1}{2}\, \overline{T(r_2(\xi)^{ab} \circ f)} \Big) \\
&= (r_1^2 \circ \xi) *_L (f,\, \overline{h}) + (r_2^2 \circ \xi) *_L (f,\, \overline{h}) + \dot{q}\big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).(h + T(h) \big) \\
&\quad + \big(0,\, \frac{1}{2}\, \overline{\gamma_{1,1;2}((r_1^2 \circ \xi)^{ab} \otimes (r_2^2 \circ \xi)^{ab} \otimes (T(H(2)) \circ f))} + \overline{r_2(\xi)^{ab} \circ f} \big) \text{, because } \overline{T(r_2(\xi)^{ab})} = \overline{r_2(\xi)^{ab}} \\
&= (r_1^2 \circ \xi) *_L (f,\, \overline{h}) + (r_2^2 \circ \xi) *_L (f,\, \overline{h}) + \dot{q}\big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).(h + T(h) \big) \\
&\quad + \big(0,\, -\frac{1}{2}\, \overline{\gamma_{1,1;2}((r_1^2 \circ \xi)^{ab} \otimes (r_2^2 \circ \xi)^{ab} \otimes (H(2) \circ f))} + \overline{\gamma_{2;1}(r_2(\xi)^{ab} \otimes f)} \big) \text{, as } T(H(2)) = -H(2) \\
&= (r_1^2 \circ \xi) *_L (f,\, \overline{h}) + (r_2^2 \circ \xi) *_L (f,\, \overline{h}) + \dot{q}\Big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(h + T(h) - \frac{1}{2}\, \gamma_{2;1}(H(2) \otimes f)\big) \Big) \\
&\quad + \dot{q}\big(\gamma_{2;1}(r_2(\xi)^{ab} \otimes f) \big) \\
&= (r_1^2 \circ \xi) *_L (f,\, \overline{h}) + (r_2^2 \circ \xi) *_L (f,\, \overline{h}) + \big(\dot{q} \circ H^L \big)\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\, \overline{h}) \big)
\end{aligned}
$$

This ends the proof. □

This provides a quadratic functor $L : \mathcal{C} \to Alg - \mathcal{P}$ by applying the functor $\mathbb{T}_2^{\mathcal{P}} : QMod_{\mathcal{C}}^{\mathcal{P}} \to Quad(\mathcal{C},\, Alg - \mathcal{P})$ defined in 2.4.27 to the quadratic $\mathcal{C}$-module over $\mathcal{P}$ given in 4.5.1. For an object X in $\mathcal{C}$, it is defined on objects by the following pushout, see 2.1.3:

$$
\begin{tikzcd}
\Big(T_1 U_E(X)^{\otimes 2} \otimes_{\Lambda \otimes \Lambda} T_{11} cr_2(U_E)(E, E) \otimes_\Lambda \mathcal{F}_{\mathcal{P}} \Big)_{\mathfrak{S}_2} \oplus (U_E(X) \otimes \mathcal{P}(2)) \arrow[r, "\overline{\phi_X}"] \arrow[d, "\overline{\psi_X^L}"'] & T_2 U_E(X) \otimes_\Lambda \mathcal{F}_{\mathcal{P}} \arrow[d, "\widehat{\psi_X^{M^L}}"] \\
\big(T_1 U_E(X)^{\otimes 2} \otimes_{\Lambda \otimes \Lambda} \mathcal{P}(2) \big)_{\mathfrak{S}_2} \arrow[r, "\widehat{\phi_X^{M^L}}"'] & L(X)
\end{tikzcd}
\tag{4.5.6}
$$

Notation 4.5.10. The quadratic functor $L : \mathcal{C} \to Alg - AbOp(\mathcal{C})$ will be called the *Lazard functor* in the sequel.

4.6 The Lazard functor is an equivalence

Here we show that the functor $L : \mathcal{C} \to Alg - AbOp(\mathcal{P})$ constructed in the previous section satisfies the equivalence criterion established in Theorem 3.5.2. This in particular requires to prove that the functor L restricted to the full-subcategory of free objects of finite rank in $\mathcal{C}$ takes values in the full subcategory of free $\mathcal{P}$-algebras of finite rank and is an equivalence.

However, as we found the criterion 3.5.2 only recently and time is too short to adapt this section to it, we will here only show that the restriction of L to $\langle E\rangle$ takes values in $\langle \mathcal{F}_{\mathcal{P}}\rangle$ (up to isomorphism) and is an isomorphism of theories by using the criterion given in 3.5.1. This is sufficient to establish an equivalence $L^*\colon Model(\langle \mathcal{F}_{\mathcal{P}}\rangle) \to Model(\langle E\rangle)$, which provides a Lazard correspondence between Alg-AbOp($\mathcal{C}$) $\simeq Model(\langle \mathcal{F}_{\mathcal{P}}\rangle)$ and $\mathcal{C} \simeq Model(\langle E\rangle)$ which will be made explicit in the next chapter, in terms of a BCH type formula.

We start with the following remark.

Remark 4.6.1. By 6.11 of [12], we know that $L(E)$ is isomorphic to $\mathcal{F}_{\mathcal{P}}$. To simplify calculations, we consider that $L(E) = \mathcal{F}_{\mathcal{P}}$, which is the same as considering that $\eta_{M^L} = id$ where $\eta_{M^L} : M^L \to \mathbb{S}_2^{\mathcal{P}}.\mathbb{T}_2^{\mathcal{P}}(M^L) = \mathbb{S}_2^{\mathcal{P}}(L)$ is the unit of the adjunction of 2.4.37 evaluated on M^L and $\mathbb{S}_2^{\mathcal{P}} : Quad(\mathcal{C}, Alg-\mathcal{P}) \to QMod_{\mathcal{C}}^{\mathcal{P}}$ is the functor defined in 2.4.27.

We recall that the morphism $\overline{\phi_X}$ in the pushout (4.5.6) is defined by $\overline{\phi_X} = \big(\overline{\phi_1' \otimes id},\, t_2 \otimes \dot{q}\big)$ (see 2.1.3), where the morphism $\phi_1' : T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \to T_2U_E(X)$ is given in (2.1.1) by:

$$\begin{tikzcd}
T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \arrow[r, "\phi_1'"] \arrow[d, "id^{\otimes 2}\otimes\overline{cr_2(t_2)}"', "\cong"] & T_2U_E(X) \\
T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} cr_2(T_2U_E)(E,E) \arrow[r, "u'_{cr_2(T_2U_E)}"'] & cr_2(T_2U_E)(X,X) \arrow[u, "S_2^{T_2U_E}"']
\end{tikzcd}$$

where $u'_{cr_2(T_2U_E)} : T_1U_E \otimes T_1U_E \Rightarrow cr_2(T_2U_E)$ is the natural transformation between bifunctors with domain $\mathcal{C}\times\mathcal{C}$ and values in *Ab* given in 3.21 of [12]. By 3.22 of [12], it is an isomorphism when restricted to $\langle E\rangle$, the full subcategory of free objects of finite rank of $\mathcal{C}$.

Lemma 4.6.2. *The natural transformation $u'_{cr_2(T_2U_E)} : T_1U_E\otimes T_1U_E \Rightarrow cr_2(T_2U_E)$ between bifunctors with domain $\mathcal{C}\times\mathcal{C}$ and values in Ab is an isomorphism.*

Proof. It suffices to see that the functors $T_1U_E, T_2U_E : \mathcal{C} \to Ab$ preserve filtered colimits and coequalizers of reflexive graphs by 6.24 of [12], and that the property given in 4.0.3 holds in $\mathcal{C}$. Then the proof is a direct consequence of 6.25 of [12]. $\square$

Now we recall that the morphism $\overline{\psi_X^L}$ in the pushout (4.5.6) is given by $\overline{\psi_X^L} = \big(\overline{id^{\otimes 2}\otimes H^L}, \pi\circ(\delta\otimes id)\big)$, where $\delta : U_E(X) \to T_1U_E(X)^{\otimes 2}, \alpha \mapsto t_1(\alpha)^{\otimes 2}$ and $\pi : T_1U_E(X)^{\otimes 2}\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2) \to \big(T_1U_E(X)^{\otimes 2}\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)\big)_{\mathfrak{S}_2}$ is the canonical quotient map. The explicit expression of the morphism $H^L : T_{11}cr_2(U_E)(E,\, E)\otimes_\Lambda \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(2)$ implies the following property on $\overline{\psi_X^L}$:

Proposition 4.6.3. *Let X be an object in $\mathcal{C}$. Then the morphism*

$$\overline{\psi_X^L} : \big(T_1U_E(X)^{\otimes 2}\otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E)\otimes_\Lambda \mathcal{F}_{\mathcal{P}}\big)_{\mathfrak{S}_2} \oplus (U_E(X)\otimes\mathcal{P}(2)) \to \big(T_1U_E(X)^{\otimes 2}\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)\big)_{\mathfrak{S}_2}$$

restricted to the first summand (in (4.5.6)) is surjective. Hence the morphism $\widehat{\psi_X^{M^L}} : T_2U_E(X)\otimes_\Lambda \mathcal{F}_{\mathcal{P}} \to L(X)$ also is surjective.

Proof. Let $f_1, f_2 \in \mathcal{C}(E, X)$ and $h \in \mathcal{P}(2)$. Then we have the following equality:

$$\begin{aligned}\overline{\psi_X^L}\big(\overline{t_1(f_1)\otimes t_1(f_2)\otimes t_{11}(\rho_2(\iota_2^{Id_\mathcal{C}}\circ h\circ ab_E))\otimes(id,\overline{0})},\ 0\big)\\ = \overline{t_1(f_1)\otimes t_1(f_2)\otimes H^L\big(t_{11}(\rho_2(\iota_2^{Id_\mathcal{C}}\circ h\circ ab_E))\otimes(id,\overline{0})\big)}\end{aligned}$$

However we get

$$H^L\big(t_{11}(\rho_2(\iota_2^{Id_\mathcal{C}}\circ h\circ ab_E))\otimes(id,\overline{0})\big) = r_2\big(\iota_2^{Id_\mathcal{C}}\circ h\circ ab_E\big)^{ab} = h$$

because, for $k = 1, 2$, we have $r_k^2 \circ \iota_2^{Id_\mathcal{C}} = 0$. It proves that

$$\overline{\psi_X^L}\big(\overline{t_1(f_1)\otimes t_1(f_2)\otimes t_{11}(\rho_2(\iota_2^{Id_\mathcal{C}}\circ h\circ ab_E))\otimes(id,\overline{0})},\ 0\big) = \overline{t_1(f_1)\otimes t_1(f_2)\otimes h}\,,$$

as desired. $\square$

Remark 4.6.4. As $\mathcal{C}$ is a semi-abelian category (in particular Mal'cev and Barr exact), the functor $L : \mathcal{C} \to Alg - \mathcal{P}$ preserves coequalizers of reflexive pairs (hence regular epimorphisms by 2.31 of [14]) because it is the quadratic tensor product (see 2.1.3) of some quadratic $\mathcal{C}$-module over $\mathcal{P}$ by 6.24 of [12].

For an object X in $\mathcal{C}$, we recall that the linearization of the Lazard functor is given in 2.3.3 by

$$T_1L(X) = T_1\big(-\otimes M^L\big)(X) \cong T_1U_E(X)\otimes_\Lambda Coker(\dot{q}) = T_1U_E(X)\otimes_\Lambda \mathcal{P}(1)$$

and M^L is the quadratic $\mathcal{C}$-module over the ring $\mathcal{P}(1)$ given in 4.5.1 and $\gamma : T_1L \Rightarrow T_1U_E \otimes_\Lambda \mathcal{P}(1)$ is the corresponding natural transformation given in (2.3).

Notation 4.6.5. We denote by $\overline{\gamma} : T_1L \Rightarrow \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big)$ the natural isomorphism that is the postcomposition of the natural isomorphism $u'_{\mathcal{C}(E^{ab},\ Ab^{\mathcal{C}})} : T_1U_E \otimes_\Lambda \mathcal{P}(1) \Rightarrow \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big)$ (see 4.2.11) with the natural isomorphism $\gamma : T_1L \Rightarrow T_1U_E \otimes_\Lambda \mathcal{P}(1)$ (see (2.3)).

In addition, the structure linear maps that make $L(X)$ a $\mathcal{P}$-algebra are given in 2.4.26. Here we recall the construction of the structure linear map encoding binary operations parametrized by $\mathcal{P}(2)$ that is given by the natural transformation $\lambda_2^L : L^{\otimes 2}\otimes\mathcal{P}(2) \Rightarrow L$ defined as follows:

$$\lambda_2^L = \mathbb{T}_2^{\mathcal{P}(1)}((i_2, id)) \circ \big((t_1^L)^{\otimes 2}\otimes_S id\big)\circ q^{M^L}\circ q_{\mathbb{Z}}^{R\otimes R}\,, \tag{4.6.1}$$

see (2.4.16) where $R = \mathcal{P}(1)$. Then, for an object X in $\mathcal{C}$, it is now possible to prove that $\lambda_2^L : L^{\otimes 2}\otimes \mathcal{P}(2) \Rightarrow L$ is entirely determined by the natural transformation $\widehat{\phi^{M^L}} : \big(T_1U_E^{\otimes 2}\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)\big)_{\mathfrak{S}_2} \Rightarrow L$ given in 2.1.3.

Notation 4.6.6. Let X be an object in $\mathcal{C}$. We consider the following notations:

- We recall that, for any $\alpha \in \mathcal{C}(E^{ab}, X^{ab})$, we denote by $\overline{\alpha} = \overline{\gamma_X}^{-1}(\alpha)$, see 4.6.5.
- We denote by $b \in L(E)$ an antecedent of $\overline{id} = \overline{\gamma_E}^{-1}(id)$ by the regular epimorphism $t_1 : L(E) \to T_1L(E)$.

Proposition 4.6.7. *Let X be an object in $\mathcal{C}$. Then we have*

$$Im\big(\widehat{\phi_X^{M^L}}\big) = Im\big((\lambda_2^L)_X\big) = [L(X), L(X)]_{Id_{Alg-\mathcal{P}}}$$

where the morphism $\widehat{\phi_X^{M^L}} : \big(T_1U_E(X)^{\otimes 2}\otimes_{\Lambda\otimes\Lambda}\mathcal{P}(2)\big)_{\mathfrak{S}_2} \to L(X)$ is present in the pushout (4.5.6) and $(\lambda_2^L)_X : L(X)^{\otimes 2}\otimes\mathcal{P}(2) \to L(X)$ is a part of the structure linear maps of $L(X)$.

Proof. First we prove that $Im\big(\widehat{\phi_X^{M^L}}\big) \subset Im\big((\lambda_2^L)_X\big) = [L(X), L(X)]_{Id_{Alg-\mathcal{P}}}$. For this, we consider $f_1, f_2 \in \mathcal{C}(E, X)$ and $h \in \mathcal{P}(2) = \mathcal{C}\big(E^{ab}, Id_{\mathcal{C}}(E|E)\big)$. For $k = 1, 2$, consider the following diagram

$$\begin{tikzcd} L(X) \arrow[r, "t_1^L"] & T_1L(X) \arrow[r, "\overline{\gamma_X}", "\cong"'] & \mathcal{C}\big(E^{ab}, X^{ab}\big) \\ L(E) \arrow[u, "L(f_k)"] \arrow[r, "t_1^L"'] & T_1L(E) \arrow[u, "T_1L(f_k)"'] \arrow[r, "\overline{\gamma_E}", "\cong"'] & \mathcal{C}\big(E^{ab}, E^{ab}\big) \arrow[u, "(f_k^{ab})_*"'] \end{tikzcd} \tag{4.6.2}$$

The above diagram commutes by naturality of $t_1^L : L(X) \to T_1L(X)$ and $\overline{\gamma_X} : T_1L(X) \to \mathcal{C}\big(E^{ab}, X^{ab}\big)$ in X. Then we have

$$\begin{aligned} \widehat{\phi_X^{M^L}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big) &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X \circ \widehat{\phi_X^{(M^L)^2}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big)\,, \text{by } (2.1.2) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X\big(T_1L(f_1)(\overline{id}) \otimes T_1L(f_2)(\overline{id}) \otimes_S h\big) \end{aligned}$$

As the morphism $t_1^L : L(E) \to T_1L(E)$ is an epimorphism, there is an element $b \in L(E)$ such that $t_1^L(b) = \overline{id} = \overline{\gamma_E}^{-1}(id)$ (see 4.6.6). Then we set $x_k = L(f_k)(b)$, for $k = 1, 2$ and we get $t_1^L(x_k) = T_1L(f_k)(\overline{id})$ because the left-hand square of the diagram (4.6.2) commutes. Then we have

$$\begin{aligned} \widehat{\phi_X^{M^L}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big) &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X\big(T_1L(f_1)(\overline{id}) \otimes T_1L(f_2)(\overline{id}) \otimes_S h\big) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X\big(t_1^L(x_1) \otimes t_1^L(x_2) \otimes_S h\big) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X \circ \big((t_1^L)_X^{\otimes 2} \otimes_S id\big) \circ q_X^{M^L} \circ q_{\mathbb{Z}}^{R\otimes R}\big(x_1 \otimes x_2 \otimes h\big) \\ &= \lambda_2^L\big(x_1 \otimes x_2 \otimes h\big)\,, \text{by } (4.6.1) \end{aligned}$$

Hence we get

$$\widehat{\phi_X^{M^L}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big) = \lambda_2^L\big(L(f_1)(b) \otimes L(f_2)(b) \otimes h\big) \tag{4.6.3}$$

Next we prove that $[L(X), L(X)]_{Id_{Alg-\mathcal{P}}} = Im\big((\lambda_2^L)_X\big) \subset Im\big(\widehat{\phi_X^{M^L}}\big)$. Let $x_1, x_2 \in L(X)$ and $h \in \mathcal{P}(2)$. As the right-hand square commutes and $\overline{\gamma_X} : T_1L(X) \to \mathcal{C}(E^{ab}, X^{ab})$ is an isomorphism (see 4.6.5), there is a unique $\alpha_k \in \mathcal{C}(E^{ab}, X^{ab})$ $(k = 1, 2)$ such that

$$t_1^L(x_k) = \overline{\gamma_X}^{-1}(\alpha_k) \tag{4.6.4}$$

As E is a regular-projective object and $ab_X : X \to X^{ab}$ is a regular epimorphism, there is a (non-unique) morphism $f_k \in \mathcal{C}(E, X)$ such that, for $k = 1, 2$, we have

$$ab_X \circ f_k = \alpha_k \circ ab_E\,, \quad \text{i.e. } \alpha_k = f_k^{ab} \tag{4.6.5}$$

By (4.6.4) and (4.6.5), it follows that, for $k = 1, 2$, we get

$$t_1^L(x_k) = T_1L(f_k)(\overline{\gamma_E}^{-1}(id)) = T_1L(f_k)(\overline{id}) \tag{4.6.6}$$

Hence we get

$$\begin{aligned} \lambda_2^L\big(x_1 \otimes x_2 \otimes h\big) &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X \circ \big((t_1^L)_X^{\otimes 2} \otimes_S id\big) \circ q_X^{M^L} \circ q_{\mathbb{Z}}^{R\otimes R}\big(x_1 \otimes x_2 \otimes h\big)\,, \text{by } (4.6.1) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X\big(t_1^L(x_1) \otimes t_1^L(x_2) \otimes_S h\big) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X\big(T_1L(f_1)(\overline{id}) \otimes T_1L(f_2)(\overline{id}) \otimes_S h\big)\,, \text{by } (4.6.6) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_X \circ \widehat{\phi_X^{(M^L)^2}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big) \\ &= \widehat{\phi_X^{M^L}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big)\,, \text{by } (2.1.2) \end{aligned}$$

as desired. Then 1.7.4 gives $[L(X), L(X)]_{Id_{Alg-\mathcal{P}}} = Im(\lambda_2^L)$ because the (linear) operad $\mathcal{P} = AbOp(\mathcal{C})$ is 2-step nilpotent. $\square$

Now we are able to prove that the linearization $T_1 L$ of the Lazard functor and the composite functors $Ab^{Alg-\mathcal{P}}.L$ both with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$ are isomorphic to each other. We first give up to isomorphism an explicit expression of the second cross-effect of the Lazard functor $L : \mathcal{C} \to Alg - \mathcal{P}$ that has been already given in 6.20 of [12]. In fact it says that there is a natural transformation $\Phi : T_1 U_E \otimes T_1 U_E \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2) \Rightarrow cr_2 L$ between bilinear bifunctors such that, for X_1 and X_2 two objects in $\mathcal{C}$, $f_k \in \mathcal{C}(E, X_k)$ $(k = 1, 2)$ and $h \in \mathcal{P}(2)$, we get

$$\Phi_{X_1,X_2}(t_1(f_1) \otimes t_1(f_2) \otimes h) = (\iota_2^L)^{-1} \circ \widehat{\phi^{M^L}_{X_1+X_2}}\big(\overline{t_1(i_1^2 \circ f_1) \otimes t_1(i_2^2 \circ f_2) \otimes h}\big) \tag{4.6.7}$$

By 4.0.3 and 6.27 of [12], it is a natural isomorphism on $\mathcal{C}\times\mathcal{C}$. Then we have the following proposition:

Proposition 4.6.8. *For an object X in $\mathcal{C}$, we have*

$$(S_2^L)_X \circ \Phi_{X,X} = \widehat{\phi^{M^L}_X} \circ \pi$$

where $S_2^L : cr_2 L.\Delta^2 \Rightarrow L$ is the natural transformation given in 1.2.7 and $\pi : T_1 U_E(X)^{\otimes 2} \otimes \mathcal{P}(2) \to \big(T_1 U_E(X)^{\otimes 2} \otimes \mathcal{P}(2)\big)_{\mathfrak{S}_2}$ is the canonical quotient.

Proof. Let $f_1, f_2 \in \mathcal{C}(E, X)$ and $h \in \mathcal{P}(2)$. Then we get

$$\begin{aligned}
&(S_2^L)_X \circ \Phi_{X,X}(t_1(f_1) \otimes t_1(f_2) \otimes h)\\
&= L(\nabla_X^2) \circ \iota_2^L \circ \Phi_{X,X}(t_1(i_1^2 \circ f_1) \otimes t_1(i_2^2 \circ f_2) \otimes h)\,, \text{by } 1.2.7\\
&= L(\nabla_X^2) \circ \widehat{\phi^{M^L}_{X+2}}(\overline{t_1(i_1^2 \circ f_1) \otimes t_1(i_2^2 \circ f_2) \otimes h})\,, \text{by } (4.6.7)\\
&= \widehat{\phi^{M^L}_X} \circ \big(t_1(\nabla_X^2)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} id\big)_{\mathfrak{S}_2}(\overline{t_1(i_1^2 \circ f_1) \otimes t_1(i_2^2 \circ f_2) \otimes h})\,, \text{by } (2.1.2)\\
&= \widehat{\phi^{M^L}_X}(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h})\,,
\end{aligned}$$

as desired. $\square$

Now we are able to give the following proposition:

Proposition 4.6.9. *The linear functors $T_1 L$ and $Ab^{Alg-\mathcal{P}} \cdot L$ with domain $\mathcal{C}$ and values in $Mod_{\mathcal{P}(1)}$ are isomorphic to each other; more precisely, there is a unique isomorphism $\vartheta_X : Ab^{Alg-\mathcal{P}} \cdot L(X) \to T_1 L(X)$ natural in X such that*

$$\vartheta_X \circ ab_{L(X)} = (t_1^L)_X \tag{4.6.8}$$

Proof. For all $X \in \mathcal{C}$, we have the following equalities:

$$\begin{aligned}
Im\Big(S_2^{Id_{Alg-\mathcal{P}}} : Id_{Alg-\mathcal{P}}\big(L(X)|L(X)\big) \to L(X)\Big) &= [L(X), L(X)]_{Id_{Alg-\mathcal{P}}}\,, \text{ by definition}\\
&= Im\big((\lambda_2^L)_X : L(X)^{\otimes} \otimes \mathcal{P}(2) \to L(X)\big)\\
&= Im\big(\widehat{\phi^{M^L}_X}\big)\,, \text{by } 4.6.7\\
&= Im\big(\widehat{\phi^{M^L}_X} \circ \pi\big)\,, \text{since } \pi \text{ is a surjection}\\
&= Im\Big((S_2^L)_X \circ \Phi_{X,X}\Big)\,, \text{b } 4.6.8\\
&= Im\big((S_2^L)_X : L(X|X) \to L(X)\big)
\end{aligned}$$

The last equality holds because $\Phi_{X,X}$ is an isomorphism (hence a surjection). As the cokernels of $S_2^{Id_{Alg-\mathcal{P}}}$ and $(S_2^L)_X$ are respectively $ab_{L(X)} : L(X) \to Ab^{Alg-\mathcal{P}} \cdot L(X)$ and $(t_1^L)_X : L(X) \to T_1L(X)$, it concludes the proof. $\square$

Remark 4.6.10. For an object X in $\mathcal{C}$ and $x \in L(X)$, we remark that we clearly have the following relation:

$$\vartheta_X(\overline{x}) = t_1^L(x)$$

where $\overline{x} = ab_{L(X)}(x)$ (see the notations given in 1.7.6).

4.6.1 The functor $L_{|\langle E\rangle} : \langle E\rangle \to Alg - AbOp(\mathcal{C})$ preserves finite coproducts

Here we check that the quadratic functor $L_{|\langle E\rangle} : \langle E\rangle \to Alg - \mathcal{P}$ restricted to $\langle E\rangle$, or simply L, preserves finite coproducts. In this case, this ensures that $L_{|\langle E\rangle} : \langle E\rangle \to Alg-\mathcal{P}$ takes values in $\langle \mathcal{F}_{\mathcal{P}}\rangle$ so that it is a quadratic functor between algebraic theories. We first recall the explicit expression of the coproduct $\mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}}$. By 1.8.5, it is given as follows:

$$\mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}} = \mathcal{F}_{\mathcal{P}} \times \mathcal{F}_{\mathcal{P}} \times \mathcal{P}(2)$$

together with its structure linear maps given by

- $\lambda_1^{\mathcal{F}_{\mathcal{P}}^{+2}} : \big(\mathcal{F}_{\mathcal{P}}^{+2}\big) \otimes \mathcal{P}(1) \to \mathcal{F}_{\mathcal{P}}$ is defined by

$$\lambda_1^{\mathcal{F}_{\mathcal{P}}^{+2}}\Big(\big((f_1, \overline{h_1}), (f_2, \overline{h_2}), h\big) \otimes g\Big) = \Big(\big(f_1 \circ g,\ \overline{h_1 \circ g)}\big), \big(f_2 \circ g,\ \overline{h_2 \circ g}\big),\ h \circ g\Big)$$

  where $f_k, g \in \mathcal{P}(1)$, $h_k, h \in \mathcal{P}(2)$ and $k = 1, 2$.

- $\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}} : (\mathcal{F}_{\mathcal{P}}^{+2})^{\otimes 2} \otimes \mathcal{P}(2) \to \mathcal{F}_{\mathcal{P}}$ is defined by

$$\lambda_2^{\mathcal{F}_{\mathcal{P}}^{+2}}\Big(\big((f_1^1, \overline{h_1^1}), (f_2^1, \overline{h_2^1}), h_1\big) \otimes \big((f_1^2, \overline{h_1^2}), (f_2^2, \overline{h_2^2}), h_2\big) \otimes h\Big)$$
$$= \Big(\big(0,\ \overline{Id_{\mathcal{C}}(f_1^1|f_1^2) \circ h)}\big),\ \big(0,\ \overline{Id_{\mathcal{C}}(f_2^1|f_2^2) \circ h)}\big),\ Id_{\mathcal{C}}(f_1^1|f_2^2) \circ h + Id_{\mathcal{C}}(f_1^2|f_2^1) \circ T(h)\Big)$$

  where $f_k \in \mathcal{P}(1)$, $f_k, h \in \mathcal{P}(2)$ and $k = 1, 2$.

Proposition 4.6.11. *The functor $L : \mathcal{C} \to Alg - \mathcal{P}$ restricted to $\langle E\rangle$ preserves finite coproducts.*

Proof. It suffices to verify that $L(E + E) \cong L(E) + L(E) = \mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}}$. For this we prove that the following diagram is a pushout:

$$\begin{tikzcd}
B(E^{+2}) \arrow[r, "\overline{\phi_{E^{+2}}}"] \arrow[d, "\overline{\psi^L_{E^{+2}}}"'] & T_2U_E(E^{+2}) \otimes_{\Lambda} \mathcal{F}_{\mathcal{P}} \arrow[d, "\widehat{\psi^{M^L}_{E^{+2}}}"] \\
\big((T_1U_E(E^{+2}) \otimes T_1U_E(E^{+2})) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \arrow[r, "\widehat{\phi^{M^L}_{E^{+2}}}"'] & \mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}}
\end{tikzcd} \tag{4.6.9}$$

where $B(E^{+2}) = \big((T_1U_E(E^{+2}) \otimes T_1U_E(E^{+2})) \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E, E) \otimes_{\Lambda} \mathcal{F}_{\mathcal{P}}\big)_{\mathfrak{S}_2} \oplus (U_E(E^{+2}) \otimes \mathcal{P}(2))$ and the maps $\widehat{\psi^{M^L}_{E^{+2}}}$ and $\widehat{\phi^{M^L}_{E^{+2}}}$ are defined as follows:

- Let $\xi \in \mathcal{C}(E,\ E^{+2})$ and $(f,\ \overline{h}) \in \mathcal{F}_{\mathcal{P}}$, then we have

$$\widehat{\psi^{M^L}_{E^{+2}}}\big(t_2(\xi) \otimes_{\Lambda} (f,\ \overline{h})\big) = \Big((r_1^2 \circ \xi) *_L (f,\ \overline{h}),\ (r_2^2 \circ \xi) *_L (f,\ \overline{h}),\ H^L\big(t_{11}(\rho_2(\xi)) \otimes_{\Lambda} (f,\ \overline{h})\big)\Big)$$

- Let $f_1, f_2 \in \mathcal{C}(E,\ E^{+2})$ and $h \in \mathcal{P}(2)$, then we have

$$\widehat{\phi^{M^L}_{E^{+2}}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes_{\Lambda\otimes\Lambda} h}\big)$$
$$= \Big(\dot{q}\big(\gamma_{1,1;2}\big((r^2_1 \circ f_1)^{ab} \otimes (r^2_1 \circ f_2)^{ab} \otimes h\big)\big),\ \dot{q}\big(\gamma_{1,1;2}\big((r^2_2 \circ f_1)^{ab} \otimes (r^2_2 \circ f_2)^{ab} \otimes h\big)\big),$$
$$\gamma_{1,1;2}\big((r^2_1 \circ f_1)^{ab} \otimes (r^2_2 \circ f_2)^{ab} \otimes h\big) + \gamma_{1,1;2}\big((r^2_1 \circ f_2)^{ab} \otimes (r^2_2 \circ f_1)^{ab}) \otimes T(h)\big)\Big)$$

First we check that the diagram (4.6.9) commutes. We denote respectively by i_1 and i_2 the injections of the first and the second summand of $B(E^{+2})$. Then it remains to prove that we have

$$\widehat{\psi^{M^L}_{E^{+2}}} \circ \overline{\phi_{E^{+2}}} \circ i_1 = \widehat{\phi^{M^L}_{E^{+2}}} \circ \overline{\psi_{E^{+2}}} \circ i_1 \quad \text{and} \quad \widehat{\psi^{M^L}_{E^{+2}}} \circ \overline{\phi_{E^{+2}}} \circ i_2 = \widehat{\phi^{M^L}_{E^{+2}}} \circ \overline{\psi_{E^{+2}}} \circ i_2$$

by the universal property of the coproduct $B(E^{+2})$. Let $f_1, f_2, f, \xi\ \in \mathcal{C}(E,\ E^{+2})$, $g\ \in \mathcal{P}(1)$ and $h\ \in \mathcal{P}(2)$. First we have

$$\big(\widehat{\psi^{M^L}_{E^{+2}}} \circ \overline{\phi_{E^{+2}}} \circ i_1\big)\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes_{\Lambda\otimes\Lambda} t_{11}(\rho_2(\xi)) \otimes_\Lambda (f, \overline{h})}\big)$$
$$= \widehat{\psi^{M^L}_{E^{+2}}}\Big(t_2\big(\nabla^2_{E^{+2}} \circ (f_1 + f_2) \circ \xi\big) \otimes_\Lambda (f, \overline{h}) - t_2\big(f_1 \circ r^2_1 \circ \xi\big) \otimes_\Lambda (f, \overline{h})$$
$$- t_2\big(f_2 \circ r^2_2 \circ \xi\big) \otimes_\Lambda (f, \overline{h})\Big) \text{, by 2.1.4}$$
$$= \Big(\big(r^2_1 \circ \nabla^2_{E^{+2}} \circ (f_1 + f_2) \circ \xi\big) *_L (f, \overline{h}),\ \big(r^2_2 \circ \nabla^2_{E^{+2}} \circ (f_1 + f_2) \circ \xi\big) *_L (f, \overline{h}),$$
$$H^L\big(t_{11}\big(\rho_2(\nabla^2_{E^{+2}} \circ (f_1 + f_2) \circ \xi)\big) \otimes_\Lambda (f, \overline{h})\big)\Big)$$
$$- \Big(\big(r^2_1 \circ f_1 \circ r^2_1 \circ \xi\big) *_L (f, \overline{h}),\ \big(r^2_2 \circ f_1 \circ r^2_1 \circ \xi\big) *_L (f, \overline{h}),\ H^L\big(t_{11}(\rho_2(f_1 \circ r^2_1 \circ \xi)) \otimes_\Lambda (f, \overline{h})\big)\Big)$$
$$- \Big(\big(r^2_1 \circ f_2 \circ r^2_2 \circ \xi\big) *_L (f, \overline{h}),\ \big(r^2_2 \circ f_2 \circ r^2_2 \circ \xi\big) *_L (f, \overline{h}),\ H^L\big(t_{11}(\rho_2(f_2 \circ r^2_2 \circ \xi)) \otimes_\Lambda (f, \overline{h})\big)\Big)$$

We compute each component of the above triplet. For $k = 1, 2$ we have

$$\big(r^2_k \circ \nabla^2_{E^{+2}} \circ (f_1 + f_2) \circ \xi\big) *_L (f, \overline{h}) = \big(\nabla^2_E \circ ((r^2_k \circ f_1) + (r^2_k \circ f_2)) \circ \xi\big) *_L (f, \overline{h}) \tag{4.6.10}$$

By 4.5.9, the relation $(QM1)$ holds and it implies that we get

$$\big(\nabla^2_E \circ ((r^2_k \circ f_1) + (r^2_k \circ f_2)) \circ \xi\big) *_L (f, \overline{h}) - \big(r^2_k \circ f_1 \circ r^2_1 \circ \xi\big) *_L (f, \overline{h}) - \big(r^2_k \circ f_2 \circ r^2_2 \circ \xi\big) *_L (f, \overline{h})$$
$$= \big(\nabla^2_E \circ ((r^2_k \circ f_1) + (r^2_k \circ f_2)) \circ \xi\big) *_L (f, \overline{h}) - \big(r^2_1 \circ ((r^2_k \circ f_1) + (r^2_k \circ f_2)) \circ \xi\big) *_L (f, \overline{h})$$
$$- \big(r^2_2 \circ ((r^2_k \circ f_1) + (r^2_k \circ f_2)) \circ \xi\big) *_L (f, \overline{h})$$
$$= \big(\dot{q} \circ H^L\big)\Big(t_{11}\big(\rho_2(((r^2_k \circ f_1) + (r^2_k \circ f_2)) \circ \xi)\big) \otimes (f, \overline{h})\Big)$$

Moreover, for $k = 1, 2$, we have

$$H^L\Big(t_{11}\big(\rho_2(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi)\big) \otimes (f, \overline{h})\Big)$$

$$= t_1(r_1^2 \circ ((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi) \otimes t_1(r_2^2 \circ ((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi).\big(h + T(h) - \frac{1}{2}\,\gamma_{2;1}(H(2) \otimes f)\big)$$

$$+ \gamma_{2;1}\big(r_2\big(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big)^{ab} \otimes f\big)$$

$$= t_1(r_k^2 \circ f_1 \circ r_1^2 \circ \xi) \otimes t_1(r_k^2 \circ f_2 \circ r_2^2 \circ \xi).\big(h + T(h) - \frac{1}{2}\,\gamma_{2;1}(H(2) \otimes f)\big)$$

$$+ \gamma_{2;1}\big(r_2\big(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big)^{ab} \otimes f\big)$$

$$= t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).\Big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(h + T(h) - \frac{1}{2}\,\gamma_{2;1}(H(2) \otimes f)\big)\Big)$$

$$+ \gamma_{2;1}\big(r_2\big(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big)^{ab} \otimes f\big)$$

Then we give another expression of $r_2\big(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big)$ as follows:

$$\begin{aligned}
&\big((r_k^2 \circ f_1) + (r_k^2 \circ f_2)\big) \circ \xi \\
&= \big((r_k^2 \circ f_1) + (r_k^2 \circ f_2)\big) \circ \Big((i_1^2 \circ r_1^2 \circ \xi) + (i_2^2 \circ r_2^2 \circ \xi) + (\iota_2^{Id_{\mathcal{C}}} \circ r_2(\xi))\Big)\text{, by (4.1.11)} \\
&= \big(i_1^2 \circ r_k^2 \circ f_1 \circ r_1^2 \circ \xi\big) + \big(i_2^2 \circ r_k^2 \circ f_2 \circ r_2^2 \circ \xi\big) + \big(\iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}\big(r_k^2 \circ f_1 | r_k^2 \circ f_2\big) \circ r_2(\xi)\big)\text{, by (4.1.21)} \\
&= \big(i_1^2 \circ r_1^2 \circ ((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big) + \big(i_2^2 \circ r_2^2 \circ ((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big) \\
&\quad + \big(\iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}\big(r_k^2 \circ f_1 | r_k^2 \circ f_2\big) \circ r_2(\xi)\big)
\end{aligned}$$

This proves that we have

$$\begin{aligned}
r_2\big(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big) &= Id_{\mathcal{C}}\big(r_k^2 \circ f_1 | r_k^2 \circ f_2\big) \circ r_2(\xi) \\
&= t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).r_2(\xi)
\end{aligned}$$

Hence we have

$$r_2\big(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi\big)^{ab} = t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).r_2(\xi)^{ab}$$

It implies that we get

$$H^L\Big(t_{11}\big(\rho_2(((r_k^2 \circ f_1) + (r_k^2 \circ f_2)) \circ \xi)\big) \otimes (f, \overline{h})\Big)$$

$$= t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).\Big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(h + T(h) - \frac{1}{2}\,\gamma_{2;1}(H(2) \otimes f)\big)\Big)$$

$$+ t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).\gamma_{2;1}(r_2(\xi)^{ab} \otimes f)$$

$$= t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).\Big(t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(h + T(h) - \frac{1}{2}\,\gamma_{2;1}(H(2) \otimes f) + \gamma_{2;1}(r_2(\xi)^{ab} \otimes f)\Big)$$

$$= t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).\, H^L\big(t_{11}(\rho_2(\xi)) \otimes (f, \overline{h})\big)$$

Consequently we obtain

$$
\begin{aligned}
&\big(r_k^2\circ\nabla^2_{E+E}\circ(f_1+f_2)\circ\xi\big)\ast_L(f,\overline{h})-\big(r_k^2\circ f_1\circ r_1^2\circ\xi\big)\ast_L(f,\overline{h})-\big(r_k^2\circ f_2\circ r_2^2\circ\xi\big)\ast_L(f,\overline{h})\\
&=\dot{q}\Big(t_1(r_k^2\circ f_1)\otimes t_1(r_k^2\circ f_2).\,H^L\big(t_{11}(\rho_2(\xi))\otimes(f,\overline{h})\big)\Big)\\
&=\dot{q}\Big(\gamma_{1,1;2}\big((r_k^2\circ f_1)^{ab}\otimes(r_k^2\circ f_2)^{ab}\otimes H^L\big(t_{11}(\rho_2(\xi))\otimes(f,\overline{h})\big)\big)\Big)
\end{aligned}
$$

Then we compute the following term:

$$
\begin{aligned}
H^L\big(t_{11}\big(\rho_2\big(\nabla_{E^{+2}}\circ(f_1+f_2)\circ\xi\big)\big)\otimes_\Lambda(f,\overline{h})\big)-H^L\big(t_{11}(\rho_2(f_1\circ r_1^2\circ\xi))\otimes_\Lambda(f,\overline{h})\big)&\\
-H^L\big(t_{11}(\rho_2(f_2\circ r_2^2\circ\xi))\otimes_\Lambda(f,\overline{h})\big)&
\end{aligned}
$$

For this we first give another expression of $r_2\big(\nabla^2_{E^{+2}}\circ(f_1+f_2)\circ\xi\big)^{ab}$. We have the following equalities:

$$
\begin{aligned}
\nabla^2_{E^{+2}}\circ(f_1+f_2)\circ\xi&=(f_1,\ f_2)\circ\xi\\
&=(f_1,\ f_2)\circ\Big((i_1^2\circ r_1^2\circ\xi)+(i_2^2\circ r_2^2\circ\xi)+(\iota_2^{Idc}\circ r_2(\xi))\Big)\text{, by (4.1.11)}\\
&=(f_1\circ r_1^2\circ\xi)+(f_2\circ r_2^2\circ\xi)+\big(c_2^{E^{+2}}\circ Id_{\mathcal{C}}(f_1|f_2)\circ r_2(\xi)\big)\text{, by (4.1.21)}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&r_2\big(\nabla^2_{E^{+2}}\circ(f_1+f_2)\circ\xi\big)\\
&=r_2\Big((f_1\circ r_1^2\circ\xi)+(f_2\circ r_2^2\circ\xi)+\big(c_2^{E^{+2}}\circ Id_{\mathcal{C}}(f_1|f_2)\circ r_2(\xi)\big)\Big)\text{, by (4.1.11)}\\
&=r_2\big((f_1\circ r_1^2\circ\xi)+(f_2\circ r_2^2\circ\xi)\big)+r_2\big(c_2^{E^{+2}}\circ Id_{\mathcal{C}}(f_1|f_2)\circ r_2(\xi)\big)\text{, by 4.1.27 and (4.1.16)}\\
&=r_2(f_1\circ r_1^2\circ\xi)+r_2(f_2\circ r_2^2\circ\xi)+\gamma_{1,1;2}\big((r_1^2\circ f_2\circ r_2^2\circ\xi)^{ab}\otimes(r_2^2\circ f_1\circ r_1^2\circ\xi)^{ab}\otimes H_E(2)\big)\\
&\qquad+\gamma_{1,1;2}\big((r_1^2\circ f_1)^{ab}\otimes(r_2^2\circ f_2)^{ab}\otimes r_2(\xi)\big)+\gamma_{1,1;2}\big((r_1^2\circ f_2)^{ab}\otimes(r_2^2\circ f_1)^{ab}\otimes T(r_2(\xi))\big)
\end{aligned}
$$

by 4.1.27 and 4.1.29. Hence we obtain

$$
\begin{aligned}
&r_2\big(\nabla^2_{E^{+2}}\circ(f_1+f_2)\circ\xi\big)^{ab}\\
&=r_2(f_1\circ r_1^2\circ\xi)^{ab}+r_2(f_2\circ r_2^2\circ\xi)^{ab}+\gamma_{1,1;2}\big((r_1^2\circ f_2\circ r_2^2\circ\xi)^{ab}\otimes(r_2^2\circ f_1\circ r_1^2\circ\xi)^{ab}\otimes H(2)\big)\\
&\qquad+\gamma_{1,1;2}\big((r_1^2\circ f_1)^{ab}\otimes(r_2^2\circ f_2)^{ab}\otimes r_2(\xi)^{ab}\big)+\gamma_{1,1;2}\big((r_1^2\circ f_2)^{ab}\otimes(r_2^2\circ f_1)^{ab}\otimes T(r_2(\xi)^{ab})\big)
\end{aligned}
$$

Now we give another expression of $t_1\big(r_k^2\circ\nabla_{E^{+2}}\circ(f_1+f_2)\circ\xi\big)$. By 2.14 of [12], we get

$$
\begin{aligned}
t_1\big(r_k^2\circ\nabla_{E^{+2}}\circ(f_1+f_2)\circ\xi\big)&=t_1\big(\nabla_E^2\circ\big((r_k^2\circ f_1)+(r_k^2\circ f_2)\big)\circ\xi\big)\\
&=t_1\big(r_1^2\circ\big((r_k^2\circ f_1)+(r_k^2\circ f_2)\big)\circ\xi\big)+t_1\big(r_2^2\circ\big((r_k^2\circ f_1)+(r_k^2\circ f_2)\big)\circ\xi\big)\\
&=t_1\big(r_k^2\circ f_1\circ r_1^2\circ\xi\big)+t_1\big(r_k^2\circ f_2\circ r_2^2\circ\xi\big)
\end{aligned}
$$

Hence we have the following equalities:

$$
\begin{aligned}
&H^L\big(t_{11}\big(\rho_2\big(\nabla^2_{E^{+2}}\circ(f_1+f_2)\circ\xi\big)\big)\otimes_\Lambda(f,\overline{h})\big)\\
&=t_1(r_1^2\circ f_1\circ r_1^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+t_1(r_1^2\circ f_1\circ r_1^2\circ\xi)\otimes t_1(r_2^2\circ f_2\circ r_2^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_2\circ r_2^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+\gamma_{2;1}\big(r_2(f_1\circ r_1^2\circ\xi)^{ab}\otimes f\big)+\gamma_{2;1}\big(r_2(f_2\circ r_2^2\circ\xi)^{ab}\otimes f\big)\\
&\qquad+\gamma_{1,1;2}\big((r_1^2\circ f_2\circ r_2^2\circ\xi)^{ab}\otimes(r_2^2\circ f_1\circ r_1^2\circ\xi)^{ab}\otimes(H(2)\circ f)\big)\\
&\qquad+\gamma_{1,1;2}\big((r_1^2\circ f_1)^{ab}\otimes(r_2^2\circ f_2)^{ab}\otimes(r_2(\xi)^{ab}\circ f)\big)\\
&\qquad+\gamma_{1,1;2}\big((r_1^2\circ f_2)^{ab}\otimes(r_2^2\circ f_1)^{ab}\otimes(T(r_2(\xi)^{ab})\circ f)\big)\\
&=t_1(r_1^2\circ f_1\circ r_1^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)+\gamma_{2;1}\big(r_2(f_1\circ r_1^2\circ\xi)^{ab}\otimes f\big)\\
&\qquad+t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_2\circ r_2^2\circ\xi).\big(h+T((h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+\gamma_{2;1}\big(r_2(f_2\circ r_2^2\circ\xi)^{ab}\otimes f\big)\\
&\qquad+t_1(r_1^2\circ f_1\circ r_1^2\circ\xi)\otimes t_1(r_2^2\circ f_2\circ r_2^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\gamma_{2;1}(H(2)\otimes f)\\
&\qquad+t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).\gamma_{2;1}(r_2(\xi)^{ab}\otimes f)\\
&\qquad+t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).\gamma_{2;1}(T(r_2(\xi)^{ab})\otimes f)\\
&=H^L\big(t_{11}(\rho_2(f_1\circ r_1^2\circ\xi))\otimes_\Lambda(f,\overline{h})\big)+H^L\big(t_{11}(\rho_2(f_2\circ r_2^2\circ\xi))\otimes_\Lambda(f,\overline{h})\big)\\
&\qquad+t_1(r_1^2\circ f_1\circ r_1^2\circ\xi)\otimes t_1(r_2^2\circ f_2\circ r_2^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\\
&\qquad+t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{1,1;2}(H(2)\otimes f)\big)\\
&\qquad+t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\gamma_{2;1}(H(2)\otimes f)\\
&\qquad+t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).\gamma_{2;1}(r_2(\xi)^{ab}\otimes f)\big)+t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).\gamma_{2;1}(T(r_2(\xi)^{ab})\otimes f)\big)
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&H^L\big(t_{11}\big(\rho_2(\nabla_{E^{+2}} \circ (f_1+f_2)\circ\xi)\big) \otimes_\Lambda (f,\overline{h})\big) - H^L\big(t_{11}(\rho_2(f_1\circ r_1^2\circ\xi)) \otimes_\Lambda (f,\overline{h})\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad - H^L\big(t_{11}(\rho_2(f_2\circ r_2^2\circ\xi)) \otimes_\Lambda (f,\overline{h})\big) \\
&= t_1(r_1^2\circ f_1\circ r_1^2\circ\xi)\otimes t_1(r_2^2\circ f_2\circ r_2^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big) \\
&\qquad + t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\big(h+T(h)+\frac{1}{2}\,T(\gamma_{2;1}(H(2)\otimes f))\big)\,\text{, by 4.1.19} \\
&\qquad - t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).T(\gamma_{2;1}(H(2)\otimes f))\,\text{, by 4.1.19} \\
&\qquad + t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).\gamma_{2;1}(r_2(\xi)^{ab}\otimes g) + t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).T(\gamma_{2;1}(r_2(\xi)^{ab}\otimes f)) \\
&= t_1(r_1^2\circ f_1\circ r_1^2\circ\xi)\otimes t_1(r_2^2\circ f_2\circ r_2^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big) \\
&\qquad + t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).\gamma_{2;1}(r_2(\xi)^{ab}\otimes f) \\
&\qquad + t_1(r_1^2\circ f_2\circ r_2^2\circ\xi)\otimes t_1(r_2^2\circ f_1\circ r_1^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,T(\gamma_{2;1}(H(2)\otimes f))\big) \\
&\qquad + t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).T(\gamma_{2;1}(r_2(\xi)^{ab}\otimes f)) \\
&= t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).\big(h+T(h)-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\Big) \\
&\qquad + t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).\gamma_{2;1}(r_2(\xi)\otimes f) \\
&\qquad + t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).\Big(t_1(r_2^2\circ\xi)\otimes t_1(r_1^2\circ\xi).T\Big(T(h)+h-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\Big)\Big) \\
&\qquad + t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).T(\gamma_{2;1}(r_2(\xi)^{ab}\otimes f)) \\
&= t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).H^L\big(t_{11}(\rho_2(\xi))\otimes_\Lambda (f,\overline{h})\big) \\
&\qquad + t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).T\Big(t_1(r_1^2\circ\xi)\otimes t_1(r_2^2\circ\xi).\big(T(h)+h-\frac{1}{2}\,\gamma_{2;1}(H(2)\otimes f)\big)\Big) \\
&\qquad + t_1(r_1^2\circ f_2)\otimes t_1(r_2^2\circ f_1).T(\gamma_{2;1}(r_2(\xi)^{ab}\otimes f)) \\
&= t_1(r_1^2\circ f_1)\otimes t_1(r_2^2\circ f_2).H^L\big(t_{11}(\rho_2(\xi))\otimes_\Lambda (f,\overline{h})\big) \\
&\qquad + t_1(r_1\circ f_2)\otimes t_1(r_2\circ f_1).T\Big(H^L\big(t_{11}(\rho_2(\xi))\otimes_\Lambda (f,\overline{h})\big)\Big) \\
&= \gamma_{2;1}\Big((r_1^2\circ f_1)^{ab}\otimes (r_2^2\circ f_2)^{ab}\otimes H^L\big(t_{11}(\rho_2(\xi))\otimes_\Lambda (f,\overline{h})\big)\Big) \\
&\qquad + \gamma_{2;1}\Big((r_1\circ f_2)^{ab}\otimes (r_2\circ f_1)^{ab}\otimes T\big(H^L\big(t_{11}(\rho_2(\xi))\otimes_\Lambda (f,\overline{h})\big)\big)\Big)
\end{aligned}
$$

Finally it proves that we have

$$\widehat{\phi^{M^L}_{E^{+2}}}\circ\overline{\psi_{E^{+2}}}\circ i_1 = \widehat{\psi^{M^L}_{E^{+2}}}\circ\overline{\phi_{E^{+2}}}\circ i_1$$

Then we have the following equalities:

$$\begin{aligned}
&\big(\widehat{\psi^{M^L}_{E^{+2}}} \circ \overline{\phi_{E^{+2}}} \circ i_2\big)(\xi \otimes_\Lambda h)\\
&= \widehat{\psi^{M^L}_{E^{+2}}}\big(t_2(\xi) \otimes_\Lambda (0, \overline{h})\big)\\
&= \Big((r_1^2 \circ \xi) *_L (0, \overline{h}),\ (r_2^2 \circ \xi) *_L (0, \overline{h}),\ H^L\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (0, \overline{h})\big)\Big)\\
&= \Big(\big(0, \overline{\gamma_{1,1;2}\big((r_1^2 \circ \xi)^{ab} \otimes (r_1^2 \circ \xi)^{ab} \otimes h\big)}\big),\ \big(0, \overline{\gamma_{1,1;2}\big((r_2^2 \circ \xi)^{ab} \otimes (r_2^2 \circ \xi)^{ab} \otimes h\big)}\big),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t_1(r_1^2 \circ \xi) \otimes t_1(r_2 \circ \xi).(h + T(h))\Big)\\
&= \widehat{\phi^{M^L}_{E^{+2}}}\big(\overline{t_1(\xi) \otimes t_1(\xi) \otimes h}\big)\\
&= \big(\widehat{\phi^{m^L}_{E^{+2}}} \circ \overline{\psi_{E^{+2}}} \circ i_2\big)(\xi \otimes_\Lambda h)
\end{aligned}$$

Hence it proves that we get

$$\widehat{\phi^{M^L}_{E^{+2}}} \circ \overline{\psi_{E^{+2}}} \circ i_2 = \widehat{\psi^{M^L}_{E^{+2}}} \circ \overline{\phi^{M^L}_{E^{+2}}} \circ i_2$$

By the universal property of the coproduct $B(E^{+2})$, the diagram (4.6.9) commutes. Next we verify that the universal property of the push-out holds. Let $\alpha : \big((T_1U_E(E^{+2}) \otimes T_1U_E(E^{+2})) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \to A$ and $\beta : T_2U_E(E^{+2}) \otimes_\Lambda \mathcal{F}_\mathcal{P} \to A$ be two morphisms in Ab such that

$$\alpha \circ \overline{\psi^L}_{E^{+2}} \ = \ \beta \circ \overline{\phi}_{E^{+2}}$$

It gives the following two equations:

$$\begin{cases}
\alpha\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes_{\Lambda\otimes\Lambda} H^L\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\ \overline{h})\big)}\big) & = \beta(t_2(\nabla^2_{E^{+2}} \circ (f_1 + f_2) \circ \xi) \otimes_\Lambda (f, \overline{h}))\\
& \quad -\beta\big(t_2(f_1 \circ r_1^2 \circ \xi) \otimes_\Lambda (f, \overline{h})\big)\\
& \quad -\beta(t_2(f_2 \circ r_2^2 \circ \xi) \otimes_\Lambda (f,\ \overline{h}))\\
\quad \beta(t_2(\xi) \otimes_\Lambda (0,\ \overline{h})) = \alpha(\overline{t_1(\xi) \otimes t_1(\xi) \otimes_{\Lambda\otimes\Lambda} h}) &
\end{cases}$$

where $f_1, f_2, \xi \in \mathcal{C}(E,\ E^{+2})$, $f \in \mathcal{P}(1)$ and $h \in \mathcal{P}(2)$. Let $\big((g_1,\ \overline{h_1}),\ (g_2,\ \overline{h_2}),\ h\big) \in \mathcal{F}_\mathcal{P} + \mathcal{F}_\mathcal{P}$. First we observe that there is a decomposition of any element in $\mathcal{F}_\mathcal{P} + \mathcal{F}_\mathcal{P}$ as follows:

$$\begin{aligned}
\big((g_1,\ \overline{h_1}),\ (g_2,\ \overline{h_2}),\ h\big) &= \widehat{\psi^{M^L}_{E^{+2}}}\big(t_2(i_1^2) \otimes_\Lambda (g_1,\ \overline{h_1})\big) \ + \ \widehat{\psi^{M^L}_{E^{+2}}}\big(t_2(i_2^2) \otimes_\Lambda (g_2,\ \overline{h_2})\big)\\
&\quad + \ \widehat{\phi^{M^L}_{E^{+2}}}\big(\overline{t_1(i_1^2) \otimes t_1(i_2^2) \otimes_{\Lambda\otimes\Lambda} h}\big)
\end{aligned}$$

Then we define the map $\delta : \mathcal{F}_\mathcal{P} + \mathcal{F}_\mathcal{P} \to A$ by

$$\delta\big((g_1,\ \overline{h_1}),\ (g_2,\ \overline{h_2}),\ h\big) = \beta\big(t_2(i_1^2) \otimes_\Lambda (g_1,\ \overline{h_1})\big) \ + \ \beta\big(t_2(i_2^2) \otimes_\Lambda (g_2,\ \overline{h_2})\big) \ + \ \alpha\big(\overline{t_1(i_1^2) \otimes t_1(i_2^2) \otimes_{\Lambda\otimes\Lambda} h}\big)$$

Now

- we have $\beta = \delta \circ \widehat{\psi^{ML}_{E^{+2}}}$ because we get

$$\begin{aligned}
&\delta \circ \widehat{\psi^{ML}_{E^{+2}}}\big(t_2(\xi) \otimes_\Lambda (f,\, \overline{h})\big)\\
&= \delta\big((r_1^2 \circ \xi) *_L (f,\, \overline{h}), (r_2^2 \circ \xi) *_L (f,\, \overline{h}),\ H^L(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\, \overline{h}))\big)\\
&= \beta\big(t_2(i_1^2) \otimes_\Lambda (r_1^2 \circ \xi) *_L (f,\, \overline{h})\big) + \beta\big(t_2(i_2^2) \otimes_\Lambda (r_2^2 \circ \xi) *_L (f,\, \overline{h})\big)\\
&\qquad + \alpha\big(\overline{t_1(i_1^2) \otimes t_1(i_2^2) \otimes_{\Lambda\otimes\Lambda} H^L(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\, \overline{h}))}\big)\\
&= \beta\big(t_2(i_1^2 \circ r_1^2 \circ \xi) \otimes_\Lambda (f,\, \overline{h})\big) + \beta\big(t_2(i_2^2 \circ r_2^2 \circ \xi) \otimes_\Lambda (f,\, \overline{h})\big)\\
&\qquad + \alpha\big(\overline{t_1(i_1^2) \otimes t_1(i_2^2) \otimes_{\Lambda\otimes\Lambda} H^L(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\, \overline{h}))}\big),
\end{aligned}$$

for $\xi \in \mathcal{C}(E,\, E^{+2})$ and $(f,\, \overline{h}) \in \mathcal{F}_{\mathcal{P}}$. But we have the following relation:

$$\begin{aligned}
&\alpha\big(\overline{t_1(i_1^2) \otimes t_1(i_2^2) \otimes_{\Lambda\otimes\Lambda} H^L(t_{11}(\rho_2(\xi)) \otimes_\Lambda (f,\, \overline{h}))}\big)\\
&= \beta\big(t_2(\nabla^2_{E^{+2}} \circ (i_1^2 + i_2^2) \circ \xi) \otimes_\Lambda (f,\, \overline{h})\big) - \beta\big(t_2(i_1^2 \circ r_1^2 \circ \xi) \otimes_\Lambda (f,\, \overline{h})\big) - \beta\big(t_2(i_2^2 \circ r_2^2 \circ \xi) \otimes_\Lambda (f, \overline{h})\big)\\
&= \beta\big(t_2(\xi) \otimes_\Lambda (f,\, \overline{h})\big) - \beta\big(t_2(i_1^2 \circ r_1^2 \circ \xi) \otimes_\Lambda (f,\, \overline{h})\big) - \beta\big(t_2(i_2^2 \circ r_2^2 \circ \xi) \otimes_\Lambda (f, \overline{h})\big)
\end{aligned}$$

It proves that we have

$$\delta \circ \widehat{\psi^{ML}_{E^{+2}}}\big(t_2(\xi) \otimes_\Lambda (f,\, \overline{h})\big) = \beta\big(t_2(\xi) \otimes_\Lambda (f,\, \overline{h})\big)$$

as desired.

- we have $\alpha = \delta \circ \widehat{\phi^{ML}_{E^{+2}}}$ because we get

$$\begin{aligned}
&\delta \circ \widehat{\phi^{ML}_{E^{+2}}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes_{\Lambda\otimes\Lambda} h}\big)\\
&= \delta\Big((0,\, \overline{t_1(r_1^2 \circ f_1) \otimes t_1(r_1^2 \circ f_2).h}),\ (0,\, \overline{t_1(r_2^2 \circ f_1) \otimes t_1(r_2^2 \circ f_2).h}),\\
&\qquad t_1(r_1^2 \circ f_1) \otimes t_1(r_2^2 \circ f_2).h + t_1(r_1^2 \circ f_2) \otimes t_1(r_2^2 \circ f_1).T(h)\Big)\\
&= \beta\Big(t_2(i_1^2) \otimes_\Lambda \big(0,\, \overline{t_1(r_1^2 \circ f_1) \otimes t_1(r_1^2 \circ f_2).h}\big)\Big) + \beta\Big(t_2(i_2^2) \otimes_\Lambda \big(0,\, \overline{t_1(r_2^2 \circ f_1) \otimes t_1(r_2^2 \circ f_2).h}\big)\Big)\\
&\qquad + \alpha\Big(\overline{t_1(i_1^2) \otimes t_1(i_2^2) \otimes_{\Lambda\otimes\Lambda} t_1(r_1^2 \circ f_1) \otimes t_1(r_2^2 \circ f_2).h}\Big)\\
&\qquad + \alpha\Big(\overline{t_1(i_1^2) \otimes t_1(i_2^2) \otimes_{\Lambda\otimes\Lambda} t_1(r_1^2 \circ f_2) \otimes t_1(r_2^2 \circ f_1).T(h)}\Big),
\end{aligned}$$

for $f_1, f_2 \in \mathcal{C}(E,\, E^{+2})$ and $h \in \mathcal{P}(2)$. Moreover we have the following relation:

$$\begin{aligned}
\beta\Big(t_2(i_k^2) \otimes_\Lambda \big(0,\, \overline{t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).h}\big)\Big) &= \alpha\big(\overline{t_1(i_k^2) \otimes t_1(i_k^2) \otimes_{\Lambda\otimes\Lambda} t_1(r_k^2 \circ f_1) \otimes t_1(r_k^2 \circ f_2).h}\big)\\
&= \alpha\big(\overline{t_1(i_k^2 \circ r_k^2 \circ f_1) \otimes t_1(i_k^2 \circ r_k^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big)
\end{aligned}$$

for $k = 1, 2$. Then we obtain

$$\begin{aligned}
&\delta\Big((0,\, \overline{t_1(r_1^2 \circ f_1) \otimes t_1(r_1^2 \circ f_2).h}),\ (0,\, \overline{t_1(r_2^2 \circ f_1) \otimes t_1(r_2^2 \circ f_2).h}),\\
&\qquad t_1(r_1^2 \circ f_1) \otimes t_1(r_2^2 \circ f_2).h + t_1(r_1^2 \circ f_2) \otimes t_1(r_2^2 \circ f_1).T(h)\Big)\\
&= \alpha\big(\overline{t_1(i_1^2 \circ r_1^2 \circ f_1) \otimes t_1(i_1^2 \circ r_1^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big) + \alpha\big(\overline{t_1(i_2^2 \circ r_2^2 \circ f_1) \otimes t_1(i_2^2 \circ r_2^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big)\\
&\quad + \alpha\big(\overline{t_1(i_1^2 \circ r_1^2 \circ f_1) \otimes t_1(i_2^2 \circ r_2^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big) + \alpha\big(\overline{t_1(i_1^2 \circ r_1^2 \circ f_2) \otimes t_1(i_2^2 \circ r_2^2 \circ f_1) \otimes_{\Lambda\otimes\Lambda} T(h)}\big)
\end{aligned}$$

Moreover we have

$$\overline{t_1(i_1^2 \circ r_1^2 \circ f_2) \otimes t_1(i_2^2 \circ r_2^2 \circ f_1) \otimes_{\Lambda\otimes\Lambda} T(h)} = \overline{T\big(t_1(i_2^2 \circ r_2^2 \circ f_1) \otimes t_1(i_1^2 \circ r_1^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h\big)}$$
$$= \overline{t_1(i_2^2 \circ r_2^2 \circ f_1) \otimes t_1(i_1^2 \circ r_1^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}$$

This implies that we obtain

$$\begin{aligned}
&\delta \circ \widehat{\phi^{M^L}_{E^{+2}}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes_{\Lambda\otimes\Lambda} h}\big) \\
&= \alpha\big(\overline{t_1(i_1^2 \circ r_1^2 \circ f_1) \otimes t_1(i_1^2 \circ r_1^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big) + \alpha\big(\overline{t_1(i_2^2 \circ r_2^2 \circ f_1) \otimes t_1(i_2^2 \circ r_2^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big) \\
&\quad + \alpha\big(\overline{t_1(i_1^2 \circ r_1^2 \circ f_1) \otimes t_1(i_2^2 \circ r_2^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big) + \alpha\big(\overline{t_1(i_2^2 \circ r_2^2 \circ f_1) \otimes t_1(i_1^2 \circ r_1^2 \circ f_2) \otimes_{\Lambda\otimes\Lambda} h}\big) \\
&= \alpha\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes_{\Lambda\otimes\Lambda} h}\big)\text{, by 3.12 of [12]}
\end{aligned}$$

The morphism $\delta : \mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}} \to A$ is necessary unique because any element of $\mathcal{F}_{\mathcal{P}} + \mathcal{F}_{\mathcal{P}}$ is decomposed as a sum of images of $\widehat{\psi^{M^L}_{E^{+2}}}$ and $\widehat{\phi^{M^L}_{E^{+2}}}$. Consequently the functor $L_{|\langle E\rangle_{\mathcal{C}}} : \langle E\rangle \to Alg - \mathcal{P}$ (restricted to $\langle E\rangle$) preserves finite coproducts. $\square$

Remark 4.6.12. By 4.6.11, the functor $L : \langle E\rangle \to Alg - \mathcal{P}$ takes values in $\langle \mathcal{F}_{\mathcal{P}}\rangle$, the full subcategory of free $\mathcal{P}$-algebras of finite rank.

Moreover it is now possible to have an explicit expression of the quadratic functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ on morphisms in $\mathcal{C}$ with source E and target E^{+2}. The explicit expression of the functor L on these kind of morphisms is important in order to provide the BCH formula later in this section.

Proposition 4.6.13. *For $\xi \in \mathcal{C}(E,\ E^{+2})$ and $(f, \overline{h}) \in \mathcal{F}_{\mathcal{P}}$, we have*

$$L(\xi)(f, \overline{h}) = \Big((r_1^2 \circ \xi) *_L (f, \overline{h}),\ (r_2^2 \circ \xi) *_L (f, \overline{h}),\ H^L\big(t_{11}(\rho_2(\xi)) \otimes (f, \overline{h})\big)\Big)$$

Proof. For this, we recall that the functor L is the quadratic tensor product (see 2.1.3) associated with the quadratic $\mathcal{C}$-module over $\mathcal{P}$ given in 4.5.9. As $L(\xi) : L(E) \to L(E^{+2})$ is given by the universal property of 2.1.3, we have in particular

$$L(\xi) \circ \widehat{\psi^{M^L}_{E}} = \widehat{\psi^{M^L}_{E^{+2}}} \circ \big(t_2(\xi) \otimes_{\Lambda} id\big)$$

Hence we get

$$\begin{aligned}
L(\xi)(f, \overline{h}) &= \big(L(\xi) \circ \widehat{\psi^{M^L}_{E}}\big)\big(t_2(id) \otimes (f, \overline{h})\big) \\
&= \Big(\widehat{\psi^{M^L}_{E^{+2}}} \circ \big(t_2(\xi) \otimes_{\Lambda} id\big)\Big)\big(t_2(id) \otimes (f, \overline{h})\big) \\
&= \widehat{\psi^{M^L}_{E^{+2}}}\big(t_2(\xi) \otimes (f, \overline{h})\big) \\
&= \Big((r_1^2 \circ \xi) *_L (f, \overline{h}),\ (r_2^2 \circ \xi) *_L (f, \overline{h}),\ H^L\big(t_{11}(\rho_2(\xi)) \otimes (f, \overline{h})\big)\Big)
\end{aligned}$$

as desired. $\square$

Now we prove that the quadratic functor $L : \mathcal{C} \to Alg - \mathcal{P}$ preserves not only the coproducts in the full subcategory $\langle E\rangle$ of $\mathcal{C}$ but also all finite coproducts in the whole of $\mathcal{C}$. We recall the natural isomorphism $u'_{\mathcal{C}(E^{ab}, Ab^{\mathcal{C}})} : T_1 U_E \otimes_{\Lambda} \mathcal{P}(1) \Rightarrow \mathcal{C}\big(E^{ab}, Ab^{\mathcal{C}}\big)$ given in 4.6.15. Consider an object X in $\mathcal{C}$. By 2.3.3, 4.2.11 and 4.2.10, we have the following isomorphisms

$$T_1 L(X) = T_1\big(- \otimes M^L\big)(X) \cong T_1 U_E(X) \otimes_{\Lambda} Coker(\dot{q}) = T_1 U_E(X) \otimes_{\Lambda} \mathcal{P}(1) \cong T_1 U_E(X) \otimes_{\Lambda} \overline{\Lambda} \cong T_1 U_E(X)$$

where M^L is the quadratic $\mathcal{C}$-module over the ring $\mathcal{P}(1)$ given in 4.5.1 and $\dot{q} = i_2 \circ q : \mathcal{P}(2) \to \mathcal{F}_{\mathcal{P}}$.

Notation 4.6.14. For an object X in $\mathcal{C}$, we set

$$\omega_X = ev \circ \left(id \otimes_\Lambda (\overline{t_1})_E^{-1}\right) \circ \gamma_X : T_1L(X) \to T_1U_E(X)$$

where $ev : T_1U_E(X) \otimes \overline{\Lambda} \to T_1U_E(X)$ is the canonical isomorphism (since $T_1U_E(X)$ is a left $\overline{\Lambda}$-module, see 2.0.3), $(\overline{t_1})_E : \overline{\Lambda} \to \mathcal{P}(1)$ is the isomorphism given in 4.2.10 and $\gamma : T_1L \Rightarrow T_1U_E \otimes_\Lambda Coker(\dot{q}) = T_1U_E \otimes_\Lambda \mathcal{P}(1)$ is the natural isomorphism in 2.3.3 (see (2.3)).

Corollary 4.6.15. *Let X be an object in $\mathcal{C}$. Then $\omega_X : T_1L \Rightarrow T_1U_E$ is a natural isomorphism in the category of functors from $\mathcal{C}$ to the category of (left) $\overline{\Lambda}$-modules such that, for any object X in $\mathcal{C}$ and $x \in L(X)$, we have*

$$\omega_X(t_1^L(x)) = t_1(\hat{f}_x)$$

where $\hat{f}_x \in \mathcal{C}(E, X)$ is a morphism (which exists because E is a regular projective object) such that $(\hat{f}_x)^{ab} \in \mathcal{C}(E^{ab}, X^{ab})$ is unique morphism satisfying $t_1^L(x) = \overline{\gamma_X}((\hat{f}_x)^{ab})$ since $\overline{\gamma} : T_1U_E \otimes_\Lambda \mathcal{P}(1) \Rightarrow \mathcal{C}\big(E^{ab},\, Ab^{\mathcal{C}}\big)$ is a natural isomorphism (see 4.6.5) in the category of functors from $\mathcal{C}$ to the category of $\overline{\Lambda}$-$\mathcal{P}(1)$-bimodules.

Proof. It is a direct consequence of 4.6.14. □

Now we are able to prove that

Proposition 4.6.16. *The Lazard functor $L : \mathcal{C} \to Alg - \mathcal{P}$ preserves finite coproducts.*

Proof. Let X and Y be objects in $\mathcal{C}$. We consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & L(X|Y) \arrow[r, tail, "\iota_2^L"] & L(X+Y) \arrow[r, "\widehat{r_2^L}"] & L(X)\times L(Y) \arrow[r] & 0 \\
0 \arrow[r] & Id_{Alg-\mathcal{P}}(L(X)|\, L(Y)) \arrow[r, tail, "\iota_2^{Id}"] \arrow[u, "\widetilde{(L(i_1^2),\, L(i_2^2))}"] & L(X)+L(Y) \arrow[r, "\widehat{r_2^{Id_{Alg-\mathcal{P}}}}"] \arrow[u, "(L(i_1^2),\, L(i_2^2))"] & L(X)\times L(Y) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \tag{4.6.11}$$

We aim at proving that $(L(i_1^2), L(i_2^2)) : L(X) + L(Y) \to L(X + Y)$ is an isomorphism. For this it suffices to check that its restriction to $Id_{Alg-\mathcal{P}}(L(X)|L(Y))$, here denoted by $\widetilde{(L(i_1^2),\, L(i_2^2))} : Id_{Alg-\mathcal{P}}(L(X)|\, L(Y)) \to L(X|Y)$ is an isomorphism. By 1.7.1, we recall that

$$\begin{aligned}
Id_{Alg-\mathcal{P}}\big(L(X)|L(Y)\big) &= \overline{L(X)} \otimes \overline{L(Y)} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \\
&= L(X)^{ab} \otimes L(Y)^{ab} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)
\end{aligned}$$

and the kernel $\iota_2^{Id_{Alg-\mathcal{P}}} : Id_{Alg-\mathcal{P}}(L(X)|L(Y)) \rightarrowtail L(X)+L(Y)$ of the comparison morphism $\widehat{r_2^{Id_{Alg-\mathcal{P}}}} : L(X) + L(Y) \to L(X) \times L(Y)$ (see 1.2.1) is given by

$$\iota_2^{Id_{Alg-\mathcal{P}}}(u) = (0,\, 0,\, u)$$

Moreover we recall that we have up to an isomorphism an explicit expression of the second cross-effect of the Lazard functor $L : \mathcal{C} \to Alg - \mathcal{P}$, namely we get the natural isomorphism

$$\Phi : T_1U_E \otimes T_1U_E \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2) \Rightarrow cr_2L$$

between bilinear bifunctors given in (4.6.7). We recall that, for $x \in L(X)$, $\overline{x} = ab_{L(X)}(x) \in L(X)^{ab}$ (see the notations given in 1.7.6). Then we define the morphism $i_{X,Y} : T_1L(X) \otimes T_1L(Y) \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \to T_1U_E(X) \otimes T_1U_E(Y) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)$ by

$$i_{X,Y}(\overline{x} \otimes \overline{y} \otimes h) = \omega_X(t_1^L(x)) \otimes \omega_Y(t_1^L(y)) \otimes h \tag{4.6.12}$$

where $x \in L(X)$, $y \in L(Y)$, $h \in \mathcal{P}(2)$ and $\omega : T_1L \Rightarrow T_1U_E$ is the natural isomorphism given in 4.6.15. We observe that

$$\begin{aligned} i_{X,Y}(\overline{x} \otimes \overline{y} \otimes h) &= \omega_X(t_1^L(x)) \otimes \omega_Y(t_1^L(y)) \otimes h \\ &= \omega_X\big(\vartheta_X(\overline{x})\big) \otimes \omega_Y\big(\vartheta_X(\overline{y})\big)\big) \otimes h \,, \text{by 4.6.10} \end{aligned}$$

Hence $i_{X,Y}$ is well-defined, and it is an isomorphism because ω and ϑ_X are natural isomorphism (see 4.6.15 and 4.6.9). We now consider the following diagram

$$\begin{tikzcd} T_1U_E(X) \otimes T_1U_E(Y) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2) \arrow[r, "\iota_2^L \circ \Phi_{X,Y}"] & L(X+Y) \\ Id_{Alg-\mathcal{P}}(L(X)|L(Y)) \arrow[u, "i_{X,Y}", "\cong"'] \arrow[r, "\iota_2^{Id_{Alg-\mathcal{P}}}"] & L(X)+L(Y) \arrow[u, "(L(i_1^2),\, L(i_2^2))"'] \end{tikzcd} \tag{4.6.13}$$

Then we prove that this diagram commutes. Let $x \in L(X)$, $y \in L(Y)$ and $h \in \mathcal{P}(2)$. As $\overline{\gamma} : T_1U_E \Rightarrow \mathcal{C}\big(E^{ab},\, Ab^{\mathcal{C}}\big)$ is a natural isomorphism (see 4.6.5), there is a unique $\alpha_x \in \mathcal{C}(E^{ab}, X^{ab})$ (resp. $\alpha_y \in \mathcal{C}(E^{ab}, Y^{ab})$) such that

$$t_1^L(x) = \overline{\gamma_X}(\alpha_x)\,, \quad \big(\text{ resp. } t_1^L(y) = \overline{\gamma_Y}(\alpha_y)\ \big) \tag{4.6.14}$$

Since E is a regular-projective object and $ab_X : X \to X^{ab}$ is a regular epimorphism, there is a (non-unique) morphism $\hat{f}_x \in \mathcal{C}(E, X)$ (resp. $\hat{f}_y \in \mathcal{C}(E, Y)$) such that

$$\alpha_x \circ ab_E = ab_X \circ \hat{f}_x\,, \quad \big(\text{ resp. } \alpha_y \circ ab_E = ab_X \circ \hat{f}_y\ \big)$$

By (4.6.14) and by naturality of $\overline{\gamma} : T_1L \Rightarrow \mathcal{C}\big(E^{ab},\, Ab^{\mathcal{C}}\big)$, we get

$$t_1^L(x) = T_1L(\hat{f}_x)(\overline{id})\,, \quad \big(\text{ resp. } t_1^L(y) = T_1L(\hat{f}_y)(\overline{id})\ \big) \tag{4.6.15}$$

We get the following equalities:

$$\begin{aligned} &(L(i_1^2), L(i_2^2)) \circ \iota_2^{Id_{Alg-\mathcal{P}}}(\overline{x} \otimes \overline{y} \otimes h) \\ &= (L(i_1^2), L(i_2^2))\big(0,\ 0,\ \overline{x} \otimes \overline{y} \otimes h\big) \\ &= (\lambda_2^L)_{X+Y}\big(L(i_1^2)(x) \otimes L(i_2^2)(y) \otimes h\big)\,, \text{by (1.8.6)} \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_{X+Y}\big((t_1^L \circ L(i_1^2))(x) \otimes (t_1^L \circ L(i_2^2))(y) \otimes_S h\big)\,, \text{by (4.6.1)} \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_{X+Y}\big(T_1L(i_1^2)(t_1^L(x)) \otimes T_1L(i_2^2)(t_1^L(y)) \otimes_S h\big) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_{X+Y}\big((T_1L(i_1^2) \circ T_1(\hat{f}_x))(\overline{id}) \otimes (T_1L(i_2^2) \circ T_1L(\hat{f}_y))(\overline{id}) \otimes_S h\big)\,, \text{by (4.6.15)} \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_{X+Y}\big(T_1L(i_1^2 \circ \hat{f}_x)(\overline{id}) \otimes T_1L(i_2^2 \circ \hat{f}_y)(\overline{id}) \otimes_S h\big) \\ &= \mathbb{T}_2^{\mathcal{P}(1)}\big((i_2, id)\big)_{X+Y} \circ \widehat{\phi_X^{(M^L)^2}}\big(\overline{t_1(i_1^2 \circ \hat{f}_x) \otimes t_1(i_2^2 \circ \hat{f}_y) \otimes h}\big) \\ &= \widehat{\phi_{X+Y}^{M^L}}\big(\overline{t_1(i_1^2 \circ \hat{f}_x) \otimes t_1(i_2^2 \circ \hat{f}_y) \otimes h}\big)\,, \text{by (2.1.2)} \\ &= \iota_2^L \circ \Phi_{X,Y}(t_1(\hat{f}_x) \otimes t_1(\hat{f}_y) \otimes h)\,, \text{by (4.6.7)} \\ &= \iota_2^L \circ \Phi_{X,Y}\big(\omega_X(t_1^L(x)) \otimes \omega_Y(t_1^L(y)) \otimes h\big)\,, \text{by 4.6.15} \\ &= \iota_2^L \circ \Phi_{X,Y} \circ i_{X,Y}(\overline{x} \otimes \overline{y} \otimes h)\,, \text{by (4.6.12)} \end{aligned}$$

It proves that $(\widetilde{L(i_1^2),\ L(i_2^2)}) : Id_{Alg-\mathcal{P}}(L(X)|\ L(Y)) \to L(X|Y)$ is an isomorphism. By applying the five lemma to the diagram (4.6.13), it follows that $(L(i_1^2), L(i_2^2)) : L(X) + L(Y) \to L(X + Y)$ is an isomorphism. It concludes the proof. □

Notation 4.6.17. For an object X in $\mathcal{C}$, we denote by

$$\overline{i_{X,X}} : \big(\overline{L(X)}^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \otimes \mathcal{P}(2)\big)_{\mathfrak{S}_2} \to \big(T_1U_E(X)^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \otimes \mathcal{P}(2)\big)_{\mathfrak{S}_2}$$

the canonical factorization between the coinvariants sets, i.e. $\pi \circ i_{X,X} = \overline{i_{X,X}} \circ \pi$ where $\pi : T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} \otimes \mathcal{P}(2) \to \big(T_1U_E(X)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} \otimes \mathcal{P}(2)\big)_{\mathfrak{S}_2}$ and $\pi : T_1L(X)^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \otimes \mathcal{P}(2) \to \big(T_1L(X)^{\otimes 2} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \otimes \mathcal{P}(2)\big)_{\mathfrak{S}_2}$ are the canonical quotient maps (which are both $\mathcal{P}(1)$-module homomorphisms).

Remark 4.6.18. Let X be an object in $\mathcal{C}$. Then $\overline{i_{X,X}}$, given in 4.6.17, is a $\mathcal{P}(1)$-module homomorphism which is an isomorphism by the five lemma applied to an appropriate diagram.

4.6.2 $L : \mathcal{C} \to Alg - \mathcal{P}$ commutes with the abelianizations

In this part, we prove that the (quadratic) functor $L : \mathcal{C} \to Alg-\mathcal{P}$ commutes with the abelianization functors, i.e.

$$L \cdot Ab^{\mathcal{C}} \cong Ab^{Alg-\mathcal{P}} \cdot L$$

We first provide the following lemma:

Lemma 4.6.19. *Let X be an object in $\mathcal{C}$. For $f_1, f_2 \in \mathcal{C}(E, X)$ and $h \in \mathcal{P}(2)$, we have the equility as follows:*

$$\widehat{\phi_X^{M^L}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big) = \widehat{\psi_X^{M^L}}\big(t_2(c_2^X \circ Id_{\mathcal{C}}(f_1|f_2) \circ h) \otimes (id, \overline{0})\big)$$

Proof. We have the following equalities:

$$\begin{aligned}
&\widehat{\phi_X^{M^L}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big)\\
&= \widehat{\phi_X^{M^L}}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes H^L\big(t_{11}(\rho_2(\iota_2^{Id_{\mathcal{C}}} \circ h)) \otimes (id, \overline{0})\big)}\big)\\
&= \widehat{\phi_X^{M^L}} \circ \overline{\psi_X^L}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes t_{11}(\iota_2^{Id_{\mathcal{C}}} \circ h) \otimes (id, \overline{0})},\ 0\big)\\
&= \widehat{\phi_X^{M^L}} \circ \overline{\phi_X}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes t_{11}(\iota_2^{Id_{\mathcal{C}}} \circ h) \otimes (id, \overline{0})},\ 0\big)\\
&= \widehat{\phi_X^{M^L}}\big(t_2\big(c_2^X \circ Id_{\mathcal{C}}(f_1|f_2) \circ h\big) \otimes (id, \overline{0})\big)\,,
\end{aligned}$$

as desied. □

Remark 4.6.20. Let Z be an abelian object in $\mathcal{C}$. Then the morphism $\widehat{\phi_Z^{M^L}} : \big(T_1U_E(Z)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \to L(Z)$ is trivial.

Next we prove that the Lazard functor has the following preservation property:

Lemma 4.6.21. *The Lazard functor $L : \mathcal{C} \to Alg - \mathcal{P}$ preserves abelian objects.*

Proof. By 4.6.20, we know that $\widehat{\phi_Z^{M^L}} : \big(T_1U_E(Z)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \to L(Z)$ is trivial because $c_2^Z = 0$ (since Z is an abelian object in $\mathcal{C}$, see 1.3.1). By 4.6.7, it implies that the linear map $\lambda_Z^L : L(Z)^{\otimes 2} \otimes \mathcal{P}(2) \to L(Z)$ is trivial. By 1.7.3, it proves that $L(Z)$ is an abelian object in $Alg - \mathcal{P}$, i.e. a (right) $\mathcal{P}(1)$-module. □

Notation 4.6.22. Let $I : Ab(\mathcal{C}) \to \mathcal{C}$ be the inclusion functor. We also denote by $L = L \cdot I : Ab(\mathcal{C}) \to Ab(Alg - \mathcal{P}) = Mod_{\mathcal{P}(1)}$ the restriction of the Lazard functor to the abelian core $Ab(\mathcal{C})$ of $\mathcal{C}$.

Now we prove that the Lazard functor $L : Ab(\mathcal{C}) \to Ab(Alg - \mathcal{C})$ restricted to the abelian core $Ab(\mathcal{C})$ is isomorphic to its linearization rectricted to the same abelian source.

Proposition 4.6.23. *The natural transformation $I^* \cdot t_1^L = t_1^L : L \cdot I \Rightarrow T_1 L \cdot I$ is an isomorphism between functors with domain $Ab(\mathcal{C})$ and values in $Ab(Alg - \mathcal{P}) = Mod_{\mathcal{P}(1)}$.*

Proof. Let Z be an abelian object in $\mathcal{C}$. We consider the pushout (4.5.6), as follows:

$$\begin{tikzcd}
\Big(T_1U_E(Z)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda \mathcal{F}_\mathcal{P}\Big)_{\mathfrak{S}_2} \oplus (U_E(Z) \otimes \mathcal{P}(2)) \arrow[r, "\overline{\phi_Z}"] \arrow[d, "\overline{\psi_Z^L}"'] & T_2U_E(Z) \otimes_\Lambda \mathcal{F}_\mathcal{P} \arrow[d, "\widehat{\psi_Z^{M^L}}"] \\
\big(T_1U_E(Z)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)\big)_{\mathfrak{S}_2} \arrow[r, "\widehat{\phi_Z^{M^L}}"'] & L(Z)
\end{tikzcd}$$

We here denote by $B(Z) = \big(T_1U_E(Z)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda \mathcal{F}_\mathcal{P}\big)_{\mathfrak{S}_2} \oplus (U_E(Z) \otimes \mathcal{P}(2))$. We denote respectively by i_1 and i_2 the injection of the first and second summand of $B(Z)$. By 4.6.20, the morphism $\widehat{\phi_Z^{M^L}}$ is trivial. Hence it follows that the pushout (4.5.6) can be seen as the following right exact sequence:

$$B(Z) \xrightarrow{\overline{\phi_Z}} T_2U_E(Z) \otimes_\Lambda \mathcal{F}_\mathcal{P} \xrightarrow{\widehat{\psi_Z^{M^L}}} L(Z) \longrightarrow 0$$

By 2.1.3, we know that $\overline{\phi_Z} = \big(\overline{\phi_1' \otimes id}, t_2 \otimes \dot{q}\big)$. Hence we have

$$coker(\overline{\phi_Z} \circ i_1) = coker(\overline{\phi_1' \otimes id}) = coker\big((\overline{\phi_1' \otimes id}) \circ \pi'\big) = coker(\phi_1' \otimes id)$$

where the map

$$\pi' : T_1U_E(Z)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda \mathcal{F}_\mathcal{P} \to \big(T_1U_E(Z)^{\otimes 2} \otimes_{\Lambda\otimes\Lambda} T_{11}cr_2(U_E)(E,E) \otimes_\Lambda \mathcal{F}_\mathcal{P}\big)_{\mathfrak{S}_2}$$

is the quotient map which is clearly a surjection. Then we obtain

$$coker(\overline{\phi_Z} \circ i_1) = coker(\phi_1' \otimes id) = coker(\phi_1') \otimes id$$

because the functor $- \otimes_\Lambda \mathcal{F}_\mathcal{P} : Mod_\Lambda \to Ab$ preserves right exact sequences. By (2.1.1), we know that

$$\phi_1' = (S_2^{T_2U_E})_Z \circ (u'_{cr_2(T_2U_E)})_{Z,Z} \circ \big(id^{\otimes 2} \otimes \overline{cr_2(t_2)}_{Z,Z}\big)$$

By 4.6.2, the morphism $(u'_{cr_2(T_2U_E)})_{Z,Z}$ is an isomorphism. As $\overline{cr_2(t_2)} : T_{11}cr_2U_E(Z,Z) \to cr_2(T_2U_E)(Z,Z)$ also is an isomorphism by 2.5 of [12], we get

$$\begin{aligned} coker(\overline{\phi_Z} \circ i_1) &= coker(\phi_1') \otimes id \\ &= coker\big((S_2^{T_2U_E})_Z\big) \otimes id \\ &= (t_1^{T_2U_E})_Z \otimes id \,, \text{see 1.2.10} \end{aligned}$$

By 2.3.2, we know that we get

$$T_1\big(T_2U_E\big) \cong T_1U_E \,,$$

more precisely the natural transformation $(T_1)_* \cdot t_2 : T_1 U_E \Rightarrow T_1\big(T_2 U_E\big)$ is an isomorphism. We prove that we have

$$\big((T_1)_* \cdot t_2\big) \circ \overline{t_1} = t_1^{T_2 U_E}$$

where $\overline{t_1} : T_2 U_E \Rightarrow T_1 U_E$ is the unique factorization of $t_1 : U_E \Rightarrow T_1 U_E$ though $t_2 : U_E \Rightarrow T_2 U_E$ by 1.2.11 (because the functor $T_1 U_E : \mathcal{C} \to Mod_\Lambda$ is linear hence quadratic), i.e.

$$t_1^{T_2 U_E} \circ t_2 = t_1 \tag{4.6.16}$$

For this we have the following equalities:

$$\begin{aligned}\big((T_1)_* \cdot t_2\big) \circ \overline{t_1} \circ t_2 &= \big((T_1)_* \cdot t_2\big) \circ t_1 \\ &= t_1^{T_2 U_E} \circ t_2 \text{ , by naturality}\end{aligned}$$

As the natural transformation $t_2 : U_E \Rightarrow T_2 U_E$ is a regular epimorphism, the relation (4.6.16) holds. Then we can choose the cokernel of $\overline{\phi_Z} \circ i_1$ to be

$$coker(\overline{\phi_Z} \circ i_1) = \overline{t_1} \otimes id$$

Now it suffices to determine the cokernel of $(\overline{t_1} \otimes id) \circ \overline{\phi_Z} \circ i_2$ to find out the cokernel of $\overline{\phi_Z}$. For this we have

$$(\overline{t_1} \otimes id) \circ \overline{\phi_Z} \circ i_2 = (\overline{t_1} \otimes id) \circ (t_2 \otimes \dot{q}) = t_1 \otimes \dot{q} \text{ , by (4.6.16)}$$

Since $Coker(\dot{q}) = \mathcal{P}(1)$ whose cokernel (morphism) is the projection $\pi_1 : \mathcal{F}_\mathcal{P} \to \mathcal{P}(1)$ onto the first summand, we get

$$coker\big((\overline{t_1} \otimes id) \circ \overline{\phi_Z} \circ i_2\big) = t_1 \otimes \pi_1$$

because the functor $T_1 U_E(X) \otimes - : Mod_\Lambda \to Ab$ preserves right sequences. Hence the cokernel of $\overline{\phi_Z}$ is $\overline{t_1} \otimes \pi_1 : T_1 U_E(X) \otimes_\Lambda \mathcal{F}_\mathcal{P} \to T_1 U_E(X) \otimes \mathcal{P}(1)$. It implies that there is an isomorphism of (right) $\mathcal{P}(1)$-modules $\varepsilon_Z : L(Z) \to T_1 U_E(Z) \otimes_\Lambda \mathcal{P}(1)$ such that

$$\varepsilon_Z \circ \widehat{\psi_Z^{ML}} = \overline{t_1} \otimes_\Lambda \pi_1 \tag{4.6.17}$$

By (4.6.17) and 2.3.3, we get the following isomorphisms:

$$L(Z) \cong T_1 U_E(Z) \otimes_\Lambda \mathcal{P}(1) \cong T_1 L(Z)$$

More precisely, we have

$$\begin{aligned}\varepsilon_Z^{-1} \circ \gamma_Z \circ (t_1^L)_Z \circ \widehat{\psi_Z^{ML}} &= \varepsilon_Z^{-1} \circ \big((\overline{t_1})_Z \otimes_\Lambda \pi_1\big) \\ &= \widehat{\psi_Z^{ML}} \text{ , by (4.6.17)}\end{aligned}$$

By 4.6.3, the morphism $\widehat{\psi_Z^{ML}} : T_2 U_E(Z) \otimes_\Lambda \mathcal{F}_\mathcal{P} \to L(Z)$ is an epimorphism. Hence we get

$$\varepsilon_Z^{-1} \circ \gamma_Z \circ (t_1^L)_Z = id$$

It implies that the morphism $(t_1^L)_Z : L(Z) \Rightarrow T_1 L(Z)$ is a monomorphism. As it is a regular epimorphism (because it is the cokernel of $S_2^L : L(Z|Z) \to L(Z)$), it proves that $(t_1)_Z$ is an isomorphism, as desired. □

Corollary 4.6.24. *The Lazard functor $L : Ab(\mathcal{C}) \to Ab(Alg-\mathcal{P})$ restricted to the abelian core $Ab(\mathcal{C})$ is linear.*

Proof. By 4.6.23, the natural transformation $I^* \cdot t_1^L : L \cdot I \Rightarrow T_1 L \cdot I$ is an isomorphism. Then the functor $T_1 L \cdot I : Ab(\mathcal{C}) \to Mod_{\mathcal{P}(1)}$ is linear because it is a linear functor postcomposed by a linear functor with abelian source and target by 1.2.6. Hence the Lazard functor $L : Ab(\mathcal{C}) \to Mod_{\mathcal{P}(1)}$ restricted to $Ab(\mathcal{C})$ is a linear functor. □

Now we prove that the Lazard functor $L : \mathcal{C} \to Alg - \mathcal{P}$ commutes with the abelianization functors:

Proposition 4.6.25. *There is a natural isomorphism $\sigma : Ab^{Alg-\mathcal{P}} \cdot L \Rightarrow L \cdot Ab^{\mathcal{C}}$ such that, for an object X in $\mathcal{C}$, we have*

$$\sigma_X \circ ab_{L(X)} = L(ab_X)$$

Proof. Let X be an object in $\mathcal{C}$. By 1.4.8, the natural transformation $T_1 L^* \cdot ab : T_1 L \Rightarrow T_1 L \cdot Ab^{\mathcal{C}}$ is an isomorphism because $T_1 L : \mathcal{C} \to Mod_{\mathcal{P}(1)}$ is a linear functor. By 4.6.9, 4.6.23 and 1.4.8, we get

$$Ab^{Alg-\mathcal{P}} \cdot L(X) \cong T_1 L(X) \cong T_1 L(X^{ab}) \cong L(X^{ab}) = L \cdot Ab^{\mathcal{C}}(X)\,,$$

We denote by $\sigma : Ab^{Alg-\mathcal{P}} \cdot L \Rightarrow L \cdot Ab^{\mathcal{C}}$ the (above) natural isomorphism such that, for an object X in $\mathcal{C}$, $\sigma_X = (t_1^L)^{-1}_{X^{ab}} \circ T_1 L(ab_X) \circ \vartheta_X : Ab^{Alg-\mathcal{P}} \cdot L(X) \to L \cdot Ab^{\mathcal{C}}(X)$, where $\vartheta_X : Ab^{Alg-\mathcal{P}} \cdot L(X) \to T_1 L(X)$ is the isomorphism given in (4.6.8). Moreover we have the following equalities:

$$\begin{aligned} \sigma_X \circ ab_{L(X)} &= (t_1^L)^{-1}_{X^{ab}} \circ T_1 L(ab_X) \circ \vartheta_X \circ ab_{L(X)} \\ &= (t_1^L)^{-1}_{X^{ab}} \circ T_1 L(ab_X) \circ (t_1^L)_X \,, \text{by (4.6.8)} \\ &= (t_1^L)^{-1}_{X^{ab}} \circ (t_1^L)_{X^{ab}} \circ L(ab_X) \\ &= L(ab_X)\,, \end{aligned}$$

as desired. □

Now we observe that, for an abelian object Z in $\mathcal{C}$, we have the following isomorphisms of (right) $\mathcal{P}(1)$-modules (natural in Z):

$$L(Z) \xrightarrow{\varepsilon_Z} T_1 U_E(Z) \otimes_\Lambda \mathcal{P}(1) \xrightarrow{\overline{\gamma_Z}} \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}(Z)\big)$$

where the isomorphisms are given in (4.6.17) and 4.6.5. It says that $\overline{\gamma} \circ \varepsilon : L \Rightarrow \mathcal{C}\big(E^{ab},\ Ab^{\mathcal{C}}\big)$ is a natural isomorphism between functors with domain $Ab(\mathcal{C})$ and values in $Ab(Alg - \mathcal{P}) = Mod_{\mathcal{P}(1)}$. Then we check that the (linear) functor $\mathcal{C}\big(E^{ab}, Ab^{\mathcal{C}}\big) : Ab(\mathcal{C}) \to Mod_{\mathcal{P}(1)}$ is an equivalence of categories. In fact, we just apply the Gabriel-Popescu theorem given in Corollary 6.4 of [35] (or also in 4.6 of [12]) by taking the set $\{E^{ab}\}$ of the small projective generator E^{ab} in the abelian category $Ab(\mathcal{C})$. Let us denote by $\mathcal{C}_{E^{ab}}$ the full subcategory of $\mathcal{C}$ whose set of objects is $\{E^{ab}\}$. The Gabriel-Popescu theorem says that the functor assigning each abelian object Z in $\mathcal{C}$ to the additive functor $Ab(\mathcal{C})(-, Z) = \mathcal{C}(-, Z)$ with domain $\mathcal{C}^{op}_{E^{ab}}$ and values in Ab is an equivalence of categories. Moreover it is a well-known fact that the category of additive functors with domain $\mathcal{C}^{op}_{E^{ab}}$ and values in Ab is isomorphic to the category of (right) $\mathcal{P}(1)$-modules, where $\mathcal{P}(1) = \mathcal{C}(E^{ab}, E^{ab})$. This isomorphism of categories assigns in particular the additive functor $\mathcal{C}(-, Z) : \mathcal{C}^{op}_{E^{ab}} \to Ab$ to the (right) $\mathcal{P}(1)$-module $\mathcal{C}(E^{ab}, Z) = \mathcal{C}\big(E^{ab}, Ab^{\mathcal{C}}(Z)\big)$. It permits us to give the following proposition:

Proposition 4.6.26. *The Lazard functor $L = L \cdot I : Ab(\mathcal{C}) \to Ab(Alg - \mathcal{P})$ restricted to $Ab(\mathcal{C})$ is a (linear) equivalence of categories.*

4.6.3 $L_{|\langle E\rangle} : \langle E\rangle \to \langle \mathcal{F}_{AbOp(\mathcal{C})}\rangle$ is an equivalence of categories

In this part, we prove that the quadratic functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ is an equivalence of categories. First we recall that $Im(Ab^{\mathcal{C}})$ is the category defined in 3.1.3 and $(Ab^{\mathcal{C}})' : \mathcal{C} \to Im(Ab^{\mathcal{C}})$ is the functor (that is the identity on objects and the abelianization functor on morphisms) given in 3.1.4. Here we consider the restriction of the functor $(Ab^{\mathcal{C}})'$ and the category $Im(Ab^{\mathcal{C}})$ to the objects of $\langle E\rangle$. Recall that the functor $Im(Ab(L)) : Im(Ab^{\langle E\rangle}) \to Im(Ab^{\langle \mathcal{F}_{\mathcal{P}}\rangle})$ is defined in 3.5.11, as follows:

- *On objects*, let X be an object in $Im(Ab^{\langle E\rangle})$ (i.e. in $\langle E\rangle$), then $Im(Ab(L))(X) = L(X)$;
- *On morphisms*, let X and Y be two objects in $Im(Ab^{\langle E\rangle})$, and $f \in Im(Ab^{\langle E\rangle})(X, Y) = \mathcal{C}(X^{ab}, Y^{ab})$ (because objects in $\langle E\rangle$ are regular-projective as a finite coproducts of the regular-projective object E), see 3.1.9. Then we set

$$Im(Ab(L))(f) = \sigma_Y^{-1} \circ L(f) \circ \sigma_X \tag{4.6.18}$$

where $\sigma : Ab^{Alg-\mathcal{P}} \cdot L \Rightarrow L \cdot Ab^{\mathcal{C}}$ is the natural isomorphism given in 4.6.25.

Proposition 4.6.27. *The functor $Im(Ab(L)) : Im(Ab^{\langle E\rangle}) \to Im(Ab^{\langle \mathcal{F}_{\mathcal{P}}\rangle})$ is a linear equivalence of categories.*

Proof. It is a direct consequence of 4.6.26. □

Now we consider the following morphism of linear extensions of categories:

$$\begin{tikzcd}
D' \arrow[r, "+"] & \langle \mathcal{F}_{\mathcal{P}}\rangle \arrow[r, "(Ab^{Alg-\mathcal{P}})'"] & Im(Ab^{\langle \mathcal{F}_{\mathcal{P}}\rangle}) \\
D \arrow[r, "+"] & \langle E\rangle \arrow[u, "L"] \arrow[r, "(Ab^{\mathcal{C}})'"'] & Im(Ab^{\langle E\rangle}) \arrow[u, "Im(Ab(L))"']
\end{tikzcd} \tag{4.6.19}$$

The bottom linear extension of categories is the restriction of the one given in 3.1.9 to the full subcategory $\langle E\rangle$ of $\mathcal{C}$. The top linear extension of categories is an in 3.1.9 replacing the category $\mathcal{C}$ with $\langle \mathcal{F}_{\mathcal{P}}\rangle$. Then we check that the criterion, given in 3.5.1, for the quadratic functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ to be an equivalence of categories is satisfied. By 4.6.4, the functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ preserves regular epimorphisms because it preserves coequalizers of reflexive pairs by 6.24 of [12] (since $\langle E\rangle$ is a semi-abelian category). Then we have a (unique) natural isomorphism $\sigma : Ab^{Alg-\mathcal{P}} \cdot L \Rightarrow L \cdot Ab^{\mathcal{C}}$ by 4.6.25 such that, for X object in $\langle E\rangle$, the triangle

$$\begin{tikzcd}
& L(X) \arrow[dl, "L(ab_X)"'] \arrow[dr, "ab_{L(X)}"] & \\
L(X^{ab}) & & L(X)^{ab} \arrow[ll, "\sigma_X", "\cong"']
\end{tikzcd} \tag{4.6.20}$$

commutes. Moreover the functor $Im(Ab(L)) : Im(Ab^{\langle E\rangle}) \to Im(Ab^{\langle \mathcal{F}_{\mathcal{P}}\rangle})$ is an equivalence of categories by 4.6.27. Finally we observe that there is just one condition left to check, namely the natural transformation $\phi^L : D \Rightarrow D^L$, defined in (3.5.3), is an isomorphism between bifunctors with domain $Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}})$ and values in Ab, where the bifunctor $D^L : Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}}) \to Ab$ is given in 3.5.5.

First we observe that up to isomorphism we give another expression of the commutator $[L(X), L(X)]_{Id_{Alg-\mathcal{P}}} = Im\big((\lambda_2^L)_X : L(X)^{\otimes 2} \otimes \mathcal{P}(2) \to L(X)\big)$, for an object X in $\langle E\rangle$, as follows:

Lemma 4.6.28. *For an object X in $\langle E\rangle$, the map*

$$\overline{\overline{(\lambda_2^L)_X}} : Id_{Alg-\mathcal{P}}\big(L(X)|L(X)\big)_{\mathfrak{S}_2} \to [L(X), L(X)]_{Id_{Alg-\mathcal{P}}}$$

given in 2.4.5 is an isomorphism of $\mathcal{P}(1)$-modules, and we have

$$S_2^{Id_{Alg-\mathcal{P}}} = i_{L(X)} \circ \overline{\overline{(\lambda_2^L)_X}} \circ \pi$$

where $S_2^{Id_{Alg-\mathcal{P}}} : Id_{Alg-\mathcal{P}}\big(L(X)|L(X)\big) \to L(X)$ is the morphism defined in 1.2.7 and $i_{L(Y)}$: $Im((\lambda_2^L)_X) = [L(X), L(X)]_{Id_{Alg-\mathcal{P}}} \to L(X)$ is the canonical inclusion (which is the image of the morphism $S_2^{Id_{Alg-\mathcal{P}}}$).

Proof. If X is an object in $\langle E\rangle$, then $L(X)$ is a free $\mathcal{P}$-algebra of finite rank because the Lazard functor $L : \mathcal{C} \to Alg - \mathcal{P}$ preserves finite coproducts by 4.6.16. Hence it is a direct consequence of 2.4.7 applying to $A = L(X)$. By 4.6.7, we recall that we have

$$[L(X), L(X)]_{Id_{Alg-\mathcal{P}}} = Im\big((\lambda_2^L)_X : L(X)^{\otimes 2} \otimes \mathcal{P}(2) \to L(X)\big)$$

Moreover, for $x, y \in L(Y)$ and $h \in \mathcal{P}(2)$, we get the equalities as follows:

$$\begin{aligned} S_2^{Id_{Alg-\mathcal{P}}}(\overline{x} \otimes \overline{x} \otimes h) &= \nabla^2_{L(X)} \circ \iota_2^{Id_{Alg-\mathcal{P}}}(\overline{x} \otimes \overline{y} \otimes h) \\ &= \nabla^2_{L(X)}\big(0, 0, \overline{x} \otimes \overline{y} \otimes h\big) \\ &= (\lambda_2^L)_X(x \otimes y \otimes h)\,, \text{by } (1.8.6) \\ &= \overline{\overline{(\lambda_2^L)_X}} \circ \pi(\overline{x} \otimes \overline{y} \otimes h)\,, \text{by } 2.4.6 \\ &= i_{L(X)} \circ \overline{\overline{(\lambda_2^L)_X}} \circ \pi(\overline{x} \otimes \overline{y} \otimes h) \end{aligned}$$

for $x, y \in L(X)$ and $h \in \mathcal{P}(2)$. Hence we obtain

$$S_2^{Id_{Alg-\mathcal{P}}} = i_{L(X)} \circ \overline{\overline{(\lambda_2^L)_X}} \circ \pi\,,$$

as desired. □

The next proposition says that the Lazard functor preserves a certain class of monomorphisms in $\mathcal{C}$, namely those of the form $i_Y : [Y, Y]_{Id_\mathcal{C}} \rightarrowtail Y$ (which is the image of the morphism $c_2^Y : Id_\mathcal{C}(Y|Y) \to Y$), for an object Y in $\langle E\rangle$.

Proposition 4.6.29. *Let Y be an object in $\langle E\rangle$. Then the coimage $e^L_{[Y,Y]_{Id_\mathcal{C}}} : L([Y, Y]_{Id_\mathcal{C}}) \to [[Y, Y]_{Id_\mathcal{C}}]_L$ of $L(i_Y) : L([Y, Y]_{Id_\mathcal{C}}) \to L(Y)$ is an isomorphism. Hence the morphism $L(i_Y)$: $L([Y, Y]_{Id_\mathcal{C}}) \to L(Y)$ is a monomorphism.*

Proof. We recall that $\overline{L(Y)} = L(Y)^{ab}$ where $\overline{L(Y)}$ is the quotient of $L(Y)$ by the ideal $L(Y)^2$. We first consider the following diagram:

$$
\begin{tikzcd}
Id_{Alg-\mathcal{P}}\big(L(Y)|L(Y)\big) \arrow[rr, "S_2^{Id_{Alg-\mathcal{P}}}"] \arrow[d, equal] & & L(Y) \arrow[dd, equal] \\
\overline{L(Y)} \otimes \overline{L(Y)} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \arrow[d, "\cong"', "i_{Y,Y}"] & & \\
T_1U_E(Y) \otimes T_1U_E(Y) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2) \arrow[rr, "(S_2^L)_Y \circ \Phi_{Y,Y}"] \arrow[d, "\big(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}\big)_{Y,Y}"', "\cong"] & & L(Y) \arrow[ddd, equal] \\
\mathcal{C}\big(E^{ab}, Id_{\mathcal{C}}(Y|Y)\big) \arrow[d, "\big(u'_{\mathcal{C}(E^{ab},Ab^{\mathcal{C}})}\big)^{-1}_{Id_{\mathcal{C}}(Y|Y)}"', "\cong"] & & \\
T_1U_E\big(Id_{\mathcal{C}}(Y|Y)\big) \otimes \mathcal{P}(1) \arrow[d, "\varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)}"', "\cong"] & & \\
L\big(Id_{\mathcal{C}}(Y|Y)\big) \arrow[rr, "L(c_2^Y)"] & & L(Y)
\end{tikzcd}
\tag{4.6.21}
$$

where $u_{\mathcal{C}(E^{ab},Ab^{\mathcal{C}})} : T_1U_E \otimes_\Lambda \mathcal{P}(1) \Rightarrow \mathcal{C}\big(E^{ab}, Ab^{\mathcal{C}}\big)$ and $u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))} : T_1U_E \otimes T_1U_E \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2) \Rightarrow \mathcal{C}(E^{ab}, Id_{\mathcal{C}}(-|-))$ are the natural isomorphisms between bifunctors respectively given in 4.2.11 and 4.1.7. The top rectangle commutes by using the relation expressed by Diagram 4.6.13. We prove that the bottom rectangle commutes. Let $f_1, f_2 \in \mathcal{C}(E, Y)$ and $h \in \mathcal{P}(2)$. Then we have

$$
\begin{aligned}
& L(c_2^Y) \circ \varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(u'_{\mathcal{C}(E^{ab},-)}\big)^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}\big)_{Y,Y}\big(t_1(f_1) \otimes t_1(f_2) \otimes h\big) \\
&= L(c_2^Y) \circ \varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(u'_{\mathcal{C}(E^{ab},-)}\big)^{-1}_{Id_{\mathcal{C}}(Y|Y)}\big(Id_{\mathcal{C}}(f_1|f_2) \circ h\big)\text{, by (4.1.6)} \\
&= L(c_2^Y) \circ \varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)}\big(t_1\big(Id_{\mathcal{C}}(f_1|f_2) \circ h \circ ab_E\big) \otimes id\big) \\
&= L(c_2^Y) \circ \varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(\overline{t_1} \otimes \pi_1\big)\big(t_2\big(Id_{\mathcal{C}}(f_1|f_2) \circ h \circ ab_E\big) \otimes (id, \overline{0})\big) \\
&= L(c_2^Y) \circ \widehat{\psi^{ML}_{Id_{\mathcal{C}}(Y|Y)}}\big(t_2\big(Id_{\mathcal{C}}(f_1|f_2) \circ h \circ ab_E\big) \otimes (id, \overline{0})\big)\text{, by (4.6.17)} \\
&= \widehat{\psi^{ML}_Y} \circ \big(t_2(c_2^Y) \otimes id\big)\big(t_2\big(Id_{\mathcal{C}}(f_1|f_2) \circ h \circ ab_E\big) \otimes (id, \overline{0})\big)\text{, by (2.1.2)} \\
&= \widehat{\psi^{ML}_Y}\big(t_2\big(c_2^Y \circ Id_{\mathcal{C}}(f_1|f_2) \circ h \circ ab_E\big) \otimes (id, \overline{0})\big) \\
&= \widehat{\phi^{ML}_Y}\big(\overline{t_1(f_1) \otimes t_1(f_2) \otimes h}\big)\text{, by 4.6.19} \\
&= (S_2^L)_Y \circ \Phi_{Y,Y}\big(t_1(f_1) \otimes t_1(f_2) \otimes h\big)
\end{aligned}
$$

where $\overline{t_1} : T_2U_E \Rightarrow T_1U_E$ is the unique factorization of $t_1 : U_E \Rightarrow T_1U_E$ through $t_2 : U_E \Rightarrow T_2U_E$ by 1.2.11 (because the functor $T_1U_E : \mathcal{C} \to Mod_{\overline{\Lambda}}$ is linear hence quadratic) and $\pi_1 : \mathcal{F}_{\mathcal{P}} \to \mathcal{P}(1)$ is the projection onto the first summand. It proves that the bottom rectangle of diagram (4.6.21) commutes. By 4.6.28, we get

$$
S_2^{Id_{Alg-\mathcal{P}}} = i_{L(Y)} \circ \overline{\overline{(\lambda_2^L)_Y}} \circ \pi \tag{4.6.22}
$$

Then we have the commutative diagram as follows:

$$\begin{tikzcd}
\overline{L(Y)} \otimes \overline{L(Y)} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \arrow[r, "\pi"] \arrow[d, "i_{Y,Y}"', "\cong"] & \Big(\overline{L(Y)} \otimes \overline{L(Y)} \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\Big)_{\mathfrak{S}_2} \arrow[d, "\cong"', "\overline{i_{Y,Y}}"] \\
T_1U_E(Y) \otimes T_1U_E(Y) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2) \arrow[r, "\pi"] \arrow[d, "\big(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}\big)_{Y,Y}"', "\cong"] & \Big(T_1U_E(Y) \otimes T_1U_E(Y) \otimes_{\Lambda\otimes\Lambda} \mathcal{P}(2)\Big)_{\mathfrak{S}_2} \arrow[d, "\cong"', "\big(\overline{u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}}\big)_{Y}"] \\
\mathcal{C}\big(E^{ab}, Id_{\mathcal{C}}(Y|Y)\big) \arrow[r, "\pi"] \arrow[dr, "(e_Y)_*"] \arrow[d, "\big(u'_{\mathcal{C}(E^{ab},Ab\mathcal{C})}\big)^{-1}_{Id_{\mathcal{C}}(Y|Y)}"', "\cong"] & \mathcal{C}\big(E^{ab}, Id_{\mathcal{C}}(Y|Y)\big)_{\mathfrak{S}_2} \arrow[d, "\cong"', "\overline{(e_Y)_*}"] \\
T_1U_E\big(Id_{\mathcal{C}}(Y|Y)\big) \otimes_{\Lambda} \mathcal{P}(1) \arrow[dr, "T_1U_E(e_Y)\otimes_{\Lambda} id"] \arrow[d, "\varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)}"', "\cong"] & \mathcal{C}\big(E^{ab}, [Y,Y]_{Id_{\mathcal{C}}}\big) \arrow[d, "\cong"', "\big(u'_{\mathcal{C}(E^{ab},Ab\mathcal{C})}\big)^{-1}_{[Y,Y]_{Id_{\mathcal{C}}}}"] \\
L\big(Id_{\mathcal{C}}(Y|Y)\big) \arrow[dr, "L(e_Y)"] & T_1U_E\big([Y,Y]_{Id_{\mathcal{C}}}\big) \otimes_{\Lambda} \mathcal{P}(1) \arrow[d, "\cong"', "\varepsilon^{-1}_{[Y,Y]_{Id_{\mathcal{C}}}}"] \\
 & L\big([Y,Y]_{Id_{\mathcal{C}}}\big)
\end{tikzcd}$$

where the isomorphism of Λ-$\mathcal{P}(1)$-bimodules $\overline{(e_Y)_*} : \mathcal{C}\big(E^{ab}, Id_{\mathcal{C}}(Y|Y)\big)_{\mathfrak{S}_2} \to \mathcal{C}\big(E^{ab}, [Y,Y]_{Id_{\mathcal{C}}}\big)$ (see 4.2.2) is defined in (4.2.1), and the isomorphisms of $\mathcal{P}(1)$-modules $\big(\overline{u_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))'}}\big)_Y$ and $\overline{i_{Y,Y}}$ are respectively given in 4.1.12 (see moreover 4.1.13) and 1.8.8 (see 4.6.18 in addition). Then we have the following equalities:

$$\begin{aligned}
& L(i_Y) \circ \varepsilon^{-1}_{[Y,Y]_{Id_{\mathcal{C}}}} \circ \big(u'_{\mathcal{C}(E^{ab},-)}\big)^{-1}_{[Y,Y]_{Id_{\mathcal{C}}}} \circ \overline{(e_Y)_*} \circ \big(\overline{u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}}\big)_Y \circ \overline{i_{Y,Y}} \circ \pi \\
&= L(i_Y) \circ L(e_Y) \circ \varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(u'_{\mathcal{C}(E^{ab},-)}\big)^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}\big)_{Y,Y} \circ i_{Y,Y} \\
&= L(c_2^L) \circ \varepsilon^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(u'_{\mathcal{C}(E^{ab},-)}\big)^{-1}_{Id_{\mathcal{C}}(Y|Y)} \circ \big(u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}\big)_{Y,Y} \circ i_{Y,Y} \\
&= (S_2^L)_Y \circ \Phi_{Y,Y} \circ i_{Y,Y} \\
&= S_2^{Id_{Alg-\mathcal{P}}} \\
&= i_{L(Y)} \circ \overline{\overline{(\lambda_2^L)_Y}} \circ \pi \text{, by (4.6.22)}
\end{aligned}$$

Hence it implies that we get

$$L(i_Y) \circ \varepsilon^{-1}_{[Y,Y]_{Id_{\mathcal{C}}}} \circ \big(u'_{\mathcal{C}(E^{ab},-)}\big)^{-1}_{[Y,Y]_{Id_{\mathcal{C}}}} \circ \overline{(e_Y)_*} \circ \big(\overline{u'_{\mathcal{C}(E^{ab},Id_{\mathcal{C}}(-|-))}}\big)_Y \circ \overline{i_{Y,Y}} = i_{L(Y)} \circ \overline{\overline{(\lambda_2^L)_Y}} \tag{4.6.23}$$

because the canonical quotient map

$$\pi : T_1L(Y) \otimes T_1L(Y) \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2) \to \Big(T_1L(Y) \otimes T_1L(Y) \otimes_{\mathcal{P}(1)\otimes\mathcal{P}(1)} \mathcal{P}(2)\Big)_{\mathfrak{S}_2}$$

is a surjection. Thus the morphism $L(i_Y) : L([Y,Y]_{Id_{\mathcal{C}}}) \to L(Y)$ is a monomorphism. By 1.3.7, we recall that $L(i_Y) = i^L_{[Y,Y]_{Id_{\mathcal{C}}}} \circ e^L_{[Y,Y]_{Id_{\mathcal{C}}}}$ implying that the coimage $e^L_{[Y,Y]_{Id_{\mathcal{C}}}} : L([Y,Y]_{Id_{\mathcal{C}}}) \to L(Y)$ of $L(i_Y)$ is a monomorphism. Hence $e^L_{[Y,Y]_{Id_{\mathcal{C}}}}$ is an isomorphism because it also is a regular epimorphism. $\square$

We now are able to prove that the Lazard functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ restricted to $\langle E\rangle$ is an equivalence of categories:

Lemma 4.6.30. *The functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ is a quadratic equivalence of categories between algebraic theories.*

Proof. Let X and Y be two objects in $\langle E\rangle$. Consider the morphism of linear extensions of categories given in (4.6.19). By 4.6.11 and 4.6.4, $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ is a quadratic functor preserving finite coproducts and coequalizers of reflexive graphs. Moreover there is a unique natural isomorphism $\sigma : Ab^{Alg-\mathcal{P}} \cdot L \Rightarrow L \cdot Ab^{\mathcal{C}}$ on $\langle E\rangle$ by 4.6.25 such that the triangle (4.6.20) commutes. By 3.5.10, the functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ preserves the action $+_D$ of $D(X,\, Y)$ on $\mathcal{C}(X,\, Y)$ to the action $+_{D'}$ of $D'(L(X),\, L(Y))$ on $Alg-\mathcal{P}(L(X),\, L(Y))$. Moreover $Im(Ab(L)) : Im(Ab^{\langle E\rangle}) \to Im(Ab^{\langle \mathcal{F}_{\mathcal{P}}\rangle})$ is an equivalences of categories by 4.6.27. Now it remains to prove that the natural transformation $\psi^L : D \Rightarrow D^L$ between bifunctors with domain $Im(Ab^{\mathcal{C}})^{op} \times Im(Ab^{\mathcal{C}})$ and values in Ab (see (3.5.3)) defined by:

$$\begin{array}{cccc} \phi^L_{X,Y} & : \mathcal{C}\big(X^{ab},\, [Y,\, Y]_{Id_{\mathcal{C}}}\big) & \longrightarrow & Alg-\mathcal{P}\big(L(X^{ab}),\, [[Y,\, Y]_{Id_{\mathcal{C}}}]_L\big) \\ & f & \longmapsto & e^L_{[Y,\, Y]_{Id_{\mathcal{C}}}} \circ L(f)\ . \end{array}$$

where $e^L_{[Y,\, Y]_{Id_{\mathcal{C}}}} : L([Y,\, Y]_{Id_{\mathcal{C}}}) \to [[Y,\, Y]_{Id_{\mathcal{C}}}]_L$ is the coimage of the morphism $L(i_Y) : L([Y,\, Y]_{Id_{\mathcal{C}}}) \to L(Y)$ (see 1.3.7); it is an isomorphism by 4.6.29. By definition of $\phi^L_{X,Y}$, we have the following commutative diagram:

$$\begin{tikzcd} & Alg-\mathcal{P}\big(L(X^{ab}), L([Y,Y]_{Id_{\mathcal{C}}})\big) \arrow[d, "\cong"', "\left(e^L_{[Y,\,Y]_{Id_{\mathcal{C}}}}\right)_*"] \\ \mathcal{C}\big(X^{ab}, [Y,Y]_{Id_{\mathcal{C}}}\big) \arrow[ur, "L_{X^{ab},[Y,Y]_{Id_{\mathcal{C}}}}", "\cong"'] \arrow[r, "\phi^L_{X,Y}"'] & Alg-\mathcal{P}\big(L(X^{ab}), [[Y,Y]_{Id_{\mathcal{C}}}]_L\big) \end{tikzcd}$$

As the Lazard functor $L : Ab(\mathcal{C}) \to Ab(Alg-\mathcal{P}) = Mod_{\mathcal{P}(1)}$ restricted to the abelian core $Ab(\mathcal{C})$ is an equivalence of categories, the map

$$L_{X^{ab},[Y,Y]_{Id_{\mathcal{C}}}} : \mathcal{C}\big(X^{ab},\ [Y,Y]_{Id_{\mathcal{C}}}\big) \to Alg-\mathcal{P}\big(L(X^{ab}), L([Y,Y]_{Id_{\mathcal{C}}})\big)$$

is a bijection. Hence the map $\phi^L_{X,Y}$ also is a bijection. By 3.5.1, the functor $L : \langle E\rangle \to \langle \mathcal{F}_{\mathcal{P}}\rangle$ is an equivalence of categories. □

Chapter 5

The Baker-Campbell-Hausdorff formula for 2-radicable 2-step nilpotent varieties

In this chapter, we first recall how to recover (concrete) operations of any arity of objects in $\mathcal{C}$, a variety supposed 2-setp nilpotent. Then we provide a decomposition of morphisms from E to E^{+n}, for $n \in \mathbb{N}^*$, as a sum (under the group law of $\mathcal{C}(E, E^{+n})$) of some morphisms. Finally, we use the latter decomposition of such morphisms and their evaluation by the Lazard functor so as to determine a Baker-Campbell-Hausdorff type formula, expressing any operation in $\mathcal{C}$ from the structure linear maps of $AbOp(\mathcal{C})$-algebras.

5.1 Decomposition of certain morphisms in $\mathcal{C}$

In this part, we provide a decomposition of any morphism with source E and target E^{+n} (with $n \geq 3$) in $\mathcal{C}$ as a sum of elements belonging to $\mathcal{C}(E, E)$ and $\mathcal{C}(E, E^{+2})$.
First we recall that, for $X \in \langle E \rangle$ and $Y \in \mathcal{C}$, the set $\mathcal{C}(X, Y)$ has a group structure (written additively) by 4.1.2 as follows:

$$f + g = (f, g) \circ \mu_X$$

where $f, g \in \mathcal{C}(X, Y)$ and $\mu_X : X \to X + X$ is the morphism in $\mathcal{C}$ given by 4.1.1. If moreover Y is an abelian object in $\mathcal{C}$, then the group $\mathcal{C}(X, Y)$ is abelian by 1.5.15 and by 4.1.5. Let $k, l \in \{1, \ldots n\}$, $k \neq l$. Here $i_k^n : E \to E^{+n}$ denotes the injection of the k-th summand. Moreover we define $i_{kl}^n : E^{+2} \to E^{+n}$ and $r_{kl}^n : E^{+n} \to E^{+2}$ the unique morphisms such that

$$\begin{cases} i_{kl}^n \circ i_1^2 = i_k^n \\ i_{kl} \circ i_2^2 = i_l^n \end{cases} \quad \text{and} \quad \begin{cases} r_{kl}^n \circ i_p^n = 0, \text{ if } p \neq k, l \\ r_{kl}^n \circ i_k^n = i_1^2 \\ r_{kl}^n \circ i_l^n = i_2^2 \end{cases} \tag{5.1.1}$$

We point out that $r_{kl}^n \circ i_{kl}^n = id$ by the universal property of the coproduct E^{+2}. For $n \in \mathbb{N}$, $n \geq 3$, we consider E^{+n} to be the coproduct $E + E^{+(n-1)}$ by choosing the appropriated injections, namely $i_1^n : E \to E^{+n}$ and $\widehat{i_{\hat{1}}^n} : E^{+(n-1)} \to E^{+n}$, where $\widehat{i_{\hat{1}}^n}$ is the unique morphism such that

$$\widehat{i_{\hat{1}}^n} \circ i_k^{n-1} = i_{k+1}^n \text{, for } 1 \leqslant k \leqslant n-1$$

and we consider $\widehat{r_{\hat{1}}^n} : E^{+n} \to E^{+(n-1)}$ the unique morphism such that

$$\begin{cases} \widehat{r_{\hat{1}}^n} \circ i_1^n = 0 \\ \widehat{r_{\hat{1}}^n} \circ i_p^n = i_{p-1}^{n-1} \text{, for } 2 \leqslant p \leqslant n \end{cases}$$

Then we recall that $\mathcal{C}(E, -|-) : \mathcal{C}^{\times 2} \to Ab$ is the bifunctor given in 4.1.8 as follows:

- *On objects*, for two objects X and Y in $\mathcal{C}$, $\mathcal{C}(E, X|Y) = \mathcal{C}\big(E, Id_{\mathcal{C}}(X|Y)\big)$, and $(\iota_2^{Id_{\mathcal{C}}})_* : \mathcal{C}(E,\ X|Y) \rightarrowtail \mathcal{C}(E,\ X+Y)$ is the kernel of the comparison morphism $\widehat{r_2} = ((r_1^2)_*,\ (r_2^2)_*)^t : \mathcal{C}(E,\ X+Y) \to \mathcal{C}(E,\ X) \times \mathcal{C}(E,\ Y)$ (see 4.1.5).

- *On morphisms*, let $f : X \to X'$ and $g : Y \to Y'$ be two morphisms in $\mathcal{C}$, then $\mathcal{C}(E, f|g) = Id_{\mathcal{C}}(f|g)_* : \mathcal{C}(E, X|Y) \to \mathcal{C}(E, X'|Y')$.

Note that, for two objects X and Y in $\mathcal{C}$, $Id_{\mathcal{C}}(X|Y)$ is an abelian object in $\mathcal{C}$ by 1.4.2 (because $\mathcal{C}$ is a 2-step nilpotent category). Hence the bifunctor $\mathcal{C}(E,\ X|Y)$ is an abelian group by 1.5.15. Then we deduce that the bifunctor $\mathcal{C}(E, -|-) : \mathcal{C}^{\times 2} \to Gr$ takes in fact values in Ab.

Remark 5.1.1. The bifunctor $\mathcal{C}(E, -|-) : \mathcal{C}^{\times 2} \to Ab$ is bilinear (i.e. linear on each variable, see 1.2.12) because $Id_{\mathcal{C}}(-|-) : \mathcal{C}^{\times 2} \to Ab(\mathcal{C})$ is a bilinear bifunctor and the representable functor $\mathcal{C}(E, -) : \mathcal{C} \to Gr$ preserves finite products.

Notation 5.1.2. For $n \in \mathbb{N}^*$, we here denote by $\widehat{r_n}$ the comparison morphism $\widehat{r_n^{\mathcal{C}(E,-)}}$.

Then the following lemma gives a decomposition of morphisms from E to E^{+n}, for $n \in \mathbb{N}^*$, as a sum (under the group law of the set morphisms $\mathcal{C}(E, E^{+n})$) of some morphisms.

Lemma 5.1.3. *There is the following short exact sequence in the category of groups:*

$$0 \longrightarrow \bigoplus_{1 \leqslant k < l \leqslant n} \mathcal{C}(E,\ E|E) \xrightarrow{k_n} \mathcal{C}(E,\ E^{+n}) \xrightarrow{\widehat{r_n}} \mathcal{C}(E,\ E)^{\times n} \longrightarrow 0$$

where $\widehat{r_n} = \big((r_1^n)_*, \ldots, (r_n^n)_*\big)^t$ *and the morphism* k_n *is defined explicitely by*

$$\begin{array}{rccc} k_n : & \displaystyle\bigoplus_{1 \leqslant k < l \leqslant n} \mathcal{C}(E,\ E|E) & \longrightarrow & \mathcal{C}(E,\ E^{+n}) \\ & (f_{kl})_{1 \leqslant k < l \leqslant n} & \longmapsto & \displaystyle\sum_{1 \leqslant k < l \leqslant n} i_{kl}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ f_{kl}\ . \end{array}$$

Moreover, for all $\xi \in \mathcal{C}(E,\ E^{+n})$, *we have*

$$\xi = \sum_{p=1}^{n} i_p^n \circ r_p^n \circ \xi \ + \sum_{1 \leqslant k < l \leqslant n} i_{kl}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ r_2(r_{kl}^n \circ \xi)$$

where the sums are for the group structure $+$ *of* $\mathcal{C}(E, E^{+n})$.

Proof. Let P_n be the property given in the statement. We prove this result by induction.

- It is clear that P_2 is verified because it corresponds to the canonical short exact sequence

$$0 \longrightarrow \mathcal{C}(E,\ E|E) \xrightarrow{(\iota_2^{Id_{\mathcal{C}}})_*} \mathcal{C}(E,\ E^{+2}) \xrightarrow{\widehat{r_2}} \mathcal{C}(E,\ E)^{\times 2} \longrightarrow 0$$

and any $\xi \in \mathcal{C}(E,\ E^{+2})$ is decomposed in the following way

$$\xi = \big(i_1^2 \circ r_1^2 \circ \xi\big) \ + \ \big(i_2^2 \circ r_2^2 \circ \xi\big) \ + \ \big(\iota_2^{Id_{\mathcal{C}}} \circ r_2(\xi)\big)$$

by 4.1.11.

- Now we suppose that P_{n-1} is satisfied, for $n \geq 3$. We consider the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{C}(E,\ E|E)^{\oplus(n-1)} \arrow[r, "\phi_n", "\cong"'] \arrow[d, "i_n"] & \mathcal{C}(E,\ E|E^{+(n-1)}) \arrow[r] \arrow[d, "(\iota_2^{Id_{\mathcal{C}}})_*"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus\limits_{1\leqslant k<l\leqslant n} \mathcal{C}(E,\ E|E) \arrow[r, "k_n"] \arrow[d, "p_n"] & \mathcal{C}(E,\ E^{+n}) \arrow[r, "\widehat{r_n}"] \arrow[d, "\widehat{r}_2=((r_1^n)_*,(\widehat{r_{\hat{1}}^n})_*)^t"] & \mathcal{C}(E,\ E)^{\times n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus\limits_{1\leqslant k<l\leqslant n-1} \mathcal{C}(E,\ E|E) \arrow[r, "(0,\, k_{n-1})^t"'] \arrow[d] & \mathcal{C}(E,\ E) \times \mathcal{C}(E,\ E^{+(n-1)}) \arrow[r, "id\times\widehat{r_{n-1}}"'] \arrow[d] & \mathcal{C}(E,\ E)^{\times n} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here

  - the morphism $\phi_n = \big(\mathcal{C}(E,\ E|i_1^{n-1}), \ldots, \mathcal{C}(E,\ E|i_{n-1}^{n-1})\big) : \mathcal{C}(E,\ E|E)^{\oplus(n-1)} \to \mathcal{C}(E,\ E|E^{+(n-1)})$ is an isomorphism of abelian groups whose inverse is

$$\phi_n^{-1} = \big(\mathcal{C}(E,\ E|r_1^{n-1}), \ldots, \mathcal{C}(E,\ E|r_{n-1}^{n-1})\big)^t : \mathcal{C}(E,\ E|E^{+(n-1)}) \to \mathcal{C}(E,\ E|E)^{\oplus(n-1)}$$

    because the bifunctor $\mathcal{C}(E,\ E|-) : \mathcal{C} \to Ab$ is linear by 5.1.1 and it is a consequence of 3.6 in [12].

  - the morphism p_n is defined by

$$\begin{array}{rccc} p_n & : & \bigoplus\limits_{1\leqslant k<l\leqslant n} \mathcal{C}(E,\ E|E) & \longrightarrow & \bigoplus\limits_{1\leqslant k<l\leqslant n-1} \mathcal{C}(E,\ E|E) \\ & & (f_{kl})_{1\leqslant k<l\leqslant n} & \longmapsto & (f_{k+1\, l+1})_{1\leqslant k<l\leqslant n-1}\ . \end{array}$$

  - the morphism i_n is given by

$$\begin{array}{rccc} i_n & : & \mathcal{C}(E,\ E|E)^{\oplus(n-1)} & \longrightarrow & \bigoplus\limits_{1\leqslant k<l\leqslant n} \mathcal{C}(E,\ E|E) \\ & & (g_1, \ldots, g_{n-1}) & \longmapsto & (f_{kl})_{1\leqslant k<l\leqslant n}\ . \end{array}$$

    where $f_{1l} = g_{l-1}$, for $l = 2, \ldots, n$, and $f_{kl} = 0$ for $k \geq 2$.

Then we prove that the four squares of the above diagram commute.

- *The bottom right-hand square commutes.* It is just an observation that, for $k = 1, \ldots n-1$, we have $r_k^{n-1} \circ \widehat{r_{\hat{1}}}^n = r_{k+1}^n$ by the uniqueness in the universal property of the coproduct E^{+n}. Consequently it also proves that the top right-hand square commutes.

- *The bottom left-hand square commutes.* Take a familly $(f_{kl})_{1\leqslant k<l\leqslant n}$ of elements of $\mathcal{C}(E,\ E|E)$. Then we have

$$\begin{aligned}
\big(\widehat{r_2}\circ k_n\big)\big((f_{kl})_{1\leqslant k<l\leqslant n}\big) &= \widehat{r_2}\Big(\sum_{1\leqslant k<l\leqslant n} i^n_{kl}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{kl}\Big)\\
&= \Big(\sum_{1\leqslant k<l\leqslant n} r^n_1\circ i^n_{kl}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{kl},\ \sum_{1\leqslant k<l\leqslant n} \widehat{r_{\hat{1}}}^{\,n}\circ i^n_{kl}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{kl}\Big)\\
&= \Big(0,\ \sum_{1\leqslant k<l\leqslant n} \widehat{r_{\hat{1}}}^{\,n}\circ i^n_{kl}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{kl}\Big)\\
&= \Big(0,\ \sum_{p=2}^{n} \widehat{r_{\hat{1}}}^{\,n}\circ i^n_{1p}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{1p}\ +\sum_{2\leqslant k<l\leqslant n} \widehat{r_{\hat{1}}}^{\,n}\circ i^n_{kl}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{kl}\Big)\\
&= \Big(0,\ \sum_{2\leqslant k<l\leqslant n} i^{n-1}_{k-1\,l-1}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{kl}\Big)\\
&= \Big(0,\ \sum_{1\leqslant k<l\leqslant n-1} i^{n-1}_{kl}\circ \iota_2^{Id_{\mathcal{C}}}\circ f_{k+1\,l+1}\Big)\\
&= (0,k_{n-1})^t\big((f_{k+1\,l+1})_{1\leqslant k<l\leqslant n-1}\big)\\
&= \big((0,k_{n-1})^t\circ p_n\big)\big((f_{kl})_{1\leqslant k<l\leqslant n}\big)
\end{aligned}$$

- *The top left-hand square commutes.* First we know that the functor $\mathcal{C}(E,\ E|-):\mathcal{C}\to Ab$ is linear. By 3.6 of [12], we have the following relation:

$$id=\sum_{k=1}^{n-1}\mathcal{C}(E,\ E|\ i^{n-1}_k\circ r^{n-1}_k)$$

Then, for $\xi\in\mathcal{C}(E,\ E|E^{+(n-1)})$, we have

$$\begin{aligned}
(\iota_2^{Id_{\mathcal{C}}})_*(\xi) &= \sum_{k=1}^{n-1}(\iota_2^{Id_{\mathcal{C}}})_*\circ\mathcal{C}(E,\ E|\ i^{n-1}_k\circ r^{n-1}_k)(\xi)\\
&= \sum_{k=1}^{n-1}(id+i^{n-1}_k)_*\circ(id+r^{n-1}_k)_*\circ(\iota_2^{Id_{\mathcal{C}}})_*(\xi)\\
&= \sum_{k=1}^{n-1}(id+i^{n-1}_k)\circ(id+r^{n-1}_k)\circ\iota_2^{Id_{\mathcal{C}}}\circ\xi\\
&= \sum_{k=1}^{n-1} i^n_{1\,k+1}\circ r^n_{1\,k+1}\circ\iota_2^{Id_{\mathcal{C}}}\circ\xi
\end{aligned}$$

where we recall that $\iota_2^{Id_{\mathcal{C}}}:\mathcal{C}(E,\ E|E^{+(n-1)})\to\mathcal{C}(E,\ E^{+n})$ is the kernel of the comparison morphism $\widehat{r_2}=\big((r^n_1)_*,(\widehat{r_{\hat{1}}}^{\,n})_*\big)^t:\mathcal{C}(E,\ E^{+n})\to\mathcal{C}(E,\ E)\times\mathcal{C}(E,\ E^{+(n-1)})$. Hence we have

$$(\iota_2^{Id_{\mathcal{C}}})_*(\xi)=\sum_{k=1}^{n-1} i^n_{1\,k+1}\circ r^n_{1\,k+1}\circ\iota_2^{Id_{\mathcal{C}}}\circ\xi \tag{5.1.2}$$

Then we have

$$\begin{aligned}
\big(k_n \circ i_n \circ \phi_n^{-1}\big)(\xi) &= \Big(k_n \circ i_n \circ \big(\mathcal{C}(E,\ E|r_1^{n-1}), \ldots, \mathcal{C}(E,\ E|r_{n-1}^{n-1})\big)^t\Big)(\xi) \\
&= \big(k_n \circ i_n\big)\Big(Id_{\mathcal{C}}(id|r_1^{n-1}) \circ \xi, \ldots, Id_{\mathcal{C}}(id|r_{n-1}^{n-1}) \circ \xi\Big) \\
&= k_n((f_{kl})_{1 \leqslant k < l \leqslant n})
\end{aligned}$$

where $(f_{kl})_{1\leqslant k<l\leqslant n}$ is the familly such that $f_{1l} = Id_{\mathcal{C}}(id|r_{l-1}^{n-1}) \circ \xi$ for $l = 2, \ldots, n$ and $f_{kl} = 0$ if $k \geq 2$. Then we obtain

$$\begin{aligned}
k_n\big((f_{kl})_{1\leqslant k<l\leqslant n}\big) &= \sum_{1\leqslant k<l\leqslant n} i_{kl}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ f_{kl} \\
&= \sum_{p=2}^{n} i_{1p}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ f_{kl} \ + \sum_{2\leqslant k<l\leqslant n} i_{kl}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ f_{kl} \\
&= \sum_{p=2}^{n} i_{1\,k+1}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(id|r_{p-1}^{n-1}) \circ \xi \\
&= \sum_{k=1}^{n-1} i_{1\,k+1}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ Id_{\mathcal{C}}(id|r_k^{n-1}) \circ \xi \\
&= \sum_{k=1}^{n-1} i_{1\,k+1}^n \circ \big(id + r_k^{n-1}\big) \circ \iota_2^{Id_{\mathcal{C}}} \circ \xi \\
&= \sum_{k=1}^{n-1} i_{1\,k+1}^n \circ r_{1k+1}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ \xi \\
&= (\iota_2^{Id_{\mathcal{C}}})_*(\xi)\,, \text{by } 5.1.2
\end{aligned}$$

It proves that we get $k_n \circ i_n \circ \phi_n^{-1} = (\iota_2^{Id_{\mathcal{C}}})_*$.

Applying the nine lemma to the above diagram, the middle short vertical sequence is exact as desired. Now we prove the decomposition of elements in $\mathcal{C}(E,\ E^{+n})$ as in the assumption. Let $\xi \in \mathcal{C}(E,\ E^{+n})$, we have

$$\xi = \big(i_1^n \circ r_1^n \circ \xi\big) \ + \ \big(\widehat{i_{\hat{1}}}^n \circ \widehat{r_{\hat{1}}}^n \circ \xi\big) \ + \ d(\xi)$$

where $d(\xi) = -\big(\widehat{i_{\hat{1}}}^n \circ \widehat{r_{\hat{1}}}^n \circ \xi\big) \ - \ \big(i_1^n \circ r_1^n \circ \xi\big) \ + \ \xi \ \in \mathcal{C}(E,\ E|E^{+(n-1)})$. By assumption, we know that

$$\begin{aligned}
\widehat{r_{\hat{1}}}^n \circ \xi &= \sum_{p=1}^{n-1} i_p^{n-1} \circ r_p^{n-1} \circ \widehat{r_{\hat{1}}}^n \circ \xi \ + \sum_{1\leqslant k<l\leqslant n-1} i_{kl}^{n-1} \circ \iota_2^{Id_{\mathcal{C}}} \circ r_2(r_{kl}^{n-1} \circ \widehat{r_{\hat{1}}}^n \circ \xi) \\
&= \sum_{p=1}^{n-1} i_p^{n-1} \circ r_{p+1}^n \circ \xi \ + \sum_{1\leqslant k<l\leqslant n-1} i_{kl}^{n-1} \circ \iota_2^{Id_{\mathcal{C}}} \circ r_2(r_{k+1\,l+1}^n \circ \xi)
\end{aligned}$$

Postcomposing by $\widehat{i_{\hat{1}}}^n$ to the above equation gives the following equality:

$$\begin{aligned}
\widehat{i_{\hat{1}}}^n \circ \widehat{r_{\hat{1}}}^n \circ \xi &= \sum_{p=1}^{n-1} i_{p+1}^n \circ r_{p+1}^n \circ \xi \ + \sum_{1\leqslant k<l\leqslant n-1} i_{k+1\,l+1}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ r_2(r_{k+1\,l+1}^n \circ \xi) \\
&= \sum_{p=2}^{n} i_p^n \circ r_p^n \circ \xi \ + \sum_{2\leqslant k<l\leqslant n} i_{kl}^n \circ \iota_2^{Id_{\mathcal{C}}} \circ r_2(r_{kl}^n \circ \xi)
\end{aligned}$$

Moreover there is another expression of $d(\xi) \in \mathcal{C}(E,\ E|E^{+(n-1)})$ that is

$$d(\xi) = \sum_{k=1}^{n-1} i^n_{1\,k+1} \circ \iota_2^{Id_\mathcal{C}} \circ r_2(r^n_{1\,k+1} \circ d(\xi)) = \sum_{k=2}^{n} i^n_{1k} \circ \iota_2^{Id_\mathcal{C}} \circ r_2(r^n_{1k} \circ d(\xi))$$

However, for $k = 2, \ldots, n$, we have

$$\begin{aligned} r^n_{1k} \circ d(\xi) &= -\left(r^n_{1k} \circ \widehat{i_{\hat{1}}}^n \circ \widehat{r_{\hat{1}}}^n \circ \xi\right) - \left(r^n_{1k} \circ i^n_1 \circ r^n_1 \circ \xi\right) + \left(r^n_{1k} \circ \xi\right) \\ &= -\left(i^2_2 \circ r^2_2 \circ (r^n_{1k} \circ \xi)\right) - \left(i^2_1 \circ r^2_1 \circ (r^n_{1k} \circ \xi)\right) + \left(r^n_{1k} \circ \xi\right) \\ &= \iota_2^{Id_\mathcal{C}} \circ r_2(r^n_{1k} \circ \xi) \end{aligned}$$

Hence we get

$$d(\xi) = \sum_{k=2}^{n} i^n_{1\,k} \circ \iota_2^{Id_\mathcal{C}} \circ r_2(r^n_{1k} \circ \xi)$$

Thus we have

$$\begin{aligned} \xi &= \left(i^n_1 \circ r^n_1 \circ \xi\right) + \left(\widehat{i_{\hat{1}}}^n \circ \widehat{r_{\hat{1}}}^n \circ \xi\right) + d(\xi) \\ &= \left(i^n_1 \circ r^n_1 \circ \xi\right) + \sum_{p=2}^{n} i^n_p \circ r^n_p \circ \xi + \sum_{2 \leqslant k < l \leqslant n} i^n_{kl} \circ \iota_2^{Id_\mathcal{C}} \circ r_2(r^n_{kl} \circ \xi) + \sum_{p=2}^{n} i^n_{1p} \circ \iota_2^{Id_\mathcal{C}} \circ r_2(r^n_{1p} \circ \xi) \\ &= \sum_{p=1}^{n} i^n_p \circ r^n_p \circ \xi + \sum_{1 \leqslant k < l \leqslant n} i^n_{kl} \circ \iota_2^{Id_\mathcal{C}} \circ r_2(r^n_{kl} \circ \xi) \end{aligned}$$

This proves the result. □

5.2 The Baker-Campbell-Hausdorff (BCH) formula

We finally deduce this desired formula for 2-step nilpotent varieties from the Lazard functor and the decomposition of operations provided in the previous section.

Recall that $AbOp(\mathcal{C})$ is the 2-step (right) operad living in the monoidal category of $\mathbb{Z}[\frac{1}{2}]$-modules as

$$AbOp(\mathcal{C})(1) = |E^{ab}| \quad \text{and} \quad AbOp(\mathcal{C})(2) = |Id_\mathcal{C}(E|E)|$$

in which the composition operations are given as follows:

$$\gamma_{1;1}(a \otimes b) = \left(\tilde{a} \circ \tilde{b}\right)(\overline{e}), \quad \gamma_{2;1}(\mu \otimes a) = \left(\tilde{\mu} \circ \tilde{a}\right)(\overline{e}), \quad \gamma_{1,1;2}(a \otimes b \otimes \mu) = \left(Id_\mathcal{C}(\tilde{a}|\tilde{b}) \circ \tilde{\mu}\right)(\overline{e})$$

where $a, b \in \mathcal{P}(1)$ and $\mu \in \mathcal{P}(2)$ and the notations are given in 4.0.2. Here $T : AbOp(\mathcal{C})(2) \to AbOp(\mathcal{C})(2)$ denotes the involution of $AbOp(\mathcal{C})(2)$ obtained by restriction of the canonical switch $\tau^2_E : |E^{+2}| \to |E^{+2}|$ to $|Id_\mathcal{C}(E|E)|$.
Now we recall the definition of "concrete" operations of the variety $\mathcal{C}$. Any (formal) n-ary operation θ of the algebraic theory $\langle E \rangle$ can be seen as a morphism $\theta^o : E \to E^{+n}$ in $\mathcal{C}$ by the Yoneda's lemma. For an object X in $\mathcal{C}$, the concrete n-ary operation $\theta_X : |X|^{\times n} \to |X|$ in $|X|$ associated with θ is given in the following way:

$$\begin{tikzcd} \mathcal{C}(E,\ X) \arrow[rr, "ev_e", "\cong"'] & & {|X|} \\ \mathcal{C}(E^{+n},\ X) \arrow[u, "(\theta^o)^*"] \arrow[r, "\cong"] & \mathcal{C}(E,\ X)^{\times n} \arrow[r, "\cong", "ev_e^{\times n}"'] & {|X|^{\times n}} \arrow[u, dashed, "\theta_X"'] \end{tikzcd}$$

Now we write $e_k = i_k^n(e)$, for $k = 1, \dots, n$. Let $i_k^n : E \to E^{+n}$ be the injection of the k-th summand, for $k = 1, \dots, n$, we point out that $\theta^o(e) = \theta_{E^{+n}}(e_1, \dots, e_n)$ which is the evaluation of $\theta_{E^{+n}}$ on the n generators e_k of the free object E^{+n} of rank n in $\mathcal{C}$. As the above diagram is natural in X, this gives rise to a natural transformation $\theta : |-|^{\times n} \Rightarrow |-|$ such that θ_X is defined by

$$\theta(x_1, \dots, x_n) = \theta_X(x_1, \dots x_n) = |(\hat{x_1}, \dots, \hat{x_n}) \circ \theta^o|(e) \tag{5.2.1}$$

where $x_1, \dots, x_n \in |X|$ and $\hat{x_k} : E \to E$ is the unique morphism such that $\hat{x_k}(e) = x_k$ (see 4.0.2), for $k = 1, \dots, n$.

Notation 5.2.1. For $k, l \in \{1, \dots, n\}$, $k \neq l$, we set $\theta_k = r_k^n \circ \theta^o$ and $\theta_{kl} = r_{kl}^n \circ \theta^o$ where $r_k^n : E^{+n} \to E$ is the retraction onto the k-th summand and $r_{kl}^n : E^{+n} \to E^{+2}$ is the morphism defined at the beginning of section 3.4.

Lemma 5.2.2. *Let X be an object in $\mathcal{C}$. For $x, x_1, x_2 \in |X|$, we have*

$$\begin{cases} \theta_k(x) = \theta_X(0, \dots, 0, x, 0, \dots, 0) \\ \theta_{kl}(x_1, x_2) = \theta_X(0, \dots, 0, x_1, 0, \dots, 0, x_2, 0, \dots, 0) \end{cases}$$

where x, x_1 are settled in the k-th place and x_2 is settled in the l-th place.

Proof. First we have the following equalities:

$$\begin{aligned}
\theta_k(x) &= |\hat{x} \circ r_k^n \circ \theta^o|(e)\,, \text{by } (5.2.1) \\
&= |\hat{x} \circ r_k^n| \circ \theta_{E^{+n}}(e_1, \dots, e_k, \dots, e_n) \\
&= \theta_X\big(\hat{x} \circ r_k^n(e_1), \dots, \hat{x} \circ r_k^n(e_k), \dots, \hat{x} \circ r_k^n(e_n)\big)\,, \text{by naturality} \\
&= \theta_X\big(\hat{x} \circ r_k^n \circ i_1^n(e), \dots, \hat{x} \circ r_k^n \circ i_k^n(e), \dots, \hat{x} \circ r_k^n \circ i_n^n(e)\big) \\
&= \theta_X(0, \dots, 0, \hat{x}(e), 0, \dots, 0) \\
&= \theta_X(0, \dots, 0, x, 0, \dots, 0)
\end{aligned}$$

Next we get

$$\begin{aligned}
\theta_{kl}(x_1, x_2) &= |(\hat{x_1}, \hat{x_2}) \circ r_{kl}^n \circ \theta^o|(e) \\
&= |(\hat{x_1}, \hat{x_2}) \circ r_{kl}^n| \circ \theta_{E^{+n}}(e_1, \dots, e_n) \\
&= \theta_X\big((\hat{x_1}, \hat{x_2}) \circ r_{kl}^n(e_1), \dots, (\hat{x_1}, \hat{x_2}) \circ r_{kl}^n(e_n)\big)\,, \text{by naturality} \\
&= \theta_X\big((\hat{x_1}, \hat{x_2}) \circ r_{kl}^n \circ i_1^n(e), \dots, (\hat{x_1}, \hat{x_2}) \circ r_{kl}^n \circ i_n^n(e)\big)
\end{aligned}$$

We recall that $r_{kl}^n : E^{+n} \to E^{+2}$ is the unique morphism such that $r_{kl}^n \circ i_k^n = i_1^2$, $r_{kl}^n \circ i_l^n = i_2^2$ and $r_{kl}^n \circ i_l^n = 0$, for $l \neq 1, 2$ (see (5.1.1)). Hence we have

$$\begin{aligned}
\theta_{kl}(x_1, x_2) &= \theta_X\big(0, \dots, 0, (\hat{x_1}, \hat{x_2}) \circ i_1^2(e), 0, \dots, 0, (\hat{x_1}, \hat{x_2}) \circ i_2^2(e), 0, \dots, 0\big) \\
&= \theta_X\big(0, \dots, 0, \hat{x_1}(e), 0, \dots, 0, \hat{x_2}(e), 0, \dots, 0\big) \\
&= \theta_X\big(0, \dots, 0, x_1, 0, \dots, 0, x_2, 0, \dots, 0\big)
\end{aligned}$$

with x_1 and x_2 being respectively settled in the k-th and l-th places. $\square$

By the uniqueness in the universal property of the coproduct E^{+n}, it is easy to check that $r_1^2 \circ \theta_{kl} = \theta_k$ and $r_2^2 \circ \theta_{kl} = \theta_l$. Hence we get

$$\iota_2^{Id_{\mathcal{C}}} \circ r_2(\theta_{kl}) = \theta_{kl} - \Big(\big(i_1^2 \circ r_2^2 \circ \theta_{kl}\big) + \big(i_2^2 \circ r_2^2 \circ \theta_{kl}\big) \Big) = \theta_{kl} - \Big(\big(i_1^2 \circ \theta_k\big) + \big(i_2^2 \circ \theta_l\big) \Big)$$

by (4.1.11). Evaluating $r_2(\theta_{kl})$ to the basis element e of E gives

$$r_2(\theta_{kl})(e) = \theta_{kl}(e_1, e_2) -_M \big(\theta_k(e_1) +_M \theta_l(e_2)\big)$$

where $+_M$ and $-_M$ respectively refers to the multiplication and to the inverse of the group structure of $|X|$. By using the notations given in 4.0.2, we get

$$\overline{r_2(\theta_{kl})(e)} = \theta_{kl}(e_1, e_2) -_M \big(\theta_k(e_1) +_M \theta_l(e_2)\big) \tag{5.2.2}$$

because we have $r_2(\xi) = r_2(\xi)^{ab} \circ ab_E$. Afterwards we recall the Lazard (quadratic) equivalence of categories $L : \langle E \rangle \to \langle \mathcal{F}_{AbOp(\mathcal{C})} \rangle$ (between algebraic theories) explicitely defined on certain morphisms in $\mathcal{C}$. Here the left action of Λ on the free $AbOp(\mathcal{C})$-algebra $\mathcal{F}_{AbOp(\mathcal{C})}$ of rank 1 making it into a (left) Λ-module (see 4.5.5) is given by

$$\alpha *_L (x,\, \overline{y}) = L(\alpha)(x,\, \overline{y}) = \Big(\overline{\alpha(x)},\ \overline{\gamma_{1,1;2}(x \otimes x \otimes y)} + \frac{1}{2}\, \overline{\gamma_{2;1}(H(\alpha) \otimes x)}\Big)$$

where $(x,\, \overline{y}) \in \mathcal{F}_{AbOp(\mathcal{C})}$ and $\overline{y}$ denotes the equivalence class of y in $AbOp(\mathcal{C})(2)_{\mathfrak{S}_2}$. We denote by $i_p : L(E) \rightarrowtail L(E)^{+n}$ the injection of the p-th summand, for $1 \leqslant p \leqslant n$. Let $s_n = (L(i_1^n), \ldots, L(i_n^n)) : L(E)^{+n} \to L(E^{+n})$ be the unique morphism obtained by the universal property of the coproduct $L(E)^{+n}$ in $Alg - AbOp(\mathcal{C})$ such that $s_n \circ i_p = L(i_p^n)$, for $1 \leqslant p \leqslant n$. It is a consequence of 4.6.16 that s_n is an isomorphism. As we have supposed that $L(E^{+2}) = L(E)^{+2}$, it remains to assume that s_2 is the identity. Let $\xi : E \to E^{+2}$ be any morphism in $\mathcal{C}$, then we have

$$L(\xi)(x,\, \overline{y}) = \Big((r_1^2 \circ \xi) *_L (x, \overline{y}),\ (r_2^2 \circ \xi) *_L (x, \overline{y}),\ H^L\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (x,\, \overline{y})\big) \Big)$$

by 4.6.13 and we recall that

$$\begin{aligned} & H^L\big(t_{11}(\rho_2(\xi)) \otimes_\Lambda (x,\, \overline{y})\big) \\ & = t_1(r_1^2 \circ \xi) \otimes t_1(r_2^2 \circ \xi).\big(y + T(y) - \frac{1}{2}\, \gamma_{2;1}(H(2) \otimes x)\big) + \gamma_{2;1}\big(\overline{r_2(\xi)(e)} \otimes x\big) \\ & = \gamma_{1,1;2}\Big(\overline{r_1^2(e)} \otimes \overline{r_2^2(e)} \otimes \big(y + T(y) - \frac{1}{2}\, \gamma_{2;1}(H(2) \otimes x)\big)\Big) + \gamma_{2;1}\big(\overline{r_2(\xi)(e)} \otimes x\big)\,, \end{aligned}$$

see 4.5.9. Now we are able to give the main theorem:

Theorem 5.2.3. *There is a Lazard correspondence*

$$L^* : Alg - AbOp(\mathcal{C}) \to \mathcal{C}$$

given by $|L^*(A)| = |A|$ and the following Baker-Campbell-Hausdorff formula holds : an n-ary operation θ of the variety $\mathcal{C}$ acts on $|L^*(A)|$ by

$$\begin{aligned} \theta(a_1, \ldots, a_n) = & \sum_{p=1}^{n} \Big(\lambda_1(a_p \otimes \overline{\theta_p(e)}) + \frac{1}{2}\, \lambda_2(a_p \otimes a_p \otimes H(\theta_p)) \Big) \\ & + \frac{1}{2} \sum_{1 \leqslant p < q \leqslant n} \lambda_2\Big(a_p \otimes a_q \otimes \gamma_{1,1;2}\big(\overline{\theta_p(e)} \otimes \overline{\theta_q(e)} \otimes [e_1,\ e_2]_M\big)\Big) \\ & + \sum_{1 \leqslant p < q \leqslant n} \lambda_2\Big(a_p \otimes a_q \otimes \big(\theta_{pq}(e_1, e_2) -_M (\theta_p(e_1) +_M \theta_q(e_2))\big)\Big) \end{aligned}$$

for $a_1, \ldots, a_n \in A$. Here

- *the multiplication maps* $\lambda_k : A^{\otimes k} \otimes \mathcal{P}(k) \to A$, $k = 1, 2$, *are the structure maps of the* $\mathcal{P}$*-algebra* A *;*

- *for* $k = 1, 2$, $e_k = i_k^2(e) \in E + E$ *where* $i_k^2 : E \to E + E$ *is the injection of the* k*-th summand. Furthermore,* $[a, b]_M = (a +_M b) -_M (b +_M a)$ *;*

- *for any unary operation* $\mathcal{V}$ *of* $\mathcal{C}$*,*

$$H(\mathcal{V}) = \mathcal{V}_{E^{+2}}\big(e_1 +_M e_2\big) -_M \big(\mathcal{V}_{E^{+2}}(e_1) +_M \mathcal{V}_{E^{+2}}(e_2)\big)$$

- θ_p *is the unary operation of* $\mathcal{C}$ *given by:*

$$\theta_p(a) = \theta\big(0, \ldots, 0, a, 0, \ldots, 0\big)$$

  where a *is placed in the* p*-th place. Similarly,* θ_{pq} *is the binary operation of* $\mathcal{C}$ *given by :*

$$\theta_{pq}(a, b) = \theta\big(0, \ldots, 0, a, 0, \ldots, 0, b, 0, \ldots, 0\big)$$

  where a, b *are respectively in the* p*-th and* q*-th place.*

Proof. We recall that $(\overline{e}, \overline{0})$ is the generator of $\mathcal{F}_{AbOp(\mathcal{C})}$ the free $AbOp(\mathcal{C})$-algebra of rank 1. Let θ be an n-ary (formal) operation of the algebraic theory $\langle E \rangle$ and $a_1, \ldots a_n \in |A|$, then finding the BCH formula for θ amounts to giving an explicit expression of the following concrete operation in $|A|$:

$$\theta(a_1, \ldots, a_n) = |(\hat{a_1}, \ldots, \hat{a_n}) \circ s_n^{-1} \circ L(\theta^o)|(\overline{e},\ 0)$$

using the structure linear maps of $|A|$. We give four main steps to prove the result:

1. *The group structure* $+_M$ in $|A|$: we use the following relation

$$a_1 +_M a_2 = |(\hat{a_1},\ \hat{a_2}) \circ s_2^{-1} \circ L(i_1^2 + i_2^2)|(\overline{e},\ \overline{0})$$

   where $a_1, a_2 \in |A|$. Let $(x,\ \overline{y}) \in \mathcal{F}_{\mathcal{P}} = L(E)$, then we have by 4.6.13

$$\begin{aligned}
& s_2^{-1} \circ L(i_1^2 + i_2^2)(x,\ \overline{y}) \\
&= \Big((r_1^2 \circ (i_1^2 + i_2^2)) *_L (x,\ \overline{y}),\ (r_2^2 \circ (i_1^2 + i_2^2)) *_L (x,\ \overline{y}),\ H^L\big(t_{11}(\rho_2(i_1^2 + i_2^2)) \otimes_\Lambda (x,\ \overline{y})\big)\Big) \\
&= \Big((id + 0) *_L (x,\ \overline{y}),\ (0 + id) *_L (x,\ \overline{y}),\ H^L\big(t_{11}(\rho_2(i_1^2 + i_2^2)) \otimes_\Lambda (x,\ \overline{y})\big)\Big) \\
&= \Big((x,\ \overline{y}),\ (x,\ \overline{y}),\ H^L\big(t_{11}(\rho_2(i_1^2 + i_2^2)) \otimes_\Lambda (x,\ \overline{y})\big)\Big)
\end{aligned}$$

   Moreover we have

$$\begin{aligned}
& H^L(t_{11}\big(\rho_2(i_1^2 + i_2^2)) \otimes_\Lambda (x,\ \overline{y})\big) \\
&= t_1\big(r_1^2 \circ (i_1^2 + i_2^2)\big) \otimes t_1\big(r_2^2 \circ (i_1^2 + i_2^2)\big).\Big(y + T(y) - \frac{1}{2}\,\gamma_{1;2}(H(2) \otimes x)\Big) \\
&\quad + \gamma_{2;1}\big(r_2(i_1^2 + i_2^2)(e) \otimes x\big) \\
&= y + T(y) - \frac{1}{2}\,\gamma_{1;2}(H(2) \otimes x)\,, \text{because } r_2(i_1^2 + i_2^2) = 0
\end{aligned}$$

   Then we obtain

$$\big(s_2^{-1} \circ L(i_1^2 + i_2^2)\big)(x,\ \overline{y}) = \Big((x,\ \overline{y}),\ (x,\ \overline{y}),\ y + T(y) - \frac{1}{2}\,\gamma_{1;2}(H(2) \otimes x)\Big)$$

Hence we have

$$\begin{aligned}
a_1 +_M a_2 &= |(\hat{a_1},\ \hat{a_2}) \circ s_2^{-1} \circ L(i_1^2 + i_2^2)|(\overline{e},\ \overline{0}) \\
&= |(\hat{a_1},\ \hat{a_2})|\Big((\overline{e},\ \overline{0}),\ (\overline{e},\ \overline{0}),\ -\frac{1}{2}\ H(2)\Big) \\
&= \hat{a_1}(\overline{e},\ \overline{0}) + \hat{a_2}(\overline{e},\ \overline{0}) + \lambda_2\big(\hat{a_1}(\overline{e},\ \overline{0}) \otimes \hat{a_2}(\overline{e},\ \overline{0}) \otimes (-\frac{1}{2}\ H(2))\big)\ , by\ 1.8.5 \\
&= a_1 + a_2 + \frac{1}{2}\ \lambda_2\big(a_1 \otimes a_2 \otimes T(H(2))\big)\ , \text{because}\ T(H(2)) = -H(2) \\
&= a_1 + a_2 + \frac{1}{2}\ \lambda_2\big(a_1 \otimes a_2 \otimes [e_1, e_2]_M\big)\ , \text{by } 4.1.19
\end{aligned}$$

Finally we obtain

$$a_1 +_M a_2 = a_1 + a_2 + \frac{1}{2}\ \lambda_2\big(a_1 \otimes a_2 \otimes [e_1, e_2]_M\big) \tag{5.2.3}$$

2. *The unary operations*: let θ be a unary (formal) operation. We have the following relation

$$\theta(a) = |\hat{a} \circ L(\theta^o)|(\overline{e},\ \overline{0})$$

where $a \in |A|$. Thus we get

$$\begin{aligned}
\theta(a) &= \hat{a}\Big(\overline{\theta(e)},\ \frac{1}{2}\ H(\theta^o)\Big) \\
&= \lambda_1\big(\hat{a}(\overline{e},\ \overline{0}) \otimes \overline{\theta(e)}\big) + \lambda_2\big(\hat{a}(\overline{e},\ \overline{0}) \otimes \hat{a}(\overline{e},\ \overline{0}) \otimes \frac{1}{2}\ H(\theta^o)\big) \\
&= \lambda_1\big(a \otimes \overline{\theta(e)}\big) + \frac{1}{2}\ \lambda_2\big(a \otimes a \otimes H(\theta^o)\big)
\end{aligned}$$

Finally we obtain

$$\theta(a) = \lambda_1\big(a \otimes \overline{\theta(e)}\big) + \frac{1}{2}\ \lambda_2\big(a \otimes a \otimes H(\theta^o)\big) \tag{5.2.4}$$

3. Let X be an object in $\langle E \rangle$ and $f, g \in \mathcal{C}(E,\ X)$. We denote by $(\lambda_1^L)_X : L(X) \otimes \mathcal{P}(1) \to L(X)$ and $(\lambda_2^L)_X : L(X)^{\otimes 2} \otimes \mathcal{P}(2) \to L(X)$ the structure linear maps of $L(X)$. Then we have

$$\begin{aligned}
&\big(s_2^{-1} \circ L(f+g)\big)(\overline{e},\ \overline{0}) \\
&= s_2^{-1} \circ L\big((f,\ g) \circ (i_1^2 + i_2^2)\big)(\overline{e}, \overline{0}) \\
&= s_2^{-1} \circ L((f,\ g)) \circ L(i_1^2 + i_2^2)(\overline{e},\ \overline{0}) \\
&= (L(f),\ L(g)) \circ s_2^{-1} \circ L(i_1^2 + i_2^2)(\overline{e},\ \overline{0}) \\
&= (L(f),\ L(g))\big((\overline{e}, \overline{0}),\ (\overline{e}, \overline{0}),\ -\frac{1}{2}\ H(2)\big) \\
&= L(f)(\overline{e}, \overline{0}) + L(g)(\overline{e}, \overline{0}) + (\lambda_2^L)_X\big(L(f)(\overline{e}, \overline{0}) \otimes L(g)(\overline{e}, \overline{0}) \otimes (-\frac{1}{2}\ H(2))\big)\ , \text{by } 1.8.5 \\
&= L(f)(\overline{e}, \overline{0}) + L(g)(\overline{e}, \overline{0}) + \frac{1}{2}\ (\lambda_2^L)_X\big(L(f)(\overline{e}, \overline{0}) \otimes L(g)(\overline{e}, \overline{0}) \otimes T(H(2))\big)\ , \text{by } 4.1.19 \\
&= L(f)(\overline{e}, \overline{0}) + L(g)(\overline{e}, \overline{0}) + \frac{1}{2}\ (\lambda_2^L)_X\big(L(f)(\overline{e}, \overline{0}) \otimes L(g)(\overline{e}, \overline{0}) \otimes [e_1, e_2]_M\big)
\end{aligned}$$

Thus we have the following relation:

$$\big(s_2^{-1} \circ L(f+g)\big)(\overline{e},\ \overline{0}) = L(f)(\overline{e},\overline{0})\ +_M\ L(g)(\overline{e},\overline{0}) \tag{5.2.5}$$

by using (5.2.3). In the rightmost term, $+_M$ refers to the group structure of $|L(X)|$, see (5.2.3).

4. *The n-ary operations*: let θ be a (formal) n-ary operation and $\theta^o : E \to E^{+n}$ be its corresponding morphism in $\mathcal{C}$ by the Yoneda's lemma. By 5.1.3, we know that θ^o can be seen as the sum (for the group structure $+$) of unary and binary operations in $\mathcal{C}(E,\ E^{+n})$ as follows:

$$\theta^o = \sum_{M,\,p=1}^{n} i_p^n \circ r_p^n \circ \theta^o +_M \sum_{1\leqslant k<l\leqslant n} i_{kl}^n \circ \iota_2^{Idc} \circ r_2(r_{kl}^n \circ \theta^o)$$

Let $a_1,\ldots,a_n \in |A|$, then we aim at giving an explicit expression of the following term:

$$\theta(a_1,\ldots,a_n) = |(\hat{a_1},\ldots,\hat{a_n}) \circ s_n^{-1} \circ L(\theta^o)|(\overline{e},\ 0)$$

By using (5.2.5), this gives the sums for the group structure $+$, given in (5.2.3), as follows:

$$\begin{aligned}
&\theta(a_1,\ldots,a_n)\\
&= \sum_{M,\,p=1}^{n} |(\hat{a_1},\ldots,\hat{a_n}) \circ s_n^{-1} \circ L(i_p^n) \circ L(r_p^n \circ \theta^o)|(\overline{e},\ \overline{0})\\
&\qquad +_M \sum_{1\leqslant k<l\leqslant n} |(\hat{a_1},\ldots,\hat{a_n}) \circ s_n^{-1} \circ L(i_{kl}^n) \circ L(\iota_2^{Idc} \circ r_2(r_{kl}^n \circ \theta^o))|(\overline{e},\ \overline{0})\\
&= \sum_{M,\,p=1}^{n} |(\hat{a_1},\ldots,\hat{a_n}) \circ i_p \circ L(r_p^n \circ \theta^o)|(\overline{e},\ \overline{0})\\
&\qquad +_M \sum_{1\leqslant k<l\leqslant n} |(\hat{a_1},\ldots,\hat{a_n}) \circ (i_k,\ i_l) \circ L(\iota_2^{Idc} \circ r_2(r_{kl}^n \circ \theta^o))|(\overline{e},\ \overline{0})\\
&= \sum_{M,\,p=1}^{n} |\hat{a_p} \circ L(r_p^n \circ \theta^o)|(\overline{e},\ \overline{0}) +_M \sum_{1\leqslant k<l\leqslant n} |(\hat{a_k},\ \hat{a_l}) \circ L(\iota_2^{Idc} \circ r_2(r_{kl}^n \circ \theta^o))|(\overline{e},\ \overline{0})\\
&= \sum_{M,\,p=1}^{n} |\hat{a_p} \circ L(\theta_p)|(\overline{e},\ \overline{0}) +_M \sum_{1\leqslant k<l\leqslant n} |(\hat{a_k},\ \hat{a_l}) \circ L(\iota_2^{Idc} \circ r_2(\theta_{kl}))|(\overline{e},\ \overline{0})\\
&= \sum_{M,\,p=1}^{n} \theta_p(a_p) +_M \sum_{1\leqslant k<l\leqslant n} |(\hat{a_k},\ \hat{a_l}) \circ L(\iota_2^{Idc} \circ r_2(\theta_{kl}))|(\overline{e},\ \overline{0})
\end{aligned}$$

However we get

$$L\big(\iota_2^{Idc} \circ r_2(r_{kl}^n \circ \theta^o)\big)(\overline{e},\ \overline{0}) = \Big((0,\ \overline{0}),\ (0,\ \overline{0}),\ \overline{r_2(\theta_{kl})(e)}\Big)\,,$$

by 4.6.13. Then we deduce that we have

$$\begin{aligned}
\theta(a_1,\ldots,a_n) &= \sum_{M,\,p=1}^{n} |\hat{a_p} \circ L(\theta_p)|(\overline{e},\ \overline{0}) +_M \sum_{1\leqslant k<l\leqslant n} |(\hat{a_k},\ \hat{a_l}) \circ L(\iota_2^{Idc} \circ r_2(\theta_{kl}))|(\overline{e},\ \overline{0})\\
&= \sum_{M,\,p=1}^{n} \theta_p(a_p) +_M \sum_{1\leqslant k<l\leqslant n} \lambda_2\big(\hat{a_k}(\overline{e},\ \overline{0}) \otimes \hat{a_l}(\overline{e},\ \overline{0}) \otimes \overline{r_2(\theta_{kl})(e)}\big)\,,\text{ by 1.8.5}\\
&= \sum_{M,\,p=1}^{n} \theta_p(a_p) +_M \sum_{1\leqslant k<l\leqslant n} \lambda_2\big(a_k \otimes a_l \otimes \overline{r_2(\theta_{kl})(e)}\big)
\end{aligned}$$

We turn it into sums for the abelian group structure of $|A|$. The leftmost term of the following relation is a sum for the group structure $+_M$. By induction, we have

$$\sum_{M,\, p=1}^{n} \theta_p(a_p) = \sum_{p=1}^{n} \theta_p(a_p) + \frac{1}{2} \sum_{1 \leqslant k < l \leqslant n} \lambda_2\big(\theta_k(a_k) \otimes \theta_l(a_l) \otimes [e_1, e_2]_M\big)$$

As $\lambda_2 \circ (\lambda_2 \otimes \lambda_1 \otimes id) = \lambda_2 \circ (\lambda_1 \otimes \lambda_2 \otimes id) = 0$ (because $AbOp(\mathcal{C})$ is a 2-step nilpotent operad), then we have

$$\begin{aligned}
\sum_{M,\, p=1}^{n} \theta_p(a_p) &= \sum_{p=1}^{n} \theta_p(a_p) + \frac{1}{2} \sum_{1 \leqslant k < l \leqslant n} \lambda_2\Big(\lambda_1(a_k \otimes \overline{\theta_k(e)}) \otimes \lambda_1(a_l \otimes \overline{\theta_l(e)}) \otimes [e_1, e_2]_M\Big) \text{, by (5.2.3)} \\
&= \sum_{p=1}^{n} \theta_p(a_p) + \frac{1}{2} \sum_{1 \leqslant k < l \leqslant n} \lambda_2\Big(a_k \otimes a_l \otimes \gamma_{1,1;2}\big(\overline{\theta_k(e)} \otimes \overline{\theta_l(e)} \otimes [e_1, e_2]_M\big)\Big) \\
&\qquad \text{, by one of the axioms of } AbOp(\mathcal{C})\text{-algebras}
\end{aligned}$$

By using (5.2.3), we obtain

$$\begin{aligned}
\theta(a_1, \ldots, a_n) &= \sum_{p=1}^{n} \Big(\lambda_1(a_p \otimes \overline{\theta_p(e)}) + \frac{1}{2}\, \lambda_2(a_p \otimes a_p \otimes H(\theta_p))\Big) \\
&\quad + \frac{1}{2} \sum_{1 \leqslant k < l \leqslant n} \lambda_2\Big(a_k \otimes a_l \otimes \gamma_{1,1;2}\big(\overline{\theta_k(e)} \otimes \overline{\theta_l(e)} \otimes [e_1, e_2]_M\big)\Big) \\
&\quad + \sum_{1 \leqslant k < l \leqslant n} \lambda_2\big(a_k \otimes a_l \otimes \overline{r_2(\theta_{kl})(e)}\big) \\
&= \sum_{p=1}^{n} \Big(\lambda_1(a_p \otimes \overline{\theta_p(e)}) + \frac{1}{2}\, \lambda_2(a_p \otimes a_p \otimes H(\theta_p))\Big) \\
&\quad + \frac{1}{2} \sum_{1 \leqslant k < l \leqslant n} \lambda_2\Big(a_k \otimes a_l \otimes \gamma_{1,1;2}\big(\overline{\theta_k(e)} \otimes \overline{\theta_l(e)} \otimes [e_1, e_2]_M\big)\Big) \\
&\quad + \sum_{1 \leqslant k < l \leqslant n} \lambda_2\Big(a_k \otimes a_l \otimes \big(\theta_{k,l}(e_1, e_2) -_M \big(\theta_k(e_1) + \theta_l(e_2)\big)\big)\Big), \text{ by (5.2.2)}
\end{aligned}$$

This proves the result. □

5.3 Application of the BCH formula to modules over a square ring

In this section, we give an example of application of the Baker-Campbell-Hausdorff type formula given in Theorem 5.2.3. First we recall the definition of a square ring and a module over a square ring introducted by Baues, Hartl and Pirashivili in [4]. Then we use the latter formula for expressing operations of any module over a given square ring from structure linear maps of algebras over a certain operad depending on the square ring.

The notion of square rings, respectively modules over a square ring, is the *quadratic* analogue of the notion of (classical) rings, respectively modules over a ring. These notions have been introduced and used by Baues, Hartl and Pirashvili in the context of metastable homotopy theory, and later in a series of papers by Baues and Muro in the study of other subjects, in particular of secondary operations in the homotopy of ring spectra. For example, it is observed in [4] that the endomorphism of the suspended projective plan $\Sigma\mathbb{R}P^2$, denoted by $End(\Sigma\mathbb{R}P^2)$, is a square ring (by a result of M.G Barrat) and the category of free modules over $End(\Sigma\mathbb{R}P^2)$ is identified with the homotopy category of Moore spaces in degree 2 whose single non-trivial homology group is of exponent 2 (see Theorem 8.1 of [4]).

Now we recall the definition of a square ring given in Definition 7.1 of [4] as follows:

Definition 5.3.1. A *square ring* $\underline{R}$ is a diagram

$$\underline{R} = \Big(R_e \xrightarrow{H} R_{ee} \xrightarrow{P} R_e\Big)$$

where

1. R_e is a (2-step nilpotent) group (whose law group is written additively) and it is a multiplicative monoid with unity denoted by 1. Moreover we have

$$r(s+s') = rs + rs'$$

2. R_{ee} is an abelian group endowed with an action of the multiplicative monoid $R_e \times R_e \times R_e^{op}$ on R_{ee}, denoted by $(r, r', s)x = (r, r')xs$, where $r, r', s \in R_e$ and $x \in R_{ee}$.

3. H and P are maps satisfying the following relations:

   (a) $(r + r')s = rs + r's + P\big((r, r')H(s)\big)$,
   (b) $H(r + r') = H(r) + H(r') + (r, r')H(2)$,
   (c) $H(rr') = (r, r)H(r') + H(r)r'$,
   (d) $P(x + x') = P(x) + P(x')$,
   (e) $P((r, r)xs) = rP(x)s$,
   (f) $\big(P(x), 1\big)x = \big(1, P(x)\big)x = yP(x) = 0$,
   (g) $P \circ H \circ P = 2P$,

   where $r, r', s \in R_e$ and $x, y \in R_{ee}$.

Remark 5.3.2. A square ring as defined in 5.3.1 is the same as a square ringoid as in Definition 3.1 of [4] with only one object by Lemma 7.4 of [4].

Notation 5.3.3. Let $\underline{R}$ be a square ring as in 5.3.1. We set $\overline{R} = Coker(P)$. For $r \in R_e$, we denote by $\overline{r}$ the equivalence class of r in $\overline{R}$.

Remark 5.3.4. Let $\underline{R}$ be a square ring as in 5.3.1. We observe that

- $\overline{R}$ is an ordinary ring,
- R_{ee} is a (left) $(\overline{R} \otimes \overline{R} \otimes \overline{R}^{op})$-module.

Let $\underline{R}$ be a square ring as in 5.3.1. Then we recall the definition of a module over $\underline{R}$, already given in Definition 7.7 of [4] as follows:

Definition 5.3.5. A *module over $\underline{R}$* is a group M (which we write additively) endowed with maps

$$M \times R_e \to M\,,\ (m,r) \longmapsto m.\,r$$
$$M \times M \times R_{ee} \longrightarrow M\,,\ (m,m',x) \longmapsto [m,m']_M.\,x$$

satisfying the following relations:

1. $m.\,1 = m, \quad (m.\,r).\,s = m.(rs), \quad m.\,(r+s) = m.\,r + m.\,s,$
2. $[m,m']_M.\,x$ is linear in m, m' and x,
3. $[m.\,r, m'.\,s]_M.\,x = [m,m']_M.\,\big((r,s)x\big)$ and $\big([m,m']_M.\,x\big).\,r = [m,m']_M.\,\big(x.\,r\big),$
4. $(m+m').\,r = m.\,r + m'.\,r + [m,m']_M.\,H(r),$
5. $m.\,P(x) = [m,m]_M.\,x,$
6. $[m,m']_M.\,T(x) = [m',m]_M.\,x,$
7. $[m,m']_M.\,x = 0$ if $m \in [M]$,

where $m,m' \in M$, $r,s \in R_e$ and $x \in R_{ee}$. Here $T = H \circ P - Id$ and $[M]$ denotes the subgroup of M generated by elements of the form $[m,m']_M.x$.

Definition 5.3.6. A morphism $f : M \to N$ of modules over $\underline{R}$ is a group homomorphism such that

$$f([m,m']_M.\,x) = [f(m), f(m')]_N.\,x \qquad \text{and} \qquad f(m.r) = f(m).r\,,$$

for $m,m' \in M$, $r \in R_e$ and $x \in R_{ee}$.

Notation 5.3.7. We denote by $Mod_{\underline{R}}$ the category of modules over the square ring $\underline{R}$. For $m \in M$, we denote by $\overline{m}$ the equivalence class of m in $\overline{M}$.

Then the category $Mod_{\underline{R}}$ has the following properties, already proved by M. Hartl and F. Goichot:

Proposition 5.3.8. *The category $Mod_{\underline{R}}$ is a semi-abelian variety, complete and cocomplete, and R_e is the free module over $\underline{R}$ of rank* 1*.*

Remark 5.3.9. The relation 7. of Definition 5.3.5 says that $[M]$ is a central subgroup of M. Moreover, for $m,m' \in M$, the commutator of m and m' is

$$m + m' - m - m' = [m',m]_M.\,H(2)$$

by relation 4. Hence M is a 2-*step nilpotent* group.

Remark 5.3.10. Now we explain the role of structure components of the square ring $\underline{R}$ on a module M over $\underline{R}$:

- the elements r of R_e encode quadratic unary operations $m \mapsto m.\,r$ on M,
- the elements x of R_{ee} encode bilinear operations $(m,m') \mapsto [m,m']_M.\,x$ on M,
- the map H assigns to every quadratic unary operation $m \mapsto m.\,r$, for $r \in R_{ee}$, its *cross-effect*

$$(m,m') \longmapsto (m+m').\,r - m.\,r - m'.\,r\,,$$

as being the bilinear operation defined by $H(r)$, see relation 4. of 5.3.5.

- the map P assigns to every bilinear operation the associated *squaring* operation, see relation 5 of 5.3.5.

Notation 5.3.11. Let M be a module over $\underline{R}$. We denote by $\overline{M}$ the quotient of M by $[M]$.

Remark 5.3.12. Let M be a module over $\underline{R}$. We clearly observe that $\overline{M}$ is a (right) $\overline{R}$-module.

Next we give an explicit expression of binary coproducts in $Mod_{\underline{R}}$, which has been already constructed by M. Hartl.

Proposition 5.3.13. *Let M and N be two modules over the square ring $\underline{R}$. Then the binary coproduct $M+N$ in $Mod_{\underline{R}}$ is the group $M \times N \times \big(\overline{M} \otimes \overline{N} \otimes_{\overline{R}\otimes\overline{R}} R_{ee}\big)$ with group law given by*

$$(m,n,u)+(m',n',u') = \big(m+m',\, n+n',\, u+u'+\overline{m'}\otimes\overline{n}\otimes H(2)\big)\,,$$

for $m,m' \in M$, $n,n' \in N$ and $u,u' \in \overline{M} \otimes \overline{N} \otimes_{\overline{R}\otimes\overline{R}} R_{ee}$. It is endowed with maps $\big(M+N\big)\times R_e \longrightarrow M+N$ given by

$$(m,n,u).\,r = \big(m.\,r,\, n.\,r,\, u.\overline{r} + \overline{m}\otimes\overline{n}\otimes H(r)\big)$$

and $\big(M+N\big)\times\big(M+N\big)\times R_{ee} \longrightarrow M+N$ given by

$$\big[(m,n,u),\,(m',n',u')\big]_{M+N}.\,x = \Big([m,m']_M.\,x,\; [n,n']_N.\,x,\; \overline{m}\otimes\overline{n'}\otimes x \;+\; \overline{m'}\otimes\overline{n}\otimes T(x)\Big)\,.$$

The universal property of the coproduct $M+N$ is the following:
Let P be a module over $\underline{R}$, let $f : M \to P$ and $g : N \to P$ be two morphisms in $Mod_{\underline{R}}$. Then the unique morphism $h : M+N \to P$ in $Mod_{\underline{R}}$ such that

$$h\circ i_M = f \quad \text{and} \quad h\circ i_M = g\,,$$

has the following explicit expression:

$$h\big(m,n,\overline{m'}\otimes\overline{n'}\otimes x\big) = f(m)+g(n)+[f(m'),g(n')]_P.\,x\,,$$

where $m,m' \in M$, $n,n' \in N$, $u,u' \in \overline{M}\otimes\overline{N}\otimes_{\overline{R}\otimes\overline{R}} R_{ee}$, $r \in R_e$ and $x \in R_{ee}$. Here $i_M : M \to M+N\, m \mapsto (m,0,0)$, respectively $i_N : N \to M+N\,, n \mapsto (0,n,0)$, is the injection of the first, respectively second, summand.

Proof. It is a straightforward verification. □

Remark 5.3.14. Let M and N be two modules over $\underline{R}$. The inverse of an element (m,n,u) in $M+N$ for the group structure of $M+N$ is $(-m,-n,-u+\overline{m}\otimes\overline{n}\otimes H(2))$.

Thus it is possible to have an explicit expression for the second cross-effect of the identity functor of $Mod_{\underline{R}}$, as follows:

Corollary 5.3.15. *Let M and N be two modules over $\underline{R}$. Then we have*

$$Id_{Mod_{\underline{R}}}(M|N) = \overline{M}\otimes\overline{N}\otimes_{\overline{R}\otimes\overline{R}} R_{ee}$$

and $\iota_2^{Id_{Mod_{\underline{R}}}} : Id_{Mod_{\underline{R}}}(M|N) \to M+N\,,\; u \mapsto (0,0,u)$ is the kernel of the comparison morphism $\widehat{r_2^{Id_{Mod_{\underline{R}}}}} : M+N \to M\times N$.

Proof. It is a direct consequence of 5.3.13. □

Let M be a module over $\underline{R}$. We now give an explicit expression of the morphism $c_2^M : Id_{Mod_{\underline{R}}}(M|M) \to M$, defined in 1.2.8.

Proposition 5.3.16. *For $m\, m' \in M$ and $x \in R_{ee}$, we have*

$$c_2^M\big(\overline{m} \otimes \overline{m'} \otimes x\big) = \big(\nabla_M^2 \circ \iota_2^{Id_{Mod_{\underline{R}}}}\big)\big(\overline{m} \otimes \overline{m'} \otimes x\big) = [m, m']_M . x \in [M]$$

where $[M]$ is the (central) subgroup of M generated by elements of the form $[m, m']_M . x$.

Proof. It is a direct consequence of 5.3.13. □

Remark 5.3.17. By 5.3.16 and by 1.4.4, $\overline{M}$ is the abelianization $Ab^{Mod_{\underline{R}}}(M)$ of M. Moreover the abelian core $Ab(Mod_{\underline{R}})$ of $Mod_{\underline{R}}$ is exactly $Mod_{\overline{R}}$, the category of (right) $\overline{R}$-modules.

From 5.3.17, we deduce that the second cross-effect of the identity functor $Id_{Mod_{\underline{R}}} : Mod_{\underline{R}} \to Mod_{\underline{R}}$ is bilinear, because the abelianization functor $Ab^{Mod_{\underline{R}}} : Mod_{\underline{R}} \to Ab(Mod_{\underline{R}}) = Mod_{\overline{R}}$ is linear by 1.4.4 and by 1.2.9. Thus the identity functor $Id_{Mod_{\underline{R}}}$ is quadratic by 1.2.13. Hence it means that the category $Mod_{\underline{R}}$ has the following additional property, already found by M. Hartl and F. Goichot:

Proposition 5.3.18. *The category $Mod_{\underline{R}}$ is a 2-step nilpotent variety.*

Thus it says that it is possible to apply Theorem 5.2.3 for $\mathcal{C} = Mod_{\underline{R}}$ and $E = R_e$. In this case, $E^{ab} = Ab^{Mod_{\underline{R}}}(R_e) = \overline{R}$ by 5.3.17 and by 5.3.7. Moreover we have

$$Id_{Mod_{\underline{R}}}(R_e|R_e) = \overline{R} \otimes \overline{R} \otimes_{\overline{R}\otimes\overline{R}} R_{ee} \cong R_{ee}\,, \tag{5.3.1}$$

by 5.3.15 and 5.3.4.

<u>**Assumption**</u>: from now on, we assume that the 2-*divisibilty condition* holds as in Chapter 5, section 2. Here it means that $\frac{1}{2} \in \overline{R}$.

Now we determine the 2-step nilpotent symmetric unitary operad $AbOp(Mod_{\underline{R}})$ in the monoidal category of $\mathbb{Z}[\frac{1}{2}]$-modules, given in (4.2.3):

$$AbOp(Mod_{\underline{R}})(1) = Mod_{\overline{R}}\big(\overline{R},\ \overline{R}\big) \cong \overline{R}\,,$$
$$AbOp(Mod_{\underline{R}})(2) = Mod_{\overline{R}}\big(\overline{R},\ Id_{Mod_{\underline{R}}}(R_e|R_e)\big) \cong Id_{Mod_{\underline{R}}}(R_e|R_e) \cong R_{ee}\,, \text{by } (5.3.1)\,.$$

Here we set $|AbOp(Mod_{\underline{R}})|(1) = \overline{R}$ and $|AbOp(Mod_{\underline{R}})|(2) = R_{ee}$. Then $|AbOp(Mod_{\underline{R}})|$ is also a 2-step nilpotent linear symmetric unitary operad whose unity is $\overline{1} \in \overline{R}$. Its structure linear composition maps are entirely determined by those of the (linear) operad $AbOp(Mod_{\underline{R}})$. More precisely, the composition map

$$\gamma_{1;1} : |AbOp(Mod_{\underline{R}})|(1) \otimes |AbOp(Mod_{\underline{R}})|(1) \to |AbOp(Mod_{\underline{R}})|(1)$$

is the multiplicative law of the ring $\overline{R}$, the linear map

$$\gamma_{2;1} : |AbOp(Mod_{\underline{R}})|(2) \otimes |AbOp(Mod_{\underline{R}})|(1) \to |AbOp(Mod_{\underline{R}})|(2)$$

is the (right) action of $\overline{R}$ on R_{ee} (see 5.3.4), and the following composition map

$$\gamma_{1,1;2} : |AbOp(Mod_{\underline{R}})|(1) \otimes |AbOp(Mod_{\underline{R}})|(1) \otimes |AbOp(Mod_{\underline{R}})|(2) \to |AbOp(Mod_{\underline{R}})|(2)$$

is the (left) action of $\overline{R} \otimes \overline{R}$ on R_{ee} (see also 5.3.4).

Let $\underline{R}$ be a square ring. Next we know that each $|AbOp(Mod_{\underline{R}})|$-algebra A can be endowed with a structure of modules over $\underline{R}$ via the Baker-Campbell-Hausdorff formula given in 5.2.3. Denote by $i_1^2 : R_e \rightarrowtail R_e^{+2}, r \mapsto (r, 0, 0)$, respectively $i_2^2 : R_e \to R_e^{+2}, r \mapsto (0, r, 0)$, the injection of the first, respectively second summand. Here $e = 1$, $e_1 = i_1^2(1)$ and $e_2 = i_2^2(1)$.

- The group structure on A: the group law $+_A$ of A is given by

$$a +_A a' = a + a' + \frac{1}{2}\,\lambda_2^A\big(a \otimes a' \otimes H(2)\big)\,,$$

for $a, a' \in A$. The group law of the variety $Mod_{\underline{R}}$ may be seen as a binary operation θ, in which we observe that the unary operations θ_1 and θ_2 are both the identity. It implies that $H(\theta_1) = H(\theta_2) = 0$. Now we check that $[e_1, e_2]_M = -H(2)$. For this we calculate the commutator of e_1 and e_2 in R_e^{+2} as follows:

$$\begin{aligned} e_1 + e_2 - (e_2 + e_1) &= (1,0,0) + (0,1,0) - \big((0,1,0) + (1,0,0)\big) \\ &= (1,1,0) - (1,1,\overline{1} \otimes \overline{1} \otimes H(2)) \\ &= (1,1,0) - (1,1,H(2)) \\ &= (1,1,0) + (-1,-1,H(2) + \overline{1} \otimes \overline{1} \otimes H(2))\,, \text{by } 5.3.14 \\ &= (1,1,0) + (-1,-1,2H(2)) \\ &= (0,0,2H(2) - \overline{1} \otimes \overline{1} \otimes H(2)) \\ &= (0,0,H(2)) \end{aligned}$$

Hence we have here $[e_1, e_2]_M = H(2)$. Then we get

$$\gamma_{1,1;2}\big(\overline{\theta_1(1)} \otimes \overline{\theta_2(1)} \otimes [e_1, e_2]_M\big) = \gamma_{1,1;2}\big(\overline{1} \otimes \overline{1} \otimes H(2)\big) = (1,1)H(2) = H(2)\,.$$

Moreover the term $\theta_{12}(e_1, e_2) -_M (\theta_1(e_1) +_M \theta_2(e_2))$ becomes

$$\theta_{12}(e_1, e_2) -_M (\theta_1(e_1) +_M \theta_2(e_2)) = (e_1 + e_2) - (e_1 + e_2) = (0,0,0)\,,$$

as desired.

- The unary operations encoded by R_e: the action of R_e on A is given by

$$a.\,r = \lambda_1^A(a \otimes \overline{r}) + \frac{1}{2}\,\lambda_2^A(a \otimes a \otimes H(r))\,,$$

for $a \in A$ and $r \in R_e$.

- The binary operations encoded by R_{ee}: these are given as follows:

$$[a, a']_A.\, x = \lambda_2^A(a \otimes a' \otimes x)\,,$$

for $a, a' \in A$ and $x \in R_{ee}$. For $x \in R_{ee}$, we consider the binary operation θ^x of the variety $Mod_{\underline{R}}$ such that $\theta^x(a, a') = [a, a'].\, x$. It is clear that the unary operations θ_1^x and θ_2^x are trivial (see the relations in 5.3.5), and that $\theta_{12}^x = \theta^x$.

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