ROBUST OPTIMIZATION FOR DISCRETE STRUCTURES AND NON-LINEAR IMPACT OF UNCERTAINTY

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Robust Optimization for Discrete structures and Non-linear impact of Uncertainty

Abstract

We address decision problems under uncertain information with non-linear structures of parameter variation, and devise solution methods in the spirit of Bertsimas and Sim's Γ-Robustness approach. Furthermore, although the non-linear impact of uncertainty often introduces discrete structures to the problem, for tractability, we provide the conditions under which the complexity class of the nominal model is preserved for the robust counterpart.

We extend the Γ-Robustness approach in three avenues. First, we propose a generic case of non-linear impact of parameter variation, and model it with a piece-wise linear approximation of the impact function. We show that the subproblem of determining the worst-case variation can be dualized despite the discrete structure of the piece-wise function. Next, we built a robust model for the location of new housing where the non-linearity is introduced by a choice model, and propose a solution combining Γ-Robustness with a scenario-based approach. We show that the subproblem is tractable and leads to a linear formulation of the robust problem. Finally, we model the demand in a Location Problem through a Poisson Process inducing, when demands are uncertain, non-linear structures of parameter variation. We propose the concept of *Nested Uncertainty Budgets* to manage uncertainty in a tractable way through a hierarchical structure and, under this framework, obtain a subproblem that includes both continuous and discrete deviation variables.

Résumé

L'objectif de cette thèse est de proposer des solutions efficaces à des problèmes de décision qui ont un impact sur la vie des citoyens, et qui reposent sur des données incertaines. Au niveau des applications, nous nous intéressons à deux problèmes de localisation qui ont un impact sur l'espace public, notamment la localisation de nouveaux logements, et la localisation de vendeurs mobiles dans l'espace urbain. Les problèmes de localisation ne sont pas un sujet récent dans la littérature, toutefois, pour ces deux problèmes qui reposent sur des modèles de choix pour le comportement d'achat des consommateurs, l'incertitude dans le modeèle génère un cas spécial qui permet d'étendre la littérature sur l'Optimisation Robuste. Les contributions de cette thèse peuvent s'appliquer à divers problèmes génériques d'optimisation.

Les décisions prises sous données incertaines ont suscité l'intérêt de la communauté des chercheurs opérationnels car l'incertitude est présente dans chaque aspect de la vie quotidienne, que ce soit les prévisions métrologiques, la valeur des stocks, des taux de change ou même l'attente du prochain train. Cette modélisation de l'incertitude est aussi prise en compte de façon croissante par les entreprises dans la planification de leurs opérations.

Plusieurs approches sont proposées dans la littérature pour traiter l'incertitude dans les problèmes de prise de décision. Les plus populaires dans le champ de la Recherche Opérationnelle sont l'optimisation stochastique et l'optimisation robuste. Dans cette thèse, nous partons de l'état de l'art de l'Optimisation Robuste,

qui est présenté dans le Chapitre 2, notamment le paradigme de budget d'incertitude introduit par Bertsimas and Sim [18]. Nous montrons que pour certains types de problèmes, la structure particulière du modèle d'incertitude peut rendre la solution obtenue par [18] irréalisable, ce qui justifie le développement d'une approche spécifique. Dans la littérature, certaines extensions du travail de [18] existent déjà: problèmes avec des structures dispersées (Poss [60]), données incertaines non distribués uniformément (Büsing and D'Andreagiovanni [23]), et incertitude sur le second membre des contraintes (Gabrel and Murat [35]).

Une structure particulière nouvelle étudiée dans cette thèse est la possibilité d'observer un impact non-linéaire de la variation de paramètres incertains. Quand les coefficients suivent une fonction non-linéaire des paramètres incertains, nous concevons une méthode de résolution dans l'esprit de [18] basée sur une approximation linéaire par morceaux de la fonction non linéaire. Nous identifions aussi les conditions sous lesquelles le degré de complexité du problème nominal est préservé pour sa contrepartie robuste.

1. Optimisation Robuste avec impact non-linéaire de la variation des données

Dans le chapitre 3, nous adaptons le paradigme dit de Γ-Robustesse développé dans [18] au cas où la variation de certains paramètres impactent les coefficients d'une contrainte de façon non-linéaire. C'est le cas si on considère la demande en fonction du prix, la Valeur Nette Présente (VNP) d'un projet d'investissement comme une fonction non-linéaire du taux d'actualisation, ou la probabilité de choix dans un modèle de choix de type MNL (*MultiNomial Logit*) comme le rapport des exponentielles des utilités (nous détaillerons plus formellement ces situations à la fin de cette introduction).

Dans l'approche classique de Γ-Robustesse, pour certaines contraintes i, les coefficients a_{ij} sujets à l'incertitude sont décrits par $a_{ij} = \bar{a}_{ij} + \hat{a}_{ij} z_{ij}$, où $\bar{a}_{ij}$ est la valeur nominale du coefficient, $\hat{a}_{ij}$ est sa variation maximale, et $z_{ij} \in [-1, 1]$. Le cadre développé dans [18] génère une contrepartie robuste qui garantit la réalisabilité de la solution robuste quelle que soit la variation des coefficients satisfaisant un niveau de conservatisme donné. Cependant, il ne permet pas de saisir directement certains types de problèmes ou de structures d'incertitude. Dans la littérature, nous pouvons notamment identifier trois familles d'extensions qui traitent (i) des variations de coefficients dépendant du temps [15, 51, 60], (ii) de la distribution de l'intervalle d'incertitude [23, 30], et (iii) de l'incertitude du membre de droite (RHS) des contraintes [20, 35, 55]; les détails peuvent être trouvés dans la section 2.3.

Dans ce chapitre, nous conservons le principe de Γ-Robustesse qui consiste à protéger contre l'irréalisabilité sur une variation contrôlée des paramètres. Cependant, nous l'étendons aux cas où les coefficients sont des fonctions non-linéaires de données potentiellement sujettes à variations. Par exemple, dans les problèmes de localisation d'installations avec capacités, les demandes D_j des sources de demande apparaissent dans le côté gauche des contraintes de capacité. S'il existe une incertitude sur le prix de marché du produit, alors nous devrions considérer $D_j = f_j(p_j)$ où p_j est le prix du produit à la source de demande j (les sources de demande peuvent être situées dans des pays différents) et f_j est généralement non-linéaire, concave décroissante. En outre, considérons un problème de *capital budgeting* avec incertitude sur les valeurs des projets (voir Meier et al. [54] pour une étude sur ce sujet). Le problème de *capital budgeting* consiste à sélectionner un portefeuille de projets d'investissement maximisant la valeur nette totale du portefeuille tout en respectant des contraintes de ressources. Ensuite, la Valeur Nette Présente (VNP) d'un projet j est la somme des cash-flows F_t^j actualisés au cours des années $t = 0, \dots, T$, c'est-à-dire, $VNP_j = \sum_{t=0}^{T} \frac{F_t^j}{(1+r_j)^t}$ où r_j est le taux d'actualisation pour le projet j. Ce taux d'actualisation dépend de plusieurs facteurs, certains sont exogènes comme le taux d'intérêt du pays où se situe le projet j, d'autres sont endogènes comme le niveau de rendement que le décideur veut compenser pour son risque (les différents projets étant plus ou moins risqués). Dans tous les cas, r_j est susceptible d'être imprécis ou incertain et la VNP est clairement non-linéaire en r_j (notons que, comme mentionné dans [18], une fonction objectif soumise à l'incertitude peut être transformée en contrainte). Dans les modèles de choix Logit Multinomial (MNL) (McFadden [53]), la probabilité p_{ij} que le client i choisisse le produit j est exprimée par $p_{ij} = e^{u_{ij}}/(\sum_k e^{u_{ik}})$ qui est une fonction non-linéaire des utilités u_{ij}, connus pour être difficiles à calibrer précisément (voir [31] et le Chapitre 4 pour une application à la localisation de programmes de nouveaux logements). Un autre exemple pourrait être des fonctions de consommation non-linéaires dépendant de températures incertaines.

La section 3.2 décrit le problème robuste générique et discute de sa capacité de résolution, car le problème robuste ne peut pas être transformé en un programme linéaire. Dans la section 3.3, nous approchons la fonction non-linéaire de variation du paramètre par une approximation binaire. Une fonction linéaire par morceaux plus raffinée est proposée dans la section 3.4. Nous montrons que l'approximation de la contrepartie robuste peut être résolue en utilisant la dualisation LP pour le sous-problème de recherche de la pire variation. La section 3.5 est dédiée aux expériences numériques et analyse les performances de notre approche de robustesse étendue et de l'approximation par morceaux. La section 3.6 conclut le chapitre.

Nous conduisons des expériences numériques sur des instances des problèmes d'affectation généralisée et du sac-à-dos (*knapsack*). L'objectif est d'analyser le compromis entre la réalisabilité et la valeur de l'objectif pour la solution robuste de l'approximation linéaire par morceaux, par rapport à la solution nominale et à une approximation binaire plus simple. Malgré l'approximation par morceaux, la solution robuste se révèle être réalisable sur les 2700 instances testées, sans beaucoup dégrader la valeur de l'objectif.

2. Localisation robuste de nouveaux logements basée sur un modèle de choix

Dans le chapitre 4, nous examinons la question du choix d'un sous-ensemble de sites pour la construction de nouveaux ensembles de logement, visant à maximiser la satisfaction des acheteurs potentiels. L'offre de logements est un problème complexe qui se pose dans des contextes multiples et variés. Driant [28], Gilbert [39], et Sandoval [66] fournissent tous des exemples de situations problématiques liées à l'offre de logements. En France, l'offre de logements est une question importante avec un déficit de 800 000 à 1 million de logements en 2012 selon Driant. Alors que les politiques de logement consistent principalement en la construction de nouveaux logements, [28] traite de l'existence d'un décalage entre la demande et l'offre. L'étude plus récente de Reed [64] confirme que la France montre une diminution constante du nombre de mises en chantier de logements neufs, avec une « tendance à la baisse progressive mais soutenue à long terme » (Institut National de la Statistique et des Études Économiques, (INSEE), 2015). En même temps, les investissements dans les bâtiments résidentiels neufs ont augmenté : en France, l'INSEE et Eurostat, le bureau statistique de l'Union européenne, affichent un nombre de logements neufs relativement stable et des coûts de construction en constante augmentation.

On constate par ailleurs une prolifération de logements inadaptés, encouragés par la hausse des prix et les politiques sociales qui exacerbent le problème initial, rendant nécessaire une meilleure solution concernant le type et le lieu de construction de nouveaux logements [28]. La question du choix des emplacements pour de nouveaux ensembles résidentiels dans un réseau de sites possibles peut être considérée comme un problème plus général de localisation. La répartition des clients vers les sites sélectionnés et les règles qui régissent cette répartition jouent un rôle fondamental dans la solution du problème.

Contrairement au secteur de la chaîne d'approvisionnement (*supply chain*) où le décideur peut affecter des sites d'approvisionnement à des clients qui sont généralement indifférents au *sourcing*, dans le développement de nouveaux logements, les clients choisissent eux-mêmes leur lieu d'achat, ce qui conduit naturellement à un modèle de choix basé sur les utilités des consommateurs (voir le document de référence de Guadagni and Little [44] pour le modèle de choix Logit Multinomial (MNL), suivant le travail précurseur de McFadden [53]).

Les modèles de choix ont déjà été utilisés dans le contexte de l'offre de logements (voir [22, 58, 59]), mais pas dans un cadre d'optimisation à notre connaissance. Ce chapitre présente la première approche dans le secteur du logement neuf combinant un modèle de choix pour le comportement des consommateurs, un modèle d'optimisation pour la sélection des sites accueillant les programmes de nouveaux logements, et une approche robuste pour traiter l'incertitude.

Notons que, bien que nous avons conduit des expériences numériques sur une zone géographique particulière, la région parisienne, notre modèle peut facilement s'adapter à d'autres contextes géographiques. Le principal ajustement consisterait à calibrer les fonctions d'utilité avec éventuellement de nouveaux critères pour tenir compte des caractéristiques locales, régionales ou nationales spécifiques. La question de l'étalonnage des fonctions d'utilité dans un modèle de choix pour le logement a déjà été étudiée dans la littérature (par exemple [22, 58, 59]).

Le chapitre est structuré comme suit. La section 4.2 présente le problème d'optimisation et le modèle de choix sous-jacent pour la probabilité d'un client à acheter dans un lieu donné. La section 4.3 décrit la version robuste du problème de localisation, où l'incertitude repose d'abord sur les volumes de demande, puis sur les préférences des clients. La section 4.4 présente les résultats numériques d'une étude réalisée en région parisienne et fournit une analyse des résultats ainsi que des perspectives managériales pour le développement de nouveaux logements.

L'allocation des demandes aux sites sélectionnés est modélisée par un modèle de choix, basé sur la distance à l'emplacement, les prix de l'immobilier et les revenus. Nous étudions deux versions robustes du problème de localisation optimale, où l'incertitude repose sur les volumes de demande pour le premier et sur les préférences des clients pour le second. Dans les deux cas, les paramètres soumis à l'incertitude apparaissent à la fois dans la fonction objectif et dans les contraintes.

Le second modèle robuste aborde la non-linéarité, introduite par le modèle de choix, combinant une approche basée sur des scénarios centrés sur les préférences des clients, et une approche de type Γ-Robustesse qui limite le nombre de clients qui peuvent s'écarter de leur scénario nominal (pour un scénario davantage centré sur le prix, ou sur la distance). Nous montrons que le sous-problème de trouver la pire variation des paramètres incertains est traitable et conduit à des formulations linéaires du problème robuste. Des expériences numériques conduites sur des instances de la région parisienne montrent que la perte de valeur moyenne de la solution robuste est raisonnablement faible par rapport à la solution optimale d'instances où on a simulé des déviations des paramètres dans le budget d'incertitude. Nous tirons également des conclusions sur question du développement de nouveaux logements.

3. Une approche de robustesse imbriquèe pour le problème de localisation avec capacités et demandes dépendantes des localisations choisies.

Dans le chapitre 5, nous examinons la localisation d'un ensemble limité de vendeurs mobiles desservant des clients selon le principe du « premier arrivé, premier servi », où la demande à chaque emplacement suit un processus de Poisson. De plus, chaque vendeur a sur place une capacité limitée (qui peut être liée à la taille du camion ou du véhicule) et où l'objectif est de maximiser le revenu total de l'entreprise qui emploie les vendeurs. Cette problématique peut s'appliquer aux camions de nourriture, vendeurs de crème glacée et autres agents mobiles tels que les stands promotionnels / publicitaires. Bien que tous les vendeurs puissent présenter les caractéristiques précitées, c'est-à-dire la capacité limitée et le principe « premier arrivé, premier servi » (par exemple dépanneurs, magasins de détail / vêtements), nous nous référons aux vendeurs mobiles car ils ont la possibilité de reconfigurer leur réseau, ce qui leur permet d'optimiser le problème d'emplacement des installations lorsque cela est nécessaire, par exemple, chaque fois qu'il y a un changement important dans les demandes. Les entreprises plus traditionnelles, telles que les magasins de détail, n'ont pas la possibilité de reconfigurer leur réseau; elles doivent donc prendre leurs décisions au niveau stratégique, avec des informations agrégées qui pourraient remettre en cause nos hypothèses opérationnelles. Dans ce cas, des modèles de choix comme celui utilisé au chapitre 3 peuvent être utilisés (voir Benati and Hansen [10] pour le problème de capture maximale avec un modèle de choix pour modéliser le comportement du consommateur).

La question de la localisation de vendeurs mobiles est abordée dans Lucan et al [52]. Les auteurs examinent la situation d'un point de vue social, où les vendeurs ambulants sont des camions verts (distribuant des produits frais entiers dans des zones marginalisées). Leur étude révèle que les vendeurs mobiles se regroupent autour de zones à forte fréquentation, qui ne sont pas nécessairement proches de la cible visée. Bien que Lucan et al soient incapables de dire si les camions verts atteignent l'objectif d'atteindre les zones les plus vulnérables, un modèle d'optimisation, comme celui qui est décrit dans les sections suivantes, pourrait offrir une option pour localiser les camions pour un tel service. Etonnamment, nous n'avons trouvé aucun autre document traitant spécifiquement de la localisation optimale de vendeurs mobiles. L'approche développée dans ce chapitre peut donc être considérée comme un nouveau problème.

En considérant la maximisation des revenus, nous proposons de calculer le revenu de chaque emplacement de cette manière

$$\sum_{i=1}^{m} \pi_{ij}\kappa_{ij}K_j y_j$$

où π_{ij} est le revenu moyen (ou profit) par source de demande i à l'emplacement j, κ_{ij} est la fraction de capacité du site j consommée par la source de demande i (qui sera définie dans la section suivante en fonction des taux d'arrivée dans le processus de Poisson).

Notons que $\sum_i \pi_{ij}\kappa_{ij}K_j$ est le revenu total (ou profit) capturé par le site j, si j est ouvert. La fonction objectif est alors le revenu total généré par les vendeurs mobiles dans les emplacements sélectionnés. Nous définissons le revenu π_{ij} comme dépendant à la fois de la source de demande i et de l'emplacement j pour inclure toutes les situations possibles: i) $\pi_{ij} = \pi_{i\cdot}$ si chaque fournisseur mobile propose le même ensemble de produits (par exemple, les habitudes de consommation, les paniers moyens des différentes sources i, mais ne varient pas sur les emplacements possibles j); et le cas général ii) π_{ij} si les deux sources de demande i et les emplacements j sont différenciés (on peut penser aux camions transportant des marchandises différenciées selon l'emplacement, en fonction de la zone géographique).

Étant donné que nous nous concentrons sur les fournisseurs de services mobiles, nous considérons que la demande n'est déterminante que dans la mesure où elle peut influencer le taux d'arrivée des clients, car le vendeur peut changer d'emplacement lorsque les taux baissent. En outre, leur capacité de stockage est limitée. Ces deux considérations nous donnent l'hypothèse d'une consommation à pleine capacité.

Dans le reste du chapitre, nous proposons un modèle nominal pour le problème (en substance, nous caractériserons la fraction κ_{ij} de capacité du site j consommée par les clients de la source i), suivi d'une formulation robuste pour prendre en compte des perturbations des taux d'arrivée. Nous présentons des expériences numériques pour évaluer la performance du modèle robuste, et comment il se compare au modèle nominal. Nous terminons le chapitre par des conclusions et des commentaires sur d'autres axes de recherche.

Le problème robuste étudié montre des structures non-linéaires dues aux demandes qui suivent un processus de Poisson, ce qui produit des fonctions fractionnaires dans l'objectif (somme de ratios). Nous abordons la non-linéarité par l'approche de Charnes and Cooper [25] pour les problèmes fractionnaires, puis appliquons Bertsimas and Sim au programme linéaire.

En raison de la non-linéarité, l'implémentation directe de l'approche de Bertsimas and Sim au problème conduirait à une solution excessivement conservatrice, car l'incertitude ne peut être gérée que localement pour chaque emplacement potentiel, sans limiter le nombre total de sites impactés. Nous introduisons donc un nouveau concept de *budgets d'incertitude imbriqués* pour résoudre ce problème et proposons une méthode pour gérer l'incertitude globale et locale de manière hiérarchique dans le réseau. Nous obtenons un problème qui inclut à la fois des variables d'incertitude continues et discrètes.

Notons que bien que les chapitres 4 et 5 traitent d'applications spécifiques (la localisation de nouveaux logements et de vendeurs mobiles), les deux problèmes ont en commun l'impact non-linéaire des variations des paramètres, et des variables discrètes de déviation du coefficient autour de sa valeur nominale. Ces approches peuvent être facilement étendues à d'autres problèmes de localisation avec une structure similaire. Nous concluons avec de possibles voies de recherche pour l'avenir.

4. Conclusion

Dans le chapitre final, nous présentons les contributions de la thèse à la littérature, ainsi qu'un résumé des conclusions de chaque chapitre et leur impact individuel sur l'état de l'art sur l'Optimisation Robuste. Bien qu'il existe de nombreuses approches pour gérer l'incertitude, même dans le paradigme de l'Optimisation Robuste, nous avons montré que la structure particulière de cette incertitude peut avoir un effet important sur la robustesse de la solution si la méthode choisie n'est pas adaptée aux caractéristiques du problème. En particulier, nous avons montré que la solution d'une approche générale (en substance, Bertsimas and Sim [18]) peut produire des solutions irréalisables aux problèmes d'impact non linéaire de la variation des données.

Nous avons étendu l'approche classique de Γ-Robustesse aux problèmes d'optimisation avec un impact non-linéaire sur les coefficients des contraintes et la fonction objectif. Nous avons proposé une contrepartie robuste pour les problèmes où l'incertitude suit une fonction continue monotone non-linéaire basée sur une approximation par une fonction linéaire par morceaux. Dans le chapitre 3, nous avons montré que la contrepartie robuste de cette approximation reste un programme linéaire. Nous avons effectué des expériences numériques pour évaluer la performance de notre méthode sur les fonctions racine carrée et inverse-quadratiques. D'après nos constatations sur les cas des problèmes d'affectation généralisée et des problèmes de type knapsack, la performance de la solution robuste dépend de la structure du problème, en particulier de la densité de la matrice de contraintes. De plus, l'approximation linéaire par morceaux de la fonction non-linéaire a fourni des solutions robustes systématiquement réalisables sur l'ensemble des instances testées, ce qui valide empiriquement l'approche.

Les applications spécifiques de localisation de nouveaux logements et de vendeurs mobiles, développées chacune dans un chapitre différent, partagent cette caractéristique de non-linéarité du problème robuste, dûe au modèle de répartition des clients entre les sites (modèle de choix ou processus de Poisson donnant des formes fractionnaires). L'approche robuste développée pour chacune de ces applications est construite sur mesure pour l'application concernée, mais participe à la contribution générale de la thèse sur les cas d'impact non-linéaire de la variation de données incertaines dans les problèmes de décision.

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1 Introduction

The purpose of this doctoral dissertation is to propose solutions to relevant decision problems that have an impact in the lives of the common citizen. For this purpose, we address two location problems with opportunities of shaping public space, namely the location of new housing developments, and of mobile vendors in a city. Location problems are not new to the literature, however, for the two problems studied we find that introducing uncertainty in the model generates a special case, formerly overlooked, offering an opportunity to expand the literature on Robust Optimization and propose generic contributions that can be applied on various optimization problems.

Decisions that have to be made under uncertain information have gathered the interest of the research community, for uncertainty is in every aspect of normal life; from weather forecasts, the value of stocks, currency exchange rates, to the waiting time for the next train, uncertainty has an important impact on everyday life. Furthermore, there are problems specifically addressed by the operations research community because of the interest of the private sector in solving decision problems under uncertainty.

Several methods have been proposed in the literature to deal with uncertainty in decision problems, the most popular in the operations research field being stochastic optimization, and robust optimization. Both approaches have strengths and drawbacks which make them more or less appropriate to solve a specific type of problem. For example, Stochastic optimization relies on a probability distribution of uncertainty to find a solution that optimizes an expected value; the main criticism to this approach is that probability distributions are hard to recognize and calibrate, moreover, even when properly identified, they can change drastically and suddenly.

Robust optimization on the other hand, relies on uncertainty sets, setting ranges in which the uncertain parameters may vary without any associated distribution of probabilities, to find a solution that optimizes the worst case scenario. The main criticism in this case is that this worst case scenario is unlikely to occur, and the solution is thus unjustly penalized. It has been argued, e.g. in [37], that some applications do need to be protected against the worst case, making a robust solution desirable for these problems. Furthermore, the works of Ben-Tal and Nemirovski [12] and Bertsimas and Sim [18] address the over-conservatism issue and offer alternatives for dealing with it. To our knowledge, only the works by Büsing and D'Andreagiovanni [23] and Düzgün and Thiele [30] have addressed a kind of probability distribution in a deterministic way under [18]'s paradigm.

For this dissertation, we further the state of the art on Robust Optimization, which is presented in Chapter 2. Particularly, we focus on the paradigm introduced in [18] to solve problems with special structures of uncertainty. Although these problems could be addressed by Bertsimas and Sim's general approach, we show that for some uncertainty models, their particular structures could render the solution obtained by [18] infeasible. For this reason, we believe these problems warrant the development of an approach tailored to their characteristics.

Bertsimas and Sim manage to propose a general framework capable of handling uncertain problems without increasing significantly their complexity, i.e. the robust counterparts remain of the same class of complexity as the nominal ones. While the approach in [18] is very powerful, achieving good results in most cases, it is not designed to handle special structures of uncertainty that may be present in particular problems. In the literature we can identify efforts to expand [18]'s work to better cover some of these cases, e.g. problems with sparse structures [60], uncertainty sets that are not uniformly distributed [23], and right-hand uncertainty [35].

A particular structure that has received little to no attention has been the possibility of observing non-linear impact of uncertain parameters. We focus our research on problems where the problem's coefficients follow explicitly a non-linear function of uncertain parameters and, according to the problem's characteristics, devise a solution method in the spirit of [18]. Furthermore, although the non-linear structure of uncertainty often leads to discrete structures of Bertsimas and Sim's protection function, we provide the conditions under which the class of the nominal problem is preserved for the robust counterpart.

In Chapter 3, we extend the Γ-Robustness approach proposed for Linear Programs to the generic case of non-linear impact of parameter variation. We study a piece-wise linear approximation of the impact function of the uncertain parameter, and show that the subproblem of determining the worst-case variation can still be dualized despite the discrete structure of the piece-wise linear function.

We conduct numerical experiments on Generalized Assignment and Knapsack instances to analyze the trade-off between feasibility and objective value for the robust solution of the piece-wise linear approximation compared to the nominal solution, and to a simpler binary approximation. Despite the piece-wise approximation, the robust solution of the problem reveals to achieve its goal to remain feasible over the 2700 runs performed in our experiments, without too much deterioration of the objective value.

In Chapter 4, we consider the issue of choosing a subset of locations to construct new housing developments maximizing the satisfaction of potential buyers, which has not been previously studied in the literature. The allocation of demands to the selected locations is modeled by a choice model, based on the distance to the location, real-estate prices and incomes. We study two robust counterparts of the optimal location problem, where uncertainty lies on demand volumes for the first one, and on customer preferences for the second one. In both cases, the parameters subject to uncertainty appear both in the objective function and constraints.

The second robust model addresses the non-linearity, introduced by the choice model, combining a scenario-based approach on customers preferences, and an uncertainty budget approach that limits the number of customers that can deviate from the nominal scenario. We show that the subproblem of finding the worst-case deviation of parameters subject to uncertainty is tractable and leads to linear formulations of the robust problem. Computational experiments conducted on instances of the Paris region show that the average loss of value of the robust solution is reasonably low when compared to the optimal solution of deviated instances. We also derive insights for the new housing development issue.

In chapter 5 we focus on a variant of the Capacitated Facility Location Problem, where the demand to each potential location is a function of the selected locations, we call this problem the Capacitated Facility Location Problem with Location-Dependent Demand (CFLP-LDD). We model demands through a Poisson Process which gives way to fractional functions in the objective (sum of ratios) and, when demands are uncertain, to a non-linear structure of uncertainty. We address the non-linearity through Charnes and Cooper [25]'s approach for fractional problems, and then apply Bertsimas and Sim on the linear program.

Due to the non-linearity, a direct implementation of Bertsimas and Sim's approach to the CFLP-LDD results in an overly conservative solution, for the uncertainty can only be bound locally, for each potential location, without limiting the total number of impacted locations. We introduce the concept of *Nested Uncertainty Budgets* to address this issue and propose a method to manage global and local uncertainty in a hierarchical way. In this case, we obtain a problem that includes both continuous and discrete uncertainty variables.

Note that although we chose to motivate Chapters 4 and 5 from a specific context (location of new housing developments, and mobile vendors), both developed approaches shared the special structures of non-linear impact of parameter variation and discrete uncertainty variables and can easily be extended to other location problems with a similar structure. We conclude with possible avenues of research for the future.

2
State of the Art

We present in this chapter a compilation of relevant works depicting the state of the art on Robust Optimization (RO) as a way to deal with uncertainty in decision problems. The works cited present various approaches to RO in general, as well as recent advances and applications made of Bertsimas and Sim's approach in particular. We structure this chapter in three sections: Section 2.1 summarizes the theory of RO, borrowing from excellent reviews already presented in [13], [37], [41], and [43]. Section 2.2 focuses on the Γ-Robustness approach developed in [18] which is the core theory behind this dissertation. Finally, Section 2.3 presents recent developments made on the Γ-Robustness paradigm, most notably the works presented in [23], [35], and [55].

2.1 Theory of Robust Optimization

According to Gabrel et al. [37], Robust Optimization refers to a set of methods designed to protect against ambiguity and uncertainty. The form of the protection, the measure protected (feasibility, objective value, distance to optimality), and the form of the uncertainty varies across the different approaches. We begin this section with an explanation of the basic concepts of RO, present in detail the main approaches to RO, and finish with some recently developed methods.

2.1.1 Basic concepts

Since the objective of RO is to protect against uncertainty, it is important to distinguish the effects of said uncertainty on a problem and the consequences it brings. [37] distinguishes between protection against uncertainty on feasibility and protection against uncertainty on the objective value, the different consequences between infeasibility and sub-optimality being the justification for the distinction.

A protection against uncertainty on feasibility looks for a solution that is feasible for all realizations of the uncertain parameters. The *price* of a feasibility guaranty is a (potentially severe) deterioration in the objective. [37] claims that in some fields of study, e.g. engineering, the search of an absolute feasibility guaranty is justified.

Against uncertainty on the objective value, the protection strives to find a solution that performs well in all scenarios. While a common criterion is to optimize the worst-case scenario, other measures like *bw*-robustness [65] and lexicographic α-robustness [49] have been developed. [38] studies the impact of evaluating an inaccurate objective function.

Structure and tractability

The general robust optimization according to Bertsimas et al. [13] and Goerigk and Schöbel [41] is shown in program (2.1)

$$
\begin{aligned}
&\text{minimize } f_0(x) \\
&\text{subject to } f_i(x, a_i) \leq 0 \qquad \forall\, a_i \in \mathscr{A}_i,\; i = 1, \ldots, m \\
&\qquad\qquad\; x \in X
\end{aligned}
\tag{2.1}
$$

where a_i is the vector of uncertain parameters for constraint i taking values in the uncertain set $\mathscr{A}_i$. It is unclear when this program is efficiently solvable and, in general, the robust counterpart of a convex optimization problem is untractable. Much of the literature has focused on defining cases when the robust program is tractable. For the most part, if the robust feasible set is defined as:

$$
X_R(\mathscr{A}) = \{\mathbf{x} \in \mathbf{X} \,|\, f_i(x, u_i) \leq 0,\ \forall a_i \in \mathscr{A}_i\} \tag{2.2}
$$

then tractability mostly depends on $X_R(\mathscr{A})$ being convex in $\mathbf{x}$.

The tractability of a robust optimization program mostly depends on the structure of the subproblem containing the uncertainty, [13] makes a classification depending on the type of deterministic problem in robust linear optimization, robust quadratic optimization, robust semidefinite optimization, and robust discrete optimization, and gives insights on the tractability of each class.

According to Ben-Tal and Nemirovski [12], the robust counterpart of a linear program is essentially always tractable for most practical uncertainty sets of interest. Different definitions of the uncertainty set result in different formulations and types of robust programs which may no longer be linear. The most common uncertainty sets are: ellipsoidal [12], polyhedral, cardinality constrained [18], and norm uncertainty [16].

Robust quadratic optimization problems can be tractable only if the uncertainty set $\mathscr{A}_i$ is a single ellipsoid, otherwise, the robust counterpart is NP-Hard. Although the problem is not convex, it can be reformulated as a semidefinite optimization problem and then solving its dual results in an exact convex reformulation of the robust subproblem.

For the last two classes, [13] finds they are in general NP-Hard, so that the search for approximate solutions is a popular avenue. For robust discrete optimization however, [17] shows that under a model of cardinality constrained uncertainty, the robust solution to a network flow problem can be computed by solving a logarithmic number of nominal problems.

2.1.2 Main Approaches to Robust Optimization

In this part we present three approaches that have had a marked impact on the field of RO. To begin with, we introduce the seminal work by Soyster [69], from where RO emerges. We follow with Ben-Tal and Nemirovski [12], where the concern for over-conservatism of the solution is addressed. Bertsimas and Sim [18] are concerned with the structure and tractability of the robust program.

Soyster's Approach

Soyster [69]'s method parts from the premise that the decision variables must be chosen before the exact value of their coefficients is known. The approach seeks to offer protection against infeasibility to solve a problem of the form

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ s.t. \ & \sum_{j=1}^{n} \bar{a}_{ij} x_j \leq b_i \qquad & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{aligned} \tag{2.3}$$

where $\bar{a}_{ij}$ are only estimates (so-called "nominal" values), and the true coefficient $a_{ij} \in \mathscr{A}_{ij}$ is known to be in a hypersphere with center at $\bar{a}_{ij}$ and radius $\hat{a}_{ij}$, such that

$$\mathscr{A}_{ij} = \{a_{ij} \mid |a_{ij} - \bar{a}_{ij}| \leq \hat{a}_{ij}\}$$

Since the feasibility sets are hypershperes, the extreme values are easily determined as $a_{ij} = \bar{a}_{ij} + \hat{a}_{ij}$, and the robust counterpart of (2.3) can be formulated by the program below:

$$\begin{aligned} \max \; & \sum_{j=1}^{n} c_j x_j \\ s.t. \; & \sum_{j=1}^{n} \left(\bar{a}_{ij} + \hat{a}_{ij}\right) x_j \leq b_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{aligned} \tag{2.4}$$

Note that this is a very conservative formulation because all a_{ij} parameters could vary to their extreme deviation $\bar{a}_{ij} + \hat{a}_{ij}$ for the worst-case scenario. This motivated the adaptation of Soyster's approach to design less conservative robust models, which are discussed in the following sections.

Ben-Tal and Nemirovski's Ellipsoidal Uncertainty

Ben-Tal and Nemirovski [12] introduce the concepts of "column-wise" and "row-wise" uncertainty, as well as the concern for over-conservatism in the robust counterpart of a linear program. The authors point out that "column-wise" uncertainty, as managed in [69], corresponds to the largest deviation possible while in "row-wise" uncertainty the robust counterpart accounts for the fact that coefficients of the constraints cannot be simultaneously at their worst values. With [12]'s approach, the tractability of the problem depends on the shape of the uncertainty set, notably, it leads to a conic quadratic problem for ellipsoidal sets.

In this approach, the uncertainty sets are considered to be ellipsoidal whose size can be interpreted as the uncertainty budget of the problem. For a nominal model as in (2.3), the uncertainty set for a given constraint i with ellipsoidal uncertainty is:

$$\mathscr{A}_i = \{a_{ij} \quad j = 1, \ldots, n | a_{ij} = \bar{a}_{ij} + \hat{a}_{ij} z_{ij}, \; ||z_i||_2 \leq \rho_i\}$$

where $||z_i||_2 = \sqrt{\sum_j z_{ij}^2}$, and ρ_i is the size of the ellipsoidal set. The robust counterpart is then

$$
\begin{aligned}
\max\ & \sum_{j=1}^{n} c_j x_j \\
s.t.\ & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \rho_i ||\hat{a}_{ij} x_j||_2 \le b_i && i = 1, \dots, m \\
& x_j \ge 0 && j = 1, \dots, n
\end{aligned}
\tag{2.5}
$$

The robust model is in this case implemented from the subproblem $\max_{a_{ij} \in \mathscr{A}_i} \sum_j a_{ij} x_j \le b_i \; i = 1, \dots, m$ which is an optimization problem over a quadratic constraint.

Bertsimas and Sim's Γ-Robustness

Bertsimas and Sim [18] takes the concept of uncertainty budget and applies it to [69]'s approach. The authors define a positive coefficient Γ_i per constraint $i = 1, \dots, m$, to control the number of parameters a_{ij} that are allowed to deviate from their nominal value. In this way, Γ_i sets the level of protection against parameter variation, just like ρ_i in [12].

The robust counterpart of (2.3) under Bertsimas and Sim's approach is stated below

$$
\begin{aligned}
\max\ & \sum_{j=1}^{n} c_j x_j \\
s.t.\ & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{\{S_i : |S_i| = \Gamma_i\}} \sum_{j \in S_i} \hat{a}_{ij} x_j \le b_i && i = 1, \dots, m \\
& x_j \ge 0 && j = 1, \dots, n
\end{aligned}
\tag{2.6}
$$

where S_i contains the elements from set j that are allowed to vary in the ith constrain. In order to simplify the notation, (2.6) presents an abridged version of Bertsimas and Sim's approach. Section 2.2 will present the Γ-robustness paradigm (as is known in the literature) in detail.

2.1.3 Recent approaches to Robust Optimization

We outlined in the previous subsection the main approaches to RO. We can see as the main features a strict robustness in Soyster's method, efforts to address the conservatism that it generates in [12] and [18], and a robust program that is of the same class as the nominal one in [18]. We present in this section other attempts to control the conservatism of a robust model as classified by Bertsimas et al. [13] and Goerigk and Schöbel [41], e.g. adaptable, recoverable, light, and *bw*-Robustness. Furthermore, we include Bertsimas and Sim [17]'s and Goh and Sim [42]'s approaches to RO. [17] deals with combinatorial optimization and [42] considers information on the uncertainty distribution into the robust problem.

Adaptable Robust Optimization

Adaptable RO is the approach of robustness to sequential decision making, situations where some of the decisions must be made before the uncertainty is observed, while others can be adjusted when the information has been revealed. The concept was introduced by Ben-Tal et al. [11] and adds flexibility to RO.

For a linear program like (2.3), one can divide the decision variables into two subsets u_j for $j = 1, \ldots, j_1$ and v_j for $j = j_1 + 1, \ldots, n$, where variables u_j are non-adjustable (must be decided a-priori), and variables v_j represent the adjustable (recourse) variables. [11] defines the adjustable robust counterpart of (2.3) as:

$$\min_u \left\{ \sum_{j=1}^{j_1} c_j u_j \mid \forall\, a_{ij} \in \mathscr{A}_{ij}\ \exists\, v\ :\ \sum_{j=1}^{j_1} a_{ij} u_j + \sum_{j=j_1+1}^{n} a_{ij} v_j \leq b_i \quad i = 1, \ldots, m \right\}$$

In general, even the two-stage linear problem with fixed recourse (where $a_{ij} = \bar{a}_{ij}\ \forall\, i,\ j = j_1 + 1, \ldots, n$) is NP-Hard, and, even if the set of first-stage variables is convex, the RO problem is in general intractable. Some variations of this approach include:

Affine Adaptability. It consists in defining future stage decisions, recourse variables, as affine functions of the revealed uncertainty affecting them. According to [11], the tractability of these problems depends on the tractability of the uncertainty set. In particular, for a fixed recourse problem the affine adaptable robust counterpart (AARC) is a linear/SOC/SD program if the uncertainty set is defined by linear/SOC/SD constraints. If the recourse is affected by uncertainty and the uncertainty sets are ellipsoids, then the AARC can be approximated in a tractable way.

Finite Adaptability. Is set forth by Bertsimas and Caramanis [14] as a middle ground between RO and ARO. It consists in selecting k second stage solutions $\{v_1, \ldots, v_k\}$ one of which is chosen after the uncertainty is realized. The level of adaptability can be adjusted by changing the number of solutions k. [14] formulates the problem in a bilineal optimization problem.

Recoverable Robust Optimization

Álvarez-Miranda et al. [3] extend the framework of adaptable robustness to include the possibility of modifying the first-stage (non-adjustable) variables paying a penalty. [3] also designs an algorithm to solve the problem. The authors use the approach on the Two-Level Network Design Problem, and adapt it for the Uncapacitated Facilities Location Problem in [2] using Bender's Decomposition.

Light and *bw* Robust Optimization

These two measures of robustness rely on two indicators to protect the solution against infeasibility and sub-optimality. Fischetti and Monaci [33]'s Light Robustness starts from the idea of the "most robust" solution, not "too far" from optimality. Roy [65]'s *bw*-Robustness is a scenario based approach that seeks to maximize the number of scenarios that are "better" than the boundary value b, while ensuring that no scenario is "worst" than the guaranteed value w.

The approach in [33] works by setting a maximum loss of the objective value, with respect to the nominal problem, enforced by a linear constraint. Among all solutions that meet this criterion, Fischetti and Monaci choose the one that minimizes the possibility of infeasibility. Feasibility is controlled by adding a slack variable per uncertain constraint. [33]'s approach provides advantages in terms of feasibility and produces solutions with comparable quality to those obtained through robust models.

Roy [65]'s concept of *bw*-Robustness characterizes a solution as robust if it guaranties a minimum value w while maximizing the number of scenarios that meet a boundary criterion b. [65] defines three robustness measures: absolute robustness (on the value of the solution per scenario), absolute deviation (on the value of the regret per scenario), and relative deviation (on the relative regret per scenario). Gabrel et al [36] further characterize the boundary values with an interval for the Shortest Path Problem

While Light and *bw*-Robustness set a lower limit for the value of the robust solution, the two approaches differ in the measure of infeasibility. Light Robustness allows for the possibility of constraint violation and minimizes a penalty for it, on the other hand, *bw*-Robustness ensures feasibility and seeks to maximize the number of scenarios that perform well.

Discrete Robust Optimization

Through an approach similar to the one in Bertsimas and Sim's Γ-Robustness, Bertsimas and Sim [17] propose a robust program for discrete problems that deals with over-conservatism in terms of the probability of constraint violation. [17] also proposes an algorithm that solves robust counterparts to polynomially solvable combinatorial problems in polynomial time when the uncertainty affects only the cost coefficients.

Álvarez-Miranda, Ljubić, and Toth [4] and Poss [62] extend the results obtained in [17]. [4] proposes an algorithm to solve combinatorial problems with uncertain cost coefficients which is faster than the one in [17]. Furthermore, [4] extends the approach to problems facing uncertainty both on costs and on the coefficients of knapsack constraints.

Similarly, [62] considers uncertainty sets defined by s knapsack-type constraints. Poss extends [17] results to the case where s is constant, and shows that in the general case the problem is NP-Hard. [62] also proposes approximation algorithms for ellipsoidal uncertainty sets.

Distributionally Robust Optimization

One of the premises of Robust Optimization is that it does not require a probabilistic distribution of the uncertain parameters. Instead, it only assumes knowledge on the set constraining the uncertain parameters. Previous literature discusses the implications of assuming an inaccurate support for uncertainty and explores the case when even the moments of the distribution are uncertain. Goh and Sim [42] study the case of partially characterized uncertainties, allow for recourse variables, and non-anticipativity requirements. The key contributions of [42] are:

- A new flexible non-anticipative decision rule suitable for multistage modeling.
- A technique of segregating uncertainties to create more flexible rules and distributionally ambiguous bounds.

[42] assumes knowledge of some descriptive statistics of the uncertain parameters: the **support**, a convex set containing all possible realizations of the uncertainty; the **mean** and **covariance**, which may be uncertain themselves; and **directional deviations**, denoted by bounds on the forward and backward deviations and characterize a family of distributions. The objective is to solve a problem of the type:

$$
\begin{aligned}
Z^*_{GEN} = \min_{x,\{y^k(.)\}_{k=1}^K} \quad & \mathbf{c}^{0'}\mathbf{x} + \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\sum_{k=1}^{K} \mathbf{d}^{0,k'}\mathbf{y}^k(\tilde{z})\right) \\
s.t. \quad & \mathbf{c}^{l'}\mathbf{x} + \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\sum_{k=1}^{K} \mathbf{d}^{l,k'}\mathbf{y}^k(\tilde{z})\right) \le b_l & \forall l \in [M] \\
& \mathbf{T}(\tilde{z})\mathbf{x} + \sum_{k=1}^{K} \mathbf{U}^k\mathbf{y}^k(\tilde{z}) = \mathbf{v} \\
& \underline{\mathbf{y}}^k \le \mathbf{y}^k(\tilde{z}) \le \overline{\mathbf{y}}^k & \forall k \in [K] \\
& \mathbf{x} \ge \mathbf{0} \\
& \mathbf{y}^k \in \mathscr{Y}(m_k, N, I_k) & \forall k \in [K]
\end{aligned}
$$

where $\mathbf{T}(\tilde{z}) = \mathbf{T}^0 + \sum_{j=1}^{N} \tilde{z}_j\mathbf{T}^j$, $\mathbf{v}(\tilde{z}) = \mathbf{v}^0 + \sum_{j=1}^{N} \tilde{z}_j\mathbf{v}^j$, and $(\mathbf{c}^l, \mathbf{d}^l, \mathbf{U}^k, \underline{\mathbf{y}}^k, \overline{\mathbf{y}}^k, \mathbf{I}_k)$ are deterministic. $\mathbf{I}_k$ is the set of dependent uncertainties for $\mathbf{y}^k(.)$, and $\mathscr{Y}(m_k, N, I_k)$ is its set of allowable recourse decisions.

Linear approximations are obtained from Linear Decision Rules (LDRs) where $\mathbf{y}^k(.)$ is restricted to a subspace of affine functions of $\tilde{\mathbf{z}}$, $\mathscr{L}(m_k, N, I_k) \subset \mathscr{Y}(m_k, N, I_k)$. Improving the approximation obtained from the LDRs, requires a finer representation of the uncertainty, for which [42] proposes a segregation of the uncertainty.

After the segmentation of the uncertainty, the support set for $\tilde{\mathbf{z}}$ is likely to be not convex, for which the corresponding sets supporting the segregated uncertainty need to be approximated. Segregated linear decision rules can be constructed in a similar fashion as the LDRs, using the approximated support and the segregated uncertainties to define $\mathscr{L}(m_k, N, I_k)$.

We offered in this section an introduction to the RO theory. There exist many more extensions and different approaches to the topic, those presented here where selected for their impact, and in an effort to provide a sample of the differences and reach of the topic. For a more detailed presentation of RO theory, we refer the reader to the reviews in [13], [37], [41], and [43]. In the next section we will present in detail Bertsimas and Sim's Γ-Robustness approach with uncertainty budgets, which is the central theory supporting the research in this dissertation.

2.2 Bertsimas and Sim's approach to Robust Optimization

In a static context where all the decisions are made before the realization of the uncertainty parameters, Bertsimas and Sim [18] develop a method using duality to obtain a robust counterpart to an uncertain problem that is of the same class as the original problem, thus maintaining the same level of complexity in the robust case. Consider a linear problem of the form

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, \ldots, m \\ & l_j \leq x_j \leq u_j \qquad j = 1, \ldots, n \end{aligned}$$

where without loss of generality only a_{ij} is subjected to uncertainty with the uncertainty set:

$$\mathscr{A} = \{A \in R^{n \times m} | a_{ij} \in [\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}] \; \forall \, i, j\}$$

The corresponding robust program protecting against infeasibility can be rewritten as:

$$
\begin{aligned}
\max \ & \sum_{j=1}^{n} c_j x_j \\
s.t. \ & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{\{S_i \cup \{t_i\} | |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \hat{a}_{ij} y_j + (\Gamma_i - \lfloor \Gamma_i \rfloor) \hat{a}_{it_i} y_{t_i} \right\} \leq b_i \quad \forall\, i \\
& - y_j \leq x_j \leq y_j \qquad \forall\, j \\
& \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \\
& \mathbf{y} \geq 0
\end{aligned}
$$

where the auxiliary variables y_j are introduced to write a general model for $x_j \in \mathbb{R}$ (i.e. x_j is allowed to be negative), and the protection function for a given vector $\mathbf{x}^*$ and an uncertainty budget Γ_i can be written as $\beta_i(\mathbf{x}, \Gamma_i) = \max_{\{S_i \cup \{t_i\} | |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \hat{a}_{ij} y_j + (\Gamma_i - \lfloor \Gamma_i \rfloor) \hat{a}_{it_i} y_{t_i} \right\}$. Furthermore, the value of the protection function equals the objective function of the following linear program:

$$
\begin{aligned}
\beta_i(\mathbf{x}^*, \Gamma_i) = \max \ & \sum_{j=1}^{m} \hat{a}_{ij} |x_j^*| z_{ij} \\
s.t. \ & \sum_{j=1}^{m} z_{ij} \leq \Gamma_i \\
& 0 \leq z_{ij} \leq 1 \quad \forall\, j
\end{aligned}
$$

Clearly the optimal solution value of the problem above consists on $\lfloor \Gamma_i \rfloor$ variables z_{ij} equal to 1 and one variable equal to $\Gamma_i - \lfloor \Gamma_i \rfloor$ which is equivalent to the selection of a subset $\{S_i \cup \{t_i\} | |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}$. The maximum protection achieved by the function $\beta_i(\mathbf{x}, \Gamma_i)$ corresponds to the worst case scenario with an uncertainty budget Γ_i.

[18]'s main result is summarized in Theorem 1 and it is obtained from the dual

of the previous program.

$$\begin{aligned}
\beta_i(\mathbf{x}^*, \Gamma_i) = \min \ & q_i\Gamma_i + \sum_{j=1}^{m} r_{ij} \\
s.t. \ & q_i + r_{ij} \geq \hat{a}_{ij}|x_j^*| \quad \forall j \\
& q_i, r_{ij} \geq 0 \qquad\qquad \forall j
\end{aligned}$$

Theorem 1 (Bertsimas and Sim): The uncertain linear programming problem has the following robust linear counterpart:

$$\begin{aligned}
\min \ & \sum_{j=1}^{m} c_j x_j \\
s.t. \ & \sum_{j=1}^{m} \bar{a}_{ij} x_j + q_i\Gamma_i + \sum_{j \in J_i} r_{ij} \leq b_i \quad \forall i \\
& q_i + r_{ij} \geq \hat{a}_{ij} y_j \qquad\qquad\qquad \forall i, \ j \in J_i \\
& -y \leq x \leq y \\
& l \leq x \leq u \\
& q, r, y \geq 0.
\end{aligned}$$

where Γ_i controls the conservatism of the solution, and J_i is the set of parameters subject to uncertainty in constraint i. Under the assumed model of data uncertainty, [18] establishes two bounds on the probability of constraint violation showing that the robust solution is feasible with high probability.

The probability of constraint violation is bounded by the probability of exceeding the uncertainty budget:

$$\Pr\left(\sum_{j=1}^{m} a_{ij} x_j^* > b_i\right) \leq \Pr\left(\sum_{j=1}^{m} \gamma_{ij}\eta_{ij} \geq \Gamma_i\right)$$

where x_j^* is the optimal solution of the robust counterpart,

$$\eta_{ij} = \frac{a_{ij} - \bar{a}_{ij}}{\hat{a}_{ij}},$$

$$\gamma_{ij} = \begin{cases} 1 & \text{if } j \in S_i^* \\ \frac{\hat{a}_{ij}|x_j^*|}{\hat{a}_{ir^*}|x_{r^*}^*|} & \text{if } j \in J_i \backslash S_i^* \end{cases},$$

$r^* = argmin_{r \in S_i^* \cup \{t_i^*\}} \hat{a}_{ir} |x_r^*|$, and

S_i^*, t_i^* are, respectively, the set and index that achieve the $\max \beta_i(x^*, \Gamma_i)$

$$\Pr\left(\sum_{j=1}^{m} \gamma_{ij}\eta_{ij} \geq \Gamma_i\right) \leq \exp\left(-\frac{\Gamma_i^2}{2|J|}\right) \qquad \textbf{Theorem 2 (Bertsimas and Sim)}$$

While the probabilistic bound in Theorem 2 (Bound 1) has the quality of being independent from x^*, Bertsimas and Sim point out that it is not as attractive for small sets J_i. For example, to guaranty a probability of constraint violation less than 1%, for $|J_i| = 5, 10, 100$ the required Γ_i should be at least 5, 9.6, 30.3, respectively. [18] develops then thighter bounds in Theorem 3.

Theorem 3 (Bertsimas and Sim)

$$\Pr\left(\sum_{j=1}^{m} \gamma_{ij}\eta_{ij} \geq \Gamma_i\right) \leq B(n, \Gamma_i) \tag{2.7}$$

$$B(n, \Gamma_i) \leq (1 - \mu)C(n, \lfloor \nu \rfloor) + \sum_{l=\lfloor nu \rfloor + 1}^{n} C(n, l) \tag{2.8}$$

$$\text{where} : B(n, \Gamma_i) = \frac{1}{2^n}\left\{(1 - \mu)\binom{n}{\lfloor \nu \rfloor} + \sum_{l=\lfloor nu \rfloor + 1}^{n} \binom{n}{l}\right\}$$

$$C(n, l) = \begin{cases} \frac{1}{2^n} & \text{if } l = 0 \text{ or } l = n \\ \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{(n-l)l}} \exp\left(n \log\left(\frac{n}{2(n-l)}\right) + \log\left(\frac{n-l}{l}\right)\right) & \text{otherwise} \end{cases}$$

while (2.7) (Bound 2)is the best bound for constraint violation, it is very expensive to compute for large n, (2.8) (Bound 3) provides a good bound easier to compute.

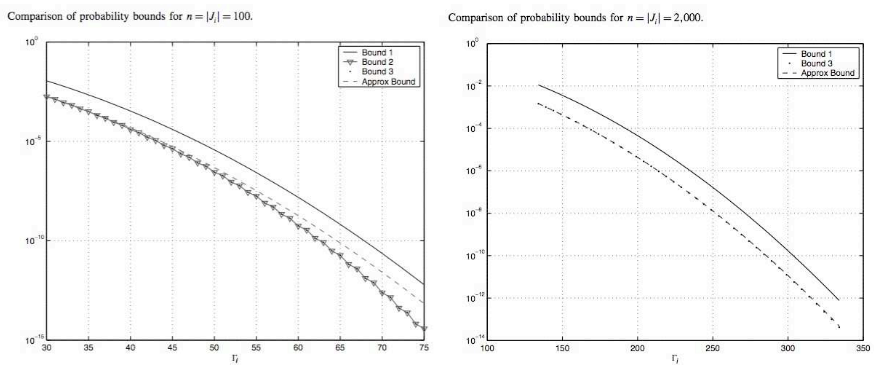

Figure 2.2.1: Comparison of bounds for different values of Γ_i

Figure 2.2.1 compares the protection achieved by the different bounds for an uncertainty set of size $|J_i|$ and different values of uncertainty budget Γ_i. As the graph shows, Bounds 2 and 3 clearly dominate Bound 1, while offering virtually the same protection.

[18] also offers a comparison of the conservativeness of the approach developed against the previous method of ellipsoidal uncertainty. The suggestion is that in the worst-case scenario, the protection level of the new approach could be smaller by at most a constant, while the ellipsoidal method could offer protections that are smaller by a factor of $n^{\frac{1}{4}}$. In the average case, the protection levels are of the same order of magnitude.

The last result of the paper is a note on the sensitivity of the protection level. Under non-degeneracy assumptions, the change to the objective function by marginal changes to the protection level Γ_i is given in terms of the primal and dual variables (z_i^* and q_i^*) and is equal to $-z_i^* q_i^*$

Experimental results on the proposed approach show that it succeeds in reducing the price of robustness. Furthermore, [18] shows that even in the case where the uncertainty exceeds the budget Γ_i, constraint i remains feasible with a "very high" probability.

2.3 Advances in Γ-Robustness

We have presented in the previous section the Γ-robustness approach, which extends the work in [69] and improves the tractability issues of [12]. However, it does not capture some specific classes of problems or uncertainty structures. In the literature, we can identify three families of extensions that deal with said special structures. We will present in this section the main works on each category.

Time-dependent uncertainty sets

Bertsimas et al. [15] and Lorca and Sun [51] address the unit commitment problem in the energy sector. In their work, the authors tackle the issue of defining a proper uncertainty set for their problem. [15] uses a set described by three parameters: a nominal value ($\bar{a}_{ij}^t$), a maximum deviation ($\hat{a}_{ij}^t$), and a budget of uncertainty (Γ^t) for each point in time. In [15], the authors follow the original setting of Γ-Robustness, and adapt it to a multi-period problem. In this way, the uncertainty set adapts to the evolution of the problem through time, considering that the nominal and deviated parameters, as well as the uncertainty, can vary at each time period.

[51] proposes the use of *dynamic uncertainty sets*, recognizing that, in some settings, the value of a parameter will depend (to a certain extend) on the value it had the previous period. Then, the uncertainty at each time period is a function of the uncertainty in previous stages, and so are the boundaries of the uncertainty sets. Alternatively, they do not consider any correlation on the budget of uncertainty between time periods. The authors use time series models to control the boundaries of the uncertainty sets for their problem.

Variable uncertainty budget $\gamma(x)$

Poss [60] presents an approach to robust combinatorial optimization where the budget of uncertainty is given by a function $\gamma(x)$, rather than a constant Γ. Poss' $\gamma(x)$ adjusts the degree of conservatism according to the cardinality of the optimal solution, especially for problems where the optimal solution is sparse. The results can be extended to different budget functions better capturing the uncertainty arising from a specific context. [60] creates a less conservative model than traditional Γ-Robustness while guarantying the same protection level, effectively reducing the price of robustness. Depending on the nominal problem, the robust counterpart in [60] could be as easy to solve as that in [18].

Poss [61] extents the approach to problems with cost uncertainty. Using the $\gamma(x)$ uncertainty budget, Poss provides a better bound for the VaR problem than the one obtained with Γ.

Distribution of the uncertainty interval

While the works previously cited mainly focus on the uncertainty budget Γ, Büsing and D'Andreagiovanni [23] propose an extension on the distribution of the uncertainty set, which they call *multi-band uncertainty*. Since the deviation probability may vary along the set, e.g. small deviations are more likely to occur than large ones, the proposed uniformity assumption could lead to an unrealistic uncertainty set. [23] proposes to break the set into multiple narrower "bands", each with a customized Γ value, according to historical data. The uncertainty set is thus built so as to approximate the shape of the distribution of deviations.

Similarly, Düzgün and Thiele [30] propose the use of multiple ranges for the uncertain parameters, limiting overly conservative solutions when the uncertainty set is too wide. [30] bounds as well the number of parameters that fall into each range (pessimistic view), but also bound the number of parameters that take the worst-case value within each range. The authors apply this setting to a project selection problem.

RHS uncertainty

Gabrel and Murat [35], Minoux [55], and Billionnet et al. [20] contribute to the literature of problems with right-hand side (RHS) uncertainty, the last two works dealing with two-stage problems.

[35] studies the relationship between RHS uncertainty and uncertainty in the objective function coefficients. Using duality theory they find a link between worst-case scenario and maximum regret criteria.

[55] studies the complexity of two-stage problems with RHS uncertainty identifying subclasses that are polynomially solvable, bounding either the cardinality of the uncertainty set or the number of recourse variables. Minoux also provides models for several problems of practical interest.

[20] solves a linear robust problem with mixed-integer first-stage variables and continuous second stage variables. They consider column wise uncertainty focusing on the RHS and a specific definition of uncertainty. Generalizations of the approach deal with left-hand side uncertainty, and uncertainty sets defined by a polytope.

After this (non exhaustive) state-of-the art on Robust Optimization, the following chapters present our original contributions which extend Γ-robustness to discrete structures and non-linear impact of data variation, either in a generic form or within specific applications.

3

Robust Optimization for Non-Linear Impact of Data Variation

3.1 Introduction

In this chapter we adapt the Γ-Robustness paradigm developed by Bertsimas and Sim [18] to the case where coefficients in constraints depend on data the variation of which impacts the coefficient in a non linear way. This is the case if one considers demand as a function of price, the Net Present Value (NPV) of an investment project as a non-linear function of the discounting rate, or the choice probability as the ratio of exponentials of utilities in a logit choice model (we will more formally detail these situations at the end of this introduction). We consider a generic linear program P of the form:

$$
\begin{aligned}
\max \ & \sum_{j=1}^{n} c_j x_j \\
s.t & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, \ldots, m \\
& x \geq 0
\end{aligned}
$$

In the classical Γ-robustness approach, for some constraints i, coefficients a_{ij} are subject to uncertainty and described by $a_{ij} = \bar{a}_{ij} + \hat{a}_{ij} z_{ij}$, where $\bar{a}_{ij}$ is the nominal value of the coefficient, $\hat{a}_{ij}$ is its maximum variation, and $z_{ij} \in [-1, 1]$. The framework developed in [18] generates a robust counterpart for P that ensures feasibility of the robust solution whatever the variation of coefficients satisfying a given level of conservatism. This level of conservatism is controlled by a so-called uncertainty budget: the number of coefficients that can vary in constraint i is supposed not to exceed a budget Γ_i, i.e., $\sum_j |z_{ij}| \leq \Gamma_i$, see chapter 2 for further details. However, it does not enable to capture in a straightforward way some specific kinds of problems or uncertainty structures. In the literature, to the best of our knowledge we can identify three families of extensions that deal with (i) time-dependent variations of coefficients [15] [51] [60], (ii) distribution of the uncertainty interval [23] [30], and (iii) Right-Hand Side (RHS) uncertainty [35] [55] [20]; the details can be found in section 2.3.

In this chapter, we keep the Γ-robustness principle that consists of protecting against infeasibility over a controlled variation of parameters. However, we extend it to cases when the coefficients are non-linear functions of given data that are potentially subject to variations. For example in capacitated Facility Location problems, demands D_j of demand sources j often appear in the left-hand side of capacity constraints. If there is uncertainty on the market price of the product, then we should consider $D_j = f_j(p_j)$ where p_j is the price of the product at demand source j (demand sources may be located in different countries) and f_j is generally non-linear, concave decreasing. Also, consider a Capital Budgeting problem with uncertainty on project values (see [54] for a study on this topic). The capital budgeting problem consists in selecting a portfolio of investment projects maximizing the total net value of the portfolio while respecting resource constraints. Then the Net Present Value (NPV) of a project j is the sum of discounting cash-flows F_t^j over the years $t = 0, \ldots, T$, i.e., $NPV_j = \sum_{t=0}^{T} \frac{F_t^j}{(1+r_j)^t}$, where r_j is the discount rate for projet j. This discount rate r_j depends on several factors, some are exogeneous like the interest rate of the country where project j is located, some are endogeneous like the level of return the decision-maker wants to compensate her risk (the various projects being more or less risky). In any case, r_j is likely to be subject to imprecision or uncertainty and the NPV is clearly non-linear in r_j (note that, as mentioned in [18], an objective function subject to uncertainty can be turned into a constraint). In Multinomial Logit (MNL) choice models ([53]), the probability p_{ij} of customer i choosing product j is expressed by $p_{ij} = e^{u_{ij}}/(\sum_k e^{u_{ik}})$ which is a non-linear function of utilities u_{ij} that are known to be hard to calibrate precisely (see [31] and Chapter 4 for an application to the location of new housing developments). Another example could be non-linear consumption functions depending on uncertain temperatures.

Section 3.2 describes the robust problem and discusses its tractability, as the robust problem cannot be turned into a linear program. In section 3.3, we approximate the non-linear variation with a binary approximation. A more refined piecewise linear function is proposed in section 3.4. We show that the approximation of the robust counterpart is tractable using LP dualization for the worst-case subproblem. Section 3.5 is devoted to numerical experiments and analyzes the performance of our extended robustness approach and of the piecewise approximation. Section 3.6 concludes the chapter.

3.2 Problem statement

We consider constraints of the form

$$\sum_j f_{ij}(a_{ij})x_j \leq b_i$$

where $f_{ij}(a_{ij})$ is not linear. We use the notation $f_{ij}(a_{ij})$ instead of $a_{ij} = f_{ij}(p_{ij})$ where p_{ij} would be the varying parameter, in order to keep the usual notation $a_{ij} = \bar{a}_{ij} + z_{ij}\bar{a}_{ij}$ where a_{ij} is the varying parameter and z_{ij} is the deviation variable. We denote by P_f the original problem P where coefficients in the constraints are of the form $f_{ij}(a_{ij})$, where a_{ij} is the coefficient subject to uncertainty and functions f are not linear. For this variant P_f, the robust transformation proposed in [18] would result in a non-linear problem. If we denote as in [18] by $\Gamma_i \in \mathbb{N}$ the maximum total variation of coefficients a_{ij} in constraint i (uncertainty budget), the robust problem associated with P_f can be expressed as:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t \;\; & \omega_i(x, \Gamma_i) \leq b_i \qquad \forall\, i = 1, \ldots, m \\ & x \geq 0 \end{aligned} \tag{3.1}$$

where the *subproblem* of determining the worst variation in the left-hand side of constraint i ensuring feasibility for a given solution x is:

$$\begin{aligned} \omega_i(x, \Gamma_i) = \max & \sum_{j=1}^{n} f_{ij}(\bar{a}_{ij} + z_{ij}\hat{a}_{ij})x_j \\ s.t. & \sum_{j=1}^{n} |z_{ij}| \leq \Gamma_i \\ & z_{ij} \in [-1, +1] \qquad \forall j = 1, \ldots, n \end{aligned}$$

which in non-linear in deviation variables z_{ij} if f is non-linear. If the above subproblem had the property to have variables z_{ij} binary 0 or 1 at optimality, the objective function of the subproblem $\omega_i(x, \Gamma_i)$ could be rewritten as $\sum_j f_{ij}(\bar{a}_{ij} + \hat{a}_{ij})z_{ij}x_j$ which is linear in variables z (it is easy to get rid of the absolute values in that case to be fully linear). However, we show in Proposition 3.2.1 below that we cannot assume $z_{ij} \in \{0, 1\}$ at optimality even for simple functions f. Therefore, the non-linear form $\sum_{j=1}^{n} f_{ij}(\bar{a}_{ij} + z_{ij}\hat{a}_{ij})x_j$ cannot be simplified without making an approximation.

Proposition 3.2.1. *There exist non-linear functions f such that an optimal solution of subproblem $\omega_i(x, \Gamma_i)$ necessarily has some fractional z_{ij}.*

Proof. We present a simple example with $j = 1, 2$, $f_{i1}(a) = f_{i2}(a) = \sqrt{a}$, $\bar{a}_{i1} = \bar{a}_{i2} = 0$, $\hat{a}_{i1} = \hat{a}_{i2} = 1$, and $\Gamma_i = 1$. For a given x, assuming $x_1, x_2 > 0$, the subproblem $\omega_i(x, \Gamma_i)$ is to maximize $\sqrt{z_{i1}}x_1 + \sqrt{z_{i2}}x_2$ submitted to $z_{i1} + z_{i2} \leq 1$. Assume that $x_1 > x_2$. The best solution of this subproblem one can obtain with $z_{i1}, z_{i2} \in \{0, 1\}$ is $z_{i1} = 1, z_{i2} = 0$ with an objective value equal to x_1. Now, take the fractional solution $z_{i1} = x_1^2/(x_1^2 + x_2^2)$, $z_{i2} = x_2^2/(x_1^2 + x_2^2)$. It satisfies the budget constraint and its objective value is

$$\frac{x_1^2}{\sqrt{x_1^2 + x_2^2}} + \frac{x_2^2}{\sqrt{x_1^2 + x_2^2}} = \sqrt{x_1^2 + x_2^2} > x_1$$

Hence, in that case a solution with $z_{i1}, z_{i2} \in \{0, 1\}$ cannot be optimal for the subproblem $\omega_i(x, \Gamma_i)$. $\square$

Since $f_{ij}(\bar{a}_{ij} + \hat{a}_{ij}z_{ij})x_j$ cannot be linearized in general if f is not linear, we propose two different approximations to deal with this complication. In section 3.3, we start by a simple binary approximation where $z_{ij} \in \{0, 1\}$ and still use duality to linearize the approximated subproblem $\omega_i(x, \Gamma_i)$, since the optimal solution of the linear relaxation is equal to that of the binary problem as in [18]. This case also matches an uncertainty structure of P_f where uncertainty is realized or not in a binary way, i.e. a_{ij} either takes value $\bar{a}_{ij} + \hat{a}_{ij}$ or $\bar{a}_{ij} - \hat{a}_{ij}$; in this case, the procedure would not be an approximation but would exactly fit the uncertainty structure.

The second alternative, studied in section 3.4, is to use a more refined piece-wise linear function as an approximation of $f_{ij}(a_{ij})$.

3.3 Binary case or binary approximation

We first address the binary case, where uncertainty is either manifested or not and the deviation variable z_{ij} can take values in $\{-1, 0, 1\}$, which is also a binary approximation of the non-linear function if deviations can be fractional. We illustrate below the separation of the protection function for the subproblem $\omega_i(x, \Gamma_i)$, and proceed with the robust counterpart.

The original constraint $\sum_j f_{ij}(\bar{a}_{ij} + \hat{a}_{ij}z_{ij})x_j \leq b_i$, where $z_{ij} \in \{-1, 0, 1\}$ can be rewritten using binary variables z_{ij}^+ and z_{ij}^- to represent the possible increase or decrease of a_{ij}:

$$\sum_{j=1}^{n} \left(f_{ij}(\bar{a}_{ij})(1 - z_{ij}^+ - z_{ij}^-) + f_{ij}(\bar{a}_{ij} + \hat{a}_{ij}) z_{ij}^+ + f_{ij}(\bar{a}_{ij} - \hat{a}_{ij}) z_{ij}^- \right) x_j \leq b_i, \text{ with } z_{ij}^+ + z_{ij}^- \leq 1. \tag{3.2}$$

In order to set the protection function $\omega_i(x, \Gamma_i)$, it is necessary to identify the changes with the most adverse impact on (3.2). Since we are looking for a protection against the worst case scenario, we only need to consider the positive variations of $f_{ij}(a_{ij})$, and identify three different cases for the deviation variables:

i. $\bar{J}_i = \{j : f_{ij}(\bar{a}_{ij}) \geq \max\left(f_{ij}(\bar{a}_{ij} - \hat{a}_{ij}), f_{ij}(\bar{a}_{ij} + \hat{a}_{ij})\right)\}$ - The corresponding variables z_{ij}^+ and z_{ij}^- are not included in $\omega_i(x, \Gamma_i)$ for any deviation from $\bar{a}_{ij}$ increases the slack of (3.2).

ii. $J_i^+ = \{j : f_{ij}(\bar{a}_{ij} + \hat{a}_{ij}) \geq \max\left(f_{ij}(\bar{a}_{ij} - \hat{a}_{ij}), f_{ij}(\bar{a}_{ij})\right)\}$ - Only variable z_{ij}^+ is included in $\omega_i(x, \Gamma_i)$ for the increase of a_{ij} represents the worst-case scenario for (3.2).

iii. $J_i^- = \{j : f_{ij}(\bar{a}_{ij} - \hat{a}_{ij}) \geq \max\left(f_{ij}(\bar{a}_{ij}), f_{ij}(\bar{a}_{ij} + \hat{a}_{ij})\right)\}$ - Only variable z_{ij}^- is included in $\omega_i(x, \Gamma_i)$ for the decrease of a_{ij} represents the worst-case scenario for (3.2).

With this consideration, at most one variable z_{ij} is needed for each pair (i, j) and we can drop the superindex $+$ or $-$, in order to simplify the notation. The worst-case subproblem can be written then as:

$$\begin{aligned} \omega_i(x, \Gamma_i) = \max & \sum_{j \in J_i^+ \cup J_i^-} a_{ij} x_j z_{ij} \\ s.t. & \sum_{j \in J_i^+ \cup J_i^-} z_{ij} \leq \Gamma_i \\ & z_{ij} \in \{0, 1\} \qquad \forall j \in J_i^+ \cup J_i^- \end{aligned} \tag{3.3}$$

where $a_{ij} = f_{ij}(\bar{a}_{ij} + \hat{a}_{ij})$ for $j \in J_i^+$ and $a_{ij} = f_{ij}(\bar{a}_{ij} - \hat{a}_{ij})$ for $j \in J_i^-$.

Proposition 3.3.1. *The robust counterpart of P_f, where coefficient $f_{ij}(a_{ij})$ can either deviate to $f_{ij}(\bar{a}_{ij} + \hat{a}_{ij})$ or $f_{ij}(\bar{a}_{ij} - \hat{a}_{ij})$, can be formulated as the following LP:*

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ s.t \ & \Gamma_i q_i + \sum_{j \in J_i^+ \cup J_i^-} r_{ij} \leq b_i && \forall i = 1, \ldots, m \\ & q_i + r_{ij} \geq a_{ij} x_j && \forall i = 1, \ldots, m,\ j \in J_i^+ \cup J_i^- \\ & q, r, x \geq 0 \end{aligned}$$

Proof. The subproblem has the same form as in [18] and is equivalent to its linear relaxation with $0 \leq z_{ij} \leq 1$. The dual of this linear relaxation is

$$\begin{aligned} \min \ & q_i \Gamma_i + \sum_{j \in J_i^+ \cup J_i^-} r_{ij} \\ s.t \ & q_i + r_{ij} \geq a_{ij} x_j && \forall j \in J_i^+ \cup J_i^- \\ & q_i, r_{ij} \geq 0 \end{aligned}$$

where q_i and r_{ij} are the dual variables associated, respectively, with the budget constraint and constraints $z_{ij} \leq 1$ in (3.3). Replacing $\omega_i(x, \Gamma_i)$ by the above dual program in (3.1) for each $i = 1, \ldots, m$, we get the formulation of the proposition. $\square$

The above procedure is able to handle non-linear functions in the constraints if uncertainty is structured in a binary way; otherwise it is only an approximation of uncertainty that may provide a too large error in computing the worst-case deviation for some functions f (see numerical experiments in section 3.5).

In section 3.4, we refine this approximation by replacing the non-linear function f by a piecewise linear function, allowing parameters to vary a fraction of their interval to achieve the worst-case scenario. We show that in this case the subproblem and thus the robust counterpart are still tractable.

3.4 Piecewise linear approximation

The approach presented in section 3.3 ignored the shape of non-linear functions f_{ij}, since there were only three possible outcomes for each uncertain parameter (nominal value, maximum increase, and maximum decrease). While this method is not more complex than the original procedure by Bertsimas and Sim, it simplifies functions f to a binary approximation. We now propose an alternative approximation that can better represent f. To simplify notation, we will consider in the remainder of this section that f_{ij} is an increasing function, however the procedure can be extended to decreasing functions.

Because f_{ij} is increasing, we can consider only positive deviations of the parameter for the worst-case in a constraint i. Hence, only the interval $[\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$ is to be segmented into l pieces, and we use binary variables z_{ij}^k, $k = 1, \ldots, l$, to identify in which segment the coefficient $f_{ij}(a_{ij})$ will fall. More precisely, $z_{ij}^k = 1$ if the true value of $f_{ij}(a_{ij})$ lies in or beyond the k^{th} segment of the piece-wise linear approximation. While in the former section uncertainty was represented in a binary way, this approach enables to obtain fractions of the uncertainty interval as solutions to subproblem $\omega_i(x, \Gamma_i)$. We note $\hat{\omega}_i(x, \Gamma_i)$ the subproblem approximated this way. Furthermore, since the binary variables z_{ij}^k are only present in $\hat{\omega}_i(x, \Gamma_i)$, the robust model will only increase the number of constraints.

The decomposition of interval $[\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$ into l sub-intervals is of the form:

$$\bigcup_{k=0}^{l-1} \left[\bar{a}_{ij} + \frac{k}{l}\hat{a}_{ij}, \bar{a}_{ij} + \frac{k+1}{l}\hat{a}_{ij}\right]$$

Rewriting the constraints of problem P_f with deviation variables to ensure feasibility, we get:

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{k=1}^{l-1} f_{ij}\left(\bar{a}_{ij} + \frac{k}{l}\hat{a}_{ij}\right)x_j\left(z_{ij}^{k} - z_{ij}^{k+1}\right) + \sum_{j=1}^{n} f_{ij}\left(\bar{a}_{ij} + \hat{a}_{ij}\right)x_j z_{ij}^{l} \leq b_i \\ &\qquad \text{with } z_{ij}^{k} \leq z_{ij}^{k-1} \text{ for } k > 1 \end{aligned} \tag{3.4}$$

and the approximate worst-case subproblem of the robust counterpart of P_f associated with constraint i is:

$$\begin{aligned} \hat{\omega}_i(x, \Gamma_i) = \max\ & \sum_{j=1}^{n}\sum_{k=1}^{l-1} f_{ij}\left(\bar{a}_{ij} + \frac{k}{l}\hat{a}_{ij}\right)x_j\left(z_{ij}^{k} - z_{ij}^{k+1}\right) \\ & + \sum_{j=1}^{n} f_{ij}\left(\bar{a}_{ij} + \hat{a}_{ij}\right)x_j z_{ij}^{l} \\ s.t \qquad & \frac{1}{l}\sum_{j=1}^{n}\sum_{k=1}^{l} z_{ij}^{k} \leq \Gamma_i & (3.5) \\ & z_{ij}^{k} \leq z_{ij}^{k-1} \qquad \forall j, k > 1 & (3.6) \\ & z_{ij}^{k} \in \{0, 1\} \qquad \forall j, k \end{aligned}$$

Note that if for a given j, $\bar{k}_{ij}$ is the largest index k such that $z_{ij}^{k} = 1$, then by (3.6) we have $\sum_{k=1}^{l} z_{ij}^{k} = \bar{k}_{ij}$. Therefore, in constraint (3.5) the variation of coefficient a_{ij} will consume exactly $\bar{k}_{ij}/l$ budget units.

Lemma 3.4.1. *Assume functions f_{ij} are non-decreasing, and concave increasing over $[\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$. Then an optimal solution z^* of subproblem $\hat{\omega}_i(x, \Gamma_i)$ verifies: for each j, at most one variable $z_{ij}^{k_{ij}}$ is fractional ($1 \leq k_{ij} \leq l$) and $z_{ij}^{k} \in \{0, 1\}\ \forall\, k \neq k_{ij}$.*

Moreover, $z_{ij}^k = 1$ *for all* $k < k_{ij}$*, and* $z_{ij}^k = 0$ *for all* $k > k_{ij}$*.*

Proof. Let us first simplify the notation by dropping indices i, j, since we work on $\hat{\omega}_i$ and must check the result for a fixed index j. So, we note $z^k = z_{ij}^k$ and $c^k = f_{ij}(\bar{a}_{ij} + \frac{k}{l}\hat{a}_{ij})$. Remark then that, due to (3.6), any feasible solution of $\hat{\omega}_i(x, \Gamma_i)$ with more than one fractional variable z^k has the form $z = \{1, \ldots, 1, z^{\underline{k}}, \ldots, z^{\bar{k}}, 0, \ldots, 0\}$ with $1 > z^{k'} > z^k > 0$ for $\underline{k} \leq k' < k \leq \bar{k}$. For indices $\underline{k}, \bar{k}$, we can extract their contribution to the objective function in ω_i from

$$c^{\underline{k}-1}\left(z^{\underline{k}-1} - z^{\underline{k}}\right) + c^{\underline{k}}\left(z^{\underline{k}} - z^{\underline{k}+1}\right) + c^{\bar{k}-1}\left(z^{\bar{k}-1} - z^{\bar{k}}\right) + c^{\bar{k}}\left(z^{\bar{k}} - z^{\bar{k}+1}\right)$$

where only the term $\left(c^{\underline{k}} - c^{\underline{k}-1}\right)z^{\underline{k}} + \left(c^{\bar{k}} - c^{\bar{k}-1}\right)z^{\bar{k}}$ depends on variables $z^{\underline{k}}, z^{\bar{k}}$. From constraint (3.5), $z^{\underline{k}}$ and $z^{\bar{k}}$ have equal contribution to the budget consumption, so one variable can increase if the other decreases by the same magnitude.

Then, for any optimal solution z^* with $1 > z^{\underline{k}} > z^{\bar{k}} > 0$, optimality ensures that $\left(c^{\underline{k}} - c^{\underline{k}-1}\right) = \left(c^{\bar{k}} - c^{\bar{k}-1}\right)$. Setting $z^{\underline{k}} = \min(z^{\underline{k}} + z^{\bar{k}}, 1)$ and $z^{\bar{k}} = \max(0, z^{\underline{k}} + z^{\bar{k}} - 1)$ does not deteriorate the objective value without violating constraints, and reduces the number of fractional z^k by one. Repeating this process iteratively for the new pair $(\underline{k}, \bar{k})$, the obtained solution remains optimal and has a single fractional z^k (or none) at the end. $\square$

Proposition 3.4.2. *For* f_{ij} *concave increasing over* $[\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$*, the linear relaxation of subproblem* $\hat{\omega}_i(x, \Gamma_i)$ *has an optimal solution* $z^* \in \{0, 1\}$*.*

Proof. From lemma 3.4.1, there are at most n fractional variables in the optimal solution of the subproblem, one fractional $z_{ij}^{k_{ij}}$ for each index j. Since these variables $z_{ij}^{k_{ij}}$ all contribute equally to the budget constraint (3.5) and do not interact in (3.6), $z_{ij}^{k_{ij}}$ can increase if $z_{ij'}^{k_{ij'}}$ decreases by the same amount. Then for z^*, $\left(c_{ij}^{k_{ij}} - c_{ij}^{k_{ij}-1}\right) = \left(c_{ij'}^{k_{ij'}} - c_{ij'}^{k_{ij'}-1}\right)$. Therefore, setting $z_{ij}^{k_{ij}} := \min(z_{ij}^{k_{ij}} + z_{ij'}^{k_{ij'}}, 1)$ and $z_{ij'}^{k_{ij'}} := \max(0, z_{ij}^{k_{ij}} + z_{ij'}^{k_{ij'}} - 1)$ does not deteriorate the objective value, nor does it violate constraints (3.5)-(3.6), but reduces the number of fractional variables by one (possibly by two if $z_{ij}^{k_{ij}} + z_{ij'}^{k_{ij'}} = 1$). Repeating this process we eliminate one by one all fractional variables without sacrificing optimality, and get an integer z^*. $\square$

From proposition 3.4.2, we can relax the integrality constraints on z_{ij}^k and substitute the continuous relaxation of $\hat{\omega}_i(x, \Gamma_i)$ with its dual to provide a linear robust counterpart for problem P_f.

Theorem 3.4.3. *The robust counterpart of problem P_f, when non-linear function f is approximated by a piece-wise linear function with l pieces, is*

$$
\begin{aligned}
\max\ & \sum_{j=1}^{n} c_j x_j \\
& s.t\ \Gamma_i q_i + \sum_{j=1}^{n} r_{ij}^1 \leq b_i && i = 1, \dots, m \\
& q_i + r_{ij}^k - r_{ij}^{k+1} \geq \left(f_{ij}\left(\bar{a}_{ij} + \frac{k}{l}\hat{a}_{ij}\right) - f_{ij}\left(\bar{a}_{ij} + \frac{k-1}{l}\hat{a}_{ij}\right)\right) x_j && \forall i, j, k \\
& r^{l+1} = 0;\ q, r, x \geq 0
\end{aligned}
$$

Proof. We use proposition 3.4.2 in order to substitute subproblem $\hat{\omega}_i(x, \Gamma_i)$ with its linear relaxation. Noting $c_{ij}^k = f_{ij}(\bar{a}_{ij} + \frac{k}{l}\hat{a}_{ij})$, by dualization we get

$$
\begin{aligned}
\max\ & \sum_{j=1}^{n} \sum_{k=1}^{l} c_{ij}^k x_j \left(z_{ij}^k - z_{ij}^{k+1} \right) & = \quad & \min \Gamma_i q_i + \sum_{j=1}^{n} r_{ij}^1 \\
s.t\ & \sum_{j=1}^{n} \sum_{k=1}^{l} z_{ij}^k \leq \Gamma_i & & s.t.\ q_i + r_{ij}^k - r_{ij}^{k+1} \geq \left(c_{ij}^k - c_{ij}^{k-1} \right) x_j \ \forall j, k \\
& 0 \leq z_{ij}^l \leq z_{ij}^{l-1} \leq \dots \leq z_{ij}^1 \leq 1\ \forall j & & q_i, r_{ij}^k \geq 0 \qquad \forall j, k
\end{aligned}
$$

where $z_{ij}^{l+1} = 0$, $c_{ij}^0 = f_{ij}(\bar{a}_{ij})$, q_i is the dual variable associated to the uncertainty budget, r_{ij}^k is the dual variable associated with the consumption of the k^{th} segment of the uncertainty interval for j. By strong duality, we can substitute subproblem $\hat{\omega}_i(x, \Gamma_i)$ with the above dual program to obtain the robust counterpart. □

In section 3.5 we carry out numerical experiments to test the above approach for robust problems with non-linear functions f of various types.

3.5 Numerical Experiments

In this section we present experiments conducted to test the performance of the robust models in sections 3 and 4. All tests were performed with commercial solver CPLEX 12.0. For testing the piece-wise linear approximation of non-linear functions f, we use benchmark or transformed instances of the Generalized Assignment Problem (GAP) and the Knapsack Problem (KP). We set a number $l = 10$ of pieces for the piecewise linear function. Preliminary tests show that this parameter l is not critical (the robust subproblem being an LP), and computation times are comparable between the binary approximation and the l-pieces approximation.

For each problem and each instance we perform 100 runs and for each run $t = 1, \ldots, 100$, we generate random values of coefficients a_{ij} denoted by a^t_{ij} (a^t_j for KP that has a single constraint), that respect the uncertainty set and the Γ-budgets of varying coefficients (i.e. the total deviation of coefficients is equal to the protection level Γ). For each run t, we compare the solution R_{pw} of the piecewise-linear approximation of the robust model to the solution R_b of the binary approximation and the nominal solution, denoted by N. For comparing the robust and nominal solutions, we use two measures of infeasibility and one measure of solution quality:

- %I : the percentage of the 100 runs where the solution is infeasible,
- % viol : the average over runs $t = 1, \ldots, 100$ of the maximum constraint violation $viol^t(x)$ (in %) of solution x on instance t, where

$$viol^t(x) = \max_i \left[\frac{\left(b_i - \sum_{j=1}^n a^t_{ij} x_j\right)^+}{b_i} \right]$$

- Δv : the average loss of objective value (in %) of the solution vs the true optimum of the run, over the restricted subset of runs t where the solution is feasible.

Note that the piecewise linear approximation of functions f in the robust model may provide a solution R_{pw} that is infeasible for some data variations in the uncertainty set, because of the substitution of the original function by another. This is a major stake of the numerical experiments to see empirically whether feasibility, which is supposed to be guaranteed by the Γ-robustness approach, is maintained or not with the approximation of f. The aim of comparing R_{pw} and R_b is to check that the lighter binary approximation does not outperform the more refined piecewise-linear approximation on both feasibility and solution quality.

We first tested instances of the Generalized Assignment Problem (GAP), considering that uncertainty occurs in resource consumption data. The GAP is to find a minimum-cost assignment of n jobs indexed by $j = 1, \ldots, n$ to m machines with limited capacities, indexed by $i = 1, \ldots, m$. The assignment cost of assigning job j to machine i is denoted by c_{ij}, the number of resource units of machine i consumed by job j, if assigned to this machine, is a_{ij}, and b_i is the number of resource units available on machine i. The classical MIP formulation for the GAP is:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
s.t \quad & \sum_{j=1}^{n} a_{ij}x_{ij} \leq b_i & i = 1, \ldots, m \\
& \sum_{i=1}^{n} x_{ij} = 1 & j = 1, \ldots, n \\
& x_{ij} \in \{0, 1\} &
\end{aligned}$$

Following a common practice in the GAP literature based on the benchmark instances of [26], we generate type A instances with assignment costs c_{ij}=U(15,25) (where U(a,b) means the Uniform distribution between a and b), resource consumptions a_{ij}=U(5,25), and machine capacities $b_i=\sum_j 0.8a_{ij}/\mathrm{m}$. Furthermore, we also consider instances B and C, with a_{ij}=U(10,25) and U(15,25), respectively. We look at three problem sizes: Small (n=15, m=5), Medium (n=60, m=5), and Large (n=60, m=10), which also differ in the average machine load (with the Medium instances being the busiest).

We test two non-linear shapes for functions f_{ij}: a square root and an inverse quadratic function, which are both concave increasing. Since the results are consistent for both kinds of functions, we left the study of the inverse quadratic function in the Appendix. For the square root function, we took $f_{ij}(a_{ij}) = \bar{a}_{ij} + \sqrt{\hat{a}_{ij}(a_{ij} - \bar{a}_{ij})}$ for $a_{ij} \in [\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$ (the results do not depend on the shape of the function for $a_{ij} < \bar{a}_{ij}$, we simply need to assume it is also non-decreasing on that side of the uncertainty interval). With this parametrization of the square root function, both a_{ij} and $f_{ij}(a_{ij})$ vary in $[\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$, and the coefficient varies as $\bar{a}_{ij} + \sqrt{z_{ij}}\hat{a}_{ij}$ for positive deviations $z_{ij} = (a_{ij} - \bar{a}_{ij})/\hat{a}_{ij}$.

Finally, we test three uncertainty protection levels Γ. The results are described in Table 3.5.1, where simulated resource consumptions a_{ij}^t were generated in the uncertainty set, for each run t, as explained in the beginning of this section.

In Table 3.5.1, we can see that the R_{pw} robust solution largely outperforms the nominal solution (N), which is very often infeasible. Moreover, the piecewise linear approximation of non-linear impact never sacrifices feasibility, which is ensured for 100% of the $9 \times 3 \times T = 2700$ instances. The average loss of value of the robust solution vs the true optimum of the run is reasonably low (an order of 5 % in average, with a range from 1 % to 10 % for the Small B-type instances). When comparing to the R_b solution, it turns out that the binary approximation provides infeasible solutions only for a low proportion of instances (up to 10 % of infeasibility for Small B-type instances and $\Gamma = 1$). R_b remains feasible for more than 99 % of all instances, with a better average objective value (higher Δv) in that case, which was expected since R_b is less protective. We could say then that the price to pay to ensure a 100 % feasibility for R_{pw} is high compared to R_b in that case.

However, we will see in the sequel that this "price to pay for 100% feasibility" compared to the binary approximation depends on the problem structure. This price was particularly high for the above GAP instances for the following reason linked to the "sparsity" of the problem that dilutes the impact of non-linear variations. Indeed, dispatching the jobs among a large number m of machines means a

Table 3.5.1: Comparison of performance measures for the GAP for square root functions f

			%I			%viol			Δv		
		Γ	$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b^*	N
S	A	1	-	-	88	-	-	3	**0.8**	0.8	0.0
		2	-	-	93	-	-	4.1	**4.7**	2.2	0.0
		3	-	-	100	-	-	4.8	**4.4**	4.4	-
	B	1	-	10	57	-	1.1	3.1	**7.6**	2.1	0
		2	-	-	89	-	-	3.9	**10**	10	0
		3	-	-	97	-	-	4.8	**10**	10.7	
	C	1	-	1	36	-	1.5	2.5	**4.7**	4.3	0.1
		2	-	-	58	-	-	3.6	**4.8**	4.8	0.2
		3	-	-	80	-	-	3.8	**4.1**	4.1	0.1
M	A	1	-	-	60	-	-	1.0	**0.4**	0.3	-
		2	-	-	80	-	-	1.2	**0.9**	0.1	-
		3	-	-	85	-	-	1.6	**1.2**	0.9	0.1
	B	1	-	6	95	-	0.2	1.0	**4.2**	1.4	0.0
		2	-	-	100	-	-	1.3	**5.9**	2.2	-
		3	-	-	100	-	-	2.0	**6.9**	3.8	-
	C	1	-	2	90	-	0.1	1.0	**4.4**	1.9	0.1
		2	-	-	95	-	-	1.2	**6.2**	3.0	0.1
		3	-	-	100	-	-	1.8	**7.0**	3.9	-
L	A	1	-	-	73	-	-	1.8	**0.9**	0.6	0.3
		2	-	-	90	-	-	1.9	**1.1**	0.8	0.4
		3	-	-	98	-	-	1.9	**1.4**	1.3	0.5
	B	1	-	-	80	-	-	1.5	**2.7**	1.4	0.1
		2	-	-	98	-	-	1.8	**3.8**	2.2	0.0
		3	-	-	100	-	-	2.3	**5.3**	3.2	-
	C	1	-	3	90	-	0.4	1.4	**6.7**	3.6	0.1
		2	-	-	95	-	-	1.9	**8.9**	5.7	0.1
		3	-	-	100	-	-	2.3	**9.7**	8.9	-

*: computed over the subset of runs where R_b is feasible.

low average number of jobs assigned to a machine, and many variables x_{ij} equal to zero for a given row i in the optimal solution. Therefore, the simulated variations in the uncertainty set are likely to hit coefficients a_{ij} with $x_{ij} = 0$ at optimality, hence with no impact. This favors feasibility for R_b which, being less constrained, provides a better value when the solution is feasible.

Another remark from Table 3.5.1 is that when the protection level is close to the average number of jobs per machine, i.e., $\Gamma = n/m$, R_{pw} and R_b behave the same way. For example, for Small GAP instances with $(n, m) = (15, 5)$ and average machine load $n/m = 3$, setting $\Gamma = 3$ achieves almost full protection for both robust models because the worst-case to be protected from will be obtained by an integer variation of 3 coefficients a_{ij} to their extreme value (assuming the load is exactly 3 jobs in that row/machine). Then, as coefficients will vary only to the bounds of their variation interval, modelling non-linearity does not help and R_b converges to R_{pw} in that case as observed in the table.

We now test another "less sparse" problem structure, namely the Knapsack Problem (KP), where the percentage of non-zero variables in the optimal solution is much higher. We shall see that this conducts to very different results, favoring more largely the piecewise-linear approximation model R_{pw}. To do so, we keep the GAP instances A,B,C and $n = 15$ but with a single machine ($m = 1$). Here, having $\Gamma = 3$ implies a protection against a variation of 20 % of the coefficients which really has an impact. A full summary of the results is shown in Table 3.5.2.

First, we can see again that R_{pw} provides a more conservative solution than R_b, as evidenced by the larger loss of value, but infeasibility is now much larger for the binary approximation R_b. Second, we see a marked difference between the type-A instances and types B and C. While the easier instances A show no infeasible runs for R_b, these are much more common in the harder ones, reaching 45% in the worst case.

We analyze this difference as follows: although the binary-approximation model protects the solution against a linear variation, it is protecting from variations on the largest coefficients in the solution; on the other hand, the random nature of our

Table 3.5.2: Comparison of performance measures for KP, for square root functions f

	Γ	%I			%viol			Δv		
		$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b^*	N
A	1	-	-	86	-	-	1.1	**1.1**	0.3	-
	2	-	-	100	-	-	2.5	**2.6**	1.4	-
	3	-	-	100	-	-	4.5	**2**	0.8	-
B	1	-	38	75	-	0.4	1	**2.7**	0.1	-
	2	-	13	100	-	0.4	2.5	**4.4**	0.6	-
	3	-	15	100	-	0.5	4.1	**4.5**	0.6	-
C	1	-	45	99	-	0.3	2	**4.9**	0.6	-
	2	-	36	100	-	0.6	3.5	**3.5**	0.8	-
	3	-	4	100	-	0.4	5.5	**2.6**	0.9	-

*: computed over the subset of runs where R_b is feasible.

experiments distributes the variations among all coefficients. Then the linear protection on the largest elements is enough to offset the uncertainty when the largest coefficients are indeed significantly *larger* than the average, as in A instances. On the B and C instances, since all coefficients are more closely distributed, the non-linear variations (even divided among all coefficients) are able to overcome the linear protection on the largest elements.

As a reminder, the (similar) results for the other inverse-quadratic function f can be found in the Appendix.

3.6 Conclusion

We extended the classical Γ-robustness approach to optimization problems where uncertainty induces variations on certain data, with non-linear impact on the coefficients of the constraints. We showed that in that case, approximating the non-linear function by a piecewise-linear function provides a robust counterpart of the problem that remains a Linear Program, hence tractable. To our opinion, the result is not trivial as the decomposition of the non-linear function into pieces induces a discrete structure that in general cannot be systematically dualized for the subproblem of finding the worst-case variation in a constraint,. We tested the performance of this robust model and a simpler binary approximation for various non-linear functions and two classical optimization problems (Generalized Assignment and Knapsack). The results show that the relative performance of the robust solution depends on the problem structure, in particular the relative sparsity or density of the constraint matrix. They also evidenced that on the 2700 instances tested for each problem, the piecewise-linear approximation of the non-linear function kept ensuring feasibility of the robust solution over data variations, which empirically validates the approach.

3.A Numerical results for inverse-quadratic functions

As mentioned before, the results for an inverse-quadratic function f being very similar to those of the square-root functions, they appear in the Appendix and similar comments as those of section 3.5 apply to tables 3.A.2 and 3.A.1. For the inverse-quadratic function, we tested coefficients $\bar{a}_{ij} + (1 - (z_{ij} - 1)^2)\hat{a}_{ij}$ which also vary in $[\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$ for positive deviations $z_{ij} \in [0, 1]$. This setting is equivalent to $f_{ij}(a_{ij}) = \bar{a}_{ij} + (1 - (\frac{a_{ij} - \bar{a}_{ij}}{\hat{a}_{ij}} - 1)^2)$.

Table 3.A.1: Comparison of performance measures for the GAP, for inverse-quadratic functions f

			%I			%viol			Δv		
		Γ	$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b^*	N
		1	-	-	82	-	-	3.1	**0.9**	0.9	0
	A	2	-	-	92	-	-	4	**4.7**	2.2	0
		3	-	-	100	-	-	5.3	**6.1**	4.5	-
		1	-	6	53	-	1.5	3.1	**7.7**	2.1	0
S	B	2	-	-	88	-	-	4.1	**10.1**	10.1	0
		3	-	-	95	-	-	5.3	**10.1**	10.8	0.1
		1	-	1	35	-	1.3	2.5	**4.9**	4.5	0.1
	C	2	-	-	57	-	-	3.6	**4.8**	4.8	0.2
		3	-	-	83	-	-	3.6	**4.1**	4.1	0.1
		1	-	1	57	-	0	1	**0.6**	0.3	0
	A	2	-	-	80	-	-	1.1	**0.9**	0.1	-
		3	-	-	85	-	-	1.5	**0.8**	0.9	0.1
		1	-	1	66	-	0.4	0.6	**1.4**	1	0.2
M	B	2	-	-	96	-	-	0.8	**2.3**	1.3	0.2
		3	-	-	100	-	-	1.2	**3.5**	2.5	-
		1	-	4	87	-	0.1	1	**2.8**	1.9	0
	C	2	-	-	95	-	-	1.2	**4.7**	3	0.1
		3	-	-	100	-	-	1.8	**6.7**	3.9	-
		1	-	-	69	-	-	2	**0.9**	0.6	0.4
	A	2	-	-	86	-	-	2.1	**1.2**	0.8	0.3
		3	-	-	98	-	-	1.9	**1.5**	1.3	0.2
		1	-	-	79	-	-	1.5	**1.7**	1.4	0.1
L	B	2	-	-	96	-	-	1.9	**4.1**	2.1	0.2
		3	-	-	100	-	-	2.3	**6.3**	3.2	-
		1	-	3	88	-	0.6	1.5	**5.3**	3.7	0.1
	C	2	-	1	94	-	0.6	2	**8.8**	5.8	-
		3	-	-	100	-	-	2.4	**11.9**	8.2	-

*: computed over the subset of runs where R_b is feasible.

Table 3.A.2: Comparison of performance measures for KP, for inverse quadratic functions f

		%I			%viol			Δv		
	Γ	$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b	N	$\mathbf{R}_{pw}$	R_b^*	N
A	1	**-**	-	85	**-**	-	1.0	**1.2**	0.4	-
	2	**-**	-	100	**-**	-	2.4	**1.4**	1.4	-
	3	**-**	1	100	**-**	0.1	4.5	**2.0**	0.8	-
B	1	**-**	33	71	**-**	0.4	1.0	**1.5**	0.1	-
	2	**-**	11	98	**-**	0.4	2.4	**1.6**	0.6	-
	3	**-**	12	100	**-**	0.5	4.0	**5.0**	0.6	-
C	1	**-**	38	99	**-**	0.3	1.9	**2.2**	0.6	-
	2	**-**	38	100	**-**	0.5	3.4	**3.3**	0.9	-
	3	**-**	3	100	**-**	0.3	5.4	**2.2**	1.0	-

*: computed over the subset of runs where BS is feasible.

4

Robust location of new housing developments using a choice model

4.1 Introduction

Housing supply is a complex problem arising in multiple and diverse contexts. Driant [28], Gilbert [39], and Sandoval [66] all provide examples of different problematic situations related to housing supply. In France, housing supply is an important issue with a shortfall of 800,000 to 1 million dwellings in 2012 according to Driant. While housing policies mainly consist in building new housing, [28] addresses the existence of a mismatch between demand and offer. The more recent study of [64] confirms for France a constant decrease in the number of commencements of new housing units, with a "gradual but sustained long-term downward trend" (French National Institute of Statistic and Economic Studies (INSEE) data

2015).

At the same time, investments in new residential buildings have been increasing. To cite an example, in the U.S., the value of residential construction by the private sector alone has increased from $208 B in 1993 to $336 B in 2013 according to information from the United States Census Bureau [70]. In France, INSEE and Eurostat, the statistical office of the European Union, paint a similar picture: a somewhat stable number of new dwellings and constantly increasing construction costs.

Failing efforts at solving this problem see a proliferation of inadequate housing encouraged by the rising prices and the social policies which exacerbate the original issue, prompting the need for a better solution as to what type of and where to build new housing units [28]. The issue of choosing locations for new housing developments in a network of possible sites can be seen as a more general facility location problem. The allocation of customers to the selected locations and the rules governing said allocation play a fundamental role in the solution of the problem.

Contrary to the supply chain sector where the decision-maker can assign supply sites to customers who are generally indifferent to the sourcing, in housing development the customers actually choose themselves their location to buy a flat, which naturally conducts to use a choice model based on consumers' utilities and willingness to buy (see the reference paper of Guadagni and Little [44] for the Multinomial Logit (MNL) choice model, following the seminal work of McFadden [53].

Choice models have already been used in the context of housing supply, but not in an optimization framework to our knowledge. Bulent and Kenneth [22] and Pagliara et al. [58] propose the use of scoring methods to address the decision of buyers in the housing context. Pagliara et al. [58] use a Stated Preference model to estimate the willingness to pay of a specific buyer for each alternative available, which is then used to estimate the probability of choosing a housing unit. Similarly, Bulent and Kenneth [22] propose a discrete choice model to compute this probability. Additionally, de Palma et al. [59] develop a model for residential location choice where capacity constraints are considered. This chapter presents the first approach in the new housing sector combining a choice model for consumers' behavior and an optimization model for housing units selection.

When dealing with optimization and especially facility location, Haase and Müller [47] have tested different linearizations of the multinomial logit model (MNL) for the problem of selecting a subset of r sites maximizing coverage, i.e. the part of the demand that does not go to competitors (the initial problem was defined in Benati and Hansen [10]). They found the formulation of Aros-Vera et al. [7], which extends the simple formulation of [46], to achieve the best performance. Later, in Müller and Haase [56] they extend this model to include customer segmentation in the retail industry. They used segmentation to reduce the predictive bias of the MNL. In [46], a linearization of the Independence of Irrelevant Alternatives (IIA) property of choice models was also proposed for the case of binary variables associated with opening a site or not.

When positioning this chapter compared to the work of Haase and Muller, beyond our specific new housing context we can distinguish two main features:

(i) Their work deals with maximizing the total demand covered, where part of the demand can flow to competitors. In our work all demands flow to the set of sites of the new housings developer as we consider well-identified customers engaged in a program, but to avoid drop-outs we maximize the average satisfaction of potential buyers.

(ii) We introduce a robustness approach where uncertainty lies on customers' demands and preferences. Due to (i), the robust model has the particularity that the demand parameters subject to uncertainty appear both in the constraints and the objective function.

Although not strictly on housing, Angilella et al [6] address the problem of developing sites among potential locations, using a Non Additive Robust Ordinal Regression approach with a Choquet integral preference model. While the authors do not use Robust Optimization in their approach, the paper does include a notion of robustness in its methodology.

Note that, although we carry numerical experiments on instances of the Paris region, our model can easily adapt to other geographical contexts. The main adjustment would be the calibration of the utility functions with possibly new criteria added to take into account specific local, regional or national features. The issue of calibrating the utility functions in a choice model for housing has already been studied in the literature (e.g. [22, 58, 59]).

Robustness approaches are natural for strategic facility location since decisions have a long term impact which is subject to uncertainty. The facility location literature typically considers uncertainty in demand quantities and service costs [1, 9, 45, 68]. Snyder [68] provides an extensive review of the literature on facility location under uncertainty where problems are classified according to the type of objective function stochastic location problems (mean-outcome models, probabilistic guaranties, where information on the underlying probability distribution of the uncertainty is required), and robust location problems.

Baron et al [9] and Gülpinar et al [45] deal with a robustness formulation of facility location problems. [9] models a location problem under box and ellipsoidal uncertainty, comparing the results in terms of number of facilities used, capacity assigned to them, and profit change w.r.t. the nominal solution. The authors also test different distribution functions for the demand and different conservativeness of the solution. [45] addresses the issue of uncertainty in the distribution functions and asymmetric uncertainty sets. The authors explore the impact of the estimated moments of the distributions and that of the cost parameters. [1] deals with the uncertainty in costs using a minimization of regret approach over a set of scenarios while the uncertainty in demands is handled using stochastic optimization. As mentioned before, this paper focuses on demand uncertainty.

As for robust optimization approaches in general, let us cite Soyster's maxmin approach [69] for constraint-wise uncertainty; minmax regret (as presented in Averbakh [8]); limited worst case scenarios: Γ-Robustness (Bertsimas and Sim [18]) and p-Robustness (as applied in Ramos et al. [63]); *bw*-robustness (Roy [65]); and refer the reader to Chapter 2 and the review by Gabrel et al. [37] for an in-depth discussion of theory and applications.

Goerigk and Schöbel [40] recently proposed a two stage approach to handle uncertainty which they call *Recovery-to-optimality*. It consists in computing recovery costs to optimal solutions and minimizing either the maximum or the average cost. See also the papers of Billionnet et al. [20] and Gabrel and Murat [35] dealing with uncertainty on the right hand-sides of constraints (column-wise uncertainty instead of row-wise); and Le and Knust [50] dealing with strict and adjustable robustness using MIP formulations. Denoyel et al. [27] also studied a robust selection problem for healthcare using a choice model, but using a fractional formulation instead of linear formulation of the IIA property.

The chapter is structured as follows. Section 4.2 presents the optimization problem and the underlying choice model for customers' willingness to buy in a given location. Section 4.3 describes the robust counterparts of the location problem, where uncertainty lies first on demand volumes, then on customers preferences. Section 4.4 reports numerical results on a study in the Paris region and provides an analysis of the results as well as managerial insights for new housing development.

4.2 Problem statement

We consider a set of demand sources indexed by $i \in \{1, \ldots, m\}$, with D_i the number of potential buyers of source i interested in new housing, and $V = \sum_i D_i$ the total number of potential buyers. Note that indices i can aggregate multiple smaller demand sources, in order to keep the optimization problem tractable. We also have a set of possible locations for constructing new housing developments, indexed by $j \in \{1, \ldots, n\}$, with a variable cost c_j per housing unit built in location $j \in J$ (which is mainly a construction cost but can include other variable costs such as local taxes and management costs). The total cost of constructed housings should not exceed a given budget B, and the number of selected locations should be no more than a given bound P, to avoid providing potential buyers a plethoric offer and to limit high fixed costs associated with opening multiple programs. The number P of housing developments is also limited by the market possibilities and the financial and debt capabilities of the company. Finally, the choice model that computes the volumes flowing from demand sources to selected locations, requires to know customer preference data, namely the utility u_{ij} of a potential buyer from demand source i for choosing location j. In a choice model where all locations would be open, the probability p_{ij} that i chooses location j is given by equation (4.1)

$$p_{ij} = \frac{e^{u_{ij}}}{\sum_{k=1}^{n} e^{u_{ik}}} \tag{4.1}$$

where the utility u_{ij} aggregates criteria or attributes based on the characteristics of the housing unit, and the preferences of the potential buyer. Pagliara et al. [58] relate u_{ij} to the price of the housing unit, cost of traveling, location in the city, type of dwelling, among others. Bulent and Kenneth [22] also consider the property tax of the location. As in the literature, we consider the utility as a linear combination of L location characteristics and their relative value to the potential customers, so that

$$u_{ij} = \sum_{l=1}^{L} w_{ij}^{l} a_{ij}^{l} \tag{4.2}$$

where a_{ij}^{l} is the value of criterion l that a customer from demand source i perceives from choosing location j, and w_{ij}^{l} is a coefficient associated with criterion l, which can be a regression parameter or a weight. In this chapter we mainly focus on price and distance attributes, as will be detailed later.

Since the subset of constructed locations is not known in advance, we introduce binary decision variables $y_j = 1$ if location j is selected for constructing new housing, 0 otherwise. The utilities are used to compute the distribution flows x_{ij}, i.e., the fraction of the demand of source i that flows from source i to location j:

$$x_{ij} = \frac{e^{u_{ij}} y_j}{\sum_{k=1}^{n} e^{u_{ik}} y_k} \tag{4.3}$$

Distribution flow x_{ij}, as the ratio of two linear functions of variables y_j, is non linear. The New Housing Location Problem (NHLP) considered can be formulated as follows:

$$\max \quad \frac{1}{V}\sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij} D_i x_{ij} \tag{4.4}$$

$$s.t. \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_j D_i x_{ij} \leq B \tag{4.5}$$

$$\sum_{j=1}^{n} y_j \leq P \tag{4.6}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \tag{4.7}$$

$$x_{ij} = \frac{e^{u_{ij}} y_j}{\sum_{k=1}^{n} e^{u_{ik}} y_k} \qquad \forall (i,j) \tag{4.8}$$

$$x_{ij} \geq 0 \qquad \forall (i,j)$$

$$y_j \in \{0, 1\} \qquad \forall j$$

The objective function (4.4) maximizes the average utility of the potential buyers, in order to avoid drop-outs as much as possible. Constraints (4.5) and (4.6) are the budget constraints for the construction of the whole set of new developments. Constraints (4.7) assign the total demand of a source node to the selected locations, and distribution flows are modeled by (4.8). In the spirit of Haase, we use the Irrelevance of Independent Alternatives (IIA) property of choice models to linearize the above formulation as follows.

Proposition 4.2.1. *The non-Linear model (4.4)-(4.8) is equivalent to the following*

Linear Program NHLP:

$$\max \quad \frac{1}{V}\sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}D_i x_{ij} \tag{4.9}$$

$$s.t. \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_j D_i x_{ij} \leq B \tag{4.10}$$

$$\sum_{j=1}^{n} y_j \leq P$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \tag{4.11}$$

$$x_{ij} \leq y_j \qquad \forall (i,j) \tag{4.12}$$

$$x_{ik}e^{u_{ij}-u_{ik}} + (2 - y_j - y_k) \geq x_{ij} \qquad \forall (i,j,k) | j \neq k \tag{4.13}$$

$$x_{ij} \geq 0 \qquad \forall (i,j)$$

$$y_j \in \{0, 1\} \qquad \forall j$$

Proof. The linearization relies on the Independence of Irrelevant Alternatives (IIA) property of the MNL. IIA has been previously identified in the literature and simply states that equations (4.8) are equivalent to $x_{ij}/x_{ik} = e^{u_{ij}}/e^{u_{ik}}$ for all (j,k) such that $y_j = y_k = 1$. This can be formulated by linear constraints (4.12) and (4.13) similarly as in [46]. Indeed, if $y_j = 0$ then $x_{ij} = 0$ by (4.12) so equations (4.8) are true for all (i,j) in that case. If $y_j = 1$, then for all k such that $y_k = 1$, by (4.13) we get $x_{ij}e^{u_{ik}} = x_{ik}e^{u_{ij}}$ so there exists $a > 0$ such that, given i, $x_{ik}/e^{u_{ik}} = a$ for all k. By summing over all k one obtains $a = \frac{\sum_k x_{ik}}{\sum_k e^{u_{ik}} y_k} = \frac{1}{\sum_k e^{u_{ik}} y_k}$ using (4.11). Since $x_{ij} = a\, e^{u_{ij}} y_j$ we get equation (4.8). □

Note that the NHLP model contains $n(m+1)$ variables and $O(mn^2)$ constraints of type (4.13). Section 4.4 will report numerical experiments to analyze the tractability of the above formulation, and the consistency of optimal locations with the attributes of the choice model, which will be described more precisely in that section. The next section presents a Robust Optimization (RO) approach for the

New Housing Development Problem.

4.3 Robust Optimization formulations for the NHLP

We consider uncertainty on the demand data of Model NHLP, both volumes and customer preferences. It is quite common in the literature of Robust Optimization to consider variations in demands. Especially in new housing development, demands are more customers intents than real orders, as the locations are not chosen yet. Moreover, given the duration of the process from development planning to building/selling the units in it, said intentions are subject to change. Decisions that are robust to uncertainty on demands are thus more than needed for protecting oneself from possible variations.

The research by Bertsimas and Sim [18], which aims at protecting the decision-maker against *infeasibility* over all possible variations of coefficients in a given uncertainty set, is the basis of this approach. This methodology has been previously used to address uncertainty in diverse contexts: Álvarez et al [5] deal with yield uncertainty; Zokaee et al. [71] tackle a supply chain design with uncertainty in demands, costs, and supplies; Bohle et al. [21] solve a scheduling problem with uncertainty in productivity; and Bertsimas and Thiele [19] address an inventory problem with uncertain demands. For an in-depth discussion of the methodology, the reader is referred to section 2.2.

We now describe the robust counterpart of the NHLP when there is uncertainty on demand volumes first, then on customer utilities. Demands and utilities have in common that they both appear in the objective function and the budget constraint, which requires to define two categories of deviation variables in each robust model.

4.3.1 Uncertainty on demand volumes

We first consider uncertainty on demands such that $D_i \in [\bar{D}_i - \hat{D}_i, \bar{D}_i + \hat{D}_i]$, where $\hat{D}_i$ is the maximum variation of demand. A robust approach with this respect seeks to maximize the worst-case average utility of potential buyers over all possible variations of the demand parameters, while ensuring feasibility of the solution in any case which, given a subset of selected locations, is only critical for the budget constraint (4.10).

Note that, since demand data D_i do not intervene in equations (4.13), the proportion of demand x_{ij} is not affected by fluctuations to $\bar{D}_i$. Model NHLP can then be rewritten with a protection function $\beta(x, \Gamma)$, where Γ is the uncertainty budget, to ensure feasibility, and a function $\delta(x, \Gamma)$ to protect the objective as:

$$\max \quad \frac{1}{V} \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} \bar{D}_i x_{ij} + \delta(x, \Gamma) \tag{4.14}$$

$$s.t. \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_j \bar{D}_i x_{ij} + \beta(x, \Gamma) \leq B \tag{4.15}$$

$$(4.11) - (4.13)$$

where

$$\begin{aligned} \beta(\mathbf{x}, \Gamma) = \max_{z} & \sum_{i=1}^{m} \sum_{j=1}^{n} c_j \hat{D}_i x_{ij} z_i \\ s.t. & \sum_{i=1}^{m} |z_i| \leq \Gamma \\ & -1 \leq z_i \leq 1 \qquad \forall i \end{aligned}$$

$$\begin{aligned} \delta(\mathbf{x}, \Gamma) = \min_{z'} & \frac{1}{V} \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} \hat{D}_i x_{ij} z'_i \\ s.t. & \sum_{i=1}^{m} |z'_i| \leq \Gamma \\ & -1 \leq z'_i \leq 1 \qquad \forall i \end{aligned}$$

Note that, despite the fact that the same parameter D_i varies in the objective function and the budget constraint, one actually needs two sets of variables z_i and z'_i in the above model since the deviation vector z that achieves the worst added cost for the budget constraint is generally not the same that achieves the worst loss of utility in the objective function.

Proposition 4.3.1. *The robust counterpart of model NHLP (4.9)-(4.13) is the following model* ***R-NHLP***

$$
\begin{aligned}
\max\ & \frac{1}{V}\sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}\bar{D}_i x_{ij} - \Gamma q^{\delta} - \sum_{i=1}^{m} r_i^{\delta} && \textbf{R-NHLP}\\
s.t. & \sum_{i=1}^{m}\sum_{j=1}^{n} c_j\bar{D}_i x_{ij} + \Gamma q^{\beta} + \sum_{i=1}^{m} r_i^{\beta} \le B \\
& q^{\delta} + r_i^{\delta} \ge \frac{1}{V}\hat{D}_i \sum_{j=1}^{n} u_{ij}x_{ij} && \forall i\\
& q^{\beta} + r_i^{\beta} \ge \hat{D}_i \sum_{j=1}^{n} c_j x_{ij} && \forall i\\
& (4.11) - (4.13)\\
& q, r, x \ge 0
\end{aligned}
$$

Proof. In the sense of [18], the protection function $\beta(\mathbf{x}, \Gamma)$ for (4.15) is the optimal value of the problem:

$$
\begin{aligned}
\max_{z}\ & \sum_{i=1}^{m}\sum_{j=1}^{n} c_j\hat{D}_i x_{ij} z_i & & & \min\ & \Gamma q^{\beta} + \sum_{i=1}^{m} r_i^{\beta} & \\
s.t. & \sum_{i=1}^{m} z_i \le \Gamma & & \iff & s.t.\ & q^{\beta} + r_i^{\beta} \ge \hat{D}_i \sum_{j=1}^{n} c_j x_{ij} & \forall i\\
& z_i \le 1 & \forall i & & & q^{\beta} \ge 0 & \\
& z_i \ge 0 & \forall i & & & r_i^{\beta} \ge 0 & \forall i
\end{aligned}
$$

where dual variable q^{β} is associated with the uncertainty budget constraint, and each dual variable r_i^{β} is associated with constraint $z_i \le 1$ for $i = 1, \ldots, m$. Similarly, the protection function $\delta(\mathbf{x}, \Gamma)$ for (4.14) is the optimal value of:

$$
\begin{aligned}
&-\max_{z'} \frac{1}{V} \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} \hat{D}_i x_{ij} z'_i & & & & -\min\ \Gamma q^{\delta} + \sum_{i=1}^{m} r_i^{\delta} & \\
&s.t. \sum_{i=1}^{m} z'_i \leq \Gamma & & \iff & & s.t.\ q^{\delta} + r_i^{\delta} \geq \frac{1}{V} \hat{D}_i \sum_{j=1}^{n} u_{ij} x_{ij} & \forall i \\
&z'_i \leq 1 & \forall i & & & q^{\delta} \geq 0 & \\
&z'_i \geq 0 & \forall i & & & r_i^{\delta} \geq 0 & \forall i
\end{aligned}
$$

$\square$

The above robust model is an application of the robustness approach of [18] to a specific case where parameters subject to uncertainty appear both in the objective function and constraints; it will provide interesting insights for new housing developments as shown in the numerical results section. The case when uncertainty lies on customer preferences or utilities is, to our mind, quite original on a purely research point of view, as we mix a scenario-based approach and a Γ-budget approach. This strategy leads to a more complex formulation where the distribution flows need to be modeled by specific variables and constraints for each scenario, given the particular structure of the choice model.

4.3.2 Uncertainty on customer utilities for NHLP

The impact of uncertainty on customers' utilities is twofold: a direct impact on the objective function, which can be addressed as in the previous section, with the added consideration of updating the proportion of demand flowing from each demand source to the chosen locations. Contrary to the previous section where demand volumes are continuous within their variation range as in [18], we choose to discretize the set of possible scenarios for utilities, for two main reasons. On one hand, considering continuous utilities leads to intractable non-linearity of the IIA constraints (4.13). On the other hand, since utilities are generally hard to calibrate in a multinomial logit choice model, we structure the uncertainty focusing on the two main criteria that drive customer preferences for new housing: distance (d) and price (p). Hence we consider for each demand source a base (nominal) scenario and two variants: one scenario that is more distance-centric than the base scenario, and one that is more price-centric.

Let us now describe more precisely the characteristics of utilities u_{ij} for our housing problem, and which part of the utility that is impacted by uncertainty. We consider in expression (4.2) that the locations differentiate from one another through the level of the attributes, a_{ij}^l which depend on both i and j, but the coefficient or weight w_i^l only varies with the customer i and not with the location j. Furthermore, only parameters w_i^l are subjected to uncertainty, e.g. from the level of aggregation of the demand sources, changing of preferences with time, errors on the estimation of the parameter. We discretize the set of possible values such as $\tilde{w}_i \in \{w_i^0, w_i^d, w_i^p\}$, where as mentioned before d and p respectively stand for distance and price, and 0 stands for the nominal scenario. This leads to the following three scenarios for a given demand source i:

- nominal (base) scenario : weight vector w_i^0 and utilities $u_{ij}^0 = \sum_{l=1}^{L} w_i^{l_0} a_{ij}^l$
- d-centric scenario : weight vector w_i^d and utilites $u_{ij}^d = \sum_{l=1}^{L} w_i^{l_d} a_{ij}^l$
- p-centric scenario : weight vector w_i^p and utilities $u_{ij}^p = \sum_{l=1}^{L} w_i^{l_p} a_{ij}^l$

Note that the d-centric and p-centric notion is relative to the base scenario. We define binary variables $z_i^d = 1$ if there is a shift from the nominal scenario to the d-centric scenario, and $z_i^p = 1$ if the shift is towards the p-centric scenario for demand source i. $z_i^d + z_i^p = 0$ means that the nominal scenario remains. The constraint $z_i^d + z_i^p \leq 1$ clearly holds, and the uncertain utility $\tilde{u}_{ij}$ can be written as:

$$\tilde{u}_{ij} = u_{ij}^0(1 - z_i^d - z_i^p) + u_{ij}^d z_i^d + u_{ij}^p z_i^p$$

The nominal scenario partly depends on the characteristics of the demand source, e.g. low income households are very sensitive to prices, whereas higher income would lead to a more distance-centric behaviour. Therefore, the nominal scenario being the most likely one, only a limited number of demand sources $\Gamma \in \mathbb{N}$ are allowed to deviate from this base scenario. This leads to add the uncertainty budget constraint $\sum_{i=1}^{n} z_i^d + z_i^p \leq \Gamma$. As uncertainty holds both in the budget constraint and the objective function, we need to duplicate the above variables z_i^d and z_i^p for feasibility into z'^d_i and z'^p_i for the objective function as in section 4.3.1.

We can rewrite model NHLP using variables z, z', and flow variables x_{ij}^s for the flow of demand in each possible scenario $s = 0, d, p$. A robust approach to Model NHLP seeking to protect the optimal value against the worst realization of the uncertainty and ensure feasibility of the solution can be written as (4.16)-(4.23):

$$\max_{x,y} \min_{\substack{\sum_s \sum_i z'^s_i \leq \Gamma \\ \sum_s z'^s_i \leq 1 \\ z \in \{0,1\}}} \frac{1}{V} \left[\sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij}^0 D_i x_{ij}^0 (1 - \sum_{s \neq 0} z'^s_i) + \sum_{s \neq 0} \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij}^s D_i x_{ij}^s z'^s_i \right] \tag{4.16}$$

$$s.t. \max_{\substack{\sum_s \sum_i z_i^s \leq \Gamma \\ \sum_s z_i^s \leq 1 \\ z \in \{0,1\}}} \sum_{i=1}^{m} \sum_{j=1}^{n} c_j D_i x_{ij}^0 (1 - \sum_{s \neq 0} z_i^s) + \sum_{s \neq 0} \sum_{i=1}^{m} \sum_{j=1}^{n} c_j D_i x_{ij}^s z_i^s \leq B \tag{4.17}$$

$$\sum_{j=1}^{n} y_j \leq P \tag{4.18}$$

$$\sum_{j=1}^{n} x_{ij}^s = 1 \qquad \forall (i,s) \tag{4.19}$$

$$x_{ij}^s \leq y_j \qquad \forall (i,j,s) \tag{4.20}$$

$$x_{ij}^s \leq x_{ik}^s e^{u_{ij}^s - u_{ik}^s} + (2 - y_j - y_k) \qquad \forall (i,j,k,s) | j \neq k \tag{4.21}$$

$$x \geq 0 \tag{4.22}$$

$$y \in \{0,1\} \tag{4.23}$$

Unlike in [18], we keep both positive and negative deviations from the estimated parameter since the direction of the objective function cannot be inferred directly from the direction of change in the parameter. This is a consequence of the behavior of variables x_{ij} in a choice model, and was confirmed by a computational analysis of optimal solutions. Furthermore, the uncertainty subproblem to be dualized is no longer a classical knapsack problem due to this fact. (4.16)-(4.23) can be linearized in a similar matter as that proposed by Bertsimas and Sim using protection functions. (4.17), and analogously (4.16), can be rewritten with a deterministic and an uncertain component as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_j D_i x_{ij}^0 + \beta(x,\Gamma) \leq B, \qquad \text{where}$$

$$\begin{aligned}
\beta(x,\Gamma) = \max \sum_{s\neq 0}\sum_{i=1}^{m}\sum_{j=1}^{n} & c_j D_i (x_{ij}^s - x_{ij}^0) z_i^s \\
s.t. \sum_{s\neq 0}\sum_{i=1}^{m} & z_i^s \leq \Gamma \\
\sum_{s\neq 0} & z_i^s \leq 1 \qquad\qquad \forall i \\
& z \in \{0,1\}
\end{aligned}$$

We now show that we can substitute the above $0-1$ subproblem with its linear relaxation. Doing so, we can replace the linear relaxation of the subproblem by its dual program in the model and find an equivalent formulation.

Lemma 4.3.2. *The linear relaxation of subproblem $\beta(x,\Gamma)$ has an optimal solution such that $z_i^d, z_i^p \in \{0,1\}$ for all $i = 1,\ldots,m$.*

Proof. We transform the constraints into $\sum_i z_i^d + z_i^p + sl_0 = \Gamma$ and $z_i^d + z_i^p + sl_i = 1$ for $i = 1,\ldots,m$, where sl_i, $i = 0,\ldots,m$ are slack variables. In this standard form the LP has $3m+1$ variables and $m+1$ constraints. So we have at most $m+1$ non-zero basis variables. If $sl_0 > 0$, then there is exactly one non-zero variable for each of the m constraints $z_i^d + z_i^p + sl_i = 1$. In that case, one of the variables z_i^d, z_i^p or sl_i is one and the others are zero, and the solution is indeed binary. If $sl_0 = 0$, then there exists a single index i such that exactly two variables are non-zero. For this constraint i, since $\Gamma \in \mathbb{N}$, we necessarily have $sl_i = 0$ otherwise we would not have $sl_0 = 0$. Therefore, $z_i^d + z_i^p = 1$ and both variables are fractional. Since the solution is optimal, we necessarily have $\sum_j D_i c_j (x_{ij}^d - x_{ij}^0) = \sum_j D_i c_j (x_{ij}^p - x_{ij}^0)$ (if we had, for example, $\sum_j D_i c_j (x_{ij}^d - x_{ij}^0) > \sum_j D_i c_j (x_{ij}^p - x_{ij}^0)$, then setting $z_i^d = 1$ and $z_i^p = 0$ would strictly increase the objective value while remaining feasible and the solution would not be optimal). Hence, setting $z_i^d = 1$ and $z_i^p = 0$, or $z_i^d = 0$ and $z_i^p = 1$, does not deteriorate the objective value. □

We can then linearize (4.16)-(4.23) in the sense of [18].

Proposition 4.3.3. *The robust counterpart of (4.16)-(4.23) is the following model **uR-NHLP***

$$
\begin{aligned}
\max \quad & \frac{1}{V}\sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}^0 D_i x_{ij}^0 - \Gamma q^\delta - \sum_{i=1}^{m} r_i^\delta && \textbf{uR-NHLP} \\
s.t. \ & \sum_{i=1}^{m}\sum_{j=1}^{n} c_j D_i x_{ij}^0 + \Gamma q^\beta + \sum_{i=1}^{m} r_i^\beta \le B && (4.24) \\
& q^\delta + r_i^\delta \ge \frac{1}{V} D_i \sum_{j=1}^{n} \left(u_{ij}^0 x_{ij}^0 - u_{ij}^s x_{ij}^s \right) \qquad \forall (i,s) | s \neq 0 \\
& q^\beta + r_i^\beta \ge D_i \sum_{j=1}^{n} c_j (x_{ij}^s - x_{ij}^0) \qquad \forall (i,s) | s \neq 0 \\
& (4.18) - (4.23) \\
& q, r \ge 0
\end{aligned}
$$

Proof. We illustrate the linearization of the equations regarding uncertainty (4.16)-(4.17) with the budget constraint (4.17) for simplicity. By Lemma 4.3.2 when dualizing the linear relaxation of subproblem $\beta(x, \Gamma)$ we indeed obtain the optimal value of its binary 0-1 counterpart. The linear relaxation of this subproblem is equivalent through dualization to:

$$\begin{aligned}
\min\ & \Gamma q^{\beta} + \sum_{i=1}^{m} r_i^{\beta} \\
s.t.\ & q^{\beta} + r_i^{\beta} \geq D_i \sum_{j=1}^{n} c_j (x_{ij}^{d} - x_{ij}^{0}) && \forall i \\
& q^{\beta} + r_i^{\beta} \geq D_i \sum_{j=1}^{n} c_j (x_{ij}^{p} - x_{ij}^{0}) && \forall i \\
& q^{\beta}, r^{\beta} \geq 0
\end{aligned}$$

which can be substituted in (4.17). The same reasoning holds for (4.16). □

We end this subsection by studying a variant of the problem where demand sources have *asymmetric uncertainty budgets*. Indeed, following the Γ-robustness framework developed by Bertsimas and Sim, we previously defined the different elements of the scenario set as deviations from a base (most likely) scenario. One can consider however in the housing context that it is more likely to deviate to the price-centric scenario than the distance-centric scenario, or the reverse. In this case, we need to differentiate uncertainty budgets, i.e., set a distinct Γ_s for each scenario s. This provides subproblem $\beta'(x, \Gamma)$.

$$\begin{aligned}
\beta'(x, \Gamma) = \max & \sum_{s \neq 0} \sum_{i=1}^{m} \sum_{j=1}^{n} c_j D_i (x_{ij}^{s} - x_{ij}^{0}) z_i^{s} \\
s.t. & \sum_{i=1}^{m} z_i^{s} \leq \Gamma_s && \forall s \neq 0 \\
& \sum_{s \neq 0} z_i^{s} \leq 1 && \forall i \\
& z \in \{0, 1\}
\end{aligned}$$

We show that, as in the general case, replacing the above $0-1$ subproblem by its linear relaxation does not change the problem, so the dualization process remains valid also in that case.

Lemma 4.3.4. *The linear relaxation of subproblem $\beta'(x, \Gamma)$ has an optimal solution such that $z_i^d, z_i^p \in \{0, 1\}$ for all $i = 1, \dots, m$.*

Proof. We transform the constraints using slack variables into $\sum_i z_i^d + sl_d = \Gamma_d$, $\sum_i z_i^p + sl_p = \Gamma_p$ and $z_i^d + z_i^p + sl_i = 1$ for $i = 1, \dots, m$. In this standard form the LP has $3m + 2$ variables and $m + 2$ constraints. Therefore there are at most $m + 2$ non-zero basis variables. There are three cases w.r.t variables (sl_d, sl_p): i) $sl_d, sl_p > 0$, ii) $sl_d, sl_p = 0$, or iii) $sl_d \oplus sl_p = 0$. If i), there is exactly one non-zero variable for each of the m constraints $z_i^d + z_i^p + sl_i = 1$, and the solution is binary as in the proof of lemma 4.3.2.

If ii), since $\Gamma_p, \Gamma_d \in \mathbb{N}$, there exists a single pair of indices (i, i') such that exactly two variables are non-zero in the constraints $z_i^d + z_i^p + sl_i = 1$. For this pair (i, i'), the quantity $z_i^d + z_i^p + z_{i'}^d + z_{i'}^p = 2 - (sl_i + sl_{i'})$ is integer as $\sum_i z_i^d$ and $\sum_i z_i^p$ are integer. If we had $sl_i + sl_i = 2$ there would be only one non-zero variable per constraint, not two, so we necessarily have either $sl_i + sl_{i'} = 0$ or $sl_i + sl_{i'} = 1$. In the first case, $z_i^d + z_i^p = 1$ and both variables are fractional. Since the solution is optimal, we necessarily have $\sum_j D_i c_j(x_{ij}^d - x_{ij}^0) = \sum_j D_i c_j(x_{ij}^p - x_{ij}^0)$, and setting $z_i^d = 1$ and $z_i^p = 0$ (or $z_i^d = 0$ and $z_i^p = 1$) does not deteriorate the objective value. The same reasoning follows for i'. If $sl_i + sl_{i'} = 1$, either $z_i^d z_{i'}^d > 0$ and $z_i^p + z_{i'}^p = 0$ or vice-versa. Since the solution is optimal, we necessarily have $\sum_j D_i c_j(x_{ij}^d - x_{ij}^0) = \sum_j D_i c_j(x_{i'j}^p - x_{i'j}^0)$, and setting $(z_i^d, z_{i'}^d, sl_i, sl_{i'}) = (1, 0, 0, 1)$, or $(z_i^d, z_{i'}^d, sl_i, sl_{i'}) = (0, 1, 1, 0)$, does not deteriorate the objective value.

If iii), $sl_d > 0$ and $sl_p = 0$ implies that there exists exactly one constraint $z_i^d + z_i^p + sl_i = 1$ where $z_i^d + sl_i = 1$ since $\Gamma_p \in \mathbb{N}$ and $z_i^d, sl_i > 0$. It necessarily follows that $\sum_j D_i c_j(x_{ij}^d - x_{ij}^0) = 0$, since the solution is optimal. Then setting $z_i^d = 1$ and $sl_i = 0$ (or $z_i^d = 0$ and $sl_i = 1$) does not deteriorate the objective value. The same is true for the case where $sl_p > 0$ and $sl_d = 0$ $\square$

Note that the above approach can be applied to a higher number of scenarios. It is particularly suited for optimization problems with choice models, for which one cannot predict that the worst-case for feasibility will be systematically for positive (or negative) deviations as in [18]. Indeed, a higher probability x_{ij} for a customer i means a lower probability x_{ij} for at least one other client i, so the impact of a deviation on the objective function is hard to predict. Therefore, the subproblem cannot be reduced to deviating or not to a single extreme scenario, which makes the above (more complex) approach relevant. We now present numerical experiments on a dataset of French cities, as well as some managerial insights for housing development.

4.4 Numerical Experiments

4.4.1 Data

Model NHLP was tested on instances built from French data of the Paris region. 30 neighborhoods were selected according to their location and population characteristics. With information gathered from the National Institute of Statistic and Economic Studies (INSEE), we built a number of instances from a combination of the following parameters:

- Number of demand sources $n \in \{30, 40, 50, 60\}$[1]
- Number of possible locations for new housings $m \in \{15, 20\}$
- Maximum number of locations allowed for development $P \in \{2, 4, 6\}$
- Customers' utilities computed from one of two behavioral traits: {distance-centric, price-centric}

The utility combines two criteria a^d and a^p with weight vector w:

- *Distance ratio* : a^d_{ij} is the ratio of the distance between i and j and the average distance from i to all sites (a small ratio is preferred)

$$a^d_{ij} = -\frac{d_{ij}}{\bar{d}_{i\cdot}}$$

[1] Each neighborhood was partitioned into two demand sources according to information available on income level.

where $\bar{d}_{i\cdot} = \sum_j d_{ij}/m$.

- *Relative deviation of Real-estate price vs Purchasing Power*: a_{ij}^p is the deviation in % between the real-estate price per meter-square at location j, denoted by p_j, and the price that demand node i would be willing to pay given its income (Purchasing Power, noted PP_i), where a small deviation is preferred:

$$a_{ij}^p = \frac{|p_j - PP_i|}{PP_i}$$

Let us give more details on how purchasing powers (PP) were estimated in our study. We denote by v_i the city of demand source i. The INSEE French institute provides income data for two different population segments in each city: non-taxed (lower-income) and taxed (higher-income) households. We set $PP_i = p_{v_i}$ if i corresponds to the lower-income population segment of city v_i, and $PP_i = (I_i/\bar{I}_i)p_{v_i}$ if i corresponds to the higher-income population of city v_i, where I_i is the average income of segment i and $\bar{I}_i$ is the average income of city v_i. Indeed, we make the assumption that the average PP of the lower-income segment is the average real-estate price of the city this population lives in, but the higher-income segment has a higher PP, with a multiplicative factor which is the ratio between the segment income and the city income.

As no survey was available with data to calibrate the parameters of the utility function, we scale the two factors a_{ij}^d and a_{ij}^p using z-scores ((value-mean)/standard-deviation), which are noted $\hat{a}_{ij}^d$ and $\hat{a}_{ij}^p$. We obtain comparable measures that can be aggregated to compute utilities u_{ij} using weights which sum over one. Thus we express the utility of a potential buyer i choosing housing in j as:

$$u_{ij} = w_i^d \hat{a}_{ij}^d + w_i^p \hat{a}_{ij}^p$$

where $w_i^d + w_i^p = 1$. We call "price-centric" the case when $(w_i^d, w_i^p) = (1/3; 2/3)$ and "distance-centric" the reverse case $(w_i^d, w_i^p) = (2/3; 1/3)$. We estimated demand volumes using INSEE data on households. The real-estate prices were obtained from MeilleursAgents.com. Table 1 reports data about cities. We selected a set of cities with high diversity in population size and income to get richer instances.

Table 4.4.1: Characteristics of the demand sources

		Price	Taxed HH		Non-Taxed HH	
City	Pop	€/m²	D	Inc (€)	D	Inc (€)
Boulogne-Billancourt	114205	7740	382	57966	146	10280
Argenteuil	103125	2811	189	31193	183	8886
Courbevoie	87469	5847	284	44904	103	9746
Cergy	56988	2534	120	31404	97	9006
Asnières-sur-Seine	82327	4795	216	42371	122	9459
Sarcelles	58614	2139	68	28385	106	8996
Nanterre	89185	4547	175	34116	151	8704
Franconville	33097	2768	86	33564	45	10112
Colombes	85398	4417	190	38949	132	9280
Garges-les-Gonesse	40012	2150	41	25742	79	8607
Rueil-Malmaison	79426	5561	240	54771	79	10993
Pontoise	29548	2594	67	33905	50	9176
Levallois-Perret	64253	7117	216	51419	81	9995
Ermont	27446	2992	67	36381	39	10244
Neuilly-sur-Seine	61754	9350	214	114418	66	13649
Bezons	27987	3085	54	30760	45	9332
Issy-les-Moulineaux	64355	6249	207	45857	73	9697
Sannois	26090	2923	62	36254	36	9695
Clichy	58916	4925	147	31640	113	8750
Taverny	26144	2765	64	36175	32	10258
		mean	155	42009	89	9743
		st. dev.	92	19186	43	1119

4.4.2 Numerical results for the non-robust model NHLP

In this section we present computational results for model NHLP run in CPLEX 12.1.0. We first analyze the consistency of optimal solutions with the d-centric vs p-centric assumption of the choice model, and then explore its tractability.

Figure 4.4.1 presents the main demographic characteristics of the area under study (Figure 4.4.1) for an example set of 15 cities. The upper level of the figure shows a higher concentration of the demand in the south of the region. Furthermore, it points to a prevalence of high income potential buyers. On the lower level of the figure, the left side shows the relative Purchasing Power (PP) for the high income-segments. A comparison with the right-hand side shows that the demand sources with the highest PP from the high income-segment also present the highest PP from the low income-segment.

Figure 4.4.1: Demographics of an instance with 15 cities (The size of the circles is proportional to corresponding D_i/PP_i). Paris is the empty space on the bottom-right corner.

Figure 4.4.2 shows the result of the optimization model with demand sources which are also potential locations (□), and locations selected for new developments (■ with some specific color). Additionally, the figure draws, in form of a pie chart, the fraction of demand (x_{ij}) flowing from each demand source to the selected locations ($y_j = 1$). As could have been expected, the choice of new development locations was influenced by the concentration of demand in the southern region. The consistency of model NHLP can be appreciated by comparing the top and bottom halves of Figure 4.4.2. When the utility is mainly driven by the distance criterion (top half of figure 4.4.2), we can see similar distributions of demands when comparing the two segments (low-income and high-income) of a given city. This is logical as these two segments mainly differ by their purchasing power but the price is not the main choice criterion for the d-centric case, so income has a minor impact contrary to the p-centric case (bottom half of the figure).

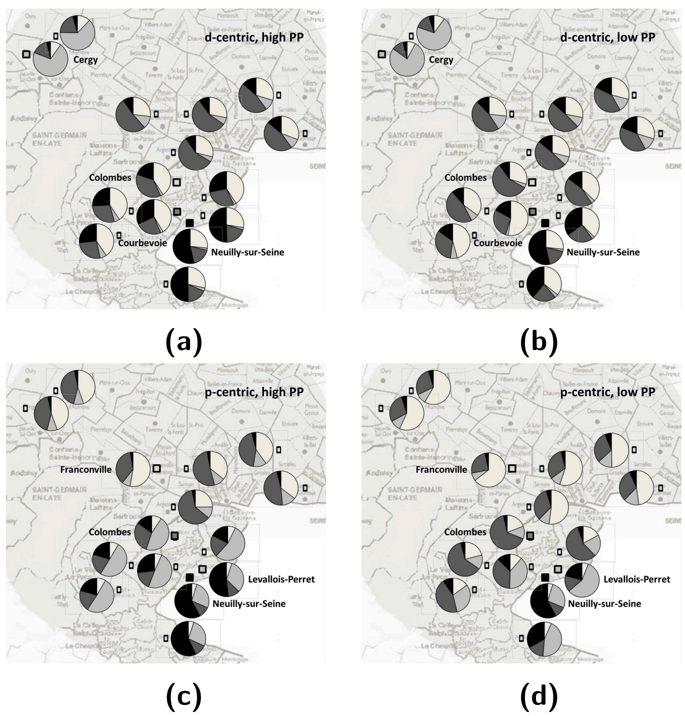

Figure 4.4.2: Distribution of the demand for an instance with 15 cities and 30 demand sources. Selected locations for development are assigned a colored square and shown with their name on the side. Demand flows from each demand source are shown in the pie charts. For example, the black fraction in a pie flows to Neuilly-sur-Seine.

Table 4.4.2 shows numerical results on instances with $n \in \{30, 60\}$ demand sources (15 or 30 cities), $m \in \{15, 20\}$ potential sites for new developments, and $P \in 2, 4, 6$. v^* is the optimal value, IGap is the ratio $\bar{v}/v^*$, where $\bar{v}$ is the value of the linear relaxation, Ts is CPU time in seconds, and #Sites is the number of sites actually selected in the optimal solution. The complexity of the problem naturally increases with each parameter, however, the number P of sites to select seems to have a particularly high impact. Additionally to the parameters described in 4.4.1, we tested the model against four budgetary levels. For a given cost vector c, we computed its median (2Q), mean ($\bar{c}$), 3rd Quartile (3Q), and mean plus standard deviation ($\bar{c} + \sigma_c$) as average budget per housing unit, so that $B/V \in \{c^{2Q}, \bar{c}, c^{3Q}, \bar{c} + \sigma_c\}$. We consistently find larger integrality gaps on instances with smaller budgets ($c^{2Q}, \bar{c}$). The computation times however are generally shorter for these instances.

4.4.3 Numerical results for uncertainty on demands

In this section we present the impact of demand uncertainty on NHLP using the **R-NHLP** robust model. The main findings in Table 4.4.3 highlight the usefulness of a robust model when dealing with uncertainty. For each original (nominal) instance with demand vector D, we generated 25 "deviated" instances noted $t = 1, \ldots, 25$ with demand vector $\tilde{D}_t$ inside uncertainty range $[\bar{D} - \hat{D}, \bar{D} + \hat{D}]$ for each instance. Let us note by y_R the robust solution and y_N the nominal solution, i.e., the optimal solution for NHLP with nominal demand $\bar{D}$. For each original instance we computed over the 25 deviated instances, for y_R and y_N: the average loss of satisfaction in the objective function, the number of infeasibilities (only for y_N as y_R is systematically feasible), and the number of times solutions achieved optimality. We call $\bar{\Delta}_R$ the average loss of customer satisfaction (utility) of the robust solution y_R over runs $t = 1, \ldots, 25$, when compared to the value v_t^* of the optimal solution of deviated instance t:

$$\bar{\Delta}_R = \frac{1}{25} \sum_{t=1}^{25} \frac{v_t(y_R) - v_t^*}{v_t^*}$$

Table 4.4.2: Numerical results for the nominal **NHLP** model

				ν^*		IGap		Ts		#Sites	
n	P	d/p	B	15	20	15	20	15	20	15	20
30	2	d	nc^{2Q}	0.582	0.498	1.328	1.298	1.3	3.4	1	2
30	2	d	$n\bar{c}$	0.582	0.517	1.350	1.513	1.5	7.7	1	2
30	2	d	nc^{3Q}	0.653	0.670	1.305	1.312	1.4	6.7	2	2
30	2	d	$n(\bar{c}+\sigma_c)$	0.653	0.672	1.305	1.308	1.5	4.3	2	2
30	2	p	nc^{2Q}	0.504	0.075	1.488	6.642	2.1	10.1	1	1
30	2	p	$n\bar{c}$	0.504	0.420	1.528	1.776	2.4	9.0	1	2
30	2	p	nc^{3Q}	0.760	0.735	1.296	1.370	1.2	5.2	2	2
30	2	p	$n(\bar{c}+\sigma_c)$	0.760	0.791	1.296	1.282	1.1	3.1	2	2
30	6	d	nc^{2Q}	0.618	0.524	1.832	1.871	39.1	126.2	4	5
30	6	d	$n\bar{c}$	0.623	0.647	1.843	1.820	42.1	271.0	5	6
30	6	d	nc^{3Q}	0.703	0.756	1.783	1.739	76.2	332.9	6	6
30	6	d	$n(\bar{c}+\sigma_c)$	0.703	0.756	1.783	1.739	72.6	310.8	6	6
30	6	p	nc^{2Q}	0.548	0.343	1.735	2.041	30.2	93.7	4	5
30	6	p	$n\bar{c}$	0.552	0.545	1.773	1.774	30.3	197.2	5	3
30	6	p	nc^{3Q}	0.775	0.820	1.689	1.652	59.5	381.8	5	6
30	6	p	$n(\bar{c}+\sigma_c)$	0.794	0.844	1.648	1.614	39.4	229.6	4	4
60	2	d	nc^{2Q}	0.569	0.462	1.361	1.342	5.6	14.3	1	2
60	2	d	$n\bar{c}$	0.569	0.490	1.386	1.571	6.8	25.5	1	2
60	2	d	nc^{3Q}	0.658	0.655	1.329	1.345	5.4	19.3	2	2
60	2	d	$n(\bar{c}+\sigma_c)$	0.667	0.672	1.311	1.311	4.8	12.7	2	2
60	2	p	nc^{2Q}	0.516	0.060	1.487	8.153	6.9	34.4	1	1
60	2	p	$n\bar{c}$	0.516	0.448	1.534	1.673	7.3	32.7	1	2
60	2	p	nc^{3Q}	0.764	0.726	1.296	1.377	3.4	20.1	2	2
60	2	p	$N(\bar{c}+\sigma_c)$	0.764	0.787	1.297	1.280	3.0	13.3	2	2
60	6	d	nc^{2Q}	0.589	0.468	1.824	1.986	133.8	516.9	6	5
60	6	d	$n\bar{c}$	0.602	0.613	1.816	1.840	187.9	1081.4	5	6
60	6	d	nc^{3Q}	0.695	0.723	1.729	1.725	205.8	1688.1	6	6
60	6	d	$n(\bar{c}+\sigma_c)$	0.695	0.723	1.729	1.725	176.4	1537.8	6	6
60	6	p	nc^{2Q}	0.575	0.338	1.648	1.996	85.2	287.7	4	5
60	6	p	$n\bar{c}$	0.583	0.552	1.671	1.731	79.2	828.9	4	6
60	6	p	nc^{3Q}	0.782	0.805	1.638	1.632	134.0	1266.6	5	4
60	6	p	$n(\bar{c}+\sigma_c)$	0.782	0.805	1.638	1.634	120.7	1138.8	5	4

$(c^{2Q}, \bar{c}, c^{3Q}, \bar{c}+\sigma_c)$:(median, mean, 3^{rd} Quartile, mean + standard deviation) values of vector c.

where $v_t(y)$ is the value of solution y on deviated instance t. Similarly, we compute the average loss of satisfaction $\bar{\Delta}_N$ achieved by the nominal solution y_N:

$$\bar{\Delta}_N = \frac{1}{25} \sum_{t=1}^{25} \frac{v_t(y_N) - v_t^*}{v_t^*}$$

We consider both the budget constraint (4.10) and the flow definition constraints (4.13) to be truly binding. In this way, if a solution y_R or y_N is infeasible, then the satisfaction derived from it is 0. Table 4.4.3 shows the effect of uncertainty on customer satisfaction with average losses $\bar{\Delta}_R, \bar{\Delta}_N$.

Generally, this loss is greater in the price-centric case. This can be explained by the interaction between the demand and the budget constraint which reduces feasibility. The initial setting of parameters Γ and $\hat{D}$ also has an impact. Not surprisingly, enlarging Γ (10 vs 5) or the maximum demand deviation $\hat{D}$ ($+30\%$ $\bar{D}$ vs $+15$ % $\bar{D}$) generates higher losses of satisfaction, mainly when the resources (B) are scarce as reported in the table. When comparing y_R, which always remains feasible, and y_N, it evidences a greater need for a robust formulation for tighter budget constraints (scenarios c^{2Q} and $\bar{c}$), as shown by the larger number of infeasible repetitions (#inf) occurring with y_N. Finally, the number of times y_R and y_N achieved optimality tells us that the advantages of using the robust model **R-NHLP** diminish when the budget constraint is less tight (scenarios c^{3Q} and $\bar{c} + \sigma_c$).

We test model **R-NHLP** for conservativeness by comparing it against **R'-NHLP**, where only the budget constraint is protected against uncertainty (and not the objective function). We find the results almost identical, with 60% of the instances having the same solution. For the remaining 40% instances, we found the difference of only 0.1 point in the average loss of value. We believe the structure of the problem favors these close results, since the model is only free to chose y_j, while x_{ij} is a byproduct of the choice model.

Table 4.4.3: Numerical results for **R-NHLP** with (n,m,P)=(30,15,4)

							# inf	# best		Time (s)	
n	Γ	U	B	$\hat{D}$	$\bar{\Delta}_R$	$\bar{\Delta}_N$	N	R	N	R	N
30	5	d	nc^{2Q}	15%D	-0.13	-0.05	1	0	24	19	23
30	5	d	nc^{2Q}	30%D	-0.12	-0.38	8	0	14	13	23
30	5	d	$n\bar{c}$	15%D	-0.09	0.00	0	0	23	18	15
30	5	d	$n\bar{c}$	30%D	-0.14	-0.06	1	0	20	16	15
30	5	d	nc^{3Q}	15%D	0.00	0.00	0	20	20	32	32
30	5	d	nc^{3Q}	30%D	-0.03	0.00	0	0	14	36	32
30	5	d	$n(\bar{c}+\sigma_c)$	15%D	0.00	0.00	0	2	23	29	30
30	5	d	$n(\bar{c}+\sigma_c)$	30%D	0.00	0.00	0	4	20	32	30
30	5	p	nc^{2Q}	15%D	-0.16	-0.31	7	0	8	27	10
30	5	p	nc^{2Q}	30%D	-0.26	-0.38	9	0	6	14	10
30	5	p	$n\bar{c}$	15%D	-0.14	-0.05	1	0	23	14	12
30	5	p	$n\bar{c}$	30%D	-0.22	-0.35	7	0	13	14	12
30	5	p	nc^{3Q}	15%D	-0.05	-0.01	0	0	10	28	26
30	5	p	nc^{3Q}	30%D	-0.07	-0.01	0	0	9	15	26
30	5	p	$n(\bar{c}+\sigma_c)$	15%D	0.00	0.00	0	11	14	11	10
30	5	p	$n(\bar{c}+\sigma_c)$	30%D	0.00	0.00	0	11	14	24	10
30	10	d	nc^{2Q}	15%D	-0.14	-0.08	2	0	21	15	23
30	10	d	nc^{2Q}	30%D	-0.20	-0.21	5	0	9	9	23
30	10	d	$n\bar{c}$	15%D	-0.13	-0.04	1	0	16	28	15
30	10	d	$n\bar{c}$	30%D	-0.21	-0.09	2	0	9	13	15
30	10	d	nc^{3Q}	15%D	-0.04	0.00	0	0	14	36	31
30	10	d	nc^{3Q}	30%D	-0.06	-0.01	0	0	12	38	31
30	10	d	$n(\bar{c}+\sigma_c)$	15%D	0.00	0.00	0	10	8	34	29
30	10	d	$n(\bar{c}+\sigma_c)$	30%D	-0.01	-0.01	0	10	8	34	29
30	10	p	nc^{2Q}	15%D	-0.21	-0.13	3	0	9	12	10
30	10	p	nc^{2Q}	30%D	-0.41	-0.22	5	0	7	13	10
30	10	p	$n\bar{c}$	15%D	-0.19	-0.09	2	0	16	23	11
30	10	p	$n\bar{c}$	30%D	-0.34	-0.22	4	0	3	7	11
30	10	p	nc^{3Q}	15%D	-0.07	-0.06	1	0	6	27	26
30	10	p	nc^{3Q}	30%D	-0.16	-0.10	2	0	3	14	26
30	10	p	$n(\bar{c}+\sigma_c)$	15%D	0.00	0.00	0	13	13	22	10
30	10	p	$n(\bar{c}+\sigma_c)$	30%D	-0.08	0.00	0	0	11	31	10

Table 4.4.4: Numerical results for **R-NHLP** with (n,m,P)=(60,15,4)

							# inf	# best		Time (s)	
n	Γ	U	B	$\hat{D}$	$\bar{\Delta}_R$	$\bar{\Delta}_N$	N	R	N	R	N
60	5	d	nc^{2Q}	15%D	-0.08	-0.43	6	0	11	49	49
60	5	d	nc^{2Q}	30%D	-0.14	-0.43	6	0	8	55	49
60	5	d	$n\bar{c}$	15%D	-0.05	-0.42	7	0	3	101	41
60	5	d	$n\bar{c}$	30%D	-0.16	-0.42	7	0	1	47	41
60	5	d	nc^{3Q}	15%D	0.00	0.00	0	25	25	99	92
60	5	d	nc^{3Q}	30%D	-0.01	0.00	0	0	24	108	92
60	5	d	$n(\bar{c}+\sigma_c)$	15%D	0.00	0.00	0	25	25	94	85
60	5	d	$n(\bar{c}+\sigma_c)$	30%D	0.00	0.00	0	24	24	93	85
60	5	p	nc^{2Q}	15%D	-0.15	-0.38	6	0	8	39	30
60	5	p	nc^{2Q}	30%D	-0.23	-0.38	6	0	4	48	30
60	5	p	$n\bar{c}$	15%D	-0.15	-0.40	6	0	10	55	37
60	5	p	$n\bar{c}$	30%D	-0.23	-0.43	6	0	7	40	37
60	5	p	nc^{3Q}	15%D	-0.02	0.00	0	0	25	88	82
60	5	p	nc^{3Q}	30%D	-0.06	-0.04	1	0	24	45	82
60	5	p	$n(\bar{c}+\sigma_c)$	15%D	0.00	0.00	0	25	25	73	75
60	5	p	$n(\bar{c}+\sigma_c)$	30%D	0.00	0.00	0	25	25	74	75

Table 4.4.5: Numerical results for u**R-NHLP** with (n,m,P)=(30,15,4)

				# inf	# best		Time (s)	
Γ	B	$\bar{\Delta}_R$	$\bar{\Delta}_N$	N	R	N	R	N
5	nc^{2Q}	0.000	0.000	0	24	24	68	15
5	$n\bar{c}$	-0.001	-0.001	0	21	21	67	33
10	nc^{2Q}	-0.053	-0.001	0	0	21	58	15
10	$n\bar{c}$	-0.015	-0.003	0	1	17	74	33
15	nc^{2Q}	-0.056	-0.120	3	0	22	70	15
15	$n\bar{c}$	-0.004	-0.004	0	19	19	87	33
20	nc^{2Q}	-0.043	-0.400	10	0	14	151	14
20	$n\bar{c}$	0.000	0.000	0	20	20	50	28
25	nc^{2Q}	-0.035	-0.680	17	0	8	120	14
25	$n\bar{c}$	0.000	0.000	0	23	23	46	28

4.4.4 Numerical results for uncertainty on utilities

In this section we focus on the robust model **uR-NHLP** where uncertainty lies on customer utilities. We classify the demand sources into two groups:

- Lower-than-average income: demand sources i with $I_i < \bar{I}$, where $\bar{I}$ is the average income over all cities, are considered more sensitive to price and better described by $w^o = \{1/3, 2/3\}$.

- Higher-than-average income: demand sources with $I_i > \bar{I}$ are considered more sensitive to distance than price, and assigned a nominal weight vector $w^o = \{2/3, 1/3\}$

In Table 4.4.5 we see again that the robust solution y_R is more interesting than the nominal solution y_N when the uncertainty budget Γ is quite large and the budget constraint is in the tightest scenario $B = nc^{2Q}$. The nominal solution starts to be infeasible for some of the 25 deviated instances from $\Gamma = 15$, i.e. $0.5n$, and is very often infeasible for greater Γ (# inf = 10 out of 25 for $\Gamma = 20$, and # inf =

17 out of 25 for $\Gamma = 25$. Nevertheless, for smaller values of Γ, the nominal solution behaves very well with no infeasible solution and a higher proportion of best solutions. We can conclude that the price of robustness, i.e. the price to pay for systematic feasibility, seems to be higher for uncertainty on utilities.

4.5 Conclusion

We have presented a robust approach for selecting new housing programs, where uncertainty lies first on demand volumes, then on customer utilities. We showed that when uncertainty lies on utilities with a discrete set of deviation scenarios from the nominal data, the robustness subproblem with binary variables can be replaced by its linear relaxation with no consequence on the original model, which enables to use the classical dualization method. Since consumer behaviors are modeled by a choice model, we used the Irrelevance of Independent Alternatives (IIA) property to describe the flows from demand sources to selected sites in our models, which are reformulated in a linear way. Numerical experiments on data of the Paris region show from which level of the uncertainty budget and which degree of tightness of the budget constraint, the robust model starts to be more accurate than just using the optimal solution of the nominal problem, that becomes more and more infeasible when these parameters increase. To our knowledge, this research is the first contribution of optimization to selecting new housing programs with a choice model, beyond the robustness aspect. The robustness approach combining an uncertainty budget and a discrete set of scenarios can be applied to a larger class of optimization problems based on a choice model.

5

A nested robustness approach for the Capacitated Facility Location Problem with Location-Dependent Demand

5.1 Introduction

In this chapter we look at the location of a limited set of Mobile Vendors serving customers on a First come First served basis, where the demand to each location follows a Poisson process with mean λ_{ij} (with index $i = 1, \ldots, m$ standing for the demand sources, and index $j = 1, \ldots, n$ refering to the possible locations of a mobile vendor). Furthermore, each vendor at location j has a bounded capacity K_j (which can be linked to the size of the truck or vehicle), and where the objective is to maximize the total revenue of the company that employs the vendors. We can find examples of this behaviour in food trucks, ice cream vendors, and other mobile agents such as promotional/publicity stands. Although all vendors could exhibit the characteristics just mentioned, i.e. bounded capacity and first-come first-served behaviour (e.g. corner stores, retail/apparel stores), we refer to mobile vendors since they have the flexibility to change the configuration of their network, thus being able to optimize the facility location problem when required, for example, every time there is a significant change in demand flows. More traditional businesses, such as retail stores, do not have the possibility of reconfiguring their network so they must make their decisions at the strategic level, with the aggregated information which might make our operational-level assumptions questionable. In this case, choice models like the one used in chapter 3 can be used (see Benati and Hansen [10] for the maximum-capture facility location problem with a choice model for modeling consumer behaviour).

The issue of mobile vendors location is addressed in Lucan et al [52]. The authors look at the situation from a social perspective where the mobile vendors are green trucks (distributing whole fresh products to marginalized areas). Their study finds that the mobile vendors cluster around high traffic areas, which are not necessarily close to their intended target. Although Lucan et al are unable to say whether the green trucks fulfil the objective of reaching the most vulnerable areas, an optimization model, as the one formulated in the following sections, could provide an option for locating the trucks to better provide their intended service. Surprisingly, we found no other paper specifically dealing with the optimal location of mobile vendors, so the approach developed in this chapter can be considered as a new problem.

When considering revenue maximization, we can model the Capacitated Facility Location Problem outlined above by the following MIP:

$$
\begin{aligned}
\max \ & \sum_{j=1}^{n} \sum_{i=1}^{m} \pi_{ij} \kappa_{ij} K_j y_j \\
s.t. \ & \sum_{j=1}^{n} y_j \leq P \\
& y \in \mathscr{Y} \subseteq \{0, 1\}^n
\end{aligned}
\tag{5.1}
$$

where π_{ij} is the average basket (or profit) per demand source i at location j, κ_{ij} is the fraction of capacity from location j consumed by demand source i (which will be defined in the next section as a function of λ_{ij}), and $\mathscr{Y}$ is a polyhedral set of constraints on the locations, possibly containing (knapsack) budget constraints of the type $\sum_j \omega_j y_j \leq B$.

Note that $\sum_i \pi_{ij} \kappa_{ij} K_j$ is the total revenue (or profit) captured by location j, if j is open, the objective function is then the total revenue generated by the mobile vendors in the selected locations. We posit π_{ij} as a parameter depending both on source i and location j to include all possible situations: i) $\pi_{ij} = \pi_{i\cdot}$ if each mobile vendor offers the same set of products (e.g. the consuming patterns, average baskets, of customers depend on the different sources i, but do not vary over the possible locations j); and the general case ii) π_{ij} if both demand sources i and locations j are differentiated (we can think of stores j carrying differentiated goods). Next we provide a few examples to visualize different profit characterizations in an easier way:

i) Differentiation on a customer basis, resulting on π_i, results naturally since different groups or segments in the population have different consuming habits. For example, depending on purchasing power and residential area, customers of type i may consume a basic product while customers from i' would go for a premium. We can refer then to an average basket for each type of customer i.

ii) In the general case, π_{ij} emerges as a consequence of both, spacial characteristics and customers' consuming habits. If a vendor's location can affect the customers average baskets, or the profit it derives from it, π_{ij} would vary in both customer type and location. There may also be specific costs on location j (e.g. rental fees for the location) that need to be substracted to the average basket to obtain a net profit π_{ij} that also depends on j.

Given our focus on mobile vendors, we consider the demand to be determinant only as far as it can influence the rate of arrival λ_{ij}, for the vendor can relocate when the rates drop. Also, as mobile vendors, their inventory capacity would be limited. These two considerations give us the assumption of full capacity consumption.

In the remainder of this chapter, we will propose a nominal model to solve the CFLP with Location-Dependent Demand (i.e. we will characterize the fraction κ_{ij} of the capacity of location j consumed by customers from source i), following with a robust formulation for disturbances for arrival rates. We present numerical experiments to evaluate the performance of the robust model and how it compares to the nominal one. We close the chapter with conclusions and comments on further research directions.

5.2 The CFLP with Location-Dependent Demand

In this section we propose a model to solve the Capacitated Facility Location Problem with Location-Dependent Demand (CFLP-LDD) under the assumptions and considerations outlined in the introduction. It is easy to see that the solution to (5.1) depends on the definition of κ_{ij}, so that will be our starting point.

Two assumptions guide our choice of the capacity allocation, namely the first-come first-served and the capacity consumption. While first-come first-served is self-explanatory, by capacity consumption we mean a demand large enough to consume all the capacity available, even when uncertainty affects the arrival rates. Since the vendor can relocate when a significant change to the demand flows occur, e.g. when demand from source i is depleted, we consider this assumption justified for our setting. The driving parameter for the capacity consumption of location j

is the rate of arrival from each demand source i to said location j, λ_{ij}. We formulate in Proposition 5.2.1 that the available capacity to each demand source should be proportional to the rate of arrival from said demand source.

Proposition 5.2.1. *If $\sum_i \lambda_{ij} \geq K_j$, then, under the first-come first-served assumption, we have:*

$$\kappa_{ij} = \frac{\lambda_{ij}}{\sum_{i'} \lambda_{i'j}}$$

.

Proof. We address the case where, for any j, $\sum_i \lambda_{ij} \geq K_j$, the left-hand side of the inequality represents the total demand on location j during a period of arbitrary length τ. In order to find κ_{ij} we need to find and interval of size τ' that exactly consumes the capacity of j, then $\tau' = K_j\tau / \sum_i \lambda_{ij}$. Necessarily we have $\tau' \leq \tau$ for the scenario under consideration. Since we have defined the demand as Poisson processes and the allocation of capacity to be on a first-come first-served basis, the mean demand at location j from source i during period τ' is

$$\lambda'_{ij} = \frac{\tau'\lambda_{ij}}{\tau} = \frac{\lambda_{ij}}{\sum_{i'} \lambda_{i'j}} K_j$$

then the fraction of capacity from j allocated to source i is simply the ratio $\kappa_{ij} = \lambda_{ij} / \sum_{i'} \lambda_{i'j}$ $\square$

Under the characterization of κ_{ij} outlined by Proposition 5.2.1, we can rewrite (5.1) as the CFLP with Location-Dependent Demand:

$$\begin{aligned} \max \; & \sum_{j=1}^{n} \frac{\sum_i \pi_{ij}\lambda_{ij}}{\sum_i \lambda_{ij}} K_j y_j \\ s.t. \; & \sum_{j=1}^{n} y_j \leq P \\ & y \in \mathscr{Y} \subseteq \{0, 1\}^n \end{aligned} \tag{5.2}$$

Model (5.2) is simple enough to solve when all parameters associated to variable y_j in the objective are known, and $\mathscr{Y} = \{y_j | y_j \in \{0, 1\} \text{ for } j = 1, \ldots, n\}$. It suffices to select the P best coefficients $v_j = K_j \sum_i \pi_{ij}\lambda_{ij} / \sum_i \lambda_{ij}$ to exactly solve the problem in linear time. If $\mathscr{Y} = \{y | \sum_j \omega_j y_j \leq B\}$ then it is a Knapsack Problem (KP) with coefficients v_j which represent the total revenue or profit captured by j. KP is known to be NP-hard but solvable in pseudo polynomial time, and admits a Fully Polynomial-Time Approximation Scheme Ibarra and Kim [48]. However, we know uncertainty can be present in any part of the model, and (5.2) is made all the more complicated by the non-linearities arising with the uncertainty, most notably if such uncertainty affects the arrival rates λ_{ij}. In the following section we will propose a robust counterpart for (5.2) in the presence of uncertainty in the network.

5.3 A Robust Model with Nested Uncertainty Budgets

To begin this section, we briefly discuss the possible sources of uncertainty in our model, following with delimitation of the uncertain parameters we will include in our study. Looking at (5.2), it can be said that all parameters could be subject to variations, with the exception perhaps of P, especially in the short term. Possible sources of uncertainty for the different parameters include customer's willingness to pay and average baskets, competition, and especially for λ_{ij}, weather and traffic conditions may impact travel times to and from a destination, while changes in demands can alter arrival rates.

Given the type of vendors we considered in our presentation, the parameters of the problem are fairly easy to forecast, with the capacity, K_j, and price, controlled by the vendor (at least in the short term). The demand is the one piece of information with the most volatility and thus the one we will focus on in the treatment of uncertainty. If we consider the arrival rates to be uncertain, e.g. due to weather (rain could alter the willingness to travel of a customer) or traffic conditions (increasing or diminishing λ_{ij} for some (location,demand-source) pairs), and distributed in $\lambda_{ij} \in [\bar{\lambda}_{ij} - \hat{\lambda}_{ij}, \bar{\lambda}_{ij} + \hat{\lambda}_{ij}]$, we look for the optimal distribution of facilities to maximize the profit. Furthermore, if we embed (5.2) in a robust optimization setting, our problem becomes:

$$\max_{\substack{y \in \{0,1\} \\ \sum_j y_j \leq P \\ y \in \mathscr{Y}}} \min_{\lambda} \sum_{j=1}^{n} \frac{\sum_i \pi_{ij} \lambda_{ij}}{\sum_i \lambda_{ij}} K_j y_j$$

$$\text{with } \lambda_{ij} \in [\bar{\lambda}_{ij} - \hat{\lambda}_{ij}, \bar{\lambda}_{ij} + \hat{\lambda}_{ij}]$$

where the expected demand from source i to location j is uncertain, and bounded in an uncertainty set containing all possible realizations of the arrival rates.

We consider, as is common practice in the main RO approaches (e.g. [12] and [18]), that not all parameters can reach their worst realization at the same time, and define the uncertainty set Λ to contain all possible realizations of λ_{ij} in a rather original way. We propose **nested uncertainty budgets** where the first tier controls the number of locations that are affected by uncertainty and bounds it with a global uncertainty budget Γ. For these locations, we limit the worst variation of the arrival rates with a local uncertainty budget Γ_j. Accordingly, if the vicinity of a potential site is affected by uncertainty, the variation of arrival rates for the customers going to j is limited by Γ_j. The corresponding uncertainty set can be formalized as follows:

$$\Lambda = \left\{ \lambda_{ij} \ : \ \begin{array}{l} \left| \{ j \in \{1, \ldots, n\} : \exists i, \lambda_{ij} \neq \bar{\lambda}_{ij} \} \right| \leq \Gamma \\ \lambda_{ij} \in [\bar{\lambda}_{ij} - \hat{\lambda}_{ij}, \bar{\lambda}_{ij} + \hat{\lambda}_{ij}] \\ \displaystyle\sum_{i=1}^{m} \frac{|\lambda_{ij} - \bar{\lambda}_{ij}|}{\bar{\lambda}_{ij}} \leq \Gamma_j \quad j = 1, \ldots, n \end{array} \right\}$$

We introduce uncertainty variables θ_j and z_{ij} to integrate the uncertainty model into our robust problem. $\theta_j \in \{0, 1\}$ controls the outer layer of our uncertainty setting, so that $\theta_j = 1$ when location j is affected by uncertainty and $\theta_j = 0$ when it is not. Similarly, $z_{ij} \in [-1, 1]$ controls the deviation of the arrival rates. We can rewrite the problem under uncertainty set Λ as:

$$\max_{\substack{y_j\in\{0,1\}\\ \sum_j y_j\le P\\ y\in\mathscr{Y}}} \min_{\substack{\theta_j\in\{0,1\}\\ \sum_j \theta_j\le\Gamma}} \sum_{j=1}^{n} \min_{\substack{\sum_i |z_{ij}|\le\Gamma_j\\ |z_{ij}|\le 1}} \frac{K_j \sum_i \pi_{ij}\left(\bar{\lambda}_{ij}+\hat{\lambda}_{ij}z_{ij}\right)}{\sum_i \bar{\lambda}_{ij}+\hat{\lambda}_{ij}z_{ij}} y_j \tag{5.3}$$

In order to deal with the non-linearities in problem (5.3), we first address the inner fractional problem, that we shall call "subproblem" in what follows, and define $\tilde{v}_{ij}$ as the minimum profit attained by j when the location is chosen, then

$$\tilde{v}_j = \min_{\substack{\sum_i |z_{ij}|\le\Gamma_j\\ -1\le z_{ij}\le 1}} \frac{K_j \sum_i \pi_{ij}\left(\bar{\lambda}_{ij}+\hat{\lambda}_{ij}z_{ij}\right)}{\sum_i \bar{\lambda}_{ij}+\hat{\lambda}_{ij}z_{ij}} \tag{5.4}$$

Proposition 5.3.1. *The optimal value $\tilde{v}_j$ of the minimization subproblem 5.4 can be obtained by solving the following linear program for $j = 1, \ldots, n$ such that $y_j = \theta_j = 1$.*

$$\begin{aligned}
\min\ & K_j \sum_{i=1}^{m} \pi_{ij}\bar{\lambda}_{ij}t + K_j \sum_{i=1}^{m} \pi_{ij}\hat{\lambda}_{ij}(\zeta_{ij}^{+} - \zeta_{ij}^{-}) \\
s.t.\ & \sum_{i=1}^{m} \bar{\lambda}_{ij}t + \sum_{i=1}^{m} \hat{\lambda}_{ij}(\zeta_{ij}^{+} - \zeta_{ij}^{-}) = 1 \\
& \sum_{i=1}^{m} (\zeta_{ij}^{+} + \zeta_{ij}^{-}) \le \Gamma_j t \\
& 0 \le \zeta_{ij}^{+}, \zeta_{ij}^{-} \le t \qquad\qquad i = 1, \ldots, m
\end{aligned} \tag{5.5}$$

Proof. We remark that (5.4) can be expressed as a maximization problem with linear constraints by changing the sign of the objective function and introducing the change of variable $z_{ij} = z_{ij}^{+}$ when $z_{ij} \ge 0$, and $z_{ij} = -z_{ij}^{-}$ otherwise (to linearize the absolute value in the constraint), with $z_{ij}^{+}, z_{ij}^{-} \ge 0$. Problem (5.4) is then equivalent to:

$$\begin{aligned} &\max \quad -\frac{\sum_i K_j \pi_{ij} \left(\bar{\lambda}_{ij} + \hat{\lambda}_{ij}(z_{ij}^{+} - z_{ij}^{-})\right)}{\sum_i \bar{\lambda}_{ij} + \hat{\lambda}_{ij}(z_{ij}^{+} - z_{ij}^{-})} \\ &s.t. \sum_{i=1}^{n} z_{ij}^{+} + z_{ij}^{-} \leq \Gamma_j \\ &\qquad 0 \leq z_{ij}^{+}, z_{ij}^{-} \leq 1 \end{aligned} \tag{5.6}$$

(5.6) is then a fractional linear program in the manner of Charnes and Cooper [25]. Indeed, if we note $c_i = -K_j \pi_{ij} \hat{\lambda}_{ij}$, $\alpha = -K_j \sum_i \pi_{ij} \bar{\lambda}_{ij}$, $d_i = \hat{\lambda}_{ij}$, and $\beta = \sum_i \bar{\lambda}_{ij}$ for a given j, we can proceed to rewrite it in the form:

$$\begin{aligned} &\max \frac{c^T z + \alpha}{d^T z + \beta} \\ &s.t.\ Az \leq b \\ &z \geq 0 \end{aligned} \quad \iff \quad \begin{aligned} &\max\ c^T \zeta + \alpha t \\ &s.t.\ A\zeta - bt \leq 0 \\ &d^T \zeta + \beta t = 1 \\ &\zeta, t \geq 0 \end{aligned}$$

where $\zeta = z/(d^T z + \beta)$ and $t = 1/(d^T z + \beta)$. Note that since we know the sign of the denominator we only need one linear program. Using Charnes and Cooper, and substituting c_i, α, d_i, and β, (5.6) becomes

$$
\begin{aligned}
\max\ & K_j \sum_{i=1}^{n} \pi_{ij}\hat{\lambda}_{ij}(\zeta_{ij}^{-} - \zeta_{ij}^{+}) - K_j \sum_{i=1}^{n} \pi_{ij}\bar{\lambda}_{ij} t \\
s.t.\ & \sum_{i=1}^{n} (\zeta_{ij}^{+} + \zeta_{ij}^{-}) - \Gamma_j t \leq 0 \\
& \zeta_{ij}^{+} - t \leq 0 && \forall\, i \\
& \zeta_{ij}^{-} - t \leq 0 && \forall\, i \\
& \sum_{i=1}^{n} \hat{\lambda}_{ij}(\zeta_{ij}^{+} - \zeta_{ij}^{-}) + \sum_{i=1}^{n} \bar{\lambda}_{ij} t = 1 \\
& \zeta_{ij}^{+}, \zeta_{ij}^{-}, t \geq 0 && \forall\, i
\end{aligned}
$$

which can easily be rearranged into (5.5). $\square$

Having a solution for $\tilde{v}_j$ (obtained by solving linear program (5.5)), and defining $\bar{v}_j$ as the nominal profit obtained by j when the location is chosen ($\bar{v}_j = K_j \sum_i \pi_{ij}\bar{\lambda}_{ij} / \sum_i \bar{\lambda}_{ij}$), we can rewrite (5.3) as the following program:

$$
\max_{\substack{y \in \{0,1\} \\ \sum_j y_j \leq P \\ y \in \mathscr{Y}}} \ \min_{\substack{\theta_j \in \{0,1\} \\ \sum_j \theta_j \leq \Gamma}} \sum_{j=1}^{n} \left[\bar{v}_j - (\bar{v}_j - \tilde{v}_j)\theta_j \right] y_j \tag{5.7}
$$

(5.7) can be linearized by: defining $\tilde{y}_{ij} = y_j \theta_j$ by constraint (5.8); changing $\sum_j \theta_j \leq \Gamma$ into constraint (5.9), since $\bar{v}_j - \tilde{v}_j \geq 0$ and the minimum of (5.7)'s inner objective is achieved when $\sum_j \theta_j = \Gamma$; and adding constraint (5.10), since locations with uncertainty must be placed for (5.7) to realize its objective. Then the robust counterpart for the CFLP-LDD is shown in the program below.

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \bar{v}_j y_j - \sum_{j=1}^{n} (\bar{v}_j - \tilde{v}_j)\tilde{y}_j && \textbf{RCFLP-LDD} \\
s.t. \quad & \tilde{y}_j \geq y_j + \theta_j - 1 \qquad j = 1, \ldots, n && (5.8) \\
& \sum_{j=1}^{n} y_j \leq P \\
& \sum_{j=1}^{n} \theta_j = \Gamma && (5.9) \\
& \theta_j \leq y_j \qquad j = 1, \ldots, n && (5.10) \\
& \tilde{y}_j, \theta_j \in \{0, 1\} \qquad j = 1, \ldots, n \\
& y \in \mathcal{Y} \subseteq \{0, 1\}^n
\end{aligned}
$$

Although **RCFLP-LDD** is easy to solve (being an LP), it requires the computation of $\tilde{v}_j$ (for each $j = 1, \ldots, n$) by solving (5.5) n times. This process is computationally expensive especially if $P \ll n$. We offer in the next section an alternative to solve **RCFLP-LDD** in the case where $\Gamma \geq P$.

5.4 Complete Global Uncertainty: special case $\Gamma \geq P$

In this section we refer to instances of the CFLP-LDD where the global uncertainty (the number of locations that can be affected by uncertainty) is so large as to be considered complete ($\Gamma \geq P$). This situation is special for a robust counterpart to the CFLP-LDD can be constructed from a simplified form of (5.3).

$$
\max_{\substack{y_j \in \{0,1\} \\ \sum_j y_j \leq P \\ y \in \mathcal{Y}}} \sum_{j=1}^{n} \min_{\substack{\sum_i |z_{ij}| \leq \Gamma_j \\ |z_{ij}| \leq 1}} \frac{K_j \sum_i \pi_{ij} \left(\bar{\lambda}_{ij} + \hat{\lambda}_{ij} z_{ij} \right)}{\sum_i \bar{\lambda}_{ij} + \hat{\lambda}_{ij} z_{ij}} y_j \tag{5.11}
$$

We can disregard the minimization w.r.t. θ_j since, being the worst case and with an uncertainty budget $\Gamma = P$, all selected locations see their profit decreased.

Problem (5.11) can then be solved in the manner of Bertsimas and Sim [18].

Proposition 5.4.1. *The robust counterpart for the CFLP-LDD under complete global uncertainty is*

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} p_j \\
s.t. \quad & p_j \sum_{i=1}^{m} \bar{\lambda}_{ij} + \Gamma_j q_j + \sum_{i=1}^{m} (r_{ij}^{+} + r_{ij}^{-}) \leq K_j y_j \sum_{i=1}^{m} \pi_{ij} \bar{\lambda}_{ij} && \forall j \\
& \hat{\lambda}_{ij} p_j - q_j - r_{ij}^{+} \leq K_j \pi_{ij} \hat{\lambda}_{ij} y_j && \forall i, j \\
& \hat{\lambda}_{ij} p_j + q_j + r_{ij}^{-} \geq K_j \pi_{ij} \hat{\lambda}_{ij} y_j && \forall i, j \\
& \sum_{j=1}^{n} y_j \leq P \\
& q_j,\ r_{ij}^{+},\ r_{ij}^{-} \geq 0 && \forall i, j \\
& y \in \mathscr{Y}
\end{aligned}
\tag{5.12}
$$

Proof. From Proposition 5.3.1 we can substitute the subproblem in (5.11) with its linear equivalent (5.5) and obtain the equivalent program (5.13):

$$
\begin{aligned}
\max_{\substack{y_j \in \{0,1\} \\ \sum_j y_j \leq P \\ y \in \mathscr{Y}}} \sum_{j=1}^{n} \min_{\zeta, t} & \left(K_j y_j \sum_{i=1}^{m} \pi_{ij} \bar{\lambda}_{ij} t + K_j y_j \sum_{i=1}^{m} \pi_{ij} \hat{\lambda}_{ij} (\zeta_{ij}^{+} - \zeta_{ij}^{-}) \right) \\
s.t. \quad & \sum_{i=1}^{m} \bar{\lambda}_{ij} t + \sum_{i=1}^{m} \hat{\lambda}_{ij} (\zeta_{ij}^{+} - \zeta_{ij}^{-}) = 1 \\
& \sum_{i=1}^{m} (\zeta_{ij}^{+} + \zeta_{ij}^{-}) \leq \Gamma_j t \\
& 0 \leq \zeta_{ij}^{+}, \zeta_{ij}^{-} \leq t \qquad i = 1, \ldots, m
\end{aligned}
\tag{5.13}
$$

However, the nonlinearity persists in the form of $y_j(\zeta_{ij}^{+} - \zeta_{ij}^{-})$ and $y_j t$ in the objective function. We address this in the manner of Bertsimas and Sim transforming the inner problem in (5.13) into its dual program:

$$
\begin{aligned}
\max \quad & p_j \\
s.t. \quad & p_j \sum_{i=1}^{m} \bar{\lambda}_{ij} - \Gamma_j q_j - \sum_{i=1}^{m} r_{ij}^{+} - \sum_{i=1}^{m} r_{ij}^{-} \leq K_j y_j \sum_{i=1}^{m} \pi_{ij} \bar{\lambda}_{ij} && (5.14) \\
& \hat{\lambda}_{ij} p_j + q_j + r_{ij}^{+} \leq K_j y_j \pi_{ij} \hat{\lambda}_{ij} \qquad \forall\, i && (5.15) \\
& -\hat{\lambda}_{ij} p_j + q_j + r_{ij}^{-} \leq -K_j y_j \pi_{ij} \hat{\lambda}_{ij} \qquad \forall\, i && (5.16) \\
& q_j, r_{ij}^{+}, r_{ij}^{-} \leq 0 \qquad \forall\, i
\end{aligned}
$$

where the set of constraints (5.15) are associated with variables ζ_{ij}^{+}, constraints (5.16) are associated with variable ζ_{ij}^{-}, and constraint (5.14) is associated with variable t. Substituting the subproblem in (5.13) by its linear dual counterpart we obtain (5.12) □

Analyzing the complexity of (5.12), and comparing it against the original problem (5.11), we can see that the linearization effort does not impose a heavy burden on the nonlinear program. In fact, it only adds $2n$ linear variables, and $n-1$ constraints; in exchange, we obtain a Mixed-Integer program of our robust problem.

We proceed in the next section to evaluate the performance of both models **RCFLP-LDD** and (5.12) for loss of optimality against an optimal solution (computed after uncertainty has been revealed).

5.5 Numerical Experiments

In this section we used benchmark instances from the OR Library proposed in Osman and Christofides [57] to test **RCFLP-LDD** and (5.12). While these instances where originally developed for the capacitated clustering problem, they have also been used for the Capacitated Facility Location Problem (CFLP) (e.g. Buzna, Koháni, and Janáček [24]).

The instances in [57] are divided into two categories by problem size, with the smaller instances formed by 50 customers and 5 potential sites, while the larger ones comprise 100 customers and 10 potential sites. Each customer location is placed by its (x,y) coordinates generated randomly from a uniform distribution, as is the demand for each site. As did [24], we consider each customer location to be a potential facility location.

We use the information provided on the OR Library to compute the distances between all customers and potential facilities location (d_{ij}), as well as their arrival rate (λ_{ij}). For d_{ij}, we use the euclidean distance between two points (from the coordinates provided). We use the demand provided (D_i) to compute $\lambda_{ij} = \frac{D_i}{d_{ij}}$ so that the arrival rate is directly proportional to the size of the node (D_i), and inversely proportional to the distance between customer and facility locations. Furthermore, we generate revenues π_{ij} randomly from a uniform distribution $U(10, 25)$.

We test three protection levels Γ_j for the robust model, limiting the sum of relative deviations z_{ij} that can occur for each potential location j, ranging from 10% to 30% variation allowed. Within these limits, we generate 100 deviated instances (indexed by $t = 1, \ldots, 100$) for each problem set in [57], and compare the optimal solution of each deviated instance (y_t^*) against the solution of the nominal problem (y_N) and that of our proposed robust counterpart (y_R). As a result we end up with 300 deviated instances for each problem set, arranged in three categories where each instance reaches an established uncertainty budget ($\Gamma_j = 10\%n,\ 20\%n$, and $30\%n$). We report in the tables below the average loss of value incurred from using the nominal solution ($\bar{\Delta}_N$), and that from the robust solution ($\bar{\Delta}_R$) on the deviated instances, and compute the relative loss of value:

$$\bar{\Delta}_{\cdot} = \frac{1}{100} \sum_{t=1}^{100} \frac{v_t(y_t^*) - v_t(y_{\cdot})}{v_t(y_t^*)}$$

where $v_t(y_t^*)$ is the value of the optimal solution of deviated instance t, and $v_t(y_{\cdot})$ is the value of the optimal solution of the nominal (N), or robust (R), problem on instance t.

Additionally, we report the number of instances where the robust solution is preferred to the nominal, $\#R \succ N$, i.e. the number of times the robust solution achieves a better value than the nominal when applied to instance t. If we consider

$$\left\lceil \frac{v_t(y_R) - v_t(y_N)}{v_t(y_R)} \right\rceil^+ = \begin{cases} 1 & \text{if } v_t(y_R) > v_t(y_N) \\ 0 & \text{otherwise} \end{cases}$$

then we can compute

$$\#R \succ N = \sum_{t=1}^{100} \left\lceil \frac{v_t(y_R) - v_t(y_N)}{v_t(y_R)} \right\rceil^+$$

We begin the discussion of the numerical experiments with those corresponding model (5.12), for the results are easier to analyze having one less dimension. Afterwards we turn to the experiments on **RCFLP-LDD** and discuss the effects of the nested uncertainty.

5.5.1 Experiments under Complete Uncertainty

We test model 5.12 for the case $\Gamma \geq P$ in Table 5.5.1. We show the results of the instances described above for the three protection levels (10%, 20%, and 30% variation), with a maximum deviation from nominal value, δ, of 50% (i.e. $\hat{\lambda}_{ij} = \delta\bar{\lambda}_{ij} = 0.5\bar{\lambda}_{ij}$), for the smaller size instances in Osman and Christofides. Furthermore, we create 3 types of instances. Type A instances take into consideration all potential locations viable for all customers (λ_{ij} as defined above). In order to reduce the dilution of the uncertainty on coefficients with little to no impact, we generate Type B and C instances, considering viable only potential locations closer than a critical distance ($d_{i\cdot}$). We set this threshold arbitrarily to include roughly half the data in type B instances and a third in type C, with:

$$d_{i\cdot} = \min_j d_{ij} + \frac{\max_j d_{ij} - \min_j d_{ij}}{2} \qquad \text{for type B instances}$$

$$d_{i\cdot} = \min_j d_{ij} + \frac{\max_j d_{ij} - \min_j d_{ij}}{3} \qquad \text{for type C instances}$$

We ran our experiments using CPLEX 12.1.0 and present the results in the tables below. Note that the solution times for the models, both nominal and robust, is less than 1 second in all cases, for this reason, that information is omitted from the results.

The first thing we can remark on Table 5.5.1, is that the loss of value for using the nominal or robust solution is very small, $\bar{\Delta} < 1\%$, with the exception of a few entries marked in bold. Furthermore, a closer look reveals a higher incidence of identical solutions (between the nominal and robust problems) in Type B instances (50%) than in Types A (30%) and C (20%). While the average loss of value is, in general, smaller for the nominal solution, $\Delta_N < \Delta_R$, Table 5.5.1 shows a significant number of instances where the robust solution is preferred to the nominal one, surpasing 50% in one case.

In Table 5.5.2 we show the impact of the maximum deviation from nominal values allowed (δ). The results in the table show clearly the impact of increasing the allowed variation in the coefficients λ_{ij}. We can see the loss of value increasing directly with the increase of δ, both for the nominal and the robust models. Furthermore, the proportion of instances where the solution of the nominal and robust models differs also increases with δ, from about 10% ($\delta = 0.25$) to 100% ($\delta = 1$).

Finally, we look into the performance of the models when compared against deviated instances where the uncertainty is directed towards a "worse case". In order to construct such a "worse case", we choose randomly the coefficients λ_{ij} with uncertainty, the sense of the variation (z_{ij}), however, is chosen in the sense that worsens the solution of the problem (i.e. $z_{ij} \in (0, 1]$ for i such that $\pi_{ij} \leq \bar{\pi}_{\cdot j}$, where $\bar{\pi}_{\cdot j} = \sum_i \pi_{ij}/n$ is the average profit for location j over all demand sources; and $z_{ij} \in [-1, 0)$ for i such that $\pi_{ij} > \bar{\pi}_{\cdot j}$). We identify this characterization of uncertainty as wc-c. We also propose a directed uncertainty with the size of variation restricted to a discrete choice (i.e. $z_{ij} \in \{-1, 0, 1\}$), and identify this characterization of uncertainty as wc-d. We show the comparison between these instances

Table 5.5.1: Case $\Gamma \geq P$: Performance measures for the RCFLP-LDD with (N,P,δ) = (50,5,0.5)

Inst #	Γ_j	A $\bar{\Delta}_N$	A $\bar{\Delta}_R$	A $\#R \succ N$	B $\bar{\Delta}_N$	B $\bar{\Delta}_R$	B $\#R \succ N$	C $\bar{\Delta}_N$	C $\bar{\Delta}_R$	C $\#R \succ N$
1	10%	0.00	0.00		0.00	0.70	1	0.00	0.00	
	20%	0.02	0.90		0.03	0.03		0.02	0.60	7
	30%	0.06	0.06		0.06	0.06		0.05	0.05	
2	10%	0.07	0.60	6	0.20	0.40	28	0.30	0.30	46
	20%	0.20	0.50	18	0.40	0.50	39	0.50	0.90	35
	30%	0.30	0.70	13	0.50	0.50		0.70	0.90	46
3	10%	0.02	0.02		0.03	0.03		0.09	0.70	2
	20%	0.09	0.09		0.09	0.09		0.20	0.90	8
	30%	0.06	0.50	13	0.10	0.10		0.30	0.80	20
4	10%	0.08	0.50	2	0.10	**1.20**		0.04	0.90	2
	20%	0.10	0.60	4	0.30	**1.30**	8	0.09	**2.40**	
	30%	0.30	0.60	14	0.40	**1.40**	9	0.20	**2.40**	
5	10%	0.02	0.30	2	0.04	0.04		0.10	**1.00**	6
	20%	0.09	0.40	8	0.08	0.08		0.20	**1.60**	
	30%	0.10	0.40	12	0.10	0.10		0.20	**1.60**	
6	10%	0.10	0.40	19	0.10	0.10	46	0.20	0.40	23
	20%	0.20	0.30	42	0.30	**1.00**	10	0.30	0.40	40
	30%	0.40	0.40	40	0.30	**1.10**	17	0.40	0.70	35
7	10%	0.07	0.20	18	0.06	0.06		0.09	0.09	
	20%	0.20	0.30	29	0.10	0.10		0.20	0.20	
	30%	0.20	0.90	3	0.20	0.20		0.40	0.40	
8	10%	0.05	0.05		0.10	0.10		0.40	0.40	46
	20%	0.10	0.10		0.30	0.30	54	0.60	0.60	48
	30%	0.20	0.20		0.30	0.30	57	0.60	0.70	45
9	10%	0.10	0.20	37	0.07	0.07		0.08	0.20	28
	20%	0.20	0.20	46	0.20	0.20		0.20	0.30	34
	30%	0.30	0.30	53	0.30	0.30		0.20	0.40	35
10	10%	0.04	0.04		0.05	0.20	16	0.20	0.40	25
	20%	0.10	0.10		0.10	0.30	30	0.30	0.50	32
	30%	0.09	0.09		0.20	0.30	37	0.50	0.50	39

*The loss of value ($\bar{\Delta}$) is shown in percentage (%)

Table 5.5.2: Case $\Gamma \geq P$: Performance measures for the RCFLP-LDD with (N,P,Type) = (50,5,B)

Inst #	Γ_j	$\delta = 0.25$ $\bar{\Delta}_N$	$\bar{\Delta}_R$	$\delta = 0.5$ $\bar{\Delta}_N$	$\bar{\Delta}_R$	$\delta = 1$ $\bar{\Delta}_N$	$\bar{\Delta}_R$
	10%	0.00	0.00	0.00	0.70	0.20	5.40
1	20%	0.00	0.00	0.03	0.03	0.50	8.00
	30%	0.00	0.00	0.06	0.06	0.90	8.00
	10%	0.04	0.20	0.20	0.40	0.80	1.30
2	20%	0.08	0.08	0.40	0.50	1.30	3.50
	30%	0.10	0.10	0.50	0.50	1.80	4.10
	10%	0.00	0.00	0.03	0.03	0.20	1.00
3	20%	0.00	0.00	0.09	0.09	0.50	1.20
	30%	0.01	0.01	0.10	0.10	0.70	3.40
	10%	0.05	0.30	0.10	1.20	0.60	1.60
4	20%	0.07	0.30	0.30	1.30	1.30	2.10
	30%	0.10	0.30	0.40	1.40	1.80	3.00
	10%	0.01	0.01	0.04	0.04	0.30	1.90
5	20%	0.02	0.02	0.08	0.08	0.50	2.10
	30%	0.03	0.03	0.10	0.10	0.90	2.10
	10%	0.05	0.05	0.10	0.10	0.40	2.30
6	20%	0.10	0.09	0.30	1.00	0.90	2.70
	30%	0.10	0.10	0.30	1.10	1.20	3.60
	10%	0.00	0.00	0.06	0.06	0.40	3.00
7	20%	0.00	0.00	0.10	0.10	0.70	3.20
	30%	0.02	0.02	0.20	0.20	1.20	3.70
	10%	0.06	0.06	0.10	0.10	0.40	1.40
8	20%	0.10	0.10	0.30	0.30	0.90	1.80
	30%	0.10	0.10	0.30	0.30	1.30	2.30
	10%	0.01	0.01	0.07	0.07	0.40	2.30
9	20%	0.02	0.02	0.20	0.20	0.80	3.30
	30%	0.04	0.04	0.30	0.30	1.20	4.20
	10%	0.01	0.20	0.05	0.20	0.30	0.80
10	20%	0.04	0.20	0.10	0.30	0.70	2.70
	30%	0.03	0.20	0.20	0.30	1.00	7.50

*The loss of value ($\bar{\Delta}$) is shown in percentage (%)

in Table 5.5.3. While we expected to see an advantage given to the robust solution (y_R) by generating the deviated instances with the directed uncertainty, the results are not significantly different among the three characterizations of uncertainty compared; neither in magnitude of loss of value, nor in nominal vs. robust comparisons. Overall, the number of instances for which the robust solution is better than the nominal, $\#R \succ N$, does increase under directed uncertainty (favoring the robust solution).

5.5.2 Experiments under Nested Uncertainty Budgets

The results in this section illustrate the diluting effect of a sparse solution on the impact of uncertainty as was previously observed in Poss [60]. In our case, however, the size of the subset containing the optimal solution is not in question, it is equal to the number of selected locations P, so an uncertainty budget as a function of the solution is not warranted.

In table 5.5.4 we can see the results of two sets of instances obtained from two different values of P where the comparisons are twofold:

- when the *proportion* of sites subject to uncertainty is equal
- when the *number* of sites subject to uncertainty remains the same

The results from the first case, where the proportion of selected sites affected by uncertainty is equal, offers no surprises. As could be expected, the number of times a robust solution is preferred to the nominal solution increases with uncertainty (i.e. is larger for the case where $P = 10, \Gamma = 4, 6, 8$). Since there is more uncertainty, it is expected to have a larger impact on the nominal solution.

When the number of selected sites affected by uncertainty remains the same, and the number of sites with unaffected revenues increases significantly however, one may expect that the profit from the unaffected sites could even out the effects of uncertainty. The results in Table 5.5.4 show this to be incorrect. Increasing the number of selected sites (even while maintaining the number of sites affected by

Table 5.5.3: Case $\Gamma \geq P$: Performance measures for the RCFLP-LDD with (N,P,δ,Type) = (50,5,0.5,A)

					wc-c			wc-d		
Inst #	Γ	$\bar{\Delta}_N$	$\bar{\Delta}_R$	$\#R \succ N$	$\bar{\Delta}_N$	$\bar{\Delta}_R$	$\#R \succ N$	$\bar{\Delta}_N$	$\bar{\Delta}_R$	$\#R \succ N$
	10%	0.00	0.00		0.00	0.00		0.01	0.01	
1	20%	0.02	0.93		0.00	0.90		0.02	0.88	2
	30%	0.06	0.06		0.01	0.01		0.04	0.04	
	10%	0.07	0.62	6	0.09	0.60	9	0.15	0.66	15
2	20%	0.18	0.47	18	0.13	0.44	22	0.20	0.59	17
	30%	0.33	0.75	13	0.14	0.55	15	0.27	0.77	23
	10%	0.02	0.02		0.02	0.02		0.03	0.03	
3	20%	0.09	0.09		0.08	0.08		0.09	0.09	
	30%	0.06	0.48	13	0.09	0.30	23	0.19	0.45	25
	10%	0.08	0.51	2	0.02	0.45		0.06	0.52	4
4	20%	0.13	0.57	4	0.04	0.49	3	0.13	0.57	7
	30%	0.25	0.58	14	0.08	0.26	30	0.18	0.40	26
	10%	0.02	0.31	2	0.00	0.23	1	0.02	0.24	3
5	20%	0.09	0.37	8	0.03	0.19	14	0.06	0.22	15
	30%	0.11	0.41	12	0.03	0.14	26	0.08	0.20	23
	10%	0.15	0.38	19	0.11	0.36	14	0.16	0.38	17
6	20%	0.25	0.29	42	0.22	0.15	49	0.25	0.23	44
	30%	0.37	0.42	40	0.20	0.17	54	0.33	0.31	44
	10%	0.07	0.22	18	0.08	0.20	23	0.09	0.23	19
7	20%	0.16	0.29	29	0.06	0.25	19	0.12	0.30	22
	30%	0.23	0.94	3	0.08	0.52	12	0.21	0.56	31
	10%	0.05	0.05		0.01	0.01		0.03	0.03	
8	20%	0.10	0.10		0.03	0.03		0.07	0.07	
	30%	0.18	0.18		0.03	0.03		0.08	0.08	
	10%	0.10	0.16	37	0.11	0.06	46	0.10	0.08	49
9	20%	0.18	0.18	46	0.20	0.05	68	0.25	0.11	59
	30%	0.28	0.28	53	0.23	0.07	62	0.29	0.15	60
	10%	0.04	0.04		0.04	0.04		0.09	0.09	
10	20%	0.11	0.11		0.10	0.10		0.16	0.16	
	30%	0.09	0.09		0.13	0.13		0.20	0.20	

*The loss of value ($\bar{\Delta}$) is shown in percentage (%)

uncertainty) increases the chances that a site suffering from uncertainty is selected in the solution, which is shown in the increase of preference for the robust solution.

5.6 Conclusion and Future research

We have proposed a model to solve the CFLP with Location-dependent Demand for mobile vendors, under assumptions on service rule and capacity. We have also proposed a robust optimization approach with nested uncertainty budgets to deal with uncertainty in the key parameter κ_{ij}, which is the fraction of the capacity from location j that will be consumed by customers from demand source i, and formulated the corresponding program as a MIP.

Extensive numerical experiments were conducted to test the performance of the robust model, and compare it against the nominal and the true optimal solutions of deviated instances. The results show a slight disadvantage of the robust model due to the fact that there is no feasibility issue with our formulation. However, when the uncertainty is expected to affect more sites than the number of vendors available, the robust solution could outperform the nominal one for over half the instances.

An important parameter that has shaped Proposition 5.4.1 is the definition of κ_{ij}, and, to a lesser degree, the choice of Γ_j and consequently the treatment of the uncertainty (and the robustness) around the locations j. These decisions reflect the original objective of our problem: to locate P facilities that maximize the total revenue or profit for the vendor. Then, since the demand is assumed large enough to consume all capacity, the worst case scenario is simply the one that impacts the revenue of each location in a negative way, that is to say, where the uncertainty budget of each location is met.

Besides the uncertainty source, changing κ_{ij} would alter the structure of the original problem and also that of the solution. We venture some scenarios that could result in such alteration of the capacity consumption mode: i) public policy, ii) asymmetric bargaining power, iii) corporate policy.

Table 5.5.4: Case $\Gamma < P$: Comparison of $\# \, R \succ N$ for the **RCFLP-LDD** with (N,δ,Type) = (50,0.5,A)

		P=5			P=10				
	$\Gamma =$	2	3	4	2	3	4	6	8
Inst #		.4P	.6P	.8P	.2P	.3P	.4P	.6P	.8P
	$\Gamma_j =$ 10%					3		8	2
1	20%					5	1	7	
	30%					10		8	3
	10%	1		2		3			1
2	20%			1		2		2	4
	30%					7		4	7
	10%							9	1
3	20%				1		6	9	1
	30%							5	6
	10%					5	3	7	4
4	20%						5	8	1
	30%						5	10	
	10%			1		8		13	
5	20%			1		8		14	2
	30%		4			4		12	1
	10%	3		4					9
6	20%	1	1	4					4
	30%						2		3
	10%	4	8	5	3			9	3
7	20%	8	7	7	1			7	7
	30%	4	4	6			4	10	10
	10%				1		7	12	5
8	20%				3		1	11	8
	30%				2		3	13	5
	10%				2		8	1	14
9	20%				2	1	2	2	5
	30%				4	2	6	2	10
	10%				5	3	10	1	14
10	20%				4	1	4		10
	30%				1	1	6	1	14

By i) public policy we refer to an outside entity with enough authority to influence the decision of the agents: e.g. limiting access to a certain demand sector, or ensuring that all sectors are serviced. With ii) asymmetric bargaining power, we refer to a demand source with enough pull to influence the decisions of the agents. Finally, with iii) corporate policy we refer to considerations other than profit maximization that could drive the decision of the agents: e.g. granting special treatment to a sector to advance a corporate goal. All these considerations could be addressed by changing our assumptions that the customers are serviced in a first-come first-served basis, however, they can also be met by choosing the appropriate locations: placing a facility close to source i would increase λ_{ij} thus ensuring service (or granting special treatment to i), while the opposite is true of a facility in a remote location.

6
Conclusion

In this final chapter we present the contributions of the dissertation to the literature as a whole, as well as a summary of the findings from each chapter and their individual impact on the state of the art on Robust Optimization. While there are many approaches to handle uncertainty, even within the RO paradigm, we showed that the particular structure of said uncertainty can have an important effect on the robustness of the solution if the method selected is not equipped for the characteristics of the problem. In particular, we showed that the solution of a general approach (i.e. Bertsimas and Sim [18]) can produce infeasible solutions to problems with non-linear impact of uncertainties.

We extended the classical Γ-robustness approach to optimization problems with non-linear impact on the coefficients of the constraints and the objective function. We proposed a robust counterpart for problems where the uncertainty follows a non-linear monotonic continuous function based on an approximation by a piecewise-linear function. In Chapter 3 we showed that the robust counterpart of this approximation remains a Linear Program. We performed numerical experiments to assess the performance of our method on square-root and inverse-quadratic functions. According to our findings on instances of the Generalized Assignment and Knapsack problems, the performance of the robust solution depends on the problem structure, in particular the sparsity of the constraint matrix. Furthermore, the piecewise-linear approximation of the non-linear function provided feasibility of the robust solution over data variations in all cases, which empirically validates the approach.

In Chapter 4, we presented a robust approach for selecting new housing programs. In this case the non-linearities were introduced in the problem's structure via a choice model, where a change in one utility parameter impacts in a non-linear way the probability of selecting each option. We addressed this problem with a discrete set of deviation scenarios from the nominal data, and showed that the binary protection subproblem can be replaced by its linear relaxation with no consequence on its solution. Numerical experiments on data of the Paris region were conducted to show the impact of using the robust solution over the nominal one. Furthermore, the robustness approach combining an uncertainty budget and a discrete set of scenarios can be applied to a larger class of optimal selection problems based on a choice model, where the decision maker has to select a subset of options to offer to customers.

In Chapter 5 we proposed a facility location model with nested uncertainty budgets to address global and local uncertainty parameters. The global budget applies to the number of impacted locations and the local one controls the number of impacted links reaching that location. We study the effects of uncertainty on the arrival rates, which affect the capacity consumption of the selected location in a non-linear (fractional) way. We showed that the robust subproblem associated with a potential location can be linearized, and the local uncertainty handled through [18]. However, without the global budget, this approach results too conservative for it assumes all alternatives to be affected by uncertainty, hence the proposed approach with nested uncertainty budgets. For this problem, due to the sparsity of the solution, the benefits of using a robust counterpart are also diluted as had been found in Chapter 3 and in Poss [60].

We finish this chapter by proposing two research avenues we believe could further the impact of RO not only in the type of problems studied in this dissertation but also on those sharing similar structures:

i. From Chapters 3 and 5, we noticed the impact that sparse structures, either on the constraint matrix or the solution set of a problem, have on the treatment of uncertainty. While this issue has already been addressed in Poss [60] [61], we believe the study could be extended beyond the cardinality of the affected set.

ii. Chapters 4 and 5 suggest that special structures, where the deviation variables from the nominal data have no obvious deteriorating direction, require further analysis. We offered two methods for dealing with this issue, using scenarios and nested uncertainty budgets, however, we encountered at least one problem structure where these methods cannot be applied, that we briefly describe in what follows.

We studied a blending problem with column-wise uncertainty in the composition of the source materials. Since the coefficients are part of a whole, a constraint of having each column of coefficients summing over 1 (100%) is natural to the problem. While we devised an algorithm to build a robust counterpart, we were unsuccessful in providing full protection against infeasibility. This is an instance of special uncertainty structures where further study could lead to interesting advancements of the theory.

References

[1] Alumur, S. A., Nickel, S., & Saldanha-da-Gama, F. (2012). Hub location under uncertainty. *Transportation Research Part B: Methodological*, 46(4), 529-543.

[2] Álvarez-Miranda, E., Fernández, E., & Ljubić, I. (2015). The recoverable robust facility location problem. *Transportation Research Part B: Methodological*, 79, 93-120.

[3] Álvarez-Miranda, E., Ljubić, I., Raghavan, S., & Toth, P. (2014). The recoverable robust two-level network design problem. *INFORMS Journal on Computing*, 27(1), 1-19.

[4] Álvarez-Miranda, E., Ljubić, I., & Toth, P. (2013). A note on the Bertsimas & Sim algorithm for robust combinatorial optimization problems. *4OR*, 11(4), 349-360.

[5] Álvarez, P. P., & Vera, J. R. (2014). Application of robust optimization to the sawmill planning problem. *Annals of Operations Research*, 219(1), 457-475.

[6] Angilella, S., Bottero, M., Corrente, S., Ferretti, V., Greco, S., & Lami, I. M. (2016). Non Additive Robust Ordinal Regression for urban and territorial planning: an application for siting an urban waste landfill. *Annals of Operations Research*, 245(1-2), 427-456.

[7] Aros-Vera, F., Marianov, V., & Mitchell, J. E. (2013). p-Hub approach for the optimal park-and-ride facility location problem. *European Journal of Operational Research*, 226(2), 277-285.

[8] Averbakh, I. (2001). On the complexity of a class of combinatorial optimization problems with uncertainty. *Mathematical Programming*, 90(2), 263-272.

[9] Baron, O., Milner, J., & Naseraldin, H. (2011). Facility location: a robust optimization approach. *Production and Operations Management*, 20(5), 772-785.

[10] Benati,S., & Hansen,P. (2002). The maximum capture problem with random utilities. Problem formulation and algorithms. *European Journal of Operational Research*, 143, 518-530.

[11] Ben-Tal, A., Goryashko, A., Guslitzer, E., & Nemirovski, A. (2004). Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2), 351-376.

[12] Ben-Tal A., & Nemirovski, A. (1999). Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1), 1-13.

[13] Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM review*, 53(3), 464-501.

[14] Bertsimas, D., & Caramanis, C. (2010). Finite adaptability in multistage linear optimization. *IEEE Transactions on Automatic Control*, 55(12), 2751-2766.

[15] Bertsimas, D., Litvinov, E., Sun, X. A., Zhao, J., & Zheng, T. (2013). Adaptive robust optimization for the security constrained unit commitment problem. *Power Systems, IEEE Transactions on*, 28(1), 52-63.

[16] Bertsimas, D., Pachamanova, D., & Sim, M. (2004). Robust linear optimization under general norms. *Operations Research Letters*, 32(6), 510-516.

[17] Bertsimas, D., & Sim, M. (2003). Robust discrete optimization and network flows. *Mathematical Programming*, 98(1), 49-71.

[18] Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations Research*, 52(1), 35-53.

[19] Bertsimas, D., & Thiele, A. (2006). A robust optimization approach to inventory theory. *Operations Research*, 54(1), 150-168.

[20] Billionnet, A., Costa, M. C., & Poirion, P. L. (2014). 2-Stage Robust MILP with continuous recourse variables. *Discrete Applied Mathematics*, 170, 21-32.

[21] Bohle, C., Maturana, S., & Vera, J. (2010). A robust optimization approach to wine grape harvesting scheduling. *European Journal of Operational Research*, 200(1), 245-252.

[22] Bulent, U., & Kenneth, B. H. (2005). Impact of local public services and taxes on dwelling choice within a single taxing jurisdiction: a discrete choice model. *Journal of Real Estate Research*, 27(4), 427-443.

[23] Büsing, C., & D'Andreagiovanni, F. (2014). A new theoretical framework for Robust Optimization under multi-band uncertainty. *In Operations Research Proceedings 2012 (pp. 115-121)*. Springer International Publishing.

[24] Buzna, L'., Koháni, M., & Janáček, J. (2014). An approximation algorithm for the facility location problem with lexicographic minimax objective. *Journal of Applied Mathematics*.

[25] Charnes, A., & Cooper, W. W. (1962). Programming with linear fractional functionals. *Naval Research Logistics Quarterly*, 9(3-4), 181-186.

[26] Chu, P. C., & Beasley, J. E. (1997). A genetic algorithm for the generalised assignment problem. *Computers & Operations Research*, 24(1), 17-23.

[27] Denoyel, V., Alfandari, L., & Thiele, A. (2017). Optimizing healthcare network design under Reference Pricing and parameter uncertainty. *European Journal of Operational Research*.

[28] Driant, Jean-Claude translated by Christina Mitrakos. Why Build New Housing? Metropolitics, 28 January 2011. URL : http://www.metropolitiques.eu/Why-Build-New-Housing.html

[29] Driant, Jean-Claude and translated by Oliver Waine. Why isn't there enough housing in France? Metropolitics, 11 January 2012. URL : http://www.metropoliti ques.eu/Why-isn-t-there-enough-housing-in.html

[30] Düzgün, R., & Thiele, A. (2015). Robust optimisation with multiple ranges: Theory and application to pharmaceutical project selection. *In Proceedings of 14th INFORMS Computing Society Conference* (pp. 103-118).

[31] Espinoza García, J. C. & Alfandari, L. (2015). Robust Location of New Housing Developments Using a Choice Model. Available at SSRN: http://ssrn.com/abstract=2692635

[32] Eurostat. Construction cost index overview, January 2015. URL : http://ec.europa.eu/eurostat/statistics-explained/ index.php/ Construction _cost _index _overview

[33] Fischetti, M., & Monaci, M. (2009). Light robustness. In *Robust and online large-scale optimization* (pp. 61-84). Springer Berlin Heidelberg.

[34] Freire, A.S., Moreno, E., & Yushimito, W.F. (2016). A column generation approach for the optimal selection of park-and-ride facilities. *In Proc. of the 9th Triennal Symp. on Transp. Analysis (TRISTAN IX)*.

[35] Gabrel, V., & Murat, C. (2010). Robustness and duality in linear programming. *Journal of the Operational Research Society*, 61(8), 1288-1296.

[36] Gabrel, V., Murat, C., & Wu, L. (2013). New models for the robust shortest path problem: complexity, resolution and generalization. *Annals of Operations Research*, 207(1), 97-120.

[37] Gabrel, V., Murat, C., & Thiele, A. (2014). Recent advances in robust optimization: An overview. *European Journal of Operational Research*, 235(3), 471-483.

[38] Gancarova, M., & Todd, M.J., 2012. A robust robust optimization result. *Operations Research Letters*, 40, 2–5.

[39] Gilbert, A. (2011). Ten Myths Undermining Latin American Housing Policy. *Revista de Ingeniería*, (35), 79-87.

[40] Goerigk, M., & Schöbel, A. (2014). Recovery-to-optimality: A new two-stage approach to robustness with an application to aperiodic timetabling. *Computers & Operations Research*, 52, 1-15.

[41] Goerigk, M., & Schöbel, A. (2016). Algorithm engineering in robust optimization. In *Algorithm engineering* (pp. 245-279). Springer International Publishing.

[42] Goh, J., & Sim, M. (2010). Distributionally robust optimization and its tractable approximations. *Operations Research*, 58(4-part-1), 902-917.

[43] Gorissen, B. L., Yanıkoğlu, İ., & den Hertog, D. (2015). A practical guide to robust optimization. *Omega*, 53, 124-137.

[44] Guadagni P.M., Little J.D.C., (1983) A logit model of branc choice calibrated on scanner data, *Marketing Science* 2(1), 203-238.

[45] Gülpinar, N., Pachamanova, D., & Çanakoglu, E. (2013). Robust strategies for facility location under uncertainty. *European Journal of Operational Research*, 225(1), 21-35.

[46] Haase, K. (2009). Discrete location planning. *Institute of Transport and Logistics Studies Working Paper*, (ITLS-WP-09-07).

[47] Haase, K., & Müller, S. (2014). A comparison of linear reformulations for multinomial logit choice probabilities in facility location models. *European Journal of Operational Research*, 232(3), 689-691.

[48] Ibarra, O. H., & Kim, C. E. (1975). Fast approximation algorithms for the knapsack and sum of subset problems. *Journal of the ACM (JACM)*, 22(4), 463-468.

[49] Kalai, R., Lamboray, C., Vanderpooten, D., 2012. Lexicographic alpha-robustness: An alternative to min-max criteria. *European Journal of Operational Research*, 220, 722–728

[50] Le, X. T., & Knust, S. (2017). MIP-based approaches for robust storage loading problems with stacking constraints. *Computers & Operations Research*, 78, 138-153.

[51] Lorca, A., & Sun, X. A. (2015). Adaptive robust optimization with dynamic uncertainty sets for multi-period economic dispatch under significant wind. *Power Systems, IEEE Transactions on*, 30(4), 1702-1713.

[52] Lucan, S. C., Maroko, A., Shanker, R., & Jordan, W. B. (2011). Green Carts (Mobile Produce Vendors) in the Bronx—Optimally Positioned to Meet Neighborhood Fruit-and-Vegetable Needs?. *Journal of Urban Health*, 88(5), 977.

[53] Mc Fadden, D. et al. (1973). Conditional logit analysis of qualitative choice behavior. Frontiers in Econometrics, ed. by P. Zarembka, New York: Academic Press. Institute of Urban and Regional Development, University of California, 105-142.

[54] Meier, H., Christofides, N., & Salkin, G. (2001). Capital budgeting under uncertainty — an integrated approach using contingent claims analysis and integer programming. *Operations Research*, 49(2), 196-206.

[55] Minoux, M. (2011). On 2-stage robust LP with RHS uncertainty: complexity results and applications. *Journal of Global Optimization*, 49(3), 521-537.

[56] Müller, S., & Haase, K. (2014). Customer segmentation in retail facility location planning. *Business Research* 7(2), 235-261.

[57] Osman, I. H., & Christofides, N. (1994). Capacitated clustering problems by hybrid simulated annealing and tabu search. *International Transactions in Operational Research*, 1(3), 317-336.

[58] Pagliara, F., Preston, J., & Simmonds, D. (2010). *Residential location choice*. Dordrecht: Springer.

[59] de Palma, A., Picard, N., & Waddell, P. (2007). Discrete choice models with capacity constraints: An empirical analysis of the housing market of the greater Paris region. *Journal of Urban Economics*, 62(2), 204-230.

[60] Poss, M. (2013). Robust combinatorial optimization with variable budgeted uncertainty. *4OR*, 11(1), 75-92..

[61] Poss, M. (2014). Robust combinatorial optimization with variable cost uncertainty. *European Journal of Operational Research*, 237(3), 836-845.

[62] Poss, M. (2017). Robust combinatorial optimization with knapsack uncertainty.

[63] Ramos, G., Pequito, S., Aguiar, A. P., & Kar, S. (2015). Analysis and design of electric power grids with p-robustness guarantees using a structural hybrid system approach. In *Control Conference (ECC)*, 2015 European IEEE (pp. 3542-3547).

[64] Reed, R. (2016). Longitudinal new housing analysis–USA, Australia, Canada and France. *International Journal of Housing Markets and Analysis* 9(4), 425-428.

[65] Roy, B. (2010). Robustness in operational research and decision aiding: A multi-faceted issue. *European Journal of Operational Research* 200, 629–638

[66] Sandoval Alarcón, Francisco. Familias abandonan 2 millones de viviendas por no poder pagarlas. Animal Politico. 8 February 2012. URL : http://www.animal politico.com/2012/02/familias-abandonan-2-millones-de-viviendas-por-no-poder-pagarlas/#ixzz2sYtFCloh

[67] Schaible, S., & Shi, J. (2003). Fractional programming: the sum-of-ratios case. *Optimization Methods and Software*, 18(2), 219-229.

[68] Snyder, L. V. (2006). Facility location under uncertainty: a review. *IIE Transactions*, 38(7), 547-564.

[69] Soyster, A.L. (1973). Technical note - convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research*, 21(5), 1154-1157.

[70] United States Census Bureau. Value of Private Residential Construction Put in Place excluding rental, vacant, and seasonal residential improvements - Not Seasonally Adjusted. URL : https://www.census.gov/construction/c30/pdf/residentialnsa.pdf.

[71] Zokaee, S., Jabbarzadeh, A., Fahimnia, B., & Sadjadi, S. J. (2014). Robust supply chain network design: an optimization model with real world application. *Annals of Operations Research*, 1-30.