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**AUTOUR DE QUELQUES CHAINES DE
MARKOV COMBINATOIRES
SOME RESULTS CONCERNING MARKOV
CHAINS ON COMBINATORIALS OBJECTS**

THESE DE DOCTORAT

par

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Résumé

On s'intéresse à deux classes de chaînes de Markov combinatoires. On commence avec les chaînes de Markov de Jonglage, inspirées du modèle de jonglage introduit par Warrington, pour lesquelles on définit des généralisations multivariées des modèles existants. On en calcule les mesures stationnaires et les facteurs de normalisation que l'on exprime par des formules explicites. On s'intéresse également au cas limite où la hauteur maximale à laquelle le jongleur peut lancer ses balles tend vers l'infini. On propose alors une reformulation de la chaîne de Markov en termes de partitions d'entiers, ce qui permet aussi de définir un modèle où le jongleur manipule une infinité de balles. Les preuves sont obtenues en utilisant une chaîne enrichie sur les partitions d'ensembles. On exhibe également, pour l'un des modèles, une propriété de convergence ultrarapide : la mesure stationnaire y est atteinte en un nombre fini d'étapes. Dans le Chapitre suivant, on s'intéresse à des généralisations multivariées de ces modèles : on considère cette fois un jongleur manipulant des balles de différents poids, et lorsqu'une balle entre en collision avec une balle plus légère, cette dernière est éjectée vers le haut, pouvant à son tour en heurter une autre plus légère, jusqu'à ce qu'une balle atteigne l'emplacement le plus élevé. On donnera ici encore une formule explicite pour les mesures stationnaires et les facteurs de normalisation. Dans le dernier Chapitre, on s'intéresse cette fois au modèle du tas de sable stochastique, pour lequel on démontre une conjecture posée par Selig, selon laquelle la mesure stationnaire ne dépend pas de la loi d'ajout des grains de sable.

Abstract

We consider two types of combinatoric Markov chains. We start with Juggling Markov chains, inspired from Warrington's model. We define multivariate generalizations of the existing models, for which we give stationary measures and normalization factors with closed-form expressions. We also investigate the case where the maximum height at which the juggler may send balls tends to infinity. We then reformulate the Markov chain in terms of integer partitions, which allows us to consider the case where the juggler interacts with infinitely many balls. Our proofs are obtained through an enriched Markov chain on set partitions. We also show that one of the models has the ultrafast convergence property : the stationary measure is reached after a finite number of steps. In the following Chapter, we consider multivariate generalizations of those models : the juggler now juggles with balls of different weights, and when a heavy ball collides with a lighter one, this light ball is bumped to a higher position, where it might collide with a lighter one, until a ball reaches the highest position. We give closed-form expressions for the stationary measures and the normalization factors. The last Chapter is dedicated to the stochastic sandpile model, for which we give a proof for a conjecture set by Selig : the stationary measure does not depend on the law governing sand grains additions.

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Chapter 1

Introduction

Cette thèse a pour sujet l'étude de Chaînes de Markov Combinatoires. La Combinatoire se situe à l'interface entre les mathématiques et l'informatique théorique, et se concentre sur l'étude de structures discrètes, parmi lesquelles figurent les arbres, les cartes ou encore les chemins de Dyck. Les Chaînes de Markov combinatoires sont des processus à temps discret opérant sur de telles structures. L'étude de ces processus est la plupart du temps motivée par des problèmes de physique statistique. On peut en particulier citer les modèles de type PASEP/TASEP, très largement étudiés par les combinatoristes et les physiciens. Les deux premiers chapitres de cette thèse sont consacrés à l'étude de chaînes de Markov de *Jonglage*, pour lesquelles nous donnons des formes explicites caractérisant la mesure stationnaire. Le troisième chapitre portera sur le *tas de sable*, pour lequel nous résoudrons une conjecture posée par Selig.

1.1 Chaînes de Markov

Cette section est en partie inspirée du cours de probabilité de Jean-François Le Gall [29].

1.1.1 Définition

Soit S un ensemble fini ou dénombrable. Une matrice stochastique sur S est une famille $(P(x, y), x, y \in S)$ de nombres réels satisfaisant :

- (i) $0 \leq P(x, y) \leq 1$ pour tout $x, y \in S$;

(ii) pour tout $x \in E$, $\sum_{y \in E} P(x, y) = 1$.

Définition 1.1.1. Soit $(X_n)_{n \in \mathbb{N}}$ une suite de variables aléatoires dans S , et P une matrice stochastique sur S . On dit que $(X_n)_{n \in \mathbb{N}}$ est une *chaîne de Markov* de matrice de transition P si et seulement si pour tout entier $n \geq 0$, la loi conditionnelle de X_{n+1} connaissant $(X_0, X_1, \dots, X_n)$ est donnée par $P(X_n, \cdot)$. On peut l'écrire de manière équivalente :

$$\mathbb{P}(X_{n+1} = y | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) = P(x_n, y).$$

Le fait que la loi conditionnelle de X_{n+1} sachant $(X_k)_{0 \leq k \leq n}$ soit fonction de X_n seul est caractéristique des chaînes de Markov. On appelle S l'*espace d'états* de la chaîne de Markov.

Dans le cas où l'ensemble S est de petite taille, il est fréquent de représenter une chaîne sous forme d'un graphe pondéré orienté : les sommets représentent les éléments de S , aussi appelés les *états* de la chaîne de Markov, et les arêtes représentent les transitions possibles d'un état à un autre : si e est une arête allant d'un état x à un état y , alors sa pondération est $P(x, y)$. Bien entendu, seules les arêtes de pondération non nulle sont représentées.

La matrice P étant stochastique, il est facile de vérifier que 1 en est valeur propre, et que toute valeur propre de P est de module inférieur ou égal à 1.

Dans le cas où la valeur propre 1 est non dégénérée (c'est à dire si le sous-espace propre associé est de dimension 1), le vecteur propre à gauche associé est appelé la *mesure stationnaire* de la chaîne de Markov.

Définition 1.1.2. Pour x et y deux éléments de S , on dit que y est *accessible* à partir de x si et seulement si il existe un chemin orienté allant de x à y dans le graphe de la chaîne de Markov. Si y est accessible à partir de x et x est accessible à partir de y , on dit que x et y *communiquent*. La relation "communiquer" est une relation d'équivalence. On appelle *classe récurrente* une classe d'équivalence pour cette relation. Si tous les états communiquent, la chaîne de Markov est dite *irréductible*. Un état est dit *récurrent* si l'espérance du temps de retour à cet état, partant de cet état, est finie.

Remarque 1.1.3. Il n'existe qu'une seule classe communicante si et seulement si la valeur propre 1 est non dégénérée pour la matrice de transition. C'est en particulier le cas si la chaîne de Markov est irréductible. De plus, dans ce cas, tous les états sont récurrents.

Définition 1.1.4. Une classe récurrente est dite *apériodique* si, partant d'un état quelconque de cette classe, le PGCD des temps possibles de retour à cette état est 1.

Remarque 1.1.5. Toutes les classes récurrentes sont apériodiques si et seulement si 1 est la seule valeur propre de module 1 pour la matrice de transition. En pratique, dans cette thèse, pour montrer l'apériodicité d'une classe récurrente, il suffira de montrer qu'il existe un état récurrent muni d'une boucle dans le graphe de transition.

1.1.2 Exemple

Nous allons ici nous intéresser à un exemple très simple inspiré par un modèle plus général, issu de la physique statistique : le TASEP (Totally Asymmetric Simple Exclusion Process).

On considère ici que l'on dispose de 2 sites, alignés et numérotés de gauche à droite, chacun de ces sites pouvant contenir une particule ou être vide. À chaque unité de temps, l'une des trois choses suivantes peut arriver :

1. Si le premier site (celui de gauche) est vide, une particule peut y entrer par la gauche (il s'agit d'une condition de bord).
2. Si le deuxième site est occupé, la particule peut en sortir par la droite (seconde condition de bord).
3. Si le premier site est occupé et le deuxième vide, la particule peut se déplacer du premier site vers le deuxième.

On va ici considérer que, à un état donné, toutes les opérations possibles sont équiprobables.

L'ensemble des états est ici l'ensemble des mots de longueur 2 sur l'alphabet $\{0, 1\}$: un 0 représente un site vide, et un 1 un site occupé. Le graphe de transition et la matrice de transition sont comme suit :

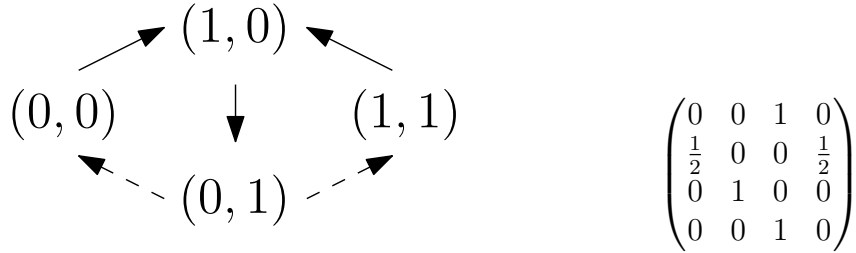


Figure 1.1 – Le graphe de transition pour le tasep à 2 sites. Les flèches pleines représentent les transitions de probabilité 1, tandis que les flèches en pointillés représentent les transitions de probabilité $\frac{1}{2}$.

La matrice de transition P pour la chaîne de Markov dans la base $\{(0,0); (0,1); (1,0); (1,1)\}$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Au vu du graphe, il est très facile de voir que la chaîne de Markov est irréductible, et périodique de période 3. Les fréquences d'apparition des états $(0,0)$, $(0,1)$, $(1,0)$ et $(1,1)$ sont respectivement $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{3}$ et $\frac{1}{6}$, et la mesure stationnaire est donnée par $(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6})$, qui est le vecteur propre à gauche associé à la valeur propre 1 pour M .

Les valeurs propres de M sont 0 et les 3 racines 3-èmes de l'unité. 1 n'est donc pas la seule valeur propre de module 1, ce qui concorde bien avec le fait que la chaîne de Markov ne soit pas apériodique.

1.1.3 Chaîne de Markov enrichie et projection

Pour déterminer la mesure stationnaire π (dans le cas où elle est unique), on intuite une formule générale à l'aide d'exemples traités par informatique. On peut ensuite vérifier directement que la formule trouvée est la bonne, en montrant que pour tout x dans S , que $\sum_{y \in S} \pi(y)P(y,x) = \pi(x)$. Cette méthode n'est cependant pas toujours efficace, car les antécédents d'un état ne sont pas toujours faciles à déterminer, et même quand elle l'est, elle reste peu éclairante sur le fonctionnement de la chaîne de Markov.

Dans la suite de cette thèse, on va souvent utiliser une chaîne enrichie, comme expliqué dans [31, Section 2.3.1]. L'idée générale est la suivante : on part d'une chaîne de Markov sur un espace d'états S , dont la matrice de transition est P , et pour laquelle on veut exprimer la mesure stationnaire, c'est à dire le vecteur ligne π dont la somme des coefficients vaut 1, et tel que $\pi P = \pi$. L'idée est d'introduire une autre chaîne de Markov, dite "enrichie", sur un espace d'états plus large $\tilde{S}$, et de matrice de transition $\tilde{P}$, vérifiant

les deux propriétés suivantes :

1. sa mesure stationnaire $\tilde{\pi}$ est facile à trouver et à exprimer
2. cette chaîne se projette sur la chaîne de Markov originale : il existe une relation d'équivalence $\sim$ sur $\tilde{S}$ telle que S soit isomorphe à l'ensemble des classes d'équivalence $\tilde{S}/\sim$, et telle que la condition de projection

$$\sum_{y' \sim y} \tilde{P}_{x,y'} = P_{[x],[y]} \quad (1.1)$$

soit vérifiée pour tout $x, y \in \tilde{S}$, où $[x] \in S$ est la classe d'équivalence de x .

Il est ensuite facile de vérifier que la mesure stationnaire π de la chaîne de Markov originale est donnée par

$$\pi_{[x]} = \sum_{x' \sim x} \tilde{\pi}_{x'}. \quad (1.2)$$

Cette méthode sera utilisée dans la section 1.2, ainsi que dans les chapitres 2 et 3. En pratique, pour deux états $[x], [y] \in S$ tels que $P_{[x],[y]} \neq 0$, et pour tout $x' \in [x]$, il existera un unique $y' \in [y]$ tel que $\tilde{P}_{x',y'} \neq 0$ et on aura $\tilde{P}_{x',y'} = P_{[x],[y]}$, ce qui rendra la condition 1.1 très facile à vérifier.

1.2 Jonglage Uniforme

Nous allons maintenant définir les Chaînes de Markov de Jonglage, introduites par Warrington dans [41], et étudiées dans les chapitres 2 et 3. Dans ces modèles, on considère un jongleur, jonglant avec un nombre fixé fini ℓ de balles. Le temps est discrétisé en pas de 1 seconde, et on suppose que à chaque seconde, le jongleur ne peut attraper au plus qu'une balle, qu'il relance aussitôt. On considère de plus qu'il jongle parfaitement, de façon à ce que chaque balle qu'il lance lui revienne après un certain nombre entier de secondes, borné par un entier fixé h .

On ignore tout autre paramètre, de telle façon que l'état des balles à un instant donné est complètement déterminé par le temps restant à chaque balle avant d'être rattrapée par le jongleur. Bien entendu, pour deux balles distinctes quelconques ces temps doivent être différents. Il y a donc $\binom{h}{\ell}$ états de jonglage possibles.

Il est naturel de décrire un tel processus comme une chaîne de Markov. On peut représenter les états comme des configurations de ℓ particules (balles) non superposées sur un réseau unidimensionnel contenant h emplacements, où le i -ème emplacement (en partant de la gauche) sera occupé si et seulement si une balle doit arriver dans la main du jongleur i secondes plus tard.

À chaque étape, toutes les balles sont déplacées d'un emplacement vers la gauche. Si le premier site était vide, rien d'autre ne se passe. Sinon, la particule qui l'occupe est réinsérée dans un des emplacements libres, uniformément au hasard. L'uniformité de cette réinsertion est propre au modèle de Warrington [41]. Dans le chapitre 2, nous nous intéresserons au cas de réinsertions non uniformes.

1.2.1 Définition formelle

Posons St_h l'ensemble des mots de longueur h sur l'alphabet $\{\bullet, \circ\}$, et $\text{St}_{h,k} \subset \text{St}_h$ le sous ensemble des mots contenant k occurrences de $\circ$ (avec $k \leq h$ nécessairement). Pour $A \in \text{St}_{h,k+1}$ et $i \in \{0, \dots, k\}$, on définit $T_i(A) \in \text{St}_{h,k}$ le mot obtenu en remplaçant la $(i+1)$ -ème occurrence de $\circ$ dans A (de gauche à droite) par $\bullet$. Chaque $\bullet$ représente une balle, et chaque $\circ$ un espace vide. Bien entendu, on a $\ell = h - k$.

Définition 1.2.1. Étant donnés h, k deux entiers positifs tels que $h \geq k$, la *Chaîne de Markov de Jonglage Uniforme* (CMJU) est la chaîne de Markov dont l'espace d'états est $\text{St}_{h,k}$ et pour laquelle les coefficients de la matrice de transition P sont donnés, pour $A = a_1 a_2 \dots a_h$ et B dans $\text{St}_{h,k}$:

$$P_{A,B} = \begin{cases} 1 & \text{si } a_1 = \circ \text{ et } B = a_2 \dots a_h \circ, \\ \frac{1}{k+1} & \text{si } a_1 = \bullet \text{ et } B = T_i(a_2 \dots a_h \circ) \text{ pour un certain } i, \\ 0 & \text{sinon.} \end{cases} \quad (1.3)$$

Exemple 1.2.2. La figure 1.2 représente la chaîne de Markov dans le cas $h = 4$ et $k = 2$.

La matrice de transition dans la base $(\bullet\bullet\circ\circ, \bullet\circ\bullet\circ, \bullet\circ\circ\bullet, \circ\bullet\bullet\circ, \circ\bullet\circ\bullet, \circ\circ\bullet\bullet)$

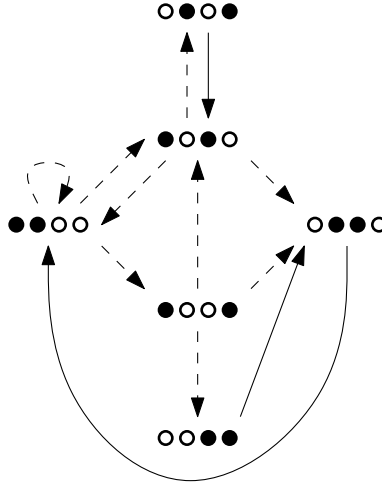


Figure 1.2 – La CMJU pour $h = 4$ and $\ell = k = 2$. Les flèches partant de chaque état sont toutes équiprobables : une flèche pleine représente une transition de probabilité 1, et une flèche en pointillés une transition de probabilité $\frac{1}{3}$.

est donnée par :

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (1.4)$$

On peut vérifier que $(9, 6, 3, 4, 2, 1)$ en est un vecteur propre à gauche pour la valeur propre 1.

Proposition 1.2.3. *Pour tout h, k dans $\mathbb{N}$ avec $h \geq k$, la CMJU est irréductible et apériodique.*

On donne ici une preuve rapide et non détaillée, inspirée de celle donnée pour le résultat correspondant apparaissant dans le chapitre 2.

Proof. Il est facile de voir que l'état minimal $C = \bullet^\ell \circ^k$ peut-être atteint en un nombre fini d'opérations depuis n'importe quel état (en envoyant les balles qui arrivent dans le premier emplacement libre à chaque fois), et que n'importe quel état peut être atteint depuis C (en plaçant les balles judicieusement),

ce qui assure l'irréductibilité de la chaîne. De plus, C peut boucler sur lui-même, ce qui assure l'apériodicité de la chaîne. $\square$

Ce résultat assure l'unicité de la mesure stationnaire.

1.2.2 La mesure stationnaire

Notation 1.2.4. For $B \in \text{St}_{h,k}$, on note

$$E_i(B) = \#\{j < i | b_j = \circ\} : \quad (1.5)$$

$E_i(B)$ compte le nombre de sites vides à gauche du i -ème site dans B .

Le théorème suivant est une reformulation d'un résultat donné dans [41].

Théorème 1.2.5. *La mesure stationnaire π pour la CMJU est donnée par*

$$\pi(B) = \frac{1}{\left\{ \begin{smallmatrix} h+1 \\ k+1 \end{smallmatrix} \right\}} \prod_{\substack{i \in \{1, \dots, h\} \\ b_i = \bullet}} (k+1 - E_i(B)) \quad (1.6)$$

où $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ dénote les nombres de Stirling de deuxième espèce : il s'agit du nombre de façons de partitionner un ensemble de n éléments distincts en m sous-ensembles non vides.

Remarque 1.2.6. On vérifie aisément que cette formule est en accord avec le vecteur propre donné dans l'exemple 1.2.2. En effet, on a $(1+2) * (1+2) = 9$, $(1+2) * (1+1) = 6$ et ainsi de suite.

Ce résultat va être montré en utilisant l'approche décrite dans la section 1.1.3.

On va étudier une chaîne de Markov enrichie, où l'espace d'états sera de cardinal $\left\{ \begin{smallmatrix} h+1 \\ k+1 \end{smallmatrix} \right\}$, dont la mesure stationnaire sera uniforme sur l'ensemble des états, et de sorte que cette nouvelle chaîne de Markov se projette sur la CMJU.

1.2.3 Chaîne de Markov enrichie sur les partitions d'ensembles

L'idée qui vient à l'esprit en observant la formule (1.6) est que pour un état donné, le nombre de façons dont chaque balle a pu arriver à sa place actuelle influe directement sur le coefficient de cet état dans la mesure stationnaire.

La chaîne de Markov enrichie doit donc permettre de différencier chacune de ces possibilités.

L'espace d'états de la chaîne enrichie sera l'ensemble $\tilde{\text{St}}_{h,k}$ des tableaux de Young de forme $(h, h-1, \dots, 1)$ contenant $\ell = h-k$ particules, avec interdiction d'avoir plus d'une particule dans chaque ligne ou dans chaque colonne. On parlera de *tableaux en escalier*. On peut facilement vérifier que le nombre des tels tableaux est $\{h+1\}_{k+1}$. On donne une idée d'une bijection Θ entre ces tableaux et les partitions d'ensemble dans la Figure 1.3. Les partitions d'entier y sont représentées sous forme de diagrammes d'arches : on a 5 particules distinctes alignées de gauche à droite, et 2 particules sont dans la même part si il existe une suite d'arches permettant de passer de l'une à l'autre.

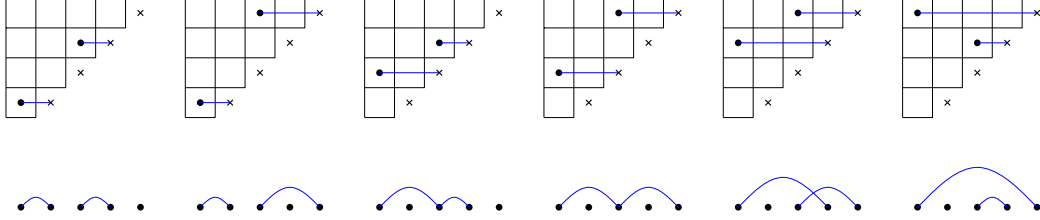


Figure 1.3 – La classe d'équivalence des tableaux en escalier associés à l'état de jonglage $\bullet \circ \bullet \circ$, ainsi que les partitions d'ensembles auxquelles ils correspondent via la bijection Θ . On passe d'un tableau à une partition en traçant, pour chaque particule dans le tableau, une ligne bleue la reliant à la croix située à droite du tableau dans la même ligne. Chaque ligne est ensuite projetée sur la partition d'ensemble comme arc entre deux éléments, en conservant sa longueur. L'opération inverse est effectuée en choisissant la bonne hauteur pour chaque ligne de manière à ce l'extrémité droite en soit une croix, puis on place une particule à l'extrémité gauche de chacune de ces lignes. On observe que le poids de $\bullet \circ \bullet \circ$ donné par la formule (1.6) est 6, qui est bien le nombre de tableaux en escalier associés à cet état.

À chaque étape, toutes les particules dans le tableau sont déplacées d'un pas vers le Sud-Ouest (en diagonale), et si une particule devait sortir du tableau, elle est alors réinsérée dans la première ligne (celle du haut), uniformément au hasard parmi les cases n'ayant pas déjà une particule dans leur colonne.

On dit que deux tableaux en escalier sont équivalents si ils ont des particules dans les mêmes colonnes. On notera $\sim$ la relation d'équivalence associée.

Chaque classe d'équivalence sera identifiée à l'état de jonglage où l'on identifie chaque colonne à un espace, vide si la colonne ne contient pas de particule, et contenant une balle sinon.

On donne un exemple de la chaîne enrichie et de la façon dont elle se projette Figure 1.4. On laissera le lecteur se convaincre que la condition (1.1) est bien vérifiée.

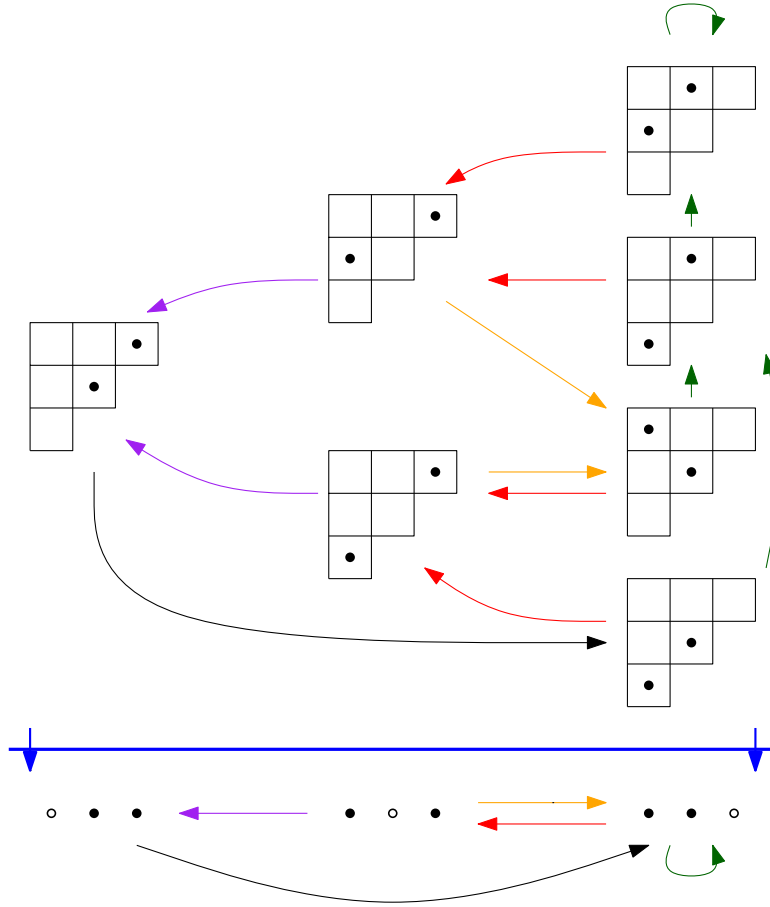


Figure 1.4 – La chaîne enrichie dans le cas $h = 3$ et $k = 1$. Les flèches colorées dans la chaîne enrichie correspondent aux flèches de la même couleur dans la chaîne de jonglage.

On peut aisément vérifier que cette chaîne enrichie est irréductible et apériodique : la preuve est similaire à celle de la Proposition 1.2.3 en prenant pour état *minimal* celui où les ℓ particules occupent tous les sites de la

diagonale partant du site situé sur la ℓ -ème ligne de la 1-ère colonne, et finissant sur le site en 1-ère ligne, ℓ -ème colonne (en comptant les lignes de haut en bas, et les colonnes de gauche à droite.)

La proposition suivante est démontrée dans [41].

Proposition 1.2.7. *La mesure stationnaire $\tilde{\pi}$ pour la chaîne de Markov enrichie est donnée par :*

$$\tilde{\pi}(T) = \frac{1}{\{h+1\}_{k+1}}. \quad (1.7)$$

pour tout $T \in \tilde{\text{St}}_{h,k}$.

Il ne reste alors plus qu'à vérifier que pour $B \in \text{St}_{h,k}$, le nombre de tableaux en escalier dans la classe d'équivalence associée à B est bien donné par le numérateur obtenu dans la formule (1.6). Il s'agit alors d'un simple problème d'énumération.

Dans le chapitre 2, nous étudierons une généralisation de ce modèle, que nous nommerons la Chaîne de Markov de Jonglage Multivariée (MJMC) : en plus des paramètres h et ℓ , on fixe $k+1$ réels positifs $x_0, \dots, x_k$ de somme 1, et lorsque le jongleur renvoie une balle dans le i -ème espace libre (en comptant à partir de l'espace le plus à gauche), il le fait avec probabilité x_{i-1} . On retrouve le modèle décrit précédemment en fixant $x_0 = \dots = x_k = \frac{1}{k+1}$.

On montrera dans le chapitre 2 le théorème suivant :

Théorème 1.2.8. *La mesure stationnaire π pour la MJMC est donnée par*

$$\frac{1}{Z_{h,k}} \prod_{\substack{i \in \{1, \dots, h\} \\ b_i = \bullet}} (x_{E_i(B)} + \dots + x_k),$$

où $B = b_1 \dots b_n$ et où $Z_{h,k}$ est le facteur de normalisation, et E_i est comme dans (1.5).

Dans ce même chapitre, on étudiera le cas où le jongleur peut envoyer ses balles aussi haut qu'il le souhaite : on oublie la paramètre h , et on se munit d'une famille infinie $(x_n)_{n \geq 0}$ de réels positifs de somme 1. Le théorème 1.2.8 reste valide dans ce cas.

Dans le chapitre 3, on s'intéressera au cas où le jongleur utilise des balles de poids différents : quand le jongleur rattrape une balle, il considère que

tout espace occupé par une balle plus légère est un espace vide. Si il choisit de lancer la balle qu'il vient de rattraper dans un tel espace, celle qui l'occupait est alors éjectée et va à son tour se loger dans un espace, situé à sa droite, et lui même occupé par une balle plus légère, et ainsi de suite, jusqu'à ce qu'une balle arrive en la position la plus élevée (ce qui est toujours possible).

On donnera également une formule pour la mesure stationnaire de ce modèle, et le MJMC est retrouvé en fixant le nombre T de poids possibles pour les balles à 2 (les balles les plus légères étant en fait les espaces vides du modèle précédent).

1.2.4 Autres modèles

On considère maintenant que le nombre de balles n'est plus fixé : la dynamique du modèle est presque la même, mais cette fois le jongleur peut aussi laisser tomber des balles, et la chaîne de Markov autorise donc des transitions vers des états avec moins de balles. Comme le nombre de balles pourrait éventuellement tomber à 0, on autorise aussi l'ajout de balles pour que le modèle reste intéressant. Ainsi, à chaque étape, le jongleur pourra décider de lancer une balle même s'il ne vient pas d'en rattraper, ou de ne rien faire (auquel cas si une balle devait lui arriver, elle tombe et est retirée à la place).

Étant donné h un entier positif, l'espace d'état sera donc St_h . On ne donne pas encore de formule pour les probabilités de transition, car on va proposer deux modèles dans lesquels bien que les transitions soient les mêmes, leurs probabilités sont différentes.

On peut facilement montrer que ces chaînes de Markov sont irréductibles et apériodiques. On donne le graphe de transition (avec les flèches seulement) pour le cas $h = 3$ Figure 1.5.

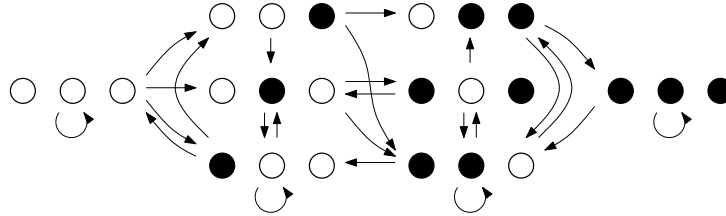


Figure 1.5 – Le graphe de transition dans le cas $h = 3$.

Le modèle add-drop

Dans ce modèle, on considère que le fait de ne pas relancer de balle est équivalent à effectuer un lancer dans le 0-ème espace, toujours libre. Pour $A = a_1 a_2 \cdots a_h$ et B dans St_h , la probabilité de transition de A à B est ainsi donnée par :

$$P_{A,B} = \begin{cases} \frac{1}{k+1} & \text{si } B = T_i(a_2 \cdots a_h \circ) \text{ pour un certain } i \text{ entre } -1 \text{ et } k-1, \\ 0 & \text{sinon.} \end{cases} \quad (1.8)$$

où k est le nombre d'occurrences de $\circ$ dans $a_2 \cdots a_h \circ$, où $(T_i(B))_{0 \leq i \leq k}$ est comme donné dans la sous-section 1.2.1, et où $T_{-1}(B) = B$.

La matrice de transition de la chaîne représentée Figure 1.5 (le cas où $h = 3$) dans la base $(\circ \circ \circ, \circ \circ \bullet, \circ \bullet \circ, \bullet \circ \circ, \circ \bullet \bullet, \bullet \circ \bullet, \bullet \bullet \circ, \bullet \bullet \bullet)$ est comme suit :

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad (1.9)$$

et on peut vérifier que $(1, 1, 2, 3, 1, 2, 4, 1)$ en est un vecteur propre à gauche pour la valeur propre 1.

Théorème 1.2.9. [41] *La mesure stationnaire pour le modèle add-drop est donnée par*

$$\Pi(B) = \frac{1}{Z_h} \prod_{\substack{i=1 \\ b_i=\bullet}}^h (\psi_i(B) + 1) \quad (1.10)$$

pour $B = b_1 \cdots b_n \in \text{St}_h$, avec $\psi_i(B) = \#\{j : i < j \leq h, b_j = \circ\}$ et Z_h le facteur de normalisation.

Comme pour le Théorème 1.2.5, ce résultat se montre en utilisant une chaîne enrichie. La preuve est effectuée dans le chapitre 2 dans le cas multi-varié.

Le modèle annihilation

Dans ce modèle, à chaque étape, le jongleur peut lancer une balle dans n'importe quel espace, y compris le 0-ème (comme pour l'annihilation) un espace déjà occupé par une balle. Pour $A = a_1 a_2 \cdots a_h$ et B dans St_h , la probabilité de transition de A à B est ainsi donnée par :

$$P_{A,B} = \begin{cases} \frac{1}{h+1} & \text{si } B = T_i(a_2 \cdots a_h \circ) \text{ pour un certain } i \text{ entre } 0 \text{ et } k-1, \\ \frac{h-k}{h} & \text{si } B = T_{-1}(a_2 \cdots a_h \circ), \\ 0 & \text{sinon.} \end{cases} \quad (1.11)$$

où k est le nombre d'occurrences de $\circ$ dans $a_2 \cdots a_h \circ$, et où $T_{-1}(B) = B$.

La matrice de transition de la chaîne représentée Figure 1.5 (le cas où $h = 3$) dans la base $(\circ \circ \circ, \circ \circ \bullet, \circ \bullet \circ, \bullet \circ \circ, \circ \bullet \bullet, \bullet \circ \bullet, \bullet \bullet \circ, \bullet \bullet \bullet)$ est comme suit :

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \quad (1.12)$$

et on peut vérifier que $(6, 6, 12, 18, 3, 6, 12, 1)$ en est un vecteur propre à gauche pour la valeur propre 1.

Théorème 1.2.10. [41] *La mesure stationnaire pour le modèle annihilation est donnée par*

$$\Pi(B) = \frac{1}{Z_h} \frac{h!}{k!} \prod_{\substack{i=1 \\ b_i = \bullet}}^h (\psi_i(B) + 1) \quad (1.13)$$

pour $B = b_1 \cdots b_h \in \text{St}_{h,k}$, avec $\psi_i(B) = \#\{j : i < j \leq h, b_j = \circ\}$ et Z_h le facteur de normalisation.

Encore une fois, on utilisera une chaîne enrichie pour montrer une version multivariée de ce résultat dans le chapitre 2. On s'intéressera également à

un phénomène de convergence ultrarapide dans ce cas : la chaîne atteint son régime stationnaire en temps fini.

Des versions multiespèces de ces modèles seront également étudiées dans le chapitre 3.

1.3 Le tas de sable abélien

Nous allons maintenant nous intéresser au modèle du tas de sable abélien, qui fut introduit par Bak, Tag et Wiesenfeld [7], et dont les aspects mathématiques sont décrits très précisément par Redig dans [36].

1.3.1 Description du modèle

Le tas de sable abélien (que l’on notera “ASM” pour “Abelian Sandpile Model” dans la suite) est une chaîne de Markov opérant sur un graphe connexe $G = (V \cup \{s\}, E)$ où chaque sommet $v \in V$ est muni d’un compteur positif $\eta(v)$, indiquant combien de grains de sable sont sur ce sommet, et où le *puits* s fonctionne différemment des autres sommets, comme expliqué plus bas. Si le nombre de grains de sable sur un sommet v est inférieur ou égal à son nombre de voisin (i.e. si $\eta(v) \leq d^G(v)$), v est dit *stable*, et *instable* dans le cas contraire. Une *configuration* η de grains de sable est *stable* si chaque sommet dans V est stable, et *instable* dans le cas contraire. L’espace d’états pour le ASM est l’ensemble des configurations stables.

À chaque étape de la chaîne de Markov, un grain de sable est ajouté sur un sommet choisi au hasard selon une distribution μ strictement positive. Si ce sommet devient instable, il s’*éboule*, donnant un de ses grains de sable à chacun de ses voisins, qui peuvent à leur tour éventuellement devenir instables et s’*ébouler*. Les éboulements s’enchaînent jusqu’à ce que l’on arrive à nouveau à une configuration stable. Le puits s se comporte différemment des autres sommets : il peut absorber n’importe quel nombre de grains de sable et ne s’*éboule* jamais, ce qui garantit la finitude du processus d’éboulement. On parlera de *configurations récurrentes* pour qualifier les états récurrents de la Chaîne de Markov. Il est facile de voir que la configuration maximale η_M définie par $\eta_M(v) = d^G(v)$ pour tout v dans V est une configuration récurrente : η_M peut être atteint depuis n’importe quelle configuration en remplissant chacun des sommets, ce qui est possible car μ est strictement

positive. On note $\mathcal{R}$ l'ensemble des configurations récurrentes.

Exemple 1.3.1. On donne un exemple du détail d'une transition pour le ASM Figure 1.6, dans le cas où le graphe G est constitué d'un triangle et du puits, lequel est relié à l'un des sommets du triangle par une arête. On donne

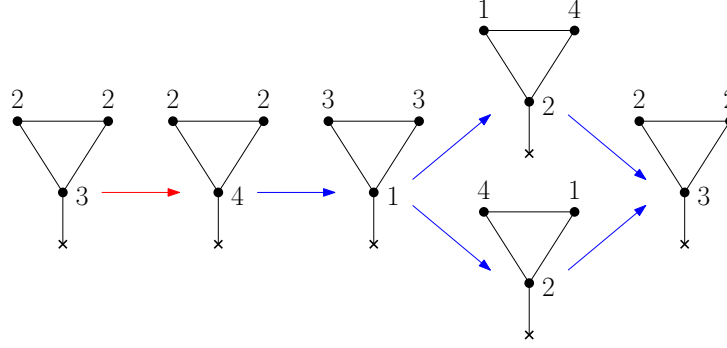


Figure 1.6 – La flèche rouge correspond à l'ajout d'un grain de sable, et les flèches bleues aux éboulements. On remarque que l'état final ne dépend pas de l'ordre dans lequel les sommets s'éboulent.

également le graphe de la chaîne de Markov sur le même graphe (en ne représentant que les états récurrents) Figure 1.7.

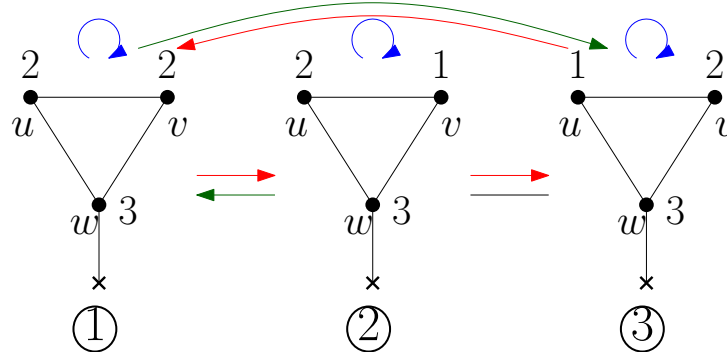


Figure 1.7 – Les flèches rouges correspondent aux transitions résultant de l'ajout d'un grain de sable en u , vertes en v et bleues en w .

Un des résultats les plus connus sur le ASM est le fait que $\mathcal{R}$ soit en bijection avec les arbres couvrants de G [36]. Un autre, que l'on va démontrer

ici, et auquel on va s'intéresser dans le chapitre 4, est le fait que la distribution stationnaire de l'ASM est la distribution uniforme sur l'ensemble des configurations récurrentes.

1.3.2 Structure de groupe, abélianité du modèle et mesure stationnaire

Soit $G = (V \cup \{s\}, E)$ un graphe connexe. Pour η une configuration de grains de sable (pas forcément stable), on note $\mathcal{S}(\eta)$ le stabilisé de η : il s'agit de l'état stable obtenu une fois que tous les éboulements possibles ont été effectués. On ne montre pas ici que cet opérateur est bien défini (c'est à dire que l'ordre dans lequel les sommets s'éboulent ne change pas le résultat), le lecteur intéressé pourra se référer à [36]. On note $\eta + \mu$ la configuration obtenue en additionnant sommet par sommet les grains de sables des configurations η et μ . Pour $v \in V$, on note $\delta(v)$ l'état ne contenant qu'un seul grain de sable, situé en v . On définit maintenant les opérateurs d'ajout de grain de sable $a_v : \mathcal{R} \rightarrow \mathcal{R}$

$$a_v(\eta) = \mathcal{S}(\eta + \delta_v)$$

pour $v \in V$. On peut vérifier que ces opérateurs commutent : pour $v, w \in V$, on a $a_v a_w = a_w a_v$, et c'est pour cette raison que l'on parle de tas de sable *abélien*.

Cette commutation sera démontrée dans le cas plus général du tas de sable stochastique (SSM pour Stochastic Sandpile Model) dans le chapitre 4.

On va maintenant s'intéresser à la proposition suivante :

Proposition 1.3.2. [36] *L'ensemble $H = \{\prod_{v \in V} a_v^{k_v}, k_v \in \mathbb{N}\}$ est un groupe abélien fini agissant sur $\mathcal{R}$.*

Proof. Le fait que H soit abélien découle immédiatement du fait que les $a_v, v \in V$ commutent.

Soit η une configuration récurrente. Comme η est récurrent, il existe des entiers $n_v, v \in V$ strictement positifs tels que

$$\eta = \prod_{v \in V} a_v^{n_v}(\eta). \quad (1.14)$$

En effet, il est possible de revenir à η une fois qu'on a atteint l'état $\prod_{v \in V} a_v(\eta)$. On nomme e l'élément $\prod_{v \in V} a_v^{n_v}$ de H . On va montrer que e est l'élément

neutre de H , c'est à dire que e laisse tout élément de $\mathcal{R}$ invariant par son action.

Soit ν une configuration récurrente. Comme ν est récurrent, il existe $h \in H$ tel que $h(\eta) = \nu$. On a alors $e(\nu) = eh(\eta) = he(\eta) = h(\eta) = \nu$, et donc e laisse bien ν invariant, et est donc l'élément neutre de H .

On peut maintenant définir, pour tout $v \in V$, l'inverse de a_v :

$$a_v^{-1} = a_v^{n_v-1} \prod_{\substack{w \in V \\ w \neq v}} a_w^{n_w} \quad (1.15)$$

et on a bien entendu $a_v a_v^{-1} = a_v^{-1} a_v = e$.

On a donc montré que H était un groupe abélien agissant sur $\mathcal{R}$. Il reste à montrer qu'il est fini. Pour celà, il suffit de montrer que son action est libre et transitive.

La transitivité découle immédiatement du fait que toute configuration récurrente puisse être atteinte à partir de toute autre configuration récurrente.

Supposons qu'il existe $\eta \in \mathcal{R}$ et $h, h' \in H$ tels que $h\eta = h'\eta$. Soit $\nu = g\eta \in \mathcal{R}$ une autre configuration récurrente, avec $g \in H$. On a alors $h\nu = hg\eta = gh\eta = gh'\eta = h'g\eta = h'\nu$. On a donc $h = h'$, et l'action est bien libre.

On en déduit donc que $|H| = |\mathcal{R}|$, et comme $\mathcal{R}$ est inclu dans l'ensemble des configurations stables qui est fini, H est bien fini. $\square$

Une fois ce résultat montré, on va facilement pouvoir déterminer la distribution stationnaire. On sait déjà que l'ASM est irréductible, car il n'existe qu'une classe récurrente : celle contenant l'état maximal η_M . On a donc unicité de la mesure stationnaire. On considère l'espace vectoriel E engendré par la base $(e_\eta)_{\eta \in \mathcal{R}}$. On définit les endomorphismes.

$$A_v : \begin{cases} E \rightarrow E \\ e_\eta \mapsto e_{a_v(\eta)} \end{cases}$$

pour tout $v \in V$. L'endomorphisme associé à la transposée de la matrice de transition de la chaîne de Markov s'écrit

$$A = \sum_{v \in V} \mu(v) A_v$$

où μ est la loi selon laquelle les grains de sable sont ajoutés au graphe.

Posons maintenant $\pi = \frac{1}{|\mathcal{R}|} \sum_{\eta \in \mathcal{R}} e_\eta$. On vérifie que π est la mesure stationnaire du ASM en écrivant :

$$\begin{aligned}
A(\pi) &= \sum_{v \in V} \mu(v) A_v \left(\frac{1}{|\mathcal{R}|} \sum_{\eta \in \mathcal{R}} e_\eta \right) \\
&= \frac{1}{|\mathcal{R}|} \sum_{v \in V} \mu(v) A_v \left(\sum_{\eta \in \mathcal{R}} e_\eta \right) \\
&= \frac{1}{|\mathcal{R}|} \sum_{v \in V} \mu(v) \sum_{\eta \in \mathcal{R}} e_{a_v(\eta)} \\
&= \frac{1}{|\mathcal{R}|} \sum_{v \in V} \mu(v) \sum_{\eta \in \mathcal{R}} e_\eta \\
&= \frac{1}{|\mathcal{R}|} \sum_{\eta \in \mathcal{R}} e_\eta \\
&= \pi
\end{aligned}$$

où le passage de la 3-ème à la 4-ème ligne se fait en invoquant le fait que pour tout v , a_v est une bijection de $\mathcal{R}$.

On a donc montré le résultat suivant :

Théorème 1.3.3. *La mesure stationnaire pour le ASM est la mesure uniforme sur l'ensemble des configurations récurrentes.*

En particulier, cette mesure ne dépend pas de la loi μ selon laquelle les grains de sable sont ajoutés sur le graphe.

Dans le chapitre 4, on s'intéressera à une généralisation de l'ASM nommée le modèle du tas de sable stochastique (ou SSM). La différence entre l'ASM et le SSM réside dans le fait que dans le cas du SSM, les éboulements sont aléatoires : lorsqu'un sommet a trop de grains de sable et doit s'écrouler, la façon dont il envoie des grains de sable à ses voisins dépend d'une loi de probabilité propre à ce sommet.

On commencera par montrer que le modèle est bien défini, puis on s'intéressera à la mesure stationnaire. Il ne s'agira plus de la mesure uniforme dans ce cas : elle dépendra des lois associées aux éboulements en chaque sommet. On pourra cependant montrer le résultat suivant :

Théorème 1.3.4. *La mesure stationnaire pour le SSM est indépendante de la loi μ selon laquelle les grains de sable sont ajoutés au graphe.*

On ne pourra pas directement transposer la structure de groupe décrite précédemment au cas du SSM. En effet, les éboulements n'y étant pas déterministes, il sera difficile de parler d'action de groupe. On pourra cependant, moyennant introduction d'un espace de probabilité adapté, montrer que les opérations d'ajout de grain de sable puis stabilisation commutent toujours, et ce sera l'élément clé de la preuve du Théorème 1.3.4.

1.4 Contributions et résultats de cette thèse

Dans le chapitre 2, inspiré d'un travail effectué en collaboration avec Ayyer, Bouttier et Corteel [3], on considère plusieurs généralisations des chaînes de Markov de Jonglage introduites par Warrington. Le premier modèle étudié est identique à celui décrit dans la partie 1.2, si ce n'est que l'on se munit cette fois de $x_0, \dots, x_k$ $k + 1$ réels positifs de somme 1, et les probabilités de transition données par l'équation 1.3 deviennent, pour A et B deux états de jonglage :

$$P_{A,B} = \begin{cases} 1 & \text{si } a_1 = \circ \text{ et } B = a_2 \cdots a_h \circ, \\ x_i & \text{si } a_1 = \bullet \text{ et } B = T_i(a_2 \cdots a_h \circ) \text{ pour un certain } i, \\ 0 & \text{sinon.} \end{cases} \quad (1.16)$$

On s'intéresse également au cas limite $h = \infty$, et on propose une reformulation du modèle en termes de partitions d'entiers. On peut alors également définir un modèle où le jongleur manipule une infinité de balles. On donne également une généralisation de la chaîne enrichie, décrite sur des partitions d'ensembles. On étudie également le cas où le nombre de balles que manipule le jongleur n'est plus fixé : à chaque étape, le jongleur peut ou non lancer une balle, indépendamment du fait qu'il vienne d'en rattraper une ou non. On définit deux modèles, différents de part leurs probabilités de transition : add-drop et annihilation. Dans ce deuxième modèle, on montre que la mesure stationnaire est atteinte en temps fini.

Dans chacun des modèles, on donne une formule close pour la mesure stationnaire, et dans l'un des cas, on exprime le facteur de normalisation comme un polynôme symétrique homogène.

Dans le chapitre 3, inspiré de [2], un travail effectué en collaboration avec Ayzer, Bouttier, Corteel et Linusson, on enrichit davantage encore le modèle étudié dans le chapitre 2. On commence par introduire le jonglage multi-espèce, dans lequel le jongleur manipule des balles de poids différents : quand une balle est lancée, elle peut se heurter à une balle plus légère, qui se déplace à son tour vers une position plus élevée, où elle peut à son tour se heurter à une balle plus légère encore, et ainsi de suite. On étudiera d'abord le cas où le nombre de balles de chaque type est fixé, puis le cas où le jongleur relance une balle du poids de son choix, indépendamment du poids de celle qu'il vient de rattraper. Dans ce dernier cas, on s'intéressera à trois modèles différents: add-drop, annihilation et overwriting. Les deux premiers sont des généralisations naturelles des modèles présentés dans le chapitre 2, tandis que le troisième, qui est une autre généralisation du modèle d'annihilation, atteint la mesure stationnaire en temps fini. Enfin, on considère le cas où plusieurs jongleurs interagissent en échangeant des balles.

Une fois encore, pour chacun des modèles, on donne une formule close pour la mesure stationnaire, et une formule explicite pour le facteur de normalisation quand on en connaît une.

Dans le chapitre 4, inspiré de [34], on s'intéresse à une variante stochastique du modèle du tas de sable abélien sur un graphe fini, introduit par Chan, Marckert et Selig. Bien qu'il s'agisse d'un modèle plus général, certaines propriétés intéressantes y sont toujours vérifiées. On montre que sur un espace de probabilités adapté, même si on perd la structure de groupe du fait que les éboulements ne sont plus uniquement déterminés, certains opérateurs commutent toujours entre eux. Le corollaire qui en découle est que la mesure stationnaire ne dépend pas de la loi selon laquelle les grains de sable sont ajoutés sur le graphe, ce qui résout une conjecture posée par Selig.

Chapter 2

Juggling Markov chains

This Chapter is very similar to [3], a joint work with Ayer, Bouttier and Corteel.

2.1 Introduction

Although juggling as a human endeavour has been around since time immemorial, it is fairly recently that mathematicians have taken an active interest in exploring the field. Combinatorialists became interested in juggling towards the end of the last century after an article in the *Amer. Math. Monthly* by Buhler, Eisenbud, Graham and Wright [10], where they enumerate what they call *juggling sequences* and relate it to other known combinatorial structures. Since then, their results have been q -ified [21] and further refined in various ways [39, 40, 13, 14, 11]. Other connections between juggling and mathematics appear for instance in algebraic geometry [17, 28]. A mathematical history of juggling is given in the fascinating book by Polster [35].

Probabilists, on the other hand, are newer to this game, possibly because no one is able to perform totally random juggling. Coincidentally, it was another popular article in the *Amer. Math. Monthly* by Warrington [41] on natural stochastic models inspired by juggling that got the attention of a few other members of the community. Exact combinatorial formulas for the stationary distribution of these models were found, and proved by a beautiful argument involving an auxiliary model defined on a larger state space whose elements may be viewed as set partitions, for which the stationary

distribution is the uniform distribution.

The simplest model considered by Warrington, where the number of balls is conserved, has been generalized by Leskelä and Varpanen as the so-called Juggler’s Exclusion Process (JEP), where the balls can be thrown arbitrarily high so that the state space is infinite [30]. These authors showed that the JEP converges to a (unique) stationary distribution under very mild assumptions, could obtain an exact expression for this stationary distribution in the particular case of a “memoryless” (geometric) height distribution, and noted an intriguing phenomenon of “ultrafast convergence” in this case. More recently, a q -deformation of Warrington’s original finite model was considered by Engström, Leskelä and Varpanen [22], who also obtained exact expressions for the stationary distribution via an enriched chain formulated in terms of rook placements (in bijection with some set partitions).

In this Chapter, we provide multivariate generalizations of all the models introduced in [41], namely *juggling*, *add-drop juggling* and *annihilation juggling*. Furthermore, in the case of the juggling model with a conserved number of balls, we investigate the limiting case where balls can be thrown arbitrarily high, which corresponds to the so-called infinite juggling model suggested by Warrington. We also consider the limiting case where the number of balls tends to infinity. In all these cases, we obtain an exact formula for the stationary distribution (in particular, in the case where balls can be thrown arbitrarily high, we find an “exactly solvable” instance of the general JEP of [30] with countably many parameters) and the normalization factor. In one case the normalization factor can be explicitly written in terms of the homogeneous symmetric polynomial. Our proofs were obtained using two approaches. The direct approach involves guessing the general formulas (for instance by a computation for small system sizes) and then proving the results in a straightforward manner by considering the juggling process itself. The other approach is more combinatorial, and consists in introducing an enriched chain whose stationary distribution is simpler, and which yields the original chain by a projection or “lumping” procedure on set partitions or words.

We remark that juggling models (especially the add-drop and annihilation versions) have a natural interpretation in statistical physics, where one could think of the balls as coming from a reservoir of particles. A model in this vein has been proposed recently, see [1].

The rest of the Chapter is organized as follows. In Section 2.2, we concentrate on the simplest version of our model, which we call the Multivariate

Juggling Markov Chain (MJMC). The model is defined in Section 2.2.1, and we also discuss the uniqueness of the stationary distribution (a technicality deemed obvious in [41]). The expression for the stationary distribution of the Markov chain is stated in Section 2.2.2. For pedagogical purposes, we decide to provide two independent proofs by the direct and the combinatorial approaches. The direct proof is given in Section 2.2.3, via an interesting reformulation of the MJMC in terms of integer partitions restricted to lie within a rectangle. The combinatorial proof comes in Section 2.2.4 and involves set partitions with a prescribed number of elements and blocks. We then turn to extended models. Extensions to infinite state spaces are considered in Section 2.3: the case of a finite number of balls but *unbounded* heights (UMJMC) is discussed in Section 2.3.1, while the case of an *infinite* number of balls (IMJMC) is considered in Section 2.3.2. In both cases, we find the stationary measure by the direct approach. Extensions to a fluctuating number of balls (but with a finite state space) are considered in Section 2.4: we provide the multivariate extension of the add-drop and the annihilation models introduced in [41], in the respective Sections 2.4.1 and 2.4.2. These models have the same transition graph, only the transitions probabilities differ. In both cases we find the stationary distribution by the combinatorial approach. In the case of the annihilation model, we further observe the interesting property that the stationary distribution is attained in bounded time.

2.2 The finite Multiparameter Juggling Markov Chain

2.2.1 Definition

In this section, we introduce our juggling model in the simplest setting, i.e. a Markov chain on a finite state space. We start by explaining the model in colloquial terms, and refer to [41] for further motivation. Consider a person, called *Magnus* with no loss of generality, who is juggling with a fixed finite number ℓ of balls. Time is discretized in, say, steps of one second and we assume that, at each second, Magnus is able to catch at most one ball, and then throws it back immediately. Besides this limitation Magnus juggles *perfectly*, i.e. in such a way that the ball will always return to him after some time (an imperfect juggler could drop a ball or throw it in a wrong direction,

for instance). Magnus controls the velocity at which he sends the ball, which determines how long it will take for the ball to return to him. We suppose for now that the launch velocity is bounded or, in other words, that the number of seconds before the ball returns to Magnus is bounded by an integer h .

Ignoring further spatial constraints, a simplified description of the state of the balls at a given time consists in associating to each ball the (integer) number of seconds remaining before it is caught by Magnus. This is now known as the *siteswap* notation. Of course, to be able to juggle for an indefinite amount of time, Magnus must choose the successive launch velocities in such a way that no two balls arrive to him at the same time. Thus the numbers associated to different balls shall be distinct and, treating the balls as indistinguishable, there are $\binom{h}{\ell}$ possible ball states. It is convenient to think of a state as a configuration of ℓ non overlapping particles on a one-dimensional lattice with h sites, where the i -th site (read from the left) is occupied if and only if a ball is scheduled to arrive i seconds in the future. However, beware that sites do not correspond to actual spatial positions, but to the “remaining flight times” of the balls. We denote by $k = h - \ell$ the number of empty (unoccupied) sites.

In this language, the time evolution of a state is easy to describe: at each time step, all particles are moved one site to the left. If there is no particle on the first site (i.e. Magnus catches no ball), then nothing else has to be done. Otherwise the particle on the first site, which would exit the lattice if moved to the left, is instead taken away and *reinserted* at one of the $k + 1$ available (empty) sites on the lattice (determined by the launch velocity chosen by Magnus). This defines the transition graph of our model, illustrated on Figure 2.1 for $h = 4$ and $\ell = k = 2$ (ignoring edge labels for now).

We now assume that Magnus juggles at random: each reinsertion is made randomly at one of the $k + 1$ available sites. In our model, we assume that the reinsertion is made at the i -th available site (read from the left) with probability x_{i-1} , independently of the past, so that our model is a Markov chain. Here, $x_0, \dots, x_k$ are fixed nonnegative real numbers such that $x_0 + \dots + x_k = 1$. This defines the *Multivariate Juggling Markov Chain* (MJMC), which generalizes the model considered in [41], obtained by taking $x_0 = \dots = x_k = 1/(k + 1)$, but is a particular case of the general “Juggling Exclusion Process” defined in [30]: beyond the extension to infinitely many empty sites which we will discuss in the next section, the main difference is that, in the model of Leskelä and Varpanen, the x_i ’s are allowed to depend

on the current state.

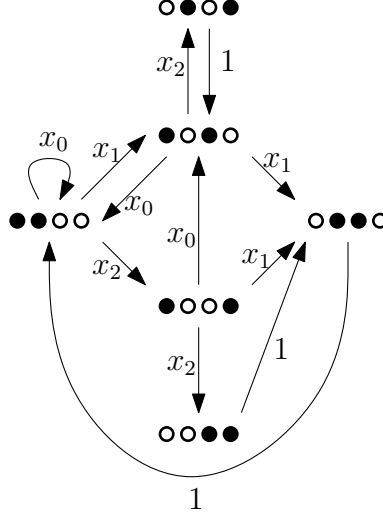


Figure 2.1 – The Markov chain with $h = 4$ and $\ell = k = 2$.

We now provide a more formal mathematical definition of the MJMC. Following Warrington's notation, let St_h denote the set of words of length h on the alphabet $\{\bullet, \circ\}$, and let $\text{St}_{h,k} \subset \text{St}_h$ be the subset of words containing exactly k occurrences of $\circ$. For $A \in \text{St}_{h,k+1}$ and $i \in \{0, \dots, k\}$, we let $T_i(A) \in \text{St}_{h,k}$ be the word obtained by replacing the $(i+1)$ -th occurrence of $\circ$ in A by $\bullet$.

Definition 2.2.1. Given h, k nonnegative integers such that $h \geq k$, and $x_0, \dots, x_k$ nonnegative real numbers such that $x_0 + \dots + x_k = 1$, the *Multi-variate Juggling Markov Chain* is the Markov chain on the state space $\text{St}_{h,k}$ for which the transition probability from $A = a_1 a_2 \dots a_h$ to B reads

$$P_{A,B} = \begin{cases} 1 & \text{if } a_1 = \circ \text{ and } B = a_2 \dots a_h \circ, \\ x_i & \text{if } a_1 = \bullet \text{ and } B = T_i(a_2 \dots a_h \circ), \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

Example 2.2.2. Figure 2.1 illustrates the Markov chain in the case $h = 4$, $k = 2$, and the transition matrix in the basis $(\bullet\bullet\circ\circ, \bullet\circ\circ\circ, \bullet\circ\circ\bullet, \circ\bullet\circ\circ, \circ\bullet\circ\bullet, \circ\circ\bullet\bullet)$

reads

$$\begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 & 0 \\ x_0 & 0 & 0 & x_1 & x_2 & 0 \\ 0 & x_0 & 0 & x_1 & 0 & x_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad (2.2)$$

Note that $(1, x_1 + x_2, x_2, (x_1 + x_2)^2, x_2(x_1 + x_2), x_2^2)$ is a left eigenvector with eigenvalue 1.

To compare our model with that of [30], note that $\text{St}_{h,k}$ may be canonically identified with the set of subsets of $\{1, \dots, h\}$ with $\ell = h - k$ elements (corresponding to the positions of the occupied sites). For two such subsets S and S' , the transition probability from S to S' reads

$$P_{S,S'} = \begin{cases} 1 & \text{if } S' = S - 1, \\ x_i & \text{if } S' = (S - 1) \setminus \{0\} \cup \{j\} \text{ with } i = |\{1, \dots, j\} \setminus S|, \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

where $S - 1$ is the set whose entries are those of S with 1 subtracted.

Proposition 2.2.3. *If $x_0 > 0$, then the MJMC has a unique closed communicating class, all of whose states are aperiodic. If furthermore $x_k > 0$, then the MJMC is irreducible.*

Proof. Assume that $x_0 > 0$ and let $\mathcal{E} = \bullet^\ell \circ^k$ be the “lowest” state. It is clear that, starting from any initial state, we may obtain $\mathcal{E}$ by a sequence of transitions of probabilities x_0 or 1, simply by doing any reinjection at the first available site. Since $\mathcal{E}$ jumps to itself with probability x_0 , there is a unique closed communicating class, and all states in this class are aperiodic.

Assume now that $x_k > 0$: to prove irreducibility we simply need to show that any state $A = a_1 \cdots a_h \in \text{St}_{h,k}$ may be obtained from $\mathcal{E}$. The idea is to build the prefix of A of length i at the i -th step so that we get A in exactly h steps. For $i \in \{0, \dots, h\}$, we let $A_i = \bullet^{\ell-i+n_i} \circ^{k-n_i} a_1 \cdots a_i$ where n_i is the number of occurrences of $\circ$ in $a_1 \cdots a_i$, so that $A_i \in \text{St}_{h,k}$, with $A_0 = \mathcal{E}$ and $A_h = A$. For $i < h$, we have

$$P_{A_i, A_{i+1}} = \begin{cases} 1 & \text{if } \ell = i - n_i, \\ x_0 & \text{if } \ell > i - n_i \text{ and } a_{i+1} = \circ, \\ x_k & \text{if } \ell > i - n_i \text{ and } a_{i+1} = \bullet. \end{cases} \quad (2.4)$$

Note that when $a_1 = \circ$, $A_1 = \mathcal{E}$, but this is allowed since there is always a self-loop from $\mathcal{E}$ to itself. Thus A may be obtained from $\mathcal{E}$ by a sequence of possible transitions. $\square$

Remark 2.2.4. When $x_0 = 0$, the chain may have several closed communicating classes. For example, see Figure 2.1 and let $x_0 = x_1 = 0$.

2.2.2 Stationary distribution

From now on we assume $x_0 > 0$. By Proposition 2.2.3, the MJMC admits a unique stationary probability distribution. Our main result is an explicit form for it, given as follows.

Theorem 2.2.5. *The stationary distribution π of the MJMC is given by*

$$\pi(B) = \frac{1}{Z_{h,k}} \prod_{\substack{i \in \{1, \dots, h\} \\ b_i = \bullet}} (x_{E_i(B)} + \dots + x_k), \quad (2.5)$$

where $B = b_1 \dots b_h \in \text{St}_{h,k}$ and $E_i(B) = \#\{j < i | b_j = \circ\}$, and where $Z_{h,k} \equiv Z_{h,k}(x_0, \dots, x_k)$ is determined by the condition that π be a probability distribution.

We will provide two proofs of this theorem, as they both lead to interesting generalizations to be studied later in Sections 2.3 and 2.4. The first proof, presented in Section 2.2.3, is a rather direct computational check that (2.5) indeed defines a stationary measure of the MJMC (but we find it clearer to first reformulate the chain in terms of integer partitions). The second proof, presented in Section 2.2.4, is more combinatorial and generalizes the approach of [41]: it consists in defining an “enriched” Markov chain on a larger state space, where the stationary probability of a given state will be given by a monomial in $x_0, \dots, x_k$. The MJMC is then recovered by lumping together states of the enriched chain, which will explain the form of (2.5). Before proceeding to the proofs, let us discuss a few interesting cases.

Example 2.2.6. If the reinsertion is made uniformly among all $k+1$ available sites, i.e. if we take $x_0 = \dots = x_k = 1/(k+1)$, then we recover [41, Theorem 1] from Theorem 2.2.5 (see Corollary 2.2.11 below for a short rederivation of the normalization factor).

Example 2.2.7. If the reinsertion site is chosen according to a geometric distribution of parameter q conditioned to be smaller than or equal to $k + 1$, i.e. if we take $x_i = (1 - q)q^i / (1 - q^{k+1})$, $i = 0, \dots, k$, then we recover the so-called bounded geometric JEP considered in [22]. It is a simple exercise to check that we recover [22, Theorem 2.1] from Theorem 2.2.5 (see also Corollary 2.2.11 below for the normalization factor).

Example 2.2.8. A slight variant of the previous example consists in picking the reinsertion site according to a “truncated” geometric distribution of parameter q , i.e. taking $x_i = (1 - q)q^i$ for $i = 0, \dots, k - 1$ and $x_k = q^k$. In that case, the stationary probability of $B \in \text{St}_{h,k}$ reads

$$\pi(B) = \frac{1}{\binom{h}{k}_q} \prod_{\substack{i \in \{1, \dots, h\} \\ b_i = \bullet}} q^{E_i(B)} \quad (2.6)$$

(see again Corollary 2.2.11 below for the normalization factor).

We now move on to discuss properties of the normalization factor $Z_{h,k}$. Observe first that the expression

$$Z_{h,k}(x_0, \dots, x_k) = \sum_{B \in \text{St}_{h,k}} \prod_{\substack{i \in \{1, \dots, h\} \\ b_i = \bullet}} (x_{E_i(B)} + \dots + x_k) \quad (2.7)$$

allows us to define it for any $x_0, \dots, x_k$ (not necessarily positive or subject to the condition $x_0 + \dots + x_k = 1$) and $h, k \geq 0$. It vanishes for $h < k$ and is otherwise a homogeneous polynomial of degree $\ell = h - k$.

Proposition 2.2.9. *We have*

$$Z_{h,k} = h_\ell(y_0, \dots, y_k) \quad (2.8)$$

with h_ℓ the complete homogeneous symmetric polynomial of degree $\ell = h - k$, and

$$y_m = \sum_{j=m}^k x_j, \quad m = 0, \dots, k. \quad (2.9)$$

Again, there are several ways to establish this result. Perhaps the most combinatorial explanation comes from the integer partition approach of Section 2.2.3, see Remark 2.2.16 below. But, starting from the definition (2.7), we may easily check (2.8) by induction, using the following:

Lemma 2.2.10. *For $h, k \geq 0$, we have*

$$Z_{h,k}(x_0, \dots, x_k) = Z_{h-1,k-1}(x_1, \dots, x_k) + (x_0 + \dots + x_k)Z_{h-1,k}(x_0, \dots, x_k) \quad (2.10)$$

where, by convention, $Z_{h,k} = \delta_{h,k}$ if $h = -1$ or $k = -1$.

Proof. Obtained immediately by distinguishing whether $B \in \text{St}_{h,k}$ starts with a $\circ$ or a $\bullet$ (for $h < k$ all terms vanish as wanted). $\square$

We readily verify by induction and homogeneity some explicit specializations:

Corollary 2.2.11. *We have*

$$Z_{h,k}(1, \dots, 1) = \left\{ \begin{matrix} h+1 \\ k+1 \end{matrix} \right\}, \quad (2.11)$$

$$Z_{h,k}(q^k, q^{k-1}, \dots, 1) = \left\{ \begin{matrix} h+1 \\ k+1 \end{matrix} \right\}_q, \quad (2.12)$$

$$Z_{h,k}(1, q, \dots, q^k) = q^{k(h-k)} \left\{ \begin{matrix} h+1 \\ k+1 \end{matrix} \right\}_{1/q}, \quad (2.13)$$

$$Z_{h,k}((1-q), (1-q)q, \dots, (1-q)q^{k-1}, q^k) = \binom{h}{k}_q, \quad (2.14)$$

where $\{\cdot\}$ denotes Stirling numbers of the second kind, $\{\cdot\}_q$ their q -analogues as defined in [26] and $\binom{\cdot}{\cdot}_q$ the q -binomial coefficients. Using homogeneity, we deduce the normalization factors for Examples 2.2.6 and 2.2.7.

For completeness, we mention another recursion relation for $Z_{h,k}$ of a different nature, as we remove x_k instead of x_0 .

Proposition 2.2.12. *For $h, k \geq 0$, we have*

$$Z_{h,k}(x_0, \dots, x_k) = \sum_{n=0}^{h-k} \binom{h}{n} x_k^n Z_{h-n-1,k-1}(x_0, \dots, x_{k-1}) \quad (2.15)$$

where, by convention, $Z_{h,k} = \delta_{h,k}$ if $h = -1$ or $k = -1$.

Proof. Write

$$Z_{h,k} = h_\ell(y'_0 + x_k, y'_1 + x_k, \dots, y'_{k-1} + x_k, x_k) \quad (2.16)$$

with $y'_m = \sum_{j=m}^{k-1} x_j$, and use the identity (easily verified e.g. by induction on $\ell + k$)

$$h_\ell(a_0 + a, a_1 + a, \dots, a_k + a) = \sum_{n=0}^{\ell} \binom{\ell + k}{n} a^n h_{\ell-n}(a_0, \dots, a_k). \quad (2.17)$$

□

2.2.3 Reformulation of the MJMC in terms of integer partitions

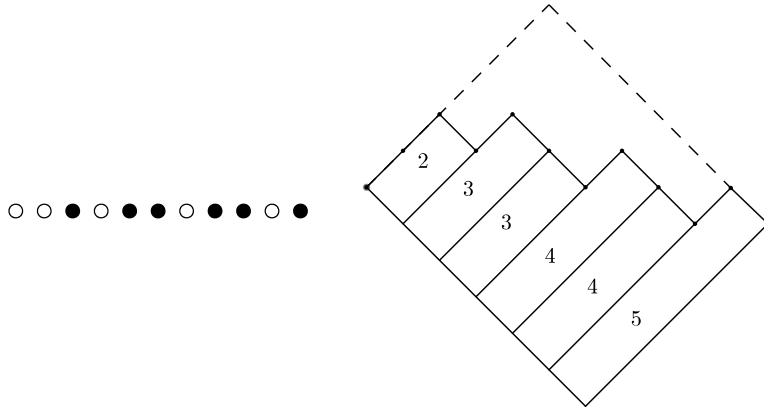


Figure 2.2 – Bijection between juggling states and integer partitions. Each $\circ$ is replaced by a north-east step, and each $\bullet$ by a south-east step. The partition thus obtained is $(5, 4, 4, 3, 3, 2)$.

There is a natural bijection between $\text{St}_{h,k}$ and $\text{Par}_{k,h-k}$, where $\text{Par}_{k,\ell}$ is the set of integer partitions whose Young diagram fits within a $k \times \ell$ rectangle. In other words, an element of $\text{Par}_{k,\ell}$ can be viewed as a nonincreasing sequence

of ℓ nonnegative integers smaller than k . The bijection is given explicitly as follows: given a state in $\text{St}_{h,k}$, we denote by $s_1 < \dots < s_\ell$ the positions of $\bullet$, then the corresponding integer partition is $(s_\ell - \ell, s_{\ell-1} - (\ell - 1), \dots, s_1 - 1)$. Alternatively, an equivalent graphical construction is displayed on Figure 2.2. Upon identifying $\text{St}_{h,k}$ and $\text{Par}_{k,h-k}$ via this bijection, we may reformulate the MJMC as follows.

Proposition 2.2.13. *For $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and μ two partitions in $\text{Par}_{k,\ell}$, the transition probability from λ to μ reads*

$$P_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda_\ell \neq 0 \text{ and } \mu = (\lambda_1 - 1, \dots, \lambda_\ell - 1), \\ x_i & \text{if } \lambda_\ell = 0 \text{ and there exists } j \in \{1, \dots, \ell\} \text{ such that} \\ & \mu = (\lambda_1 - 1, \dots, \lambda_{j-1} - 1, i, \lambda_j, \dots, \lambda_{\ell-1}), \\ 0 & \text{otherwise.} \end{cases} \quad (2.18)$$

The proof of this proposition is left to the reader. The interest of the reformulation in terms of integer partitions is that the stationary distribution of the MJMC takes a particularly simple form: indeed, Theorem 2.2.5 is equivalent to

Theorem 2.2.14. *The stationary distribution π of the MJMC is given by*

$$\pi(\lambda) = \frac{1}{Z_{h,k}} \prod_{i=1}^{\ell} y_{\lambda_i} \quad (2.19)$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is an element of $\text{Par}_{k,\ell}$ and where the y_m are as in (2.9).

Proof. Let $w(\lambda) = \prod_{i=1}^{\ell} y_{\lambda_i}$ be the unnormalized weight of $\lambda \in \text{Par}_{k,\ell}$. We simply check that

$$w(\lambda) = \sum_{\mu \in \text{Par}_{k,\ell}} w(\mu) P_{\mu,\lambda} \quad (2.20)$$

by considering the possible predecessors of λ :

- If $\lambda_1 = k$ then the only possible predecessor for λ is $\mu = (\lambda_2, \dots, \lambda_\ell, 0)$, so that $w(\mu) = \prod_{i=2}^{\ell} y_{\lambda_i}$ (because $y_0 = 1$) and $P_{\mu,\lambda} = x_k = y_k$. Thus we have $w(\lambda) = P_{\mu,\lambda} w(\mu)$ as wanted.
- We now assume that $\lambda_1 < k$. Let μ be a predecessor of λ , μ can

- either be of the form $\mu = (\lambda_1 + 1, \dots, \lambda_\ell + 1)$, i.e. is associated with a word that starts with a $\circ$, in which case we have $w(\mu) = \prod_{i=1}^{\ell} y_{\lambda_i+1} = \prod_{i=1}^{\ell} (y_{\lambda_i} - x_{\lambda_i})$ and $P_{\mu,\lambda} = 1$;
- or be of the form $\mu = (\lambda_1 + 1, \dots, \lambda_{j-1} + 1, \lambda_{j+1}, \dots, \lambda_\ell, 0)$ for any $1 \leq j \leq \ell$, i.e. is associated with a word that starts with a $\bullet$, in which case we have $w(\mu) = \prod_{i=1}^{j-1} (y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\ell} y_{\lambda_i}$ and $P_{\mu,\lambda} = x_{\lambda_j}$.

Thus, we have

$$\begin{aligned} \sum_{\mu \in \text{Par}_{k,\ell}} w(\mu) P_{\mu,\lambda} &= \prod_{i=1}^{\ell} (y_{\lambda_i} - x_{\lambda_i}) + \sum_{j=1}^{\ell} \left(x_{\lambda_j} \prod_{i=1}^{j-1} (y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\ell} y_{\lambda_i} \right) \\ &= \prod_{i=1}^{\ell} y_{\lambda_i} = w(\lambda) \end{aligned} \tag{2.21}$$

where, to go from the first line to the second, we write $x_{\lambda_j} = y_{\lambda_j} - (y_{\lambda_j} - x_{\lambda_j})$ so that the sum on the right hand side becomes telescopic.

□

Example 2.2.15. We may now revisit Examples 2.2.6, 2.2.7 and 2.2.8 in the language of integer partitions. The corresponding stationary probabilities of $\lambda \in \text{Par}_{k,\ell}$ read

$$\pi(\lambda) = \begin{cases} \frac{1}{\left\{ \frac{h+1}{k+1} \right\}} \prod_{i=1}^{\ell} (k+1 - \lambda_i) & \text{for } x_i = \frac{1}{k+1}, \\ \frac{1}{q^{k\ell} \left\{ \frac{h+1}{k+1} \right\}_{1/q}} \prod_{i=1}^{\ell} q^{\lambda_i} [k+1 - \lambda_i]_q & \text{for } x_i = \frac{q^i}{[k+1]_q}, \\ \frac{1}{\binom{h}{k}_q} \prod_{i=1}^{\ell} q^{\lambda_i} & \text{for } x_i = (1-q)^{1-\delta_{i,k}} q^i. \end{cases} \tag{2.22}$$

Remark 2.2.16. One can immediately recover Proposition 2.2.9 from Theorem 2.2.14, as

$$Z_{h,k} = \sum_{\lambda \in \text{Par}_{k,\ell}} \prod_{j=1}^{\ell} y_{\lambda_j} = \sum_{\substack{(m_0, \dots, m_k) \in \mathbb{N}^{k+1} \\ \sum_{i=0}^k m_i = \ell}} \prod_{i=0}^k y_i^{m_i} = h_{\ell}(y_0, \dots, y_k) \tag{2.23}$$

where the m_i correspond to the part multiplicities. A slight extension of this reasoning consists in restricting the sum to partitions λ such that, say, $\lambda_j = n$ (with $1 \leq j \leq \ell$ and $0 \leq n \leq k$), so as to obtain the probability that the j -th part in the stationary distribution π is n :

$$\pi(\{\lambda_j = n\}) = \frac{y_n h_{j-1}(y_n, \dots, y_k) h_{\ell-j}(y_0, \dots, y_n)}{h_\ell(y_0, \dots, y_k)}. \quad (2.24)$$

More generally, the joint distribution for a fixed number of parts reads

$$\pi(\{\lambda_{j_1} = n_1, \dots, \lambda_{j_m} = n_m\}) = \frac{\prod_{s=1}^{m+1} y_{n_s} h_{j_s - j_{s-1} - 1}(y_{n_s}, \dots, y_{n_{s-1}})}{h_\ell(y_0, y_1, \dots, y_k)} \quad (2.25)$$

where $0 = j_0 < \dots < j_{m+1} = \ell + 1$ and $k = n_0 \geq \dots \geq n_{m+1} = 0$ (with $y_0 = 1$).

2.2.4 Enriched Markov chain on set partitions

We now provide another, more combinatorial, proof of Theorem 2.2.5 whose rough idea goes as follows: consider the stationary probability (2.5) and expand the product in the right hand side as a sum of monomials in the x_i 's. We will interpret each of these monomials as the stationary probability of an “enriched” state belonging to the larger state space of another Markov chain. The MJMC will then be obtained as a projection of this enriched chain (see e.g. [31, Section 2.3.1] for a definition of this notion).

Definitions and basic properties

Our construction is a generalization of that of Warrington's. Rather than working with “landing/throwing-states”, we prefer to work directly in the language of set partitions, see [41, Lemma 2]. Recalling that h stands for the total number of sites and k for the number of empty sites in the particle picture of the MJMC, let us introduce the shorthand notations

$$H = h + 1 \quad \text{and} \quad K = k + 1. \quad (2.26)$$

An *enriched state* will then be a partition of the set $\{1, \dots, H\}$ into K subsets or *blocks*. We denote by $\mathcal{S}(H, K)$ the set of enriched states, and recall that $\left\{ \begin{smallmatrix} H \\ K \end{smallmatrix} \right\} = |\mathcal{S}(H, K)|$ is a Stirling number of the second kind. To each enriched state σ , we associate a word $\psi(\sigma) = a_1 \dots a_h$ by setting, for all i between

1 and h , $a_i = \circ$ if i is a *block maximum* of σ (i.e. the largest element of its block in σ), and $a_i = \bullet$ otherwise. Observe that ψ is a surjection from $\mathcal{S}(H, K)$ onto $\text{St}_{h,k}$.

We now define the enriched Markov chain on $\mathcal{S}(H, K)$, which requires some notations. For an enriched state σ , we denote by $\sigma^\downarrow$ the partition of the set $\{1, \dots, h\}$ obtained by removing 1 from σ (i.e. removing 1 from its block in σ , and removing this block from σ if it becomes empty), and shifting all the remaining elements of all blocks down by 1. Note that $\sigma \mapsto \sigma^\downarrow$ is a mapping from $\mathcal{S}(H, K)$ to $\mathcal{S}(h, k) \cup \mathcal{S}(h, K)$ (which is surjective). For $\tau \in \mathcal{S}(h, K)$ and $i \in \{0, \dots, k\}$, we denote by $I_i(\tau)$ the set partition of $\{1, \dots, H\}$ obtained by inserting H into the $(i+1)$ -th block of τ , where the blocks are numbered by ascending order of their maxima (i.e. the first block has the smallest maximum among all blocks, etc. – this differs from the so-called standard form which consists in ordering blocks by ascending order of their minima). Note that $I_i(\tau) \in \mathcal{S}(H, K)$ and that the mapping $(\tau, i) \mapsto I_i(\tau)$ is injective.

Definition 2.2.17. The *enriched chain* is the Markov chain on $\mathcal{S}(H, K)$ for which the transition probability from σ to τ is given by

$$\tilde{P}_{\sigma, \tau} = \begin{cases} 1 & \text{if } \{1\} \in \sigma \text{ and } \tau = \sigma^\downarrow \cup \{H\}, \\ x_i & \text{if } \{1\} \notin \sigma \text{ and } \tau = I_i(\sigma^\downarrow) \text{ for some } i \in \{0, \dots, k\}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.27)$$

The condition $x_0 + \dots + x_k = 1$ and the above remarks ensure that $\tilde{P}$ is indeed a right stochastic matrix (i.e. each of its rows sums to 1).

Example 2.2.18. For $H = 8$ and $K = 3$:

- the enriched state $1 \mid 3, 5, 6 \mid 2, 4, 7, 8$ jumps with probability 1 to $2, 4, 5 \mid 1, 3, 6, 7 \mid 8$,
- the enriched state $\sigma = 3, 5 \mid 2, 6, 7 \mid 1, 4, 8$ reaches the intermediate state $\sigma^\downarrow = 2, 4 \mid 1, 5, 6 \mid 3, 7$ and jumps with probability:
 - x_0 to $1, 5, 6 \mid 3, 7 \mid 2, 4, 8$,
 - x_1 to $2, 4 \mid 3, 7 \mid 1, 5, 6, 8$,
 - x_2 to $2, 4 \mid 1, 5, 6 \mid 3, 7, 8$.

Note that here we write blocks in ascending order of their maxima, which differs from the standard notation of writing set partitions in ascending order of their minima.

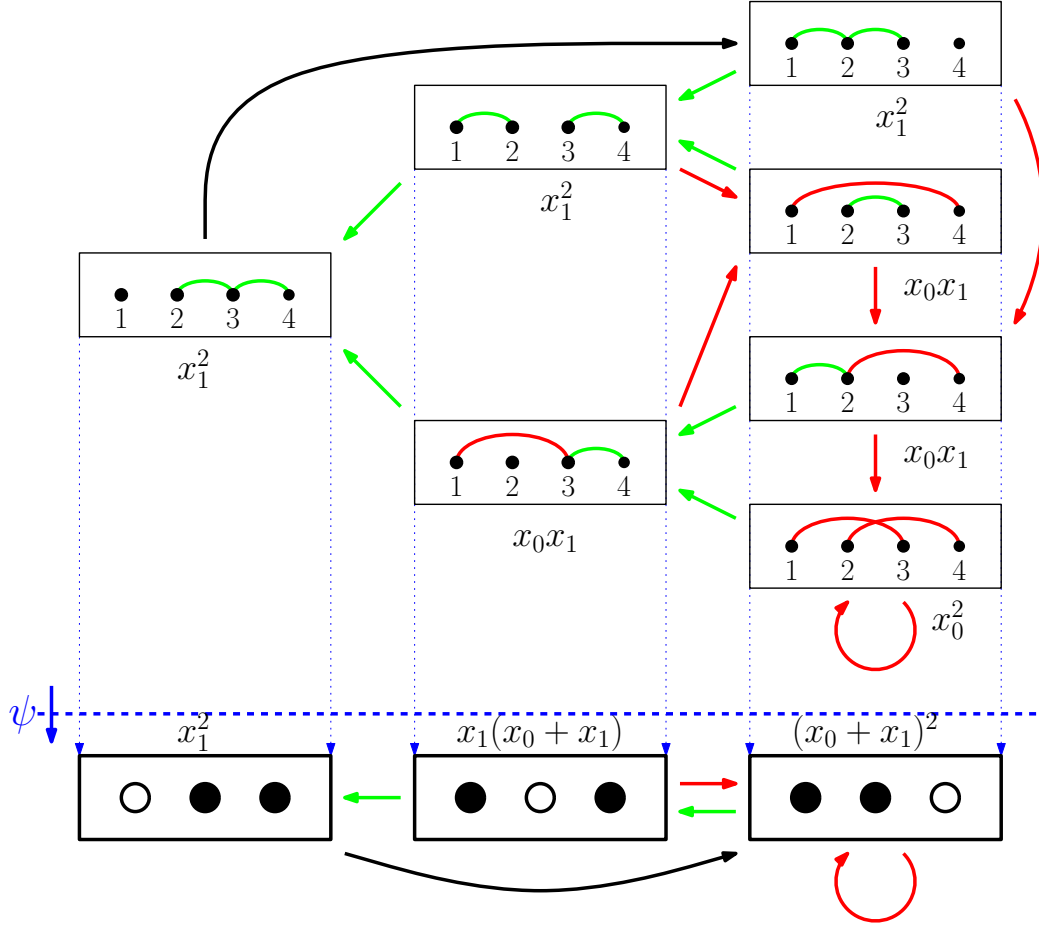


Figure 2.3 – Transition graphs of the enriched chain (top part, above blue horizontal dashed line) and of the MJMC (bottom part) for $H = 4$ and $K = 2$ ($h = 3$ and $k = 1$). Enriched states (set partitions) are represented via “arches”, see Definition 2.2.23. Red, green and grey arrows represent transitions of respective probability x_0 , x_1 and 1. Arches covering 1 and 2 blocks are represented in green and red respectively. The reader may easily check that attaching a weight x_0 (resp. x_1) to each red (resp. green) arch yields the stationary measure of the enriched chain. Enriched states in a same column map to the same MJMC state (displayed below) via ψ .

Example 2.2.19. For $H = 4$ and $K = 2$, the transition graph of the enriched chain is illustrated on Figure 2.3.

The existence and uniqueness of the stationary distribution of the enriched chain results from:

Proposition 2.2.20. *For $x_0 > 0$, the enriched chain has a unique closed communicating class, all of whose states are aperiodic. The chain is irreducible if and only if all x_i 's are nonzero.*

Proof. The first assertion is proved in the same way as that of Proposition 2.2.3, the “lowest” state being now the set partition

$$\tilde{\mathcal{E}} = \{\{1, \dots, H\} \cap (j + K\mathbb{N})\}_{j=1, \dots, K} \quad (2.28)$$

(precisely, starting from any initial state, we eventually obtain $\tilde{\mathcal{E}}$ by applying only transitions of probabilities x_0 and 1). Note that for $H = 4$ and $K = 2$, we have $\tilde{\mathcal{E}} = 1, 3 \mid 2, 4$ (which is the state with two red arches in Figure 2.3).

For the second assertion, suppose that $x_i > 0$ for all i : we want to prove that the transition graph is strongly connected. We first replace the transitions of probability 1 by K transitions of probabilities $x_0, \dots, x_k$, so that each state has K outgoing transitions, counting transitions to itself. It is then not difficult to check from the definition of the enriched chain that each state also has K incoming transitions. It follows that for any set of states, the number of incoming and outgoing transitions are the same. Since the state space is finite, all communicating classes are closed. As there is a unique closed communicating class, irreducibility follows. Conversely, if $x_i = 0$ for some i , then clearly some set partitions, e.g. $1 \mid 2 \mid \dots \mid i \mid i+2 \mid \dots \mid K-1 \mid K \mid (K+1) \dots h \mid (i+1) \mid H$, are never produced. $\square$

Projection, stationary distribution and second proof of Theorem 2.2.5

From the surjection ψ introduced above, we define a rectangular matrix Ψ with rows indexed by elements of $\mathcal{S}(H, K)$ and columns indexed by elements of $\text{St}_{h,k}$ in the obvious manner (namely an entry of Ψ is 1 if its column index is the image by ψ of its row index, and 0 otherwise).

Lemma 2.2.21. *We have the intertwining relation*

$$\tilde{P}\Psi = \Psi P. \quad (2.29)$$

For clarity, the proofs of all lemmas in this section are postponed to Section 2.2.4. Note that, if we view the preimages by ψ of elements of $\text{St}_{h,k}$ as equivalence classes for an equivalence relation on $\mathcal{S}(H, K)$, then (2.29) shows that the MJMC is the projection of the enriched chain [31, Lemma 2.5]. Noting that the sum of each row of Ψ is 1, we immediately deduce:

Corollary 2.2.22. *We have*

$$\pi = \tilde{\pi}\Psi \quad (2.30)$$

where $\tilde{\pi}$ is the stationary probability distribution of the enriched chain (viewed as a row vector), and π is that of the MJMC.

Our route to Theorem 2.2.5 is now clear. We want to find an explicit expression for $\tilde{\pi}$, then compute $\tilde{\pi}\Psi$. We first need to introduce some definitions and notations.

Definition 2.2.23. Let σ be an enriched state and s, t two integers such that $1 \leq s < t \leq H$. We say that the pair (s, t) is an *arch* of σ if s and t belong to the same block β of σ , while no integer strictly between s and t belongs to β .

Note that $\sigma \in \mathcal{S}(H, K)$ has exactly $\ell = H - K$ arches.

Notation 2.2.24. For $1 \leq s < t \leq H$, we denote by $C_\sigma(s, t)$ the number of blocks containing at least one element in $\{s, s+1, \dots, t\}$ (when (s, t) is an arch, then we say that these blocks are those *covered* by (s, t)).

We are now ready to express the stationary distribution of the enriched chain, see Figure 2.3 again for an illustration in the case $H = 4$, $K = 2$.

Lemma 2.2.25. *For $\sigma \in \mathcal{S}(H, K)$, the monomial*

$$\tilde{w}(\sigma) = \prod_{(s,t) \text{ arch of } \sigma} x_{K-C_\sigma(s,t)} \quad (2.31)$$

defines an unnormalized stationary measure of the enriched chain.

Remark 2.2.26. It is here natural to introduce the notation $z_i = x_{K-i}$, $i = 1, \dots, K$, so that z_i is simply the weight per arch covering i blocks. In the juggling language, z_i is the probability of doing an insertion at the i -th available site starting from the right. This notation will be useful for the “add-drop” and “annihilation” generalizations of the MJMC, see Sections 2.4.1 and 2.4.2. However, the x_i notation is more convenient to study the $k \rightarrow \infty$ limit, as we do in Sections 2.3.1 and 2.3.2.

Noting that, for $\tilde{\mathcal{E}}$ as in (2.28), we have $\tilde{w}(\tilde{\mathcal{E}}) = x_0^\ell$, we find that

$$Z_{h,k} = \sum_{\sigma \in \mathcal{S}(H,K)} \tilde{w}(\sigma) \quad (2.32)$$

is positive whenever $x_0 > 0$, so that $\tilde{\pi} = \tilde{w}/Z_{h,k}$ is the stationary probability distribution of the enriched chain. For $B \in \text{St}_{h,k}$, we set

$$w(B) = \sum_{\sigma \in \psi^{-1}(B)} \tilde{w}(\sigma), \quad (2.33)$$

that is $w = \tilde{w}\Psi$. This implies that $\pi = w/Z_{h,k}$ is the stationary probability distribution of the MJMC by Corollary 2.2.22. Theorem 2.2.5 then follows immediately from:

Lemma 2.2.27. *For $B = b_1 \cdots b_h \in \text{St}_{h,k}$, we have*

$$w(B) = \prod_{\substack{i \in \{1, \dots, h\} \\ b_i = \bullet}} (x_{E_i(B)} + \cdots + x_k). \quad (2.34)$$

where $E_i(B) = \#\{j < i | b_j = \circ\}$.

Example 2.2.28. Returning again to the case $H = 4$, $K = 2$ (i.e. $h = 3$, $k = 1$) illustrated on Figure 2.3, we find

$$w(\circ \bullet \bullet) = x_1^2, \quad w(\bullet \circ \bullet) = (x_0 + x_1)x_1 = x_1, \quad w(\bullet \bullet \circ) = (x_0 + x_1)^2 = 1. \quad (2.35)$$

Remark 2.2.29. The unnormalized weights $\tilde{w}(\sigma)$ are well-defined for arbitrary $x_0, \dots, x_k$ (not necessarily positive or subject to the condition $x_0 + \cdots + x_k = 1$). For $x_0 = \cdots = x_k = 1$, we recover Warrington's combinatorial proof of the identity (2.11). For $x_i = q^{k-i}$, we have $\tilde{w}(\sigma) = q^{N(\sigma)}$ with

$$N(\sigma) = \sum_{(s,t) \text{ arch of } \sigma} (C_\sigma(s,t) - 1). \quad (2.36)$$

Technical proofs

We now prove Lemmas 2.2.21, 2.2.25 and 2.2.27.

Proof of Lemma 2.2.21. We check the relation (2.29) coefficientwise. Fix $\sigma \in \mathcal{S}(H, K)$ and $B \in \text{St}_{h,k}$, and let $A = a_1 \cdots a_h = \psi(\sigma)$, where $\psi(\sigma)$ is defined in the beginning of Section 2.2.4. Observe that

$$\psi(\sigma^\downarrow \cup \{H\}) = a_2 \cdots a_h \circ \quad (2.37)$$

and that, for all i ,

$$\psi(I_i(\sigma^\downarrow)) = T_i(a_2 \cdots a_h \circ) \quad (2.38)$$

(since inserting H into the $(i+1)$ -th block of $\sigma^\downarrow$ transforms its maximum into a non-maximum, which means replacing the $(i+1)$ -th occurrence of $\circ$ in $\psi(\sigma^\downarrow) \circ = a_2 \cdots a_h \circ$ by a $\bullet$). If $\{1\} \in \sigma$ (i.e. $a_1 = \circ$), then, by the definition of the transition matrices and (2.37),

$$(\tilde{P}\Psi)_{\sigma,B} = \delta_{\psi(\sigma^\downarrow \cup \{H\}),B} = \delta_{a_2 \cdots a_h \circ, B} = P_{A,B} = (\Psi P)_{\sigma,B}. \quad (2.39)$$

Otherwise, if $\{1\} \notin \sigma$ (i.e. $a_1 = \bullet$), we may write

$$\begin{aligned} (\tilde{P}\Psi)_{\sigma,B} &= \sum_{\substack{\tau \in \mathcal{S}(H,K) \\ i \in \{0, \dots, k\}}} x_i \delta_{I_i(\sigma^\downarrow), \tau} \delta_{\psi(\tau), B} = \sum_{i=0}^k x_i \delta_{\psi(I_i(\sigma^\downarrow)), B} \\ &= \sum_{i=0}^k x_i \delta_{T_i(a_2 \cdots a_h \circ), B} = P_{A,B} = (\Psi P)_{\sigma,B} \end{aligned} \quad (2.40)$$

where we use (2.38) to go from the first to the second line. In both cases, we have the wanted relation. $\square$

Proof of Lemma 2.2.25. We need to show that

$$\tilde{w}(\sigma) = \sum_{\tau \in \mathcal{S}(H,K)} \tilde{w}(\tau) \tilde{P}_{\tau, \sigma} \quad (2.41)$$

which is done by considering the possible predecessors of σ in the enriched chain. If $\{H\} \in \sigma$ then σ has a unique predecessor τ such that $\sigma = \tau^\downarrow \cup \{H\}$. The arches of σ and τ are clearly in one-to-one correspondence, and their weights are unchanged, thus $\tilde{w}(\sigma) = \tilde{w}(\tau)$ as wanted. Otherwise, there exists a unique pair $(\rho, i) \in \mathcal{S}(h, K) \times \{0, \dots, k\}$ such that $\sigma = I_i(\rho)$ and it is easily seen that $\tilde{w}(\sigma) = x_i \tilde{w}(\rho)$ (since inserting H in the $(i+1)$ -th block of ρ amounts to creating an arch covering $K-i$ blocks, the other arch

weights being unaffected). The predecessors of σ are the τ such that $\tau^\downarrow = \rho$ and $\{1\} \notin \tau$, and we have $\tilde{P}_{\tau,\sigma} = x_i$ regardless of τ . There are exactly K predecessors, whose weights are $x_j \tilde{w}(\rho)$ with $j \in \{0, \dots, k\}$ (indeed all these predecessors are obtained by shifting all elements of all blocks of ρ up by one, which preserves the arch weights, then inserting 1 into one of the K blocks, which creates a new arch covering an arbitrary number of blocks between 1 and K). From the condition $x_0 + \dots + x_k = 1$, we conclude that (2.41) holds as wanted. $\square$

proof of Lemma 2.2.27. We proceed by induction on $\ell = h - k = H - K$. For $\ell = 0$, the statement is true since $B = \circ^h$ has only one preimage, namely the set partition consisting of singletons $\{\{1\}, \{2\}, \dots, \{H\}\}$ which has no arch, thus has a weight 1 consistent with (2.34). Let us now assume that $\ell > 0$ and consider the smallest i such that $b_i = \bullet$, i.e. $B = \circ^{i-1} \bullet b_{i+1} \dots b_h$. Let $B' \in \text{St}_{h,k+1}$ be the word obtained by replacing the i -th letter of B by $\circ$, i.e. $B' = \circ^i b_{i+1} \dots b_h$.

Consider $\tau \in \psi^{-1}(B)$: it is easily seen from the definition of ψ that, for all $j < i$, the singleton $\{j\}$ is necessarily a block of τ , but the block β containing i contains at least another (larger) element, thus τ has an arch of the form (i, i') with weight $\tilde{w}_{i,i'} = x_n$ for some $n \geq E_i(B)$.

Splitting β in two blocks $\{i\}$ and $\beta \setminus \{i\}$, we obtain a set partition $\tau' \in \mathcal{S}(H, K+1)$, and since the inverse operation is obvious, this shows that $\tau \mapsto (\tau', n)$ is a bijection between $\psi^{-1}(B)$ and $\psi^{-1}(B') \times \{E_i(B), \dots, k\}$ such that $\tilde{w}(\tau) = x_n \tilde{w}(\tau')$. By the induction hypothesis we have

$$w(B') = \sum_{\tau' \in \psi^{-1}(B')} \tilde{w}(\tau') = \prod_{\substack{j \in \{i+1, \dots, h\} \\ b_j = \bullet}} (x_{E_j(B)} + \dots + x_k). \quad (2.42)$$

and, multiplying this relation by x_n and summing over n , the desired relation (2.34) follows. $\square$

2.3 Extensions to infinite state spaces

In this section, we discuss extensions of the Multivariate Juggling Markov Chain to an infinite setting. More precisely, we first let the number of available sites k tend to infinity, keeping the number of balls ℓ fixed. This is the Unbounded MJMC discussed in Section 2.3.1. Further, we consider the case

where the number of balls ℓ tends to infinity (with k infinite or not), which corresponds to the Infinite MJMC discussed in Section 2.3.2.

2.3.1 Unbounded heights

As suggested in the conclusion of [41], a first natural extension is to allow Magnus to throw balls arbitrarily high, so that the ball flight times are unbounded. This corresponds to taking the limit $h \rightarrow \infty$ of the MJMC, keeping the number of balls ℓ fixed. In the particle picture, the sites are now labelled by the set of positive integers, and exactly ℓ sites are occupied by a particle. The time evolution is essentially unchanged: at each time step, all particles are moved one site to the left, and if there was a particle on the first site, it is reinserted at an available site anywhere on the lattice. We keep the MJMC prescription of choosing the i -th available site with probability x_{i-1} , but since there are now infinitely many available sites, we have an infinite sequence $(x_i)_{i \geq 0}$ of nonnegative real numbers such that $\sum_{i=0}^{\infty} x_i = 1$.

Formally, states can be viewed as infinite words on the alphabet $\{\bullet, \circ\}$ containing exactly ℓ occurrences of $\bullet$. We denote by $\text{St}^{(\ell)}$ the set of such states (which can be viewed as the direct limit of the set sequence $(\text{St}_{h,h-\ell})_{h \geq \ell}$). For $A \in \text{St}^{(\ell-1)}$ and i a nonnegative integer, we let $T_i(A) \in \text{St}^{(\ell)}$ be the word obtained by replacing the $(i+1)$ -th occurrence of $\circ$ in A by $\bullet$.

Definition 2.3.1. Given a nonnegative integer ℓ and a sequence $(x_i)_{i \geq 0}$ of nonnegative real numbers such that $\sum_{i=0}^{\infty} x_i = 1$, the *Unbounded Multivariate Juggling Markov Chain* (UMJMC) is the Markov chain on the state space $\text{St}^{(\ell)}$ for which the transition probability from $A = a_1 a_2 a_3 \cdots$ to B reads

$$P_{A,B} = \begin{cases} 1 & \text{if } a_1 = \circ \text{ and } B = a_2 a_3 \cdots, \\ x_i & \text{if } a_1 = \bullet \text{ and } B = T_i(a_2 a_3 \cdots), \\ 0 & \text{otherwise.} \end{cases} \quad (2.43)$$

Again, we might identify $\text{St}^{(\ell)}$ with the set of ℓ -element subsets of $\mathbb{N}$ (corresponding to the positions of the occupied sites). For two such subsets S and S' , the UMJMC transition probability from S to S' is still given by (2.3). Our model is thus a particular case of the general model considered in [30]. The following proposition is an immediate extension of Proposition 2.2.3.

Proposition 2.3.2. *If $x_0 > 0$, then the UMJMC has a unique closed communicating class all of whose states are aperiodic. Furthermore, if infinitely many x_i 's are nonzero, then the UMJMC is irreducible.*

From now on, we assume that $x_0 > 0$. Note that, if $k = \sup\{i : x_i > 0\}$ is finite, then no insertion is ever made at a position larger than k . Thus, the states with particles at positions larger than k are transient, and upon removing them we recover the finite MJMC on $\text{St}_{\ell+k,k}$. In this degenerate case, all the forthcoming statements remain true, but were already established in Section 2.2.

We are again interested in the stationary distribution, but the fact that the state space is now countably infinite requires a bit of care. Still, we might extend Theorem 2.2.5 in the following form.

Theorem 2.3.3. *The unique invariant measure (up to constant of proportionality) of the UMJMC is given by*

$$w(B) = \prod_{i \in \mathbb{N}, b_i = \bullet} y_{E_i(B)} \quad (2.44)$$

where $B = b_1 b_2 \dots \in \text{St}^{(\ell)}$, $E_i(B) = \#\{j < i | b_j = \circ\}$ and $y_m = \sum_{j=m}^{\infty} x_j$.

We prove this theorem by a straightforward extension of the discussion of Section 2.2.3. In the integer partition language, the extension simply consists in lifting the bound on part sizes. We denote by $\text{Par}_{\infty,\ell}$ the set of integer partitions with at most ℓ parts and unbounded part sizes (which we can view as nonincreasing sequences of ℓ nonnegative integers). This set is naturally identified with $\text{St}^{(\ell)}$: given a state in $\text{St}^{(\ell)}$, we denote by $s_1 < s_2 < \dots < s_{\ell-1} < s_{\ell}$ the positions of $\bullet$, so that the corresponding integer partition is $(s_{\ell} - \ell, s_{\ell-1} - (\ell - 1), \dots, s_1 - 1)$. We let the reader verify that, in the integer partition language, the UMJMC transitions are still given by (2.18). Theorem 2.3.3 may then be reformulated as follows.

Theorem 2.3.4. *The unique invariant measure (up to constant of proportionality) of the UMJMC is given by*

$$w(\lambda) = \prod_{i=1}^{\ell} y_{\lambda_i} \quad (2.45)$$

where $\lambda = (\lambda_1, \dots, \lambda_{\ell}) \in \text{Par}_{\infty,\ell}$.

Proof. We simply check that

$$w(\lambda) = \sum_{\mu \in \text{Par}_{\infty,\ell}} w(\mu) P_{\mu,\lambda}. \quad (2.46)$$

Any predecessor μ of λ is either:

- of the form $\mu = (\lambda_1 + 1, \dots, \lambda_\ell + 1)$ in which case we have $w(\mu) = \prod_{i=1}^\ell y_{\lambda_i+1} = \prod_{i=1}^\ell (y_{\lambda_i} - x_{\lambda_i})$ and $P_{\mu,\lambda} = 1$;
- of the form $\mu = (\lambda_1 + 1, \dots, \lambda_{j-1} + 1, \lambda_{j+1}, \dots, \lambda_\ell, 0)$ for some $j \in \{1, \dots, \ell\}$ in which case we have $w(\mu) = \prod_{i=1}^{j-1} (y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^\ell y_{\lambda_i}$ and $P_{\mu,\lambda} = x_{\lambda_j}$.

The verification of (2.46) is then done exactly as in (2.21). $\square$

We have exhibited an invariant measure of the UMJMC which is clearly σ -finite. We might wonder whether it is actually finite, so that it may be normalized into a probability distribution.

Proposition 2.3.5. *The invariant measure w of the UMJMC is finite if and only if*

$$\sum_{i=0}^{\infty} ix_i < \infty \quad (2.47)$$

in which case its total mass reads

$$Z^{(\ell)} = h_\ell(y_0, y_1, \dots). \quad (2.48)$$

Proof. Let $Z^{(\ell)} \in [0, \infty]$ be the total mass of w . Observe that

$$Z^{(1)} = \sum_{\lambda_1=0}^{\infty} y_{\lambda_1} = \sum_{i=0}^{\infty} (i+1)x_i \quad (2.49)$$

thus the first assertion is obviously true for $\ell = 1$. It remains true for general ℓ as we have the inequalities

$$Z^{(1)} \leq Z^{(\ell)} \leq (Z^{(1)})^\ell \quad (2.50)$$

since $\text{Par}_{\infty,\ell}$ contains the set of partitions with at most one nonzero part (which has mass $Z^{(1)}$ by (2.45) and the fact that $y_0 = 1$), but may be viewed as a subset of $\mathbb{N}^\ell$, for which we have $\sum_{\lambda \in \mathbb{N}^\ell} w(\lambda) = (Z^{(1)})^\ell$ by (2.45).

The identity (2.48) is obtained by letting $k \rightarrow \infty$ in (2.23), since $(\text{Par}_{k,\ell})_{k \geq 0}$ forms an increasing family of sets with union $\text{Par}_{\infty,\ell}$ so that $Z_{k+\ell,k} \nearrow Z^{(\ell)}$. Note that the notation $h_\ell(y_0, y_1, \dots)$ makes sense: h_ℓ , being a symmetric function, can be expressed as a polynomial in the power sum symmetric functions $(p_m)_{m \geq 1}$ that does not depend on its number of variables, and

$$p_m(y_0, y_1, \dots) = y_0^m + y_1^m + \dots \quad (2.51)$$

is a finite real number for any $m \geq 1$ by (2.47). $\square$

By standard results from the theory of Markov chains, see e.g. [29, Chapter 13] or [31, Chapter 21], we deduce:

Corollary 2.3.6. *The UMJMC is positive recurrent if and only if (2.47) holds. In that case, there is a unique stationary probability distribution, and the chain started from any initial state converges to it in total variation as time tends to infinity.*

Example 2.3.7. Fix $q \in (0, 1)$ and pick $x_i = (1 - q)q^i$. We recover the “JEP with memoryless height distribution” with parameter q considered in [30]. Clearly (2.47) holds and we recover from Theorem 2.3.3 Leskelä-Varpanen’s expression for the stationary probability distribution. Interestingly, in the integer partition language, we find that $w(\lambda) = q^{|\lambda|}$ for $\lambda \in \text{Par}_{\infty, \ell}$, where $|\lambda|$ stands for the size (sum of all parts) of λ .

Remark 2.3.8. It is interesting to note that (2.47) is a necessary and sufficient condition for the UMJMC to be uniformly integrable in the sense of [30]. When (2.47) does not hold, the chain may either be null recurrent or transient. For $\ell = 1$, the state with the first site occupied is clearly recurrent, thus the chain is null recurrent. Figuring out the situation for $\ell > 1$ is an intriguing question. By somewhat heuristic arguments inspired by Polyá’s theorem on recurrence of random walks, we expect that, for a sequence x_i decaying asymptotically as $i^{-1-\alpha}$ with $\alpha \in (0, 1)$ (so that $\sum ix_i = \infty$), the UMJMC is null recurrent if $\ell(1 - \alpha) \leq 1$ and transient if $\ell(1 - \alpha) > 1$. A formal proof of this statement is beyond the scope of this Chapter.

2.3.2 Infinitely many balls

We now consider another limit in which Magnus juggles with infinitely many balls ($\ell \rightarrow \infty$). Let us first heuristically discuss this limit in the particle picture, a precise mathematical statement (Proposition 2.3.13) being given at the end of this section. Since the particles tend to accumulate on the left side of the lattice (i.e. balls tend to have low remaining flight times), we expect that, as ℓ becomes large, all sites at a finite distance from the first site will be occupied with high probability (in particular, Magnus receives a ball at every time step), and the first available site will then be typically at a position of order ℓ . It is convenient to relabel the sites by arbitrary (not necessarily positive) integers, so that the leftmost site has label $-\ell$, and the lowest state corresponds to having particles on sites $-\ell, -\ell + 1, \dots, -1$.

In the limit $\ell \rightarrow \infty$, the leftmost site is sent to $-\infty$, and the lowest state corresponds to having all negative sites filled and all nonnegative sites empty.

A general state is obtained by moving finitely many particles from occupied to empty sites, in other words it is a bi-infinite word $B = (b_i)_{i \in \mathbb{Z}} \in \{\bullet, \circ\}^{\mathbb{Z}}$ such that

$$\{i \geq 0 : b_i = \bullet\} = \{i < 0 : b_i = \circ\} < \infty \quad (2.52)$$

(such a word is sometimes called a “Maya diagram”). The time evolution is now easy to describe: at each time step, all particles are moved one site to the left and a new particle is inserted at an available site (so that the condition (2.52) is preserved). Again, we keep the MJMC prescription of picking the i -th available site from the left with probability x_{i-1} (this is well-defined since the set of available positions is bounded from below).

Rather than writing down the transitions formally, we prefer to work directly in the language of integer partitions. It is well-known that bi-infinite words subject to (2.52) are in one-to-one correspondence with arbitrary integer partitions (i.e. non-increasing sequences of integers that vanish eventually), see again Figure 2.2 and think about extending the displayed juggling state by adding infinitely many $\bullet$ on the left and infinitely many $\circ$ on the right, which does not change the corresponding integer partition. We denote by Par the set of all integer partitions and, for $\lambda = (\lambda_j)_{j \geq 1} \in \text{Par}$ and $i \geq 0$, we set

$$\lambda^{(i)} = (\lambda_1 - 1, \dots, \lambda_j - 1, i, \lambda_j, \lambda_{j+1}, \dots) \quad (2.53)$$

with j the smallest index such that $\lambda_j \leq i$. This is the Par equivalent of the above time evolution with insertion at the $(i+1)$ -th available site.

Definition 2.3.9. Given a sequence $(x_i)_{i \geq 0}$ of nonnegative real numbers such that $\sum_{i=0}^{\infty} x_i = 1$, the *Infinite Multivariate Juggling Markov Chain* (IMJMC) is the Markov chain on the state space Par for which the transition probability from λ to μ reads

$$P_{\lambda, \mu} = \begin{cases} x_i & \text{if } \mu = \lambda^{(i)}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.54)$$

The existence and uniqueness (up to normalization) of an invariant measure of the IMJMC is ensured by the following:

Proposition 2.3.10. *If $x_0 > 0$, then the IMJMC has a unique closed communicating class, all of whose states are aperiodic. They are precisely the*

integer partitions whose parts are smaller than or equal to $\sup\{i : x_i > 0\}$. In particular, if infinitely many x_i 's are nonzero, then the IMJMC is irreducible.

Proof. It is easily seen that, starting from any initial state and applying repeatedly the transition x_0 , we end up with the empty partition (i.e. the partition corresponding to an empty Young diagram), which is a fixed point. This proves the first statement. For the second statement, let us prove that, for any k such that $x_k > 0$, we may obtain any partition μ with $\mu_1 \leq k$ from the empty partition using only the transitions x_0 and x_k . We proceed by double induction on the number of parts p of μ (i.e. the largest p such that $\mu_p > 0$) and on $k - \mu_1$. If $p = 0$, then μ is already the empty partition. Otherwise, if $k - \mu_1 = 0$, then μ is obtained from $(\mu_2, \mu_3, \dots)$ (which has one less part) by the transition x_k . Finally, if $k - \mu_1 > 0$, then μ is obtained from $\mu' = (\mu_1 + 1, \dots, \mu_p + 1, 0, 0, \dots)$ (which still has p parts and $k - \mu'_1 < k - \mu_1$) by the transition x_0 . $\square$

Theorem 2.3.11. *An invariant measure of the IMJMC is given by*

$$w(\lambda) = \prod_{i=1}^{\infty} y_{\lambda_i} \quad (2.55)$$

where $\lambda \in \text{Par}$ and $y_m = \sum_{j=m}^{\infty} x_j$. Its total mass reads

$$Z = \prod_{m=1}^{\infty} \frac{1}{1 - y_m} \quad (2.56)$$

and Z is finite (i.e. the IMJMC is positive recurrent) if and only if (2.47) holds.

Proof. Note that the right-hand side of (2.55) is well-defined since $y_0 = 1$ and λ_i vanishes eventually. By (2.53), any predecessor of $\lambda \in \text{Par}$ is necessarily of the form $\mu = (\lambda_1 + 1, \dots, \lambda_{j-1} + 1, \lambda_{j+1}, \lambda_{j+2}, \dots)$ for some j , and $\lambda = \mu^{(\lambda_j)}$. We then have $w(\mu) = \prod_{i=1}^{j-1} (y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\infty} y_{\lambda_i}$ and $P_{\mu, \lambda} = x_{\lambda_j}$. Telescoping as in (2.21), we deduce

$$\sum_{\mu \in \text{Par}} w(\mu) P_{\mu, \lambda} = \sum_{j=1}^{\infty} \left(x_{\lambda_j} \prod_{i=1}^{j-1} (y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\infty} y_{\lambda_i} \right) = \prod_{i=1}^{\infty} y_{\lambda_i} = w(\lambda) \quad (2.57)$$

(note that $\prod_{i=1}^{j-1}(y_{\lambda_i} - x_{\lambda_i}) \prod_{i=j+1}^{\infty} y_{\lambda_i} \leq y_1^{j-1}$ with $y_1 < 1$). This establishes the invariance of w .

The expression (2.56) follows from standard considerations on integer partitions, and is finite if and only if $\sum y_m < \infty$, i.e. if (2.47) holds. $\square$

Again, when (2.47) does not hold, the IMJMC is either null recurrent or transient, and it would be interesting to know which situation occurs.

Example 2.3.12. Consider again the geometric case $x_i = (1 - q)q^i$, $q \in (0, 1)$. Then, the stationary distribution is nothing but the “ q^{size} ” measure over arbitrary integer partitions. Note that, contrary to the case of finitely many balls (Example 2.3.7), the phenomenon of ultrafast convergence to stationarity observed in [30] cannot occur: the stationary distribution is supported on the set of all integer partitions with arbitrarily many parts, but since we may create at most one new part at each time step, the distribution at a finite time starting from a given initial partition is supported on a strictly smaller set.

So far we have not given a precise mathematical sense to the fact that the IMJMC is the limit as $\ell \rightarrow \infty$ of the (U)MJMC. This is actually the case according to a certain notion of “local convergence”, which again is easier to state in the language of integer partitions. Let us fix the sequence $(x_i)_{i \geq 0}$ and, for $\ell \in \mathbb{N} \cup \{\infty\}$, denote by $\Lambda(\ell; 0), \Lambda(\ell; 1), \dots$ the Par-valued Markov chain corresponding to the UMJMC with ℓ balls for $\ell \in \mathbb{N}$, or the IMJMC for $\ell = \infty$, started at an arbitrary (deterministic or random) initial state $\Lambda(\ell, 0)$. (Note that the state space $\text{Par}_{\infty, \ell}$ of $\Lambda(\ell, \cdot)$ with $\ell \in \mathbb{N}$ may naturally be viewed as a subset of Par , upon appending an infinite zero sequence to its elements.)

Proposition 2.3.13. *For any nonnegative integer t and any fixed partition ν , we have the convergence in distribution*

$$(\Lambda(\ell; 0), \dots, \Lambda(\ell; t)) \xrightarrow[\ell \rightarrow \infty]{(d)} (\Lambda(\infty; 0), \dots, \Lambda(\infty; t)) \quad (2.58)$$

when each chain is started at the deterministic state $\Lambda(\ell, 0) = \nu$ (assuming that ℓ is larger than the number of parts of ν).

Furthermore, if the condition (2.47) for positive recurrence is satisfied, then the convergence (2.58) also holds when each chain is started at its stationary probability distribution.

Proof. The transition probabilities for the chain $\Lambda(\ell, \cdot)$ are given by (2.18) when $\ell \in \mathbb{N}$ and by (2.54) when $\ell = \infty$: observe that they are equal whenever λ has strictly less than ℓ parts, and furthermore that any transition increases the number of parts by at most one. It follows that, when each chain is started at the deterministic state $\Lambda(\ell, 0) = \nu$, then the law of $(\Lambda(\ell; 0), \dots, \Lambda(\ell; t))$ does not depend on ℓ as soon as $\ell - t$ is larger than the number of parts of ν , which immediately implies the first statement.

To prove the second statement we note that the stationary distribution of $\Lambda(\ell, \cdot)$ converges in total variation to that of $\Lambda(\infty, \cdot)$ as $\ell \rightarrow \infty$ (so the statement holds for $t = 0$). This is a simple consequence of Theorems 2.3.4 and 2.3.11: the unnormalized invariant measure of $\Lambda(\ell, \cdot)$ for $\ell \in \mathbb{N}$ is the restriction of that of $\Lambda(\infty, \cdot)$ to the set of integer partitions with at most ℓ parts and, when (2.47) holds, their total masses are finite and tend to one another as $\ell \rightarrow \infty$. Since a partition has finitely many parts, the previous argument allows us to conclude that (2.58) holds for any t . $\square$

Remark 2.3.14. It is easier to state the convergence in terms of integer partitions since the state spaces of the chains $\Lambda(\ell, \cdot)$ are naturally included in one another. In terms of particles, this corresponds to embedding the state space $\text{St}^{(\ell)}$ of the UMJMC with ℓ balls into that of the IMJMC by prepending infinitely many $\bullet$'s on the left, then shifting all letters in the resulting bi-infinite word by ℓ positions to the left so as to satisfy (2.52). This allows us to translate Proposition 2.3.13 in the particle language, and justifies the heuristic discussion at the beginning of this section.

2.4 Extensions with a fluctuating number of balls

We now consider extensions of the MJMC where the number of balls is not fixed but is allowed to fluctuate. These extensions are the natural multivariate generalizations of the so-called *add-drop* and *annihilation* models introduced in [41, Section 4] (to which we refer for motivations), and we thus keep the same denomination here. Both models are defined on the same state space and have the same transition graph, only the transitions probabilities differ. In both cases, we will provide an exact expression for the stationary distribution, whose validity will be proved by considering the enriched version of the chain.

Since many definitions and notations will be common to both models, we factorize their discussion here. The basic state space of the model will be $\text{St}_h = \{\circ, \bullet\}^h$, with h a fixed nonnegative integer. As hinted in Remark 2.2.26, it will be convenient here to read a word from right to left.

Notation 2.4.1. For $A \in \bigcup_{h \geq 0} \text{St}_h$ and i a nonnegative integer, we let $S_i(A)$ be the word obtained by replacing the i -th occurrence of $\circ$ in A by $\bullet$, upon reading the word *from the right* (if i is equal to 0 or larger than the number of occurrences of $\circ$ in A then we set $S_i(A) = A$ by convention).

Note that $S_i(A) = T_{k-i}(A)$ for $A \in \text{St}_{h,k}$ and $1 \leq i \leq k$, with T as in Section 2.2.1. We now turn to the enriched model, whose state space is the set $\mathcal{S}(H)$ of all set partitions of $\{1, \dots, H\}$, with $H = h + 1$ (the number of enriched states is thus a Bell number). Recall the notations from Section 2.2.4: ψ is a surjection from $\mathcal{S}(H)$ onto St_h and $\sigma \mapsto \sigma^\downarrow$ is a mapping from $\mathcal{S}(H)$ to $\mathcal{S}(h)$.

Notation 2.4.2. For $\tau \in \bigcup_{h \geq 0} \mathcal{S}(h)$ and i a nonnegative integer, we define $J_i(\tau)$ as follows. If $\tau \in \mathcal{S}(h)$ then we let $J_i(\tau) \in \mathcal{S}(h + 1)$ be the set partition obtained by inserting $h + 1$ into the i -th block of τ , now numbered by *decreasing* order of maxima (if i is equal to 0 or larger than the number of blocks of τ then we set $J_i(\tau) = \tau \cup \{h + 1\}$ by convention).

Note that $J_i(\tau) = I_{K-i}(\tau)$ for $\tau \in \mathcal{S}(h, K)$ and $1 \leq i \leq K$. Observe that, for any $i \geq 0$, we have the fundamental “intertwining” relation

$$\psi(J_i(\tau)) = S_i(\psi(\tau)\circ) \quad (2.59)$$

which is nothing but a compact rewriting of (2.37) and (2.38). Note also that, when τ is non empty ($h \geq 1$), we have the commutation relation

$$J_i(\tau)^\downarrow = J_i(\tau^\downarrow). \quad (2.60)$$

The basic transition graph is defined as follows: for any $A = a_1 a_2 \dots a_h \in \text{St}_h$ and $i \geq 0$, we have an oriented edge from A to $S_i(a_2 \dots a_h \circ)$ (we ignore edge multiplicities). See Figure 2.4 for $h = 2$. Similarly, the enriched transition graph is obtained by connecting each $\sigma \in \mathcal{S}(H)$ to $J_i(\sigma^\downarrow)$ for all $i \geq 0$. It is not difficult to check that both transition graphs are strongly connected using Proposition 2.2.3 and Proposition 2.2.20.

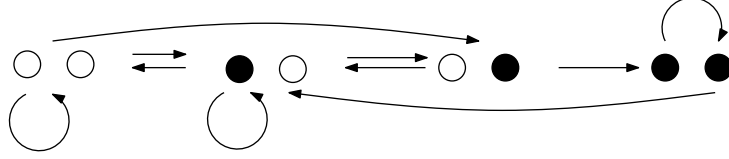


Figure 2.4 – The transition graph for $h = 2$

2.4.1 Add-drop juggling

Definition 2.4.3. Given h a nonnegative integer and $a = z_0, \dots, z_h$ nonnegative real numbers, the (multivariate) *add-drop model* is the Markov chain on the state space St_h for which the transition probability from $A = a_1 \dots a_h$ to B reads

$$P_{A,B} = \begin{cases} \frac{z_i}{z_0 + \dots + z_k} & \text{if } B = S_i(a_2 \dots a_h \circ) \text{ for some } i \in \{0, \dots, k\}, \\ 0 & \text{otherwise,} \end{cases} \quad (2.61)$$

with k the number of occurrences of $\circ$ in $a_2 \dots a_h \circ$.

Warrington's add-drop model is recovered by taking $a = z_1 = \dots = z_h = 1$. It is easily seen that, in general, the chain is aperiodic with a unique communicating class whenever $a > 0$, so that the stationary distribution is unique.

Theorem 2.4.4. *The stationary distribution of the add-drop model is given by*

$$\Pi(B) = \frac{a^k}{Z_h} \prod_{\substack{i=1 \\ b_i = \bullet}}^h (z_1 + \dots + z_{\psi_i(B)+1}), \quad (2.62)$$

for $B = b_1 \dots b_h \in \text{St}_{h,k}$, with $\psi_i(B) = \#\{j : i < j \leq h, b_j = \circ\}$ and

$$Z_h = \sum_{k=0}^h a^k h_{h-k}(z_1, z_1 + z_2, \dots, z_1 + \dots + z_{k+1}), \quad (2.63)$$

where h_ℓ is the complete homogeneous symmetric polynomial of degree ℓ . (Note that z_{h+1} never appears in (2.62) since the product is empty for $k = h$.)

Example 2.4.5. The transition matrix for $h = 2$ in the lexicographically-ordered basis $(\bullet\bullet, \circ\bullet, \bullet\circ, \circ\circ)$ is given by

$$P = \begin{pmatrix} \frac{z_1}{a+z_1} & 0 & \frac{a}{a+z_1} & 0 \\ \frac{z_1}{a+z_1} & 0 & \frac{a}{a+z_1} & 0 \\ 0 & \frac{z_1}{a+z_1+z_2} & \frac{z_2}{a+z_1+z_2} & \frac{a}{a+z_1+z_2} \\ 0 & \frac{z_1}{a+z_1+z_2} & \frac{z_2}{a+z_1+z_2} & \frac{a}{a+z_1+z_2} \end{pmatrix} \quad (2.64)$$

and the stationary probabilities are given by the normalized coordinates of the row eigenvector with eigenvalue 1, namely

$$\frac{1}{Z_2} (z_1^2, az_1, a(z_1 + z_2), a^2), \quad (2.65)$$

where $Z_2 = a^2 + az_2 + 2az_1 + z_1^2$.

Theorem 2.4.4 is the natural multivariate generalization of [41, Theorem 3, item 1]. The parameter a has the physical interpretation of the fugacity for a $\circ$. In other words the distribution Π is the grand-canonical version of the stationary distribution π of Theorem 2.2.5. We prove Theorem 2.4.4 by a straightforward extension of the construction of Section 2.2.4, which we now detail.

Definition 2.4.6. The (multivariate) *enriched add-drop model* is the Markov chain on the state space $\mathcal{S}(H)$ for which the transition probability from σ to τ is given by

$$\tilde{P}_{\sigma,\tau} = \begin{cases} \frac{z_i}{z_0 + \dots + z_k} & \text{if } \tau = J_i(\sigma^\downarrow) \text{ for some } i \in \{0, \dots, k\}, \\ 0 & \text{otherwise,} \end{cases} \quad (2.66)$$

with k the number of blocks of $\sigma^\downarrow$.

By (2.59) it is immediate that the add-drop model is indeed the projection of the enriched chain, and furthermore it is easily seen that the enriched chain is aperiodic with a unique communicating class for $z_0 = a > 0$.

Lemma 2.4.7. For $\sigma \in \mathcal{S}(H)$ with, say, K blocks, the monomial

$$\tilde{W}(\sigma) = a^{K-1} \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)}, \quad (2.67)$$

where $C_\sigma(s,t)$ is as in Notation 2.2.24, defines an unnormalized stationary distribution of the enriched chain.

Proof. We want to show that $\tilde{W}(\tau) = \sum_{\sigma \in \mathcal{S}(H)} \tilde{W}(\sigma) \tilde{P}_{\sigma, \tau}$ for any τ . Let $\tau^\uparrow$ be the set partition obtained by removing the element H from τ , so that the possible predecessors of τ are the σ such that $\sigma^\downarrow = \tau^\uparrow$. More precisely, any predecessor is obtained by shifting all elements of $\tau^\uparrow$ up by 1, then inserting the element 1 either as a singleton or into a pre-existing block. Hence, there are $k+1$ predecessors where k is the number of blocks of $\tau^\uparrow$, and we readily see that their weights are $z_i \tilde{W}(\tau^\uparrow)$ with $i = 0, \dots, k$. On the other hand, let j be equal to 0 if $\{H\}$ is a singleton of τ , or equal to $C_\tau(r, H)$ where r is such that (r, H) is an arch of τ . We have $\tilde{W}(\tau) = z_j \tilde{W}(\tau^\uparrow)$ and $\tilde{P}_{\sigma, \tau} = z_j / (z_0 + \dots + z_k)$ for any predecessor σ . We deduce that, as wanted,

$$\sum_{\sigma \in \mathcal{S}(H)} \tilde{W}(\sigma) \tilde{P}_{\sigma, \tau} = \sum_{i=0}^k z_i \tilde{W}(\tau^\uparrow) \frac{z_j}{z_0 + \dots + z_k} = \tilde{W}(\tau). \quad (2.68)$$

□

Example 2.4.8. The transition matrix for $H = 3$ in the ordered basis $(1, 2, 3; \ 1 | 2, 3; \ 2 | 1, 3; \ 1, 2 | 3; \ 1 | 2 | 3)$ is given by

$$\tilde{P} = \begin{pmatrix} \frac{z_1}{a+z_1} & 0 & 0 & \frac{a}{a+z_1} & 0 \\ \frac{z_1}{a+z_1} & 0 & 0 & \frac{a}{a+1} & 0 \\ 0 & \frac{z_1}{a+z_1+z_2} & \frac{z_2}{a+z_1+z_2} & 0 & \frac{a}{a+z_1+z_2} \\ 0 & \frac{z_1}{a+z_1+z_2} & \frac{z_2}{a+z_1+z_2} & 0 & \frac{a}{a+z_1+z_2} \\ 0 & \frac{z_1}{a+z_1+z_2} & \frac{z_2}{a+z_1+z_2} & 0 & \frac{a}{a+z_1+z_2} \end{pmatrix}, \quad (2.69)$$

and admits the row vector $(z_1^2, az_1, az_2, az_1, a^2)$ as a left eigenvector of eigenvalue 1. Compare this with the stationary distribution in Example 2.4.5.

proof of Theorem 2.4.4. For $B \in \text{St}_{h,k}$, let

$$W(B) = \sum_{\sigma \in \psi^{-1}(B)} \tilde{W}(\sigma). \quad (2.70)$$

By Lemma 2.2.27 and Remark 2.2.26 (note that $x_{E_i(B)} = z_{\psi_i(B)+1}$), we readily deduce

$$W(B) = a^k \prod_{\substack{i=1 \\ b_i = \bullet}}^h (z_1 + \dots + z_{\psi_i(B)+1}). \quad (2.71)$$

The total mass of W is clearly equal to $\sum_{k=0}^h a^k Z_{h,k}(z_{k+1}, z_k, \dots, z_1)$ with $Z_{h,k}$ the normalization factor of the MJMC defined in Proposition 2.2.9. □

2.4.2 Annihilation juggling

As the reader is by now familiar with our approach, we define the basic model and its enriched version at the same time.

Definition 2.4.9. Given h a nonnegative integer and $z_1, \dots, z_{h+1} = a$ non-negative real numbers such that $z_1 + \dots + z_h + a = 1$, the (multivariate) *annihilation model* is the Markov chain on the state space St_h for which the transition probability from $A = a_1 a_2 \dots a_h$ to B reads

$$P_{A,B} = \begin{cases} z_i & \text{if } B = S_i(a_2 \dots a_h \circ) \text{ for some } i \in \{1, \dots, k\}, \\ z_{k+1} + \dots + z_h + a & \text{if } B = a_2 \dots a_h \circ, \\ 0 & \text{otherwise,} \end{cases} \quad (2.72)$$

with k the number of occurrences of $\circ$ in $a_2 \dots a_h \circ$. Similarly, the (multivariate) *enriched annihilation model* is the Markov chain on the state space $\mathcal{S}(H)$ for which the transition probability from σ to τ is given by

$$\tilde{P}_{\sigma,\tau} = \begin{cases} z_i & \text{if } \tau = J_i(\sigma^\downarrow) \text{ for some } i \in \{1, \dots, k\}, \\ z_{k+1} + \dots + z_h + a & \text{if } \tau = \sigma^\downarrow \cup \{H\}, \\ 0 & \text{otherwise,} \end{cases} \quad (2.73)$$

with k the number of blocks of $\sigma^\downarrow$.

Remark 2.4.10. By our convention for $S_i(A)$ (resp. $J_i(\tau)$) when i is larger than the number of occurrences of $\circ$ in A (resp. the number of blocks of τ), we have the more compact expression $P_{A,B} = \sum z_i$ (resp. $\tilde{P}_{\sigma,\tau} = \sum z_i$) where the sum runs over all $i \in \{1, \dots, h+1\}$ such that $B = S_i(a_2 \dots a_h \circ)$ (resp. $\tau = J_i(\sigma^\downarrow)$).

Remark 2.4.11. Warrington's annihilation model is recovered by taking $z_1 = \dots = z_h = a = 1/(h+1)$. We still call our multivariate generalization the annihilation model, but this requires some clarification. Indeed, in Warrington's uniform case, one can interpret the dynamics by saying that ball insertions are made at arbitrary (empty or occupied) sites, and that a ball inserted at an occupied site is annihilated. However, in our multivariate generalization the correct interpretation is to say that we pick an i between 1 and $h+1$ and insert the ball at the i -th available site from the right, and that the ball is annihilated if there is no such site on the lattice.

Here is the multivariate generalization of [41, Theorem 3, item 2]:

Theorem 2.4.12. *The stationary distribution of the annihilation model is given by*

$$\Pi(B) = \prod_{\substack{i=1 \\ b_i = \bullet}}^h (z_1 + \cdots + z_{\psi_i(B)+1}) \prod_{j=1}^k (z_{j+1} + \cdots + z_h + a), \quad (2.74)$$

for $B = b_1 \dots b_h \in St_{h,k}$, with $\psi_i(B) = \#\{j : i < j \leq h, b_j = \circ\}$ as before. Similarly, the stationary distribution of the enriched annihilation model is given by

$$\tilde{\Pi}(\sigma) = \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)} \prod_{i=1}^{K-1} (z_{i+1} + \cdots + z_{H-1} + a), \quad (2.75)$$

with $\sigma \in \mathcal{S}(H)$ and K its number of blocks. There is no normalization factor, as Π and $\tilde{\Pi}$ are already normalized for $z_1 + \cdots + z_h + a = 1$.

Example 2.4.13. The transition matrix of the basic annihilation model for $h = 2$ in the lexicographically-ordered basis $(\bullet\bullet, \circ\bullet, \bullet\circ, \circ\circ)$ is given by

$$P = \begin{pmatrix} z_1 & 0 & z_2 + a & 0 \\ z_1 & 0 & z_2 + a & 0 \\ 0 & z_1 & z_2 & a \\ 0 & z_1 & z_2 & a \end{pmatrix}, \quad (2.76)$$

and admits the row vector $(z_1^2, z_1(z_2 + a), (z_1 + z_2)(z_2 + a), a(z_2 + a))$ as a left eigenvector of eigenvalue 1. The sum of the coordinates of this row vector is $(z_1 + z_2 + a)^2 = 1$. Similarly, the transition matrix of the enriched annihilation model for $H = 3$ in the same ordered basis as in Example 2.4.8 is given by

$$\tilde{P} = \begin{pmatrix} z_1 & 0 & 0 & z_2 + a & 0 \\ z_1 & 0 & 0 & z_2 + a & 0 \\ 0 & z_1 & z_2 & 0 & a \\ 0 & z_1 & z_2 & 0 & a \\ 0 & z_1 & z_2 & 0 & a \end{pmatrix}, \quad (2.77)$$

and admits the row vector $(z_1^2, z_1(z_2 + a), z_2(z_2 + a), z_1(z_2 + a), a(z_2 + a))$ as a left eigenvector of eigenvalue 1.

Proving Theorem 2.4.12 can be done by checking “by hand” the stationarity of $\tilde{\Pi}$, then deducing that of Π using the projection property (2.59). However, the fact that $\tilde{\Pi}(\sigma)$ is not a monomial suggests that there exists a “doubly enriched” chain that yields both P and $\tilde{P}$ by projection. It can be seen that, if we relax the condition $z_1 + \cdots + z_h + a = 1$, then the total mass of Π (or of $\tilde{\Pi}$) is equal to $(z_1 + \cdots + z_h + a)^h$: this suggests that doubly enriched states should consist of h -tuples of elements in a set of cardinality $h + 1$. From now on we will write z_{h+1} in lieu of a .

Definition 2.4.14. Given h a nonnegative integer and $z_1, \dots, z_{h+1}$ nonnegative real numbers such that $z_1 + \cdots + z_{h+1} = 1$, the *doubly enriched annihilation model* is the Markov chain on the state space $\{1, \dots, h+1\}^h$ for which the transition probability from $W = w_1 w_2 \dots w_h$ to W' reads

$$\hat{P}_{W,W'} = \begin{cases} z_i & \text{if } W' = w_2 \cdots w_h i, \\ 0 & \text{otherwise.} \end{cases} \quad (2.78)$$

Remark 2.4.15. The doubly enriched annihilation model can be seen a specialization of the de Bruijn process considered in [6].

It is obvious that the stationary probability of $W = w_1 \cdots w_h$ in the doubly enriched annihilation model is

$$\hat{\Pi}(W) = z_{w_1} \cdots z_{w_h} \quad (2.79)$$

since we are basically moving a window of size h within a sequence of independent identically distributed random variables. The nontrivial fact is that this chain may be projected to the annihilation and the enriched annihilation models.

Theorem 2.4.16. Let ϕ and $\tilde{\phi}$ be the mappings from $\bigcup_{j \geq 0} \{1, \dots, h+1\}^j$ to respectively $\bigcup_{j \geq 0} \text{St}_j$ and $\bigcup_{j \geq 0} \mathcal{S}(j+1)$ defined inductively by

$$\phi(w_1 \cdots w_j) = \begin{cases} \emptyset & \text{for } j = 0, \\ S_{w_j}(\phi(w_1 \cdots w_{j-1}) \circ) & \text{for } j \geq 1, \end{cases} \quad (2.80)$$

where $\emptyset$ denotes the empty word, and

$$\tilde{\phi}(w_1 \cdots w_j) = \begin{cases} \{\{1\}\} & \text{for } j = 0, \\ J_{w_j}(\tilde{\phi}(w_1 \cdots w_{j-1})) & \text{for } j \geq 1. \end{cases} \quad (2.81)$$

Then, the restrictions of ϕ and $\tilde{\phi}$ to $\{1, \dots, h+1\}^h$ yield projections of the doubly enriched chain onto respectively the annihilation model and the enriched annihilation model.

In other words, the projection ϕ (resp. $\tilde{\phi}$) consists in applying the composition of the mappings $B \mapsto S_{w_i}(B \circ)$ (resp. $\sigma \mapsto J_{w_i}(\sigma)$) with i running from 1 to h , to the “seed” $\emptyset$ (resp. $\{\{1\}\}$).

Example 2.4.17. The transition matrix of the doubly enriched annihilation model for $h = 2$ in the ordered basis $(11, 21, 31, 12, 22, 32, 13, 23, 33)$ is

$$\hat{P} = \begin{pmatrix} z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 & 0 \\ z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 & 0 \\ z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 & 0 \\ 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 \\ 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 \\ 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 & 0 \\ 0 & 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 \\ 0 & 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 \\ 0 & 0 & z_1 & 0 & 0 & z_2 & 0 & 0 & z_3 \end{pmatrix}, \quad (2.82)$$

The Markov chains with transition matrices $\hat{P}$, $\tilde{P}$ and P for $h = 2$ are displayed on Figure 2.5. The words 12 and 13 project via $\tilde{\phi}$ to the set partition $3|1, 2$ while 22 and 32 project to $1, 3|2$, and all four project via ϕ to the juggling state $\bullet\circ$.

Remark 2.4.18. The cardinality $h+1$ of the “alphabet” in the doubly enriched annihilation model is optimal in the sense that some states in St_h or $\mathcal{S}(h+1)$ are not attained with a smaller alphabet while, in a word of length h , all letters larger than $h+1$ can be replaced by $h+1$ without changing its image by ϕ or $\tilde{\phi}$. To be specific, if the alphabet has ℓ letters with $\ell \leq h$, the corresponding stationary probability of $B = b_1 \cdots b_h \in \text{St}_{h,k}$ in the annihilation model is

$$\Pi(B) = \prod_{\substack{i=1 \\ b_i=\bullet}}^h (z_1 + \cdots + z_{\psi_i(B)+1}) \prod_{j=1}^k (z_{j+1} + \cdots + z_\ell), \quad (2.83)$$

if $k < \ell$, and 0 otherwise. The corresponding stationary distribution of $\sigma \in \mathcal{S}(H, K)$ in the enriched annihilation model

$$\tilde{\Pi}(\sigma) = \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)} \prod_{i=1}^{K-1} (z_{i+1} + \cdots + z_\ell) \quad (2.84)$$

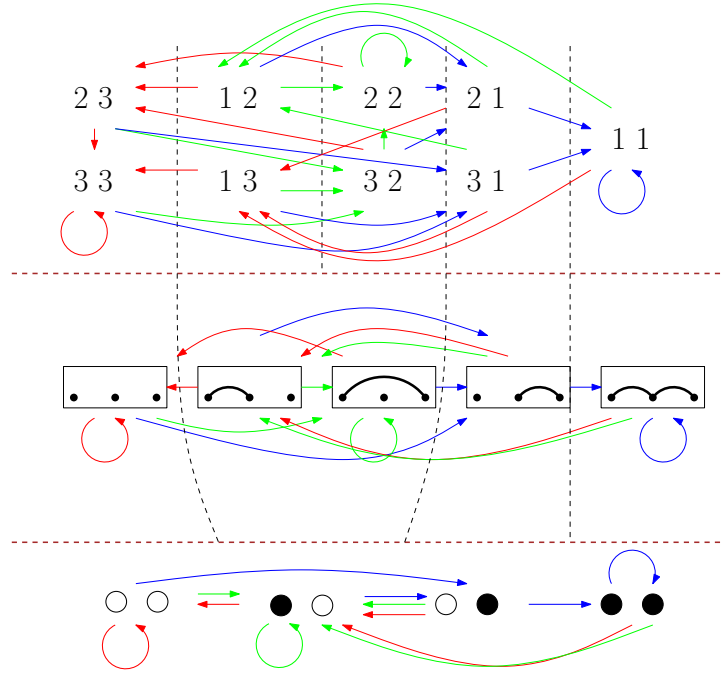


Figure 2.5 – Transition graphs of the doubly enriched annihilation model (top), the enriched annihilation model (middle) and the annihilation model (bottom) for $h = 2$. Blue, green and red arrow represent transitions of respective probabilities z_1, z_2, z_3 (when several arrows have the same endpoints, the corresponding probabilities should be added).

if $K \leq \ell$, and 0 otherwise. When $\ell \geq h + 1$, we might do the substitutions $z_{h+1} \leftarrow z_{h+1} + \dots + z_\ell$ and $z_j \leftarrow 0$ for $j > h + 1$ without affecting Π and $\tilde{\Pi}$.

proof of Theorem 2.4.16. It is not difficult to see that $\phi = \psi \circ \tilde{\phi}$ from (2.59); thus we need only check the statement for $\tilde{\phi}$. Consider the transition of probability z_i from $W = w_1 \dots w_h$ to $W' = w_2 \dots w_h i$ in the doubly enriched annihilation model: it is sufficient to have that $\tilde{\phi}(W)$ is sent to $\tilde{\phi}(W')$ by the corresponding transition of the enriched annihilation model, namely that

$$\tilde{\phi}(W') = J_i(\tilde{\phi}(W)^\downarrow). \quad (2.85)$$

But this is an easy consequence from the commutation relation (2.60) and the definition of $\tilde{\phi}$. $\square$

proof of Theorem 2.4.12. Let us first prove (2.75). We extend the definition (2.79) of $\tilde{\Pi}$ to words of arbitrary length by setting $\hat{\Pi}(w_1 \dots w_j) = z_{w_1} \dots z_{w_j}$. We will prove by induction on $j \leq h$ that, for any $\sigma \in \mathcal{S}(j + 1)$, we have

$$\sum_{W \in \tilde{\phi}^{-1}(\sigma)} \hat{\Pi}(W) = \prod_{(s,t) \text{ arch of } \sigma} z_{C_\sigma(s,t)} \prod_{i=1}^{K-1} (z_{i+1} + \dots + z_{h+1}) \quad (2.86)$$

with K the number of blocks of σ . The relation is true for $j = 0$ since both sides equal 1. For $j \geq 1$, let $\sigma' \in \mathcal{S}(j)$ be the set partition obtained from σ by removing the element $j + 1$. The preimages by $\tilde{\phi}$ of σ are then obtained from those of σ' by

- appending a unique letter $w \leq K$ if $j + 1$ is not a singleton in σ (note that σ has one arch more than σ' , and w is the number of blocks that σ covers),
- appending an arbitrary letter $w \geq K$ if $j + 1$ is a singleton in σ (note that σ has one block more than σ').

Then (2.86) follows from the induction hypothesis: it is deduced from the relation for σ' by adding an extra factor to the first product in the first case, and to the second product in the second case. We finally deduce (2.75) by taking $j = h$ (recall that $H = h + 1$ and $a = z_{h+1}$), noting that the left hand side of (2.86) is nothing but the wanted $\tilde{\Pi}(\sigma)$.

We then deduce (2.74) from the relation $\phi = \psi \circ \tilde{\phi}$, using again Lemma 2.2.27. $\square$

Another easy property of the doubly enriched annihilation model is that it is “memoryless”: after h transitions we end up with a perfect sample of the stationary distribution, since all the initial letters have been flushed out. In other words, for any initial probability distribution $\hat{\eta}$ over $\{1, \dots, h+1\}^h$, we have

$$\hat{\eta}\hat{P}^h = \hat{\Pi}, \quad (2.87)$$

and this implies that the only eigenvalues of the transition matrix $\hat{P}$ are 1 (with multiplicity 1) and 0. These properties are clearly preserved by projection, which implies the following nontrivial and remarkable property of the annihilation model, that makes it distinct from the generic MJMC and add-drop models.

Theorem 2.4.19. *For any initial probability distributions η over St_h and $\tilde{\eta}$ over $\mathcal{S}(H)$, the distribution at time h is equal to the stationary distribution, namely*

$$\eta P^h = \Pi, \quad \tilde{\eta} \tilde{P}^h = \tilde{\Pi}. \quad (2.88)$$

In particular, the only eigenvalues of P and $\tilde{P}$ are 1 (with multiplicity 1) and 0.

Remark 2.4.20. The fact that all other eigenvalues of the doubly enriched annihilation model are zero can also be proved directly from [6, Theorem 12] by setting $x_{a,m} = z_a$ so that $\beta_{a,m} = z_1 + \dots + z_{h+1}$ for all $a \in [n], m \in [L]$ for words of length L on an alphabet of size n . However, the fact that the distribution at time L is the stationary distribution in general is not remarked in [6].

This result is stronger than a statement about mixing times since we reach the exact stationary state in bounded time! Another “ultrafast convergence to equilibrium” was observed in [30] for the model of Example 2.3.7, but its combinatorial origin seems rather different from that of the annihilation model (in our model the time needed to reach the stationary distribution is constant, while in Leskelä and Varpanen’s model it depends on the initial state, and can be arbitrarily large).

Chapter 3

Bumping sequences and multispecies juggling

This Chapter is very similar to [2], a joint work with Ayyer, Bouttier, Corteel and Linusson.

3.1 Introduction

Several Markov chains studied in nonequilibrium statistical physics are known to have, despite nontrivial dynamics, an explicit and sometimes remarkably simple stationary state. The most famous examples of these are one-dimensional models of hopping particles such as the asymmetric exclusion process [15], where the stationary state satisfies the so-called matrix product representation [9] and the zero-range process, where the stationary state is factorized [23]. The main reason for this simplicity is the underlying combinatorial structure of these processes. Of the two examples mentioned above, a variant of the former known as the totally asymmetric simple exclusion process (TASEP), solved first in [16], has a rich combinatorial structure even when the system is generalized to include several types of particles. The latter system is known as the *multispecies TASEP*, and its stationary state has an explicit solution which comes from queuing theory [25].

The multispecies TASEP has the further exceptional property that the stationary state can also be calculated if the hopping probabilities of particles depend on their location, known as the *inhomogeneous multispecies TASEP*. This was first done for the three-species case in [4] and the result

for arbitrary species has been announced in [32]. While the stationary state of the general inhomogeneous multispecies TASEP has an explicit description in principle, the actual formulas for the stationary probabilities can be considerably complicated.

In this Chapter, we will first study the multispecies variants of the basic juggling process introduced in [41] then extended to their inhomogeneous versions in [22, 3]. In contrast to the TASEP, as we will show in Theorem 3.2.4, the stationary probabilities and the partition function have elegant and compact expressions. We then study the multispecies variants of two other juggling processes, which were also introduced in [41], where the number of balls of each type can vary. In all of these cases, we prove analogous results; see Theorems 3.3.3 and 3.3.5. We also introduce a new model where the number of balls of each type can vary that we call the overwriting model. This model has the nice property that it converges to its stationary distribution in deterministic finite time.

The rest of the Chapter is organized as follows. In Section 3.2, we discuss in some detail the first model, the so-called Multispecies Juggling Markov Chain (MSJMC): Section 3.2.1 provides its definition and the expression for its stationary distribution, and Section 3.2.2 is devoted to the enriched chain. Other models with a fluctuating number of balls of each type (but with a finite state space) are considered in Section 3.3: we introduce the multispecies extension of the add-drop and the annihilation models studied in [3] in the respective Sections 3.3.1 and 3.3.2. In Section 3.3.3, we present the overwriting model. Finally, in Section 3.4, we describe another possible extension of the juggling Markov chain of [41], that involves several jugglers. For all the juggling models, we prove that the Markov chain is aperiodic and irreducible. This implies that the stationary distribution is unique. For the enriched chains, we do not prove irreducibility (even though all the chains are conjectured to be irreducible). Computing a stationary distribution and lumping is enough to get the unique stationary distribution of the original chain.

3.2 Multispecies juggling

3.2.1 Definition and stationary distribution

The first model that we consider in this Chapter, and for which we give the most details, is a “multispecies” generalization of the so-called Multivariate Juggling Markov Chain (MJMC) [3]. Colloquially speaking, the juggler (Magnus) is now using balls of different weights, or numbers, and when a heavy ball (represented by a low number) collides with a lighter one (represented by a higher number), the lighter ball is bumped to a higher position, where it can itself bump a lighter ball, and so on, until a ball arrives at the topmost position. Should the reader find this model unrealistic, she may instead think of a lazy referee “juggling” with a stack of papers of varying priorities to review: every day the referee takes the paper on the top of the stack but, after spending his time on other duties, decides to postpone it to a later date, possibly bumping a less important paper further down the stack, etc. Formally, our *Multispecies Juggling Markov Chain (MSJMC)* is defined as follows.

Let T be a fixed positive integer, and $n_1, \dots, n_T$ be a sequence of positive integers. The state space $St_{n_1, \dots, n_T}$ of the MSJMC is the set of words on the alphabet $\{1, \dots, T\}$ containing, for all $i = 1, \dots, T$, n_i occurrences of the letter i (the letter 1 represents the heaviest ball and T the lightest one). Of course those words have length $n = n_1 + \dots + n_T$, and there are $\binom{n}{n_1, \dots, n_T}$ different states.

To understand the transitions, it is perhaps best to start with an example, by considering the word 132132 (i.e. $T = 3$, $n_1 = n_2 = n_3 = 2$). The first letter 1 corresponds to the ball received by Magnus: it can be thrown either directly to the rightmost position, i.e. to the top (resulting in the word 321321), or in the place of any lighter ball. Say we throw it in place of the first 2. This 2 can in turn be thrown either to the rightmost position (resulting in the word 311322), or in the place of a lighter ball on its right: here it can only “bump” the second 3, which in turn has no choice but to go to the rightmost position, resulting in the word 311223. This latter transition is represented on Figure 3.1.

We now give a formal definition of the transitions. Let $w = w_1 \dots w_n$ be a state in $St_{n_1, \dots, n_T}$, and set, by convention, $w_{n+1} = \infty$. A *bumping sequence* for w is an increasing sequence of integers $(a(1), \dots, a(k))$ with length at most $T + 1$ such that $a(1) = 1$, $a(k) = n + 1$ and, for all j between 1 and

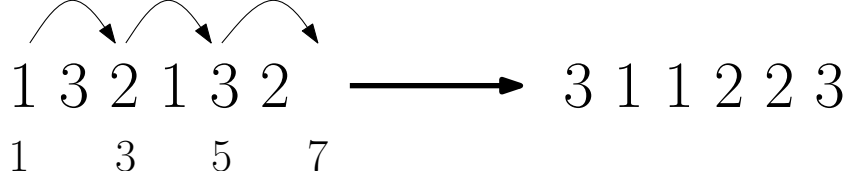


Figure 3.1 – A possible transition from the state 132132, corresponding to the bumping sequence (1, 3, 5, 7). Magnus catches the 1 in first position, and sends it to the third position, where it bumps the 2. The 2 then goes to the fifth position, where it bumps the 3. Finally, the 3 can only go to the seventh and last position, thus ending the bumping sequence.

$k - 1$, $w_{a(j)} < w_{a(j+1)}$ (that is to say, the ball at position $a(j)$ is heavier than that at position $a(j + 1)$). We denote by $\mathcal{B}_w$ the set of bumping sequences for w . For $a \in \mathcal{B}_w$, we define the state w^a resulting from w via the bumping sequence a by

$$w_i^a = \begin{cases} w_{a(\ell)-1} & \text{if } i = a(\ell) - 1 \text{ for some } \ell, \\ w_{i+1} & \text{otherwise,} \end{cases} \quad (3.1)$$

which is easily seen to belong to $St_{n_1, \dots, n_T}$. Returning to the example in Figure 3.1 with $w = 132132$, the longest possible bumping sequence is $a = (1, 3, 5, 7)$ and indeed $w^a = 311223$.

We now turn to defining the transition probabilities, which means assigning a probability to each bumping sequence. As in the MJMC, these probabilities will depend on a sequence $z_1, z_2, \dots$ of nonnegative real parameters, whose interpretation is now the following. Suppose that we have constructed the $i - 1$ first positions $(a(1), \dots, a(i - 1))$ of a random bumping sequence, so that $a(i)$ has to be chosen in the set $\{\ell \mid a(i - 1) < \ell \leq n + 1, w_\ell > w_{a(i-1)}\}$: z_j is then proportional to the probability that we pick $a(i)$ as the j^{th} largest¹ element in that set. Upon normalizing, we find that the actual probability of picking a specific $a(i)$ can be written

$$Q_{w,a}(i) = \frac{z_{J_w(a(i), w_{a(i-1)})}}{y_{J_w(a(i-1), w_{a(i-1)})}}, \quad (3.2)$$

¹If we replace “largest” by “smallest” here, then the MSJMC does not seem to have a simple stationary distribution anymore.

where we introduce the useful notation

$$y_i = z_1 + \cdots + z_i, \quad (3.3)$$

$$J_w(m, t) = 1 + \#\{\ell \mid m \leq \ell \leq n, w_\ell > t\}, \quad (3.4)$$

for $m \in \{1, \dots, n\}$, $t \in \{1, \dots, T\}$ and $i \in \{2, \dots, k\}$. All in all, the global probability assigned to the bumping sequence a is $\prod_{i=2}^k Q_{w,a}(i)$. Noting that, for all states $w, w' \in St_{n_1, \dots, n_T}$, there is at most one $a \in \mathcal{B}_w$ such that $w' = w^a$, we define the transition probability from w to w' as

$$P_{w,w'} = \begin{cases} \prod_{i=2}^k Q_{w,a}(i) & \text{if } w' = w^a \text{ for some } a \in \mathcal{B}_w, \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

For instance, the transition of Figure 3.1 has probability $z_4/y_5 \times z_2/y_2 \times z_1/y_1$.

Remark 3.2.1. The MJMC [3] is recovered by taking $T = 2$, upon identifying 1's with balls ($\bullet$) and 2's with vacant positions ($\circ$).

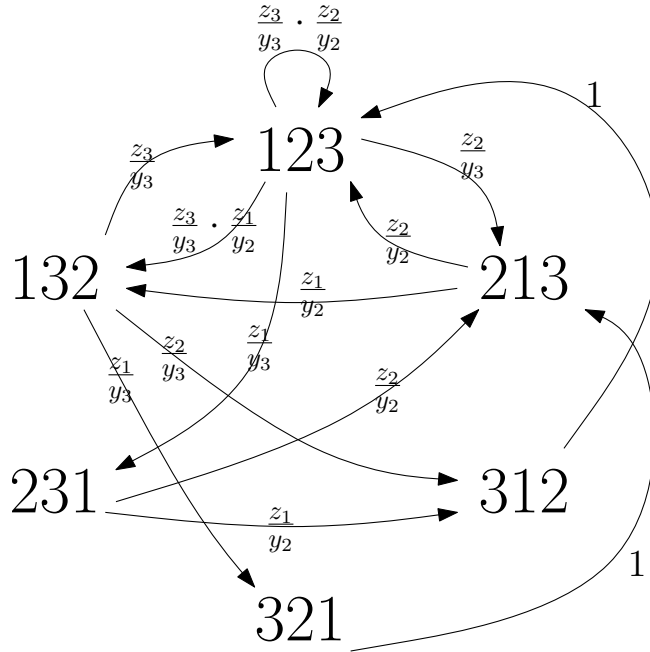


Figure 3.2 – Transition graph of the MSJMC with $T = 3$ and $n_1 = n_2 = n_3 = 1$.

Example 3.2.2. Figure 3.2 illustrates the MSJMC on $St_{1,1,1}$, and the corresponding transition matrix in the basis $\{123, 132, 213, 231, 312, 321\}$ reads

$$\begin{pmatrix} \frac{z_3}{y_3} & \frac{z_2}{y_2} & \frac{z_3}{y_3} & \frac{z_1}{y_2} & \frac{z_2}{y_3} & \frac{z_1}{y_3} & 0 & 0 \\ \frac{z_3}{y_3} & 0 & 0 & 0 & \frac{z_2}{y_3} & \frac{z_1}{y_3} & 0 & 0 \\ \frac{z_2}{y_2} & \frac{z_1}{y_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{z_2}{y_2} & 0 & \frac{z_1}{y_2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.6)$$

Observe that $(y_1 y_2 y_3, y_1^2 y_3, y_1 y_2^2, y_1^2 y_2, y_1^2 y_2, y_1^3)$ is a left eigenvector with eigenvalue 1, and thus is proportional to the stationary distribution.

Remark 3.2.3. It is not difficult to check that generically (i.e. when all z_i are nonzero), the MSJMC is irreducible and aperiodic, and thus admits a unique stationary distribution π .

Indeed, we can reach any state π by the following procedure. Note that it suffices to reach a cyclic shift of π , since we can always throw balls to the rightmost position. Start by positioning the heaviest balls, that is the balls labelled 1 as is done in π . This can be done by the irreducibility of the unlabelled chain, studied in [3]. Assume now by induction that balls labelled $1, \dots, i$ have been positioned as in π . Magnus will now throw all balls labelled $1, \dots, i$ to the rightmost position and balls labelled $i+1$ to the positions given by π . Those positions will be occupied by lighter balls since all heavier balls already are sorted according to π . Those lighter balls can bounce to anywhere, the rightmost position for instance. After T transitions all balls labelled $i+1$ have been sorted according to π . By induction we are done.

Furthermore, the state $1^{n_1} 2^{n_2} \dots T^{n_T}$ can be sent to itself through the bumping sequence $(1, n_1 + 1, n_1 + n_2 + 1, \dots, n + 1)$, which proves the aperiodicity of the model.

Our main result for this section is an explicit expression for π .

Theorem 3.2.4. *The stationary probability of $w \in St_{n_1, \dots, n_T}$ is given by*

$$\pi(w) = \frac{1}{Z} \prod_{i=1}^n y_{E_w(i)} \quad (3.7)$$

with the notation

$$E_w(i) = 1 + \#\{j | i \leq j \leq n, w_j > w_i\} = J_w(i, w_i) \quad (3.8)$$

where the normalization factor Z reads

$$Z = \prod_{i=1}^T h_{n_i}(y_1, \dots, y_{n-n_1-\dots-n_i+1}) \quad (3.9)$$

with h_ℓ the complete homogeneous symmetric polynomial of degree ℓ .

Returning again to the example $w = 132132$, we have $\pi(w) = y_1^3 y_2 y_3 y_5$. According to the general lumping strategy outlined in section 1.2.3, Theorem 3.2.4 is proved in Section 3.2.2 by introducing a suitable enriched Markov chain.

3.2.2 The enriched Markov chain

The first idea to define the enriched Markov chain comes from expanding the product on the right-hand side of (3.7) using the definition (3.3) of the y_j 's, resulting in a sum of monomials in the z_j 's which is naturally indexed by the set of sequences $v = v_1 \cdots v_n$ of positive integers such that $v_i \leq E_w(i)$ (with E as defined in (3.8)) for all $i \in \{1, \dots, n\}$. Let us call such v an *auxiliary word* for w . This suggests that we can define an enriched state as a pair (w, v) where $w \in St_{n_1, \dots, n_T}$ and v is an auxiliary word for w . We denote by $\mathcal{S}_{n_1, \dots, n_T}$ the set of enriched states.

The second idea, needed to define the transitions, is to use the auxiliary word to “record” some information about the past, in such a way that all transitions leading to a given enriched state have the same probability (this will be a key ingredient in the proof of Theorem 3.2.5 below). More precisely, given an enriched state (w, v) , we consider as before a bumping sequence $a \in \mathcal{B}_w$, and we define the resulting enriched state $(w, v)^a = (w', v')$ by updating of course the basic state as before, i.e. we set $w' = w^a$ as in (3.1), while we update the auxiliary word as

$$v'_i = \begin{cases} E_{w'}(i) & \text{if } i = a(\ell) - 1 \text{ for some } \ell, \\ v_{i+1} & \text{otherwise.} \end{cases} \quad (3.10)$$

For instance, in our running example $w = 132132$ and $a = (1, 3, 5, 7)$, for $v = 412211$ we have $v' = 142211$. We may think of the auxiliary word as “labels” carried by the balls, that are modified (maximized) for the bumped

balls and preserved otherwise. The transition probability from (w, v) to (w', v') is as before

$$\tilde{P}_{(w,v),(w',v')} = \begin{cases} \prod_{i=2}^k Q_{w,a}(i) & \text{if } (w', v') = (w, v)^a \text{ for some } a \in \mathcal{B}_w, \\ 0 & \text{otherwise.} \end{cases} \quad (3.11)$$

It is clear that the enriched chain projects to the MSJMC. Indeed, we define an equivalence relation over $\mathcal{S}_{n_1, \dots, n_T}$ by simply “forgetting” the auxiliary word, so that the equivalence classes may be identified with $St_{n_1, \dots, n_T}$ (note that 1^n is a valid auxiliary word for any element of $St_{n_1, \dots, n_T}$). The lumping condition (1.1) is trivially satisfied, since we have $\tilde{P}_{(w,v),(w,v)^a} = P_{w,w^a}$ for all (w, v) in $\mathcal{S}_{n_1, \dots, n_T}$ and a in $\mathcal{B}_w$, and $\tilde{P}_{(w,v),(w^a,v')} = 0$ whenever $(w^a, v') \neq (w, v)^a$.

Theorem 3.2.5. *The stationary distribution of (w, v) in $\mathcal{S}_{n_1, \dots, n_T}$ for the enriched chain is*

$$\tilde{\pi}(w, v) = \frac{1}{Z} \prod_{i=1}^n z_{v_i} \quad (3.12)$$

where Z is the normalization factor.

Proof. We have to check that, for all $(w', v') \in \mathcal{S}_{n_1, \dots, n_T}$, we have

$$\sum_{(w,v) \in \mathcal{S}_{n_1, \dots, n_T}} \tilde{P}_{(w,v),(w',v')} \tilde{\pi}(w, v) = \tilde{\pi}(w', v'), \quad (3.13)$$

which is done by characterizing the possible predecessors of (w', v') . Let (w, v) be such that $(w', v') = (w, v)^a$ for some bumping sequence $a \in \mathcal{B}_w$. We will show in particular that a and w are uniquely determined from the data of (w', v') hence, as claimed above, all transitions to (w', v') have the same probability.

We start by explaining how to recover the bumping sequence $a = (a(1), \dots, a(k))$ or, more precisely, its set of values $A = \{a(1), \dots, a(k)\}$. Recall that 1 and $n+1$ belong to A by definition. We claim that $j \in \{2, \dots, n\}$ belongs to A if and only if the following two conditions hold:

(i) $v'_{j-1} = E_{w'}(j-1)$,

(ii) $w'_{j-1} < w'_{j'-1}$ where j' is the smallest element of $A \cap \{j+1, \dots, n+1\}$.

Indeed, these two conditions are clearly necessary: (i) by (3.10), and (ii) by (3.1) and the requirement that $w_j < w_{j'}$ when $j < j'$ are both in the bumping sequence. Conversely, assume that $j \notin A$, so that $w_j = w'_{j-1}$ and $v_j = v'_{j-1}$. By the definition of the MSJMC transitions, the subword $w'_j \cdots w'_n$ is a permutation of $w_{j+1} \cdots w_n w'_{j'-1}$. Hence, recalling (3.4), $E_{w'}(j-1) - E_w(j)$ is equal to 1 if (ii) holds and to 0 otherwise. If (i) holds, we have $E_{w'}(j-1) = v'_{j-1} = v_j \leq E_w(j)$, hence (ii) cannot hold. This completes the proof of our claim, which fully determines A (hence a) by reverse induction.

Once we have recovered a , it is clear that w is uniquely determined, while we have $v_j = v'_{j-1}$ for $j \notin A$. All predecessors of (w', v') are then obtained by picking, for each $j \in A \setminus \{n+1\}$, v_j an arbitrary integer between 1 and $E_w(j)$. This shows that

$$\sum_{v:(w',v')=(w,v)^a} \tilde{\pi}(w,v) = \frac{1}{Z} \prod_{j \notin A} z_{v'_{j-1}} \prod_{j \in A \setminus \{n+1\}} y_{E_w(j)}. \quad (3.14)$$

The last observation we need is that

$$J_w(a(i), w_{a(i)-1}) = J_{w'}(a(i), w'_{a(i)-1}) = E_{w'}(a(i) - 1) = v'_{a(i)-1} \quad (3.15)$$

for all $i \in \{2, \dots, k\}$, since $w'_{a(i)} \cdots w'_n$ is a permutation of $w_{a(i)} \cdots w_n$ and since $w_{a(i)-1} = w'_{a(i)-1}$. By (3.2) and (3.5) we find that, for any predecessor (w, v) of (w', v') ,

$$\tilde{P}_{(w,v),(w',v')} = \frac{\prod_{j \in A \setminus \{1\}} z_{v'_{j-1}}}{\prod_{j \in A \setminus \{n+1\}} y_{E_w(j)}}. \quad (3.16)$$

Combined with (3.14), the desired stationarity condition (3.13) follows. $\square$

Proof of Theorem 3.2.4. The expression (3.7) is immediately obtained by applying the general lumping expression (1.2) for the stationary state, Theorem 3.2.5 and the definition of enriched states. It remains to check the expression (3.9), which we do by induction on T . Let $\phi(w) = \prod_{i=1}^n y_{E_w(i)}$, so that Z is the sum of $\phi(w)$ over all $w \in St_{n_1, \dots, n_T}$. The expression (3.9) holds for $T = 0$, as $Z = \phi(\epsilon) = 1$ where ϵ is the empty word. For $T \geq 1$, let w be a word in $St_{n_1, \dots, n_T}$, and let $\hat{w} \in St_{n_2, \dots, n_T}$ be the word obtained by removing all occurrences of 1 in w , and shifting all remaining letters down by 1. Denote by $i_1 > \dots > i_{n_1}$ the positions of 1's in w , and let $j_\ell = n + 2 - i_\ell - \ell$, so that $1 \leq j_1 \leq \dots \leq j_{n_1} \leq n - n_1 + 1$. The mapping $w \mapsto (\hat{w}, (j_1, \dots, j_{n_1}))$ is

bijjective, and it is not difficult to see from the definition (3.8) of E that

$$\phi(w) = \phi(\hat{w}) \prod_{\ell=1}^{n_1} y_{j_\ell}. \quad (3.17)$$

Summing the product on the right-hand side over all sequences $(j_1, \dots, j_{n_1})$ yields the complete homogeneous symmetric polynomial $h_{n_1}(y_1, \dots, y_{n-n_1+1})$, and (3.9) follows by induction. $\square$

3.3 Multispecies juggling with fluctuating types

Our goal is here to introduce multispecies generalizations of the add-drop and annihilation models developed in [41, 3]. Both models have the same state space and the same transition graph, but different transition probabilities. The state space is St_n^T , the set of words of length n on the alphabet $\mathcal{A} = \{1, \dots, T\}$. The number of balls of each type is not fixed anymore and thus there are T^n possible states. The transitions are similar to the ones in the MSJMC, except that the type of the ball Magnus throws is independent of the type of the ball he just caught. This ball then initiates a bumping sequence as defined before. More precisely, starting with a state $w = w_1 \cdots w_n \in St_n^T$, we let $w^- = w_2 \cdots w_n$. Transitions involve replacing the first letter of w by an arbitrary $j \in \mathcal{A}$, resulting in the intermediate state jw^- , then applying a bumping sequence $a \in \mathcal{B}_{jw^-}$, resulting in the final state $(jw^-)^a$, where $(\cdot)^a$ is defined as in (3.1). Defining transitions probabilities requires specifying how we pick j and a . The multispecies add-drop and annihilation models differ in the way that we pick the new ball of type j and the position $a(2)$ where it is inserted, while the subsequent elements $a(3), \dots, a(k)$ of the bumping sequence are then chosen in the same way as for the MSJMC. Figure 3.3 shows all allowed transitions for St_2^3 .

Remark 3.3.1. Both chains are irreducible, since a state $w_1 \cdots w_n$ can be reached from any state in n steps by just putting a w_i in the rightmost position at the i -th step. The models are also aperiodic since the state 1^n can be sent to itself by putting a 1 at the rightmost position. This remark also applies to the overwriting model, described later in Section 3.3.3.

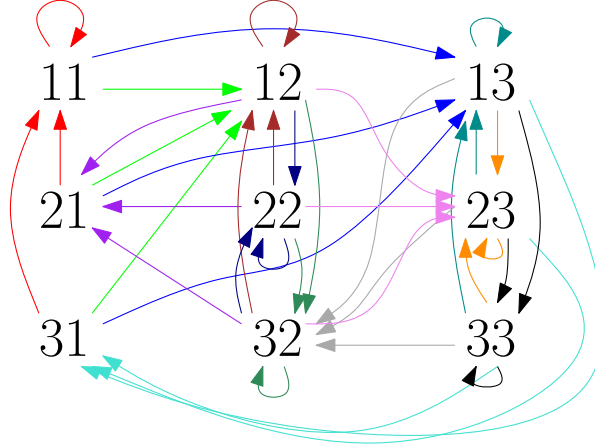


Figure 3.3 – The basic transition graph on St_2^3 . Note that the first letter has no effect on which states can be reached.

3.3.1 Add-drop model

In the add-drop model, choosing a ball of type j and sending it to the ℓ -th available position from the right is done with probability proportional to $c_j z_\ell$ where, in addition to the previous parameters $z_1, z_2, \dots$, we introduce new nonnegative real parameters $c_1, \dots, c_T$ that can be interpreted as “activities” for each type of ball. Because lighter balls can be inserted in fewer possible positions, the actual probability of choosing j and ℓ has to be normalized, and reads $c_j z_\ell / (\sum_{t=1}^T c_t y_{J_w(2,t)})$ where w is the initial state and where we use the same notations (3.3) and (3.4) as before (note that $J_w(m, t) = J_{j_w-}(m, t)$ for all $m > 1$). As the position where the new ball is inserted is $a(2) > 1$, saying that it is the ℓ -th available position from the right means that $\ell = J_w(a(2), j)$. As described above, subsequent elements $a(3), \dots, a(k)$ of the bumping sequence a are chosen in the same way as for the MSJMC, so that the global probability of picking a new ball of type j and a bumping sequence $a \in \mathcal{B}_{j_w-}$ is

$$p_w(j, a) = \frac{c_j z_{J_w(a(2), j)}}{\sum_{t=1}^T c_t y_{J_w(2, t)}} \prod_{i=3}^k Q_{w, a}(i) \quad (3.18)$$

where we recall the notation (3.2). The *multispecies add-drop juggling Markov chain* is then the Markov chain on the state space St_n^T for which the transition

probability from w to w' reads

$$P_{w,w'} = \begin{cases} p_w(j, a) & \text{if } w' = (jw^-)^a \text{ for some } j \in \mathcal{A} \text{ and } a \in \mathcal{B}_{jw^-}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.19)$$

Note that we recover the add-drop juggling model [3] when we set $T = 2$.

Example 3.3.2. The transition matrix of the multispecies add-drop Markov chain on the state space St_2^3 in the ordered basis $\{11, 21, 31, 12, 22, 32, 13, 23, 33\}$ reads

$$\begin{pmatrix} \frac{c_1 z_1}{\lambda_1} & 0 & 0 & \frac{c_2 z_1}{\lambda_1} & 0 & 0 & \frac{c_3 z_1}{\lambda_1} & 0 & 0 \\ \frac{c_1 z_1}{\lambda_1} & 0 & 0 & \frac{c_2 z_1}{\lambda_1} & 0 & 0 & \frac{c_3 z_1}{\lambda_1} & 0 & 0 \\ \frac{c_1 z_1}{\lambda_1} & 0 & 0 & \frac{c_2 z_1}{\lambda_1} & 0 & 0 & \frac{c_3 z_1}{\lambda_1} & 0 & 0 \\ 0 & \frac{c_1 z_1}{\lambda_2} & 0 & \frac{c_1 z_2}{\lambda_2} & \frac{c_2 z_1}{\lambda_2} & 0 & 0 & \frac{c_3 z_1}{\lambda_2} & 0 \\ 0 & \frac{c_1 z_1}{\lambda_2} & 0 & \frac{c_1 z_2}{\lambda_2} & \frac{c_2 z_1}{\lambda_2} & 0 & 0 & \frac{c_3 z_1}{\lambda_2} & 0 \\ 0 & \frac{c_1 z_1}{\lambda_2} & 0 & \frac{c_1 z_2}{\lambda_2} & \frac{c_2 z_1}{\lambda_2} & 0 & 0 & \frac{c_3 z_1}{\lambda_2} & 0 \\ 0 & 0 & \frac{c_1 z_1}{\lambda_3} & 0 & 0 & \frac{c_2 z_1}{\lambda_3} & \frac{c_1 z_2}{\lambda_3} & \frac{c_2 z_2}{\lambda_3} & \frac{c_3 z_1}{\lambda_3} \\ 0 & 0 & \frac{c_1 z_1}{\lambda_3} & 0 & 0 & \frac{c_2 z_1}{\lambda_3} & \frac{c_1 z_2}{\lambda_3} & \frac{c_2 z_2}{\lambda_3} & \frac{c_3 z_1}{\lambda_3} \\ 0 & 0 & \frac{c_1 z_1}{\lambda_3} & 0 & 0 & \frac{c_2 z_1}{\lambda_3} & \frac{c_1 z_2}{\lambda_3} & \frac{c_2 z_2}{\lambda_3} & \frac{c_3 z_1}{\lambda_3} \end{pmatrix} \quad (3.20)$$

with the notation $\lambda_1 = (c_1 + c_2 + c_3)y_1$, $\lambda_2 = c_1 y_2 + (c_2 + c_3)y_1$ and $\lambda_3 = (c_1 + c_2)y_2 + c_3 y_1$. One can check that $(c_1^2 y_1^2, c_1 c_2 y_1^2, c_1 c_3 y_1^2, c_1 c_2 y_1 y_2, c_2^2 y_1^2, c_2 c_3 y_1^2, c_1 c_3 y_1 y_2, c_2 c_3 y_1 y_2, c_3^2 y_1^2)$ is a left eigenvector for the eigenvalue 1.

Theorem 3.3.3. *The stationary probability of $w = w_1 \cdots w_n \in St_n^T$ for the add-drop model is given by*

$$\pi(w) = \frac{1}{Z} \prod_{i=1}^n c_{w_i} y_{E_w(i)}, \quad (3.21)$$

where the normalization factor Z reads

$$Z = \sum_{n_1 + \cdots + n_T = n} \left(c_1^{n_1} \cdots c_T^{n_T} \prod_{i=1}^T h_{n_i}(y_1, \dots, y_{n-n_1-\cdots-n_{i+1}}) \right) \quad (3.22)$$

with h_ℓ the complete homogeneous symmetric polynomial of degree ℓ .

Proof. We will again follow the strategy described in Section 1.2.3. We consider the enriched Markov chain whose state space is the set $\mathcal{S}_n^T$ of pairs of words (w, v) with $w \in St_n^T$ and $v = v_1 \cdots v_n$ such that $v_i \leq E_w(i)$ for all

$i \in \{1, \dots, n\}$. Given $(w, v) \in \mathcal{S}_n^T$, $j \in \mathcal{A}$ and $a \in \mathcal{B}_{jw^-}$, we define the resulting enriched state $(w, v)_j^a = (w', v')$ by setting $w' = (jw^-)^a$,

$$v'_i = \begin{cases} E_{w'}(i) & \text{if } i = a(l) - 1 \text{ for some } l, \\ v_{i+1} & \text{otherwise,} \end{cases} \quad (3.23)$$

and the transition probabilities are of course given by :

$$\tilde{P}_{(w,v),(w',v')} = \begin{cases} p_w(j, a) & \text{if } (w', v') = (w, v)_j^a \text{ for some } j \in \mathcal{A} \text{ and } a \in \mathcal{B}_{jw^-}, \\ 0 & \text{otherwise,} \end{cases} \quad (3.24)$$

with p as defined in (3.18).

We will now show that the stationary probability of $(w, v) \in \mathcal{S}_n^T$ for the enriched add-drop model is given by

$$\tilde{\pi}(w, v) = \frac{1}{Z} \prod_{i=1}^n c_{w_i} z_{v_i}, \quad (3.25)$$

which will give us equation (3.21) by lumping.

We have to check that, for all $(w', v') \in \mathcal{S}_n^T$, we have

$$\sum_{(w,v) \in \mathcal{S}_n^T} \tilde{P}_{(w,v),(w',v')} \tilde{\pi}(w, v) = \tilde{\pi}(w', v'). \quad (3.26)$$

Let (w', v') be a state in $\mathcal{S}_n^T$. For a given (w', v') we can deduce most of a possible predecessor (w, v) , which allows to describe precisely the set of predecessors. As in the proof of Theorem 3.2.5 we can first deduce the bumping sequence a , then the type j of the added ball. This means that $w_2, \dots, w_n$ and $v_i, i \notin A$ are uniquely determined. Recall that for $a = (a(1), \dots, a(k))$ we defined $A = \{a(1), \dots, a(k)\}$. Let $W = jw_2 \cdots w_n$. We have :

$$\sum_{(w,v):(w',v')=(w,v)_j^a} \tilde{\pi}(w, v) = \frac{1}{Z} \sum_{i=1}^T c_i y_{J_W(2,i)} \prod_{\ell \notin A} z_{v'_{\ell-1}} \prod_{\ell \in A \setminus \{1, n+1\}} y_{E_W(\ell)}. \quad (3.27)$$

Furthermore, by observing that $J_w(a(i), w_{a(i-1)}) = v'_{\ell-1}$ (as in Equation (3.15)) we have, for all (w, v) such that $(w, v)_j^a = (w', v')$,

$$\tilde{P}_{(w,v),(w',v')} = \frac{c_j z_{J_W(a(2),j)} \prod_{\ell \in A \setminus \{1, a(2)\}} z_{v'_{\ell-1}}}{\sum_{t \in \{1 \dots T\}} c_t y_{J_W(2,t)} \prod_{\ell \in A \setminus \{1, n+1\}} y_{E_W(\ell)}} \quad (3.28)$$

Combined with (3.27), the desired stationarity condition (3.26) follows. $\square$

3.3.2 Annihilation model

In this section, we assume that hopping parameters are probabilities, $z_1 + \dots + z_{n+1} = 1$. In this model, we consider that Magnus first tries to send a ball of type 1. He chooses $\ell \in \{1, \dots, n+1\}$ with probability z_ℓ , and tries to send the ball at the ℓ -th available position (counted from the right as before). If ℓ is a valid position (that is, it is not larger than the number of available positions for the 1), there is a bumping sequence whose subsequent elements are drawn in the same way as for the MSJMC. Otherwise, he tries instead to send a 2 according to the same procedure, etc. In the end, if he did not manage to send any ball of type in $\{1, \dots, T-1\}$, he just sends a T to the rightmost position. Note that failing to send a ball of type t for an initial state w is done with probability

$$1 - y_{J_w(2,t)}. \quad (3.29)$$

Globally, the probability of picking a new ball of type j and a bumping sequence $a \in \mathcal{B}_{jw^-}$ reads

$$q_w(j, a) = \begin{cases} z_{J_w(a(2),j)} \prod_{t=1}^{j-1} (1 - y_{J_w(2,t)}) \prod_{i=3}^k Q_{w,a}(i) & \text{if } j < T, \\ \prod_{t=1}^{T-1} (1 - y_{J_w(2,t)}) & \text{if } j = T, \end{cases} \quad (3.30)$$

note that $j = T$ implies $a = (1, n+1)$. The transition probabilities of the *multispecies annihilation juggling Markov chain* are obtained by replacing $p_w(j, a)$ with $q_w(j, a)$ in (3.19). Note that we recover the annihilation juggling model [3] when we set $T = 2$.

Example 3.3.4. The transition matrix of the multispecies annihilation Markov chain on the set space St_2^3 in the ordered basis $\{11, 21, 31, 12, 22, 32, 13, 23, 33\}$

reads

$$\begin{pmatrix} z_1 & 0 & 0 & z_1(z_2 + z_3) & 0 & 0 & (z_2 + z_3)^2 & 0 & 0 \\ z_1 & 0 & 0 & z_1(z_2 + z_3) & 0 & 0 & (z_2 + z_3)^2 & 0 & 0 \\ z_1 & 0 & 0 & z_1(z_2 + z_3) & 0 & 0 & (z_2 + z_3)^2 & 0 & 0 \\ 0 & z_1 & 0 & z_2 & z_1 z_3 & 0 & 0 & (z_2 + z_3)z_3 & 0 \\ 0 & z_1 & 0 & z_2 & z_1 z_3 & 0 & 0 & (z_2 + z_3)z_3 & 0 \\ 0 & z_1 & 0 & z_2 & z_1 z_3 & 0 & 0 & (z_2 + z_3)z_3 & 0 \\ 0 & 0 & z_1 & 0 & 0 & z_1 z_3 & z_2 & z_2 z_3 & z_3^2 \\ 0 & 0 & z_1 & 0 & 0 & z_1 z_3 & z_2 & z_2 z_3 & z_3^2 \\ 0 & 0 & z_1 & 0 & 0 & z_1 z_3 & z_2 & z_2 z_3 & z_3^2 \end{pmatrix} \quad (3.31)$$

Note that $(z_1^2, z_1^2(z_2 + z_3), z_1(z_2 + z_3)^2, z_1(z_1 + z_2)(z_2 + z_3), z_1^2 z_3(z_2 + z_3), z_1 z_3(z_2 + z_3)^2, (z_1 + z_2)(z_2 + z_3)^2, z_3(z_1 + z_2)(z_2 + z_3)^2, z_3^2(z_2 + z_3)^2)$ is a left eigenvector for the eigenvalue 1.

Theorem 3.3.5. *The stationary probability of $w = w_1 \cdots w_n \in St_n^T$ for the annihilation model is given by*

$$\pi(w) = \left(\prod_{i=1, w_i < T}^n y_{E_w(i)} \right) \left(\prod_{\ell=2}^T \prod_{p=1}^{\#\{m | w_m \geq \ell\}} (1 - y_p) \right). \quad (3.32)$$

Moreover, here no normalization factor is needed as

$$\sum_{w \in St_n^T} \pi(w) = (z_1 + \cdots + z_{n+1})^{n(T-1)} = 1. \quad (3.33)$$

The stationary probabilities of enriched states are no longer monomials in the z_i 's, which suggest that a further enrichment is possible as already observed for the case $T = 2$ in Section 4.2 of [3].

Proof. The theorem is proved by enriching the chain as before. We again use the state space $\mathcal{S}_n^T$ for the enriched multispecies annihilation Markov chain, just as for the enriched multispecies add-drop Markov chain in Section 3.3.1. The transitions are defined in the same way too, only the transition probabilities change :

$$\tilde{P}_{(w,v),(w',v')} = \begin{cases} q_w(j, a) & \text{if } (w', v') = (w, v)_j^a \text{ for some } j \in \mathcal{A} \text{ and } a \in \mathcal{B}_{jw-}, \\ 0 & \text{otherwise,} \end{cases} \quad (3.34)$$

with q as defined in (3.30). We will now show that the stationary probability of $(w, v) \in \mathcal{S}_n^T$, $w = w_1 \cdots w_n$, $v = v_1 \cdots v_n$ is given by

$$\tilde{\pi}(w, v) = \prod_{i=1, w_i < T}^n z_{v_i} \prod_{\ell=2}^T \prod_{p=1}^{\#\{m | w_m \geq \ell\}} (1 - y_p). \quad (3.35)$$

Once we prove this, we will obtain a proof of (3.32) by lumping. To do so, we have to check that $\tilde{\pi}$ satisfies

$$\sum_{(w, v) \in \mathcal{S}_n^T} \tilde{P}_{(w, v), (w', v')} \tilde{\pi}(w, v) = \tilde{\pi}(w', v'). \quad (3.36)$$

Let (w', v') be a state in $\mathcal{S}_n^T$. We do not have a lot of choice in choosing a predecessor (w, v) of (w', v') ; the bumping sequence a , the type j of the added ball, $w_2, \dots, w_n$ and $v_i, i \notin A$, (where as before A is the set of values in a), are uniquely determined (this works exactly as for the proof of Theorem 3.2.5). Let $W = jw_2 \cdots w_n$. We define :

$$\begin{aligned} C &= \prod_{i=1, w'_i < T}^n z_{v'_i} \\ C' &= \begin{cases} z_{J_W(a(2), j)} \frac{\prod_{i \in A \setminus \{1, a(2)\}} z_{v'_{i-1}}}{\prod_{i \in A \setminus \{1, n+1\}} y_{E_W(i)}} & \text{if } j < T, \\ 1 & \text{otherwise,} \end{cases} \\ C'' &= \prod_{\substack{i \in A \setminus \{1, n+1\} \\ w_i < T}} y_{E_W(i)} \prod_{\substack{i \notin A \\ w_i < T}} z_{v_i} \\ D &= \prod_{\ell=2}^T \prod_{p=1}^{\#\{m | w'_m \geq \ell\}} (1 - y_p) \\ D' &= \prod_{i=1}^{j-1} (1 - y_{J_W(2, i)}) \\ D'' &= \prod_{\ell=2}^T \prod_{p=1}^{\#\{m \geq 2 | w_m \geq \ell\}} (1 - y_p) \\ K &= \sum_{w_1=1}^T y_{E_{w_1 w_2 \cdots w_n}(1)} \prod_{\ell=2}^{w_1} (1 - y_{\#\{m \geq 2 | w_m \geq \ell\} + 1}) \end{aligned}$$

First, we note that $C'C'' = C$. Recall that $j = T$ implies $A = \{1, n+1\}$. Secondly, we have $D'D'' = D$, since $J_W(2, i) = \#\{m \geq 2 | w_m > i\} + 1$. For all (w, v) such that $\tilde{P}_{(w,v),(w',v')} \neq 0$, we have

$$\tilde{P}_{(w,v),(w',v')} = C'D', \quad (3.37)$$

where the C' follows from (3.15) and (3.2) and D' follows from (3.29). Note that the transfer probability does not depend on the choice of (w, v) . Now,

$$\sum_{(w,v):(w',v')=(w,v)_j^a} \tilde{\pi}(w, v) = C''D''K, \quad (3.38)$$

where the $i = 1$ case has been removed from C'' and collected in K . Furthermore, $K = 1$, since it is telescopic sum using that $E_{w_1 w_2 \dots w_n}(1) = \#\{m \geq 2 | w_m > w_1\} + 1$, (start with adding the cases $w_1 = T$ and $w_1 = T - 1$ together). The desired condition (3.36) follows. $\square$

3.3.3 Overwriting model

The aim is here to describe a multispecies generalisation of the annihilation model studied in [3] in which the *ultrafast convergence* property holds, namely we want the stationary distribution to be reached in a finite number of steps, independent of the starting distribution. We will show it by using a doubly enriched chain on which the property will be obvious, and use the fact that lumpings preserve the ultrafast convergence.

Model description

The set space St_n^T is, as described in Section 3.3, is the set of words in $\{1, \dots, T\}^n$, and the transition probabilities are dictated by the indeterminates $z_1, \dots, z_{n+1}$ satisfying $z_1 + z_2 + \dots + z_{n+1} = 1$ as in Section 3.3.2.

In the *overwriting model*, each integer gets a chance to overwrite an integer larger than it. More formally, the transitions are described by the following process. Initially, the 1-st letter is erased, and everything is moved to the left by 1 step. Now Magnus first tries to send a ball of type 1; he chooses $i \in \{1, n+1\}$ with probability z_i , and aims for the i^{th} available position, meaning those positions which contain an integer greater than 1 (counting from right as before). If i is greater than the number of available positions

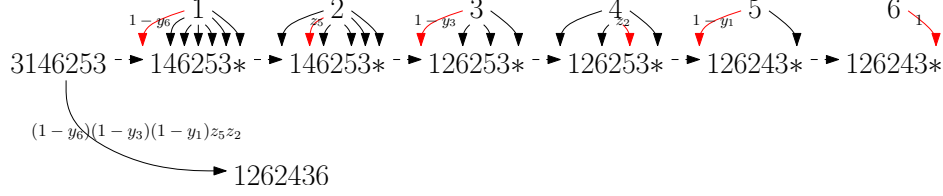


Figure 3.4 – A transition in the overwriting multispecies Markov chain on St_7^6 .

for a 1 to land, Magnus simply fails to send a 1, otherwise the 1 lands at some position, destroying the ball previously occupying it if it isn't the rightmost position. He then tries to send a 2, which can only land in a higher position than the 1 did (any available position if the 1 didn't land), or fail to land. He then tries to send a 3, and so on. If, after trying to send a ball of each type in $\{1, \dots, T-1\}$, no ball landed to the rightmost position, he puts a T in that position. Otherwise, the new state is reached as soon as a ball is sent to the rightmost position.

For $w \in St_n^T$, we define an *overwriting sequence* $B = ((b_1, t_1), \dots, (b_k, t_k))$ for w as follows : $1 < b_1 < b_2 < \dots < b_k = n+1$, $1 \leq t_1 < t_2 < \dots < t_k \leq T$, and for all $i = 1, \dots, k$, $t_i < w_{b_i}$ with, by convention, $w_{n+1} = +\infty$. We denote by $\mathcal{B}_w$ the set of overwriting sequences for w .

For $w \in St_n^T$ and $B \in \mathcal{B}_w$, the state in St_n^T obtained by applying the overwriting sequence B to the word w , denoted w^B is defined as

$$w_i^B = \begin{cases} t_j & \text{if } i = b_j - 1 \text{ for some } j \\ w_{i+1} & \text{otherwise.} \end{cases} \quad (3.39)$$

Note that the probability for Magnus to fail to send a ball of type ℓ during the j -th part of the overwriting is given by

$$1 - y_{J_w(b_{j-1}+1, \ell)} \quad (3.40)$$

with the convention that $b_0 = 1$ and where J is as in (3.4). Note that this returns 1 if a ball has already been sent to the rightmost position. The probability for Magnus to succeed in sending a ball of type t_j to position $b_j - 1$ during the j -th part of the overwriting is given by

$$z_{J_w(b_j, t_j)}. \quad (3.41)$$

Example 3.3.6. In Figure 3.4, we describe the transition from the state 3146253 to the state 1262436 in the state space St_7^6 . The corresponding overwriting sequence is given by $((3, 2), (6, 4), (8, 6))$ and the probability of this transition is $(1 - y_6)z_5(1 - y_3)z_2(1 - y_1)$.

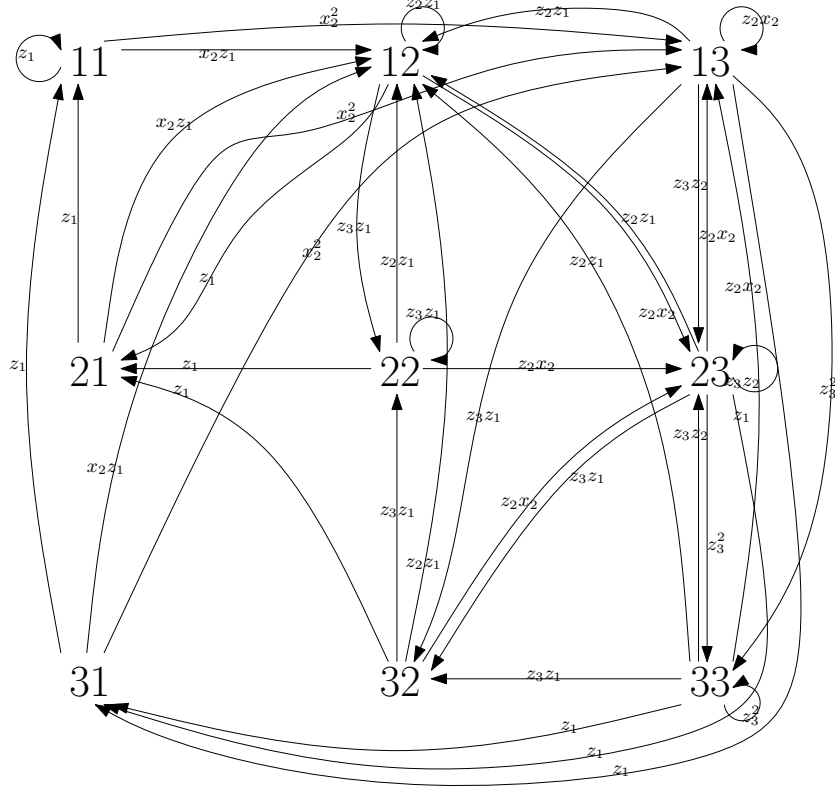


Figure 3.5 – The overwriting multispecies Markov chain on St_2^3 . Here $x_i = 1 - y_{i-1}$.

Given $z_1, \dots, z_{n+1}$ nonnegative real numbers summing to 1, we are now able to define the transition probabilities for the annihilation multispecies Markov chain : for $w, w' \in St_n^T$ if there exists $B \in \mathcal{B}_w$ such that $w' = w^B$, then the transition probability from w to w' reads :

$$\mathcal{P}_{w,w'} = \prod_{j=1}^k \left(\prod_{\ell=t_{j-1}+1}^{t_j-1} (1 - y_{J_w(b_{j-1}+1, \ell)}) \right) \prod_{j|t_j \neq T} z_{J_w(b_j, t_j)}, \quad (3.42)$$

and it reads 0 otherwise.

Example 3.3.7. The transitions of the overwriting Markov chain on St_2^3 are represented in Figure 3.5.

The stationary distribution of the overwriting chain does not have a simple formula, unlike the add-drop and annihilation multispecies variants. However, we do obtain an indirect formula in terms of the enriched chain, which we state as Corollary 3.3.11 in the next section. It turns out that the occupation probability for the last site and the joint occupation distributions at the last two sites have particularly simple expressions, which is what we state next.

Theorem 3.3.8. *The stationary probability of having a j at the last site is given by*

$$\mathbb{P}(w_n = j) = \begin{cases} z_1(1 - z_1)^{j-1} & \text{if } j < T, \\ (1 - z_1)^{T-1} & \text{if } j = T. \end{cases}$$

The joint probability of having an i at the $(n - 1)$ -th site and a j at the n -th site is given by

$$\mathbb{P}(w_{n-1} = i, w_n = j) = (1 - z_1)^{\max(i,j)-1} (1 - y_2)^{\min(i,j)-1} \times \begin{cases} z_1 y_2 & \text{if } i < j < T, \\ z_1^2 & \text{if } j \leq i < T, \\ y_2 & \text{if } i < j = T, \\ z_1 & \text{if } j < i = T, \\ 1 & \text{if } i = j = T. \end{cases}$$

The proof is given in the following section, by constructing an enriched chain, and analysing the transitions therein.

Staircase tableaux enrichment

It is now natural to look at staircase tableaux, as was done for the original juggling model in [22]. The state space for the enriched version of the overwriting chain on St_n^T is the set of Young tableaux of shape $(n, n - 1, \dots, 2, 1)$, with the following conditions on the entries in cells.

1. Entries belong to the set $\{1, \dots, T - 1\}$.
2. Entries appear in increasing order from left to right, and from bottom to top.

3. Empty cells are allowed.

This set of tableaux is denoted by $\mathcal{T}_n^T$. For $V \in \mathcal{T}_n^T$, we will denote by $V_{*,k}$ the k -th column, from the left, of V (which has $n + 1 - k$ cells).

The transitions of the enriched Markov chain are as follows. At each step, the entries in the bottom row are deleted, all remaining entries are moved one step down and one step right, and we add entries to the leftmost column so that the tableaux conditions above still hold. More precisely, we proceed in the following way. We first try to add a 1 by choosing a number k_1 in $\{1, \dots, n + 1\}$ and placing a 1 in the k_1 -th free position (from top to bottom; a position is “free” if and only if there is no 1 in the same row). If k_1 is greater than the number of free positions, no 1 is added. We then similarly try to add a 2 by choosing k_2 in $\{1, \dots, n + 1\}$ and placing a 2 in the k_2 -th free position, a position being free if there is no 1 or 2 in the same row, and having no 1 above it. We continue this way until all numbers between 1 and $T - 1$ have been tried.

For $V \in \mathcal{T}_n^T$, for $i \in \{1, \dots, T\}$ and $k \in \{1, \dots, n\}$, we introduce the useful notation :

$$C_V(i, k) = \begin{cases} z_{1+\#\{\text{cells above entry } i \text{ in } V_{*,k}, \text{ with no entry } j \leq i \text{ to the right}\}}, & \text{if } i \text{ is in } V_{*,k} \\ 1 - y_{\#\{\text{cells in } V_{*,k} \text{ with no entry } j \leq i \text{ in, to the right or on top}\}}, & \text{otherwise.} \end{cases} \quad (3.43)$$

Here we use the convention that $y_0 = 0$.

Example 3.3.9. Figure 3.6 gives the example of a state in $\mathcal{T}_4^4$, and all possible states that it can transition to (with transition probabilities below the corresponding arrows). If we call V the topmost tableau, we have for example $C_V(3, 1) = 1 - y_0 = 1$ and $C_V(2, 2) = z_2$.

The probability of such a transition $V \rightarrow W$ is then given by

$$\mathcal{P}_{V,W} = \begin{cases} 0 & \text{if } W_{i,j} \neq V_{i-1,j-1} \text{ for some } 2 \leq j \leq i \leq n \\ \prod_{i=1}^{T-1} C_W(i, 1) & \text{otherwise.} \end{cases} \quad (3.44)$$

This chain lumps to the overwriting model by the following procedure. Let the rows of the tableaux be numbered from bottom to top. For V be a

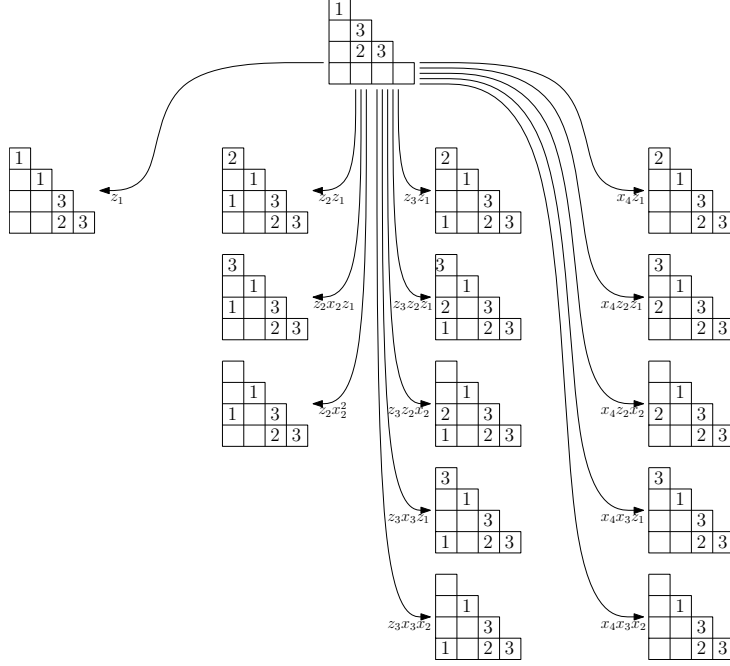


Figure 3.6 – A state and its successors. Here $x_i = 1 - y_{i-1}$.

tableau in $\mathcal{T}_n^T$, we define, for $k \in \{1, \dots, n\}$,

$$a_V(k) = \begin{cases} \text{leftmost entry on the } n - k + 1\text{-th row of } V \\ \text{if this row is not empty} \\ T \\ \text{otherwise.} \end{cases} \quad (3.45)$$

The resulting lumped word $w \in St_n^T$ is then given by

$$w = a(V) := a_V(1) \cdots a_V(n). \quad (3.46)$$

One can check that this procedure satisfies all the conditions for lumping, see Section 1.2.3. We are now in a position to prove Theorem 3.3.8.

Proof of Theorem 3.3.8. The probability of having a j in the last site of $w \in St_n^T$ in the overwriting chain is the same as the probability of having a j in the topmost cell of $V \in \mathcal{T}_n^T$ if $j < T$, or of having nothing in this cell if $j = T$. This means that any number $i < j$ failed to reach the first

available cell, which happens with probability $1 - z_1$ for each one of them, and if $j < T$, j reached that cell, which happens with probability z_1 , which proves the first part of the theorem.

For the second part of the theorem, we will only treat the case $i < j < T$, since all the cases are proved in a similar fashion. The joint probability of having a i in the $(n-1)^{th}$ site and a j in the n^{th} site of $w \in St_n^T$ is the same as the probability of being in one of the two configurations of Figure 3.7.

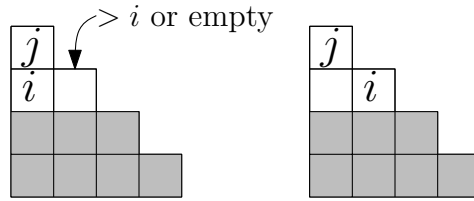


Figure 3.7 – The two possible cases for $i < j < T$.

The transition probability into the first configuration is

$$\begin{aligned}
 & \underbrace{(1 - z_1)^i}_{\text{no } k \leq i \text{ in the topmost cell of the 2nd column}} \times \underbrace{(1 - y_2)^{i-1}}_{\text{no } k < i \text{ in the top 2 cells of the 1st column}} \times \underbrace{z_2}_{i \text{ is in the 2nd cell of the 1st column}} \\
 & \times \underbrace{(1 - z_1)^{j-i-1}}_{\text{no } k \text{ between } i+1 \text{ and } j-1 \text{ in the topmost cell of the 1st column}} \times \underbrace{z_1}_{j \text{ is in the topmost cell of the 1st column}} = (1 - z_1)^{j-1} (1 - y_2)^{i-1} z_1 z_2,
 \end{aligned} \tag{3.47}$$

while the transition probability into the second configuration is

$$\begin{aligned}
 & \underbrace{(1 - z_1)^{i-1}}_{\text{no } k < i \text{ in the topmost cell of the 2nd column}} \times \underbrace{z_1}_{i \text{ is in the topmost cell of the 2nd column}} \times \underbrace{(1 - y_2)^{i-1}}_{\text{no } k < i \text{ in the top 2 cells of the 1st column}} \\
 & \times \underbrace{(1 - z_1)^{j-i}}_{\text{no } k \text{ between } i \text{ and } j-1 \text{ in the topmost cell of the 1st column}} \times \underbrace{z_1}_{j \text{ is in the topmost cell of the 1st column}} = (1 - z_1)^{j-1} (1 - y_2)^{i-1} z_1^2.
 \end{aligned} \tag{3.48}$$

By summing (3.47) and (3.48), we get the desired probability. □

The idea of this proof also hints at why the stationary distribution of the overwriting Markov chain is not of a simple form. However, we will show that the stationary distribution of the enriched Markov chain on staircase tableaux has a particularly nice structure.

Theorem 3.3.10. *The stationary distribution of $V \in \mathcal{T}_n^T$ for the staircase tableaux enriched Markov chain is given by*

$$\Pi(V) = \prod_{k=1}^n \prod_{i=1}^{T-1} C_V(i, k)$$

with the normalisation factor $(z_1 + \cdots + z_{n+1})^{n(T-1)} = 1$.

We will prove this formula by considering an even larger enlargement of the Markov chain on staircase tableaux, analogous to the doubly enriched chain of the single species annihilation model in [3, Definition 4.14]. An immediate corollary of this result is the formula for the stationary distribution of the overwriting Markov chain.

Corollary 3.3.11. *The stationary probability of $w = w_1 \cdots w_n \in St_n^T$ for the overwriting model is given by*

$$\pi(w) = \sum_{\substack{V \in \mathcal{T}_n^T \\ a(V)=w}} \Pi(V). \quad (3.49)$$

where $a(V)$ is defined in (3.46).

The doubly enriched chain

In this section we will construct a generalization of the doubly enriched chain of the single species annihilation model in [3, Definition 4.14], which was a Markov chain on words of length n . In this case, the natural extension of this Markov chain is to matrices. The state space $\tilde{\mathcal{T}}_n^T$ is the set of matrices with $T - 1$ rows and n columns with entries in $\{1, \dots, n + 1\}$.

It is clear there are $(n+1)^{n(T-1)}$ different states. As usual, given $z_1, \dots, z_{n+1}$ nonnegative real numbers summing to 1, the transitions are defined as follows. For $M \in \tilde{\mathcal{T}}_n^T$, all transitions from M are obtained by deleting the last column in M , shifting all the remaining columns to the right by one, and adding an arbitrary column on the left. The probability of this transition

is given by the product of factors z_i for each element i in the resulting first column. More precisely,

$$\tilde{\mathcal{P}}_{M,N} = \begin{cases} \prod_{i=1}^{T-1} z_{N_{i,1}} & \text{if } N_{k,l} = M_{k,l-1} \text{ for all } k \in \{1, \dots, T-1\} \\ & \text{and all } l \in \{2, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.50)$$

Since this is a product of $T-1$ independent copies of the single row Markov chain, it is clear this Markov chain is recurrent. Further, the stationary distribution is given, for $M \in \tilde{\mathcal{T}}_n^T$, by the product

$$\tilde{\Pi}(M) = \prod_{i=1}^{T-1} \prod_{j=1}^n z_{M_{i,j}}. \quad (3.51)$$

with normalisation $(z_1 + \dots + z_n)^{n(T-1)} = 1$.

Remark 3.3.12. The dynamics of the doubly enriched chain guarantee that the stationary distribution is reached after n steps (indeed, the first state has been completely forgotten after n steps), which is the desired ultrafast convergence property. Equivalently, n is a deterministic strong stationary time for this chain.

An immediate consequence of Remark 3.3.12 is a complete description of the spectrum of the transition matrix, given by the following theorem.

Theorem 3.3.13. *Let $n, T \in \mathbb{N}$, and let M be the transition matrix of the doubly enriched chain on $\tilde{\mathcal{T}}_n^T$. The eigenvalues for M are 1 with multiplicity 1 and 0 with multiplicity $n-1$.*

Proof. As stated before, the stationary distribution is reached after n steps. This means that M^n is the matrix with all the rows being the left normalised eigenvector for M (which represents the stationary distribution). Thus, $M^{n+1} = M^n$, and therefore $X^{n+1} - X^n$ is a nullifying polynomial for M . This shows that 1 is an eigenvalue of multiplicity 1 (we already knew that its multiplicity was at least 1) and that 0 is the only other eigenvalue. $\square$

The doubly enriched Markov chain on matrices described above lumps onto the singly-enriched chain on staircase tableaux. Let M be in $\tilde{\mathcal{T}}_n^T$. We

will construct a tableau T associated to M by starting with an empty tableau. We then fill T according to the following pseudocode.

- for k decreasing from n down to 1:
 - for i increasing from 1 to $T - 1$:
 - if $M_{i,k}$ is less than or equal to the number of available positions in the k 'th column of T :
 - insert i in the $M_{i,k}$ 'th position from the top
 - else:
 - do not insert i .

We then set $A(M) := T$. One can check that this algorithm leads to an actual lumping between these two Markov chains, see Section 1.2.3.

Example 3.3.14. Figure 3.8 gives the example of a matrix M in $\tilde{\mathcal{T}}_4^3$ and the tableau T in $\mathcal{T}_4^3$ it lumps onto. This tableau T can then be lumped onto $w = 1243$ in St_4^3 .

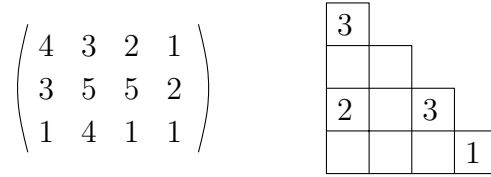


Figure 3.8 – A matrix and the tableau it lumps to.

Remark 3.3.15. Since lumpings preserve the ultrafast convergence property, both the chain on staircase tableaux on $\mathcal{T}_n^T$ and the overwriting Markov chain on St_n^T converge in n steps.

We now prove the formula for the stationary distribution of the Markov chain on staircase tableaux.

Proof of Theorem 3.3.10. We have to check that, for each $V \in \mathcal{T}_n^T$,

$$\sum_{\substack{M \in \mathcal{T}_n^T \\ M \in A^{-1}(V)}} \tilde{\Pi}(M) = \Pi(V)$$

To do so, let $k \in \{1, \dots, n\}$ and $i \in \{1, \dots, T - 1\}$. If i appears in the k^{th} column of V , then every M in $\tilde{\mathcal{T}}_n^T$ projecting to V , must verify

$$M_{i,k} = \#\{\text{cells above entry } i \text{ in } V_{*,k}, \text{ with no entry } j \leq i \text{ to the right}\}$$

because of how the projection is defined. Similarly if i does not appear in the k^{th} column of V , then $M_{i,k}$ must be greater than

$$\#\{\text{cells in } V_{*,k} \text{ with no entry } j \leq i \text{ in, to the right or atop of them}\}.$$

In both cases, summing z_ℓ over all the possible values ℓ of $M_{i,k}$ gives us $C_V(i, k)$, and thus proves the result. $\square$

3.4 Several jugglers

We now consider a completely different generalization of Warrington's model [41]. Instead of a multivariate or multispecies generalization, we will now consider that there are several jugglers, and that each one of them can send the balls he catches to any other juggler. We model this situation as follows. For r, c, ℓ nonnegative integers such that $\ell \leq rc$, we denote by $S_{r \times c}$ the set of rectangular arrays with r rows and c columns, such that each cell either is empty or contains a ball, and by $S_{r \times c, \ell} \subset S_{r \times c}$ the subset of arrays containing exactly ℓ balls. Each column represents the balls that are sent to a specific juggler. For A and B two arrays in $S_{r \times c}$, we denote A^- the array obtained by removing all the balls in the lowest row, and moving all the other balls down one row (hence the topmost row of A^- is always empty). We write $A \subset B$ if all the balls in A are also in B . For i between 1 and r , we denote by A_i the number of balls in the i -th row (rows are numbered from top to bottom).

The several jugglers Markov chain is the Markov chain on the state space $S_{r \times c, \ell}$ whose transition probabilities read, for $A, B \in S_{r \times c, \ell}$,

$$\mathcal{P}_{A,B} = \begin{cases} \frac{1}{\binom{rc-\ell+A_r}{A_r}} & \text{if } A^- \subset B, \\ 0 & \text{otherwise.} \end{cases} \quad (3.52)$$

Here, A_r is the number of balls in the lowest row of A , which is exactly the number of balls the jugglers will have to send back. These A_r balls are reinjected uniformly in the $rc - \ell + A_r$ available positions, under the constraint that no two balls go to the same position. Note that there are no balls reinjected when $A_r = 0$, and $\mathcal{P}_{A,A^-} = 1$ in this case. The irreducibility and aperiodicity of the several jugglers Markov chain are easy to check.

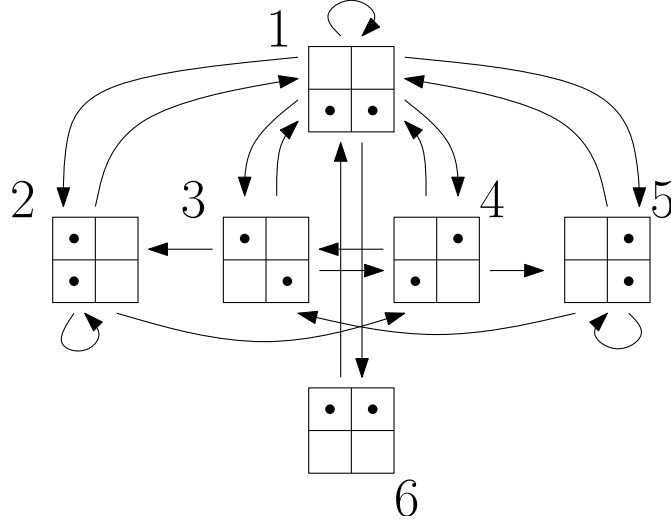


Figure 3.9 – The several jugglers Markov chain on the set space $S_{2 \times 2, 2}$.

Example 3.4.1. The transition matrix of the several jugglers Markov chain on the set space $S_{2 \times 2, 2}$ in the basis ordered $(1, 2, 3, 4, 5, 6)$ in Figure 3.9 reads

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.53)$$

Note that $(6, 3, 3, 3, 3, 1)$ is a left eigenvector for the eigenvalue 1.

Again, we have an explicit expression for the stationary distribution of this Markov chain.

Theorem 3.4.2. *The stationary probability of $A \in S_{r \times c, \ell}$ for the several jugglers Markov chain reads*

$$\pi(A) = \frac{1}{Z_{r \times c, \ell}} \prod_{i=1}^r (ci - A_{<i})_{A_i} \quad (3.54)$$

where $A_{<i} = A_1 + \dots + A_{i-1}$ is the number of balls strictly above row i , $(x)_n = x(x-1)\dots(x-n+1)$ is the Pochhammer symbol and $Z_{r \times c, \ell}$ is the normalization factor.

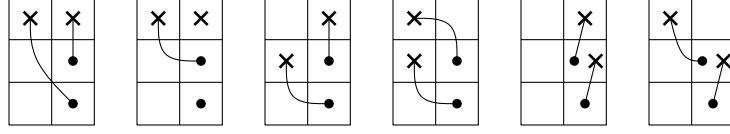


Figure 3.10 – All the states in the enriched chain projecting to the state in $S_{2 \times 2, 2}$ with both balls in the right column.

Remark 3.4.3. We didn't find a simple expression for the normalization factor $Z_{r \times c, \ell}$.

Proof. We introduce an enriched chain as follows. A state in the enriched chain is an $(r+1) \times c$ array with ℓ arcs, each arc going between two cells that are not in the same row. For each arc we mark the top cell with a cross and the bottom cell with a ball. Each cell can contain at most one cross and at most one ball, but could have one ball and one cross belonging to different arcs. The projection to $S_{r \times c, \ell}$ is obtained by simply removing the top row, all arcs and crosses but leaving the balls in place. See Figure 3.10 for all the states of the enriched chain projecting down to the rightmost state in Figure 3.9.

The transitions in the enriched chain are obtained by first moving all arcs, balls and crosses down one row in the array. Secondly, if there were balls in the bottom row, they and the corresponding arcs and crosses are removed. The balls are reinjected uniformly into the array except for the top row and under the condition that no two balls may be in the same cell, just like in the several jugglers Markov chain. For each of these balls an arc is inserted from the ball and up to a cross positioned uniformly in the top row under the condition that no two crosses can be in the same cell. (Alternatively we could define the transitions such that the new crosses in the top row appear in the same columns as the removed balls.)

Now we note that if we run the enriched chain backwards, it will be an identical chain with the roles of balls and crosses exchanged (turned upside down). The number of balls in the bottom row in a state is equal to the number of crosses in the top row for every state it may transition to. Also the number of ways to inject balls is the same as the number of ways of removing crosses. It follows that the number of transitions out of any state is equal to the number of transitions into the same state. Thus the stationary distribution is uniform for the enriched chain.

The uniformity of the enriched chain means that, to evaluate the stationary probability $\pi(A)$ of a state of the several jugglers Markov chain, it suffices to count the number of states in the enriched chain projecting to it. For each ball we can place the cross in any position in a row above with the constraint that no two crosses can be in the same cell. The number of possibilities can be counted row by row: assuming that the crosses corresponding to the balls strictly above row j have been chosen, there remains $cj - A_{<j}$ cells without crosses that may be matched with the A_j balls in row j , hence there are $(cj - A_{<j})_{A_j}$ possible choices for row j . □

Remark 3.4.4. In [27], J.S. Kim studied the model of a juggler with each site being allowed to contain up to a certain number $c > 1$ of balls. This model can be obtained from the several jugglers Markov chain by a lumping.

Chapter 4

Stochastic Sandpile Model

This Chapter is very similar to [34].

4.1 Introduction

The abelian sandpile model (ASM in the rest of the Chapter) was introduced by Bak, Tang and Wiesenfeld [7], as an example of a dynamical system displaying self-organized criticality. A very similar model, known as the chip-firing game, had been introduced by Spencer [38] a little earlier : we consider a pile of N chips in the center of a long path, and at each step, the player moves $\lfloor \frac{N}{2} \rfloor$ to the left and $\lfloor \frac{N}{2} \rfloor$ to the right. He then does the same thing with the two new piles obtained, in whatever order, and so on, until no site contains more than one chip. This game was then refined, allowing the player to move only two chips (one to the left and one to the right) at each step, and starting with any configuration, and then a generalization to simple graphs was studied by Björner, Lovász and Shor in [8]. The ASM is a Markov chain and can be seen as a succession of chipfiring games on a graph, the states being configurations where no vertex can chipfire : at each step, we add a chip (or sand grain) on one of the vertices of the graph, we then proceed as if we were playing a chipfiring game, until no vertex can chipfire (or topple), thus getting to our new state. Many Chapters are dedicated to the mathematical aspects of the ASM, and we will just cite Redig's [36] and Dhar's review [18]. A first stochastic extension, known as the Manna model, introduced in [33], and studied further by Dahr [19], lead to an extension of ASM, called the stochastic sandpile model (SSM from now on), studied by

Chan, Marckert and Selig [12]. In this model, when a vertex topples, it only gives sand grains to some of its neighbors, in such a way that every partial toppling may occur. The authors of [12] studied the recurrent states, and the Lacking polynomial which is very similar to the Tutte polynomial. In his thesis [37], Thomas Selig stated a conjecture :

Conjecture 4.1.1. *Let $G = (V, E)$ be a graph and μ be a probability distribution on V . Then the stationary distribution for the SSM where grain additions are made according to the distribution μ does not depend on μ .*

The aim of our Chapter is to prove this conjecture. We will do it to the more general case where only a subset of the set of partial topplings may occur. In order to do that, we will need to describe the Markov chain as some sort of game, with decks of cards on each site, governing the way those sites will topple. The deck presentation was used by Diaconis and Fulton in [20], in which they describe a game where particles are randomly moved on a set of sites until no site contains more than one particle. We will do it in a slightly more general case than the one described in [12], since we will not suppose that all partial topplings can occur.

4.2 Informal model description

In this section, we will recall some well known results about the abelian sandpile model (ASM), as described for instance in [36], thus giving motivation for the search of similar results in a more general model.

4.2.1 The abelian sandpile model

Quick description

The ASM is a Markov chain operating on a connected graph $G = (V \cup \{s\}, E)$ where each vertex $v \in V$ has a positive counter $\eta(v)$, indicating how many sand grains are on it, and where s is a special vertex called the sink, playing a different role described below. If the number of grains on a vertex $v \in V$ is smaller than or equal to its number $d^G(v)$ of neighbors, v is said to be *stable*, and it is said to be *unstable* otherwise. A *configuration* η of sand grains on G is *stable* if each vertex in V is stable (i.e. $\eta(v) \leq d^G(v)$ for all $v \in V$), and *unstable* otherwise. The state space for the ASM is the set of stable configurations.

At each step of the Markov chain, a sand grain is added at a random vertex $v \in V$ (according to a distribution μ). If this vertex becomes unstable, it then *topples*, giving a sand grain to each of its neighbors, which may in turn become unstable and topple. One of the vertices, called the *sink*, and denoted s , behaves in a different way : it can absorb any number of sand grains and never topples. The topplings go on until we reach a new stable configuration (which will be our new state). The recurrent states of the Markov Chain will be called *recurrent configurations*.

Example 4.2.1. An example of the ASM is given in Figure 4.1. Only the recurrent configurations are being displayed here. The graph is a triangle with one of the three vertices being the sink, and sand grains can be added on both vertices in V . The transition matrix in the basis $(\textcircled{1}, \textcircled{2}, \textcircled{3})$ reads

$$\begin{pmatrix} 0 & \alpha & \beta \\ \beta & 0 & \alpha \\ \alpha & \beta & 0 \end{pmatrix}.$$

Note that $(1, 1, 1)$ is a left eigenvector with eigenvalue $\alpha + \beta = 1$.

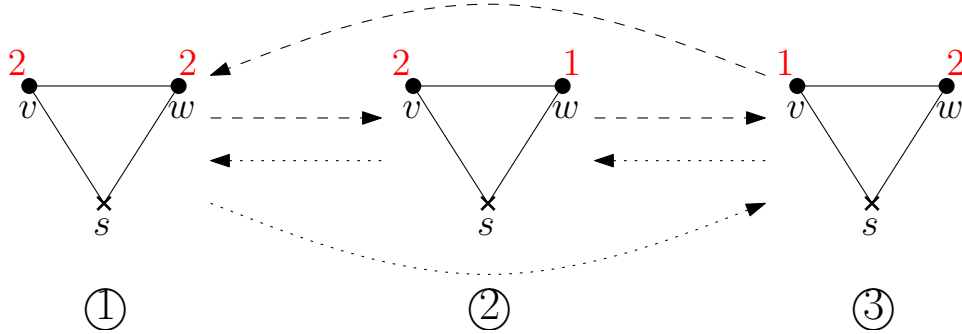


Figure 4.1 – Dashed arrows correspond to transitions where we add a sand grain on v (which happens with probability α), and dotted arrows on w (probability $\beta = 1 - \alpha$).

Some results

There are two important things to notice : first, the fact that the graph is connected ensures that the toppling process will end (if there are too many sand grains, some of them will necessarily reach the sink), and second, the

resulting stable state does not depend on the order in which vertices topple. Those two facts ensure that the Markov chain described above is well defined.

We talk about the *abelian* sandpile model because of the following result which can be found in [36].

Proposition 4.2.2. *For $v \in V$, let A_v be the operator “add a sand grain on v and make topplings until you reach a stable configuration”. Then $\{A_v, v \in V\}$ is a commutative group acting on the set of recurrent configurations.*

As a corollary, we have the following property :

Corollary 4.2.3. *The stationary distribution for the ASM is the uniform distribution on recurrent configurations, and thus it does not depend on how we add sand grains on the graph (it does not depend on the distribution μ).*

4.2.2 The Stochastic Sandpile Model (SSM)

The Bernoulli Stochastic Sandpile Model (BSSM)

This is the model described in [8], and is a generalization of the ASM. Here, randomness not only happens when sand grains are added onto the graph, but topplings are also random. More precisely, we fix $p \in [0, 1]$, and when a vertex is unstable, each neighbor independently has a probability p of receiving a grain from that vertex. The important thing about this model is that given an unstable vertex v and a subset D_v of v 's neighbors, there is a nonzero probability D_v sends a sand grain exactly to the vertices of D_v . We can rephrase the last statement as follows : every partial toppling may occur.

Quick description of the SSM

The model described here and studied in the rest of the Chapter is a generalization of the BSSM in which only a subset of the partial topplings may occur. The Markov chain is still operating on a graph $G = (V \cup s, E)$ with sand grains, but this time for each $v \in V$, we have a probability distribution λ_v on partial topplings of v and a toppling value M_v indicating how many sand grains are needed for v to topple (M_v will be the greatest number of sand grains v can give away when toppling). Each time v topples, it sends sand grains to only a subset of its neighbors, chosen at random according to λ_v . One can notice that the graph structure becomes irrelevant, and we can

instead consider the complete graph, or just the set of vertices, since all the needed information is contained in $(\lambda_v, v \in V)$.

Configurations are defined in the same way as for the SSM, but for η a configuration, a vertex v will be said to be *stable* if $\eta(v)$ is at most M_v , and *unstable* otherwise.

Note that the Markov chain becomes a lot more complicated, since it is much harder to compute the transition probability from one stable state to another. There is however a relatively simple way to do it by considering an extended Markov chain. This is explained in [37], will be detailed in Section 4.4.1, and the following example will be helpful.

Example 4.2.4. An example of the SSM is given in Figure 4.2. The graph is again a triangle with one of the three vertices being the sink. We consider that each vertex has 3 ways to topple, either by giving one sand grain to one of its two neighbors, or by giving one sand grain to both of them (with probabilities detailed on the figure). Here we represent all the unstable configurations appearing while making topplings, before finally reaching a stable configuration (stable configurations are the ones in the blue rectangle).

Results to be investigated in the rest of the Chapter

To ensure that the chain is well defined, we need the two following results :

Proposition 4.2.5. *The toppling process almost surely ends.*

For this proposition to be true, we will need some additional hypothesis on the supports of the probability distributions $(\lambda_v, v \in V)$ on partial topplings.

Proposition 4.2.6. *The order in which the vertices topple does not matter.*

Those two results will be needed to prove Conjecture 4.1.1.

4.3 Model description.

We now give a more formal definition for the SSM.

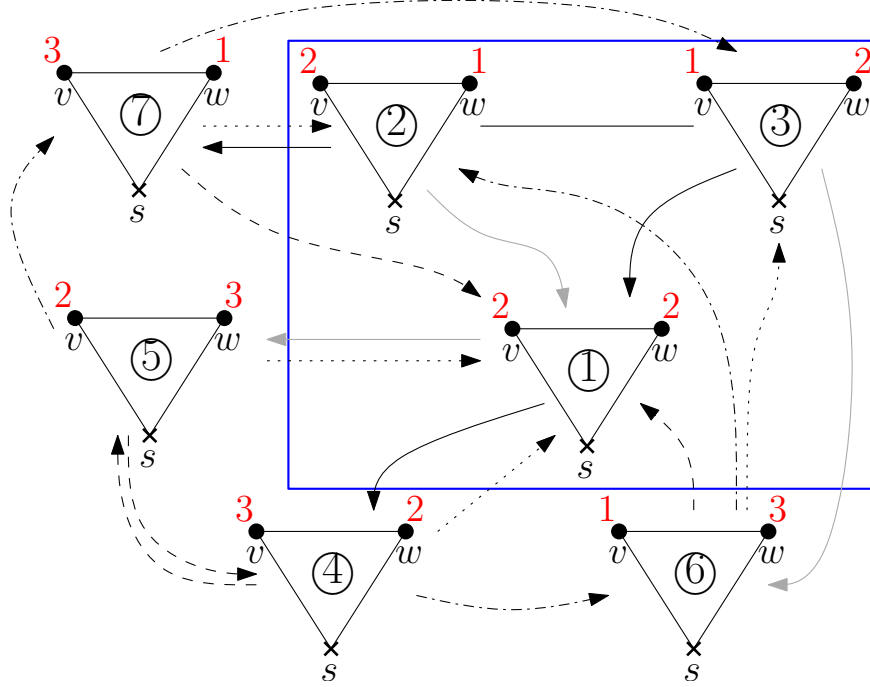


Figure 4.2 – Solid black arrows correspond to transitions where a sand grain has just been added on v (probability a), and solid gray arrows when a sand grain has been added on w (probability $b = 1 - a$). Dotted arrows correspond to transitions where the unstable vertex gives a sand grain to the sink (which happens with probability α), dashed arrows to transitions where it gives a sand grain to the other vertex (with probability β), and dashed-dotted arrows to transitions where it gives a sand grain to both (probability $\gamma = 1 - \alpha - \beta$). Stable configurations are the ones in the blue rectangle, the others are just intermediate unstable states.

4.3.1 Notations and definitions.

Let N be a nonnegative integer. The set $[N] := \{1, \dots, N\}$ represents the *sites* (or the vertices, except the sink). A sandpile configuration on $[N]$ is a vector $\eta = (\eta_k, k \in [N]) \in (\mathbb{N} - \{0\})^{[N]}$: for v in $[N]$, η_v represents the number of sand grains on the site v .

For v in $[N]$, we define

$$\mathcal{S}_v = \{\nu \in \mathbb{Z}^{[N]} | \nu_v < 0, \nu_w \geq 0 \text{ for } w \neq v, \text{ and } \sum_{i=1}^n \nu_i \leq 0\}$$

the set of possible topplings of v . If a toppling $\nu \in \mathcal{S}_v$ occurs, the state η

becomes $\eta + \nu$: v loses $-\nu_v$ sand grains, and any other site $w \neq v$ receives ν_w sand grains from v . If $\sum_{i=1}^n \nu_i < 0$, it means some sand grains were sent to the sink (Here the sink is not represented. Sending sand grains to is equivalent to losing sand grains).

For $v \in V$, let λ_v be a probability distribution with finite support on $\mathcal{S}_v$ that describes the law of the random topplings of v . We define $M_v = \max\{-\nu_v, \nu \in \mathcal{S}_v | \lambda_v(\nu) \neq 0\}$ to be the *threshold* of site v , i.e. the highest number of sand grains v can lose during a toppling. If ν is a configuration, the site v is said to be *stable* if $\nu_v \leq M_v$, otherwise it is said to be *unstable*. The configuration ν is said to be stable if and only if all its sites are stable.

For $v \in [N]$, we also define δ^v to be the vector with a 1 in v^{th} position and zeros everywhere else : $\delta_i^v = \delta_{i,v}$ (where $\delta_{i,v}$ is the Kronecker symbol).

Finally, we need a probability distribution μ on $[N]$ which governs the way sand grains are added on the set of sites.

4.3.2 State space and transitions

The state space for our SSM is the set $St = [M_1] \times [M_2] \times \cdots \times [M_N]$ of stable configurations.

To properly describe the transitions, we need to define the random stabilization operator $\mathcal{RS}$: let η be a sandpile configuration, we construct $\mathcal{RS}(\eta)$ by the following procedure :

1. If η is stable, return $\mathcal{RS}(\eta) = \eta$ and exit the procedure.
2. Pick an arbitrary unstable site $v \in [N]$.
3. Draw a random toppling $D_v \in \mathcal{S}_v$ according to the probability distribution λ_v , independently from the past.
4. Replace η by $\eta + D_v$.
5. Go to step 1.

As we shall see, this procedure terminates almost surely under some natural hypotheses on the λ_v 's, and the law of the resulting $\mathcal{RS}(\eta)$ does not depend on which unstable site we pick at each execution of the step 2.

Now the *Stochastic Sandpile Model* is the Markov chain in which, at each time step, we add a sand grain at a random site drawn according to the probability distribution μ , and then perform random stabilization on the

resulting configuration. For η, η' two states, then the transition probability from η to η' reads

$$P_{\eta, \eta'} = \sum_{k=1}^N \mu(k) \mathbb{P}(\mathcal{RS}(\eta + \delta^k) = \eta'). \quad (4.1)$$

We now give a result about the finiteness of the stabilization process :

Lemma 4.3.1. *The two following assertions are equivalent :*

- (i) *The random stabilization process is always almost surely finite regardless of μ .*
- (ii) *For all nonempty subset H of $[N]$, there exists a ℓ in H such that*

$$\lambda_\ell(\{\nu \in \mathcal{S}_\ell \mid \sum_{k \in H} \nu_k < 0\}) > 0. \quad (4.2)$$

Note that we can rephrase condition (4.2) as follows : For all nonempty subset H of $[N]$, H contains a vertex which has a nonzero probability to send sand grains outside of H when toppling.

Proof. Let us first prove that (i) implies (ii). Here we proceed by contraposition. Let us assume that there exists a nonempty subset $H \subset [N]$ from which sand grains can not escape. If the distribution μ allows sand grains to be added on some sites in H , then the total number of sand grains on H will eventually become greater than $\sum_{k \in H} M_k$, ensuring that there is always at least one unstable site in H and that the toppling process never ends.

Let us now prove that (ii) implies (i). Let us denote $G_0 = \emptyset$ and $H_0 = [N]$. We recursively define $G_n, n \in \mathbb{N}$ and $H_n, n \in \mathbb{N}$ by

$$G_{n+1} = \{\ell \in H_n \mid \lambda_\ell(\{\nu \in \mathcal{S}_\ell \mid \sum_{k \in H_n} \nu_k < 0\}) > 0\},$$

and

$$H_{n+1} = H_n \setminus G_{n+1}.$$

G_1 is the set of sites in H_0 having a nonzero probability to send sand grains to the sink (outside of $[N]$), and H_1 is its complement. Similarly, G_{n+1} is the set of sites in H_n having a nonzero probability to send sand grains to

G_n . The condition (ii) ensures that $\cup_{n \in \mathbb{N}} G_n = [N]$. We define the *depth* of the Markov chain to be $D = \max\{n \in \mathbb{N} | G_n \neq \emptyset\}$. We will now proceed by induction on D .

If $D = 1$, it means that every site has a nonzero probability to send grains to the sink, and one easily checks that the stabilization occurs almost surely in that case, using the fact that the total number of sand grains on the graph will decrease during the stabilization process.

Assume that the property is true for any case where $D \leq n$, which means that if $D \leq n$, the stabilization process always eventually ends. Consider the case $D = n+1$. Let us consider an unstable configuration. Each site in G_1 has a positive probability to send sand grains to the sink, and the total number of sand grains on the graph is finite, so the number of topplings in G_1 is almost surely finite. Furthermore, between two topplings in G_1 , the number of topplings in H_1 is also almost surely finite (by induction hypothesis, since the process restricted to H_1 is of depth n). Therefore the number of topplings in the entire stabilization process is almost surely finite. $\square$

From now on, we consider that the condition (4.2) is satisfied. This condition was used by Diaconis and Fulton in [20], when they defined their *growth model*.

We will now need to show that the result of the stabilization process does not depend on the choices we make when picking unstable sites to topple.

4.3.3 The Abelian property

We now prove the fact that the law of $\mathcal{RS}(\eta)$ is independent of which unstable site is toppled at each step. This is done by a coupling argument or “deck of cards” picture as in [20].

Let $(D_v^{(i)}, i \geq 1, v \in [N])$ be a collection of independent random variables such that, for any $v \in [N]$ and $i \geq 1$, $D_v^{(i)}$ takes its value in $\mathcal{S}_v$ according to the probability distribution λ_v . It is convenient to think of each site as being bound to an infinite deck of cards, where each card indicates a possible toppling at the site at hand. Each time we want to topple a vertex $v \in V$, we draw the top card of its deck, make the toppling according to the instructions on this card, and then throw the card away. For $i \geq 1$ and $v \in [N]$, $D_v^{(i)}$ represents the card that was originally at the i^{th} position of the deck bound to the vertex v .

Given a realization of the $D_v^{(i)}$'s, the topplings are deterministic. More precisely, each sandpile configuration η shall be supplemented with a collection of *counters* $c \in \mathbb{N}^{[N]}$ keeping track of how many cards have been drawn at each site. For $v \in [N]$, we define the toppling function

$$\begin{aligned} T_v : \mathbb{Z}^N \times \mathbb{N}^N &\rightarrow \mathbb{Z}^N \times \mathbb{N}^N \\ (\eta, c) &\mapsto (\eta + D_v^{(c_v+1)}, c + \delta^v) \end{aligned}$$

We perform the toppling $D_v^{(c_v+1)}$ at v and increment the counter c_v by one.

As one can easily check, for all $v, w \in [N]$, $T_v \circ T_w = T_w \circ T_v$.

We now define the determinized stabilization operator $\mathcal{DS}$: for any sandpile configuration with counters (η, c) :

1. If η is stable, return $\mathcal{DS}(\eta, c) = (\eta, c)$ and exit the procedure.
2. Pick an arbitrary unstable site $v \in [N]$.
3. Replace (η, c) by $T_v(\eta, c)$.
4. Go to step 1.

It looks like $\mathcal{DS}(\eta, c)$ depends on the sequence of vertices $v_1, v_2, \dots$ that we pick at each execution of the step 2. But it turns out that, if there exists one such sequence for which the procedure terminates, then the procedure also terminates for any other sequence and the resulting configuration is identical. To prove this, we need to introduce some further terminology.

Given (η, c) a sandpile configuration with counters, a sequence of sites $(v_1, \dots, v_n)$ is said to be *legal* if v_i is an unstable vertex in $(T_{v_{i-1}} \circ T_{v_{i-2}} \circ \dots \circ T_{v_1})(\eta, c)$ for all i (in particular, v_1 must be an unstable site in η). In other words, a legal sequence is a possible choice of sites to topple in the first n iterations of the procedure $\mathcal{DS}(\eta, c)$. A legal sequence is said *stabilizing* if $(T_{v_n} \circ \dots \circ T_{v_1})(\eta, c)$ is stable.

Lemma 4.3.2. *Let (η, c) be a sandpile configuration with counters which admits a stabilizing sequence $(v_1, \dots, v_n)$, and let $(\xi, d) = (T_{v_n} \circ \dots \circ T_{v_1})(\eta, c)$. Then the final counters d are maximal in the sense that, for any legal sequence $(w_1, \dots, w_p)$ for (η, c) , if $(\xi', d') = (T_{w_p} \circ \dots \circ T_{w_1})(\eta, c)$ then we have $d'_v \leq d_v$ for all $v \in [N]$.*

This lemma and its proof are inspired from [36].

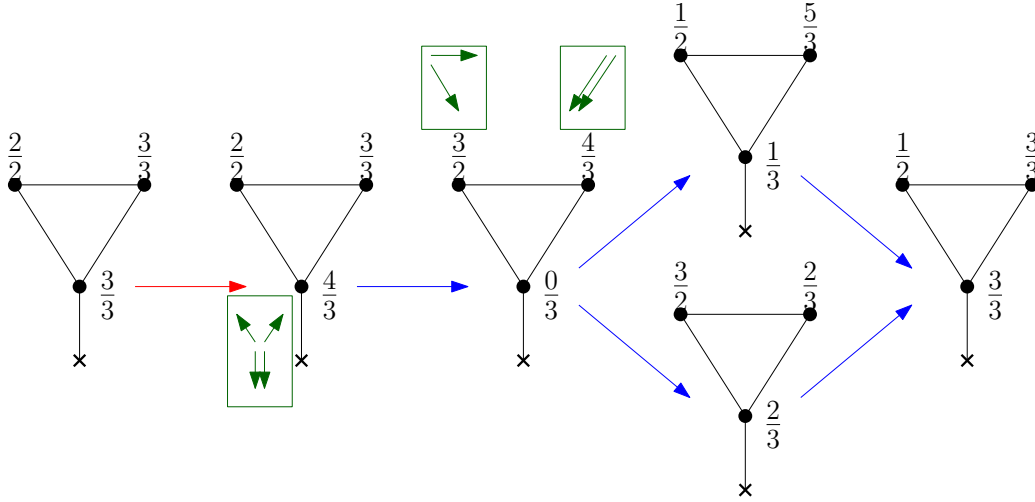


Figure 4.3 – A detailed transition for the SSM on a triangle linked to the sink. The red arrow corresponds to the addition of a sand grain (on the lowest site here), and blue arrows to topplings. The cards on top of the decks have been represented in green. The above numbers near each site indicate how many sand grains are on this site, while the below number is the threshold of the site. Note that the resulting state does not depend on the order in which the vertices topple.

Proof. Let $(\xi, d) = T_{v_n} \circ T_{v_{n-1}} \circ \dots \circ T_{v_1}(\eta, c)$ be the stable configuration we obtained. We have

$$(\xi, d) = (\eta + \sum_{v=1}^N \sum_{i=c_v+1}^{d_v} D_v^{(i)}, d).$$

Suppose that the resulting state of a sequence of legal topplings starting from (η, c) has counters $d'_v \leq d_v$ (this is always possible since we can choose $d'_v = c_v$ for all v), and for a site $\ell \in V$ an extra legal toppling can be performed. We put

$$(\zeta, d') = (\eta + \sum_{v=1}^N \sum_{i=d'_v+1}^{d'_v} D_v^{(i)}, d').$$

Since an extra toppling can be performed at site ℓ , we have $\zeta_\ell - \xi_\ell > 0$. This can be rewritten as

$$\sum_{v=1}^N \sum_{i=d'_v+1}^{d_v} (D_v^{(i)})_\ell < 0,$$

which gives us

$$\sum_{i=d'_\ell+1}^{d_\ell} (D_\ell^{(i)})_\ell < - \sum_{v \neq \ell} \sum_{i=d'_v+1}^{d_v} (D_v^{(i)})_\ell \leq 0.$$

The first sum being nonzero, it necessarily has at least one term, ensuring that $d'_\ell < d_\ell$ and thus $d'_\ell + 1 \leq d_\ell$ still holds once we make another toppling on ℓ . $\square$

This result, along with Lemma 4.3.1 proves that the stabilization operator is well defined.

4.3.4 Some specific cases

1. If for all $i \geq 1$, for all $k \in [N]$, we always have $(D_k^{(i)})_k = -1$, this corresponds to the case where a toppling vertex will always give exactly one sand grain to either an other vertex or the sink. It is easy to see that in this case, the only possible recurrent state is the one with exactly one sand grain on every site : Each time we add a sand grain, it will move from site to site until it falls into the sink.
2. For all k, ℓ in $[N]$, let $d_{k,\ell} = d_{\ell,k}$ and be fixed positive integers. Let $p \in]0, 1]$ be a fixed real number.

For $i \geq 1$, k, ℓ in $[N]$ and j in $\{1, \dots, d_{k,\ell}\}$, $B_{k,\ell}^{(i)}(j)$ are independent Bernoulli random variables with parameter p .

The case where for all $i \geq 1$ and for all $k \neq \ell$, we have

$$(D_k^{(i)})_\ell = \sum_{j=1}^{d_{k,\ell}} B_{k,\ell}^{(i)}(j)$$

and

$$(D_k^{(i)})_k = - \left(\sum_{m \neq k} (D_k^{(i)})_m + \sum_{j=1}^{d_{k,k}} B_{k,k}^{(i)}(j) \right)$$

corresponds to the Stochastic Sandpile Model on a graph defined by Chan, Marckert and Selig in [12] : indeed, for $k \neq \ell$, $d_{k,\ell}$ is the number of edges between the vertices k and ℓ , and $d_{k,k}$ is the number of edges between k and the sink.

3. The previous case with p fixed to 1 corresponds to the abelian sandpile model on a graph.

4.4 Around the stationary distribution

We now show how to precisely compute the stationary distribution, and we then show Conjecture 4.1.1.

4.4.1 The transition matrix

We here quickly explain how to compute the transition matrix on simple examples. We follow the same procedure as in [37, Section 5.7.1.1], by defining an extended Markov chain.

We first extend the state space : the new state space is the set $St' = (\mathbb{N} - \{0\})^{[N]}$ of sandpile configurations.

The transitions are defined as follows : Let η, η' be two sandpile configurations, the transition probability from η to η' reads :

$$\tilde{P}_{\eta, \eta'} = \begin{cases} \mu(k) & \text{if } \eta \text{ is stable and } \eta' = \eta + \delta^k \text{ for some } k \in [N], \\ \mathbb{P}(\bar{T}_k(\eta) = \eta') & \text{if } \eta \text{ is unstable and } k \text{ is the smallest unstable site in } \eta, \\ 0 & \text{otherwise.} \end{cases} \quad (4.3)$$

This Markov chain is much simpler, because only one operation (be it a toppling or the addition of a sand grain) will happen at each transition.

The transition matrix for this Markov chain reads $\tilde{P} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A = (\tilde{P}_{\eta, \eta'} | \eta, \eta' \in St)$, $B = (\tilde{P}_{\eta, \eta'} | (\eta, \eta') \in St \times St')$, $C = (\tilde{P}_{\eta, \eta'} | (\eta, \eta') \in St' \times St)$ and $D = (\tilde{P}_{\eta, \eta'} | \eta, \eta' \in St')$.

We are now able to describe the transition matrix P for the SSM. Indeed, for all $\eta, \eta' \in St$, we have

$$P_{\eta, \eta'} = \tilde{P}_{\eta, \eta'} + \sum_{\eta_1, \eta_2, \dots, \eta_k \in St'} \tilde{P}_{\eta, \eta_1} \tilde{P}_{\eta_1, \eta_2} \dots \tilde{P}_{\eta_{k-1}, \eta_k} \tilde{P}_{\eta_k, \eta'}$$

which can be written as :

$$P = A + B \left(\sum_{n \geq 0} D^n \right) C,$$

or more conveniently

$$P = A + B(I - D)^{-1}C. \quad (4.4)$$

Example 4.4.1. The transition matrix of the extended Markov chain represented in Figure 4.2 in the basis $\{\textcircled{1}, \dots, \textcircled{7}\}$ reads :

$$\begin{pmatrix} 0 & 0 & 0 & a & b & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & a \\ a & 0 & 0 & 0 & 0 & b & 0 \\ \alpha & 0 & 0 & 0 & \beta & \gamma & 0 \\ \alpha & 0 & 0 & \beta & 0 & 0 & \gamma \\ \beta & \gamma & \alpha & 0 & 0 & 0 & 0 \\ \beta & \alpha & \gamma & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.5)$$

And if we write $A = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ a & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \end{pmatrix}$, $C = \begin{pmatrix} \alpha & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & \alpha \\ \beta & \alpha & \gamma \end{pmatrix}$ and

$D = \begin{pmatrix} 0 & \beta & \gamma & 0 \\ \beta & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and by substituting those matrices in Equation (4.4),

we find the transition matrix for the SSM in the basis $\{\textcircled{1}, \textcircled{2}, \textcircled{3}\}$:

$$\begin{pmatrix} \frac{\alpha+\beta\gamma}{1-\beta} & \frac{\gamma(a(\gamma+\alpha\beta)+b(\alpha+\beta\gamma))}{1-\beta^2} & \frac{\gamma(a(\alpha+\beta\gamma)+b(\gamma+\alpha\beta))}{1-\beta^2} \\ b+a\beta & a\alpha & a\gamma \\ a+b\beta & b\gamma & b\alpha \end{pmatrix}.$$

Note that $(\frac{\beta+1}{\gamma}, 1, 1)$ is a left eigenvector for that matrix. The fact that this vector does not depend on (a, b) is quite remarkable, and we will now show that this fact is true in general.

4.4.2 Commuting matrices and stationary distribution.

We have explained how to compute the transition matrix P for the SSM. This matrix of course depends on the distribution μ governing the additions of sand grains on the sites, but our aim is to show that the stationary distribution does not. For k in $[N]$, we will note $P^{(k)}$ is the matrix in St with coefficients reading :

$$P_{\eta, \eta'}^{(k)} = \mathbb{P}(\mathcal{RS}(\eta + \delta^k) = \eta') \quad (4.6)$$

The rows of this matrix give the probability to go from one state to another by adding a sand grain on k and then making topplings until we reach a stable state.

Proposition 4.4.2. *For all $k, \ell \in [N]$, we have $P^{(k)}P^{(\ell)} = P^{(\ell)}P^{(k)}$.*

Proof. Let (η, c) be a stable state sandpile configuration with counters, and let k, ℓ be in $[N]$. If $k = \ell$, then $P^{(k)}P^{(\ell)} = P^{(\ell)}P^{(k)}$ and the result holds.

If $k \neq \ell$, let $v_1, \dots, v_n$ be a stabilizing sequence for $(\eta + \delta^{(k)}, c)$: we note $(\xi, d) = \mathcal{DS}(\eta + \delta^{(k)}, c) = T_{v_n} \circ \dots \circ T_{v_1}(\eta + \delta^{(k)}, c)$. Let $w_1, \dots, w_m$ be a stabilizing sequence for $(\xi + \delta^{(\ell)}, d)$. Then $v_1, \dots, v_n, w_1, \dots, w_m$ is a stabilizing sequence for $(\eta + \delta^{(k)} + \delta^{(\ell)}, c)$.

Thus we have $\mathcal{DS}(\mathcal{DS}(\eta + \delta^{(k)}, c) + (\delta^{(\ell)}, 0)) = \mathcal{DS}(\eta + \delta^{(k)} + \delta^{(\ell)}, c)$. The same argument can be used after switching the roles of k and ℓ , which shows that $\mathcal{DS}(\mathcal{DS}(\eta + \delta^{(k)}, c) + (\delta^{(\ell)}, 0)) = \mathcal{DS}(\mathcal{DS}(\eta + \delta^{(\ell)}, c) + (\delta^{(k)}, 0))$, allowing to conclude that $P^{(k)}P^{(\ell)} = P^{(\ell)}P^{(k)}$. $\square$

Proposition 4.4.3. *The stationary distribution for the SSM does not depend on μ .*

Proof. Equation (4.1) allows us to write $P = \sum_{k \in [N]} \mu(k)P^{(k)}$. Let us now consider $P' = \sum_{k \in [N]} \mu'(k)P^{(k)}$ with μ' an other probability distribution on $[N]$. Since for all k, ℓ in $[N]$, $P^{(k)}P^{(\ell)} = P^{(\ell)}P^{(k)}$, we can conclude that $PP' = P'P$. Thus, if π is the stationary distribution of the Markov chain when sand grains are added according to μ , we have $\pi PP' = \pi P' = \pi P'P$ which shows that $\pi P'$ is an eigenvector with eigenvalue 1 for P , and since P' is a probability matrix and 1 is a non degenerated eigenvalue for P , we necessarily have $P'\pi = \pi$. $\square$

Remark 4.4.4. Note that we do not know necessary and sufficient conditions for the chain to be irreducible and aperiodic.

However, if μ is strictly positive, the Markov chain will be irreducible on the recurrent class containing the maximal state $(M_1, M_2, \dots, M_N)$.

In [12], the SSM is defined on a connected graph with a sink, and topplings are such that any partial toppling can happen. The sand grain addition law μ is also strictly positive (any site can receive a sand grain). Those conditions ensure that the SSM is irreducible on the recurrent class containing the maximal configuration $(M_1, \dots, M_N)$. The authors of [12] also were able to describe the set of recurrent states in this case.

An example of a case where the SSM is not aperiodic is the following : if for all $k \in V$, and for all $\nu \in \mathcal{S}_k$, $\sum_{i \in V} \nu_i$ is even, then the process will alternate between states with an even number of sand grains and states with an odd number of them. This of course works with any similar condition (if the GCD of all possible values for $\sum_{i \in V} \nu_i$ is not 1 for example).

Chapter 5

Conclusion

We end this thesis with speculative ideas for further work.

Both the annihilation Markov chains (on juggling sequences, on set partitions and on words) described in Section 2.4.2 and the overwriting Markov chain (on words and on staircase tableaux) described in Section 3.3.3 have the property that all the eigenvalues of the transition matrices are nonnegative and trivially linear in the parameters z_i . Such Markov chains typically arise from an underlying structure which we now describe. If one considers the matrices P_i (resp. $\tilde{P}_i$) obtained by setting $z_i = 1$ and $z_j = 0$ for $j \neq i$ in the transition matrix P (resp. $\tilde{P}$) in (2.72) (resp. (2.73)) then the monoid generated by these matrices is $\mathcal{R}$ -trivial. For a recent monograph on the connection between Markov chains and $\mathcal{R}$ -trivial monoids, see [5]. One could reprove our statement about the eigenvalues in Theorem 2.4.19 using this connection. The general theory of free tree monoids described in [5], however, does not directly apply to the annihilation chains because the generators P_i (resp. $\tilde{P}_i$) do not always square to themselves (i.e. are not idempotents). It should be interesting to study when the results in [5] can be applied to situations when the generators are not idempotents. This work suggests that it is worth studying a general framework for a class of problems in statistical physics, which we tentatively call “boundary-driven Markov chains”. Just as the juggling and set-partition chains (2.1) and (2.27) studied in this Chapter, they have the property that the motion is deterministic for the most part (that is, in the bulk). Only when a ball (in the juggling context) reaches the boundary, i.e. Magnus’ hand, something stochastic happens, namely the ball is thrown to a randomly chosen height. Similar things happen with the set-

partition Markov chain.

A natural idea is to combine the juggling process with an exclusion process, for instance the Totally Asymmetric Exclusion Process (TASEP) where particles do not all move at the same time but, instead, at each time step one particle is selected randomly and moves if the next site is empty. When a particle on the first site is selected, it is then reinserted, say, uniformly at any available site on the lattice. This model was investigated via a physical hydrodynamic approach in [1], where a nontrivial phase diagram was found. It would be interesting to obtain more precise results on this model, by looking for instance for a possible exact solution. Finally, the fact that the stationary distributions of the Markov chains considered in this Chapter all admit product forms is reminiscent of the Zero-Range Process admitting a “factorized stationary state” [23, 24], and one might wonder whether a connection exists.

Several questions remain open in the multispecies juggling context. We have not found an expression for the normalization factor for the juggling chain with several jugglers. We have also not yet found a multiparameter version for the latter model, as the possibility of catching more than one ball at a time changes the behavior quite drastically. A multispecies model with several jugglers is one possible extension of our model. From a probabilistic point of view, it would also be natural to look at the extension to infinite models, such as a Markov chain on the set space $St_{n_1, \dots, n_{T-1}, \infty}$. This would contain as special cases, the infinite and unbounded juggling models described in Section 2.3.

A natural thing to investigate in the case of the stochastic sandpile model is the set of recurrent states. In the case of the abelian sandpile model, the “burning algorithm” described in [36] gives a bijection between this set and the set of spanning trees of the underlying graph. Marckert, Chan and Selig gave a characterization of the recurrent states in terms of graph orientations in the case where all partial topplings are possible. In our case, the set of recurrent states is a subset of the one Marckert, Chan and Selig obtained, but nothing more can be said for now.

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