UNIVERSITÉ DE TOULON

ÉCOLE DOCTORALE Mer et Sciences (ED548)

Institut de mathématiques de Toulon

THÈSE présentée par :

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soutenue le : **07 Décembre 2016**

pour obtenir le grade de Docteur en Mathématiques

Spécialité : Mathématiques appliquées

Quelques résultats en analyse théorique et numérique pour les équations de Navier-Stokes compressibles

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Remerciements

Mes premiers remerciements s'adressent bien enetendu à Antonín Novotný et Thierry Gallouët.

Je remercie Didier Bresch et Endre Suli pour avoir accepté de rapporter ma thèse.

Je tiens également à remercier Robert Eymard, Cedric Galusinski et Eduard Feireisl d'avoir accepté d'être membre de mon jury.

Je remercie les membres du laboratoire IMATH pour leur accueil.

Je remercie tout les membres de ma famille.

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Première partie .

Document de synthèse

1. Introduction

Cette thèse est composée de plusieurs articles traitant de l'analyse mathématique et numérique des équations de Navier-Stokes compressibles en regime barotrope. La plupart des travaux présentés ici combinent des méthodes d'analyse des équations aux dérivées partielles et des méthodes d'analyse numériques afin de clarifier la notion de solutions faibles ainsi que les mécanismes de convergence de méthodes numériques approximant ces solutions faibles. En effet les équations de Navier-Stokes compressibles sont fortement non linéaires et leur analyse mathématique repose nécessairement sur la structure des ces équations.

Le chapitre 2 est consacré à la présentation des équations de Navier-Stokes compressibles en regime barotrope dans le contexte de la thermodynamique des fluides compressibles. Ce chapitre est une vue d'ensemble de resúltats théoriques sur les équations de Navier-Stokes compressibles qui sont souvent citeś et utilisés dans les articles théoriques et numériques présentés dans cette thèse : nous rappelons ici particulièrement

1. Les théorèmes d'existence global des solutions faibles (voir par exemple [?], [?], [?], [?], [?]).
2. Les théorèmes d'existence local de solutions fortes (voir par exemple [?] et [?]).
3. Critère d'explosion pour les solutions fortes dans la classe de [?] (voir [?])
4. La méthode d'énergie relative provenant de [?] et [?].

Les chapitres 3 et 4 décrivent nos résultats théoriques. Le chapitre 3 résume les résultats de l'article "Compressible Navier-Stokes equations on thin domains" (voir Annexe A) traitant de le problématique de réduction de dimension tandis que la section 4 correspond à l'article "Existence of weak solutions for compressible Navier-Stokes equations with entropy transport." traitant de l'existence de solutions faibles d'un modèle de fluide compressible en régime barotrope avec transport d'entropie (voir Annexe B).

Les chapitres 5 et 6 concernent l'analyse numérique des équations de Navier-Stokes compressibles en régime barotrope. Plus précisement les résultats présentés dans ces chapitres portent sur des estimations d'erreur inconditionnelles entre une solution discrète d'un schéma numérique approximant les solutions faibles des équations de Navier-Stokes compressibles en régime barotrope et une solution forte de ces équations. Dans le chapitre 5 nous présentons des estimations d'erreur pour un schéma numérique de type volumes finis/éléments finis. Ces estimations d'erreur sont données dans le cas d'un domaine polyédrique et dans la cas d'un ouvert suffisament régulier garantissant l'existence de solutions fortes. Ces résultats correspondent aux articles "Error estimates for a numerical approximation to the compressible barotropic Navier-Stokes equations" (voir Annexe C) et "Error estimates for a numerical method for the compressible Navier-Stokes sysem on sufficiently smooth domains" (voir Annexe D). Dans le chapitre 6 nous présentons des résultats d'estimations d'erreur pour le schéma Marker-and-Cell. Le premier résultat de ce chapitre concerne des estimations d'erreur sur un domaine adapté au schèma Marker-and-Cell et correspond au à l'article "Implicit MAC scheme for compressible Navier-Stokes equations : Unconditional error estimates" (voir Annexe E). Le second résultat, correspondant à l'article "Implicit MAC scheme for compressible Navier-Stokes equations : Low Mach asymptotic error estimates" (voir Annexe F), concerne le caractère inconditionnellement et uniformément asymptotiquement stable en régime bas Mach du schéma Marker-and-Cell.

This thesis is composed of several papers dealing with mathematical and numerical analysis of compressible Navier-Stokes equations in barotropic regime. Most of these works combine mathematical analysis of parital differential equations and numerical methods with aim to shred more light on the construction of weak solutions on one side and on the convergence mechanisms of numerical methods approximating these weak solutions on the other side. Indeed, the compressible Navier-Stokes equations are strongly nonlinear and their mathematical analysis necessarily relies on the structure of equations. The papers collected in this thesis are preceded by a synthetic document whose goal is to present the main results and their mutual and corrections. This synthetic document is divided into several chapters.

The chapter 2 is devoted to the presentation of the compressible Navier-Stokes equations within the context of thermodynamics of compressible fluids. It is an overview of theorical results on compressible Navier-Stokes equations in barotropic regime that are often quote and used in both theorical and numerical papers presented in the thesis : we recall here especially theorems on

1. global existence of weak solutions (taken over from papers [?],[?] and monographs [?], [?], [?]).
2. local existence of strong solutions (taken over form [?], [?] for example).
3. blow-up criteria for strong solution in the class of [?] (taken over from [?]).
4. relative energy inequality (taken over from [?], [?]).

Chapter 3 summerizes the results of "Compressible Navier-Stokes on thin domains " (see Annexe A) wihich deals with the 3-D/2-D dimension reduction of equations while the chapter 4, corresponding to paper "Existence of weak solutions for compressible Navier-Stokes equations with entropy transport ", deals with the existence of weak solutions for a model of a compressible fluid in barotropic regime with entropy transport (see Annexe B).

Chapters 5 and 6 concerns the numerical analysis of the equations. Chapter 5 deals with a finite element/ finite volume method for the compressible Navier-Stokes equations. In chapter 5 we present results concerning error estimates for this numerical scheme. These error estimates are given on a bounded polyhedral domain and on a sufficiently smooth domain for which we are guaranteed of the existence of strong solutions . These results correspond to papers "Error estimates for a numerical approximation to the compressible barotropic Navier-Stokes equations" (see Annexe C) and "Error estimates for a numerical method for the compressible Navier-Stokes sysem on sufficiently smooth domains" (see Annexe D). Chapter 6 deals with the with the Marker-and-Cell scheme for compressible Navier-Stokes equations. In paper "Implicit MAC scheme for compressible Navier-Stokes equations : Unconditional error estimates " we derive unconditional error estimates for the MAC scheme in a bounded domain adapted to the MAC scheme. In paper "Implicit MAC scheme for compressible Navier-Stokes equations : Low Mach asymptotic error estimates" (see Annexe F) we derive uniform unconditional error estimates for the same MAC scheme in the low Mach number regime.

2. Géneralités sur les équations de Navier-Stokes compressible en regime barotrope

Ce chapitre commence avec une introduction dans laquelle nous rappelons quelques idées fondamentales sur le mouvement d' un fluide compressible, visqueux et conducteur de chaleur (elle sera en particulier utile pour la comprehension du chapitre 4). Après cette breve introduction, nous traitons le système de Navier-Stokes compressible en régime barotrope. Les résultats présentées dans ce chapitre sont basés sur les articles [?], [?] ainsi que sur les monographes [?], [?], [?], [?]. Ce chapitre apporte des améliorations mineures et examine à chaque étape comment affaiblir les hypothèses des théorèmes, en particulier en terme de loi constitutive pour la pression.

2.1. Modèle physique

Nous décrivons ici le mouvement d'un fluide compressible, visqueux et conducteur de chaleur appelé aussi quelque fois gaz visqueux. Dans un but de simplicité, nous supposerons que le fluide occupe un domaine fixe $\Omega \subset \mathbb{R}^3$ et nous nous intéressons à son évolution au cours d'un intervalle de temps de taille arbitraire $(0, T)$, $T > 0$. Son mouvement sera décrit au moyen de trois variables d'états : sa densité $\varrho = \varrho(t, \boldsymbol{x})$, sa vitesse $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$, et sa température absolue $\vartheta = \vartheta(t, \boldsymbol{x})$, où $t \in (0, T)$ représente le temps et $\boldsymbol{x} \in \mathbb{R}^3$ représente la variable spatiale en coordonnées eulériennes. La nature physique de la densité et de la température impose à la densité d'être une fonction positive sur $(0, T) \times \Omega$ et à la température absolue d'être une fonction strictement positive sur $(0, T) \times \Omega$. L'évolution temporelle de ces quantités, en l'absence de forces extérieures spécifiques et de source externe de chaleur, est décrite par des lois de conservation de la physique exprimées au travers des équations aux dérivées partielles suivantes :

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ dans } (0, T) \times \Omega, \tag{2.1.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\varrho, \vartheta) = \operatorname{div} \mathbb{S}(\varrho, \vartheta, \nabla \boldsymbol{u}) \text{ dans } (0, T) \times \Omega, \tag{2.1.1b}$$

$$\partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}(\varrho e(\varrho, \vartheta)\boldsymbol{u}) + \operatorname{div} \boldsymbol{q}(\varrho, \vartheta, \nabla \vartheta) + p(\varrho, \vartheta) \operatorname{div} \boldsymbol{u} = \mathbb{S}(\varrho, \vartheta, \nabla \boldsymbol{u}) : \nabla \boldsymbol{u} \text{ dans } (0, T) \times \Omega. \tag{2.1.1c}$$

Dans ces équations $p = p(\varrho, \vartheta)$ est la pression, $e = e(\varrho, \vartheta)$ est l'énergie interne spécifique, $\mathbb{S}(\varrho, \vartheta, \nabla \boldsymbol{u})$ est le tenseur des contraintes visqueuses tandis que $\boldsymbol{q}(\varrho, \vartheta, \nabla \vartheta)$ est le flux de chaleur. Ce sont des fonctions données caractérisant le gaz.

En physique, il y a au moins deux autres manières d'écrire la conservation de l'énergie (2.1.1*c*) : en terme d'énergie totale spécifique, et en terme d'entropie spécifique.

Formulation de la première loi en terme d'énergie cinétique. L'énergie totale spécifique est la somme de l'énergie cinétique spécifique $e_{\text{cin}} = \frac{1}{2}\boldsymbol{u}^2$ et de l'énergie interne spécifique $e(\varrho, \vartheta)$

$$e_{\text{tot}} = \frac{1}{2}\boldsymbol{u}^2 + e(\varrho, \vartheta). \tag{2.1.2}$$

A cause de (2.1.1*a*)-(2.1.1*b*), elle obéit à l'équation

$$\partial_t(\varrho e_{\text{tot}}(\varrho, \vartheta)) + \operatorname{div}\Big(\Big(\varrho e_{\text{tot}}(\varrho, \vartheta) + p(\varrho, \vartheta)\Big)\boldsymbol{u}\Big) + \operatorname{div} \boldsymbol{q}(\varrho, \vartheta, \nabla\vartheta) = \operatorname{div}\Big(\mathbb{S}(\varrho, \vartheta, \nabla\boldsymbol{u}) \cdot \boldsymbol{u}\Big). \tag{2.1.3}$$

Formulation de la première loi en terme d'entropie spécifique. La seconde loi de la thermodynamique postule l'existence de l'entropie spécifique $s = s(\varrho, \vartheta)$ définie par la relation de Gibbs.

$$\vartheta \, \mathrm{d}s(\varrho, \vartheta) = \mathrm{d}e(\varrho, \vartheta) \quad \frac{p(\varrho, \vartheta)}{\varrho^2} \, \mathrm{d}\,\varrho \tag{2.1.4}$$

qui doit obéir à l'équation de conservation d'entropie

$$\partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}(\varrho s(\varrho, \vartheta)\boldsymbol{u}) + \operatorname{div}\left(\frac{\boldsymbol{q}(\varrho, \vartheta, \nabla\vartheta)}{\vartheta}\right) = \sigma, \tag{2.1.5}$$

où la quantité σ doit être positive. Elle est appelée taux de production d'entropie. Dans la présente situation,

$$\sigma = \frac{1}{\vartheta}\Big(\mathbb{S}(\varrho, \vartheta, \nabla\boldsymbol{u}) : \nabla\boldsymbol{u} \quad \frac{\boldsymbol{q}(\varrho, \vartheta, \nabla\vartheta) \cdot \nabla\vartheta}{\vartheta}\Big). \tag{2.1.6}$$

Si $p, e, \mathbb{S}, \boldsymbol{q}$ sont des fonctions différentiables de leurs arguments respectifs, si la densité ϱ et la température ϑ sont positives et suffisament regulières sur $(0, T) \times \Omega$ et si la vitesse $\boldsymbol{u}$ est suffisament regulière sur $(0, T) \times \Omega$, alors les équations (2.1.1*c*), (2.1.3) et (2.1.5)-(2.1.6) sont équivalentes. Cette équivalence n'est pas nécessairement vraie si les fonctions précédemment citées ne sont pas suffisament régulières.

Ainsi, malgré le fait que la formulation faible de la conservation de l'énergie basée respectivement sur chacune des équations (2.1.1*c*), (2.1.3) et (2.1.5), est justifiable comme étant physiquement égale, cette formulation peut mener à des solutions faibles possédant différentes propriétés. Il peut ainsi arriver, selon par exemple le régime d'écoulement considéré et les lois constitutives caractérisant le gaz, que certaines des possibles définitions de solutions faibles peuvent être plus avantageuses dans certaines situations et peuvent mêmes mener à des résultats d'existence global en temps alors que d'autres définitions ne posséderons pas cette propriété. Nous renvoyons le lecteur intéressé à ces questions à [**?**].

Nous supposons que le tenseur des contraintes visqueuses $\mathbb{S}$ est décrit par la loi de Newton

$$\mathbb{S}(\varrho, \vartheta, \nabla\boldsymbol{u}) = \mu(\varrho, \vartheta)(\nabla\boldsymbol{u} + \nabla^t\boldsymbol{u} \quad \frac{2}{3}\operatorname{div}\boldsymbol{u}\mathbb{I}_3) + \eta(\varrho, \vartheta)\operatorname{div}\boldsymbol{u}\mathbb{I}_3 \tag{2.1.7}$$

où $\mathbb{I}_3$ est le tenseur identité, tandis que $\boldsymbol{q}$ est le flux de chaleur satisfaisant la loi de Fourier

$$\boldsymbol{q} = \quad \kappa(\varrho,\vartheta)\nabla\vartheta. \tag{2.1.8}$$

Les quantités μ, η, and κ sont appelées coefficients de transport, plus précisément et respectivement, viscosité de cisaillement, viscosité volumique et conductivité de chaleur. D'aprés la seconde loi de la thermodynamique elles sont toutes positives. Comme nous travaillons avec des fluides visqueux nous supposerons que les coefficients de viscosité satisfont au moins

$$\mu(\varrho,\vartheta) > 0, \;\; \eta(\varrho,\vartheta) \geq 0. \tag{2.1.9}$$

Le système (2.1.1) (ou (2.1.1c) peut être remplacé par (2.1.3) ou (2.1.5)-(2.1.6) avec les relations constitutives (2.1.7) et (2.1.8) est appelé système de Navier-Stokes-Fourier. De plus lorsque les coefficients de transports sont pris égales à zero nous obtenons le système d'Euler compressible.

2.2. Ecoulements barotropes

Un fluide en écoulement est dit être en régime barotrope ou que le fluide est qualifié de barotrope si la pression p dépend seulement de la densité et si l'énergie interne spécifique est donnée par

$$e(\varrho,\vartheta) = e_{\text{el}}(\varrho) + e_{\text{th}}(\vartheta), \;\; e_{\text{el}}(\varrho) = \int_1^{\varrho} \frac{p(z)}{z^2}\,\mathrm{d}z. \tag{2.2.1}$$

Le système (2.1.1) dans cette situation s'écrit

$$\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0 \text{ dans } (0,T)\times\Omega, \tag{2.2.2a}$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p(\varrho) = \operatorname{div}\mathbb{S}(\nabla\boldsymbol{u}) \text{ dans } (0,T)\times\Omega, \tag{2.2.2b}$$

$$\partial_t(\varrho e_{\text{th}}(\vartheta)) + \operatorname{div}(\varrho e_{\text{th}}(\vartheta)\boldsymbol{u}) + \operatorname{div}\boldsymbol{q}(\varrho,\vartheta,\nabla\vartheta) = \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{u} \text{ dans } (0,T)\times\Omega. \tag{2.2.2c}$$

où nous avons utilisé l'identité (2.2.1) afin de transformer (2.1.1c) en (2.2.2c). Nous remarquons que l'équation (2.2.2c) et le système (2.2.2a)-(2.2.2b) sont découplés dans le sens que une fois le couple $(\varrho,\boldsymbol{u})$ determiné à partir des équations (2.2.2a)-(2.2.2b), la température ϑ peut être obtenue en résolvant (2.2.2c) avec certaines conditions au bord ;

Le système d'équations aux dérivées partielles (2.2.5a)-(2.2.5b) est appelé système de Navier-Stokes compressible en regime barotrope. Il ne décrit pas de façon satisfaisante les situations physiquement réalistes. Cependant, il est consistant du point de vue thermodynamique et il contient déjà beaucoup de difficultés mathématiques rencontrées quand nous traitons le système complet de Navier-Stokes-Fourier. Son étude n'est pas seulement sans intérêt mais peut aussi être vue comme un premier "toy problem" avant de s'attaquer au système complet. Les exemples les plus usuels d'écoulement barotrope sont les écoulement isothermes où

$$p(\varrho) = R\overline{\vartheta}\varrho \tag{2.2.3}$$

décrivant les écoulements des gaz parfaits de température constante $\vartheta > 0$ et les écoulements isentropiques

$$p(\varrho) = Re^{\overline{s}}\varrho^{\gamma},\ \gamma = \frac{R + c_v}{c_v} \tag{2.2.4}$$

décrivant les écoulements des gaz parfaits d'entropie constante $\overline{s} \in \mathbb{R}$. Notons cependant, que les conditions de température constante ou d'entropie constante violent la conservation de l'énergie (2.2.2c) sauf si aucune source de chaleur spécifique externe n'est ajoutée dans (2.2.2c). Les valeurs typiques de γ, appelé exposant adiabatique, se situent entre un maximum de $\frac{5}{3}$ pour gaz monoatomiques, passant par $\frac{7}{5}$ pour gaz diatomiques incluant l'air et allant jusqu'à des valeurs proches de 1 pour les gaz polyatomiques à haute température.

2.2.1. Existence de solutions faibles aux équations de Navier-Stokes compressibles en regime barotrope

Lorsque les changements de température ne sont pas pris en compte, l'écoulement d'un fluide compressible en régime barotrope en l'absence de forces extérieures est décrit par le système d'équations aux dérivées partielles suivant

$$\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0 \text{ dans } (0,T)\times\Omega, \tag{2.2.5a}$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p(\varrho) = \operatorname{div}\mathbb{S}(\nabla\boldsymbol{u}) \text{ dans } (0,T)\times\Omega, \tag{2.2.5b}$$

Nous supposons que la pression satisfait

$$p = p(\varrho) \in C(\mathbb{R}_+)\cap C^1(\mathbb{R}_+^\star),\ p(0) = 0, \tag{2.2.6}$$

et que le tenseur des contraintes visqueuses est donné par

$$\mathbb{S}(\nabla\boldsymbol{u}) = \mu(\nabla\boldsymbol{u} + \nabla^t\boldsymbol{u} \quad \frac{2}{3}\operatorname{div}\boldsymbol{u}\mathbb{I}_3) + \eta\operatorname{div}\boldsymbol{u}\mathbb{I}_3 \tag{2.2.7}$$

où les coefficients de viscosité sont supposés constants (ceci est considéré comme étant satisfaisant dans plusieurs situations) et vérifient

$$\mu > 0 \text{ et } \eta \geq 0. \tag{2.2.8}$$

Le système est complété avec les conditions initiales pour la densité et la quantité de mouvement

$$\varrho(0,\cdot) = \varrho_0,\ \varrho\boldsymbol{u}(0,\cdot) = \varrho_0\boldsymbol{u}_0, \tag{2.2.9}$$

où ϱ_0 et $\boldsymbol{u}_0$ sont deux fonctions données respectivement de Ω dans $\mathbb{R}_+$ et $\mathbb{R}^d$, et la condition au bord

$$\boldsymbol{u} = 0 \text{ dans } (0,T)\times\partial\Omega. \tag{2.2.10}$$

De plus, en prenant le produit scalaire de l'équation (2.2.5b) avec $\boldsymbol{u}$ et en intégrant sur Ω (sous l'hypothèse de regularité suffisante de $(\varrho,\boldsymbol{u})$ et de positivité de ϱ) nous obtenons l'égalité d'énergie (ou identité de dissipation)

$$\partial_t\int_\Omega \frac{1}{2}\varrho|\boldsymbol{u}|^2 + \mathcal{H}(\varrho)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = 0, \tag{2.2.11}$$

où la fonction de Helmholtz $\mathcal{H}$ est définie par

$$\mathcal{H}(\varrho) = \varrho \int_1^{\varrho} \frac{p(t)}{t^2}\, \mathrm{d}t, \ \ \varrho \geq 0, \tag{2.2.12}$$

ou nous avons utilisé la condition au bord pour la vitesse (2.2.10). Dans la suite, nous sommes intéressés par les propriétés des solutions au système d'équations (2.2.5)-(2.2.10).

Nous commençons par la définition de solutions faibles au sens de Leray au problème (2.2.5)-(2.2.10). Elle consiste en une formulation faible standard des équations (2.2.5). L'identité de dissipation (2.2.11) sera remplacé par l'inégalité de dissipation " $\leq$ " dans sa forme intégrale. En effet l'identité (2.2.11) intégrée sur le temps contient la fonctionnelle $\mathbb{Z} \to \int_0^{\tau} \int_{\Omega} \mathbb{S}(\mathbb{Z}) : \mathbb{Z}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t$, $\mathbb{Z} = \nabla \boldsymbol{u}$, qui n'est pas continue mais seulement semi-continue inférieurerement par rapport à la topologie faible sur $L^2((0,T)\times\Omega)^{3\times 3}$.

Definition 1 (Solutions faibles)**.** *Soit Ω un ouvert borné $\mathbb{R}^3$ et soient $\varrho_0 : \Omega \to \mathbb{R}_+$, $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^3$ deux fonctions mesurables de masse finie et d'énergie finie c'est-à-dire*

$$0 < M_0 = \int_{\Omega} \varrho_0 \,\mathrm{d}\boldsymbol{x} < \infty \ \ \textit{et} \ \ \mathcal{E}_0 = \int_{\Omega} \frac{1}{2}\varrho_0 |\boldsymbol{u}_0|^2 + \mathcal{H}(\varrho_0)\,\mathrm{d}\boldsymbol{x} \in (\ \ \infty, \infty). \tag{2.2.13}$$

Nous dirons que le couple $(\varrho, \boldsymbol{u})$ est une solution faible à énergie finie au problème (2.2.5)-(2.2.10) émanant de la donnée initiale $(\varrho_0, \boldsymbol{u}_0)$ si :

1. *Le couple $(\varrho, \boldsymbol{u})$ appartient à la classe*

$$\varrho \in L^{\infty}(0,T;L^1(\Omega)), \ \ \varrho \geq 0 \ \textit{p.p dans} \ (0,T)\times\Omega, \ \ p(\varrho) \in L^1((0,T)\times\Omega),$$

$$\boldsymbol{u} \in L^2(0,T;H_0^1(\Omega)^3), \ \ \varrho\boldsymbol{u}, \ \ \frac{1}{2}\varrho|\boldsymbol{u}|^2, \ \ \mathcal{H}(\varrho) \in L^{\infty}(0,T;L^1(\Omega)).$$

2. *$\varrho \in C_w([0,T];L^1(\Omega))$ et l'équation de continuité (2.1.1a) est satisfaite au sens faible suivant*

$$\int_{\Omega} \varrho(\tau)\varphi(\tau,\cdot)\,\mathrm{d}\boldsymbol{x} \quad \int_{\Omega} \varrho_0\varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x} = \int_0^{\tau}\int_{\Omega} \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{2.2.15}$$

pour tout $\tau \in [0,T]$ et pour toute fonction $\varphi \in C_c^{\infty}([0,T]\times\overline{\Omega})$.

3. *$\varrho\boldsymbol{u} \in C_w([0,T];L^1(\Omega)^3)$ et l'équation de quantité de mouvement (2.1.1b) est satisfaite au sens faible*

$$\int_{\Omega} \varrho\boldsymbol{u}(\tau)\cdot\boldsymbol{\psi}(\tau,\cdot)\,\mathrm{d}\boldsymbol{x} \quad \int_{\Omega} \varrho_0\boldsymbol{u}_0\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}\boldsymbol{x} =$$
$$\int_0^{\tau}\int_{\Omega}\Big(\varrho\boldsymbol{u}\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi} + p(\varrho)\,\mathrm{div}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{2.2.16}$$

pour tout $\tau \in [0,T]$ et pour toute fonction $\boldsymbol{\psi} \in C_c^{\infty}([0,T]\times\Omega)^3$.

4. L'inégalité d'énergie suivante est satisfaite

$$\int_\Omega \frac{1}{2}\varrho(\tau)|\boldsymbol{u}|^2(\tau) + \mathcal{H}(\varrho(\tau))\,\mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \mathcal{E}_0, \tag{2.2.17}$$

pour presque tout $\tau \in (0, T)$.

L'existence de solutions faibles au problème (2.2.5)-(2.2.10) est connue sous l'hypothèse que la pression satisfait en plus de (2.2.6) les hypothèses

$$p'(\varrho) \geq a_1\varrho^{\gamma\ \ 1}\ \ \ b,\ \ \varrho > 0, \tag{2.2.18a}$$

$$p(\varrho) \leq a_2\varrho^\gamma + b,\ \ \varrho \geq 0, \tag{2.2.18b}$$

où $\gamma > \frac{3}{2}$, $a_1 > 0$, $a_2, b \in \mathbb{R}$. Nous renvoyons le lecteur interessé à [?] pour le cas des pressions monotones, à [?] pour les cas des pressions non monotones. Des details supplémentaires concernant ce problème sont disponibles dans les monographes [?], [?], [?]. La pression non monotone dans [?] doit être strictement croissante en dehors d'un ensemble compact de $[0, +\infty)$ comme stipulé dans (2.2.18a). La condition (2.2.18) a été récemment affaiblie par Bresch et Jabin dans [?] nécessitant seulement

$$|p'(\varrho)| \leq C\varrho^{\gamma\ \ 1},\ \ \varrho > 0, \tag{2.2.19a}$$

$$\frac{1}{C}\varrho^\gamma\ \ \ C \leq p(\varrho) \leq C\varrho^\gamma + C,\ \ \varrho \geq 0, \tag{2.2.19b}$$

où $C > 0$ et $\gamma > \frac{9}{5}$.

2.2.2. Energie relative pour les écoulements barotropes

Cette partie est basée sur les articles [?], [?] introduisant l'énergie relative pour les équations de Navier-Stokes compressibles en régime barotrope.

Energie relative et fonction d'energie relative

Introduisons maintenant la notion d'énergie relative. Nous introduisons premièrment la fonction d'énergie relative

$$\begin{aligned} E:\ \ & [0,\infty) \times (0,\infty) \to \mathbb{R}, \\ & (\varrho, r) \mapsto E(\varrho|r) = \mathcal{H}(\varrho)\ \ \ \mathcal{H}'(r)(\varrho\ \ \ r)\ \ \ \mathcal{H}(r), \end{aligned} \tag{2.2.20}$$

où $\mathcal{H}$ est définie par (2.2.12). Si la pression vérifie la condition de stabilité thermodynamique

$$p'(\varrho) > 0 \text{ pour tout } \varrho > 0, \tag{2.2.21}$$

la fonction de Helmholtz $\mathcal{H}$ est strictement convexe sur $\mathbb{R}_+^\star$, et donc

$$E(\varrho|r) \geq 0 \quad \text{et} \quad E(\varrho|r) = 0 \ \Leftrightarrow\ \varrho = r. \tag{2.2.22}$$

Afin de mesurer la “distance” entre une solution faible $(\varrho, \boldsymbol{u})$ des équations de Navier-Stokes compressible et n'importe quel état $(r, \boldsymbol{U})$ du fluide, nous introduisons la fonctionnelle suivante, appelée énergie relative et définie par

$$\mathcal{E}(\varrho, \boldsymbol{u} \Big| r, \boldsymbol{U}) = \int_\Omega \Big(\frac{1}{2} \varrho |\boldsymbol{u} \quad \boldsymbol{U}|^2 + E(\varrho \,|\, r) \Big) \,\mathrm{d}\boldsymbol{x}. \tag{2.2.23}$$

Il apparait que n'importe quelle solution faible saitsfait une inégalité impliquant l'énergie relative appelée inégalité d'énergie relative et ce indépendamment de toute hypothèse de monotonicité satisfaite par la pression. Cependant, il est important de souligner que l'énergie relative mesure une “distance” entre une solution faible $(\varrho, \boldsymbol{u})$ des équations Navier-Stokes compressibles et n'importe quel état $(r, \boldsymbol{U})$ du fluide seulement si la pression vérifie la condition de stabilité thermodynamique (2.2.21).

Ceci est l'objet du théorème suivant

Théorème 1 (Inégalité d'énergie relative)**.** *Si $(\varrho, \boldsymbol{u})$ une solution faible du problème (2.2.5)-(2.2.10) émanant de la condition initale d'énergie finie $(\varrho_0, \boldsymbol{u}_0)$ comme spécifiée dans (2.2.13) alors*

$$\begin{aligned}
&\mathcal{E}(\varrho, \boldsymbol{u} \Big| r, \boldsymbol{U})(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla(\boldsymbol{u} \quad \boldsymbol{U})) : \nabla(\boldsymbol{u} \quad \boldsymbol{U}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\leq \mathcal{E}(\varrho_0, \boldsymbol{u}_0 \Big| r(0), \boldsymbol{U}(0)) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla \boldsymbol{U}) : \nabla(\boldsymbol{U} \quad \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau \int_\Omega \varrho \partial_t \boldsymbol{U} \cdot (\boldsymbol{U} \quad \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + \int_0^\tau \int_\Omega \varrho \boldsymbol{u} \cdot \nabla \boldsymbol{U} \cdot (\boldsymbol{U} \quad \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \int_0^\tau \int_\Omega p(\varrho) \operatorname{div} \boldsymbol{U} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad\qquad + \int_0^\tau \int_\Omega \frac{r \quad \varrho}{r} \partial_t p(r) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \int_0^\tau \int_\Omega \frac{\varrho}{r} \nabla p(r) \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned} \tag{2.2.24}$$

pour presque tout $\tau \in (0, T)$, et pour tout couple de fonctions $(r, \boldsymbol{U})$ tel que

$$r \in C^1([0,T] \times \overline{\Omega}), \; r > 0, \; \boldsymbol{U} \in C^1([0,T] \times \overline{\Omega})^3, \; \boldsymbol{U}_{|(0,T)\times\partial\Omega} = 0. \tag{2.2.25}$$

Inegalité d'énergie relative avec solution forte comme fonctions tests

Si le couple de fonctions $(r, \boldsymbol{U})$ dans l'inégalité d'énergie relative (2.2.24) vérifient les équations presque partout dans $(0,T) \times \Omega$, le membre de droite de cette inégalité devient alors quadratique en la différence $(\varrho \quad r, \boldsymbol{u} \quad \boldsymbol{U})$. Cette observation est le sujet du lemme suivant :

Lemme 1. *Soit Ω un ouvert borné Lipschitzien de $\mathbb{R}^3$. Soit $(\varrho, \boldsymbol{u})$ une solution faible aux équations de Navier-Stokes de conditions initiales et de bord (2.2.9)-(2.2.10). Soit $(r, \boldsymbol{U})$ apartenant à la classe*

$$0 < \underline{r} \leq r \leq \overline{r}, \; r \in L^\infty((0,T) \times \Omega), \; \boldsymbol{U} \in L^2(0,T; H_0^1(\Omega)^3), \tag{2.2.26a}$$

$$\partial_t r, \; \partial_t \boldsymbol{U}, \; \nabla r, \; \nabla \boldsymbol{U} \in L^2(0,T; L^\infty(\Omega)), \; \operatorname{div} \boldsymbol{U} \in L^\infty((0,T) \times \Omega), \tag{2.2.26b}$$

une solution forte de ces mêmes équations de condition initiale $(r(0), \boldsymbol{U}(0)) = (r_0, \boldsymbol{U}_0)$. *Alors nous avons pour presque tout* $\tau \in (0, T)$

$$\mathcal{E}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U})(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla(\boldsymbol{u} - \boldsymbol{U})) : \nabla(\boldsymbol{u} - \boldsymbol{U}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ \leq \mathcal{E}(\varrho_0, \boldsymbol{u}_0 \big| r(0), \boldsymbol{U}(0)) + \int_0^\tau \mathcal{R}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U}) \,\mathrm{d}t$$

où

$$\mathcal{R}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U}) = \int_\Omega (\varrho - r)(\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}) \cdot (\boldsymbol{U} - \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} + \int_\Omega \varrho(\boldsymbol{u} - \boldsymbol{U}) \cdot \nabla \boldsymbol{U} \cdot (\boldsymbol{U} - \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} \\ + \int_\Omega \frac{\nabla p(r)}{r}(r - \varrho) \cdot (\boldsymbol{u} - \boldsymbol{U}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega (p(\varrho) - p'(r)(\varrho - r) - p(r)) \operatorname{div} \boldsymbol{U} \,\mathrm{d}\boldsymbol{x}. \quad (2.2.27)$$

Stabilité, principe d'unicité fort-faible

En général, pour un problème donné, les solutions faibles ne sont malheureusement pas unique (voir par exemple l'article [?] traitant des équations de Navier-Stokes incompressibles en dimension 3) et peuvent aussi faire l'objet de propriétés gênantes, comme l'ont montré Hoff et Serre dans [?]. La plus grande avancée dans le domaine de l'unicité des solutions faibles reste le principe d'unicité fort-faible stipulant qu'une solution faible coincide avec la solution forte issue de même donnée initiale, pourvue que cette dernière existe. Le principe d'unicité fort-faible a déjà été employé pour les écoulements de Navier-Stokes incompressibles par Prodi [?] en 1959, Serrin [?] en 1962. Une approche plus moderne avec des resultats plus pointus pour les équations de Navier-Stokes incompressibles peuvent être trouvés dans [?]. Ce n'est que 50 ans plus tard que ce principe a trouvé son application dans le cas des équations de Navier-Stokes compressibles. Les premiers travaux remontent à Dejardins [?] et Germain [?] qui obtinrent des résultats partiels et conditionnels. Finalement, le principe d'unicité fort-faible dans sa version inconditionnelle a été prouvé dans [?] (voir aussi [?]). Ce n'est que trés récemment dans [?] que le principe d'unicité fort-faible a été prouvé dans le cas du système complet de Navier-Stokes Fourier en formulation entropique introduite dans [?]. Dans tout ces exemples le principe d'unicité fort-faible a été obtenu à l'aide de l'énergie relative.

Nous présentons trois théorèmes de stabilité des solutions fortes dans la classe des solutions faibles.

Dans le premier théorème, nous imposons seulement à la pression de satisfaire la condition de stabilité thermodynamique (2.2.21), tandis que nous supposons à la densité d'être une fonction bornée (essentiellement) "loin" de zéro.

Théorème 2. *Soit* $\Omega \subset \mathbb{R}^3$ *un ouvert borné Lipschitzien. Supposons que la pression* p *appartenant à la classe (2.2.6) est deux fois continuement différentiable sur* $\mathbb{R}_+^\star$ *et vérifie la condition de stabilité thermodynamique (2.2.21). Soit* $(\varrho, \boldsymbol{u})$ *une solution faible aux équations de Navier-Stokes (2.2.5)-(2.2.10) émanant de la donnée initiale* $(\varrho_0, \boldsymbol{u}_0)$ *comme spécifiée dans (2.2.13) sur l'intervalle de temps* $[0, T)$, $T > 0$ *telle que*

$$0 < \underline{\varrho} \leq \varrho(t, \boldsymbol{x}) \leq \overline{\varrho} \text{ p.p dans } (0, T) \times \Omega. \quad (2.2.28)$$

Soit $(r, \boldsymbol{U})$ une solution forte des mêmes équations appartenant à la classe (2.2.26), de donnée initiale $(r_0, \boldsymbol{U}_0)$ vérifiant (2.2.13). Alors il existe $c > 0$, dépendant seulement de

$$\mu, T, \Omega, \underline{\varrho}, \overline{\varrho}, \underline{r}, \overline{r}, \min_{[\underline{r},\overline{r}]} p, \min_{[\underline{r}/2, 2\overline{r}]} p', \|p\|_{C^2([\underline{r}/2,2\overline{r}])}, \|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \|\partial_t \boldsymbol{U}, \nabla \boldsymbol{U}, \nabla r\|_{L^2(0,T;L^\infty(\Omega)^{15})},$$

mais indépendant de la solution faible elle même, telle que pour presque tout τ dans $(0, T)$

$$\int_\Omega \frac{1}{2}\varrho(\tau)|\boldsymbol{u} \quad \boldsymbol{U}|^2(\tau) + (\varrho \quad r)^2(\tau)\,\mathrm{d}\boldsymbol{x} \leq c\Big(\int_\Omega \frac{1}{2}\varrho_0|\boldsymbol{u}_0 \quad \boldsymbol{U}_0|^2 + (\varrho_0 \quad r_0)^2\,\mathrm{d}\boldsymbol{x}\Big). \tag{2.2.29}$$

Remarque 1. *Bien qu'intéressant, ce théorème a un inconvenient majeur. C'est un résultat conditionel dans la mesure où nous ne sommes pas capables de construire une solution faible globale en temps saitsfaisant la condition (2.2.28).*

Dans les deux théorèmes suivants, nous imposons à la pression de satisfaire des hypthèses en plus de la condition de stabilité thermodynamique (2.2.21). En contrepartie nous pouvons travailler avec des solutions faibles sans hypothèses supplementaires sur celles-ci.

Théorème 3. *Soit Ω un ouvert borné Lipschitzien. Supposons que la pression p appartenant à la classe (2.2.6) est deux fois continuement différentiable sur $\mathbb{R}_+^\star$, vérifie la condition de stabilité thermodynamique (2.2.21) ainsi que la condition*

$$c_p(1 + \mathcal{H}(\varrho)) \geq p(\varrho), \ \forall \varrho \geq R_p, \tag{2.2.30}$$

où R_p et c_p sont des constantes positives. Soit $(\varrho, \boldsymbol{u})$ une solution faible aux équations de Navier-Stokes (2.2.5)-(2.2.10), émanant de la donnée initiale $(\varrho_0, \boldsymbol{u}_0)$ spécifiée dans (2.2.13). Soit $(r, \boldsymbol{U})$ une solution forte des mêmes équations appartenant à la classe (2.2.26) de donnée initiale $(r_0, \boldsymbol{U}_0)$ comme dans (2.2.13). Alors il existe une constante positive c, dépendant

$$\mu, T, \Omega, \underline{r}, \overline{r}, \min_{[\underline{r},\overline{r}]} p, \min_{[\underline{r}/2, 2\overline{r}]} p', \|p\|_{C^2([\underline{r}/2,2\overline{r}])}, \|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \|\partial_t \boldsymbol{U}, \nabla \boldsymbol{U}, \nabla r\|_{L^2(0,T;L^\infty(\Omega)^{15})},$$

mais indépendant de la solution faible elle même, telle que pour presque tout τ in $(0, T)$

$$\mathcal{E}(\varrho, \boldsymbol{u} \Big| r, \boldsymbol{U})(\tau) \leq c(\mathcal{E}(\varrho_0, \boldsymbol{u}_0 \Big| r_0, \boldsymbol{U}_0). \tag{2.2.31}$$

En particulier, si $(\varrho_0, \boldsymbol{u}_0) = (r_0, \boldsymbol{U}_0)$, alors

$$(\varrho, \boldsymbol{u}) = (r, \boldsymbol{U}) \text{ dans } [0, T] \times \Omega. \tag{2.2.32}$$

La troisième variante du principe d'unicité fort-faible est la suivante et nous renvoyons le lecteur à [?] pour une preuve.

Théorème 4. *Les conclusions du théorème 3 restent vraies si nous remplaçons la classe de la solution forte (2.2.26) par la plus grande classe*

$$0 < \underline{r} \leq r \leq \overline{r}, \;\; r \in L^\infty((0,T) \times \Omega), \tag{2.2.33a}$$

$$\boldsymbol{U} \in L^2(0,T; H_0^1(\Omega)^3), \tag{2.2.33b}$$

$$\nabla r, \nabla^2 \boldsymbol{U} \in L^2(0,T; L^q(\Omega)), \;\; q > \max(3, \frac{6\gamma}{5\gamma \quad 6}), \tag{2.2.33c}$$

et l'hypothèse (2.2.30) par l'hypothèse plus forte (2.2.18) avec $\gamma > \frac{6}{5}$.

Remarque 2. *La fonction donnée de pression joue un role fondamental d'équilibre dans les équations de Navier-Stokes compressibles. Elle donne la régularité de la solution faible au travers la fonction* $\mathcal{H}$ *et l'égalité d'énergie (2.2.11). En conséquence une pression moins restrictive donne lieu à une solution faible dans une large classe. Cette apparente plus faible hypothèse sur la pression est alors compensée par la regularité plus restrictive de la solution forte pour le principe d'unicité fort-faible.*

L'idée générale de la preuve des théorèmes de stabilité est d'utiliser l'inégalité d'énergie relative avec comme fonction $(r, \boldsymbol{U})$ une solution forte au système dans la forme obtenue dans le lemme 1. Le but est alors d'obtenir une minoration du membre de gauche de (2.2.24) par

$$c \int_0^\tau \|\boldsymbol{u} \quad \boldsymbol{U}\|^2_{W^{1,2}(\Omega)^3} \,\mathrm{d}\boldsymbol{x} \quad \overline{c}' \int_0^\tau \mathcal{E}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U})(t) \,\mathrm{d}t + \mathcal{E}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U})(\tau) \tag{2.2.34}$$

et une majoration du membre de droite par

$$\mathcal{E}(\varrho_0, \boldsymbol{u}_0 \big| r(0), \boldsymbol{U}(0)) + \delta \int_0^\tau \|\boldsymbol{u} \quad \boldsymbol{U}\|^2_{W^{1,2}(\Omega)^3} \,\mathrm{d}\boldsymbol{x} + c'(\delta) \int_0^\tau a(t)\, \mathcal{E}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U})(t) \,\mathrm{d}t \tag{2.2.35}$$

pour tout $\delta > 0$, où $c > 0$ est indépendant de δ, $\overline{c}' \geq 0$, $c' = c'(\delta) > 0$ et $a \in L^1(0,T)$. Nous obtenons alors l'inegalité

$$\mathcal{E}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U})(\tau) \leq \mathcal{E}(\varrho_0, \boldsymbol{u}_0 \big| r(0), \boldsymbol{U}(0)) + c \int_0^\tau a(t)\, \mathcal{E}(\varrho, \boldsymbol{u} \big| r, \boldsymbol{U})(t) \,\mathrm{d}t. \tag{2.2.36}$$

Il suffit alors de conclure en utilisant le lemme de Gronwall.

Applications

Comme mentionné précédemment, l'existence de solutions faibles au problème (2.2.5)-(2.2.10) est connue sous l'hypothèse que la pression satisfait en plus de l'hypothèse (2.2.6) les hypothèses (2.2.18) avec $\gamma > \frac{3}{2}$. D'un autre coté, il existe une vaste litterature concernant l'existence de solutions fortes au système (2.2.5)-(2.2.10) (ou même pour des systèmes plus complexes impliquant aussi la conductivité thermique du fluide) sur un domaine fixe, de telles solutions existant localement en temps (voir [?], [?], [?] pour l'espace entier, [?] [?], [?], [?] pour un domaine borné avec les conditions au bord de Dirichlet) ou globalement dès que la donnée initiale est suffisament proche d'un état d'équilibre (voir [?], [?] pour l'espace entier, voir [?], [?], [?] pour un domaine borné avec les conditions au bord de Dirichlet). Le cas du demi-espace et d'un domaine extérieure a été étudié dans [?]. Ces résultats d'existence nous permettent d'obtenir des résultats d'unicité fort-faible non vides. Nous utilisons ici [?], [?] et [?]. Introduisons

$$
\begin{aligned}
S^q_{[0,T]\times\Omega} = \{(\varrho,\boldsymbol{u}),\ & \varrho \in C([0,T];W^{q,2}(\Omega)) \cap C^1([0,T];W^{q\quad 1,2}(\Omega)),\\
& \boldsymbol{u} \in L^2(0,T;W^{q+1,2}(\Omega)^3) \cap W^{1,2}(0,T;W^{2,2}(\Omega)^3) \cap W^{2,2}(0,T;L^2(\Omega)^3) \text{ si } q=2,\\
& \qquad \boldsymbol{u} \in W^{4,2}((0,T)\times\Omega)^3 \text{ si } q=3,\ \min_{[0,T]\times\overline{\Omega}} \varrho > 0,\ \boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}\}.
\end{aligned}
$$

Le théorème suivant peut être déduit du théorème A dans Valli [?] et du théorème 2.5 dans Valli, Zajaczkowski [?] :

Théorème 5. *Soit Ω un ouvert borné de classe C^{q+1}. Supposons que la pression $p \in C^q(\mathbb{R}^\star_+)$. Soit $q = 2,3$. Finalement soit*

$$(\varrho_0, \boldsymbol{u}_0) \in W^{q,2}(\Omega) \times W^{3,2}(\Omega)^3,\ 0 < m \leq \varrho_0 \leq M,\ \boldsymbol{u}_0 = \boldsymbol{0} \text{ on } \partial\Omega. \tag{2.2.37}$$

satisfaisant la condition de compatiblité au bord

$$\frac{1}{\varrho_0}\Big(\quad \nabla p(\varrho_0) + \operatorname{div}(\mathbb{S}(\nabla \boldsymbol{u}_0)) \quad \varrho_0 \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0\Big)_{|\partial\Omega} = \boldsymbol{0}. \tag{2.2.38}$$

Alors il existe T suffisament petit tel que le problème (2.2.5)-(2.2.10) admet une unique solution forte $(\varrho, \boldsymbol{u}) \in S^q_{[0,T]\times\Omega}$ émanant de $(\varrho_0, \boldsymbol{u}_0)$.

Corollaire 1. *Toute solution faible émanant d'une donnée initiale au sens de Valli-Zajaczkowski (c'est-à-dire appartenant à (2.2.37)) avec $q = 3$ coincide avec l'unique solution forte maximale au sens de Valli-Zajaczkowski sur au moins l'intervalle maximal existence $[0, T_M)$ de la solution forte maximale au sens de Valli-Zajaczkowski dès lors que la pression satisfait $p \in C^3(\mathbb{R}^\star_+)$ et les hypothèses du théorème 3 et Ω est un ouvert borné de classe C^4.*

Corollaire 2. *Toute solution faible émanant d'une donnée initiale au sens de Valli-Zajaczkowski (c'est-à-dire appartenant à (2.2.37)) avec $q = 2$ coincide avec l'unique solution forte maximale au sens de Valli-Zajaczkowski sur au moins l'intervalle maximale d'existence $[0, T_M)$ de la solution forte maximale au sens de Valli-Zajaczkowski dès lors que la pression satisfait les hypothèses du théroème 4 où γ satisfait la restriction d'existence $\gamma > \frac{3}{2}$ et Ω est un ouvert borné de classe C^3.*

Dans le théorème suivant, nous énonçons un résultat de solutions fortes pour lequel nous pouvons utiliser des conditions intiales qui ne satisfont pas la condition de compatibilité (2.2.38). Dans la seconde partie du théorème, nous rappelons un critère d'explosion de type Beale-Kato-Majda en terme de borne supérieure de la densité de la solution forte des équations de Navier-Stokes compressibles en régime isentropique. Introduisons

$$\begin{aligned}\tilde{S}^q_{[0,T]\times\Omega} = \{(\varrho, \boldsymbol{u}),\ \varrho \in C([0,T]; W^{1,q_0}(\Omega)) \cap C^1([0,T]; L^{q_0}(\Omega)),\\ \boldsymbol{u} \in L^2(0,T; W^{2,q_0}(\Omega)^3) \cap C([0,T]; W^{2,2}(\Omega)^3) \cap W^{1,2}(0,T; W^{1,2}_0(\Omega)^3),\\ \sqrt{\varrho}\partial_t \boldsymbol{u} \in L^\infty(0,T, L^2(\Omega)^3),\ \boldsymbol{u}_{|(0,T)\times\partial\Omega} = \mathbf{0}\}.\end{aligned}$$

où $q_0 = \min(q, 6)$. Notons que $S^q_{[0,T]\times\Omega} \subset \tilde{S}^q_{[0,T]\times\Omega}$ pour tout $3 < q < \infty$. Alors nous avons (voir [?, Proposition 5] pour l'existence local en temps de la solution forte et [?, Theorem 1.3] pour le critère d'explosion)

Théorème 6. *Soit Ω un ouvert borné de classe C^3. Supposons que la pression $p \in C^1(\mathbb{R}_+)$. Finalement soit*

$$(\varrho_0, \boldsymbol{u}_0) \in W^{1,q}(\Omega) \times W^{2,2}(\Omega)^3,\ 0 < m \leq \varrho_0 \leq M,\ \boldsymbol{u}_0 = \mathbf{0} \text{ sur } \partial\Omega, \tag{2.2.39}$$

où $3 < q < \infty$. Alors il existe T suffisament petit tel que le problème (2.2.5)-(2.2.10) admet une unique solution classique $(\varrho, \boldsymbol{u}) \in \tilde{S}^q_{[0,T]\times\Omega}$ émanant de $(\varrho_0, \boldsymbol{u}_0)$. De plus, si le temps maximal d'existence de la solution T_M satisfait $T_M < \infty$, si la pression vérifie (2.2.18) et si les coefficients de viscosité satisfont

$$\eta < \frac{23}{3}\mu, \tag{2.2.40}$$

alors on a :

$$\limsup_{T\to T_M} \|\varrho\|_{L^\infty(0,T;L^\infty(\Omega))} = +\infty.$$

Corollaire 3. *Toute solution faible émanant d'une donnée initiale au sens de Cho, Choe, Kim (c'est-à-dire appartenant à (2.2.39)) avec $q \geq 6$ coincide avec l'unique solution forte maximale au sens de Cho, Choe, Kim maximal strong solution sur au moins l'intervalle d'existence $[0, T_M)$ de la solution forte maximale au sens de Cho, Choe, Kim dès lors que la pression satisfait les hypothèses du théorème 4 où γ satisfait la restriction d'existence $\gamma > \frac{3}{2}$ et Ω est un ouvert borné de classe C^3.*

Corollaire 4. *Toute solution faible émanant d'une donnée initiale au sens de Cho, Choe, Kim sur un ouvert borné de classe C^3 est en fait une solution forte tant que la densité reste borné et dès lors que la pression satisfait les hypothèses du théorème 4 et $p \in C^1(\mathbb{R}^\star_+)$ où γ satisfait la restriction d' existence $\gamma > \frac{3}{2}$ et les coefficients de viscosité satisfont (2.2.40).*

Remarque 3. *Nous pouvons aussi montrer que toute solution faible au système (2.2.5)-(2.2.10) émanant d'une donnée initiale appartenant à l'espace de Sobolev $W^{3,2}$ reste regulière tant que la gradient de vitesse est borné. La preuve est basé sur le principe d'unicité fort-faible et sur des estimations paraboliques a priori pour les solutions fortes locales en temps, voir [?].*

3. Limites singulières

3.1. Introduction

Bien que les écoulements de fluide sont en général tri-dimensionnels, dans plusieurs situations la forme spécifique du domaine physique impose des changements majeurs dans la densité et la vitesse dans seulement deux directions et même éventuellement seulement dans une direction. Un exemple typique est celui d'un fluide confiné dans une couche plane et qui peut être décrit en utilisant seulement deux variables. Plus précisement, dans [**?**], nous avons considéré les équations de Navier-Stokes compressibles dans le domaine mince

$$\Omega_\epsilon = \omega \times (0, \epsilon), \tag{3.1.1}$$

où $\omega \subset \mathbb{R}^2$ est un domaine régulier du plan. Plus précisément, nous considérons le système suivant

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \quad \text{dans } (0,T) \times \Omega_\epsilon, \tag{3.1.2}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\varrho) = \operatorname{div}\mathbb{S}(\nabla \boldsymbol{u}) \quad \text{dans } (0,T) \times \Omega_\epsilon. \tag{3.1.3}$$

où le tenseur des contraintes visqueuses $\mathbb{S}(\nabla \boldsymbol{u})$ est décrit par la loi de Newton

$$\mathbb{S}(\nabla \boldsymbol{u}) = \mu(\nabla \boldsymbol{u} + \nabla^t \boldsymbol{u} \quad \frac{2}{3} \operatorname{div} \boldsymbol{u} \mathbb{I}_3) + \eta \operatorname{div} \boldsymbol{u} \mathbb{I}_3, \tag{3.1.4}$$

les coefficients de viscosité étant supposés constants et vérifiant

$$\mu > 0, \ \eta \geq 0, \tag{3.1.5}$$

et la pression satisfait

$$p \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^\star), \ p(0) = 0, \tag{3.1.6}$$

$$p'(\varrho) > 0 \text{ pour tout } \varrho > 0, \ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma\ 1}} = p_\infty > 0, \ \gamma > \frac{3}{2}.$$

Ces équations sont complétées avec les conditions initiales

$$\varrho(0, \boldsymbol{x}) = \tilde{\varrho}_{0,\epsilon}(\boldsymbol{x}), \quad \boldsymbol{u}(0, \boldsymbol{x}) = \tilde{\boldsymbol{u}}_{0,\epsilon}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_\epsilon, \tag{3.1.7}$$

tandis que les conditions au bord seront prescrites plus tard en fonction de ω.

Le but est d'étudier la limite quand $\epsilon \to 0$.

La justification rigoureuse du passage à la limite du mouvement 3D d'un fluide vers un mouvement plan semble d'une importance pratique évidente. Cependant, pour les écoulements compressibles, à notre connaissance, il n'y avait pas de résultat concernant le passage à la limite 3D-2D et seulement quelques résultats concernant la réduction 3D-1D, voir [?] ou plus récemment [?]. Il y a plusieurs études d'écoulements de fluides incompressibles dans des domaines fins, où le mouvement limite devient plan, voir Iftimie, Raugel et Sell [?], Raugel et Sell [?] et les références qui y figurent. Ce travail développe et adapte les idées introduites dans [?] à la problématique de la réduction 3D-2D dans les écoulements de fluides compressibles.

L'analyse de problème similaire de réduction en théorie de l'élasticité s'appuie sur des variantes de l'inégalité de Korn qui fournit des estimations sur le gradient d'une fonction vectorielle $\boldsymbol{v}$ en terme de son gradient symétrique, plus précisement,

$$\|\nabla \boldsymbol{v}\|_{L^2(\Omega_\epsilon)^{3\times 3}} \leq c(\epsilon)\|\nabla \boldsymbol{v} + \nabla^t \boldsymbol{v}\|_{L^2(\Omega_\epsilon)^{3\times 3}}, \tag{3.1.8}$$

De façon évidente, la validité de (3.1.8) implique que le noyau de l'opérateur linéaire $\boldsymbol{v} \mapsto \nabla \boldsymbol{v} + \nabla^t \boldsymbol{v}$ soit non réduit à zéro sur l'espace des champs de vecteur satsifaisant la condition au bord donnée. De plus, même si l'inégalité (3.1.8) est satisfaite à $\epsilon > 0$ fixé, la constante $c(\epsilon)$ peut exploser pour $\epsilon \to 0$ à moins que quelques restrictions nécessaires soient imposées sur le champ $\boldsymbol{v}$, et ceci est vraie même si l'ensemble ω n'est pas à symétrie centrale, cf. Lewicka et Müller [?]. Il n'est pas diffcile de voir que les problèmes similaires survenant dans le cadre des fluides compressibles nécessiterons un anologue plus fort de (3.1.8) à savoir

$$\|\nabla \boldsymbol{v}\|_{L^2(\Omega_\epsilon)^{3\times 3}} \leq c(\epsilon)\|\nabla \boldsymbol{v} + \nabla^t \boldsymbol{v} \quad \frac{2}{3}\operatorname{div} \boldsymbol{v}\mathbb{I}_3\|_{L^2(\Omega_\epsilon)^{3\times 3}}. \tag{3.1.9}$$

évidemment relié à la composante visqueuse de cisaillement dans le tenseur des contraintes visqueuses, voir Dain [?], Reshetnyak [?]. En vue des difficultés mentionnées au dessus en rapport avec la validité de (3.1.8) ou (3.1.9), notre approche se repose sur la stabilité structurelle de la famille des solutions faibles au système de Navier-Stokes compressible barotrope encodée dans l'inégalité d'énergie relative. Cette méthode est en gros indépendante de la forme spécifique du tenseur des contraintes visqueuses et de possible estimations dissipatives pour le système de Navier-Stokes.

Dans cette étude nous faisons le choix d'effectuer un changement d'échelle dans les équations et de se ramener à un domaine fixe. Considérant le changement de variables

$$\Omega_\epsilon \ni (\boldsymbol{x}_h, \epsilon x_3) \mapsto (\boldsymbol{x}_h, x_3) \in \Omega := \Omega_1, \quad \text{où } \boldsymbol{x}_h = (x_1, x_2), \tag{3.1.10}$$

et en notant respectivement la nouvelle densité et la nouvelle vitesse encore par ϱ et $\boldsymbol{u}$, nous pouvons réécrire le système (3.1.2)-(3.1.3) et les conditions initiales (3.1.7) de la manière suivante :

$$\partial_t \varrho + \operatorname{div}_\epsilon(\varrho \boldsymbol{u}) = 0 \quad \text{dans } (0,T)\times\Omega, \tag{3.1.11}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_\epsilon(\varrho \boldsymbol{u}\otimes \boldsymbol{u}) + \nabla_\epsilon p(\varrho) = \operatorname{div}_\epsilon \mathbb{S}(\nabla_\epsilon \boldsymbol{u}) \quad \text{dans } (0,T)\times\Omega, \tag{3.1.12}$$

$$\varrho(0,\boldsymbol{x}) = \varrho_{0,\epsilon}(\boldsymbol{x}) \quad \boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_{0,\epsilon}(\boldsymbol{x}), \quad x\in\Omega \tag{3.1.13}$$

(où $\varrho_{0,\epsilon}(\boldsymbol{x}) = \tilde{\varrho}_{0,\epsilon}(\boldsymbol{x}_h, \epsilon x_3)$, $\boldsymbol{u}_{0,\epsilon}(\boldsymbol{x}) = \tilde{\boldsymbol{u}}_{0,\epsilon}(\boldsymbol{x}_h, \epsilon x_3)$, cf. (3.1.7)). Les conditions au bord seront prescrites plus tard en fonction de ω

Ici et plus tard, nous notons

$$\nabla_\epsilon = (\nabla_h, \frac{1}{\epsilon}\partial_{x_3}),\ \nabla_h = (\partial_{x_1}, \partial_{x_2}),$$

$$\mathrm{div}_\epsilon \boldsymbol{v} = \mathrm{div}_h \boldsymbol{v}_h + \frac{1}{\epsilon}\partial_{x_3} v_3,\ \boldsymbol{v}_h = (v_1, v_2),\ \mathrm{div}_h \boldsymbol{v}_h = \partial_{x_1} v_1 + \partial_{x_2} v_2.$$

Le but de ce travail est d'étudier la limite quand $\epsilon \to 0$ dans le système d'équations (3.1.11)–(3.1.13), dés que la donnée initiale $[\varrho_{0,\epsilon}, \boldsymbol{u}_{0,\epsilon}](\boldsymbol{x})$ converge en un certain sens vers $[r_0, \boldsymbol{v}_0](\boldsymbol{x}) = [r_0, \boldsymbol{v}_{0,h}, 0](\boldsymbol{x}_h)$. Puisque la donnée initlale limite ne dépend pas de la variable verticale x_3, il est naturel d'attendre que la suite $[\varrho_\epsilon, \boldsymbol{u}_\epsilon](t, \boldsymbol{x})$ de solutions faibles à (3.1.11)–(3.1.13) converge vers $[r, \boldsymbol{V}](t, \boldsymbol{x}_h)$, $\boldsymbol{V} = [\boldsymbol{w}, 0]$, où le couple $[r(t, \boldsymbol{x}_h), \boldsymbol{w}(t, \boldsymbol{x}_h)]$ est solution des équations de Navier-Stokes compressibles 2 D dans le domaine ω :

$$\partial_t r + \mathrm{div}_h(r\boldsymbol{w}) = 0 \text{ dans } (0,T)\times\omega, \tag{3.1.14}$$

$$r\partial_t \boldsymbol{w} + r\boldsymbol{w}\cdot\nabla_h \boldsymbol{w} + \nabla_h p(r) = \mathrm{div_h}\mathbb{S}_h(\nabla_h \boldsymbol{w}) \text{ dans } (0,T)\times\omega, \tag{3.1.15}$$

$$r(0, \boldsymbol{x}_h) = r_0(\boldsymbol{x}_h),\ \boldsymbol{w}(0, \boldsymbol{x}_h) = \boldsymbol{w}_0 := \boldsymbol{v}_{0,h}(\boldsymbol{x}_h),\ \boldsymbol{x}_h \in \omega, \tag{3.1.16}$$

où

$$\mathbb{S}_h(\nabla_h \boldsymbol{w}) = \mu\Big(\nabla_h \boldsymbol{w} + (\nabla_h \boldsymbol{w})^T \quad \mathrm{div}_h \boldsymbol{w}\Big) + (\eta + \frac{\mu}{3})\mathrm{div}_h \boldsymbol{w}\mathbb{I}_h,$$

et $\mathbb{I}_h$ est la matrice identité en dimension 2.

Notre but est de justifier la précédente limite formelle dans le cadre des solutions faibles du système (3.1.11), (3.1.12), (3.1.13).

Nous avons consideré dans [?] plusieurs situations géométriques. Dans la suite nous présentons le cas des solutions périodiques.

3.2. Résultats

Nous considérons donc le système (3.1.2–3.1.3), (3.1.7) dans le domaine mince Ω_ϵ (voir (3.1.1)), où $\omega = [0,1]^2\Big|_{0,1}$ est une cellule périodique de dimension 2 et de période 1 dans chaque direction. Il est complété avec les conditions au bord de non glissement en haut et en bas du domaine

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega_\epsilon} = 0,\ [\mathbb{S}(\nabla\boldsymbol{u})\cdot\boldsymbol{n}]\times\boldsymbol{n}|_{\partial\Omega_\epsilon} = 0. \tag{3.2.1}$$

Aprés un changement d'échelle, cette situation correspond au système (3.1.11–3.1.13) sur Ω (voir (3.1.10)) completé avec les conditions aux bords

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0,\ [\mathbb{S}(\nabla_\epsilon\boldsymbol{u})\cdot\boldsymbol{n}]\times\boldsymbol{n}|_{\partial\Omega} = 0, \tag{3.2.2}$$

Puisque ω est une cellule périodique, les conditions (3.2.2) signifient que

$$u_3|_{\omega\times\{0,1\}} = 0,\ \ \Big(\partial_{x_k} u_3 + \frac{1}{\epsilon}\partial_{x_3} u_k\Big)|_{\omega\times\{0,1\}} = 0,\ k = 1, 2, \tag{3.2.3}$$

où $\boldsymbol{u}(\cdot, x_3)$, $x_3 \in (0,1)$ sont des fonctions 1-périodique en la variable $\boldsymbol{x}_h$.

Solutions faibles et énergie relative

Definition 2 (Solutions faibles)**.** *Notons*

$$W^{1,2}_{\boldsymbol{n}}(\Omega) = \{\boldsymbol{v} \in W^{1,2}(\Omega)^3,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}.$$

Soient $\varrho_0 : \Omega \to \mathbb{R}_+$, $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^3$ *deux fonctions mesurables de masse et d'énergie finies c'est-à-dire*

$$0 < M_0 = \int_\Omega \varrho_0 \,\mathrm{d}\boldsymbol{x} < \infty \text{ and } \mathcal{E}_0 = \int_\Omega \frac{1}{2}\varrho_0|\boldsymbol{u}_0|^2 + \mathcal{H}(\varrho_0)\,\mathrm{d}\boldsymbol{x} \in (\ \ \infty,\infty). \tag{3.2.4}$$

Nous dirons que le couple $(\varrho, \boldsymbol{u})$ *est une solution faible à énergie finie au problème (3.1.11)-(3.1.12) émanant de la condition initiale* $(\varrho_0, \boldsymbol{u}_0)$ *et de condition au bord (3.2.2) si :*

1. *Le couple* $(\varrho, \boldsymbol{u})$ *appartient à la classe suivante*

$$\varrho \in L^\infty(0,T;L^1(\Omega)),\ \varrho \geq 0 \text{ p.p dans } (0,T)\times\Omega,\ p(\varrho) \in L^1((0,T)\times\Omega),$$

$$\boldsymbol{u} \in L^2(0,T;W^{1,2}_{\boldsymbol{n}}(\Omega)),\ \varrho\boldsymbol{u},\ \frac{1}{2}\varrho|\boldsymbol{u}|^2,\ \mathcal{H}(\varrho) \in L^\infty(0,T;L^1(\Omega)).$$

2. $\varrho \in C_w([0,T];L^1(\Omega))$ *et l'équation de continuité (2.1.1a) est satisfaite dans le sens faible*

$$\int_\Omega \varrho(\tau)\varphi(\tau,\cdot)\,\mathrm{d}\boldsymbol{x} \quad \int_\Omega \varrho_0\varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla_\epsilon\varphi\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{3.2.6}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\varphi \in C^\infty_c([0,T]\times\overline{\Omega})$.

3. $\varrho\boldsymbol{u} \in C_w([0,T];L^1(\Omega)^3)$ *et l'équation de quantité de mouvement (2.1.1b) est satisfaite au sens faible*

$$\int_\Omega \varrho\boldsymbol{u}(\tau)\cdot\boldsymbol{\psi}(\tau,\cdot)\,\mathrm{d}\boldsymbol{x} \quad \int_\Omega \varrho_{\epsilon,0}\boldsymbol{u}_{\epsilon,0}\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}\boldsymbol{x} =$$
$$\int_0^\tau \int_\Omega \Big(\varrho\boldsymbol{u}\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_\epsilon\boldsymbol{\psi} + p(\varrho)\,\mathrm{div}_\epsilon\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \mathbb{S}(\nabla_\epsilon\boldsymbol{u}) : \nabla_\epsilon\boldsymbol{\psi}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{3.2.7}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\boldsymbol{\psi} \in C^\infty_c([0,T]\times\overline{\Omega})^3$ *vérifiant* $\boldsymbol{\psi}\cdot\boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0$.

4. *L' inegalité d'énergie suivante*

$$\int_\Omega \frac{1}{2}\varrho(\tau)|\boldsymbol{u}|^2(\tau) + \mathcal{H}(\varrho(\tau))\,\mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_\epsilon\boldsymbol{u}) : \nabla_\epsilon\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq \mathcal{E}_0\,. \tag{3.2.8}$$

est satisfaite pour presque tout $\tau \in (0,T)$.

Notons que la définition de solution faible pour le système (3.1.2), (3.1.3), (3.1.7), (3.2.1) avant changement d'échelle peut être obtenue en replacant Ω par Ω_ϵ, ∇_ϵ par ∇, div_ϵ par div, $\varrho_{0,\epsilon}$ par $\tilde{\varrho}_{0,\epsilon}$ et $\boldsymbol{u}_{0,\epsilon}$ par $\tilde{\boldsymbol{u}}_{0,\epsilon}$. De ce fait, le système (3.1.11)-(3.1.13), (3.2.2) aprés changement d'échelle admet une solution faible à énergie finie (voir Chapter 2). De plus, comme dans le chapitre 2 toute solution faible vérifie l'inégalité d'énergie relative suivante.

Théorème 7. *Soit $(\varrho_\epsilon, \boldsymbol{u}_\epsilon)$ une solution faible à énergie finie au problème (3.1.11)-(3.1.12), (3.2.2) émanant de la condition initiale $(\varrho_{0,\epsilon}, \boldsymbol{u}_{0,\epsilon})$ comme spécifiée dans (3.2.4) et de condition au bord (3.2.2). Alors*

$$\begin{aligned}
&\mathcal{E}(\varrho, \boldsymbol{u} \Big| r, \boldsymbol{U})(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_\epsilon(\boldsymbol{u} \quad \boldsymbol{U})) : \nabla_\epsilon(\boldsymbol{u} \quad \boldsymbol{U}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\leq \mathcal{E}(\varrho_{0,\epsilon}, \boldsymbol{u}_{0,\epsilon} \Big| r(0), \boldsymbol{U}(0)) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_\epsilon \boldsymbol{U}) : \nabla_\epsilon(\boldsymbol{U} \quad \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^\tau \int_\Omega \varrho \partial_t \boldsymbol{U} \cdot (\boldsymbol{U} \quad \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\qquad + \int_0^\tau \int_\Omega \varrho \boldsymbol{u} \cdot \nabla_\epsilon \boldsymbol{U} \cdot (\boldsymbol{U} \quad \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \quad \int_0^\tau \int_\Omega p(\varrho) \, \mathrm{div}_\epsilon \boldsymbol{U} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\qquad\qquad + \int_0^\tau \int_\Omega \frac{r \quad \varrho}{r} \partial_t p(r) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \quad \int_0^\tau \int_\Omega \frac{\varrho}{r} \nabla_\epsilon p(r) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t
\end{aligned}$$

pour presque tout $\tau \in (0, T)$ et pour tout couple de fonctions $(r, \boldsymbol{U})$ telle que

$$r \in C^1([0,T] \times \overline{\Omega}), \ r > 0, \ \boldsymbol{U} \in C^1([0,T] \times \overline{\Omega})^3, \ \boldsymbol{U} \cdot \boldsymbol{n}_{|(0,T) \times \partial\Omega} = 0. \tag{3.2.9}$$

Problème limite et résultat principal

Le théorème d'existence suivant concerne l'existence de solution forte au problème (3.1.14)-(3.1.16) en domaine périodique. Il peut être déduit du théorème 2.5 dans Valli et Zajackzkowski [?] :

Proposition 1. *Soit D une constante positive. Supposons que $p \in C^2(\mathbb{R}_+^\star)$ et que*

$$r_0 \in W^{2,2}(\omega), \ \inf_\omega r_0 > 0, \ \boldsymbol{w}_0 \in W^{3,2}(\omega)^2. \tag{3.2.10}$$

Alors il existe $T = T_{max}(D)$ tel que si

$$\|r_0\|_{W^{2,2}(\omega)} + \|\boldsymbol{w}_0\|_{W^{3,2}(\omega)^2} + \frac{1}{\inf_\omega r_0} \leq D, \tag{3.2.11}$$

alors le problème (3.1.14)-(3.1.16) admet une unique solution forte maximale dans la classe

$$\begin{aligned}
r \in C([0,T]; W^{2,2}(\omega)), \ \boldsymbol{w} \in C([0,T]; W^{2,2}(\omega)^2) \cap L^2(0,T; W^{3,2}(\omega)^2) \qquad (3.2.12)\\
\partial_t r \in C([0,T]; W^{1,2}(\omega)), \ \partial_t \boldsymbol{w} \in L^2(0,T; W^{2,2}(\omega)^2).
\end{aligned}$$

En particulier,

$$0 < \underline{r} \equiv \inf_{(t,\boldsymbol{x}_h) \in (0,T) \times \omega} r(t, \boldsymbol{x}_h) \leq \sup_{(t,\boldsymbol{x}_h) \in (0,T) \times \omega} r(t, x_h) \equiv \overline{r}. \tag{3.2.13}$$

Nous sommes maintenant en mesure de formuler the résultat principal de cette section (voir [**?**, Theorem 2.1]).

Théorème 8 (Maltese, Novotný, 2014, [**?**])**.** *Supposons que la pression p vérifie les hypothèses (3.1.6), et que le tenseur des contraintes visqueuses est donné par (3.1.4). Soit $(r_0, \boldsymbol{w}_0)$ satisfaisant les hypothèses (3.2.10) et soit $T_{\max} > 0$ le temps maximale d'existence de la solution forte au problème (3.1.14)-(3.1.16) émanat de $[r_0, \boldsymbol{w}_0]$ determinée dans la proposition 1. Soit $(\varrho_\epsilon, \boldsymbol{u}_\epsilon)$ une suite de solution faibles aux équations de Navier-Stokes compressibles 3 D (3.1.11)-(3.1.13), (3.2.2) émanant de la donnée initiale $[\varrho_{0,\epsilon}, \boldsymbol{u}_{0,\epsilon}]$. Supposons que*

$$\mathcal{E}\Big(\varrho_{0,\epsilon}, \boldsymbol{u}_{0,\epsilon}\Big| r_0, \boldsymbol{V}_0\Big) \to 0, \tag{3.2.14}$$

où $\boldsymbol{V}_0 = [\boldsymbol{w}_0, 0]$.

Alors pour tout $T < T_{\max}$,

$$\operatorname{supess}_{t\in(0,T)} \mathcal{E}(\varrho_\epsilon, \boldsymbol{u}_\epsilon \Big| r, \boldsymbol{V}) \to 0, \tag{3.2.15}$$

où $\boldsymbol{V}(t, \boldsymbol{x}) = [\boldsymbol{w}(t, \boldsymbol{x}_h), 0]$ et où le couple $(r, \boldsymbol{w})$ satisfait le système de Navier-Stokes compressible 2 D (3.1.14)–(3.1.16) dans la cellule périodique ω et sur l'intervalle de temps $[0, T_{\max})$.

Remarque 4. *Afin d'y voir plus clair dans le sens de la limite au dessus, nous remarquons que (3.2.15) implique, par exemple*

$$\varrho_\epsilon \to r \text{ fortement dans } L^\infty(0, T; L^\gamma(\Omega)),$$

$$\sqrt{\varrho_\epsilon}\boldsymbol{u}_\epsilon \to \sqrt{r}\boldsymbol{V} \text{ fortement dans } L^\infty(0, T; L^2(\Omega)^3),$$

$$\varrho_\epsilon \boldsymbol{u}_\epsilon \to r\boldsymbol{V} \text{ fortement dans } L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)^3).$$

Remarque 5. *Nous remarquons que la reduction 3D-2D augmente la viscosité volumique bidimensionel du fluide de η à $\eta + \frac{\mu}{3}$, cf. (3.1.14)-(3.1.16).*

Corollaire 5. *Supposons que la pression p, le tenseur des contraintes visqueuses $\mathbb{S}$ vérifient les hypothèses du Théorème 8. Supposons que $[\varrho_{\epsilon,0}, \boldsymbol{u}_{\epsilon,0}]$, $\varrho_\epsilon \geq 0$ vérifie*

$$\int_0^1 \varrho_{\epsilon,0}(\boldsymbol{x}) \mathrm{d}x_3 \rightharpoonup r_0 \text{ faiblement dans } L^\gamma(\omega)^3, \tag{3.2.16}$$

$$\int_0^1 \varrho_{\epsilon,0}\boldsymbol{u}_{\epsilon,0} \mathrm{d}x_3 = r_0 \boldsymbol{V}_0 \text{ faiblement dans } L^{\frac{2\gamma}{\gamma+1}}(\omega)^3,$$

où $\boldsymbol{V}_0 = [\boldsymbol{w}_0, 0]$ et $[r_0, \boldsymbol{w}_0]$ appartient à la classe (3.2.10), et

$$\int_\Omega \Big[\frac{1}{2}\varrho_{\epsilon,0}\boldsymbol{u}_{\epsilon,0}^2 + \mathcal{H}(\varrho_{\epsilon,0})\Big] \mathrm{d}\boldsymbol{x} \to \int_\omega \Big[\frac{1}{2} r_0 \boldsymbol{w}_0^2 + \mathcal{H}(r_0)\Big] \mathrm{d}\boldsymbol{x}_h. \tag{3.2.17}$$

Finalement, soit $[\varrho_\epsilon, \boldsymbol{u}_\epsilon]$ une suite de solutions faibles aux équations de Navier-Stokes compressibles 3 D (3.1.11)-(3.1.13), (3.2.2) emmanant de la condition initiale $[\varrho_{0,\epsilon}, \boldsymbol{u}_{0,\epsilon}]$. Alors (3.2.15) est satisfait.

Le corollaire 5 peut être reformulé en terme de suite de solutions $[\tilde{\varrho}_\epsilon, \tilde{\boldsymbol{u}}_\epsilon]$ au problème original sans changment d'échelle (3.1.2–3.1.7), (3.2.1).

Corollaire 6. *Supposons que la pression p, le tenseur des contraintes visqueuses $\mathbb{S}$ vérifient les hypothèses du Théorème 8. Supposons que $[\tilde{\varrho}_{\epsilon,0}, \tilde{\boldsymbol{u}}_{\epsilon,0}]$, $\tilde{\varrho}_\epsilon \geq 0$ vérifie*

$$\frac{1}{\epsilon}\int_0^\epsilon \tilde{\varrho}_{\epsilon,0}(\boldsymbol{x})\mathrm{d}x_3 \rightharpoonup r_0 \text{ faiblement dans } L^1(\omega), \tag{3.2.18}$$

$$\frac{1}{\epsilon}\int_0^\epsilon \tilde{\varrho}_{\epsilon,0}\tilde{\boldsymbol{u}}_{\epsilon,0}\mathrm{d}x_3 = r_0\boldsymbol{V}_0 \text{ faiblement dans } L^1(\omega)^3,$$

où $\boldsymbol{V}_0 = [\boldsymbol{w}_0, 0]$ et $[r_0, \boldsymbol{w}_0]$ appartiennent à la classe (3.2.10), et

$$\frac{1}{\epsilon}\int_{\Omega_\epsilon}\left[\frac{1}{2}\tilde{\varrho}_{\epsilon,0}\tilde{\boldsymbol{u}}_{\epsilon,0}^2 + H(\tilde{\varrho}_{\epsilon,0})\right]\mathrm{d}\boldsymbol{x} \to \int_\omega \left[\frac{1}{2}r_0\boldsymbol{w}_0^2 + H(r_0)\right]\mathrm{d}\boldsymbol{x}_h. \tag{3.2.19}$$

Finalement, soit $(\tilde{\varrho}_\epsilon, \tilde{\boldsymbol{u}}_\epsilon)$ une suite de solutions faibles aux équations de Navier-Stokes compressibles 3 D (3.1.2)-(3.1.7), (3.2.1) emmanant de la condition initiale $[\tilde{\varrho}_{0,\epsilon}, \tilde{\boldsymbol{u}}_{0,\epsilon}]$. Alors

$$\operatorname{supess}_{t\in(0,T_{\max})}\mathcal{E}_\epsilon\left(\tilde{\varrho}_\epsilon, \tilde{\boldsymbol{u}}_\epsilon \middle| r, \boldsymbol{V}\right) \to 0 \tag{3.2.20}$$

avec

$$\mathcal{E}_\epsilon\left(\varrho, \boldsymbol{u}\middle| r, \boldsymbol{U}\right) = \frac{1}{\epsilon}\int_{\Omega_\epsilon}\left[\frac{1}{2}\varrho|\boldsymbol{u} \quad \boldsymbol{U}|^2 + H(\varrho) \quad H'(r)(\varrho \quad r) \quad H(r)\right]\mathrm{d}\boldsymbol{x},$$

où $\boldsymbol{V}(t,\boldsymbol{x}) = [\boldsymbol{w}(t,\boldsymbol{x}_h), 0]$ et où the couple $[r, \boldsymbol{w}]$ satisfait le système de Navier-Stokes compressible 2 D (3.1.14–3.1.16) dans la cellule périodique ω et sur l'intervalle de temps $[0, T_{\max})$.

L'idée générale de la preuve du théorème 8 est relativement similaire à la preuve des théorèmes de stabilité dans le chapitre 2. L'idée est d'utiliser l'inégalité d'énergie relative avec comme fonction $(r, \boldsymbol{V})$ construite dans le théorème 1. Le but est alors d'obtenir une minoration du membre de gauche de (2.2.24) par

$$c\int_0^\tau \|\boldsymbol{u}_\epsilon \quad \boldsymbol{V}\|^2_{W^{1,2}(\Omega)^3}\,\mathrm{d}t \quad \overline{c}'\int_0^\tau \mathcal{E}\left(\varrho_\epsilon, \boldsymbol{u}_\epsilon\middle| r, \boldsymbol{V}\right)(t)\,\mathrm{d}t + \mathcal{E}\left(\varrho_\epsilon, \boldsymbol{u}_\epsilon\middle| r, \boldsymbol{V}\right)(\tau) \tag{3.2.21}$$

et une majoration du membre de droite par

$$h_\epsilon(\tau) + \delta\int_0^\tau \|\boldsymbol{u}_\epsilon \quad \boldsymbol{V}\|^2_{W^{1,2}(\Omega)^3}\,\mathrm{d}t + c'(\delta)\int_0^\tau a(t)\,\mathcal{E}\left(\varrho_\epsilon, \boldsymbol{u}_\epsilon\middle| r, \boldsymbol{V}\right)(t)\,\mathrm{d}t \tag{3.2.22}$$

pour tout $\delta > 0$, où $c > 0$ est indépendant de δ, $\overline{c}' \geq 0$, $c' = c'(\delta) > 0$, $a \in L^1_{\mathrm{loc}}(0, T_M)$ et

$$h_\epsilon(\tau) \to 0 \text{ dans } L^\infty_{\mathrm{loc}}(0, T_M). \tag{3.2.23}$$

Nous obtenons alors l'inegalité

$$\mathcal{E}\left(\varrho_\epsilon, \boldsymbol{u}_\epsilon\middle| r, \boldsymbol{V}\right)(\tau) \leq h_\epsilon(\tau) + c\int_0^\tau a(t)\,\mathcal{E}\left(\varrho_\epsilon, \boldsymbol{u}_\epsilon\middle| r, \boldsymbol{U}\right)(t)\,\mathrm{d}t. \tag{3.2.24}$$

Il suffit alors de conclure en utilisant le lemme de Gronwall.

4. Existence de solutions faibles pour les équations de Navier-Stokes compressibles avec transport d'entropie

4.1. Introduction

L'objet de ce chapitre est d'analyser un modèle d'écoulement d'un fluide compressible visqueux d'entropie variable. Un tel écoulement peut être décrit par les équations de Navier-Stokes compressibles couplées avec une équation supplémentaire décrivant l'évolution de l'entropie du fluide. Dans le cas où la conductivité est négligée (voir (2.1.8) et (2.1.5)), les changements d'entropie sont seulement dus au transport et le système entier peut être écrit comme suivant :

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ dans } (0,T) \times \Omega, \tag{4.1.1a}$$

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s \boldsymbol{u}) = 0 \text{ dans } (0,T) \times \Omega, \tag{4.1.1b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \mathbb{S} \text{ dans } (0,T) \times \Omega, \tag{4.1.1c}$$

où les inconnues sont la densité $\varrho\colon (0,T) \times \Omega \to \mathbb{R}_+$, l'entropie $s\colon (0,T) \times \Omega \to \mathbb{R}_+^\star$ et la vitesse du fluide $\boldsymbol{u}\colon (0,T) \times \Omega \to \mathbb{R}^3$ et où Ω est un domaine de $\mathbb{R}^3$ de frontière régulière $\partial\Omega$.

L'équation de la quantité de mouvement, de continuité et d'entropie sont liées ensembles à travers la forme de la pression p dont nous supposons qu'elle satisfait

$$p(\varrho, s) = \varrho^\gamma \mathcal{T}(s), \quad \gamma > 1, \tag{4.1.2}$$

où $\mathcal{T}(\cdot)$ est une fonction regulière donnée strictement monotone de $\mathbb{R}_+^\star$ dans $\mathbb{R}_+^\star$, en particulier $\mathcal{T}(s) > 0$ pour $s > 0$.

Nous supposons que le fluide est Newtonien et que le tenseur de contraintes visqueuses est de la forme suivante

$$\mathbb{S} = \mathbb{S}(\nabla \boldsymbol{u}) = \mu\Big(\frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \quad \frac{2}{3} \operatorname{div} \boldsymbol{u} \mathbb{I}_3\Big) + \eta \operatorname{div}_x \boldsymbol{u} \mathbb{I}_3$$

Les coefficients de viscosité μ and η sont supposés être constants et vérifient

$$\mu > 0, \quad \eta \geq 0. \tag{4.1.3}$$

Le système est completé par les conditions intiales et par les conditions au bord :

$$\varrho(0,\boldsymbol{x}) = \varrho_0(\boldsymbol{x}),\ (\varrho s)(0,\boldsymbol{x}) = S_0(\boldsymbol{x}),\ (\varrho\boldsymbol{u})(0,\boldsymbol{x}) = \boldsymbol{q}_0(\boldsymbol{x}), \tag{4.1.4}$$

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}. \tag{4.1.5}$$

Le système (4.1.1) est un modèle décrivant le mouvement d'un gaz visqueux compressible d'entropie variable transportée par l'écoulement. La quantité $\theta = [\mathcal{T}(s)]^{1/\gamma}$ peut être aussi interprétée comme une température potentielle auquel cas la pression (4.1.2) prend la forme $(\rho\theta)^\gamma$ et a été étudié dans [?] et [?].

Dans [?], le but est de prouver l'existence de solutions faibles globales en temps au système (4.1.1). Notons au moins pour une solution régulière l'équation de continuité (4.1.1a) nous autorise à reformuler (4.1.1b) comme une équation de transport pure pour s et nous avons alors

$$\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0 \text{ dans } (0,T)\times\Omega, \tag{4.1.6a}$$

$$\partial_t s + \boldsymbol{u}\cdot\nabla s = 0 \text{ dans } (0,T)\times\Omega, \tag{4.1.6b}$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \operatorname{div}\mathbb{S} \text{ dans } (0,T)\times\Omega. \tag{4.1.6c}$$

Contrairement à l'équation d'entropie dans le système (4.1.1) la forme au dessus est insensible à l'apparence d'états de vide ; en effet elle est complètement découplée de l'équation de continuité. La régularité de la densité dans les sytèmes de Navier-Stokes compressible est en général une affaire délicate. De plus, on peut attendre qua la preuve d'existence de solutions au système (4.1.1) requiert des hypothèses plus restrictives que celle pour obtenir des solutions à (4.1.6). Cette observation sera reflétée par le biais de la valeur du paramètre γ qui determine la qualité des estimations *a priori* pour l'argument de pression $Z = \varrho[\mathcal{T}(s)]^{\frac{1}{\gamma}}$ d' aprés les notations au dessus.

Afin de clarifier un peu plus cette question introduisons une troisième formulation au problème (4.1.1) décrivant l'évolution de l'argument de pression $Z = \varrho[\mathcal{T}(s)]^{\frac{1}{\gamma}}$ au lieu de l'entropie elle-même. Nous avons :

$$\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0 \text{ dans } (0,T)\times\Omega, \tag{4.1.7a}$$

$$\partial_t Z + \operatorname{div}(Z\boldsymbol{u}) = 0 \text{ dans } (0,T)\times\Omega, \tag{4.1.7b}$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla Z^\gamma = \operatorname{div}\mathbb{S} \text{ dans } (0,T)\times\Omega. \tag{4.1.7c}$$

Une fois de plus, la formulation au dessus est équivalente à la précédente dés que la solution est suffisament régulière, ce qui, cependant, peut ne pas être vraie dans le cas des solutions faibles.

La discussion au dessus nous a alors incité à faire la distinction entre le cas où l'évolution de l'entropie est décrite par l'équation de continuité (4.1.1*b*), l'équation de transport (4.1.6*b*) ou l'équation de transport renormalisée (4.1.7*b*). En effet, la forme de l'équation d'entropie, bien qu'utilisée pour décrire le même phénomène, est un marqueur de diagnostic indiquant la notion plausible de solution faible au système entier considéré. L' article [?] contient une analyse d'existence ainsi qu'une définition convenable de solutions faibles pour chacun des trois systèmes : (4.1.1), (4.1.6) et (4.1.7). Une telle approche nous autorise à souligner les implications entre les solutions et de mieux comprendre les restrictions des techniques de renormalisation. Ces questions, absentes dans l'analyse des systèmes standards ne présentant qu'une seule densité, sont d'une trés grande importance pour des écoulements multicomposents ou multiphasiques plus complexes. Nos résultats montrent des applications des outils classiques actuels dans l'analyse du système de Navier-Stokes compressible à des problèmes difficiles comme par exemple des équations constitutives impliquant des combinaisons nonlineaires de quantités solutions de problèmes hyperboliques : densités, concentrations, etc.

4.2. Solutions faibles, résultats d'existence

Au cours de notre analyse nous distinguons deux situations différentes. Elles sont associées à la valeur de l'exposant adiabatique γ. D'un point de vue de la théorie d'exisence des solutions faibles globales en temps (voir chapitre 2), il est raisonnable de supposer que

$$\gamma > \frac{3}{2}. \tag{4.2.1}$$

Cette hypothèse fournit une borne L^1 sur le terme convectif et elle est nécessaire pour appliquer les techniques actuelles. Sous cette condition nous prouvons premièrement l'existence de solutions faibles au système (4.1.7), voir Théorème 9 et [?, Theorem 3]. Nous déduisons de ce résultat l'existence de solutions faibles pour la formulation (4.1.6) toujours sous l' hypothèse (4.2.1), voir Théorème 11 et [?, Theorem 2]. Cependant, ce résultat n'est pas équivalent à l'existence de solutions faibles au système (4.1.1), voir Théorème 10. L'existence pour ce dernier système peut être prouvée seulement sous la restriction

$$\gamma \geq \frac{9}{5}. \tag{4.2.2}$$

En effet, cette dernière restriction sur la valeur de γ nous autorise à obtenir une estimation L^2 sur la densité et, comme mentionné dans l'introduction, rend possible l'application de la théorie de Diperna-Lions des solutions renormalisées de l'équation de transport (voir [?]) et de multipler l'équation (4.1.6*b*) par ϱ dans la classe des solutions faibles, voir théorème 10 et [?, Theorem 1]. Nous présentons dans la suite les définitions de ces solutions faibles ainsi que les résultats d'existence.

4.2.1. Solutions faibles au système (4.1.7)

Le système (4.1.7) est un bon point de départ pour nos considérations. En effet, pour des conditions initiales et au bord raisonnables il est possible de montrer qu'il possède une solution faible pour $\gamma > \frac{3}{2}$, utilisant une approche plus ou moins standard.

Nous supposons que les conditions initiales pour le système (4.1.7) sont

$$\varrho_0 : \Omega \to \mathbb{R}_+, \;\; Z_0 : \Omega \to \mathbb{R}_+, \;\; \boldsymbol{u}_0 : \Omega \to \mathbb{R}^3,$$
$$\varrho(0, \boldsymbol{x}) = \varrho_0(\boldsymbol{x}), \;\; Z(0, \boldsymbol{x}) = Z_0(\boldsymbol{x}), \;\; (\varrho \boldsymbol{u})(0, \boldsymbol{x}) = \boldsymbol{q}_0(\boldsymbol{x}) = \varrho_0 \boldsymbol{u}_0(\boldsymbol{x}), \tag{4.2.3}$$

et qu'elles vérifient

$$\begin{aligned} (\varrho_0, Z_0) \in L^\gamma(\Omega)^2, \quad \varrho_0, Z_0 \geq 0 \text{ p.p dans } \Omega, \quad \int_\Omega \varrho_0 \, \mathrm{d}\boldsymbol{x} > 0, \\ 0 \leq c_\star \varrho_0 \leq Z_0 \leq c^\star \varrho_0 \text{ p.p dans } \Omega, \quad 0 < c_\star \leq c^\star < \infty, \quad \boldsymbol{q}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega)^3. \end{aligned} \tag{4.2.4}$$

Alors nous avons

Definition 3. *Supposons que les conditions initiales vérifient (4.2.4). Nous dirons que le triplet $(\varrho, Z, \boldsymbol{u})$ est une solution faible au problème (4.1.7) de donnée intiale et de conditions au bord (4.1.5), (4.2.3) si*

$$(\varrho, Z, \boldsymbol{u}) \in L^\infty(0,T; L^\gamma(\Omega)) \times L^\infty(0,T; L^\gamma(\Omega)) \times L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{4.2.5}$$

et nous avons :

(i) $\varrho \in C_w([0,T]; L^\gamma(\Omega))$ *et l'équation de continuité (4.1.7a) est satisfaite dans le sens faible*

$$\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot) \, \mathrm{d}\boldsymbol{x} \quad \int_\Omega \varrho_0 \varphi(0,\cdot) \, \mathrm{d}\boldsymbol{x} = \int_0^t \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\,\tau, \tag{4.2.6}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\varphi \in C^1([0,T] \times \overline{\Omega})$.

(ii) $Z \in C_w([0,T]; L^\gamma(\Omega))$ *et l'équation de continuité (4.1.7b) est satisfaite dans le sens faible*

$$\int_\Omega Z(\tau,\cdot)\varphi(\tau,\cdot) \, \mathrm{d}\boldsymbol{x} \quad \int_\Omega Z_0 \varphi(0,\cdot) \, \mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(Z \partial_t \varphi + Z \boldsymbol{u} \cdot \nabla \varphi \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \tag{4.2.7}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\varphi \in C^1([0,T] \times \overline{\Omega})$.

(iii) $\varrho \boldsymbol{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$ *et l'équation de la quantité de mouvement (4.1.1c) est satisfaite dans le sens faible*

$$\begin{aligned} \int_\Omega (\varrho \boldsymbol{u})(\tau,\cdot) \cdot \boldsymbol{\psi}(\tau,\cdot) \, \mathrm{d}\boldsymbol{x} \quad \int_\Omega \boldsymbol{q}_0 \cdot \boldsymbol{\psi}(0,\cdot) \, \mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\psi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\psi} \\ + Z^\gamma \operatorname{div} \boldsymbol{\psi} \quad \mathbb{S}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{\psi} \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \end{aligned} \tag{4.2.8}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\boldsymbol{\psi} \in C_c^1([0,T] \times \Omega, \mathbb{R}^3)$.

(iv) l'inégalité d'énergie

$$\mathcal{E}^2(\varrho, Z, \boldsymbol{u})(\tau) + \int_0^\tau \int_\Omega \Big(\mu |\nabla \boldsymbol{u}|^2 + (\mu + \lambda)(\operatorname{div} \boldsymbol{u})^2 \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\,\tau \leq \mathcal{E}^2(\varrho_0, Z_0, \boldsymbol{u}_0) \tag{4.2.9}$$

est satisfaite pour presque tout $\tau \in (0,T)$, *où*

$$\mathcal{E}^2(\varrho, Z, \boldsymbol{u}) = \int_\Omega \Big(\frac{1}{2} \varrho |\boldsymbol{u}|^2 + \frac{Z^\gamma}{\gamma \quad 1} \Big) \, \mathrm{d}\boldsymbol{x}. \tag{4.2.10}$$

Nous avons le résultat suivant d'existence pour les solutions faibles définies dans la définition 3

Théorème 9 (Maltese, Michálek, Mucha, Novotný, Zatorska, 2016, [**?**])**.** *Supposons que μ, η vérifient (4.1.3), $\gamma > \frac{3}{2}$, et que la condition initiale $(\varrho_0, Z_0, \boldsymbol{q}_0)$ vérifie (4.2.4).*

Alors il existe une solution faible $(\varrho, Z, \boldsymbol{u})$ au problème (4.1.7) de condition au bord (4.1.5), au sens de la Définition 3. De plus, $(Z, \boldsymbol{u})$ est solution (4.1.7b) au sens renormalisé et

$$0 \leq c_\star \varrho \leq Z \leq c^\star \varrho$$

presque partout dans $(0,T) \times \Omega$.

4.2.2. Solutions faibles au système (4.1.1)

Formellement, en prenant l'entropie s telle que $Z = \varrho(\mathcal{T}(s))^{\frac{1}{\gamma}}$ dans (4.1.7) nous pouvons retrouver notre système original (4.1.1). Cependant, pour les solutions faibles, cet argument formel ne peut pas être fait rigoureusement à moins de supposer que $\gamma \geq \frac{9}{5}$. Introduisons premièrement la définition de solutions faibles au système original (4.1.1). Nous supposons que la condition initiale (4.1.4) vérifie :

$$\begin{gathered} \varrho_0 : \Omega \to \mathbb{R}_+, \quad s_0 : \Omega \to \mathbb{R}_+^\star, \quad \boldsymbol{u}_0 : \Omega \to \mathbb{R}^3, \\ \varrho_0 \in L^\gamma(\Omega), \quad \int_\Omega \varrho_0 \, \mathrm{d}\boldsymbol{x} > 0, \\ S_0 = \varrho_0 s_0, \ s_0 \in L^\infty(\Omega), \quad \boldsymbol{q}_0 = \varrho_0 \boldsymbol{u}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega)^3. \end{gathered} \tag{4.2.11}$$

Nous considérons

Definition 4. *Supposons que les conditions initiales vérifient (4.2.11). Nous dirons que le triplet $(\varrho, s, \boldsymbol{u})$ est une solution faible au problème (4.1.1)–(4.1.5) si :*

$$(\varrho, s, \boldsymbol{u}) \in L^\infty(0,T; L^\gamma(\Omega)) \times L^\infty((0,T) \times \Omega) \times L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{4.2.12}$$

et nous avons :

(i) $\varrho \in C_w([0,T]; L^\gamma(\Omega))$ *et l'équation de continuité (4.1.1a) est satisfaite au sens faible*

$$\int_\Omega \varrho(\tau, \cdot)\varphi(\tau, \cdot) \, \mathrm{d}\boldsymbol{x} \quad \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \tag{4.2.13}$$

pour tout $\tau \in [0,T]$ et pour toute fonction $\varphi \in C^1([0,T] \times \overline{\Omega})$.

(ii) $\varrho s \in C_w([0,T]; L^\gamma(\Omega))$ *et l'équation (4.1.1b) est satisfaite au sens faible*

$$\int_\Omega (\varrho s)(\tau, \cdot)\varphi(\tau, \cdot) \, \mathrm{d}\boldsymbol{x} \quad \int_\Omega S_0 \varphi(0, \cdot) \, \mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(\varrho s \partial_t \varphi + \varrho s \boldsymbol{u} \cdot \nabla \varphi \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \tag{4.2.14}$$

pour tout $\tau \in [0,T]$ et pour toute fonction $\varphi \in C^1([0,T] \times \overline{\Omega})$.

(iii) $\varrho\boldsymbol{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$ *et l'équation de la quantité de mouvement (4.1.1c) est satisfaite au sens faible*

$$\int_\Omega (\varrho\boldsymbol{u})(\tau,\cdot)\cdot\boldsymbol{\psi}(\tau,\cdot)\,\mathrm{d}\boldsymbol{x} \quad \int_\Omega \boldsymbol{q}_0\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}\boldsymbol{x} = \int_0^\tau\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi} + \varrho^\gamma\mathcal{T}(s)\,\mathrm{div}\,\boldsymbol{\psi} \quad \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{4.2.15}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\boldsymbol{\psi} \in C_c^1([0,T]\times\Omega, \mathbb{R}^3)$*.*

(iv) l'inégalité d'énergie

$$\mathcal{E}^1(\varrho, s, \boldsymbol{u})(\tau) + \int_0^\tau\int_\Omega \Big(\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)(\mathrm{div}\,\boldsymbol{u})^2\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \le \mathcal{E}^1(\varrho_0, s_0, \boldsymbol{u}_0) \tag{4.2.16}$$

est satisfaite pour presque tout $\tau \in (0,T)$*, où*

$$\mathcal{E}^1(\varrho, s, \boldsymbol{u}) = \int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{\varrho^\gamma\mathcal{T}(s)}{\gamma \quad 1}\Big)\,\mathrm{d}\boldsymbol{x}.$$

Le premier résultat principal concernant les solutions faibles au sens de la définition 4 s'énonce ainsi.

Théorème 10 (Maltese, Michálek, Mucha, Novotný, Zatorska, 2016, [?])**.** *Supposons que* μ, η *vérifient (4.1.3),* $\gamma \ge \frac{9}{5}$ *et que la donnée initiale* $(\varrho_0, S_0, \boldsymbol{q}_0)$ *vérifie (4.2.11). Alors il existe une solution faible* $(\varrho, s, \boldsymbol{u})$ *au problème (4.1.1)–(4.1.5) au sens de la définition 4.*

Remarque 6. *La restriction sur* γ *dans le théorème 10 est évidemment pas satisfaisante puisque les valeurs physiquement raisonnables de* γ *sont inférieures ou égales à* $\frac{5}{3}$ *(voir chapitre 2).*

4.2.3. Solutions faibles au système (4.1.6)

Si nous remplaçons (4.1.1b) par (4.1.6b) (utilisant aussi la renormalisation de cette dernière), le résultat est alors bien meilleur que dans le théorème 10, en fait optimal d'un point de vue de la théorie actuelle des équations de Navier-Stokes compressibles concernant l'existence de solutions faibles. Afin de formuler précisement le résultat, nous réécrivons premièrement le système (4.1.6) d' une manière légèrement différente. Nous cherchons un triplet $(\varrho, \zeta, \boldsymbol{u})$ solution du système d'équations

$$\partial_t\varrho + \mathrm{div}(\varrho\boldsymbol{u}) = 0, \tag{4.2.17a}$$

$$\partial_t\zeta + \boldsymbol{u}\cdot\nabla\zeta = 0, \tag{4.2.17b}$$

$$\partial_t(\varrho\boldsymbol{u}) + \mathrm{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla\left(\frac{\varrho}{\zeta}\right)^\gamma = \mathrm{div}\,\mathbb{S}(\nabla\boldsymbol{u}), \tag{4.2.17c}$$

de conditions initiales

$$\varrho(0,\boldsymbol{x}) = \varrho_0(\boldsymbol{x}),\ \zeta(0,\boldsymbol{x}) = \zeta_0(\boldsymbol{x}),\ (\varrho\boldsymbol{u})(0,\boldsymbol{x}) = \boldsymbol{q}_0(\boldsymbol{x}), \tag{4.2.18}$$

telle que

$$0 < m \leq \zeta_0 \leq M \text{ p.p dans } \Omega. \tag{4.2.19}$$

Alors les solutions faibles sont définies de la manière suivante.

Definition 5. *Supposons que la condition initiale* $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ *vérifie (4.2.19) et (4.2.4) (pour* ϱ_0 *et* $\boldsymbol{q}_0$*), Nous dirons que le triplet* $(\varrho, \zeta, \boldsymbol{u})$ *est une solution faible au problème (4.2.17) émanant de la condition initiale* $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ *si*

$$(\varrho, \zeta, \boldsymbol{u}) \in L^\infty(0, T; L^\gamma(\Omega)) \times L^\infty((0,T)\times\Omega) \times L^2(0, T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{4.2.20}$$

et nous avons :

(i) $\varrho \in C_w([0,T]; L^\gamma(\Omega))$ *et l'équation de continuité (4.2.17a) est satisfaite au sens faible*

$$\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)\ \mathrm{d}\boldsymbol{x} \quad \int_\Omega \varrho_0\varphi(0,\cdot)\ \mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi\Big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t, \tag{4.2.21}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\varphi \in C^1([0,T]\times\overline{\Omega})$.

(ii) $\zeta \in C_w([0,T]; L^\infty(\Omega))$ *et l'équation (4.2.17b) est satisfaite au sens faible*

$$\int_\Omega \zeta(\tau,\cdot)\varphi(\tau,\cdot)\ \mathrm{d}\boldsymbol{x} \quad \int_\Omega \zeta_0\varphi(0,\cdot)\ \mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(\zeta\partial_t\varphi + \zeta\operatorname{div}(\boldsymbol{u}\varphi)\Big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t, \tag{4.2.22}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\varphi \in C^1([0,T]\times\overline{\Omega})$.

(iii) $\varrho\boldsymbol{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$ *et l'équation de la quantité de mouvement (4.2.17c) est satisfaite dans le sens faible*

$$\int_\Omega (\varrho\boldsymbol{u})(\tau,\cdot)\cdot\boldsymbol{\psi}(\tau,\cdot)\ \mathrm{d}\boldsymbol{x} \quad \int_\Omega \boldsymbol{q}_0\cdot\boldsymbol{\psi}(0,\cdot)\ \mathrm{d}\boldsymbol{x} = \int_0^\tau \int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi}$$
$$+ \left(\frac{\varrho}{\zeta}\right)^\gamma \operatorname{div}\boldsymbol{\psi} \quad \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t, \tag{4.2.23}$$

pour tout $\tau \in [0,T]$ *et pour toute fonction* $\boldsymbol{\psi} \in C_c^1([0,T]\times\Omega, \mathbb{R}^3)$.

(iv) l'inégalité d' 'energie

$$\mathcal{E}^2(\varrho, \varrho/\zeta, \boldsymbol{u})(\tau) + \int_0^\tau \int_\Omega \Big(\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)(\operatorname{div}\boldsymbol{u})^2\Big)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t \leq \mathcal{E}^2(\varrho_0, \varrho_0/\zeta_0, \boldsymbol{u}_0) \tag{4.2.24}$$

est satisfaite pour presque tout $\tau \in (0,T)$ *où* $\mathcal{E}^2$ *est définie par (4.2.10).*

Le dernier résultat concerne l'existence de solutions faibles au sens de la définition 5.

Théorème 11 (Maltese, Michálek, Mucha, Novotný, Zatorska, 2016, [?])**.** *Supposons que* μ, η *vérifient (4.1.3),* $\gamma > \frac{3}{2}$*, et que la condition initiale* $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ *vérifie (4.2.19) et (4.2.4) (pour* ϱ_0 *et* $\boldsymbol{q}_0$*).*

Alors il existe une solution faible $(\varrho, \zeta, \boldsymbol{u})$ *au problème (4.2.17) de conditions au bord (4.1.5), au sens de la définition 5. De plus,* $(\varrho, \boldsymbol{u})$ *est solution de (4.2.17a) et* $(\zeta, \boldsymbol{u})$ *est solution de (4.2.17b) au sens renormalisé.*

5. Estimations d'erreur pour une méthode volumes finis/éléments finis pour les équations de Navier-Stokes compressibles

5.1. De l'inégalité d'énergie relative aux estimations d'erreur

Le but de ce chapitre est d'établir des estimations d'erreur pour un schéma numérique de type volumes finis/éléments finis. Nous considérons donc le système de Navier-Stokes compressible en regime barotrope suivant

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ dans } (0,T) \times \Omega, \tag{5.1.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\varrho) = \mu \Delta \boldsymbol{u} + (\mu + \lambda) \nabla \operatorname{div} \boldsymbol{u} \text{ dans } (0,T) \times \Omega, \tag{5.1.1b}$$

où nous supposons que la pression satisfait

$$p \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^\star),\ p(0) = 0,\ p'(\varrho) > 0 \text{ pour tout } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma \quad 1}} = p_\infty > 0 \tag{5.1.2}$$

où $\gamma \geq 1$. Notons que si γ existe alors il est unique. De plus, si $\gamma < 2$ dans (5.1.2), nous avons besoin d'une condition additionnelle pour les petites densités :

$$\lim_{\varrho \to 0} \frac{p'(\varrho)}{\varrho^{\alpha+1}} = p_0 > 0,\ \alpha \leq 0. \tag{5.1.3}$$

Remarquons que les hypothèses (5.1.2)-(5.1.3) sont compatibles avec la loi de pression isentropique $p(\varrho) = \varrho^\gamma$ dès que $\gamma \geq 1$. Les coefficients de viscosité sont supposés constants et vérifient

$$\mu > 0 \text{ et } \lambda + \mu \geq 0. \tag{5.1.4}$$

Le système est complété avec les conditions initiales pour la densité et la quantité de mouvement

$$\varrho(0, \cdot) = \varrho_0,\ \varrho \boldsymbol{u}(0, \cdot) = \varrho_0 \boldsymbol{u}_0, \tag{5.1.5}$$

où ϱ_0 et $\boldsymbol{u}_0$ sont deux fonctions données respectivement de Ω dans $\mathbb{R}_+^\star$ et $\mathbb{R}^3$, et la condition au bord

$$\boldsymbol{u} = 0 \text{ dans } (0,T) \times \partial\Omega. \tag{5.1.6}$$

Comme expliqué dans le chapitre 2, sous les hypothèses de stabilité thermodynamique pour la pression (2.2.21), l'énergie relative (2.2.23) mesure la "distance" entre une solution faible $(\varrho, \boldsymbol{u})$ des équations de Navier-Stokes compressibles et n'importe quel état $(r, \boldsymbol{U})$ du fluide.

Dans [?] nous avons développé une méthodologie pour obtenir des estimations d'erreur inconditionnelles pour les schémas numériques des équations de Navier Stokes compressibles (5.1.1)-(5.1.6). Nous avons alors appliqué cette méthodologie à un schéma numerique proposé par Karper dans [?] et présenté dans la suite. L'idée générale a été de reproduire la demarche en continue proposée dans [?] pour aboutir à une version discrète du théorème 3. Nous avons alors obtenu des estimations d'erreur pour les solutions discrètes par rapport à une solution classique du système (5.1.1)-(5.1.6) possédant une certaine régularité sur le même domaine polyédrique. Nous présentons par la suite deux schémas numériques pour la discrétisation des équations de Navier-Stokes compressibles.

5.2. Schéma numérique

Le schéma numérique suivant a été proposé par Karper dans [?] où la preuve de convergence à été effectué dans le cas isentropique pour $\gamma > 3$.

5.2.1. Discrétisation spatiale et temporelle

On suppose que le problème est posé sur un domaine Ω polyédrique de $\mathbb{R}^3$, c'est-à-dire que $\overline{\Omega}$ est réunion finie de polyèdres de $\mathbb{R}^d$. Rappelons qu'un polyhèdre de $\mathbb{R}^d$ est une intersection de demi-espaces de $\mathbb{R}^3$ et que les parties de son bord qui appartiennent à un seul hyperplan sont appelées ses faces. La raison de cette hypothèse est qu'il n'est possible de mailler exactement que de tels ouverts.

Soit $\mathcal{M}$ une décomposition du domaine polyédrique Ω en symplexes, appelée aussi triangulation de Ω. Par $\mathcal{E}(K)$, nous désignons l'ensemble des faces $(d = 3)$ σ de l'élément $K \in \mathcal{M}$. L'ensemble de toutes les faces est noté $\mathcal{E}$. L'ensemble des faces contenue dans $\partial\Omega$ est noté $\mathcal{E}_{\text{ext}}$ et son complémentaire dans $\mathcal{E}$ est noté $\mathcal{E}_{\text{int}}$. La triangulation est supposée $\mathcal{M}$ être regulière au sens de la litterature des élements finis (voir [?]), et, en particulier, $\mathcal{M}$ satisfait les proprietes suivantes : $\overline{\Omega} = \cup_{K \in \mathcal{M}} \overline{K}$; si $K, L \in \mathcal{M}$, alors $\overline{K} \cap \overline{L} = \emptyset$, $\overline{K} \cap \overline{L}$ est un sommet ou $\overline{K} \cap \overline{L}$ est une face commune de K et L , qui est notée par $K|L$. Pour chaque face intéreur du maillage $\sigma = K|L$, $\boldsymbol{n}_{KL}$ représente le vecteur normal à σ, orienté de K vers L (en particulier $\boldsymbol{n}_{KL} = \quad \boldsymbol{n}_{LK}$). Nous notons respectivement par $|K|$ et $|\sigma|$ la mesure de K et d'une face σ, et par h_K le diamètre de K. Nous mesurons la regularité du maillage au travers du paramètre $\theta_{\mathcal{M}}$ définie par :

$$\theta_{\mathcal{M}} = \min\{\frac{\xi_K}{h_K},\ K \in \mathcal{M}\} \tag{5.2.1}$$

où ξ_K représente le diamètre de la plus grosse sphère incluse dans K.

La discrétisation spatiale utilise une technique d'élément fini non conforme à savoir l'élément fini de Crouzeix-Raviart (voir [?] pour l' article originel et, par exemple, [?, p. 83-85] pour une présentation synthétique)

L'élément de référence pour l'élément fini de Crouzeix-Raviart est le simplexe unitaire de $\mathbb{R}^3$ et l'espace des fonctions de forme est l'espace P_1 des polynômes affines. Les degrés de liberté sont déterminés par l'ensemble des fonctions de forme globale suivantes :

$$\{m_{\sigma,i},\ \sigma \in \mathcal{E}(K),\ i = 1,2,3\}, \quad m_{\sigma,i}(\boldsymbol{v}) = \frac{1}{|\sigma|}\int_\sigma v_i \,\mathrm{d}\boldsymbol{x},\ \boldsymbol{v} = (v_1, v_2, v_3)\}. \tag{5.2.2}$$

La transformation de l'élément de référence en un élément quelconque du maillage est la transformation affine. Enfin, on demande que la valeur moyenne des vitesses discrètes (c'est-à-dire pour tout champ de vitesse discret $\boldsymbol{v}$, $m_{\sigma,i}(\boldsymbol{v})$) soit continue sur chaque face du maillage. En prenant en compte les conditions aux limites de Dirichlet homogènes, l'espace discret $\boldsymbol{W}_{\mathcal{E},0}(\Omega)$ est définie de la facon suivante :

$$\boldsymbol{W}_{\mathcal{E},0}(\Omega) = [W_{\mathcal{E},0}(\Omega)]^3 = \{\boldsymbol{v} \in L^2(\Omega)^3,\ \forall K \in \mathcal{M},\ \boldsymbol{v}_{|K} \in W(K)^3 \text{ and } \forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \\ m_{\sigma,i}(\boldsymbol{v}_{|K}) = m_{\sigma,i}(\boldsymbol{v}_{|L}), \forall \sigma \in \mathcal{E}_{\text{ext}}, m_{\sigma,i}(\boldsymbol{v}) = 0\}$$

Puisque seulement la continuité de l'intégrale sur chaque face du maillage est imposée, les fonctions de $\boldsymbol{W}_{\mathcal{E},0}(\Omega)$ sont discontinues à travers chaque face ; la discrétisation est donc non conforme dans $H^1(\Omega)^3$. Nous définissonns, pour $1 \leq i \leq 3$ et $u_i \in W_{\mathcal{E},0}(\Omega)$, $\partial_{h,i} u_i$ comme étant la fonction de $L^2(\Omega)$ qui est égal presque partout à la dérivé de u_i par rapport a la i^{eme} variable d'espace. Cette notation nous autorise à définir un gradient discret, noté comme dans le cas continue par ∇ que ce soit dans le cas scalaire ou vectoriel et une divergence discrète pour les fonctions vectorielles discrètes, notée comme dans le cas continue par div.

A partir de la définition (5.2.2), chaque degré de liberté de la vitesse peut être univoquement associé à une face. Nous notons alors l'ensemble des degrés de liberté pour la vitesse par :

$$\{\boldsymbol{u}_\sigma,\ \sigma \in \mathcal{E}\}.$$

Pour $\boldsymbol{u} \in \boldsymbol{W}_{\mathcal{E},0}(\Omega)$, nous introduisons

$$\boldsymbol{u}_K = \frac{1}{3}\sum_{\sigma \in \mathcal{E}(K)} \boldsymbol{u}_\sigma = \frac{1}{|K|}\int_K \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}.$$

et nous notons

$$\hat{\boldsymbol{u}} = \sum_{K \in \mathcal{M}} \boldsymbol{u}_K \mathcal{X}_K(\boldsymbol{x})$$

où $\mathcal{X}_K$ est la fonction caractéristique de $K \in \mathcal{M}$. Finalement, nous devons traiter la condition au bord de Dirichlet. Puisque les inconnues de la vitesse se situent sur le bord de Ω (et non à l'intérieur des cellules), ces conditions sont prises en compte dans la définition des espaces discrets en posant zéro pour les inconnues de la vitesse qui se situent sur le bord de Ω

$$\forall \sigma \in \mathcal{E}_{\text{ext}}, \qquad \boldsymbol{u}_\sigma = 0. \tag{5.2.3}$$

Les degrés de liberté pour la densité (c'est-à-dire les inconnues pour la densité discrète) sont associés aux cellules du maillage $\mathcal{M}$, et sont notés :

$$\big\{\varrho_K,\ K \in \mathcal{M}\big\}.$$

et nous notons

$$L_{\mathcal{M}} = \{q \in L^2(\Omega),\ q_{|K} = \text{cste}\}. \tag{5.2.4}$$

Concernant la discrétisation en temps du problème (5.1.1)-(5.1.6), nous considérons, dans un but de simplcité, une partition $0 = t^0 < t^1 < \cdots < t^N = T$ de l'intervalle de temps $(0, T)$ de pas constant $\delta t = t^n \quad t^{n\ \ 1}$; donc $t^n = n\delta t$ pour $n \in \{0, \cdots, N\}$. Nous notons respectivement par $\{\boldsymbol{u}_\sigma^n, \sigma \in \mathcal{E}_{\text{int}}, n \in \{0, \cdots, N\}\}$, et $\{\varrho_K^n, K \in \mathcal{M}, n \in \{1, \cdots, N\})$ l'ensemble des inconnues discrètes pour la vitesse et la densité.

Finalement, la condition initiale discrète $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ est telle que $\varrho^0 > 0$. La masse totale discrète et l'énergie totale discrète sont respectivement définies par

$$M_{0,\mathcal{M}} = \int_\Omega \varrho^0 \,\mathrm{d}\boldsymbol{x}, \qquad \mathcal{E}_{0,\mathcal{M}} = \int_\Omega \frac{1}{2}\varrho^0 |\boldsymbol{u}^0|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathcal{H}(\varrho^0)\,\mathrm{d}\boldsymbol{x}. \tag{5.2.5}$$

5.2.2. Schéma numérique

Pour approximer l'équation de continuité, la méthode introduite dans [?] utilise une méthode de Galerkin discontinue avec des fonctions constantes par morceaux et un flux classique de type upwind. Pour l'équation de quantité de mouvement, la méthode utilisée est une combinaison d'une méthode de Galerkin discontinue et d'une méthode d'élement finis approximant la vitesse avec l'élḿent fini de Crouzeix-Raviart. Tandis que l'opérateur de diffusion est discrétisé de manière standard, le terme convective et les termes de derivations temporelles sont discrétisés en utilisant une méthode de Galerkin discontinue avec les valeurs moyennes sur le maillage primal de la vitesse discrète et un flux de type Lax-Friedrich.

Etant donné $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \boldsymbol{W}_{\mathcal{E},0},\ \varrho^0 > 0,$ *nous cherchons* $(\varrho^n, \boldsymbol{u}^n)_{n=1,\ldots,N}$ *solution du système algebrique suivant (schèma numérique) :*

$$\varrho^n \in L_{\mathcal{M}}(\Omega),\ \varrho^n > 0,\ \boldsymbol{u}^n \in \boldsymbol{W}_{\mathcal{E},0}(\Omega), \quad n = 0, 1, \ldots, N, \tag{5.2.6}$$

$$\sum_{K\in\mathcal{M}} |K| \frac{\varrho_K^n \quad \varrho_K^{n\ \ 1}}{\delta t} \phi_K + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} (\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}) \phi_K = 0 \textit{ pour tout } \phi \in L_{\mathcal{M}}(\Omega) \textit{ et } n = 1, \ldots, N, \tag{5.2.7}$$

$$\sum_{K\in\mathcal{M}} \frac{|K|}{\delta t} \Big(\varrho_K^n \boldsymbol{u}_K^n \quad \varrho_K^{n\ \ 1} \boldsymbol{u}_K^{n\ \ 1} \Big) \cdot \boldsymbol{v}_K + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} \hat{\boldsymbol{u}}_\sigma^{n,\text{up}} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] \cdot \boldsymbol{v}_K \tag{5.2.8}$$

$$\sum_{K\in\mathcal{T}} p(\varrho_K^n) \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{v}_\sigma \cdot \boldsymbol{n}_{\sigma,K} + \mu \sum_{K\in\mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v} \ \ \mathrm{d}\boldsymbol{x}$$

$$+(\mu+\lambda)\sum_{K\in\mathcal{T}}\int_K \mathrm{div}\boldsymbol{u}^n \mathrm{div}\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}=0,\ \textit{pour tout } \boldsymbol{v}\in \boldsymbol{W}_{\mathcal{E},0}(\Omega)\ \textit{et } n=1,\dots,N.$$

où les quantités de type upwind $\varrho_\sigma^{n,\mathrm{up}}$ et $\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}$ sont respectivement définies pour $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ par

$$\varrho_\sigma^{\mathrm{up}} = \left|\begin{array}{ll} \varrho_K & \text{si } \boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K}\geq 0, \\ \varrho_L & \text{sinon.}\end{array}\right. \qquad \hat{\boldsymbol{u}}_\sigma^{\mathrm{up}} = \left|\begin{array}{ll} \boldsymbol{u}_K & \text{si } \boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K}\geq 0, \\ \boldsymbol{u}_L & \text{sinon.}\end{array}\right. \tag{5.2.9}$$

Notons que la positivité de la densité n'est pas imposé dans le schéma mais est une conséquence du choix d'une discrétisation de type upwind dans l'équation de continuité (5.2.6). Mentionnons aussi que le schéma numérique présenté ici admet une solution. En effet ce schéma numérique étant nonlinéaire et implicite en temps, l'existence d'une soluton n'est pas triviale. Le résultat d'existence provient d'arguments standard de gradient topologique (voir [?] pour la théorie, [?] pour la première application à un schéma numérique nonlinéaire).

5.3. Estimations d'erreur

5.3.1. Esimations d'erreur pour le système de Navier-Stokes compressible sur des domaines polyédriques

La méthodologie employé, inspirée du cas continue dans [?], peut se résumer de la manière suivante

1. Inégalité d'énergie relative discrète pour la solution discrète par rapport à un état discret quelconque (voir [?, Theorem 5.1]).
2. Identité discrète satisfaite par n'importe quelle solution classique des équations de Navier-Stokes compressible. Elle sera utilisée afin d'obtenir une version discrète du lemme 1 dans le chapitre 2 (voir [?, Lemma 7.1])
3. Version discrète de l'inégalité (3.2.24) dans le chapitre 2 (voir [?, Lemma 8.1]).
4. Utilisation du lemme de Gronwall discret pour obtenir une version discrète du théorème 4.

Introduisons l'espace fonctionnel suivant :

$$\begin{aligned}\mathcal{F} = \Big\{(r,\boldsymbol{U})\in C^1([0,T]\times\overline{\Omega})^4,\ 0<\underline{r} = \inf_{(t,x)\in\overline{Q}_T} r(t,x), \nabla^2\boldsymbol{U}\in C([0,T]\times\overline{\Omega})^3,\\ \partial_t^2 r\in L^1(0,T;L^{\gamma'}(\Omega)),\ \partial_t\nabla r\in L^2(0,T;L^{6\gamma/(5\gamma\ \ 6)}(\Omega)^3),\\ (\partial_t^2\boldsymbol{U},\partial_t\nabla\boldsymbol{U})\in L^2(0,T;L^{6/5}(\Omega)^{12})\Big\},\end{aligned} \tag{5.3.1}$$

équipé de la norme suivante

$$\begin{aligned}\|(r,\boldsymbol{U})\|_{\mathcal{F}} = \|(r,\boldsymbol{U})\|_{C^1([0,T]\times\overline{\Omega})^4} + \|\nabla^2\boldsymbol{U}\|_{C([0,T]\times\overline{\Omega})^3} + \|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}\\ + \|\partial_t\nabla r\|_{L^2(0,T;L^{6\gamma/(5\gamma\ \ 6)}(\Omega)^3)} + \|\partial_t^2\boldsymbol{U}\|_{L^2(0,T;L^{6/5}(\Omega)^{12})}\\ + \|\partial_t\nabla\boldsymbol{U}\|_{L^2(0,T;L^{6/5}(\Omega)^{12})}.\end{aligned} \tag{5.3.2}$$

Le théorème suivant est le résultat principal de l'article [**?**]. Il peut être vu comme une version discrète du théorème 4 dans une classe plus forte pour la solution forte que la classe (2.2.33).

Théorème 12 (Gallouët, Herbin, Maltese, Novotný, 2015, [**?**])**.** *Soit $\mathcal{M}$ une décomposition du domaine polyédrique $\Omega \subset \mathbb{R}^3$ en symplexes, de taille $h_{\mathcal{M}}$ et de regularité $\theta_{\mathcal{M}}$ définie par (5.2.1). Soit p satisfaisant (5.1.2) avec $\gamma \geq \frac{3}{2}$ et l'hypothèse supplementaire (5.1.3) lorsque $\gamma < 2$. Considérons une parition $0 = t^0 < t^1 < ... < t^N = T$ de l'intervalle $[0,T]$, dont, dans un but de simplicité, nous supposons uniforme où δt désigne le pas de temps. Soit $(\varrho^n, \boldsymbol{u}^n)_{1\leq n\leq N} \in L_{\mathcal{M}}(\Omega) \times \boldsymbol{W}_{\mathcal{E},0}(\Omega)$ une solution discrète au problème (5.2.6)-(5.2.8) émanant de $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ tel que $\varrho^0 > 0$ et soit $(r, \boldsymbol{U}) \in \mathcal{F}$ une solution forte au problème (5.1.1)-(5.1.6). Alors il existe une constante $c > 0$ dépendant seulement de $T, \Omega, p_0, p_\infty, \mu, \gamma, \underline{r}, \min_{[\underline{r},\overline{r}]} p, \min_{[\underline{r}/2, 2\overline{r}]} p'$, de$\|(r, \boldsymbol{U})\|_{\mathcal{F}}$ de manière croissante, de $\mathcal{E}_{0,\mathcal{M}}$ de manière croissante et de $\theta_{\mathcal{M}}$ de manière décroissante telle que*

$$\max_{0\leq n\leq N} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r(t^n,\cdot)), \boldsymbol{U}(t^n,\cdot)) \leq c\Big(\mathcal{E}(\varrho^0, \boldsymbol{u}^0 \Big| r^0, \boldsymbol{U}^0) + h_{\mathcal{M}}^A + \sqrt{\delta t}\Big), \tag{5.3.3}$$

où

$$A = \min(\frac{2\gamma - 3}{\gamma}, \frac{1}{2}). \tag{5.3.4}$$

Remarque 7. *En constraste avec n'importe qu'elle autre estimation d'erreur traitant des méthodes volumes/éléments finis pour les fluides compressibles (voir Yovanovic [**?**], Cancès et al [**?**], Eymard et al. [**?**], Villa, Villedieu [**?**], Rohde, Yovanovich [**?**] et autres), ce résultat ne requiert aucune estimation sur la solution discrète : les seules estimations nécessaires pour le résultat sont celles fournies par le schéma numérique.*

Remarque 8. *Nous renvoyons le lecteur à [**?**] pour obtenir des estimations d'erreur en terme d'espace classique de Lebesgue obtenue à partir de l'inegalité (5.3.3).*

Remarque 9. *Notons que pour $\gamma = \frac{3}{2}$, le théorème 12 donne seulement une borne uniforme en la différence de la solution exacte avec la solution numérique, et non la convergence. Ce résultat est à mettre en comparaison avec la théorie d'existence de solutions faibles.*

5.3.2. Estimations d'erreur pour le système de Navier-Stokes compressible sur des domaines suffisament réguliers

Malheureusement la regularité Lipchitzienne pour le domaine Ω n'est pas suffisante pour obtenir l'existence d'une solution forte aux équations de Navier-Stokes compressibles (voir les théorèmes 5 et 6 dans le chapitre 2). Dans [**?**] notre but est alors de comparer une solution numérique construite sur un domaine polyédrique convenable avec une solution forte à densité bornée aux équations de Navier-Stokes sur un ouvert borné suffisament régulier pour lequel nous sommes assurés de l'existence de celle-ci.

Dans cette partie nous supposons que le tenseur des contraintes visqueuses est donné par

$$\mathbb{S}(\nabla \boldsymbol{u}) = \mu(\nabla \boldsymbol{u} + \nabla^t \boldsymbol{u} \quad \frac{2}{3} \operatorname{div} \boldsymbol{u} \mathbb{I}_3), \tag{5.3.5}$$

ce qui donne

$$\operatorname{div} \mathbb{S}(\nabla \boldsymbol{u}) = \mu \nabla \boldsymbol{u} + \frac{\mu}{3} \nabla \operatorname{div} \boldsymbol{u},$$

Nous supposons de plus que la pression en plus des hypothéses (5.1.2)-(5.1.3) vérifie

$$p \in C^1(\mathbb{R}_+). \tag{5.3.6}$$

Plus précisement, dans [?], nous étendons le résultat du théorème 12 dans deux directions :

1. Le domaine physique Ω occupé par le fluide et le domaine numérique Ω_{approx}, approximant le domaine physique Ω ne coincident pas.
2. Si le domaine physique est suffisament régulier (au moins de classe C^3) et si la donnée initale de classe C^3 satisfait la condition de compatibilté, nous sommes alors capable d'obtenir des erreurs inconditionnelles par rapport à n'importe qu'elle solution exacte de densité bornée.

En contraste avec [?] et tout les articles mentionnés dans la remarque 7, la solution exacte dans [?] est seulement une solution faible (au sens de la définition 1 dans le chapitre 2) de densité borné. Cette apparente plus faible hypothèse est compensée par la régularité et les conditions de compatibilité imposées sur la donnée initiale ce qui rend possible un argument de bootstraping sophistiqué montrant que toute solution faible à densité bornée èmanant d'une donnée initiale suffisament régulière est en fait une solution forte dans la classe étudiée dans [?] c'est-à-dire dans la classe (5.3.1).

Ces résultats ont été accomplis à l'aide des outils suivants.

1. La technique introduite dans [?] est modifiée afin de s'accomoder avec des vitesses non nulles de la solution exacte au bord du domaine numérique. Rappelons que le domaine physique Ω occupé par le fluide et le domaine numérique Ω_{approx}, approximant le domaine physique Ω ne coincident pas. Pour ceci nous utilisons une famille de domaines polyédriques particulière pour approximer le domaine physique.
2. Trois résultats fondamentaux de la théorie des équations de Navier-Stokes compressibles, à savoir
   - Existence locale en temps de solutions fortes dans la classe (5.3.1) par Cho, Choe, Kim [?], voir théorème 6 dans le chapitre 2.
   - Principe d'unicité fort-faible , voir corollaire 4 dans le chapitre 2.
   - Critère d'explosion pour les solutions fortes dans la classe (5.3.1) par Sun, Wang, Zhang [?], voir théorème 5 dans le chapitre 2.

   Les trois résultas mentionnés au dessus nous autorise à montrer qu'une solution faible à densité bornée émanant d'une condition initiale suffisament régulière est en faite une solution forte définie sur un large intervalle de temps $[0, T)$ (voir corollaire 4 dans le chapitre 2). Notons que l'hypothèse (5.3.6) sur la pression est une conséquence de l'utilisation du théorème 6.

3. Argument de bootstraping utilisant des résultats récents sur la régularité maximale des systèmes paraboliques par Danchin [?], Denk, Pruess, Hieber [?] et Krylov [?]. Ce dernier point nous autorise à utliser une technique de bootstraping de la solution forte dans la classe de Cho, Choe, Kim [?] dans la classe nécessaire pour les estimations d'erreur dans [?], dès lors que la condition initiale satisfait une certaine condition de compatibilité.

Les deux derniers points sont formulés dans la proposition suivante (voir [?, Proposition 2.1]).

Proposition 2. *Soit $\Omega \subset \mathbb{R}^3$ un ouvert borné de classe C^3. Soit $(r, \boldsymbol{U})$ une solution faible au problème (5.1.1)-(5.1.6) dans $(0,T) \times \Omega$, émanant de la donnée initiale $(r_0, \boldsymbol{U}_0)$ satisfaisant*

$$r_0 \in C^3(\overline{\Omega}),\ \ r_0 > 0 \ \textit{dans}\ \overline{\Omega}, \tag{5.3.7}$$

$$\boldsymbol{U}_0 \in C^3(\overline{\Omega})^3, \tag{5.3.8}$$

et la condition de compatibilité

$$\boldsymbol{U}_0|_{\partial\Omega} = 0,\ \ \nabla p(r_0)|_{\partial\Omega} = \operatorname{div} \mathbb{S}(\nabla \boldsymbol{U}_0)|_{\partial\Omega} \tag{5.3.9}$$

et telle que

$$0 \le r \le \overline{r}\ \textit{p.p dans}\ \ (0,T) \times \Omega. \tag{5.3.10}$$

Alors $(r, \boldsymbol{U})$ est une solution classique satisfaisant les estimations suivantes :

$$\|1/r\|_{C([0,T]\times\overline{\Omega})} + \|r\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t \nabla r\|_{C([0,T];L^6(\Omega)^3)} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(\Omega))} \le D, \tag{5.3.11}$$

$$\|\boldsymbol{U}\|_{C^1([0,T]\times\overline{\Omega})^3} + \|\boldsymbol{U}\|_{C([0,T];C^2(\overline{\Omega})^3)} + \|\partial_t \nabla \boldsymbol{U}\|_{C([0,T];L^6(\Omega)^{3\times 3})} + \|\partial^2_{t,t} \boldsymbol{U}\|_{L^2(0,T;L^6(\Omega)^3)} \le D, \tag{5.3.12}$$

où D dépend de Ω, T, $\overline{r}$, et de la donnée initiale $(r_0, \boldsymbol{U}_0)$ (au travers de $\|(r_0, \boldsymbol{U}_0)\|_{C^3(\overline{\Omega})^4}$ et $\min_{\boldsymbol{x}\in\overline{\Omega}} r_0(\boldsymbol{x})$).

Puisque le domaine physique Ω occupé par le fluide et le domaine numérique Ω_{approx} approximant le domaine physique ne coincident pas, nous devons étendre proprement la solution forte construite dans la proposition 2.

Lemme 2. *Sous les hypothèses de la proposition 2, les fonctions r et $\boldsymbol{U}$ peuvent être prolongés en dehors de Ω de telles sorte que :*

(1) Les fonctions prolongées (toujours notées r et $\boldsymbol{U}$) sont telles que $\boldsymbol{U}$ est à support compact dans $[0,T] \times \mathbb{R}^3$ et $r \ge \underline{r} > 0$.

(2)

$$\|\boldsymbol{U}\|_{C^1([0,T]\times\mathbb{R}^3)^3} + \|\boldsymbol{U}\|_{C([0,T];C^2(\mathbb{R}^3)^3)} + \|\partial_t \nabla \boldsymbol{U}\|_{C([0,T];L^6(\mathbb{R}^3)^{3\times 3})} + \|\partial^2_{t,t} \boldsymbol{U}\|_{L^2(0,T;L^6(R^3))} \tag{5.3.13}$$

$$\lesssim \|\boldsymbol{U}\|_{C^1([0,T]\times\overline{\Omega})^3} + \|\boldsymbol{U}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla \boldsymbol{U}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t} \boldsymbol{U}\|_{L^2(0,T;L^6(\Omega))};$$

(3)

$$\|r\|_{C^1([0,T]\times R^3)} + \|\partial_t \nabla r\|_{C([0,T];L^6(R^3;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(R^3))} \tag{5.3.14}$$
$$\lesssim \|r\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t \nabla r\|_{C([0,T];L^6(\Omega;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(\Omega))} +$$
$$\|\boldsymbol{U}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\boldsymbol{U}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla \boldsymbol{U}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t}\boldsymbol{U}\|_{L^2(0,T;L^6(\Omega))};$$

(4)

$$\partial_t r + \operatorname{div}(r\boldsymbol{U}) = 0 \text{ dans } (0,T)\times\mathbb{R}^3. \tag{5.3.15}$$

Domaine physique, maillage

Comme mentionné précédemment, puisque le domaine physique Ω occupé par le fluide et le domaine numérique Ω_{approx} approximant le domaine physique ne coincident pas, la technique introduite dans [?] doit être modifié afin de s'adapter avec des vitesse non nulles de la solution exacte sur le bord du domaine numérique. Afin de surmonter cette difficulté, nous introduisons une famille de domaines polyédriques $(\Omega_{\mathcal{M}}, \mathcal{M})$ où $\mathcal{M}$ est une une triangulation de $\Omega_{\mathcal{M}}$ supposée être régulière au sens de la litterature des éléments finis (voir [?]), et qui vérifie la propriété suivante

$$\mathcal{V} \in \partial\Omega_{\mathcal{M}} \text{ est un sommet } \Rightarrow \mathcal{V} \in \partial\Omega. \tag{5.3.16}$$

Les propriétés du maillage nécessaires pour les estimations d'erreur afin de s'adapter avec des vitesse non nulles de la solution exacte sur le bord du domaine numérique sont formulées dans le lemme suivant, dont la preuve (évidente) est laissé au lecteur (voir [?, Theorem 1]).

Lemme 3. *Il existe une constante d_Ω dépendant seulement des propriétés géométriques de $\partial\Omega$ telle que*

$$\operatorname{dist}[\boldsymbol{x}, \partial\Omega] \leq d_\Omega h^2_{\mathcal{M}}$$

pour tout $\boldsymbol{x} \in \partial\Omega_{\mathcal{M}}$. De plus,

$$|(\Omega_h \setminus \Omega) \cup (\Omega \setminus \Omega_h)| \lesssim h^2_{\mathcal{M}}.$$

Remarque 10. *Il est important de souligner que $\Omega_{\mathcal{M}} \not\subset \Omega$, en général.*

Le résultat principal de l'article [?] est le suivant (voir [?, Theorem 3.1]) :

Théorème 13 (Feireisl, Hosek, Maltese, Novotný, 2015, [?])**.** *Soit $\Omega \subset \mathbb{R}^3$ une ouvert borné de classe C^3. Supposons que la pression satisfait (5.1.2) avec $\gamma \geq 3/2$, et l'hypothèse supplementaire (5.1.3) lorsque $\gamma < 2$. Soit $(\Omega_{\mathcal{M}}, \mathcal{M})$ une famille de domaine polyhédrique où $\mathcal{M}$ est une une triangulation de $\Omega_{\mathcal{M}}$ supposée être régulière, de taille $h_{\mathcal{M}}$ et de rgularité $\theta_{\mathcal{M}}$ vérifiant $\theta_{\mathcal{M}} \geq \theta_0$ où $\theta_0 > 0$. Supposons que cette famille vérifie (5.3.16). Considérons une parition $0 = t^0 < t^1 < ... < t^N = T$ de l'intervalle $[0,T]$, dont, dans un but de simplicité, nous supposons uniforme où δt désigne le pas de temps. Soit $\{\varrho^n, \boldsymbol{u}^n\}_{0\leq n\leq N}$ une solution numérique au problème (5.2.6–5.2.8) sur le domaine polyhédrique $\Omega_{\mathcal{M}}$. Considérons une donnée initiale $(r_0, \boldsymbol{U}_0)$ appartenant à la classe spécifié* *dans la proposition 2 et donnant lieu à une solution faible* $(r, \boldsymbol{U})$ *au problème (5.1.1)-(5.1.6) sur* $(0,T)\times\Omega$ *satisfaisant*

$$0 \leq r(t,\boldsymbol{x}) \leq \overline{r} \text{ p.p dans } (0,T)\times\Omega.$$

Alors $(r, \boldsymbol{V})$ *est régulière et il existe une constante positive*

$$C = C\Big(M_0, \mathcal{E}_0, \theta_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{U}, \partial_t \boldsymbol{U}, \nabla\boldsymbol{U}, \nabla^2\boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{45})},$$

$$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma\quad 6}(\Omega;\mathbb{R}^3))}, \|\partial_t^2\boldsymbol{U}, \partial_t\nabla\boldsymbol{U}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))}\Big)$$

telle que

$$\sup_{1\leq n\leq N}\int_{\Omega\cap\Omega_{\mathcal{M}}}\frac{1}{2}\varrho^n|\hat{\boldsymbol{u}}^n \quad \boldsymbol{U}(t^n,\cdot)|^2 + E(\varrho^n|r(t^n,\cdot))\ \mathrm{d}\boldsymbol{x}$$

$$\leq C\left(\sqrt{\delta t} + h_{\mathcal{M}}^A + \int_{\Omega\cap\Omega_{\mathcal{M}}}\frac{1}{2}\varrho^0|\hat{\boldsymbol{u}}^0 \quad \boldsymbol{U}_0|^2 + E(\varrho^0|r_0)\ \mathrm{d}\boldsymbol{x}\right),$$

où

$$A = \min(\frac{2\gamma \quad 3}{\gamma}, \frac{1}{2}).$$

Remarque 11. *Notons que l'existence d'une famille* $(\Omega_{\mathcal{M}}, \mathcal{M})$ *vérfiant les propriétés du théorème précédent est assurée par [**?**, Theorem 1] dés lors que* Ω *est un ouvert borné de classe* C^2.

6. Estimations d'erreur pour le schema Marker-and-Cell pour les équations de Navier-Stokes compressibles

6.1. Présentation

Le but de ce chapitre, dont les résultats sont issus de l'article [?] (voir Annexe E), est d'établir des estimations d'erreur pour le schéma Marker-and-Cell pour le système de Navier-Stokes compressible en regime barotrope. Nous considérons donc le système suivant

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ dans } (0,T) \times \Omega, \tag{6.1.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\varrho) = \mu \Delta \boldsymbol{u} + (\mu + \lambda) \nabla \operatorname{div} \boldsymbol{u} \text{ dans } (0,T) \times \Omega, \tag{6.1.1b}$$

où nous supposons, comme dans le chapitre 5, que la pression satisfait

$$p \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^\star),\ p(0) = 0,\ p'(\varrho) > 0 \text{ pour tout } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma\;1}} = p_\infty > 0 \tag{6.1.2}$$

où $\gamma \geq 1$ et si $\gamma < 2$ dans (6.1.2), nous avons besoin d'une condition additionnelle pour les petites densités :

$$\lim_{\varrho \to 0} \frac{p'(\varrho)}{\varrho^{\alpha+1}} = p_0 > 0,\ \alpha \leq 0. \tag{6.1.3}$$

Les coefficients de viscosité sont supposés constants et vérifient

$$\mu > 0 \text{ et } \lambda + \mu \geq 0. \tag{6.1.4}$$

Le système est complété avec les conditions initiales pour la densité et la quantité de mouvement

$$\varrho(0, \cdot) = \varrho_0,\ \varrho \boldsymbol{u}(0, \cdot) = \varrho_0 \boldsymbol{u}_0, \tag{6.1.5}$$

où ϱ_0 et $\boldsymbol{u}_0$ sont deux fonctions données respectivement de Ω dans $\mathbb{R}_+^\star$ et $\mathbb{R}^d$, et la condition au bord

$$\boldsymbol{u} = 0 \text{ dans } (0,T) \times \partial\Omega. \tag{6.1.6}$$

Depuis l'introduction du schéma Marker-and-cell (MAC) [?], il est communément accepté que cette discretisation est convenable à la fois pour les problèmes d'écoulement incompressible ou compressible (see [?, ?] pour les articles phares, [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] pour des développements subséquents et [?] pour un état de l'art). L'utilisation du schéma dans la cas incompressible est maintenant standard, et la preuve de convergence du schema MAC en variables primitives a été récemment effectué dans [?].

6.2. Schéma numérique

6.2.1. Discretisation spatiale

Nous supposons $\Omega \subset \mathbb{R}^d$ est un domaine dont la fermeture est une union de rectangles fermés ($d = 2$) ou de parallélépipèdes fermés ($d = 3$) d'intérieurs mutuellement disjoints, et, sans perdre de généralité, nous supposons que les arêtes (ou faces) de ces rectangles (ou parallélépipèdes) sont orthogonales aux vecteurs de la base canonique noté par $(\boldsymbol{e}^{(1)}, \ldots, \boldsymbol{e}^{(d)})$. Introduisons alors la notion de grille MAC. Nous renvoyons le lecteur à [?, Definition 2] pour plus de détails concernant ce maillage.

Definition 6 (Grille MAC)**.** *Une discretisation de Ω avec une grille MAC, dénotée par $\mathcal{D}$, est donnée par $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, où :*

- *$\mathcal{M}$ est une grille conforme de Ω, constituée d'une union de rectangles fermés ($d = 2$) ou de parallélépipèdes fermés ($d = 3$) non nécéssairement uniformes et d'intérieurs mutuellement disjoints et telle que les arêtes (ou faces) de ces rectangles (ou parallélépipèdes) sont orthogonales aux vecteurs de la base canonique. Une cellule générique de cette grille est notée K. Cette grille est la grille de discretisation de la densité.*
- *$\mathcal{E}$ est l'ensemble des faces de la grille primale $\mathcal{M}$. L'ensemble des faces qui appartiennent à l'interieur de Ω est notée par $\mathcal{E}_{\rm int}$. L'ensemble des faces qui appartiennent au bord de Ω est notée par $\mathcal{E}_{\rm ext}$. En particulier nous avons $\mathcal{E} = \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$. L'ensemble des faces orthogonales au i eme vecteur unitaire $\mathbf{e}^{(i)}$ de la base canonique $\mathbb{R}^d$ est noté par $\mathcal{E}^{(i)}$, for $i = 1, \ldots, d$. Nous avons alors $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}_{\rm int} \cup \mathcal{E}^{(i)}_{\rm ext}$, where $\mathcal{E}^{(i)}_{\rm int}$ (resp. $\mathcal{E}^{(i)}_{\rm ext}$) sont les éléments de $\mathcal{E}^{(i)}$ qui appartiennent à l'interieur (resp. au bord) du domaine Ω.*
- *Pour chaque $\sigma \in \mathcal{E}$, nous écrivons $\sigma = K|L$ if $\sigma = \partial K \cap \partial L$ est une face commune.*
- *Une cellule duale D_σ associée à la face $\sigma \in \mathcal{E}$ est définie de la manière suivante :*
  - *si $\sigma = K|L \in \mathcal{E}_{\rm int}$ alors $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$, où l'ensemble fermé $D_{\sigma,K}$ (resp. $D_{\sigma,L}$) est la moitié de K (resp. de L) adjacent à σ (voir Fig. 6.1 pour le cas bidimensionel) ;*
  - *if $\sigma \in \mathcal{E}_{\rm ext}$ est adjacent à la cellule K, alors $D_\sigma = D_{\sigma,K}$.*

  La grille duale $\{D_\sigma\}_{\sigma \in \mathcal{E}^{(i)}}$ de Ω (appelée quelquefois grille de la i-eme composante de la vitesse) est une grille conforme et vérifie en particulier pour chaque $i \in \{1, \ldots, d\}$

$$\overline{\Omega} = \cup_{\sigma \in \mathcal{E}^{(i)}} D_\sigma, \;\; \mathrm{int}(D_\sigma) \cap \mathrm{int}(D_{\sigma'}) = \emptyset, \; \sigma, \sigma' \in \mathcal{E}^{(i)}, \; \sigma \neq \sigma'. \tag{6.2.1}$$

  Notons aussi que l'ensemble des faces $\{D_\sigma\}_{\sigma \in \mathcal{E}^{(i)}}$ sont orthogonales aux vecteurs de la base canonique.
- *Nous définissons la taille maillage par*

$$h_{\mathcal{M}} = \max\{h_K, K \in \mathcal{M}\} \tag{6.2.2}$$

où h_K représente le diamètre de K. Nous mesurons la regularité du maillage au travers du paramètre θ définie par :

$$\theta_{\mathcal{M}} = \min\{\frac{\xi_K}{h_K},\ K \in \mathcal{M}\} \tag{6.2.3}$$

où ξ_K représente le diamètre de la plus grosse sphère incluse dans K.

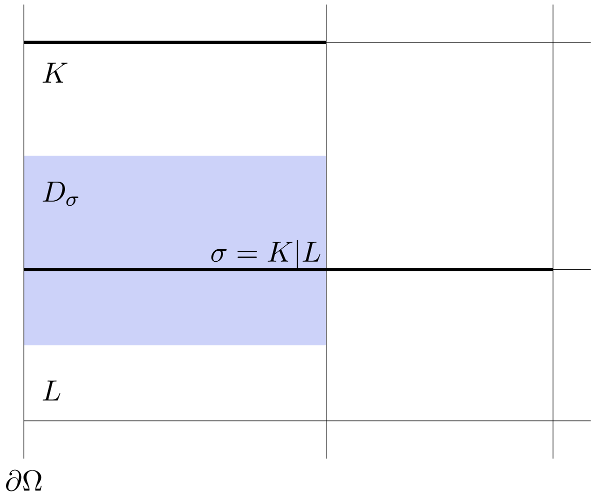

FIGURE 6.1. – Notations pour les volumes de controle et les cellules duales

Definition 7 (Espaces discrets)**.** *Soit $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ une grille MAC au sens de la définition 6. L'espace de la densité discrète $L_{\mathcal{M}}$ est alors défini comme l'ensemble des fonctions constantes par morceaux sur chaque cellule K de $\mathcal{M}$, et le i ème espace de la vitesse discrète $H_{\mathcal{E}}^{(i)}$ comme l'ensemble des fonctions constantes par morceaux sur chaque cellule D_σ , $\sigma \in \mathcal{E}^{(i)}$. Comme dans le cas continue, les conditions de Dirichlet homogène (6.1.6) sont partiellement incorporées dans la définition des espaces discrets pour la vitesse, et, à cet effet, nous introduisons $H_{\mathcal{E},0}^{(i)} \subset H_{\mathcal{E}}^{(i)}, i = 1, \ldots, d$, définis de la manière suivante :*

$$H_{\mathcal{E},0}^{(i)} = \Big\{ v \in H_{\mathcal{E}}^{(i)},\ v(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \widetilde{\mathcal{E}}_{\text{ext}}^{(i)},\ i = 1, \ldots, d \Big\}.$$

Nous notons alors $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^{d} H_{\mathcal{E},0}^{(i)}$.

6.2.2. Discrétisation temporelle

Concernant la discrétisation en temps du problème (6.1.1)-(6.1.6), nous considérons, dans un but de simplcité, une partition $0 = t^0 < t^1 < \cdots < t^N = T$ de l'intervalle de temps $(0, T)$ de pas constant $\delta t = t^n \quad t^{n \ \ 1}$; donc $t^n = n\delta t$ pour $n \in \{0, \cdots, N\}$.

Finalement, la condition initiale discrète $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ est telle que $\varrho^0 > 0$. La masse totale discrète et l'énergie totale discrète sont respectivement définies par

$$M_{0,\mathcal{M}} = \int_\Omega \varrho^0 \,\mathrm{d}\boldsymbol{x}, \qquad \mathcal{E}_{0,\mathcal{M}} = \int_\Omega \frac{1}{2} \varrho^0 |\boldsymbol{u}^0|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathcal{H}(\varrho^0) \,\mathrm{d}\boldsymbol{x}. \tag{6.2.4}$$

6.2.3. Schéma numérique

Etant donné $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$, $\varrho^0 > 0$, *nous cherchons* $(\varrho^n, \boldsymbol{u}^n)_{n=1,\ldots,N}$ *solution du système algebrique suivant (schèma numérique) :*

$$\varrho^n \in L_{\mathcal{M}}, \;\; \varrho^n > 0, \;\; \boldsymbol{u}^n \in \mathbf{H}_{\mathcal{E},0}, \quad n = 0, 1, \ldots, N, \tag{6.2.5a}$$

$$\frac{1}{\delta t}(\varrho^n \quad \varrho^{n\ \ 1}) + \mathrm{div}^{\mathrm{up}}_{\mathcal{M}}(\varrho^n \boldsymbol{u}^n) = 0, \tag{6.2.5b}$$

$$\frac{1}{\delta t}(\widehat{\varrho^n}^{(i)} u_i^n \quad \widehat{\varrho^{n\ \ 1}}^{(i)} u_i^{n\ \ 1}) + \mathrm{div}^{(i)}_{\mathcal{E}}(\varrho^n \boldsymbol{u}^n u_i^n) \quad \mu \Delta^{(i)}_{\mathcal{E}} u_i^n$$
$$(\mu + \lambda)\eth_i \,\mathrm{div}_{\mathcal{M}} \boldsymbol{u}^n + \eth_i p(\varrho^n) = 0, \tag{6.2.5c}$$

et nous renvoyons le lecteur à [?] pour une définiion des opérateurs differentiels introduits dans le schéma numérique précédent. Notons aussi que la positivité de la densité n'est pas imposé dans le schéma mais est une conséquence du choix d'une discrétisation de type upwind dans l'équation de continuité (5.2.6). Mentionnons aussi que le schéma numérique présenté ici admet une solution. Le résultat d'existence provient aussi d'arguments standard de gradient topologique.

6.3. Estimations d'erreur

La méthodologie employée pour obtenir des estimations d'erreur inconditionnelles pour le schéma numérique (6.2.5) est la même que celle utilisée dans pour le sch`ema numérique introduit dans le chapitre 5 à savoir

1. Inégalité d'énergie relative discrète pour la solution discrète par rapport à un état discret quelconque (voir [?, Proposition 2]).
2. Identité discrète satisfaite par n'importe quelle solution classique des équations de Navier-Stokes compressible. Elle sera utilisée afin d'obtenir une version discrète du lemme 1 dans le chapitre 2 (voir [?, Lemma 7])
3. Version discrète de l'inégalité (3.2.24) dans le chapitre 2 (voir [?, Lemma 8]).
4. Utilisation du lemme de Gronwall discret pour obtenir une version discrète du théorème 4.

Le résultat suivant est alors le résultat principal de l'article [?] (voir aussi Annexe E).

Théorème 14 (Gallouët, Herbin, Maltese, Novoyný, 2016, [?]). *Soit* $\Omega \subset \mathbb{R}^3$ *une domaine dont la fermeture est une union de parallélépipèdes fermés d'intérieurs mutuellement disjoints, et, sans perdre de généralité, de faces orthogonales aux vecteurs de la base canonique. Soit* $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ *une grille MAC de* Ω *de taille* $h_{\mathcal{M}}$ *(voir (6.2.2)) et de régularité* $\theta_{\mathcal{M}}$ *where* $\theta_{\mathcal{M}}$ *définie en (6.2.3). Soit* p *satisfaisant (6.1.2) avec* $\gamma \geq \frac{3}{2}$ *et l'hypothèse supplementaire (6.1.3) lorsque* $\gamma < 2$. *Considérons une parition* $0 = t^0 < t^1 < \ldots < t^N = T$ *de l'intervalle* $[0, T]$, *dont, dans un but de simplicité, nous supposons uniforme* *où δt désigne le pas de temps. Soit $(\varrho^n, \boldsymbol{u}^n)_{1\le n\le N} \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ une solution au problème discret (6.2.5) émanant de $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ tel que $\varrho^0 > 0$ et soit $(r, \boldsymbol{U}) \in \mathcal{F}$ (voir (5.3.1) une solution forte au problème (6.1.1)-(6.1.6). Alors il existe une constante $c > 0$ dépendant seulement de $T, \Omega, p_0, p_\infty, \mu, \gamma, \underline{r}, \min_{[\underline{r},\overline{r}]} p, \min_{[\underline{r}/2, 2\overline{r}]} p'$, de$\|(r, \boldsymbol{U})\|_{\mathcal{F}}$ de manière croissante, de $\mathcal{E}_{0,\mathcal{M}}$ de manière croissante et de $\theta_{\mathcal{M}}$ de manière décroissante telle que*

$$\max_{0\le n\le N} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r(t^n,\cdot), \boldsymbol{U}(t^n,\cdot)) \le c\Big(\mathcal{E}(\varrho^0, \boldsymbol{u}^0 \Big| r(0,\cdot), \boldsymbol{U}(0,\cdot)) + h_{\mathcal{M}}^A + \sqrt{\delta t}\Big), \tag{6.3.1}$$

où

$$A = \min(\frac{2\gamma \quad 3}{\gamma}, \frac{1}{2}). \tag{6.3.2}$$

Remarque 12. *Nous renvoyons le lecteur à [?] (voir aussi Annexe F) pour obtenir des estimations d'erreur en terme d'espace classique de Lebesgue obtenue à partir de l'inegalité (5.3.3).*

6.4. Estimations d'erreur uniformes en régime bas Mach

6.4.1. Présentation du problème

Dans [?], nous avons obtenu des estimations d'erreur inconditionnelles pour le schéma Marker-and-Cell pour les équations de Navier-Stokes compressibles sans considérer le nombre de Mach. Let but de l'article [?] est de prendre en compte le nombre de Mach et d'étudier le caractère asymptotiquement préservant du schéma Marker-and-Cell en régime bas Mach. Le but est de donner des estimations d'erreur entre une solution discrète du schéma Marker-and-Cell pour les équations de Navier-Stokes compressibles en régime bas Mach et l'unique solution forte limite aux équations de Navier-Stokes incompressibles en termes de puissances positives des paramètres de discrétisation et du nombre de Mach. La constante dans les estimations d'erreurs doit aussi être indépendante de la solution discrète afin d'obtenir comme dans le théorème 14 une estimation d'erreur inconditionnelle mais peut dépendre de la solution forte. Les schémas numériques possédant ce type d'estimations d'erreur sont qualifiés d'uniformément asymptotiquement préservant. Malgré l'importance de cette propriété dans les applications, la littérature mathématique sur ce sujet est peu abondante, probablement du à la complexité du problème : le caractère asymptotiquement préservant des schémas numériques est seulement connu au niveau de l'approximation numérique, et dans ce cas la constante dans les estimations d'erreur dépent des paramètres spatiaux et temporels de la discrétisation (voir [?], [?], [?], [?], [?], [?]) Ce type d'estimations ne donnent aucune information sur la convergence du schéma numérique, et c'est un sérieux incovénient. En notant $\epsilon > 0$ le nombre de Mach, nous considérons alors le système de Navier-Stokes compressible en regime barotrope et en régime bas Mach

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ dans } (0,T)\times\Omega, \tag{6.4.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u}\otimes \boldsymbol{u}) + \frac{1}{\epsilon^2}\nabla p(\varrho) = \mu \Delta \boldsymbol{u} + (\mu+\lambda)\nabla \operatorname{div} \boldsymbol{u} \text{ dans } (0,T)\times\Omega. \tag{6.4.1b}$$

Nous supposons toujours que la pression satisfait les hypothèses précédentes utilisées pour les estimations d'erreur à savoir

$$p \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^\star),\ p(0) = 0,\ p'(\varrho) > 0 \text{ pour tout } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma\ \ 1}} = p_\infty > 0 \quad (6.4.2)$$

où $\gamma \geq 1$ et si $\gamma < 2$ dans (6.4.2), nous supposons de plus :

$$\lim_{\varrho \to 0} \frac{p'(\varrho)}{\varrho^{\alpha+1}} = p_0 > 0,\ \ \alpha \leq 0. \tag{6.4.3}$$

Les coefficients de viscosité sont supposés constants et vérifient

$$\mu > 0 \text{ et } \lambda + \mu \geq 0. \tag{6.4.4}$$

Le système est complété avec les conditions initiales pour la densité et la quantité de mouvement

$$\varrho(0, \cdot) = \varrho_0,\ \ \varrho \boldsymbol{u}(0, \cdot) = \varrho_0 \boldsymbol{u}_0, \tag{6.4.5}$$

où ϱ_0 et $\boldsymbol{u}_0$ sont deux fonctions données respectivement de Ω dans $\mathbb{R}_+^\star$ et $\mathbb{R}^d$, et la condition au bord

$$\boldsymbol{u} = 0 \text{ dans } (0, T) \times \partial\Omega. \tag{6.4.6}$$

Parallèlement, nous considérons la solution forte aux équations de Navier-Stokes incompressibles, limite du problème (6.4.1)-(6.4.6),

$$\overline{\varrho}(\partial_t \boldsymbol{V} + \nabla \boldsymbol{V} \cdot \nabla \boldsymbol{V}) + \Pi = \mu \Delta \boldsymbol{V} \text{ dans } (0, T) \times \partial\Omega. \tag{6.4.7}$$

$$\operatorname{div} \boldsymbol{V} = 0 \text{ dans } (0, T) \times \partial\Omega, \tag{6.4.8}$$

telle que

$$\overline{\varrho} = \frac{M_0}{|\Omega|} > 0, \qquad M_0 = \int_\Omega \varrho_0 \,\mathrm{d}\boldsymbol{x}. \tag{6.4.9}$$

Le système est complété avec la condition initiale pour la vitesse

$$\boldsymbol{V}(0, \cdot) = \boldsymbol{V}_0, \tag{6.4.10}$$

et de condition au bord

$$\boldsymbol{V} = 0 \text{ dans } (0, T) \times \partial\Omega. \tag{6.4.11}$$

6.4.2. Schéma numérique

Concernant la discrétisation en temps du problème (6.4.1)-(6.4.6), nous considérons, dans un but de simplcité, une partition $0 = t^0 < t^1 < \cdots < t^N = T$ de l'intervalle de temps $(0, T)$ de pas constant $\delta t = t^n \quad t^{n\ \ 1}$; donc $t^n = n\delta t$ pour $n \in \{0, \cdots, N\}$. Nous notons respectivement par $\{u_\sigma^n, \sigma \in \mathcal{E}_{\text{int}}^{(i)}, i \in \{1, \cdots, d\}, n \in \{0, \cdots, N\}\}$, and $\{\varrho_K^n, K \in \mathcal{M}, n \in \{1, \cdots, N\}\}$ l'ensemble des inconnues discrètes de la i-eme composante de la vitesse et la densité.

Finalement, la condition initiale discrète $(\varrho_\epsilon^0, \boldsymbol{u}_\epsilon^0) \in L_\mathcal{M} \times \mathbf{H}_{\mathcal{E},0}$ est telle que $\varrho_\epsilon^0 > 0$. La masse totale discrète et l'énergie totale discrète sont respectivement définies par

$$M_{0,\mathcal{M},\epsilon} = \int_\Omega \varrho_\epsilon^0 \,\mathrm{d}\boldsymbol{x}, \qquad \mathcal{E}_{0,\mathcal{M},\epsilon} = \int_\Omega \frac{1}{2}\varrho^0|\boldsymbol{u}^0|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{\epsilon^2}\int_\Omega E(\varrho_\epsilon^0|\overline{\varrho}) \,\mathrm{d}\boldsymbol{x}. \tag{6.4.12}$$

Etant donné $(\varrho_\epsilon^0, \boldsymbol{u}_\epsilon^0) \in L_\mathcal{M} \times \mathbf{H}_{\mathcal{E},0}$, $\varrho_\epsilon^0 > 0$, nous cherchons $(\varrho_\epsilon^n, \boldsymbol{u}_\epsilon^n)_{n=1,\dots,N}$ solution du système algebrique suivant (schèma numérique) :

$$\varrho_\epsilon^n \in L_\mathcal{M}(\Omega),\ \varrho_\epsilon^n > 0,\ \boldsymbol{u}_\epsilon^n \in \boldsymbol{W}_{\mathcal{E},0}(\Omega), \quad n = 0, 1, \dots, N, \tag{6.4.13a}$$

$$\frac{1}{\delta t}(\varrho_\epsilon^n \quad \varrho_\epsilon^{n\ \ 1}) + \mathrm{div}_\mathcal{M}^{\mathrm{up}}(\varrho_\epsilon^n \boldsymbol{u}_\epsilon^n) = 0, \tag{6.4.13b}$$

$$\begin{aligned}\frac{1}{\delta t}(\widehat{\varrho_\epsilon^n}^{(i)} u_{\epsilon,i}^n \quad \widehat{\varrho_\epsilon^{n\ \ 1}}^{(i)} u_{\epsilon,i}^{n\ \ 1}) + \mathrm{div}_\mathcal{E}^{(i)}(\varrho_\epsilon^n \boldsymbol{u}_\epsilon^n u_{\epsilon,i}^n) \quad \mu\Delta_\mathcal{E}^{(i)} u_{\epsilon,i}^n \\ (\mu+\lambda)\eth_i \,\mathrm{div}_\mathcal{M}\, \boldsymbol{u}_\epsilon^n + \frac{1}{\epsilon^2}\eth_i p(\varrho_\epsilon^n) = 0,\end{aligned} \tag{6.4.13c}$$

et nous renvoyons le lecteur à [?] pour une définiion des opérateurs differentiels introduits dans le schéma numérique.

6.4.3. Estimations d'erreur

La solution forte $(\Pi, \boldsymbol{V})$ du problème limite incompressible (6.4.7)-(6.4.11) est supposée appartenir à la classe suivante

$$\Pi \in \mathcal{Y}_T^p(\Omega) \equiv \{\Pi \in C([0,T]; C^1(\overline{\Omega})),\quad \partial_t\Pi \in L^1(0,T;L^p(\Omega))\}, 2 \le p \le \infty\} \tag{6.4.14a}$$

$$\begin{aligned}\boldsymbol{V} \in \mathcal{X}_T(\Omega) \equiv \{\boldsymbol{V} \in C^1([0,T]\times\overline{\Omega};\mathbb{R}^3),\ \nabla^2\boldsymbol{V} \in C([0,T]\times\overline{\Omega};\mathbb{R}^3),\\ (\partial_t^2\boldsymbol{V}, \partial_t\nabla\boldsymbol{V}) \in L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12})).\end{aligned} \tag{6.4.14b}$$

où les espaces $\mathcal{Y}_T^p(\Omega)$ et $\mathcal{X}_T(\Omega)$ sont respectivement equipés des normes suivantes

$$\|\Pi\|_{\mathcal{Y}_T^p(\Omega)} = \|\Pi\|_{C([0,T];C^1(\overline{\Omega}))} + \|\partial_t\Pi\|_{L^1(0,T;L^p(\Omega))}$$

$$\|\boldsymbol{V}\|_{\mathcal{X}_T(\Omega)} = \|\boldsymbol{V}\|_{C^1([0,T]\times\overline{\Omega};\mathbb{R}^3)} + \|\nabla^2\boldsymbol{V}\|_{C([0,T]\times\overline{\Omega};\mathbb{R}^3)} + \|\partial_t^2\boldsymbol{V}, \partial_t\nabla\boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))}.$$

Afin de prendre en compte le nombre de Mach nous introduisons la fonctionnelle d'énergie relative suivante

$$\mathcal{E}_\epsilon(\varrho, \boldsymbol{u}\Big| r, \boldsymbol{U}) = \int_\Omega \frac{1}{2}\varrho|\boldsymbol{u} \quad \boldsymbol{U}|^2 + \frac{1}{\epsilon^2}E(\varrho|r)\,\mathrm{d}\boldsymbol{x}. \tag{6.4.15}$$

Le résultat suivant concernant le caractère uniformément asymptotiquement préservant du schéma Marker-and-Cell est le résultat principal de l'article [?].

Théorème 15 (Gallouët, Herbin, Maltese, Novotný, 2016, [**?**]). *Soit* $\Omega \subset \mathbb{R}^3$ *une domaine dont la fermeture est une union de parallélépipèdes fermés d'intérieurs mutuellement disjoints, et, sans perdre de généralité, de faces orthogonales aux vecteurs de la base canonique. Soit* $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ *une grille MAC de* Ω *de taille* $h_{\mathcal{M}}$ *(voir (6.2.2)) et de régularité* $\theta_{\mathcal{M}}$ *où* $\theta_{\mathcal{M}}$ *est définie en (6.2.3). Soit* p *satisfaisant (6.4.2) avec* $\gamma \geq \frac{3}{2}$ *et l'hypothèse supplementaire (6.4.3) lorsque* $\gamma < 2$. *Considérons une parition* $0 = t^0 < t^1 < ... < t^N = T$ *de l'intervalle* $[0, T]$, *dont, dans un but de simplicité, nous supposons uniforme où* δt *désigne le pas de temps. Soit* $(\varrho_\epsilon^n, \boldsymbol{u}_\epsilon^n)_{1 \leq n \leq N} \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ *une solution au problème discret (6.4.13) émanant de* $(\varrho_\epsilon^0, \boldsymbol{u}_\epsilon^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ *tel que* $\varrho^0 > 0$ *et*

$$\frac{M_0}{2} \leq M_{0,\mathcal{M},\epsilon} \leq 2M_0, \ \overline{\varrho} M_0 = |\Omega|, \ \mathcal{E}_{0,\mathcal{M},\epsilon} \leq \mathcal{E}_0, \ \mathcal{E}_0 > 0.$$

Soit $(\Pi, \boldsymbol{V}) \in \mathcal{Y}_T^p(\Omega) \times \mathcal{X}_T(\Omega)$ *avec* $p = \max(2, \gamma')$ *(voir (6.4.14)) une solution forte au problème (6.4.7)-(6.4.11) émanant d'une donnée initiale* $\boldsymbol{V}_0 \in C^1(\overline{\Omega})^3$ *telle que* $\operatorname{div} \boldsymbol{V}_0 = 0$ *dans* $\overline{\Omega}$. *Alors il existe une constante* $c > 0$ *dépendant seulement de* $T, \Omega, p_0, p_\infty, \mu, \gamma, \underline{r}, \min_{[\underline{r}, \overline{r}]} p, \min_{[\underline{r}/2, 2\overline{r}]} p'$ *ainsi que de* $\|\Pi\|_{\mathcal{Y}_T^p(\Omega)}, \|\boldsymbol{V}\|_{\mathcal{X}_T(\Omega)}$, $\overline{\varrho}$, M_0, $\mathcal{E}_0$ *et de* $\theta_{\mathcal{M}}$ *de manière décroissante telle que*

$$\max_{0 \leq n \leq N} \mathcal{E}_\epsilon(\varrho^n, \boldsymbol{u}^n \Big| \overline{\varrho}, \boldsymbol{V}(t^n, \cdot)) \leq c \Big(\mathcal{E}_\epsilon(\varrho^0, \boldsymbol{u}^0 \Big| \overline{\varrho}, \boldsymbol{V}_0) + h_{\mathcal{M}}^A + \sqrt{\delta t} + \epsilon \Big), \tag{6.4.16}$$

où

$$A = \min(\frac{2\gamma \quad 3}{\gamma}, 1). \tag{6.4.17}$$

Remarque 13. *Si la donnée initiale est mal préparée, c'est-à-dire il existe une constante* $c > 0$ *indépendante de* ϵ *telle que*

$$\int_\Omega E(\varrho_\epsilon^0 | \overline{\varrho}) \, \mathrm{d}\boldsymbol{x} \leq c\epsilon^2, \quad \int_\Omega \varrho_\epsilon^0 |\boldsymbol{u}_\epsilon^0 \quad \boldsymbol{V}_0|^2 \, \mathrm{d}\boldsymbol{x} \leq c,$$

nous obtenons dans le théorème 15 seulement une borne indépendante de ϵ. *D'un autre coté, si la donnée initiale est bien préparée, avec un taux de convergence* ϵ^ξ, $\xi > 0$, *c'est-à-dire il existe une constante* $c > 0$ *indépendante de* ϵ *telle que*

$$\int_\Omega E(\varrho_\epsilon^0 | \overline{\varrho}) \, \mathrm{d}\boldsymbol{x} \leq c\epsilon^{2+\xi}, \quad \int_\Omega \varrho_\epsilon^0 |\boldsymbol{u}_\epsilon^0 \quad \boldsymbol{V}_{\mathcal{E}}^0|^2 \, \mathrm{d}\boldsymbol{x} \leq c\epsilon^\xi,$$

alors le théorème 15 donne une convergence uniforme lorsque les paramètres $(h_{\mathcal{M}}, \delta t, \epsilon)$ *tendent vers zéro de la solution numérique vers la solution forte des équations de Navier-Stokes incompressibles, dés lors que cette solution forte existe, incluant les taux de convergence. Ces résultats sont en accord avec la théorie des limites faible Mach dans le cas continue, voir [**?**].*

Deuxième partie .

Publications

J. Math. Fluid Mech. 16 (2014), 571–594

1422-6928/14/030571-24
DOI 10.1007/s00021-014-0177-2

Journal of Mathematical Fluid Mechanics

Compressible Navier–Stokes Equations on Thin Domains

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Communicated by E. Feireisl

Abstract. We consider the barotropic Navier–Stokes system describing the motion of a compressible viscous fluid confined to a straight layer $\Omega_\varepsilon = \omega \times (0, \varepsilon)$, where ω is a particular 2-D domain (a periodic cell, bounded domain or the whole 2-D space). We show that the weak solutions in the 3D domain converge to a (strong) solutions of the 2-D Navier–Stokes system on ω as $\varepsilon \to 0$ on the maximal life time of the strong solution.

Keywords. Compressible Navier–Stokes system, dimension reduction, thin domains.

1. Introduction

This paper is devoted to the problem of the limit passage from 3D two 2D in the compressible fluid flows. We shall consider compressible Navier–Stokes equations in thin domains

$$\Omega_\varepsilon = \omega \times (0, \varepsilon),\ \varepsilon > 0, \tag{1.1}$$

where ω is a fixed domain in R^2, and investigate the situation when $\varepsilon \to 0$.

These domains are supposed to be filled with a compressible viscous gas, whose evolution through the time interval $[0, T]$, $T > 0$ is described by the isentropic compressible Navier–Stokes system for the unknown functions, density $\varrho = \varrho(t, x)$ and velocity $\mathbf{u} = \mathbf{u}(t, x)$, $t \in [0, T]$, $x \in \Omega_\varepsilon$:

$$\partial t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } (0, T) \times \Omega_\varepsilon, \tag{1.2}$$

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div}\mathbb{S}(\nabla\mathbf{u}) \quad \text{in } (0, T) \times \Omega_\varepsilon. \tag{1.3}$$

Equations are completed with the initial conditions

$$\varrho(0, x) = \tilde{\varrho}_{0,\varepsilon}(x), \quad \mathbf{u}(0, x) = \tilde{\mathbf{u}}_{0,\varepsilon}(x), \quad x \in \Omega_\varepsilon \tag{1.4}$$

and boundary conditions that will be specified later.

In (1.3), $\mathbb{S}$ is the viscous stress tensor given by the *Newton law*

$$\mathbb{S}(\nabla\mathbf{u}) = \mu\left(\mathbb{D}(\nabla\mathbf{u}) - \frac{2}{3}\operatorname{div}\mathbf{u}\mathbb{I}\right) + \eta\operatorname{div}\mathbf{u}\mathbb{I},\ \mathbb{D}(\nabla\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^T \tag{1.5}$$

with the shear viscosity coefficient $\mu > 0$, the bulk viscosity coefficient $\eta \geq 0$ and $\mathbb{I}$ the identity matrix.

Finally, p denotes the pressure, a given function of density ϱ characterizing the gaz. Anticipating the existence theory of weak solutions in 3-D domains, we shall suppose that

$$p \in C[0, \infty) \cap C^2(0, \infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,$$
$$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,\ \gamma > \frac{3}{2}. \tag{1.6}$$

A. Novotný work was supported by the MODTERCOM project within the APEX programme of the region Provence-Alpe-Côte d'Azur.

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Rigorous justification of the limit passage from the $3D$-fluid motion to a planar one seems to be of obvious practical importance. However, for the compressible fluid flows, to the best of our knowledge, there are no results concerning the 3D-2D limit passage and only a few results concerning the 3D-1D reduction, see Vodák [22] and recently [2]. There are numerous studies of the *incompressible* fluid flows on thin domains, where the limit motion becomes planar, see Iftimie et al. [10], Raugel and Sell [19] and the references therein. This work develops and adapts the ideas introduced in [2] to the problematics of 3D-2D reduction in the compressible fluid flows.

Analysis of similar dimension reduction problems in the elasticity theory leans on variants of the Korn inequality that provides estimates on the gradient of a vector function $\mathbf{v}$ in terms of its symmetric part, specifically,

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_\varepsilon;R^{3\times 3})} \leq c(\varepsilon) \left\|\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right\|_{L^2(\Omega_\varepsilon;R^{3\times 3})}. \tag{1.7}$$

Clearly, the validity of (1.7) requires the kernel of the linear operator $\mathbf{v} \mapsto \nabla \mathbf{v} + (\nabla \mathbf{v})^T$ to be empty on the space of vector fields satisfying the given boundary data. Even if (1.7) holds for any fixed $\varepsilon > 0$, the constant $c(\varepsilon)$ blows up for $\varepsilon \to 0$ unless some necessary restrictions are imposed on the field $\mathbf{v}$, and this is true even if the set ω is not rotationally symmetric, cf. Lewicka and Müller [13].

It is not difficult to see that the problems arising in the context of *compressible* fluids would need a stronger analogue of (1.7), namely

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_\varepsilon;R^{3\times 3})} \leq c(\varepsilon) \left\|\nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3}\mathrm{div}\mathbf{v}\mathbb{I}\right\|_{L^2(\Omega_\varepsilon;R^{3\times 3})}, \tag{1.8}$$

obviously related to the shear viscosity component of the viscous stress tensor, see Dain [4], Reshetnyak [20]. In view of the above mentioned difficulties related to the validity of (1.7) or (1.8), our approach relies on the structural stability of the family of solutions of the barotropic Navier–Stokes system encoded in the *relative entropy inequality* introduced in [6,8]. This method is basically independent of the specific form of the viscous stress and of possible "dissipative" bounds for the Navier–Stokes system.

In this investigation we make a choice to rescale the equations to a fixed domain. Introducing the change of variables

$$\Omega_\varepsilon \ni (x_h, \varepsilon x_3) \mapsto (x_h, x_3) \in \Omega := \Omega_1, \quad \text{where } x_h = (x_1, x_2), \tag{1.9}$$

and denoting the new density and velocity again by ϱ, $\mathbf{u}$, we may rewrite system (1.2–1.4) as follows:

$$\partial_t \varrho + \mathrm{div}_\varepsilon(\varrho \mathbf{u}) = 0 \quad \text{in } (0,T)\times\Omega, \tag{1.10}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_\varepsilon(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_\varepsilon p(\varrho) = \mathrm{div}_\varepsilon \mathbb{S}(\nabla_\varepsilon \mathbf{u}) \quad \text{in}(0,T)\times\Omega, \tag{1.11}$$

$$\varrho(0,x) = \varrho_{0,\varepsilon}(x) \quad \mathbf{u}(0,x) = \mathbf{u}_{0,\varepsilon}(x), \quad x \in \Omega \tag{1.12}$$

[where $\varrho_{0,\varepsilon}(x) = \tilde{\varrho}_{0,\varepsilon}(x_h, \varepsilon x_3)$, $\mathbf{u}_{0,\varepsilon}(x) = \tilde{\mathbf{u}}_{0,\varepsilon}(x_h, \varepsilon x_3)$, cf. (1.4)]. The boundary conditions will be specified later.

Here and hereafter, we denote

$$\nabla_\varepsilon = \left(\nabla_h, \frac{1}{\varepsilon}\partial_{x_3}),\ \nabla_h = (\partial_{x_1}, \partial_{x_2}\right),$$

$$\mathrm{div}_\varepsilon \mathbf{u} = \mathrm{div}_h \mathbf{v}_h + \frac{1}{\varepsilon}\partial_{x_3} v_3,\ \mathbf{v}_h = (v_1, v_2),\ \mathrm{div}_h \mathbf{v}_h = \partial_{x_1} v_1 + \partial_{x_2} v_2.$$

The goal of this work is to investigate the limit process $\varepsilon \to 0$ in the system of equations (1.10–1.12), provided the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}](x)$ converge in a certain sense to $[r_0, \mathbf{v}_0](x) = [\mathbf{v}_{0,h}, 0](x_h)$. Since the target initial data do not depend on the vertical variable x_3, it is natural to expect that the sequence $[\varrho_\varepsilon, \mathbf{u}_\varepsilon](t,x)$ of (weak) solutions to (1.10–1.12) will converge to $[r, \mathbf{V}](t, x_h)$, $\mathbf{V} = [\mathbf{w}, 0]$, where the couple

$[r(t,x_h),\mathbf{w}(t,x_h)]$ solves the 2-D compressible Navier–Stokes equations on the domain ω:

$$\partial_t r + \mathrm{div}_h(r\mathbf{w}) = 0 \text{ in } (0,T)\times\omega, \tag{1.13}$$

$$r\partial_t\mathbf{w} + r\mathbf{w}\cdot\nabla_h\mathbf{w} + \nabla_h p(r) = \mathrm{div_h}\mathbb{S}_h(\nabla_h\mathbf{w}) \text{ in } (0,T)\times\omega, \tag{1.14}$$

$$r(0,x_h) = r_0(x_h),\ \mathbf{w}(0,x_h) = \mathbf{w}_0 := \mathbf{v}_{0,h}(x_h),\ x_h\in\omega, \tag{1.15}$$

where

$$\mathbb{S}_h(\nabla_h\mathbf{w}) = \mu\Big(\nabla_h\mathbf{w} + (\nabla_h\mathbf{w})^T - \mathrm{div}_h\mathbf{w}\Big) + (\eta + \frac{\mu}{3})\mathrm{div}_h\mathbf{w}\mathbb{I}_h,$$

and $\mathbb{I}_h$ is the identity matrix.

Our goal is to justify the above (formal) limit in the framework of weak solutions of the *primitive system* (1.10), (1.11), (1.12).

We shall consider several geometrical situations: periodic layers in Sect. 2, layers over bounded domains ω with no-slip conditions on $\partial\omega$ in Sect. 3, layers over bounded domains with slip conditions in Sect. 4, and some particular cases of unbounded layers in Sect. 5. Finally, in the last section we discuss the possible generalizations of the pressure law (1.6) in order accommodate adiabatic coefficients $\gamma\geq 1$.

We finish the introduction with some remarks on the notation. As far as the functional spaces are concerned, we deal with the classical Lebesgue and Sobolev spaces and we use the standard notation that can be find e.g. in the book of Adams [1]. The notation of Bocher types spaces is again standard, the same as in the book [17]. Finally, in various estimates, the symbols c, c' denote generic positive constants always independent of the small parameter ε; they may take different values in different formulas.

2. Periodic Layers

We consider system (1.2–1.3), (1.4) of thin domains Ω_ε [see (1.1)], where $\omega = [0,1]^2\Big|_{0,1}$ is a 2-D periodic cell of period 1 in both directions. It is completed with the no slip boundary conditions on the bottom and top of the layer

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{2.1}$$

After rescaling, this situation corresponds to system (1.10–1.12) on Ω [see (1.9)] completed with boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}(\nabla_\varepsilon\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0, \tag{2.2}$$

Since ω is a periodic cell, conditions (2.2) means that

$$u_3|_{\omega\times\{0,1\}} = 0,\ \ \Big(\partial_{x_k}u_3 + \frac{1}{\varepsilon}\partial_{x_3}u_k\Big)|_{\omega\times\{0,1\}} = 0,\ k=1,2, \tag{2.3}$$

where $\mathbf{u}(\cdot,x_3)$, $x_3\in(0,1)$ are 1-periodic functions in x_h-variable.

2.1. Preliminaries, Main Results

2.1.1. Weak Solutions to the Primitive System.

Denote

$$W^{1,2}_{\mathbf{n}}(\Omega;R^3) = \{\mathbf{v}\in W^{1,2}(\Omega;R^3)\big|\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}.$$

Definition 2.1. We say that $[\varrho,\mathbf{u}]$ is a *finite energy weak solution* to the compressible Navier–Stokes (1.10–1.11) with initial conditions (1.12) and boundary conditions (2.2) in the space time cylinder $(0,T)\times\Omega$ if the following holds:

- the functions $[\varrho, \mathbf{u}]$ belong to the regularity class

$$\left\{ \begin{array}{l} \varrho \in L^\infty([0,T]; L^\gamma(\Omega)),\ \varrho \geq 0 \text{ a.a. in } (0,T)\times\Omega,\ \gamma > \frac{3}{2}, \\ \\ \mathbf{u} \in L^2(0,T; W^{1,2}_{\mathbf{n}}(\Omega; R^3)),\ \varrho \mathbf{u}^2 \in L^\infty([0,T]; L^1(\Omega)); \end{array} \right\} \tag{2.4}$$

- $\varrho \in C_{\text{weak}}([0,T]; L^\gamma(\Omega))$ and the *continuity equation* (1.10) is satisfied in the weak sense,

$$\int_\Omega \varrho\varphi \ \mathrm{d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\varphi(0,\cdot)\ \mathrm{d}x = \int_0^\tau \int_\Omega \varrho\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_\varepsilon\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t. \tag{2.5}$$

for all $\tau \in [0,T]$ and any test function $\varphi \in C^\infty_c([0,T]\times\overline{\Omega})$;
- $\varrho\mathbf{u} \in C_{\text{weak}}([0,T]; L^{2\gamma/(\gamma+1)}(\Omega))$ and the *momentum equation* (1.11) holds in the sense that

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi\ \mathrm{d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\varphi(0,\cdot)\ \mathrm{d}x$$

$$= \int_0^T\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_\varepsilon\varphi + p(\varrho_\varepsilon)\mathrm{div}_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{2.6}$$

for all $\tau \in [0,T]$ and for any $\varphi \in C^\infty_c([0,T]\times\overline{\Omega}; R^3)$, $\varphi_3|_{\omega\times\{0,1\}} = 0$;
- the *energy inequality*

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right](\tau)\ \mathrm{d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \leq \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + H(\varrho_{0,\varepsilon})\right]\ \mathrm{d}x \tag{2.7}$$

holds for a.a. $\tau \in (0,T)$.

Notice that the definition of weak solutions for the system (1.2–1.3), (1.4), (2.1) before rescaling can be obtained replacing Ω by Ω_ε, ∇_ε by ∇, div_ε by div and $\varrho_{\varepsilon,0}$ by $\tilde{\varrho}_{\varepsilon,0}$, $\mathbf{u}_{\varepsilon,0}$ by $\tilde{\mathbf{u}}_{\varepsilon,0}$.

The reader may consult the monograph [15] by Lions, Feireisl [5] and [17], for the mathematical theory of compressible viscous fluids in the framework of weak solutions. In particular, the weak solutions for the system (1.2–1.4), (2.1) before rescaling are known to exist globally in time for any finite energy initial data.

Consequently, the system (1.10–1.12), (2.2) after rescaling possesses finite energy weak solutions as well. The corresponding theorem reads:

Proposition 2.1. *Let $\mathbb{S}$ satisfy* (1.5) *and p verify* (1.6). *Suppose that the initial data satisfy*

$$\varrho_{0,\varepsilon} \geq 0,\ \int_\Omega \varrho_{0,\varepsilon} = M_\varepsilon > 0,\ \int_\Omega \Big(\frac{1}{2}\varrho_{0,\varepsilon}\mathbf{u}^2_{0,\varepsilon} + H(\varrho_{0,\varepsilon})\Big)\ \mathrm{d}x < \infty. \tag{2.8}$$

Then the problem (1.10–1.12), (2.2) *admits at least one finite energy weak solution on the arbitrary time interval* $(0,T)$.

2.1.2. Relative Entropy Inequality. Motivated by [8] we introduce the relative entropy functional

$$\mathcal{E}\left(\varrho,\mathbf{u}\Big|r,\mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho,r)\right]\ \mathrm{d}x, \tag{2.9}$$

where

$$E(\varrho,r) = H(\varrho) - H'(r)(\varrho - r) - H(r),\ H(\varrho) = \varrho\int_1^\varrho \frac{p(s)}{s^2}\mathrm{d}s.$$

(The reader can consult Dafermos [3], Germain [9], Mellet and Vasseur [16] for the utilisation of the notion of relative entropies in other contexts in the mathematical fluid mechanics).

Now, the crucial observation is that *any* finite energy weak solution defined through (2.4–2.7) satisfies the so called relative entropy inequality. More precisely, we have the following theorem (that follows by rescaling from see [6, Section 3.2.1]):

Proposition 2.2. *Let all assumptions of Proposition 2.1 be satisfied and let* $[\varrho, \mathbf{u}]$ *be a finite energy weak solution of the system* (1.10)–(1.12), (2.2)*. Then* $[\varrho, \mathbf{u}]$ *satisfies the* relative entropy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right)(\tau) + \int_0^\tau \int_\Omega \left(\mathbb{S}\Big(\nabla_\varepsilon(\mathbf{u}-\mathbf{U})\Big) : \nabla_\varepsilon(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t\right.$$
$$\leq \mathcal{E}\left([\varrho, \mathbf{u}]\middle|[r, \mathbf{U}]\right)(0) + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})\ \mathrm{d}t, \tag{2.10}$$

with the remainder term

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) := \int_\Omega \varrho\Big(\partial_t \mathbf{U} + \mathbf{u}\cdot\nabla_\varepsilon \mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{U}) : \nabla_\varepsilon(\mathbf{U}-\mathbf{u})\ \mathrm{d}x$$
$$+ \int_\Omega ((r-\varrho)\partial_t H'(r) + \nabla_\varepsilon H'(r)\cdot(r\mathbf{U}-\varrho\mathbf{u}))\ \mathrm{d}x - \int_\Omega \mathrm{div}_\varepsilon \mathbf{U}\Big(p(\varrho)-p(r)\Big)\ \mathrm{d}x, \tag{2.11}$$

with any pair of test functions

$$r \in C^1([0,T]\times\overline{\Omega}),\ r > 0,\ \mathbf{U} \in C^1([0,T]\times\overline{\Omega}; R^3),\ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Remark 2.1. Notice that the set of test functions in the relative entropy inequality can be enlarged by density argument to:

$$\begin{aligned} &0 < \underline{r} < r < \overline{r} < \infty,\ \partial_t r \in L^1(0,T; L^{\frac{\gamma}{\gamma-1}}(\Omega)),\ \nabla r \in L^2(0,T; L^{\frac{2\gamma}{\gamma-1}}(\Omega; R^3))\\ &\mathbf{U} \in L^2(0,T; W^{1,2}(\Omega),\ \partial_t \mathbf{U} \in L^2(0,T; L^6(\Omega; R^3)),\\ &\nabla\mathbf{U} \in L^2(0,T; L^{\frac{3\gamma}{2\gamma-3}}(\Omega; R^{3\times 3})),\ \mathrm{div}\mathbf{U} \in L^1(0,T; L^\infty(\Omega)).\end{aligned} \tag{2.12}$$

We remark that this class is not optimal. Nevertheless, we shall see that it is sufficient for our purposes.

2.1.3. Solutions of the Target System. It is well known since eighties that the target system (1.13–1.15) admits a unique strong solution on a maximal time interval $[0, T_{\max})$ that depends on the size of the initial data. The following theorem can be deduced as Theorem 2.5 in Valli and Zajaczkowski [21]:

Proposition 2.3. *Let D be a positive constant. Suppose that $p \in C^2(0,\infty)$ and that*

$$r_0 \in W^{2,2}(\omega),\ \inf_\omega r_0 > 0,\ \mathbf{w}_0 \in W^{3,2}(\omega; R^2). \tag{2.13}$$

Then there exists $T = T_{\max}(D)$ such that if

$$\|r_0\|_{W^{2,2}(\omega)} + \|\mathbf{w}_0\|_{W^{3,2}(\omega;R^2)} + 1/\inf_\omega r_0 \leq D, \tag{2.14}$$

then the problem (1.13–1.15) *admits a unique strong solution (in the sense a.e. in $(0,T)\times\omega$) in the class*

$$\begin{aligned} r \in C([0,T); W^{2,2}(\omega)),\ \mathbf{w} \in C([0,T); W^{2,2}(\omega; R^2)) \cap L^2(0,T; W^{3,2}(\omega; R^2))\\ \partial_t r \in C([0,T); W^{1,2}(\omega)),\ \partial_t \mathbf{w} \in L^2(0,T; W^{2,2}(\omega; R^2)).\end{aligned} \tag{2.15}$$

In particular,

$$0 < \underline{r} \equiv \inf_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \leq \sup_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \equiv \overline{r}. \tag{2.16}$$

2.1.4. Main Result. We are now in a position to formulate the main result of this section.

Theorem 2.1. *Suppose that the pressure p satisfies hypotheses (1.6) and that the stress tensor is given by (1.5). Let r_0, $\mathbf{w}_0$ satisfy assumptions (2.13) and let $T_{\max} > 0$ be the life time of the strong solution to problem (1.13–1.15) corresponding to $[r_0, \mathbf{w}_0]$ determined in Proposition 2.3. Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.10–1.12), (2.2) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Suppose that*

$$\mathcal{E}\Big(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\Big| r_0, \mathbf{V}_0\Big) \to 0, \tag{2.17}$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$.

Then

$$\mathrm{esssup}_{t\in(0,T_{\max})}\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}) \to 0, \tag{2.18}$$

where $\mathbf{V}(t,x) = [\mathbf{w}(t,x_h), 0]$ and where the couple $(r, \mathbf{w})$ satisfies the 2-D compressible Navier–Stokes system (1.13–1.15) on the periodic cell ω on the time interval $[0, T_{\max})$.

Before coming to the proof in the next section, we comment the above theorem and formulate two additional Corollaries and two Remarks which shed more light on the result.

Remark 2.2. In order to see more clearly the sense of the limit above, we notice that (2.18) implies, for example

$$\begin{gathered} \varrho_\varepsilon \to r \text{ strongly in } L^\infty(0,T;L^\gamma(\Omega)), \\ \sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{r}\mathbf{V} \text{ strongly in } L^\infty(0,T;L^2(\Omega;R^3)), \\ \varrho_\varepsilon\mathbf{u}_\varepsilon \to r\mathbf{V} \text{ strongly in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3)). \end{gathered}$$

Remark 2.3. We notice that the 3D-2D reduction increases the two dimensional bulk viscosity of the fluid from η to $\eta + \mu/3$, cf. (1.13–1.15).

Corollary 2.1. *Suppose that the pressure p, the stress tensor $\mathbb{S}$ satisfy assumptions of Theorem 2.1. Assume that $[\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0}]$, $\varrho_\varepsilon \geq 0$ verify*

$$\int_0^1 \varrho_{\varepsilon,0}(x)\mathrm{d}x_3 \rightharpoonup r_0 \textit{ weakly in } L^1(\omega;R^3), \tag{2.19}$$

$$\int_0^1 \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\mathrm{d}x_3 = r_0\mathbf{V}_0 \textit{ weakly in } L^1(\omega;R^3),$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$ and $[r_0, \mathbf{w}_0]$ belongs to the regularity class (2.13), and

$$\int_\Omega \Big[\frac{1}{2}\varrho_{\varepsilon,0}\mathbf{u}^2_{\varepsilon,0} + H(\varrho_{\varepsilon,0})\Big]\mathrm{d}x \to \int_\omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + H(r_0)\Big]\mathrm{d}x_h. \tag{2.20}$$

Let finally $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.10–1.12), (2.2) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Then (2.18) holds.

Corollary 2.1 follows directly from Theorem 2.1. It is enough to observe that (2.19–2.20) imply (2.17). Indeed, recalling that $[r_0, \mathbf{w}_0]$ is independent of x_3 and realizing that

$$\begin{aligned} \mathcal{E}(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\Big|r_0, \mathbf{V}_0) &= \int_\Omega \Big[\frac{1}{2}\varrho_{0,\varepsilon}\mathbf{u}^2_{0,\varepsilon} + H(\varrho_{0,\varepsilon})\Big]\mathrm{d}x - \int_\Omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + H(r_0)\Big]\mathrm{d}x \\ &- \int_\Omega H'(r_0)(\varrho_\varepsilon - r_0)\ \mathrm{d}x + \int_\Omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + \frac{1}{2}\varrho_{0,\varepsilon}\mathbf{w}_0^2 - \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\cdot\mathbf{V}_0\Big]\ \mathrm{d}x, \end{aligned}$$

we may use (2.20) at the first line and (2.19) at the second line, to show the both lines converge to 0.

Corollary 2.1 can be reformulated in terms of the sequence of solutions $[\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon]$ of the non rescaled original problem (1.2–1.4), (2.1).

Corollary 2.2. *Suppose that the pressure p, the stress tensor $\mathbb{S}$ satisfy assumptions of Theorem 2.1. Assume that $[\tilde{\varrho}_{\varepsilon,0}, \tilde{\mathbf{u}}_{\varepsilon,0}]$, $\tilde{\varrho}_\varepsilon \geq 0$ verify*

$$\frac{1}{\varepsilon}\int_0^\varepsilon \tilde{\varrho}_{\varepsilon,0}(x)\mathrm{d}x_3 \rightharpoonup r_0 \text{ weakly in} L^1(\omega), \tag{2.21}$$

$$\frac{1}{\varepsilon}\int_0^\varepsilon \tilde{\varrho}_{\varepsilon,0}\tilde{\mathbf{u}}_{\varepsilon,0}\mathrm{d}x_3 = r_0\mathbf{V}_0 \text{ weakly in} L^1(\omega; R^3),$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$ and $[r_0, \mathbf{w}_0]$ belongs to the regularity class (2.13), and

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big[\frac{1}{2}\tilde{\varrho}_{\varepsilon,0}\tilde{\mathbf{u}}_{\varepsilon,0}^2 + H(\tilde{\varrho}_{\varepsilon,0})\Big]\mathrm{d}x \to \int_\omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + H(r_0)\Big]\mathrm{d}x_h. \tag{2.22}$$

Let finally $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon)$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.2–1.4), (2.1) emanating from the initial data $[\tilde{\varrho}_{0,\varepsilon}, \tilde{\mathbf{u}}_{0,\varepsilon}]$.

Then

$$\mathrm{esssup}_{t\in(0,T_{\max})}\mathcal{E}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon \Big| r, \mathbf{V}) \to 0 \tag{2.23}$$

with

$$\mathcal{E}_\varepsilon(\varrho, \mathbf{u}\Big| r, \mathbf{U}) = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)\Big]\mathrm{d}x,$$

where $\mathbf{V}(t,x) = [\mathbf{w}(t,x_h), 0]$ and where the couple $[r, w]$ satisfies the 2-D compressible Navier–Stokes system (1.13–1.15) on the periodic cell ω on the time interval $[0, T_{\max})$.

2.2. Proof of Theorem 2.1

2.2.1. Korn and Poincaré Type Inequalities. We start by the following Korn type inequality.

Lemma 2.1. *Let ω be a periodic cell and $\Omega = \omega \times (0,1)$. Then there exists $c > 0$ such that for all $\mathbf{v} \in W^{1,2}_{\mathbf{n}}(\Omega; R^3)$)*

$$\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x\mathrm{d}x = \mu\|\nabla_\varepsilon \mathbf{v}\|^2_{L^2(\Omega;R^{3\times 3})} + (\frac{1}{3}\mu + \eta)\|\mathrm{div}_\varepsilon \mathbf{v}\|^2_{L^2(\Omega)},$$

where $\|\mathbb{A}\|^2_{L^2(\Omega;R^{3\times 3})} = \sum_{i,j=1}^3 \int_\Omega |A_{ij}|^2\mathrm{d}x$.

Proof of Lemma 2.1. We write

$$\mathbb{S}(\nabla_\varepsilon \mathbf{v}) = \mu\mathbb{T}(\nabla_\varepsilon \mathbf{v}) + \eta\mathrm{div}_\varepsilon\mathbf{v}\mathbb{I},\ \text{where } \mathbb{T}(\mathbb{A}) = \mathbb{D}(\mathbb{A}) - \frac{2}{3}\mathrm{tr}(\mathbb{A}).$$

Employing the integration by parts, taking advantage of the periodicity of domain ω and of the Navier boundary conditions on the boundary of the layer, we get by an easy calculation

$$\int_\Omega \mathbb{T}(\nabla_\epsilon \mathbf{v}) : \nabla_\epsilon \mathbf{v}\mathrm{d}x = \mu\left(\int_\Omega |\nabla_\epsilon \mathbf{v}|^2 dx + \frac{1}{3}\int_\Omega (\mathrm{div}_\epsilon \mathbf{v})^2 dx\right);$$

whence

$$\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x = \mu ||\nabla_\varepsilon \mathbf{v}||^2_{L^2(\Omega)} + (\frac{1}{3}\mu + \eta)||\mathrm{div}_\varepsilon \mathbf{v}||^2_{L^2(\Omega)}.$$

Lemma 2.1 is proved.

The next Poincaré type inequality follows from [7, Appendix, Theorem 10.14] by an easy straightforward argument.

Lemma 2.2. *Let Ω be the same as in Lemma 2.1. Let $K > 0$, $M > 0$, $\gamma > 6/5$. There exists $c = c(K, M, \gamma) > 0$ such that for any*

$$\mathbf{v} \in W^{1,2}(\Omega; R^3),\ \ \varrho \geq 0,\ \int_\Omega \varrho \mathrm{d}x \geq M,\ \int_\Omega \varrho^\gamma \mathrm{d}x \leq K \tag{2.24}$$

and $\varepsilon \in (0, 1)$, there holds

$$\|\mathbf{v}\|^2_{L^2(\Omega;R^3)} \leq c \left(\|\nabla_\varepsilon \mathbf{v}\|^2_{L^2(\Omega;R^{3\times 3})} + \int_\Omega \varrho \mathbf{v}^2 \mathrm{d}x \right). \tag{2.25}$$

Putting together both Lemmas 2.1 and 2.2 we obtain:

Lemma 2.3. *Let $\gamma > 6/5$. For the same domain as in Lemma 2.2, and for any $K > 0$, $M > 0$, there exists $c = c(K, M, \gamma) > 0$ such that for any couple in the class (2.25) and for any $\varepsilon \in (0, 1)$ we have*

$$\|\mathbf{v}\|^2_{W^{1,2}(\Omega;R^3)} \leq c \left(\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x + \int_\Omega \varrho \mathbf{v}^2 \mathrm{d}x \right). \tag{2.26}$$

This Lemma will be useful later in Sect. 2.2.5 for estimating the remainder.

2.2.2. Relative Entropy Inequality With Special Test Functions. In view of Remark 2.1, we may use the couple $[r, \mathbf{V}]$, where $\mathbf{V} = [\mathbf{w}, 0]$ and $[r, \mathbf{w}]$ is the strong solution of the target problem (1.13–1.15) on the periodic cell ω, as the test in the relative entropy inequality (2.10).

We start by observing that the couple $(r, \mathbf{V})$ satisfies, by virtue of of (1.13–1.14),

$$\partial_t r + \mathrm{div}(r\mathbf{V}) = 0 \text{ in } (0, T) \times \Omega, \tag{2.27}$$

$$r\partial_t \mathbf{V} + r\mathbf{V} \cdot \nabla \mathbf{V} + \nabla p(r) = \mathrm{div}\mathbb{S}(\nabla \mathbf{V}) \text{ in } (0, T) \times \Omega. \tag{2.28}$$

Multiplying (2.28) scalarly by $\mathbf{u}_\varepsilon - \mathbf{V}$ and integrating over Ω, we get

$$\int_\Omega \Big(r\partial_t \mathbf{V} + r\mathbf{V} \cdot \nabla_\varepsilon \mathbf{V} + \nabla_\varepsilon p(r) \Big) \cdot (\mathbf{u}_\varepsilon - \mathbf{V})\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{V}) : \nabla_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{V})\ \mathrm{d}x = 0, \tag{2.29}$$

where we have used the integration by parts in the last integral.

Next we rewrite the relative entropy inequality (2.10) with test functions r, $\mathbf{U} = \mathbf{V}$. In view of (2.29), we may rewrite the remainder (2.11) as follows

$$\begin{aligned}\mathcal{R}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon, r, \mathbf{V}\right) := &\int_\Omega \varrho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{V}) \cdot \nabla_\varepsilon \mathbf{V} \cdot (\mathbf{V} - \mathbf{u}_\varepsilon)\ \mathrm{d}x \\ &+ \int_\Omega \rho_\varepsilon (\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_\varepsilon \mathbf{V}) \cdot (\mathbf{V} - \mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega (r\partial_t \mathbf{V} + r\mathbf{V} \cdot \nabla_\varepsilon \mathbf{V} + \nabla_\varepsilon p(r)) \cdot (\mathbf{u}_\varepsilon - \mathbf{V})\ \mathrm{d}x \\ &+ \int_\Omega \frac{r - \rho_\varepsilon}{r} \partial_t p(r) + \frac{\nabla_\varepsilon p(r)}{r} (r\mathbf{V} - \rho_\varepsilon \mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega (p(r) - p(\rho_\varepsilon)) \mathrm{div}_\varepsilon \mathbf{V}\ \mathrm{d}x\end{aligned}$$

$$= \int_\Omega (\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x$$
$$+ \int_\Omega \frac{r-\rho_\varepsilon}{r}\partial_t p(r) + \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\mathbf{u}_\varepsilon\ \mathrm{d}x + \int_\Omega (p(r)-p(\rho_\varepsilon))\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x.$$

Using additionally (2.27), one gets

$$\mathcal{R}(\rho_\varepsilon,\mathbf{u}_\varepsilon,r,\mathbf{V}) = \int_\Omega (\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x$$
$$+ \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\mathbf{u}_\varepsilon\ \mathrm{d}x + \int_\Omega (p(r)-p(\rho_\varepsilon))\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x$$
$$+ \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}\cdot\mathbf{V}(\rho_\varepsilon - r)\ \mathrm{d}x + \int_\Omega p'(r)(\rho_\varepsilon - r)\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x.$$

Resuming the above calculation, we rewrite the relative entropy inequality (2.10) in the form

$$\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V}\Big)(\tau) + \int_0^\tau\int_\Omega \mathbb{S}\Big(\nabla_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{V})\Big):\nabla_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{V})\ \mathrm{d}x\mathrm{d}t$$
$$\le \mathcal{E}\Big(\varrho_{0,\varepsilon},\mathbf{u}_{0,\varepsilon}\Big|r_0,\mathbf{V}_0\Big) + \int_0^\tau \mathcal{R}(\varrho_\varepsilon,\mathbf{u}_\varepsilon,r,\mathbf{V})\mathrm{d}t \tag{2.30}$$

where the remainder reads

$$\mathcal{R}(\rho_\varepsilon,\mathbf{u}_\varepsilon,r,\mathbf{V}) = \int_\Omega (\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x$$
$$+ \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\cdot(\mathbf{u}_\varepsilon-\mathbf{V})\ \mathrm{d}x - \int_\Omega (p(\rho_\varepsilon)-p'(r)(\rho_\varepsilon-r)-p(r))\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x. \tag{2.31}$$

2.2.3. Main Ideas: Towards the Gronwall Inequality. The goal now is to use Lemma 2.3 in order to find an estimate of the left hand side of (2.10) from below in the form

$$\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})(\tau) + c\int_0^\tau \|\mathbf{u}_\varepsilon-\mathbf{V}\|^2_{W^{1,2}(\Omega)}\mathrm{d}t - c'\int_0^\tau \mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})\mathrm{d}t \tag{2.32}$$

and of the right hand side from above in the form

$$h_\varepsilon(\tau) + \delta\int_0^\tau \|\mathbf{u}_\varepsilon-\mathbf{V}\|^2_{W^{1,2}(\Omega)}\mathrm{d}t + c'(\delta)\int_0^\tau a(t)\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})\mathrm{d}t \tag{2.33}$$

with any $\delta > 0$, where $c > 0$ is independent of δ, $c' = c'(\delta) > 0$, $a \in L^1(0,T)$ and

$$h_\varepsilon \to 0 \text{ in } L^\infty(0,T).$$

If we succeed to establish these bounds, we deduce from the relative entropy inequality (2.10) estimate

$$\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})(\tau) \le h_\varepsilon(\tau) + c\int_0^\tau a(t)\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})\mathrm{d}t \tag{2.34}$$

that implies (2.18) by using the Gronwall inequality. In the rest of this section, we shall perform this programme.

2.2.4. Essential and Residual Sets. We begin with an algebraic inequality whose straightforward proof is left to the reader.

Lemma 2.4. *Let $0 < a < b < \infty$. Then there exists $c = c(a,b) > 0$ such that for all $\varrho \in [0,\infty)$ and $r \in [a,b]$ there holds*

$$E(\varrho|r) \geq c(a,b)\Big(1_{\mathcal{O}_{\rm res}} + \varrho^\gamma 1_{\mathcal{O}_{\rm res}} + (\varrho - r)^2 1_{\mathcal{O}_{\rm ess}}\Big),$$

where $E(\varrho|r)$ is defined in (2.9) and

$$\mathcal{O}_{\rm ess} = [a/2, 2b],\ \mathcal{O}_{\rm res} = R_+ \backslash [a/2, 2b].$$

Next we introduce essential and residual sets in Ω. Let $0 < \underline{\varrho} < \overline{\varrho} < \infty$. We define for a.e. $t \in (0,T)$ the residual and essential subsets of Ω as follows:

$$N^\varepsilon_{\rm ess}(t) = \{x \in \Omega \,\Big|\, \frac{1}{2}\underline{\varrho} \leq \varrho_\varepsilon(t,x) \leq 2\overline{\varrho}\},\ N^\varepsilon_{\rm res}(t) = \Omega \setminus N^\varepsilon_{\rm ess}(t) \tag{2.35}$$

and denote for a function h defined a.e. in $(0,T)\times\Omega$,

$$[h]_{\rm ess} = h 1_{N^\varepsilon_{\rm ess}},\ [h]_{\rm res} = h 1_{N^\varepsilon_{\rm res}}.$$

Here and hereafter we shall use essential and residual sets with

$$\underline{\varrho} = \underline{r}, \quad \overline{\varrho} = \overline{r}, \tag{2.36}$$

where $\underline{r}$ and $\overline{r}$ are defined in Proposition 2.3. In particular, Lemma 2.4 implies,

$$c\int_\Omega \left([1]_{\rm res} + [\varrho_\varepsilon^\gamma]_{\rm res} + [\varrho_\varepsilon - r]^2_{\rm ess}\right)\,{\rm d}x \leq \int_\Omega E(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{U})\,{\rm d}x, \tag{2.37}$$

with some $c = c(\underline{\varrho}, \overline{\varrho}) > 0$.

2.2.5. Estimates of the Remainder. We are now in position to estimate the remainder (2.31). We shall do it in four steps.

Step 1: We shall first estimate the "essential part" of the first term: Since $V_3 = 0$ and $\partial_{x_3}\mathbf{V} = 0$, we have

$$\begin{aligned}
&\int_0^\tau\int_\Omega 1_{\rm ess}(\rho_\varepsilon - r)(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon\mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\,{\rm d}x{\rm d}t \\
&\leq \int_0^\tau \Big\|\partial_t\mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h\mathbf{V}_h\|_{L^\infty(\Omega;R^2)}\Big\|\Big[\rho_\varepsilon - r\Big]_{\rm ess}\Big\|_{L^2(\Omega)}\Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|_{L^2(\Omega;R^2)}{\rm d}t \\
&\leq \delta\int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{L^2(\Omega;R^2)}{\rm d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big|r, \mathbf{V}\Big){\rm d}t,
\end{aligned} \tag{2.38}$$

where

$$a = \|\partial_t\mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h\mathbf{V}_h\|^2_{L^\infty(\Omega;R^2)} \in L^1(0,T).$$

For the "residual" part of the first term, we get

$$\begin{aligned}
&\int_0^\tau \int_\Omega 1_{\rm res}(\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ {\rm d}x{\rm d}t \\
&\quad \le \int_0^\tau \Big\|\partial_t \mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h \mathbf{V}_h\|_{L^\infty(\Omega;R^3)}\Big\|\Big[\rho_\varepsilon - r\Big]_{\rm res}\Big\|_{L^{6/5}(\Omega)}\Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|_{L^6(\Omega;R^2)}{\rm d}t \\
&\quad \le \delta \int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{W^{1,2}(\Omega;R^2)}{\rm d}t + c(\delta)\int_0^\tau a(t)\Big[\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big)\Big]^{10/3}{\rm d}t \\
&\quad \le \delta \int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{W^{1,2}(\Omega;R^2)}{\rm d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t
\end{aligned} \tag{2.39}$$

provided $\gamma \ge 6/5$. Here we have used Hölder and Young inequalities, continuous imbedding $W^{1,2}(\Omega) \subset L^6(\Omega)$ and the fact that

$$\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big) \in L^\infty(0,T),$$

by virtue of (2.7), (2.15–2.16).

Step 2:

$$\int_0^\tau \int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ {\rm d}x{\rm d}t \le c\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t \tag{2.40}$$

with

$$a = \|\nabla_h \mathbf{V}_h\|_{L^\infty(\Omega;R^{2\times 2})} \in L^\infty(0,T).$$

Step 3: Similarly as in Step 1,

$$\begin{aligned}
&\int_0^\tau \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\cdot(\mathbf{u}_\varepsilon - \mathbf{V})\ {\rm d}x{\rm d}t \\
&\quad \le \delta \int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{W^{1,2}(\Omega;R^2)}{\rm d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t,
\end{aligned} \tag{2.41}$$

where

$$a = \Big\|\frac{\nabla_h p(r)}{r}\Big\|^2_{L^\infty(\Omega;R^2)} \in L^\infty(0,T).$$

Step 4:

As far as the last term is concerned, we use: 1) The Taylor formula together with the regularity C^2 of the pressure p [see (1.6)], in order to estimate the essential part

$$\begin{aligned}
-\int_0^\tau \int_\Omega \Big[p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)\Big]_{\rm ess}{\rm div}_\varepsilon \mathbf{V}\ {\rm d}x{\rm d}t &\le c\int_0^\tau \|{\rm div}_h \mathbf{V}_h\|_{L^\infty(\Omega)}\Big\|\Big[\varrho_\varepsilon - r\Big]_{\rm ess}\Big\|^2_{L^2(\Omega)} \\
&\le c\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t, \quad a = \|{\rm div}_h \mathbf{V}_h\|_{L^\infty(\Omega)}.
\end{aligned} \tag{2.42}$$

2) The pointwise bound

$$\Big|[p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)]_{\rm res}\Big| \leq c\Big(1_{\rm res} + [\varrho]^\gamma_{\rm res}\Big)$$

[cf. (1.6), (2.15–2.16)], in order to estimate the residual part

$$-\int_0^\tau\int_\Omega \Big[p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)\Big]_{\rm res} {\rm div}_h\mathbf{V}\ {\rm d}x{\rm d}t$$

$$\leq c\int_0^\tau \|{\rm div}_h\mathbf{V}_h\|_{L^\infty(\Omega)}\int_\Omega \Big(1_{\rm res}\varrho^\gamma_\varepsilon + 1_{\rm res}\Big)\ {\rm d}x{\rm d}t \leq c\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t. \tag{2.43}$$

Coming back with these estimates to the relative entropy inequality (2.30), we easily verify the validity of (2.34) with

$$a = \|\partial_t\mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h\mathbf{V}_h\|^2_{L^\infty(\Omega;R^2)} + \|\nabla_h\mathbf{V}\|_{L^\infty(\Omega;R^{2\times2})} + \Big\|\frac{\nabla_h p(r)}{r}\Big\|^2_{L^\infty(\Omega;R^2)} + \|{\rm div}_h\mathbf{V}_h\|_{L^\infty(\Omega)}$$

and

$$h_\varepsilon = \mathcal{E}(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\Big| r, \mathbf{V}).$$

This finishes the proof of Theorem 2.1.

3. Bounded Layers With No-Slip Boundary Conditions

In this section we consider the compressible Navier–Stokes system (1.2–1.4) on $\Omega_\varepsilon = \omega\times(0,\varepsilon)$, where $\omega\subset R^2$ is a bounded domain. It is completed with the slip boundary conditions on the boundary $\omega\times\{0,\varepsilon\}$ and no slip boundary conditions on the boundary $\partial\omega\times(0,\varepsilon)$:

$$\mathbf{u}|_{\partial\omega\times(0,\varepsilon)} = 0,\ \mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,\varepsilon\}} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,\varepsilon\}} = 0. \tag{3.1}$$

After rescaling, this problem is equivalent to the system (1.10–1.12) on the domain $\Omega = \omega\times(0,1)$ the with boundary conditions

$$\mathbf{u}|_{\partial\omega\times(0,1)} = 0,\ \mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,1\}} = 0. \tag{3.2}$$

3.1. Weak Solutions and Relative Entropy

Definition 3.1. The couple $[\varrho,\mathbf{u}]$ is a finite energy weak solution of the problem (1.10–1.12) with boundary conditions (3.2) if:

- $[\varrho,\mathbf{u}]$ belongs to the functional spaces (2.4), where we replace $W^{1,2}_{\mathbf{n}}(\Omega;R^3))$ with

$$W^{1,2}_{0,\mathbf{n}}(\Omega;R^3) \equiv \{\mathbf{v}\in W^{1,2}(\Omega;R^3)\Big|\mathbf{v}|_{\partial\omega\times(0,1)} = 0,\ \mathbf{v}\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0\}.$$

- Weak formulation (2.5) of the continuity Eq. (1.10) remains without changes;
- Weak formulation (2.6) of the momentum Eq. (1.11) holds with test functions

$$\varphi\in C^\infty_c([0,T]\times\overline{\Omega};R^3),\ \varphi|_{[0,T]\times\partial\omega\times(0,1)} = 0,\ \varphi_3|_{[0,T]\times\omega\times\{0,1\}} = 0;$$

- Energy inequality (2.7) holds.

Again, from Lions [15], Feireisl, [5,17] and [11], one can deduce *existence of weak solutions* for the above problem with $\mathbb{S}$, p and $[\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0}]$ verifying assumptions (1.5), (1.6), (2.8), provided the domain ω is Lipschitz. Moreover, due to [6], any weak solution satisfies relative entropy inequality (2.10) with the remainder (2.11), where the test functions satisfy

$$\begin{aligned} &r \in C^1([0,T] \times \overline{\Omega}),\ r > 0 \\ &\mathbf{U} \in C^1([0,T] \times \overline{\Omega}; R^3),\ \mathbf{U}|_{\partial\omega\times(0,1)} = 0,\ U_3|_{\omega\times\{0,1\}} = 0. \end{aligned} \tag{3.3}$$

Finally, as in Remark 2.1, it can be shown by the density argument, that the test function $(r, \mathbf{U})$ can be taken in the larger regularity class (2.12).

3.2. Target System and the Main Result

The expected target system is system (1.13–1.15) endowed with the no slip boundary conditions:

$$\mathbf{w}|_{\partial\omega} = 0. \tag{3.4}$$

Theorem 2.5 in Valli and Zajaczkowski [21] states:

Proposition 3.1. *Let D be a positive constant. Suppose that $p \in C^2(0,\infty)$, $\partial\omega \in C^3$. Assume that the initial conditions $[r_0, \mathbf{w}_0]$ belong to the regularity class (2.13) and satisfy the compatibility condition*

$$\frac{1}{r_0}\Big(\nabla_h p(r_0) - \mathrm{div}_h \mathbb{S}_h(\nabla_h \mathbf{w}_0) + r_0 \mathbf{w}_0 \cdot \nabla_h \mathbf{w}_0\Big)\Big|_{\partial\omega} = 0. \tag{3.5}$$

Then there exists $T = T_{\max}(D)$ such that if

$$\|r_0\|_{W^{2,2}(\omega)} + \|\mathbf{w}_0\|_{W^{3,2}(\omega;R^2)} + 1/\inf_{\omega} r_0 \le D,$$

then the problem (1.13–1.15), (3.4) admits a unique strong solution (in the sense a.e. in $(0,T)\times\omega$) in the class (2.15), (2.16).

We are now in a position to formulate the main result of Sect. 3.

Theorem 3.1. *Let $\partial\omega \in C^3$. Suppose that the pressure p satisfies hypotheses (1.6) and that the stress tensor is given by (1.5). Let r_0, $\mathbf{w}_0$ satisfy assumptions (2.13), (3.5) and let $T_{\max} > 0$ be the life time of the strong solution to problem (1.13–1.15), (3.4) corresponding to $[r_0, \mathbf{w}_0]$ determined in Proposition 3.1. Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.10–1.12), (3.2) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Suppose that initial data satisfy (2.17).*

Then

$$\mathrm{esssup}_{t\in(0,T_{\max})} \mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \big| r, \mathbf{V}) \to 0, \tag{3.6}$$

where $\mathbf{V}(t,x) = [\mathbf{w}(t,x_h), 0]$ and where the couple $(r, \mathbf{w})$ satisfies the 2-D compressible Navier–Stokes system (1.13–1.15) with the boundary conditions (3.4) on the time interval $[0, T_{\max})$.

Remark 3.1. We notice that all conclusions of Remarks 2.2–2.3 and Corollaries 2.1–2.2 remain valid also in the case of bounded layers with the no slip boundary conditions.

3.3. Proof of Theorem 3.1

The great lines of the proof follow the proof of Theorem 2.1. However, we are able to obtain only a weaker uniform lower bound for the integrals involving the quantity $\mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}$. Consequently, finding the convenient upper bound will be more involved. In this Section, we focus essentially on these two points

3.3.1. Some Auxiliary Results. We observe that the following identity holds pointwise on Ω:

$$\mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} = \eta |\mathrm{div}_\varepsilon \mathbf{v}|^2 + \frac{\mu}{2} \Big| \mathbb{D}(\nabla_\varepsilon \mathbf{v}) - \frac{2}{3} \mathrm{div}_\varepsilon \mathbf{v} \mathbb{I} \Big|^2 \tag{3.7}$$

This implies the following lemma:

Lemma 3.1. *Let $\mathbb{S}$, $\mathbb{D}$ be defined in (1.5), where $\mu > 0$, $\eta > 0$. Then we have: For all $\mathbf{v} \in W^{1,2}(\Omega; R^3)$ and $\varepsilon \in (0,1)$, there holds*

$$\eta \|\mathrm{div}_\varepsilon \mathbf{v}\|^2_{L^2(\Omega)} \le \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \ \mathrm{d}x$$

and

$$\frac{\mu}{2} \|\mathbb{D}(\nabla_\varepsilon \mathbf{v})\|^2_{L^2(\Omega)} \le 2 \left(1 + \frac{\mu}{3\eta}\right) \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \ \mathrm{d}x.$$

Next, simply by integration by parts, we show the identity

$$\|\mathbb{D}_h(\nabla_h \mathbf{v})\|^2_{L^2(\omega,R^2)} = 2 \Big(\|\nabla_h \mathbf{v}\|^2_{L^2(\omega,R^{2\times 2})} + \|\mathrm{div_h} \mathbf{v}\|^2_{\mathrm{L}^2(\omega)} \Big) \tag{3.8}$$

Finally, we recall the standard Poincaré inequality,

$$\|\mathbf{v}\|_{L^2(\omega;R^2} \le c \|\nabla_h \mathbf{v}\|^2_{L^2(\omega,R^{2\times 2})} \tag{3.9}$$

Both formulas (3.8), (3.9) hold for all $\mathbf{v} \in W^{1,2}_0(\omega; R^2)$ provided ω is a Lipschitz domain.

Putting together these results, we may write the following lemma.

Lemma 3.2. *Let $\Omega = \omega \times (0,1)$, where ω is a bounded Lipschitz domain and let $\eta > 0$. Then there exists $c = c(\omega) > 0$ such that for all $\mathbf{v} \in W^{1,2}(\Omega; R^3)$, $\mathbf{v}|_{\partial\omega \times (0,1)} = 0$ and $\varepsilon \in (0,1)$,*

$$\|\mathbf{v}_h\|^2_{L^2(\Omega,R^2)} + \|\nabla_h \mathbf{v}_h\|^2_{L^2(\Omega,R^{2\times 2})} \le c \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \ \mathrm{d}x, \tag{3.10}$$

where the tensor $\mathbb{S}$ is defined in (1.5).

Remark 3.2. Besides (3.10) we have also trivially

$$\|\partial_3 v_3\|^2_{L^2(\Omega)} \le c\varepsilon^2 \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \ \mathrm{d}x,$$

and moreover, if $v_3|_{\omega \times \{0,1\}} = 0$,

$$\|v_3\|^2_{L^2(\Omega)} \le c\varepsilon^2 \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \ \mathrm{d}x.$$

These additional estimates will not be exploited throughout the proof.

Remark 3.3. In contrast to Lemma 2.3 dealing with the periodic case, Lemma 3.2 does not provide the estimate for the whole $W^{1,2}$- norm of neither $\mathbf{v}$ nor $\mathbf{v}_h$. The challenge now is to estimate the remainder in the relative entropy inequality in a different way than in Sect. 2 by using only the above "incomplete" estimate.

3.3.2. Application of the Relative Entropy. In view of Remark 3.3 we shall modify the procedure of Sect. 2.2.3 as follows: We shall estimate the left hand side of the relative entropy inequality (2.10) with test functions $[r, \mathbf{V}]$, $\mathbf{V} = [\mathbf{w}, 0]$ from below by

$$\mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{V}\right)(\tau) + c\int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega;R^2)} \mathrm{d}t - c'\int_0^\tau \mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon\middle| r, \mathbf{V}\right) \mathrm{d}t \tag{3.11}$$

[this follows trivially from (2.32)], and the right hand side from above by

$$h_\varepsilon(\tau) + \delta\int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega)} \mathrm{d}t + c'(\delta)\int_0^\tau a(t)\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon\big| r, \mathbf{V}) \mathrm{d}t \tag{3.12}$$

with any $\delta > 0$, where $c > 0$ is independent of δ, $c' = c'(\delta) > 0$, $a \in L^1(0,T)$ and

$$h_\varepsilon \to 0 \text{ in } L^\infty(0,T).$$

This process leads again to the inequality (2.34), that finishes the proof. The "only" point in the proof is therefore to find the bound (3.12) of the remainder (2.31). This will be done in the next section.

3.3.3. Estimate of the Remainder. In order to get the estimate (3.12) of the remainder (2.31), we proceed in three steps. As in Sect. 2.2.5, we shall systematically use that $V_3 = 0$ and $\partial_{x_3}\mathbf{V} = 0$.

Step 1: The essential part of the first term $\int_0^\tau \int_\Omega 1_{\mathrm{ess}}(\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\, \mathrm{d}x\mathrm{d}t$ is estimated in the same way as in formula (2.38).

Concerning the residual part, we shall estimate the integrals over the sets $\{\varrho \le \underline{\varrho}/2\}$ and $\{\varrho \ge 2\overline{\varrho}\}$ separately. [Numbers $\underline{\varrho}$, $\overline{\varrho}$ are defined in (2.36) and essential/residual sets in (2.35)].

$$\begin{aligned}
&\int_0^\tau \int_\Omega 1_{\{\varrho \le \underline{\varrho}/2\}}(\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\, \mathrm{d}x\mathrm{d}t \\
&\quad \le 2\overline{\varrho}\int_0^\tau \int_\Omega 1_{\mathrm{res}}\left|\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V}\right| \left|\mathbf{V} - \mathbf{u}_\varepsilon\right| \mathrm{d}x\mathrm{d}t \\
&\quad \le \int_0^\tau \left\|\partial_t \mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h \mathbf{V}_h\|_{L^\infty(\Omega;R^2)}\right\|1_{\mathrm{res}}\Big\|_{L^2(\Omega)}\Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|_{L^2(\Omega;R^2)} \mathrm{d}t \\
&\quad \le \delta\int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{L^2(\Omega;R^2)} \mathrm{d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big)\mathrm{d}t,
\end{aligned}$$

where a is given in (2.38).

Finally,

$$\begin{aligned}
&\int_0^\tau \int_\Omega 1_{\{\varrho \ge 2\overline{\varrho}\}}(\varrho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\, \mathrm{d}x\mathrm{d}t \\
&\quad \le 2\int_0^\tau \int_\Omega 1_{\mathrm{res}}\sqrt{\varrho_\varepsilon}\left|\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V}\right|\sqrt{\varrho_\varepsilon}\left|\mathbf{V} - \mathbf{u}_\varepsilon\right| \mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$\leq \int_0^\tau \Big\|\partial_t \mathbf{V}_h + \mathbf{V}_h \cdot \nabla_h \mathbf{V}_h \|_{L^\infty(\Omega;R^2)} \Big\| \Big[\varrho\Big]_{\rm res} \|^{1/2}_{L^1(\Omega)} \Big\| \varrho_\varepsilon \Big(\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big)^2 \Big\|^{1/2}_{L^1(\Omega} {\rm d}t$$

$$\leq c \int_0^\tau a(t) \mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}\Big) {\rm d}t$$

with the same a as before. In all above three formulas, we have employed (2.37) in the passage to their last lines.

Step 2: Estimates of the second

$$\int_0^\tau \int_\Omega \rho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{V}) \cdot \nabla_\varepsilon \mathbf{V} \cdot (\mathbf{V} - \mathbf{u}_\varepsilon) \ {\rm d}x{\rm d}t$$

and the fourth

$$\int_0^\tau \int_\Omega (p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)) {\rm div}_\varepsilon \mathbf{V} \ {\rm d}x{\rm d}t$$

terms of the remainder are the same as (2.40) and (2.42–2.43) in Sect. 2.2.5.

Step 3: Estimate of the third term folows the same lines as the estimates effectuated in Step 1, namely

$$\int_0^\tau \int_\Omega \frac{\nabla_\varepsilon p(r)}{r} (r - \rho_\varepsilon) \cdot (\mathbf{u}_\varepsilon - \mathbf{V}) \ {\rm d}x{\rm d}t \leq \delta \int_0^\tau \Big\| \mathbf{u}_{\varepsilon h} - \mathbf{V}_h \Big\|^2_{L^2(\Omega;R^2)} {\rm d}t + c(\delta) \int_0^\tau a(t) \mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}\Big) {\rm d}t,$$

where

$$a(t) = \Big\| \frac{1}{r} \nabla_h p(r) \Big\|^2_{L^\infty(\Omega;R^2)}.$$

Collecting these estimates in the relative entropy inequality (2.30), we obtain formula (2.34) with

$$h_\varepsilon = \mathcal{E}\Big(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \Big| r_0, \mathbf{V}_0\Big),$$

and conclude by the Gronwall lemma.

4. Bounded Layers With Slip Boundary Conditions

We start again with system (1.2–1.4) on the domain $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ where $\omega \in R^2$ is bounded. The boundary conditions read

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \ [\mathbb{S}(\nabla \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{4.1}$$

Again, after rescaling, this problem is equivalent to the system (1.10–1.12) on the domain $\Omega = \omega \times (0, 1)$ with boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\omega \times \{0,1\} \cup \partial\omega \times (0,1)\}} = 0, \ [\mathbb{S}(\nabla \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\omega \times \{0,1\} \cup \partial\omega \times (0,1)} = 0. \tag{4.2}$$

4.1. Weak Solutions, Relative Entropy, Target System and the Main Result

4.2. Weak Solutions and Relative Entropy

Definition 4.1. The couple $[\varrho, \mathbf{u}]$ is a finite energy weak solution of the problem (1.10–1.12) with boundary conditions (3.2) if:

- $[\varrho, \mathbf{u}]$ belongs to the functional spaces (2.4), where we replace $W^{1,2}_{\mathbf{n}}(\Omega; R^3))$ with

$$W^{1,2}_{\mathbf{n}}(\Omega; R^3) \equiv \{\mathbf{v} \in W^{1,2}(\Omega; R^3) \Big| \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

- Weak formulation (2.5) of the continuity Eq. (1.10) remains without changes;
- Weak formulation (2.6) of the momentum Eq. (1.11) holds with test functions

$$\varphi \in C^\infty_c([0,T] \times \overline{\Omega}; R^3), \;\; \varphi \cdot \mathbf{n}|_{[0,T]\times\partial\Omega} = 0.$$

- Energy inequality (2.7) holds.

Existence of weak solutions for the above problem with $\mathbb{S}$, p and $[\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0}]$ verifying assumptions (1.5), (1.6), (2.8) can be deduced from [15], Feireisl [5,17] and [11], provided the domain ω is of class $C^{2,\nu}$, $\nu \in (0,1)$. Moreover, due to [6], any weak solution satisfies relative entropy inequality (2.10) with the remainder (2.11), where the test functions satisfy

$$\begin{aligned} &r \in C^1([0,T] \times \overline{\Omega}), \;\; r > 0 \\ &\mathbf{U} \in C^1([0,T] \times \overline{\Omega}; R^3), \;\; \mathbf{U} \cdot \mathbf{n}|_{\partial\omega\times(0,1)} = 0, \end{aligned} \tag{4.3}$$

Finally, as in Remark 2.1, it can be shown by the density argument, that the test function $[r, \mathbf{U}]$ can be taken in the larger regularity class (2.12).

The expected target system is system (1.13–1.15) endowed with the no slip boundary conditions:

$$\mathbf{w} \cdot \mathbf{n}|_{\partial\omega} = 0, \quad \Big(\mathbb{S}_h(\nabla_h \mathbf{w})\mathbf{n}\Big) \times \mathbf{n}|_{\partial\omega} = 0. \tag{4.4}$$

One can again deduce from Valli and Zajaczkowski [21, Theorem 2.5] that under assumptions $p \in C^2(0,\infty)$, $\partial\omega \in C^3$ with initial conditions $[r_0, \mathbf{w}_0]$ belonging to the regularity class (2.13) and satisfying the compatibility condition

$$\frac{1}{r_0}\Big(\nabla_h p(r_0) - \mathrm{div}_h \mathbb{S}_h(\nabla_h \mathbf{w}_0) + r_0 \mathbf{w}_0 \cdot \nabla_h \mathbf{w}_0\Big) \cdot \mathbf{n}\Big|_{\partial\omega} = 0, \tag{4.5}$$

the problem (1.13–1.15), (4.4) admits a unique strong solution in the class (2.15), (2.16) on the maximal time interval $[0, T = T_{\max})$ [that depends on the size of initial data as described in formula (2.14)].

We are now in a position to formulate the main result of Sect. 4.

Theorem 4.1. *Let $\partial\omega \in C^3$. Suppose that the pressure p satisfies hypotheses* (1.6) *and that the stress tensor is given by* (1.5) *with $\eta > 0$. Let r_0, $\mathbf{w}_0$ satisfy assumptions* (2.13), (4.5). *Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations* (1.10–1.12), (4.2) *emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$ that converge to $[r_0, \mathbf{w}_0]$ in the sense* (2.17).

Then $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ converges to $[r, \mathbf{w}]$ in the sense (2.18), *where $[\varrho, \mathbf{w}]$ is the unique strong solution of the 2-D compressible Navier–Stokes system* (1.13–1.15) *with boundary conditions* (4.4) *on the maximal existence time interval of the strong solution $[0, T = T_{\max})$.*

Remark 4.1. Remarks 2.2–2.3 and Corollaries 2.1–2.2 remain valid also in the case of bounded layers with the slip boundary conditions.

4.3. Proof of Theorem 4.1

First, recall the classical Korn inequality on a bounded Lipschitz domain ω. In particular, if $\omega \subset R^2$ it reads: There exists $c = c(\omega)$ such that for all $\mathbf{v} \in W^{1,2}(\omega; R^2)$, there holds

$$\|\mathbf{v}\|_{W^{1,2}(\omega;R^2)} \leq c(\Big(\|\mathbb{D}(\nabla_h \mathbf{v})\|_{L^2(\omega,R^{2\times 2})} + \|\mathbf{v}\|^2_{L^2(\omega;R^2)}\Big). \tag{4.6}$$

Next, we deduce from this inequality, the following result:

Lemma 4.1. *Let $\omega \in R^2$ be a bounded Lipschitz domain that is not radially symmetric. Then there exists $c = c(\omega) > 0$ such that for all $\mathbf{v} \in W^{1,2}(\omega; R^2)$, $\mathbf{w} \cdot \mathbf{n}|_{\partial\omega} = 0$ there holds:*

$$\|\mathbf{v}\|_{W^{1,2}(\omega;R^2)} \leq c\|\mathbb{D}_h(\nabla \mathbf{v})\|_{L^2(\omega;R^{2\times 2})}.$$

Proof of Lemma 4.1. If the conclusion is not true then there exits a sequence $\mathbf{v}_n \in W^{1,2}_{\mathbf{n}}(\omega; R^2)$ such that

$$||\mathbf{v}_n||_{W^{1,2(\omega;R^2)}} = 1$$
$$\|\mathbb{D}_h(\nabla \mathbf{v}_n)\|_{L^2(\omega;R^{2\times 2})} < \tfrac{1}{n}$$

Consequently, there is $\mathbf{v} \in W^{1,2}(\omega, R^2)$ such that $\mathbf{v}_n \to \mathbf{v}$ in $L^2(\omega, R^2)$ and $\mathbf{v}_n \to \mathbf{v}$ weakly in $W^{1,2}(\omega, R^2)$. Using (4.6) we deduce that $\mathbf{v}_n$ is a Cauchy sequence in $W^{1,2}(\omega, R^2)$. Therefore, $\mathbf{v} \in W^{1,2}(\omega, R^2)$ and

$$\mathbf{v}_n \to \mathbf{v} \in W^{1,2}(\omega, R^2).$$

On the other hand

$$\nabla \mathbf{v} + (\nabla \mathbf{v})^T = 0;$$

whence $\mathbf{v}$ is a rigid rotation $\mathbf{v} = \mathbf{b} \times x$, $\mathbf{b} \in R^3$. This contradicts condition $\mathbf{v} \cdot \mathbf{n}|_{\partial\omega} = 0$ for the non circular domains. Lemma 4.1 is proved.

Collecting results of Lemmas 3.1 and 4.1 we obtain an estimate that will be used later for the estimating of the remainder of in the relative entropy inequality, namely:

Lemma 4.2. *Let $\mathbb{S}$ satisfy (1.5), where $\mu > 0$, $\eta > 0$. Let $\Omega = \omega \times (0, 1)$, where ω is a bounded Lipschitz domain that is not a circle. Then there exists $c = c(\omega) > 0$ such that for all $\mathbf{v} \in W^{1,2}(\Omega; R^3)$, $\mathbf{v}_h \cdot \mathbf{n}_h|_{\partial\omega\times(0,1)} = 0$ and $\varepsilon \in (0, 1)$,*

$$\|\mathbf{v}_h\|^2_{L^2(\Omega;R^2)} + \|\nabla_h \mathbf{v}_h\|^2_{L^2(\Omega,R^{2\times 2})} \leq \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\, dx.$$

Since Lemma 4.2 provides exactly the same estimate as Lemma 3.2, the proof of Theorem 4.1 is exactly the same as the proof of Theorem 3.1.

5. Unbounded Layers

In this Section we consider the compressible Navier–Stokes system (1.2–1.4) on unbounded layers with slip boundary conditions on the boundary of the layer, more precisely

$$\Omega_\varepsilon = R^2 \times (0, \varepsilon),\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}(\nabla \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0 \tag{5.1}$$

Since the domain is unbounded, the system has to be completed with conditions at infinity,

$$\mathbf{u}(t, x) \to 0,\ \varrho(t, x) \to \overline{\varrho} > 0 \text{ as } |x| \to \infty. \tag{5.2}$$

After rescaling, we get system (1.10–1.12) with the same boundary conditions on the boundary of Ω:

$$\Omega = R^2 \times (0, 1),\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}(\nabla \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{5.3}$$

and conditions (5.2) as $|x| \to \infty$.

5.1. Weak Solutions to the Primitive System

5.1.1. Weak Solutions. We introduce again the Hilbert space

$$W^{1,2}_{\mathbf{n}}(\Omega; R^3) = \{\mathbf{v} \in W^{1,2}(\Omega; R^3),\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\} = W^{1,2}(\Omega, R^2) \times H^1_0(\Omega).$$

As we are on an unbounded domain, the definition of weak solutions has to be modified as follows, cf. e.g. [17].

Definition 5.1. We say that $[\varrho, \mathbf{u}]$ is a *finite energy weak solution* to the compressible Navier–Stokes (1.10–1.11) with initial conditions (1.12) and boundary conditions (5.2–5.3) in the space time cylinder $(0,T)\times\Omega$ if the following holds:

- the functions $[\varrho, \mathbf{u}]$ belong to the regularity class
$$\left\{ \begin{array}{l} \varrho - \overline{\varrho} \in L^\infty([0,T]; L^\gamma(\Omega) + L^2(\Omega)),\ \varrho \geq 0 \text{ a.a. in } (0,T)\times\Omega,\ \gamma > \frac{3}{2}, \\ \\ \mathbf{u} \in L^2(0,T; W^{1,2}_{\mathbf{n}}(\Omega; R^3)),\ \varrho\mathbf{u} \in L^\infty(0,T; L^2(\Omega) + L^{\frac{2\gamma}{\gamma+1}}(\Omega)); \end{array} \right\} \tag{5.4}$$
- $\varrho - \overline{\varrho} \in C_{\rm weak}([0,T]; L^\gamma(\Omega) + L^2(\Omega))$ and the *continuity Eq.* (1.10) is satisfied in the weak sense,
$$\int_\Omega \varrho\varphi \ {\rm d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\varphi(0,\cdot)\ {\rm d}x = \int_0^\tau \int_\Omega \varrho\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_\varepsilon\varphi\Big)\ {\rm d}x\ {\rm d}t. \tag{5.5}$$
for all $\tau \in [0,T]$ and any test function $\varphi \in C^\infty_c([0,T]\times\overline{\Omega})$;
- $\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^{2\gamma/(\gamma+1)}(\Omega) + L^2(\Omega))$ and the *momentum equation* (1.11) holds in the sense that
$$\int_\Omega \varrho\mathbf{u}\cdot\varphi\ {\rm d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\varphi(0,\cdot)\ {\rm d}x$$
$$= \int_0^T\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_\varepsilon\varphi + p(\varrho_\varepsilon){\rm div}_\varepsilon\varphi\ {\rm d}x\ {\rm d}t - \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\varphi\ {\rm d}x\ {\rm d}t \tag{5.6}$$
for all $\tau \in [0,T]$ and for any $\varphi \in C^\infty_c([0,T]\times\overline{\Omega}; R^3)$, $\varphi_3|_{R^2\times\{0,1\}} = 0$;
- the *energy inequality*
$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + E(\varrho,\overline{\varrho})\right](\tau)\ {\rm d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\mathbf{u}\ {\rm d}x\ {\rm d}t$$
$$\leq \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + E(\varrho_{0,\varepsilon},\overline{\varrho})\right]\ {\rm d}x \tag{5.7}$$
holds for a.a. $\tau \in (0,T)$.

5.1.2. Existence of Weak Solutions and the Relative Entropy Inequality.

Proposition 5.1. *Let* $\mathbb{S}$ *satisfy* (1.5) *and* p *verify* (1.6). *Suppose that the initial data satisfy*

$$\varrho_{\varepsilon,0} \geq 0,\quad \int_\Omega \left(\frac{1}{2}\varrho_{\varepsilon,0}\mathbf{u}^2_{\varepsilon,0} + E(\varrho_{\varepsilon,0},\overline{\varrho})\right)\ {\rm d}x < \infty. \tag{5.8}$$

Then the problem (1.10–1.12), (5.2–5.3) *admits at least one finite energy weak solution on the arbitrary time interval* $(0,T)$.

Moreover $[\varrho, \mathbf{u}]$ *satisfy the relative entropy inequality* (2.10) *with the remainder term* (2.11) *for any couple* $(r, \mathbf{U})$ *such that*

$$r > 0,\ r - \overline{\varrho} \in C^\infty_c([0,T]\times\overline{\Omega}),\ \mathbf{U} \in C^\infty_c([0,T]\times\overline{\Omega}, R^3),\ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Existence of weak solutions is a particular case of [11, Theorem 6.3] that treats more general heat conducting case. The relative entropy inequality is proved in [6].

5.2. Target System and the Main Result

The expected target system is system (1.13–1.15) on $\omega = R^2$ with conditions at infinity

$$r(x_h) \to \overline{\varrho},\ \mathbf{w}(x_h) \to 0 \text{ as } |x| \to \infty.$$

If the initial data satisfy conditions

$$\int\limits_{R^2} E(r_0, \overline{\varrho})\mathrm{d}x < \infty \tag{5.9}$$

$$\nabla_h r_0 \in W^{1,2}(R^2),\ \inf_{R^2} r_0 > 0,\ \mathbf{w}_0 \in W^{3,2}(R^2; R^2), \tag{5.10}$$

then there exists an unique (classical) solutions to this system on a short time interval $[0, T = T_{\max})$ (depending on the initial data) in the class

$$\nabla r \in C([0,T); W^{1,2}(\omega)),\ \mathbf{w} \in C([0,T); W^{2,2}(\omega; R^2)) \cap L^2(0,T; W^{3,2}(\omega; R^2))$$

$$\partial_t r \in C([0,T); W^{1,2}(\omega)),\ \partial_t \mathbf{w} \in L^2(0,T; W^{2,2}(\omega; R^2)), \tag{5.11}$$

$$0 < \underline{r} \equiv \inf_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \le \sup_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \equiv \overline{r}, \tag{5.12}$$

$$\int\limits_{R^2} E(r, \overline{\varrho})\mathrm{d}x \in L^\infty(0,T). \tag{5.13}$$

This fact can be obtained following the proof of [21, Theorem 2.5].

We have the following Theorem

Theorem 5.1. *Suppose that p and S satisfy hypotheses (1.6) and (1.5), respectively. Let $(r, \mathbf{w})$ a solution of the target system (1.13–1.15) on $\omega = R^2$, on the time interval $(0,T)$ belonging to the class (5.11–5.13) emanating from the initial data $[r_0, \mathbf{w}_0]$ satisfying (5.9–5.10). Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be a weak solution of the compressible Navier–Stokes equations (1.10–1.12), (5.2–5.3) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$ satisfying (5.8). Finally assume that*

$$\mathcal{E}(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \,\Big|\, r_0, \mathbf{V}_0) \to 0,$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$.

Then

$$\mathrm{esssup}_{t\in(0,T)} \mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, r, \mathbf{V}) \to 0,$$

where $\mathbf{V} = [\mathbf{w}, 0]$.

Remark 5.1. In accordance with Remark 2.2 we have

$$\varrho_\varepsilon \to r \ in\ L^\infty(0,T; L^2 + L^\gamma(\Omega))$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to r\mathbf{V} \ in\ L^\infty(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega))$$

5.3. Proof of Theorem 5.1

5.3.1. Uniform Estimates. The bounds that can be deduced from the inequality (5.7) are collected in the following lemma.

Lemma 5.1. *There exits c independent of* $\epsilon \in (0,1)$ *such that*

$$\|[1]_{res}\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{5.14}$$

$$\|[\varrho_\varepsilon - \overline{\varrho}]_{ess}\|_{L^\infty(0,T;L^2(\Omega))} \leq c, \tag{5.15}$$

$$\|[\varrho_\varepsilon]_{res}\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq c, \tag{5.16}$$

$$\mathrm{esssup}_{t\in(0,T)} \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon^2 \, \mathrm{d}x \leq c,$$

$$\int_0^T \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{u}_\varepsilon) : \nabla_\varepsilon \mathbf{u}_\varepsilon \, \mathrm{d}x\mathrm{d}t \leq c.$$

In spite of the fact that we shall need in the sequel solely estimates (5.14–5.16), we have collected in the above lemma all uniform estimates available for the sequence of the finite energy weak solutions.

The straightforward proof of the following algebraic lemma is left to the reader.

Lemma 5.2. *Let* $0 < a < b < \infty$. *There exits* $c = c(a,b) > 0$ *such that*

$$\forall \varrho \geq 0, (r_1, r_2) \in [a,b]^2, \; E(\varrho, r_2) \leq c\Big(E(\varrho, r_1) + E(r_1, r_2)\Big).$$

We deduce from Lemma 5.1 and Lemma 5.2 the following result that will be used during the estimates of the remainder.

Lemma 5.3. *Let the couple* r, $\mathbf{V} = [\mathbf{w}, 0]$ *belong to the class (5.11–5.13). Then*

$$\|\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, r, \mathbf{V})\|_{L^\infty(0,T)} \leq c$$

uniformly with respect to $\varepsilon \in (0,1)$.

5.3.2. Korn and Poincaré Type Inequalities. We start with a Korn type inequality:

Lemma 5.4. *There exists* $c > 0$ *such that for all* $\mathbf{v} \in W^{1,2}_{\mathbf{n}}(\Omega; R^3)$

$$\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \, \mathrm{d}x = \mu \|\nabla_\varepsilon \mathbf{v}\|^2_{L^2(\Omega;R^{3\times 3})} + \left(\frac{1}{3}\mu + \eta\right) \|\mathrm{div}_\varepsilon \mathbf{v}\|^2_{L^2(\Omega)}.$$

Proof of Lemma 5.4. Since the set $\{\mathbf{v} \in C^\infty_c(\overline{\Omega}; R^3) \,|\, v_3|_{R^2\times\{0,1\}}\}$ is dense in $W^{1,2}_{\mathbf{n}}(\Omega; R^3)$, it is enough to prove lemma with $\mathbf{v} \in C^\infty_c(\overline{\Omega}, R^2) \times C^\infty_c(\Omega, R)$. The proof is the same as the proof of Lemma 2.1.

Next we show a Poincaré type inequality including the sequence ϱ_ε.

Lemma 5.5. *There exists* $c > 0$ *such that for all* $\epsilon \in (0,1)$ *we have,*

$$\forall \mathbf{v} \in W^{1,2}(\Omega, R^3), \; ||\mathbf{v}||^2_{L^2(\Omega)} \leq c\left(||\nabla_\varepsilon \mathbf{v}||^2_{L^2(\Omega)} + \int_\Omega \varrho_\varepsilon \mathbf{v}^2 dx\right), a.a \; t \in (0,T).$$

Proof of Lemma 5.5. Let $A \in R^*_+$ and $\mathcal{V} = (0,A)^2 \times (0,1)$. We denote $\tilde{N}^\varepsilon_{res}(t) = \{x \in \Omega, \; \varrho_\varepsilon(x,t) \leq \frac{\overline{\varrho}}{2}\} \subset N^\varepsilon_{res}(t)$. Due to (5.14), there exists $\Gamma > 0$ such that

$$\mathrm{esssup}_{t\in(0,T)} |N^\varepsilon_{res}(t)| < \Gamma.$$

We chose A in such a way that

$$|\mathcal{V}| \geq 2\Gamma.$$

We denote

$$\mathcal{V}_b = \mathcal{V} + Ab,\ b \in Z^2 \times \{0\}$$

and we remark that

$$\Omega = \Big(\cup_{b \in Z^2 \times \{0\}} \mathcal{V}_b \Big) \cup \mathcal{N},\ |\mathcal{N}| = 0,\ \mathcal{V}_b \cap \mathcal{V}_{b'} = \emptyset \text{ if } b \neq b'.$$

In view of (5.14)

$$|\mathcal{V}_b \cap N^{\varepsilon}_{ess}(t)| \geq \Gamma, \text{ for a.a } t \in (0,T);$$

whence

$$\int_{\mathcal{V}_b} \varrho_\varepsilon(t,x) dx \geq \frac{\overline{\varrho}}{2}\Gamma, \text{ for a.a. } t \in (0,T).$$

Moreover

$$\int_{\mathcal{V}_b} \varrho_\varepsilon^\gamma(t,x) dx = \int_{\mathcal{V}_b \cap N^{\varepsilon}_{res}(t)} \varrho_\varepsilon^\gamma(t,x) dx + \int_{\mathcal{V}_b \cap N^{\varepsilon}_{ess}(t)} \varrho_\varepsilon^\gamma(t,x) dx \leq C + (2\overline{\varrho})^\gamma |\mathcal{V}|.$$

Now, according to Lemma using 2.2, there exits $c = c(\Gamma, \gamma, |\mathcal{V}|) > 0$ such that for $t \in (0,T)$ and for all $\mathbf{v} \in W^{1,2}(\mathcal{V}_b, R^3)$,

$$||\mathbf{v}||^2_{L^2(\mathcal{V}_b, R^3)} \leq c \left(||\nabla_\varepsilon \mathbf{v}||^2_{L^2(\mathcal{V}_b)} + \int_{\mathcal{V}_b} \varrho_\varepsilon \mathbf{v}^2 dx \right).$$

We obtain the statement of Lemma 5.5 by summing the above estimates over $b \in Z^2 \times \{0\}$. This completes the proof.

5.3.3. Estimates of the Remainder. One can show by a density argument that the solution $[r, \mathbf{V}]$, $\mathbf{V} = [\mathbf{w}, 0]$ of the target system in the class (5.11), (5.13) may be used as a test function in the relative entropy inequality (2.10). Lemmas 5.4 and Lemma 5.5 provide the same estimates as Lemma 2.3 in the case of the periodic layer. When estimating the remainder (2.31) we can proceed step by step as in Sect. 2.2.5.

6. Relaxing the Hypotheses on the Pressure

Estimates effectuated in Sect. 3.3.2 indicate that one can considerably relax the hypotheses (1.6) imposed on the pressure, provided one takes the existence of weak solutions to the primitive system (1.2–1.5) for granted. In fact, in all cases considered in this paper, it is enough to suppose that the pressure satisfies the hypotheses $(1.6)_1$ and

$$c_1 + c_2\varrho + c_3 H(\varrho) \geq p(\varrho), \text{ for } \varrho > \overline{R}, \tag{6.1}$$

where $\overline{R}$, c_1, c_2, c_3 are some fixed positive constants. Similar observation in the case of 3−D–1−D dimension reduction has been done in Bella et al. [2].

Indeed, without any growth condition like $(1.6)_2$ [and without (6.1)], we still get the following lemma:

Lemma 6.1. *Let*

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p'(\varrho) > 0.$$

Then, with the notation of Lemma 2.4, we have for all $\varrho \in [0,\infty)$ and $r \in [a,b]$:

$$E(\varrho|r) \geq c(a,b)\Big(1_{\mathcal{O}_{\rm res}} + \varrho 1_{\mathcal{O}_{\rm res}} + (\varrho - r)^2 1_{\mathcal{O}_{\rm ess}} \Big).$$

Proof of Lemma 6.1. If $\varrho \in [a/2, 2b]$ we use the strict convexity of H to obtain that

$$E(\varrho|r) \geq c|\varrho - r|^2 \text{ where } c = c(a,b) > 0.$$

If $\varrho \in R_+ \setminus [a/2, 2b]$, we observe that

$$\partial_\varrho E(\varrho|r) = H'(\varrho) - H'(r), \quad \partial_r E(\varrho|r) = H''(r)(r - \varrho),$$

where $s \to H'(s)$ is an increasing function on $(0, \infty)$. Now, relying on the monotonicity of functions $s \to E(s|r)$ and $s \to E(\varrho|s)$ induced by the above formulas, we consider two situations. 1) If $\varrho > 2b$, we observe that $E(\varrho|2b) > 0$, whence $H(\varrho) + p(2b) > H'(2b)\varrho$. Consequently,

$$p(2b) - p(b) + E(\varrho|r) \geq p(2b) - p(b) + E(\varrho|b) = H(\varrho) + p(2b) - H'(b)\varrho \geq \Big(H'(2b) - H'(b)\Big)\varrho.$$

This inequality and the fact that $E(\varrho, r) \geq E(2b, b) > 0$, $p(2b) > p(b)$, $H'(2b) > H'(b)$ yield

$$E(\varrho|r) \geq c(1 + \varrho)$$

with some $c = c(b) > 0$. 2) If $\varrho < a/2$ then

$$E(\varrho|r) \geq E(a/2|a) \geq \frac{E(a/2|a)}{a}\varrho + \frac{E(a/2|a)}{2} \geq c(1 + \varrho)$$

with some $c = c(a) > 0$. This completes the proof.

The above relaxed hypothesis can be investigated in combination with the situations studied in Sects. 3, 4 and 5. For the sake of brevity, we choose the system (1.2–1.5) with no slip boundary conditions (3.2) to illustrate the modifications needed in the proofs.

We shall start with a possible definition of a weak solution, modifying slightly Definition 3.1: We shall require that

- $\varrho \geq 0$, $\varrho, p(\varrho) \in L^\infty(0, T; L^1(\Omega))$, $\mathbf{u} \in L^2(0, T; W^{1,2}_{0,\mathbf{n}}(\Omega; R^3))$, $\varrho\mathbf{u} \in L^\infty(0, T; L^1(\Omega; R^3))$;
- $\varrho \in C_{\text{weak}}([0, T]; L^1(\Omega))$ and weak formulation (2.5) of the continuity Eq. (1.10) is valid;
- $\varrho\mathbf{u} \in C_{\text{weak}}([0, T]; L^1(\Omega; R^3))$ and weak formulation (2.6) of the momentum Eq. (1.11) holds with test functions

$$\varphi \in C_c^\infty([0, T] \times \overline{\Omega}; R^3), \ \varphi|_{[0,T]\times\partial\omega\times(0,1)} = 0, \ \varphi_3|_{[0,T]\times\omega\times\{0,1\}} = 0;$$

- Energy inequality (2.7) holds.

We shall suppose the existence of weak solutions to system (1.2–1.5), (3.2) for granted. It is easy to verify that any weak solution satisfies the relative entropy inequality (2.10–2.11) with test functions (3.3) whose regularity can be relaxed. In particular, we can take for the test functions the couple r, $\mathbf{U} = \mathbf{V}$, $\mathbf{V} = [\mathbf{w}, 0]$, where $[r, \mathbf{w}]$ is the strong solution of the system (1.13–1.15), (3.4) in the regularity class (2.15–2.16) guaranteed by the Proposition 3.1. We set in the definition of the residual and essential sets $\underline{\varrho} = \underline{r}$, $\overline{\varrho} = \max\{\overline{r}, \overline{R}\}$. Hence, in view of Lemma 6.1, we have the following bound

$$c\int_\Omega \left(\Big[1\Big]_{\text{res}} + \Big[\varrho_\varepsilon\Big]_{\text{res}} + \Big[\varrho_\varepsilon - r\Big]^2_{\text{ess}}\right) \, \mathrm{d}x \leq \int_\Omega E(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{U}) \, \mathrm{d}x, \tag{6.2}$$

that will replace the bound (2.37) in the estimate of the remainder. The only point, where we need the additional assumption (6.1) is the estimate of the residual part of the expression in Step 2 of Sect. 3.3.3.

After this preparation, we can repeat the argumentation of Sect. 3.3.2. We can conclude that *Theorem* 3.1 *remains valid*, if we replace $(1.6)_{1-2}$ by *a weaker assumption* $(1.6)_1$, (6.1). Likewise, one can reformulate in the same way *Theorem* 2.1 *dealing with periodic layers*, *Theorem* 4.1 *dealing with slip boundary conditions* and *Theorem* 5.1 *dealing with unbounded layers*.

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(accepted: March 12, 2014; published online: May 6, 2014)

Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 261 (2016) 4448–4485
Journal of Differential Equations
www.elsevier.com/locate/jde

Existence of weak solutions for compressible Navier–Stokes equations with entropy transport

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Received 14 April 2016
Available online 15 July 2016

Abstract

We consider the compressible Navier–Stokes system with variable entropy. The pressure is a nonlinear function of the density and the entropy/potential temperature which, unlike in the Navier–Stokes–Fourier system, satisfie only the transport equation. We provide existence results within three alternative weak formulations of the corresponding classical problem. Our constructions hold for the optimal range of the adiabatic coefficient from the point of view of the nowadays existence theory.

1. Introduction

The purpose of this paper is to analyze the model of fl w of compressible viscous flui with variable entropy. Such fl w can be described by the compressible Navier–Stokes equations coupled with an additional equation describing the evolution of the entropy. In case when

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http://dx.doi.org/10.1016/j.jde.2016.06.029

the conductivity is neglected, the changes of the entropy are solely due to the transport and the whole system can be written as:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } (0,T)\times\Omega, \tag{1.1a}$$

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s \boldsymbol{u}) = 0 \text{ in } (0,T)\times\Omega, \tag{1.1b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \operatorname{div}\mathbb{S} \text{ in } (0,T)\times\Omega, \tag{1.1c}$$

where the unknowns are the density $\varrho\colon (0,T)\times\Omega \to \mathbb{R}_+\cup\{0\}$, the entropy $s\colon (0,T)\times\Omega\to\mathbb{R}_+$ and the velocity of flui $\boldsymbol{u}\colon (0,T)\times\Omega\to\mathbb{R}^3$, and where Ω is a three dimensional domain with a smooth boundary $\partial\Omega$.

The momentum, the continuity, and the entropy equations are additionally coupled by the form of the pressure p, so we assume that

$$p(\varrho, s) = \varrho^\gamma \mathcal{T}(s), \quad \gamma > 1, \tag{1.2}$$

where $\mathcal{T}(\cdot)$ is a given smooth and strictly monotone function from $\mathbb{R}_+$ to $\mathbb{R}_+$, in particular $\mathcal{T}(s) > 0$ for $s > 0$.

We assume that the flui is Newtonian and that the viscous part of the stress tensor is of the following form

$$\mathbb{S} = \mathbb{S}(\nabla\boldsymbol{u}) = 2\mu\Big(\mathbb{D}(\boldsymbol{u}) - \frac{1}{3}\operatorname{div}\boldsymbol{u}\mathbb{I}\Big) + \eta\operatorname{div}_x\boldsymbol{u}\mathbb{I}$$

with $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$. Viscosity coefficient μ and η are assumed to be constant, hence we can write

$$\operatorname{div}\mathbb{S}(\nabla\boldsymbol{u}) = \mu\Delta\boldsymbol{u} + (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u}$$

with $\lambda = \eta - \frac{2}{3}\mu$. To keep the ellipticity of the Lamé operator we require that

$$\mu > 0, \quad 3\lambda + 2\mu > 0. \tag{1.3}$$

The system is supplemented by the initial and the boundary conditions:

$$\varrho(0,x) = \varrho_0(x),\ (\varrho s)(0,x) = S_0(x),\ (\varrho\boldsymbol{u})(0,x) = \boldsymbol{q}_0(x), \tag{1.4}$$

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}. \tag{1.5}$$

System (1.1) is a model of motion of compressible viscous gas with variable entropy transported by the fl w. The quantity $\theta = [\mathcal{T}(s)]^{1/\gamma}$ can be also interpreted as a potential temperature in which case the pressure (1.2) takes the form $(\rho\theta)^\gamma$ and has been studied in [7,9].

We aim at proving the existence of global in time weak solutions to system (1.1). Note that at least for smooth solution the continuity equation (1.1a) allows us to reformulate (1.1b) as a pure transport equation for s, so we have

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } (0,T)\times\Omega, \tag{1.6a}$$

$$\partial_t s + \boldsymbol{u} \cdot \nabla s = 0 \text{ in } (0,T) \times \Omega, \tag{1.6b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \mathbb{S} \text{ in } (0,T) \times \Omega. \tag{1.6c}$$

In contrast to entropy equation in system (1.1) the above form is insensitive to appearance of vacuum states; in fact it is completely decoupled from the continuity equation. The regularity of the density in the compressible Navier–Stokes-type systems is in general rather delicate matter. Therefore, one can expect that proving the existence of solutions to system (1.1) requires more severe assumptions than to get a relevant solution to (1.6). This observation will be reflecte in the range of parameter γ which determines the quality of a priori estimates for the argument of the pressure – $Z = \varrho[\mathcal{T}(s)]^{\frac{1}{\gamma}}$ according to the notation from above.

In order to clarify this issue a little more let us introduce a third formulation of system (1.1) describing the evolution of the pressure argument $Z = \varrho[\mathcal{T}(s)]^{\frac{1}{\gamma}}$ instead of the entropy itself. We have:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } (0,T) \times \Omega, \tag{1.7a}$$

$$\partial_t Z + \operatorname{div}(Z \boldsymbol{u}) = 0 \text{ in } (0,T) \times \Omega, \tag{1.7b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla Z^{\gamma} = \operatorname{div} \mathbb{S} \text{ in } (0,T) \times \Omega. \tag{1.7c}$$

Again, the above formulation is equivalent with the previous ones provided the solution is regular enough, which, however, may not be true in case of weak solutions.

The above discussion motivates distinction between the cases when the evolution of the entropy is described by the continuity, the transport or the renormalized transport equation. Indeed, the form of the entropy equation, although used to describe the same phenomena, is a diagnostic marker indicating the notion of plausible solution to the whole system. Our paper contains an existence analysis for all three systems: (1.1), (1.6) and (1.7) within suitably adjusted definition of weak solutions. Such an approach allows us to emphasize the implications between the solutions and to better understand the restrictions of renormalization technique. These issues, absent in the analysis of the standard single density systems, are of great importance for more complex multi-component or multi-phase fl ws. Our results show possible applications of nowadays classical tools in the analysis of the Navier–Stokes system to challenging problems, e.g. constitutive equation involving nonlinear combinations of hyperbolic quantities: densities, concentrations, etc.

The outline of the paper is the following. We firs consider system (1.7), for which we are able to show the existence of a weak solution using standard technique available for the compressible Navier–Stokes system, see [6]. Next, using a special form of renormalization, and division of equation (1.7b) by ϱ, we show that we may replace (1.7b) by (1.6b) and finall by (1.1b). We are able to handle (1.6b) as well as (1.7b) for the optimal range of γ's (i.e. $\gamma > \frac{3}{2}$), while getting equation (1.1b) requires the assumption $\gamma \geq \frac{9}{5}$. This is a restriction under which the renormalization theory of DiPerna–Lions [1] can be applied.

In Section 2, we introduce the definitio of the weak solutions to all three systems mentioned above and present our main existence theorems. Then, in Section 3 we recall some specifi classical results which are then used in the proof. Further, in Sections 4 and 5 we prove the existence of weak solutions to system (1.7); we introduce several levels of approximations and prove the existence of solutions at each step by performing relevant limit passages in Sections 6 and 7. Finally, in Section 8 we prove the existence of a weak solution to systems (1.1) and (1.6).

2. Weak solutions, existence results

Throughout our analysis we naturally distinguish two different situations. They are associated to the magnitude of the adiabatic exponent γ. From the point of view of theory of global in time weak solutions, it is reasonable to assume that

$$\gamma > \frac{3}{2}. \tag{2.1}$$

This assumption provides L^1 bound of the convective term and is necessary for application of nowadays techniques. Under this condition we will firs prove the existence of a weak solution to system (1.7), see Theorem 2. Then we shall deduce from this result existence of weak solutions for the formulation (1.6) still under assumption (2.1), see Theorem 2. This result is not equivalent to the existence of weak solutions to system (1.1) though. The latter can be proved solely under the restriction

$$\gamma \geq \frac{9}{5}. \tag{2.2}$$

Indeed, the latter more restricted range of γ enables to obtain L^2 estimate of the density and, as mentioned in the introduction, makes it possible to apply the DiPerna–Lions theory of the renormalized solutions to the transport equation (1.6b) and to multiply it by ϱ within the class of weak solutions.

2.1. Weak solutions to system (1.1)

Let us firs introduce the definitio of a weak solution to our original system (1.1). We assume that the initial data (1.4) satisfy:

$$\begin{gathered} \varrho_0 : \Omega \to \mathbb{R}_+, \ \ s_0 : \Omega \to \mathbb{R}_+, \ \ \boldsymbol{u}_0 : \Omega \to \mathbb{R}^3, \\ \varrho_0 \in L^\gamma(\Omega), \ \ \int_\Omega \varrho_0 \mathrm{d}x > 0, \\ S_0 = \varrho_0 s_0, \ \ s_0 \in L^\infty(\Omega), \ \ \boldsymbol{q}_0 = \varrho_0 \boldsymbol{u}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3). \end{gathered} \tag{2.3}$$

The choice of nontrivial initial condition for s on the set $\{\varrho_0 = 0\}$ will play an important role in the last section. Indeed, there is a certain difference in the proof of the case $s_0 = const$, and s_0 non-constant on this set. We consider

Definition 1. Suppose the initial conditions satisfy (2.3). We say that the triplet $(\varrho, s, \boldsymbol{u})$ is a weak solution of problem (1.1)–(1.5) if:

$$(\varrho, s, \boldsymbol{u}) \in L^\infty(0, T; L^\gamma(\Omega)) \times L^\infty((0, T) \times \Omega) \times L^2(0, T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{2.4}$$

and for any $t \in [0, T]$ we have:

(i) $\varrho \in C_w([0,T]; L^\gamma(\Omega))$ and the continuity equation (1.1a) is satisfie in the weak sense

$$\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0\varphi(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall\varphi \in C^1([0,T]\times\overline{\Omega}); \tag{2.5}$$

(ii) $\varrho s \in C_w([0,T]; L^\gamma(\Omega))$ and equation (1.1b) is satisfie in the weak sense

$$\int_\Omega (\varrho s)(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega S_0\varphi(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho s\partial_t\varphi + \varrho s\boldsymbol{u}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall\varphi \in C^1([0,T]\times\overline{\Omega}); \tag{2.6}$$

(iii) $\varrho\boldsymbol{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3))$ and the momentum equation (1.1c) is satisfie in the weak sense

$$\int_\Omega (\varrho\boldsymbol{u})(t,\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x - \int_\Omega \boldsymbol{q}_0\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla\boldsymbol{\psi}$$
$$+\varrho^\gamma\mathcal{T}(s)\operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}):\nabla\boldsymbol{\psi}\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \forall\boldsymbol{\psi}\in C_c^1([0,T]\times\Omega,\mathbb{R}^3); \tag{2.7}$$

(iv) the energy inequality

$$\mathcal{E}^1(\varrho,s,\boldsymbol{u})(t) + \int_0^t\int_\Omega \Big(\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)(\operatorname{div}\boldsymbol{u})^2\Big)\,\mathrm{d}x\,\mathrm{d}\tau \le \mathcal{E}^1(\varrho_0,s_0,\boldsymbol{u}_0) \tag{2.8}$$

holds for a.a. $t \in (0,T)$, where

$$\mathcal{E}^1(\varrho,s,\boldsymbol{u}) = \int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{\varrho^\gamma\mathcal{T}(s)}{\gamma-1}\Big)\,\mathrm{d}x.$$

The firs main result concerning solutions meant by Definitio 1 reads.

Theorem 1. *Let μ, λ satisfy (1.3), $\gamma \ge \frac{9}{5}$ and the initial data $(\varrho_0, S_0, \boldsymbol{q}_0)$ satisfy (2.3). Then there exists a weak solution $(\varrho, s, \boldsymbol{u})$ to problem (1.1)–(1.5) in the sense of Definition 1.*

2.2. Weak solution to system (1.7)

The restriction on γ in Theorem 1 is obviously not satisfactory as all the physically reasonable values of γ are less than or equal to $\frac{5}{3}$. We are able to relax this constraint for system (1.7). Formally, taking $Z = \varrho(\mathcal{T}(s))^{\frac{1}{\gamma}}$ in (1.7) one can recover our original system (1.1). However, for the weak solution this formal argument cannot be made rigorous unless we assume that $\gamma \geq \frac{9}{5}$. Nevertheless, system (1.7) is a good starting point for our considerations. Indeed, for reasonable initial and boundary conditions it can be shown that it possesses a weak solution for $\gamma > \frac{3}{2}$, using more or less standard approach. Proving existence of solutions directly for system (1.1) seems not to be so simple.

We assume that the initial data for system (1.7) are

$$\varrho_0 : \Omega \to \mathbb{R}_+, \ \ s_0 : \Omega \to \mathbb{R}_+, \ \ \boldsymbol{u}_0 : \Omega \to \mathbb{R}^3,$$
$$\varrho(0,x) = \varrho_0(x), \ \ Z(0,x) = Z_0(x), \ \ (\varrho \boldsymbol{u})(0,x) = \boldsymbol{q}_0(x) = \varrho_0 \boldsymbol{u}_0(x), \tag{2.9}$$

and they satisfy

$$\begin{gathered}(\varrho_0, Z_0) \in L^\gamma(\Omega)^2, \quad \varrho_0, Z_0 \geq 0 \text{ a.e. in } \Omega, \quad \int_\Omega \varrho_0 \,\mathrm{d}x > 0, \\ 0 \leq c_\star \varrho_0 \leq Z_0 \leq c^\star \varrho_0 \text{ a.e. in } \Omega, \quad 0 < c_\star \leq c^\star < \infty, \quad \boldsymbol{q}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3).\end{gathered} \tag{2.10}$$

Then we have

Definition 2. Suppose that the initial conditions satisfy (2.10). We say that the triplet $(\varrho, Z, \boldsymbol{u})$ is a weak solution of problem (1.7) with the initial and boundary conditions (1.5), (2.9) if

$$(\varrho, Z, \boldsymbol{u}) \in L^\infty(0,T; L^\gamma(\Omega)) \times L^\infty(0,T; L^\gamma(\Omega)) \times L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{2.11}$$

and for any $t \in (0,T]$ we have:

(i) $\varrho \in C_w([0,T]; L^\gamma(\Omega))$ and the continuity equation (1.7a) is satisfie in the weak sense

$$\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0 \varphi(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall \varphi \in C^1([0,T] \times \overline{\Omega}); \tag{2.12}$$

(ii) $Z \in C_w([0,T]; L^\gamma(\Omega))$ and equation (1.7b) is satisfie in the weak sense

$$\int_\Omega Z(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega Z_0 \varphi(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(Z \partial_t \varphi + Z \boldsymbol{u} \cdot \nabla \varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall \varphi \in C^1([0,T] \times \overline{\Omega}); \tag{2.13}$$

(iii) $\varrho \boldsymbol{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$ and the momentum equation (1.1c) is satisfie in the weak sense

$$\int_\Omega (\varrho \boldsymbol{u})(t,\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x - \int_\Omega \boldsymbol{q}_0\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi}$$

$$+ Z^\gamma \operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \forall\boldsymbol{\psi}\in C_c^1([0,T]\times\Omega,\mathbb{R}^3); \quad (2.14)$$

(iv) the energy inequality

$$\mathcal{E}^2(\varrho, Z, \boldsymbol{u})(t) + \int_0^t\int_\Omega \Big(\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)(\operatorname{div}\boldsymbol{u})^2\Big)\,\mathrm{d}x\,\mathrm{d}\tau \le \mathcal{E}^2(\varrho_0, Z_0, \boldsymbol{u}_0) \quad (2.15)$$

holds for a.a. $t\in(0,T)$, where

$$\mathcal{E}^2(\varrho, Z, \boldsymbol{u}) = \int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{Z^\gamma}{\gamma-1}\Big)\mathrm{d}x. \quad (2.16)$$

Before presenting the existence result for the auxiliary problem, let us recall the definitio of a renormalized solution to equation (1.7b).

Definition 3. We say that equation (1.7b) holds in the sense of renormalized solutions, provided $(Z, \boldsymbol{u})$, extended by zero outside of Ω, satisfy

$$\partial_t b(Z) + \operatorname{div}(b(Z)\boldsymbol{u}) + \big(b'(Z)Z - b(Z)\big)\operatorname{div}\boldsymbol{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3), \quad (2.17)$$

where

$$b\in C^1(\mathbb{R}), \quad b'(z)=0, \quad \forall z\in\mathbb{R} \text{ large enough}. \quad (2.18)$$

We have the following existence result for solutions define by Definitio 2.

Theorem 2. *Let μ, λ satisfy (1.3), $\gamma > \frac{3}{2}$, and the initial data $(\varrho_0, Z_0, \boldsymbol{q}_0)$ satisfy (2.10).*
Then there exists a weak solution $(\varrho, Z, \boldsymbol{u})$ to problem (1.7) with boundary conditions (1.5), in the sense of Definition 2. Moreover, $(Z, \boldsymbol{u})$ solves (1.7b) in the renormalized sense and

$$0 \le c_\star\varrho \le Z \le c^\star\varrho$$

a.e. in $(0,T)\times\Omega$.

2.3. Weak solution to system (1.6)

If we replace (1.1b) by (1.6b) (using also the renormalization of the latter), the result is also much better than in Theorem 1, in fact optimal from the point of view of nowadays theory of compressible Navier–Stokes equations. In order to formulate the result precisely, we firs rewrite system (1.6) in a slightly different way. We look for a triplet $(\rho, \zeta, \boldsymbol{u})$ solving the system of equations

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \tag{2.19a}$$

$$\partial_t \zeta + \boldsymbol{u} \cdot \nabla \zeta = 0, \tag{2.19b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \left(\frac{\varrho}{\zeta}\right)^\gamma = \operatorname{div} \mathbb{S}(\nabla \boldsymbol{u}), \tag{2.19c}$$

with initial conditions

$$\varrho(0,x) = \varrho_0(x),\ \zeta(0,x) = \zeta_0(x),\ (\varrho \boldsymbol{u})(0,x) = \boldsymbol{q}_0(x), \tag{2.20}$$

such that $\zeta_0 = \frac{\varrho_0}{Z_0}$ and satisfying assumptions (2.10), in particular

$$\zeta_0 \in \Big((c^\star)^{-1}, (c_\star)^{-1}\Big). \tag{2.21}$$

Then the weak solution is define as follows.

Definition 4. Suppose the initial conditions $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ satisfy (2.21) and (2.10) (for ϱ_0 and $\boldsymbol{q}_0$). We say that the triplet $(\varrho, \zeta, \boldsymbol{u})$ is a weak solution of problem (2.19) emanating from the initial data $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ if

$$(\varrho, \zeta, \boldsymbol{u}) \in L^\infty(0,T; L^\gamma(\Omega)) \times L^\infty((0,T) \times \Omega) \times L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{2.22}$$

and for any $t \in (0,T]$ we have:

(i) $\varrho \in C_w([0,T]; L^\gamma(\Omega))$ and the continuity equation (2.19a) is satisfie in the weak sense

$$\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0 \varphi(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall \varphi \in C^1([0,T] \times \overline{\Omega}); \tag{2.23}$$

(ii) $\zeta \in C_w([0,T]; L^\infty(\Omega))$ and equation (2.19b) is satisfie in the weak sense

$$\int_\Omega \zeta(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \zeta_0 \varphi(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(\zeta \partial_t \varphi + \zeta \operatorname{div}(\boldsymbol{u}\varphi)\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall \varphi \in C^1([0,T] \times \overline{\Omega}); \tag{2.24}$$

(iii) $\varrho \boldsymbol{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$ and the momentum equation (2.19c) is satisfie in the weak sense

$$\int_\Omega (\varrho \boldsymbol{u})(t,\cdot) \cdot \boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x - \int_\Omega \boldsymbol{q}_0 \cdot \boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\psi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\psi}$$

$$+\left(\frac{\varrho}{\zeta}\right)^{\gamma} \operatorname{div} \boldsymbol{\psi} - \mathbb{S}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{\psi}\Big) \,\mathrm{d}x \,\mathrm{d}\tau, \forall \boldsymbol{\psi} \in C_c^1([0,T] \times \Omega, \mathbb{R}^3); \quad (2.25)$$

(iv) the energy inequality

$$\mathcal{E}^2(\varrho, \varrho/\zeta, \boldsymbol{u})(t) + \int_0^t \int_\Omega \left(\mu |\nabla \boldsymbol{u}|^2 + (\mu+\lambda)(\operatorname{div} \boldsymbol{u})^2\right) \mathrm{d}x \,\mathrm{d}\tau \le \mathcal{E}^2(\varrho_0, \varrho_0/\zeta_0, \boldsymbol{u}_0) \quad (2.26)$$

holds for a.a. $t \in (0,T)$, where $\mathcal{E}^2$ is define through (2.16).

The last result concerns the existence of solutions meant by Definitio 4.

Theorem 3. *Let μ, λ satisfy (1.3), $\gamma > \frac{3}{2}$, and the initial data $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ satisfy (2.21) and (2.10) (for ϱ_0 and $\boldsymbol{q}_0$).*

Then there exists a weak solution $(\varrho, \zeta, \boldsymbol{u})$ to problem (2.19) with boundary conditions (1.5), in the sense of Definition 4. Moreover, $(\varrho, \boldsymbol{u})$ solves (2.19a) and $(\zeta, \boldsymbol{u})$ solves (2.19b) in the renormalized sense.

Using the result of Theorem 3, we may easily obtain a solution to system (1.6). Indeed, we may defin

$$s = \mathcal{T}^{-1}(\zeta^{-\gamma}),$$

and use the fact that equation (2.19b) holds in the renormalized sense.

Remark 1. Note that in two space dimensions, all results hold for any $\gamma > 1$. In both two and three space dimensions, we can also include a non-zero external force on the right-hand side of the momentum equation, i.e. we have additionally the term $\varrho \boldsymbol{f}$ on the right-hand side of (1.1c), (1.6c) and (1.7c). For $\boldsymbol{f} \in L^\infty((0,T) \times \Omega, \mathbb{R}^3)$ we would get the same results as in Theorems 1, 2 and 3.

3. Auxiliary results

Before proving our main theorems, we recall several auxiliary results used in this paper. These are mostly standard results and we include them only for the sake of clarity of presentation.

Lemma 1. *Let $\mu > 0$, $\lambda + 2\mu > 0$. Then there exists a positive constant c such that*

$$\mu \|\nabla \boldsymbol{u}\|^2_{L^2(\Omega, \mathbb{R}^{3\times 3})} + (\lambda + \mu)\|\operatorname{div} \boldsymbol{u}\|_{L^2(\Omega)} \ge c \|\nabla \boldsymbol{u}\|_{L^2(\Omega, \mathbb{R}^{3\times 3})}. \quad (3.1)$$

Lemma 2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. If $g_n \to g$ in $C_w([0,T]; L^q(\Omega))$, $1 < q < \infty$ then $g_n \to g$ strongly in $L^p(0,T; W^{-1,r}(\Omega))$ provided $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,r}(\Omega)$.*

Note that $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,r}(\Omega)$ holds for Ω a Lipschitz domain in $\mathbb{R}^3$ for $1 \le r \le \frac{3}{2}$ if $q > 1$ arbitrary or for $\frac{3}{2} < r < \infty$ provided $q > \frac{3r}{3+r}$.

Lemma 3. *Let $1 \le q < \infty$. Let the sequence $g_n \in C_w([0,T], L^q(\Omega))$ be bounded in $L^\infty(0,T; L^q(\Omega))$. Then it is uniformly bounded on $[0,T]$. More precisely, we have*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|g_n(t)\|_{L^q(\Omega)} \le C \Rightarrow \sup_{t\in[0,T]} \|g_n(t)\|_{L^q(\Omega)} \le C, \tag{3.2}$$

where c is a positive constant independent of n.

Lemma 4. *Let $1 < p, q < \infty$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on $[0,T]$ with values in $L^q(\Omega)$ such that*

$$g_n \in C_w([0,T], L^q(\Omega)),\ g_n \text{ is uniformly continuous in } W^{-1,p}(\Omega) \text{ and uniformly bounded in } L^q(\Omega). \tag{3.3}$$

Then, at least for a chosen subsequence

$$g_n \to g \text{ in } C_w([0,T], L^q(\Omega)). \tag{3.4}$$

If, moreover, $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,p}(\Omega)$, then

$$g_n \to g \qquad \text{in } C([0,T]; W^{-1,p}(\Omega)). \tag{3.5}$$

Next, let us consider weak solutions to the continuity equation

$$\partial_t Z + \operatorname{div}(Z\boldsymbol{u}) = 0, \quad Z(0,\cdot) = Z_0(\cdot). \tag{3.6}$$

As a result of the DiPerna–Lions [1] theory we have

Lemma 5. *Assume $Z \in L^q((0,T)\times\Omega)$ and $\boldsymbol{u} \in L^2(0,T; W_0^{1,2}(\Omega))$, where $\Omega \subset \mathbb{R}^3$ is a domain with Lipschitz boundary. Let $(Z, \boldsymbol{u})$ be a weak solution to (3.6) and $q \ge 2$. Then $(Z, \boldsymbol{u})$ is also a renormalized solution to (3.6), i.e. it solves (2.17) in the sense of distributions on $(0,T)\times\mathbb{R}^3$ provided Z, $\boldsymbol{u}$ are extended by zero outside of Ω.*

Remark 2. By density argument and standard approximation technique, we may extend the validity of (2.17) to functions $b \in C([0,\infty)) \cap C^1(0,\infty))$ such that

$$|b'(t)| \le Ct^{-\lambda_0}, \qquad \lambda_0 < -1, \quad t \in (0,1],$$
$$|b'(t)| \le Ct^{\lambda_1}, \qquad -1 < \lambda_1 \le \frac{q}{2} - 1, \quad t \ge 1.$$

Lemma 6. *Let*

$$(s, \mathbf{u}) \in \Big(L^\infty((0,T)\times\Omega) \cap C_w([0,T]; L^q(\Omega))\Big) \times L^2(0,T; W^{1,2}(\Omega,\mathbb{R}^3))$$

be a weak solution to (1.6b) with $s(0,\cdot) = s_0 \in L^\infty(\Omega)$. Then for every $B \in C(\mathbb{R})$, $(B(s), \boldsymbol{u})$ is a distributional solution to (1.6b), i.e.

$$\partial_t B(s) + \boldsymbol{u} \cdot \nabla B(s) = 0$$

in $\mathcal{D}'((0,T)\times\Omega)$. *Moreover,* s *and* $B(s) \in C([0,T]; L^r(\Omega))$ *for all* $r < \infty$ *and* $B(s)(0,\cdot) = B(s_0)$.

In some situations when the DiPerna–Lions theory is not applicable, i.e. when $q < 2$ in Lemma 5, we can still prove that the solution is in fact a renormalized one using the approach from [3]. To this purpose one has to consider the oscillation defect measure of the sequence Z_δ approximating Z, i.e.

$$\operatorname{osc}_q(Z_\delta - Z) = \sup_{k\in\mathbb{N}} \limsup_{\delta\to 0^+} \|T_k(Z_\delta) - T_k(Z)\|_{L^q((0,T)\times\Omega)}, \tag{3.7}$$

where

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad z \in \mathbb{R}, \quad k \ge 1, \tag{3.8}$$

with $T \in C^\infty(\mathbb{R})$ such that

$$T(z) = z \text{ for } z \le 1, \quad T(z) = 2 \text{ for } z \ge 3, \quad T \text{ concave, non-decreasing.} \tag{3.9}$$

We have

Lemma 7. *Let* $\Omega \subset \mathbb{R}^3$ *be a domain with Lipschitz boundary. Assume that* $(Z_\delta, \boldsymbol{u}_\delta)$ *is a sequence of renormalized solutions to the continuity equation such that*

$$\begin{aligned} Z_\delta \to Z \qquad & \textit{weakly in } L^1((0,T)\times\Omega), \\ \boldsymbol{u}_\delta \to \boldsymbol{u} \qquad & \textit{weakly in } L^2(0,T; W_0^{1,2}(\Omega,\mathbb{R}^3)) \end{aligned}$$

such that $\operatorname{osc}_q(Z_\delta - Z) < \infty$ *for some* $q > 2$. *Then* $(Z, \boldsymbol{u})$ *is a renormalized solution to the continuity equation.*

We further need the following well-known result [2,13] concerning the solution operator to the problem

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= f, \\ \boldsymbol{v}|_{\partial\Omega} &= \mathbf{0}. \end{aligned} \tag{3.10}$$

Lemma 8. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^3$. *For any* $1 < p < \infty$ *there exists a solution operator* $\mathcal{B}\colon \{f \in L^p(\Omega); \int_\Omega f \,\mathrm{d}x = 0\} \to W_0^{1,p}(\Omega,\mathbb{R}^3)$ *to (3.10) such that for* $\boldsymbol{v} = \mathcal{B}f$ *it holds*

$$\|\boldsymbol{v}\|_{W^{1,p}(\Omega)} \le C(\Omega, p)\|f\|_{L^p(\Omega)}.$$

Next, we report the following general result concerning the compensated compactness (see [12] or [16])

Lemma 9. *Let $\boldsymbol{U}_n$, $\boldsymbol{V}_n$ be two sequences such that*

$$\boldsymbol{U}_n \to \boldsymbol{U} \quad \text{weakly in } L^p(\Omega, \mathbb{R}^3),$$
$$\boldsymbol{V}_n \to \boldsymbol{V} \quad \text{weakly in } L^q(\Omega, \mathbb{R}^3),$$

where $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1$, *and*

$$\operatorname{div} \boldsymbol{U}_n \quad \text{is precompact in } W^{-1,r}(\Omega),$$
$$\operatorname{curl} \boldsymbol{V}_n \quad \text{is precompact in } W^{-1,r}(\Omega, \mathbb{R}^{3\times 3})$$

for a certain $r > 0$. *Then*

$$\boldsymbol{U}_n \cdot \boldsymbol{V}_n \to \boldsymbol{U} \cdot \boldsymbol{V} \qquad \text{weakly in } L^s(\Omega).$$

We will further need the following operators

$$\boldsymbol{\mathcal{A}}[\cdot] = \{\mathcal{A}_i\}_{i=1,2,3}[\cdot] = \nabla \Delta^{-1}[\cdot], \tag{3.11}$$

where Δ^{-1} stands for the inverse of the Laplace operator on $\mathbb{R}^3$. To be more specific the Fourier symbol of $\mathcal{A}_j$ is

$$\mathcal{F}(\mathcal{A}_j)(\xi) = \frac{-\mathrm{i}\xi_j}{|\xi|^2}. \tag{3.12}$$

Note that for a sufficientl smooth v

$$\sum_{i=1}^{3} \partial_i \mathcal{A}_i[v] = v \tag{3.13}$$

and, by virtue of the classical Marcinkiewicz multiplier theorem,

$$\|\nabla \boldsymbol{\mathcal{A}}[v]\|_{L^s(\Omega,\mathbb{R}^3)} \le C(s,\Omega)\|v\|_{L^s(\Omega)},\ 1 < s < \infty. \tag{3.14}$$

Note that (see [5]) if $v, \partial_t v \in L^p((0,T) \times \mathbb{R}^3)$, then

$$\partial_t \boldsymbol{\mathcal{A}}[v(t,\cdot)](x) = \boldsymbol{\mathcal{A}}[\partial_t v(t,\cdot)](x) \text{ for a.a. } (t,x) \in (0,T) \times \mathbb{R}^3. \tag{3.15}$$

Next, let us also introduce the so-called *Riesz operators*

$$\mathcal{R}_{ij}[\cdot] = \partial_j \mathcal{A}_i[\cdot] = \partial_j \partial_i \Delta^{-1}[\cdot], \tag{3.16}$$

or, in terms of Fourier symbols, $\mathcal{F}(\mathcal{R}_{ij})(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}$. We recall some of its evident properties needed in the sequel. We have

$$\sum_{i=1}^{3} \mathcal{R}_{ii}[g] = g, \quad g \in L^r(\mathbb{R}^3), \quad 1 < r < \infty, \tag{3.17}$$

$$\int_{\mathbb{R}^3} \mathcal{R}_{ij}[u]v \,\mathrm{d}x = \int_{\mathbb{R}^3} u\,\mathcal{R}_{ij}[v]\mathrm{d}x, \quad u \in L^r(\mathbb{R}^3), v \in L^{r'}(\mathbb{R}^3), \quad 1 < r < \infty, \tag{3.18}$$

and

$$\| \mathcal{R}_{ij}[u]\|_{L^p(\mathbb{R}^3)} \le c(p)\|u\|_{L^p(\mathbb{R}^3)}, \qquad 1 < p < \infty. \tag{3.19}$$

4. Approximation

We firs focus on the proof of the auxiliary result, i.e. on Theorem 2. The problem can be viewed as compressible Navier–Stokes system with two densities, where one is connected with inertia of the flui and the other one with the pressure. The proof of Theorem 2 is hence very similar to the construction of solutions to the usual barotropic Navier–Stokes equations.

The purpose of this section is to introduce subsequent levels of approximation and to formulate relevant existence theorems for each of them. The proofs of these theorems are presented afterwards by performing several limit passages when corresponding approximation parameters vanish. We firs regularize the pressure in order to get higher integrability of Z (and also of ϱ) in order to obtain the renormalized continuity equations using the DiPerna–Lions technique [1]. Next we regularize the continuity equations (for both ϱ and Z). The construction of a solution is done at another level of approximation, the Galerkin approximation for the velocity.

4.1. First approximation level

A weak solution of problem (1.7), (1.5) is obtained as a limit when $\delta \to 0^+$ of the solutions to following problem

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \tag{4.1a}$$

$$\partial_t Z + \operatorname{div}(Z\boldsymbol{u}) = 0, \tag{4.1b}$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla Z^\gamma + \delta\nabla Z^\beta = \operatorname{div}\mathbb{S}(\nabla\boldsymbol{u}) \tag{4.1c}$$

with the boundary conditions

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}, \tag{4.2}$$

and modifie initial data

$$(\varrho(0,\cdot), Z(0,\cdot)) = (\varrho_{0,\delta}(\cdot), Z_{0,\delta}(\cdot)) \in C^\infty(\overline{\Omega}, \mathbb{R}^2), \quad 0 < c_\star\varrho_{0,\delta} \le Z_{0,\delta} \le c^\star\varrho_{0,\delta} \text{ in } \overline{\Omega}, \tag{4.3a}$$

$$\nabla\varrho_{0,\delta}\cdot\boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \quad \nabla Z_{0,\delta}\cdot\boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \tag{4.3b}$$

$$(\varrho\boldsymbol{u})(0,\cdot) = \boldsymbol{q}_{0,\delta}(\cdot) \in C^\infty(\overline{\Omega}, \mathbb{R}^3). \tag{4.3c}$$

The specifi assumption on the initial data (4.3b) is not needed here, at this approximation level we would be satisfie with less regular approximation without this condition. However, more regular approximation with the above mentioned compatibility condition is needed at another approximation level and we prefer to regularize the initial condition just once.

Note that we require $\boldsymbol{q}_{0,\delta} \to \boldsymbol{q}_0$ in $L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)$ and $\varrho_{0,\delta} \to \varrho_0$, $Z_{0,\delta} \to Z_0$, both in $L^\gamma(\Omega)$. While the firs part, i.e. the initial condition for the linear momentum, is easy to ensure by standard mollification the regularization of the initial condition for Z and ϱ is more complex. However, we may multiply Z_0 by a suitable cut-off function (to set the function to be zero near the boundary), then add a small constant to this function and finall mollify it; i.e.

$$Z_{0,\delta} = (\varphi_\delta Z_0 + \delta) * \omega_\delta.$$

It is not difficul to see that for suitably chosen cut-off function φ_δ[1] all properties connected with $Z_{0,\delta}$ in (4.3a)–(4.3b) will be fulfille as well as $Z_{0,\delta} \to Z_0$ in $L^\gamma(\Omega)$ for $\delta \to 0^+$. Similarly we proceed for ϱ_0. By a suitable regularization of the initial linear momentum we may also ensure that

$$\frac{|\boldsymbol{q}_{0,\delta}|^2}{\varrho_{0,\delta}} 1_{\{\varrho_0>0\}} \to \frac{|\boldsymbol{q}_0|^2}{\varrho_0} 1_{\{\varrho_0>0\}}$$

in $L^1(\Omega)$.

4.2. Second approximation level

We prove the existence of a solution to problem (4.1)–(4.3) by letting $\epsilon \to 0^+$ in the following approximate system. Given $\epsilon, \delta > 0$, we consider

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = \epsilon \Delta \varrho, \tag{4.4a}$$

$$\partial_t Z + \operatorname{div}(Z \boldsymbol{u}) = \epsilon \Delta Z, \tag{4.4b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla Z^\gamma + \delta \nabla Z^\beta + \epsilon \nabla \boldsymbol{u} \cdot \nabla \varrho = \operatorname{div} \mathbb{S}(\nabla \boldsymbol{u}), \tag{4.4c}$$

supplemented with the boundary conditions

$$\nabla_x \varrho \cdot \boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \quad \nabla_x Z \cdot \boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \tag{4.5}$$

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}, \tag{4.6}$$

and modifie initial data (4.3) (see the comments above).

4.3. Existence results for the approximate systems

Let us present now the existence result for the firs approximation level

[1] We may take $\varphi_\delta \in C_c^\infty(\Omega)$ such that $0 \le \varphi_\delta \le 1$ in Ω with $\varphi_\delta(x) = 1$ if (for $x \in \Omega$) $\operatorname{dist}\{x, \partial\Omega\} \ge \frac{\delta}{2}$ and $\varphi_\delta(x) = 0$ if $\operatorname{dist}\{x, \partial\Omega\} \le \frac{\delta}{4}$.

Proposition 1. *Let $\beta \geq \max(\gamma, 4)$, $\delta > 0$. Then, given initial data $(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})$ as in (4.3), there exists a finite energy weak solution $(\varrho, Z, \boldsymbol{u})$ to problem (4.1)–(4.3) such that*

$$(\varrho, Z, \boldsymbol{u}) \in [L^\infty(0,T; L^\beta(\Omega))]^2 \times L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{4.7}$$

$$0 \leq c_\star \varrho \leq Z \leq c^\star \varrho \text{ a.e. in } (0,T) \times \Omega, \tag{4.8}$$

and for any $t \in (0,T)$ we have:

(i) $\varrho \in C_w([0,T]; L^\beta(\Omega))$ *and the continuity equation (4.1a) is satisfied in the weak sense*

$$\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi\big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall \varphi \in C^1([0,T]\times\overline{\Omega}); \tag{4.9}$$

(ii) $Z \in C_w([0,T]; L^\beta(\Omega))$ *and equation (4.1b) is satisfied in the weak sense*

$$\int_\Omega Z(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega Z_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \big(Z\partial_t\varphi + Z\boldsymbol{u}\cdot\nabla\varphi\big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall \varphi \in C^1([0,T]\times\overline{\Omega}); \tag{4.10}$$

(iii) $\varrho\boldsymbol{u} \in C_w([0,T]; L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3))$ *and the momentum equation (4.1c) is satisfied in the weak sense*

$$\int_\Omega \varrho\boldsymbol{u}(t,\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x - \int_\Omega \boldsymbol{q}_{0,\delta}\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi} + Z^\gamma \operatorname{div}\boldsymbol{\psi}$$
$$+ \delta Z^\beta \operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \forall \boldsymbol{\psi} \in C_c^1([0,T]\times\Omega,\mathbb{R}^3); \tag{4.11}$$

(iv) the energy inequality

$$\mathcal{E}_\delta(\varrho,\boldsymbol{u},Z)(t) + \int_0^t \int_\Omega \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}\tau \leq \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}) \tag{4.12}$$

holds for a.a. $t \in (0,T)$, where $\mathcal{E}_\delta(\varrho,\boldsymbol{u},Z) = \int_\Omega \big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{\delta}{\beta-1}Z^\beta + \frac{1}{\gamma-1}Z^\gamma\big)\,\mathrm{d}x$;

(v) the following estimates hold with constants independent of δ

$$\sup_{t\in[0,T]} \|\varrho(t)\|_{L^\gamma(\Omega)}^\gamma + \sup_{t\in[0,T]} \|Z(t)\|_{L^\gamma(\Omega)}^\gamma \leq C(\gamma, c_\star)\,\mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.13}$$

$$\delta \sup_{t\in[0,T]} \|\varrho(t)\|_{L^\beta(\Omega)}^\beta + \delta \sup_{t\in[0,T]} \|Z(t)\|_{L^\beta(\Omega)}^\beta \leq C(\beta, c_\star)\,\mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.14}$$

$$\|\boldsymbol{u}\|_{L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3))} \le C\,\mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.15}$$

$$\sup_{t\in[0,T]} \|\varrho\boldsymbol{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3))} + \sup_{t\in[0,T]} \|Z\boldsymbol{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3)} \le C(\gamma, c^\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.16}$$

$$\|\varrho\boldsymbol{u}\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega,\mathbb{R}^3))} + \|Z\boldsymbol{u}\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega,\mathbb{R}^3))} \le C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.17}$$

$$\|\varrho|\boldsymbol{u}|^2\|_{L^1(0,T;L^{\frac{3\gamma}{\gamma+3}}(\Omega))} + \|Z|\boldsymbol{u}|^2\|_{L^1(0,T;L^{\frac{3\gamma}{\gamma+3}}(\Omega))} \le C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.18}$$

$$\|\varrho|\boldsymbol{u}|^2\|_{L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega))} + \|Z|\boldsymbol{u}|^2\|_{L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega))} \le C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.19}$$

$$\begin{aligned}&\|\varrho\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega)} + \delta\|\varrho\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega)} + \|Z\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega)}\\ &+ \delta\|Z\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega)} \le C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})),\end{aligned} \tag{4.20}$$

where $\theta = \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2}\}$. Moreover, equations (4.1a), (4.1b) hold in the sense of renormalized solutions in $\mathcal{D}'((0,T)\times\Omega)$ and $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ provided ϱ, Z, $\boldsymbol{u}$ are prolonged by zero outside Ω.

We have for the second approximation level

Proposition 2. *Suppose $\beta \ge \max(4,\gamma)$. Let ϵ, $\delta > 0$. Assume the initial data $(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})$ satisfy (4.3). Then there exists a weak solution $(\varrho, Z, \boldsymbol{u})$ to problem (4.3)–(4.6) such that*

$$(\varrho, Z, \boldsymbol{u}) \in [L^\infty(0,T;L^\beta(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))]^2 \times L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3)), \tag{4.21}$$

$$0 \le c_\star\varrho \le Z \le c^\star\varrho \text{ a.e. in } (0,T)\times\Omega, \tag{4.22}$$

and for any $t \in (0,T)$ we have:

(i) $\varrho \in C_w([0,T];L^\beta(\Omega))$ and the continuity equation (4.4a) is satisfied in the weak sense

$$\begin{aligned}&\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x\\ &= \int_0^t\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi - \epsilon\nabla\varrho\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \quad \forall\varphi\in C^1([0,T]\times\overline{\Omega});\end{aligned} \tag{4.23}$$

(ii) $Z \in C_w([0,T];L^\beta(\Omega))$ and equation (4.4b) is satisfied in the weak sense

$$\begin{aligned}&\int_\Omega Z(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega Z_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x\\ &= \int_0^t\int_\Omega \Big(Z\partial_t\varphi + Z\boldsymbol{u}\cdot\nabla\varphi - \epsilon\nabla Z\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \quad \forall\varphi\in C^1([0,T]\times\overline{\Omega});\end{aligned} \tag{4.24}$$

(iii) $\varrho \boldsymbol{u} \in C_w([0,T]; L^{\frac{2\beta}{\beta+1}}(\Omega, \mathbb{R}^3))$ *and the momentum equation (4.4c) is satisfied in the weak sense*

$$\int_\Omega \varrho \boldsymbol{u}(t,\cdot) \cdot \boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x - \int_\Omega \boldsymbol{q}_{0,\delta} \cdot \boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\psi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\psi} + Z^\gamma \operatorname{div} \boldsymbol{\psi}$$
$$+ \delta Z^\beta \operatorname{div} \boldsymbol{\psi} - \mathbb{S}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{\psi} + \epsilon \nabla \varrho \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\psi} \Big)\,\mathrm{d}x\,\mathrm{d}\tau, \quad \forall \boldsymbol{\psi} \in C^1_c([0,T] \times \Omega, \mathbb{R}^3); \quad (4.25)$$

(iv) *the energy inequality*

$$\mathcal{E}_\delta(\varrho, \boldsymbol{u}, Z)(t) + \int_0^t \int_\Omega \Big(\mathbb{S}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{u} + \frac{\epsilon\gamma}{\gamma-1} Z^{\gamma-2} |\nabla Z|^2$$
$$+ \frac{\epsilon\delta\beta}{\beta-1} Z^{\beta-2} |\nabla Z|^2 \Big)\,\mathrm{d}x\,\mathrm{d}\tau \le \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}) \quad (4.26)$$

holds for a.a. $t \in (0,T)$, *where* $\mathcal{E}_\delta(\varrho, \boldsymbol{u}, Z)$ *is the same as in Proposition 1;*

(v) *the following estimates hold with constants independent of* ϵ

$$\sup_{t\in[0,T]} \|\varrho(t)\|^\beta_{L^\beta(\Omega)} + \sup_{t\in[0,T]} \|Z(t)\|^\beta_{L^\beta(\Omega)} \le C(\beta, c_\star)\, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \quad (4.27)$$

$$\|\boldsymbol{u}\|_{L^2(0,T;W^{1,2}_0(\Omega,\mathbb{R}^3))} \le C\, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \quad (4.28)$$

$$\sup_{t\in[0,T]} \|\varrho \boldsymbol{u}\|_{L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3))} + \sup_{t\in[0,T]} \|Z \boldsymbol{u}\|_{L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)} \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \quad (4.29)$$

$$\epsilon \Big(\|\nabla \varrho\|^2_{L^2((0,T)\times\Omega,\mathbb{R}^3)} + \|\nabla Z\|^2_{L^2((0,T)\times\Omega,\mathbb{R}^3)} \Big) \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \quad (4.30)$$

$$\|\varrho |\boldsymbol{u}|^2\|_{L^2(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega))} + \|Z |\boldsymbol{u}|^2\|_{L^2(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega))} \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \quad (4.31)$$

$$\|\varrho\|_{L^{\beta+1}((0,T)\times\Omega)} + \|Z\|_{L^{\beta+1}((0,T)\times\Omega)} \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})). \quad (4.32)$$

5. Existence for the second approximation level

We are not going to present detailed proof of Proposition 2, as it is similar to the corresponding step in the existence proof for the barotropic Navier–Stokes equations, cf. [13]. In what follows we only explain main ideas as well as how to obtain the crucial estimate (4.22).

We introduce another approximation level, the Galerkin approximation for the velocity. We take a suitable basis $\{\boldsymbol{\Phi}_j\}_{j=1}^\infty$ in $W^{1,2}_0(\Omega, \mathbb{R}^3)$, orthonormal in $L^2(\Omega, \mathbb{R}^3)$, and replace (4.25) by

$$\int_\Omega \partial_t(\varrho \boldsymbol{u}^n) \cdot \boldsymbol{\Phi}_j\,\mathrm{d}x = \int_\Omega \Big(\varrho \boldsymbol{u}^n \otimes \boldsymbol{u}^n : \nabla \boldsymbol{\Phi}_j + Z^\gamma \operatorname{div} \boldsymbol{\Phi}_j + \delta Z^\beta \operatorname{div} \boldsymbol{\Phi}_j$$
$$- \mathbb{S}(\nabla \boldsymbol{u}^n) : \nabla \boldsymbol{\Phi}_j + \epsilon \nabla \varrho \cdot \nabla \boldsymbol{u}^n \cdot \boldsymbol{\Phi}_j \Big)\,\mathrm{d}x, \quad \forall j = 1, 2, \ldots, n, \quad (5.1)$$

where ϱ and Z solve (4.4a) and (4.4b), respectively, with $\boldsymbol{u}$ replaced by $\boldsymbol{u}^n$, and

$$\boldsymbol{u}^n(t,x) = \sum_{j=1}^{n} a_j^n(t)\boldsymbol{\Phi}_j(x).$$

The initial condition for the momentum equation reads

$$\varrho(0,\cdot)\boldsymbol{u}^n(0,\cdot) = P^n(\boldsymbol{q}_{0,\delta})(\cdot)$$

with P^n the corresponding orthogonal projection on the space spanned by $\{\boldsymbol{\Phi}_j\}_{j=1}^n$. We construct the solutions to the n-th Galerkin approximation by means of a version of the Schauder fi ed point theorem. The fundamental step in this procedure is derivation of the a priori estimates. They can be obtained by using the solution $\boldsymbol{u}^n$ as a test function in (5.1) and combining it with (4.4b) as well as with (4.4a). We then deduce

$$\mathcal{E}_\delta(\varrho, Z, \boldsymbol{u}^n)(t) + \int_0^t \int_\Omega \left(\mathbb{S}(\nabla \boldsymbol{u}^n) : \nabla \boldsymbol{u}^n + \epsilon\gamma Z^{\gamma-2}|\nabla Z|^2 + \epsilon\delta\beta Z^{\beta-2}|\nabla Z|^2\right) \mathrm{d}x\,\mathrm{d}\tau$$
$$\le \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, P^n(\boldsymbol{q}_{0,\delta})/\varrho_{0,\delta}) \le C \quad (5.2)$$

with C independent of n (also of ϵ and δ). Next, testing equations (4.4a) and (4.4b) by ϱ and Z, respectively, we also have

$$\|\varrho\|^2_{L^2(\Omega)}(t) + \|Z\|^2_{L^2(\Omega)}(t) + \epsilon \int_0^t \left(\|\nabla\varrho\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla Z\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)\mathrm{d}\tau \le C \quad (5.3)$$

provided $\beta \ge 4$. Note also that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega Z\,\mathrm{d}x = 0.$$

To prove inequalities (4.22) we use a simple comparison principle between ϱ and Z. Taking $c_\star$, $c^\star$ as in (4.3a) we may write

$$\partial_t(Z - c_\star\varrho) + \operatorname{div}\left(\boldsymbol{u}^n(Z - c_\star\varrho)\right) - \epsilon\Delta(Z - c_\star\varrho) = 0$$

and

$$\partial_t(c^\star\varrho - Z) + \operatorname{div}\left(\boldsymbol{u}^n(c^\star\varrho - Z)\right) - \epsilon\Delta(c^\star\varrho - Z) = 0.$$

As both equations have non-negative initial conditions, it is easy to see that also the solutions are non-negative and due to the uniqueness of solutions we deduce that

$$0 < c_\star\varrho \le Z \le c^\star\varrho < \infty \quad (5.4)$$

a.e. in $(0,T)\times\Omega$. Combining (5.4) with (5.2) we also have

$$\|\varrho\|_{L^\infty(0,T;L^\beta(\Omega))} \le C \tag{5.5}$$

with $C = C(c_\star, \delta, \mathcal{E}_\delta)$. The regularity of solutions to parabolic problems allows us to deduce that we have independently of n

$$\|\partial_t \varrho\|_{L^q(0,T;L^q(\Omega))} + \|\partial_t Z\|_{L^q(0,T;L^q(\Omega))} + \|\varrho\|_{L^q(0,T;W^{2,q}(\Omega))} + \|Z\|_{L^q(0,T;W^{2,q}(\Omega))} \le C(\epsilon) \tag{5.6}$$

for all $q\in(1,\infty)$. These estimates are sufficien to apply the fi ed point argument, but also to pass to the limit $n\to\infty$. To this aim, recall also that ϱ and Z belong to $C_w([0,T];L^\beta(\Omega))$ and $\varrho\boldsymbol{u}^n$ to $C_w([0,T];L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3))$. Hence, using several general results from Section 3 (see Lemmas 2–4) we may pass to the limit with $n\to\infty$ to recover system (4.3)–(4.6) as stated in Proposition 2. To finis the proof of this proposition, we have to show estimate (4.32). To this aim, we use as test function in (4.25) $\boldsymbol{\psi}$, solution to (cf. Lemma 8 in Section 3)

$$\operatorname{div}\boldsymbol{\psi} = Z - \frac{1}{|\Omega|}\int_\Omega Z\,\mathrm{d}x$$

with homogeneous Dirichlet boundary conditions. Due to properties of the Bogovskii operator we may prove

$$\|Z\|_{L^{\beta+1}((0,T)\times\Omega)} \le C$$

which, together with (5.4), finishe the proof of Proposition 2.

6. Vanishing viscosity limit: proof of Proposition 1

6.1. Limit passage based on the a priori estimates

At this stage, we are ready to pass to the limit for $\epsilon\to0^+$ to get rid of the diffusion term in the equations (4.4a), (4.4b) as well as of the ϵ-dependent term in (4.4c). Note that the parameter δ is kept fi ed throughout this procedure so that we may use the estimates derived above, except (5.6). Accordingly, the solution of problem (4.3)–(4.6) obtained in Proposition 2 above will be denoted $(\varrho_\epsilon, Z_\epsilon, \boldsymbol{u}_\epsilon)$.

First of all, by virtue of (4.28) and (4.30), we obtain

$$\epsilon\nabla\varrho_\epsilon\cdot\nabla\boldsymbol{u}_\epsilon \to 0 \text{ in } L^1((0,T)\times\Omega),$$

and, analogously,

$$\epsilon\nabla Z_\epsilon,\ \epsilon\nabla\varrho_\epsilon \to 0 \text{ in } L^2((0,T)\times\Omega).$$

From estimates (4.27)–(4.32) we further deduce

$$\varrho_\epsilon \to \varrho \text{ weakly-}\star \text{ in } L^\infty(0,T;L^\beta(\Omega)) \text{ and weakly in } L^{\beta+1}((0,T)\times\Omega), \tag{6.1a}$$

$$Z_\epsilon \to Z \text{ weakly-}\star \text{ in } L^\infty(0,T;L^\beta(\Omega)) \text{ and weakly in } L^{\beta+1}((0,T)\times\Omega), \tag{6.1b}$$

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3)), \tag{6.1c}$$

passing to subsequences if necessary.

By virtue of (4.22) and the weak $L^{\beta+1}$-convergence derived above we obtain

$$0 \le c_\star\varrho \le Z \le c^\star\varrho \text{ a.e. in } (0,T)\times\Omega. \tag{6.2}$$

Due to (4.23), (4.24), (4.30) and (4.32), ϱ_ϵ and Z_ϵ are uniformly continuous in $W^{-1,\frac{2\beta}{\beta+1}}(\Omega)$. Since they belong to $C_w([0,T];L^\beta(\Omega))$ and they are uniformly bounded in $L^\beta(\Omega)$ (by virtue of (4.27)), we use Lemma 4, in order to get at least for a chosen subsequence

$$\varrho_\epsilon \to \varrho, \quad Z_\epsilon \to Z \text{ in } C_w([0,T];L^\beta(\Omega)). \tag{6.3}$$

Once we realize that the imbedding $L^s(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ is compact for $s > \frac{6}{5}$, we apply Lemma 2 to ϱ_ϵ and Z_ϵ, and obtain

$$\varrho_\epsilon \to \varrho, \quad Z_\epsilon \to Z \text{ in } L^p(0,T;W^{-1,2}(\Omega)), \quad 1 \le p < \infty. \tag{6.4}$$

Consequently, by virtue of the previous formula, (4.29) and (6.1c) we obtain

$$\varrho_\epsilon\boldsymbol{u}_\epsilon \to \varrho\boldsymbol{u}, \quad Z_\epsilon\boldsymbol{u}_\epsilon \to Z\boldsymbol{u} \text{ weakly-}\star \text{ in } L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)). \tag{6.5}$$

Taking into account (4.25) and (4.27)–(4.32) we conclude that $\varrho_\epsilon\boldsymbol{u}_\epsilon$ is uniformly continuous in $W^{-1,s}(\Omega,\mathbb{R}^3)$, where $s = \frac{\beta+1}{\beta}$. Since it belongs to $C_w([0,T];L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3))$ and since it is uniformly bounded in $L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)$ (see (4.29)), Lemma 4 yields

$$\varrho_\epsilon\boldsymbol{u}_\epsilon \to \varrho\boldsymbol{u} \text{ in } C_w([0,T];L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)). \tag{6.6}$$

The imbedding $L^{\frac{2\beta}{\beta+1}}(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ is compact, hence we deduce from Lemma 2

$$\varrho_\epsilon\boldsymbol{u}_\epsilon \to \varrho\boldsymbol{u} \text{ strongly in } L^p(0,T;W^{-1,2}(\Omega,\mathbb{R}^3)). \tag{6.7}$$

It implies, together with (6.1c) that

$$\varrho_\epsilon\boldsymbol{u}_\epsilon \otimes \boldsymbol{u}_\epsilon \to \varrho\boldsymbol{u}\otimes\boldsymbol{u} \text{ in } L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3}) \tag{6.8}$$

for some $q > 1$.

We have proven that the limits ϱ, Z and $\boldsymbol{u}$ satisfy for any $t \in [0,T]$ the following system of equations

$$\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall\varphi\in C^1([0,T]\times\overline{\Omega}); \tag{6.9}$$

$$\int_\Omega Z(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega Z_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(Z\partial_t\varphi + Z\boldsymbol{u}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall\varphi\in C^1([0,T]\times\overline{\Omega}); \tag{6.10}$$

$$\int_\Omega \varrho\boldsymbol{u}(t.\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \boldsymbol{q}_{0,\delta}\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla\boldsymbol{\psi} + \overline{p}\,\mathrm{div}\,\boldsymbol{\psi}$$
$$- \mathbb{S}(\nabla\boldsymbol{u}):\nabla\boldsymbol{\psi}\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \forall\boldsymbol{\psi}\in C_c^1([0,T]\times\Omega,\mathbb{R}^3), \tag{6.11}$$

where, by virtue of (4.32),

$$Z_\epsilon^\gamma + \delta Z_\epsilon^\beta \to \overline{p} \text{ weakly in } L^{\frac{\beta+1}{\beta}}((0,T)\times\Omega). \tag{6.12}$$

In particular, equations (4.4a), (4.4b) and (4.4c) (with $\overline{p}$ instead of $Z^\gamma + \delta Z^\beta$) are satisfie in the sense of distributions and the limit functions satisfy the initial condition

$$\varrho(0,\cdot) = \varrho_{0,\delta}(\cdot), \quad Z(0,\cdot) = Z_{0,\delta}(\cdot), \quad (\varrho\boldsymbol{u})(0,\cdot) = \boldsymbol{q}_{0,\delta}(\cdot), \tag{6.13}$$

where $(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{q}_{0,\delta})$ are define in (4.3).

Thus our ultimate goal is to show that

$$\overline{p} = Z^\gamma + \delta Z^\beta \tag{6.14}$$

which is equivalent to the strong convergence of Z_ϵ in $L^1((0,T)\times\Omega)$.

6.2. Effective viscous ux

We introduce the quantity $Z^\gamma + \delta Z^\beta - (\lambda + 2\mu)\,\mathrm{div}\,\boldsymbol{u}$ called usually the effective viscous flux This quantity enjoys remarkable properties for which we refer to Hoff [8], Lions [10], or Serre [15]. We have the following crucial result.

Lemma 10. *Let ϱ_ϵ, Z_ϵ, $\boldsymbol{u}_\epsilon$ be the sequence of approximate solutions, the existence of which is guaranteed by Proposition 2, and let ϱ, Z, $\boldsymbol{u}$ and $\overline{p}$ be the limits appearing in (6.1a), (6.1b), (6.1c) and (6.12) respectively. Then*

$$\lim_{\epsilon\to0^+}\int_0^T \psi\int_\Omega \phi\Big(Z_\epsilon^\gamma + \delta Z_\epsilon^\beta - (\lambda+2\mu)\,\mathrm{div}\,\boldsymbol{u}_\epsilon\Big)Z_\epsilon\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \psi\int_\Omega \phi\Big(\overline{p} - (\lambda+2\mu)\,\mathrm{div}\,\boldsymbol{u}\Big)Z\,\mathrm{d}x\,\mathrm{d}t$$

for any $\psi\in C_c^\infty((0,T))$ and $\phi\in C_c^\infty(\Omega)$, passing to subsequences, if necessary.

The proof of Lemma 10 is based on the Div–Curl Lemma of compensated compactness, see Lemma 9. We will not present it here, as it is a relatively standard result in the theory of weak solutions to the compressible Navier–Stokes equations; see e.g. [13] for more details. The basic tools for the proof can be found in Section 3. We shall give more details to the proof of a similar result used in the limit passage $\delta \to 0$, where, moreover, several arguments are more subtle than here.

We conclude this section by showing (6.14) and, consequently, strong convergence of the sequence Z_ϵ in $L^1((0,T)\times\Omega)$.

Recall that Z solves (6.10) in the sense of renormalized equations, see Lemma 5. Thus, we take $b(Z) = Z\ln Z$ (see Remark 2) to get

$$\int_0^T\int_\Omega Z \operatorname{div}_x \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega Z_{0,\delta}\ln(Z_{0,\delta})\,\mathrm{d}x - \int_\Omega Z(T)\ln\big(Z(T)\big)\,\mathrm{d}x. \tag{6.15}$$

On the other hand, Z_ϵ solves (4.4b) a.e. on $(0,T)\times\Omega$, in particular,

$$\partial_t b(Z_\epsilon) + \operatorname{div}_x(b(Z_\epsilon)\boldsymbol{u}_\epsilon) + \big(b'(Z_\epsilon)Z_\epsilon - b(Z_\epsilon)\big)\operatorname{div}\boldsymbol{u}_\epsilon - \epsilon\Delta b(Z_\epsilon) \le 0$$

for any b convex and globally Lipschitz on $\mathbb{R}^+$; whence

$$\int_0^T\int_\Omega \Big(b'(Z_\epsilon)Z_\epsilon - b(Z_\epsilon)\Big)\operatorname{div}\boldsymbol{u}_\epsilon \,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega b(Z_{0,\delta})\,\mathrm{d}x - \int_\Omega b\big(Z_\epsilon(T)\big)\,\mathrm{d}x$$

from which we easily deduce

$$\int_0^T\int_\Omega Z_\epsilon \operatorname{div}\boldsymbol{u}_\epsilon \,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega Z_{0,\delta}\ln(Z_{0,\delta})\,\mathrm{d}x - \int_\Omega Z_\epsilon(T)\ln\big(Z_\epsilon(T)\big)\,\mathrm{d}x. \tag{6.16}$$

Note that

$$\int_\Omega Z(T)\ln(Z(T))\,\mathrm{d}x \le \liminf_{\epsilon\to 0^+}\int_\Omega Z_\epsilon(T)\ln\big(Z_\epsilon(T)\big)\,\mathrm{d}x.$$

Take two non-decreasing sequences ψ_n, ϕ_n of non-negative functions such that

$$\psi_n \in C_c^\infty(0,T),\ \psi_n \to 1,\ \phi_n \in C_c^\infty(\Omega),\ \phi_n \to 1. \tag{6.17}$$

Lemma 10 implies that

$$\limsup_{\epsilon\to 0^+}\int_0^T \psi_m \int_\Omega \phi_m(Z_\epsilon^\gamma + \delta Z_\epsilon^\beta)Z_\epsilon \,\mathrm{d}x\,\mathrm{d}t \le \limsup_{\epsilon\to 0^+}\int_0^T \psi_n \int_\Omega \phi_n(Z_\epsilon^\gamma + \delta Z_\epsilon^\beta)Z_\epsilon \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \lim_{\epsilon\to 0^+} \int_0^T \psi_n \int_\Omega \phi_n \big(Z_\epsilon^\gamma + \delta Z_\epsilon^\beta - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}_\epsilon\big) Z_\epsilon \,\mathrm{d}x\,\mathrm{d}t$$

$$+ (\lambda+2\mu)\limsup_{\epsilon\to 0^+} \int_0^T \psi_n \int_\Omega \phi_n Z_\epsilon \operatorname{div}\boldsymbol{u}_\epsilon \,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \psi_n \int_\Omega \phi_n(\overline{p} - (\lambda+2\mu)\operatorname{div}_x \boldsymbol{u}) Z \,\mathrm{d}x\,\mathrm{d}t$$

$$+ (\lambda+2\mu)\limsup_{\epsilon\to 0^+} \int_0^T \int_\Omega Z_\epsilon |1-\psi_n\phi_n||\operatorname{div}\boldsymbol{u}_\epsilon| \,\mathrm{d}x\,\mathrm{d}t + (\lambda+2\mu)\limsup_{\epsilon\to 0^+} \int_0^T \int_\Omega Z_\epsilon \operatorname{div}\boldsymbol{u}_\epsilon \,\mathrm{d}x\,\mathrm{d}t.$$

Using also (6.15) and (6.16), we observe that

$$\limsup_{\epsilon\to 0^+} \int_0^T \psi_m \int_\Omega \phi_m (Z_\epsilon^\gamma + \delta Z_\epsilon^\beta) Z_\epsilon \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_0^T \int_\Omega \overline{p} Z \,\mathrm{d}x\,\mathrm{d}t + \eta(n) + (\lambda+2\mu)\Big[\int_\Omega Z(T)\ln(Z(T))\,\mathrm{d}x - \limsup_{\epsilon\to 0^+} \int_\Omega Z_\epsilon(T)\ln(Z_\epsilon(T))\,\mathrm{d}x\Big]$$

for all $m \leq n$, where

$$\eta(n) \to 0 \text{ for } n \to \infty.$$

Thus we have proved

$$\limsup_{\epsilon\to 0^+} \int_0^T \psi_m \int_\Omega \phi_m (Z_\epsilon^\gamma + \delta Z_\epsilon^\beta) Z_\epsilon \,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \int_\Omega \overline{p} Z \,\mathrm{d}x\,\mathrm{d}t, \quad \forall m \geq 1.$$

To conclude the proof of (6.14), we make use of a (slightly modified Minty's trick. Since the nonlinearity $P(Z) = Z^\gamma + \delta Z^\beta$ is monotone, we have for any $v \in L^{\beta+1}((0,T)\times\Omega)$

$$\int_0^T \psi_m \int_\Omega \phi_m (P(Z_\epsilon) - P(v))(Z_\epsilon - v) \,\mathrm{d}x\,\mathrm{d}t \geq 0$$

and, consequently,

$$\int_0^T \int_\Omega \overline{p} Z \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi_m \int_\Omega \phi_m P(v) v \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \psi_m \int_\Omega \phi_m (\overline{p} v + P(v) Z) \,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Now, letting $m \to \infty$, we get

$$\int_0^T \int_\Omega (\overline{p} - P(v))(Z - v)\, \mathrm{d}x\, \mathrm{d}t \geq 0$$

and the choice $v = Z + \eta\varphi$, $\eta \to 0$, $\varphi \in C_c^\infty((0,T)\times\Omega)$ arbitrary, yields the desired conclusion

$$\overline{p} = Z^\gamma + \delta Z^\beta.$$

To finis the proof of Proposition 1 we have to show (4.20). To this aim, we use as test function in (4.1c) solution to (cf. Lemma 8 in Section 3)

$$\operatorname{div} \boldsymbol{\psi} = Z^\theta - \frac{1}{|\Omega|}\int_\Omega Z^\theta \mathrm{d}x$$

with homogeneous Dirichlet boundary conditions, where $\theta > 0$ is a constant. Due to properties of the Bogovskii operator we may show (the proof is similar to the case of compressible Navier–Stokes equations, see e.g. [13])

$$\|Z\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega)} + \delta\|Z\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega)} \leq C$$

with $\theta \leq \min\{\frac{\gamma}{2}, \frac{2}{3}\gamma - 1\}$. Other estimates can be obtained easily. The proof of Proposition 1 is finished

7. Passing to the limit in the artificial pressure term. Proof of Theorem 2

Our next goal is to let $\delta \to 0^+$. We will relax the assumptions on the growth of the pressure and on the regularity of the initial data. We are again confronted with a missing estimate for the sequence of densities which would guarantee the strong convergence. Additional problems will arise from the fact that the a priori bounds for the density do not allow us to apply the DiPerna–Lions transport theory, see Lemma 5. To overcome these difficulties we will apply to system (4.1) Feireisl's approach. Accordingly, the solution of problem (4.1) obtained in Proposition 1 above will be denoted ϱ_δ, Z_δ, $\boldsymbol{u}_\delta$.

7.1. Limit passage based on a priori estimates

Using estimates independent of the parameter δ, i.e. (4.13)–(4.20), as well as the procedure at the beginning of the previous section we show (see also [13])

$$\varrho_\delta \to \varrho \text{ in } C_w([0,T]; L^\gamma(\Omega)), \tag{7.1a}$$

$$Z_\delta \to Z \text{ in } C_w([0,T]; L^\gamma(\Omega)), \tag{7.1b}$$

$$\boldsymbol{u}_\delta \to \boldsymbol{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega,\mathbb{R}^3)), \tag{7.1c}$$

$$\varrho_\delta\boldsymbol{u}_\delta \to \varrho\boldsymbol{u} \text{ in } C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3)), \tag{7.1d}$$

$$\varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta \to \varrho\boldsymbol{u}\otimes\boldsymbol{u} \text{ weakly in } L^q((0,T)\times\Omega,\mathbb{R}^{3\times3}) \quad \text{for some } q > 1, \tag{7.1e}$$

$$\varrho_\delta^\gamma \to \overline{\varrho^\gamma} \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\Omega), \tag{7.1f}$$

$$Z_\delta^\gamma \to \overline{Z^\gamma} \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\Omega), \tag{7.1g}$$

$$\delta Z_\delta^\beta \to 0 \text{ weakly in } L^q((0,T)\times\Omega), \quad \text{for some } q>1, \tag{7.1h}$$

passing to subsequences as the case may be.

Consequently, ϱ, Z, $\boldsymbol{u}$ satisfy

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3), \tag{7.2}$$

$$\partial_t Z + \operatorname{div}(Z\boldsymbol{u}) = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3), \tag{7.3}$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla\overline{Z^\gamma} = \mu\Delta\boldsymbol{u} + (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u} \text{ in } \mathcal{D}'((0,T)\times\Omega,\mathbb{R}^3). \tag{7.4}$$

Thus the only thing to complete the proof of Theorem 2 is to show the strong convergence of Z_δ in $L^1((0,T)\times\Omega)$ which is actually equivalent to identifying $\overline{Z^\gamma} = Z^\gamma$.

7.2. Strong convergence of Z_δ

Recall that the cut-off functions T and T_k were introduced in (3.8)–(3.9).

7.2.1. Effective viscous ux

As in Section 6, we need the following auxiliary result:

Lemma 11. *Let ϱ_δ, Z_δ, $\boldsymbol{u}_\delta$ be the sequence of approximate solutions constructed by means of Proposition 1. Then*

$$\lim_{\delta\to 0^+}\int_0^T \psi \int_\Omega \phi\big(Z_\delta^\gamma - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}_\delta\big)T_k(Z_\delta)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \psi \int_\Omega \phi\big(\overline{Z^\gamma} - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}\big)\overline{T_k(Z_\delta)}\,\mathrm{d}x\,\mathrm{d}t \tag{7.5}$$

for any $\psi \in C_c^\infty((0,T))$ and $\phi \in C_c^\infty(\Omega)$, passing to subsequences, if necessary.

Proof. Recall that we have for $\delta>0$ the renormalized form of equation (4.1b)

$$\partial_t(T_k(Z_\delta)) + \operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta) + (Z_\delta T_k'(Z_\delta) - T_k(Z_\delta))\operatorname{div}\boldsymbol{u}_\delta = 0, \tag{7.6}$$

however, for the limit we only have

$$\partial_t(\overline{T_k(Z)}) + \operatorname{div}(\overline{T_k(Z)}\boldsymbol{u}) + \overline{(ZT_k'(Z) - T_k(Z))\operatorname{div}\boldsymbol{u}} = 0, \tag{7.7}$$

both in the sense of distributions.

We use as the test function in the approximated momentum equation (4.1c) the function

$$\boldsymbol{\varphi}_\delta = \psi\phi\nabla\Delta^{-1}[1_\Omega T_k(Z_\delta)] = \psi\phi\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)],\ k \in \mathbb{N},$$

and for the limit equation (7.4) the test function

$$\boldsymbol{\varphi} = \psi\phi\nabla\Delta^{-1}[1_\Omega \overline{T_k(Z)}] = \psi\phi\boldsymbol{\mathcal{A}}[1_\Omega \overline{T_k(Z)}],\ k \in \mathbb{N}.$$

Here, $\psi \in C_c^\infty(0,\infty)$ and $\phi \in C_c^\infty(\Omega)$, for the definitio of $\boldsymbol{\mathcal{A}}$ see Section 3. Note that thanks to properties of ψ and ϕ we indeed extend our domain from Ω onto the whole space $\mathbb{R}^3$. It allows then to work with $\boldsymbol{\mathcal{A}}$ define in terms of Fourier multipliers.

We get

$$\begin{aligned}
&\lim_{\delta\to 0^+}\int_0^T \psi \int_\Omega \left(\phi Z_\delta^\gamma T_k(Z_\delta) + Z_\delta^\gamma \nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega T_k(\varrho_\delta)]\right) \mathrm{d}x\,\mathrm{d}t \qquad (7.8)\\
&\quad - \lim_{\delta\to 0^+}\int_0^T \psi \int_\Omega \phi\Big(\mu\nabla\boldsymbol{u}_\delta : \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}_\delta T_k(Z_\delta)\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \lim_{\delta\to 0^+}\int_0^T \psi \int_\Omega \Big(\mu\nabla\boldsymbol{u}_\delta\cdot\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}_\delta\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)]\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T \psi\int_\Omega \left(\phi\overline{Z^\gamma}\ \overline{T_k(Z)} - \overline{Z^\gamma}\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}]\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_0^T \psi\int_\Omega \phi\Big(\mu\nabla\boldsymbol{u} : \boldsymbol{\mathcal{R}}[1_\Omega\overline{T_k(Z)}] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}\overline{T_k(Z)}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_0^T \psi\int_\Omega \Big(\mu\nabla\boldsymbol{u}\cdot\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}]\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \lim_{\delta\to 0^+}\int_0^T \psi\int_\Omega \Big(\phi\varrho_\delta\boldsymbol{u}_\delta\cdot\boldsymbol{\mathcal{A}}[\operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta) + (Z_\delta T_k'(Z_\delta) - T_k(Z_\delta))\operatorname{div}\boldsymbol{u}_\delta]\\
&\quad - \varrho_\delta(\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta) : \nabla\left(\phi\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)]\right)\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_0^T \psi\int_\Omega \Big(\phi\varrho\boldsymbol{u}\cdot\boldsymbol{\mathcal{A}}[\operatorname{div}(\overline{T_k(Z)}\boldsymbol{u}) + \overline{(ZT_k'(Z) - T_k(Z))\operatorname{div}\boldsymbol{u}}]\\
&\quad - \varrho(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\left(\phi\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}]\right)\Big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$- \lim_{\delta\to 0^+} \int_0^T \partial_t \psi \int_\Omega \phi \varrho_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\mathcal{A}}(T_k(Z_\delta))\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \partial_t \psi \int_\Omega \phi \varrho \boldsymbol{u} \cdot \boldsymbol{\mathcal{A}}[\overline{T_k(Z)}]\,\mathrm{d}x\,\mathrm{d}t.$$

We have

$$\begin{aligned}
\int_\Omega \phi \nabla \boldsymbol{u}_\delta : \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)]\,\mathrm{d}x &= \int_\Omega \phi \sum_{i,j=1}^3 \Big(\partial_{x_j} u_\delta^i \mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)]\Big)\,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j=1}^3 \Big(\partial_{x_j}(\phi u_\delta^i)\mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)]\Big)\,\mathrm{d}x \\
&\quad - \int_\Omega \sum_{i,j=1}^3 \Big(\partial_{x_j}\phi u_\delta^i \mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)]\Big)\,\mathrm{d}x \\
&= \int_\Omega \phi \operatorname{div} \boldsymbol{u}_\delta T_k(Z_\delta)\,\mathrm{d}x + \int_\Omega \nabla\phi \cdot \boldsymbol{u}_\delta T_k(Z_\delta)\mathrm{d}x \\
&\quad - \int_\Omega \sum_{i,j=1}^3 \Big(\partial_{x_j}\phi u_\delta^i \mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)]\Big)\,\mathrm{d}x.
\end{aligned}$$

Consequently, going back to (7.8) and dropping the compact terms, where we use

$$\mathcal{A}[1_\Omega T_k(\varrho_\delta)] \to \mathcal{A}[1_\Omega \overline{T_k(\varrho)}] \text{ in } C([0,T]\times\overline{\Omega}),$$

we obtain

$$\begin{aligned}
&\lim_{\delta\to 0^+} \int_0^T \psi \int_\Omega \phi\Big(Z_\delta^\gamma T_k(Z_\delta) - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}_\delta T_k(Z_\delta)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \psi \int_\Omega \phi\Big(\overline{Z^\gamma}\ \overline{T_k(Z)} - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}\overline{T_k(Z)}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\delta\to 0^+} \int_0^T \psi \int_\Omega \Big(\varrho_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\mathcal{A}}[\operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta)] - \varrho_\delta(\boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta) : \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)]\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \psi \int_\Omega \Big(\phi\varrho\boldsymbol{u} \cdot \boldsymbol{\mathcal{A}}[\operatorname{div}(\overline{T_k(Z)}\boldsymbol{u})] - \varrho(\boldsymbol{u}\otimes\boldsymbol{u}) : \boldsymbol{\mathcal{R}}[1_\Omega \overline{T_k(Z)}]\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{7.9}$$

Our goal is to show that the right-hand side of (7.9) vanishes. We write

$$\int_\Omega \phi\Big[\varrho_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\mathcal{A}}[1_\Omega \operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta)] - \varrho_\delta(\boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta) : \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)]\Big]\mathrm{d}x$$
$$= \int_\Omega \phi \boldsymbol{u}_\delta \cdot \Big[T_k(Z_\delta)\boldsymbol{\mathcal{A}}[\operatorname{div}(1_\Omega \varrho_\delta \boldsymbol{u}_\delta)] - \varrho_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)]\Big]\mathrm{d}x + l.o.t.,$$

where l.o.t. denotes lower order terms (with derivatives on ϕ) and appear due to the integration by parts in the firs term on the left-hand side. We consider the bilinear form

$$[\boldsymbol{v}, \boldsymbol{w}] = \sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{ij}[w^j] - w^i \mathcal{R}_{ij}[v^j]\Big),$$

where

$$\boldsymbol{v} = \boldsymbol{v}(Z) = (T_k(Z), T_k(Z), T_k(Z)), \qquad \boldsymbol{w} = \boldsymbol{w}(\varrho, \boldsymbol{u}) = \varrho \boldsymbol{u}.$$

We may write

$$\sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{ij}[w^j] - w^i \mathcal{R}_{ij}[v^j]\Big)$$
$$= \sum_{i,j=1}^{3} \Big((v^i - \mathcal{R}_{ij}[v^j])\mathcal{R}_{ij}[w^j] - (w^i - \mathcal{R}_{ij}[w^j])\mathcal{R}_{ij}[v^j]\Big) = \boldsymbol{U} \cdot \boldsymbol{V} - \boldsymbol{W} \cdot \boldsymbol{Z},$$

where

$$U^i = \sum_{j=1}^{3}(v^i - \mathcal{R}_{ij}[v^j]),\ W^i = \sum_{j=1}^{3}(w^i - \mathcal{R}_{ij}[w^j]),\ \operatorname{div}\boldsymbol{U} = \operatorname{div}\boldsymbol{W} = 0,$$

and

$$V^i = \partial_{x_i}\left(\sum_{j=1}^{3}\Delta^{-1}\partial_{x^j} w^j\right),\ Z^i = \partial_{x_i}\left(\sum_{j=1}^{3}\Delta^{-1}\partial_{x^j} v^j\right),\ i = 1, 2, 3.$$

Therefore we may apply the Div–Curl lemma (Lemma 9) and using

$$T_k(Z_\delta) \to \overline{T_k(Z)} \text{ in } C_{\text{weak}}([0, T]; L^q(\Omega)),\ 1 \le q < \infty,$$
$$\varrho_\delta \boldsymbol{u}_\delta \to \varrho \boldsymbol{u} \text{ in } C_{\text{weak}}([0, T]; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)),$$

we conclude that

$$\begin{gathered} T_k(Z_\delta)(t,\cdot)\boldsymbol{\mathcal{A}}[1_\Omega \operatorname{div}(\varrho_\delta \boldsymbol{u}_\delta)(t,\cdot)] - (\varrho_\delta \boldsymbol{u}_\delta)(t,\cdot)\cdot \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)(t,\cdot)] \\ \to \\ \overline{T_k(Z)}(t,\cdot)\boldsymbol{\mathcal{A}}[1_\Omega \operatorname{div}(\varrho \boldsymbol{u})(t,\cdot)] - (\varrho \boldsymbol{u})(t,\cdot)\cdot \boldsymbol{\mathcal{R}}[1_\Omega \overline{T_k(Z)}(t,\cdot)] \\ \text{weakly in } L^s(\Omega;\mathbb{R}^3) \text{ for all } t\in[0,T], \end{gathered} \tag{7.10}$$

with

$$s < \frac{2\gamma}{\gamma+1}.$$

Note that $s > \frac{6}{5}$ since $\gamma > \frac{3}{2}$ and thus the convergence in (7.10) takes place in the space

$$L^q(0,T;W^{-1,2}(\Omega)) \text{ for any } 1 \le q < \infty;$$

going back to (7.9), we have

$$\begin{aligned} &\lim_{\delta\to 0^+} \int_0^T \psi \int_\Omega \phi\Big(Z_\delta^\gamma T_k(Z_\delta) - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}_\delta T_k(Z_\delta)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \psi \int_\Omega \phi\Big(\overline{Z^\gamma}\ \overline{T_k(Z)} - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}\overline{T_k(Z)}\Big)\,\mathrm{d}x\,\mathrm{d}t. \quad \square \end{aligned} \tag{7.11}$$

Remark 3. Observe that an analogue of equality (7.6) holds also when we consider σ_δ instead of $T_k(Z_\delta)$, where σ_δ are uniformly essentially bounded and satisfy

$$\partial_t \sigma_\delta + \operatorname{div}(\sigma_\delta \boldsymbol{u}_\delta) = f_\delta$$

where f_δ are bounded in $L^2((0,T)\times\Omega)$ (see [11] and [14]). This generalization will be necessary in Section 8.

7.2.2. Oscillation defect measure and renormalized solutions

The main results of this part are essentially taken over from [3]:

Lemma 12. *There exists a constant c independent of k such that*

$$\limsup_{\delta\to 0^+} \|T_k(Z_\delta) - T_k(Z)\|_{L^{\gamma+1}((0,T)\times\Omega)} \le c \tag{7.12}$$

with c independent of $k \ge 1$.

Proof. One has

$$\limsup_{\delta\to 0^+}\int_0^T\int_\Omega \left(Z_\delta^\gamma T_k(Z_\delta) - \overline{Z^\gamma}\,\overline{T_k(Z)}\right)\,\mathrm{d}x\,\mathrm{d}t = \limsup_{\delta\to 0^+}\int_0^T\int_\Omega (Z_\delta^\gamma - Z^\gamma)(T_k(Z_\delta) - T_k(Z))\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^T\int_\Omega (\overline{Z^\gamma} - Z^\gamma)(T_k(Z) - \overline{T_k(Z)})\,\mathrm{d}x\,\mathrm{d}t \geq \limsup_{\delta\to 0^+}\int_0^T\int_\Omega (Z_\delta^\gamma - Z^\gamma)(T_k(Z_\delta) - T_k(Z))\,\mathrm{d}x\,\mathrm{d}t$$

$$\geq \limsup_{\delta\to 0^+}\int_0^T\int_\Omega |T_k(Z_\delta) - T_k(Z)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t, \tag{7.13}$$

as $Z \mapsto Z^\gamma$ is convex, T_k concave on $\mathbb{R}_+$, and

$$(z^\gamma - y^\gamma)(T_k(z) - T_k(y)) \geq |T_k(z) - T_k(y)|^{\gamma+1} \tag{7.14}$$

for all $z, y \geq 0$. Hence,

$$\limsup_{\delta\to 0^+}\int_0^T\int_\Omega |T_k(Z_\delta) - T_k(Z)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T\int_\Omega (\overline{Z^\gamma T_k(Z)} - \overline{Z^\gamma}\,\overline{T_k(Z)})\,\mathrm{d}x\,\mathrm{d}t. \tag{7.15}$$

On the other hand,

$$\limsup_{\delta\to 0^+}\int_0^T\int_\Omega \operatorname{div}\boldsymbol{u}_\delta\left(T_k(Z_\delta) - \overline{T_k(Z)}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \limsup_{\delta\to 0^+}\int_0^T\int_\Omega \left(T_k(Z_\delta) - T_k(Z) + T_k(Z) - \overline{T_k(Z)}\right)\operatorname{div}\boldsymbol{u}_\delta\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq 2\sup_{\delta>0}\|\operatorname{div}\boldsymbol{u}_\delta\|_{L^2((0,T)\times\Omega)}\limsup_{\delta\to 0^+}\|T_k(Z_\delta) - T_k(Z)\|_{L^2((0,T)\times\Omega)}. \tag{7.16}$$

Relations (7.14), (7.15) combined with Lemma 11 yield the desired conclusion. □

Using the result of Lemma 12 one has the following crucial assertion (see Lemma 7):

Lemma 13. *The limit functions* $(Z, \boldsymbol{u})$ *solve (1.7b) in the sense of renormalized solutions, i.e.,*

$$\partial_t b(Z) + \operatorname{div}(b(Z)\boldsymbol{u}) + ((b'(Z)Z - b(Z))\operatorname{div}\boldsymbol{u} = 0 \tag{7.17}$$

holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ *for any* $b \in C^1(\mathbb{R})$ *satisfying (2.18) provided* $(Z, \boldsymbol{u})$ *are extended by zero outside* Ω.

7.2.3. Strong convergence of the density

We are going to complete the proof of Theorem 2. To this end, we introduce a family of functions $(L_k)_{k\geq 1}$:

$$L_k(z) = z\int_1^z \frac{T_k(s)}{s^2}\,\mathrm{d}s.$$

Note that L_k is convex for any $k \geq 1$ and

$$ZL_k'(Z) - L_k(Z) = T_k(Z). \tag{7.18}$$

We can use the fact that $(Z_\delta, \boldsymbol{u}_\delta)$ are renormalized solutions of (4.1b) to deduce

$$\partial_t L_k(Z_\delta) + \operatorname{div}\big(L_k(Z_\delta)\boldsymbol{u}_\delta\big) + T_k(Z_\delta)\operatorname{div}\boldsymbol{u}_\delta = 0 \tag{7.19}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ with Z_δ, $\boldsymbol{u}_\delta$ extended by zero outside of Ω. Similarly, by virtue of (7.3) and Lemma 13 (as above, we may justify the use of $L_k(\cdot)$ by density argument)

$$\partial_t L_k(Z) + \operatorname{div}\big(L_k(Z)\boldsymbol{u}\big) + T_k(Z)\operatorname{div}\boldsymbol{u} = 0 \tag{7.20}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

In view of (7.19), we have

$$L_k(Z_\delta) \to \overline{L_k(Z)} \;\text{ in }\; C_w([0,T]; L^q(\Omega)) \tag{7.21}$$

for all $1 \leq q < \infty$. Hence (7.19) yields

$$\partial_t \overline{L_k(Z)} + \operatorname{div}(\overline{L_k(Z)}\boldsymbol{u}) + \overline{T_k(Z)\operatorname{div}\boldsymbol{u}} = 0 \tag{7.22}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$. Therefore, (7.20) and (7.22) imply

$$\int_\Omega \Big(\overline{L_k(Z(T))} - L_k(Z(T))\Big)\,\mathrm{d}x = \int_0^T\int_\Omega \Big(T_k(Z)\operatorname{div}\boldsymbol{u} - \overline{T_k(Z)\operatorname{div}\boldsymbol{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

Due to convexity of $L_k(\cdot)$ we have

$$\begin{aligned} 0 \leq &\int_0^T\int_\Omega \Big(T_k(Z)\operatorname{div}\boldsymbol{u} - \overline{T_k(Z)\operatorname{div}\boldsymbol{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_0^T\int_\Omega \Big(T_k(Z) - \overline{T_k(Z)}\Big)\operatorname{div}\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \Big(\overline{T_k(Z)}\operatorname{div}\boldsymbol{u} - \overline{T_k(Z)\operatorname{div}\boldsymbol{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Now, the effective viscous flu equality (7.5) and (7.15) imply

$$\frac{1}{2\mu+\lambda}\limsup_{\delta\to 0^+}\int_0^T\int_\Omega |T_k(Z_\delta)-T_k(Z)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega \left(\overline{T_k(Z)\operatorname{div}\boldsymbol{u}}-\overline{T_k(Z)}\operatorname{div}\boldsymbol{u}\right)\mathrm{d}x\,\mathrm{d}t;$$

whence

$$\frac{1}{2\mu+\lambda}\limsup_{\delta\to 0^+}\int_0^T\int_\Omega |T_k(Z_\delta)-T_k(Z)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega |T_k(Z)-\overline{T_k(Z)}||\operatorname{div}\boldsymbol{u}|\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\|T_k(Z)-\overline{T_k(Z)}\|_{L^1((0,T)\times\Omega)}^{\frac{\gamma-1}{2\gamma}}\|T_k(Z)-\overline{T_k(Z)}\|_{L^{\gamma+1}((0,T)\times\Omega)}^{\frac{\gamma+1}{2\gamma}}.$$

Recall that

$$\|T_k(Z)-\overline{T_k(Z)}\|_{L^1((0,T)\times\Omega)}\le\|T_k(Z)-Z\|_{L^1((0,T)\times\Omega)}+\|\overline{T_k(Z)}-Z\|_{L^1((0,T)\times\Omega)},$$

yielding

$$\lim_{k\to\infty}\|T_k(Z)-\overline{T_k(Z)}\|_{L^1((0,T)\times\Omega)}=0.$$

As

$$\sup_{k\ge 1}\limsup_{\delta\to 0^+}\|T_k(Z_\delta)-T_k(Z)\|_{L^{\gamma+1}((0,T)\times\Omega)}<+\infty,$$

we also have

$$\lim_{k\to\infty}\limsup_{\delta\to 0^+}\|T_k(Z_\delta)-T_k(Z)\|_{L^{\gamma+1}((0,T)\times\Omega)}=0.$$

Therefore, one verifie that

$$Z_\delta\to Z \text{ strongly in } L^q((0,T)\times\Omega)$$

for any $q<\gamma+\theta$. The proof of Theorem 2 is finished

8. Proof of equivalent formulations

From Theorem 2 it follows that for any $\gamma>\frac{3}{2}$ there exists a triple of functions

$$(\varrho,Z,\boldsymbol{u})\in L^\infty(0,T;L^\gamma(\Omega))\times L^\infty(0,T;L^\gamma(\Omega))\times L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3)) \tag{8.1}$$

satisfying equations (1.7) in the sense specifie in Definitio 2. However, in what follows, we will use the result only for $\gamma\ge\frac{9}{5}$.

Our aim will be to deduce from this the existence of $s\in L^\infty((0,T)\times\Omega)$ such that the pressure in the momentum equation equals $p=\varrho^\gamma\mathcal{T}(s)$ satisfying either equality (2.13) or the distributional formulation of (1.6b) with corresponding initial data in a similar way as suggested in Feireisl et al. [7].

8.1. The case $\gamma \geq \frac{9}{5}$

We firs present the main ideas of the proof which corresponds to the situation

$$\frac{Z_0}{\varrho_0} 1_{\{\varrho_0=0\}} = T(s_0)^{\frac{1}{\gamma}} 1_{\{\varrho_0=0\}} = 1.$$

Due to the construction we know that functions $(\varrho, Z, \boldsymbol{u})$ extended by zero outside of Ω fulfil equations (1.7a), (1.7b) in the sense of distributions on the whole $(0, T) \times \mathbb{R}^3$. Therefore, we may test both of these equations by $\xi_\eta(x - \cdot)$, where ξ_η is a standard mollifie . We obtain the following equations

$$\partial_t \varrho_\eta + \operatorname{div}(\varrho_\eta \boldsymbol{u}) = r_\eta^1, \tag{8.2}$$

$$\partial_t Z_\eta + \operatorname{div}(Z_\eta \boldsymbol{u}) = r_\eta^2, \tag{8.3}$$

satisfie a.e. in $(0, T) \times \mathbb{R}^3$, where by a_η we denoted $a * \xi_\eta$. From the Friedrichs lemma (see e.g. [13]) we know that r_η^1, r_η^2 converge to 0 strongly in $L^1((0, T) \times \mathbb{R}^3)$ as $\eta \to 0^+$ (the strong convergence of r_η^1 requires the stronger assumption on γ). Now we multiply the firs equation by $-\frac{(Z_\eta+\lambda)}{(\varrho_\eta+\lambda)^2}$ and the second by $\frac{1}{\varrho_\eta+\lambda}$ with $\lambda > 0$, respectively. Note that for η fi ed $\partial_t \varrho_\eta$ and $\partial_t Z_\eta$ belong to $L^\infty(0, T; C_c^\infty(\mathbb{R}^3))$, so these are sufficientl regular test functions. After some manipulations, we obtain the following equation

$$\partial_t \left(\frac{Z_\eta + \lambda}{\varrho_\eta + \lambda}\right) + \operatorname{div}\left[\left(\frac{Z_\eta + \lambda}{\varrho_\eta + \lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z_\eta + \lambda)\varrho_\eta}{(\varrho_\eta + \lambda)^2} + \frac{\lambda}{\varrho_\eta + \lambda}\right]\operatorname{div} \boldsymbol{u}$$
$$= -r_\eta^1 \frac{Z_\eta + \lambda}{(\varrho_\eta + \lambda)^2} + r_\eta^2 \frac{1}{\varrho_\eta + \lambda}$$

satisfie a.e. in $(0, T) \times \mathbb{R}^3$. Note that $Z_\eta(t, x) = \int_{\mathbb{R}^3} Z(t, y)\xi_\eta(x - y)\mathrm{d}y \leq c^\star \int_{\mathbb{R}^3} \varrho(t, y)\xi_\eta(x - y)\mathrm{d}y = c^\star \varrho_\eta$, therefore

$$\frac{Z_\eta + \lambda}{\varrho_\eta + \lambda} \leq \frac{c^\star \varrho_\eta + \lambda}{\varrho_\eta + \lambda} \leq \max\{1, c^\star\}, \qquad \frac{1}{\varrho_\eta + \lambda} \leq \frac{1}{\lambda}.$$

So, for λ fi ed, we may use the strong convergence of $\varrho_\eta \to \varrho$, $Z_\eta \to Z$ and the dominated convergence theorem to let $\eta \to 0$ and to obtain the following equation

$$\partial_t \left(\frac{Z + \lambda}{\varrho + \lambda}\right) + \operatorname{div}\left[\left(\frac{Z + \lambda}{\varrho + \lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z + \lambda)\varrho}{(\varrho + \lambda)^2} + \frac{\lambda}{\varrho + \lambda}\right]\operatorname{div} \boldsymbol{u} = 0$$

which is satisfie in the sense of distributions on $(0, T) \times \mathbb{R}^3$. Before we pass to the limit with $\lambda \to 0^+$ note that we may distinguish two situations

- for $\varrho = 0$ we have $Z = 0$ and therefore $\frac{Z+\lambda}{\varrho+\lambda} = 1$, while $\frac{(Z+\lambda)\varrho}{(\varrho+\lambda)^2} + \frac{\lambda}{\varrho+\lambda} = 1$,

- for $\varrho > 0$ we have $\frac{Z+\lambda}{\varrho+\lambda} \leq \max\{1, c^\star\}$, $\varrho + \lambda \to \varrho$, $Z + \lambda \to Z$ strongly in $L^\infty(0;T; L^2_{\mathrm{loc}}(\mathbb{R}^3))$, therefore $\frac{Z+\lambda}{\varrho+\lambda} \to \frac{Z}{\varrho}$ strongly in $L^\infty((0,T)\times\mathbb{R}^3)$ and so $\frac{(Z+\lambda)\varrho}{(\varrho+\lambda)^2} + \frac{\lambda}{\varrho+\lambda} \to \frac{Z}{\varrho}$.

Recall that this construction corresponds to the choice $\zeta_0 = 1$ in (2.21). In the more general case, for $\mathcal{T}(s_0)^{\frac{1}{\gamma}} = A_0$ we would have to replace λ in the numerator by $A_0\lambda$.

The case of non-constant $A_0 1_{\varrho_0=0} = \mathcal{T}(s_0)^{\frac{1}{\gamma}} 1_{\varrho_0=0} \in L^\infty(\Omega)$ demands a bit more technical treatment. Let A be any solution of the transport equation (1.6b) with the initial data A_0. Such solution can be found using smoothing of $\boldsymbol{u}$ and solving the transport equation by the method of trajectories. Along with (8.2) we also test the transport equation for A by the same family of mollifier obtaining

$$\partial_t A_\eta + \boldsymbol{u}\nabla A_\eta = r^3_\eta$$

with $r^3_\eta \to 0$ in $L^1((0,T)\times\mathbb{R}^3)$ as $\eta \to 0^+$. In combination with the continuity equation we obtain

$$\partial_t(Z_\eta + \lambda A_\eta) + \operatorname{div}((Z_\eta + \lambda A_\eta)\boldsymbol{u}) = r^2_\eta + \lambda(r^3_\eta + A_\eta \operatorname{div}\boldsymbol{u}). \tag{8.4}$$

We multiply the last equality by $\frac{1}{\varrho_\eta+\lambda}$ and mimicking the previous approach we obtain

$$\begin{aligned}\partial_t\left(\frac{Z_\eta+\lambda A_\eta}{\varrho_\eta+\lambda}\right) &+ \operatorname{div}\left[\left(\frac{Z_\eta+\lambda A_\eta}{\varrho_\eta+\lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z_\eta+\lambda A_\eta)\varrho_\eta}{(\varrho_\eta+\lambda)^2} + \frac{\lambda A_\eta}{\varrho_\eta+\lambda}\right]\operatorname{div}\boldsymbol{u}\\ &= -r^1_\eta\frac{Z_\eta+\lambda A_\eta}{(\varrho_\eta+\lambda)^2} + r^2_\eta\frac{1}{\varrho_\eta+\lambda} + r^3_\eta\frac{\lambda}{\varrho_\eta+\lambda}.\end{aligned}$$

Next, we let $\eta \to 0^+$ and get

$$\partial_t\left(\frac{Z+\lambda A}{\varrho+\lambda}\right) + \operatorname{div}\left[\left(\frac{Z+\lambda A}{\varrho+\lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z+\lambda A)\varrho}{(\varrho+\lambda)^2} + \frac{\lambda A}{\varrho+\lambda}\right]\operatorname{div}\boldsymbol{u} = 0.$$

Let us denote $\theta = \frac{Z}{\varrho}$ for $\varrho > 0$ and $\theta = A$ for $\varrho = 0$. Observe that $c_* \leq \theta \leq c^*$ almost everywhere in $(0,T)\times\Omega$. Once again, we use the uniform boundedness of $\frac{Z+A\lambda}{\varrho+\lambda}$ and send $\lambda \to 0^+$ obtaining

$$\partial_t\theta + \operatorname{div}(\theta\boldsymbol{u}) - \theta\operatorname{div}\boldsymbol{u} = 0$$

or, equivalently,

$$\partial_t\theta + \boldsymbol{u}\cdot\nabla\theta = 0 \tag{8.5}$$

in the sense of distributions on $(0,T)\times\mathbb{R}^3$. The initial condition A_0 is attained in the sense of weak solutions for the transport equation. In addition, we can renormalize this equation, using any $G \in C^1(\mathbb{R})$ and deduce that

$$\partial_t G(\theta) + \boldsymbol{u}\cdot\nabla G(\theta) = 0 \tag{8.6}$$

is also satisfie in the sense of distributions on $(0, T) \times \mathbb{R}^3$. Taking for example $G(\theta) = \mathcal{T}^{-1}(\theta^\gamma)$, we obtain equation for s

$$\partial_t s + \boldsymbol{u} \cdot \nabla s = 0, \tag{8.7}$$

and, for $G(\theta) = B(\mathcal{T}^{-1}(\theta^\gamma))$, also its renormalized version

$$\partial_t B(s) + \boldsymbol{u} \cdot \nabla B(s) = 0 \tag{8.8}$$

satisfie in the sense of distributions on $(0, T) \times \mathbb{R}^3$ for any $B \in C^1(\mathbb{R})$.

In order to obtain the weak solution to problem (1.1b) we need to test equation (8.7) by ϱ. This is, however, not allowed due to low regularity of ϱ. Instead we will use $\varphi\varrho_\eta$, where $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^3)$ and ϱ_η satisfie (8.2). Here we essentially use the fact that $\varrho \in L^2((0, T) \times \Omega)$, hence this step cannot be repeated for γ less than $\frac{9}{5}$. Then we also multiply (8.2) by φs and sum up the obtained expressions to deduce

$$\int_0^T \int_{\mathbb{R}^3} \varrho_\eta s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} (\varrho_\eta s \boldsymbol{u}) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \int_\Omega r_\eta^1 s \varphi \, \mathrm{d}x \, \mathrm{d}t.$$

Having this formulation we pass to the limit with $\eta \to 0^+$, note that the term on the r.h.s. vanishes and therefore we obtain

$$\int_0^T \int_{\mathbb{R}^3} \varrho s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} (\varrho s \boldsymbol{u}) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{8.9}$$

Note that if we start from (8.8), we can also get

$$\int_0^T \int_{\mathbb{R}^3} \varrho B(s) \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} (\varrho B(s) \boldsymbol{u}) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{8.10}$$

for any $B \in C^1(\mathbb{R})$.

Thus we have almost our formulation from Definitio 1, except the initial condition. Indeed, for the moment we only know that equation $\partial_t(\varrho s) + \operatorname{div}(\varrho s \boldsymbol{u}) = 0$ is satisfie in the sense of distributions on $(0, T) \times \mathbb{R}^3$. Moreover, from the $L^\infty((0, T) \times \mathbb{R}^3)$ bound on s and the above equation, we deduce using Arzelà–Ascoli theorem that $\varrho s \in C_w([0, T]; L^\gamma(\Omega))$.

To recover the initial and the terminal condition, we need to use a test function φ from the space $C^1([0, T] \times \overline{\Omega})$ instead of $C_c^\infty((0, T) \times \Omega)$. To this purpose we defin the following function

$$\varphi_\tau(t, x) = \begin{cases} \frac{t}{\tau}\varphi(\tau, x) & \text{for} \quad t \le \tau \\ \varphi(t, x) & \text{for} \quad \tau \le t \le T - \tau, \\ \frac{T-t}{\tau}\varphi(T-\tau, x) & \text{for} \quad T - \tau \le t \end{cases}$$

for $\varphi \in C^1([0,T] \times \overline{\Omega})$. Note that φ_τ is an admissible test function for (8.10), we can write

$$\int_\tau^T \int_\Omega \varrho s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega (\varrho s \boldsymbol{u}) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t$$
$$= -\frac{1}{\tau} \int_0^\tau \int_\Omega \varrho s \varphi(\tau, x) \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\tau} \int_{T-\tau}^T \int_\Omega \varrho s \varphi(T-\tau, x) \, \mathrm{d}x \, \mathrm{d}t. \tag{8.11}$$

We represent function $\varphi(t,x)$ as $\varphi(t,x) = \psi(t)\zeta(x)$ (or approximate by such sums), where $\psi \in C_c^\infty((0,T))$, $\zeta \in C_c^\infty(\overline{\Omega})$, then the r.h.s. of (8.11) equals

$$-\frac{1}{\tau} \int_0^\tau \int_\Omega \varrho s \varphi(\tau, x) \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\tau} \int_{T-\tau}^T \int_\Omega \varrho s \varphi(T-\tau, x) \, \mathrm{d}x \, \mathrm{d}t$$
$$= -\frac{\psi(\tau)}{\tau} \int_0^\tau \int_\Omega \varrho s \zeta(x) \, \mathrm{d}x \, \mathrm{d}t + \frac{\psi(T-\tau)}{\tau} \int_{T-\tau}^T \int_\Omega \varrho s \zeta(x) \, \mathrm{d}x \, \mathrm{d}t,$$

and by the weak continuity of ϱs, letting $\tau \to 0$, we conclude that

$$\int_0^T \int_\Omega \varrho s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega (\varrho s \boldsymbol{u}) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{8.12}$$
$$= -\int_\Omega (\varrho s)(0,\cdot)\varphi(0,\cdot)\mathrm{d}x + \int_\Omega (\varrho s)(T,\cdot)\varphi(T,\cdot)\mathrm{d}x$$
$$= -\int_\Omega S_0(\cdot)\varphi(0,\cdot)\mathrm{d}x + \int_\Omega (\varrho s)(T,\cdot)\varphi(T,\cdot)\mathrm{d}x$$

and so the statement of Theorem 1 is proven. Similarly we may get the initial condition for $s(t,\cdot)$.

8.2. *The case $\gamma > \frac{3}{2}$*

The case of general γ has to be treated differently, due to the lack of $L^2((0,T) \times \Omega)$ estimate on ϱ and Z. The latter is necessary to apply the DiPerna–Lions technique of renormalization of the transport equation [1]. In the general case, we have to use more subtle technique developed by Feireisl, see e.g. [4] and used recently in [11] to study stability of solutions to system (1.6). In this section we will extend the stability result and prove existence of solutions to system (1.6) by giving a suitable sequence of approximative problems.

As a starting point for the further analysis we take system (4.1) with $\beta > \max\{\gamma, 4\}$ and initial data $Z_{0,\delta} = \frac{\rho_{0,\delta}}{\zeta_{0,\delta}}$, with $\zeta_{0,\delta}$ satisfying (2.21). At this stage, we are able to repeat the procedure described in the previous section in order to recover equation (8.5) for θ_δ and its renormalized version (8.6) in the sense of distributions on $(0,T)\times\Omega$. Moreover, θ_δ, θ_δ^{-1} are bounded in $L^\infty((0,T)\times\Omega)$ uniformly with respect to δ. Thus, our system

$$\partial_t\varrho_\delta+\operatorname{div}(\varrho_\delta\boldsymbol{u}_\delta)=0, \tag{8.13a}$$

$$\partial_t B(\theta_\delta)+\boldsymbol{u}_\delta\cdot\nabla B(\theta_\delta)=0, \tag{8.13b}$$

$$\partial_t(\varrho_\delta\boldsymbol{u}_\delta)+\operatorname{div}(\varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta)+\nabla(\varrho_\delta\theta_\delta)^\gamma+\delta\nabla(\varrho_\delta\theta_\delta)^\beta=\operatorname{div}\mathbb{S}(\nabla\boldsymbol{u}_\delta), \tag{8.13c}$$

where $\theta_\delta=\frac{Z_\delta}{\varrho_\delta}$, is satisfie in the sense of distributions on $(0,T)\times\Omega$. Observe that ϱ_δ belongs (not necessarily uniformly with respect to δ) to $L^\beta((0,T)\times\Omega)$ for each $\delta>0$. At this stage we can use the stability result given by Theorem 3.1 in [11] and finis the proof of Theorem 3.

For the sake of completeness, we will provide the limit process $\delta\to0^+$ following the arguments from [11]. We take $\zeta_\delta=\theta_\delta^{-1}$ and denote ζ the weak-$\star$ limit of ζ_δ (or its subsequence) in $L^\infty((0,T)\times\Omega)$. For any $\delta>0$ the pair $(\zeta_\delta,\boldsymbol{u}_\delta)$ satisfie the transport equation in the weak sense (see Definitio 4) along with the initial data $\zeta_{0,\delta}=\frac{Z_{0,\delta}}{\varrho_{0,\delta}}$. As we know from Section 7, sequence $Z_\delta=\frac{\varrho_\delta}{\zeta_\delta}$ (or its subsequence) converges strongly in $L^q((0,T)\times\Omega)$ to Z for any $q<\gamma+\theta$. Hence for the same q we have

$$\varrho_\delta=Z_\delta\zeta_\delta\to Z\zeta \text{ weakly in } L^q((0,T)\times\Omega).$$

Therefore ζ, ϱ and $\boldsymbol{u}$ satisfy in the weak sense

$$\partial_t\varrho+\operatorname{div}(\varrho\boldsymbol{u})=0, \tag{8.14a}$$

$$\partial_t(\varrho\boldsymbol{u})+\operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u})+\nabla\left(\frac{\varrho}{\zeta}\right)^\gamma=\operatorname{div}\mathbb{S}(\nabla\boldsymbol{u}). \tag{8.14b}$$

The next step is to show that the pair $(\zeta,\boldsymbol{u})$ satisfie the transport equation

$$\partial_t\zeta+\boldsymbol{u}\cdot\nabla\zeta=0 \tag{8.14c}$$

in the weak sense. We apply the Div–Curl lemma (Lemma 9) with

$$\boldsymbol{U}_\delta=(\zeta_\delta,\zeta_\delta\boldsymbol{u}_\delta),\quad \boldsymbol{V}_\delta=(\boldsymbol{u}_\delta^j,0,0,0),$$

where $j\in\{1,2,3\}$. We know that $\operatorname{div}\boldsymbol{U}_\delta$ and $\operatorname{curl}\boldsymbol{V}_\delta$ are bounded in $L^2((0,T)\times\Omega)$, hence precompact in $W^{-1,2}((0,T)\times\Omega)$. Therefore we obtain $\zeta_\delta\boldsymbol{u}_\delta\to\zeta\boldsymbol{u}$ weakly in $L^2((0,T)\times\Omega,\mathbb{R}^3)$. Due to the strong convergence of the pressure terms Z_δ^γ we get by the means of Lemma 11 (and Remark 3)

$$\zeta_\delta\operatorname{div}\boldsymbol{u}_\delta\to\zeta\operatorname{div}\boldsymbol{u} \text{ weakly in } L^2((0,T)\times\Omega).$$

Therefore (8.14c) is satisfie in the weak sense and due to the boundedness of ζ it is also a renormalized solution. The proof of Theorem 3 is complete.

Acknowledgments

The work of D.M. and A.N. has been partially supported by the MODTERCOM project within the APEX program of the Provence-Alpes-Côte d'Azur region. The work of P.B.M. and E.Z. has been partly supported by the National Science Centre grant 2014/14/M/ST1/00108 (Harmonia). M.M. acknowledges the support of the GACR (Czech Science Foundation) project 13-00522S in the framework of RVO: 67985840. The work of M.P. was partly supported by the GACR (Czech Science Foundation) project 16-03230S.

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IMA Journal of Numerical Analysis (2016) **36**, 543–592
doi:10.1093/imanum/drv028
Advance Access publication on June 19, 2015

Error estimates for a numerical approximation to the compressible barotropic Navier–Stokes equations

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[Received on 11 November 2014; revised on 19 April 2015]

We present here a general method based on the investigation of the relative energy of the system that provides an unconditional error estimate for the approximate solution of the barotropic Navier–Stokes equations obtained by time and space discretization. We use this methodology to derive an error estimate for a specifi DG/finit element scheme for which the convergence was proved in Karper (2013, *Numer. Math.*, **125**, 441–510).

Keywords: compressible f uids; Navier–Stokes equations; relative energy; error estimates; f nite element methods; f nite volume methods.

1. Introduction

The aim of this paper is to derive an error estimate for approximate solutions of the compressible barotropic Navier–Stokes equations obtained by a discretization scheme. These equations are posed on the time–space domain $Q_T = (0,T) \times \Omega$, where Ω is a bounded polyhedral domain of $\mathbb{R}^d$, $d = 2, 3$ and $T > 0$, and read

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} - (\mu + \lambda)\nabla \operatorname{div} \boldsymbol{u} + \nabla_x p(\varrho) = \boldsymbol{0}, \tag{1.1b}$$

supplemented with the initial conditions

$$\varrho(0,x) = \varrho_0(x), \ \varrho \boldsymbol{u}(0,x) = \varrho_0 \boldsymbol{u}_0, \tag{1.2}$$

where ϱ_0 and $\boldsymbol{u}_0$ are given functions from Ω to $\mathbb{R}_+$ and $\mathbb{R}^d$, respectively, and boundary conditions

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = 0. \tag{1.3}$$

In the above equations, the unknown functions are the scalar density fiel $\varrho(t,x) \geqslant 0$ and vector velocity fiel $\boldsymbol{u} = (u_1, \ldots, u_d)(t,x)$, where $t \in (0,T)$ denotes the time and $x \in \Omega$ is the space variable. The viscosity coefficient μ and λ are such that

$$\mu > 0, \quad \lambda + \frac{2}{d}\mu \geqslant 0. \tag{1.4}$$

The pressure p is given by an equation of state, that is, a function of density which satisfie

$$p \in C([0,\infty)) \cap C^1(0,\infty), \quad p(0) = 0, \quad p'(\varrho) > 0. \tag{1.5}$$

In addition to (1.5), in the error analysis, we shall need to prescribe the asymptotic behaviour of the pressure at large densities:

$$\lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \quad \text{with some } \gamma \geqslant 1; \tag{1.6}$$

furthermore, if $\gamma < 2$ in (1.6), we need the additional condition (for small densities),

$$\liminf_{\varrho\to 0} \frac{p'(\varrho)}{\varrho^{\alpha+1}} = p_0 > 0 \quad \text{with some } \alpha \leqslant 0. \tag{1.7}$$

The main underlying idea of this paper is to derive the error estimates for approximate solutions of problem (1.1–1.3) obtained by time and space discretization by using the discrete version of the *relative energy method* introduced on the continuous level in Feireisl *et al.* (2011, 2012) and Feireisl & Novotný (2012). In spite of the fact that the relative energy method looks at the f rst glance pretty much similar to the widely used *relative entropy method* (and both approaches translate the same thermodynamic stability conditions), they are very different in appearance and formulation and may provide different results. The notions of relative entropy and relative entropy inequality were firs introduced by Dafermos (1979) in the context of systems of conservation laws and in particular for the compressible Euler equations. The relative energy functional was suggested and successfully used for the investigation of the stability of weak solutions to the equations of viscous compressible and heat-conducting fluid in Feireisl & Novotný (2012). In contrast with the relative entropy of Dafermos, for the viscous and heat conducting fluids the relative energy approach is able to provide the structural stability of weak solutions, while the relative entropy approach fails in this case.

Both functionals coincide (modulo a change of variables) in the case of (viscous) compressible flow in the barotropic regime. The relative energy functional and the intrinsic version of the relative energy inequality have been recently employed to obtain several stability results for the weak solutions to these equations, including the weak–strong uniqueness principle; see Feireisl *et al.* (2011, 2012). Note that particular versions of the relative entropy inequality with particular specifi test functions had been previously derived in the context of low Mach number limits; see, e.g. Masmoudi (2001) and Wang *et al.* (2010).

The discrete version of the Dafermos relative entropy was employed in the nonviscous case to derive an error estimate for the numerical approximation to a hyperbolic system of conservation laws and, in particular, to the compressible Euler equations (Cancès *et al.*, 2013). In this latter paper, the authors assume an L^∞-bound for the discrete solution, which is uniform with respect to the size of the space and time discretization (usually called a stability hypothesis), that is not provided by the discrete equations. The same method with the same severe hypotheses has been used in Yovanovic (2007) to treat the compressible Navier–Stokes equations. The error analysis in the present paper relies on the theoretical background introduced in Feireisl *et al.* (2012) and yields an *unconditional result*; in particular, *we do not need any assumed bound* on the solution to get the error estimate.

The mathematical analysis of numerical schemes for the discretization of the steady and/or nonsteady compressible Navier–Stokes and/or compressible Stokes equations has been the object of some recent works. The convergence of the discrete solutions to the weak solutions of the compressible stationary Stokes was shown for a finit volume–nonconforming P1 finit element (Gallouët *et al.*, 2009; Eymard *et al.*, 2010a,b) and for the well-known Marker and Cell (MAC) scheme which was introduced in Harlow & Welsh (1965) and is widely used in computational flui dynamics (see, e.g. Li & Sun, 2014). The unsteady Stokes problem was also discretized by some other discretization schemes on a reformulation of the problem, which were proved to be convergent (Karlsen & Karper, 2010, 2011, 2012). The unsteady barotropic Navier–Stokes equations were recently investigated in Karper (2013) in the case $\gamma > 3$ (there is a real difficult in the realistic case $\gamma \leqslant 3$ arising from the treatment of the nonlinear convective term). However, in these works, the rate of convergence is not provided; in fact, to the best of our knowledge, no error analysis has yet been performed for any of the numerical schemes that have been designed for the compressible Navier–Stokes equations, in spite of its great importance for the numerical analysis of the equations and for the mathematical simulations of compressible flui flows We present here a general technique to obtain an error analysis and apply it to one of the available numerical schemes. To the best of our knowledge, this is the f rst result of this type in the mathematical literature on the subject.

To achieve the goal, we systematically use the relative energy method on the discrete level. From this point of view, this paper is as valuable for the introduced methodology as for the result itself. Here, we apply the method to the scheme of Karper (2013). In spite of the fact that this latter scheme is not used in practice (see, e.g. Kheriji *et al.*, 2013 for related schemes used in industrial codes), we begin the error analysis with the scheme of Karper (2013) because of its readily available convergence proof. In fact, we aim to use this approach to investigate the numerical errors of less academic numerical schemes, such as the finit volume–nonconforming P1 finit element (Gallouët et al., 2008; Gastaldo *et al.*, 2010, 2011; Kheriji *et al.*, 2013) or the MAC scheme (Babik *et al.*, 2011; Herbin *et al.*, 2014).

The paper is organized as follows. After recalling the fundamental setting of the problem and the relative energy inequality in the continuous case in Section 2, we proceed in Section 3 to the discretization: we introduce the discrete functional spaces and the definitio of the numerical scheme, and state the main result of the paper, that is, the error estimate formulated in Theorem 3.2. The remaining sections are devoted to the proof of Theorem 3.2.

- In Section 4, we recall the existence theorem for the numerical scheme (Proposition 3.1) and derive estimates provided by the scheme.
- In Section 5, we derive the discrete intrinsic version of the relative energy inequality for the solutions of the numerical scheme (see Theorem 5.1).
- The relative energy inequality is transformed into a more convenient form in Section 6; see Lemma 6.1.
- Finally, in Section 7, we investigate the form of the discrete relative energy inequality with the test function being a strong solution to the original problem. This investigation is formulated in Lemma 7.1 and finall leads to a Gronwall-type estimate formulated in Lemma 8.1. The latter yields the error estimates and completes the proof of the main result.

Fundamental properties of the discrete functional spaces needed throughout the paper are reported in the Appendix. Some of them (especially those referring to the L^p setting, $p \neq 2$, that are not currently available in the mathematical literature) are proved. Section 8 is therefore of independent interest.

2. The continuous problem

The aim of this section is to recall some fundamental notions and results. We begin with the definitio of weak solutions to problem (1.1–1.3).

DEFINITION 2.1 (Weak solutions) Let $\varrho_0 : \Omega \to [0, +\infty)$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^d$ with finit energy $E_0 = \int_\Omega (\frac{1}{2}\varrho_0|\boldsymbol{u}_0|^2 + H(\varrho_0))\,\mathrm{d}x$ and finit mass $0 < M_0 = \int_\Omega \varrho_0\,\mathrm{d}x$. We shall say that the pair $(\varrho, \boldsymbol{u})$ is a weak solution to the problem (1.1–1.3) emanating from the initial data $(\varrho_0, \boldsymbol{u}_0)$ if the following conditions are satisfied

(a) $\varrho \in L^\infty(0, T; L^1(\Omega))$, $\varrho \geqslant 0$ a.e. in $(0, T)$, and $\boldsymbol{u} \in L^2(0, T; W_0^{1,2}(\Omega))$;

(b) $\varrho \in C_{\mathrm{weak}}([0, T]; L^1(\Omega))$, and the continuity equation (1.1a) is satisfie in the following weak sense:

$$\int_\Omega \varrho\varphi\,\mathrm{d}x\Big|_0^\tau = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \quad \forall\tau \in [0, T],\ \forall\varphi \in C_c^\infty([0, T] \times \bar{\Omega}); \tag{2.1}$$

(c) $\varrho\boldsymbol{u} \in C_{\mathrm{weak}}([0, T]; L^1(\Omega))$, and the momentum equation (1.1b) is satisfie in the weak sense,

$$\begin{aligned}\int_\Omega \varrho\boldsymbol{u}\cdot\varphi\,\mathrm{d}x\Big|_0^\tau &= \int_0^\tau \int_\Omega (\varrho\boldsymbol{u}\cdot\partial_t\varphi + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\varphi + p(\varrho)\mathrm{div}\,\varphi)\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_0^\tau \int_\Omega (\mu\nabla\boldsymbol{u} : \nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t + (\mu+\lambda)\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\varphi)\,\mathrm{d}x\,\mathrm{d}t\\ &\forall\tau \in [0, T],\ \forall\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3);\end{aligned} \tag{2.2}$$

(d) the following energy inequality is satisfie

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + H(\varrho)\right)\mathrm{d}x\Big|_0^\tau\\ &+ \int_0^\tau \int_\Omega (\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)|\mathrm{div}\,\boldsymbol{u}|^2)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0 \quad \text{for a.a.} \tau \in (0, T),\end{aligned} \tag{2.3}$$

$$\text{with } H(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z. \tag{2.4}$$

Here and hereafter the symbol $\int_\Omega g\,\mathrm{d}x|_0^\tau$ is meant for $\int_\Omega g(\tau, x)\,\mathrm{d}x - \int_\Omega g_0(x)\,\mathrm{d}x$.

In the above definition we tacitly assume that all the integrals in the formulas (2.1–2.3) are define and we recall that $C_{\mathrm{weak}}([0, T]; L^1(\Omega))$ is the space of functions of $L^\infty([0, T]; L^1(\Omega))$ which are continuous for the weak topology.

We note that the function $\varrho \mapsto H(\varrho)$ is a solution of the ordinary differential equation $\varrho H'(\varrho) - H(\varrho) = p(\varrho)$ with the constant of integration fixe such that $H(1) = 0$.

Note that the existence of weak solutions emanating from the finit energy initial data are well known on bounded Lipschitz domains under assumptions (1.5) and (1.6) provided $\gamma > d/(d-1)$; see Lions (1998) for 'large' values of γ, Feireisl *et al.* (2001) for $\gamma > d/(d-1)$.

Let us now introduce the notion of relative energy. We f rst introduce the function

$$\begin{aligned} E:\ &[0,\infty)\times(0,\infty)\to\mathbb{R},\\ &(\varrho,r)\mapsto E(\varrho|r)=H(\varrho)-H'(r)(\varrho-r)-H(r), \end{aligned} \tag{2.5}$$

where H is define by (2.4). Owing to the monotonicity hypothesis in (1.5), H is strictly convex on $[0,\infty)$, and therefore

$$E(\varrho|r)\geqslant 0 \quad\text{and}\quad E(\varrho|r)=0 \Leftrightarrow \varrho=r.$$

In order to measure a 'distance' between a weak solution $(\varrho,\boldsymbol{u})$ of the compressible Navier–Stokes system and any other state $(r,\boldsymbol{U})$ of the fluid we introduce the relative energy functional, define by

$$\mathcal{E}(\varrho,\boldsymbol{u}\,|\,r,\boldsymbol{U})=\int_\Omega\left(\frac{1}{2}\varrho|\boldsymbol{u}-\boldsymbol{U}|^2+E(\varrho\,|\,r)\right)\mathrm{d}x. \tag{2.6}$$

It was proved recently in Feireisl *et al.* (2012) that, provided assumption (1.5) holds, any weak solution satisfie the following so-called relative energy inequality:

$$\begin{aligned} &\mathcal{E}(\varrho,\boldsymbol{u}\,|\,r,\boldsymbol{U})(\tau)-\mathcal{E}(\varrho,\boldsymbol{u}\,|\,r,\boldsymbol{U})(0)+\int_0^\tau\int_\Omega(\mu|\nabla(\boldsymbol{u}-\boldsymbol{U})|^2+(\mu+\lambda)|\mathrm{div}(\boldsymbol{u}-\boldsymbol{U})|^2)\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqslant\int_0^\tau\int_\Omega(\mu\nabla\boldsymbol{U}:\nabla(\boldsymbol{U}-\boldsymbol{u})+(\mu+\lambda)\mathrm{div}\,\boldsymbol{U}\,\mathrm{div}\,(\boldsymbol{U}-\boldsymbol{u}))\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_0^\tau\int_\Omega\varrho\partial_t\boldsymbol{U}\cdot(\boldsymbol{U}-\boldsymbol{u})\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega\varrho\boldsymbol{u}\cdot\nabla\boldsymbol{U}\cdot(\boldsymbol{U}-\boldsymbol{u})\mathrm{d}x\,\mathrm{d}t\\ &\qquad-\int_0^\tau\int_\Omega p(\varrho)\mathrm{div}\,\boldsymbol{U}\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega(r-\varrho)\partial_tH'(r)\mathrm{d}x\,\mathrm{d}t-\int_0^\tau\int_\Omega\varrho\nabla H'(r)\cdot\boldsymbol{u}\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.7}$$

for a.a. $\tau\in(0,T)$, and for any pair of test functions

$$r\in C^1([0,T]\times\overline{\Omega}),\quad r>0,\quad \boldsymbol{U}\in C^1([0,T]\times\overline{\Omega};\mathbb{R}^3),\quad \boldsymbol{U}|_{\partial\Omega}=0.$$

The stability of strong solutions in the class of weak solutions is stated in the following proposition.

Proposition 2.2 (Estimate on the relative energy) Let Ω be a Lipschitz domain. Assume that the viscosity coefficient satisfy assumptions (1.4), that the pressure p is a twice continuously differentiable function on $(0,\infty)$ satisfying (1.5) and (1.6), and that $(\varrho,\boldsymbol{u})$ is a weak solution to problem (1.1–1.3) emanating from initial data $(\varrho_0\geqslant 0,\boldsymbol{u}_0)$, with finit energy E_0 and finit mass $M_0=\int_\Omega\varrho_0\mathrm{d}x>0$. Let $(r,\boldsymbol{U})$ in the class

$$\begin{cases} r\in C^1([0,T]\times\bar{\Omega}),\ 0<\underline{r}=\min_{(t,x)\in\bar{Q}_T}r(t,x)\leqslant r(t,x)\leqslant\bar{r}=\max_{(t,x)\in\bar{Q}_T}r(t,x),\\ \boldsymbol{U}\in C^1([0,T]\times\bar{\Omega};\mathbb{R}^3),\quad \boldsymbol{U}|_{(0,T)\times\partial\Omega}=0 \end{cases} \tag{2.8}$$

be a (strong) solution of problem (1.1) emanating from the initial data $(r_0,\boldsymbol{U}_0)$. Then there exists

$$c=c(T,\Omega,M_0,E_0,\underline{r},\bar{r},|p'|_{C^1([\underline{r},\bar{r}])},\|(\nabla r,\partial_t r,\boldsymbol{U},\nabla\boldsymbol{U},\partial_t\boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{19})})>0$$

such that, for almost all $t \in (0,T)$,

$$\mathcal{E}(\varrho, \boldsymbol{u} \,|\, r, \boldsymbol{U})(t) \leqslant c\mathcal{E}(\varrho_0, \boldsymbol{u}_0 \,|\, r_0, \boldsymbol{U}_0). \tag{2.9}$$

This estimate (implying among other results the weak–strong uniqueness) was proved in Feireisl *et al.* (2012) (see also Feireisl *et al.*, 2011) for pressure laws (1.6) with $\gamma > d/(d-1)$. It remains valid under a weaker hypothesis on the pressure, such as (1.6) with $\gamma \geqslant 1$; this can be proved using ideas introduced in Bella *et al.* (2014) and Maltese & Novotny (2014).

3. The numerical scheme

3.1 *Partition of the domain*

We suppose that Ω is a bounded domain of $\mathbb{R}^d$, polygonal if $d=2$ and polyhedral if $d=3$. Let $\mathcal{T}$ be a decomposition of the domain Ω in tetrahedra, which we call hereafter a triangulation of Ω, regardless of the space dimension. By $\mathcal{E}(K)$, we denote the set of the edges ($d=2$) or faces ($d=3$) σ of the element $K \in \mathcal{T}$ called hereafter faces, regardless of the dimension. The set of all faces of the mesh is denoted by $\mathcal{E}$; the set of faces included in the boundary $\partial\Omega$ of Ω is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal faces (i.e. $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$. The triangulation $\mathcal{T}$ is assumed to be regular in the usual sense of the finit element literature (see, e.g. Ciarlet, 1991), and in particular, $\mathcal{T}$ satisfie the following properties:

- $\bar{\Omega} = \bigcup_{K\in\mathcal{T}} \bar{K}$;
- if $(K,L) \in \mathcal{T}^2$, then $\bar{K} \cap \bar{L} = \emptyset$ or $\bar{K} \cap \bar{L}$ is a vertex or $\bar{K} \cap \bar{L}$ is a common face of K and L; in the latter case it is denoted by $K|L$.

For each internal face of the mesh $\sigma = K|L$, $\boldsymbol{n}_{\sigma,K}$ stands for the normal vector of σ, oriented from K to L (so that $\boldsymbol{n}_{\sigma,K} = -\boldsymbol{n}_{\sigma,L}$). We denote by $|K|$ and $|\sigma|$ the (d- and $(d-1)$-dimensional) Lebesgue measure of the tetrahedron K and of the face σ, respectively, and by h_K and h_σ the diameter of K and σ, respectively. We measure the regularity of the mesh by the parameter θ define by

$$\theta = \inf\left\{\frac{\xi_K}{h_K}, K \in \mathcal{T}\right\}, \tag{3.1}$$

where ξ_K stands for the diameter of the largest ball included in K. Last but not least, we denote by h the maximal size of the mesh,

$$h = \max_{K\in\mathcal{T}} h_K. \tag{3.2}$$

The triangulation $\mathcal{T}$ is said to be regular if it satisfie

$$\theta \geqslant \theta_0 > 0. \tag{3.3}$$

3.2 *Discrete function spaces*

Let $\mathcal{T}$ be a mesh of Ω. We denote by $L_h(\Omega)$ the space of piecewise constant functions on the cells of the mesh; the space $L_h(\Omega)$ is the approximation space for the pressure and density. For $1 \leqslant p < \infty$, the mapping

$$q \mapsto \|q\|_{L_h^p(\Omega)} = \|q\|_{L^p(\Omega)} = \left(\sum_{K\in\mathcal{T}} |K||q_K|^p\right)^{1/p}$$

is a norm on $L_h(\Omega)$. We also introduce spaces of non-negative and positive functions:

$$L_h^+(\Omega) = \{q \in L_h(\Omega),\ q_K > 0,\ \forall K \in \mathcal{T}\}.$$

The approximation space for the velocity fiel is the space $\boldsymbol{W}_h(\Omega) = V_h(\Omega;\mathbb{R}^d)$, where $V_h(\Omega)$ is the nonconforming piecewise linear finit element space (Crouzeix & Raviart, 1973; Ern & Guermond, 2004) define by

$$V_h(\Omega) = \Big\{ v \in L^2(\Omega);\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K);\ \forall \sigma \in \mathcal{E}_{\text{int}},\ \sigma = K|L,$$
$$\int_\sigma v_{|K}\,\mathrm{d}S = \int_\sigma v_{|L}\,\mathrm{d}S;\ \forall \sigma \in \mathcal{E}_{\text{ext}},\ \int_\sigma v\,\mathrm{d}S = 0 \Big\}, \tag{3.4}$$

where $\mathbb{P}_1(K)$ denotes the space of affin functions on K and $\mathrm{d}S$ the integration with respect to the $(d-1)$-dimensional Lebesgue measure on the face σ. Each element $v \in V_h(\Omega)$ can be written in the form

$$v(x) = \sum_{\sigma\in\mathcal{E}_{\text{int}}} v_\sigma \varphi_\sigma(x), \quad x \in \Omega, \tag{3.5}$$

where the set $\{\varphi_\sigma\}_{\sigma\in\mathcal{E}_{\text{int}}} \subset V_h(\Omega)$ is the classical basis determined by

$$\forall (\sigma,\sigma') \in \mathcal{E}_{\text{int}}^2,\quad \int_{\sigma'} \varphi_\sigma\,\mathrm{d}S = \delta_{\sigma,\sigma'};\quad \forall \sigma' \in \mathcal{E}_{\text{ext}},\quad \int_{\sigma'} \varphi_\sigma\,\mathrm{d}S = 0 \tag{3.6}$$

and $\{v_\sigma\}_{\sigma\in\mathcal{E}_{\text{int}}} \subset R$ is the set of degrees of freedom relative to v. Note that $V_h(\Omega)$ approximates the functions with zero traces in the sense that, for all elements in $V_h(\Omega)$, $v_\sigma = 0$ provided $\sigma \in \mathcal{E}_{\text{ext}}$. Since only the continuity of the integral over each face of the mesh is imposed, the functions in $V_h(\Omega)$ may be discontinuous through each face; the discretization is thus nonconforming in $W^{1,p}(\Omega;\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$. Finally, we note that, for any $1 \leqslant p < \infty$, the expression

$$|v|_{V_h^p(\Omega)} = \left(\sum_{K\in\mathcal{T}} \|\nabla v\|^p_{L^p(K;\mathbb{R}^d)}\right)^{1/p}$$

is a norm on $V_h(\Omega)$ and we denote by $V_h^p(\Omega)$ the space $V_h(\Omega)$ endowed with this norm.

We finis this section by introducing some notation. For a function v in $L^1(\Omega)$, we set

$$v_K = \frac{1}{|K|}\int_K v\,\mathrm{d}x \quad \text{for } K \in \mathcal{T} \quad \text{and} \quad \hat{v}(x) = \sum_{K\in\mathcal{T}} v_K 1_K(x),\quad x \in \Omega \tag{3.7}$$

so that $\hat{v} \in L_h(\Omega)$. Here and in what follows, 1_K is the characteristic function of K.

If $v \in W^{1,p}(\Omega)$, we set

$$v_\sigma = \frac{1}{|\sigma|} \int_\sigma v \, \mathrm{d}S \quad \text{for } \sigma \in \mathcal{E}. \tag{3.8}$$

Finally, if $v \in W_0^{1,p}(\Omega)$, we set

$$v_h(x) = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} v_\sigma \varphi_\sigma(x), \quad x \in \Omega, \tag{3.9}$$

so that $v_h \in V_h(\Omega)$. In accordance with the above notation, for $v \in W_0^{1,p}(\Omega)$, the symbol $\hat{v}_h$ means $\widehat{v_h}(x) = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} v_\sigma \hat{\phi}_\sigma(x)$, the symbol $v_{h,K} = (1/|K|) \int_K v_h(x)\, \mathrm{d}x$ and the symbol $\hat{v}_{h,\sigma}^{\mathrm{up}} = [\widehat{(v_h)}]_\sigma^{\mathrm{up}}$.

3.3 *Discrete equations*

Let us consider a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $[0, T]$, which, for the sake of simplicity, we suppose uniform. Let k be the constant time step $k = t_n - t_{n-1}$ for $n = 1, \ldots, N$. The density fiel $\varrho(t_n, x)$ and the velocity fiel $\boldsymbol{u}(t_n, x)$ will be approximated by the quantities

$$\varrho^n(x) = \sum_{K \in \mathcal{T}} \varrho_K^n 1_K(x), \quad \boldsymbol{u}^n(x) = \sum_{\sigma \in \mathcal{E}} \boldsymbol{u}_\sigma^n \varphi_\sigma(x), \tag{3.10}$$

where the approximate densities $(\varrho_K^n)_{K \in \mathcal{T}, n=1,\ldots,N}$ and velocities $(\boldsymbol{u}_\sigma^n)_{\sigma \in \mathcal{E}_{\mathrm{int}}, n=1,\ldots,N}$ are the discrete unknowns (with $\varrho_K^n \in \mathbb{R}^+$ and $\boldsymbol{u}_\sigma^n \in \mathbb{R}^d$).

For future convenience, we defin here and hereafter,

$$\varrho(t, x) = \sum_{n=1}^N \varrho^n(x) 1_{[n-1,n)}(t), \quad \boldsymbol{u}(t, x) = \sum_{n=1}^N \boldsymbol{u}^n(x) 1_{[n-1,n)}(t) \tag{3.11}$$

and recall that the usual Lebesgue norms of these functions read

$$\|\varrho\|_{L^\infty(0,T;L^p(\Omega))} \equiv \max_{n=1,\ldots,N} \|\varrho^n\|_{L^p(\Omega)}, \quad \|\boldsymbol{u}\|_{L^p(0,T;L^q(\Omega;\mathbb{R}^3))} \equiv k \left(\sum_{n=1}^N \|\boldsymbol{u}^n\|_{L^q(\Omega;\mathbb{R}^3)}^p \right)^{1/p}. \tag{3.12}$$

Starting from this point, unlike in Section 1, here and hereafter, the couple $(\varrho, \mathbf{u})$, respectively $(\varrho^n, \mathbf{u}^n)$, introduced in (3.10–3.12) always denote, exclusively, a *discrete numerical solution*.

The numerical scheme consists in writing the equations that are solved to determine these discrete unknowns. In order to ensure the positivity of the approximate densities, we shall use an upwinding technique for the density in the mass equation. For $q \in L_h(\Omega)$ and $\boldsymbol{u} \in \mathbf{W}_h(\Omega)$, the upwinding of q with respect to $\boldsymbol{u}$ is defined for $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, by

$$q_\sigma^{\mathrm{up}} = \begin{cases} q_K & \text{if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} > 0, \\ q_L & \text{if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} \leqslant 0, \end{cases} \tag{3.13}$$

so that

$$\sum_{\sigma\in\mathcal{E}(K)} q_\sigma^{\mathrm{up}} \boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K} = \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} (q_K[\boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K}]^+ - q_L[\boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K}]^-),$$

where $a^+ = \max(a,0)$, $a^- = -\min(a,0)$.

Let us then consider the following numerical scheme (Karper, 2013).

Given $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$ *find* $(\varrho^n)_{1\leqslant n\leqslant N} \subset (L_h(\Omega))^N$, $(\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \subset (\boldsymbol{W}_h(\Omega))^N$ *such that, for all* $n = 1,\dots,N$,

$$|K|\frac{\varrho_K^n - \varrho_K^{n-1}}{k} + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] = 0 \quad \forall K \in \mathcal{T}, \tag{3.14a}$$

$$\begin{aligned}\sum_{K\in\mathcal{T}} \frac{|K|}{k}(\varrho_K^n \boldsymbol{u}_K^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1}) \cdot \boldsymbol{v}_K &+ \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] \cdot \boldsymbol{v}_K \\ &- \sum_{K\in\mathcal{T}} p(\varrho_K^n) \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{v}_\sigma \cdot \mathbf{n}_{\sigma,K} + \mu \sum_{K\in\mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v}\, \mathrm{d}x \\ &+ (\mu+\lambda)\sum_{K\in\mathcal{T}} \int_K \operatorname{div} \boldsymbol{u}^n \operatorname{div} \boldsymbol{v}\, \mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{W}_h(\Omega).\end{aligned} \tag{3.14b}$$

Note that the boundary condition $\boldsymbol{u}_\sigma^n = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}}$ is ensured by the definitio of the space $V_h(\Omega)$. Note also that if $\sigma \in \mathcal{E}_{\mathrm{int}}$, $\sigma = K|L$, one has, following (3.7) and (3.13),

$$\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} = \boldsymbol{u}_K^n = \frac{1}{|K|}\int_K \boldsymbol{u}^n(x)\,\mathrm{d}x \quad \text{if } \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} > 0$$

and

$$\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} = \boldsymbol{u}_L^n = \frac{1}{|L|}\int_L \boldsymbol{u}^n(x)\,\mathrm{d}x \quad \text{if } \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} < 0.$$

It is well known that any solution $(\varrho^n)_{1\leqslant n\leqslant N} \subset (L_h(\Omega))^N$ satisfie $\varrho^n > 0$ owing to the upwind choice in (3.14a) (see, e.g. Gallouët et al., 2008; Karper, 2013). Furthermore, summing (3.14a) over $K \in \mathcal{T}$ immediately yields the total conservation of mass, which reads

$$\forall n = 1,\dots,N, \quad \int_\Omega \varrho^n\,\mathrm{d}x = \int_\Omega \varrho^0\,\mathrm{d}x. \tag{3.15}$$

We finall state in this section the existence result, which can be proved by a topological degree argument (Gallouët et al., 2008; Karper, 2013).

Proposition 3.1 (Existence) Let $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$. Under assumptions (1.4) and (1.5), problem (3.14) admits at least one solution:

$$(\varrho^n)_{1\leqslant n\leqslant N} \in [L_h^+(\Omega)]^N, \quad (\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \in [\boldsymbol{W}_h(\Omega)]^N.$$

3.4 *Main result: error estimate*

Let $(r, \boldsymbol{U}) : [0,T] \times \bar{\Omega} \mapsto (0,\infty) \times \mathbb{R}^3$ be C^2 functions such that $\boldsymbol{U} = \boldsymbol{0}$ on $\partial\Omega$. Let $(\varrho, \boldsymbol{u})$ be a solution of the discrete problem (3.14). Inspired by (2.6), we introduce the discrete relative energy functional

$$\mathcal{E}(\varrho^n, \boldsymbol{u}^n \mid r^n, \boldsymbol{U}^n) = \int_\Omega \left(\frac{1}{2}\varrho^n|\hat{\boldsymbol{u}}^n - \hat{\boldsymbol{U}}_h^n|^2 + E(\varrho^n|\hat{r}^n)\right) \mathrm{d}x$$
$$= \sum_{K\in\mathcal{T}} |K| \left(\frac{1}{2}\varrho_K|\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n|^2 + E(\varrho_K^n|r_K^n)\right), \tag{3.16}$$

where

$$r^n(x) = r(t_n, x), \quad \boldsymbol{U}^n(x) = \boldsymbol{U}(t_n, x), \quad n = 0, \ldots, N, \tag{3.17}$$

$(\varrho^n, \boldsymbol{u}^n)$ is define in (3.10) and E is define by (2.5). Let us finall introduce the notation

$$M_0 = \sum_{K\in\mathcal{K}} |K|\varrho_K^0 \quad \text{and} \quad E_0 = \sum_{K\in\mathcal{K}} |K| \left(\frac{1}{2}\varrho_K^0|\boldsymbol{u}_K^0|^2 + H(\varrho_K^0)\right).$$

Now, we are ready to state the main result of this paper. For the sake of clarity, we shall state the theorem and perform the proofs only in the most interesting three-dimensional case. The modification to be done for the two-dimensional case, which is in fact more simple, are mostly due to the different Sobolev embeddings, and are left to the interested reader.

THEOREM 3.2 (Error estimate) Let $\theta_0 > 0$ and $\mathcal{T}$ be a regular triangulation of a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$ introduced in Section 3.1 such that $\theta \geqslant \theta_0$, where θ is define in (3.1). Let p be a twice continuously differentiable function satisfying assumptions (1.5), (1.6) with $\gamma \geqslant 3/2$, and the additional assumption (1.7) in the case $\gamma < 2$. Let the viscosity coefficient satisfy assumptions (1.4). Suppose that $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$ and that $(\varrho^n)_{1\leqslant n\leqslant N} \subset [L_h^+(\Omega)]^N$, $(\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \subset [\boldsymbol{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.14). Let $(r, \boldsymbol{U})$ in the class

$$r \in C^2([0,T] \times \bar{\Omega}), \quad 0 < \underline{r} := \min_{(t,x)\in\bar{Q}_T} \leqslant r(t,x) \leqslant \bar{r} := \max_{(t,x)\in\bar{Q}_T} r(t,x), \tag{3.18a}$$

$$\boldsymbol{U} \in C^2([0,T] \times \bar{\Omega}; \mathbb{R}^3), \quad \boldsymbol{U}|_{\partial\Omega} = 0 \tag{3.18b}$$

be a (strong) solution of problem (1.1). Then there exists

$$c = c(T, |\Omega|, \mathrm{diam}(\Omega), \theta_0, \gamma, M_0, E_0, \underline{r}, \bar{r},$$
$$|p'|_{C^1([\underline{r},\bar{r}])}, \|(\nabla r, \partial_t r, \partial_t\nabla r, \partial_t^2 r, \boldsymbol{U}, \nabla\boldsymbol{U}, \nabla^2\boldsymbol{U}, \partial_t\boldsymbol{U}, \partial_t^2\boldsymbol{U}, \partial_t\nabla\boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{68})}) \in (0, +\infty)$$

(independent of h, k) such that, for any $m = 1, \ldots, N$,

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m \mid r^m, \boldsymbol{U}^m) + k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\int_K |\nabla_x(u^n - \mathbf{U}_h^n)|^2 \mathrm{d}x \leqslant c(\mathcal{E}(\varrho^0, \boldsymbol{u}^0 \mid r^0, \boldsymbol{U}^0) + h^A + \sqrt{k}), \tag{3.19}$$

where

$$A = \begin{cases} \dfrac{2\gamma - 3}{\gamma} & \text{if } \gamma \in (3/2, 2], \\ \dfrac{1}{2} & \text{if } \gamma > 2. \end{cases} \tag{3.20}$$

Starting from this point, unlike in Section 1, here and hereafter, the symbol $\mathcal{E}$ refers always to the *discrete* relative energy functional define in (3.16).

REMARK 3.3 Assumptions (3.18) on the regularity of the strong solution $(r, \mathbf{U})$ in Theorem 3.2 may be slightly relaxed: it is enough to suppose

$$(r, \mathbf{U}) \in C^1([0,T] \times \bar{\Omega}; \mathbb{R}^4), \ \nabla^2 \mathbf{U} \in C([0,T] \times \bar{\Omega}; \mathbb{R}^3), \quad 0 < \inf_{(t,x) \in \bar{Q}_T} r(t,x),$$

$$\partial_t^2 r \in L^1(0,T; L^{\gamma'}(\Omega)), \quad \partial_t \nabla r \in L^2(0,T; L^{6\gamma/(5\gamma-6)}(\Omega; \mathbb{R}^3)),$$

$$(\partial_t^2 \mathbf{U}, \partial_t \nabla \mathbf{U}) \in L^2(0,T; L^{6/5}(\Omega; \mathbb{R}^{12})).$$

The constant in the error estimate depends on $\underline{r}$ and the norms of r and $\mathbf{U}$ in these spaces. This improvement is at the price of more technicalities in estimates of several residual terms, namely in estimates (6.3–6.5), (6.14), (6.21), (7.9), (7.11–7.13) and (8.2). The details are available in the extended version of the present paper available in Gallouët *et al.* (2015).

REMARK 3.4 (1) Theorem 3.2 holds also for two-dimensional bounded polygonal domains under the assumption that $\gamma \geqslant 1$. Assumption (1.7) on the asymptotic behaviour of pressure near 0 is no more necessary in this case. The value of A in the error estimate (3.19) is

$$A = \begin{cases} \dfrac{2\gamma - 2}{\gamma} & \text{if } \gamma \in (1, 2], \\ 1 & \text{if } \gamma > 2. \end{cases}$$

(2) Suppose that the discrete initial data $(\varrho^0, \mathbf{u}^0)$ coincide with the projection $(\hat{r}^0, \hat{\mathbf{U}}_h^0)$ of the initial data determining the strong solution. Then formula (3.19) provides, in terms of classical Lebesgue spaces, the following bounds:

$$\|\varrho^m - r^m\|^2_{L^2(\Omega \cap \{\underline{r}/2 \leqslant \varrho^m \leqslant 2\bar{r}\})} + \|\hat{\mathbf{u}}^m - \mathbf{U}^m\|^2_{L^2(\Omega \cap \{\underline{r}/2 \leqslant \varrho^m \leqslant 2\bar{r}\})} \leqslant c(h^A + \sqrt{k})$$

for the ‘essential part’ of the solution (where the numerical density remains bounded from above and from below outside zero), and

$$|\{\varrho^m \leqslant \underline{r}/2\}| + |\{\varrho^m \geqslant 2\bar{r}\}| + \|\varrho^m\|^\gamma_{L^\gamma(\Omega \cap \{\varrho^m \geqslant 2\bar{r}\})} + \|\varrho^m |\hat{\mathbf{u}}^m - \mathbf{U}^m|^2\|_{L^1(\Omega \cap \{\varrho^m \geqslant 2\bar{r}\})} \leqslant c(h^A + \sqrt{k})$$

for the ‘residual part’ of the solution, where the numerical density can be ‘close’ to zero or infinit . (In the above formula, for $B \subset \Omega$, $|B|$ denotes the Lebesgue measure of B.)

Moreover, in the particular case of $p(\varrho) = \varrho^2$ (that, however, represents a nonphysical situation) $E(\varrho|r) = (\varrho - r)^2$ and the error estimate (3.19) gives

$$\|\varrho^m - r^m\|^2_{L^2(\Omega)} + \|\varrho^m |\hat{\mathbf{u}}^m - \mathbf{U}^m|^2\|_{L^1(\Omega)} \leqslant c(\sqrt{h} + \sqrt{k}).$$

(3) Theorem 3.2 can be viewed as a discrete version of Proposition 2.2. It is to be noted that the assumptions on the constitutive law for pressure guaranteeing the error estimates for the scheme (3.14) are somewhat stronger ($\gamma \geqslant 3/2$) than the assumptions needed for the stability in the continuous case ($\gamma \geqslant 1$). The threshold value $\gamma = 3/2$ is, however, in accordance with the existence theory of weak solutions. The assumptions on the regularity of the strong solution to be compared with the discrete solution in the scheme are slightly stronger than those needed to establish the stability estimates in the continuous case.

(4) If $d = 3$, we note that the assumptions on the pressure (as function of the density) in Theorem 3.2 are compatible with the isentropic case $p(\varrho) = \varrho^\gamma$ for all values $\gamma \geqslant 3/2$.

(5) The scheme Karper (2013) contains in addition artificia stabilizing terms both in the continuity and momentum equations. These terms are necessary for the convergence proof in Karper (2013) even for the large values of γ. It is to be noted that the error estimate in Theorem 3.2 is formulated for the numerical scheme without these stabilizing terms. Of course, a similar error estimate is *a fortiori* valid also for the scheme with the stabilizing terms; however, this issue is not discussed in the present paper.

The rest of the paper is devoted to the proof of Theorem 3.2. For the sake of simplicity, and in order to simplify notation, we present the proof for the uniformly regular mesh, meaning that there exist positive numbers $c_i = c_i(\theta_0)$ such that

$$c_1 h_K \leqslant h \leqslant c_2 h_\sigma \leqslant c_3 h_K, \quad c_1|K| \leqslant |\sigma| h \leqslant c_2|\sigma| h_K \leqslant c_3|\sigma| h_\sigma \leqslant c_4|K| \tag{3.21}$$

for any $K \in \mathcal{T}$ and any $\sigma \in \mathcal{E}$. The necessary (small) modification needed to accommodate the regular mesh satisfying only (3.3) are straightforward. Even with this simplificatio the proof is quite involved, and some details have to be necessarily omitted to keep its length within reasonable bounds. The reader can eventually fin them in the extended version of this paper available on Gallouët *et al.* (2015).

4. Mesh-independent estimates

We start with a remark on the notation. From now on, the letter c denotes positive numbers that may tacitly depend on T, $|\Omega|$, diam(Ω), γ, α, θ_0, λ and μ, and on other parameters; the dependency on these other parameters (if any) is always explicitly indicated in the arguments of these numbers. These numbers can take different values even in the same formula. They are always independent of the size of the discretization k and h.

4.1 *Energy identity*

Our analysis starts with an energy inequality, which is crucial both in the convergence analysis and in the error analysis. We recall this energy estimate which is already given in Karper (2013), along with its proof for the sake of completeness.

LEMMA 4.1 Let $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1\leqslant n\leqslant N} \in [L_h^+(\Omega)]^N$, $(\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \in [\boldsymbol{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.14) with the pressure p satisfying condition (1.5).

Then there exist

$$\bar{\varrho}_\sigma^n \in [\min(\varrho_K^n, \varrho_L^n), \max(\varrho_K^n, \varrho_L^n)], \quad \sigma = K|L \in \mathcal{E}_{\text{int}},\ n = 1, \ldots, N,$$
$$\bar{\varrho}_K^{n-1,n} \in [\min(\varrho_K^{n-1}, \varrho_K^n), \max(\varrho_K^{n-1}, \varrho_K^n)], \quad K \in \mathcal{T},\ n = 1, \ldots, N$$

such that

$$\begin{aligned}\sum_{K\in\mathcal{T}} |K| \left(\frac{1}{2}\varrho_K^m |\boldsymbol{u}_K^m|^2 + H(\varrho_K^m)\right) - \sum_{K\in\mathcal{T}} |K| \left(\frac{1}{2}\varrho_K^0 |\boldsymbol{u}_K^0|^2 + H(\varrho_K^0)\right) \\ + k\sum_{n=1}^m \sum_{K\in\mathcal{T}} \left(\mu \int_K |\nabla_x \boldsymbol{u}^n|^2 \,\mathrm{d}x + (\mu+\lambda) \int_K |\operatorname{div} \boldsymbol{u}^n|^2 \,\mathrm{d}x\right) \\ + [D_{\text{time}}^{m,|\Delta \boldsymbol{u}|}] + [D_{\text{time}}^{m,|\Delta \varrho|}] + [D_{\text{space}}^{m,|\Delta \boldsymbol{u}|}] + [D_{\text{space}}^{m,|\Delta \varrho|}] = 0,\end{aligned} \tag{4.1}$$

for all $m = 1, \ldots, N$, where

$$[D_{\text{time}}^{m,|\Delta \boldsymbol{u}|}] = \sum_{n=1}^m \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{|\boldsymbol{u}_K^n - \boldsymbol{u}_K^{n-1}|^2}{2}, \tag{4.2a}$$

$$[D_{\text{time}}^{m,|\Delta \varrho|}] = \sum_{n=1}^m \sum_{K\in\mathcal{T}} |K| H''(\bar{\varrho}_K^{n-1,n}) \frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}, \tag{4.2b}$$

$$[D_{\text{space}}^{m,|\Delta \boldsymbol{u}|}] = k\sum_{n=1}^m \sum_{\sigma=K|L\in\mathcal{E}_{\text{int}}} |\sigma| \varrho_\sigma^{n,\text{up}} \frac{(\boldsymbol{u}_K^n - \boldsymbol{u}_L^n)^2}{2} |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|, \tag{4.2c}$$

$$[D_{\text{space}}^{m,|\Delta \varrho|}] = k\sum_{n=1}^m \sum_{\sigma=K|L\in\mathcal{E}_{\text{int}}} |\sigma| H''(\bar{\varrho}_\sigma^n) \frac{(\varrho_K^n - \varrho_L^n)^2}{2} |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|. \tag{4.2d}$$

Proof. Mimicking the formal derivation of the total energy conservation in the continuous case, we take as test function $\boldsymbol{v} = \boldsymbol{u}^n$ in the discrete momentum equation (3.14b)^n and obtain

$$I_1 + I_2 + I_3 + I_4 = 0, \tag{4.3}$$

where

$$I_1 = \sum_{K\in\mathcal{T}} \frac{|K|}{k} (\varrho_k^n \boldsymbol{u}_K^n - \varrho_K^{n-1} \boldsymbol{u}_K^{n-1}) \cdot \boldsymbol{u}_K^n a, \quad I_2 = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} \hat{\boldsymbol{u}}_\sigma^{n,\text{up}} \cdot \boldsymbol{u}_K^n\, [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}],$$

$$I_3 = -\sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| p(\varrho_K^n) [\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}], \quad I_4 = \sum_{K\in\mathcal{T}} \int_K (\mu \nabla \boldsymbol{u}^n : \nabla \boldsymbol{u}^n + (\mu+\lambda) \operatorname{div} \boldsymbol{u}^n \operatorname{div} \boldsymbol{u}^n) \,\mathrm{d}x.$$

Next, we multiply the continuity equation $(3.14\text{a})_K^n$ by $\frac{1}{2}|\boldsymbol{u}_K^n|^2$ and sum over all $K \in \mathcal{T}$. We obtain

$$I_5 + I_6 = 0 \tag{4.4}$$

$$\text{with } I_5 = -\sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{k}(\varrho_K^n - \varrho_K^{n-1})|\boldsymbol{u}_K^n|^2 \quad \text{and} \quad I_6 = -\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} \frac{1}{2}|\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]|\boldsymbol{u}_K^n|^2.$$

Finally, we multiply the continuity equation $(3.14a)_K^n$ by $H'(\varrho_K^n)$ and sum over all $K\in\mathcal{T}$. We obtain

$$I_7 + I_8 = 0, \tag{4.5}$$

$$\text{with } I_7 = \sum_{K\in\mathcal{T}} \frac{|K|}{k}(\varrho_K^n - \varrho_K^{n-1})H'(\varrho_K^n) \quad \text{and} \quad I_8 = \sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]H'(\varrho_K^n).$$

We now sum formulas (4.3–4.5) in several steps.
Step 1*: term* $I_1 + I_7$. We verify by a direct calculation that

$$I_1 = \sum_{K\in\mathcal{T}} \frac{|K|}{k}\left(\frac{1}{2}\varrho_K^n|\boldsymbol{u}_K^n|^2 - \frac{1}{2}\varrho_K^{n-1}|\boldsymbol{u}_K^{n-1}|^2\right) + \sum_{K\in\mathcal{T}} \frac{|K|}{k}\varrho_K^{n-1}\frac{|\boldsymbol{u}_K^n - \boldsymbol{u}_K^{n-1}|^2}{2}.$$

In order to transform the term I_7, we employ the Taylor formula,

$$H'(\varrho_K^n)(\varrho_K^n - \varrho_K^{n-1}) = H(\varrho_K^n) - H(\varrho_K^{n-1}) + \tfrac{1}{2}H''(\bar\varrho_K^{n-1,n})(\varrho_K^n - \varrho_K^{n-1})^2,$$

where $\bar\varrho_K^{n-1,n} \in [\min(\varrho_K^{n-1},\varrho_K^n), \max(\varrho_K^{n-1},\varrho_K^n)]$. Consequently,

$$\begin{aligned} I_1 + I_7 = &\sum_{K\in\mathcal{T}} \frac{|K|}{k}\left(\frac{1}{2}\varrho_K^n|\boldsymbol{u}_K^n|^2 - \frac{1}{2}\varrho_K^{n-1}|\boldsymbol{u}_K^{n-1}|^2\right) + \sum_{K\in\mathcal{T}} \frac{|K|}{k}(H(\varrho_K^n) - H(\varrho_K^{n-1})) \\ &+ \sum_{K\in\mathcal{T}} \frac{|K|}{k}\varrho_K^{n-1}\frac{|\boldsymbol{u}_K^n - \boldsymbol{u}_K^{n-1}|^2}{2} + \sum_{K\in\mathcal{T}} \frac{|K|}{k}H''(\bar\varrho_K^{n-1,n})\frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}. \end{aligned} \tag{4.6}$$

Step 2*: term* $I_2 + I_6$. The contribution of the face $\sigma = K|L$ to the sum $I_2 + I_6$ reads, by virtue of (3.13),

$$\begin{aligned} &|\sigma|[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]^+\varrho_K\left(|\boldsymbol{u}_K^n|^2 - \boldsymbol{u}_K^n\cdot\boldsymbol{u}_L^n - \frac{1}{2}|\boldsymbol{u}_K^n|^2 + \frac{1}{2}|\boldsymbol{u}_L^n|^2\right) \\ &\quad + |\sigma|[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,L}]^+\varrho_L\left(|\boldsymbol{u}_L^n|^2 - \boldsymbol{u}_K^n\cdot\boldsymbol{u}_L^n - \frac{1}{2}|\boldsymbol{u}_L^n|^2 + \frac{1}{2}|\boldsymbol{u}_K^n|^2\right). \end{aligned}$$

Consequently,

$$I_2 + I_6 = \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} |\sigma||\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|\varrho_\sigma^{n,\mathrm{up}}\frac{(\boldsymbol{u}_K^n - \boldsymbol{u}_L^n)^2}{2}. \tag{4.7}$$

Step 3*: term* $I_3 + I_8$. We have

$$I_8 = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}](H'(\varrho_K^n)(\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n) + H(\varrho_K^n))$$
$$+ \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}](\varrho_K^n H'(\varrho_K^n) - H(\varrho_K^n)).$$

Recalling (3.13), we may write the contribution of the face $\sigma = K|L$ to the first sum in I_8; it reads

$$|\sigma|[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]^+(H(\varrho_K^n) - H'(\varrho_L^n)(\varrho_K^n - \varrho_L^n) - H(\varrho_L^n))$$
$$+ |\sigma|[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,L}]^+(H(\varrho_L^n) - H'(\varrho_K^n)(\varrho_L^n - \varrho_K^n) - H(\varrho_K^n)).$$

Recalling that $rH'(r) - H(r) = p(r)$, we obtain, employing the Taylor formula,

$$I_3 + I_8 = \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| H''(\bar\varrho_\sigma^n)\frac{(\varrho_K^n - \varrho_L^n)^2}{2}$$

with some $\bar\varrho_\sigma^n \in [\min(\varrho_K^n, \varrho_L^n), \max(\varrho_K^n, \varrho_L^n)]$.

Step 4*: Conclusion.* Collecting the results of Steps 1–3, we arrive at

$$\sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{k}(\varrho_K^n|\boldsymbol{u}_K^n|^2 - \varrho_K^{n-1}|\boldsymbol{u}_K^{n-1}|^2) + \sum_{K\in\mathcal{T}}\frac{|K|}{k}(H(\varrho_K^n) - H(\varrho_K^{n-1})) + \sum_{K\in\mathcal{T}}\left(\mu\int_K |\nabla_x \boldsymbol{u}^n|^2\,\mathrm{d}x\right.$$
$$\left. + (\mu+\lambda)\int_K |\mathrm{div}\,\boldsymbol{u}^n|^2\,\mathrm{d}x\right) + \sum_{K\in\mathcal{T}}\frac{|K|}{k}\varrho_K^{n-1}\frac{|\boldsymbol{u}_K^n - \boldsymbol{u}_K^{n-1}|^2}{2} + \sum_{K\in\mathcal{T}}\frac{|K|}{k}H''(\bar\varrho_K^{n-1,n})\frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}$$
$$+ \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\frac{(\boldsymbol{u}_K^n - \boldsymbol{u}_L^n)^2}{2}|\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| + \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma|H''(\bar\varrho_\sigma^n)\frac{(\varrho_K^n - \varrho_L^n)^2}{2}|\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| = 0. \tag{4.8}$$

At this stage, we get the statement of Lemma 4.1 by multiplying $(4.8)^n$ by k and summing from $n = 1$ to $n = m$. Lemma 4.1 is proved. □

4.2 *Estimates*

We have the following corollary of Lemma 4.1.

COROLLARY 4.2 (1) Under the assumptions of Lemma 4.1, there exists $c = c(M_0, E_0) > 0$ (independent of h and k) such that

$$|\boldsymbol{u}|_{L^2(0,T;V_h^2(\Omega;\mathbb{R}^3))} \leqslant c, \tag{4.9}$$
$$\|\boldsymbol{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \leqslant c, \tag{4.10}$$
$$\|\varrho\hat{\boldsymbol{u}}^2\|_{L^\infty(0,T;L^1(\Omega))} \leqslant c. \tag{4.11}$$

(2) If in addition the pressure satisfie assumption (1.6), then

$$\|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} \leqslant c. \tag{4.12}$$

(3) If the pair $(r, \boldsymbol{U})$ belongs to the class (3.18), there exists $c = c(M_0, E_0, \underline{r}, \bar{r}, \|\boldsymbol{U}, \nabla \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{12})}) > 0$ such that, for all $n = 1, \ldots, N$,

$$\mathcal{E}(\varrho^n, \boldsymbol{u}^n | r^n, \boldsymbol{U}^n) \leqslant c, \tag{4.13}$$

where the discrete relative energy $\mathcal{E}$ is define in (3.16).

Proof. Recall that

$$|\boldsymbol{u}|^2_{L^2(0,T;V_h^2(\Omega;\mathbb{R}^3))} = k \sum_{n=1}^{N} \sum_{K\in\mathcal{T}} \int_K |\nabla_x \boldsymbol{u}^n|^2 \,\mathrm{d}x;$$

the estimate (4.9) follows from (4.1). The estimate (4.10) holds due to embedding (A.29) in Lemma A6 and bound (4.9). The estimate (4.11) is just a short transcription of the bound for the kinetic energy in (4.1).

We prove estimate (4.12). First, we deduce from (1.5) and the definitio (2.4) of H that $0 \leqslant -H(\varrho) \leqslant c_1$ with some $c_1 > 0$, provided $0 < \varrho \leqslant 1$ and $H(\varrho) > 0$ if $\varrho > 1$. This fact in combination with the bound for $\int_\Omega H(\varrho)\mathrm{d}x$ derived in (4.1) yields

$$\int_\Omega |H(\varrho)| \,\mathrm{d}x \leqslant c < \infty. \tag{4.14}$$

Second, relations (1.5–1.7) imply that there are $\bar{\varrho} > 1$, $\underline{c} = \underline{c}(\alpha) > 0$ and $0 < \underline{p} < \bar{p} < \infty$ such that

$$\left\{ \begin{array}{c} \varrho^\alpha \underline{c} p_0/2 \leqslant \dfrac{p(\varrho)}{\varrho^2} \text{ if } 0 < \varrho < 1/\bar{\varrho}, \\ \underline{p} \leqslant \dfrac{p(\varrho)}{\varrho^2} \leqslant \bar{p} \text{ if } 1/\bar{\varrho} \leqslant \varrho \leqslant \bar{\varrho}, \\ \varrho^{\gamma-2} p_\infty/2 \leqslant \dfrac{p(\varrho)}{\varrho^2} \text{ if } \varrho > \bar{\varrho}. \end{array} \right\}$$

Using these bounds and the definitio (2.4) of H, we verify that

$$\varrho^\gamma \leqslant c(|H(\varrho)| + \varrho + 1)$$

with a convenient positive constant c. Now, bound (4.12) follows readily from the boundedness of $\int_\Omega \varrho^m \mathrm{d}x \equiv \sum_{K\in\mathcal{T}} |K| \varrho_K^m$ and $\int_\Omega H(\varrho^m)\,\mathrm{d}x \equiv \sum_{K\in\mathcal{T}} |K| H(\varrho_K^m)$ established in (3.15) and (4.1).

Finally, to get (4.13), we have employed (2.5), (3.16), (3.15), (4.14) to estimate $\int_\Omega E(\varrho^n|\hat{r}^n)\mathrm{d}x$ and (4.11), (A.3), (A.21) to evaluate $\sum_{K\in\mathcal{T}} \int_K \varrho_K^n |\boldsymbol{U}_{h,K}^n - \boldsymbol{u}_K^n|^2 \mathrm{d}x$. □

The follow ing estimates are obtained owing to the numerical diffusion due to the upwinding, as is classical in the framework of hyperbolic conservation laws; see, e.g. Eymard *et al.* (2000).

LEMMA 4.3 (Dissipation estimates on the density) Let $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$. Suppose that $(\varrho^n)_{1\leqslant n\leqslant N} \subset [L_h^+(\Omega)]^N$, $(\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \subset [\boldsymbol{W}_h]^N(\Omega)$ is a solution of problem (3.14). Finally, assume that the pressure satisfie hypotheses (1.5) and (1.6). Then the following conditions are satisfied

(1) If $\gamma \geqslant 2$, then there exists $c = c(\gamma, \theta_0, E_0) > 0$ such that

$$k \sum_{n=1}^{N} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma| \frac{(\varrho_K^n - \varrho_L^n)^2}{\max(\varrho_K^n, \varrho_L^n)} |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leqslant c. \tag{4.15}$$

(2) If $\gamma \in [1,2)$ and the pressure satisfie additionally assumption (1.7), then there exists $c = c(M_0, E_0) > 0$ such that

$$k \sum_{n=1}^{N} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma| \frac{(\varrho_K^n - \varrho_L^n)^2}{[\max(\varrho_K^n, \varrho_L^n)]^{2-\gamma}} 1_{\{\bar{\varrho}_\sigma^n \geqslant 1\}} |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| + k \sum_{n=1}^{N} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma| (\varrho_K^n - \varrho_L^n)^2 1_{\{\bar{\varrho}_\sigma^n < 1\}} |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leqslant c, \tag{4.16}$$

where the numbers $\bar{\varrho}_\sigma^n$ are define in Lemma 4.1.

Proof. We start by proving the simpler statement (2). Taking into account the continuity of the pressure, we deduce from assumptions (1.6) and (1.7) that there exist numbers $\bar{p}_0 > 0$, $\bar{p}_\infty > 0$ such that

$$H''(s) \geqslant \begin{cases} \dfrac{\bar{p}_\infty}{s^{2-\gamma}} & \text{if } s \geqslant 1, \\ \bar{p}_0 s^\alpha \geqslant \bar{p}_0 & \text{if } s < 1; \end{cases}$$

then, splitting the sum in the definitio of the term $[D_{\rm space}^{N,\Delta\varrho}]$ (see (4.2d)) into two sums, where (σ, n) satisfie $\bar{\varrho}_\sigma^n \geqslant 1$ for the f rst one and $\bar{\varrho}_\sigma^n < 1$ for the second, we obtain the desired result.

Let us now turn to the proof of statement (1). Multiplying the discrete continuity equation $(3.14a)_K^n$ by $\ln \varrho_K^n$ and summing over $K \in \mathcal{T}$, we obtain

$$\sum_{K\in\mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{k} \ln \varrho_K^n + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K),\sigma=K|L} (\ln \varrho_K^n) \varrho_\sigma^{n,\rm up} \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} = 0.$$

By virtue of the convexity of the function $\varrho \mapsto \varrho \ln \varrho - \varrho$ on the positive real line, and due to the Taylor formula, we have

$$\varrho_K^n \ln \varrho_K^n - \varrho_K^{n-1} \ln \varrho_K^{n-1} - (\varrho_K^n - \varrho_K^{n-1}) \leqslant \ln \varrho_K^n (\varrho_K^n - \varrho_K^{n-1})$$

then, owing to the mass conservation (3.15) and the definitio of $\varrho_\sigma^{\rm up}$, we arrive at

$$\sum_{K\in\mathcal{T}} |K| \frac{\varrho_K^n \ln \varrho_K^n - \varrho_K^{n-1} \ln \varrho_K^{n-1}}{k} + \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma| \varrho_K^n [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]^+ (\ln \varrho_K^n - \ln \varrho_L^n)$$
$$+ \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma| \varrho_L^n [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,L}]^+ (\ln \varrho_L^n - \ln \varrho_K^n) \leqslant 0,$$

or equivalently,

$$\begin{aligned}
k &\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma| [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]^+ (\varrho_K^n (\ln \varrho_K^n - \ln \varrho_L^n) - (\varrho_K^n - \varrho_L^n)) \\
&\quad + k \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma| [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,L}]^+ (\varrho_L^n (\ln \varrho_L^n - \ln \varrho_K^n) - (\varrho_L^n - \varrho_K^n)) \\
&\leqslant - \sum_{K\in\mathcal{T}} |K| (\varrho_K^n \ln \varrho_K^n - \varrho_K^{n-1} \ln \varrho_K^{n-1}) \\
&\quad + k \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma| ([\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]^+ (\varrho_L^n - \varrho_K^n) + [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,L}]^+ (\varrho_K^n - \varrho_L^n)).
\end{aligned} \tag{4.17}$$

From Fettah & Gallouët (2013, Lemma C.5), we know that if φ and ψ are functions in $C^1((0,\infty);\mathbb{R})$ such that $s\psi'(s) = \varphi'(s)$ for all $s \in (0,\infty)$, then, for any $(a,b) \in (0,\infty)^2$, there exists $c \in [a,b]$ such that

$$(\psi(b) - \psi(a))b - (\varphi(b) - \varphi(a)) = \tfrac{1}{2}(b-a)^2 \psi'(c).$$

Applying this result with $\psi(s) = \ln s$, $\varphi(s) = s$, we obtain that the left-hand side of (4.17) is greater than or equal to

$$k \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma| ([\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]^+ + [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,L}]^+) \frac{(\varrho_K^n - \varrho_L^n)^2}{\max(\varrho_K^n, \varrho_L^n)}.$$

On the other hand, the f rst term on the right-hand side is bounded from above by $\|\varrho^n\|_{L^\gamma(\Omega)}^\gamma$. Finally, the second term on the right-hand side is equal to

$$-k \sum_{K\in\mathcal{T}} \int_K \varrho_K^n \operatorname{div} \boldsymbol{u}^n \leqslant k \sum_{K\in\mathcal{T}} \|\varrho_K^n\|_{L^2(K)} \|\operatorname{div} \mathbf{u}^n\|_{L^2(K)},$$

hence bounded from above by $k\|\boldsymbol{u}^n\|_{V_h^2(\Omega;\mathbb{R}^3)} \|\varrho^n\|_{L^2(\Omega)}$, where we have used the Hölder inequality and the definitio of the $V_h^2(\Omega)$-norm. Statement (1) of Lemma 4.3 now follows from the estimates of Corollary 4.2. □

5. Exact relative energy inequality for the discrete problem

The goal of this section is to prove the discrete version of the relative energy inequality.

THEOREM 5.1 Suppose that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain and $\mathcal{T}$ is its regular triangulation introduced in Section 3.1. Let p satisfy hypotheses (1.5) and the viscosity coefficient μ, λ obey (1.4). Let $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1\leqslant n\leqslant N} \in [L_h^+(\Omega)]^N$, $(\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \in [\boldsymbol{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.14). Then there holds, for all $m = 1, \ldots, N$,

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} \frac{1}{2}|K|(\varrho_K^m|\boldsymbol{u}_K^m - \boldsymbol{U}_{h,K}^m|^2 - \varrho_K^0|\boldsymbol{u}_K^0 - \boldsymbol{U}_{h,K}^0|^2) + \sum_{K\in\mathcal{T}}|K|(E(\varrho_K^m|r_K^m) - E(\varrho_K^0|r_K^0)) \\
&\quad + k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\left(\mu\int_K|\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2\,\mathrm{d}x + (\mu+\lambda)\int_K|\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2\,\mathrm{d}x\right) \\
&\quad\leqslant k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\left(\mu\int_K\nabla_x\boldsymbol{U}_h^n : \nabla_x(\boldsymbol{U}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x + (\mu+\lambda)\int_K\mathrm{div}\,\boldsymbol{U}_h^n\,\mathrm{div}(\boldsymbol{U}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x\right) \\
&\qquad + k\sum_{n=1}^m\sum_{K\in\mathcal{T}}|K|\varrho_K^{n-1}\frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k}\cdot\left(\frac{\boldsymbol{U}_{h,K}^{n-1} + \boldsymbol{U}_{h,K}^n}{2} - \boldsymbol{u}_K^{n-1}\right) \\
&\qquad - k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\left(\frac{\boldsymbol{U}_{h,K}^n + \boldsymbol{U}_{h,L}^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\right)\cdot\boldsymbol{U}_{h,K}^n[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}] \\
&\qquad - k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|p(\varrho_K^n)[\boldsymbol{U}_{h,\sigma}^n\cdot\boldsymbol{n}_{\sigma,K}] \\
&\qquad + k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\frac{|K|}{k}(r_K^n - \varrho_K^n)(H'(r_K^n) - H'(r_K^{n-1})) \\
&\qquad + k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\varrho_\sigma^{n,\mathrm{up}}H'(r_K^{n-1})[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}],
\end{aligned}
\tag{5.1}
$$

for any $0 < r \in C^1([0,T]\times\bar{\Omega})$, $\boldsymbol{U} \in C^1([0,T]\times\bar{\Omega})$, $\boldsymbol{U}|_{\partial\Omega} = 0$, where we have used notation (3.17) for r^n, $\boldsymbol{U}^n$ and (3.7–3.8) for $\mathbf{U}_h^n$, $\mathbf{U}_{h,K}^n$, r_K^n, $\mathbf{u}_\sigma^n$.

We note, comparing the terms in the 'discrete' formula (5.1) with the terms in the 'continuous' formula (2.7), that Theorem 5.1 represents a discrete counterpart of the 'continuous' relative energy inequality (2.7). The rest of this section is devoted to its proof. To this end, we shall follow the proof of the 'continuous' relative energy inequality (see Feireisl *et al.*, 2011, 2012) and adapt it to the discrete case.

Proof. First, noting that the numerical diffusion represented by terms (4.2a–4.2d) in the energy identity (4.1) is positive, we infer

$$I_1 + I_2 + I_3 \leqslant 0, \tag{5.2}$$

with

$$I_1 := \sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{k}(\varrho_K^n|\boldsymbol{u}_K^n|^2 - \varrho_K^{n-1}|\boldsymbol{u}_K^{n-1}|^2), \quad I_2 := \sum_{K\in\mathcal{T}} \frac{|K|}{k}(H(\varrho_K^n) - H(\varrho_K^{n-1})),$$
$$I_3 := \sum_{K\in\mathcal{T}} \left(\mu\int_K |\nabla_x \boldsymbol{u}^n|^2 \mathrm{d}x + (\mu+\lambda)\int_K |\operatorname{div}\boldsymbol{u}^n|^2\,\mathrm{d}x\right).$$

Next, we multiply the discrete continuity equation $(3.14\mathrm{a})_K^n$ by $\frac{1}{2}|\mathbf{U}_{h,K}^n|^2$ and sum over $K\in\mathcal{T}$ to obtain

$$I_4 := \sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{k}(\varrho_K^n - \varrho_K^{n-1})|\boldsymbol{U}_{h,K}^n|^2 = -\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} \frac{1}{2}|\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]|\boldsymbol{U}_{h,K}^n|^2 := J_1. \tag{5.3}$$

In the next step, taking $-\boldsymbol{U}^n$ as test function in the discrete momentum equation (3.14b), we obtain

$$I_5 = -\sum_{K\in\mathcal{T}} \frac{|K|}{k}(\varrho_K^n\boldsymbol{u}_K^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1})\cdot\boldsymbol{U}_{h,K}^n = J_2 + J_3 + J_4,$$

with

$$J_2 = \sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\cdot\boldsymbol{U}_{h,K}^n\,[\boldsymbol{u}_\sigma^n\cdot\mathbf{n}_{\sigma,K}],$$
$$J_3 = \mu\sum_{K\in\mathcal{T}}\int_K \nabla\boldsymbol{u}^n : \nabla\boldsymbol{U}_h^n\,\mathrm{d}x + (\mu+\lambda)\sum_{K\in\mathcal{T}}\int_K \operatorname{div}\boldsymbol{u}^n\operatorname{div}\boldsymbol{U}_h^n\,\mathrm{d}x$$

and

$$J_4 = -\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|p(\varrho_K^n)[\boldsymbol{U}_\sigma^n\cdot\mathbf{n}_{\sigma,K}].$$

We then multiply the continuity equation $(3.14\mathrm{a})_K^n$ by $H'(r_K^{n-1})$ and sum over all $K\in\mathcal{T}$ to obtain

$$-\sum_{K\in\mathcal{T}} \frac{|K|}{k}(\varrho_K^n - \varrho_K^{n-1})H'(r_K^{n-1}) = \sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]H'(r_K^{n-1}).$$

Observing that $\varrho_K^n H'(r_K^n) - \varrho_K^{n-1}H'(r_K^{n-1}) = \varrho_K^n(H'(r_K^n) - H'(r_K^{n-1})) + (\varrho_K^n - \varrho_K^{n-1})H'(r_K^{n-1})$, we rewrite the last identity in the form

$$I_6 := -\sum_{K\in\mathcal{T}} \frac{|K|}{k}(\varrho_K^n H'(r_K^n) - \varrho_K^{n-1}H'(r_K^{n-1})) = J_5 + J_6$$

$$\text{with } J_5 = -\sum_{K\in\mathcal{T}} \frac{|K|}{k}\varrho_K^n(H'(r_K^n) - H'(r_K^{n-1})) \quad\text{and}\quad J_6 = \sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]H'(r_K^{n-1}). \tag{5.4}$$

Finally, owing to the convexity of the function H, we have

$$\begin{aligned} I_7 &:= \sum_{K\in\mathcal{T}} \frac{|K|}{k}[(r_K^n H'(r_K^n) - H(r_K^n)) - (r_K^{n-1}H'(r_K^{n-1}) - H(r_K^{n-1}))] \\ &= \sum_{K\in\mathcal{T}} \frac{|K|}{k} r_K^n (H'(r_K^n) - H'(r_K^{n-1})) - \sum_{K\in\mathcal{T}} \frac{|K|}{k}(H(r_K^n) - (r_K^n - r_K^{n-1})H'(r_K^{n-1}) - H(r_K^{n-1})) \\ &\leqslant \sum_{K\in\mathcal{T}} \frac{|K|}{k} r_K^n (H'(r_K^n) - H'(r_K^{n-1})) := J_7. \end{aligned} \tag{5.5}$$

Now, we gather the expressions (5.2–5.5); this is performed in several steps.
Step 1*: term* $I_1 + I_4 + I_5$. We observe that

$$\begin{cases} \dfrac{|\boldsymbol{U}_{h,K}^n|^2}{2}(\varrho_K^n - \varrho_K^{n-1}) = \dfrac{\varrho_K^n|\boldsymbol{U}_{h,K}^n|^2 - \varrho_K^{n-1}|\boldsymbol{U}_{h,K}^{n-1}|^2}{2} + \varrho_K^{n-1}\dfrac{\boldsymbol{U}_{h,K}^{n-1} + \boldsymbol{U}_{h,K}^n}{2}\cdot(\boldsymbol{U}_{h,K}^{n-1} - \boldsymbol{U}_{h,K}^n), \\ -(\varrho_K^n \boldsymbol{u}_K^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1})\cdot \boldsymbol{U}_{h,K}^n = -(\varrho_K^n \boldsymbol{u}_K^n \cdot \boldsymbol{U}_{h,K}^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1}\cdot\boldsymbol{U}_{h,K}^{n-1}) - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1}\cdot(\boldsymbol{U}_{h,K}^{n-1} - \boldsymbol{U}_{h,K}^n). \end{cases}$$

Consequently,

$$\begin{aligned} I_1 + I_4 + I_5 = &\sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{k}(\varrho_K^n|\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n|^2 - \varrho_K^{n-1}|\boldsymbol{u}_K^{n-1} - \boldsymbol{U}_{h,K}^{n-1}|^2) \\ &- \sum_{K\in\mathcal{T}} |K|\varrho_K^{n-1}\frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k}\cdot\left(\frac{\boldsymbol{U}_{h,K}^{n-1} + \boldsymbol{U}_{h,K}^n}{2} - \boldsymbol{u}_K^{n-1}\right). \end{aligned} \tag{5.6}$$

Step 2*: term* $J_1 + J_2$. The contribution of the face $\sigma = K|L$ to J_1 reads

$$\begin{aligned} &- |\sigma|\varrho_K^n \frac{\boldsymbol{U}_{h,K}^n + \boldsymbol{U}_{h,L}^n}{2}\cdot(\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,L}^n)[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]^+ \\ &- |\sigma|\varrho_L^n \frac{\boldsymbol{U}_{h,K}^n + \boldsymbol{U}_{h,L}^n}{2}\cdot(\boldsymbol{U}_{h,L}^n - \boldsymbol{U}_{h,K}^n)[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,L}]^+. \end{aligned}$$

Similarly, the contribution of the face $\sigma = K|L$ to J_2 is

$$|\sigma|\varrho_K^n\boldsymbol{u}_K^n\cdot(\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,L}^n)[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]^+ + |\sigma|\varrho_L^n\boldsymbol{u}_L^n\cdot(\boldsymbol{U}_{h,L}^n - \boldsymbol{U}_{h,K}^n)[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,L}]^+.$$

Consequently,

$$J_1 + J_2 = -\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}_K} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\left(\frac{\boldsymbol{U}_{h,K}^n + \boldsymbol{U}_{h,L}^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\right)\cdot\boldsymbol{U}_{h,K}^n[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]. \tag{5.7}$$

Step 3*: term* $I_3 - J_3$. This term can be written in the form

$$I_3 - J_3 = \sum_{K\in\mathcal{T}} \left(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2 \,\mathrm{d}x + (\mu+\lambda)\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2 \,\mathrm{d}x \right)$$
$$- \sum_{K\in\mathcal{T}} \mu \int_K \left(\nabla \boldsymbol{U}_h^n : \nabla(\boldsymbol{U}_h^n - \boldsymbol{u}^n) + (\mu+\lambda)\int_K \mathrm{div}\,\boldsymbol{U}_h^n\,\mathrm{div}(\boldsymbol{U}_h^n - \boldsymbol{u}^n) \right). \tag{5.8}$$

Step 4*: term* $I_2 + I_6 + I_7$. By virtue of (5.2), (5.4–5.5), we easily fin that

$$I_2 + I_6 + I_7 = \sum_{K\in\mathcal{T}} \frac{|K|}{k}\left(E(\varrho_K^n \,|\, r_K^n) - E(\varrho_K^{n-1} \,|\, r_K^{n-1})\right), \tag{5.9}$$

where the function E is define in (2.5).

Step 5*: term* $J_5 + J_6 + J_7$. Coming back to (5.4–5.5), we deduce that

$$J_5 + J_6 + J_7 = \sum_{K\in\mathcal{T}} \frac{|K|}{k}(r_K^n - \varrho_K^n)(H'(r_K^n) - H'(r_K^{n-1}))$$
$$+ \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]H'(r_K^{n-1}). \tag{5.10}$$

Step 6*: conclusion*. According to (5.2–5.5), we have

$$\sum_{i=1}^{7} I_i \leqslant \sum_{i=1}^{7} J_i;$$

then, writing this inequality by using expressions (5.6–5.10) calculated in Steps 1–5, we obtain

$$\sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{k}\left(\varrho_K^n|\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n|^2 - \varrho_K^{n-1}|\boldsymbol{u}_K^{n-1} - \boldsymbol{U}_{h,K}^{n-1}|^2\right) + \sum_{K\in\mathcal{T}} \frac{|K|}{k}\left(E(\varrho_K^n|r_K^n) - E(\varrho_K^{n-1}|r_K^{n-1})\right)$$
$$+ \sum_{K\in\mathcal{T}} \left(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2 \,\mathrm{d}x + (\mu+\lambda)\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2 \,\mathrm{d}x \right)$$
$$\leqslant \sum_{K\in\mathcal{T}} \left(\mu \int_K \nabla_x \boldsymbol{U}_h^n : \nabla_x(\boldsymbol{U}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x + (\mu+\lambda)\int_K \mathrm{div}\,\boldsymbol{U}_h^n\,\mathrm{div}(\boldsymbol{U}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x \right)$$
$$+ \sum_{K\in\mathcal{T}} |K|\varrho_K^{n-1}\frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot \left(\frac{\boldsymbol{U}_{h,K}^{n-1} + \boldsymbol{U}_{h,K}^n}{2} - \boldsymbol{u}_K^{n-1} \right)$$
$$- \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}_K} |\sigma|\varrho_\sigma^{n,\mathrm{up}} \left(\frac{\boldsymbol{U}_{h,K}^n + \boldsymbol{U}_{h,L}^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \boldsymbol{U}_{h,K}^n[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]$$

$$
- \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}_K} |\sigma| p(\varrho_K^n)[\boldsymbol{U}_{h,\sigma}^n \cdot \boldsymbol{n}_{\sigma,K}] + \sum_{K\in\mathcal{T}} \frac{|K|}{k}(r_K^n - \varrho_K^n)(H'(r_K^n) - H'(r_K^{n-1}))
$$
$$
+ \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} H'(r_K^{n-1})[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]. \tag{5.11}
$$

We obtain formula (5.1) by summing $(5.11)^n$ from $n=1$ to $n=m$ and multiplying the resulting inequality by k. □

6. Approximate relative energy inequality for the discrete problem

The exact relative energy inequality as stated in Section 5 is a general inequality for the given numerical scheme, however, it does not immediately provide a comparison of the approximate solution with the strong solution of the compressible Navier–Stokes equations. Its right-hand side has to be conveniently transformed (modulo the possible appearance of residual terms vanishing as the space and time steps tend to 0) to provide such a comparison tool via a Gronwall-type argument.

The goal of this section is to derive a version of the discrete relative energy inequality, still with arbitrary (sufficientl regular) test functions $(r, \boldsymbol{U})$, that will be convenient for the comparison of the discrete solution with the strong solution.

LEMMA 6.1 (Approximate relative energy inequality) Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain and $\mathcal{T}$ is its regular triangulation introduced in Section 3.1. Let the pressure p be a $C^2(0,\infty)$ function satisfying hypotheses (1.5), (1.6) with $\gamma \geqslant 3/2$ and satisfying the additional condition (1.7) if $\gamma < 2$.

Let $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1\leqslant n\leqslant N} \in [L_h^+(\Omega)]^N$, $(\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \in [\boldsymbol{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.14) with the viscosity coefficient μ, λ obeying (1.4).

Then there exists

$$
c = c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1[\underline{r},\bar{r}]}, \|(\partial_t r, \partial_t^2 r, \nabla r, \partial_t \nabla r, \boldsymbol{U}, \partial_t \boldsymbol{U}, \nabla \boldsymbol{U}, \partial_t \nabla \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{31})}) > 0
$$

(where $\bar{r} = \max_{(t,x)\in\overline{Q_T}} r(t,x)$, $\underline{r} = \min_{(t,x)\in\overline{Q_T}} r(t,x)$), such that for all $m = 1, \ldots, N$, we have

$$
\begin{aligned}
&\mathcal{E}(\varrho^m, \boldsymbol{u}^m \mid r^m, \boldsymbol{U}^m) - \mathcal{E}(\varrho^0, \boldsymbol{u}^0 \mid r(0), \boldsymbol{U}(0)) \\
&\quad + k \sum_{n=1}^m \sum_{K\in\mathcal{T}} \left(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2 \,\mathrm{d}x + (\mu+\lambda) \int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}_h^n)|^2 \,\mathrm{d}x \right) \\
&\quad \leqslant k \sum_{n=1}^m \sum_{K\in\mathcal{T}} \left(\mu \int_K \nabla_x \boldsymbol{U}_h^n : \nabla_x(\boldsymbol{U}_h^n - \boldsymbol{u}^n) \,\mathrm{d}x + (\mu+\lambda) \int_K \mathrm{div}\, \boldsymbol{U}_h^n \,\mathrm{div}(\boldsymbol{U}_h^n - \boldsymbol{u}^n) \,\mathrm{d}x \right) \\
&\qquad + k \sum_{n=1}^m \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot (\boldsymbol{U}_{h,K}^n - \boldsymbol{u}_K^n) \\
&\qquad + k \sum_{n=1}^m \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}) \cdot (\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n) \hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}
\end{aligned}
$$

$$
\begin{aligned}
& - k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K p(\varrho_K^n) \operatorname{div} \boldsymbol{U}^n \, \mathrm{d}x + k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} [\partial_t r]^n \, \mathrm{d}x \\
& - k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \frac{\varrho_K^n}{r_K^n} p'(r_K^n) \boldsymbol{u}^n \cdot \nabla r^n \, \mathrm{d}x + R_{h,k}^m + G^m \qquad (6.1)
\end{aligned}
$$

for any pair $(r, \boldsymbol{U})$ belonging to the class (3.18), where

$$
|G^m| \leqslant ck \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \,|\, r^n, U^n), \quad |R_{h,k}^m| \leqslant c(\sqrt{k} + h^A),
$$

$$
\text{and} \quad A = \begin{cases} \dfrac{2\gamma - 3}{\gamma} & \text{if } \gamma \in [3/2, 2), \\ \dfrac{1}{2} & \text{if } \gamma \geqslant 2, \end{cases} \qquad (6.2)
$$

and where we have used notation (3.17) for r^n, $\boldsymbol{U}^n$ and (3.7–3.8) for $\mathbf{U}_h^n$, $\mathbf{U}_{h,K}^n$, r_K^n, $\mathbf{u}_\sigma^n$.

Proof. The right-hand side of the relative energy inequality (5.1) is a sum $\sum_{i=1}^{6} T_i$, where

$$
\begin{aligned}
T_1 &= k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \left(\mu \int_K \nabla_x \boldsymbol{U}_h^n : \nabla_x (\boldsymbol{U}_h^n - \boldsymbol{u}^n) \, \mathrm{d}x + (\mu + \lambda) \int_K \operatorname{div} \boldsymbol{U}_h^n \operatorname{div}(\boldsymbol{U}_h^n - \boldsymbol{u}^n) \, \mathrm{d}x \right), \\
T_2 &= k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot \left(\frac{\boldsymbol{U}_{h,K}^{n-1} + \boldsymbol{U}_{h,K}^n}{2} - \boldsymbol{u}_K^{n-1} \right), \\
T_3 &= -k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \sum_{\sigma = K|L \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\frac{\boldsymbol{U}_{h,K}^n + \boldsymbol{U}_{h,L}^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \boldsymbol{U}_{h,K}^n [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}], \\
T_4 &= -k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \sum_{\sigma = K|L \in \mathcal{E}(K)} |\sigma| p(\varrho_K) [\boldsymbol{U}_{h,\sigma}^n \cdot \boldsymbol{n}_{\sigma,K}], \\
T_5 &= k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} |K| (r_K^n - \varrho_K^n) \frac{H'(r_K^n) - H'(r_K^{n-1})}{k}, \\
T_6 &= k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \sum_{\sigma = K|L \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} H'(r_K^{n-1}) [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}].
\end{aligned}
$$

The term T_1 will be kept as it is; all the other terms T_i will be transformed to a more convenient form, as described in the following steps.

Step 1*: term* T_2. We have

$$
T_2 = T_{2,1} + R_{2,1},
$$

with

$$
T_{2,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot (\boldsymbol{U}_{h,K}^{n-1} - \boldsymbol{u}_K^{n-1})
$$

and

$$R_{2,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} R_{2,1}^{n,K},$$

where $R_{2,1}^{n,K} = (|K|/2)\varrho_K^{n-1}((\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1})^2/k) = (|K|/2)\varrho_K^{n-1}(([\boldsymbol{U}^n - \boldsymbol{U}^{n-1}]_{h,K})^2/k)$; owing to the mass conservation (3.15), the f rst-order Taylor formula applied to function $t \mapsto \mathbf{U}(t,x)$ on the interval (t_{n-1}, t_n) and the property (A.20) of the projection onto the space $V_h(\Omega)$, we obtain

$$|R_{2,1}^{n,K}| \leqslant \frac{M_0}{2}|K|k\|\partial_t \boldsymbol{U}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}. \tag{6.3}$$

Let us now decompose the term $T_{2,1}$ as

$$\begin{aligned} & T_{2,1} = T_{2,2} + R_{2,2}, \\ \text{with} \quad & T_{2,2} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} |K|\varrho_K^{n-1} \frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot (\boldsymbol{U}_{h,K}^n - \boldsymbol{u}_K^n) \\ \text{and} \quad & R_{2,2} = k \sum_{n=1}^{m} R_{2,2}^n, \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} R_{2,2}^n = & \sum_{K \in \mathcal{T}} |K|\varrho_K^{n-1} \frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot (\boldsymbol{U}_{h,K}^{n-1} - \boldsymbol{U}_{h,K}^n) \\ & - \sum_{K \in \mathcal{T}} |K|\varrho_K^{n-1} \frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot (\boldsymbol{u}_K^{n-1} - \boldsymbol{u}_K^n). \end{aligned}$$

By the same token as above, we may estimate the residual term as follows:

$$\begin{aligned} |R_{2,2}^n| \leqslant & \, kcM_0 \|\partial_t \boldsymbol{U}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))} \\ & + cM_0^{1/2} \left(\sum_{K \in \mathcal{T}} |K|\varrho_K^{n-1} |\boldsymbol{u}_K^{n-1} - \boldsymbol{u}_K^n|^2 \right)^{1/2} \|\partial_t \boldsymbol{U}\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}; \end{aligned}$$

then, by virtue of estimate (4.2a),

$$|R_{2,2}| \leqslant \sqrt{k}\, c(M_0, E_0, \|(\partial_t \boldsymbol{U}, \partial_t \nabla \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \tag{6.5}$$

Step 2: term T_3. Employing the definitio (3.13) of upwind quantities, we easily establish that

$$T_3 = T_{3,1} + R_{3,1},$$

with

$$T_{3,1} = k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}) \cdot \boldsymbol{U}_{h,K}^n \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K},$$

$$R_{3,1} = k \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} R_{3,1}^{n,\sigma}$$

and

$$R_{3,1}^{n,\sigma} = |\sigma| \varrho_K^n \frac{|\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,L}^n|^2}{2} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]^+ + |\sigma| \varrho_L^n \frac{|\boldsymbol{U}_{h,L}^n - \boldsymbol{U}_{h,K}^n|^2}{2} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,L}]^+ \quad \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}.$$

Employing estimates (A.1) and (A.21)$_{s=1}$ and the continuity of the mean value $\boldsymbol{U}_\sigma^n = \boldsymbol{U}_{h,\sigma}^n$ of $\boldsymbol{U}_h^n$ over faces σ, we infer

$$|R_{3,1}^{n,\sigma}| \leqslant h^2 c \|\nabla \boldsymbol{U}\|^2_{L^\infty(Q_T;\mathbb{R}^9)} |\sigma| (\varrho_K^n + \varrho_L^n) |\boldsymbol{u}_\sigma^n| \quad \forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}},$$

whence

$$\begin{aligned} |R_{3,1}| &\leqslant hc \|\nabla \boldsymbol{U}\|^2_{L^\infty(Q_T;\mathbb{R}^9)} \left(\sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} h|\sigma| (\varrho_K^n + \varrho_L^n)^{6/5} \right)^{5/6} \\ &\quad \times \left[k \sum_{n=1}^{m} \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| |\boldsymbol{u}_\sigma^n|^6 \right)^{1/3} \right]^{1/2} \\ &\leqslant hc(M_0, E_0, \|\nabla \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^9)}), \end{aligned} \tag{6.6}$$

provided $\gamma \geqslant 6/5$, owing to the Hölder inequality, the equivalence relation (3.21), the equivalence of norms (A.34) and energy bounds listed in Corollary 4.2.

Evidently, for each face $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, $\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} + \boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L} = 0$; then, finall ,

$$T_{3,1} = k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}) \cdot (\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_\sigma^n) \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}. \tag{6.7}$$

Let us now decompose the term $T_{3,1}$ as

$$T_{3,1} = T_{3,2} + R_{3,2}, \quad \text{with } R_{3,2} = k \sum_{n=1}^{m} R_{3,2}^n,$$

$$T_{3,2} = k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}) \cdot (\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n) \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}$$

and

$$R^n_{3,2} = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho^{n,\mathrm{up}}_\sigma (\hat{\boldsymbol{U}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma) \cdot (\boldsymbol{U}^n_\sigma - \boldsymbol{U}^n_{h,K})(\boldsymbol{u}^n_\sigma - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma) \cdot \boldsymbol{n}_{\sigma,K}.$$

By virtue of the Hölder inequality and the f rst-order Taylor formula applied to function $x \mapsto \mathbf{U}^n(x)$ in order to evaluate the difference $\boldsymbol{U}^n_\sigma - \boldsymbol{U}^n_{h,K}$, we obtain

$$\begin{aligned}|R^n_{3,2}| \leqslant c\|\nabla\mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^9)} &\left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| \varrho^{n,\mathrm{up}}_\sigma |\hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma - \hat{\boldsymbol{U}}^{n,\mathrm{up}}_{h,\sigma}|^2\right)^{1/2} \\ &\times \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\varrho^{n,\mathrm{up}}_\sigma|^{\gamma_0}\right)^{1/(2\gamma_0)} \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}^n_\sigma - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma|^q\right)^{1/q},\end{aligned}$$

where $\frac{1}{2} + \frac{1}{2\gamma_0} + \frac{1}{q} = 1$, $\gamma_0 = \min\{\gamma, 2\}$ and $\gamma \geqslant 3/2$. For the sum in the last term of the above product, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}^n_\sigma - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma|^q &\leqslant c \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}^n_\sigma - \boldsymbol{u}^n_K|^q \\ &\leqslant c\left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \|\boldsymbol{u}^n_\sigma - \boldsymbol{u}^n\|^q_{L^q(K;\mathbb{R}^3)} + \sum_{K\in\mathcal{T}} \|\boldsymbol{u}^n - \boldsymbol{u}^n_K\|^q_{L^q(K;\mathbb{R}^3)}\right) \\ &\leqslant ch^{((2\gamma_0-3)/2\gamma_0)q} |\boldsymbol{u}^n|^q_{V^2_h(\Omega;\mathbb{R}^3)},\end{aligned}$$

where we have used the definitio (3.13), the Minkowski inequality and the interpolation inequalities (A.18–A.19). Now, we can go back to the estimate of $R^n_{3,2}$ taking into account the upper bounds (4.9), (4.12–4.13), in order to obtain

$$|R_{3,2}| \leqslant h^A c(M_0, E_0, \|\nabla\mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^9)}) \tag{6.8}$$

provided $\gamma \geqslant 3/2$, where A is given in (6.2).

Finally, we rewrite term $T_{3,2}$ as

$$\begin{aligned}&T_{3,2} = T_{3,3} + R_{3,3}, \quad \text{with } R_{3,3} = k\sum_{n=1}^m R^n_{3,3}, \\ &T_{3,3} = k\sum_{n=1}^m \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho^{n,\mathrm{up}}_\sigma (\hat{\boldsymbol{U}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma) \cdot (\boldsymbol{U}^n_\sigma - \boldsymbol{U}^n_{h,K}) \hat{\boldsymbol{U}}^{n,\mathrm{up}}_{h,\sigma} \cdot \boldsymbol{n}_{\sigma,K} \quad \text{and} \\ &R^n_{3,3} = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho^{n,\mathrm{up}}_\sigma (\hat{\boldsymbol{U}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma) \cdot (\boldsymbol{U}^n_\sigma - \boldsymbol{U}^n_{h,K})(\hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma - \hat{\boldsymbol{U}}^{n,\mathrm{up}}_{h,\sigma}) \cdot \boldsymbol{n}_{\sigma,K};\end{aligned} \tag{6.9}$$

then,

$$|R_{3,3}| \leqslant c(\|\nabla\boldsymbol{U}\|_{L^\infty(Q_T,\mathbb{R}^9)}) k\sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n \mid r^n, \boldsymbol{U}^n). \tag{6.10}$$

Step 3*: term* T_4. Using the Stokes formula and the property (A.22) in Lemma A4, we easily see that

$$T_4 = -k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K p(\varrho_K^n) \operatorname{div} \boldsymbol{U}^n \, \mathrm{d}x. \tag{6.11}$$

Step 4*: term* T_5. Using the Taylor formula, we obtain

$$H'(r_K^n) - H'(r_K^{n-1}) = H''(r_K^n)(r_K^n - r_K^{n-1}) - \tfrac{1}{2} H'''(\bar{r}_K^n)(r_K^n - r_K^{n-1})^2,$$

where $\bar{r}_K^n \in [\min(r_K^{n-1}, r_K^n), \max(r_K^{n-1}, r_K^n)]$; we infer

$$T_5 = T_{5,1} + R_{5,1},$$

with

$$T_{5,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} |K| (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \frac{r_K^n - r_K^{n-1}}{k}, \qquad R_{5,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} R_{5,1}^{n,K}$$

and

$$R_{5,1}^{n,K} = \frac{1}{2} |K| H'''(\bar{r}_K^n) \frac{(r_K^n - r_K^{n-1})^2}{k} (\varrho_K^n - r_K^n).$$

Consequently, by the first-orde Taylor formula applied to function $t \mapsto r(t,x)$ on the interval (t_{n-1}, t_n) and owing to the mass conservation (3.15),

$$|R_{5,1}| \leqslant kc(M_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r},\bar{r}])}, \|\partial_t r\|_{L^\infty(Q_T)}), \tag{6.12}$$

where $\underline{r}$, $\bar{r}$ are define in (3.18).

Let us now decompose $T_{5,1}$ as follows:

$$\begin{aligned}
&T_{5,1} = T_{5,2} + R_{5,2}, \quad \text{with } T_{5,2} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} [\partial_t r]^n \mathrm{d}x, \\
&R_{5,2} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} R_{5,2}^{n,K} \\
&\text{and} \quad R_{5,2}^{n,K} = \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \left(\frac{r_K^n - r_K^{n-1}}{k} - [\partial_t r]^n \right) \mathrm{d}x.
\end{aligned} \tag{6.13}$$

In accordance with (3.17), here and in the sequel, $[\partial_t r]^n(x) = \partial_t r(t_n, x)$.

By the same means as the preceding residual term, we may estimate

$$|R_{5,2}| \leqslant kc(M_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r},\bar{r}])}, \|(\partial_t^2 r, \partial_t \nabla r)\|_{L^\infty(Q_T;\mathbb{R}^4)}). \tag{6.14}$$

Step 5*: term* T_6. Using the same argumentation as in formula (6.7), we may write

$$\begin{aligned} &T_6 = T_{6,1} + R_{6,1}, \quad R_{6,1} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} R_{6,1}^{n,\sigma,K}, \quad \text{with} \\ &T_{6,1} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma|\varrho_K^n(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}))\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K} \quad \text{and} \\ &R_{6,1}^{n,\sigma,K} = |\sigma|(\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n)(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}))\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K} \quad \text{for } \sigma = K|L. \end{aligned} \tag{6.15}$$

We estimate this term separately for $\gamma \leqslant 2$ and $\gamma > 2$. If $\gamma \leqslant 2$, motivated by Lemma 4.3, we may write

$$\begin{aligned} |R_{6,1}^{n,\sigma,K}| &\leqslant \sqrt{h}\|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)}|\sigma| \\ &\times\left(\frac{|\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n|}{\max(\varrho_K,\varrho_L)^{(2-\gamma)/2}}\sqrt{|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|}1_{\bar\varrho_\sigma^n\geqslant 1}\sqrt{h}(\varrho_K^n+\varrho_L^n)^{(2-\gamma)/2}\sqrt{|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|}\right. \\ &\left. +|\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n|\sqrt{|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|}1_{\bar\varrho_\sigma^n<1}\sqrt{h}\sqrt{|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|}\right), \end{aligned} \tag{6.16}$$

where we again use the f rst-order Taylor formula applied to function H' between end points r_K^{n-1}, r_σ^{n-1}, and where the numbers $\bar\varrho_\sigma^n$ are define in Lemma 4.1. Consequently, an application of the Hölder and Young inequalities yields

$$\begin{aligned} |R_{6,1}| &\leqslant \sqrt{h}c\|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)}k\sum_{n=1}^{m}\left[\left(\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|\frac{(\varrho_\sigma^{n,\mathrm{up}}-\varrho_K^n)^2}{\max(\varrho_K,\varrho_L)^{(2-\gamma)}}|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|1_{\bar\varrho_\sigma^n\geqslant 1}\right)^{1/2}\right. \\ &\times\left(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h\varrho_K^{2-\gamma}|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|\right)^{1/2} \\ &+\left(\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|h(\varrho_\sigma^{n,\mathrm{up}}-\varrho_K^n)^2|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|1_{\bar\varrho_\sigma^n<1}\right)^{1/2} \\ &\left.\times\left(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|\right)^{1/2}\right] \\ &\leqslant \sqrt{h}c\|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)}k\sum_{n=1}^{m}\left[\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|\frac{(\varrho_\sigma^{n,\mathrm{up}}-\varrho_K^n)^2}{\max(\varrho_K,\varrho_L)^{(2-\gamma)}}|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|1_{\bar\varrho_\sigma^n\geqslant 1}\right. \\ &+\left(\sum_{K\in\mathcal{T}}|K|\varrho_K^{6(2-\gamma)/5}\right)^{5/6}\left(\sum_{\sigma\in\mathcal{E}}|\sigma|h|\boldsymbol{u}_\sigma^n|^6\right)^{1/6} \end{aligned}$$

$$+\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|h(\varrho_\sigma^{n,\mathrm{up}}-\varrho_K^n)^2|\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}|1_{\bar{\varrho}_K^n<1}+|\Omega|^{5/6}\left(\sum_{\sigma\in\mathcal{E}}|\sigma|h|\boldsymbol{u}_\sigma^n|^6\right)^{1/6}\Bigg]$$
$$\leqslant\sqrt{h}\,c(M_0,E_0,\underline{r},\bar{r},|p'|_{C([\underline{r},\bar{r}])},\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}) \tag{6.17}$$

provided $\gamma\geqslant 12/11$, where we use estimate (4.16), estimates (4.10), (4.12) of Corollary 4.2 and equivalence relation (A.34). In the case $\gamma>2$, the same fina bound may be obtained by a similar argument, replacing estimate (4.16) by (4.15).

Let us now decompose the term $T_{6,1}$ as

$$T_{6,1}=T_{6,2}+R_{6,2},$$

with

$$T_{6,2}=k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|\varrho_K^nH''(r_K^{n-1})(r_K^{n-1}-r_\sigma^{n-1})[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}],$$
$$R_{6,2}=k\sum_{n=1}^m\sum_{K\in\mathcal{K}}\sum_{\sigma\in\mathcal{E}(K)}R_{6,2}^{n,\sigma,K}$$

and

$$R_{6,2}^{n,\sigma,K}=|\sigma|\varrho_K^n(H'(r_K^{n-1})-H'(r_\sigma^{n-1})-H''(r_K^{n-1})(r_K^{n-1}-r_\sigma^{n-1}))[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}].$$

Therefore, by virtue of the second-order Taylor formula applied to function H', the Hölder inequality, (A.29), (A.34), and (4.9), (4.12) in Corollary 4.2, we have, provided $\gamma\geqslant 6/5$,

$$|R_{6,2}|\leqslant hc(|H''|_{C([\underline{r},\bar{r}])}+|H'''|_{C([\underline{r},\bar{r}])})\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}\|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))}\|\boldsymbol{u}\|_{L^2(0,T;V_h^2(\Omega;\mathbb{R}^3))}$$
$$\leqslant hc(M_0,E_0,\underline{r},\bar{r},|p'|_{C^1([\underline{r},\bar{r}])},\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}), \tag{6.18}$$

where in the firs line we have used notation (3.11).

Let us now deal with the term $T_{6,2}$. Noting that $\int_K\nabla r^{n-1}\,\mathrm{d}x=\sum_{\sigma\in\mathcal{E}(K)}|\sigma|(r_\sigma^{n-1}-r_K^{n-1})\boldsymbol{n}_{\sigma,K}$, we may write $T_{6,2}=T_{6,3}+R_{6,3}$, with

$$T_{6,3}=-k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K\varrho_K^nH''(r_K^{n-1})\boldsymbol{u}^n\cdot\nabla r^{n-1}\,\mathrm{d}x,$$
$$R_{6,3}=k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K\varrho_K^nH''(r_K^{n-1})(\boldsymbol{u}^n-\boldsymbol{u}_K^n)\cdot\nabla r^{n-1}\,\mathrm{d}x$$
$$+k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_K^nH''(r_K^{n-1})(r_K^{n-1}-r_\sigma^{n-1})(\boldsymbol{u}_\sigma^n-\boldsymbol{u}_K^n)\cdot\boldsymbol{n}_{\sigma,K},$$

where, by virtue of the Hölder inequality, (A.18), (A.19), and (4.9), (4.12) in Corollary 4.2,

$$|R_{6,3}|\leqslant h^A\,c(M_0,E_0,\underline{r},\bar{r},|p'|_{C^1([\underline{r},\bar{r}])}\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}), \tag{6.19}$$

where A is define in (6.2).

Finally, we write $T_{6,3} = T_{6,4} + R_{6,4}$, with

$$\begin{aligned} T_{6,4} &= -k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \varrho_K^n \frac{p'(r_K^n)}{r_K^n} \boldsymbol{u}^n \cdot \nabla r^n \,\mathrm{d}x, \\ R_{6,4} &= k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \varrho_K^n (H''(r_K^n)\nabla r^n - H''(r_K^{n-1})\nabla r^{n-1}) \cdot \boldsymbol{u}^n \,\mathrm{d}x, \end{aligned} \tag{6.20}$$

where

$$|R_{6,4}| \leqslant k\, c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r},\bar{r}])}, \|\nabla r, \partial_t r, \partial_t \nabla r\|_{L^\infty(Q_T;\mathbb{R}^7)}). \tag{6.21}$$

We are now in position to conclude the proof of Lemma 6.1: we obtain inequality (6.1) by gathering the principal terms (6.4), (6.9), (6.11), (6.13), (6.20) and the residual terms estimated in (6.3), (6.5), (6.6), (6.8), (6.10), (6.12), (6.14), (6.16–6.19), (6.21) on the right-hand side $\sum_{i=1}^{6} T_i$ of the discrete relative energy inequality (5.1). □

7. A discrete identity satisfied by the strong solution

This section is devoted to the proof of a discrete identity satisfie by any strong solution. This identity is stated in Lemma 7.1. It will be used in combination with the approximate relative energy inequality stated in Lemma 6.1 to deduce the convenient form of the relative energy inequality verifie by any function being a strong solution to the compressible Navier–Stokes system. This last step is performed in the next section.

LEMMA 7.1 (A discrete identity for strong solutions) Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain and $\mathcal{T}$ is a regular triangulation introduced in Section 3.1. Let the pressure p be a $C^2(0,\infty)$ function satisfying hypotheses (1.5) and (1.6) with $\gamma \geqslant 3/2$. Let $(r, \boldsymbol{U})$ belong to the class (3.18) and satisfy equation (1.1) with the viscosity coefficient μ, λ obeying (1.4).

Let $(\varrho^0, \boldsymbol{u}^0) \in L_h^+(\Omega) \times \boldsymbol{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1\leqslant n\leqslant N} \in [L_h^+(\Omega)]^N$, $(\boldsymbol{u}^n)_{1\leqslant n\leqslant N} \in [\boldsymbol{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.14). Then there exists

$$c = c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r},\bar{r}])}, \|(\nabla r, \partial_t r, \boldsymbol{U}, \nabla \boldsymbol{U}, \nabla^2 \boldsymbol{U}, \partial_t \boldsymbol{U}, \partial_t^2 \boldsymbol{U}, \partial_t \nabla \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{58})}) > 0,$$

such that, for any $m = 1, \ldots, N$, the following identity holds:

$$\begin{aligned} & k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K (\mu \nabla \boldsymbol{U}_h^n \cdot \nabla(\boldsymbol{u}^n - \boldsymbol{U}_h^n) + (\mu+\lambda)\,\mathrm{div}\,\boldsymbol{U}_h^n\,\mathrm{div}\,(\boldsymbol{u}^n - \boldsymbol{U}_h^n))\,\mathrm{d}x \\ & + k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r_K^{n-1} \frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k} \cdot (\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n)\,\mathrm{d}x \\ & + k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\mathrm{up}} [\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}] (\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n) \cdot (\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}) \\ & + k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K p(r_K^n)\,\mathrm{div}\,\boldsymbol{U}^n\,\mathrm{d}x + k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K p'(r_K^n) \boldsymbol{u}^n \cdot \nabla r^n\,\mathrm{d}x + \mathcal{R}_{h,k}^m = 0, \end{aligned} \tag{7.1}$$

where

$$|\mathcal{R}_{h,k}^m| \leqslant c(h+k)$$

and where we have used notation (3.17) for r^n, $\boldsymbol{U}^n$ and (3.7–3.8) for $\mathbf{U}_h^n$, $\mathbf{U}_{h,K}^n$, r_K^n, $\mathbf{u}_\sigma^n$.

Before starting the proof, we recall an auxiliary algebraic inequality whose straightforward proof is left to the reader, and introduce some notation.

LEMMA 7.2 Let p satisfy assumptions (1.5) and (1.6). Let $0 < a < b < \infty$. Then there exists $c = c(a,b) > 0$ such that, for all $\varrho \in [0,\infty)$ and $r \in [a,b]$, there holds

$$E(\varrho|r) \geqslant c(a,b)(1_{R_+\setminus[a/2,2b]}(\varrho) + \varrho^\gamma 1_{R_+\setminus[a/2,2b]}(\varrho) + (\varrho - r)^2 1_{[a/2,2b]}(\varrho)),$$

where $E(\varrho|r)$ is define in (2.5).

If we take, in Lemma 7.2, $\varrho = \varrho^n(x)$, $\varrho^n \in L_h^+(\Omega)$, $r = \hat{r}^n(x)$, $a = \underline{r}$, $b = \bar{r}$ (where r is a function belonging to class (3.18) and $\underline{r}$, $\bar{r}$ are its lower and upper bounds, respectively), we obtain

$$\begin{aligned} E(\varrho^n(x)|\hat{r}^n(x)) \geqslant c(\underline{r},\bar{r})(1_{R_+\setminus[\underline{r}/2,2\bar{r}]}(\varrho^n(x)) + (\varrho^n)^\gamma(x) 1_{R_+\setminus[\underline{r}/2,2\bar{r}]}(\varrho^n(x)) \\ + (\varrho^n(x) - \hat{r}^n(x))^2 1_{[\underline{r}/2,2\bar{r}]}(\varrho^n(x))). \end{aligned} \tag{7.2}$$

Now, for fixe numbers $\underline{r}$ and $\bar{r}$ and fixe functions $\varrho^n \in L_h^+(\Omega)$, $n = 0,\dots,N$, we introduce the residual and essential subsets of Ω (relative to ϱ^n) as follows:

$$N_{\rm ess}^n = \{x \in \Omega \mid \tfrac{1}{2}\underline{r} \leqslant \varrho^n(x) \leqslant 2\bar{r}\}, \quad N_{\rm res}^n = \Omega \setminus N_{\rm ess}^n, \tag{7.3}$$

and we set

$$[g]_{\rm ess}(x) = g(x) 1_{N_{\rm ess}^n}(x), \quad [g]_{\rm res}(x) = g(x) 1_{N_{\rm res}^n}(x), \quad x \in \Omega,\ g \in L^1(\Omega).$$

Integrating inequality (7.2), we deduce

$$c(\underline{r},\bar{r}) \sum_{K\in\mathcal{T}} \int_K ([1]_{\rm res} + [(\varrho^n)^\gamma]_{\rm res} + [\varrho^n - \hat{r}^n]_{\rm ess}^2)\,{\rm d}x \leqslant \mathcal{E}(\varrho^n, \boldsymbol{u}^n \mid r^n, \boldsymbol{U}^n) \tag{7.4}$$

for any pair $(r, \boldsymbol{U})$ belonging to the class (3.18) and any $\varrho^n \in L_h(\Omega)$.

We are now ready to proceed to the proof of Lemma 7.1.

Proof. We start by projecting the momentum equation to the discrete spaces. Since $(r, \boldsymbol{U})$ satisfie (1.1) and belongs to the class (3.18), equation (1.1b) can be rewritten in the form

$$r\partial_t \boldsymbol{U} + r\boldsymbol{U}\cdot\nabla\boldsymbol{U} + \nabla p(r) = \mu\Delta\boldsymbol{U} + (\mu+\lambda)\nabla\,{\rm div}\,\boldsymbol{U}. \tag{7.5}$$

We write equation (7.5) at $t = t_n$, multiply scalar by $\boldsymbol{u}^n - \boldsymbol{U}_h^n$ and integrate over Ω. We get, after summation from $n = 1$ to m,

$$\sum_{i=1}^{5} \mathcal{T}_i = 0, \quad \text{with} \quad \mathcal{T}_1 = -k \sum_{n=1}^{m} \int_{\Omega} (\mu \Delta \boldsymbol{U}^n + (\mu+\lambda) \nabla \operatorname{div} \boldsymbol{U}^n) \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x,$$
$$\mathcal{T}_2 = k \sum_{n=1}^{m} \int_{\Omega} r^n [\partial_t \boldsymbol{U}]^n \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x, \quad \mathcal{T}_3 = k \sum_{n=1}^{m} \int_{\Omega} r^n \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x, \tag{7.6}$$
$$\mathcal{T}_4 = k \sum_{n=1}^{m} \int_{\Omega} \nabla p(r^n) \cdot \boldsymbol{u}^n\, \mathrm{d}x, \quad \mathcal{T}_5 = -k \sum_{n=1}^{m} \int_{\Omega} \nabla p(r^n) \cdot \boldsymbol{U}_h^n\, \mathrm{d}x.$$

In the steps below, we deal with each of the terms $\mathcal{T}_i$.

Step 1*: term* $\mathcal{T}_1$. Integrating by parts, we obtain

$$\mathcal{T}_1 = \mathcal{T}_{1,1} + \mathcal{R}_{1,1}, \quad \text{with}$$
$$\mathcal{T}_{1,1} = k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K (\mu \nabla \boldsymbol{U}_h^n : \nabla(\boldsymbol{u}^n - \boldsymbol{U}_h^n) + (\mu+\lambda) \operatorname{div} \boldsymbol{U}_h^n \operatorname{div}(\boldsymbol{u}^n - \boldsymbol{U}_h^n))\, \mathrm{d}x \quad \text{and}$$
$$\mathcal{R}_{1,1} = I_1 + I_2, \quad \text{with} \tag{7.7}$$
$$I_1 = k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K (\mu \nabla(\boldsymbol{U}^n - \boldsymbol{U}_h^n) : \nabla(\boldsymbol{u}^n - \boldsymbol{U}_h^n) + (\mu+\lambda) \operatorname{div}(\boldsymbol{U}^n - \boldsymbol{U}_h^n) \operatorname{div}(\boldsymbol{u}^n - \boldsymbol{U}_h^n))\, \mathrm{d}x,$$
$$I_2 = -k \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_{\sigma} (\mu \boldsymbol{n}_{\sigma,K} \cdot \nabla \boldsymbol{U}^n \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n) + (\lambda+\mu) \operatorname{div} \boldsymbol{U}^n (\boldsymbol{u}^n - \boldsymbol{U}_h^n) \cdot \boldsymbol{n}_{\sigma,K})\, \mathrm{d}S,$$

where in the last line $\mathbf{n}_\sigma$ is a unit normal to σ and $[\cdot]_{\sigma,\mathbf{n}_\sigma}$ is the jump over sigma (with respect to $\mathbf{n}_\sigma$) define in Lemma A9. Employing estimate (A.21), we easily verify that

$$|I_1| \leqslant h\, c(M_0, E_0, \|\nabla \mathbf{U}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}).$$

Since the integral over any face σ of the jump of a function from $V_h(\Omega)$ is zero, we may write

$$I_2 = -k \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}} \int_{\sigma} (\mu \boldsymbol{n}_\sigma \cdot (\nabla \boldsymbol{U}^n - [\nabla \boldsymbol{U}^n]_\sigma) \cdot [\boldsymbol{u}^n - \boldsymbol{U}_h^n]_{\sigma,\mathbf{n}_\sigma}$$
$$+ (\lambda+\mu)(\operatorname{div} \boldsymbol{U}^n - [\operatorname{div} \boldsymbol{U}^n]_\sigma)[\boldsymbol{u}^n - \boldsymbol{U}_h^n]_{\sigma,\mathbf{n}_\sigma} \cdot \boldsymbol{n}_\sigma)\, \mathrm{d}S;$$

then, by using the first-orde Taylor formula applied to functions $x \mapsto \nabla \mathbf{U}^n(x)$ to evaluate the differences $\nabla \boldsymbol{U}^n - [\nabla \boldsymbol{U}^n]_\sigma$, $\operatorname{div} \boldsymbol{U}^n - [\operatorname{div} \boldsymbol{U}^n]_\sigma$, and the Hölder inequality,

$$|I_2| \leqslant khc \|\nabla^2 \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{27})} \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}} \sqrt{|\sigma|}\sqrt{h} \left(\frac{1}{\sqrt{h}} \|[\boldsymbol{u}^n - \boldsymbol{U}_h^n]_{\sigma,\mathbf{n}_\sigma}\|_{L^2(\sigma;\mathbb{R}^3)} \right)$$
$$\leqslant khc \|\nabla^2 \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{27})} \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}} \left(|\sigma| h + \frac{1}{h} \|[\boldsymbol{u}^n - \boldsymbol{U}_h^n]_{\sigma,\mathbf{n}_\sigma}\|^2_{L^2(\sigma;\mathbb{R}^3)} \right).$$

Therefore,

$$|\mathcal{R}_{1,1}| \leqslant hc(M_0, E_0, \|\mathbf{U}, \nabla \boldsymbol{U}, \nabla^2 \boldsymbol{U}\|_{L^\infty(Q_T, \mathbb{R}^{39})}), \tag{7.8}$$

where we have employed Lemma A9 and (4.9) in Corollary 4.2.

Step 2*: term* $\mathcal{T}_2$. Let us now decompose the term $\mathcal{T}_2$ as

$$\mathcal{T}_2 = \mathcal{T}_{2,1} + \mathcal{R}_{2,1},$$

$$\text{with} \quad \mathcal{T}_{2,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K r^{n-1} \frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k} \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x, \quad \mathcal{R}_{2,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \mathcal{R}_{2,1}^{n,K}$$

$$\text{and} \quad \mathcal{R}_{2,1}^{n,K} = \int_K (r^n - r^{n-1})[\partial_t \mathbf{U}]^n \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x + \int_K r^{n-1} \left([\partial_t \boldsymbol{U}]^n - \frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k} \right) \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x.$$

We have

$$\begin{aligned} |\mathcal{R}_{2,1}^{n,K}| \leqslant k[(\|r\|_{L^\infty(Q_T)} + \|\partial_t r\|_{L^\infty(Q_T)})(\|\partial_t \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^3)} \\ + \|\partial_t^2 \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^3)})\sqrt{|K|}(\|\boldsymbol{u}^n\|_{L^2(K)} + \|\boldsymbol{U}_h^n\|_{L^2(K)}); \end{aligned}$$

then,

$$|\mathcal{R}_{2,1}| \leqslant kc(M_0, E_0, \bar{r}, \|(\partial_t r, \boldsymbol{U}, \partial_t \boldsymbol{U}, \nabla \boldsymbol{U}, \partial_t^2 \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{19})}), \tag{7.9}$$

owing to the Hölder and Young inequalities, to the estimates (A.20), (A.23), (A.28), (A.29) and to the energy bound (4.9) from Corollary 4.2.

Step 2a*: term* $\mathcal{T}_{2,1}$. We decompose the term $\mathcal{T}_{2,1}$ as

$$\mathcal{T}_{2,1} = \mathcal{T}_{2,2} + \mathcal{R}_{2,2},$$

$$\text{with} \quad \mathcal{T}_{2,2} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K r_K^{n-1} \frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k} \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x,$$

$$\mathcal{R}_{2,2} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \mathcal{R}_{2,2}^{n,K}$$

$$\text{and} \quad \mathcal{R}_{2,2}^{n,K} = \int_K (r^{n-1} - r_K^{n-1}) \frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k} \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\, \mathrm{d}x;$$

therefore,

$$|\mathcal{R}_{2,2}^n| = \left| \sum_{K \in \mathcal{T}} \mathcal{R}_{2,2}^{n,K} \right| \leqslant hc \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)} \|\partial_t \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^3)} \|\boldsymbol{u}^n - \boldsymbol{U}_h^n\|_{L^6(\Omega;\mathbb{R}^3)}.$$

Consequently, by virtue of formula (4.10) in Corollary 4.2 and estimates (A.29), (A.24),

$$|\mathcal{R}_{2,2}| \leqslant hc(M_0, E_0, \|(\nabla r, \boldsymbol{U}, \partial_t \boldsymbol{U}, \nabla \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{18})}). \tag{7.10}$$

Step 2b*: term* $\mathcal{T}_{2,2}$. We decompose the term $\mathcal{T}_{2,2}$ as

$$
\begin{aligned}
&\mathcal{T}_{2,2} = \mathcal{T}_{2,3} + \mathcal{R}_{2,3},\\
\text{with}\quad &\mathcal{T}_{2,3} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\int_K r_K^{n-1}\frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k}\cdot(\boldsymbol{u}^n - \boldsymbol{U}_h^n)\,\mathrm{d}x,\\
&\mathcal{R}_{2,3} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\mathcal{R}_{2,3}^{n,K}\\
\text{and}\quad &\mathcal{R}_{2,3}^{n,K} = \int_K r_K^{n-1}\left(\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k} - \left[\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k}\right]_h\right)\cdot(\boldsymbol{u}^n - \boldsymbol{U}_h^n)\,\mathrm{d}x\\
&\qquad + \int_K r_K^{n-1}\left(\left[\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k}\right]_h - \left[\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{k}\right]_{h,K}\right)\cdot(\boldsymbol{u}^n - \boldsymbol{U}_h^n)\,\mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
|\mathcal{R}_{2,3}^n| = \left|\sum_{K\in\mathcal{T}}\mathcal{R}_{2,3}^{n,K}\right| \leqslant hc\|r\|_{L^\infty(Q_T)}\|\partial_t\boldsymbol{U},\partial_t\nabla\boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{13})}\|\boldsymbol{u}^n - \boldsymbol{U}_h^n\|_{L^6(\Omega;\mathbb{R}^3)},
$$

where we have used the first-order Taylor formula applied to function $t \mapsto U(t,x)$ on the interval (t_{n-1},t_n), the Hölder inequality, (A.1), (A.21)$_{s=1}$, (A.14) and (A.28). Consequently, by virtue of formula (4.10) in Corollary 4.2 and estimates (A.29), (A.24),

$$
|\mathcal{R}_{2,3}| \leqslant hc(M_0,E_0,\|(\nabla r,\boldsymbol{U},\nabla\boldsymbol{U},\partial_t\boldsymbol{U},\partial_t\nabla\boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{25})}). \tag{7.11}
$$

Step 2c*: term* $\mathcal{T}_{2,3}$. We rewrite this term in the form

$$
\begin{aligned}
&\mathcal{T}_{2,3} = \mathcal{T}_{2,4} + \mathcal{R}_{2,4},\quad \mathcal{R}_{2,4} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\mathcal{R}_{2,4}^{n,K},\\
\text{with}\quad &\mathcal{T}_{2,4} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\int_K r_K^{n-1}\frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k}\cdot(\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n)\,\mathrm{d}x\\
\text{and}\quad &\mathcal{R}_{2,4}^{n,K} = \int_K r_K^{n-1}\frac{\boldsymbol{U}_{h,K}^n - \boldsymbol{U}_{h,K}^{n-1}}{k}\cdot((\boldsymbol{u}^n - \boldsymbol{u}_K^n) - (\boldsymbol{U}_h^n - \boldsymbol{U}_{h,K}^n))\,\mathrm{d}x.
\end{aligned} \tag{7.12}
$$

Owing to the Hölder inequality, to the estimates (A.1), (A.14), (A.21)$_{s=1}$, (A.28) and finall to estimate (4.9) in Corollary 4.2, we then obtain

$$
|\mathcal{R}_{2,4}| \leqslant hc(M_0,E_0,\bar{r},\|(\partial_t\boldsymbol{U},\boldsymbol{U},\nabla\boldsymbol{U},\partial_t\nabla\boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{24})}). \tag{7.13}
$$

Step 3: *term* $\mathcal{T}_3$. Let us firs decompose $\mathcal{T}_3$ as

$$\mathcal{T}_3 = \mathcal{T}_{3,1} + \mathcal{R}_{3,1}, \quad \text{with}$$

$$\mathcal{T}_{3,1} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\int_K r_K^n \boldsymbol{U}_{h,K}^n \cdot \nabla \boldsymbol{U}^n \cdot (\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n)\,\mathrm{d}x,$$

$$\mathcal{R}_{3,1} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}} \mathcal{R}_{3,1}^{n,K} \quad \text{and}$$

$$\begin{aligned}\mathcal{R}_{3,1}^{n,K} = &\int_K (r^n - r_K^n)\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n)\,\mathrm{d}x + \int_K r_K^n(\boldsymbol{U}^n - \boldsymbol{U}_h^n)\cdot\nabla\boldsymbol{U}^n\cdot(\boldsymbol{u}^n - \boldsymbol{U}_h^n)\,\mathrm{d}x\\ &+ \int_K r_K^n(\boldsymbol{U}_h^n - \boldsymbol{U}_{h,K}^n)\cdot\nabla\boldsymbol{U}^n\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{d}x\\ &+ \int_K r_K^n\boldsymbol{U}_{h,K}^n\cdot\nabla\boldsymbol{U}^n\cdot(\boldsymbol{u}^n - \boldsymbol{u}_K^n - (\boldsymbol{U}_h^n - \boldsymbol{U}_{h,K}^n))\,\mathrm{d}x.\end{aligned}$$

We fin that

$$\begin{aligned}|\mathcal{R}_{3,1}^{n,K}| \leqslant\ & h[|K|^{1/2}(\|\boldsymbol{u}^n\|_{L^2(K;\mathbb{R}^3)} + \|\boldsymbol{U}_h^n\|_{L^2(K;\mathbb{R}^3)}) + |K|^{1/2}(\|\nabla\boldsymbol{u}^n\|_{L^2(K;\mathbb{R}^3)} + \|\nabla\boldsymbol{U}_h^n\|_{L^2(K;\mathbb{R}^3)})]\\ &\times(\|r\|_{L^\infty(Q_T)} + \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)})(\|\boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^3)} + \|\nabla\boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^9)})^2,\end{aligned}$$

where several times we have used the Hölder inequality and the standard f rst-order Taylor formula (to evaluate $r^n - r_K^n$), along with the estimates (A.20) (to evaluate $\boldsymbol{U}^n - \boldsymbol{U}_h^n$), (A.1), $(\text{A.21})_{s=1}$ (to evaluate $\boldsymbol{U}_h^n - \boldsymbol{U}_{h,K}^n$) and (A.1) (to evaluate $\boldsymbol{u}^n - \boldsymbol{u}_K^n$).

Consequently, again using $(\text{A.21})_{s=1}$ (to estimate $\|\nabla\boldsymbol{U}_h^n\|_{L^2(K;\mathbb{R}^3)}$), the definitio of the $|\cdot|_{V_h^2(\Omega)}$-norm, the Sobolev inequality (A.29) and the energy bound (4.9) from Corollary 4.2, we conclude that

$$|\mathcal{R}_{3,1}| \leqslant hc(M_0, E_0, \bar{r}, \|(\nabla r, \boldsymbol{U}, \nabla\boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{15})}). \tag{7.14}$$

Now we shall deal with the term $\mathcal{T}_{3,1}$. Integrating by parts, we obtain

$$\begin{aligned}\int_K r_K^n\boldsymbol{U}_{h,K}^n\cdot\nabla\boldsymbol{U}^n\cdot(\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n)\,\mathrm{d}x &= \sum_{\sigma\in\mathcal{E}(K)}|\sigma| r_K^n[\boldsymbol{U}_{h,K}^n\cdot\mathbf{n}_{\sigma,K}]\boldsymbol{U}_\sigma^n\cdot(\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n)\\ &= \sum_{\sigma\in\mathcal{E}(K)}|\sigma| r_K^n[\boldsymbol{U}_{h,K}^n\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n)\cdot(\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n),\end{aligned}$$

owing to the fact that $\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma \boldsymbol{U}_{h,K}^n\cdot\mathbf{n}_{\sigma,K}\mathrm{d}S = 0$.

Next we write

$$\mathcal{T}_{3,1} = \mathcal{T}_{3,2} + \mathcal{R}_{3,2}, \quad \mathcal{R}_{3,2} = k\sum_{n=1}^{m}\mathcal{R}_{3,2}^n,$$

$$\mathcal{T}_{3,2} = k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\hat{r}_\sigma^{n,\mathrm{up}}[\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n)\cdot(\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}) \tag{7.15}$$

and

$$\mathcal{R}_{3,2}^n = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|(r_K^n - \hat{r}_\sigma^{n,\mathrm{up}})[\boldsymbol{U}_{h,K}^n\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n)\cdot(\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n)$$
$$+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}_\sigma^{n,\mathrm{up}}[(\boldsymbol{U}_{h,K}^n - \hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}})\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n)\cdot(\boldsymbol{u}_K^n - \boldsymbol{U}_{h,K}^n)$$
$$+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}_\sigma^{n,\mathrm{up}}[\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{U}_\sigma^n - \boldsymbol{U}_{h,K}^n)\cdot((\boldsymbol{u}_K^n - \hat{\boldsymbol{u}}_{h,\sigma}^{n,\mathrm{up}}) - (\boldsymbol{U}_{h,K}^n - \hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}})).$$

We may use the Taylor formula several times (in order to estimate $r_K^n - \hat{r}_\sigma^{n,\mathrm{up}}$, $\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n$, $\mathbf{U}_{h,K}^n - \hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}}$) to get the bound

$$|\mathcal{R}_{3,2}^n| \leqslant hc\|r\|_{W^{1,\infty}(\Omega)}(1+\|\boldsymbol{U}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)})^3\sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n|$$
$$+ c\|r\|_{W^{1,\infty}(\Omega)}(1+\|\boldsymbol{U}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)})^2\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}_K^n - \boldsymbol{u}_\sigma^n|,$$

where by virtue of the Hölder inequality, (A.16), (A.30), (A.18) and (A.19),

$$\sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n| \leqslant c\left[\left(\sum_{K\in\mathcal{T}}\|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|_{L^6(K;\mathbb{R}^3)}^6\right)^{1/6} + \left(\sum_{K\in\mathcal{T}}\|\boldsymbol{u}^n\|_{L^6(K;\mathbb{R}^3)}\right)^{1/6}\right] \leqslant c|\boldsymbol{u}_n|_{V_h^2(\Omega;\mathbb{R}^3)},$$

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}_k^n - \boldsymbol{u}_\sigma^n| \leqslant c\left[\left(\sum_{K\in\mathcal{T}}\|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|_{L^2(K;\mathbb{R}^3)}^2\right)^{1/2}\right.$$
$$\left.+\left(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}\|\boldsymbol{u}^n - \boldsymbol{u}_\sigma^n\|_{L^6(K;\mathbb{R}^3)}^2\right)^{1/2}\right] \leqslant h\,c|\boldsymbol{u}_n|_{V_h^2(\Omega;\mathbb{R}^3)}.$$

Consequently, we may use (4.9) to conclude

$$|\mathcal{R}_{3,2}| \leqslant hc(M_0, E_0, \bar{r}, \|\nabla r, \boldsymbol{U}, \nabla\boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{15})}). \tag{7.16}$$

Step 4*: terms* $\mathcal{T}_4$ and $\mathcal{T}_5$. We decompose $\mathcal{T}_4$ as

$$\mathcal{T}_4 = \mathcal{T}_{4,1} + \mathcal{R}_{4,1},$$
$$\text{with}\quad \mathcal{T}_{4,1} = k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K p'(r_K^n)\boldsymbol{u}^n\cdot\nabla r^n\,\mathrm{d}x,$$
$$\mathcal{R}_{4,1} = k\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K (p'(r^n) - p'(r_K^n))\boldsymbol{u}^n\cdot\nabla r^n\,\mathrm{d}x; \tag{7.17}$$

then,

$$|\mathcal{R}_{4,1}| \leqslant h\,c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r},\bar{r}])}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}). \tag{7.18}$$

Employing integration by parts, we infer

$$\mathcal{T}_5 = \mathcal{T}_{5,1} + \mathcal{R}_{5,1}, \quad \text{with } \mathcal{T}_{5,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K p(r_K^n) \operatorname{div} \boldsymbol{U}^n \, \mathrm{d}x,$$
$$\mathcal{R}_{5,1} = k \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K [\nabla p(r^n) \cdot (\boldsymbol{U}^n - \boldsymbol{U}_h^n) + (p(r^n) - p(r_K^n)) \operatorname{div} \boldsymbol{U}^n] \, \mathrm{d}x \tag{7.19}$$

and

$$|\mathcal{R}_{5,1}| \leqslant h\, c(\bar{r}, |p'|_{C([\underline{r},\bar{r}])}, \|\nabla r, \nabla \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \tag{7.20}$$

Gathering the formulae (7.7), (7.12), (7.15), (7.17), (7.19) and estimates for the residual terms (7.8), (7.9–7.13), (7.14–7.16), (7.18), (7.20) concludes the proof of Lemma 7.1. □

8. End of the proof of the error estimate (Theorem 3.2)

In this section, we put together the relative energy inequality (6.1) and the identity (7.1) derived in the previous section. The fina inequality resulting from this manipulation is formulated in the following lemma.

LEMMA 8.1 Under the assumptions of Theorem 3.2 there exists a positive number

$$c = c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r},\bar{r}])}, \|(\nabla r, \partial_t r, \partial_t \nabla r, \partial_t^2 r, \boldsymbol{U}, \nabla \boldsymbol{U}, \nabla^2 \boldsymbol{U}, \partial_t \boldsymbol{U}, \partial_t \nabla \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{65})})$$

(depending tacitly also on T, θ_0, γ, diam(Ω), $|\Omega|$), such that for all $m = 1, \ldots, N$, there holds

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m | r^m, \boldsymbol{u}^m) + k\frac{\mu}{2} \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K |\nabla_x(\mathbf{u}^n - \mathbf{U}_h^n)|^2 \, \mathrm{d}x$$
$$\leqslant c[h^A + \sqrt{k} + \mathcal{E}(\varrho^0, \boldsymbol{u}^0 | r^0, \boldsymbol{U}^0)] + ck \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n | r^n, \boldsymbol{u}^n),$$

where A is define in (6.2).

Proof. Gathering the formulae (6.1) and (7.1), one gets

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m \mid r^m, U^m) - \mathcal{E}(\varrho^0, \boldsymbol{u}^0 \mid r(0), \boldsymbol{U}(0)) + \mu k \sum_{n=1}^{m} |\boldsymbol{u}^n - \boldsymbol{U}_h^n|^2_{V_h^2(\Omega;\mathbb{R}^3)} \leqslant \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{Q}, \tag{8.1}$$

where

$$
\begin{aligned}
\mathcal{P}_1 &= k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}|K|(\varrho_K^{n-1}-r_K^{n-1})\frac{\boldsymbol{U}_{h,K}^n-\boldsymbol{U}_{h,K}^{n-1}}{k}\cdot(\boldsymbol{U}_{h,K}^n-\boldsymbol{u}_K^n),\\
\mathcal{P}_2 &= k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|(\varrho_\sigma^{n,\mathrm{up}}-\hat{r}_\sigma^{n,\mathrm{up}})(\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}-\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}})\cdot(\boldsymbol{U}_\sigma^n-\boldsymbol{U}_{h,K}^n)\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}\cdot\boldsymbol{n}_{\sigma,K},\\
\mathcal{P}_3 &= k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\int_K(p(r_K^n)-p(\varrho_K^n))\,\mathrm{div}\,\boldsymbol{U}^n\,\mathrm{d}x\\
&\quad+k\sum_{n=1}^{m}\sum_{K\in\mathcal{T}}\left[\int_K\frac{r_K^n-\varrho_K^n}{r_K^n}p'(r_K^n)\boldsymbol{u}^n\cdot\nabla r^n\,\mathrm{d}x+\int_K\frac{r_K^n-\varrho_K^n}{r_K^n}p'(r_K^n)[\partial_t r]^n\,\mathrm{d}x\right],\\
\mathcal{Q} &= \mathcal{R}_{h,k}^m+R_{h,k}^m+G^m.
\end{aligned}
$$

Now, we conveniently estimate the terms $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ in three steps.

Step 1*: term* $\mathcal{P}_1$. Owing to the Hölder inequality and to the Taylor formula applied to function $t\mapsto U(t,x)$ on the interval (t_{n-1},t_n), we have

$$
\begin{aligned}
&\left|\sum_{K\in\mathcal{T}}|K|(\varrho_K^{n-1}-r_K^{n-1})\frac{\boldsymbol{U}_{h,K}^n-\boldsymbol{U}_{h,K}^{n-1}}{k}\cdot(\boldsymbol{U}_{h,K}^n-\boldsymbol{u}_K^n)\right|\leqslant c(\|\partial_t\boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^3)}+\|\partial_t\nabla\boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^9)})\\
&\times\left[\left(\sum_{K\in\mathcal{T}}|K||\varrho_K^{n-1}-r_K^{n-1}|^2 1_{[\underline{r}/2,2\overline{r}]}(\varrho_K)\right)^{1/2}+\left(\sum_{K\in\mathcal{T}}|K||\varrho_K^{n-1}-r_K^{n-1}|^{6/5}1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho_K)\right)^{5/6}\right]\\
&\times\left(\sum_{K\in\mathcal{T}}|K||\boldsymbol{U}_{h,K}^n-\boldsymbol{u}_K^n|^6\right)^{1/6}\leqslant c(\|(\partial_t\boldsymbol{U},\partial_t\nabla\boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{12})})(\mathcal{E}^{1/2}(\varrho^{n-1},\boldsymbol{u}^{n-1}|r^{n-1},\boldsymbol{U}^{n-1})\\
&+\mathcal{E}^{5/6}(\varrho^{n-1},\boldsymbol{u}^{n-1}|r^{n-1},\boldsymbol{U}^{n-1}))\left(\sum_{K\in\mathcal{T}}\|\boldsymbol{U}_{h,K}^n-\boldsymbol{u}_K^n\|_{L^6(K;\mathbb{R}^3)}^6\right)^{1/6},
\end{aligned}
$$

where we have used Lemma 7.2 and estimate (4.13) to obtain the last line. Now, by the Minkowski inequality,

$$
\begin{aligned}
\left(\sum_{K\in\mathcal{T}}\|\boldsymbol{U}_{h,K}^n-\boldsymbol{u}_K^n\|_{L^6(K;\mathbb{R}^3)}^6\right)^{1/6}&\leqslant\left(\sum_{K\in\mathcal{T}}\|(\boldsymbol{U}_{h,K}^n-\boldsymbol{u}_K^n)-(\boldsymbol{U}_h^n-\boldsymbol{u}^n)\|_{L^6(K;\mathbb{R}^3)}^6\right)^{1/6}\\
&\quad+\|\boldsymbol{U}_h^n-\boldsymbol{u}^n\|_{L^6(\Omega;\mathbb{R}^3)}\leqslant c|\boldsymbol{u}^n-\boldsymbol{U}_h^n|_{V_h^2(\Omega;\mathbb{R}^3)},
\end{aligned}
$$

where we have used estimate (A.14) and the Sobolev inequality (A.29). Finally, employing the Young inequality and estimate (4.13), we arrive at

$$|\mathcal{P}_1| \leqslant c(\delta, M_0, E_0, \underline{r}, \bar{r}, \|(\boldsymbol{U}, \nabla \boldsymbol{U}, \partial_t \boldsymbol{U}, \partial_t \nabla \boldsymbol{U})\|_{L^\infty(Q_T, \mathbb{R}^{24})}) \times \left(k\mathcal{E}(\varrho^0, \boldsymbol{u}^0|r^0, \boldsymbol{U}^0) + k\sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \mid r^n, \boldsymbol{U}^n)\right) + \delta k \sum_{n=1}^{m} |\boldsymbol{u}^n - \boldsymbol{U}_h^n|^2_{V_h^2(\Omega;\mathbb{R}^3)} \tag{8.2}$$

with any $\delta > 0$.

Step 2*: term* $\mathcal{P}_2$. We write $\mathcal{P}_2 = k\sum_{n=1}^{m} \mathcal{P}_2^n$ where Lemma 7.2 and the Hölder inequality yield, similarly to the previous step,

$$\begin{aligned} |\mathcal{P}_2^n| &\leqslant c(\underline{r}, \bar{r}, \|\nabla \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^9)}) \\ &\quad \times \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h (E^{1/2}(\varrho_\sigma^{n,\mathrm{up}}, \hat{r}_\sigma^{n,\mathrm{up}}) + E^{2/3}(\varrho_\sigma^{n,\mathrm{up}}, \hat{r}_\sigma^{n,\mathrm{up}})) |\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}}| |\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}| \\ &\leqslant c(\underline{r}, \bar{r}, \|(\boldsymbol{U}, \nabla \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{12})}) \left[\left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h (E(\varrho_\sigma^{n,\mathrm{up}} | \hat{r}_\sigma^{n,\mathrm{up}}))^{1/2} \right.\right. \\ &\quad \left.\left. + \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h E(\varrho_\sigma^{n,\mathrm{up}} | \hat{r}_\sigma^{n,\mathrm{up}}) \right)^{2/3} \right] \times \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h |\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}|^6 \right)^{1/6} \end{aligned}$$

provided $\gamma \geqslant 3/2$. Next, we observe that the contribution of the face $\sigma = K|L$ to the sums $\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h E(\varrho_\sigma^{n,\mathrm{up}} | \hat{r}_\sigma^{n,\mathrm{up}})$ and $\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h |\hat{\boldsymbol{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}|^6$ is less than or equal to $2|\sigma| h (E(\varrho_K^n | \hat{r}_K^n) + E(\varrho_L^n | \hat{r}_L^n))$ and less than or equal to $2|\sigma| h (|\boldsymbol{U}_{h,K}^n - \boldsymbol{u}_K^n|^6 + |\boldsymbol{U}_{h,L}^n - \boldsymbol{u}_L^n|^6)$, respectively. Consequently, we get by the same reasoning as in the previous step, under the assumption $\gamma \geqslant 3/2$,

$$|\mathcal{P}_2| \leqslant c(\delta, M_0, E_0, \underline{r}, \bar{r}, \|(\boldsymbol{U}, \nabla \boldsymbol{U})\|_{L^\infty(Q_T;\mathbb{R}^{12})}) k \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n | r^n, \boldsymbol{U}^n) + \delta k \sum_{n=1}^{m} |\boldsymbol{u}^n - \boldsymbol{U}_h^n|^2_{V_h^2(\Omega;\mathbb{R}^3)}. \tag{8.3}$$

Step 3*: term* $\mathcal{P}_3$. Since the pair $(r, \boldsymbol{U})$ satisfie continuity equation (1.1a) in the classical sense, we have, for all $n = 1, \ldots, N$,

$$[\partial_t r]^n + \boldsymbol{U}^n \cdot \nabla r^n = -r^n \operatorname{div} \boldsymbol{U}^n,$$

where we recall that $[\partial_t r]^n(x) = \partial_t r(t_n, x)$ in accordance with (3.17). Using this identity, we write

$$\begin{aligned} &\mathcal{P}_3^n = \mathcal{P}_{3,1} + \mathcal{P}_{3,2}, \quad \mathcal{P}_{3,i} = k\sum_{n=1}^{m} \mathcal{P}_{3,i}^n, \\ \text{with} \quad &\mathcal{P}_{3,1}^n = -\sum_{K\in\mathcal{T}} \int_K (p(\varrho_K^n) - p'(r_K^n)(\varrho_K^n - r_K^n) - p(r_K^n)) \operatorname{div} \boldsymbol{U}^n \,\mathrm{d}x \\ \text{and} \quad &\mathcal{P}_{3,2}^n = \sum_{K\in\mathcal{T}} \int_K \frac{r_K^n - \varrho_K^n}{r_K^n} p'(r_K^n)(\boldsymbol{u}^n - \boldsymbol{U}^n) \cdot \nabla r^n \,\mathrm{d}x. \end{aligned}$$

Now, we apply Lemma 7.2 in combination with assumption (1.6) to deduce

$$|\mathcal{P}_{3,1}| \leqslant c\|\operatorname{div} \boldsymbol{U}\|_{L^\infty(Q_T)} k \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n | r^n, \boldsymbol{U}^n). \tag{8.4}$$

Finally, the same reasoning as in Step 2 leads to the estimate

$$\begin{aligned} |\mathcal{P}_{3,2}| \leqslant{} & hc(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C([\underline{r},\bar{r}])} \|(\nabla r, \nabla \boldsymbol{U})\|_{L^\infty(\Omega;\mathbb{R}^9)}) \\ & + c(\delta, \|\underline{r}, \bar{r}, |p'|_{C([\underline{r},\bar{r}])} \|\nabla r\|_{L^\infty(\Omega;\mathbb{R}^3)}) k \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n | r^n, \boldsymbol{U}^n) \\ & + \delta k \sum_{n=1}^{m} |\boldsymbol{u}^n - \boldsymbol{U}_h^n|^2_{V_h^2(\Omega;\mathbb{R}^3)}. \end{aligned} \tag{8.5}$$

Gathering the formulae (8.1) and (8.2–8.5) with δ sufficientl small (with respect to μ), we conclude the proof of Lemma 8.1. □

Finally, Lemma 8.1 in combination with the bound (4.13) yields

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m | r^m, \boldsymbol{U}^m) \leqslant c[h^A + \sqrt{k} + k + \mathcal{E}(\varrho^0, \boldsymbol{u}^0 | r^0, \boldsymbol{U}^0)] + ck \sum_{n=1}^{m-1} \mathcal{E}(\varrho^n, \boldsymbol{u}^n | r^n, \boldsymbol{U}^n);$$

then Theorem 3.2 is a direct consequence of the standard discrete version of Gronwall's lemma. Theorem 3.2 is thus proved.

Acknowledgements

The work was partially supported by the MODTERCOM project within the APEX programme of the Provence-Alpes-Côte d'Azur region.

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Appendix. Fundamental auxiliary lemmas and estimates

In this section, we report several results related to the properties of the Sobolev spaces on tetrahedra and of the Crouzeix–Raviart (C-R) space. We refer the reader to the book Brezzi & Fortin (1991) for a general introduction to the subject.

We start with the inequalities that can be obtained by rescaling from the standard inequalities on a reference tetrahedron of size equivalent to 1.

LEMMA A1 (Poincaré, Sobolev and interpolation inequalities on tetrahedra) Let $1 \leqslant p \leqslant \infty$. Let $\theta_0 > 0$ and $\mathcal{T}$ be a triangulation of Ω such that $\theta \geqslant \theta_0$ where θ is define in (3.1). Then we have the following estimates.

(1) Poincaré-type inequalities on tetrahedra.
Let $1 \leqslant p \leqslant \infty$. There exists $c = c(\theta_0, p) > 0$ such that for all $K \in \mathcal{T}$ and for all $v \in W^{1,p}(K)$ we have

$$\|v - v_K\|_{L^p(K)} \leqslant ch\|\nabla v\|_{L^p(K)}, \tag{A.1}$$

$$\forall \sigma \in \mathcal{E}(K), \quad \|v - v_\sigma\|_{L^p(K)} \leqslant ch\|\nabla v\|_{L^p(K)}. \tag{A.2}$$

(2) Sobolev-type inequalities on tetrahedra.
Let $1 \leqslant p < d$. There exists $c = c(\theta_0, p) > 0$ such that, for all $K \in \mathcal{T}$ and for all $v \in W^{1,p}(K)$, we have

$$\|v - v_K\|_{L^{p^*}(K)} \leqslant c\|\nabla v\|_{L^p(K)}, \tag{A.3}$$

$$\forall \sigma \in \mathcal{E}(K), \quad \|v - v_\sigma\|_{L^{p^*}(K)} \leqslant c\|\nabla v\|_{L^p(K)}, \tag{A.4}$$

where $p^* = dp/(d-p)$.

(3) Interpolation inequalities on the tetrahedra.
Let $1 \leqslant p < d$ and $p \leqslant q \leqslant p^*$. There exists $c = c(\theta_0, p) > 0$ such that, for all $K \in \mathcal{T}$ and $v \in W^{1,p}(K)$, we have

$$\|v - v_K\|_{L^q(K)} \leqslant ch^\beta\|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{A.5}$$

$$\|v - v_\sigma\|_{L^q(K)} \leqslant ch^\beta\|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{A.6}$$

where $1/q = \beta/p + (1-\beta)/p^*$.

Combining estimates (A.1–A.6) with the algebraic inequality

$$\left(\sum_{i=1}^{L} |a_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{L} |a_i|^q\right)^{1/q} \tag{A.7}$$

for all $a = (a_1, \ldots, a_L) \in \mathbb{R}^L$, $1 \leqslant q \leqslant p < \infty$, we obtain the following corollaries.

Corollary A2 (Poincaré- and Sobolev-type inequalities on the Sobolev spaces) Under the assumptions of Lemma A1, we have the following estimates.

(1) Poincaré-type inequalities on the domain Ω.
Let $1 \leqslant p \leqslant \infty$. There exists $c = c(\theta_0, p) > 0$ such that, for all $v \in W^{1,p}(\Omega)$, we have

$$\|v - \hat{v}\|_{L^p(\Omega)} \equiv \left(\sum_{K \in \mathcal{T}} \|v - v_K\|^p_{L^p(K)}\right)^{1/p} \leqslant ch\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}, \tag{A.8}$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|^p_{L^p(K)}\right)^{1/p} \leqslant ch\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}, \tag{A.9}$$

where $\hat{v}$ and v_σ are define by (3.7) and (3.8).

(2) Sobolev-type inequalities on the domain Ω.
Let $1 \leqslant p < d$. There exists $c = c(\theta_0, p) > 0$ such that, for all $v \in W^{1,p}(\Omega)$, we have

$$\|v - \hat{v}\|_{L^{p^*}(\Omega)} \leqslant c\|\nabla v\|_{L^p(\Omega)}, \tag{A.10}$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|^{p^*}_{L^{p^*}(K)}\right)^{1/p^*} \leqslant c\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}. \tag{A.11}$$

(3) Interpolation inequalities on the domain Ω.
Let $1 \leqslant p < d$ and $p \leqslant q \leqslant p^*$. There exists $c = c(\theta_0, p) > 0$ such that, for all $v \in W^{1,p}(\Omega)$, we have

$$\|v - \hat{v}\|_{L^q(\Omega)} \leqslant ch^\beta\|\nabla v\|_{L^p(\Omega)}, \tag{A.12}$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|^q_{L^q(K)}\right)^{1/q} \leqslant ch^\beta\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}, \tag{A.13}$$

where $1/q = \beta/p + (1 - \beta)/p^*$.

Corollary A3 (Poincaré- and Sobolev-type inequalities on V_h) Under the assumptions of Lemma A1, the following estimates hold.

(1) Poincaré-type inequality in $V_h(\Omega)$.

Let $1 \leqslant p < \infty$. There exists $c = c(\theta_0, p)$ such that, for all $v \in V_h$,

$$\|v - \hat{v}\|_{L^p(\Omega)} \leqslant ch|v|_{V_h^p(\Omega)}, \tag{A.14}$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|^p_{L^p(K)}\right)^{1/p} \leqslant ch|v|_{V_h^p(\Omega)}. \tag{A.15}$$

(2) Sobolev-type inequality in $V_h(\Omega)$.
Let $1 \leqslant p < d$. There exists $c = c(\theta_0, p)$ such that, for all $v \in V_h(\Omega)$,

$$\|v - \hat{v}\|_{L^{p^*}(\Omega)} \leqslant c|v|_{V_h^p(\Omega)}, \tag{A.16}$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|^{p^*}_{L^{p^*}(K)}\right)^{1/p^*} \leqslant c|v|_{V_h^p(\Omega)}. \tag{A.17}$$

(3) Interpolation-type inequalities in $V_h(\Omega)$.
Let $1 \leqslant p < d$ and $p \leqslant q \leqslant p^*$. There exists $c = c(\theta_0, p) > 0$ such that, for all $v \in V_h(\Omega)$, we have

$$\|v - \hat{v}\|_{L^q(\Omega)} \leqslant ch^\beta|v|_{V_h^p(\Omega)}, \tag{A.18}$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|^q_{L^q(K)}\right)^{1/q} \leqslant ch^\beta|v|_{V_h^p(\Omega)}, \tag{A.19}$$

where $1/q = \beta/p + (1-\beta)/p^*$.

The next fundamental lemma deals with the properties of the projection v_h define by (3.9).

Lemma A4 (Projection on V_h) Let $\theta_0 > 0$ and $\mathcal{T}$ be a triangulation of Ω such that $\theta \geqslant \theta_0$, where θ is define in (3.1).

(1) Approximation estimates on tetrahedra.
Let $1 \leqslant p \leqslant \infty$. There exists $c = c(\theta_0, p) > 0$ such that

$$\forall v \in W_0^{1,p} \cap W^{s,p}(\Omega),\ \forall K \in \mathcal{T}, \quad \|v - v_h\|_{L^p(K)} \leqslant ch^s\|\nabla^s v\|_{L^p(K;\mathbb{R}^{d^s})}, \tag{A.20}$$

$$\|\nabla(v - v_h)\|_{L^p(K;\mathbb{R}^d)} \leqslant ch^{s-1}\|\nabla^s v\|_{L^p(K;\mathbb{R}^{d^s})}, \quad s = 1, 2. \tag{A.21}$$

(2) Preservation of divergence.

$$\forall \boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^d),\ \forall q \in L_h(\Omega), \quad \sum_{K \in \mathcal{T}} \int_K q \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x = \int_\Omega q \operatorname{div} \boldsymbol{v} \, \mathrm{d}x. \tag{A.22}$$

(3) Approximation estimates of Poincaré type on the whole domain.
Let $1 \leqslant p < \infty$. There exists $c = c(\theta_0, p) > 0$ such that, for all $v \in W_0^{1,p}(\Omega)$,

$$\|v - v_h\|_{L^p(\Omega)} \leqslant ch\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}. \tag{A.23}$$

(4) Approximation estimates of Sobolev type on the whole domain.
Let $1 \leqslant p < d$. There exists $c = c(\theta_0, p) > 0$ such that, for all $v \in W_0^{1,p}(\Omega)$,

$$\|v - v_h\|_{L^{p^*}(\Omega)} \leqslant c\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}. \tag{A.24}$$

Statement (2) of Lemma A4 is proved in Crouzeix & Raviart (1973), where one can also fin the proof of item (1) for $p = 2$. We present here the proof of statements (1), (3), (4) for arbitrary p for the reader's convenience, since a straightforward reference is not available.

Proof. *Step* 1. We start with some generalities. First, we complete the Crouzeix–Raviart basis (3.6) by functions ϕ_σ indexed also with $\sigma \in \mathcal{E}_{\text{ext}}$ saying

$$\frac{1}{|\sigma'|}\int_{\sigma'} \phi_\sigma \, \mathrm{d}S = \delta_{\sigma,\sigma'}, \quad (\sigma,\sigma') \in \mathcal{E}^2$$

and observe that

$$\sum_{\sigma \in \mathcal{E}(K)} \phi_\sigma(x) = 1 \quad \text{for any } x \in K. \tag{A.25}$$

A scaling argument yields

$$\|\phi_\sigma\|_{L^\infty(\Omega)} \leqslant c(\theta_0), \quad h\|\nabla\phi_\sigma\|_{L^\infty(\Omega;\mathbb{R}^d)} \leqslant c(\theta_0). \tag{A.26}$$

Second, we defin the projection $v \to v_h$ for any $v \in W^{1,p}(\Omega)$ by saying

$$v_h = \sum_{\sigma \in \mathcal{E}} v_\sigma \phi_\sigma.$$

We note that if $v \in W_0^{1,p}(\Omega)$, then v_h coincides with (3.5). Moreover,

$$v_h = v \text{ for any affin function } v. \tag{A.27}$$

Third, due to the density argument, it is enough to show the remaining statements (1), (3), (4) for $v \in W_0^{1,p}(\Omega) \cap W^{s,\infty}(\Omega)$, $s = 1, 2$, according to the case.

Step 2. We denote by $x_K = (1/|K|)\int_K x \, \mathrm{d}x$ the centre of gravity of the tetrahedron K. We calculate by using (A.27) and the f rst-order Taylor formula,

$$\begin{aligned} v(x) - v_h(x) &= v(x) - v(x_K) - [v - v(x_K)]_h(x) \\ &= (x - x_K)\cdot\int_0^1 \nabla v(x_K + t(x - x_K))\,\mathrm{d}t \\ &\quad - \sum_{\sigma\in\mathcal{E}(K)} \phi_\sigma(x)\frac{1}{|\sigma|}\int_\sigma \left[(x - x_K)\cdot\int_0^1 \nabla v(x_K + t(x - x_K))\,\mathrm{d}t\right] \mathrm{d}S, \end{aligned}$$

where $x \in K$. This formula yields immediately the upper bound stated in $(\text{A.20})_{s=1}$ if $p = \infty$. If $1 \leqslant p < \infty$, we calculate the upper bound of the L^p-norm of each term on the right-hand side separately by using (A.26), Fubini's theorem, the Hölder inequality and the change of variables $y = x_K + t(x - x_K)$ together with the convexity of K.

The same reasoning can be applied to prove $(A.20)_{s=2}$. Indeed, we observe that

$$v(x) - v_h(x) = v(x) - (x - x_K) \cdot \nabla v(x_K) - v(x_K) - [v - (x - x_K) \cdot \nabla v(x_K) - v(x_K)]_h(x)$$

by virtue of (A.27). Now, we apply to the right-hand side of the last expression the second-order Taylor formula in integral form, and proceed exactly as described before.

Finally, one applies the same straightforward argumentation to get (A.21). This completes the proof of statement (1).

Step 3. Statement (3) follows easily from $(A.20)_{s=1}$ and the algebraic inequality (A.7).

Step 4. We use (A.25) and (A.27) to write

$$v(x) - v_h(x) = \sum_{\sigma \in \mathcal{E}(K)} (v(x) - v_\sigma)\phi_\sigma(x), \quad x \in K;$$

then

$$\|v - v_h\|_{L^{p^*}(K)} \leqslant c\|\nabla v\|_{L^p(K;\mathbb{R}^d)}$$

where we have used the Sobolev inequality (A.4) on the tetrahedron $K \in \mathcal{T}$ and the L^∞-bound (A.26). We conclude the proof of statement (4) by using the relation (A.7). The proof of Lemma A4 is complete. □

The following corollary is a direct consequence of (A.21).

Corollary A5 (Continuity of the projection onto V_h) Under the assumptions of Lemma A4, there exists $c = c(\theta_0, p) > 0$ such that

$$\forall v \in W_0^{1,p}(\Omega), \quad |v_h|_{V_h^p(\Omega)} \leqslant c\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}, \tag{A.28}$$

where $1 \leqslant p < \infty$.

Although the nonconforming finit element space V_h is not a subspace of any Sobolev space, its elements enjoy Sobolev-type inequalities. This important fact is formulated in the next lemma.

Lemma A6 (Sobolev inequality on V_h) Let Ω be a bounded domain of $\mathbb{R}^d$. Let $\mathcal{T}$ be a triangulation of the domain Ω in simplices such that $\theta \geqslant \theta_0 > 0$ where θ is define in (3.1). Then we have the following estimates.

(1) Sobolev inequality in $V_h(\Omega)$ (case $1 \leqslant p < d$).
There exists $c = c(\theta_0, p)$ such that, for all $v \in V_h(\Omega)$,

$$||v||_{L^{p^*}(\Omega)} \leqslant c|v|_{V_h^p(\Omega)}. \tag{A.29}$$

(2) Sobolev inequality in $V_h(\Omega)$, case $p \geqslant d$.
Let $1 \leqslant q < \infty$. There exists $c = c(\theta_0, p, q) > 0$ such that for all $v \in V_h(\Omega)$,

$$\|v\|_{L^q(\Omega)} \leqslant c|v|_{V_h^p(\Omega)}. \tag{A.30}$$

Proof. *Step* 1. Let $1 \leqslant r \leqslant \alpha < \infty$. Let $u \in V_h$. We call v the element of V_h such that $v_\sigma = |u_\sigma|^\alpha$. Then there exists C depending only on d, r, α such that

$$\|u\|^\alpha_{L^r(\Omega)} \leqslant c\|u\|_{L^{r/\alpha}(\Omega)}. \tag{A.31}$$

To prove (A.31), we remark that, using a change of variable, it is enough to show the existence of C for only the unit simplex $\hat{K}$. Let $u \in \mathbb{P}_1(\hat{K})$ and we call v the element of $\mathbb{P}_1(\hat{K})$ such that $v_\sigma = |u_\sigma|^\alpha$. Let $T(u) = \|u\|_{L^r(\hat{K})}$ and $S(u) = \|u\|^{1/\alpha}_{L^{r/\alpha}(\hat{K})}$. These two functions are continuous, homogeneous of degree 1 and nonzero if $u \neq 0$. Since $\mathbb{P}_1(\hat{K})$ is a finite-dimensiona space, we can choose a norm on $\mathbb{P}_1(\hat{K})$ and take $C = (M/m)^\alpha$ where $M = \max\{T(u), \|u\|_{\mathbb{P}_1(\hat{K})} = 1\}$ and $m = \min\{T(u), \|u\|_{\mathbb{P}_1(\hat{K})} = 1\}$.

Step 2. Proof for $p = 1$.
We set $u = 0$ outside Ω. For $\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L$, we set $|[u(x)]| = |u_K(x) - u_L(x)|$ for $x \in \sigma$. For $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$, we set $|[u(x)]| = |u_K(x)|$ for $x \in \sigma$. We f rst remark that there exists $C_{1,1}$ and $C_{1,2}$ depending only on d such that

$$\|u\|_{L^{d/(d-1)}(\Omega)} \leqslant C_{1,1}\|u\|_{BV(\mathbb{R}^d)} \leqslant C_{1,2}\|\nabla_h u\|_{L^1(\Omega)} + C_{1,2}\sum_{\sigma\in\mathcal{E}}\int_\sigma |[u]|\,\mathrm{d}S,$$

where we refer the reader to Ambrosio *et al.* (2000) for the definitio of the BV norm and more details.

We now prove that there exists $C_{1,3}$ depending only on d and θ_0 such that

$$\sum_{\sigma\in\mathcal{E}}\int_\sigma |[u]|\,\mathrm{d}S \leqslant C_{1,3}\|\nabla_h u\|_{L^1(\Omega)}.$$

Let $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$. Let x_σ be the centre of mass of σ. We have, with $u_K = u$ in K,

$$u_K(x) - u(x_\sigma) = \int_0^1 \nabla u_K \cdot (x - x_\sigma)\,\mathrm{d}x.$$

Then if $\sigma = K|L$, we have

$$|u_K(x) - u_L(x)| \leqslant h_\sigma(|\nabla u_K| + |\nabla u_L|).$$

Integrating this inequality on σ gives

$$\int_\sigma |[u]|\,\mathrm{d}S \leqslant |\sigma|h_\sigma(|\nabla u_K| + |\nabla u_L|) \leqslant \frac{2}{\theta_0^d}(\|\nabla u\|_{L^1(K)} + \|\nabla u\|_{L^1(L)}).$$

Similarly, for $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$ we have

$$\int_\sigma |[u]|\mathrm{d}S \leqslant \frac{2}{\theta_0^d}\|\nabla u\|_{L^1(K)}.$$

Then there exists $C_{1,3} = C(d, \theta_0)$ such that

$$\sum_{i,\sigma\in\mathcal{E}}\int_\sigma |[u]|\,\mathrm{d}S \leqslant C_{1,3}\|\nabla_h u\|_{L^1(\Omega)}$$

and, then,

$$\|u\|_{L^{1^*}(\Omega)} \leqslant c(d, \theta_0)\|\nabla_h u\|_{L^1(\Omega)}.$$

Step 3: Proof for $1 < p < d$.

Let $1 < p < d$ and $p^* = pd/(d-p)$ and let $u \in V_h$. We set $u = 0$ outside Ω. Let $\alpha = p(d-1)/(d-p)$, so that $\alpha > 1$ and $\alpha 1^* = p^*$. We call v the element of V_h such that $v_\sigma = |u_\sigma|^\alpha$ for $\sigma \in \mathcal{E}$. One has $v \neq |u|^\alpha$ but there exists $C_{2,1}$ depending only on d and p (see (A.31)) such that

$$\|u\|^\alpha_{L^{p^*}(\Omega)} \leqslant C_{2,1}\|v\|_{L^{1^*}(\Omega)} \leqslant c(d,p,\theta_0)\|\nabla_h v\|_{L^1(\Omega)}.$$

Moreover, using a scaling argument, we obtain

$$\|\nabla_h v\|_{L^1(K)} \leqslant c(d,p,\theta_0) \sum_{\sigma \in \mathcal{E}(K)} |u_\sigma|^{\alpha-1}|\nabla u_K||K|.$$

Then, using the Hölder inequality, we have, with $q = p/(p-1)$ (so that $q(\alpha-1) = p^*$),

$$\|\nabla_h v\|_{L^1(K)} \leqslant c(d,p,\theta_0)\|\nabla u\|_{L^p(K)}\|u\|^{p^*/q}_{L^{p^*}(K)}.$$

Summing on $K \in \mathcal{T}$, we obtain

$$\|u\|_{L^{p^*}(\Omega)} \leqslant C_2\|\nabla_h u\|_{L^p(\Omega)}.$$

Step 4. Proof for $p \geqslant d$.

Let $1 \leqslant q < \infty$. There exists $r = r(d,q)$ such that $r < d$ and $r^* \geqslant q$. We have

$$\|u\|_{L^{r^*}(\Omega)} \leqslant c(r,d,q,\theta_0)\|\nabla_h u\|_{L^r(\Omega)}.$$

Moreover,

$$\|u\|_{L^q(\Omega)} \leqslant |\Omega|^{1/q-1/r^*}\|u\|_{L^{r^*}(\Omega)} \leqslant c(d,q,\theta_0)|\Omega|^{1/q-1/r^*}\|\nabla_h u\|_{L^r(\Omega)}$$

and

$$\|\nabla_h u\|_{L^r(\Omega)} \leqslant |\Omega|^{1/r-1/p}\|\nabla_h u\|_{L^p(\Omega)}.$$

Finally,

$$\|u\|_{L^q(\Omega)} \leqslant c(\Omega,d,p,q,\theta_0)\|\nabla_h u\|_{L^p(\Omega)}.$$

□

A combination of Lemma A6 with estimates (A.14), (A.16) and the Hölder inequality yields the following corollary.

COROLLARY A7 (Estimates of the norms of mean values) We have the following under the assumptions of Lemma A6.

(1) Poincaré-type inequality involving mean values on tetrahedra.
There exists $c = c(\theta_0, p)$ such that, for all $v \in V_h$,

$$\|\hat{v}\|_{L^p(\Omega)} \equiv \left(\sum_{K\in\mathcal{T}} |K||v_K|^p\right)^{1/p} \leqslant c(\|v\|_{L^p(\Omega)} + h|v|_{V_h^p(\Omega)}). \tag{A.32}$$

(2) Sobolev-type inequality involving mean values on tetrahedra.
Let $1 \leqslant p < d$; there exists $c = c(\theta_0, p)$ such that, for all $v \in V_h$,

$$\|\hat{v}\|_{L^{p^*}(\Omega)} \equiv \left(\sum_{K\in\mathcal{T}} |K||v_K|^{p^*}\right)^{1/p^*} \leqslant c(\|v\|_{L^{p^*}(\Omega)} + |v|_{V_h^p}). \tag{A.33}$$

Note that, last but not least, we recall a result on the equivalence of norms in the space $V_h(\Omega)$ which is a consequence of a discrete Poincaré inequality on the broken Sobolev space V_h (Temam, 1984, proposition 4.13).

LEMMA A8 (Discrete and continuous norms in V_h) Let $1 \leqslant p < \infty$. Let $\theta_0 > 0$ and $\mathcal{T}$ be a triangulation of Ω such that $\theta \geqslant \theta_0$, where θ is define in (3.1). Then the norms

$$\left(\sum_{\sigma\in\mathcal{E}_{\text{int}}} |\sigma||h|v_\sigma|^p\right)^{1/p} \quad \text{and} \quad \|v\|^p_{L^p(\Omega)} \tag{A.34}$$

are equivalent on $V_h(\Omega)$ uniformly with respect to $h > 0$.

The last lemma in this overview deals with the estimates of jumps over faces. The reader can consult Ern & Guermond (2004, Lemma 3.32) or Gallouët *et al.* (2009, Lemma 2.2) for its proof.

LEMMA A9 (Jumps over faces in the Crouzeix–Raviart space) Let $\theta_0 > 0$ and $\mathcal{T}$ be a triangulation of Ω such that $\theta \geqslant \theta_0$, where θ is define in (3.1). Then there exists $c = c(\theta_0) > 0$ such that, for all $v \in V_h(\Omega)$,

$$\sum_{\sigma\in\mathcal{E}} \frac{1}{h} \int_\sigma [v]^2_{\sigma,\boldsymbol{n}_\sigma} \,\mathrm{d}S \leqslant c|v|^2_{V_h^2(\Omega)}, \tag{A.35}$$

where $[v]_{\sigma,\boldsymbol{n}_\sigma}$ is a jump of v with respect to a normal $\boldsymbol{n}_\sigma$ to the face σ,

$$\forall x \in \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad [v]_{\sigma,\boldsymbol{n}_\sigma}(x) = \begin{cases} v|_K(x) - v|_L(x) & \text{if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,K}, \\ v|_L(x) - v|_K(x) & \text{if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,L} \end{cases}$$

and

$$\forall x \in \sigma \in \mathcal{E}_{\text{ext}}, \quad [v]_{\sigma,\boldsymbol{n}_\sigma}(x) = v(x), \quad \text{with } \boldsymbol{n}_\sigma \text{ an exterior normal to } \partial\Omega.$$

Error estimates for a numerical method for the compressible Navier-Stokes system on sufficiently smooth domains

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Abstract

We derive an *a priori* error estimate for the numerical solution obtained by time and space discretization by the finite volume/finite element method of the barotropic Navier–Stokes equations. The numerical solution on a convenient polyhedral domain approximating a sufficiently smooth bounded domain is compared with an exact solution of the barotropic Navier–Stokes equations with a bounded density. The result is unconditional in the sense that there are no assumed bounds on the numerical solution. It is obtained by the combination of discrete relative energy inequality derived in [22] and several recent results in the theory of compressible Navier-Stokes equations concerning blow up criterion established in [41] and weak strong uniqueness principle established in [11].

Key words: Navier-Stokes system, finite element numerical method, finite volume numerical method, error estimates

AMS classification 35Q30, 65N12, 65N30, 76N10, 76N15, 76M10, 76M12

1 Introduction

We consider the compressible Navier-Stokes equations in the barotropic regime in a space-time cylinder $Q_T = (0,T) \times \Omega$, where $T > 0$ is arbitrarily large and $\Omega \subset R^3$ is a bounded domain:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

In equations (1.1–1.2) $\varrho = \varrho(t,x) \geq 0$ and $\mathbf{u} = \mathbf{u}(t,x) \in R^3$, $t \in [0,T)$, $x \in \Omega$ are the unknown density and velocity fields, while $\mathbb{S}$ and p are the viscous stress and pressure characterizing the fluid via the constitutive relations

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right), \ \mu > 0, \tag{1.3}$$

[*]The research of E.F. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

[†]The research of R.H. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

[‡]The work of D.M. and A.N. has been supported by the MODTERCOM project within the APEX programme of the Provence-Alpes-Côte d'Azur region

$$p \in C^2(0,\infty) \cap C^1[0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho \geq 0,\ \lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0, \tag{1.4}$$

where $\gamma \geq 1$.

The assumption $p'(0) > 0$ in (1.4) excludes the constitutive laws for pressure behaving as ϱ^γ as $\varrho \to 0^+$. The error estimates stated in Theorem 3.1 however still hold in the case $\lim_{\varrho\to 0^+} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = 0$, in particular for the isentropic pressure laws $p(\varrho) = \varrho^\gamma$. The proof contains some additional technical difficulties, see also Remark 3.2.

Equations (1.1–1.2) are completed with the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.5}$$

and initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \varrho_0 > 0 \text{ in } \overline{\Omega}. \tag{1.6}$$

We notice that under assumption (1.3), we may write

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \mu\Delta\mathbf{u} + \frac{\mu}{3}\nabla_x \mathrm{div}_x \mathbf{u}. \tag{1.7}$$

The results on error estimates for numerical schemes for the compressible Navier-Stokes equations are in the mathematical literature on short supply. We refer the reader to papers of Liu [39], [40], Jovanović [28], Gallouet et al. [22].

In [22] the authors have developed a methodology of deriving unconditional error estimates for the numerical schemes to the compressible Navier-Stokes equations (1.1–1.6) and applied it to the numerical scheme (3.5–3.7) discretizing the system on polyhedral domains. They have obtained error estimates for the discrete solution with respect to a *classical solution* of the system on the same (polyhedral) domain. In spite of the fact that [22] provides the first and to the best of our knowledge so far the sole error estimate for discrete solutions of a finite volume/finite element approximation to a model of compressible fluids that does not need any assumed bounds on the numerical solution itself, it has two weak points: 1) The existence of classical solutions on at least a short time interval to the compressible Navier-Stokes equations is known for smooth C^3 domains (see Valli, Zajaczkowski [43] or Cho, Choe, Kim [4]) but may not be in general true on the polyhedral domains. 2) The numerical solutions are compared with the classical exact solutions (as is usual in any previous existing mathematical literature). In this paper we address both points raised above and to a certain extent remove the limitations of the theory presented in [22].

More precisely, we generalize the result of Gallouet et al. [22, Theorem 3.1] in two directions:

(1) The physical domain Ω filled by the fluid and the numerical domain Ω_h, $h > 0$ approximating the physical domain do not need to coincide.

(2) If the physical domain is sufficiently smooth (at least of class C^3) and the C^3- initial data satisfy natural compatibility conditions, we are able to obtain the unconditional error estimates with respect to any *weak exact solution with bounded density*.

As in [22], and in contrast with any other literature dealing with finite volume or mixed finite volume/finite element methods for compressible fluids (Feistauer et al. [16], [17], [18], [19], Kröner et al. [32], [33], [34], [35], Jovanović [28], Cancès et al [3], Eymard et al. [10], Villa, Villedieu [44], Rohde, Jovanović [29], Gastaldo et al. [24], Herbin et al. [25], [23] and others) this result does not require any assumed bounds on the discrete solution: the sole bounds needed for the result are those provided by the numerical scheme. Moreover, in contrast with [22] and with all above mentioned papers, the exact solution is solely weak solution with bounded density. This seemingly weak hypothesis is compensated by the regularity and compatibility conditions imposed on initial data that make possible a (sophisticated) bootstrapping argument showing that weak solutions with bounded density are in fact strong solutions in the class investigated in [22].

These results are achieved by using the following tools:

(1) The technique introduced in [22] modified in order to accommodate non-zero velocity of the exact sample solution on the boundary of the numerical domain.

(2) Three fundamental recent results from the theory of compressible Navier-Stokes equations, namely

- Local in time existence of strong solutions in class (2.11–2.12) by Cho, Choe, Kim [4].
- Weak strong uniqueness principle proved in [11] (see also [15]).
- Blow up criterion for strong solutions in the class (2.11–2.12) by Sun, Wang, Zhang [41].

The three above mentioned items allow to show that the weak solution with bounded density emanating from the sufficiently smooth initial data is in fact a strong solution defined on the large time interval $[0,T)$.

(3) Bootstrapping argument using recent results on maximal regularity for parabolic systems by Danchin [8], Denk, Pruess, Hieber [5] and Krylov [36]. The last item allows to bootstrap the strong solution in the class Cho, Choe, Kim [4] to the class needed for the error estimates in [22], provided a certain compatibility condition for the initial data is satisfied.

2 Preliminaries

2.1 Weak and strong solutions to the Navier-Stokes system

We introduce the notion of the weak solution to system (1.1–1.4):

Definition 2.1 (Weak solutions). *Let $\varrho_0 : \Omega \to [0,+\infty)$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^3$ with finite energy $E_0 = \int_\Omega (\frac{1}{2}\varrho_0|\boldsymbol{u}_0|^2 + H(\varrho_0))\,\mathrm{d}x$ and finite mass $0 < M_0 = \int_\Omega \varrho_0\,\mathrm{d}x$. We shall say that the pair $(\varrho, \boldsymbol{u})$ is a weak solution to the problem (1.1)–(1.6) emanating from the initial data $(\varrho_0, \boldsymbol{u}_0)$ if:*

(a) $\varrho \in C_{\rm weak}([0,T]; L^a(\Omega))$, for a certain $a > 1$, $\varrho \geq 0$ a.e. in $(0,T)$, and $\boldsymbol{u} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$.

(b) the continuity equation (1.1) is satisfied in the following weak sense

$$\int_\Omega \varrho\varphi\,\mathrm{d}x\Big|_0^\tau = \int_0^\tau\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\,\mathrm{dt},\ \forall\tau\in[0,T],\ \forall\varphi\in C_c^\infty([0,T]\times\overline{\Omega}). \tag{2.1}$$

(c) $\varrho\boldsymbol{u} \in C_{\rm weak}([0,T]; L^b(\Omega;\mathbb{R}^3))$, for a certain $b > 1$, and the momentum equation (1.2) is satisfied in the weak sense,

$$\begin{aligned}\int_\Omega \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\Big|_0^\tau &= \int_0^\tau\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}\,\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{dt}\\ &- \int_0^\tau\int_\Omega \Big(\mu\nabla\boldsymbol{u} : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{dt} + (\mu+\lambda)\mathrm{div}\boldsymbol{u}\mathrm{div}\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{dt},\ \forall\tau\in[0,T],\ \forall\boldsymbol{\varphi}\in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3).\end{aligned} \tag{2.2}$$

(d) The following energy inequality is satisfied

$$\int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + H(\varrho)\Big)\,\mathrm{d}x\Big|_0^\tau + \int_0^\tau\int_\Omega \Big(\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)|\,\mathrm{div}\,\boldsymbol{u}|^2\Big)\,\mathrm{d}x\,\mathrm{dt} \leq 0,\ \textit{for a.a. }\tau\in(0,T), \tag{2.3}$$

$$\textit{with}\ \ H(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\mathrm{d}z. \tag{2.4}$$

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Here and hereafter the symbol $\int_\Omega g \,\mathrm{d}x \,|_0^\tau$ *is meant for* $\int_\Omega g(\tau, x)\,\mathrm{d}x - \int_\Omega g_0(x)\,\mathrm{d}x$.

In the above definition, we tacitly assume that all the integrals in the formulas (2.1)–(2.3) are defined and we recall that $C_{\rm weak}([0,T]; L^a(\Omega))$ is the space of functions of $L^\infty([0,T]; L^a(\Omega))$ which are continuous as functions of time in the weak topology of the space $L^a(\Omega)$.

We notice that the function $\varrho \mapsto H(\varrho)$ is a solution of the ordinary differential equation $\varrho H'(\varrho) - H(\varrho) = p(\varrho)$ with the constant of integration fixed such that $H(1) = 0$.

Note that the existence of weak solutions emanating from the finite energy initial data is well-known on bounded Lipschitz domains provided $\gamma > 3/2$, see Lions [38] for 'large' values of γ, Feireisl and coauthors [14] for $\gamma > 3/2$.

Proposition 2.1. *Suppose the* $\Omega \subset R^3$ *is a bounded domain of class* C^3. *Let* r, $\mathbf{V}$ *be a weak solution to problem (1.1–1.6) in* $(0,T) \times \Omega$, *originating from the initial data*

$$r_0 \in C^3(\overline{\Omega}),\ \ r_0 > 0 \ \ in \ \overline{\Omega}, \tag{2.5}$$

$$\mathbf{V}_0 \in C^3(\overline{\Omega}; R^3), \tag{2.6}$$

satisfying the compatibility conditions

$$\mathbf{V}_0|_{\partial\Omega} = 0,\ \ \nabla_x p(r_0)|_{\partial\Omega} = {\rm div}_x \mathbb{S}(\nabla_x \mathbf{V}_0)|_{\partial\Omega}, \tag{2.7}$$

and such that

$$0 \le r \le \overline{r} \ \ a.a. \ in \ (0,T) \times \Omega. \tag{2.8}$$

Then r, $\mathbf{V}$ *is a classical solution satisfying the bounds:*

$$\|1/r\|_{C([0,T]\times\overline{\Omega})} + \|r\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t \nabla_x r\|_{C([0,T];L^6(\Omega;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(\Omega))} \le D, \tag{2.9}$$

$$\|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(\Omega;R^{3\times3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(\Omega))} \le D, \tag{2.10}$$

where D *depends on* Ω, T, $\overline{r}$, *and the initial data* r_0, $\mathbf{V}_0$ *(via* $\|(r_0, \mathbf{V}_0)\|_{C^3(\overline{\Omega};R^4)}$ *and* $\min_{x\in\overline{\Omega}} r_0(x)$*).*

Proof:
The proof will be carried over in several steps.

Step 1
According to Cho, Choe, and Kim [4], problem (1.1–1.6) admits a strong solution unique in the class

$$r \in C([0,T_M); W^{1,6}(\Omega)),\ \ \partial_t r \in C([0,T_M); L^6(\Omega)),\ 1/r \in L^\infty(Q_T), \tag{2.11}$$

$$\mathbf{V} \in C([0,T_M]; W^{2,2}(\Omega;R^3)) \cap L^2(0,T_M; W^{2,6}(\Omega;R^3)),\ \ \partial_t \mathbf{V} \in L^2(0,T_M; W^{1,2}_0(\Omega;R^3)). \tag{2.12}$$

defined on a time interval $[0,T_M)$, where $T_M > 0$ is finite or infinite and depends on the initial data. Moreover, for any $T^*_M < T_M$, there is a constant $c = c(T^*_M)$ such that

$$\begin{aligned} &\|r\|_{L^\infty(0,T^*_M;W^{1,6}(\Omega))} + \|\partial_t r\|_{L^\infty(0,T^*_M;L^6(\Omega))} + \|1/r\|_{L^\infty(Q_T)} \\ &+ \|\mathbf{V}\|_{L^\infty(0,T^*_M;W^{2,2}(\Omega;R^3))} + \|\mathbf{V}\|_{L^2(0,T^*_M;W^{2,6}(\Omega;R^3))} + \|\partial_t \mathbf{V}\|_{L^2(0,T^*_M;W^{1,2}(\Omega))} \\ &\le c\Big(\|r_0\|_{W^{1,6}(\Omega)} + \|\mathbf{V}_0\|_{W^{2,2}(\Omega)}\Big). \end{aligned} \tag{2.13}$$

Step 2
By virtue of the weak-strong uniqueness result stated in [11, Theorem 4.1] (see also [15, Theorem 4.6]), the weak solution r, $\mathbf{V}$ coincides on the time interval $[0,T_M)$ with the strong solution, the existence

of which is claimed in the previous step. According to Sun, Wang, Zhang [41, Theorem 1.3], if $T_M < \infty$ then

$$\limsup_{t \to T_M -} \|r(t)\|_{L^\infty(\Omega)} = \infty.$$

Since (2.8) holds, we infer that $T_M = T$. At this point we conclude that couple $(r, \mathbf{V})$ possesses regularity (2.11–2.12) and that that the bound (2.13) holds with c dependent solely on T.

Step 3

Since the initial data enjoy the regularity and compatibility conditions stated in (2.5–2.7), a straightforward bootstrap argument gives rise to better bounds, specifically, the solution belongs to the Valli-Zajaczkowski (see [43, Theorem 2.5]) class

$$r \in C([0,T]; W^{3,2}(\Omega)), \ \partial_t r \in L^2(0,T; W^{2,2}(\Omega)), \tag{2.14}$$

$$\mathbf{V} \in C([0,T]; W^{3,2}(\Omega)) \cap L^2(0,T; W^{4,2}(\Omega; R^3)), \ \partial_t \mathbf{V} \in L^2(0,T; W^{2,2}(\Omega; R^3)), \tag{2.15}$$

where, similarly to the previous step, the norms depend only on the initial data, $\overline{r}$, and T.

Step 4

We write equation (1.2) in the form

$$\partial_t \mathbf{V} - \frac{1}{r} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{V}) = -\mathbf{V} \cdot \nabla_x \mathbf{V} + \frac{1}{r} \nabla_x p(r), \tag{2.16}$$

where, by virtue of (2.15) and a simple interpolation argument, $\mathbf{V} \in C^{1+\nu}([0,T] \times \overline{\Omega}; R^{3\times 3})$, and, by the same token $r \in C^{1+\nu}([0,T] \times \overline{\Omega})$ for some $\nu > 0$. Consequently, by means of the standard theory of parabolic equations, see for instance Ladyzhenskaya et al. [37], we may infer that r, $\mathbf{V}$ is a classical solution,

$$\partial_t \mathbf{V}, \ \nabla_x^2 \mathbf{V} \text{ Hölder continuous in } [0,T] \times \overline{\Omega}. \tag{2.17}$$

and, going back to (1.1),

$$\partial_t r \text{ Hölder continuous in } [0,T] \times \overline{\Omega}. \tag{2.18}$$

Step 5

We write

$$\nabla_x \partial_t r = -\nabla_x \mathbf{V} \cdot \nabla_x r - \mathbf{V} \cdot \nabla_x^2 r - \nabla_x r \mathrm{div}_x \mathbf{V} - r \nabla_x \mathrm{div}_x \mathbf{V};$$

whence, by virtue (2.14), (2.17), (2.18), and the Sobolev embedding $W^{1,2} \hookrightarrow L^6$,

$$\partial_t r \in C([0,T]; W^{1,6}(\Omega)). \tag{2.19}$$

Next, we differentiate (2.16) with respect to t. Denoting $\mathbf{Z} = \partial_t \mathbf{V}$ we therefore obtain

$$\partial_t \mathbf{Z} - \frac{1}{r} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{Z}) + \mathbf{V} \cdot \nabla_x \mathbf{Z} = \partial_t \left(\frac{1}{r} \right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{V}) - \partial_t \mathbf{V} \cdot \nabla_x \mathbf{V} + \partial_t \left(\frac{1}{r} \nabla_x p(r) \right), \tag{2.20}$$

where, in view of (2.19) and the previously established estimates, the expression on the right-hand side is bounded in $C([0,T]; L^6(\Omega; R^3))$. Thus using the L^p-maximal regularity (see Denk, Hieber, and Pruess [5], Krylov [36] or Danchin [8, Theorem 2.2]), we deduce that

$$\partial^2_{t,t} \mathbf{V} = \partial_t \mathbf{Z} \in L^2(0,T; L^6(\Omega; R^3)), \ \partial_t \mathbf{V} = \mathbf{Z} \in C([0,T]; W^{1,6}(\Omega; R^3)). \tag{2.21}$$

Finally, writing

$$\partial^2_{t,t} r = -\partial_t \mathbf{V} \cdot \nabla_x r - \mathbf{V} \cdot \partial_t \nabla_x r - \partial_t r \mathrm{div}_x \mathbf{V} - r \partial_t \mathrm{div}_x \mathbf{V},$$

and using (2.19), (2.21), we obtain the desired conclusion

$$\partial^2_{t,t} r \in C([0,T]; L^6(\Omega)).$$

□

Here and hereafter, we shall use notation $a \lesssim b$ and $a \approx b$. the symbol $a \lesssim b$ means that there exists $c = c(\Omega, T, \mu, \gamma) > 0$ such that $a \leq cb$; $a \approx b$ means $a \lesssim b$ and $b \lesssim a$.

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2.2 Extension lemma

Lemma 2.1. *Under the hypotheses of Proposition 2.1, the functions r and $\mathbf{V}$ can be extended outside Ω in such a way that:*

(1) The extended functions (still denoted by r and $\mathbf{V}$) are such that $\mathbf{V}$ is compactly supported in $[0,T] \times \mathbb{R}^3$ and $r \geq \underline{r} > 0$.

(2)

$$\|\mathbf{V}\|_{C^1([0,T]\times R^3;R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(R^3;R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(R^3;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(R^3))} \tag{2.22}$$
$$\lesssim \|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(\Omega))};$$

(3)

$$\|r\|_{C^1([0,T]\times R^3)} + \|\partial_t \nabla_x r\|_{C([0,T];L^6(R^3;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(R^3))} \tag{2.23}$$
$$\lesssim \|r\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t \nabla_x r\|_{C([0,T];L^6(\Omega;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(\Omega))} +$$
$$\|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(\Omega))};$$

(4)

$$\partial_t r + \mathrm{div}_x(r\mathbf{V}) = 0 \ \text{in} \ (0,T) \times R^3. \tag{2.24}$$

Proof: We first construct the extension of the vector field $\mathbf{V}$. To this end, we follow the standard construction in the flat domain, see Adams [1, Chapter 5, Theorem 5.22] and combine it with the standard procedure of 'flattening' of the boundary and the partition of unity technique, we get (2.22) Once this is done, we solve on the whole space the transport equation (2.24). It is easy to show that the unique solution r of this equation possesses regularity and estimates stated in (2.23). □

Remark 2.1. *Here and hereafter, we denote $X_T(\mathbb{R}^3)$ a subset of $L^2((0,T)\times\mathbb{R}^3)$ of couples $(r,\mathbf{V})$, $r > 0$ with finite norm*

$$\|(r,\mathbf{V})\|_{X_T(\mathbb{R}^3)} \equiv \|r\|_{C^1([0,T]\times R^3)} + \|\partial_t \nabla_x r\|_{C([0,T];L^6(R^3;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(R^3))} \tag{2.25}$$
$$\|\mathbf{V}\|_{C^1([0,T]\times R^3;R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(R^3;R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(R^3;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(R^3))}$$

We notice that if r, $\mathbf{V}$ are interrelated through (2.7), then the first component of the couple belonging to $X_T(\mathbb{R}^3)$ is always strictly positive on $[0,T] \times R^3$. We set

$$0 < \underline{r} = \min_{(t,x)\in[0,T]\times\mathbb{R}^3} r(t,x), \quad \overline{r} = \max_{(t,x)\in[0,T]\times\mathbb{R}^3} r(t,x) < \infty \tag{2.26}$$

2.3 Physical domain, mesh approximation

The physical space is represented by a bounded domain $\Omega \subset R^3$ of class C^3. The numerical domains Ω_h are polyhedral domains,

$$\overline{\Omega}_h = \cup_{K\in\mathcal{T}} K, \tag{2.27}$$

where $\mathcal{T}$ is a set of tetrahedra which have the following property: If $K \cap L \neq \emptyset$, $K \neq L$, then $K \cap L$ is either a common face, or a common edge, or a common vertex. By $\mathcal{E}(K)$, we denote the set of the faces σ of the element $K \in \mathcal{T}$. The set of all faces of the mesh is denoted by $\mathcal{E}$; the set of faces included in the boundary $\partial\Omega_h$ of Ω_h is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal faces (i.e $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$.

Further, we ask

$$\mathcal{V}_h \in \partial\Omega_h \text{ a vertex } \Rightarrow \mathcal{V}_h \in \partial\Omega. \tag{2.28}$$

Furthermore, we suppose that each K is a tetrahedron such that

$$\xi[K] \approx \text{diam}[K] \approx h, \tag{2.29}$$

where $\xi[K]$ is the radius of the largest ball contained in K.

The properties of this mesh needed in the sequel are formulated in the following lemma, whose proof is left to the reader, see Johnson and Nedelec [27] for the $2D$ case, and [26] for the general $3D$ case.

Lemma 2.2. *There exists a positive constant d_Ω depending solely on the geometric properties of $\partial\Omega$ such that*

$$\text{dist}[x, \partial\Omega] \leq d_\Omega h^2,$$

for any $x \in \partial\Omega_h$. Moreover,

$$|(\Omega_h \setminus \Omega) \cup (\Omega \setminus \Omega_h)| \lesssim h^2.$$

We find important to emphasize that $\Omega_h \not\subset \Omega$, in general.

2.4 Numerical spaces

We denote by $Q_h(\Omega_h)$ the space of piecewise constant functions:

$$Q_h(\Omega_h) = \{q \in L^2(\Omega_h) \,|\, \forall K \in \mathcal{T},\ q_{|K} \in \mathbb{R}\}. \tag{2.30}$$

For a function v in $C(\overline{\Omega}_h)$, we set

$$v_K = \frac{1}{|K|}\int_K v\,\mathrm{d}x \text{ for } K \in \mathcal{T} \text{ and } \Pi_h^Q v(x) = \sum_{K\in\mathcal{T}} v_K 1_K(x),\ x \in \Omega. \tag{2.31}$$

Here and in what follows, 1_K is the characteristic function of K.

We define the Crouzeix-Raviart space with 'zero traces':

$$V_{h,0}(\Omega_h) = \{v \in L^2(\Omega_h),\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K), \tag{2.32}$$

$$\forall\sigma \in \mathcal{E}_{\text{int}},\ \sigma = K|L,\ \int_\sigma v_{|K}\,\mathrm{d}S = \int_\sigma v_{|L}\,\mathrm{d}S, \quad \forall\sigma' \in \mathcal{E}_{\text{ext}},\ \int_{\sigma'} v\,\mathrm{d}S = 0\},$$

and 'with general traces'

$$V_h(\Omega_h) = \{v \in L^2(\Omega),\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K),\ \forall\sigma \in \mathcal{E}_{\text{int}},\ \sigma = K|L,\ \int_\sigma v_{|K}\,\mathrm{d}S = \int_\sigma v_{|L}\,\mathrm{d}S\}. \tag{2.33}$$

We denote by Π_h^V the standard Crouzeix-Raviart projection, and $\Pi_{h,0}^V$ the Crouzeix-Raviart projection with 'zero trace', specifically,

$$\Pi_h^V : C(\overline{\Omega}_h) \to V_h(\Omega_h),\ \int_\sigma \Pi_h^V[\phi]\ \mathrm{dS}_x = \int_\sigma \phi\ \mathrm{dS}_x \text{ for all } \sigma \in \mathcal{E},$$

$$\Pi_{h,0}^V : C(\overline{\Omega}_h) \to V_{h,0}(\Omega_h),\ \int_\sigma \Pi_{h,0}^V[\phi]\ \mathrm{dS}_x = \int_\sigma \phi\ \mathrm{dS}_x \text{ for all } \sigma \in \mathcal{E}_{\text{int}}, \tag{2.34}$$

$$\int_\sigma \Pi_{h,0}^V[\phi]\ \mathrm{dS}_x = 0 \text{ whenever } \sigma \in \mathcal{E}_{\text{ext}}.$$

If $v \in W^{1,1}(\Omega_h)$, we set

$$v_\sigma = \frac{1}{|\sigma|}\int_\sigma v\mathrm{d}S \text{ for } \sigma \in \mathcal{E}. \tag{2.35}$$

(See e.g. [9, Section 4.3] for the definition of traces of functions in $W^{1,1}$.)

Each element $v \in V_h(\Omega_h)$ can be written in the form

$$v(x) = \sum_{\sigma\in\mathcal{E}} v_\sigma \varphi_\sigma(x), \quad x \in \Omega_h, \tag{2.36}$$

where the set $\{\varphi_\sigma\}_{\sigma\in\mathcal{E}} \subset V_h(\Omega_h)$ is the classical Crouzeix-Raviart basis determined by

$$\forall(\sigma,\sigma') \in \mathcal{E}^2,\ \frac{1}{|\sigma'|}\int_{\sigma'} \varphi_\sigma \, \mathrm{d}S = \delta_{\sigma,\sigma'}. \tag{2.37}$$

Similarly, each element $v \in V_{h,0}(\Omega_h)$ can be written in the form

$$v(x) = \sum_{\sigma\in\mathcal{E}_{\rm int}} v_\sigma \varphi_\sigma(x), \quad x \in \Omega_h. \tag{2.38}$$

We first recall in Lemmas 2.3–2.7 the standard properties of the projection Π_h^V. The collection of their proofs in the requested generality can be found in the Appendix of [22] with exception of Lemma 2.8 and its Corollary 2.1. We refer to the monograph of Brezzi, Fortin [2], the Crouzeix's and Raviart's paper [6], Gallouet, Herbin, Latché [21] for the original versions of some of these proofs. We present the proof of Lemma 2.8 dealing with the comparison of projections Π_h^V and $\Pi_{h,0}^V$ that we did not find in the literature.

Lemma 2.3. *The following estimates hold true:*

$$\|\Pi_h^V[\phi]\|_{L^\infty(K)} + \|\Pi_{h,0}^V[\phi]\|_{L^\infty(K)} \stackrel{<}{\sim} \|\phi\|_{L^\infty(K)}, \tag{2.39}$$

for all $K \in \mathcal{T}$ and $\phi \in C(K)$;

$$\|\phi - \Pi_h^V[\phi]\|_{L^p(K)} \stackrel{<}{\sim} h^s \|\nabla^s\phi\|_{L^p(K;\mathbb{R}^{d^s})},\ s = 1,2,\ 1 \le p \le \infty, \tag{2.40}$$

and

$$\|\nabla(\phi - \Pi_h^V[\phi])\|_{L^p(K;\mathbb{R}^d)} \le ch^{s-1}\|\nabla^s\phi\|_{L^p(K;\mathbb{R}^{d^s})},\ s = 1,2,\ 1 \le p \le \infty, \tag{2.41}$$

for all $K \in \mathcal{T}$ and $\phi \in C^s(K)$.

Lemma 2.4. *Let $1 \le p < \infty$. Then*

$$\sum_{\sigma\in\mathcal{E}} |\sigma| h |v_\sigma|^p \approx \|v\|^p_{L^p(\Omega_h)}, \tag{2.42}$$

with any $v \in V_h(\Omega_h)$.

Lemma 2.5. *The following Sobolev-type inequality holds true:*

$$\|v\|^2_{L^6(\Omega_h)} \stackrel{<}{\sim} \sum_{K\in\mathcal{T}} \int_K |\nabla_x v|^2 \mathrm{d}x, \tag{2.43}$$

with any $v \in V_{h,0}(\Omega_h)$.

Lemma 2.6. *There holds:*

$$\sum_{K\in\mathcal{T}} \int_K q \ \mathrm{div}\, \Pi_h^V[\boldsymbol{v}] \, \mathrm{d}x = \int_\Omega q \ \mathrm{div}\, \boldsymbol{v} \, \mathrm{d}x, \tag{2.44}$$

for all $\boldsymbol{v} \in C^1(\overline{\Omega}_h, \mathbb{R}^d)$ and all $q \in Q_h(\Omega_h)$.

Lemma 2.7 (Jumps over faces in the Crouzeix-Raviart space). *For all $v \in V_{h,0}(\Omega_h)$ there holds*

$$\sum_{\sigma\in\mathcal{E}} \frac{1}{h}\int_\sigma [v]^2_{\sigma,\mathrm{n}_\sigma}\,\mathrm{d}S \lesssim \sum_{K\in\mathcal{T}}\int_K |\nabla_x v|^2 \mathrm{d}x, \tag{2.45}$$

where $[v]_{\sigma,\mathrm{n}_\sigma}$ is a jump of v with respect to a normal $\boldsymbol{n}_\sigma$ to the face σ,

$$\forall x \in \sigma = K|L \in \mathcal{E}_{\rm int}, \quad [v]_{\sigma,\mathrm{n}_\sigma}(x) = \begin{cases} v|_K(x) - v|_L(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,K} \\ v|_L(x) - v|_K(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,L} \end{cases},$$

($\boldsymbol{n}_{\sigma,K}$ is the normal of σ, that is outer w.r. to element K) and

$$\forall x \in \sigma \in \mathcal{E}_{\rm ext}, \quad [v]_{\sigma,\mathrm{n}_\sigma}(x) = v(x), \text{ with } \boldsymbol{n}_\sigma \text{ an exterior normal to } \partial\Omega.$$

We will need to compare the projections Π_h^V and $\Pi_{h,0}^V$. Clearly they coincide on 'interior' elements meaning $K \in \mathcal{T}$, $K \cap \partial\Omega_h = \emptyset$. We have the following lemma for the tetrahedra with non void intersection with the boundary.

Lemma 2.8. *We have*

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^\infty(K)} + h\|\nabla_x(\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi])\|_{L^\infty(K;R^3)} \lesssim \sup_{\sigma\subset K\cap\partial\Omega_h} \|\phi\|_{L^\infty(\sigma)} \text{ if } K \in \mathcal{T},\ K\cap\partial\Omega_h \neq \emptyset, \tag{2.46}$$

for any $\phi \in C(K)$.

Proof: We recall the Crouzeix-Raviart basis (2.37) and the fact that Π_h^V and $\Pi_{h,0}^V$ differ only in basis functions corresponding to $\sigma \in \mathcal{E}_{\rm ext}$. We have

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^v[\phi]\|_{L^\infty(K)} \le \left\| \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \varphi_\sigma \frac{1}{|\sigma|}\int_\sigma \phi\,\mathrm{d}S \right\|_{L^\infty(K)} \le c(K)\cdot \sup_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \|\phi\|_{L^\infty(\sigma)}, \tag{2.47}$$

and

$$h\|\nabla_x(\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi])\|_{L^\infty(K)} \le h\left\| \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \nabla_x\varphi_\sigma \frac{1}{|\sigma|}\int_\sigma \phi\,\mathrm{d}S \right\|_{L^\infty(K)}$$
$$\le ch \sup_{\sigma\subseteq K\cap\partial\Omega_h} \|\phi\|_{L^\infty(\sigma)} \left\| \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \nabla_x\varphi_\sigma \right\|_{L^\infty(K)}.$$

The proof is completed by $\| \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} \nabla_x\varphi_\sigma\|_{L^\infty(K)} \le c(K)h^{-1}$. □

In fact, in the derivation of the error estimates we will use the consequence of the above observations formulated in the following two corollaries.

Corollary 2.1. *Let $\phi \in C^1(R^3)$ such that $\phi|_{\partial\Omega} = 0$. Then we have,*

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^\infty(K)} = 0 \text{ if } K \in \mathcal{T},\ K\cap\partial_{\Omega_h} = \emptyset, \tag{2.48}$$

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^\infty(K)} + h\|\nabla_x(\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi])\|_{L^\infty(K;R^3)} \lesssim h^2\|\nabla_x\phi\|_{L^\infty(R^3;R^3)}, \tag{2.49}$$

if $K \in \mathcal{T}_h$, $K\cap\partial\Omega_h \neq \emptyset$, $\partial K \not\subset \partial\Omega$.

Proof: Relation (2.48) follows immediately from (2.46), as there is an empty sum on the right hand side for 'interior' elements ($K \cap \partial\Omega_h = \emptyset$).

For any $x \in \partial\Omega_h$ there exists $y \in \partial\Omega$ (and thus $\phi(y) = 0$) such that

$$|\phi(x)| \leq \mathrm{dist}[x, y] \|\nabla_x \phi\|_{L^\infty(R^3;R^3)} \stackrel{<}{\sim} h^2 \|\nabla_x \phi\|_{L^\infty(R^3;R^3)}, \tag{2.50}$$

where we used Lemma 2.2 for the latter inequality. The proof is completed by taking supremum over $K \in \mathcal{T}_h$ and combining with (2.50). Note that the mesh regularity property (2.29) supplies a uniform estimate of constants $c(K)$ from the previous lemma, which enables to write the latter inequality in (2.50). □

Corollary 2.2. *For any $\phi \in C(R^3)$,*

$$\|\Pi_h^V[\phi] - \Pi_{h,0}^V[\phi]\|_{L^p(\Omega_h)} \stackrel{<}{\sim} h^{1/p} \|\phi\|_{L^\infty(\Omega_h)},\ 1 \leq p < \infty. \tag{2.51}$$

Proof: Apply inverse estimates (see e.g. [31, Lemma 2.9]) to (2.46). □

We will frequently use the Poincaré, Sobolev and interpolation inequalities on tetrahedra reported in the following lemma.

Lemma 2.9.

(1) We have,

$$\|v - v_K\|_{L^p(K)} \stackrel{<}{\sim} h \|\nabla v\|_{L^p(K)}, \tag{2.52}$$

$$\forall \sigma \in \mathcal{E}(K),\ \|v - v_\sigma\|_{L^p(K)} \stackrel{<}{\sim} h \|\nabla v\|_{L^p(K)}, \tag{2.53}$$

for any $v \in W^{1,p}(K)$, where $1 \leq p \leq \infty$.

(2) There holds

$$\|v - v_K\|_{L^{p^*}(K)} \stackrel{<}{\sim} \|\nabla v\|_{L^p(K)}, \tag{2.54}$$

$$\forall \sigma \in \mathcal{E}(K),\ \|v - v_\sigma\|_{L^{p^*}(K)} \stackrel{<}{\sim} \|\nabla v\|_{L^p(K)}, \tag{2.55}$$

for any $v \in W^{1,p}(K)$, $1 \leq p < d$, where $p^* = \frac{dp}{d-p}$.

(3) We have,

$$\|v - v_K\|_{L^q(K)} \leq c h^\beta \|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{2.56}$$

$$\|v - v_\sigma\|_{L^q(K)} \leq c h^\beta \|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{2.57}$$

for any $v \in W^{1,p}(K)$, $1 \leq p < d$, where $\frac{1}{q} = \frac{\beta}{p} + \frac{1-\beta}{p^*}$, $p \leq q \leq p^*$.

We finish the section of preliminaries by recalling two algebraic inequalities 1) the 'imbedding' inequality

$$\Big(\sum_{i=1}^L |a_i|^p \Big)^{1/p} \leq \Big(\sum_{i=1}^L |a_i|^q \Big)^{1/q}, \tag{2.58}$$

for all $a = (a_1, \ldots, a_L) \in R^L$, $1 \leq q \leq p < \infty$ and the discrete Hölder inequality

$$\sum_{i=1}^L |a_i||b_i| \leq \Big(\sum_{i=1}^L |a_i|^q \Big)^{1/q} \Big(\sum_{i=1}^L |a_i|^p \Big)^{1/p}, \tag{2.59}$$

for all $a = (a_1, \ldots, a_L) \in R^L$, $b = (b_1, \ldots, b_L) \in R^L$, $\frac{1}{q} + \frac{1}{p} = 1$.

3 Main result

Here and hereafter we systematically use the following abbreviated notation:

$$\hat{\phi} = \Pi_h^Q[\phi], \quad \phi_h = \Pi_h^V[\phi], \quad \phi_{h,0} = \Pi_{h,0}^V[\phi], \tag{3.1}$$

where projections Π_h^Q, Π_h^V and $\Pi_{h,0}^V$ are defined in (2.31) and (2.34). For a function $v \in C([0,T], L^1(\Omega))$ we set

$$v^n(x) = v(t_n, x), \tag{3.2}$$

where $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n < t_{n+1} < \ldots t_N = T$ is a partition of the interval $[0,T]$. Finally, for a function $v \in V_h(\Omega_h)$ we denote

$$\nabla_h v(x) = \sum_{K \in \mathcal{T}} \nabla_x v(x) 1_K(x), \quad \mathrm{div}_h \mathbf{v}(x) = \sum_{K \in \mathcal{T}} \mathrm{div}_x \mathbf{v}(x) 1_K(x). \tag{3.3}$$

In order to ensure the positivity of the approximate densities, we shall use an upwinding technique for the density in the mass equation. For $q \in Q_h(\Omega_h)$ and $\boldsymbol{u} \in \mathbf{V}_{h,0}(\Omega_h;\mathbb{R}^3)$, the upwinding of q with respect to $\boldsymbol{u}$ is defined, for $\sigma = K|L \in \mathcal{E}_{\rm int}$ by

$$q_\sigma^{\rm up} = \begin{cases} q_K \text{ if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} > 0 \\ q_L \text{ if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} \leq 0 \end{cases}, \tag{3.4}$$

and we denote

$$\mathrm{Up}_K(q, \mathbf{u}) \equiv \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm int}} q_\sigma^{\rm up} \boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K} = \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm int}} \Big(q_K [\boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K}]^+ + q_L [\boldsymbol{u}_\sigma \cdot \mathbf{n}_{\sigma,K}]^- \Big),$$

where $a^+ = \max(a,0)$, $a^- = \min(a,0)$.

3.1 Numerical scheme

We consider a couple $(\varrho^n, \boldsymbol{u}^n) = (\varrho^{n,(\Delta t,h)}, \boldsymbol{u}^{n,(\Delta t,h)})$ of (numerical) solutions of the following algebraic system (numerical scheme):

$$\varrho^n \in Q_h(\Omega_h), \ \varrho^n > 0, \ \mathbf{u}^n \in V_{h,0}(\Omega_h; R^3), \quad n = 0,1,\ldots,N, \tag{3.5}$$

$$\sum_{K \in \mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} \phi_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\rm up} (\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}) \phi_K = 0 \textit{ for any } \phi \in Q_h(\Omega_h) \textit{ and } n = 1,\ldots,N, \tag{3.6}$$

$$\sum_{K \in \mathcal{T}} \frac{|K|}{\Delta t} \Big(\varrho_K^n \hat{\boldsymbol{u}}_K^n - \varrho_K^{n-1} \hat{\boldsymbol{u}}_K^{n-1} \Big) \cdot \boldsymbol{v}_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\rm up} \hat{\boldsymbol{u}}_\sigma^{n,\rm up} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] \cdot \boldsymbol{v}_K \tag{3.7}$$

$$- \sum_{K \in \mathcal{T}} p(\varrho_K^n) \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{v}_\sigma \cdot \mathbf{n}_{\sigma,K} + \mu \sum_{K \in \mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v} \ {\rm d}x$$

$$+ \frac{\mu}{3} \sum_{K \in \mathcal{T}} \int_K \mathrm{div} \boldsymbol{u}^n \mathrm{div} \boldsymbol{v} \ {\rm d}x = 0, \ \textit{for any } \boldsymbol{v} \in V_{h,0}(\Omega; R^3) \textit{ and } n = 1,\ldots,N.$$

The numerical solutions depend on the size h of the space discretization and on the time step Δt. For the sake of clarity and in order to simplify notation we will always systematically write in all formulas $(\varrho^n, \boldsymbol{u}^n)$ instead of $(\varrho^{n,(\Delta t,h)}, \boldsymbol{u}^{n,(\Delta t,h)})$.

The numerical method (3.5–3.7) has been suggested in [31, Definition 3.1]; it is *strongly nonlinear and implicit*. It is therefore not a trivial question whether this (finite dimensional) problem admits a solution. The problem of the well posedness of this numerical scheme is investigated in Karper [31, Proposition 3.3]. Karper's result states that:

For each fixed $h > 0$, $\Delta t > 0$, problem (3.5–3.7) admits a solution $(\varrho_h^n, \mathbf{u}_h^n)$:

$$\varrho_h^n \in Q_h(\Omega_h),\ \mathbf{u}_h^n \in V_{h,0}(\Omega_h; R^3),\ n = 0, 1, \dots, N,$$

and $\varrho_h^n > 0$, $n = 1, \dots, N$, provided $\varrho_h^0 > 0$.

The proof uses topological degree theory in the spirit suggested in [20]. All its details are available in Section 11 of [31]. Notice that the above result does not guarantee the *uniqueness* of numerical solutions.

Remark 3.1 Throughout the paper, $q_\sigma^{\rm up}$ is defined in (3.4), where $\mathbf{u}$ is the numerical solution constructed in (3.5–3.7).

3.2 Error estimates

The main result of this paper is announced in the following theorem:

Theorem 3.1. *Let $\Omega \subset R^3$ be a bounded domain of class C^3 and let the pressure satisfy (1.4) with $\gamma \geq 3/2$. Let $\{\varrho^n, \mathbf{u}^n\}_{0 \leq n \leq N}$ be a family of numerical solutions resulting from the scheme (3.5–3.7). Moreover, suppose there are initial data $[r_0, \mathbf{V}_0]$ belonging to the regularity class specified in Proposition 2.1 and giving rise to a weak solution $[r, \mathbf{V}]$ to the initial-boundary value problem (1.1–1.6) in $(0,T) \times \Omega$ satisfying*

$$0 \leq r(t,x) \leq \overline{r} \ \textit{a.a. in}\ (0,T) \times \Omega.$$

Then $[r, \mathbf{V}]$ is regular and there exists a positive number

$$C = C\Big(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{45})},$$

$$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}, \|\partial_t^2 \boldsymbol{V}, \partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))}, \Big)$$

such that

$$\sup_{1 \leq n \leq N} \int_{\Omega \cap \Omega_h} \left[\frac{1}{2}\varrho^n |\hat{\mathbf{u}}^n - \mathbf{V}(t_n, \cdot)|^2 + H(\varrho^n) - H'(r(t_n, \cdot))(\varrho^n - r(t_n, \cdot)) - H(r(t_n))\right] \, \mathrm{d}x \tag{3.8}$$

$$+\Delta t \sum_{1 \leq n \leq N} \int_{\Omega \cap \Omega_h} |\nabla_h \mathbf{u}^n - \nabla_x \mathbf{V}(t_n, \cdot)|^2 \, \mathrm{d}x$$

$$\leq C \left(\sqrt{\Delta t} + h^a + \int_{\Omega \cap \Omega_h} \left[\frac{1}{2}\varrho^0 |\hat{\mathbf{u}}^0 - \mathbf{V}_0|^2 + H(\varrho^0) - H'(r_0)(\varrho^0 - r_0) - H(r_0)\right] \, \mathrm{d}x\right),$$

where

$$a = \frac{2\gamma - 3}{\gamma} \ \textit{if}\ \tfrac{3}{2} \leq \gamma \leq 2,\ \ a = \frac{1}{2}\ \textit{otherwise.} \tag{3.9}$$

Note that for $\gamma = 3/2$ Theorem 3.1 gives only uniform bounds on the difference of exact and numerical solution, not the convergence.

Remark 3.2
The constitutive assumptions for the pressure (1.4) in Theorem 3.1 require, in particular, $p'(0) > 0$. This condition excludes the isentropic pressure laws

$$p(\varrho) = \varrho^\gamma,\ \gamma > 1. \tag{3.10}$$

Nevertheless, Theorem 3.1 *holds under the same assumptions also for the isentropic pressure laws (3.10)*. Here, we have adopted the more restrictive condition (1.4) (in particular $p'(0) > 0$) only for the sake of simplicity and clarity, in order to avoid some unnecessary technical difficulties. It allows to simplify proofs of some estimates: for example estimates (4.7), (4.10) are in this case immediate consequences of the energy inequality (4.2), while in the general case of pressure laws vanishing at 0, the derivation of the same estimates requires more effort, see [22, Corollary 4.1 and Lemma 4.2], where the proofs of these estimates are performed in the general case.

4 Uniform estimates

If we take $\phi = 1$ in formula (3.6) we get immediately the conservation of mass:

$$\forall n = 1, ...N, \quad \int_{\Omega_h} \varrho^n \, \mathrm{d}x = \int_{\Omega_h} \varrho^0 \, \mathrm{d}x. \tag{4.1}$$

The next Lemma reports the standard energy estimates for the numerical scheme (3.5–3.7). The reader can consult Section 4.1 in Gallouet et al. [22, Lemma 4.1] for its laborious but straightforward proof.

Lemma 4.1. *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5–3.7) with the pressure p satisfying (1.4). Then there exist*

$$\overline{\varrho}_\sigma^n \in [\min(\varrho_K^n, \varrho_L^n), \max(\varrho_K^n, \varrho_L^n)], \ \sigma = K|L \in \mathcal{E}_{\rm int}, \ n = 1, \dots, N,$$
$$\overline{\varrho}_K^{n-1,n} \in [\min(\varrho_K^{n-1}, \varrho_K^n), \max(\varrho_K^{n-1}, \varrho_K^n)], \ K \in \mathcal{T}, \ n = 1, \dots, N,$$

such that

$$\sum_{K\in\mathcal{T}} |K| \Big(\frac{1}{2}\varrho_K^m |\boldsymbol{u}_K^m|^2 + H(\varrho_K^m)\Big) - \sum_{K\in\mathcal{T}} |K| \Big(\frac{1}{2}\varrho_K^0 |\boldsymbol{u}_K^0|^2 + H(\varrho_K^0)\Big)$$
$$+ \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x \boldsymbol{u}^n|^2 \, \mathrm{d}x + \frac{\mu}{3} \int_K |\mathrm{div} \boldsymbol{u}^n|^2 \, \mathrm{d}x\Big)$$
$$+ [D_{\rm time}^{m,|\Delta \mathrm{u}|}] + [D_{\rm time}^{m,|\Delta \varrho|}] + [D_{\rm space}^{m,|\Delta \mathrm{u}|}] + [D_{\rm space}^{m,|\Delta \varrho|}] = 0, \tag{4.2}$$

for all $m = 1, \dots, N$, where

$$[D_{\rm time}^{m,|\Delta \mathrm{u}|}] = \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{|\boldsymbol{u}_K^n - \boldsymbol{u}_K^{n-1}|^2}{2}, \tag{4.3a}$$

$$[D_{\rm time}^{m,|\Delta \varrho|}] = \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| H''(\overline{\varrho}_K^{n-1,n}) \frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}, \tag{4.3b}$$

$$[D_{\rm space}^{m,|\Delta \mathrm{u}|}] = \Delta t \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma| \varrho_\sigma^{n,\rm up} \frac{(\boldsymbol{u}_K^n - \boldsymbol{u}_L^n)^2}{2} \, |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|, \tag{4.3c}$$

$$[D_{\rm space}^{m,|\Delta \varrho|}] = \Delta t \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma| H''(\overline{\varrho}_\sigma^n) \frac{(\varrho_K^n - \varrho_L^n)^2}{2} \, |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}|. \tag{4.3d}$$

We have the following corollary of Lemma 4.1.

Corollary 4.1. *Under assumptions of Lemma 4.1, we have:*

(1) There exists $c = c(M_0, E_0) > 0$ (independent of n, h and Δt) such that

$$k \sum_{n=1}^{N} \int_K |\nabla_x \boldsymbol{u}^n|^2 \,\mathrm{d}x \leq c, \tag{4.4}$$

$$k \sum_{n=1}^{N} \|\boldsymbol{u}^n\|^2_{L^6(\Omega_h;\mathbb{R}^3)} \leq c, \tag{4.5}$$

$$\sup_{n=0,\dots N} \|\varrho^n |\hat{\boldsymbol{u}}^n|^2\|_{L^1(\Omega_h)} \leq c. \tag{4.6}$$

(2)

$$\sup_{n=0,\dots N} \|\varrho^n\|_{L^\gamma(\Omega_h)} \leq c, \tag{4.7}$$

(3) If the pair $(r, \boldsymbol{U})$ belongs to the class (2.25) there is $c = c(M_0, E_0, \underline{r}, \overline{r}, \|\boldsymbol{U}, \nabla \boldsymbol{U}\|_{L^\infty(Q_T;\mathbb{R}^{12})}) > 0$ such that for all $n = 1, \dots, N$,

$$\sup_{n=0,\dots N} \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n | \hat{r}(t_n), \hat{\boldsymbol{U}}(t_n)) \leq c, \tag{4.8}$$

where

$$\mathcal{E}(\varrho, \boldsymbol{u}|z, \mathbf{v}) = \int_{\Omega_h} \Big(\varrho|\boldsymbol{u} - \mathbf{v}|^2 + E(\varrho|z)\Big)\mathrm{d}x, \quad E(\varrho|z) = H(\varrho) - H'(z)(\varrho - z) - H(z). \tag{4.9}$$

(4) There exists $c = c(M_0, E_0, \underline{r}, |p'|_{C[\underline{r},\overline{r}]}) > 0$ such that

$$\Delta t \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma|(\varrho_K^n - \varrho_L^n)^2 \Big[\frac{1_{\{\overline{\varrho}_\sigma^n \geq 1\}}}{[\max\{\varrho_K, \varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n < 1\}}\Big] \, |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leq c \quad \text{if } \gamma \in [1,2), \tag{4.10}$$

$$\Delta t \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma|(\varrho_K^n - \varrho_L^n)^2 \, |\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leq c \quad \text{if } \gamma \geq 2$$

Items (1)-(3) of Corollary 4.1 are direct consequences of Lemma 4.1. Item (4) represents the convenient expression for the numerical dissipation (4.3d). The interested reader can consult Section 4.2 in Gallouet et al. [22, Corollary 4.1, Lemma 4.2] for the detailed proofs of these estimates.

5 Discrete relative energy inequality

The starting point of our error analysis is the discrete relative energy inequality for the numerical scheme (3.5–3.7) formulated in the following lemma.

Lemma 5.1. *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5–3.7) with the pressure p satisfying (1.4). Then there holds for all $m = 1, \dots, N$,*

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} \frac{1}{2}|K|\Big(\varrho_K^m|\boldsymbol{u}_K^m - \boldsymbol{U}_K^m|^2 - \varrho_K^0|\boldsymbol{u}_K^0 - \boldsymbol{U}_K^0|^2\Big) + \sum_{K\in\mathcal{T}} |K|\Big(E(\varrho_K^m|r_K^m) - E(\varrho_K^0|r_K^0)\Big) \\
&\quad + \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x + \frac{\mu}{3}\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x\Big) \leq \sum_{i=1}^{6} T_i,
\end{aligned} \tag{5.1}$$

for any $0 < r^n \in Q_h(\Omega_h)$, $\boldsymbol{U}^n \in V_{h,0}(\Omega_h;\mathbb{R}^3)$, $n = 1,\dots,N$, *where*

$$
\begin{aligned}
T_1 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \Big(\mu \int_K \nabla_x \boldsymbol{U}^n : \nabla_x(\boldsymbol{U}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \frac{\mu}{3}\int_K \mathrm{div}\boldsymbol{U}^n \mathrm{div}(\boldsymbol{U}^n - \boldsymbol{u}^n)\,\mathrm{d}x \Big),\\
T_2 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{U}_K^n - \boldsymbol{U}_K^{n-1}}{\Delta t} \cdot \Big(\frac{\boldsymbol{U}_K^{n-1} + \boldsymbol{U}_K^n}{2} - \boldsymbol{u}_K^{n-1} \Big),\\
T_3 &= -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\frac{\boldsymbol{U}_K^n + \boldsymbol{U}_L^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \Big) \cdot \boldsymbol{U}_K^n [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}],\\
T_4 &= -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| p(\varrho_K^n) [\boldsymbol{U}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}],\\
T_5 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} (r_K^n - \varrho_K^n) \Big(H'(r_K^n) - H'(r_K^{n-1}) \Big),\\
T_6 &= \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\mathrm{up}} H'(r_K^{n-1}) [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}].
\end{aligned}
\tag{5.2}
$$

Proof: Lemma 5.1 is proved in Section 5 in Gallouet et al. [22, Theorem 5.1]. We provide here the proof for the sake of completeness.

First, noting that the numerical diffusion represented by terms (4.3a–4.3d) in the energy identity (4.2) is positive, we infer

$$I_1 + I_2 + I_3 \le 0, \tag{5.3}$$

with

$$
\begin{aligned}
I_1 &:= \sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n |\boldsymbol{u}_K^n|^2 - \varrho_K^{n-1} |\boldsymbol{u}_K^{n-1}|^2 \Big), \qquad I_2 := \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big(H(\varrho_K^n) - H(\varrho_K^{n-1}) \Big),\\
I_3 &:= \sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x \boldsymbol{u}^n|^2\,\mathrm{d}x + \frac{\mu}{3}\int_K |\mathrm{div}\boldsymbol{u}^n|^2\,\mathrm{d}x \Big).
\end{aligned}
$$

Next, we consider the discrete continuity equation (3.6) with $\phi = \frac{1}{2}|\hat{\mathrm{U}}^n|^2$ as test function in order to obtain

$$I_4 := \sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{\Delta t}(\varrho_K^n - \varrho_K^{n-1})|\boldsymbol{U}_K^n|^2 = -\sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} \frac{1}{2}|\sigma|\varrho_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]|\boldsymbol{U}_K^n|^2 := J_1. \tag{5.4}$$

In the next step, taking $-\boldsymbol{U}^n$ as test function $\mathbf{v}$ in the discrete momentum equation (3.7) one gets

$$I_5 = -\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big(\varrho_K^n \boldsymbol{u}_K^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1} \Big) \cdot \boldsymbol{U}_K^n = J_2 + J_3 + J_4,$$

with

$$J_2 = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \cdot \boldsymbol{U}_K^n \, [\boldsymbol{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}],$$

$$J_3 = \mu \sum_{K\in\mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{U}^n \,\mathrm{d}x + \frac{\mu}{3} \sum_{K\in\mathcal{T}} \int_K \mathrm{div}\boldsymbol{u}^n \mathrm{div}\boldsymbol{U}^n \,\mathrm{d}x$$

and

$$J_4 = -\sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| p(\varrho_K^n)[\boldsymbol{U}_\sigma^n \cdot \mathbf{n}_{\sigma,K}].$$

We then consider the discrete continuity equation (3.6) with a test function $\phi = H'(r^{n-1})$ and obtain

$$-\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}(\varrho_K^n - \varrho_K^{n-1}) H'(r_K^{n-1}) = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\mathrm{up}} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] H'(r_K^{n-1}).$$

Observing that $\varrho_K^n H'(r_K^n) - \varrho_K^{n-1} H'(r_K^{n-1}) = \varrho_K^n \Big(H'(r_K^n) - H'(r_K^{n-1}) \Big) + (\varrho_K^n - \varrho_K^{n-1}) H'(r_K^{n-1})$, we rewrite the last identity in the form

$$\begin{aligned} &I_6 := -\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} \Big(\varrho_K^n H'(r_K^n) - \varrho_K^{n-1} H'(r_K^{n-1}) \Big) = J_5 + J_6 \\ &\text{with } J_5 = -\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} \varrho_K^n \Big(H'(r_K^n) - H'(r_K^{n-1}) \Big) \text{ and } J_6 = \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\mathrm{up}} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] H'(r_K^{n-1}). \end{aligned} \tag{5.5}$$

Finally, thanks to the convexity of the function H, we have

$$\begin{aligned} I_7 &:= \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} \Big[\Big(r_K^n H'(r_K^n) - H(r_K^n) \Big) - \Big(r_K^{n-1} H'(r_K^{n-1}) - H(r_K^{n-1}) \Big) \Big] \\ &= \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} r_K^n \Big(H'(r_K^n) - H'(r_K^{n-1}) \Big) - \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} \Big(H(r_K^n) - (r_K^n - r_K^{n-1}) H'(r_K^{n-1}) - H(r_K^{n-1}) \Big) \\ &\le \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} r_K^n \Big(H'(r_K^n) - H'(r_K^{n-1}) \Big) := J_7. \end{aligned} \tag{5.6}$$

Now, we gather the expressions (5.3)-(5.6); this is performed in several steps.

Step 1: *Term* $I_1 + I_4 + I_5$. We obtain by direct calculation,

$$\begin{aligned} I_1 + I_4 + I_5 = \sum_{K\in\mathcal{T}} \frac{1}{2} \frac{|K|}{\Delta t} \Big(\varrho_K^n |\boldsymbol{u}_K^n - \boldsymbol{U}_K^n|^2 - \varrho_K^{n-1} |\boldsymbol{u}_K^{n-1} - \boldsymbol{U}_K^{n-1}|^2 \Big) \\ - \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{U}_K^n - \boldsymbol{U}_K^{n-1}}{\Delta t} \cdot \Big(\frac{\boldsymbol{U}_K^{n-1} + \boldsymbol{U}_K^n}{2} - \boldsymbol{u}_K^{n-1} \Big). \end{aligned} \tag{5.7}$$

Step 2: *Term* $J_1 + J_2$. Employing the definition (3.4) of the upwinding, one gets

$$J_1 + J_2 = -\sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\frac{\boldsymbol{U}_K^n + \boldsymbol{U}_L^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \Big) \cdot \boldsymbol{U}_K^n [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]. \tag{5.8}$$

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Step 3: *Term* $I_3 - J_3$. This term can be written in the form

$$
\begin{aligned}
I_3 - J_3 = &\sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x + \frac{\mu}{3}\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x\Big) \\
&- \sum_{K\in\mathcal{T}} \mu \int_K \Big(\nabla \boldsymbol{U}^n : \nabla(\boldsymbol{U}^n - \boldsymbol{u}^n) + \frac{\mu}{3}\int_K \mathrm{div}\boldsymbol{U}^n \mathrm{div}(\boldsymbol{U}^n - \boldsymbol{u}^n)\Big).
\end{aligned}
\tag{5.9}
$$

Step 4: *Term* $I_2 + I_6 + I_7$. By virtue of (5.3), (5.5–5.6), we easily find that

$$
I_2 + I_6 + I_7 = \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big(E(\varrho_K^n \,|\, r_K^n) - E(\varrho_K^{n-1} \,|\, r_K^{n-1})\Big),
\tag{5.10}
$$

where the function E is defined in (4.9).

Step 5: *Term* $J_5 + J_6 + J_7$. Coming back to (5.5–5.6), we deduce that

$$
J_5 + J_6 + J_7 = \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}(r_K^n - \varrho_K^n)\Big(H'(r_K^n) - H'(r_K^{n-1})\Big) + \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\, \varrho_\sigma^{n,\mathrm{up}} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] H'(r_K^{n-1}).
\tag{5.11}
$$

Step 6: *Conclusion*

According to (5.3)–(5.6), we have

$$
\sum_{i=1}^7 I_i \le \sum_{i=1}^7 J_i;
$$

whence, writing this inequality by using expressions (5.7)–(5.11) calculated in steps 1-5, we get

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n |\boldsymbol{u}_K^n - \boldsymbol{U}_K^n|^2 - \varrho_K^{n-1}|\boldsymbol{u}_K^{n-1} - \boldsymbol{U}_K^{n-1}|^2\Big) + \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big(E(\varrho_K^n|r_K^n) - E(\varrho_K^{n-1}|r_K^{n-1})\Big) \\
&\quad + \sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x + \frac{\mu}{3}\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x\Big) \\
&\le \sum_{K\in\mathcal{T}} \Big(\mu \int_K \nabla_x \boldsymbol{U}_h^n : \nabla_x(\boldsymbol{U}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \frac{\mu}{3}\int_K \mathrm{div}\boldsymbol{U}^n \mathrm{div}(\boldsymbol{U}^n - \boldsymbol{u}^n)\,\mathrm{d}x\Big) \\
&\quad + \sum_{K\in\mathcal{T}} |K|\varrho_K^{n-1} \frac{\boldsymbol{U}_K^n - \boldsymbol{U}_K^{n-1}}{\Delta t} \cdot \Big(\frac{\boldsymbol{U}_K^{n-1} + \boldsymbol{U}_K^n}{2} - \boldsymbol{u}_K^{n-1}\Big) \\
&\quad - \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}} \Big(\frac{\boldsymbol{U}_K^n + \boldsymbol{U}_L^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big) \cdot \boldsymbol{U}_K^n [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] \\
&\quad - \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| p(\varrho_K^n)[\boldsymbol{U}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] + \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}(r_K^n - \varrho_K^n)\Big(H'(r_K^n) - H'(r_K^{n-1})\Big) \\
&\quad + \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}_K} |\sigma|\varrho_\sigma^{n,\mathrm{up}} H'(r_K^{n-1})[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}].
\end{aligned}
\tag{5.12}
$$

We obtain formula (5.1) by summing $(5.12)^n$ from $n = 1$ to $n = m$ and multiplying the resulting inequality by Δt.

6 Approximate discrete relative energy inequality

In this section, we transform the right hand side of the relative energy inequality (5.1) to a form that is more convenient for the comparison with the strong solution. This transformation is given in the following lemma.

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Lemma 6.1 (Approximate relative energy inequality). *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5–3.7), where the pressure satisfies (1.4) with $\gamma \geq 3/2$. Then there exists*

$$c = c\Big(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{18})},$$

$$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}\Big) > 0,$$

such that for all $m = 1, \ldots, N$, we have:

$$\int_{\Omega_h} \Big(\varrho^m |\hat{\boldsymbol{u}}^m - \hat{\boldsymbol{V}}^m_{h,0}|^2 + E(\varrho^m|\hat{r}^m)\Big)\mathrm{d}x - \int_{\Omega_h} \Big(\varrho^0 |\hat{\boldsymbol{u}}^0 - \hat{\boldsymbol{V}}^0_{h,0}|^2 + E(\varrho^0|\hat{r}^0)\Big)\mathrm{d}x$$
$$+ \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{V}^n_{h,0})|^2 \,\mathrm{d}x + \frac{\mu}{3}\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{V}^n_{h,0})|^2 \,\mathrm{d}x\Big) \leq \sum_{i=1}^6 S_i + R^m_{h,\Delta t} + G^m, \tag{6.1}$$

for any couple $(r, \mathbf{V})$ belonging to the class (2.25), where

$$\begin{aligned}
S_1 &= \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \Big(\mu \int_K \nabla_x \boldsymbol{V}^n_{h,0} : \nabla_x(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,\mathrm{d}x + \frac{\mu}{3}\int_K \mathrm{div}\boldsymbol{V}^n_{h,0}\mathrm{div}(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,\mathrm{d}x\Big),\\
S_2 &= \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} |K|\varrho_K^{n-1} \frac{\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t} \cdot \Big(\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\Big),\\
S_3 &= \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\Big(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma\Big)\cdot\Big(\boldsymbol{V}^n_{h,0,\sigma} - \boldsymbol{V}^n_{h,0,K}\Big)\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}\cdot \boldsymbol{n}_{\sigma,K},\\
S_4 &= -\Delta t \sum_{n=1}^m \int_{\Omega_h} p(\varrho^n)\,\mathrm{div}\,\boldsymbol{V}^n\,\mathrm{d}x,\\
S_5 &= \Delta t \sum_{n=1}^m \int_{\Omega_h} (\hat{r}^n - \varrho^n)\frac{p'(\hat{r}^n)}{\hat{r}^n}[\partial_t r]^n\,\mathrm{d}x,\\
S_6 &= -\Delta t \sum_{n=1}^m \int_{\Omega_h} \frac{\varrho^n}{\hat{r}^n} p'(\hat{r}^n)\boldsymbol{u}^n\cdot\nabla r^n\,\mathrm{d}x,
\end{aligned} \tag{6.2}$$

and

$$|G^m| \leq c\,\Delta t \sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n \Big| \hat{r}^n, \hat{\boldsymbol{V}}^n),\ \ |R^m_{h,\Delta t}| \leq c(\sqrt{\Delta t} + h^a), \tag{6.3}$$

with the power a defined in (3.9) and with the functional $\mathcal{E}$ introduced in (4.9). (Recall that in agreement with the notation (2.35), (3.1-3.3), $\mathbf{V}^n_{h,0} = \Pi^V_{h,0}[\mathbf{V}(t_n)]$, $\mathbf{V}^n_{h,0,K} = \Pi^Q_h \Pi^V_{h,0}\mathbf{V}(t_n)|_K$, $\mathbf{V}^n_{h,0,\sigma} = \frac{1}{|\sigma|}\int_\sigma \mathbf{V}^n_{h,0}$, $\hat{r}^n = \Pi^Q_h[r(t_n)]$, where the projections Π^Q, Π^V are defined in (2.31) and (2.34).)

Proof: We take as test functions $\mathbf{U}^n = \mathbf{V}^n_{h,0}$ and $r^n = \hat{r}^n$ in the discrete relative energy inequality (5.1). We keep the left hand side and the first term (term T_1) at the right hand side as they stay. The transformation of the remaining terms at the right hand side (terms T_2–T_6) is performed in the following steps:

Step 1: *Term T_2.* We have

$$T_2 = T_{2,1} + R_{2,1} + R_{2,2},\ \text{with}\ T_{2,1} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} |K|\varrho_K^{n-1}\frac{\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t}\cdot\Big(\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\Big), \tag{6.4}$$

and

$$R_{2,1} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} R^{n,K}_{2,1},\ \ R_{2,2} = \Delta t \sum_{n=1}^m R^n_{2,2},$$

where

$$R_{2,1}^{n,K} = -\frac{|K|}{2}\varrho_K^{n-1}\frac{(\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1})^2}{\Delta t} = -\frac{|K|}{2}\varrho_K^{n-1}\frac{([\boldsymbol{V}^n - \boldsymbol{V}^{n-1}]_{h,0,K})^2}{\Delta t},$$

and

$$R_{2,2}^n = -\sum_{K\in\mathcal{T}}|K|\varrho_K^{n-1}\frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t}\cdot\Big(\boldsymbol{u}_K^{n-1} - \boldsymbol{u}_K^n\Big).$$

We may write by virtue of the first order Taylor formula applied to function $t\mapsto \mathbf{V}(t,x)$,

$$\Big|\frac{[\boldsymbol{V}^n - \boldsymbol{V}^{n-1}]_{h,0,K}}{\Delta t}\Big| = \Big|\frac{1}{|K|}\int_K\Big[\frac{1}{\Delta t}\Big[\int_{t_{n-1}}^{t_n}\partial_t\mathbf{V}(z,x)\mathrm{d}z\Big]_{h,0}\Big]\mathrm{d}x\Big|$$

$$= \Big|\frac{1}{|K|}\int_K\Big[\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\Big[\partial_t\mathbf{V}(z)\Big]_{h,0}(x)\mathrm{d}z\Big]\mathrm{d}x\Big| \le \|[\partial_t\mathbf{V}]_{h,0}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} \le \|\partial_t\mathbf{V}\Big\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))},$$

where we have used the property (2.39) of the projection $\Pi_{h,0}^V$ on the space $V_{h,0}(\Omega_h)$. Therefore, thanks to the mass conservation (4.1), we get

$$|R_{2,1}^{n,K}| \le \frac{M_0}{2}|K|\Delta t\|\partial_t\boldsymbol{V}\|^2_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}. \tag{6.5}$$

To treat term $R_{2,2}^n$ we use the discrete Hölder inequality and identity (4.1) in order to get

$$|R_{2,2}^n| \le \Delta t\, cM_0\|\partial_t\boldsymbol{V}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))} + cM_0^{1/2}\Big(\sum_{K\in\mathcal{T}}|K|\varrho_K^{n-1}|\boldsymbol{u}_K^{n-1} - \boldsymbol{u}_K^n|^2\Big)^{1/2}\|\partial_t\boldsymbol{V}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))};$$

whence, by virtue of estimate (4.2) for the upwind dissipation term (4.3a), one obtains

$$|R_{2,2}| \le \sqrt{\Delta t}\, c(M_0, E_0, \|\partial_t\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}). \tag{6.6}$$

Step 2: *Term* T_3. Employing the definition (3.4) of upwind quantities, we easily establish that

$T_3 = T_{3,1} + R_{3,1}$,

with $T_{3,1} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\Big(\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}}\Big)\cdot\boldsymbol{V}_{h,0,K}^n\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}$, $\quad R_{3,1} = \Delta t\sum_{n=1}^m\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}}R_{3,1}^{n,\sigma}$,

and $R_{3,1}^{n,\sigma} = |\sigma|\varrho_K^n\dfrac{|\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,L}^n|^2}{2}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]^+ + |\sigma|\varrho_L^n\dfrac{|\boldsymbol{V}_{h,0,L}^n - \boldsymbol{V}_{h,0,K}^n|^2}{2}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,L}]^+$, $\forall\sigma = K|L\in\mathcal{E}_{\mathrm{int}}$.

Writing

$$\begin{aligned}\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,L}^n &= [\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_h^n]_K + \boldsymbol{V}_{h,K}^n - \boldsymbol{V}_h^n + \boldsymbol{V}_h^n - \boldsymbol{V}_{h,\sigma}^n\\ &+\boldsymbol{V}_{h,\sigma}^n - \boldsymbol{V}_h^n + \boldsymbol{V}_h^n - \boldsymbol{V}_{h,L}^n + [\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_L,\ \sigma = K|L\in\mathcal{E}_{\mathrm{int}},\end{aligned}$$

and employing estimates (2.48) (if $K\cap\partial\Omega_h = \emptyset$), (2.49) (if $K\cap\partial\Omega_h \neq \emptyset$) to evaluate the L^∞-norm of the first term, (2.52) then $(2.41)_{s=1}$ and (2.53) after $(2.41)_{s=1}$ to evaluate the L^∞-norm of the second and third terms, and performing the same tasks at the second line, we get

$$\|\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,L}^n\|_{L^\infty(K\cup L;\mathbb{R}^3)} \le ch\|\nabla\boldsymbol{V}\|_{L^\infty(K\cup L;\mathbb{R}^9)}; \tag{6.7}$$

consequently

$$|R_{3,1}^{n,\sigma}| \le h^2\, c\|\nabla\boldsymbol{V}\|^2_{L^\infty((0,T)\times\Omega;\mathbb{R}^9)}|\sigma|(\varrho_K^n + \varrho_L^n)|\boldsymbol{u}_\sigma^n|,\ \forall\sigma = K|L\in\mathcal{E}_{\mathrm{int}},$$

whence

$$
\begin{aligned}
|R_{3,1}| \le h\, c\|\nabla \boldsymbol{V}\|^2_{L^\infty((0,T)\times\Omega;\mathbb{R}^9)}\Big(\sum_{K\in\mathcal{T}}\ \sum_{\sigma=K|L\in\mathcal{E}(K)} h|\sigma|(\varrho_K^n+\varrho_L^n)^{6/5}\Big)^{5/6}\times\\
\Big[\Delta t\sum_{n=1}^m\Big(\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}_\sigma^n|^6\Big)^{1/3}\Big]^{1/2} \le h\, c(M_0,E_0,\|\nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}),
\end{aligned}
\tag{6.8}
$$

provided $\gamma\ge 6/5$, thanks to the discrete Hölder inequality, the equivalence relation (2.29), the equivalence of norms (2.42) and energy bounds listed in Corollary 4.1.

Clearly, for each face $\sigma=K|L\in\mathcal{E}_{\rm int}$, $\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}+\boldsymbol{u}_\sigma^n\cdot\mathbf{n}_{\sigma,L}=0$; whence, finally

$$
T_{3,1}=\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_\sigma^{n,\rm up}\Big(\hat{\boldsymbol{u}}_\sigma^{n,\rm up}-\hat{\boldsymbol{V}}^{n,\rm up}_{h,0,\sigma}\Big)\cdot\Big(\boldsymbol{V}^n_{h,0,K}-\boldsymbol{V}^n_{h,0,\sigma}\Big)\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}. \tag{6.9}
$$

Before the next transformation of term $T_{3,1}$, we realize that

$$
\mathbf{V}^n_{h,0,K}-\mathbf{V}^n_{h,0,\sigma}=[\mathbf{V}^n_{h,0}-\mathbf{V}^n_h]_K+\mathbf{V}^n_{h,K}-\mathbf{V}^n_h+\mathbf{V}^n_h-\mathbf{V}^n_{h,\sigma}+[\mathbf{V}^n_h-\mathbf{V}^n_{h,0}]_\sigma;
$$

whence by virtue of (2.48–2.49), (2.52–2.53) and $(2.41)_{s=1}$, similarly as in (6.7),

$$
\|\boldsymbol{V}^n_{h,0,K}-\boldsymbol{V}^n_{h,0,\sigma}\|_{L^\infty(K;\mathbb{R}^3)}\le ch\|\nabla_x\boldsymbol{V}\|_{L^\infty(0,T;L^\infty(\Omega:R^3))},\ \ \sigma\subset K. \tag{6.10}
$$

Let us now decompose the term $T_{3,1}$ as

$$
\begin{aligned}
&T_{3,1}=T_{3,2}+R_{3,2},\ \text{ with } R_{3,2}=\Delta t\sum_{n=1}^m R^n_{3,2},\\
&T_{3,2}=\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_\sigma^{n,\rm up}\Big(\hat{\boldsymbol{V}}^{n,\rm up}_{h,0,\sigma}-\hat{\boldsymbol{u}}_\sigma^{n,\rm up}\Big)\cdot\Big(\boldsymbol{V}^n_{h,0,\sigma}-\boldsymbol{V}^n_{h,0,K}\Big)\hat{\boldsymbol{u}}_\sigma^{n,\rm up}\cdot\boldsymbol{n}_{\sigma,K},\ \text{and}\\
&R^n_{3,2}=\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_\sigma^{n,\rm up}\Big(\hat{\boldsymbol{V}}^{n,\rm up}_{h,0,\sigma}-\hat{\boldsymbol{u}}_\sigma^{n,\rm up}\Big)\cdot\Big(\boldsymbol{V}^n_{h,0,\sigma}-\boldsymbol{V}^n_{h,0,K}\Big)\Big(\boldsymbol{u}_\sigma^n-\hat{\boldsymbol{u}}_\sigma^{n,\rm up}\Big)\cdot\boldsymbol{n}_{\sigma,K}.
\end{aligned}
$$

By virtue of discrete Hölder's inequality and estimate (6.10), we get

$$
\begin{aligned}
|R^n_{3,2}|\le c\|\nabla\mathbf{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}\Big(\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}h|\sigma|\varrho_\sigma^{n,\rm up}\Big|\hat{\boldsymbol{u}}_\sigma^{n,\rm up}-\hat{\boldsymbol{V}}^{n,\rm up}_{h,0,\sigma}\Big|^2\Big)^{1/2}\\
\times\Big(\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}h|\sigma||\varrho_\sigma^{n,\rm up}|^{\gamma_0}\Big)^{1/(2\gamma_0)}\Big(\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}h|\sigma|\Big|\boldsymbol{u}_\sigma^n-\hat{\boldsymbol{u}}_\sigma^{n,\rm up}\Big|^q\Big)^{1/q},
\end{aligned}
$$

where $\frac12+\frac{1}{2\gamma_0}+\frac1q=1$, $\gamma_0=\min\{\gamma,2\}$ and $\gamma\ge 3/2$. For the sum in the last term of the above product, we have

$$
\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}h|\sigma|\Big|\boldsymbol{u}_\sigma^n-\hat{\boldsymbol{u}}_\sigma^{n,\rm up}\Big|^q\le c\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}h|\sigma||\boldsymbol{u}_\sigma^n-\boldsymbol{u}_K^n|^q
$$

$$
\le c\Big(\sum_{K\in\mathcal{T}}\ \sum_{\sigma\in\mathcal{E}(K)}\Big(\|\boldsymbol{u}_\sigma^n-\boldsymbol{u}^n\|^q_{L^q(K;\mathbb{R}^3)}+\sum_{K\in\mathcal{T}}\|\boldsymbol{u}^n-\boldsymbol{u}_K^n\|^q_{L^q(K;\mathbb{R}^3)}\Big)\le ch^{\frac{2\gamma_0-3}{2\gamma_0}q}\Big(\sum_{K\in\mathcal{T}}\|\nabla_x\boldsymbol{u}^n\|^2_{L^2(K;\mathbb{R}^9)}\Big)^{q/2},
$$

where we have used the definition (3.4), the discrete Minkowski inequality, interpolation inequalities (2.56–2.57) and the discrete 'imbedding' inequality (2.58). Now we can go back to the estimate of $R^n_{3,2}$ taking into account the upper bounds (4.4), (4.7–4.8), in order to get

$$
|R_{3,2}|\le h^a\, c(M_0,E_0,\|\nabla\mathbf{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}), \tag{6.11}
$$

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provided $\gamma \geq 3/2$, where a is given in (6.3).

Finally, we rewrite term $T_{3,2}$ as

$$T_{3,2} = T_{3,3} + R_{3,3}, \text{ with } R_{3,3} = \Delta t \sum_{n=1}^{m} R_{3,3}^{n},$$

$$T_{3,3} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_{\sigma}^{n,\mathrm{up}} \Big(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_{\sigma}^{n,\mathrm{up}}\Big) \cdot \Big(\boldsymbol{V}_{h,0,\sigma}^{n} - \boldsymbol{V}_{h,0,K}^{n}\Big) \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}, \text{ and} \tag{6.12}$$

$$R_{3,3}^{n} = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_{\sigma}^{n,\mathrm{up}} \Big(\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_{\sigma}^{n,\mathrm{up}}\Big) \cdot \Big(\boldsymbol{V}_{h,0,\sigma}^{n} - \boldsymbol{V}_{h,0,K}^{n}\Big) \Big(\hat{\boldsymbol{u}}_{\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}}\Big) \cdot \boldsymbol{n}_{\sigma,K};$$

whence

$$|R_{3,3}| \leq c(\|\nabla \boldsymbol{V}\|_{L^{\infty}(Q_T,\mathbb{R}^9)})\, \Delta t \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \hat{\mathbf{u}}^n \,|\, \hat{r}^n, \hat{\boldsymbol{V}}_{h,0}^{n}). \tag{6.13}$$

Step 3: *Term T_4.* Integration by parts over each $K \in \mathcal{T}$ gives

$$T_4 = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_{\mathrm{K}} p(\varrho_K^n) \mathrm{div}_x \boldsymbol{V}_{h,0}^{n} \,\mathrm{d}x.$$

We may write

$$\|\mathrm{div}_x(\boldsymbol{V}_{0,h}^{n} - \boldsymbol{V}_{h}^{n})\|_{L^{\infty}(K)} \leq ch \|\nabla_x \boldsymbol{V}\|_{L^{\infty}(0,T;L^{\infty}(\Omega;\mathbb{R}^9))}, \tag{6.14}$$

where we have used (2.48–2.49). Therefore, employing identity (2.44) we obtain

$$T_4 = T_{4,1} + R_{4,1}, \quad T_{4,1} = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_{\mathrm{K}} p(\varrho_K^n) \mathrm{div}_x \boldsymbol{V}^{n} \,\mathrm{d}x, \tag{6.15}$$

$$R_{4,1} = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_{\mathrm{K}} p(\varrho_K^n) \mathrm{div}_x (\boldsymbol{V}_{h,0}^{n} - \boldsymbol{V}_{h}^{n}) \,\mathrm{d}x.$$

Due to (1.4) and (4.7), $p(\varrho^n)$ is bounded uniformly in $L^{\infty}(L^1(\Omega))$; employing this fact and (6.14) we immediately get

$$|R_{4,1}| \leq h\; c(E_0, M_0, \|\nabla \mathbf{V}\|_{L^{\infty}(0,T;L^{\infty}(\Omega;\mathrm{R}^9))}). \tag{6.16}$$

Step 4: *Term T_5.* Using the Taylor formula, we get

$$H'(r_K^n) - H'(r_K^{n-1}) = H''(r_K^n)(r_K^n - r_K^{n-1}) - \frac{1}{2} H'''(\overline{r}_K^n)(r_K^n - r_K^{n-1})^2,$$

where $\overline{r}_K^n \in [\min(r_K^{n-1}, r_K^n), \max(r_K^{n-1}, r_K^n)]$. We infer

$$T_5 = T_{5,1} + R_{5,1}, \text{ with } T_{5,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \frac{r_K^n - r_K^{n-1}}{\Delta t},\ R_{5,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} R_{5,1}^{n,K}, \text{ and}$$

$$R_{5,1}^{n,K} = \frac{1}{2} |K| H'''(\overline{r}_K^n) \frac{(r_K^n - r_K^{n-1})^2}{\Delta t} (\varrho_K^n - r_K^n).$$

Consequently, by the first order Taylor formula applied to function $t \mapsto r(t,x)$ on the interval (t_{n-1}, t_n) and thanks to the mass conservation (4.1)

$$|R_{5,1}| \leq \Delta t\; c(M_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}]}, \|\partial_t r\|_{L^{\infty}(Q_T)}). \tag{6.17}$$

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Let us now decompose $T_{5,1}$ as follows:

$$T_{5,1} = T_{5,2} + R_{5,2}, \text{ with } T_{5,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} [\partial_t r]^n \mathrm{d}x, \; R_{5,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} R_{5,2}^{n,K}, \text{ and}$$
$$R_{5,2}^{n,K} = \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \Big(\frac{r_K^n - r_K^{n-1}}{\Delta t} - [\partial_t r]^n \Big) \mathrm{d}x. \tag{6.18}$$

In accordance with (3.2), here and in the sequel, $[\partial_t r]^n(x) = \partial_t r(t_n, x)$. We write using twice the Taylor formula in the integral form and the Fubini theorem,

$$\begin{aligned} |R_{5,2}^{n,K}| &= \frac{1}{\Delta t} \Big| p'(r_K^n) r_K^n (\varrho_K^n - r_K^n) \int_K \int_{t_{n-1}}^{t_n} \int_s^{t_n} \partial_t^2 r(z) \mathrm{d}z \mathrm{d}s \mathrm{d}x \Big| \\ &\le \frac{p'(r_K^n)}{r_K^n} \int_{t_{n-1}}^{t_n} \int_K |\varrho_K^n - r_K^n| \Big|\partial_t^2 r(z)\Big| \mathrm{d}x \mathrm{d}z \mathrm{d}s \\ &\le \frac{p'(r_K^n)}{r_K^n} \|\varrho^n - \hat{r}^n\|_{L^\gamma(K)} \int_{t_{n-1}}^{t_n} \|\partial_t^2 r(z)\|_{L^{\gamma'}(K)} \mathrm{d}z \mathrm{d}s. \end{aligned}$$

Therefore, by virtue of Corollary 4.1, we have estimate

$$|R_{5,2}| \le \Delta t \; c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}]}, \|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega)}). \tag{6.19}$$

Step 5: *Term* T_6. We decompose this term as follows:

$$T_6 = T_{6,1} + R_{6,1}, \quad R_{6,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} R_{6,1}^{n,\sigma,K}, \text{ with}$$
$$T_{6,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| \varrho_K^n \Big(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) \Big) \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}, \text{ and}$$
$$R_{6,1}^{n,\sigma,K} = |\sigma| \Big(\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n \Big) \Big(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) \Big) \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}, \text{ for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}.$$

We will now estimate the term $R_{6,1}^{n,\sigma,K}$. We shall treat separately the cases $\gamma < 2$ and $\gamma \ge 2$. The 'simple' case $\gamma \ge 2$ is left to the reader. The more complicated case $\gamma < 2$ will be treated as follows: We first write

$$\begin{aligned} |R_{6,1}^{n,\sigma,K}| \le \sqrt{h} \, \|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} |\sigma| |\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n| \Big[\frac{1_{\{\overline{\varrho}_\sigma^n \ge 1\}}}{[\max\{\varrho_K, \varrho_L\}]^{(2-\gamma)/2}} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \Big] \sqrt{|\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|} \times \\ \Big[1_{\{\overline{\varrho}_\sigma^n \ge 1\}} [\max\{\varrho_K, \varrho_L\}]^{(2-\gamma)/2} + 1_{\{\overline{\varrho}_\sigma^n < 1\}} \Big] \sqrt{h} \sqrt{|\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|}, \end{aligned}$$

where we have employed the first order Taylor formula applied to function $x \mapsto H'(r(t_{n-1}, x))$. Consequently, the application of the discrete Hölder and Young inequalities yield

$$
\begin{aligned}
|R_{6,1}| &\le \sqrt{h}\, c \|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} \times \\
&\Delta t \sum_{n=1}^m \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \Big[1_{\{\overline{\varrho}_\sigma^n \ge 1\}} [\max\{\varrho_K,\varrho_L\}]^{2-\gamma} + 1_{\{\overline{\varrho}_\sigma^n<1\}} \Big] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| \Big)^{1/2} \times \\
&\Big(\sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n)^2 \Big[\frac{1_{\{\overline{\varrho}_\sigma^n\ge 1\}}}{[\max\{\varrho_K,\varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n<1\}} \Big] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| \Big)^{1/2} \\
&\le \sqrt{h}\, c \|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} \times \\
&\Delta t \sum_{n=1}^m \Big\{ \Big[|\Omega_h|^{\frac56} + \Big(\sum_{K\in\mathcal{T}} |\sigma| h (\varrho_K^n)^{\frac65(2-\gamma)} \Big)^{\frac56} \Big] \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|^6 \Big)^{\frac16} \\
&\sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n)^2 \Big[\frac{1_{\{\overline{\varrho}_\sigma^n\ge 1\}}}{[\max\{\varrho_K,\varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n<1\}} \Big] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} \Big\}^{1/2} \\
&\le \sqrt{h}\, c \|\nabla H'(r)\|_{L^\infty(Q_T;\mathbb{R}^3)} \Big\{ \Delta t \sum_{n=1}^m \Big[|\Omega_h|^{\frac56} + \Big(\sum_{K\in\mathcal{T}} |\sigma| h (\varrho_K^n)^{\frac65(2-\gamma)} \Big)^{\frac56} \Big] \Big(\sum_{\sigma\in\mathcal{E}} |\sigma| h |\boldsymbol{u}_\sigma^n|^6 \Big)^{1/6} \\
&+ \Delta t \sum_{n=1}^m \Big[\sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\mathrm{up}} - \varrho_K^n)^2 \Big[\frac{1_{\{\overline{\varrho}_\sigma^n\ge 1\}}}{[\max\{\varrho_K,\varrho_L\}]^{2-\gamma}} + 1_{\{\overline{\varrho}_\sigma^n<1\}} \Big] |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| \Big] \Big\} \\
&\le \sqrt{h}\; c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C([\underline{r},\overline{r}])}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}),
\end{aligned}
$$

where, in order to get the last line, we have used the estimate (4.10) of the numerical dissipation to evaluate the second term, and finally equivalence of norms $(2.42)_{p=6}$ together with (4.5) and (4.7), under assumption $\gamma \ge 12/11$, to evaluate the first term.

Let us now decompose the term $T_{6,1}$ as

$$
\begin{aligned}
&T_{6,1} = T_{6,2} + R_{6,2}, \text{ with } T_{6,2} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma| \varrho_K^n H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1})[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}], \\
&R_{6,2} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{K}} \sum_{\sigma\in\mathcal{E}(K)} R_{6,2}^{n,\sigma,K}, \text{ and} \\
&R_{6,2}^{n,\sigma,K} = |\sigma| \varrho_K^n \Big(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) - H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1}) \Big) [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}].
\end{aligned}
$$

Therefore, by virtue of the second order Taylor formula applied to function H', the Hölder inequality, (2.42), and (4.5), (4.7) in Corollary 4.1, we have, provided $\gamma \ge 6/5$,

$$
\begin{aligned}
|R_{6,2}| &\le hc \Big(|H''|_{C([\underline{r},\overline{r}])} + |H'''|_{C([\underline{r},\overline{r}])} \Big) \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}\, \Delta t \sum_{n=1}^m \|\varrho^n\|_{L^\gamma(\Omega_h)} \|\boldsymbol{u}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \\
&\le h\; c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}).
\end{aligned} \tag{6.20}
$$

Let us now deal with the term $T_{6,2}$. Noting that $\int_K \nabla r^{n-1}\,\mathrm{d}x = \sum_{\sigma\in\mathcal{E}(K)} |\sigma| (r_\sigma^{n-1} - r_K^{n-1}) \boldsymbol{n}_{\sigma,K}$, we

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may write $T_{6,2} = T_{6,3} + R_{6,3}$, with

$$T_{6,3} = -\Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \varrho_K^n H''(r_K^{n-1}) \boldsymbol{u}^n \cdot \nabla r^{n-1} \, \mathrm{d}x,$$
$$R_{6,3} = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \varrho_K^n H''(r_K^{n-1})(\boldsymbol{u}^n - \boldsymbol{u}_K^n) \cdot \nabla r^{n-1} \, \mathrm{d}x$$
$$+ \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_K^n H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1})(\boldsymbol{u}_\sigma^n - \boldsymbol{u}_K^n) \cdot \boldsymbol{n}_{\sigma,K}.$$

Consequently, by virtue of Hölder's inequality, interpolation inequality (2.56) (to estimate $\|\mathbf{u}^n - \mathbf{u}_K^n\|_{L^{\gamma_0'}(K;\mathbb{R}^3)}$ by $h^{(5\gamma_0-6)/(2\gamma_0)} \|\nabla_x \mathbf{u}^n\|_{L^2(K;\mathbb{R}^9)}$, $\gamma_0 = \min\{\gamma, 2\}$) in the first term, and by the Taylor formula applied to function $x \mapsto r(t_{n-1}, x)$, then Hölder's inequality and (2.56–2.57) (to estimate $\|\mathbf{u}_\sigma^n - \mathbf{u}_K^n\|_{L^{\gamma_0'}(K;\mathbb{R}^3)}$ by $h^{(5\gamma_0-6)/(2\gamma_0)} \|\nabla_x \mathbf{u}^n\|_{L^2(K;\mathbb{R}^9)}$), we get

$$|R_{6,3}| \le h^b \, c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])} \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}), \;\; b = \frac{5\gamma_0 - 6}{2\gamma_0}, \tag{6.21}$$

provided $\gamma \ge 6/5$, where we have used at the end the discrete imbedding and Hölder inequalities (2.58–2.59) and finally estimates (4.4) and (4.7).

Finally we write $T_{6,3} = T_{6,4} + R_{6,4}$, with

$$\begin{aligned} T_{6,4} &= -\Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \varrho_K^n \frac{p'(r_K^n)}{r_K^n} \boldsymbol{u}^n \cdot \nabla r^n \, \mathrm{d}x, \\ R_{6,4} &= \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \varrho_K^n \Big(H''(r_K^n) \nabla r^n - H''(r_K^{n-1}) \nabla r^{n-1} \Big) \cdot \boldsymbol{u}^n \, \mathrm{d}x, \end{aligned} \tag{6.22}$$

where by the same token as in (6.19),

$$|R_{6,4}| \le \Delta t \; c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}, \|\nabla r, \partial_t r\|_{L^\infty(Q_T;\mathbb{R}^4)}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/(5\gamma-6)}(\Omega;\mathbb{R}^3))}), \tag{6.23}$$

provided $\gamma \ge 6/5$.

We are now in position to conclude the proof of Lemma 6.1: we obtain the inequality (6.1) by gathering the principal terms (6.4), (6.12), (6.15), (6.18), (6.22) and the residual terms estimated in (6.5), (6.6), (6.8), (6.11), (6.13), (6.17), (6.19), (6.20), (6.21), (6.23) at the right hand side $\sum_{i=1}^{6} T_i$ of the discrete relative energy inequality (5.1).

□

7 A discrete identity satisfied by the strong solution

This section is devoted to the proof of a discrete identity satisfied by any strong solution of problem (1.1–1.6) in the class (2.9–2.10) extended eventually to $\mathbb{R}^3$ according to Lemma 2.1. This identity is stated in Lemma 7.1 below. It will be used in combination with the approximate relative energy inequality stated in Lemma 6.1 to deduce the convenient form of the relative energy inequality verified by any function being a strong solution to the compressible Navier-Stokes system. This last step is performed in the next section.

Lemma 7.1 (A discrete identity for strong solutions). *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5–3.7) with the pressure satisfying (1.4), where $\gamma \ge 3/2$. There exists*

$$c = c\Big(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{45})},$$

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$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}, \|\partial_t^2 \boldsymbol{V}, \partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))}\Big) > 0,$

such that for all $m = 1, \ldots, N$, we have:

$$\sum_{i=1}^{6} \mathcal{S}_i + \mathcal{R}^m_{h,\Delta t} = 0, \tag{7.1}$$

where

$$\mathcal{S}_1 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \Big(\mu \int_K \nabla_x \boldsymbol{V}^n_{h,0} : \nabla_x(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,\mathrm{d}x + \frac{\mu}{3}\int_K \mathrm{div}\boldsymbol{V}^n_{h,0}\mathrm{div}(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,\mathrm{d}x\Big),$$

$$\mathcal{S}_2 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| r^{n-1}_K \frac{\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t} \cdot \Big(\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\Big),$$

$$\mathcal{S}_3 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r^{n,\mathrm{up}}_\sigma \Big(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma\Big) \cdot \Big(\boldsymbol{V}^n_{h,0,\sigma} - \boldsymbol{V}^n_{h,0,K}\Big) \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} \cdot \boldsymbol{n}_{\sigma,K}$$

$$\mathcal{S}_4 = -\Delta t \sum_{n=1}^{m} \int_{\Omega_h} p(\hat{r}^n)\,\mathrm{div}\,\boldsymbol{V}^n\,\mathrm{d}x,$$

$$\mathcal{S}_5 = 0,$$

$$\mathcal{S}_6 = -\Delta t \sum_{n=1}^{m} \int_{\Omega_h} p'(\hat{r}^n)\boldsymbol{u}^n \cdot \nabla r^n\,\mathrm{d}x,$$

and

$$|\mathcal{R}^m_{h,\Delta t}| \le c\Big(h^{5/6} + \Delta t\Big),$$

for any couple $(r, \mathbf{V})$ belonging to (2.25) and satisfying the continuity equation (1.1) on $(0,T)\times\mathbb{R}^3$ and momentum equation (1.2) with boundary conditions (1.5) on $(0,T)\times\Omega$ in the classical sense. (Recall that in agreement with notation (2.35), (3.1-3.3), $\mathbf{V}^n_{h,0} = \Pi^V_{h,0}[\mathbf{V}(t_n)]$, $\mathbf{V}^n_{h,0,K} = [\mathbf{V}^n_{h,0}]_K$, $\mathbf{V}^n_{h,0,\sigma} = [\mathbf{V}^n_{h,0}]_\sigma$, $\hat{r}^n = \Pi^Q_h[r(t_n)]$, where projections Π^Q, Π^V are defined in (2.31) and (2.34)).

Before starting the proof we recall an auxiliary algebraic inequality whose straightforward proof is left to the reader, and introduce some notations.

Lemma 7.2. *Let p satisfies assumptions(1.4). Let $0 < a < b < \infty$. Then there exists $c = c(a,b) > 0$ such that for all $\varrho \in [0,\infty)$ and $r \in [a,b]$ there holds*

$$E(\varrho|r) \ge c(a,b)\Big(1_{R_+\setminus[a/2,2b]}(\varrho) + \varrho^\gamma 1_{R_+\setminus[a/2,2b]}(\varrho) + (\varrho - r)^2 1_{[a/2,2b]}(\varrho)\Big),$$

where $E(\varrho|r)$ is defined in (4.9).

If we consider Lemma 7.2 with $\varrho = \varrho^n(x)$, $r = \hat{r}^n(x)$, $a = \underline{r}$, $b = \overline{r}$ (where r is a function belonging to class (2.25) and $\underline{r}$, $\overline{r}$ are its lower and upper bounds, respectively), we obtain

$$E(\varrho^n(x)|\hat{r}^n(x)) \ge c(\underline{r},\overline{r})\Big(1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho^n(x)) + (\varrho^n)^\gamma(x) 1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho^n(x)) + (\varrho^n(x) - \hat{r}^n(x))^2 1_{[\underline{r}/2,2\overline{r}]}(\varrho^n(x))\Big). \tag{7.2}$$

Now, for fixed numbers $\underline{r}$ and $\overline{r}$ and fixed functions ϱ^n, $n = 0, \ldots, N$, we introduce the residual and essential subsets of Ω (relative to ϱ^n) as follows:

$$N^n_{\mathrm{ess}} = \{x \in \Omega \Big| \frac{1}{2}\underline{r} \le \varrho^n(x) \le 2\overline{r}\},\ N^n_{\mathrm{res}} = \Omega \setminus N^n_{\mathrm{ess}}, \tag{7.3}$$

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and we set

$$[g]_{\rm ess}(x) = g(x)1_{N^n_{\rm ess}}(x),\ [g]_{\rm res}(x) = g(x)1_{N^n_{\rm res}}(x),\ \ x \in \Omega,\ \ g \in L^1(\Omega).$$

Integrating inequality (7.2) we deduce

$$c(\underline{r}, \overline{r}) \sum_{K\in\mathcal{T}} \int_K \Big(\Big[1\Big]_{\rm res} + \Big[(\varrho^n)^\gamma\Big]_{\rm res} + \Big[\varrho^n - \hat{r}^n\Big]^2_{\rm ess} \Big)\,{\rm d}x \le \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| \hat{r}^n, \boldsymbol{V}^n), \tag{7.4}$$

for any pair $(r, \boldsymbol{V})$ belonging to the class (2.25) and any $\varrho^n \in Q_h(\Omega_h)$, $\varrho^n \ge 0$.

We are now ready to proceed to the proof of Lemma 7.1.

Proof: Since $(r, \boldsymbol{V})$ satisfies (1.1) on $(0,T)\times\Omega$ and belongs to the class (2.25), Equation (1.2) can be rewritten in the form

$$r\partial_t \boldsymbol{V} + r\boldsymbol{V}\cdot\nabla\boldsymbol{V} + \nabla p(r) - \mu\Delta\boldsymbol{V} - \mu/3\nabla\,{\rm div}\,\boldsymbol{V} = 0 \ \text{ in } (0,T)\times\Omega.$$

From this fact, we deduce the identity

$$\sum_{i=1}^5 \mathcal{T}_i = \mathcal{R}_0, \tag{7.5}$$

where

$$\mathcal{R}_0 = -\Delta t \sum_{n=1}^m \int_{\Omega_h\setminus\Omega} \Big(r^n[\partial_t \boldsymbol{V}]^n + r\boldsymbol{V}^n\cdot\nabla\boldsymbol{V}^n + \nabla p(r^n) - \mu\Delta\boldsymbol{V}^n - \frac{\mu}{3}\nabla\,{\rm div}\,\boldsymbol{V}^n\Big)\cdot(\boldsymbol{V}^n_{h,0} - \mathbf{u}^n){\rm d}x,$$

$$\mathcal{T}_1 = -\Delta t \sum_{n=1}^m \int_{\Omega_h} \Big(\mu\Delta\boldsymbol{V}^n + \frac{\mu}{3}\nabla\,{\rm div}\,\boldsymbol{V}^n\Big)\cdot(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,{\rm d}x, \quad \mathcal{T}_2 = \Delta t \sum_{n=1}^m \int_{\Omega_h} r^n[\partial_t\boldsymbol{V}]^n\cdot(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,{\rm d}x,$$

$$\mathcal{T}_3 = \Delta t \sum_{n=1}^m \int_{\Omega_h} r^n\boldsymbol{V}^n\cdot\nabla\boldsymbol{V}^n\cdot(\boldsymbol{V}^n_{h,0} - \boldsymbol{u}^n)\,{\rm d}x, \quad \mathcal{T}_4 = \Delta t \sum_{n=1}^m \int_{\Omega_h} \nabla p(r^n)\cdot\boldsymbol{V}^n_{h,0}\,{\rm d}x,$$

$$\mathcal{T}_5 = 0, \qquad \mathcal{T}_6 = -\Delta t \sum_{n=1}^m \int_{\Omega_h} \nabla p(r^n)\cdot\boldsymbol{u}^n\,{\rm d}x.$$

In the steps below, we deal with each of the terms $\mathcal{R}_0$ and $\mathcal{T}_i$.

Step 0: *Term* $\mathcal{R}_0$. By the Hölder inequality

$$|\mathcal{R}_0| \le |\Omega_h\setminus\Omega|^{5/6}\, c(\overline{r}, |p'|_{C[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \mathbf{V}, \nabla\mathbf{V}, \nabla^2\mathbf{V})\|_{L^\infty(Q_T;\mathbb{R}^{43})}\Delta t \sum_{n=1}^m (\|\boldsymbol{u}^n\|_{L^6(\Omega_h)} + \|\boldsymbol{V}^n_{h,0}\|_{L^6(\Omega_h)})$$

$$\le h^{5/3}\, c(M_0, E_0, \overline{r}, |p'|_{C[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \mathbf{V}, \nabla\mathbf{V}, \nabla^2\mathbf{V})\|_{L^\infty(Q_T;\mathbb{R}^{43})}, \tag{7.6}$$

where we have used (4.5) and (2.48–2.49), (2.39).

Step 1: *Term* $\mathcal{T}_1$. Integrating by parts, we get:

$$\mathcal{T}_1 = \mathcal{T}_{1,1} + \mathcal{R}_{1,1},$$

$$\text{with } \mathcal{T}_{1,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \Big(\mu \nabla \boldsymbol{V}_{h,0}^n : \nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) + \frac{\mu}{3} \operatorname{div} \boldsymbol{V}_{h,0}^n \operatorname{div}(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \Big) \,\mathrm{d}x,$$

and $\mathcal{R}_{1,1} = I_1 + I_2$, with

$$I_1 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K \Big(\mu \nabla (\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n) : \nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) + \frac{\mu}{3} \operatorname{div}(\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n) \operatorname{div}(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \Big) \,\mathrm{d}x, \tag{7.7}$$

$$I_2 = -\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \Big(\mu \boldsymbol{n}_{\sigma,K} \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) + \frac{\mu}{3} \operatorname{div} \boldsymbol{V}^n (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \cdot \boldsymbol{n}_{\sigma,K} \Big) \,\mathrm{d}S$$

$$= -\Delta t \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}} \int_\sigma \Big(\mu \boldsymbol{n}_\sigma \cdot \nabla \boldsymbol{V}^n \cdot \Big[\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\Big]_{\sigma,\mathsf{n}_\sigma} + \frac{\mu}{3} \operatorname{div} \boldsymbol{V}^n \Big[\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\Big]_{\sigma,\mathsf{n}_\sigma} \cdot \boldsymbol{n}_\sigma \Big) \,\mathrm{d}S,$$

where in the last line $\mathbf{n}_\sigma$ is the unit normal to the face σ and $[\cdot]_{\sigma,\mathsf{n}_\sigma}$ is the jump over sigma (with respect to $\mathbf{n}_\sigma$) defined in Lemma 2.7.

To estimate I_1, we use the Cauchy-Schwartz inequality, decompose $\mathbf{V}^n - \mathbf{V}_{h,0}^n = \mathbf{V}^n - \mathbf{V}_h^n + \mathbf{V}_h^n - \mathbf{V}_{h,0}^n$ and employ estimates $(2.41)_{s=2}$, (2.48–2.49) to evaluate the norms involving $\nabla(\mathbf{V}^n - \mathbf{V}_{h,0}^n)$, and decompose $\mathbf{V}_{h,0}^n = \mathbf{V}_{h,0}^n - \mathbf{V}_h^n + \mathbf{V}_h^n$ use (2.48–2.49), $(2.40)_{s=1}$, (4.4), the Minkowski inequality to estimate the norms involving $\nabla(\mathbf{V}_{h,0}^n - \mathbf{u}^n)$. We get

$$|I_1| \le h\, c(M_0, E_0, \|\nabla \mathbf{V}, \nabla^2 \boldsymbol{V}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{36}))}).$$

Since the integral over any face $\sigma \in \mathcal{E}_{\rm int}$ of the jump of a function from $V_{h,0}(\Omega_h)$ is zero, we may write

$$I_2 = \Delta t \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \Big(\mu \boldsymbol{n}_\sigma \cdot \Big(\nabla \boldsymbol{V}^n - (\nabla \boldsymbol{V}^n)_\sigma \Big) \cdot \Big[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\Big]_{\sigma,\mathsf{n}_\sigma}$$

$$+ \frac{\mu}{3} \Big(\operatorname{div} \boldsymbol{V}^n - (\operatorname{div} \boldsymbol{V}^n)_\sigma \Big) \Big[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\Big]_{\sigma,\mathsf{n}_\sigma} \cdot \boldsymbol{n}_\sigma \Big) \,\mathrm{d}S;$$

whence by using the first order Taylor formula applied to functions $x \mapsto \nabla \mathbf{V}^n(x)$ to evaluate the differences $\nabla \boldsymbol{V}^n - (\nabla \boldsymbol{V}^n)_\sigma$, $\operatorname{div} \boldsymbol{V}^n - [\operatorname{div} \boldsymbol{V}^n]_\sigma$, and Hölder's inequality,

$$|I_2| \le \Delta t\, h\, c \|\nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{27})} \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{\rm int}} \sqrt{|\sigma|}\sqrt{h} \left(\frac{1}{\sqrt{h}} \Big\| \Big[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\Big]_{\sigma,\mathsf{n}_\sigma} \Big\|_{L^2(\sigma;\mathbb{R}^3)} \right)$$

$$\le \Delta t\, h\, c \|\nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{27})} \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}_{\rm int}} \Big(|\sigma| h + \frac{1}{h} \Big\| \Big[\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\Big]_{\sigma,\mathsf{n}_\sigma} \Big\|^2_{L^2(\sigma;\mathbb{R}^3)} \Big).$$

Therefore,

$$|\mathcal{R}_{1,1}| \le h\, c(M_0, E_0, \|\mathbf{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T,\mathbb{R}^{39})}), \tag{7.8}$$

where we have employed Lemma 2.7, (4.4) and (2.48–2.49), (2.40).

Step 2: *Term* $\mathcal{T}_2$. Let us now decompose the term $\mathcal{T}_2$ as

$$\mathcal{T}_2 = \mathcal{T}_{2,1} + \mathcal{R}_{2,1},$$

$$\text{with } \mathcal{T}_{2,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r^{n-1} \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \,\mathrm{d}x, \quad \mathcal{R}_{2,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mathcal{R}_{2,1}^{n,K},$$

$$\text{and } \mathcal{R}_{2,1}^{n,K} = \int_K (r^n - r^{n-1}) [\partial_t \mathbf{V}]^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \,\mathrm{d}x + \int_K r^{n-1} \Big([\partial_t \boldsymbol{V}]^n - \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \Big) \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n) \,\mathrm{d}x.$$

The remainder $\mathcal{R}_{2,1}^{n,K}$ can be rewritten as follows

$$\mathcal{R}_{2,1}^{n,K} = \int_K \Big[\int_{t_{n-1}}^{t_n} \partial_t r(t,\cdot)\mathrm{d}t \Big] [\partial_t \mathbf{V}]^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \frac{1}{\Delta t}\int_K r^{n-1}\Big[\int_{t_{n-1}}^{t_n}\int_s^{t_n} \partial_t^2 \boldsymbol{V}(z,\cdot)\mathrm{d}z\mathrm{d}s\Big]\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x;$$

whence, by the Hölder inequality,

$$|\mathcal{R}_{2,1}^{n,K}| \le \Delta t\Big[(\|r\|_{L^\infty(Q_T)} + \|\partial_t r\|_{L^\infty(Q_T)})(\|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}|K|^{5/6}(\|\boldsymbol{u}^n\|_{L^6(K)} + \|\boldsymbol{V}_{h,0}^n\|_{L^6(K)})$$
$$+\|\partial_t^2 \boldsymbol{V}^n\|_{L^{6/5}(\Omega;\mathbb{R}^3))}(\|\boldsymbol{u}^n\|_{L^6(K)} + \|\boldsymbol{V}_{h,0}^n\|_{L^6(K)})\Big].$$

Consequently, by the same token as in (6.19) or (6.23),

$$|\mathcal{R}_{2,1}| \le \Delta t\, c\Big(M_0, E_0, \overline{r}, \|(\partial_t r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{16})}, \|\partial_t^2 \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^3))}\Big), \tag{7.9}$$

where we have used the discrete Hölder and Young inequalities, the estimates (2.39), (2.48–2.49) and the energy bound (4.4) from Corollary 4.1.

Step 2a: *Term* $\mathcal{T}_{2,1}$. We decompose the term $\mathcal{T}_{2,1}$ as

$$\mathcal{T}_{2,1} = \mathcal{T}_{2,2} + \mathcal{R}_{2,2},$$
$$\text{with } \mathcal{T}_{2,2} = \Delta t\sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K r_K^{n-1}\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x,\ \mathcal{R}_{2,2} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\mathcal{R}_{2,2}^{n,K},$$
$$\text{and } \mathcal{R}_{2,2}^{n,K} = \int_K (r^{n-1} - r_K^{n-1})\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x;$$

therefore,

$$|\mathcal{R}_{2,2}^n| = |\sum_{K\in\mathcal{T}}\mathcal{R}_{2,2}^{n,K}| \le h\, c\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}\|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}\|\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\|_{L^6(\Omega;\mathbb{R}^3)}.$$

Consequently, by virtue of formula (4.5) for $\mathbf{u}^n$ and estimates (2.39), (2.48–2.49),

$$|\mathcal{R}_{2,2}| \le h\, c(M_0, E_0, \|(\nabla r, \boldsymbol{V}, \partial_t\boldsymbol{V}, \nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{18})}). \tag{7.10}$$

Step 2b: *Term* $\mathcal{T}_{2,2}$. We decompose the term $\mathcal{T}_{2,2}$ as

$$\mathcal{T}_{2,2} = \mathcal{T}_{2,3} + \mathcal{R}_{2,3},$$
$$\text{with } \mathcal{T}_{2,3} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K r_K^{n-1}\frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t}\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x,\ \mathcal{R}_{2,3} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\mathcal{R}_{2,3}^{n,K},$$
$$\text{and } \mathcal{R}_{2,3}^{n,K} = \int_K r_K^{n-1}\Big(\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} - \Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_h\Big)\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x$$
$$+\int_K r_K^{n-1}\Big(\Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_h - \Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_{h,K}\Big)\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x$$
$$+\int_K r_K^{n-1}\Big(\Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_{h,K} - \Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_{h,0,K}\Big)\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}_n)\,\mathrm{d}x = I_1^K + I_2^K + I_3^K.$$

We calculate carefully

$$|I_3^K| = \frac{1}{\Delta t}r_K^{n-1}\int_K\Big\{\int_{t_{n-1}}^{t_n}\Big[[\partial_t\boldsymbol{V}(z)]_h - [\partial_t\boldsymbol{V}(z)]_{h,0}\Big]_K\cdot(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\mathrm{d}z\Big\}\,\mathrm{d}x$$

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$$\leq \frac{1}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \Big\| \Big[[\partial_t \boldsymbol{V}(z)]_h - [\partial_t \boldsymbol{V}(z)]_{h,0} \Big]_K \Big\|_{L^{6/5}(K;\mathbb{R}^3)} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(K;\mathbb{R}^3)} \mathrm{d}z.$$

Summing over polyhedra $K \in \mathcal{T}$ we get simply by using the discrete Sobolev inequality

$$\sum_{K\in\mathcal{T}} |I_3^K| \leq \frac{1}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \Big\{ \Big(\sum_{K\in\mathcal{T}} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(K;\mathbb{R}^3)}^6 \Big)^{1/6} \Big(\sum_{K\in\mathcal{T}} \|[\partial_t \boldsymbol{V}(z)]_h - [\partial_t \boldsymbol{V}(z)]_{h,0}\Big\|_{L^{6/5}(K;\mathbb{R}^3)}^{6/5} \Big)^{5/6} \Big\} \mathrm{d}z$$

$$\leq \frac{1}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \|[\partial_t \boldsymbol{V}(z)]_h - [\partial_t \boldsymbol{V}(z)]_{h,0}\Big\|_{L^{6/5}(\Omega_h;\mathbb{R}^3)} \mathrm{d}z$$

$$\leq \frac{h^{5/6}}{\Delta t} \int_{t_{n-1}}^{t_n} \|\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \|\partial_t \boldsymbol{V}(z)\|_{L^\infty(\Omega_h;\mathbb{R}^3)} \mathrm{d}z,$$

where we have used estimate (2.51) to obtain the last line.

As far as the term I_2^K is concerned, we write

$$|I_2^K| = \frac{1}{\Delta t} r_K^{n-1} \Big| \int_K \Big(\Big[\int_{t_{n-1}}^{t_n} \partial_t \boldsymbol{V}(z) \mathrm{d}z \Big]_h - \Big[\int_{t_{n-1}}^{t_n} \partial_t \boldsymbol{V}(z) \mathrm{d}z \Big]_{h,K} \Big) \cdot (\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n) \, \mathrm{d}x \Big|$$

$$\leq \frac{h}{\Delta t} r_K^{n-1} \int_{t_{n-1}}^{t_n} \Big\| \nabla_x \Big[\partial_t \boldsymbol{V}(z) \Big]_h \Big\|_{L^{6/5}(K;\mathbb{R}^3)} \|\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\|_{L^6(K;\mathbb{R}^3)},$$

where we have used the Fubini theorem, Hölder's inequality and (2.52), $(2.41)_{s=1}$. Further, employing the Sobolev inequality on the Crouzeix-Raviart space $V_{h,0}(\Omega_h)$ (2.43), the Hölder inequality and estimate $(2.41)_{s=1}$, we get

$$\sum_{K\in\mathcal{T}} |I_2^K| \leq \frac{h}{\Delta t} r_K^{n-1} \|\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\|_{L^6(\Omega_h;\mathbb{R}^3)} \int_{t_{n-1}}^{t_n} \Big\| \nabla_x \partial_t \boldsymbol{V}(z) \Big\|_{L^{6/5}(\Omega_h;\mathbb{R}^3)} \mathrm{d}z.$$

We reserve the similar treatment to the term I_1^K. Resuming these calculations and summing over n from 1 to m we get by using Corollary 4.1 and estimates (2.48–2.49), (2.39),

$$|\mathcal{R}_{2,3}| \leq h^{5/6} c(M_0, E_0, \|(r, \boldsymbol{V}, \nabla \boldsymbol{V}, \partial_t \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{16})}, \|\partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^9))}). \tag{7.11}$$

Step 2c: *Term* $\mathcal{T}_{2,3}$. We rewrite this term in the form

$$\begin{aligned} &\mathcal{T}_{2,3} = \mathcal{T}_{2,4} + \mathcal{R}_{2,4}, \ \mathcal{R}_{2,4} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \mathcal{R}_{2,4}^{n,K}, \\ &\text{with } \mathcal{T}_{2,4} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K r_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot (\boldsymbol{u}_K^n - \boldsymbol{V}_{h,0,K}^n) \, \mathrm{d}x, \\ &\text{and } \mathcal{R}_{2,4}^{n,K} = \int_K r_K^{n-1} \frac{\boldsymbol{V}_{h,0,K}^n - \boldsymbol{V}_{h,0,K}^{n-1}}{\Delta t} \cdot \Big((\boldsymbol{u}^n - \boldsymbol{u}_K^n) - (\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n) \Big) \, \mathrm{d}x. \end{aligned} \tag{7.12}$$

First, we estimate the L^∞ norm of $\frac{\mathsf{V}_{h,0,K}^n - \mathsf{V}_{h,0,K}^{n-1}}{\Delta t}$ as in (6.5). Next, we decompose

$$\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n = \boldsymbol{V}_{h,0}^n - \boldsymbol{V}_h^n + \boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n + [\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_K,$$

and use $(2.52)_{p=2}$ to estimate $\boldsymbol{u}^n - \boldsymbol{u}_K^n$, $(2.52)_{p=\infty}$, $(2.41)_{s=1}$ to estimate $\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n$ and (2.48–2.49) to evaluate $\|[\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_K\|_{L^\infty(K;\mathbb{R}^3)} \leq \|\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n\|_{L^\infty(K;\mathbb{R}^3)}$. Thanks to the Hölder inequality and (4.4) we finally deduce

$$|\mathcal{R}_{2,4}| \leq h \, c(M_0, E_0, \overline{r}, \|(V, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{15})}). \tag{7.13}$$

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Step 3: *Term* $\mathcal{T}_3$. Let us first decompose $\mathcal{T}_3$ as

$$\mathcal{T}_3 = \mathcal{T}_{3,1} + \mathcal{R}_{3,1},$$

$$\text{with } \mathcal{T}_{3,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K r_K^n \boldsymbol{V}_{h,0,K}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x, \quad \mathcal{R}_{3,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mathcal{R}_{3,1}^{n,K},$$

$$\begin{aligned}
\text{and } \mathcal{R}_{3,1}^{n,K} = & \int_K (r^n - r_K^n)\boldsymbol{V}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \int_K r_K^n(\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n) \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x \\
& + \int_K r_K^n(\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n) \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x \\
& + \int_K r_K^n \boldsymbol{V}_{h,0,K}^n \cdot \nabla \boldsymbol{V}^n \cdot \Big(\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n - (\boldsymbol{u}^n - \boldsymbol{u}_K^n)\Big)\,\mathrm{d}x.
\end{aligned}$$

We have

$$\|r^n - r_K^n\|_{L^\infty(K)} \lesssim h\|\nabla r^n\|_{L^\infty(K)},$$

by the Taylor formula,

$$\|\boldsymbol{V}^n - \boldsymbol{V}_{h,0}^n\|_{L^\infty(K;\mathbb{R}^3)} \lesssim h\|\nabla \boldsymbol{V}^n\|_{L^\infty(K;\mathbb{R}^9)},$$

by virtue of $(2.40)_{s=1}$ and (2.48–2.49),

$$\begin{aligned}
\|\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,0,K}^n\|_{L^\infty(K;\mathbb{R}^3)} \le & \|\boldsymbol{V}_{h,0}^n - \boldsymbol{V}_{h,}^n\|_{L^\infty(K;\mathbb{R}^3)} + \|\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n\|_{L^\infty(K;\mathbb{R}^3)} \\
& + \|[\boldsymbol{V}_h^n - \boldsymbol{V}_{h,0}^n]_K\|_{L^\infty(K;\mathbb{R}^3)} \lesssim h\|\nabla \boldsymbol{V}^n\|_{L^\infty(K;\mathbb{R}^9)}
\end{aligned}$$

by virtue of (2.52), $(2.40)_{s=1}$ $(2.41)_{s=1}$ and (2.48–2.49),

$$\|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|_{L^\infty(K;\mathbb{R}^3)} \lesssim h\|\nabla \boldsymbol{u}^n\|_{L^\infty(K;\mathbb{R}^9)}.$$

Consequently by employing several times the Hölder inequality (for integrals over K) and the discrete Hölder inequality (for the sums over $K \in \mathcal{T}$), and using estimate (4.4), we arrive at

$$|\mathcal{R}_{3,1}| \le h\, c(M_0, E_0, \overline{r}, \|(\nabla r, \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{15})}). \tag{7.14}$$

Now we shall deal wit term $\mathcal{T}_{3,1}$. Integrating by parts, we get:

$$\begin{aligned}
\int_K r_K^n \boldsymbol{V}_{h,0,K}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x &= \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r_K^n [\boldsymbol{V}_{h,0,K}^n \cdot \mathbf{n}_{\sigma,K}] \boldsymbol{V}_\sigma^n \cdot (\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n) \\
&= \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r_K^n [\boldsymbol{V}_{h,0,K}^n \cdot \mathbf{n}_{\sigma,K}] (\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n) \cdot (\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n),
\end{aligned}$$

thanks to the the fact that $\sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \boldsymbol{V}_{h,K}^n \cdot \mathbf{n}_{\sigma,K} \mathrm{d}S = 0$.

Next we write

$$\mathcal{T}_{3,1} = \mathcal{T}_{3,2} + \mathcal{R}_{3,2}, \quad \mathcal{R}_{3,2} = \Delta t \sum_{n=1}^{m} \mathcal{R}_{3,2}^n,$$

$$\mathcal{T}_{3,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\mathrm{up}} [\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}] (\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n) \cdot (\hat{\boldsymbol{V}}_{h,0,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}), \tag{7.15}$$

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$$\text{and } \mathcal{R}^n_{3,2} = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|(r^n_K - \hat{r}^{n,\mathrm{up}}_\sigma)[\boldsymbol{V}^n_{h,0,K}\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{V}^n_\sigma - \boldsymbol{V}^n_{h,K})\cdot(\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K)$$
$$+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}^{n,\mathrm{up}}_\sigma\Big[\Big(\boldsymbol{V}^n_{h,0,K} - \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}\Big)\cdot\mathbf{n}_{\sigma,K}\Big](\boldsymbol{V}^n_\sigma - \boldsymbol{V}^n_{h,K})\cdot(\boldsymbol{V}^n_{h,K} - \boldsymbol{u}^n_K)$$
$$+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}^{n,\mathrm{up}}_\sigma[\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{V}^n_\sigma - \boldsymbol{V}^n_{h,K})\cdot\Big((\boldsymbol{V}^n_{h,0,K} - \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}) - (\boldsymbol{u}^n_K - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_{h,\sigma})\Big).$$

We may write

$$\mathbf{V}^n_\sigma - \mathbf{V}^n_{h,0,K} = \mathbf{V}^n_\sigma - \mathbf{V}^n + \mathbf{V}^n - \mathbf{V}^n_h + \mathbf{V}^n_h - \mathbf{V}^n_{h,K} + [\mathbf{V}^n_h - \mathbf{V}^n_{h,0}]_K,$$

and use several times the Taylor formula along with $(2.40)_{s=1}$, (2.52), $(2.41)_{s=1}$, $(2.48$–$2.49)$(in order to estimate $r^n_K - \hat{r}^{n,\mathrm{up}}_\sigma$, $\mathbf{V}^n_\sigma - \mathbf{V}^n_{h,0,K}$, $\mathbf{V}^n_{h,K} - \hat{\mathbf{V}}^{n,\mathrm{up}}_{h,\sigma}$) to get the bound

$$|\mathcal{R}^n_{3,2}| \le h\,c\|r\|_{W^{1,\infty}(\Omega)}\Big(1 + \|\boldsymbol{V}\|_{W^{1,\infty}(Q_T;\mathbb{R}^3)}\Big)^3 \sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}^n_K|$$
$$+c\|r\|_{W^{1,\infty}(\Omega)}\Big(1 + \|\boldsymbol{V}\|_{W^{1,\infty}(Q_T;\mathbb{R}^3)}\Big)^2 \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}^n_K - \boldsymbol{u}^n_\sigma|.$$

We have by the Hölder inequality

$$\sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}^n_K| \le c\Big(\sum_{\sigma\in\mathcal{T}} h|\sigma||\boldsymbol{u}^n_K|^6\Big)^{1/6} \le c\Big[\Big(\sum_{K\in\mathcal{T}}\|\boldsymbol{u}^n - \boldsymbol{u}^n_K\|^6_{L^6(K;\mathbb{R}^3)}\Big)^{1/6}$$
$$+\Big(\sum_{K\in\mathcal{T}}\|\boldsymbol{u}^n\|^6_{L^6(K;\mathbb{R}^3)}\Big)^{1/6}\Big] \le c\Big(\sum_{K\in\mathcal{T}}\|\nabla\boldsymbol{u}_n\|^2_{L^2(K;\mathbb{R}^9)}\Big)^{1/2},$$
$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}^n_K - \boldsymbol{u}^n_\sigma| \le c\Big[\Big(\sum_{K\in\mathcal{T}}\|\boldsymbol{u}^n - \boldsymbol{u}^n_K\|^2_{L^2(K;\mathbb{R}^3)}\Big)^{1/2}$$
$$+\Big(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}\|\boldsymbol{u}^n - \boldsymbol{u}^n_\sigma\|^2_{L^2(K;\mathbb{R}^3)}\Big)^{1/2}\Big] \le h\,c\Big(\sum_{K\in\mathcal{T}}\|\nabla\boldsymbol{u}_n\|^2_{L^2(K;\mathbb{R}^9)}\Big)^{1/2},$$

where we have used $(2.54)_{p=2}$, $(2.52$–$2.53)_{p=2}$. Consequently, we may use (4.4) to conclude

$$|\mathcal{R}_{3,2}| \le h\,c\Big(M_0, E_0, \overline{r}, \|\nabla r, \boldsymbol{V}, \nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{15})}\Big). \tag{7.16}$$

Finally, we replace in $\mathcal{T}_{3,2}$ $\boldsymbol{V}^n_\sigma - \boldsymbol{V}^n_{h,K}$ by $\boldsymbol{V}^n_{h,0,\sigma} - \boldsymbol{V}^n_{h,0,K}$. We get

$$\mathcal{T}_{3,2} = \mathcal{T}_{3,3} + \mathcal{R}_{3,3}, \quad \mathcal{R}_{3,3} = \Delta t\sum_{n=1}^m \mathcal{R}^n_{3,3},$$

$$\mathcal{T}_{3,3} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}^{n,\mathrm{up}}_\sigma[\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}\cdot\mathbf{n}_{\sigma,K}](\boldsymbol{V}^n_{h,0,\sigma} - \boldsymbol{V}^n_{h,0,K})\cdot(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma), \tag{7.17}$$

and

$$\mathcal{R}^n_{3,3} = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\hat{r}^{n,\mathrm{up}}_\sigma\boldsymbol{V}^n_{h,0,K}\cdot\mathbf{n}_{\sigma,K}\Big([\boldsymbol{V}^n - \boldsymbol{V}_{h,0}]^n_\sigma - [\boldsymbol{V}^n_h - \boldsymbol{V}^n_{h,0}]_K\Big)\cdot(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma),$$

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committing error

$$|\mathcal{R}^n_{3,3}| \le h\, c\Big(M_0, E_0, \overline{r}, \|\nabla r, \boldsymbol{V}, \nabla \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{15})}\Big), \tag{7.18}$$

as in the previous step.

Step 4: *Terms $\mathcal{T}_4$* We write

$$\mathcal{T}_4 = \mathcal{T}_{4,1} + \mathcal{R}_{4,1}, \quad \mathcal{T}_{4,1} = -\int_{\Omega_h} \nabla p(r^n) \cdot \mathbf{V}^n \mathrm{d}x,$$

$$\mathcal{R}_{4,1} = \int_{\Omega_h} \nabla p(r^n) \cdot (\mathbf{V}^n - \boldsymbol{V}^n_{h,0})\,\mathrm{d}x;$$

whence

$$|\mathcal{R}_{4,1}| \le hc(\overline{r}, |p'|_{C[\underline{r},\overline{r}]}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}), \tag{7.19}$$

by virtue of $(2.40)_{s=1}$, (2.48–2.49).

Next, employing the integration by parts

$$\mathcal{T}_{4,2} = \mathcal{T}_{4,2} + \mathcal{R}_{4,2}, \quad \mathcal{T}_{4,2} = \int_{\Omega_h} p(r^n)\,\mathrm{div}\,\mathbf{V}^n\,\mathrm{d}x,$$

$$\mathcal{R}_{4,2} = -\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K),\sigma\in\partial\Omega_h} \int_\sigma p(r^n)\mathbf{V}^n \cdot \mathbf{n}_{\sigma,K}\mathrm{d}S = -\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K),\sigma\in\partial\Omega_h} \int_\sigma p(r^n)\Big(\mathbf{V}^n - \boldsymbol{V}^n_{h,0,\sigma}\Big)\cdot \mathbf{n}_{\sigma,K}\mathrm{d}S.$$

Writing

$$\mathbf{V}^n - \boldsymbol{V}^n_{h,0,\sigma} = \mathbf{V}^n - \boldsymbol{V}^n_h + \mathbf{V}^n_h - \boldsymbol{V}^n_{h,\sigma} + [\boldsymbol{V}^n_h - \boldsymbol{V}^n_{h,0}]_\sigma,$$

we deduce by using $(2.40)_{s=1}$, $(2.41)_{s=1}$, $(2.53)_{p=\infty}$, (2.48), (2.49),

$$\|\mathbf{V}^n - \boldsymbol{V}^n_{h,0,\sigma}\|_{L^\infty(K;\mathbb{R}^3)} \stackrel{<}{\sim} h\|\nabla \boldsymbol{V}^n\|_{L^\infty(K;\mathbb{R}^3)},\ \sigma\in K.$$

Now, we employ the fact that

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K),\sigma\in\partial\Omega_h} \int_\sigma \mathrm{d}S \approx 1;$$

whence

$$|\mathcal{R}_{4,2}| \le hc(\overline{r}, |p|_{C[\underline{r},\overline{r}]}, \|\nabla \mathbf{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}) \tag{7.20}$$

Finally,

$$\mathcal{T}_{4,2} = \mathcal{T}_{4,3} + \mathcal{R}_{4,3}, \quad \mathcal{T}_{4,3} = \int_{\Omega_h} p(\hat{r}^n)\,\mathrm{div}\,\mathbf{V}^n\,\mathrm{d}x, \quad \mathcal{R}_{4,3} = \int_{\Omega_h} (p(r^n) - p(\hat{r}^n))\,\mathrm{div}\,\mathbf{V}^n\,\mathrm{d}x; \tag{7.21}$$

whence

$$|\mathcal{R}_{4,3}| \le hc(|p'|_{C[\underline{r},\overline{r}]}, \|(\nabla r, \nabla \mathbf{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \tag{7.22}$$

Step 5: *Term $\mathcal{T}_6$* We decompose $\mathcal{T}_6$ as

$$\begin{aligned} \mathcal{T}_6 &= \mathcal{T}_{6,1} + \mathcal{R}_{6,1}, \text{with } \mathcal{T}_{6,1} = -\Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K p'(\hat{r}^n)\boldsymbol{u}^n \cdot \nabla r^n\,\mathrm{d}x, \\ \mathcal{R}_{6,1} &= \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K (p'(\hat{r}^n) - p'(r^n)) \cdot \boldsymbol{u}^n \cdot \nabla r^n\,\mathrm{d}x; \end{aligned} \tag{7.23}$$

Consequently, by the Taylor formula, Hölder inequality and estimate (4.5),

$$|\mathcal{R}_{6,1}| \le h\, c(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1([\underline{r},\overline{r}])}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}). \tag{7.24}$$

Gathering the formulae (7.7), (7.12), (7.17), (7.21), (7.23) and estimates for the residual terms (7.8), (7.9–7.13), (7.14–7.18), (7.19), (7.20), (7.22), (7.24) concludes the proof of Lemma 7.1. □

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8 A Gronwall inequality

In this Section we put together the relative energy inequality (6.1) and the identity (7.1) derived in the previous section. The final inequality resulting from this manipulation is formulated in the following lemma.

Lemma 8.1. *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5–3.7) with the pressure satisfying (1.4), where $\gamma \geq 3/2$. Then there exists a positive number*

$$c = c\Big(M_0, E_0, \underline{r}, \overline{r}, |p'|_{C^1[\underline{r},\overline{r}]}, \|(\partial_t r, \nabla r, \boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{45})},$$

$$\|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}, \|\partial_t \nabla r\|_{L^2(0,T;L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3))}, \|\partial_t^2 \boldsymbol{V}, \partial_t \nabla \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^{12}))}\Big),$$

such that for all $m = 1, \ldots, N$, there holds:

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m | \hat{r}^m, \hat{\boldsymbol{V}}^m_{h,0}) + \Delta t \frac{\mu}{2} \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K |\nabla_x (u^n - \mathbf{V}^n_{h,0})|^2 \mathrm{dx}$$

$$\leq c\Big[h^a + \sqrt{\Delta t} + \mathcal{E}(\varrho^0, \boldsymbol{u}^0 | \hat{r}(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big] + c\,\Delta t \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n | \hat{r}^n, \hat{\boldsymbol{V}}^n_{h,0}),$$

with any couple $(r, \mathbf{V})$ belonging to (2.25) and satisfying the continuity equation (1.1) on $(0,T)\times\mathbb{R}^3$ and momentum equation (1.2) with boundary conditions (1.5) on $(0,T)\times\Omega$ in the classical sense, where a is defined in (3.9) and $\mathcal{E}$ is given in (4.9).

Proof. We observe that

$$S_6 - \mathcal{S}_6 = \Delta t \sum_{n=1}^{m} \int_{\Omega_h} p'(\hat{r}^n) \frac{\hat{r}^n - \varrho^n}{\hat{r}^n} \mathbf{V}^n \cdot \nabla r^n \, \mathrm{d}x + \Delta t \sum_{n=1}^{m} \int_{\Omega_h} p'(\hat{r}^n) \frac{\hat{r}^n - \varrho^n}{\hat{r}^n} (\mathbf{u}^n - \mathbf{V}^n) \cdot \nabla r^n \, \mathrm{d}x.$$

Gathering the formulae (6.1) and (6.2), one gets

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m \Big| \hat{r}^m, \hat{\boldsymbol{V}}^m_{h,0}) - \mathcal{E}(\varrho^0, \boldsymbol{u}^0 \Big| \hat{r}(0), \hat{\boldsymbol{V}}_{h,0}(0)) + \mu\Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \Big|\nabla(\boldsymbol{u}^n - \boldsymbol{V}^n_{0,h})\Big|^2_{L^2(K;\mathbb{R}^3)} \leq \sum_{i=1}^{4} \mathcal{P}_i + \mathcal{Q}, \quad (8.1)$$

where

$$\mathcal{P}_1 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| (\varrho_K^{n-1} - r_K^{n-1}) \frac{\boldsymbol{V}^n_{h,0,K} - \boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t} \cdot \Big(\boldsymbol{V}^n_{h,0,K} - \boldsymbol{u}^n_K\Big),$$

$$\mathcal{P}_2 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma=K|L\in\mathcal{E}_K} |\sigma| \Big(\varrho_\sigma^{n,\mathrm{up}} - \hat{r}_\sigma^{n,\mathrm{up}}\Big) \Big(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma\Big) \cdot \Big(\boldsymbol{V}^n_{h,0,\sigma} - \boldsymbol{V}^n_{h,0,K}\Big) \boldsymbol{V}^{n,\mathrm{up}}_{h,0,\sigma} \cdot \boldsymbol{n}_{\sigma,K},$$

$$\mathcal{P}_3 = -\Delta t \sum_{n=1}^{m} \int_{\Omega_h} \Big(p(\varrho^n) - p'(\hat{r}^n)(\varrho^n - \hat{r}^n) - p(\hat{r}^n)\Big) \mathrm{div} \mathbf{V}^n,$$

$$\mathcal{P}_4 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \int_K p'(\hat{r}^n) \frac{\hat{r}^n - \varrho^n}{\hat{r}^n} (\mathbf{u}^n - \mathbf{V}^n) \cdot \nabla r^n \, \mathrm{d}x,$$

$$\mathcal{Q} = \mathcal{R}^m_{h,\Delta t} + R^m_{h,\Delta t} + G^m.$$

Now, we estimate conveniently the terms $\mathcal{P}_i$, $i = 1, \ldots, 4$ in four steps.

Step 1: *Term* $\mathcal{P}_1$. We estimate the L^∞ norm of $\frac{\mathrm{V}^{n}_{h,0,K}-\mathrm{V}^{n-1}_{h,0,K}}{\Delta t}$ by L^∞ norm of $\partial_t \mathbf{V}$ in the same manner as in (6.5). According to Lemma 7.2, $|\varrho - r|^\gamma 1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho) \le c(p)E^p(\varrho|r)$, with any $p\ge 1$; in particular,

$$|\varrho - r|^{6/5} 1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho) \le cE(\varrho|r) \tag{8.2}$$

provided $\gamma \ge 6/5$.

We get by using the Hölder inequality,

$$\Big|\sum_{K\in\mathcal{T}}|K|(\varrho_K^{n-1}-r_K^{n-1})\frac{\boldsymbol{V}^n_{h,0,K}-\boldsymbol{V}^{n-1}_{h,0,K}}{\Delta t}\cdot\Big(\boldsymbol{V}^n_{h,K}-\boldsymbol{u}^n_K\Big)\Big| \le c\|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}\times$$

$$\Big[\Big(\sum_{K\in\mathcal{T}}|K||\varrho_K^{n-1}-r_K^{n-1}|^2 1_{[\underline{r}/2,2\overline{r}]}(\varrho_K)\Big)^{1/2}+\Big(\sum_{K\in\mathcal{T}}|K||\varrho_K^{n-1}-r_K^{n-1}|^{6/5}1_{R_+\setminus[\underline{r}/2,2\overline{r}]}(\varrho_K)\Big)^{5/6}\Big]\times$$

$$\Big(\sum_{K\in\mathcal{T}}|K|\Big|\boldsymbol{V}^n_{h,0,K}-\boldsymbol{u}^n_K\Big|^6\Big)^{1/6} \le c(\|(\partial_t\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^3)})\Big(\mathcal{E}^{1/2}(\varrho^{n-1},\hat{\boldsymbol{u}}^{n-1}|\hat{r}^{n-1},\hat{\boldsymbol{V}}^{n-1}_{h,0})$$

$$+\mathcal{E}^{5/6}(\varrho^{n-1},\hat{\boldsymbol{u}}^{n-1}|\hat{r}^{n-1},\hat{\boldsymbol{V}}^{n-1}_{h,0})\Big)\Big(\sum_{K\in\mathcal{T}}\|\boldsymbol{V}^n_{h,0,K}-\boldsymbol{u}^n_K\|^6_{L^6(K;\mathbb{R}^3)}\Big)^{1/6},$$

where we have used (8.2) and estimate (4.8) to obtain the last line. Now, we write $\boldsymbol{V}^n_{h,0,K}-\boldsymbol{u}^n_K = ([\boldsymbol{V}^n_{h,0}-\boldsymbol{u}^n]_K-(\boldsymbol{V}^n_{h,0}-\boldsymbol{u}^n))+(\boldsymbol{V}^n_{h,0}-\boldsymbol{u}^n)$ and use the Minkowski inequality together with formulas (2.54), (2.43) to get

$$\Big(\sum_{K\in\mathcal{T}}\|\boldsymbol{V}^n_{h,0,K}-\boldsymbol{u}^n_K\|^6_{L^6(K;\mathbb{R}^3)}\Big)^{1/6}\le\Big(\sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{V}^n_{h,0}-\boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)}\Big)^{1/2}.$$

Finally, employing Young's inequality, and estimate (4.8), we arrive at

$$|\mathcal{P}_1|\le\ c(\delta,M_0,E_0,\underline{r},\overline{r},\|(\boldsymbol{V},\nabla\boldsymbol{V},\partial_t\boldsymbol{V})\|_{L^\infty(Q_T,\mathbb{R}^{15})})$$
$$\times\Big(\Delta t\mathcal{E}(\varrho^0,\hat{\boldsymbol{u}}^0|\hat{r}^0,\hat{\boldsymbol{V}}^0_{h,0})+\Delta t\sum_{n=1}^m\mathcal{E}(\varrho^n,\hat{\boldsymbol{u}}^n|\hat{r}^n,\hat{\boldsymbol{V}}^n_{h,0})\Big)+\delta\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{V}^n_{h,0}-\boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)},\tag{8.3}$$

with any $\delta>0$.

Step 2: *Term* $\mathcal{P}_2$. We rewrite $\boldsymbol{V}^n_{h_0,\sigma}-\boldsymbol{V}^n_{h_0,K}=\boldsymbol{V}^n_{h,\sigma}-\boldsymbol{V}^n_{h,K}+[\boldsymbol{V}^n_{h,0}-\boldsymbol{V}^n_h]_\sigma+[\boldsymbol{V}^n_{h,0}-\boldsymbol{V}^n_h]_K$ and estimate the L^∞ norm of this expression by $h\|\nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}$ by virtue of (2.48–2.49), (2.52–2.53), $(2.41)_{s=1}$. Now we write $\mathcal{P}_2=\Delta t\sum_{n=1}^m\mathcal{P}_2^n$ where Lemma 7.2 and the Hölder inequality yield, similarly as in the previous step,

$$|\mathcal{P}_2^n|\le c(\underline{r},\overline{r},\|\nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)})\times$$
$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h\Big(E^{1/2}(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma)+E^{2/3}(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma)\Big)|\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}||\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}-\hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma|$$
$$\le c(\underline{r},\overline{r},\|(\boldsymbol{V},\nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})})\Big[\Big(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h\Big(E(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma)\Big)^{1/2}$$
$$+\Big(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|hE(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma)\Big)^{2/3}\Big]\times\Big(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h\Big|\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}-\hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma\Big|^6\Big)^{1/6},$$

provided $\gamma\ge 3/2$. Next, we observe that the contribution of the face $\sigma=K|L$ to the sums $\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|hE(\varrho^{n,\mathrm{up}}_\sigma|\hat{r}^{n,\mathrm{up}}_\sigma)$ and $\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h|\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}-\hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma|^6$ is less or equal than $2|\sigma|h(E(\varrho^n_K|\hat{r}^n_K)+$

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$E(\varrho_L^n|\hat r_L^n))$, and than $2|\sigma|h(|\boldsymbol{V}_{h,0,K}^n - \boldsymbol{u}_K^n|^6 + |\boldsymbol{V}_{h,0,L}^n - \boldsymbol{u}_L^n|^6)$, respectively. Consequently,we get by the same reasoning as in the previous step, under assumption $\gamma \geq 3/2$,

$$|\mathcal{P}_2| \leq c(\delta, M_0, E_0, \underline{r}, \overline{r}, \|(\boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})})\, \Delta t \sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat r^n, \hat{\boldsymbol{V}}_{h,0}^n) + \delta \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \|\nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)}. \tag{8.4}$$

Step 3: *Term $\mathcal{P}_3$.* We realize that

$$p(\varrho_K^n) - p'(r_K^n)(\varrho_K^n - r_K^n) - p(r_K^n) \leq c(\underline{r}, \overline{r}) E(\varrho_K|r_K),$$

by virtue of Lemma 7.2 in combination with assumption (1.4). Consequently,

$$|\mathcal{P}_3| \leq c\|\operatorname{div} \boldsymbol{V}\|_{L^\infty(Q_T)} \Delta t \sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat r^n, \hat{\boldsymbol{V}}_{h,0}^n). \tag{8.5}$$

Step 4: *Term $\mathcal{P}_4$.* We write $\boldsymbol{u}^n - \boldsymbol{V}^n$ as the sum $(\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n) + (\boldsymbol{V}_{h,0}^n - \boldsymbol{V}^n)$ accordingly splitting $\mathcal{P}_4$ into two terms

$$\Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K p'(\hat r^n) \frac{\hat r^n - \varrho^n}{\hat r^n} (\mathbf{u}^n - \boldsymbol{V}_{h,0}^n) \cdot \nabla r^n \,\mathrm{d}x \quad \text{and} \quad \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K p'(\hat r^n) \frac{\hat r^n - \varrho^n}{\hat r^n} (\mathbf{V}_{h,0}^n - \mathbf{V}^n) \cdot \nabla r^n \,\mathrm{d}x.$$

Reasoning similarly as in Step 2, we get

$$\begin{aligned}|\mathcal{P}_4| &\leq h^2\, c(\delta, M_0, E_0, \underline{r}, \overline{r}, |p'|_{C([\underline{r},\overline{r}])} \|(\nabla r, \nabla \boldsymbol{V})\|_{L^\infty(\Omega;\mathbb{R}^9)}) \\ &+ c(\delta, \|\underline{r}, \overline{r}, |p'|_{C([\underline{r},\overline{r}])} \|\nabla r\|_{L^\infty(\Omega;\mathbb{R}^3)})\, \Delta t \sum_{n=1}^m \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat r^n, \hat{\boldsymbol{V}}_{h,0}^n) + \delta\, \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \|\nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\|^2_{L^2(K;\mathbb{R}^3)}.\end{aligned} \tag{8.6}$$

Gathering the formulae (8.1) and (8.3)-(8.6) with δ sufficiently small (with respect to μ), we conclude the proof of Lemma 8.1. □

9 End of the proof of the error estimate (Theorem 3.1)

Finally, Lemma 8.1 in combination with the bound (4.8) yields

$$\mathcal{E}(\varrho^m, \hat{\boldsymbol{u}}^m|\hat r^m, \hat{\boldsymbol{V}}_{h,0}^m) \leq c\Big[h^A + \sqrt{\Delta t} + \Delta t + \mathcal{E}(\varrho^0, \hat{\boldsymbol{u}}^0|\hat r(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big] + c\,\Delta t \sum_{n=1}^{m-1} \mathcal{E}(\varrho^n, \hat{\boldsymbol{u}}^n|\hat r^n, \hat{\boldsymbol{V}}_{h,0}^n);$$

whence by the discrete standard version of the Gronwall lemma one gets at the first step

$$\mathcal{E}(\varrho^m, \hat{\boldsymbol{u}}^m|\hat r^m, \hat{\boldsymbol{V}}_{h,0}^m) \leq c\Big[h^a + \sqrt{\Delta t} + \mathcal{E}(\varrho^0, \hat{\boldsymbol{u}}^0|\hat r(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big].$$

Going with this information back to Lemma 8.1, one gets finally

$$\mathcal{E}(\varrho^m, \hat{\boldsymbol{u}}^m|\hat r^m, \hat{\boldsymbol{V}}_{h,0}^m) + \Delta t \frac{\mu}{2} \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K |\nabla_x(\mathbf{u}^n - \mathbf{V}_{h,0}^n)|^2 \mathrm{dx} \leq c\Big[h^a + \sqrt{\Delta t} + \mathcal{E}(\varrho^0, \hat{\boldsymbol{u}}^0|\hat r(0), \hat{\boldsymbol{V}}_{h,0}(0))\Big]. \tag{9.1}$$

Now, we write

$$\varrho_K^n(\boldsymbol{u}_K^n - \mathbf{V}_{h,0,K}^n)^2 = \varrho_K^n(\boldsymbol{u}_K^n - \mathbf{V}^n)^2 + 2\varrho_K^n \mathbf{V}^n(\boldsymbol{u}_K^n - \mathbf{V}_{h,0,K}^n) + \varrho_K^n(\mathbf{V}^n - \mathbf{V}_{h,0,K}^n)^2,$$

where

$$\|\mathbf{V}^n - \mathbf{V}_{h,0,K}^n\|_{L^\infty(K;\mathbb{R}^3)} \lesssim \|\mathbf{V}^n - \mathbf{V}_h^n\|_{L^\infty(K;\mathbb{R}^3)} + \|\mathbf{V}_h^n - \mathbf{V}_{h,K}^n\|_{L^\infty(K;\mathbb{R}^3)} + \|[\mathbf{V}_h^n - \mathbf{V}_{h,0}^n]_K\|_{L^\infty(K;\mathbb{R}^3)}$$

$$\lesssim h\Big(\|\nabla_x \mathbf{V}^n\|_{L^\infty(K;\mathbb{R}^9)} + \|\nabla_x \mathbf{V}_h^n\|_{L^\infty(K;\mathbb{R}^9)} + \|\mathbf{V}_h^n - \mathbf{V}_{h,0}^n\|_{L^\infty(K;\mathbb{R}^3)} \lesssim h\|\nabla \mathbf{V}^n\|_{L^\infty(K;\mathbb{R}^9)}.$$

In the above calculation we have employed formula (2.40) to estimate the first term, estimates $(2.52)_{s=1}$, $(2.41)_{s=1}$ to estimate the second term, and formulas (2.48) and (2.49) for $K \cap \partial\Omega_h = \emptyset$ and $K \cap \partial\Omega_h \neq \emptyset$, respectively, to evaluate the last term. We conclude that

$$\sum_{K\in\mathcal{T}} \frac{1}{2}|K|\Big(\varrho_K^m|\boldsymbol{u}_K^m - \boldsymbol{V}_{h,0,K}^m|^2 - \varrho_K^0|\boldsymbol{u}_K^0 - \boldsymbol{V}_{h,0,K}^0|^2\Big) \tag{9.2}$$

$$\geq \int_{\Omega\cap\Omega_h} \varrho^m(\hat{\boldsymbol{u}}^m - \mathbf{V}^m)^2 \mathrm{d}x - \int_{\Omega\cap\Omega_h} \varrho^0(\hat{\boldsymbol{u}}^0 - \mathbf{V}^0)^2 \mathrm{d}x + L_1,$$

where

$$|L_1| \lesssim h\, M_0 \|\nabla_x \mathbf{V}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^9)}.$$

Similarly, we find with help of (4.8),

$$\|E(\varrho_K^n|\hat{r}^n) - E(\varrho_K^n, r^n)\|_{L^\infty(K)} \leq h\, c(M_0, \underline{r}, \overline{r}, |p|_{C^1[\underline{r},\overline{r}]}\|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)});$$

whence

$$\sum_{K\in\mathcal{T}} |K|\Big(E(\varrho_K^n|\hat{r}^n) - E(\varrho_K^0|\hat{r}^0\Big) \geq \int_{\Omega\cap\Omega_h} E(\varrho^m|r^m)\mathrm{d}x - \int_{\Omega\cap\Omega_h} E(\varrho^0|r^0)\mathrm{d}x + L_2, \tag{9.3}$$

where

$$|L_2| \leq h\, c(M_0, \underline{r}, \overline{r}, |p|_{C^1[\underline{r},\overline{r}]}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}).$$

Finally, by virtue of (2.48–2.49) and $(2.41)_{s=2}$

$$\|\nabla(\boldsymbol{V}_{h,0}^n - \boldsymbol{V}^n)\|_{L^2(K;\mathbb{R}^3)} \lesssim h\|(\nabla \boldsymbol{V}^n, \nabla^2 \boldsymbol{V}^n)\|_{L^\infty(K;\mathbb{R}^{12})};$$

whence

$$\Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K |\nabla_x(\mathbf{u}^n - \mathbf{V}_{h,0}^n)|^2 \mathrm{dx} \geq \Delta t \sum_{n=1}^m \int_{\Omega\cap\Omega_h} |(\nabla_h \mathbf{u}^n - \nabla_x \mathbf{V}^n)|^2 \mathrm{d}x + L_3, \tag{9.4}$$

where

$$|L_3| \leq h^2 c(\|(\nabla \boldsymbol{V}^n, \nabla^2 \boldsymbol{V}^n)\|_{L^\infty(K;\mathbb{R}^{12})}).$$

Theorem 3.1 is a direct consequence of estimate (9.1) and identities (9.2–9.4). Theorem 3.1 is thus proved.

10 Concluding remarks

In the convergence proofs one usually needs to complete the numerical scheme by stabilizing terms, so that the new numerical scheme reads

$$\sum_{K\in\mathcal{T}_h} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t}\phi_K + \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}(\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K})\phi_K + T_c(\phi) = 0, \tag{10.1}$$

for any $\phi \in Q_h(\Omega_h)$ and $n = 1, \ldots, N$,

$$\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big(\varrho_K^n \boldsymbol{u}_K^n - \varrho_K^{n-1}\boldsymbol{u}_K^{n-1}\Big)\cdot \boldsymbol{v}_K + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]\cdot \boldsymbol{v}_K \tag{10.2}$$

$$-\sum_{K\in\mathcal{T}} p(\varrho_K^n) \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\boldsymbol{v}_\sigma \cdot \mathbf{n}_{\sigma,K} + \mu \sum_{K\in\mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v}\ \mathrm{d}x$$

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$$+\frac{\mu}{3}\sum_{K\in\mathcal{T}}\int_K \mathrm{div}\boldsymbol{u}^n \mathrm{div}\boldsymbol{v}\ \mathrm{d}x + T_m(\phi) = 0,\ \text{for any } \boldsymbol{v}\in V_{h,0}(\Omega;R^3) \text{ and } n=1,\ldots,N,$$

where

$$T_c(\phi) = h^{1-\varepsilon}\sum_{\sigma\in\mathcal{E}_{\rm int}}|\sigma|[\varrho^n]_{\sigma,\mathsf{n}_\sigma}[\phi]_{\sigma\mathsf{n}_\sigma},\quad T_m(\phi)=\sum_{\sigma\in\mathcal{E}_{\rm int}}|\sigma|[\varrho^n]_{\sigma,\mathsf{n}_\sigma}\{\hat{\boldsymbol{u}}^n\}_\sigma[\hat{\phi}]_{\sigma,\mathsf{n}_\sigma},\ \varepsilon\in[0,1),$$

see Karlsen, Karper [30], Gallouet, Gastaldo, Herbin, Latché [20]. These terms are designed to provide the supplementary positive term

$$h^{1-\varepsilon}\sum_{\sigma\in\mathcal{E}_{\rm int}}|\sigma|[\varrho^n]^2_{\sigma,\mathsf{n}_\sigma},$$

to the left hand side of the discrete energy identity (4.2). They contribute to the right hand side of the discrete relative energy (5.1) by supplementary terms whose absolute value is bounded from above by

$$h^{(1-\varepsilon)/2}\,c\Big(M_0,E_0,\sup_{n=0,\ldots,N}\|r^n,\mathbf{U}^n,\nabla\mathbf{U}^n\|_{L^\infty(\Omega_h;\mathbb{R}^{13})},\sup_{n=0,\ldots,N}\sup_{\sigma\in\mathcal{E}\rm int}[r^n]_{\sigma,\mathsf{n}_\sigma}/h\Big).$$

Consequently, they give rise to the contributions at the right hand side of the approximate relative energy inequality (6.1) whose bound is

$$h^{(1-\varepsilon)/2}\,c\Big(M_0,E_0,\|r,\nabla r,\mathbf{U},\nabla\mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^{16})}\Big).$$

Similar estimates are true, if we replace in the numerical scheme everywhere classical upwind formula (3.4)

$$\mathrm{Up}_K(q,\mathbf{u}) = \sum_{\sigma\in\mathcal{E}(K)} q_\sigma^{\rm up}\boldsymbol{u}_\sigma\cdot\mathbf{n}_{\sigma,K} = \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}\Big(q_K[\boldsymbol{u}_\sigma\cdot\mathbf{n}_{\sigma,K}]^+ + q_L[\boldsymbol{u}_\sigma\cdot\mathbf{n}_{\sigma,K}]^-\Big),$$

by the modified upwind suggested in [12]:

$$\mathrm{Up}_K(q,\mathbf{u}) = \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}\frac{q_K}{2}\Big([\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K}+h^{1-\varepsilon}]^+ + [\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K}-h^{1-\varepsilon}]^+\Big)+\frac{q_L}{2}\Big([\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K}+h^{1-\varepsilon}]^- + [\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K}-h^{1-\varepsilon}]^-\Big), \tag{10.3}$$

where $\sigma = K|L\in\mathcal{E}_{\rm int}$. We will finish by formulating the error estimate for the numerical problem (3.5), (10.1), (10.2) or for (3.5), (3.6), (3.7) with modified upwind (10.3).

Theorem 10.1. *Let Ω, p, $[r_0,\mathbf{V}^0]$, $[r,V]$ satisfy assumptions of Theorem 3.1. Let $(\varrho^n,\mathbf{u}^n)_{n=0,\ldots,N}$ be a family of numerical solutions to the scheme (3.5), (10.1), (10.2) or to the scheme (3.5), (3.6), (3.7) with modified upwind (10.3), where $\varepsilon\in[0,1)$. Then error estimate (3.8) holds true with the exponent*

$$a=\min\Big\{\frac{2\gamma-3}{\gamma},\frac{1-\varepsilon}{2}\Big\}\ \textit{if}\ \tfrac{3}{2}\le\gamma<2,\quad a=\frac{1-\varepsilon}{2}\ \textit{if}\ \gamma\ge 2.$$

Finally, a natural question arises as top what extent the obtained error estimates are optimal. In the light of the results obtained in [28], [29], it may seem we loose, in particular in terms of the spatial discretization parameter h for $\gamma\to 3/2$. On the other hand, however, it is worth noting we do not make any extra assumption concerning boundedness of the numerical solutions in contrast with [28].

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Implicit MAC scheme for compressible Navier-Stokes equations: Unconditional error estimates

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October 13, 2016

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Abstract

We derive in this paper error estimates of the marker-and-cell (MAC) scheme for the discretization of the compressible and isentropic Navier-Stokes equations on two or three dimensional Cartesian grids. Existence of a solution to the scheme is proved. Error estimates are obtained by using the discrete version of the *relative energy method*.

Keywords: Compressible fluids, Navier-Stokes equations, Cartesian grids, Marker and Cell scheme, Error estimates.
AMS classification 35Q30, 65N12, 76N10, 76N15, 76M12, 76M20

1 Introduction

The aim of this paper is to derive error estimates for approximate solutions of the compressible barotropic Navier-Stokes equations obtained by the Marker-and-cell scheme. These equations are posed on the time-space domain $Q_T = (0,T)\times\Omega$, where Ω is a bounded domain of $\mathbb{R}^d$, $d = 2, 3$, adapted to the MAC scheme (see section 3), and $T > 0$, and read:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\varrho) = \mu \Delta \boldsymbol{u} + (\mu + \lambda)\nabla \operatorname{div} \boldsymbol{u}, \tag{1.1b}$$

supplemented with the initial conditions

$$\varrho(0, \boldsymbol{x}) = \varrho_0(\boldsymbol{x}),\ \ \varrho \boldsymbol{u}(0, \boldsymbol{x}) = \varrho_0 \boldsymbol{u}_0, \tag{1.2}$$

where ϱ_0 and $\boldsymbol{u}_0$ are given functions from Ω to $\mathbb{R}^*_+$ and $\mathbb{R}^d$ respectively, and boundary conditions

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = 0. \tag{1.3}$$

In the above equations, the unknown functions are the scalar density field $\varrho(t, \boldsymbol{x}) \geq 0$ and vector velocity field $\boldsymbol{u} = (u_1, \ldots, u_d)(t, \boldsymbol{x})$, where $t \in (0, T)$ denotes the time and $\boldsymbol{x} \in \Omega$ is the space variable. The viscosity coefficients μ and λ, assumed to be constant, are such that

$$\mu > 0,\ \ \lambda + \mu \geq 0. \tag{1.4}$$

In the compressible barotropic Navier-Stokes equations, the pressure is a given function of the density. Here we assume that the pressure satisfies

$$p \in C([0, \infty)),\ \ p \in C^2(0, \infty),\ \ p(0) = 0,\ \ p'(\varrho) > 0 \text{ for all } \varrho > 0, \tag{1.5a}$$

*This work was supported by the MODTERCOM project within the APEX programme of the Provence-Alpes-Côte d'Azur region

$$\lim_{\varrho\to\infty}\frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,\quad \lim_{\varrho\to 0+}\frac{p'(\varrho)}{\varrho^{\alpha+1}} = p_0 > 0 \tag{1.5b}$$

where $\gamma \geq 1$ and $\alpha \leq 0$. We notice that assumptions (1.5) are compatible with the isentropic pressure law $p(\varrho) = \varrho^\gamma$ provided $1 \leq \gamma \leq 2$.

The main underlying idea of this paper is to derive the error estimates for approximate solutions of problem (1.1)–(1.5) obtained by time and space discretization by using the discrete version of the *relative energy method* introduced for equations (1.1a), (1.1b) on the continuous level in [16, 18, 19].

The discrete relative energy method was suggested in [23] in the context of rather academic finite-volume:finite element scheme proposed in [34]. The method provides unconditional estimate of error between any numerical solution of the scheme [34] and a classical solution of equations (1.1)–(1.5), without any additional assumption on the numerical solution. This is highly wanted result, first of its type in the mathematical literature. The natural question arises whether a similar method can lead to similar unconditional error estimates for the less academic and more practical numerical schemes.

The main goal of this paper is to get unconditional error estimates for the MAC scheme implicit in time.

In spite of the fact that we stuck to the [23] methodology, the proofs remain still difficult. Not speaking on technicalities linked to the approximations, interpolations and projections to the relevant function spaces related to the MAC discretization, the most involved part is the treatment of transport terms in the continuity and momentum equations which requires derivation of quite sophisticated formulas involving primal and dual fluxes. In this part, our approach is reminiscent to the recent work of Therme [29] devoted to the staggered space approximations to the Euler equations.

Since the very beginning of the introduction of the Marker-and-Cell (MAC) scheme [28], it is claimed that this discretization is suitable for both incompressible and compressible flow problems (see [26, 27] for the seminal papers, [2, 5, 7, 30, 31, 33, 39–41, 43, 44] for subsequent developments and [45] for a review). The use of the MAC scheme in the incompressible case is now standard, and the proof of convergence for the MAC scheme in primitive variables has been recently been completed [22].

The paper is organized as follows. After recalling the fundamental setting of the problem and the relative energy inequality in the continuous case in Section 2, we proceed in Section 3 to the discretization: we introduce the discrete meshes and functional spaces and the definition of the numerical scheme, and state a known existence result, along with the main result of the paper, that is the error estimate, which is stated in Theorem 3.3. The remaining sections are devoted to the proof of Theorem 3.3:

- In Section 4 we derive estimates provided by the scheme.
- In Section 5, we derive the discrete intrinsic version of the relative energy inequality for the solutions of the numerical scheme (see Theorem 5.1). We then transform this inequality to a more convenient form, see Lemma 5.1.
- Finally, in Section 6, we investigate the form of the discrete relative energy inequality with the test functions being strong solutions to the original problem. This investigation is formulated in Lemma 6.1 and finally leads to a Gronwall type estimate formulated in Lemma 7.1. The latter yields the error estimates and finishes the proof of the main result.

The theorem 3.3 remains valid for other finite volume schemes with staggered space discretization as e.g. non conforming Rannacher-Turek finite elements or the lowest degree Crouzeix-Raviart finite elements on simplical meshes. These results are formulated without proofs, that are similar and simpler than those for the MAC scheme, in Appendix B.

2 The continuous problem

The aim of this section is to recall some fundamental notions and results for the continuous problem. We begin by the definition of weak solutions to problem (1.1)– (1.5). Let us introduce the Helmholtz's function defined by

$$\mathcal{H}(\varrho) = \varrho\int_1^\varrho \frac{p(t)}{t^2}\,\mathrm{d}t,\ \ \varrho \geq 0. \tag{2.1}$$

Note that $\mathcal{H} \in C(\mathbb{R}_+)$, $\mathcal{H}(0) = 0$ and that $\mathcal{H}$ is a solution on $\mathbb{R}_+^\star$ of the ordinary differential equation

$$\varrho\, \mathcal{H}' - \mathcal{H} = p \tag{2.2}$$

with the constant of integration fixed such that $\mathcal{H}(1) = 0$. Note also that

$$\mathcal{H}''(\varrho) = \frac{p'(\varrho)}{\varrho}. \tag{2.3}$$

Definition 2.1 (Weak solutions)**.** *Let $\varrho_0 : \Omega \to (0, +\infty)$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^d$ with finite energy $\mathcal{E}_0 = \int_\Omega (\frac{1}{2}\varrho_0 |\boldsymbol{u}_0|^2 + \mathcal{H}(\varrho_0))\,\mathrm{d}\boldsymbol{x}$ and finite mass $0 < M_0 = \int_\Omega \varrho_0\,\mathrm{d}\boldsymbol{x}$. We shall say that the pair $(\varrho, \boldsymbol{u})$ is a weak solution to the problem (1.1)–(1.5) emanating from the initial data $(\varrho_0, \boldsymbol{u}_0)$ if:*

1. *$\varrho \in L^\infty(0,T; L^\gamma(\Omega))$, $\varrho \geq 0$ a.e. in $(0,T) \times \Omega$, $\boldsymbol{u} \in L^2(0,T; H_0^1(\Omega)^d)$ and $\varrho|\boldsymbol{u}|^2 \in L^\infty(0,T;L^1(\Omega))$.*
2. *The continuity equation (1.1a) is satisfied in the following weak sense*

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = -\int_\Omega \varrho_0 \varphi(0, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{2.4}$$

   for any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$ such that $\varphi(T, \cdot) = 0$.

3. *The momentum equation (1.1b) is satisfied in the weak sense,*

$$\int_0^T \int_\Omega \Big(\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\psi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\psi} + p(\varrho)\,\mathrm{div}\,\boldsymbol{\psi}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}$$
$$- \int_0^T \int_\Omega \Big(\mu \nabla \boldsymbol{u} : \nabla \boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} + (\mu + \lambda)\mathrm{div}\boldsymbol{u}\mathrm{div}\boldsymbol{\psi}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = -\int_\Omega \varrho_0 \boldsymbol{u}_0 \cdot \boldsymbol{\psi}(0, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{2.5}$$

   for any $\boldsymbol{\psi} \in C_c^\infty([0,T] \times \Omega)^d$ such that $\boldsymbol{\psi}(T, \cdot) = \boldsymbol{0}$.

4. *The following energy inequality is satisfied a.e in $(0,T)$*

$$\int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \mathcal{H}(\varrho)\Big)(\tau)\,\mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \Big(\mu |\nabla \boldsymbol{u}|^2 + (\mu+\lambda)|\,\mathrm{div}\,\boldsymbol{u}|^2\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \leq \mathcal{E}_0\,. \tag{2.6}$$

Note that the existence of weak solutions emanating from the finite energy initial data is well-known on bounded Lipschitz domains under assumptions (1.4) and (1.5) provided $\gamma > d/(d-1)$, see Lions [35] for "large" values of γ, Feireisl and coauthors [17] for $\gamma > d/(d-1)$. More details about this problem are avaible in monographs [35], [14], [37].

Remark 1. *The density ϱ satisfy the conservation of mass that is*

$$\int_\Omega \varrho(t)\,\mathrm{d}\boldsymbol{x} = M_0 \text{ a.e in } (0,T). \tag{2.7}$$

Let us now introduce the notion of relative energy. We first introduce the function

$$\begin{aligned} E:\ & [0,\infty) \times (0,\infty) \to \mathbb{R}, \\ & (\varrho, r) \mapsto E(\varrho|r) = \mathcal{H}(\varrho) - \mathcal{H}'(r)(\varrho - r) - \mathcal{H}(r), \end{aligned} \tag{2.8}$$

where $\mathcal{H}$ is defined by (2.1). Since $\mathcal{H}$ is strictly convex on $[0,\infty)$, we have

$$E(\varrho|r) \geq 0 \quad \text{and} \quad E(\varrho|r) = 0 \Leftrightarrow \varrho = r.$$

More precisely, E satisfy the following algebraic inequality whose straightforward proof is left to the reader:

Lemma 2.1. *Let $0 < a < b < \infty$. Then there exists a number $c = c(\gamma, p_\infty, a, b, \min_{[a,b]} p, \min_{[a/2,2b]} p') > 0$ such that for all $\varrho \in [0,\infty)$ and $r \in [a,b]$*

$$E(\varrho|r) \geq c(a,b)\Big(\varrho^\gamma 1_{\mathbb{R}_+\setminus[a/2,2b]}(\varrho) + 1_{\mathbb{R}_+\setminus[a/2,2b]}(\varrho) + (\varrho - r)^2 1_{[a/2,2b]}(\varrho)\Big)$$
$$\geq c(a,b,\gamma)\Big(|\varrho - r|^\gamma 1_{\mathbb{R}_+\setminus[a/2,2b]}(\varrho) + (\varrho - r)^2 1_{[a/2,2b]}(\varrho)\Big). \quad (2.9)$$

In order to measure a "distance" between a weak solution $(\varrho, \boldsymbol{u})$ of the compressible Navier-Stokes system and any other state $(r, \boldsymbol{U})$ of the fluid , we introduce the relative energy functional, defined by

$$\mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U}) = \int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u} - \boldsymbol{U}|^2 + E(\varrho\,|\,r)\Big)\,\mathrm{d}\boldsymbol{x}. \quad (2.10)$$

It was proved recently in [16] that, provided assumption (1.5) holds, any weak solution satisfies the following so-called relative energy inequality

$$\begin{aligned}
&\mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(\tau) - \mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(0)\\
&\quad + \int_0^\tau \int_\Omega \Big(\mu|\nabla(\boldsymbol{u} - \boldsymbol{U})|^2 + (\mu+\lambda)|\operatorname{div}(\boldsymbol{u} - \boldsymbol{U})|^2\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\\
&\quad \leq \int_0^\tau \int_\Omega \Big(\mu\nabla\boldsymbol{U} : \nabla(\boldsymbol{U} - \boldsymbol{u}) + (\mu+\lambda)\operatorname{div}\boldsymbol{U}\operatorname{div}(\boldsymbol{U} - \boldsymbol{u})\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\\
&\quad + \int_0^\tau \int_\Omega \varrho\partial_t\boldsymbol{U}\cdot(\boldsymbol{U} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} + \int_0^\tau \int_\Omega \varrho\boldsymbol{u}\cdot\nabla\boldsymbol{U}\cdot(\boldsymbol{U} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\\
&\quad + \int_0^\tau \int_\Omega \frac{r-\varrho}{r}p'(r)\partial_t r\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} - \int_0^\tau \int_\Omega \frac{\varrho}{r}p'(r)\nabla r\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\\
&\quad - \int_0^\tau \int_\Omega p(\varrho)\operatorname{div}\boldsymbol{U}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\,. \quad (2.11)
\end{aligned}$$

for a.a. $\tau \in (0,T)$, and for any pair of test functions

$$r \in C^1([0,T]\times\overline{\Omega}),\ r > 0,\ \boldsymbol{U} \in C^1([0,T]\times\overline{\Omega})^3,\ \boldsymbol{U}|_{\partial\Omega} = \boldsymbol{0}.$$

Moreover if $(r, \boldsymbol{U})$ is a sufficiently strong solution to problem (1.1)–(1.5) emanating from initial data $(r_0, \boldsymbol{U}_0)$, the right member becomes quadratic in difference $(\varrho - r, \boldsymbol{u} - \boldsymbol{U})$ and inequality (2.11) reduces

$$\begin{aligned}
&\mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(\tau) + \int_0^\tau \int_\Omega \Big(\mu|\nabla(\boldsymbol{u} - \boldsymbol{U})|^2 + (\mu+\lambda)|\operatorname{div}(\boldsymbol{u} - \boldsymbol{U})|^2\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\\
&\quad \leq \mathcal{E}(\varrho_0, \boldsymbol{u}_0\Big|r(0), \boldsymbol{U}(0)) + \int_0^\tau \mathcal{R}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})\,\mathrm{dt} \quad (2.12)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U}) &= \int_\Omega (\varrho - r)(\partial_t\boldsymbol{U} + \boldsymbol{U}\cdot\nabla\boldsymbol{U})\cdot(\boldsymbol{U} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \varrho(\boldsymbol{u} - \boldsymbol{U})\cdot\nabla\boldsymbol{U}\cdot(\boldsymbol{U} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_\Omega \frac{\nabla p(r)}{r}(r - \varrho)\cdot(\boldsymbol{u} - \boldsymbol{U})\,\mathrm{d}\boldsymbol{x} - \int_\Omega (p(\varrho) - p'(r)(\varrho - r) - p(r))\operatorname{div}\boldsymbol{U}\,\mathrm{d}\boldsymbol{x}. \quad (2.13)
\end{aligned}$$

In order to obtain a stability result of strong solutions in the class of weak solutions, the goal is to find an estimate of the left hand side of from below by (2.12) by

$$c\int_0^\tau \|\boldsymbol{u} - \boldsymbol{U}\|^2_{W^{1,2}(\Omega)^3}\,\mathrm{d}\boldsymbol{x} - \overline{c}'\int_0^\tau \mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(t)\,\mathrm{dt} + \mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(\tau) \quad (2.14)$$

and the right hand side from above by

$$\mathcal{E}(\varrho_0, \boldsymbol{u}_0\Big|r(0), \boldsymbol{U}(0)) + \delta\int_0^\tau \|\boldsymbol{u} - \boldsymbol{U}\|^2_{W^{1,2}(\Omega)^3}\,\mathrm{d}\boldsymbol{x} + c'(\delta)\int_0^\tau a(t)\,\mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(t)\,\mathrm{dt} \quad (2.15)$$

with any $\delta > 0$, where $c > 0$ is independant of δ, $\overline{c}' \geq 0$, $c' = c'(\delta) > 0$ and $a \in L^1(0,T)$. This process leads to the estimate

$$\mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(\tau) \leq \mathcal{E}(\varrho_0, \boldsymbol{u}_0\Big|r(0), \boldsymbol{U}(0)) + c\int_0^\tau a(t)\,\mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(t)\,\mathrm{d}t\,. \tag{2.16}$$

It remains to conclude by using Gronwall Lemma.

The stability of strong solutions in the class of weak solutions is stated in the following proposition (see [16] for a proof).

Proposition 2.1 (Estimate on the relative energy)**.** *Let Ω be a bounded Lipschitz domain. Assume that the viscosity coefficients satisfy assumptions (1.4), that the pressure p satisfy (1.5) with $\gamma > \frac{6}{5}$, and that $(\varrho, \boldsymbol{u})$ is a weak solution to problem (1.1)–(1.5) emanating from initial data $(\varrho_0 > 0, \boldsymbol{u}_0)$, with finite energy $\mathcal{E}_0$ and finite mass $M_0 = \int_\Omega \varrho_0 \mathrm{d}x > 0$. Let $(r, \boldsymbol{U})$ that belongs to the class*

$$0 < \underline{r} \leq r \leq \overline{r},\ \ r \in L^\infty((0,T)\times\Omega), \tag{2.17a}$$

$$\boldsymbol{U} \in L^2(0,T; H^1_0(\Omega)^3), \tag{2.17b}$$

$$\nabla r, \nabla^2 \boldsymbol{U} \in L^2(0,T; L^q(\Omega)),\ \ q > \max(3, \frac{6\gamma}{5\gamma-6}), \tag{2.17c}$$

be a strong solution of the same equations with initial data $(r(0), \boldsymbol{U}(0)) = (r_0, \boldsymbol{U}_0)$. Then there exists

$$c = c(T, \Omega, M_0, \mathcal{E}_0, \underline{r}, \overline{r}, \gamma, \|\operatorname{div}\boldsymbol{U}\|_{L^1(0,T;L^\infty(\Omega))}, \|(\nabla r, \nabla^2\boldsymbol{U})\|_{L^2(0,T;L^q(\Omega)^{12})}) > 0$$

such that for almost all $t \in (0,T)$,

$$\mathcal{E}(\varrho, \boldsymbol{u}\Big|r, \boldsymbol{U})(t) \leq c\mathcal{E}(\varrho_0, \boldsymbol{u}_0\Big|r_0, \boldsymbol{U}_0). \tag{2.18}$$

The goal of the present paper is to obtain estimate of type (2.18) for $(\varrho, \boldsymbol{u})$ being a numerical solution of Problem (1.1)-(1.5) obtained by the MAC discretization.

3 The numerical scheme

3.1 Space discretization

We assume that the closure of the domain Ω is a union of closed rectangles ($d = 2$) or closed orthogonal parallelepipeds ($d = 3$) with mutually disjoint interiors, and, without loss of generality, we assume that the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors, denoted by $(\boldsymbol{e}^{(1)}, \ldots, \boldsymbol{e}^{(d)})$,

Definition 3.1 (MAC grid)**.** *A discretization of Ω with MAC grid, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, where:*

- *The density and pressure (or primal) grid denoted by $\mathcal{M}$, consists of a union of possibly non uniform (closed) rectangles (d=2) or (closed) parallelpipeds ($d = 3$), the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors; a generic cell of this grid is denoted by K (a closed set), and its mass center $\boldsymbol{x}_K$. It is a conforming grid, meaning that*

$$\overline{\Omega} = \cup_{K\in\mathcal{M}} K,\ \text{where}\ \mathrm{int}(K)\cap\mathrm{int}(L) = \emptyset\ \text{whenever}\ (K,L)\in\mathcal{M}^2,\ K \neq L, \tag{3.1}$$

and if $K \cap L \neq \emptyset$ then $K \cap L$ is a common face or edge or vertex of K and L. A generic face (or edge in the two-dimensional case) of such a cell is denoted by $\sigma \in \mathcal{E}(K)$ (a closed set), and its mass center $\boldsymbol{x}_\sigma$, where $\mathcal{E}(K)$ denotes the set of all faces of K. We denote by $\boldsymbol{n}_{\sigma,K}$ the unit normal vector to σ outward K. The set of all faces of the mesh is denoted by $\mathcal{E}$; we have $\mathcal{E} = \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{ext}}$, where $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$) are the edges of $\mathcal{E}$ that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to the i^{th} unit vector $\boldsymbol{e}^{(i)}$ of the canonical basis of $\mathbb{R}^d$ is denoted by $\mathcal{E}^{(i)}$, for $i = 1, \ldots, d$. We then have $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}_{\mathrm{int}} \cup \mathcal{E}^{(i)}_{\mathrm{ext}}$, where $\mathcal{E}^{(i)}_{\mathrm{int}}$ (resp. $\mathcal{E}^{(i)}_{\mathrm{ext}}$) are the edges of $\mathcal{E}^{(i)}$ that lie in the interior (resp. on the boundary) of the domain. Finally, for $i = 1, ..., d$ and $K \in \mathcal{M}$, we denote $\mathcal{E}^{(i)}(K) = \mathcal{E}(K) \cap \mathcal{E}^{(i)}$ and $\mathcal{E}^{(i)}_{\mathrm{int}}(K) = \mathcal{E}(K) \cap \mathcal{E}^{(i)}_{\mathrm{int}}$.

- *For each $\sigma \in \mathcal{E}$, we write that $\sigma = K|L$ if $\sigma = \partial K \cap \partial L$ and we write that $\sigma = \overrightarrow{K|L}$ if, furthermore, $\sigma \in \mathcal{E}^{(i)}$ and $(\boldsymbol{x}_L - \boldsymbol{x}_K) \cdot \boldsymbol{e}^{(i)} > 0$ for some $i \in [\![1, d]\!] \equiv \{1, \dots, d\}$. A primal cell K will be denoted $K = [\overrightarrow{\sigma\sigma'}]$ if $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i = 1, \dots, d$ are such that $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}^{(i)} > 0$. For a face $\sigma \in \mathcal{E}$, the distance d_σ is defined by:*

$$d_\sigma = \begin{cases} d(\boldsymbol{x}_K, \boldsymbol{x}_L) & \text{if } \sigma = K|L \in \mathcal{E}_{\rm int}, \\ d(\boldsymbol{x}_K, x_\sigma) & \text{if } \sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K) \end{cases} \tag{3.2}$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^d$.

- *A dual cell D_σ associated to a face $\sigma \in \mathcal{E}$ is defined as follows:*
  - *if $\sigma = K|L \in \mathcal{E}_{\rm int}$ then $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$, where $D_{\sigma,K}$- a closed set (resp. $D_{\sigma,L}$ -a closed set) is the half-part of K (resp. L) adjacent to σ (see Fig. 1 for the two-dimensional case) ;*
  - *if $\sigma \in \mathcal{E}_{\rm ext}$ is adjacent to the cell K, then $D_\sigma = D_{\sigma,K}$.*

  The dual grid $\{D_\sigma\}_{\sigma\in\mathcal{E}^{(i)}}$ of Ω (sometimes called the i-th velocity component grid) verifies for each fixed $i \in \{1, \dots, d\}$

$$\overline{\Omega} = \cup_{\sigma\in\mathcal{E}^{(i)}} D_\sigma,\ \ \mathrm{int}(D_\sigma) \cap \mathrm{int}(D_{\sigma'}) = \emptyset,\ \sigma, \sigma' \in \mathcal{E}^{(i)},\ \sigma \neq \sigma'. \tag{3.3}$$

- *A dual face separating two neighboring dual cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$ or $\epsilon = D_\sigma|D_{\sigma'}$ (a closed set) or $\epsilon = \overrightarrow{\sigma|\sigma'}$ when specifying its orientation: more precisely we write that $\epsilon = \overrightarrow{\sigma|\sigma'}$ if $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}^{(j)} > 0$ for some $j \in [\![1, d]\!]$. The set of all faces of D_σ is denoted $\tilde{\mathcal{E}}(D_\sigma)$; it is decomposed to the set of external faces $\tilde{\mathcal{E}}_{\rm ext}(D_\sigma) = \{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) | \epsilon \subset \partial\Omega\}$ and the set of internal faces $\tilde{\mathcal{E}}_{\rm int}(D_\sigma) = \{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) | \mathrm{int}_{d-1}\epsilon \subset \Omega\}$, where int_{d-1} denote the interior in the topology of $\mathbb{R}^{d-1}$. The set of all dual faces of the dual grid $\{D_\sigma\}_{\sigma\in\mathcal{E}^{(i)}}$ is decomposed into the internal and boundary edges: $\widetilde{\mathcal{E}}^{(i)} = \widetilde{\mathcal{E}}^{(i)}_{\rm int} \cup \widetilde{\mathcal{E}}^{(i)}_{\rm ext}$, where $\widetilde{\mathcal{E}}^{(i)}_{\rm int} = \{\epsilon = \sigma|\sigma' \,|\, \sigma, \sigma' \in \mathcal{E}^{(i)}\}$ and $\widetilde{\mathcal{E}}^{(i)}_{\rm ext} = \{\epsilon = \partial D_\sigma \cap \partial\Omega \,|\, \sigma \in \mathcal{E}^{(i)}, \partial D_\sigma \cap \partial\Omega \neq \emptyset\}$. We denote by $\boldsymbol{n}_{\epsilon,D_\sigma}$ the unit normal vector to $\epsilon \in D_\sigma$ outward D_σ.*

  We denote for further convenience $\mathbf{n}_\epsilon$ and $\mathbf{n}_\sigma$ a normal unit vector to face ϵ and σ, respectively. We write $\epsilon \perp \sigma$ resp. $\sigma_\perp\sigma'$ iff $\mathbf{n}_\epsilon \cdot \mathbf{n}_\sigma = 0$ resp. $\mathbf{n}_\sigma \cdot \mathbf{n}_{\sigma'} = 0$. Similarly we write $\epsilon \perp \mathbf{e}^{(j)}$ resp. $\sigma \perp \mathbf{e}^{(j)}$ iff $\mathbf{n}_\epsilon$ and $\mathbf{e}^{(j)}$ resp. $\mathbf{n}_\sigma$ and $\mathbf{e}^{(j)}$ are parallel. We also denote by $\mathbf{ab}$ the segment $\{\mathbf{a} + t(\mathbf{b} - \mathbf{a}) | t \in [0, 1]\}$, where $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2d}$, and by $\boldsymbol{x}_\epsilon$ resp. $\boldsymbol{x}_{\sigma\cap\epsilon}$ the mass centers of the face ϵ resp. of the set $\sigma \cap \epsilon$ (provided it is not empty).

- *In order to define bi-dual grid, we introduce the set $\tilde{\mathcal{E}}^{(i,j)} = \{\epsilon \in \tilde{\mathcal{E}}^{(i)} \,|\, \epsilon \perp \boldsymbol{e}^{(j)}\}$ of dual faces of the i-th component velocity grid that are orthogonal to $\boldsymbol{e}^{(j)}$. A bi-dual cell D_ϵ associated to a face $\epsilon \in \tilde{\mathcal{E}}$ is defined as follows:*
  - *If $\epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\rm int}$ then $D_\epsilon = \epsilon \times \boldsymbol{x}_\sigma\boldsymbol{x}_{\sigma'}$ (see Figure 2). (We notice that, if $\sigma, \sigma' \in \mathcal{E}^{(i)}$ with $K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$ and $\epsilon = \sigma|\sigma'$ then $D_\epsilon = K$.)*
  - *If $\epsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\rm ext}$ with $\epsilon \in \tilde{\mathcal{E}}(D_\sigma)$ and $i \neq j$ then $D_\epsilon = \epsilon \times \boldsymbol{x}_\sigma\boldsymbol{x}_{\sigma\cap\epsilon}$.*

  In the list above we did not consider the sitution $\epsilon \in \tilde{\mathcal{E}}^{(i,i)} \cap \tilde{\mathcal{E}}^{(i)}_{\rm ext}$ with $\epsilon \in \tilde{\mathcal{E}}(D_\sigma)$. In this case $\epsilon = \sigma \subset \partial\Omega$, and we set for completeness $D_\epsilon = \emptyset$.

  It is to be noticed that, for each fixed couple $(i, j) \in \{1, \dots, d\}^2$

$$\cup_{\epsilon\in\tilde{\mathcal{E}}^{(i,j)}} D_\epsilon = \overline{\Omega},\ \ \mathrm{int}(D_\epsilon) \cap \mathrm{int}(D_{\epsilon'}) = \emptyset,\ \epsilon \neq \epsilon',\ \epsilon, \epsilon' \in \tilde{\mathcal{E}}^{(i,j)}. \tag{3.4}$$

  To any dual face $\epsilon \in \tilde{\mathcal{E}}^{(i)}$, we associate a distance d_ϵ

$$d_\epsilon = \begin{cases} d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}) \text{ if } \epsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\rm int}, \\ d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma\cap\epsilon}) \text{ if } \epsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\rm ext} \text{ with } \epsilon \in \tilde{\mathcal{E}}(D_\sigma) \text{ and } i \neq j, \\ d_\sigma \text{ if } \epsilon \in \tilde{\mathcal{E}}^{(i,i)} \cap \tilde{\mathcal{E}}^{(i)}_{\rm ext} \text{ with } \epsilon \in \tilde{\mathcal{E}}(D_\sigma). \end{cases} \tag{3.5}$$

  (We notice that the last line in the above definition is irrelevant and pure convention, since in that case $D_\epsilon = \emptyset$.)

- *We also define the size of the mesh by*

$$h_{\mathcal{M}} = \max\{h_K, K \in \mathcal{M}\} \tag{3.6}$$

where h_K stands for the diameter of K. Moreover if $K = [\overrightarrow{\sigma\sigma'}]$ where $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i = 1, \dots, d$ we will denote

$$h_K^{(i)} = \frac{|K|}{|\sigma|} = \frac{|K|}{|\sigma'|}. \tag{3.7}$$

We mesure the regularity of the mesh through the positive real number $\theta_{\mathcal{M}}$ defined by

$$\theta_{\mathcal{M}} = \min\{\frac{\xi_K}{h_K},\ K \in \mathcal{M}\} \tag{3.8}$$

where ξ_K stands for the diameter of the largest ball included in K. Finally, we denote by h_σ the diameter of the face $\sigma \in \mathcal{E}$.

- *Some geometric notions introduced in this definition are exposed in the figures 1 and 2:*

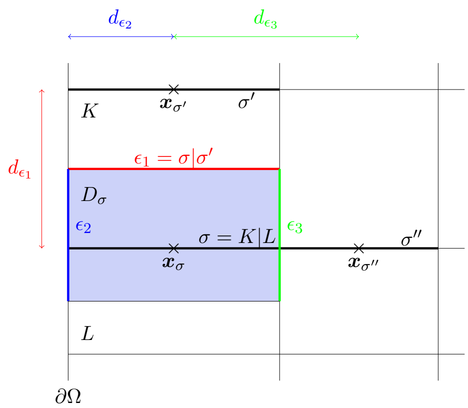

Figure 1: Notations for control volumes and dual cells

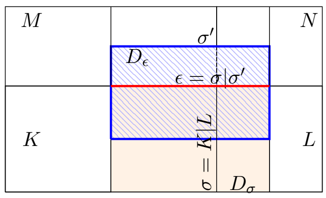

Figure 2: Notations for bi-dual cells

Remark 2. *For all $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, we have $h_\sigma \geq \xi_K \geq \theta_{\mathcal{M}} h_K$ and $h_\sigma \leq h_L$ and so $\theta_{\mathcal{M}} h_K \leq h_L \leq \frac{1}{\theta_{\mathcal{M}}} h_K$. Note also that for all $K \in \mathcal{M}$ and for all $\sigma \in \mathcal{E}(K)$, the inequality $h_\sigma|\sigma| \leq 2\frac{1}{\theta_{\mathcal{M}}^d}|K|$ holds and if $\sigma = K|L$ a rough estimate gives $|K| \leq (\frac{2}{\theta_{\mathcal{M}}})^d |L|$. These relations will be used throughout this paper.*

Definition 3.2 (Discrete spaces)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid in the sense of Definition 3.1. The discrete density and pressure space $L_{\mathcal{M}}$ is defined as the set of piecewise constant functions over each of the grid cells K of $\mathcal{M}$, and the discrete $i-th$ velocity space $H_{\mathcal{E}}^{(i)}$ as the set of piecewise constant functions over each of the grid cells D_σ ,$\sigma \in \mathcal{E}^{(i)}$. As in the continuous case, the Dirichlet boundary conditions*

(1.3) are (partly) incorporated in the definition of the velocity spaces, and, to this purpose, we introduce $H^{(i)}_{\mathcal{E},0} \subset H^{(i)}_{\mathcal{E}}, i = 1, \ldots, d$, defined as follows:

$$H^{(i)}_{\mathcal{E},0} = \Big\{ v \in H^{(i)}_{\mathcal{E}},\ v(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \widetilde{\mathcal{E}}^{(i)}_{\rm ext},\ i = 1, \ldots, d \Big\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H^{(i)}_{\mathcal{E},0}$. Since we are dealing with piecewise constant functions, it is useful to introduce the characteristic functions $\mathcal{X}_K, K \in \mathcal{M}$ and $\mathcal{X}_{D_\sigma}, \sigma \in \mathcal{E}$ of the density (or pressure) and velocity cells, defined by

$$\mathcal{X}_K(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in K, \\ 0 \text{ if } \boldsymbol{x} \notin K, \end{cases} \quad \mathcal{X}_{D_\sigma}(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in D_\sigma, \\ 0 \text{ if } \boldsymbol{x} \notin D_\sigma. \end{cases}$$

We can then write a function $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ as $\boldsymbol{v} = (v_1, \ldots, v_d)$ with $v_i = \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}} v_\sigma \mathcal{X}_{D_\sigma},\ i \in [\![1, d]\!]$ and a function $q \in L_{\mathcal{M}}$ as $q = \sum_{K \in \mathcal{M}} q_K \mathcal{X}_K$.

3.2 Time discretization

Let us now turn to the time discretization of Problem 1.1. We consider a partition $0 = t^0 < t^1 < \cdots < t^N = T$ of the time interval $(0, T)$, and, for the sake of simplicity, a constant time step $\delta t = t^n - t^{n-1}$; hence $t^n = n\delta t$ for $n \in \{0, \cdots, N\}$. We denote respectively by $\{u^n_\sigma, \sigma \in \mathcal{E}^{(i)}_{\rm int}, i \in \{1, \cdots, d\}, n \in \{0, \cdots, N\}\}$, and $\{\varrho^n_K, K \in \mathcal{M}, n \in \{1, \cdots, N\}\}$ the sets of discrete i-th component of velocity and density unknowns. For $\sigma \in \mathcal{E}^{(i)}_{\rm int}$, $i \in \{1, \cdots, d\}$ the value u^n_σ is an expected approximation of the mean value over $(t^{n-1}, t^n) \times D_\sigma$ of the i-th component of the velocity of a weak solution, while for $K \in \mathcal{M}$ the value ϱ^n_K is an expected approximation of the mean value over $(t^{n-1}, t^n) \times K$ of the density of a weak solution. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, which are expected approximation of weak solutions, For the velocities, these constant functions are of the form:

$$u_i(t, \boldsymbol{x}) = \sum_{n=1}^N \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}} u^n_\sigma\, \mathcal{X}_{D_\sigma}(\boldsymbol{x})\, \mathcal{X}_{(t^{n-1}, t^n)}(t),$$

where $\mathcal{X}_{(t^{n-1}, t^n)}$ is the characteristic function of the interval (t^{n-1}, t^n). We denote by $X_{i,\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells, and we set $\boldsymbol{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. For the density, the constant function is of the form:

$$\varrho(t, \boldsymbol{x}) = \varrho^n_K \text{ for } \boldsymbol{x} \in K \text{ and } t \in (t^{n-1}, t^n),$$

and we denote by $Y_{\mathcal{M},\delta t}$ the space of such piecewise constant functions.

For a given $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ associated to the set of discrete velocity unknowns $\{u^n_\sigma, \sigma \in \mathcal{E}^{(i)}_{\rm int}, i \in \{1, \cdots, d\}, n \in \{1, \cdots, N\}\}$, and for $n \in \{1, \cdots, N\}$, we denote by $u^n_i \in H^{(i)}_{\mathcal{E},0}$ the piecewise constant function defined by $u^n_i(\boldsymbol{x}) = u^n_\sigma$ for $\boldsymbol{x} \in D_\sigma, \sigma \in \mathcal{E}^{(i)}_{\rm int}$, and set $\boldsymbol{u}^n = (u^n_1, \ldots, u^n_d)^t \in \mathbf{H}_{\mathcal{E},0}$. We sometimes write $u^n_{i,\sigma}$ instead of u^n_σ in order to avoid all confusion. Notice that $u^n_{i,\sigma}$ may be non zero for $\sigma \in \mathcal{E}^{(i)}_{\rm int}$ while it is zero whenever $\sigma \in \mathcal{E}^{(i)}_{\rm ext}$. In the same way, given $\varrho \in Y_{\mathcal{M},\delta t}$ associated to the discrete density unknowns $\{\varrho^n_K, K \in \mathcal{M}, n \in \{1, \cdots, N\}\}$ we denote by $\varrho^n \in L_{\mathcal{M}}$ the piecewise constant function defined by $\varrho^n(\boldsymbol{x}) = \varrho^n_K$ for $\boldsymbol{x} \in K$, $K \in \mathcal{M}$.

Finally, the discrete initial condition $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ is such that $\varrho^0 > 0$ and the discrete initial total mass and energy are respectively defined by

$$M_{0,\mathcal{M}} = \int_\Omega \varrho^0 \,\mathrm{d}\boldsymbol{x}, \qquad \mathcal{E}_{0,\mathcal{M}} = \int_\Omega \frac{1}{2} \varrho^0 |\boldsymbol{u}^0|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathcal{H}(\varrho^0) \,\mathrm{d}\boldsymbol{x}. \tag{3.9}$$

3.3 The numerical scheme

We consider an implicit-in-time scheme, which reads in its fully discrete form, for $1 \leq n \leq N$ and $1 \leq i \leq d$:

$$\frac{1}{\delta t}(\varrho^n - \varrho^{n-1}) + \mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\varrho^n \boldsymbol{u}^n) = 0, \tag{3.10a}$$

$$\frac{1}{\delta t}(\widehat{\varrho^n}^{(i)} u_i^n - \widehat{\varrho^{n-1}}^{(i)} u_i^{n-1}) + \mathrm{div}_{\mathcal{E}}^{(i)}(\varrho^n \boldsymbol{u}^n u_i^n) - \mu \Delta_{\mathcal{E}}^{(i)} u_i^n \\ - (\mu+\lambda)\eth_i \,\mathrm{div}_{\mathcal{M}}\, \boldsymbol{u}^n + \eth_i p(\varrho^n) = 0, \tag{3.10b}$$

where the terms introduced for each discrete equation are defined hereafter.

3.3.1 Mass balance equation

Equation (3.10a) is a finite volume discretization of the mass balance ($1.1a$) over the primal mesh. The discrete "upwind" divergence is defined by

$$\mathrm{div}_{\mathcal{M}}^{\mathrm{up}} : \left| \begin{array}{l} L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow L_{\mathcal{M}} \\ (\varrho, \boldsymbol{u}) \longmapsto \mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\varrho \boldsymbol{u}) = \displaystyle\sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{\sigma,K}(\varrho, \boldsymbol{u})\; \mathcal{X}_K, \end{array} \right. \tag{3.11}$$

where $F_{\sigma,K}(\varrho, \boldsymbol{u})$ stands for the mass flux across σ outward K, which, because of the Dirichlet boundary conditions, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad F_{\sigma,K}(\varrho, \boldsymbol{u}) = |\sigma|\; \varrho_\sigma^{\mathrm{up}}\; u_{\sigma,K}, \tag{3.12}$$

where $u_{\sigma,K}$ is an approximation of the normal velocity to the face σ outward K, defined by:

$$u_{\sigma,K} = u_\sigma\; \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}_{\sigma,K} \text{ for } \sigma \in \mathcal{E}^{(i)} \cap \mathcal{E}(K). \tag{3.13}$$

Thanks to the boundary conditions, $u_{\sigma,K}$ vanishes for any external face σ. The density at the internal face $\sigma = K|L$ is obtained by an upwind technique:

$$\varrho_\sigma^{\mathrm{up}} = \left| \begin{array}{ll} \varrho_K & \text{if } u_{\sigma,K} \geq 0, \\ \varrho_L & \text{otherwise.} \end{array} \right. \tag{3.14}$$

Note that any solution $(\varrho^n, \boldsymbol{u}^n) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ to ($3.10a$) satisfy $\varrho_K^n > 0$, $\forall K \in \mathcal{M}$ provided $\varrho_K^{n-1} > 0$, $\forall K \in \mathcal{M}$ and in particular $p(\varrho^n)$ makes sense. The positivity of the density ϱ^n in ($3.10a$) is not enforced in the scheme but results from the above upwind choice. Indeed, for a given velocity field, the discrete mass balance ($3.10a$) is a linear system for ϱ^n the matrix of which is an invertible matrix with a non negative inverse [20, Lemma C.3].

Note also that, with this definition, we have the usual finite volume property of local conservativity of the flux through a primal face $\sigma = K|L$ *i.e.*

$$F_{\sigma,K}(\varrho, \boldsymbol{u}) = -F_{\sigma,L}(\varrho, \boldsymbol{u}). \tag{3.15}$$

Consequently, summing (3.10a) over $K \in \mathcal{M}$ immediately yields the total conservation of mass, which reads:

$$\forall n = 1, ...N, \quad \int_\Omega \varrho^n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho^0 \,\mathrm{d}\boldsymbol{x}, \tag{3.16}$$

which is nothing than a discrete version of (2.7).

3.3.2 The momentum equation

We now turn to the discrete momentum balances (3.10b), which are obtained by discretizing the momentum balance equation (1.1b) on the dual cells associated to the faces of the mesh.

The discrete convective operator - The discrete divergence of the convective term $\varrho \boldsymbol{u} \otimes \boldsymbol{u}$ is defined by

$$\mathrm{div}_{\mathcal{E}}^{\mathrm{up}} : \left| \begin{array}{l} L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ (\varrho, \boldsymbol{u}) \longmapsto \mathrm{div}_{\mathcal{E}}^{\mathrm{up}}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) = (\mathrm{div}_{\mathcal{E}}^{(1)}(\varrho \boldsymbol{u} u_1), ..., \mathrm{div}_{\mathcal{E}}^{(d)}(\varrho \boldsymbol{u} u_d)), \end{array} \right. \tag{3.17}$$

where for any $1 \leq i \leq d$, the i^{th} component of the above operator reads:

$$\mathrm{div}_{\mathcal{E}}^{(i)} : \left| \begin{array}{l} L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ (\varrho, \boldsymbol{u}) \longmapsto \mathrm{div}_{\mathcal{E}}^{(i)}(\varrho \boldsymbol{u} u_i) = \displaystyle\sum_{\sigma \in \mathcal{E}_{\mathrm{int}}^{(i)}} \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) \; u_\epsilon \; \mathcal{X}_{D_\sigma}, \end{array} \right. \tag{3.18}$$

the quantity $F_{\epsilon,\sigma} = F_{\epsilon,\sigma}(\varrho, \boldsymbol{u})$ stands for a mass flux through the dual faces of the mesh and u_ϵ stands for an approximation of i^{th} component of the velocity over ϵ are defined hereafter (see (3.19), (3.20), (3.25).

Let $\sigma \in \mathcal{E}^{(i)}$ (without loss of generality). The dual flux $F_{\epsilon,\sigma}(\varrho, \boldsymbol{u})$ is defined as follows

- If $\epsilon \in \tilde{\mathcal{E}}_{\mathrm{ext}}$ then $F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) = 0$.
  Otherwise:
- First case – The vector $\boldsymbol{e}^{(i)}$ is normal to ϵ, so ϵ is included in a primal cell K, and we denote by σ' the second face of K which, in addition to σ, is normal to $\boldsymbol{e}^{(i)}$. We thus have $\epsilon = D_\sigma | D_{\sigma'}$. Then the mass flux through ϵ is given by:

$$F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) = \frac{1}{2} \left[F_{\sigma,K}(\varrho, \boldsymbol{u}) \; \boldsymbol{n}_{\epsilon,D_\sigma} \cdot \boldsymbol{n}_{\sigma,K} + F_{\sigma',K}(\varrho, \boldsymbol{u}) \; \boldsymbol{n}_{\epsilon,D_\sigma} \cdot \boldsymbol{n}_{\sigma',K} \right]. \tag{3.19}$$

  where $\boldsymbol{n}_{\epsilon,D_\sigma}$ stands for the unit normal vector to ϵ outward D_σ.
- Second case – The vector $\boldsymbol{e}^{(i)}$ is tangent to ϵ, and ϵ is the union of the halves of two primal faces τ and τ' such that $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ϵ is then given by:

$$F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) = \frac{1}{2} \left[F_{\tau,K}(\varrho, \boldsymbol{u}) + F_{\tau',L}(\varrho, \boldsymbol{u}) \right]. \tag{3.20}$$

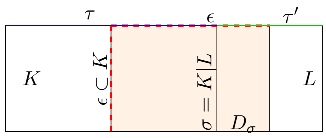

Figure 3: Notations for the dual fluxes of the first component of the velocity.

We notice that the sum over $\epsilon \in \tilde{\mathcal{E}}(D_\sigma)$ in formula (3.18) can be replaced by the sum over $\epsilon \in \tilde{\mathcal{E}}_{\mathrm{int}}(D_\sigma)$.

Note that, with this definition, we have the usual finite volume property of local conservativity of the flux through a dual face $\epsilon = D_\sigma | D_{\sigma'}$ *i.e.*

$$F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) = -F_{\epsilon,\sigma'}(\varrho, \boldsymbol{u}). \tag{3.21}$$

In what follows we shall often use the abbreviated notation

$$F_{\epsilon,\sigma}(\varrho^n, \mathbf{u}^n) = F_{\epsilon,\sigma}^n, \quad F_{\sigma,K}(\varrho^n, \mathbf{u}^n) = F_{\sigma,K}^n. \tag{3.22}$$

The density on a dual cell is given by:

$$\begin{array}{ll} \text{for } \sigma \in \mathcal{E}_{\mathrm{int}}, \; \sigma = K|L & |D_\sigma| \; \varrho_{D_\sigma} = |D_{\sigma,K}| \; \varrho_K + |D_{\sigma,L}| \; \varrho_L, \\ \text{for } \sigma \in \mathcal{E}_{\mathrm{ext}}, \; \sigma \in \mathcal{E}(K), & \varrho_{D_\sigma} = \varrho_K. \end{array} \tag{3.23}$$

and we denote

$$\text{for } 1 \le i \le d, \quad \widehat{\varrho}^{(i)} = \sum_{\sigma \in \mathcal{E}^{(i)}} \varrho_{D_\sigma} \mathcal{X}_{D_\sigma}.$$

These definitions of the dual mass fluxes and the dual densities ensures that a finite volume discretization of the mass balance equation over the diamond cells holds:

$$\forall 1 \le i \le d,\ \forall \sigma \in \mathcal{E}_{\rm int}^{(i)}, \qquad \frac{1}{\delta t}(\varrho_{D_\sigma}^n - \varrho_{D_\sigma}^{n-1}) + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}^n = 0. \tag{3.24}$$

This is a necessary condition to be able to derive a discrete kinetic energy balance (see Theorem 4.1).

Since the flux across a dual face lying on the boundary is zero, the values u_ϵ are only needed at the internal dual faces, and we make the centered choice for their discretization, i.e., for $\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}_{\rm int}^{(i)}$,

$$u_\epsilon \equiv u_{i,\epsilon} = \frac{u_\sigma + u_{\sigma'}}{2} \equiv \frac{u_{i,\sigma} + u_{i,\sigma'}}{2}. \tag{3.25}$$

Discrete divergence and gradient - The discrete divergence operator $\mathrm{div}_\mathcal{M}$ is defined by:

$$\mathrm{div}_\mathcal{M} : \left| \begin{array}{l} \mathbf{H}_\mathcal{E} \longrightarrow L_\mathcal{M} \\ \boldsymbol{u} \longmapsto \mathrm{div}_\mathcal{M}\, \boldsymbol{u} = \displaystyle\sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{\sigma,K}\ \mathcal{X}_K, \end{array} \right. \tag{3.26}$$

where $u_{\sigma,K}$ is defined in (3.13).

The discrete divergence of $\boldsymbol{u} = (u_1, \dots, u_d) \in \mathbf{H}_{\mathcal{E},0}$ may also be written as

$$\mathrm{div}_\mathcal{M}\, \boldsymbol{u} = \sum_{i=1}^d \sum_{K \in \mathcal{M}} (\eth_i u_i)_K \mathcal{X}_K, \tag{3.27}$$

where the discrete derivative $(\eth_i u_i)_K$ of u_i on K is defined by

$$(\eth_i u_i)_K = \frac{|\sigma|}{|K|}(u_{\sigma'} - u_\sigma) \text{ with } K = [\overrightarrow{\sigma\sigma'}], \sigma, \sigma' \in \mathcal{E}^{(i)}. \tag{3.28}$$

The gradient in the discrete momentum balance equation is defined as follows:

$$\boldsymbol{\nabla}_\mathcal{E} : \left| \begin{array}{l} L_\mathcal{M} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ p \longmapsto \boldsymbol{\nabla}_\mathcal{E} p \\ \boldsymbol{\nabla}_\mathcal{E} p(\boldsymbol{x}) = (\eth_1 p(\boldsymbol{x}), \dots, \eth_d p(\boldsymbol{x}))^t, \end{array} \right. \tag{3.29}$$

where $\eth_i p \in H_{\mathcal{E},0}^{(i)}$ is the discrete derivative of p in the i-th direction, defined by:

$$\eth_i p(\boldsymbol{x}) = \frac{|\sigma|}{|D_\sigma|}\ (p_L - p_K) \quad \forall \boldsymbol{x} \in D_\sigma,\ \text{for } \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}^{(i)},\ i = 1, \dots, d. \tag{3.30}$$

Note that in fact, the discrete gradient of a function of $L_\mathcal{M}$ should only defined on the internal faces, and does not need to be defined on the external faces; we set it here in $\mathbf{H}_{\mathcal{E},0}$ (that is zero on the external faces) for the sake of simplicity.

The gradient in the discrete momentum balance equation is built as the dual operator of the discrete divergence which means:

Lemma 3.1 (Discrete div $-\,\boldsymbol{\nabla}$ duality). *Let $q \in L_\mathcal{M}$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ then we have:*

$$\int_\Omega q\ \mathrm{div}_\mathcal{M} \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_\mathcal{E} q \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = 0. \tag{3.31}$$

Discrete Laplace operator - For $i = 1\dots,d$, we classically define the discrete Laplace operator on the i-th velocity grid by:

$$-\Delta_{\mathcal{E}}^{(i)} : \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ u_i \longmapsto -\Delta_{\mathcal{E}}^{(i)} u_i \end{array} \right.$$

$$-\Delta_{\mathcal{E}}^{(i)} u_i(\boldsymbol{x}) = \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} \phi_{\epsilon,\sigma}(u_i), \qquad \forall \boldsymbol{x} \in D_\sigma, \quad \text{for } \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \tag{3.32}$$

where $\tilde{\mathcal{E}}(D_\sigma)$ denotes the set of faces of D_σ, and

$$\phi_{\epsilon,\sigma}(u_i) = \begin{cases} \dfrac{|\epsilon|}{d_\epsilon}(u_\sigma - u_{\sigma'}) & \text{if } \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}_{\text{int}}^{(i)}, \\[2ex] \dfrac{|\epsilon|}{d_\epsilon} u_\sigma & \text{if } \epsilon \in \widetilde{\mathcal{E}}_{\text{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases} \tag{3.33}$$

where d_ϵ is defined by (3.5). Note that we have the usual finite volume property of local conservativity of the flux through an interface $\epsilon = \sigma\ |\sigma'$:

$$\phi_{\epsilon,\sigma}(u_i) = -\phi_{\epsilon,\sigma'}(u_i), \quad \forall \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}_{\text{int}}^{(i)}. \tag{3.34}$$

Then the discrete Laplace operator of the full velocity vector is defined by

$$\begin{aligned} -\Delta_{\mathcal{E}} :\ & \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ & \boldsymbol{u} \mapsto -\Delta_{\mathcal{E}} \boldsymbol{u} = (-\Delta_{\mathcal{E}}^{(1)} u_1, \dots, -\Delta_{\mathcal{E}}^{(d)} u_d)^t. \end{aligned} \tag{3.35}$$

Let us now recall the definition of the discrete H_0^1 inner product [12]; it is obtained by multiplying the discrete Laplace operator scalarly by a test function $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ and integrating over the computational domain. A simple reordering of the sums (which may be seen as a discrete integration by parts) yields, thanks to the conservativity of the diffusion flux (3.34):

$$\begin{aligned} &\forall (\boldsymbol{u},\boldsymbol{v}) \in {\mathbf{H}_{\mathcal{E},0}}^2, \qquad \int_\Omega -\Delta_{\mathcal{E}} \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = [\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0} = \sum_{i=1}^d [u_i, v_i]_{1,\mathcal{E}^{(i)},0}, \\ &\text{with } [u_i,v_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon}\ (u_\sigma - u_{\sigma'})\ (v_\sigma - v_{\sigma'}) + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\text{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon}\ u_\sigma\ v_\sigma \end{aligned} \tag{3.36}$$

The bilinear forms $\left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \times H_{\mathcal{E},0}^{(i)} \to \mathbb{R} \\ (u,v) \mapsto [u_i,v_i]_{1,\mathcal{E}^{(i)},0} \end{array} \right.$ and $\left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0} \to \mathbb{R} \\ (\boldsymbol{u},\boldsymbol{v}) \mapsto [\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0} \end{array} \right.$ are inner products on $H_{\mathcal{E},0}^{(i)}$ and $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following discrete H_0^1 norms:

$$\|u_i\|_{1,\mathcal{E}^{(i)},0}^2 = [u_i,u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\text{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon}\ (u_\sigma - u_{\sigma'})^2 + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\text{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon}\ u_\sigma^2 \text{ for } \ i = 1,...,d, \tag{3.37a}$$

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0}^2 = [\boldsymbol{u},\boldsymbol{u}]_{1,\mathcal{E},0} = \sum_{i=1}^d \|u_i\|_{1,\mathcal{E}^{(i)},0}^2. \tag{3.37b}$$

Since we are working on Cartesian grids, this inner product may be formulated as the L^2 inner product of discrete gradients. Indeed, consider the following discrete gradient of each velocity component u_i.

$$\nabla_{\mathcal{E}^{(i)}} u_i = (\eth_1 u_i, \dots, \eth_d u_i) \text{ with } \eth_j u_i = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)} \\ \epsilon \perp \boldsymbol{e}^{(j)}}} (\eth_j u_i)_{D_\epsilon}\ \mathcal{X}_{D_\epsilon}, \tag{3.38}$$

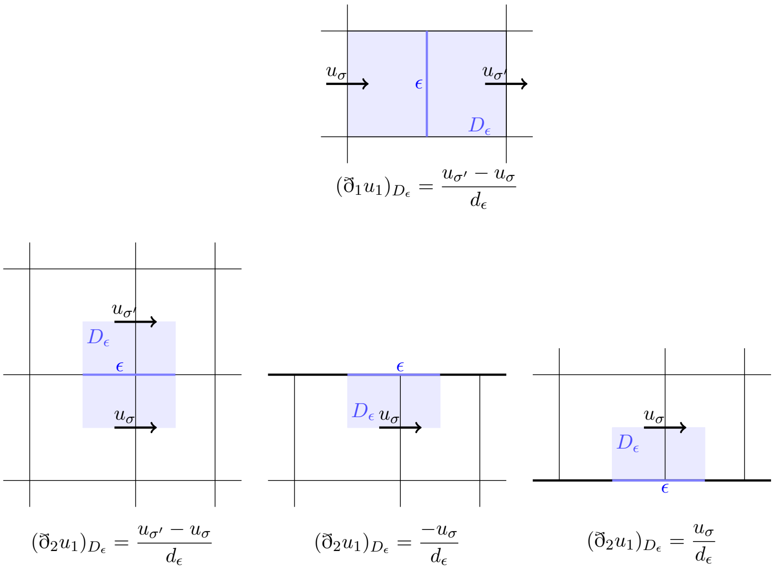

Figure 4: Notations for the definition of the partial space derivatives of the first component of the velocity, in two space dimensions.

where the elements D_ϵ of the bi-dual grid are defined in (3.4) (see also Figure 2) and

$$(\eth_j u_i)_{D_\epsilon} = \begin{cases} \dfrac{u_{\sigma'} - u_\sigma}{d_\epsilon} \text{ with } \epsilon = \overrightarrow{\sigma|\sigma'}, \\[2ex] -\dfrac{u_\sigma}{d_\epsilon} \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\epsilon, D_\sigma} \text{ with } \epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \cap \tilde{\mathcal{E}}(D_\sigma), \end{cases} \tag{3.39}$$

where $\boldsymbol{n}_{\epsilon,D_\sigma}$ stands for the unit normal vector to ϵ outward D_σ, see Figure 4. This definition is compatible with the definition of the discrete derivative $(\eth_i u_i)_K$ given by (3.28), since, if $\epsilon \subset K$ then $D_\epsilon = K$. Note that the second line in (3.39) is zero provide $i = j$, or provided $\sigma \in \mathcal{E}^{(i)}_{\text{ext}}$, $\epsilon \perp \mathbf{e}^{(j)}$ with $i \neq j$. With this definition, it is easily seen that

$$\int_\Omega \nabla_{\mathcal{E}^{(i)}} u \cdot \nabla_{\mathcal{E}^{(i)}} v \,\mathrm{d}\boldsymbol{x} = [u, v]_{1,\mathcal{E}^{(i)},0}, \forall u, v \in H^{(i)}_{\mathcal{E},0}, \forall i = 1, \ldots, d. \tag{3.40}$$

where $[u, v]_{1,\mathcal{E}^{(i)},0}$ is the discrete H^1_0 inner product defined by (3.36). We may then define

$$\nabla_{\mathcal{E}} \boldsymbol{u} = (\nabla_{\mathcal{E}^{(1)}} u_1, \ldots, \nabla_{\mathcal{E}^{(d)}} u_d),$$

so that

$$\int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u} : \nabla_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0}.$$

We will need discrete Sobolev inequalites for the discrete approximations. The following is proved in [12].

Theorem 3.1 (Discrete Sobolev inequalities)**.** *Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2$ or $d = 3$, adapted to the MAC-scheme (that is any finite union of rectangles in 2D or rectangular in 3D), and let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of Ω. Let $q < +\infty$ if $d = 2$ and $q = 6$ if $d = 3$.Then there exists $c = c(q, |\Omega|, \theta_\mathcal{M})$ depending on $\theta_\mathcal{M}$ in a nonincreasing way such that, for all $u \in \mathbf{H}_{\mathcal{E},0}$,*

$$\|\boldsymbol{u}\|_{L^q(\Omega)} \leq c \|\boldsymbol{u}\|_{1,\mathcal{E},0}.$$

3.4 Projection operators

In this section we introduce several projection operators. We first define the mean-value interpolator over $L_{\mathcal{M}}$:

$$\mathcal{P}_{\mathcal{M}}: \quad \left| \begin{array}{lll} L^1_{\mathrm{loc}}(\Omega) & \longrightarrow & L_{\mathcal{M}} \\ \varphi & \mapsto & \mathcal{P}_{\mathcal{M}}\varphi = \displaystyle\sum_{K\in\mathcal{M}} \varphi_K \ \mathcal{X}_K, \end{array} \right. \tag{3.41}$$

with

$$\varphi_K = \frac{1}{|K|}\int_K \varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \ \forall K \in \mathcal{M}. \tag{3.42}$$

This operator satisfies for any $1 \le p \le \infty$ and for any $r \in L^p(\Omega)$,

$$\| \mathcal{P}_{\mathcal{M}}\, r\|_{L^p(\Omega)} \le \|r\|_{L^p(\Omega)}. \tag{3.43}$$

We also define over $H^{(i)}_{\mathcal{E},0}$ the following interpolation operator $\mathcal{P}^{(i)}_{\mathcal{E}}$:

$$\mathcal{P}^{(i)}_{\mathcal{E}}: \quad \left| \begin{array}{lll} H^1_0(\Omega) & \longrightarrow & H^{(i)}_{\mathcal{E},0} \\ \varphi & \mapsto & \mathcal{P}^{(i)}_{\mathcal{E}}\varphi = \displaystyle\sum_{\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}} \varphi_\sigma \ \mathcal{X}_{D_\sigma}, \end{array} \right. \tag{3.44}$$

with

$$\varphi_\sigma = \frac{1}{|\sigma|}\int_\sigma \varphi(\boldsymbol{x})\mathrm{d}\gamma(\boldsymbol{x}), \ \forall \sigma \in \mathcal{E}^{(i)}_{\mathrm{int}}, \tag{3.45}$$

and we denote

$$\mathcal{P}_{\mathcal{E}} = (\mathcal{P}^{(1)}_{\mathcal{E}}, ..., \mathcal{P}^{(d)}_{\mathcal{E}}) \in \mathcal{L}(H^1_0(\Omega)^d, \mathbf{H}_{\mathcal{E},0}) \tag{3.46}$$

the vector valued extension. This operator preserves the divergence in the following sense:

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \forall q \in L_{\mathcal{M}}, \int_\Omega q \ \mathrm{div}_{\mathcal{M}}\, \mathcal{P}_{\mathcal{E}}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega q \ \mathrm{div}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}. \tag{3.47}$$

This operator satisfies for any $1 \le p \le \infty$ and for any $\boldsymbol{U} \in W^{1,p}_0(\Omega)^d$,

$$\| \mathcal{P}_{\mathcal{E}}\, \boldsymbol{U}\|_{L^p(\Omega)^d} \le c\|\boldsymbol{U}\|_{L^p(\Omega)^d}. \tag{3.48}$$

In the following Lemma we recall some classical mean value inequalities. These inequalities are used to obtain estimates involving the projector operators $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_{\mathcal{E}}$ previously defined.

Lemma 3.2. *[Mean value inequalities] Let D be an open bounded convex subset of $\mathbb{R}^d$, $d \ge 1$. Let $\sigma \subset \partial D$ such that $|\sigma| > 0$. Let $1 \le p \le \infty$. There exists c only depending on d and p such that $\forall v \in C^1(\overline{D})$,*

$$||v - v_\sigma||_{L^p(D)} \le c \ \mathrm{diam}(D)\|\nabla v\|_{L^p(D)^d}, \tag{3.49}$$

$$||v - v_D||_{L^p(D)} \le c \ \mathrm{diam}(D)\|\nabla v\|_{L^p(D)^d}, \tag{3.50}$$

where $v_D = \frac{1}{|D|}\int_D v\,\mathrm{d}\boldsymbol{x}$ and $v_\sigma = \frac{1}{|\sigma|}\int_\sigma v\,\mathrm{d}\,\gamma(\boldsymbol{x})$ ($\mathrm{d}\,\gamma(\boldsymbol{x})$ is the $d-1$-Lebesgue measure on σ). Moreover if $v \in C^2(\overline{\Omega})$ then

$$|v_D - v(\boldsymbol{x}_D)| \le \|\nabla^2 v\|_{L^\infty(\Omega)^{d\times d}}\,\mathrm{diam}(D)^2, \tag{3.51}$$

where $\boldsymbol{x}_D$ stands for the center of mass of D and

$$|v(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}) - \frac{1}{2}(v(\boldsymbol{x}) + v(\boldsymbol{y}))| \le \frac{1}{8}\|\nabla^2 v\|_{L^\infty(\Omega)^{d\times d}}|\boldsymbol{x}-\boldsymbol{y}|^2, \forall \boldsymbol{x}, \boldsymbol{y} \in \overline{D}. \tag{3.52}$$

Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ a MAC grid of the computational domain Ω, let $1 \le p \le \infty$. Then there exsits c only depending on p and on d such that for any $(r, \boldsymbol{U}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})^d$, $\boldsymbol{U}_{|\partial\Omega} = \boldsymbol{0}$,

$$||r - \mathcal{P}_{\mathcal{M}}\, r||_{L^p(\Omega)} \le ch_{\mathcal{M}}\|\nabla v\|_{L^p(\Omega)^d}, \tag{3.53}$$

$$||\boldsymbol{U} - \mathcal{P}_{\mathcal{E}}\, \boldsymbol{U}||_{L^p(\Omega)^d} \le ch_{\mathcal{M}}\|\nabla \boldsymbol{U}\|_{L^p(\Omega)^{d\times d}}, \tag{3.54}$$

$$\|\nabla_{\mathcal{E}}\, \mathcal{P}_{\mathcal{E}}\, \boldsymbol{U}\|_{L^p(\Omega)^{d\times d}} \le c\|\nabla \boldsymbol{U}\|_{L^p(\Omega)^{d\times d}}, \tag{3.55}$$

Moreover if $\boldsymbol{U} \in C^2(\overline{\Omega})^d$,

$$\|\eth_j \mathcal{P}_{\mathcal{E}} U_i - \partial_j U_i\|_{L^\infty(\Omega)} \le c\|\nabla^2 \boldsymbol{U}_i\|_{L^\infty(\Omega)^{d\times d}}. \tag{3.56}$$

Proof. Let us prove (3.50). By virtue of the connexity of $\overline{D}$ there exists $\boldsymbol{x}_0 \in \overline{D}$ such that

$$v_D = v(\boldsymbol{x}_0).$$

Let $(\boldsymbol{x}_n)_{n\in\mathbb{N}} \in D^{\mathbb{N}}$ be a sequence such that $\boldsymbol{x}_n \to \boldsymbol{x}_0$ as $n \to \infty$. Using the convexity of D we obtain

$$v(\boldsymbol{x}) - v(\boldsymbol{x}_n) = \int_0^1 \nabla v(t\boldsymbol{x} + (1-t)\boldsymbol{x}_n) \cdot (\boldsymbol{x} - \boldsymbol{x}_n)\,\mathrm{d}\boldsymbol{x},\ \forall \boldsymbol{x} \in D.$$

If $p = +\infty$ we have for any $\boldsymbol{x} \in D$,

$$|v(\boldsymbol{x}) - v_D| = \lim_{n\to\infty} |v(\boldsymbol{x}) - v(\boldsymbol{x}_n)| \le \operatorname{diam}(D)\|\nabla \boldsymbol{v}\|_{L^\infty(D)^d}.$$

which gives (3.50). If $1 \le p < \infty$, using Jensen inequality and Fubini theorem we obtain

$$\int_D |v(\boldsymbol{x}) - v(\boldsymbol{x}_n)|^p\,\mathrm{d}\boldsymbol{x} \le \int_D \int_0^1 \|\nabla v(t\boldsymbol{x} + (1-t)\boldsymbol{x}_n)\|^p \|\boldsymbol{x} - \boldsymbol{x}_n\|^p\,\mathrm{d}t\ \mathrm{d}\boldsymbol{x}$$
$$\le \operatorname{diam}(D)^p \int_1^{+\infty} \frac{1}{t^{d+2}}\,\mathrm{d}t \int_D \|\nabla v(\boldsymbol{x})\|^p\,\mathrm{d}t\ \mathrm{d}\boldsymbol{x} \le c\,\operatorname{diam}(D)^p \|\nabla \boldsymbol{v}\|^p_{L^p(D)^d}.$$

which also gives (3.50). The proof of (3.49) is similar. Let us prove (3.51). We have

$$v_D - v(\boldsymbol{x}_D) = \int_0^1 \nabla v(t\boldsymbol{x}_0 + (1-t)\boldsymbol{x}_D)\cdot(\boldsymbol{x}_0 - \boldsymbol{x}_D)\,\mathrm{d}\boldsymbol{x} = \int_0^1 (\nabla v(t\boldsymbol{x}_0 + (1-t)\boldsymbol{x}_D) - \nabla v(\boldsymbol{x}_D))\cdot(\boldsymbol{x}_0 - \boldsymbol{x}_D)\,\mathrm{d}t$$
$$= \int_0^1 \int_0^1 t(\nabla^2 v(zt\boldsymbol{x}_0 + z(1-t)\boldsymbol{x}_D + (1-z)\boldsymbol{x}_D)(\boldsymbol{x}_0 - \boldsymbol{x}_D)) \cdot (\boldsymbol{x}_0 - \boldsymbol{x}_D)\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}t\,.$$

Consequently

$$|v_D - v(\boldsymbol{x}_D)| \le \|\nabla^2 v\|_{L^\infty(D)^{d\times d}} \operatorname{diam}(D)^2. \tag{3.57}$$

Let us prove (3.52). A Taylor expansion of the function $t \to \boldsymbol{v}(t\boldsymbol{x} + (1-t)\boldsymbol{y})$ gives

$$v(\boldsymbol{x}) = v(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}) + \nabla v(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}) \cdot \frac{\boldsymbol{x}-\boldsymbol{y}}{2} + \frac{1}{8}(\nabla^2 v(\xi)(\boldsymbol{x}-\boldsymbol{y})) \cdot (\boldsymbol{x}-\boldsymbol{y}).$$
$$v(\boldsymbol{y}) = v(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}) + \nabla v(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}) \cdot \frac{\boldsymbol{y}-\boldsymbol{x}}{2} + \frac{1}{8}(\nabla^2 v(\tilde{\xi})(\boldsymbol{x}-\boldsymbol{y})) \cdot (\boldsymbol{x}-\boldsymbol{y}).$$

where $\xi, \tilde{\xi} \in [\boldsymbol{x}, \boldsymbol{y}]$. The expected result follows from the summation of the two previous identity. The proof of (3.53) and (3.54) are trivial consequences of (3.49) and (3.50). The proof of (3.55) is consequence of (3.49). Let us prove (3.56). By virtue of (3.4) one has

$$\|\eth_j \mathcal{P}_{\mathcal{E}} U_i - \partial_j U_i\|_{L^\infty(\Omega)} \le \max_{\epsilon\in\tilde{\mathcal{E}}^{(i,j)}} \|\eth_j \mathcal{P}_{\mathcal{E}} U_i - \partial_j U_i\|_{L^\infty(D_\epsilon)}.$$

Moreover by virtue of (3.51) and (3.52) we can write for $\epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}^{(i)}_{\text{int}},\ \epsilon \perp \boldsymbol{e}^{(j)},\ \boldsymbol{x} \in D_\epsilon$,

$$\eth_j(\mathcal{P}^{(i)}_{\mathcal{E}} U_i)_{D_\epsilon} - \frac{\partial}{\partial x_j} U_i(\boldsymbol{x}) = \frac{1}{d_\epsilon}(U_i(\boldsymbol{x}_{\sigma'}) - U_i(\boldsymbol{x}_\sigma)) - \frac{\partial}{\partial x_j} U_i(\boldsymbol{x}) + R_\epsilon$$
$$= \frac{\partial}{\partial x_j} U_i^n(\boldsymbol{x}_{\sigma,\sigma'}) - \frac{\partial}{\partial x_j} U_i(\boldsymbol{x}) + R^n_\epsilon$$

where $\boldsymbol{x}_{\sigma,\sigma'} \in [\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}]$ and where the remainder R^n_ϵ satisfies

$$|R_\epsilon| \le c h_{\mathcal{M}}.$$

Note that the case $\epsilon \in \tilde{\mathcal{E}}^{(i)}_{\text{ext}}$ can be treated in the same way. Consequently we have inequality

$$\|\eth_j \mathcal{P}_{\mathcal{E}} U_i - \partial_j U_i\|_{L^\infty(D_\epsilon)} \le c h_{\mathcal{M}},\ \forall (i,j) \in \{1,2,3\}^2, \forall \epsilon \in \tilde{\mathcal{E}}^{(i)}_{\text{int}},\ \epsilon \perp \boldsymbol{e}^{(j)},$$

where the constant c depends on $\|\nabla^2 \boldsymbol{U}\|_{L^\infty(\Omega)}$ and we obtain the expected result. □

In the following defintion, we introduce two velocity interpolators.

Definition 3.3 (Velocity interpolators). *For a given MAC gris $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, we define, for $i, j = 1, ..., d$, the full grid velocity reconstruction operator with respect to (i, j) by*

$$\begin{aligned} \mathcal{R}_{\mathcal{E}}^{(i,j)} :\ & H_{\mathcal{E},0}^{(i)} \to H_{\mathcal{E},0}^{(j)} \\ & v \mapsto \mathcal{R}_{\mathcal{E}}^{(i,j)} v = \sum_{\sigma \in \mathcal{E}_{\rm int}^{(j)}} (R_{\mathcal{E}}^{(i,j)} v)_\sigma \mathcal{X}_{D_\sigma}, \end{aligned} \tag{3.58}$$

where

$$(R_{\mathcal{E}}^{(i,j)} v)_\sigma = v_\sigma \text{ if } \sigma \in \mathcal{E}_{\rm int}^{(i)}, \quad (R_{\mathcal{E}}^{(i,j)} v)_\sigma = \frac{1}{\operatorname{card}(\mathcal{N}_\sigma)} \sum_{\sigma' \in \mathcal{N}_\sigma} v_{\sigma'} \text{ otherwise,} \tag{3.59}$$

$$\text{where, for any } \sigma \in \mathcal{E}_{\rm int} \setminus \mathcal{E}_{\rm int}^{(i)},\ \mathcal{N}_\sigma = \{\sigma' \in \mathcal{E}^{(i)}, D_\sigma \cap \sigma' \neq \emptyset\}. \tag{3.60}$$

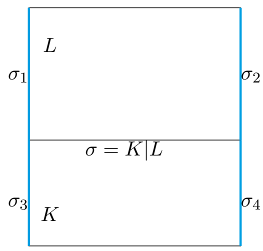

Figure 5: Set $\mathcal{N}_\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ with $\sigma \in \mathcal{E}_{\rm int}^{(j)}(K)$, $j \neq i$ in two dimensions ($i = 1$, $j = 2$)

For any $i = 1, ...d$, we also define a projector from $H_{\mathcal{E}}^{(i)}$ into $L_{\mathcal{M}}$ by

$$\begin{aligned} \mathcal{R}_{\mathcal{M}}^{(i)} :\ & H_{\mathcal{E}}^{(i)} \to L_{\mathcal{M}} \\ & v \mapsto \mathcal{R}_{\mathcal{M}}^{(i)} v = \sum_{K \in \mathcal{M}} (\mathcal{R}_{\mathcal{M}}^{(i)} v)_K \mathcal{X}_K, \end{aligned} \tag{3.61}$$

where

$$(\mathcal{R}_{\mathcal{M}}^{(i)} v)_K = \frac{1}{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} v_\sigma. \tag{3.62}$$

We then define

$$\begin{aligned} \mathcal{R}_{\mathcal{M}} :\ & \mathbf{H}_{\mathcal{E}} \to L_{\mathcal{M}}^d \\ & \boldsymbol{v} = (v_1, ...v_d) \mapsto \mathcal{R}_{\mathcal{M}} \boldsymbol{v} = (\mathcal{R}_{\mathcal{M}}^{(1)} v_1, ..., \mathcal{R}_{\mathcal{M}}^{(d)} v_d). \end{aligned} \tag{3.63}$$

Lemma 3.3. *Then there exists $c > 0$, depending only on d, p such that for any $1 \leq p < \infty$, for any $i, j = 1, ..., d$ and for any $v \in H_{\mathcal{E},0}^{(i)}$,*

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j)} v - v\|_{L^p(\Omega)} \leq c h_{\mathcal{M}} \|\nabla_{\mathcal{E}^{(i)}} v\|_{L^p(\Omega)}. \tag{3.64}$$

Then there exists $c > 0$ such that for any $i = 1, ...d$, , for any $v \in H_{\mathcal{E},0}^{(i)}$ and for any $1 \leq p < \infty$ one has

$$\| \mathcal{R}_{\mathcal{M}}^{(i)} v - v\|_{L^p(\Omega)} \leq c h_{\mathcal{M}} \|\eth_i v\|_{L^p(\Omega)}. \tag{3.65}$$

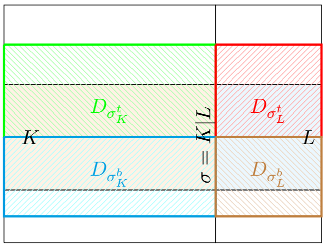

Figure 6: Full grid velocity interpolate.

Proof. Let us prove (3.65). From the definition of $\mathcal{R}_{\mathcal{M}}^{(i)}$ (see (3.61)) we have

$$\| \mathcal{R}_{\mathcal{M}}^{(i)} v - v\|^p_{L^p(\Omega)} = \sum_{K\in\mathcal{M}} \| \mathcal{R}_{\mathcal{M}}^{(i)} v - v\|^p_{L^p(K)} = \sum_{\substack{K=[\overrightarrow{\sigma\sigma'}]\\ \sigma,\sigma'\in\mathcal{E}^{(i)}}} \Big(\| \mathcal{R}_{\mathcal{M}}^{(i)} v - v\|^p_{L^p(D_{\sigma,K})} + \| \mathcal{R}_{\mathcal{M}}^{(i)} v - v\|^p_{L^p(D_{\sigma',K})}\Big)$$
$$\leq \frac{1}{2^p} h^p_{\mathcal{M}} \sum_{K\in\mathcal{M}} \|\eth_i v\|^p_{L^p(K)} \leq c h^p_{\mathcal{M}} \|\eth_i v\|^p_{L^p(\Omega)}$$

which gives the expected result.

Let us now prove (3.64). For the sake of simplicity we assume that $d = 2$, $i = 1$ and $j = 2$. Other cases are similar. First we write

$$\mathcal{R}_{\mathcal{E}}^{(i,j)} v - v = \mathcal{R}_{\mathcal{E}}^{(i,j)} v - \mathcal{R}_{\mathcal{M}}^{(i)} v + \mathcal{R}_{\mathcal{M}}^{(i)} v - v$$

The second term in the right hand side of the previous equality is estimated uisng (3.65). Now using the decomposition of D_σ established in figure 7 we can write

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j)} v - \mathcal{R}_{\mathcal{M}}^{(i)} v\|^p_{L^p(\Omega)} = \sum_{\sigma\in\mathcal{E}^{(j)}} \|\mathcal{R}_{\mathcal{E}}^{(i,j)} v - \mathcal{R}_{\mathcal{M}}^{(i)} v\|^p_{L^p(D_\sigma)}$$
$$= \sum_{\sigma\in\mathcal{E}^{(j)}} \|\mathcal{R}_{\mathcal{E}}^{(i,j)} v - \mathcal{R}_{\mathcal{M}}^{(i)} v\|^p_{L^p(D^l_\sigma)} + \|\mathcal{R}_{\mathcal{E}}^{(i,j)} v - \mathcal{R}_{\mathcal{M}}^{(i)} v\|^p_{L^p(D^r_\sigma)}$$
$$\leq c h^p_{\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}} (\|\eth_j v\|^p_{L^p(D_{\sigma^l})} + \|\eth_j v\|^p_{L^p(D_{\sigma^r})}) \leq c h^p_{\mathcal{M}} \|\nabla_{\mathcal{E}^{(i)}} v\|^p_{L^p(\Omega)}.$$

This finishes the proof of Lemma 3.3.

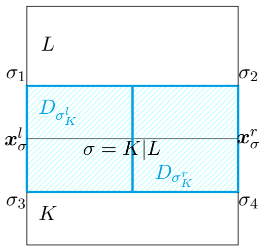

Figure 7: Decomposition of D_σ

□

The following algebraic identity is used to transform some terms involving the dual fluxes into terms involving primal fluxes for which the expression is easier to manipulate.

Lemma 3.4. *Let $\varrho \in L_{\mathcal{M}}$ and $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$. Let $i \in \{1, \ldots, d\}$ Let $\varphi = \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}} \varphi_\sigma \chi_{D_\sigma} \in H^{(i)}_{\mathcal{E},0}$ be a discrete scalar function. Let the primal fluxes be given by (3.12) and let the dual fluxes $F_{\epsilon,\sigma}$ be given by (3.19) or (3.20) (depending on the direction of $\mathbf{n}_\epsilon$ with respect to $\mathbf{e}^{(i)}$). Then we have:*

$$\sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma} u_\epsilon \varphi_\sigma = \sum_{K \in \mathcal{M}} (\mathcal{R}^{(i)}_{\mathcal{M}} \varphi)_K \sum_{j=1}^{d} \sum_{\sigma \in \mathcal{E}^{(j)}_{\rm int}(K)} F_{\sigma,K} (\mathcal{R}^{(i,j)}_{\mathcal{E}} u_i)_\sigma + R^i(u_i, \varphi)$$

where

$$\begin{aligned} R^i(u_i,\varphi) = & \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}(K)} (\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}} \varphi)_K) F_{\sigma,K} (u_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}} u_i)_K) \\ & + \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}(K)} (\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}} \varphi)_K) \sum_{j=1, j \neq i}^{d} \sum_{\tau \in \mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma' \in \mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \Big(\frac{u_{i,\sigma} + u_{i,\sigma'}}{2} - (\mathcal{R}^{(i)}_{\mathcal{M}} u_i)_K \Big). \end{aligned}$$

In the last sum we have denoted

$$\mathcal{N}_{\tau,\sigma} = \{\sigma' \in \mathcal{E}^{(i)} \,|\, {\rm int}_{d-1} \tau \cap {\rm int}_{d-1}(D_\sigma | D_{\sigma'}) \neq \emptyset\},$$

where $\sigma \in \mathcal{E}^{(i)}_{\rm int}(K)$, $\tau \in \mathcal{E}^{(j)}_{\rm int}(K)$, $j \neq i$, ${\rm int}_{d-1}$ means interior in the topology of $\mathbb{R}^{d-1}$.

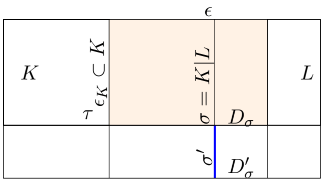

Figure 8: Set $\mathcal{N}_{\tau,\sigma} = \{\sigma'\}$ with $\tau \in \mathcal{E}^{(j)}_{\rm int}(K)$, $\sigma \in \mathcal{E}^{(i)}(K)$, $j \neq i$ in two dimensions ($i = 1$, $j = 2$)

Proof. We split the sum at the left hand side of the identity in Lemma 3.4 to two sums, first one over faces ϵ parallel to faces σ and second one over faces ϵ orthogonal to sigma:

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}} \varphi_\sigma \Bigg[\sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma} u_{i,\epsilon} \Bigg] &= \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}} \varphi_\sigma \Bigg[\sum_{\epsilon \in \tilde{\mathcal{E}}_{\rm int}(D_\sigma)} F_{\epsilon,\sigma} u_{i,\epsilon} \Bigg] \\ = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}(K)} \varphi_\sigma \Big[& \sum_{\epsilon \in \tilde{\mathcal{E}}_{\rm int}(D_\sigma), \epsilon \in K} F_{\epsilon,\sigma}\ u_{i,\epsilon} + \sum_{j=1, j\neq i}^{d} \sum_{\tau \in \mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\epsilon \in \tilde{\mathcal{N}}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} u_{i,\epsilon} \Big] \end{aligned}$$

where we have used definition (3.20) of $F_{\epsilon,\sigma}$ and denoted for fixed $\sigma \in \mathcal{E}^{(i)}_{\rm int}(K)$ and $\tau \in \mathcal{E}_{\rm int}(K) \setminus \mathcal{E}^{(i)}_{\rm int}(K)$.

$$\tilde{\mathcal{N}}_{\tau,\sigma} = \{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \,|\, \epsilon \perp \sigma, \emptyset \neq \epsilon \cap K \subset \tau\}.$$

For further calculation we notice that

$$\mathcal{N}_\tau = \cup_{\sigma \in \mathcal{E}^{(i)}(K)} \mathcal{N}_{\tau,\sigma} \cup \mathcal{E}^{(i)}(K), \ \ \tilde{\mathcal{N}}_{\tau,\sigma} = \{D_\sigma | D_{\sigma'} \,|\, \sigma' \in \mathcal{N}_{\tau,\sigma}\}, \tag{3.66}$$

where $\mathcal{N}_\tau$ is defined in (3.60).

Realizing that the set $\{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) | \epsilon \in K\}$ contains exactly one face denoted ϵ_K (see Figure 8), we rewrite the above expression using definitions (3.25), (3.66) as follows

$$\sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}} \varphi_\sigma \Bigg[\sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma} u_{i,\epsilon} \Bigg] = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}^{(i)}_{\rm int}(K)} \varphi_\sigma \Big[F_{\epsilon_K,\sigma}\ u_{i,\epsilon_K} + \sum_{j=1, j\neq i}^{d} \sum_{\tau \in \mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma' \in \mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \frac{u_{i,\sigma} + u_{i,\sigma'}}{2} \Big]$$

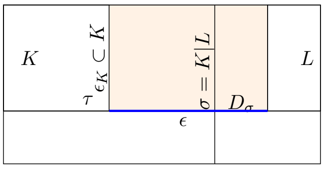

Figure 9: Set $\tilde{\mathcal{N}}_{\tau,\sigma} = \{\epsilon\}$ with $\tau \in \mathcal{E}^{(j)}_{\rm int}(K)$, $\sigma \in \mathcal{E}^{(i)}(K)$, $j \neq i$ in two dimnsions ($i=1$, $j=2$)

$$= \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \varphi_\sigma \Big[F_{\epsilon_K,\sigma}\; u_{i,\epsilon_K} + F_{\sigma,K} u_{i,\sigma} + \sum_{j=1,j\neq i}^{d} \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \frac{u_{i,\sigma}+u_{i,\sigma'}}{2}\Big]$$

$$= \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K \Big[F_{\sigma,K}\; u_{i,\sigma} + \sum_{j=1,j\neq i}^{d} \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \frac{u_{i,\sigma}+u_{i,\sigma'}}{2}\Big]$$

$$+ \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \Big[\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K\Big] \Big[F_{\epsilon_k,\sigma} u_{i,\epsilon_K} + F_{\sigma,K} u_{i,\sigma} + \sum_{j=1,j\neq i}^{d} \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \frac{u_{i,\sigma}+u_{i,\sigma'}}{2}\Big].$$

In the above we have used the conservation (3.15) of primal fluxes (which gives in particular $\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \varphi_\sigma F_{\sigma,K} u_{i,\sigma} = 0$) to pass from the first to the second expression, and the conservation of (3.21) of dual fluxes (in particular $\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K F_{\epsilon_K,\sigma} u_{i,\epsilon_K} = 0$). Consequently,

$$\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}} \varphi_\sigma \left[\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma} u_{i,\epsilon}\right] = \sum_{K\in\mathcal{M}} (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K I_i^K + \sum_{K\in\mathcal{M}} J_i^K, \tag{3.67}$$

where

$$I_i^K := \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} F_{\sigma,K}\; u_{i,\sigma} + \sum_{j=1,j\neq i}^{d} \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \frac{u_{i,\sigma}+u_{i,\sigma'}}{2}$$

and

$$J_i^K := \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \Big[\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K\Big] \Big[F_{\epsilon_k,\sigma} u_{i,\epsilon_K} + F_{\sigma,K} u_{i,\sigma} + \sum_{j=1,j\neq i}^{d} \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \frac{u_{i,\sigma}+u_{i,\sigma'}}{2}\Big].$$

Employing (3.66) together with (3.59) (see also Figures 5 and 8) we easily find that

$$I_i^K = \sum_{j=1}^{d} \sum_{\sigma\in\mathcal{E}^{(j)}_{\rm int}(K)} F_{\sigma,K} (\mathcal{R}^{(i,j)}_{\mathcal{E}}(u_i))_\sigma \tag{3.68}$$

In order to transform J_i^K, we first remark with help of (3.19) the identity

$$F_{\sigma,K} + F_{\epsilon_K,\sigma} + \sum_{j=1,j\neq i}^{d} \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} = \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{\rm int}(K)} F_{\sigma,K}; \tag{3.69}$$

consequently,

$$\sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} (\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}\varphi)_K) \left[F_{\sigma,K} + F_{\epsilon_K,\sigma} + \sum_{j=1,j\neq i}^{d} \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2}\right] =$$

$$\left[\sum_{\sigma\in\mathcal{E}_{\rm int}(K)} F_{\sigma,K}\right] \left[\left(\sum_{\sigma\in\mathcal{E}^{(i)}(K)} \frac{\varphi_\sigma}{2}\right) - (\mathcal{R}^{(i)}_{\mathcal{M}}\varphi)_K\right] = 0,$$

where, we have used (3.62). Next we write

$$J_i^K = \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \Big[\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K\Big]\Big[F_{\sigma,K}\Big(u_{i,\sigma} - (\mathcal{R}^{(i)}_{\mathcal{M}}(u_i))_K\Big) + F_{\epsilon_K,\sigma}\Big(u_{i,\epsilon_K} - (\mathcal{R}^{(i)}_{\mathcal{M}}(u_i))_K\Big)$$
$$+ \sum_{j=1,j\neq i}^{d}\ \sum_{\tau\in\mathcal{E}^{(j)}_{rmint}(K)}\ \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2}\Big(\frac{u_{i,\sigma}+u_{i,\sigma'}}{2} - (\mathcal{R}^{(i)}_{\mathcal{M}}(u_i))_K\Big)\Big]$$
$$+ \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \Big[\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K\Big]\Big[F_{\epsilon_k,\sigma} + F_{\sigma,K} + \sum_{j=1,j\neq i}^{d}\ \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)}\ \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2}\Big](\mathcal{R}^{(i)}_{\mathcal{M}}(u_i))_K,$$

where

$$u_{i,\epsilon_K} - (\mathcal{R}^{(i)}_{\mathcal{M}}(u_i))_K = 0$$

(due to (3.62) and (3.25)) and

$$\sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} \Big[\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}(\varphi))_K\Big]\Big[F_{\epsilon_k,\sigma} + F_{\sigma,K} + \sum_{j=1,j\neq i}^{d}\ \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)}\ \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2}\Big](\mathcal{R}^{(i)}_{\mathcal{M}}(u_i))_K = 0$$

due to (3.69). Consequently,

$$R^i(u_i,\varphi) = \sum_{K\in\mathcal{M}} J_i^K = \sum_{K\in\mathcal{M}}\ \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} (\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}\varphi)_K) F_{\sigma,K}(u_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}u_i)_K)$$
$$+ \sum_{K\in\mathcal{M}}\ \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}(K)} (\varphi_\sigma - (\mathcal{R}^{(i)}_{\mathcal{M}}\varphi)_K) \sum_{j=1,j\neq i}^{d}\ \sum_{\tau\in\mathcal{E}^{(j)}_{\rm int}(K)}\ \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2}\Big(\frac{u_{i,\sigma}+u_{i,\sigma'}}{2} - (\mathcal{R}^{(i)}_{\mathcal{M}}(u_i))_K\Big) \quad (3.70)$$

Putting together formulas (3.67), (3.68) and (3.70) concludes the proof of Lemma 3.4. □

3.5 Main result: error estimates

Now, we are ready to state the main result of this paper. For the sake of clarity, we shall state the theorem and perform the proofs only in the most interesting three dimensional case. The modifications to be done for the two dimensional case, which is in fact more simple, are mostly due to the different Sobolev embedings and are left to the interested reader.

Let us introduce the following functional space:

$$\mathcal{F} = \Big\{(r,\boldsymbol{U}) \in C^1([0,T]\times\overline{\Omega})^4,\ 0 < \underline{r} = \inf_{(t,x)\in\overline{Q}_T} r(t,x), \nabla^2\boldsymbol{U} \in C([0,T]\times\overline{\Omega})^3,\ \partial_t^2 r \in L^1(0,T;L^{\gamma'}(\Omega)),$$
$$\partial_t\nabla r \in L^2(0,T;L^{6\gamma/(5\gamma-6)}(\Omega)^3),\ (\partial_t^2\boldsymbol{U}, \partial_t\nabla\boldsymbol{U}) \in L^2(0,T;L^{6/5}(\Omega)^{12})\Big\}, \quad (3.71)$$

endowed with the following norm

$$\|(r,\boldsymbol{U})\|_{\mathcal{F}} = \|(r,\boldsymbol{U})\|_{C^1([0,T]\times\overline{\Omega})^4} + \|\nabla^2\boldsymbol{U}\|_{C([0,T]\times\overline{\Omega})^3} + \|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))} + \|\partial_t\nabla r\|_{L^2(0,T;L^{6\gamma/(5\gamma-6)}(\Omega)^3)}$$
$$+ \|\partial_t^2\boldsymbol{U}\|_{L^2(0,T;L^{6/5}(\Omega)^{12})} + \|\partial_t\nabla\boldsymbol{U}\|_{L^2(0,T;L^{6/5}(\Omega)^{12})}. \quad (3.72)$$

Let $(r,\boldsymbol{U}) \in \mathcal{F}$ such that $\boldsymbol{U} = \boldsymbol{0}$ on $\partial\Omega$. Let us consider a MAC grid $\mathcal{D} = (\mathcal{M},\mathcal{E})$ of size $h_{\mathcal{M}}$ and regularity $\theta_{\mathcal{M}}$ of the computational domain Ω , a partition $0 = t^0 < t^1 < ... < t^N = T$ of the time interval $[0,T]$, which, for the sake of simplicity, we suppose uniform (where δt stands for the constant time step) and $(\varrho,\boldsymbol{u}) \in Y_{\mathcal{M},\delta t}\times \boldsymbol{X}_{\mathcal{E},\delta t}$ a solution of the discrete problem (3.10). Inspired by (2.10), we introduce the discrete relative energy functional

$$\mathcal{E}(\varrho^n,\boldsymbol{u}^n\Big|r^n_{\mathcal{M}},\boldsymbol{U}^n_{\mathcal{E}}) = \int_\Omega \Big(\frac{1}{2}\varrho^n|\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}|^2 + E(\varrho^n|r^n_{\mathcal{M}})\Big)\,{\rm d}\boldsymbol{x} \quad (3.73)$$
$$= \sum_{i=1}^{3}\ \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}} \frac{1}{2}|D_\sigma|\varrho^n_{D_\sigma}|u^n_\sigma - U^n_\sigma|^2 + \sum_{K\in\mathcal{M}} |K|E(\varrho^n_K|r^n_K)$$

where

$$r^n = r(t^n, \cdot),\ \boldsymbol{U}^n = \boldsymbol{U}(t^n, \cdot),\ r^n_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}(r^n),\ \boldsymbol{U}^n_{\mathcal{E}} = \mathcal{P}_{\mathcal{E}}(\boldsymbol{U}^n), \tag{3.74}$$

where $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_{\mathcal{E}}$ are respectively defined in (3.41) and (3.46). Finally we denote

$$0 < \underline{r} = \min_{((0,T)\times\Omega} r, \quad \overline{r} = \max_{(0,T)\times\Omega} r, \quad [\partial_t r]^n = \partial_t r(t^n, \cdot). \tag{3.75}$$

Let us now state that the discrete problem (3.10) admits at least one solution. This existence result follows from standard arguments of the topological degree theory (see [9] for the theory, [11] for the first application to a nonlinear scheme). We refer to Appendix A for a proof.

Theorem 3.2. *Let $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ such that $\varrho^0 > 0$ (that is $\varrho^0_K > 0$ for any $K \in \mathcal{M}$). There exists a solution $(\boldsymbol{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$ of Problem (3.10). Moreover any solution is such that $\varrho > 0$ a.e in Ω (meaning that $\varrho^n_K > 0$ for any $n = 1, ..., N$ and for any $K \in \mathcal{M}$).*

The following Theorem is the main result of the paper. It can be seen as a discrete version of inequality (2.18).

Theorem 3.3 (Error estimate)**.** *Let $\Omega \subset \mathbb{R}^3$ be a domain which is a union of orthogonal closed parallelepipeds with mutually disjoint interiors, and, without loss of generality, such that the faces of these parallelepipeds are orthogonal to the canonical basis vectors. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of Ω (see Definition 3.1 in Section 3), with step size $h_{\mathcal{M}}$ (see (3.6)) and regularity $\theta_{\mathcal{M}}$ where $\theta_{\mathcal{M}}$ is defined in (3.8). Let us consider a partition $0 = t^0 < t^1 < ... < t^N = T$ of the time interval $[0, T]$, which, for the sake of simplicity, we suppose uniform, where δt stands for the constant time step. Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{M},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution of the discrete problem (3.10) emanating from $(\varrho^0, \boldsymbol{u}^0) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ such that $\varrho^0 > 0$ (the existence of which granted by Theorem 3.2), and $(r, \boldsymbol{U}) \in \mathcal{F}$ be a (strong) solution of problem (1.1). Then there exists a constant $c > 0$ only depending on $T, \Omega, p_0, p_\infty, \mu, \lambda, \gamma, \alpha, \underline{r}, \min_{[\underline{r},\overline{r}]} p, \min_{[\underline{r}/2, 2\overline{r}]} p', \|p\|_{C^2([\underline{r},\overline{r}])}$, on $\|(r, \boldsymbol{U})\|_{\mathcal{F}}$ in a non decreasing way, on $\mathcal{E}_{0,\mathcal{M}}$ in a nondecreasing way and on $\theta_{\mathcal{M}}$ in a non increasing way such that*

$$\max_{0\le n\le N} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r^n_{\mathcal{M}}, \boldsymbol{U}^n_{\mathcal{E}}) \le c\Big(\mathcal{E}(\varrho^0, \boldsymbol{u}^0 \Big| r^0_{\mathcal{M}}, \boldsymbol{U}^0_{\mathcal{E}}) + h^A_{\mathcal{M}} + \sqrt{\delta t}\Big), \tag{3.76}$$

where

$$A = \min(\frac{2\gamma - 3}{\gamma}, \frac{1}{2}). \tag{3.77}$$

Remark 3.

1. *As mentioned previously, Theorem 3.3 holds also in dimension 2 under the assumption that $\gamma > 1$. The value of A in the error estimate (3.76) can be chosen such that*

$$\left\{ \begin{array}{l} A < \min(\frac{2\gamma-2}{\gamma}, 1) \ \text{if}\ \gamma \in (1, 2], \\ A = 1 \ \text{if}\ \gamma > 2. \end{array} \right. \tag{3.78}$$

2. *Suppose that the discrete initial data $(\varrho^0, \boldsymbol{u}^0)$ coincides with the projection $(\mathcal{P}_{\mathcal{M}} r_0, \mathcal{P}_{\mathcal{E}} \boldsymbol{u}_0)$ of the initial data determining the strong solution. Then formula (3.76), combined with Lemma 2.1, provides in terms of classical Lebesgue spaces the following bounds:*

$$\|\varrho^n - r^n\|^2_{L^2(\{\underline{r}/2 \le \varrho^n \le 2\overline{r}\})} + \|\boldsymbol{u}^n - \boldsymbol{U}^n\|^2_{L^2(\{\underline{r}/2 \le \varrho^n \le 2\overline{r}\})} \le c\Big(h^A_{\mathcal{M}} + \sqrt{\delta t}\Big)$$

for the "essential part" of the solution (where the numerical density remains bounded from above and from below outside zero), and

$$|\{\varrho^n \le \underline{r}/2\}| + |\{\varrho^n \ge 2\overline{r}\}| + \|\varrho^n\|^\gamma_{L^\gamma(\Omega\cap\{\varrho^n \ge 2\overline{r}\})} + \|\varrho^n |\boldsymbol{u}^n - \boldsymbol{U}^n|^2\|_{L^1(\{\varrho^n \ge 2\overline{r}\})} \le c\Big(h^A_{\mathcal{M}} + \sqrt{\delta t}\Big)$$

for the "residual part" of the solution, where the numerical density can be "close" to zero or infinity. (In the above formula, for $B \subset \Omega$, $|B|$ denotes the Lebesgue measure of B.) In particular, we obtain

$$\|\varrho - r\|^2_{L^2(\{\underline{r}/2 \le \varrho \le 2\overline{r}\})} + \|\boldsymbol{u} - \boldsymbol{U}\|^2_{L^2(\{\underline{r}/2 \le \varrho \le 2\overline{r}\})} \le c\Big(h^A_{\mathcal{M}} + \sqrt{\delta t}\Big)$$

Moreover, in the particular case of $p(\varrho) = \varrho^2$ *(that however represents a non physical situation)* $E(\varrho|r) = (\varrho - r)^2$ *and the error estimate (3.76) gives*

$$\|\varrho - r\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\varrho|\boldsymbol{u} - \boldsymbol{U}|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c\Big(\sqrt{h_{\mathcal{M}}} + \sqrt{\delta t}\Big)$$

3. *If we assume that the discrete density* ϱ *is bounded from above uniformly with respect to* $(h_{\mathcal{M}}, \delta t)$*, the growth condition at infinity in (1.5) becomes irrelevant. In this case, following step by step proof of Theorem 3.3 we obtain error estimate (3.76) with* $A = \frac{1}{2}$ *for any* $\gamma \geq 1$*. Compare with [15], where the similar problem is treated for a Finite volume/Finite element method. This is qualitatively better result than any other conditional error estimate in the mathematical literature dealing with finite volume or mixed finite volume/finite element methods for compressible fluids (see [4], [11], [32], [42], [46]), where the authors need to assume other bounds for the numerical solution, in addition to the upper bound for the density.*

4. *Theorem 3.3 can be viewed as a discrete version of Proposition 2.1. It is to be noticed that the assumptions on the constitutive law for pressure guaranteeing the error estimates for the scheme* (3.10) *are somewhat stronger* ($\gamma \geq 3/2$) *than the assumptions needed for the stability in the continuous case* ($\gamma > 1$)*. The threshold value* $\gamma = 3/2$ *is however in accordance with the existence theory of weak solutions. The assumptions on the regularity of the strong solution to be compared with the discrete solution in the scheme are slightly stronger than those needed to establish the stability estimates in the continuous case.*

5. *The assumption on the asymptotic behaviour of the pressure for small densities in (*1.5*) can be relaxed for* $\gamma \geq 2$*, see [23]. In particular Theorem 3.3 also holds for the isentropic pressure law* $p(\varrho) = \varrho^\gamma$ *where* $\gamma \geq 2$*.*

The rest of the paper is devoted to the proof of Theorem 3.3. We employ the methodology inspired by that one suggested in [16] in the continuous case. It can be summarized as follows

1. We establish the energy inequality for discrete solutions of the numerical scheme - see Theorem 4.1, formula (4.1). This correspond to energy inequality (2.6) in the continuous case.

2. Knowing (4.1) we establish the discrete relative energy inequality for the discrete solution of the numerical scheme with test functions taken in the discrete spaces introduced in Definition 3.2 - see formula (5.1) in Proposition 5.1. This is a numerical counterpart of relative energy inequality (2.11) in the continuous case.

3. We take in the discrete relative energy inequality as test functions We derive a consistency error for the strong solution above, see equality (6.1) in Lemma 6.1. Combining Lemma 5.1 and Lemma 6.1 we obtain inequality (7.1). This inequality is a numerical counterpart of relative energy inequality (2.12) in the continuous case.

4. We estimate conveniently the right hand side of inequality (7.1) in order to get the Gronwall type estimate, see Lemma 7.1. The rather Lemma implies the result.

4 Mesh independent estimates

4.1 Energy Inequality

Our analysis starts with an energy equality (which can be seen as a discrete differential version of (2.6)), which is crucial both in the convergence analysis and in the error analysis.

Theorem 4.1 (Energy estimate)**.** *Let* $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{M},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ *be a solution of (*3.10*). Then for any* $n = 1, ..., N$*, there exists* $\varrho^{n-1,n} \in L_{\mathcal{M}}$ *such that* $\min(\varrho^{n-1}, \varrho^n) \leq \varrho^{n-1,n} \leq \max(\varrho^{n-1}, \varrho^n)$ *and* $\overline{\varrho}^n_\sigma \in [\min(\varrho^n_K, \varrho^n_L), \max(\varrho^n_K, \varrho^n_L)]$*,* $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ *such that*

$$\frac{1}{\delta t}\int_\Omega \mathcal{H}(\varrho^n)-\mathcal{H}(\varrho^{n-1})\,\mathrm{d}\boldsymbol{x}+\frac{1}{2\delta t}\int_\Omega \varrho^n|\boldsymbol{u}^n|^2-\varrho^{n-1}|\boldsymbol{u}^{n-1}|^2\,\mathrm{d}\boldsymbol{x}$$
$$+\mu||\boldsymbol{u}^n||^2_{1,\mathcal{E},0}+(\mu+\lambda)||\operatorname{div}_{\mathcal{M}}\boldsymbol{u}^n||^2_{L^2(\Omega)}+\frac{1}{2\delta t}\int_\Omega \varrho^{n-1}|\boldsymbol{u}^n-\boldsymbol{u}^{n-1}|^2\,\mathrm{d}\boldsymbol{x}$$
$$+\int_\Omega \frac{1}{2\delta t}\mathcal{H}''(\varrho^{n-1,n})(\varrho^n-\varrho^{n-1})^2\,\mathrm{d}\boldsymbol{x}+\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\sigma=K|L}|\sigma|\,\mathcal{H}''(\overline{\varrho}^n_\sigma)(\varrho^n_K-\varrho^n_L)^2|\boldsymbol{u}^n_{\sigma,K}|=0. \quad (4.1)$$

Proof. Multiplying (3.10*a*) by $\mathcal{H}'(\varrho^n)$ and using a Taylor expansion we obtain the existence of $\varrho^{n-1,n}\in L_{\mathcal{M}}$ such that $\min(\varrho^{n-1},\varrho^n)\le\varrho^{n-1,n}\le\max(\varrho^{n-1},\varrho^n)$ and

$$\int_\Omega \frac{\mathcal{H}(\varrho^n)-\mathcal{H}'(\varrho^{n-1})}{\delta t}\,\mathrm{d}\boldsymbol{x}+\int_\Omega\frac{1}{2\delta t}\mathcal{H}''(\varrho^{n-1,n})(\varrho^n-\varrho^{n-1})^2\,\mathrm{d}\boldsymbol{x}+\int_\Omega \operatorname{div}^{\mathrm{up}}_{\mathcal{M}}(\varrho^n\boldsymbol{u}^n)\,\mathcal{H}'(\varrho^n)\,\mathrm{d}\boldsymbol{x}=0. \quad (4.2)$$

Using again a Taylor expansion (see for instance [21]) one has

$$\int_\Omega \operatorname{div}^{\mathrm{up}}_{\mathcal{M}}(\varrho^n\boldsymbol{u}^n)\,\mathcal{H}'(\varrho^n)\,\mathrm{d}\boldsymbol{x}=\int_\Omega p(\varrho^n)\operatorname{div}_{\mathcal{M}}\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}+\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\sigma=K|L}|\sigma|\,\mathcal{H}''(\varrho^n_\sigma)(\varrho^n_K-\varrho^n_L)^2|\boldsymbol{u}^n_{\sigma,K}| \quad (4.3)$$

where $\varrho^n_\sigma\in[\min(\varrho^n_K,\varrho^n_L),\max(\varrho^n_K,\varrho^n_L)]$. Consequently

$$\int_\Omega \frac{\mathcal{H}(\varrho^n)-\mathcal{H}'(\varrho^{n-1})}{\delta t}\,\mathrm{d}\boldsymbol{x}+\int_\Omega\frac{1}{2\delta t}\mathcal{H}''(\varrho^{n-1,n})(\varrho^n-\varrho^{n-1})^2\,\mathrm{d}\boldsymbol{x}$$
$$+\int_\Omega p(\varrho^n)\operatorname{div}_{\mathcal{M}}\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}+\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\sigma=K|L}|\sigma|\,\mathcal{H}''(\varrho^n_\sigma)(\varrho^n_K-\varrho^n_L)^2|\boldsymbol{u}^n_{\sigma,K}|=0. \quad (4.4)$$

Mutliplying (3.10*b*) by u^n_σ, summing over $\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}$ and $i=1,2,3$ and using (3.31) we infer that

$$\int_\Omega \frac{\varrho^n\boldsymbol{u}^n-\varrho^{n-1}\boldsymbol{u}^{n-1}}{\delta t}\cdot\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}+\mu\|\boldsymbol{u}^n\|^2_{1,\mathcal{E},0}+(\mu+\lambda)\|\operatorname{div}_{\mathcal{M}}\boldsymbol{u}^n\|^2_{L^2(\Omega)}$$
$$+\sum_{i=1}^3\sum_{\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)}F^n_{\epsilon,\sigma}u^n_\epsilon u^n_\sigma-\int_\Omega p(\varrho^n)\operatorname{div}_{\mathcal{M}}\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}=0. \quad (4.5)$$

By virtue of the centered choice for u^n_ϵ (see (3.25)) we have

$$\sum_{i=1}^3\sum_{\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)}F^n_{\epsilon,\sigma}u^n_\epsilon u^n_\sigma=\sum_{i=1}^3\sum_{\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)}F^n_{\epsilon,\sigma}\frac{(u^n_\sigma)^2}{2}. \quad (4.6)$$

Multiplying (3.24) by $\frac{(u^n_\sigma)^2}{2}$ and summing over $\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}$ and $i=1,2,3$ we infer that

$$\frac{1}{2\delta t}\int_\Omega(\varrho^n-\varrho^{n-1})|\boldsymbol{u}^n|^2\,\mathrm{d}\boldsymbol{x}+\sum_{i=1}^3\sum_{\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)}F^n_{\epsilon,\sigma}\frac{(u^n_\sigma)^2}{2}=0 \quad (4.7)$$

Subtracting (4.7) to (4.5) gives

$$\frac{1}{2\delta t}\int_\Omega \varrho^n|\boldsymbol{u}^n|^2-\varrho^{n-1}|\boldsymbol{u}^{n-1}|^2\,\mathrm{d}\boldsymbol{x}+\mu\|\boldsymbol{u}^n\|^2_{1,\mathcal{E},0}+(\mu+\lambda)\|\operatorname{div}_{\mathcal{M}}\boldsymbol{u}^n\|^2_{L^2(\Omega)}-\int_\Omega p(\varrho^n)\operatorname{div}_{\mathcal{M}}\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}=0. \quad (4.8)$$

Consequently adding (4.4) to (4.8) and using (4.3) gives (4.1).

□

Remark 4. *The above computation shows that this numerical scheme is unconditionaly stable meaning that the discrete energy inequality holds without any estra assumptions on the discrete solution.*

The following estimates are obtained thanks to the identity (4.1). In particular the numerical diffusion (4.16) is due to the upwinding and assumptions (1.5), as is classical in the framework of hyperbolic conservation laws, see e.g. [12].

Corollary 4.1. *Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{M},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution of (3.10). Then we have*

1. *There exists $c > 0$ only depending on $\mathcal{E}_{0,\mathcal{M}}$ in a nondecreasing way (independent of $h_{\mathcal{M}}$ and δt) such that*

$$\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0}(\Omega))} \leq c, \tag{4.9}$$

$$\|\boldsymbol{u}\|_{L^2(0,T;L^6(\Omega)^3)} \leq c, \tag{4.10}$$

$$\|\varrho|\boldsymbol{u}|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{4.11}$$

$$\|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq c, \tag{4.12}$$

$$\|\varrho\boldsymbol{u}\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega)^3)} \leq c. \tag{4.13}$$

2. *If $(r, \boldsymbol{U}) \in L^\infty((0,T)\times\Omega) \times L^\infty((0,T)\times\Omega)^3$, then*

$$\max_{1\leq n\leq N} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r^n_{\mathcal{M}}, \boldsymbol{U}^n_{\mathcal{E}}) \leq c, \tag{4.14}$$

where c depends on $\overline{r}, \|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \mathcal{E}_{0,\mathcal{M}}$ in a nondecreasing way.

3. *There exists c only depending on $\mathcal{E}_{0,\mathcal{M}}$ in a nondecreasing way such that for any $m = 1, ..., N$*

$$\sum_{n=1}^{m}\sum_{i=1}^{3}\sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}} |D_\sigma|\varrho^{n-1}_{D_\sigma}|u^n_\sigma - u^{n-1}_\sigma|^2 \leq c. \tag{4.15}$$

4. *The following dissipation estimate due to the upwinding of the density in (3.10a) and (1.5) holds*

$$\delta t\sum_{n=1}^{N}\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma|\frac{(\varrho^n_K - \varrho^n_L)^2}{[\max(\varrho^n_K, \varrho^n_L)]^{(2-\gamma)^+}} 1_{\{\overline{\varrho}^n_\sigma \geq 1\}}\, |u^n_{\sigma,K}| \\ + \delta t\sum_{n=1}^{N}\sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |\sigma|(\varrho^n_K - \varrho^n_L)^2 1_{\{\overline{\varrho}^n_\sigma < 1\}}\, |u^n_{\sigma,K}| \leq c. \tag{4.16}$$

where c depends on $\mathcal{E}_{0,\mathcal{M}}$ in a nondecreasing way and where the quantity $\overline{\varrho}^n_\sigma$ is defined in Theorem 4.1.

5 Relative energy inequality for the discrete problem

5.1 Exact relative energy inequality for the discrete problem

The goal of this section is to prove the discrete (differential) version of the relative energy inequality (2.11).

Proposition 5.1 (Exact discrete relative energy)**.** *Any solution $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{M},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ of the discrete problem (3.10) satisfy*

$$\frac{1}{\delta t}\Big(\mathcal{E}(\varrho^n, \boldsymbol{u}^n\Big|r^n, \boldsymbol{U}^n) - \mathcal{E}(\varrho^{n-1}, \boldsymbol{u}^{n-1}\Big|r^{n-1}, \boldsymbol{U}^{n-1})\Big) + \mu||\boldsymbol{u}^n - \boldsymbol{U}^n||^2_{1,\mathcal{E},0} + (\mu+\lambda)||\operatorname{div}_{\mathcal{M}}(\boldsymbol{u}^n - \boldsymbol{U}^n)||^2_{L^2(\Omega)}$$
$$\leq \int_\Omega (r^n - \varrho^n)\frac{\mathcal{H}'(r^n) - \mathcal{H}'(r^{n-1})}{\delta t}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{div}^{\rm up}_{\mathcal{M}}(\varrho^n\boldsymbol{u}^n)\,\mathcal{H}'(r^{n-1})\,\mathrm{d}\boldsymbol{x}$$
$$+ \mu[\boldsymbol{U}^n - \boldsymbol{u}^n, \boldsymbol{U}^n]_{1,\mathcal{E},0} + (\mu+\lambda)\int_\Omega \operatorname{div}_{\mathcal{M}}(\boldsymbol{U}^n - \boldsymbol{u}^n)\operatorname{div}_{\mathcal{M}}\boldsymbol{U}^n\,\mathrm{d}\boldsymbol{x}$$
$$-\int_\Omega p(\varrho^n)\operatorname{div}_{\mathcal{M}}\boldsymbol{U}^n\,\mathrm{d}\boldsymbol{x} + \int_\Omega \varrho^{n-1}\frac{\boldsymbol{U}^{n-1} - \boldsymbol{U}^n}{\delta t}\cdot\Big(\boldsymbol{u}^{n-1} - \frac{1}{2}(\boldsymbol{U}^{n-1} + \boldsymbol{U}^n)\Big)\,\mathrm{d}\boldsymbol{x}$$
$$+\sum_{i=1}^{3}\sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F^n_{\epsilon,\sigma}U^n_\sigma\Big(u^n_\epsilon - U^n_\epsilon\Big) \tag{5.1}$$

for any $0 < r \in Y_{\mathcal{M},\delta t}$, $\boldsymbol{U} \in \boldsymbol{X}_{\mathcal{E},\delta t}$.

We notice, comparing the terms in the "discrete" formula (5.1) with the terms in the "continuous" formula (2.11), that Theorem 5.1 represents a discrete counterpart of the "continuous" relative energy inequality (2.11). The rest of this section is devoted to its proof. To this end, we shall follow the proof of the "continuous" relative energy inequality (see [16] and [19]) and adapt it to the discrete case.

Proof. We proceed in several steps.

Investigation of the momentum equation (3.10b) : Multiplying (3.10b) by $\boldsymbol{U}^n$ and integrating over Ω we obtain

$$\int_\Omega \frac{1}{\delta t}(\varrho^n \boldsymbol{u}^n - \varrho^{n-1}\boldsymbol{u}^{n-1}\cdot) \cdot \boldsymbol{U}^n \,\mathrm{d}\boldsymbol{x} + \sum_{i=1}^{3} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \sum_{\epsilon\in\widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}^n u_\epsilon^n U_\sigma^n \\ + \mu[\boldsymbol{u}^n, \boldsymbol{U}^n]_{1,\mathcal{E},0} + (\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}^n\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{U}^n \,\mathrm{d}\boldsymbol{x} - \int_\Omega p^n\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{U}^n\,\mathrm{d}\boldsymbol{x} = 0$$

We observe that

$$(\varrho^n \boldsymbol{u}^n - \varrho^{n-1}\boldsymbol{u}^{n-1})\cdot \boldsymbol{U}^n = \varrho^n \boldsymbol{u}^n \cdot U^n - \varrho^{n-1}\boldsymbol{u}^{n-1}\cdot \boldsymbol{U}^{n-1} + \varrho^{n-1}\boldsymbol{u}^{n-1}\cdot(\boldsymbol{U}^{n-1} - \boldsymbol{U}^n).$$

Consequently

$$-\frac{1}{\delta t}\int_\Omega \varrho^n \boldsymbol{u}^n\cdot\boldsymbol{U}^n - \varrho^{n-1}\boldsymbol{u}^{n-1}\cdot\boldsymbol{U}^{n-1}\,\mathrm{d}\boldsymbol{x} = \frac{1}{\delta t}\int_\Omega \varrho^{n-1}\boldsymbol{u}^{n-1}\cdot(\boldsymbol{U}^{n-1} - \boldsymbol{U}^n)\,\mathrm{d}\boldsymbol{x} + \sum_{i=1}^{3} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \sum_{\epsilon\in\widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}^n u_\epsilon^n U_\sigma^n \\ + \mu[\boldsymbol{u}^n, \boldsymbol{U}^n]_{1,\mathcal{E},0} + (\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}^n\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{U}^n \,\mathrm{d}\boldsymbol{x} - \int_\Omega p^n\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{U}^n\,\mathrm{d}\boldsymbol{x}. \quad (5.2)$$

Investigation of the dual continuity equation (3.24) : Multiplying (3.24) by $\frac{1}{2}|U_\sigma^n|^2$ we obtain

$$\frac{1}{2\delta t}\int_\Omega (\varrho^n - \varrho^{n-1})|\boldsymbol{U}^n|^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\sum_{i=1}^{3} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \sum_{\epsilon\in\widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}^n |U_\sigma^n|^2 = 0. \quad (5.3)$$

Moreover due to (3.21)

$$\frac{1}{2}\sum_{i=1}^{3} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \sum_{\epsilon\in\widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}^n U_\sigma^n U_{\sigma'}^n = 0. \quad (5.4)$$

We observe that

$$\int_\Omega (\varrho^n - \varrho^{n-1})|\boldsymbol{U}^n|^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho^n|\boldsymbol{U}^n|^2 - \varrho^{n-1}|\boldsymbol{U}^{n-1}|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \varrho^{n-1}(\boldsymbol{U}^{n-1} + \boldsymbol{U}^n)\cdot(\boldsymbol{U}^{n-1} - \boldsymbol{U}^n)\,\mathrm{d}\boldsymbol{x}.$$

which gives

$$\int_\Omega \frac{1}{2\delta t}\Big(\varrho^n|\boldsymbol{U}^n|^2 - \varrho^{n-1}|\boldsymbol{U}^{n-1}|^2\Big)\,\mathrm{d}\boldsymbol{x} = -\frac{1}{2\delta t}\int_\Omega \varrho^{n-1}(\boldsymbol{U}^{n-1} + \boldsymbol{U}^n)\cdot(\boldsymbol{U}^{n-1} - \boldsymbol{U}^n)\,\mathrm{d}\boldsymbol{x} \\ - \frac{1}{2}\sum_{i=1}^{3} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \sum_{\epsilon\in\widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}^n |U_\sigma^n|^2. \quad (5.5)$$

Investigation of the primal continuity equation (3.10a) : Multiplying the continuity equation by $\mathcal{H}'(r^{n-1})$ and integrating over Ω we obtain

$$-\frac{1}{\delta t}\int_\Omega (\varrho^n\mathcal{H}'(r^n) - \varrho^{n-1}\mathcal{H}'(r^{n-1}))\,\mathrm{d}\boldsymbol{x} = -\frac{1}{\delta t}\int_\Omega \varrho^n(\mathcal{H}'(r^n) - \mathcal{H}'(r^{n-1}))\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\varrho^n\boldsymbol{u}^n)\,\mathcal{H}'(r^{n-1})\,\mathrm{d}\boldsymbol{x}. \quad (5.6)$$

Finally, thanks to the the convexity of the function $\mathcal{H}$, we have

$$\frac{1}{\delta t}\int_\Omega \Big[\Big(r^n\,\mathcal{H}'(r^n)-\mathcal{H}(r^n)\Big)-\Big(r^{n-1}\,\mathcal{H}'(r^{n-1})-\mathcal{H}(r^{n-1})\Big)\Big]\,\mathrm{d}\boldsymbol{x} = \frac{1}{\delta t}\int_\Omega r^n\Big(\mathcal{H}'(r^n)-\mathcal{H}'(r^{n-1})\Big)\,\mathrm{d}\boldsymbol{x}$$
$$-\frac{1}{\delta t}\int_\Omega \Big(\mathcal{H}(r^n)-(r^n-r^{n-1})\,\mathcal{H}'(r^{n-1})-\mathcal{H}(r^{n-1})\Big)\,\mathrm{d}\boldsymbol{x}$$
$$\leq \frac{1}{\delta t}\int_\Omega r^n\Big(\mathcal{H}'(r^n)-\mathcal{H}'(r^{n-1})\Big)\,\mathrm{d}\boldsymbol{x}. \quad (5.7)$$

Conclusion : Summing (4.1), (5.2), (5.4), (5.5), (5.6) and (5.7) we obtain (5.1). □

5.2 Approximate relative energy inequality for the discrete problem

The exact relative energy inequality as stated in Section 5.1 is a general inequality for the given numerical scheme, however it does not immediately provide a comparison of the approximate solution with the strong solution of the compressible Navier-Stokes equations. Its right hand side has to be conveniently transformed (modulo the possible appearance of residual terms vanishing as the space and time steps tend to 0) to provide such comparison tool via a Gronwall type argument.

The goal of this section is to derive a version of the discrete relative energy inequality, still with arbitrary (sufficiently regular) test functions $(r,\boldsymbol{U})$, that will be convenient for the comparison of the discrete solution with the strong solution.

Let us introduce some notations useful for the rest of the paper. Considering a solution $(\varrho,\boldsymbol{u})$ of Problem 3.10, and $(r,\boldsymbol{U})\in\mathcal{F}$ we define for $\sigma=K|L\in\mathcal{E}_{\rm int}$:

$$r_\sigma^{n,\rm up} = \left|\begin{array}{ll} r_K^n & \text{if } u_{\sigma,K}^n\geq 0,\\ r_L^n & \text{otherwise,}\end{array}\right. \quad (5.8)$$

where r_K^n and $u_{\sigma,K}$ are respectively defined in (3.42) and (3.13). Note that $r_\sigma^{n,\rm up}$ will be not prescribe for $\sigma\in\mathcal{E}_{\rm ext}$ (it will be a consequence of the fact $u_{\sigma,K}$ that vanishes for $\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}$). Similarly to (3.23) we define

$$\begin{array}{ll} \text{for } \sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L & |D_\sigma|\ r_{D_\sigma}^n = |D_{\sigma,K}|\ r_K^n + |D_{\sigma,L}|\ r_L^n,\\ \text{for } \sigma\in\mathcal{E}_{\rm ext},\ \sigma\in\mathcal{E}(K), & r_{D_\sigma}^n = r_K^n.\end{array} \quad (5.9)$$

For $i=1,...,d$ and $\epsilon=D_\sigma|D_{\sigma'}\in\tilde{\mathcal{E}}_{\rm int}^{(i)}$ we define

$$U_\epsilon^n = \frac{(\mathcal{P}_\mathcal{E}^{(i)}\,U_i^n)_\sigma + (\mathcal{P}_\mathcal{E}^{(i)}\,U_i^n)_{\sigma'}}{2} \quad (5.10)$$

where $\mathcal{P}_\mathcal{E}^{(i)}$ is defined in (3.44).

Starting from now, we shall use the following convention for the constants in estimates: We shall denote by c a positive number which can take different values even in the same formula. It always depend tacitly on the geometric and structural coefficients

$$T,\Omega,p_0,p_\infty,\mu,\lambda,\gamma,\alpha, \quad (5.11)$$

and if not stated explicitly otherwise, on the characteristics of the strong solution

$$\underline{r},\ \min_{[\underline{r},\overline{r}]}p,\ \min_{[\underline{r}/2,2\overline{r}]}p',\ \|p\|_{C^2([\underline{r},\overline{r}])},\ \|(r,\boldsymbol{U})\|_\mathcal{F} \quad (5.12)$$

and on

$$\mathcal{E}_{0,\mathcal{M}} \text{ in a non decreasing way,} \quad \theta_\mathcal{M} \text{ in a non increasing way.} \quad (5.13)$$

It is always independent of the size of the discretisation δt and $h_\mathcal{M}$.

Lemma 5.1 (Approximate discrete relative energy). *Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{M},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution of the discrete problem (3.10) and $(r, \boldsymbol{U}) \in \mathcal{F}$ such that $\boldsymbol{U}_{|\partial\Omega} = \boldsymbol{0}$. Then there exists c only depending on parameters (5.11–5.13) such that for all $m = 1, \dots, N$:*

$$
\begin{aligned}
&\mathcal{E}(\varrho^m, \boldsymbol{u}^m \big| r^m_{\mathcal{M}}, \boldsymbol{U}^m_{\mathcal{E}}) - \mathcal{E}(\varrho^0, \boldsymbol{u}^0 \big| r^0_{\mathcal{M}}, \boldsymbol{U}^0_{\mathcal{E}}) \\
&\qquad + \delta t \sum_{n=1}^{m} \Big(\mu ||\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}||^2_{1,\mathcal{E},0} + (\mu+\lambda) || \operatorname{div}_{\mathcal{M}}(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})||^2_{L^2(\Omega)} \Big) \\
&\leq \delta t \sum_{n=1}^{m} \Big(\mu [\boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{u}^n, \boldsymbol{U}^n_{\mathcal{E}}]_{1,\mathcal{E},0} + (\mu+\lambda) \int_\Omega \operatorname{div}_{\mathcal{M}}(\boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{u}^n) \operatorname{div}_{\mathcal{M}} \boldsymbol{U}^n_{\mathcal{E}} \, \mathrm{d}\boldsymbol{x} \Big) \\
&+ \delta t \sum_{n=1}^{m} \int_\Omega \varrho^{n-1} \Big(\frac{\boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{U}^{n-1}_{\mathcal{E}}}{\delta t} \Big) \cdot \Big(\boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{u}^n \Big) + \delta t \sum_{n=1}^{m} \int_\Omega (r^n_{\mathcal{M}} - \varrho^n) \frac{p'(r^n_{\mathcal{M}})}{r^n_{\mathcal{M}}} [\partial_t r]^n \, \mathrm{d}\boldsymbol{x} \\
&\qquad + \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{K \in \mathcal{M}} \sum_{j=1}^{3} \sum_{\sigma \in \mathcal{E}^{(j)}(K)} |\sigma| \varrho^{n,\mathrm{up}}_\sigma (\mathcal{P}^{(j)}_{\mathcal{E}} \, U^n_j)_\sigma \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} \\
&\qquad \times (\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} \, U^n_i)_\sigma) (\mathcal{R}^{(i)}_{\mathcal{M}} (\mathcal{P}^{(i)}_{\mathcal{E}} \, U^n_i)_K - (\mathcal{P}^{(j)}_{\mathcal{E}} \, U^n_i)_\sigma) \\
&- \delta t \sum_{n=1}^{m} \int_\Omega \frac{\varrho^n}{r^n_{\mathcal{M}}} p'(r^n_{\mathcal{M}}) \, \mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n) \cdot \nabla r^n \, \mathrm{d}\boldsymbol{x} - \delta t \sum_{n=1}^{m} \int_\Omega p(\varrho^n) \operatorname{div} \boldsymbol{U}^n \, \mathrm{d}\boldsymbol{x} + \mathcal{R}^m_{\mathcal{M},\delta t} + \mathcal{G}^m_{\mathcal{M},\delta t}
\end{aligned} \tag{5.14}
$$

for any pair $(r, \boldsymbol{U})$ belonging to the class (3.71) such that $\boldsymbol{U}_{|\partial\Omega} = \boldsymbol{0}$, where

$$
|\mathcal{G}^m_{\mathcal{M},\delta t}| \leq \frac{c}{\delta} \, \delta t \sum_{n=1}^{m} \mathcal{E}(\varrho^m, \boldsymbol{u}^m \big| r^m_{\mathcal{M}}, \boldsymbol{U}^m_{\mathcal{E}}) + \delta \sum_{n=1}^{m} \|\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}\|^2_{1,\mathcal{E},0}, \tag{5.15}
$$

with any $\delta > 0$,

$$
|\, \mathcal{R}^m_{\mathcal{M},\delta t} \,| \leq c(\sqrt{\delta t} + h^A_{\mathcal{M}}), \tag{5.16}
$$

and where A is given by (3.77).

Proof. The right hand side of the relative energy inequality (5.1), after a summation over n and a multiplication by δt, is a sum $\sum_{i=1}^{6} T_i$, where

$$
\begin{aligned}
T_1 &= \delta t \sum_{n=1}^{m} \Big(\mu [\boldsymbol{U}^n_{\mathcal{E}}, \boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{u}^n]_{1,\mathcal{E},0} + (\mu+\lambda) \int_\Omega \operatorname{div}_{\mathcal{M}} \boldsymbol{U}^n_{\mathcal{E}} \operatorname{div}_{\mathcal{M}}(\boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{u}^n) \, \mathrm{d}\boldsymbol{x} \Big), \\
T_2 &= \delta t \sum_{n=1}^{m} \int_\Omega \varrho^{n-1} \frac{\boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{U}^{n-1}_{\mathcal{E}}}{\delta t} \cdot \Big(\frac{\boldsymbol{U}^{n-1}_{\mathcal{E}} + \boldsymbol{U}^n_{\mathcal{E}}}{2} - \boldsymbol{u}^{n-1} \Big) \, \mathrm{d}\boldsymbol{x}, \\
T_3 &= \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{\sigma \in \mathcal{E}^{(i)}_{\mathrm{int}}} \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma) \\ \epsilon = D_\sigma | D_{\sigma'}}} F^n_{\epsilon,\sigma} U^n_\sigma \Big(u^n_\epsilon - U^n_\epsilon \Big) \\
T_4 &= -\delta t \sum_{n=1}^{m} \int_\Omega p(\varrho^n) \operatorname{div} \boldsymbol{U}^n \, \mathrm{d}\boldsymbol{x}, \\
T_5 &= \delta t \sum_{n=1}^{m} \int_\Omega (r^n_{\mathcal{M}} - \varrho^n) \frac{\mathcal{H}'(r^n_{\mathcal{M}}) - \mathcal{H}'(r^{n-1}_{\mathcal{M}})}{\delta t} \, \mathrm{d}\boldsymbol{x}, \\
T_6 &= \delta t \sum_{n=1}^{m} \int_\Omega \operatorname{div}^{\mathrm{up}}_{\mathcal{M}}(\varrho^n \boldsymbol{u}^n) \, \mathcal{H}'(r^{n-1}_{\mathcal{M}}) \, \mathrm{d}\boldsymbol{x}
\end{aligned}
$$

The term T_1 and T_4 will be kept as they are; all the other terms T_i will be transformed to a more convenient form, as described in the following steps.

Step 1: *Term* T_2. We have $T_2 = T_{2,1} + R_{2,1}$ with

$$\begin{cases} T_{2,1} = \delta t \sum_{n=1}^{m} \int_\Omega \varrho^{n-1}\Big(\frac{\boldsymbol{U}_\mathcal{E}^n - \boldsymbol{U}_\mathcal{E}^{n-1}}{\delta t}\Big) \cdot \Big(\boldsymbol{U}_\mathcal{E}^{n-1} - \boldsymbol{u}^{n-1}\Big)\,\mathrm{d}\boldsymbol{x}, \\ R_{2,1} = \sum_{n=1}^{m} \int_\Omega \frac{1}{2}\varrho^{n-1} |U_\mathcal{E}^n - U_\mathcal{E}^{n-1}|^2 \,\mathrm{d}\boldsymbol{x}. \end{cases} \tag{5.17}$$

Thanks to the mass conservation (3.16), (4.12) and the Taylor formula applied to the function $t \to \boldsymbol{U}(t, \boldsymbol{x})$ between t^{n-1} and t^n we easily get

$$|R_{2,1}| \le c\delta t \tag{5.18}$$

where c depends on $||\partial_t \boldsymbol{U}||_{L^\infty([0,T]\times\overline{\Omega})^3}$ and on $\mathcal{E}_{0,\mathcal{M}}$. Let us now decompose the term $T_{2,1}$ as

$$T_{2,1} = T_{2,2} + R_{2,2}, \text{ with } T_{2,2} = \delta t \sum_{n=1}^{m} \int_\Omega \varrho^{n-1}\Big(\frac{\boldsymbol{U}_\mathcal{E}^n - \boldsymbol{U}_\mathcal{E}^{n-1}}{\delta t}\Big) \cdot \Big(\boldsymbol{U}_\mathcal{E}^n - \boldsymbol{u}^n\Big)\,\mathrm{d}\boldsymbol{x} \tag{5.19}$$

and $R_{2,2} = \delta t \sum_{n=1}^m R_{2,2}^n$ where

$$R_{2,2}^n = \int_\Omega \varrho^{n-1}\Big(\frac{\boldsymbol{U}_\mathcal{E}^n - \boldsymbol{U}_\mathcal{E}^{n-1}}{\delta t}\Big) \cdot \Big(\boldsymbol{U}_\mathcal{E}^{n-1} - \boldsymbol{U}_\mathcal{E}^n\Big)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \varrho^{n-1}\Big(\frac{\boldsymbol{U}_\mathcal{E}^n - \boldsymbol{U}_\mathcal{E}^{n-1}}{\delta t}\Big) \cdot \Big(\boldsymbol{u}^{n-1} - \boldsymbol{u}^n\Big)\,\mathrm{d}\boldsymbol{x}.$$

By the same token as above, and using estimate (4.15) we may estimate the residual term as follows

$$|R_{2,2}| \le c\sqrt{\delta t} \tag{5.20}$$

where c depends on $||\partial_t \boldsymbol{U}||_{L^\infty((0,T)\times\Omega)^3}$ and on $\mathcal{E}_{0,\mathcal{M}}$.

Step 2: *Term* T_3. Using Lemma 3.4 we can write

$$\begin{aligned} T_3 = \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{K\in\mathcal{M}} \mathcal{R}_\mathcal{M}^{(i)}(\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_K \sum_{j=1}^{3} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(j)}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} u_{\sigma,K}^n\, \mathcal{R}_\mathcal{E}^{(i,j)}(u_i^n - \mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_\sigma \\ + \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} (R_{3,1,1,i}^n + R_{3,1,2,i}^n) = T_{3,1} + R_{3,1} \end{aligned} \tag{5.21}$$

where the reminder $R_{3,1,i}^n$ and $R_{3,2,i}^n$ are respectively given by

$$R_{3,1,1,i}^n = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}(K)} \Big((\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_\sigma - \mathcal{R}_\mathcal{M}^{(i)}\, \mathcal{P}_\mathcal{E}^{(i)}(U_i^n)_K\Big) F_{\sigma,K}^n \Big(u_\sigma^n - (\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_\sigma - \mathcal{R}_\mathcal{M}^{(i)}(u_i^n - \mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_K\Big),$$

and

$$R_{3,1,2,i}^n = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}(K)} \Big((\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_\sigma - \mathcal{R}_\mathcal{M}^{(i)}\, \mathcal{P}_\mathcal{E}^{(i)}(U_i^n)_K\Big)$$

$$\times \sum_{j=1, j\neq i}^{d} \sum_{\tau\in\mathcal{E}_{\mathrm{int}}^{(j)}(K)} \sum_{\sigma'\in\mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}^n}{2} \Big(\frac{u_{i,\sigma}^n + u_{i,\sigma'}^n - (\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_\sigma - (\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_{\sigma'}}{2} - \mathcal{R}_\mathcal{M}^{(i)}(u_i^n - \mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_K\Big).$$

From the definition of $\mathcal{R}_\mathcal{M}^{(i)}$ and $\mathcal{P}_\mathcal{E}^{(i)}$ we infer that

$$R_{3,1,1,i}^n = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}(K)} \frac{1}{4}\frac{|K|^2}{|\sigma|^2} \eth_i(\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_K \eth_i(u_i - \mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_K F_{\sigma,K}$$

From (3.49) and the defintion the discrete derivative (3.28), we infer that for any $i \in \{1, 2, 3\}$ and $K \in \mathcal{M}$

$$|\eth_i(\mathcal{P}_\mathcal{E}^{(i)}\, U_i^n)_K| \le c$$

where c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^{3\times3}}$. Using the geometric relations established in Remark 2 and the Hölder's inequality we infer that

$$|R^n_{3,1,1,i}| \le c \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int},\sigma=K|L} \sqrt{h_\sigma}|\sigma|\varrho^n_{D_\sigma}|u^n_\sigma|\|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))\|_{L^2(K\cup L)}$$

$$\le c \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int},\sigma=K|L} \sqrt{h_\sigma}|\sigma||D_\sigma|^{-\frac{\gamma+6}{6\gamma}}\|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))\|_{L^2(K\cup L)}\|\varrho^n u^n_i\|_{L^{\frac{6\gamma}{\gamma+6}}(D_\sigma)}$$

where c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^{3\times3}}$ and on $\theta_{\mathcal{M}}$. Using again the geometric relation established in Remark 2 we can write for all $\sigma\in\mathcal{E}^{(i)}_{\rm int}$,

$$c_1 h^3_\sigma \le |D_\sigma| \le c_2 h^3_\sigma, \qquad c_3 h^2_\sigma \le |\sigma| \le c_4 h^2_\sigma$$

where c_1 and c_3 depend on $\theta_{\mathcal{M}}$ in a nonincreasing way and c_2 and c_4 depend on $\theta_{\mathcal{M}}$ in a nondecreasing way, which gives

$$\sqrt{h_\sigma}|\sigma||D_\sigma|^{-\frac{\gamma+6}{6\gamma}} \le c h^A_{\mathcal{M}}.$$

where c depends on $\theta_{\mathcal{M}}$ and where A is given by (3.77). Consequently

$$|R^n_{3,1,1,i}| \le c h^A_{\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int},\sigma=K|L} \|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))\|_{L^2(K\cup L)}\|\varrho^n u^n_i\|_{L^{\frac{6\gamma}{\gamma+6}}(D_\sigma)}.$$

$$|R^n_{3,1,1,i}| \le c h^A_{\mathcal{M}}\Big(\sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int},\sigma=K|L} \|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))\|^p_{L^2(K\cup L)}\Big)^{\frac{1}{p}}\Big(\sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int},\sigma=K|L}\|\varrho^n u^n_i\|^q_{L^{\frac{6\gamma}{\gamma+6}}(D_\sigma)})\Big)^{\frac{1}{q}}.$$

where $1 < p, q < +\infty$ and $\frac{1}{p}+\frac{1}{q} = 1$.

If $\frac{3}{2} \le \gamma \le 3$ and then $\frac{6\gamma}{5\gamma-6} \ge 2$ we take $p = \frac{6\gamma}{5\gamma-6}$ and $q = \frac{6\gamma}{\gamma+6}$ and we obtain

$$|R^n_{3,1,1,i}| \le c h^A_{\mathcal{M}}\|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))\|_{L^2(\Omega)}\|\varrho^n u^n_i\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega)}$$

Now if $\gamma \ge 3$ and then $\frac{6\gamma}{\gamma+6} \ge 2$ we take $p = 2$ and $q = 2$ and we obtain

$$|R^n_{3,1,1,i}| \le c h^A_{\mathcal{M}}\|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))\|_{L^2(\Omega)}\|\varrho^n u^n_i\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega)}$$

where the constant c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}$ and on $\theta_{\mathcal{M}}$. Finally from the estimates (3.55), (4.9) and (4.13) we deduce that

$$\delta t\sum_{n=1}^{N}\sum_{i=1}^{3}|R^n_{3,1,1,i}| \le c h^A_{\mathcal{M}}.$$

where c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \mathcal{E}_{0,\mathcal{M}}$ and on $\theta_{\mathcal{M}}$. Let us now estimate the remainder $R^n_{3,1,2,i}$. Let $K\in\mathcal{M}$ and let us consider $\sigma\in\mathcal{E}^{(i)}(K)$. Without loss of generality we assume that $\sigma = K|L \in \mathcal{E}^{(i)}_{\rm int}$. Let $\epsilon\in\tilde{\mathcal{E}}(D_\sigma)$ such that $\epsilon \ne \epsilon_K$ and $\epsilon\cap K \subset \sigma' \in \mathcal{E}^{(j)}$ for $j \ne i$ that is $\epsilon \in \tilde{\mathcal{N}}_{\sigma',\sigma}$. Since the primal fluxes vanish on external faces we can assume that $\epsilon\in\tilde{\mathcal{E}}^{(i)}_{\rm int}$ saying $\epsilon = \sigma|\sigma''$ where $\sigma''\in\tilde{\mathcal{E}}^{(i)}$. Let $\tilde K\in\mathcal{M}$ such that $\sigma' = K|\tilde K$. We define $\sigma'''\in\mathcal{E}^{(i)}$ such that $\tilde K = [\sigma'''\sigma'']$ and $\sigma''''\in\mathcal{E}^{(i)}$ such that $K = [\sigma''''\sigma]$. Finally let $\tilde L$ be the primal cell such that $\sigma'' = \tilde K|\tilde L$. We summarize the above notations in the figure 10.

In accordance with the defintion of u^n_ϵ and U^n_ϵ we can write

$$\begin{aligned}
&|\frac{F^n_{\sigma',K}}{2}(u^n_\epsilon - U^n_\epsilon - (u^n_\epsilon - U^n_\epsilon - \mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}U^n_i)_K)|\\
&\qquad\le c|\sigma'|\varrho^n_{D_{\sigma'}}|u^n_{\sigma'}|\Big(\frac{|\tilde K|}{|\sigma''|}|\eth_i u_i|_{\tilde K} + d_{\epsilon'}|\eth_j u_i|_{D_{\epsilon'}}\Big)\\
&\le c\sqrt{h_{\sigma'}}|\sigma'|\varrho^n_{D_{\sigma'}}|u^n_{\sigma'}|\Big(\|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}U^n_i)\|_{L^2(D_{\sigma'})} + \|\eth_j(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}U^n_i)\|_{L^2(D_{\sigma'})}\Big)\\
&\qquad\le c h^A_{\mathcal{M}}\|\varrho^n u^n_j\|_{L^{\frac{6\gamma}{\gamma+6}}(D_{\sigma'})}\Big(\|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}U^n_i)\|_{L^2(D_{\sigma'})} + \|\eth_j(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}U^n_i)\|_{L^2(D_{\sigma'})}\Big)
\end{aligned}$$

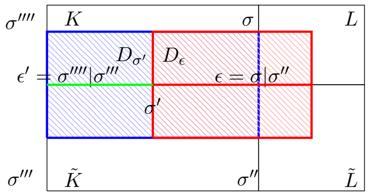

Figure 10: Decomposition of the dual grid

We deduce from the previous computation that

$$|R^n_{3,1,2,i}| \leq ch^A_{\mathcal{M}} \sum_{j\neq i} \sum_{\sigma'\in\mathcal{E}^{(j)}} \|\varrho^n u^n_j\|_{L^{\frac{6\gamma}{\gamma+6}}(D_{\sigma'})} \Big(\|\eth_i(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)\|_{L^2(D_{\sigma'})} + \|\eth_j(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)\|_{L^2(D_{\sigma'})} \Big)$$

where the constant c depends on $\theta_{\mathcal{M}}$. Finally from the estimates (3.55), (4.9) and (4.13) and thanks to computation established to estimate $R_{3,1,1,i}$ we deduce that

$$\delta t \sum_{n=1}^{N} \sum_{i=1}^{3} |R^n_{3,1,2,i}| \leq ch^A_{\mathcal{M}},$$

where c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \mathcal{E}_{0,\mathcal{M}}$ and on $\theta_{\mathcal{M}}$. Consequently we have

$$\delta t \sum_{n=1}^{N} \sum_{i=1}^{3} (|R^n_{3,1,1,i}| + |R^n_{3,1,2,i}|) \leq ch^A_{\mathcal{M}}. \tag{5.22}$$

Evidently, for each face $\sigma = K|L \in \mathcal{E}_{\rm int}, u^n_{\sigma,K} + u^n_{\sigma,L} = 0$; whence,

$$T_{3,1} = \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{K\in\mathcal{M}} \sum_{j=1}^{3} \sum_{\sigma\in\mathcal{E}^{(j)}(K)} |\sigma| \varrho^{n,\rm up}_\sigma u^n_\sigma \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} \times (\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_\sigma)(\mathcal{R}^{(i)}_{\mathcal{M}} (\mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_K - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_i)_\sigma)$$

Consequently

$$T_{3,1} = \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{K\in\mathcal{M}} \sum_{j=1}^{3} \sum_{\sigma\in\mathcal{E}^{(j)}(K)} |\sigma| \varrho^{n,\rm up}_\sigma (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_j)_\sigma \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} \times (\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_\sigma)(\mathcal{R}^{(i)}_{\mathcal{M}} (\mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_K - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_i)_\sigma) + R_{3,2} \tag{5.23}$$

where $R_{3,2} = \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{K\in\mathcal{M}} \sum_{j=1}^{3} \sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}_{\rm int}} R^n_{3,2,i,K,j,\sigma}$

$$R^n_{3,2,i,K,j,\sigma} = |\sigma| \varrho^{n,\rm up}_\sigma (u^n_\sigma - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_j)_\sigma) \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} (\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_\sigma)(\mathcal{R}^{(i)}_{\mathcal{M}} (\mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_K - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_i)_\sigma)$$

Using (3.49) combined with the geometric relations listed in Remark 2 we infer that for any $(i,j) \in \{1,2,3\}^2, K \in \mathcal{M}, \sigma \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\rm int}$,

$$|\,\mathcal{R}^{(i)}_{\mathcal{M}} (\mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_K - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_i)_\sigma| \leq ch_\sigma,$$

where c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^{3\times3}}$ and on $\theta_{\mathcal{M}}$. Consequently we can estimate the general term of $R_{3,2}$ as follows

$$\begin{aligned} |R^n_{3,2,i,K,j,\sigma}| &= \Big| |\sigma| \varrho^{n,\rm up}_\sigma (u^n_\sigma - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_j)_\sigma) \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} (\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_\sigma)(\mathcal{R}^{(i)}_{\mathcal{M}} (\mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_K - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_i)_\sigma) \Big| \\ &\leq c|D_\sigma||\varrho^{n,\rm up}_\sigma||(u^n_\sigma - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_j)_\sigma)||(\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_\sigma)| \\ &\leq c|D_\sigma|(\varrho^n_K + \varrho^n_L)|(u^n_\sigma - (\mathcal{P}^{(j)}_{\mathcal{E}} U^n_j)_\sigma)||(\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} U^n_i)_\sigma)| \\ &\leq c \int_{D_\sigma} \varrho^n |u^n_j - \mathcal{P}^{(j)}_{\mathcal{E}} (U^n_j)||\,\mathcal{R}^{(i,j)}_{\mathcal{E}} (u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}} (U^n_i))|\,{\rm d}\boldsymbol{x}, \end{aligned}$$

where $\sigma = K|L \in \mathcal{E}^{(j)}(K) \cap \mathcal{E}_{\text{int}}$. Consequently using (3.64) and Hölder's inequality and then Young's inequality, we infer that

$$|R_{3,2,1}| \leq c\delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega \varrho^n |u_j^n - \mathcal{P}_\mathcal{E}^{(j)}(U_j^n)| |\mathcal{R}_\mathcal{E}^{(i,j)}(u_i^n - \mathcal{P}_\mathcal{E}^{(i)}(U_i^n))| \, \mathrm{d}\boldsymbol{x}$$
$$\leq \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{j=1}^{3} \|\varrho^n\|_{L^{3/2}(\Omega)}^{1/2} \Big(\mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r_\mathcal{M}^n, \boldsymbol{U}_\mathcal{E}^n) \Big)^{1/2} \|u_i^n - \mathcal{P}_\mathcal{E}^{(i)}(U_i^n)\|_{1,\mathcal{E}^{(i)},0}$$
$$\leq \frac{c}{\delta} \, \delta t \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r_\mathcal{M}^n, \boldsymbol{U}_\mathcal{E}^n) + \delta \, \delta t \sum_{n=1}^{m} \|\boldsymbol{u}^n - \boldsymbol{U}_\mathcal{E}^n\|_{1,\mathcal{E},0}^2$$

where c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}, \mathcal{E}_{0,\mathcal{M}}$ and on $\theta_\mathcal{M}$.

Step 3: *Term* T_5. Using the Taylor formula since $p \in C^2(\mathbb{R}_+^\star)$ we get

$$\mathcal{H}'(r_K^n) - \mathcal{H}'(r_K^{n-1}) = \mathcal{H}''(r_K^n)(r_K^n - r_K^{n-1}) - \frac{1}{2} \mathcal{H}'''(\overline{r}_K^n)(r_K^n - r_K^{n-1})^2,$$

where $\overline{r}_K^n \in [\min(r_K^{n-1}, r_K^n), \max(r_K^{n-1}, r_K^n)]$. Consequently $T_5 = T_{5,1} + R_{5,1}$ with

$$T_{5,1} = \delta t \sum_{n=1}^{m} \int_\Omega (r_\mathcal{M}^n - \varrho^n) \frac{p'(r_\mathcal{M}^n)}{r_\mathcal{M}^n} \frac{r_\mathcal{M}^n - r_\mathcal{M}^{n-1}}{\delta t} \, \mathrm{d}\boldsymbol{x} \tag{5.24}$$

and

$$R_{5,1} = \delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{M}} R_{5,1}^{n,K}, \ R_{5,1}^{n,K} = \frac{1}{2}|K| \, \mathcal{H}'''(\overline{r}_K^n) \frac{(r_K^n - r_K^{n-1})^2}{\delta t} (\varrho_K^n - r_K^n).$$

By the first order Taylor formula applied to function $t \mapsto r(t,x)$ on the interval (t^{n-1}, t^n), thanks to the relation (2.3), to the mass conservation (3.16) and (4.12) we have

$$|R_{5,1}| \leq c \, \delta t \tag{5.25}$$

where c depends on $\|\partial_t r\|_{L^\infty(Q_T)}$ and on $\mathcal{E}_{0,\mathcal{M}}$.

Let us now decompose $T_{5,1}$ as follows: $T_{5,1} = T_{5,2} + R_{5,2}$ with

$$T_{5,2} = \delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{M}} \int_K (r_\mathcal{M}^n - \varrho^n) \frac{p'(r_\mathcal{M}^n)}{r_\mathcal{M}^n} [\partial_t r]^n \mathrm{d}x, \tag{5.26}$$

and

$$R_{5,2} = \delta t \sum_{n=1}^{m} \int_\Omega (r_\mathcal{M}^n - \varrho^n) \frac{p'(r_\mathcal{M}^n)}{r_\mathcal{M}^n} \Big(\frac{r_\mathcal{M}^n - r_\mathcal{M}^{n-1}}{\delta t} - [\partial_t r]^n \Big) \, \mathrm{d}\boldsymbol{x}$$

Using twice the Taylor formula, the Fubini Theorem and Hölder's inequality (see [23]) we can obtain

$$|R_{5,2}| \leq c\delta t, \tag{5.27}$$

where c depends on $\underline{r}, \overline{r}, \|\partial_t^2 r\|_{L^1(0,T;L^{\gamma'}(\Omega))}$ and on $\mathcal{E}_{0,\mathcal{M}}$.

Step 4: *Term* T_6. Using the local conservativity of the flux through a primal face, we may write

$$\begin{aligned} & T_6 = T_{6,1} + R_{6,1}, \quad R_{6,1} = \delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} R_{6,1}^{n,\sigma,K}, \ \text{with} \\ & T_{6,1} = \delta t \sum_{n=1}^{m} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\text{int}}} |\sigma| \varrho_K^n \Big(\mathcal{H}'(r_K^{n-1}) - \mathcal{H}'(r_\sigma^{n-1}) \Big) u_{\sigma,K}^n, \ \text{and} \\ & R_{6,1}^{n,\sigma,K} = |\sigma| \Big(\varrho_\sigma^{n,\text{up}} - \varrho_K^n \Big) \Big(\mathcal{H}'(r_K^{n-1}) - \mathcal{H}'(r_\sigma^{n-1}) \Big) u_{\sigma,K}^n, \ \sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\text{int}}\,. \end{aligned} \tag{5.28}$$

Motivated by (4.16), we may write for $\sigma = K|L \in \mathcal{E}_{\rm int}$

$$
\begin{aligned}
|R_{6,1}^{n,\sigma,K}| \le c\sqrt{h_{\mathcal{M}}}|\sigma| & \\
\times\Big(\frac{|\varrho_\sigma^{n,\rm up}-\varrho_K^n|}{\max(\varrho_K^n,\varrho_L^n)^{(2-\gamma)^+/2}}\sqrt{|u_{\sigma,K}^n|1_{\overline{\varrho}_\sigma^n\ge 1}}\sqrt{h_K}(\varrho_K^n+\varrho_L^n)^{(2-\gamma)^+/2}\sqrt{|u_{\sigma,K}^n|} & \\
+|\varrho_\sigma^{n,\rm up}-\varrho_K^n|\sqrt{|u_{\sigma,K}^n|1_{\overline{\varrho}_\sigma^n<1}}\sqrt{h_K}\sqrt{|u_{\sigma,K}^n|}\Big), & \quad (5.29)
\end{aligned}
$$

where c depends on $\underline{r}$, $\|\nabla r\|_{L^\infty((0,T)\times\Omega)^3}$ and where the numbers $\overline{\varrho}_\sigma^n$ are defined in Theorem 4.1. Here we have used the first order Taylor formula applied to function $\mathcal{H}'$ between endpoints r_K^{n-1}, r_σ^{n-1}. Consequently, application of the geometric relations established in Remark 2, the Hölder and Young inequalities gives

$$
\begin{aligned}
|R_{6,1}| \le c\sqrt{h_{\mathcal{M}}}\delta t\sum_{n=1}^m\Big[\Big(\sum_{K\in\mathcal{M}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|\frac{(\varrho_\sigma^{n,\rm up}-\varrho_K^n)^2}{\max(\varrho_K^n,\varrho_L^n)^{(2-\gamma)^+}}|u_{\sigma,K}^n|1_{\overline{\varrho}_\sigma^n\ge1}\Big)^{1/2} & \\
\times\Big(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h_\sigma(\varrho_K^n)^{(2-\gamma)^+}|u_{\sigma,K}^n|\Big)^{1/2} & \\
+\Big(\sum_{K\in\mathcal{M}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|h_\sigma(\varrho_\sigma^{n,\rm up}-\varrho_K^n)^2|u_{\sigma,K}^n|1_{\overline{\varrho}_\sigma^n<1}\Big)^{1/2}\Big(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|h_\sigma|u_{\sigma,K}^n|\Big)^{1/2}\Big] & \\
\le c\sqrt{h_{\mathcal{M}}}\delta t\sum_{n=1}^m\Big[\Big(\sum_{K\in\mathcal{M}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|\frac{(\varrho_\sigma^{n,\rm up}-\varrho_K^n)^2}{\max(\varrho_K^n,\varrho_L^n)^{(2-\gamma)^+}}|u_{\sigma,K}^n|1_{\overline{\varrho}_\sigma^n\ge1} & \\
+\Big(\sum_{K\in\mathcal{M}}|K|(\varrho_K^n)^{6(2-\gamma)^+/5}\Big)^{5/6}\Big(\sum_{\sigma\in\mathcal{E}}|\sigma|h_\sigma|u_{\sigma,K}^n|^6\Big)^{1/6} & \\
+\sum_{K\in\mathcal{M}}\sum_{\sigma=K|L\in\mathcal{E}(K)}|\sigma|h_\sigma(\varrho_\sigma^{n,\rm up}-\varrho_K^n)^2|u_{\sigma,K}^n|1_{\overline{\varrho}_K^n<1}+|\Omega|^{5/6}\Big(\sum_{\sigma\in\mathcal{E}}|\sigma|h_\sigma|u_{\sigma,K}^n|^6\Big)^{1/6}\Big] & \\
\le c\sqrt{h_{\mathcal{M}}} & \quad (5.30)
\end{aligned}
$$

where c depends on $\underline{r},\overline{r},\|\nabla r\|_{L^\infty((0,T)\times\Omega)^3},\mathcal{E}_{0,\mathcal{M}}$ provided $\gamma\ge 12/11$. Here we have used estimate (4.16), estimate (4.10), (4.12) of Corollary 4.1.

Let us now decompose the term $T_{6,1}$ as $T_{6,1}=T_{6,2}+R_{6,2}$ with

$$
T_{6,2}=\delta t\sum_{n=1}^m\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm int}}|\sigma|\varrho_K^n\,\mathcal{H}''(r_K^{n-1})(r_K^{n-1}-r_\sigma^{n-1})u_{\sigma,K}^n, \quad (5.31)
$$

where

$$
R_{6,2}=\delta t\sum_{n=1}^m\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm int}}R_{6,2}^{n,\sigma,K},
$$

and

$$
R_{6,2}^{n,\sigma,K}=|\sigma|\varrho_K^n\Big(\mathcal{H}'(r_K^{n-1})-\mathcal{H}'(r_\sigma^{n-1})-\mathcal{H}''(r_K^{n-1})(r_K^{n-1}-r_\sigma^{n-1})\Big)u_{\sigma,K}^n
$$

Therefore, by virtue of the second order Taylor formula applied to function $\mathcal{H}'$, Hölder's inequality, (3.49), (3.50), (4.9), (4.13) in Corollary 4.1 and the geometric relations established in 2 we have,

$$
|R_{6,2}|\le ch_{\mathcal{M}} \quad (5.32)
$$

where c depends on $\underline{r},\overline{r},\|\nabla r\|_{L^\infty((0,T)\times\Omega)^3},\mathcal{E}_{0,\mathcal{M}}$ and on $\theta_{\mathcal{M}}$.

Let us now deal with the term $T_{6,2}$. First of all, let us remark that $\int_K\nabla r^{n-1}\,{\rm d}\boldsymbol{x}=\sum_{\sigma\in\mathcal{E}(K)}|\sigma|(r_\sigma^{n-1}-r_K^{n-1})\boldsymbol{n}_{\sigma,K}$. Therefore we may write

$$
T_{6,2}=T_{6,3}+R_{6,3},
$$

with

$$T_{6,3} = -\delta t \sum_{n=1}^{m} \int_{\Omega} \varrho^n \, \mathcal{H}''(r_{\mathcal{M}}^{n-1}) \, \mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n) \cdot \nabla r^{n-1} \, \mathrm{d}\boldsymbol{x}, \tag{5.33}$$

where $\mathcal{R}_{\mathcal{M}}$ is defined in (3.63) and where the remainder $R_{6,3}$ is given by

$$R_{6,3} = \delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{M}} \sum_{j=1}^{3} \sum_{\sigma \in \mathcal{E}^{(j)}(K)} |\sigma| \varrho_K^n \, \mathcal{H}''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1})(u_\sigma^n - (\mathcal{R}_{\mathcal{M}}^{(j)} u_j^n)_K) \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K},$$

Now in accordance with the proof of the remainder $R_{6,3}$ in Lemma 6.1 of [23] and by virtue of the Hölder's inequality, (3.49), (3.50), and (4.9), (4.12) in Corollary 4.1,

$$|R_{6,3}| \le c h_{\mathcal{M}}^{A} \tag{5.34}$$

where c depends on $\underline{r}, \overline{r}, \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)}, \mathcal{E}_{0,\mathcal{M}}$ and where A is defined in (3.77).

Finally we write $T_{6,3} = T_{6,4} + R_{6,4}$, with

$$\begin{aligned} T_{6,4} &= -\delta t \sum_{n=1}^{m} \int_{\Omega} \varrho^n \frac{p'(r_{\mathcal{M}}^n)}{r_{\mathcal{M}}^n} \mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n) \cdot \nabla r^n \, \mathrm{d}\boldsymbol{x}, \\ R_{6,4} &= \delta t \sum_{n=1}^{m} \int_{\Omega} \varrho^n \Big(\mathcal{H}''(r_{\mathcal{M}}^n) \nabla r^n - \mathcal{H}''(r_{\mathcal{M}}^{n-1}) \nabla r^{n-1} \Big) \cdot \mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n) \, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{5.35}$$

where by the same token as above the remainder $R_{6,4}$ satisfies

$$|R_{6,4}| \le c \, \delta t. \tag{5.36}$$

Here the constant c depends on $\underline{r}, \overline{r}, \|\nabla r, \partial_t r\|_{L^\infty(Q_T)^7}, \|\partial_t \nabla r\|_{L^2(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega)^3}$ and on $\mathcal{E}_{0,\mathcal{M}}$.

We are now in position to conclude the proof of Lemma 5.1: we obtain the inequality (5.14) by gathering the principal terms (5.19), (5.23), (5.26), (5.35) and the residual terms estimated in (5.18), (5.20), (5.22), (5.25), (5.27), (5.30), (5.32), (5.34), (5.36) at the right hand side $\sum_{i=1}^{6} T_i$ of the discrete relative energy inequality (5.1).

□

6 A consistency error

This section is devoted to the derivation of a discrete identity satisfied by any strong solution. This identity is stated in Lemma 6.1 below. It will be used in combination with the approximate relative energy inequality stated in Lemma 5.1 to deduce the convenient form of the relative energy inequality verified by any function being a strong solution to the compressible Navier-Stokes system. This last step is performed in the next section.

Lemma 6.1 (Consistency error). *Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{M},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution of the discrete problem (3.10). Let $(r, \boldsymbol{U})$ belonging to the class (3.71) such that $\boldsymbol{U}_{|(0,T)\times\partial\Omega} = 0$ and satisfying (1.1). Then there exists c only depending on parameters (5.11–5.13) such that for any $m = 1, \ldots, N$, the following identity holds:*

$$\begin{aligned} \delta t \sum_{n=1}^{m} \Big(\mu [\boldsymbol{U}_{\mathcal{E}}^n, \boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n]_{1,\mathcal{E},0} + (\mu+\lambda) \int_{\Omega} \operatorname{div} \boldsymbol{U}^n \operatorname{div}_{\mathcal{M}}(\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n) \, \mathrm{d}\boldsymbol{x} \Big) \\ + \delta t \sum_{n=1}^{m} \int_{\Omega} r_{\mathcal{M}}^{n-1} \frac{\boldsymbol{U}_{\mathcal{E}}^n - \boldsymbol{U}_{\mathcal{E}}^{n-1}}{\delta t} \cdot (\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n) \, \mathrm{d}\boldsymbol{x} \\ + \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{K \in \mathcal{M}} \sum_{j=1}^{3} \sum_{\sigma \in \mathcal{E}^{(j)}(K)} \Big[|\sigma| r_\sigma^{n,\mathrm{up}} (\mathcal{P}_{\mathcal{E}}^{(j)} U_j^n)_\sigma \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} \, \mathcal{R}_{\mathcal{E}}^{(i,j)} (u_i^n - \mathcal{P}_{\mathcal{E}}^{(i)} U_i^n)_\sigma \\ \times ((\mathcal{R}_{\mathcal{E}}^{(i,j)} U_i^n)_\sigma - \mathcal{R}_{\mathcal{M}}^{(i)} (\mathcal{P}_{\mathcal{E}}^{(i)} U_i^n)_K) \Big] \\ + \delta t \sum_{n=1}^{m} \int_{\Omega} p(r_{\mathcal{M}}^n) \operatorname{div} \boldsymbol{U}^n \, \mathrm{d}\boldsymbol{x} + \delta t \sum_{n=1}^{m} \int_{\Omega} p'(r_{\mathcal{M}}^n) \, \mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n) \cdot \nabla r^n \, \mathrm{d}\boldsymbol{x} + \mathcal{K}_{\mathcal{M},\delta t}^m = 0, \end{aligned} \tag{6.1}$$

where the remainder $\mathcal{K}^m_{\mathcal{M},\delta t}$ *satisfies*

$$|\mathcal{K}^m_{\mathcal{M},\delta t}| \le c\Big(h_{\mathcal{M}} + \delta t\Big).$$

Proof. Since $(r, \boldsymbol{U})$ satisfies (1.1) and belongs to the class (3.71), Equation (1.1*b*) can be rewritten in the form

$$r\partial_t \boldsymbol{U} + r\boldsymbol{U}\cdot\nabla\boldsymbol{U} + \nabla p(r) = \mu\Delta\boldsymbol{U} + (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u} \text{ in } (0,T)\times\Omega. \tag{6.2}$$

We write equation (6.2) at $t = t^n$, multiply scalarly by $\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}$, and integrate over Ω. We get, after summation from $n = 1$ to m, $\sum_{i=1}^5 Q_i$ where

$$\begin{aligned}
Q_1 &= \delta t\sum_{n=1}^m \int_\Omega r^n\partial_t\boldsymbol{U}^n\cdot(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})\,\mathrm{d}\boldsymbol{x},\\
Q_2 &= \delta t\sum_{n=1}^m \int_\Omega r^n\boldsymbol{U}^n\cdot\nabla\boldsymbol{U}^n\cdot(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})\,\mathrm{d}\boldsymbol{x},\\
Q_3 &= \delta t\sum_{n=1}^m \int_\Omega \nabla p(r^n)\cdot(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})\,\mathrm{d}\boldsymbol{x}\\
Q_4 &= -\delta t\sum_{n=1}^m \int_\Omega \mu\Delta\boldsymbol{U}^n\cdot(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})\,\mathrm{d}\boldsymbol{x},\\
Q_5 &= -\delta t\sum_{n=1}^m \int_\Omega (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u}^n\cdot(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

An adaptation of the proof of Lemma 7.1 in [23] gives

$$\begin{aligned}
Q_1 + Q_3 = \delta t\sum_{n=1}^m \int_\Omega r^{n-1}_{\mathcal{M}}\frac{\boldsymbol{U}^n_{\mathcal{E}} - \boldsymbol{U}^{n-1}_{\mathcal{E}}}{\delta t}(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})\,\mathrm{d}\boldsymbol{x} + \delta t\sum_{n=1}^m\int_\Omega p(r^n_{\mathcal{M}})\operatorname{div}\boldsymbol{U}^n\,\mathrm{d}\boldsymbol{x}\\
+ \delta t\sum_{n=1}^m\int_\Omega p'(r^n_{\mathcal{M}})\,\mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n)\cdot\nabla r^n\,\mathrm{d}\boldsymbol{x} + R^m_{\mathcal{M},\delta t},
\end{aligned}\tag{6.3}$$

where the remainder $R^m_{\mathcal{M},\delta t}$ satisfies

$$|R^m_{\mathcal{M},\delta t}| \le c\Big(h_{\mathcal{M}} + \delta t\Big).$$

and where the constant c depends on $\overline{r}, |p|_{C^2([\underline{r},\overline{r}])}, \|\nabla r\|_{L^\infty((0,T)\times\Omega)}, \|\partial_t\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}\|\nabla\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}$ and on $\|\partial^2_t\boldsymbol{U}\|_{L^2(0,T;L^{\frac{6}{5}}(\Omega))}, \|\partial_t\nabla\boldsymbol{U}\|_{L^2(0,T;L^{\frac{6}{5}}(\Omega))}, \mathcal{E}_{0,\mathcal{M}}$.

By virtue of the Stoke's formula we transform the term T_4 as follows, using (3.38), (3.40) and $d_\epsilon|\epsilon| = |D_\epsilon|$,

$$\begin{aligned}
Q_4 = -\delta t\sum_{n=1}^m \int_\Omega \mu\Delta\boldsymbol{U}^n\cdot(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}})\,\mathrm{d}\boldsymbol{x} &= \delta t\mu\sum_{n=1}^m\sum_{i=1}^3\sum_{j=1}^3\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}\\ \epsilon\perp\boldsymbol{e}^{(j)}}} d_\epsilon|\epsilon|\eth_j(u^n_i - (\mathcal{P}^{(i)}_{\mathcal{E}}\,U^n_i))_{D_\epsilon}\frac{1}{|\epsilon|}\int_\epsilon\frac{\partial}{\partial x_j}U^n_i\,\mathrm{d}\gamma\\
&= \delta t\mu\sum_{n=1}^m\sum_{i=1}^3\sum_{j=1}^3\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}\\ \epsilon\perp\boldsymbol{e}^{(j)}}} d_\epsilon|\epsilon|\eth_j(u^n_i - (\mathcal{P}^{(i)}_{\mathcal{E}}\,U^n_i))_{D_\epsilon}\eth_j(\mathcal{P}^{(i)}_{\mathcal{E}}\,U^n_i)_{D_\epsilon} + R^m_{\mathcal{M},\delta t}\\
&= \delta t\sum_{n=1}^m\mu[\boldsymbol{U}^n_{\mathcal{E}}, \boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}]_{1,\mathcal{E},0} + R^m_{\mathcal{M},\delta t}
\end{aligned}$$

where the remainder $R^m_{\mathcal{M},\delta t}$ is given by

$$\delta t\mu\sum_{n=1}^m\sum_{i=1}^3\sum_{j=1}^3\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}\\ \epsilon\perp\boldsymbol{e}^{(j)}}} d_\epsilon|\epsilon|\eth_j(u^n_i - (\mathcal{P}^{(i)}_{\mathcal{E}}\,U^n_i))_{D_\epsilon}\Big(\frac{1}{|\epsilon|}\int_\epsilon\frac{\partial}{\partial x_j}U^n_i\,\mathrm{d}\gamma - \eth_j(\mathcal{P}^{(i)}_{\mathcal{E}}\,U^n_i)_{D_\epsilon}\Big)$$

Moreover by virtue of (3.51) and (3.52) in Lemma 3.2 we can write for $\epsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}^{(i)}_{\text{int}},\ \epsilon \perp \boldsymbol{e}^{(j)}$,

$$\eth_j(\mathcal{P}^{(i)}_{\mathcal{E}}\, U_i^n)_{D_\epsilon} - \frac{1}{|\epsilon|}\int_\epsilon \frac{\partial}{\partial x_j} U_i^n \,\mathrm{d}\gamma = \frac{1}{d_\epsilon}(U_i(\boldsymbol{x}_{\sigma'}) - U_i(\boldsymbol{x}_\sigma)) - \frac{\partial}{\partial x_j} U_i^n(\boldsymbol{x}_\epsilon) + R^n_\epsilon$$
$$= \frac{\partial}{\partial x_j} U_i^n(\boldsymbol{x}_{\sigma,\sigma'}) - \frac{\partial}{\partial x_j} U_i^n(\boldsymbol{x}_\epsilon) + R^n_\epsilon$$

where $\boldsymbol{x}_{\sigma,\sigma'} \in [\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}]$ and where the remainder R^n_ϵ satisfies

$$|R^n_\epsilon| \le c h_{\mathcal{M}}.$$

Note that the case $\epsilon \in \tilde{\mathcal{E}}^{(i)}_{\text{ext}}$ can be treated in the same way. Consequently we have inequality

$$\Big|\frac{1}{|\epsilon|}\int_\epsilon \frac{\partial}{\partial x_j} U_i^n \,\mathrm{d}\gamma - \eth_j(\mathcal{P}^{(i)}_{\mathcal{E}}\, U_i^n)_{D_\epsilon}\Big| \le c h_{\mathcal{M}},\ \forall (i,j) \in \{1,2,3\}^2, \forall \epsilon \in \widetilde{\mathcal{E}}^{(i)},\ \epsilon \perp \boldsymbol{e}^{(j)},$$

where the constant c depends on $\|\nabla^2 \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}$. Therefore

$$|R^m_{\mathcal{M},\delta t}| \le c h_{\mathcal{M}} \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)} \\ \epsilon \perp \boldsymbol{e}^{(j)}}} |D_\epsilon| |\eth_j(u_i^n - (\mathcal{P}^{(i)}_{\mathcal{E}}\, U_i^n))_{D_\epsilon}| \le c h_{\mathcal{M}} \delta t \sum_{n=1}^m \|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}\|_{1,\mathcal{E},0}$$

where c depends on $\|\nabla^2 \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}$. Consequently by virtue of (3.55) and (4.9) we have

$$Q_4 = \delta t \sum_{n=1}^m \mu[\boldsymbol{U}^n_{\mathcal{E}}, \boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}]_{1,\mathcal{E},0} + R^m_{\mathcal{M},\delta t}, \tag{6.4}$$

where the remainder $R^m_{\mathcal{M},\delta t}$ satisfies

$$|R^m_{\mathcal{M},\delta t}| \le c h_{\mathcal{M}}$$

and where c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}, \|\nabla^2 \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}$, $\theta_{\mathcal{M}}$ and on $\mathcal{E}_{0,\mathcal{M}}$.

The term Q_5 can be treated exactly in the same way as Q_4 in order to obtain

$$Q_5 = -\delta t \sum_{n=1}^m (\mu + \lambda) \int_\Omega \operatorname{div} \boldsymbol{U}^n \operatorname{div}_{\mathcal{M}}(\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}) \,\mathrm{d}\boldsymbol{x} + R^m_{\mathcal{M},\delta t}, \tag{6.5}$$

where the remainder $R^m_{\mathcal{M},\delta t}$ satisfies

$$|R^m_{\mathcal{M},\delta t}| \le c h_{\mathcal{M}}.$$

where the constant c depends on $\|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}, \|\nabla^2 \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}$, $\theta_{\mathcal{M}}$ and on $\mathcal{E}_{0,\mathcal{M}}$.

Let us deal with the term Q_2. We have

$$Q_2 = \delta t \sum_{n=1}^m \sum_{i=1}^3 \int_\Omega r^n U^n \cdot \nabla U_i^n (u_i^n - \mathcal{P}^{(i)}_{\mathcal{E}}(U_i^n)) \,\mathrm{d}\boldsymbol{x}$$
$$= \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K \in \mathcal{M}} \int_K r^n_K \,\mathcal{R}^{(j)}_{\mathcal{M}}(\mathcal{P}^{(j)}_{\mathcal{E}}\, U_j^n)_K \partial_j U_i^n \,\mathcal{R}^{(i)}_{\mathcal{M}}(u_i^n - \mathcal{P}^{(i)}_{\mathcal{E}}(U_i^n))_K \,\mathrm{d}\boldsymbol{x} + R^m_{\mathcal{M},\delta t}$$

where the remainder $R^m_{\mathcal{M},\delta t}$ is given by

$$R^m_{\mathcal{M},\delta t} = \delta t \sum_{n=1}^m \int_\Omega (r^n - \mathcal{P}_{\mathcal{M}}(r^n)) \boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n \cdot (\boldsymbol{u}^n - \mathcal{P}_{\mathcal{E}}(\boldsymbol{U}^n)) \,\mathrm{d}\boldsymbol{x}$$
$$+ \delta t \sum_{n=1}^m \int_\Omega \mathcal{P}_{\mathcal{M}}(r^n)(\boldsymbol{U}^n - \mathcal{R}_{\mathcal{M}}\,\mathcal{P}_{\mathcal{E}}\,\boldsymbol{U}^n) \cdot \nabla \boldsymbol{U}^n \cdot (\boldsymbol{u}^n - \mathcal{P}_{\mathcal{E}}(\boldsymbol{U}^n)) \,\mathrm{d}\boldsymbol{x}$$
$$+ \delta t \sum_{n=1}^m \int_\Omega \mathcal{P}_{\mathcal{M}}(r^n)\,\mathcal{R}_{\mathcal{M}}\,\mathcal{P}_{\mathcal{E}}\,\boldsymbol{U}^n \cdot \nabla \boldsymbol{U}^n \cdot (\boldsymbol{u}^n - \mathcal{P}_{\mathcal{E}}(\boldsymbol{U}^n) - \mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n - \mathcal{P}_{\mathcal{E}}(\boldsymbol{U}^n))) \,\mathrm{d}\boldsymbol{x}.$$

By virtue of (3.53), (3.54), (3.65) and (4.9) the remainder $R^m_{\mathcal{M},\delta t}$ satisfies

$$|R^m_{\mathcal{M},\delta t}| \le c h_{\mathcal{M}},$$

where the constant c depends on $\overline{r}, \|\nabla r\|_{L^\infty((0,T)\times\Omega)}, \|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}$ and on $\mathcal{E}_{0,\mathcal{M}}$. Using the Stoke's formula we infer that

$$\begin{aligned}
\delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{M}} \int_K r^n_K\, \mathcal{R}^{(j)}_{\mathcal{M}}(\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_K \partial_j U^n_i\, \mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K \,\mathrm{d}\boldsymbol{x} \\
= \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}(K)} \Big[|\sigma| r^n_K\, \mathcal{R}^{(j)}_{\mathcal{M}}(\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_K \\
\times ((\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K)\boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K}\, \mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K \Big],
\end{aligned}$$

where we have used the identity

$$\int_K \partial_j U^n_i \,\mathrm{d}\boldsymbol{x} = \int_K \partial_j (U^n_i - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K)\,\mathrm{d}\boldsymbol{x}.$$

Finally keeping in mind the definition of the quantity $r^{n,\mathrm{up}}_\sigma$ (see (5.8)) we obtain

$$\begin{aligned}
\delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}(K)} \Big[|\sigma| r^n_K\, \mathcal{R}^{(j)}_{\mathcal{M}}(\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_K ((\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K)\boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K} \\
\times \mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K \Big] \\
= \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}_{\mathrm{int}}(K)} \Big[|\sigma| r^{n,\mathrm{up}}_\sigma (\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_\sigma ((\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K)\boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K} \\
\times \mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K \Big] + R^m_{\mathcal{M},\delta t}
\end{aligned}$$

where by virtue of (3.49), (3.50) the remainder $R^m_{\mathcal{M},\delta t}$ satisfies

$$|R^m_{\mathcal{M},\delta t}| \le c\delta t \sum_{n=1}^m \int_\Omega \|\,\mathcal{P}_{\mathcal{E}}\, \boldsymbol{U}^n\| \|\,\mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n - \mathcal{P}_{\mathcal{E}}(\boldsymbol{U}^n))\| \,\mathrm{d}\boldsymbol{x}$$

where c depends on $\|\nabla r\|_{L^\infty((0,T)\times\Omega)}$ and on $\|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}$. Consequently by virtue of (3.65) and (4.9)

$$|R^m_{\mathcal{M},\delta t}| \le c h_{\mathcal{M}}.$$

Now we write

$$\begin{aligned}
\delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}_{\mathrm{int}}(K)} \Big[|\sigma| r^{n,\mathrm{up}}_\sigma (\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_\sigma ((\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K)\boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K} \\
\times \mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K \Big] \\
= \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}_{\mathrm{int}}(K)} \Big[|\sigma| r^{n,\mathrm{up}}_\sigma (\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_\sigma \boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K} \\
\times \mathcal{R}^{(i,j)}_{\mathcal{E}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_\sigma ((\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K) \Big] + R^m_{\mathcal{M},\delta t}
\end{aligned}$$

where by virtue of (3.49), (3.64), (3.65) and (4.9) the remainder $R^m_{\mathcal{M},\delta t}$ satisfies

$$\begin{aligned}
R^m_{\mathcal{M},\delta t} = \delta t \sum_{n=1}^m \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}_{\mathrm{int}}(K)} \Big[|\sigma| r^{n,\mathrm{up}}_\sigma (\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_\sigma \boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K} ((\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K) \\
\times (\mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K - \mathcal{R}^{(i,j)}_{\mathcal{E}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_\sigma) \Big]
\end{aligned}$$

Consequently

$$|R^m_{\mathcal{M},\delta t}| \leq c\delta t \sum_{n=1}^{m}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}^{(j)}_{\rm int}(K)} |\sigma||(\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K| \\ \times |\,\mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K - \mathcal{R}^{(i,j)}_{\mathcal{E}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_\sigma|$$

From (3.49) we infer that for any $(i,j)\in\{1,2,3\}^2$, $K\in\mathcal{M}$, $\sigma\in\mathcal{E}^{(j)}_{\rm int}(K)$,

$$|(\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K| \leq c h_K,$$

which gives

$$|R^m_{\mathcal{M},\delta t}| \leq c\delta t \sum_{n=1}^{m}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}^{(j)}_{\rm int}(K)} |\sigma| h_\sigma |\,\mathcal{R}^{(i)}_{\mathcal{M}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}(U^n_i))_K - \mathcal{R}^{(i,j)}_{\mathcal{E}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_\sigma| \\ \leq c h_{\mathcal{M}}\delta t \sum_{n=1}^{m}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}^{(j)}_{\rm int}(K)} \|\eth_j(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)\|_{L^1(D_\sigma)},$$

which gives by virtue of (4.9) and (3.55),

$$|R^m_{\mathcal{M},\delta t}| \leq c h_{\mathcal{M}}$$

where c depends on $\overline{r}, \|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \|\nabla r\|_{L^\infty((0,T)\times\Omega)}, \|\nabla\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}, \mathcal{E}_{0,\mathcal{M}}$ and on $\theta_{\mathcal{M}}$. Consequently

$$Q_2 = \delta t \sum_{n=1}^{m}\sum_{i=1}^{3}\sum_{K\in\mathcal{M}}\sum_{j=1}^{3}\sum_{\sigma\in\mathcal{E}^{(j)}(K)} \Big(|\sigma| r^{n,\rm up}_\sigma (\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_j)_\sigma \boldsymbol{e}^{(j)}\cdot\boldsymbol{n}_{\sigma,K}\,\mathcal{R}^{(i,j)}_{\mathcal{E}}(u^n_i - \mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_\sigma \\ \times ((\mathcal{P}^{(j)}_{\mathcal{E}}\, U^n_i)_\sigma - \mathcal{R}^{(i)}_{\mathcal{M}}(\mathcal{P}^{(i)}_{\mathcal{E}}\, U^n_i)_K)\Big) + R^m_{\mathcal{M},\delta t} \quad (6.6)$$

where the remainder $R^m_{\mathcal{M},\delta t}$ satisfies

$$|R^m_{\mathcal{M},\delta t}| \leq c h_{\mathcal{M}}.$$

Summing (6.3) to (6.6) we obtain the expected result that is (6.1).

□

7 End of the proof of the error estimate Theorem 3.3

In this Section we put together the relative energy inequality (5.14) and the identity (6.1) derived in the previous section to obtain a discrete version of inequality (2.16). The final inequality resulting from this manipulation is formulated in the following lemma.

Lemma 7.1. *Under assumptions of Theorem 3.3 there exists c depending on parameters (5.11–5.13) such that for all $m = 1,\ldots,N$, there holds:*

$$\mathcal{E}(\varrho^m,\boldsymbol{u}^m\Big|r^m_{\mathcal{M}},\boldsymbol{U}^m_{\mathcal{E}}) + \delta t\frac{\mu}{2}\sum_{n=1}^{m}\|\boldsymbol{u}^n - \boldsymbol{U}^n_{\mathcal{E}}\|^2_{1,\mathcal{E},0} \\ \leq c\Big[h^A_{\mathcal{M}} + \sqrt{\delta t} + \mathcal{E}(\varrho^0,\boldsymbol{u}^0\Big|r^0_{\mathcal{M}},\boldsymbol{U}^0_{\mathcal{E}})\Big] + c\,\delta t\sum_{n=1}^{m}\mathcal{E}(\varrho^n,\boldsymbol{u}^n\Big|r^n_{\mathcal{M}},\boldsymbol{U}^n_{\mathcal{E}}),$$

where A is defined in (3.77).

Proof. Gathering the formulae (5.14) and (6.1), one gets

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m \Big| r_{\mathcal{M}}^m, U_{\mathcal{E}}^m) - \mathcal{E}(\varrho^0, \boldsymbol{u}^0 \Big| r_{\mathcal{M}}^0, U_{\mathcal{E}}^0) + \mu\delta t \sum_{n=1}^{m} \|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}^2 \leq \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{Q} \tag{7.1}$$

where

$$\mathcal{P}_1 = \delta t \sum_{n=1}^{m} \int_\Omega (\varrho^{n-1} - r_{\mathcal{M}}^{n-1}) \frac{\boldsymbol{U}_{\mathcal{E}}^n - \boldsymbol{U}_{\mathcal{E}}^{n-1}}{\delta t} \cdot \Big(\boldsymbol{U}_{\mathcal{E}}^n - \boldsymbol{u}^n\Big) \,\mathrm{d}\boldsymbol{x},$$

$$\mathcal{P}_2 = \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{K\in\mathcal{M}} \sum_{j=1}^{3} \sum_{\sigma\in\mathcal{E}^{(j)}(K)} \Big(|\sigma| (\varrho_\sigma^{n,\mathrm{up}} - r_\sigma^{n,\mathrm{up}}) (\mathcal{P}_{\mathcal{E}}^{(j)} U_j^n)_\sigma \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K}$$
$$\times (\mathcal{R}_{\mathcal{E}}^{(i,j)} (\mathcal{P}_{\mathcal{E}}^{(i)}(U_i^n) - u_i^n)_\sigma) (\mathcal{R}_{\mathcal{M}}^{(i)} (\mathcal{P}_{\mathcal{E}}^{(i)} U_i^n)_K - (\mathcal{P}_{\mathcal{E}}^{(j)} U_i^n)_\sigma) \Big),$$

$$\mathcal{P}_3 = \delta t \sum_{n=1}^{m} \int_\Omega \Big(p(r_{\mathcal{M}}^n) - p(\varrho^n) \Big) \operatorname{div} \boldsymbol{U}^n \,\mathrm{d}\boldsymbol{x}$$
$$+ \delta t \sum_{n=1}^{m} \int_\Omega \Big(\frac{r_{\mathcal{M}}^n - \varrho^n}{r_{\mathcal{M}}^n} p'(r_{\mathcal{M}}^n) \,\mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n) \cdot \nabla r^n + \frac{r_{\mathcal{M}}^n - \varrho_{\mathcal{M}}^n}{r_{\mathcal{M}}^n} p'(r_{\mathcal{M}}^n) [\partial_t r]^n \Big) \,\mathrm{d}\boldsymbol{x},$$

$$\mathcal{Q} = \mathcal{R}_{\mathcal{M},\delta t}^m + \mathcal{G}_{\mathcal{M},\delta t}^m + \mathcal{K}_{\mathcal{M},\delta t}^m,$$

and where the remainders $\mathcal{R}_{\mathcal{M},\delta t}^m, \mathcal{G}_{\mathcal{M},\delta t}^m$ and $\mathcal{K}_{\mathcal{M},\delta t}^m$ are explicited in Lemma 5.1 and Lemma 6.1.

Step 1: *Term* $\mathcal{P}_1$. Writing $\mathcal{P}_1 = \delta t \sum_{n=1}^m \mathcal{P}_1^n$, an application of the Taylor formula and of Lemma 2.1 gives, since $\gamma \geq \frac{6}{5}$:

$$|\mathcal{P}_1^n| \leq c \int_\Omega |\varrho^{n-1} - r_{\mathcal{M}}^{n-1}| |\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n| \,\mathrm{d}\boldsymbol{x}$$
$$\leq c \int_\Omega (\sqrt{E(\varrho^{n-1}, r_{\mathcal{M}}^{n-1})} + (E(\varrho^{n-1}, r_{\mathcal{M}}^{n-1}))^{5/6}) |\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n| \,\mathrm{d}\boldsymbol{x}$$
$$\leq c \Big(\int_\Omega E(\varrho^{n-1}, r_{\mathcal{M}}^{n-1})^{3/5} + E(\varrho^{n-1}, r_{\mathcal{M}}^{n-1}) \,\mathrm{d}\boldsymbol{x} \Big)^{5/6} \| \boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n \|_{L^6(\Omega,\mathbb{R}^3)}$$
$$\leq \frac{c}{\delta} \Big[\Big(\int_\Omega (E(\varrho^{n-1}, r_{\mathcal{M}}^{n-1}) \,\mathrm{d}\boldsymbol{x} \Big)^{5/3} + \int_\Omega E(\varrho^{n-1}, r_{\mathcal{M}}^{n-1})) \,\mathrm{d}\boldsymbol{x} \Big] + \delta \|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{L^6(\Omega,\mathbb{R}^3)}^2$$
$$\leq \frac{c}{\delta} \mathcal{E}(\varrho^{n-1}, \boldsymbol{u}^{n-1} \Big| r_{\mathcal{M}}^{n-1}, \boldsymbol{U}_{\mathcal{E}}^{n-1}) + \delta \|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}^2,$$

with any $\delta > 0$, where c depends on $\underline{r}, \overline{r}, \|\partial_t \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}$. Here we have used Theorem 3.1 to get a bound on $\|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{L^6(\Omega)^3}^2$ by $\|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}^2$, the Jensen inequality and the Young inequality to perform the before last inequality. Consequently

$$|\mathcal{P}_1| \leq \frac{c}{\delta} \Big(\mathcal{E}(\varrho^0, r_{\mathcal{M}}^0 \Big| \boldsymbol{u}^0, \boldsymbol{U}_{\mathcal{E}}^0) + \delta t \sum_{n=1}^{m} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r_{\mathcal{M}}^n, \boldsymbol{U}_{\mathcal{E}}^n) \Big) + \delta\, t\delta \sum_{n=1}^{m} \|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}^2. \tag{7.2}$$

Step 2: *Term* $\mathcal{P}_2$. We write $\mathcal{P}_2 = \delta t \sum_{n=1}^m \mathcal{P}_2^n$ where Lemma 2.1, the Hölder inequality and the geometric relations listed in Remark 2 yield, since $\gamma \geq \frac{3}{2}$,

$$|\mathcal{P}_2^n| \leq c \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}_{\mathrm{int}}} h_K |\sigma| |\varrho_\sigma^{n,\mathrm{up}} - r_\sigma^{n,\mathrm{up}}| \,|\, \mathcal{R}_{\mathcal{E}}^{(i,j)} (u_i^n - \mathcal{P}_{\mathcal{E}}^{(i)}(U_i^n))_\sigma|$$
$$\leq c \Big[\Big(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{int}}} |\sigma| h_\sigma \Big(E(\varrho_\sigma^{n,\mathrm{up}} | r_\sigma^{n,\mathrm{up}}) \Big)^{1/2} +$$
$$\Big(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{int}}} |\sigma| h_\sigma E(\varrho_\sigma^{n,\mathrm{up}} | r_\sigma^{n,\mathrm{up}}) \Big)^{2/3} \Big]$$
$$\times \Big(\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(j)}(K)} h_\sigma |\sigma| |\, \mathcal{R}_{\mathcal{E}}^{(i,j)} (u_i^n - \mathcal{P}_{\mathcal{E}}^{(i)}(U_i^n))_\sigma|^6 \Big)^{1/6}$$

where c depends on $\|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \|\nabla \boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^9}$ and on $\theta_{\mathcal{M}}$. Next, we observe that the contribution of the face $\sigma = K|L$ to the sums $\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)} |\sigma| h_\sigma E(\varrho_\sigma^{n,\rm up}|r_\sigma^{n,\rm up})$ is less or equal than $2|\sigma|h_\sigma(E(\varrho_K^n|r_K^n) + E(\varrho_L^n|r_L^n))$. Moreover using (3.58) and (3.64) we have

$$\Big(\sum_{i=1}^3\sum_{j=1}^3\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}^{(j)}_{\rm int}(K)} h_K|\sigma||\,\mathcal{R}_{\mathcal{E}}^{(i,j)}(u_i^n - \mathcal{P}_{\mathcal{E}}^{(i)}(U_i^n))_\sigma|^6\Big)^{1/6} \le c\|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{L^6(\Omega)^3} \le c\|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}.$$

where the constant c depends on $\theta_{\mathcal{M}}$ in a non increasing way.

Consequently, we get by the same reasoning as in the previous step, under assumption $\gamma \ge 3/2$,

$$|\mathcal{P}_2| \le \frac{c}{\delta}\delta t\sum_{n=1}^m \mathcal{E}(\varrho^n, \boldsymbol{u}^n\Big|r_{\mathcal{M}}^n, \boldsymbol{U}_{\mathcal{E}}^n) + \delta\,\delta t\sum_{n=1}^m \|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}^2. \tag{7.3}$$

where c depends on $\underline{r}, \overline{r}$ $\|\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}, \|\nabla\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^9}$ and $\theta_{\mathcal{M}}$ in a non increasing way.

Step 3: *Term* $\mathcal{P}_3$. Since the pair $(r,\boldsymbol{U})$ satisfies continuity equation (1.1a) in the classical sense, we have for all $n = 1,\dots,N$,

$$[\partial_t r]^n + \boldsymbol{U}^n\cdot\nabla r^n = -r^n\,{\rm div}\,\boldsymbol{U}^n,$$

where we recall that $[\partial_t r]^n(x) = \partial_t r(t^n, x)$ in accordance with (3.75). Using this identity we write

$$\begin{aligned}
&\mathcal{P}_3 = \mathcal{P}_{3,1} + \mathcal{P}_{3,2} + \mathcal{P}_{3,3}, \quad \mathcal{P}_{3,i} = \delta t\sum_{n=1}^m \mathcal{P}_{3,i}^n,\\
&\text{with } \mathcal{P}_{3,1}^n = -\int_\Omega \Big(p(\varrho^n) - p'(r_{\mathcal{M}}^n)(\varrho^n - r_{\mathcal{M}}^n) - p(r_{\mathcal{M}}^n)\Big)\,{\rm div}\,\boldsymbol{U}^n\,{\rm d}\boldsymbol{x}\\
&\mathcal{P}_{3,2}^n = \int_\Omega \frac{r_{\mathcal{M}}^n - \varrho^n}{r_{\mathcal{M}}^n}p'(r_{\mathcal{M}}^n)(\mathcal{R}_{\mathcal{M}}(\boldsymbol{u}^n) - \boldsymbol{U}^n)\cdot\nabla r^n\,{\rm d}\boldsymbol{x},\\
&\text{and } \mathcal{P}_{3,3}^n = \int_\Omega \frac{r_{\mathcal{M}}^n - \varrho^n}{r_{\mathcal{M}}^n}p'(r_{\mathcal{M}}^n)(r_{\mathcal{M}}^n - r^n)\,{\rm div}\,\boldsymbol{U}^n\,{\rm d}\boldsymbol{x}
\end{aligned}$$

From the asympototic behaviour (1.5) for large values of the pressure and Lemma 2.1 we easily deduce that

$$|\mathcal{P}_{3,1}| \le c\delta t\sum_{n=1}^m \mathcal{E}(\varrho^n, \boldsymbol{u}^n\Big|r_{\mathcal{M}}^n, \boldsymbol{U}_{\mathcal{E}}^n). \tag{7.4}$$

where c depends on $\underline{r}, \overline{r}, \min_{[\underline{r},\overline{r}]} p, \min_{[\underline{r}/2, 2\overline{r}]} p'$ and $\|\nabla\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)}$. From the total mass conservation (3.16) and (3.50) we deduce

$$|\mathcal{P}_{3,3}| \le ch_{\mathcal{M}} \tag{7.5}$$

where c depends on $\underline{r}, \overline{r}$, $\|\nabla r\|_{L^\infty((0,T)\times\Omega)^3}$, $\|\nabla\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}$ and on $\mathcal{E}_{0,\mathcal{M}}$.

Last but not least, the same reasoning as in Step 2 leads to the estimate

$$|\mathcal{P}_{3,2}| \le \frac{c}{\delta}\Big(h_{\mathcal{M}} + \delta t\sum_{n=1}^m \mathcal{E}(\varrho^n, \boldsymbol{u}^n\Big|r_{\mathcal{M}}^n, \boldsymbol{U}_{\mathcal{E}}^n)\Big) + \delta\,\delta t\sum_{n=1}^m \|\boldsymbol{u}^n - \boldsymbol{U}_{\mathcal{E}}^n\|_{1,\mathcal{E},0}^2. \tag{7.6}$$

with any $\delta > 0$, where c depends on $\underline{r}, \overline{r}$, $\|\nabla r\|_{L^\infty((0,T)\times\Omega)^3}$, $\|\nabla\boldsymbol{U}\|_{L^\infty((0,T)\times\Omega)^3}$ and on $\mathcal{E}_{0,\mathcal{M}}$ in a non increasing way. Gathering the formulae (7.1)-(7.6) with δ sufficiently small (with respect to μ), we conclude the proof of Lemma 7.1. □

Finally, Lemma 7.1 in combination with the bound (4.14) yields

$$\mathcal{E}(\varrho^m, \boldsymbol{u}^m\Big|r_{\mathcal{M}}^m, \boldsymbol{U}_{\mathcal{E}}^m) \le c\Big(h_{\mathcal{M}}^A + \delta t + \mathcal{E}(\varrho^0, \boldsymbol{u}^0\Big|r_{\mathcal{M}}^0, \boldsymbol{U}_{\mathcal{E}}^0)\Big) + c\delta t\sum_{n=1}^{m-1}\mathcal{E}(\varrho^n, \boldsymbol{u}^n\Big|r_{\mathcal{M}}^n, \boldsymbol{U}_{\mathcal{E}}^n)$$

whence Theorem 3.3 is a direct consequence of the standard discrete version of Gronwall's lemma. Theorem 3.3 is thus proved.

A Existence of a discrete solution

This section is devoted to the proof of Theorem 3.2. More precisely we are going to prove the following Proposition

Proposition A.1. *Consider a MAC grid $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ of Ω of size $h_{\mathcal{M}}$. Let $\delta t > 0$. Let $p : \mathbb{R} \to \mathbb{R}$ such that $p \in C^1(\mathbb{R}_+^\star)$. Let $(\varrho^\star, \boldsymbol{u}^\star) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ such that $\varrho^\star > 0$ a.e in Ω. Then there exists $(\varrho, \boldsymbol{u}) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ such that $\varrho > 0$ a.e in Ω which satisfies*

$$\frac{1}{\delta t}(\varrho - \varrho^\star) + \mathrm{div}^{\mathrm{up}}_{\mathcal{M}}(\varrho \boldsymbol{u}) = 0, \tag{A.1a}$$

$$\frac{1}{\delta t}(\widehat{\varrho}^{(i)} u_i - \widehat{\varrho^\star}^{(i)} u_i^\star) + \mathrm{div}^{(i)}_{\mathcal{E}}(\varrho \boldsymbol{u} u_i) - \mu \Delta^{(i)}_{\mathcal{E}} u_i \\ - (\mu + \lambda)\eth_i \,\mathrm{div}_{\mathcal{M}} \boldsymbol{u} + \eth_i p(\varrho) = 0, \ \forall i = 1, ...d. \tag{A.1b}$$

Proof. Let us state the abstract theorem which will be used hereafter.

Theorem A.1. *Let N and M be two positive integers and V be defined as follows:*

$$V = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M, \ y > 0\},$$

where, for any real number c, the notation $y > c$ means that each component of y is greater than c. Let F be a continuous function from $V \times [0, 1]$ to $\mathbb{R}^N \times \mathbb{R}^M$ satisfying:

1. *$\forall \zeta \in [0, 1]$, if $v \in V$ is such that $F(v, \zeta) = 0$ then $v \in W$ where W is defined as follows:*
$$W = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M, \ \|x\| \le C_1, \ \text{and } \epsilon < y < C_2\},$$
with C_1 , C_2 and ϵ three positive constants and $\|\cdot\|$ a norm defined over $\mathbb{R}^N$;
2. *the topological degree of $F(\cdot, 0)$ with respect to 0 and W is equal to $d_0 \ne 0$.*

Then the topological degree of $F(\cdot, 1)$ with respect to 0 and W is also equal to $d_0 \ne 0$; consequently, there exists at least a solution $v \in W$ such that $F(v, 1) = 0$.

We shall now prove the existence of a solution to (3.10). Let us define

$$V = \{(\boldsymbol{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}, \ \varrho_K > 0 \ \forall K \in \mathcal{M}\}.$$

and consider the mapping

$$F : V \times [0, 1] \longrightarrow \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}} \\ (\boldsymbol{u}, \varrho, \zeta) \mapsto F(\boldsymbol{u}, \varrho, \zeta) = (\hat{\boldsymbol{u}}, \hat{\varrho}),$$

where $(\hat{\boldsymbol{u}}, \hat{\varrho}) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$ is such that

$$\int_\Omega \hat{\boldsymbol{u}} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{\varrho \boldsymbol{u} - \varrho^\star \boldsymbol{u}^\star}{\delta t} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \mu[\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} + (\mu + \lambda) \int_\Omega \mathrm{div}_{\mathcal{M}} \boldsymbol{u} \,\mathrm{div}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \\ + \zeta \int_\Omega \mathrm{div}_{\mathcal{E}}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \zeta \int_\Omega p(\varrho) \,\mathrm{div}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \ \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}, \tag{A.2a}$$

$$\int_\Omega \hat{\varrho} \, q \,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{\varrho - \varrho^\star}{\delta t} q \,\mathrm{d}\boldsymbol{x} + \zeta \int_\Omega \mathrm{div}^{\mathrm{up}}_{\mathcal{M}}(\varrho \boldsymbol{u}) \, q \,\mathrm{d}\boldsymbol{x}, \ \forall q \in L_{\mathcal{M}}. \tag{A.2b}$$

Any solution of $F(\boldsymbol{u}, \varrho, 1) = 0$ is a solution of Problem A.1. Note also that in (*A.2a*) the fluxes $F_{\epsilon,\sigma}(\varrho, \boldsymbol{u})$ which determine $\mathrm{div}_{\mathcal{E}}(\varrho \boldsymbol{u} \otimes \boldsymbol{u})$ are constructed from the fluxes $F_{\sigma,K}(\varrho, \boldsymbol{u})$ which determine $\mathrm{div}^{\mathrm{up}}_{\mathcal{M}}(\varrho \boldsymbol{u})$ as in (3.19) and (3.20).

It is easily checked that F is indeed a one to one mapping, since the values of $\hat{u}_i; i = 1, \cdots, d$, and $\hat{\varrho}$ are readily obtained by setting in this system $v_i = 1_{D_\sigma}$, $v_j = 0, j \ne i$ in (A.2a) and $q = 1_K$ in (A.2b). Moreover, the mapping F is clearly continuous.

Let $(\boldsymbol{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$ and $\zeta \in [0,1]$ such that $F(\boldsymbol{u}, \varrho, \zeta) = (0,0)$ (in particular $\varrho > 0$). Then for any $(\boldsymbol{v}, q) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$,

$$\int_\Omega \frac{\varrho \boldsymbol{u} - \varrho^\star \boldsymbol{u}^\star}{\delta t} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \zeta \int_\Omega \mathrm{div}_{\mathcal{E}}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \mu[\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0}$$
$$+ (\mu + \lambda) \int_\Omega \mathrm{div}_{\mathcal{M}} \boldsymbol{u} \, \mathrm{div}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \zeta \int_\Omega p(\varrho) \, \mathrm{div}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = 0 \tag{A.3a}$$
$$\int_\Omega \frac{\varrho - \varrho^\star}{\delta t} q \,\mathrm{d}\boldsymbol{x} + \zeta \int_\Omega \mathrm{div}_{\mathcal{M}}^{\mathrm{up}} (\varrho \boldsymbol{u}) \, q \,\mathrm{d}\boldsymbol{x} = 0. \tag{A.3b}$$

Taking $q = 1$ as a test function in (*A.3b*), and using the conservativity of the fluxes we obtain

$$\int_\Omega \varrho \,\mathrm{d}\boldsymbol{x} = \|\varrho\|_{L^1(\Omega)} = \int_\Omega \varrho^\star \,\mathrm{d}\boldsymbol{x} > 0. \tag{A.4}$$

This relation provides a bound for ϱ in the L^1 norm, and therefore in all norms since the problem is of finite dimension.

Taking $\boldsymbol{u}$ as a test function in (*A.3a*) and following the proof of Theorem 4.1 gives

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0} \le C_1 \tag{A.5}$$

where the constant C_1 depends only on the data of the problem. Now a straightforward computation gives

$$\varrho_K \ge \frac{\min_{K \in \mathcal{M}} |K| \min_{K \in \mathcal{M}} \varrho_K^\star}{|\Omega| + \delta t \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}, \sigma = K|L} |u_{\sigma,K}|}.$$

Consequently by virtue of (*A.5*) there exists $\epsilon > 0$ such that

$$\varrho_K \ge \epsilon, \ \forall K \in \mathcal{M}, \tag{A.6}$$

where the constant ϵ depends only on the data of the problem. Clearly from (*A.4*) one has also

$$\varrho_K \le \frac{\int_\Omega \varrho^\star \,\mathrm{d}\boldsymbol{x}}{\min_{K \in \mathcal{M}} |K|} = C_2, \ \forall K \in \mathcal{M}. \tag{A.7}$$

Moreover $\zeta = 0$ the system $F(\boldsymbol{u}, \varrho, 0) = 0$ reads:

$$\int_\Omega \frac{\varrho \boldsymbol{u} - \varrho^\star \boldsymbol{u}^\star}{\delta t} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \mu[\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} + (\mu + \lambda) \int_\Omega \mathrm{div}_{\mathcal{M}} \boldsymbol{u} \, \mathrm{div}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = 0, \ \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}, \tag{A.8a}$$
$$\varrho_K = \varrho_K^\star, \ \forall K \in \mathcal{M}. \tag{A.8b}$$

which has clearly one and only one solution. Let W defined by

$$W = \{(\boldsymbol{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}} \text{ such that } \|\boldsymbol{u}\| \le C_1, \ \epsilon \le \varrho_K \le C_2\}$$

It is quite easy to see that the determinant of the Jacobian matrix does not vanish for the solution of the system (*A.8*). Therefore the topological degree d_0 of $F(\cdot,\cdot,0)$ with respect to 0 and W is not zero. Since the topological degree of $F(\cdot,\cdot,0)$ with respect to 0 and W does not vanish and by virtue of inequalities (*A.5*), (*A.6*), (*A.7*), Theorem A.1 applies, which concludes the proof.

□

B Error estimates for a class of staggered schemes

In this section we present some alternative numerical schemes for the approximation of problem (1.1)-(1.5), called staggered schemes. The space discretization in these schemes is staggered using nonconforming low-order finite element approximations, namely the Rannacher and Turek element (RT) [36] for quadrilateral or hexahedric meshes, or the lowest degree Crouzeix-Raviart element (CR) [8] for simplicial meshes. By the approach presented in this paper, it is possible to establish for these schemes similar error estimates as those established in Theorem 3.3 for the MAC scheme. The exact result is stated in Theorem B.1. We invite the reader wishing to read more about the discretizations of compressible flows via the staggered schemes to consult [21], [13], [20], [25].

B.1 Space and time discretization

From now, let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain. Let $\mathcal{M}$ be a decomposition of the domain Ω in simplices, wich we call hereafter a triangulation of Ω, regardless of the space dimension. By $\mathcal{E}(K)$, we denote the set of the edges (d=2) or faces (d=3) σ of the element $K \in \mathcal{M}$; for short, each edge or face will be caled an edge hereafter. The set of all edges of the mesh is denoted by $\mathcal{E}$; the set of edges included in the boundary of Ω is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal edges (i.e $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$. The triangulation $\mathcal{M}$ verifies the following assumption: : $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$; if $K, L \in \mathcal{T}$,then $\overline{K}\cap\overline{L} = \emptyset$, $\overline{K}\cap\overline{L}$ is a vertex or $\overline{K}\cap\overline{L}$ is a common edge of K and L , wich is denoted by $K|L$. For each internal edge of the mesh $\sigma = K|L$, $\boldsymbol{n}_{KL}$ stands for the normal vector of σ, oriented form K to L (so that $\boldsymbol{n}_{KL} = -\boldsymbol{n}_{LK}$). By $|K|$ and $|\sigma|$ we denote the (d and $d-1$ dimensional) measure, respectively, of an element K and of an edge σ, and h_K and h_σ stand for the diameter of K and σ, respectively. As in the MAC case, we measure the size of the mesh through the parameter $h_{\mathcal{M}}$ defined by

$$h_{\mathcal{M}} = \max\{h_K, K \in \mathcal{M}\}, \tag{B.1}$$

where h_K stands for the diameter of K. We measure the regularity of the mesh through the parameter $\theta_{\mathcal{M}}$ defined by

$$\theta_{\mathcal{M}} = \min\{\frac{\xi_K}{h_K},\ K \in \mathcal{M}\}, \tag{B.2}$$

where ξ_K stands for the diameter of the largest ball included in K.

Let us briefly describe the Crouzeix-Raviart element for simplicial meshes (see [8] for the seminal paper and, for instance, [10, p. 83-85], for a synthetic presentation), and the so-called 'rotated bilinear element' introduced by Rannacher and Turek for quadrilateral or hexahedric meshes [36]. The reference element for the Crouzeix-Raviart element is the unit d-simplex and the discrete function space is the space $\mathbb{P}_1$ of affine polynomials. The reference element $\hat{K}$ for the rotated bilinear element is the unit d-cube (with edges parallel to the coordinate axes); the discrete function space on $\hat{K}$ is $\tilde{\mathbb{Q}}_1(\hat{K})$, where $\tilde{\mathbb{Q}}_1(\hat{K})$ is defined as follows

$$\tilde{\mathbb{Q}}_1(\hat{K}) = \operatorname{span}\{1, (x_i)_{i=1,\dots,d}, (x_i^2 - x_{i+1}^2)_{i=1,\dots,d-1}\}.$$

For both velocity elements used here, the degrees of freedom are determined by the following set of nodal functionals:

$$\{m_{\sigma,i},\ \sigma \in \mathcal{E}(K),\ i = 1, ...d\}, \quad m_{\sigma,i}(\boldsymbol{v}) = \frac{1}{|\sigma|}\int_\sigma v_i \,\mathrm{d}\boldsymbol{x},\ \boldsymbol{v} = (v_1, ..., v_d)\}. \tag{B.3}$$

The mapping from the reference element to the actual one is, for the Rannacher-Turek element, the standard Q_1 mapping and, for the Crouzeix-Raviart element, the standard affine mapping. Finally, in both cases, the continuity of the average value of discrete velocities (i.e., for a discrete velocity field $\boldsymbol{v}$, $m_{\sigma,i}(\boldsymbol{v})$, $1 \le i \le d$) across each edge of the mesh is required, thus the discrete space $\boldsymbol{W}_{\mathcal{E},0}(\Omega)$ is defined as follows:

$$\begin{aligned}\boldsymbol{W}_{\mathcal{E},0}(\Omega) = [W_{\mathcal{E},0}(\Omega)]^d = \{&\boldsymbol{v} \in L^2(\Omega)^d,\ \forall K \in \mathcal{M},\ \boldsymbol{v}_{|K} \in W(K)^d \text{ and } \forall\sigma = K|L \in \mathcal{E}_{\rm int},\\ &m_{\sigma,i}(\boldsymbol{v}_{|K}) = m_{\sigma,i}(\boldsymbol{v}_{|L}), \forall\sigma \in \mathcal{E}_{\rm ext}, m_{\sigma,i}(\boldsymbol{v}) = 0\}\end{aligned}$$

where $W(K)$ is the space of functions on K generated by $\tilde{\mathbb{Q}}_1(\hat{K})$ through the Q_1 mapping from $\hat{K}$ to K for the Rannacher-Turek element and the space of affine functions on K for the Crouzeix-Raviart element.

From the definition ($B.3$), each velocity degree of freedom can be uniquely associated to an element edge. More precisely the degrees of freedom for the velocity components are located at the center of the faces of the mesh. Hence, the velocity degrees of freedom may be indexed by the number of the component and the associated edge, and the set of velocity degrees of freedom reads:

$$\{\boldsymbol{u}_\sigma,\ \sigma \in \mathcal{E}\}.$$

Finally, we need to deal with the Dirichlet boundary condition. Since the velocity unknowns lie on the boundary (and not inside the cells), these conditions are taken into account in the definition of the discrete spaces by setting zero to the velocity unknows that lie on the boundary

$$\forall \sigma \in \mathcal{E}_{\text{ext}}, \qquad \boldsymbol{u}_\sigma = 0. \tag{B.4}$$

Since only the continuity of the integral over each edge of the mesh is imposed, the functions of $\boldsymbol{W}_{\mathcal{E},0}(\Omega)$ are discontinous through each edge; the discretization is thus nonconforming in $H^1(\Omega)^d$.

We denote by φ_σ the function of $W_{\mathcal{E},0}(\Omega)$ such that

$$\int_{\sigma'} \varphi_\sigma \,\mathrm{d}\gamma = |\sigma'|\delta_{\sigma,\sigma'} \text{ for any } \sigma, \sigma' \in \mathcal{E}_{\text{int}}\,. \tag{B.5}$$

The degrees of freedom for the density (*i.e.* the discrete density unknowns) are associated to the cells of the mesh $\mathcal{M}$, and are denoted by:

$$\big\{\varrho_K,\ K \in \mathcal{M}\big\}.$$

We now introduce a dual mesh, which will be used for the finite volume approximation of the time derivative and convection terms in the momentum balance equation. In contrast with the MAC scheme, the dual mesh is the same for all velocity components. When $K \in \mathcal{M}$ is a simplex, a rectangle or a cuboid, for $\sigma \in \mathcal{E}(K)$, we define $D_{\sigma,K}$ as the cone with basis σ and with vertex the mass center of K (see Figure 1). We thus obtain a partition of K in m sub-volumes, where m is the number of faces of the mesh, each sub-volume having the same measure $|D_{\sigma,K}| = |K|/m$. We extend this definition to general quadrangles and hexahedra, by supposing that we have built a partition still of equal-volume sub-cells, and with the same connectivities. Note that this is of course always possible, but that such a volume $D_{\sigma,K}$ may be no longer a cone; indeed, if K is far from a parallelogram, it may not be possible to build a cone having σ as basis, the opposite vertex lying in K and a volume equal to $|K|/m$. The volume $D_{\sigma,K}$ is referred to as the half-diamond cell associated to K and σ.
For $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we now define the diamond cell D_σ associated to σ by $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$; for an external face $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$, D_σ is just the same volume as $D_{\sigma,K}$. We define the space $\boldsymbol{S}_{\mathcal{E}}(\Omega)$ of vector valued functions constant on every D_σ, $\sigma \in \mathcal{E}$. We denote by $\boldsymbol{S}_{\mathcal{E},0}(\Omega)$ the subspace of functions from $\boldsymbol{S}_{\mathcal{E}}(\Omega)$ that are zero on every D_σ, $\sigma \in \mathcal{E}_{\text{ext}}$. We then introduce the following operator

$$\mathcal{P}_{\mathcal{E}} : \left| \begin{array}{l} \boldsymbol{W}_{\mathcal{E},0}(\Omega) \longrightarrow \boldsymbol{S}_{\mathcal{E},0}(\Omega) \\ \boldsymbol{u} \longmapsto \mathcal{P}_{\mathcal{E}}\, \boldsymbol{u} = \displaystyle\sum_{\sigma \in \mathcal{E}_{\text{int}}} \boldsymbol{u}_\sigma\ \mathcal{X}_{D_\sigma}(\boldsymbol{x}), \end{array} \right. \tag{B.6}$$

which is clearly a one to one mapping.

The density on a dual cell is given by:

$$\begin{array}{ll} \text{for } \sigma \in \mathcal{E}_{\text{int}},\ \sigma = K|L & |D_\sigma|\ \varrho_{D_\sigma} = |D_{\sigma,K}|\ \varrho_K + |D_{\sigma,L}|\ \varrho_L, \\ \text{for } \sigma \in \mathcal{E}_{\text{ext}},\ \sigma \in \mathcal{E}(K), & \varrho_{D_\sigma} = \varrho_K. \end{array} \tag{B.7}$$

and we denote

$$\hat{\varrho} = \sum_{\sigma \in \mathcal{E}} \varrho_{D_\sigma} \mathcal{X}_{D_\sigma}(\boldsymbol{x}).$$

For the the time discretization of problem (1.1)-(1.5), we consider a partition $0 = t^0 < t^1 < \cdots < t^N = T$ of the time interval $(0, T)$, and, for the sake of simplicity, a constant time step $\delta t = t^n - t^{n-1}$; hence $t^n = n\delta t$ for $n \in \{0, \cdots, N\}$. We denote respectively by $\{\boldsymbol{u}^n_\sigma, \sigma \in \mathcal{E}_{\text{int}}, n \in \{0, \cdots, N\}\}$, and $\{\varrho^n_K, K \in \mathcal{M}, n \in \{1, \cdots, N\}\}$ the sets of discrete velocity and density unknowns. For $\sigma \in \mathcal{E}_{\text{int}}$, the value $\boldsymbol{u}^n_\sigma$ is an expected approximation of the mean value over $(t^{n-1}, t^n) \times D_\sigma$ of the velocity of a weak solution, while for $K \in \mathcal{M}$ the value ϱ^n_K is an expected approximation of the mean value over $(t^{n-1}, t^n) \times K$ of the density of a weak solution. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, which are expected approximation of weak solutions, For the velocity, this constant function is of the form:

$$\boldsymbol{u}(t,\boldsymbol{x}) = \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{E}_{\rm int}} \boldsymbol{u}_\sigma^n\, \mathcal{X}_{D_\sigma}(\boldsymbol{x})\, \mathcal{X}_{(t^{n-1},t^n)}(t),$$

where $\mathcal{X}_{(t^{n-1},t^n)}$ is the characteristic function of the interval (t^{n-1},t^n). We denote by $\boldsymbol{X}_{\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells. For the density, the constant function is of the form:

$$\varrho(t,\boldsymbol{x}) = \varrho_K^n \text{ for } \boldsymbol{x}\in K \text{ and } t\in(t^{n-1},t^n),$$

and we denote by $Y_{\mathcal{M},\delta t}$ the space of such piecewise constant functions.

For a given $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ associated to the set of discrete velocity unknowns $\{\boldsymbol{u}_\sigma^n, \sigma\in\mathcal{E}_{\rm int}, n\in\{1,\cdots,N\}\}$, and for $n\in\{1,\cdots,N\}$, we denote by $\boldsymbol{u}^n \in \boldsymbol{S}_{\mathcal{E},0}(\Omega)$ the piecewise constant function defined by $\boldsymbol{u}^n(\boldsymbol{x}) = \boldsymbol{u}_\sigma^n$ for $\boldsymbol{x}\in D_\sigma, \sigma\in\mathcal{E}_{\rm int}$. In a same way, given $\varrho\in Y_{\mathcal{M},\delta t}$ associated to the discrete density unknows $\{\varrho_K^n, K\in\mathcal{M}, n\in\{1,\cdots,N\}\}$ we denote by $\varrho^n\in L_{\mathcal{M}}$ the piecewise constant function defined by $\varrho^n(\boldsymbol{x}) = \varrho_K^n$ for $\boldsymbol{x}\in K$, $K\in\mathcal{M}$.

We consider an implicit-in-time scheme, which reads in its fully discrete form, for $1\le n\le N$ and $1\le i\le d$:

$$\frac{1}{\delta t}(\varrho^n - \varrho^{n-1}) + {\rm div}_{\mathcal{M}}^{\rm up}(\varrho^n\boldsymbol{u}^n) = 0, \tag{B.8a}$$

$$\frac{1}{\delta t}(\widehat{\varrho^n}\boldsymbol{u}^n - \widehat{\varrho^{n-1}}\boldsymbol{u}^{n-1}) + {\rm div}_{\mathcal{E}}(\varrho^n\boldsymbol{u}^n\otimes\boldsymbol{u}^n) - \mu\Delta_{\mathcal{E}}\boldsymbol{u}^n \\ - (\mu+\lambda)\nabla_{\mathcal{E}}\,{\rm div}_{\mathcal{M}}\,\boldsymbol{u}^n + \nabla_{\mathcal{E}}p(\varrho^n) = 0, \tag{B.8b}$$

where the terms introduced for each discrete equation are defined hereafter.

B.1.1 Mass balance equation

As for the MAC scheme, equation (3.10a) is a finite volume discretization of the mass balance (1.1a) over the primal mesh. The discrete "upwind" divergence is defined by

$$\left.{\rm div}_{\mathcal{M}}^{\rm up} : \quad \right| \begin{array}{l} S_{\mathcal{M}}(\Omega)\times\boldsymbol{S}_{\mathcal{E},0}(\Omega) \longrightarrow S_{\mathcal{M}}(\Omega) \\ (\varrho,\boldsymbol{u}) \longmapsto {\rm div}_{\mathcal{M}}^{\rm up}(\varrho\boldsymbol{u}) = \displaystyle\sum_{K\in\mathcal{M}} \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} F_{\sigma,K}(\varrho,\boldsymbol{u})\ \mathcal{X}_K, \end{array} \tag{B.9}$$

where $F_{\sigma,K}(\varrho,\boldsymbol{u})$ stands for the mass flux across σ outward K, which, because of the Dirichlet boundary conditions, vanishes on external faces and is given on the internal faces by:

$$\forall\sigma = K|L\in\mathcal{E}_{\rm int}, \qquad F_{\sigma,K}(\varrho,\boldsymbol{u}) = |\sigma|\ \varrho_\sigma^{\rm up}\ u_{\sigma,K}, \tag{B.10}$$

where $u_{\sigma,K}$ is an approximation of the normal velocity to the face σ outward K, defined by:

$$u_{\sigma,K} = \boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K} \text{ for } \sigma\in\mathcal{E}(K). \tag{B.11}$$

Thanks to the boundary conditions, $u_{\sigma,K}$ vanishes for any external face σ. The density at the internal face $\sigma = K|L$ is obtained by an upwind technique:

$$\varrho_\sigma^{\rm up} = \left| \begin{array}{ll} \varrho_K & \text{if } u_{\sigma,K}\ge 0, \\ \varrho_L & \text{otherwise.} \end{array} \right. \tag{B.12}$$

B.1.2 The momentum equation

We now turn to the discrete momentum balances (3.10b), which are obtained by discretizing the momentum balance equation (1.1b) on the dual cells associated to the faces of the mesh.

The discrete convective operator - The discrete divergence of the convective term $\varrho \boldsymbol{u} \otimes \boldsymbol{u}$ is defined by

$$\mathrm{div}_{\mathcal{E}} : \left| \begin{array}{l} S_{\mathcal{M}}(\Omega) \times \boldsymbol{S}_{\mathcal{E},0}(\Omega) \longrightarrow \boldsymbol{S}_{\mathcal{E},0}(\Omega) \\ (\varrho, \boldsymbol{u}) \longmapsto \mathrm{div}_{\mathcal{E}}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) = \displaystyle\sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \frac{1}{|D_\sigma|} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}(\varrho, \boldsymbol{u})\, \boldsymbol{u}_\epsilon\, \mathcal{X}_{D_\sigma}, \end{array} \right. \tag{B.13}$$

where for $\sigma \in \mathcal{E}_{\mathrm{int}}$ and $\epsilon \in \mathcal{E}(D_\sigma)$ the quantity $F_{\epsilon,\sigma} = F_{\epsilon,\sigma}(\varrho, \boldsymbol{u})$ stands for a mass flux through the dual faces of the mesh and are defined hereafter while u_ϵ stands for an approximation of i^{th} component of the velocity over ϵ in the case of $\sigma \in \mathcal{E}^{(i)}$. First of all by virtue of the Dirichlet boundary condition, that the flux through a dual face included in the boundary is taken equal to zero. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, let ξ_K^σ be given by:

$$\xi_K^\sigma = \frac{|D_{\sigma,K}|}{|K|}. \tag{B.14}$$

With the definition of the dual mesh adopted here, the value of the coefficients ξ_K^σ is independant of the cell and the face. For the Rannacher-Turek elements, we have $\xi_K^\sigma = 1/(2d)$ and, for the Crouzeix-Raviart elements, $\xi_K^\sigma = 1/(d+1)$. We suppose first that the flux through the external dual faces, which are also faces of the primal mesh, is equal to zero.

Then the mass fluxes through the inner dual faces are supposed to satisfy the following properties.

Definition B.1 (Definition of the dual fluxes from the primal ones)**.** *The fluxes through the faces of the dual mesh are defined so as to satisfy the following three constraints:*

(H1) The discrete mass balance over the half-diamond cells is satisfied, in the following sense. For all primal cell K in $\mathcal{M}$, the set $(F_{\epsilon,\sigma})_{\epsilon \subset K}$ of dual fluxes included in K solves the following linear system

$$F_{\sigma,K} + \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_{\epsilon,\sigma} = \xi_K^\sigma \sum_{\sigma' \in \mathcal{E}(K)} F_{\sigma',K}, \qquad \sigma \in \mathcal{E}(K). \tag{B.15}$$

(H2) The dual fluxes are conservative, i.e. *for any dual face $\epsilon = D_\sigma | D'_\sigma$, we have $F_{\epsilon,\sigma} = -F_{\epsilon,\sigma'}$.*

(H3) The dual fluxes are bounded with respect to the primal fluxes $(F_{\sigma,K})_{\sigma \in \mathcal{E}(K)}$, in the sense that there exists a constant real number C such that:

$$|F_{\epsilon,\sigma}| \leq C\ \max\left\{|F_{\sigma,K}|,\ \sigma \in \mathcal{E}(K)\right\}, \quad K \in \mathcal{M},\ \sigma \in \mathcal{E}(K),\ \epsilon \in \widetilde{\mathcal{E}}(D_\sigma),\ \epsilon \subset K. \tag{B.16}$$

In fact, definition B.1 is not complete, since the system of equations (B.15) has an infinite number of solutions, which makes necessary to impose in addition the constraint (B.16); however, assumptions (H1)-(H3) are sufficient for the subsequent developments of this paper. A detailed process of the dual fluxes construction can be found in [1, 24].

Since the flux across a dual face lying on the boundary is zero, the values u_ϵ are only needed at the internal dual faces, and we make the centered choice for their discretization, i.e. for $\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}_{\mathrm{int}}$,

$$\boldsymbol{u}_\epsilon = \frac{\boldsymbol{u}_\sigma + \boldsymbol{u}_{\sigma'}}{2}. \tag{B.17}$$

The discrete divergence and gradient - The discrete divergence $\mathrm{div}_{\mathcal{M}} \in \mathcal{L}(\boldsymbol{S}_{\mathcal{E},0}(\Omega), S_{\mathcal{M}}(\Omega))$ of the velocity (or more generally of a function $\boldsymbol{S}_{\mathcal{E},0}(\Omega)$) has a natural approximation:

$$\text{for } K \in \mathcal{M}, \qquad (\mathrm{div}_{\mathcal{M}}\, \boldsymbol{u})_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma|\ u_{\sigma,K}. \tag{B.18}$$

The term $(\boldsymbol{\nabla}_{\mathcal{E}} p)_\sigma$ stands for the discrete pressure gradient at the face σ. This gradient operator, which belongs to $\mathcal{L}(S_{\mathcal{M}}(\Omega), \boldsymbol{S}_{\mathcal{E},0}(\Omega))$ is built as the transpose of the discrete operator for the divergence of the velocity, *i.e.* in such a way that the following duality relation with respect to the L^2 inner product holds:

$$\sum_{K\in\mathcal{M}} |K|\ p_K\ (\mathrm{div}_{\mathcal{M}}\,\boldsymbol{u})_K + \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma|\ \boldsymbol{u}_\sigma \cdot (\boldsymbol{\nabla}_{\mathcal{E}} p)_\sigma = 0. \tag{B.19}$$

This yields to the following expression:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad (\boldsymbol{\nabla}_{\mathcal{E}} p)_\sigma = \frac{|\sigma|}{|D_\sigma|}\ (p_L - p_K)\ \boldsymbol{n}_{\sigma,K}. \tag{B.20}$$

Note that, because of the Dirichlet boundary conditions, the discrete gradient is not defined at the external faces.

Discrete Laplace operator - The discrete Laplace operator $\Delta_{\mathcal{E}} \in \mathcal{L}(\boldsymbol{S}_{\mathcal{E},0}(\Omega), \boldsymbol{S}_{\mathcal{E},0}(\Omega))$ reads for $\boldsymbol{u} \in \boldsymbol{S}_{\mathcal{E},0}(\Omega)$ and $\sigma \in \mathcal{E}_{\mathrm{int}}$:

$$(-\Delta_{\mathcal{E}}\boldsymbol{u})_\sigma = \int_\Omega \nabla_{\mathcal{M}}\,\mathcal{P}_{\mathcal{E}}^{-1}\,\boldsymbol{u} : \nabla_{\mathcal{M}}\boldsymbol{\varphi}_\sigma\,\mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{\varphi}_\sigma = (\varphi_\sigma, ..., \varphi_\sigma) \in \boldsymbol{W}_{\mathcal{E},0}(\Omega)$, where the shape function φ_σ is introduced in ($B.5$) and where $\mathcal{P}_{\mathcal{E}}$ is defined in ($B.6$). In the above formula and for a function $\boldsymbol{v} \in \boldsymbol{W}_{\mathcal{E},0}(\Omega)$, the quantity $\nabla\boldsymbol{v}$ is equal to the gradient of the function $\boldsymbol{v}$ almost everywhere in Ω.

Here again let us introduce the discrete relative energy functional

$$\mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r^n_{\mathcal{M}}, \boldsymbol{U}^n_{\mathcal{E}}) = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \frac{1}{2}|D_\sigma|\varrho^n_{D_\sigma}|\boldsymbol{u}^n_\sigma - \boldsymbol{U}^n_\sigma|^2 + \sum_{K\in\mathcal{M}} |K|E(\varrho^n_K|r^n_K) \tag{B.21}$$

where

$$r^n = r(t^n,\cdot),\ \boldsymbol{U}^n = \boldsymbol{U}(t^n,\cdot),\ r^n_K = \frac{1}{|K|}\int_K r^n\,\mathrm{d}\boldsymbol{x},\ \boldsymbol{U}^n_\sigma = \frac{1}{|\sigma|}\int_\sigma \boldsymbol{U}^n\,\mathrm{d}\gamma\,. \tag{B.22}$$

Now, we are ready to state the result about the error estimate for these alternative discretizations.

Theorem B.1 (Error estimate)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain. Let $\mathcal{M}$ be a decomposition of the domain Ω in simplices, with step size $h_{\mathcal{M}}$ (see ($B.1$)) and regularity $\theta_{\mathcal{M}}$ where $\theta_{\mathcal{M}}$ is defined in ($B.2$). Let us consider a partition $0 = t^0 < t^1 < ... < t^N = T$ of the time interval $[0,T]$, which, for the sake of simplicity, we suppose uniform where δt stands for the constant time step. Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{M},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution of the discrete problem ($B.8$) emanating from $(\varrho^0, \boldsymbol{u}^0) \in S_{\mathcal{M}}(\Omega) \times \boldsymbol{S}_{\mathcal{E},0}(\Omega)$ such that $\varrho^0 > 0$ and $(r, \boldsymbol{U}) \in \mathcal{F}$ (see (3.71)) be a (strong) solution of problem (1.1)-(1.5). Then there exists a constant $c > 0$ only depending on $T, \Omega, p_0, p_\infty, \mu, \gamma, \alpha, \underline{r}, \min_{[\underline{r},\overline{r}]} p, \min_{[\underline{r}/2,2\overline{r}]} p', on\|(r,\boldsymbol{U})\|_{\mathcal{F}}$ in a nondecreasing way, on $\mathcal{E}_{0,\mathcal{M}}$ in a nondecreasing way and on $\theta_{\mathcal{M}}$ in a nonincreasing way such that*

$$\max_{0\le n\le N} \mathcal{E}(\varrho^n, \boldsymbol{u}^n \Big| r^n_{\mathcal{M}}, \boldsymbol{U}^n_{\mathcal{E}}) \le c\Big(\mathcal{E}(\varrho^0, \boldsymbol{u}^0 \Big| r^0_{\mathcal{M}}, \boldsymbol{U}^0_{\mathcal{E}}) + h^A_{\mathcal{M}} + \sqrt{\delta t}\Big), \tag{B.23}$$

where A is given by

$$A = \min(\frac{2\gamma-3}{\gamma}, \frac{1}{2}). \tag{B.24}$$

Remark 5. *1. The discrete problem ($B.8$) admits a solution. As for the MAC case, the proof is based on a topological degree argument.*

2. Note that the exponent A is the same for all discretizations investigated in this paper.

does not differ from the discretization used. It is a consequence of the used for the continuity equation which is the same for each discretization.

3. The items listed in Remark 3 remain valid also for discretizations described in this section.

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Implicit MAC scheme for compressible Navier-Stokes equations: Low Mach asymptotic error estimates

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Abstract

We investigate error between any discrete solution of the implicit Marker and Cell (MAC) numerical scheme for compressible Navier-Stokes equations in low Mach number regime and an exact strong solution of the incompressible Navier-Stokes equations. The main tool is the relative energy method suggested on the continuous level in [7], whose discrete numerical version has been developed in [19]. We get unconditional error estimate in terms of explicitly determined positive powers of the space-time disretization parameters and Mach number in the case of well prepared initial data, and the boundedness of the error if the initial data are ill prepared. The multiplicative constant in the error estimate depends on the suitable norm of the strong solution but is independent on the numerical solution itself (and of course, on the disretization parameters and the Mach number). This is the first proof ever that the MAC scheme is unconditionally and uniformly asymptotically stable at the low Mach number regime.

Key words: Navier-Stokes system, finite difference numerical method, finite volume numerical method, Marker and Cell scheme, error estimate

AMS classification 35Q30, 65N12, 65N30, 76N10, 76N15, 76M10, 76M12

1 Introduction

In [20], we have derived unconditional error estimates for the Marker and Cell (MAC) numerical scheme for the compressible Navier-Stokes equations. The goal of this paper is to investigate the low Mach number asymptotic for this discretization. The aim is to estimate the error of the MAC discrete numerical solution on a mesh of size h and time step δt in the MAC discrete function space with respect to a convenient projection to the discrete numerical space of the unique strong solution of the incompressible Navier-Stokes equations in terms of the (positive) powers of h, δt and Mach number ε. The multiplicative constant in this estimate must be independent of the numerical solution (and of course of h, δt and ε); it may however depend on the norm of the strong solution $(\Pi, \mathbf{V})$ of the target problem in a convenient functional space of sufficiently regular functions. In particular, we shall not require any additional information on the numerical solution than the information provided by the algebraic numerical scheme itself.

Such type of estimates are referred as (unconditional) error estimates in the numerical analysis of PDEs. The numerical schemes possessing this type of error estimates are referred as (uniformly) asymptotic preserving. In spite of the importance of this property for applications, the mathematical literature on this subject is in a short supply, mostly due to the complexity of the problem: the rigorous asymptotic preserving error estimates are known solely on the level of the numerical schemes, and, in this case the error estimate depends on the space-time discretization. This philosophy is pursued for example in papers [1], [4], [16], [25], [33], [34], [35], [36]. This type of estimates does not provide any information on the convergence of the scheme, and this is a serious drawback. To the best of our knowledge, we present here the first *unconditional and uniform* result providing quantitatively *an uniform convergence rate* in terms of space-time discretization $(h, \delta t)$ and Mach number ε for the MAC scheme (compare with [8] establishing asymptotic preserving estimates for an academic FEM/DG scheme). Its importance and interest is underlined by the fact that the Marker and Cell scheme in its explicit or semi-implicit form constitutes the basis for many industrially ran codes in fluid mechanics.

The relative energy method introduced on the continuous level in [11], [7], [9] and its numerical counterpart developed in Gallouët et al. [19] seem to provide the convenient strategy to achieve this goal.

We consider the compressible Navier-Stokes equations in the low Mach number regime in a space-time cylinder $Q_T = (0,T) \times \Omega$, where $T > 0$ is arbitrarily large and $\Omega \subset R^3$ is a bounded Lipschitz domain:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1} \quad \texttt{\{i1\}}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mu \Delta \mathbf{u} + (\mu + \lambda)\nabla_x \mathrm{div}_x \mathbf{u}, \tag{1.2} \quad \texttt{\{i2\}}$$

In equations (1.1–1.2) $\varrho = \varrho(t,x) \geq 0$ and $\mathbf{u} = \mathbf{u}(t,x) \in R^3$, $t \in [0,T)$, $x \in \Omega$ are unknown density and velocity fields, μ, λ are viscosity coefficients

$$\mu > 0, \quad \lambda + \frac{2}{d}\mu \geq 0, \tag{1.3} \quad \texttt{\{i3\}}$$

p is a pressure characterizing the fluid via the constitutive relations

$$p \in C^2(0,\infty) \cap C[0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0, \tag{1.4} \quad \texttt{\{i4\}}$$

$$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = p_\infty > 0,\ \ \lim_{\varrho \to 0+} \frac{p'(\varrho)}{\varrho^{\alpha+1}} = p_0 > 0$$

where $\gamma \geq 1$ and $\alpha \leq 0$. The (small) number $\varepsilon > 0$ is the Mach number. We notice that assumptions (1.4) are compatible with the isentropic pressure law $p(\varrho) = \varrho^\gamma$ provided $1 \leq \gamma \leq 2$.

Equations (1.1–1.2) are completed with the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.5} \quad \texttt{\{i6\}}$$

and initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \varrho_0 > 0 \text{ in } \overline{\Omega}. \tag{1.6} \quad \texttt{\{i7\}}$$

In parallel, we consider a strong solution of the incompressible Navier-Stokes equation

$$\overline{\varrho}\Big(\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V}\Big) + \nabla_x \Pi = \mu \Delta \mathbf{V}, \quad \mathrm{div} \mathbf{V} = 0, \tag{1.7} \quad \texttt{\{ns1\}}$$

$$\mathbf{V}|_{\partial\Omega} = 0, \qquad \overline{\varrho} = \mathrm{const} > 0 \tag{1.8} \quad \texttt{\{ns1+\}}$$

endowed with initial data

$$\mathbf{V}(0) = \mathbf{V}_0, \tag{1.9} \quad \texttt{\{ns2\}}$$

The solution of the incompressible target problem (1.7–1.9) is supposed to belong to the regularity class

$$\Pi \in \mathcal{Y}^p_T(\Omega) \equiv \{\Pi \in C([0,T]; C^1(\overline{\Omega})),\ \ \partial_t \Pi \in L^1(0,T; L^p(\Omega))\}, 2 \leq p \leq \infty, ;\ \mathbf{V} \in X_T(\Omega), \tag{1.10} \quad \texttt{\{ns3\}}$$

$$\mathcal{X}_T(\Omega) \equiv \{\mathbf{V} \in C^1([0,T] \times \overline{\Omega}; \mathbb{R}^3),\ \nabla^2 \mathbf{V} \in C([0,T] \times \overline{\Omega}; \mathbb{R}^3),\ (\partial_t^2 \mathbf{V}, \partial_t \nabla \mathbf{V}) \in L^2(0,T; L^{6/5}(\Omega; \mathbb{R}^{12})).$$

2 The numerical scheme

2.1 MAC space and time discretization

2.1.1 Space discretization

We assume that the closure of the domain Ω is a union of closed rectangles ($d = 2$) or closed orthogonal parallelepipeds ($d = 3$) with mutually disjoint interiors, and, without loss of generality, we assume that the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors, denoted by $(\boldsymbol{e}^{(1)}, \ldots, \boldsymbol{e}^{(d)})$,

Definition 2.1 (MAC grid - definition notations and basic properties)**.** *A discretization of Ω with MAC grid, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{T}, \mathcal{E})$, where:*

- *The primal (or density or pressure) grid of domain Ω denoted by $\mathcal{T}$ consists of union of possibly non uniform (closed) rectangles (d=2) or (closed) parallelpipeds ($d = 3$), the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors; a generic cell of this grid is denoted by K (a closed set), and its mass center $\boldsymbol{x}_K$. It is a conforming grid, meaning that*
$$\overline{\Omega} = \cup_{K \in \mathcal{M}} K, \text{ where } \operatorname{int}(K) \cap \operatorname{int}(L) = \emptyset \text{ whenever } (K, L) \in \mathcal{M}^2,\ K \neq L, \tag{2.1}$$
{primalgrid}
and if $K \cap L \neq \emptyset$ then $K \cap L$ is a common face or edge or vertex of K and L. A generic face (or edge in the two-dimensional case) of such a cell is denoted by σ (a closed set), its interior in the $\mathbb{R}^{d-1}$ topology is denoted by $\operatorname{int}_{d-1}(\sigma)$ and its mass center $\boldsymbol{x}_\sigma$. Symbol $\mathcal{E}(K)$ denotes the set of all faces of K. We denote by $\boldsymbol{n}_{\sigma,K}$ the unit normal vector to σ outward K. The set of all faces of the mesh is denoted by $\mathcal{E}$; we have $\mathcal{E} = \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$, where $\mathcal{E}_{\rm int}$ (resp. $\mathcal{E}_{\rm ext}$) are the edges of $\mathcal{E}$ that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to the i^{th} unit vector $\mathbf{e}^{(i)}$ of the canonical basis of $\mathbb{R}^d$ is denoted by $\mathcal{E}^{(i)}$, for $i = 1, \ldots, d$. We then have $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}_{\rm int} \cup \mathcal{E}^{(i)}_{\rm ext}$, where $\mathcal{E}^{(i)}_{\rm int}$ (resp. $\mathcal{E}^{(i)}_{\rm ext}$) are the edges of $\mathcal{E}^{(i)}$ that lie in the interior (resp. on the boundary) of the domain. Finally, for $i = 1, ..., d$ and $K \in \mathcal{T}$, we denote $\mathcal{E}^{(i)}(K) = \mathcal{E}(K) \cap \mathcal{E}^{(i)}$ and $\mathcal{E}^{(i)}_{\rm int}(K) = \mathcal{E}(K) \cap \mathcal{E}^{(i)}_{\rm int}$, $\mathcal{E}^{(i)}_{\rm ext}(K) = \mathcal{E}(K) \cap \mathcal{E}^{(i)}_{\rm ext}$.
- *For each $\sigma \in \mathcal{E}$, we write that $\sigma = K|L$ if $\sigma = K \cap L$ and we write that $\sigma = \overrightarrow{K|L}$ if, furthermore, $\sigma \in \mathcal{E}^{(i)}$ and $(\boldsymbol{x}_L - \boldsymbol{x}_K) \cdot \boldsymbol{e}^{(i)} > 0$ for some $i \in [\![1, d]\!] = \{1, \ldots, d\}$. A primal cell K will be denoted $K = [\overrightarrow{\sigma\sigma'}]$ if $\sigma, \sigma' \in \mathcal{E}^i(K)$ for some $i = 1, \ldots, d$ are such that $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}^{(i)} > 0$. For a face $\sigma \in \mathcal{E}$, the distance d_σ is defined by:*
$$d_\sigma = \begin{cases} d(\boldsymbol{x}_K, \boldsymbol{x}_L) & \text{if } \sigma = K|L \in \mathcal{E}_{\rm int}, \\ d(\boldsymbol{x}_K, \boldsymbol{x}_\sigma) & \text{if } \sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K) \end{cases} \tag{2.2}$$
{dsigma}
where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{R}^d$.
- *A dual cell D_σ associated to a face $\sigma \in \mathcal{E}$ is defined as follows:*
  - *if $\sigma = K|L \in \mathcal{E}_{\rm int}$ then $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$, where $D_{\sigma,K}$ - a closed set (resp. $D_{\sigma,L}$ - a closed set) is the half-part of K (resp. L) adjacent to σ (see Fig. 1 for the two-dimensional case) ;*
  - *if $\sigma \in \mathcal{E}_{\rm ext}$ is adjacent to the cell K, then $D_\sigma = D_{\sigma,K}$.*

  The dual grid $\{D_\sigma\}_{\sigma \in \mathcal{E}^{(i)}}$ of Ω (sometimes called the i-th velocity component grid) verifies for each fixed $i \in \{1, \ldots, d\}$
$$\overline{\Omega} = \cup_{\sigma \in \mathcal{E}^{(i)}} D_\sigma,\ \ \operatorname{int}(D_\sigma) \cap \operatorname{int}(D_{\sigma'}) = \emptyset,\ \sigma, \sigma' \in \mathcal{E}^{(i)},\ \sigma \neq \sigma'. \tag{2.3}$$
{dualgrid}
- *A dual face separating two neighboring dual cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$ or $\epsilon = D_\sigma|D_{\sigma'}$ (a closed set). Symbol $\tilde{\mathcal{E}}(D_\sigma)$ denotes the set of the faces of D_σ; it is decomposed to the set of external faces $\tilde{\mathcal{E}}_{\rm ext}(D_\sigma) = \{\varepsilon \in \tilde{\mathcal{E}}(D_\sigma) | \varepsilon \subset \partial\Omega\}$ and the set of internal faces $\tilde{\mathcal{E}}_{\rm int}(D_\sigma) = \{\varepsilon \in$*

$\tilde{\mathcal{E}}(D_\sigma)|\mathrm{int}_{d-1}\varepsilon \subset \Omega\}$. Symbol $\widetilde{\mathcal{E}}^{(i)}$ denotes the set of the faces of the i-th dual grid (associated to the i-th velocity component). It is decomposed into the internal and boundary edges: $\widetilde{\mathcal{E}}^{(i)} = \widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}} \cup \widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}}$, where $\widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}} = \{\varepsilon = \sigma|\sigma' \,|\, \sigma, \sigma' \in \mathcal{E}^i\}$ and $\widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}} = \{\varepsilon = \partial D_\sigma \cap \partial\Omega \,|\, \sigma \in \mathcal{E}^{(i)}, \partial D_\sigma \cap \partial\Omega \neq \emptyset\}$. Finally, for $\varepsilon \in \widetilde{\mathcal{E}}^{(i)}$ we write $\epsilon = \overrightarrow{\sigma|\sigma'}$ if $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \mathbf{e}^{(i)} > 0$. We denote by $\boldsymbol{n}_{\varepsilon, D_\sigma}$ the unit normal vector to $\varepsilon \in D_\sigma$ outward D_σ.

We denote for further convenience $\mathbf{n}_\epsilon$ and $\mathbf{n}_\sigma$ a normal unit vector to face ϵ and σ, respectively. We write $\epsilon \perp \sigma$ resp. $\sigma \perp \sigma'$ iff $\mathbf{n}_\epsilon \cdot \mathbf{n}_\sigma = 0$ resp. $\mathbf{n}_\sigma \cdot \mathbf{n}_{\sigma'} = 0$. Similarly we write $\epsilon \perp \mathbf{e}^{(j)}$ resp. $\sigma \perp \mathbf{e}^{(j)}$ iff $\mathbf{n}_\epsilon$ and $\mathbf{e}^{(j)}$ resp. $\mathbf{n}_\sigma$ and $\mathbf{e}^{(j)}$ are parallel. We also denote by $\mathbf{ab}$ the segment $\{\mathbf{a} + t(\mathbf{b} - \mathbf{a})|\mathbf{t} \in [\mathbf{0}, \mathbf{1}]\}$, where $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2d}$, and by $\boldsymbol{x}_\varepsilon$ resp. $\boldsymbol{x}_{\sigma\cap\epsilon}$ the mass centers of the face ε resp. of the set $\sigma \cap \epsilon$ (provided it is not empty).

- *In order to define bi-dual grid, we introduce the set $\tilde{\mathcal{E}}^{(i,j)} = \{\varepsilon \in \tilde{\mathcal{E}}^{(i)} \,|\, \varepsilon \perp \mathbf{e}^{(j)}\}$ of dual faces of the i-th component velocity grid that are orthogonal to $\mathbf{e}^{(j)}$. A bi-dual cell D_ε associated to a face $\varepsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}$ is defined as follows:*
  - *If $\varepsilon = \sigma|\sigma' \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\mathrm{int}}$ then $D_\varepsilon = \varepsilon \times \boldsymbol{x}_\sigma \boldsymbol{x}_{\sigma'}$ (see Figure 2). (We notice that, if $\sigma, \sigma' \in \mathcal{E}^{(i)}$ with $K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{T}$ and $\varepsilon = \sigma|\sigma'$ then $D_\varepsilon = K$.)*
  - *If $\varepsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\mathrm{ext}}$ with $\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)$ and $i \neq j$ then $D_\varepsilon = \varepsilon \times \boldsymbol{x}_\sigma \boldsymbol{x}_{\sigma\cap\epsilon}$.*

  In the list above we did not consider the sitution $\varepsilon \in \tilde{\mathcal{E}}^{(i,i)} \cap \tilde{\mathcal{E}}^{(i)}_{\mathrm{ext}}$ with $\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)$. In this case $\varepsilon = \sigma \subset \partial\Omega$, and we set for completeness $D_\varepsilon = \emptyset$.

  It is to be noticed that, for each fixed couple $(i,j) \in \{1, \dots, d\}^2$

$$\cup_{\varepsilon \in \tilde{\mathcal{E}}^{(i,j)}} D_\varepsilon = \overline{\Omega}, \quad \mathrm{int}(D_\varepsilon) \cap \mathrm{int}(D_{\varepsilon'}) = \emptyset,\ \varepsilon \neq \varepsilon',\ \varepsilon, \varepsilon' \in \tilde{\mathcal{E}}^{(i,j)}. \tag{2.4}$$ {bidualgrid}

  To any dual face $\epsilon \in \tilde{\mathcal{E}}^{(i)}$, we associate a distance d_ϵ

$$d_\epsilon = \begin{cases} d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}) \text{ if } \varepsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\mathrm{int}}, \\ d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma\cap\epsilon}) \text{ if } \varepsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\mathrm{ext}} \text{ with } \varepsilon \in \tilde{\mathcal{E}}(D_\sigma) \text{ and } i \neq j, \\ d_\sigma \text{ if } \varepsilon \in \tilde{\mathcal{E}}^{(i,i)} \cap \tilde{\mathcal{E}}^{(i)}_{\mathrm{ext}} \text{ with } \varepsilon \in \tilde{\mathcal{E}}(D_\sigma). \end{cases} \tag{2.5}$$ {depsilon}

  (We notice that the last line in the above definition is irrelevant and pure convention, since in that case $D_\varepsilon = \emptyset$.)

- *We define the size of the mesh by*

$$h = \max\{h_K, K \in \mathcal{T}\} \tag{2.6}$$ {stepsize}

  where h_K stands for the diameter of K. Moreover if $K = [\overrightarrow{\sigma\sigma'}]$ where $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i = 1, \dots, d$ we will denote

$$h_K^{(i)} = \frac{|K|}{|\sigma|} = \frac{|K|}{|\sigma'|}. \tag{2.7}$$

  We measure the regularity of the mesh through the positive real number $\theta_{\mathcal{T}}$ defined by

$$\theta_{\mathcal{T}} = \min\{\frac{\xi_K}{h_K},\ K \in \mathcal{T}\} \tag{2.8}$$ {reg}

  where ξ_K stands for the diameter of the largest ball included in K. Finally, we denote by h_σ the diameter of the face $\sigma \in \mathcal{E}$.

Some geometric notions presented in Definition 2.1 are sketched on Figures 1 and 2 below.

Remark 1. *For all $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, we have $h_\sigma \geq \xi_K \geq \theta_\mathcal{T} h_K$ and $h_\sigma \leq h_L$ and so $\theta_\mathcal{T} h_K \leq h_L \leq \frac{1}{\theta_\mathcal{T}} h_K$. Note also that for all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, the inequality $h_\sigma|\sigma| \leq 2\frac{1}{\theta_\mathcal{T}^d}|K|$ holds and if $\sigma = K|L$ a rough estimate gives $|K| \leq (\frac{2}{\theta_\mathcal{T}})^d|L|$. These relations will be used throughout this paper.*

Definition 2.2 (Discrete spaces)**.** *Let $\mathcal{D} = (\mathcal{T}, \mathcal{E})$ be a MAC grid in the sense of Definition 2.1. The discrete density and pressure space $L_\mathcal{T}$ is defined as the set of piecewise constant functions over each of the grid cells K of $\mathcal{T}$, and the discrete $i-th$ velocity space $H_\mathcal{E}^{(i)}$ as the set of piecewise constant functions over each of the grid cells D_σ , $\sigma \in \mathcal{E}^{(i)}$. As in the continuous case, the Dirichlet boundary conditions (1.5) are (partly) incorporated in the definition of the velocity spaces, and, to this purpose, we introduce $H_{\mathcal{E},0}^{(i)} \subset H_\mathcal{E}^{(i)}, i = 1, \dots, d$, defined as follows:*

$$H_{\mathcal{E},0}^{(i)} = \Big\{ v \in H_\mathcal{E}^{(i)},\ v(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \widetilde{\mathcal{E}}_{\rm ext}^{(i)}, \Big\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H_{\mathcal{E},0}^{(i)}$. Since we are dealing with piecewise constant functions, it is useful to introduce the characteristic functions $\mathcal{X}_K, K \in \mathcal{T}$ and $\mathcal{X}_{D_\sigma}, \sigma \in \mathcal{E}$ of the density (or pressure) and velocity cells. We can then write a function $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ as $\boldsymbol{v} = (v_1, \dots, v_d)$ with $v_i = \sum_{\sigma \in \mathcal{E}_{\rm int}^{(i)}} v_{i,\sigma} \mathcal{X}_{D_\sigma}$, $i \in [\![1, d]\!]$ and a function $q \in L_\mathcal{T}$ as $q = \sum_{K \in \mathcal{T}} q_K \mathcal{X}_K$. If there is no confusion possible we shall write v_σ instead of $v_{i,\sigma}$, where $\sigma \in \mathcal{E}^{(i)}$.

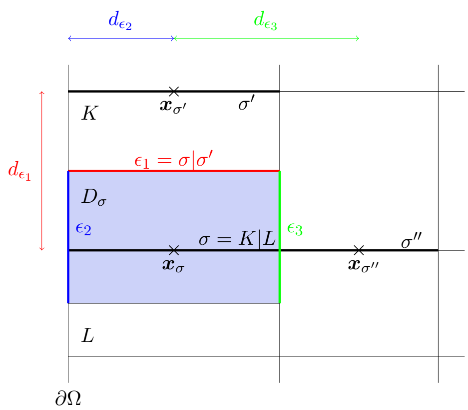

Figure 1: Notations for control volumes and dual cells

2.1.2 Time discretization

We consider a partition $0 = t^0 < t^1 < \cdots < t^N = T$ of the time interval $(0, T)$, and, for the sake of simplicity, a constant time step $\delta t = t^n - t^{n-1}$; hence $t^n = n\delta t$ for $n \in \{0, \cdots, N\}$. We denote respectively by $\{u_{i,\sigma}^n \equiv u_\sigma^n, \sigma \in \mathcal{E}_{\rm int}^{(i)}, i \in \{1, \cdots, d\}, n \in \{0, \cdots, N\}\}$, and $\{\varrho_K^n, K \in \mathcal{T}, n \in \{1, \cdots, N\})$ the sets of discrete velocity and density unknowns. For $\sigma \in \mathcal{E}_{\rm int}^{(i)}$, $i \in \{1, \cdots, d\}$ the value u_σ^n is an expected approximation of the mean value over $(t^{n-1}, t^n) \times D_\sigma$ of the i-th component of the velocity of a weak solution, while for $K \in \mathcal{T}$ the value ϱ_K^n is an expected approximation of the mean value over $(t^{n-1}, t^n) \times K$ of the density of a weak solution. To the discrete unknowns, we associate piecewise

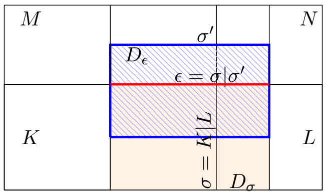

Figure 2: Notations for bi-dual cells

constant functions on time intervals and on primal or dual meshes, which are expected approximation of weak solutions, For the velocities, these constant functions are of the form:

$$u_i(t,\boldsymbol{x}) = \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}} u_\sigma^n\, \mathcal{X}_{D_\sigma}(\boldsymbol{x})\, \mathcal{X}_{(t^{n-1},t^n)}(t), \tag{2.9}$$ {du}

where $\mathcal{X}_{(t^{n-1},t^n)}$ is the characteristic function of the interval (t^{n-1},t^n). We denote by $X_{i,\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells, and we set $\boldsymbol{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. For the density, the piecewise constant function is of the form:

$$\varrho(t,\boldsymbol{x}) = \sum_{K\in\mathcal{T}} \varrho_K^n(\boldsymbol{x})\mathcal{X}_K(\boldsymbol{x})\, \mathcal{X}_{(t^{n-1},t^n)}(t) \tag{2.10}$$ {drho}

and we denote by $Y_{\mathcal{T},\delta t}$ the space of such piecewise constant functions.

For a given $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ associated to the set of discrete velocity unknowns $\{u_\sigma^n, \sigma \in \mathcal{E}_{\rm int}^{(i)}, i \in \{1,\cdots,d\}, n\in\{1,\cdots,N\}\}$, and for $n\in\{1,\cdots,N\}$, we denote by $u_i^n \in H_{\mathcal{E},0}^{(i)}$ the piecewise constant function defined by $u_i^n(\boldsymbol{x}) = u_\sigma^n \equiv u_{i,\sigma}^n$ for $\boldsymbol{x}\in D_\sigma, \sigma\in\mathcal{E}_{\rm int}^{(i)}$, and set $\boldsymbol{u}^n = (u_1^n,\ldots,u_d^n)^t \in \mathbf{H}_{\mathcal{E},0}$. In a same way, given $\varrho \in Y_{\mathcal{T},\delta t}$ associated to the discrete density unknowns $\{\varrho_K^n, K\in\mathcal{T}, n\in\{1,\cdots,N\}\}$ we denote by $\varrho^n \in L_{\mathcal{T}}$ the piecewise constant function defined by $\varrho^n(\boldsymbol{x}) = \varrho_K^n$ for $\boldsymbol{x}\in K$, $K\in\mathcal{T}$.

2.2 MAC discretization of differential operators

2.2.1 Upwind divergence and primal fluxes

The discrete "upwind" divergence is defined by

$$\mathrm{div}_{\mathcal{T}}^{\rm up} : \left|\begin{array}{l} L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow L_{\mathcal{T}} \\ (\varrho,\boldsymbol{u}) \longmapsto \mathrm{div}_{\mathcal{T}}^{\rm up}(\varrho\boldsymbol{u}) = \displaystyle\sum_{K\in\mathcal{T}} \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} F_{\sigma,K}(\varrho,\boldsymbol{u})\, \mathcal{X}_K, \end{array}\right. \tag{2.11}$$ {eq:divup}

where $F_{\sigma,K}(\varrho,\boldsymbol{u})$ stands for the mass flux across σ outward K, which, because of the Dirichlet boundary conditions, vanishes on external faces and is given on the internal faces by:

$$\forall\sigma = K|L \in \mathcal{E}_{\rm int}, \qquad F_{\sigma,K}(\varrho,\boldsymbol{u}) = |\sigma|\ \varrho_\sigma^{\rm up}\ u_{\sigma,K}, \tag{2.12}$$ {eq:massflux}

where $u_{\sigma,K}$ is an approximation of the normal velocity to the face σ outward K, defined by:

$$u_{\sigma,K} = u_\sigma\ \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}_{\sigma,K} \text{ for } \sigma\in\mathcal{E}^{(i)}\cap\mathcal{E}(K). \tag{2.13}$$ {eq:edge_velo}

Thanks to the boundary conditions, $u_{\sigma,K}$ vanishes for any external face σ. The density at the internal face $\sigma = K|L$ is obtained by an upwind technique:

$$\varrho_\sigma^{\rm up} = \left|\begin{array}{ll} \varrho_K & \text{if } u_{\sigma,K}\geq 0, \\ \varrho_L & \text{otherwise.} \end{array}\right. \tag{2.14}$$ {eq:rho_upwind}

Note that any solution $(\varrho^n, \boldsymbol{u}^n) \in L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0}$ to (2.38a) satisfy $\varrho^n_K > 0$, $\forall K \in \mathcal{T}$ provided $\varrho^{n-1}_K > 0$, $\forall K \in \mathcal{T}$ and in particular $p(\varrho^n)$ makes sense. The positivity of the density ϱ^n in (2.38a) is not enforced in the scheme but results from the above upwind choice, see Proposition 2.1.

Note also that, with this definition, we have the usual finite volume property of local conservation of the flux through a primal face

$$F_{\sigma,K}(\varrho, \boldsymbol{u}) = -F_{\sigma,L}(\varrho, \boldsymbol{u}), \text{ where } \sigma = K|L. \tag{2.15}$$ {dod1}

2.2.2 Discrete convective operator and dual fluxes

The discrete divergence of the convective term $\varrho \boldsymbol{u} \otimes \boldsymbol{u}$ is defined by

$$\mathrm{div}^{\mathrm{up}}_{\mathcal{E}} : \left| \begin{array}{l} L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ (\varrho, \boldsymbol{u}) \longmapsto \mathrm{div}^{\mathrm{up}}_{\mathcal{E}}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) = (\mathrm{div}^{(1)}_{\mathcal{E}}(\varrho \boldsymbol{u} u_1), ..., \mathrm{div}^{(d)}_{\mathcal{E}}(\varrho \boldsymbol{u} u_d)), \end{array} \right. \tag{2.16}$$

where for any $1 \leq i \leq d$, the i^{th} component of the above operator reads:

$$\mathrm{div}^{(i)}_{\mathcal{E}} : \left| \begin{array}{l} L_{\mathcal{T}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow H^{(i)}_{\mathcal{E},0} \\ (\varrho, \boldsymbol{u}) \longmapsto \mathrm{div}^{(i)}_{\mathcal{E}}(\varrho \boldsymbol{u} u_i) = \displaystyle\sum_{\sigma \in \mathcal{E}^{(i)}_{\mathrm{int}}} \frac{1}{|D_\sigma|} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}(\varrho, \boldsymbol{u})\, u_\epsilon\, \mathcal{X}_{D_\sigma}. \end{array} \right. \tag{2.17}$$ {FLU}

Here for $\sigma \in \mathcal{E}^{(i)}_{\mathrm{int}}$ and $\epsilon \in \mathcal{E}(D_\sigma)$ the quantity $F_{\epsilon,\sigma} = F_{\epsilon,\sigma}(\varrho, \boldsymbol{u})$ stands for a mass flux through the dual faces of the mesh and are defined hereafter while u_ϵ stands for the centered approximation of i^{th} component of the velocity over the face ϵ: For internal dual face $\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}^{(i)}_{\mathrm{int}}$,

$$u_\epsilon \equiv u_{i,\varepsilon} = \frac{u_{i,\sigma} + u_{i,\sigma'}}{2} \equiv \frac{u_\sigma + u_{\sigma'}}{2}. \tag{2.18}$$ {centeredchoice}

The dual fluxes $F_{\epsilon,\sigma}$ are defined as follows:

Since we consider homogenous Dirichlet boundary condition, the flux through a dual face ε included in the boundary is taken equal to zero. (For this reason $\tilde{\mathcal{E}}(D_\sigma)$ in the sum (2.17) can be replaced by $\tilde{\mathcal{E}}_{\mathrm{int}}(D_\sigma)$, and it is not necessary to define the value u_ε at the external dual faces ε.)

Otherwise, we have to distinguish two cases (see Figure 2.2.2):

- First case – The vector $\boldsymbol{e}^{(i)}$ is normal to ϵ, so ϵ is included in a primal cell K, and we denote by σ' the second face of K which, in addition to σ, is normal to $\boldsymbol{e}^{(i)}$. We thus have $\epsilon = D_\sigma | D_{\sigma'}$. Then the mass flux through ϵ is given by:

$$F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) = \frac{1}{2}\, \big[F_{\sigma,K}(\varrho, \boldsymbol{u})\, \boldsymbol{n}_{\epsilon,D_\sigma} \cdot \boldsymbol{n}_{\sigma,K} + F_{\sigma',K}(\varrho, \boldsymbol{u})\, \boldsymbol{n}_{\epsilon,D_\sigma} \cdot \boldsymbol{n}_{\sigma',K}\big]. \tag{2.19}$$ {eq:flux_eK}

- Second case – The vector $\boldsymbol{e}^{(i)}$ is tangent to ϵ, and ϵ is the union of the halves of two primal faces τ and τ' such that $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ϵ is then given by:

$$F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) = \frac{1}{2}\, \big[F_{\tau,K}(\varrho, \boldsymbol{u}) + F_{\tau',L}(\varrho, \boldsymbol{u})\big]. \tag{2.20}$$ {eq:flux_eorth}

Note that, with this definition, we have the usual finite volume property of local conservativity of the flux through a dual face, $D_\sigma | D_{\sigma'}$,

$$F_{\epsilon,\sigma}(\varrho, \boldsymbol{u}) = -F_{\epsilon,\sigma'}(\varrho, \boldsymbol{u}).$$

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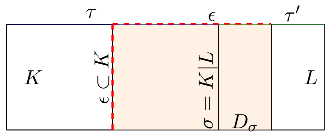

Figure 3: Notations for the dual fluxes of the first component of the velocity.

The density on a dual cell is given by:

$$\begin{aligned} &\text{for } \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L \qquad |D_\sigma|\ \varrho_{D_\sigma} = |D_{\sigma,K}|\ \varrho_K + |D_{\sigma,L}|\ \varrho_L, \\ &\text{for } \sigma \in \mathcal{E}_{\rm ext},\ \sigma \in \mathcal{E}(K), \qquad \varrho_{D_\sigma} = \varrho_K. \end{aligned} \tag{2.21}$$ {eq:rho_edge}

and we denote

$$\text{for } 1 \le i \le d, \quad \widehat{\varrho}^{(i)} = \sum_{\sigma \in \mathcal{E}^{(i)}} \varrho_{D_\sigma} \mathcal{X}_{D_\sigma}. \tag{2.22}$$ {rohat}

2.2.3 Discrete divergence and gradient

The discrete divergence operator $\mathrm{div}_\mathcal{T}$ is defined by:

$$\mathrm{div}_\mathcal{T} : \left| \begin{aligned} &\mathbf{H}_\mathcal{E} \longrightarrow L_\mathcal{T} \\ &\boldsymbol{u} \longmapsto \mathrm{div}_\mathcal{T}\, \boldsymbol{u} = \sum_{K \in \mathcal{T}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{\sigma,K}\ \mathcal{X}_K, \end{aligned} \right. \tag{2.23}$$ {eq:div}

where $u_{\sigma,K}$ is defined in (2.13).

The discrete divergence of $\boldsymbol{u} = (u_1, \dots, u_d) \in \mathbf{H}_{\mathcal{E},0}$ may also be written as

$$\mathrm{div}_\mathcal{T}\, \boldsymbol{u} = \sum_{i=1}^{d} \sum_{K \in \mathcal{T}} (\eth_i u_i)_K \mathcal{X}_K, \tag{2.24}$$

where the discrete derivative $(\eth_i u_i)_K$ of u_i on K is defined by

$$(\eth_i u_i)_K = \frac{|\sigma|}{|K|}(u_{\sigma'} - u_\sigma) \text{ with } K = [\overrightarrow{\sigma\sigma'}], \sigma, \sigma' \in \mathcal{E}^{(i)}. \tag{2.25}$$ {discrete-derivative-i-ui}

The gradient in the discrete momentum balance equation is defined as follows:

$$\boldsymbol{\nabla}_\mathcal{E} : \left| \begin{aligned} &L_\mathcal{T} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ &p \longmapsto \boldsymbol{\nabla}_\mathcal{E} p \\ &\boldsymbol{\nabla}_\mathcal{E} p(\boldsymbol{x}) = (\eth_1 p(\boldsymbol{x}), \dots, \eth_d p(\boldsymbol{x}))^t, \end{aligned} \right. \tag{2.26}$$ {eq:grad}

where $\eth_i p \in H^{(i)}_{\mathcal{E},0}$ is the discrete derivative of p in the i-th direction, defined by:

$$\eth_i p = \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}^{(i)}_{\rm int}} \frac{|\sigma|}{|D_\sigma|}\ (p_L - p_K) \chi_{D_\sigma}, \quad i = 1, \dots, d. \tag{2.27}$$ {discderive}

Note that in fact, the discrete gradient of a function of $L_\mathcal{T}$ should only defined on the internal faces, and does not need to be defined on the external faces; we set it here in $\mathbf{H}_{\mathcal{E},0}$ (that is zero on the external faces) for the sake of simplicity.

The gradient in the discrete momentum balance equation is built as the dual operator of the discrete divergence. Indeed, we have the following lemma:

Lemma 2.1. *[Discrete div − $\boldsymbol{\nabla}$ duality]*
Let $q \in L_{\mathcal{T}}$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ then we have:

$$\int_\Omega q \, \mathrm{div}_{\mathcal{T}} \boldsymbol{v} \, \mathrm{d}x + \int_\Omega \nabla_{\mathcal{E}} q \cdot \boldsymbol{v} \, \mathrm{d}x = 0. \tag{2.28}$$ {Ndiscret}

2.2.4 Discrete Laplace operator

For $i = 1 \ldots, d$, we classically define the discrete Laplace operator on the i-th velocity grid by:

$$-\Delta_{\mathcal{E}}^{(i)} : \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ u_i \longmapsto -\Delta_{\mathcal{E}}^{(i)} u_i \end{array} \right.$$

$$-\Delta_{\mathcal{E}}^{(i)} u_i = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}^{(i)}} \Big[\frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} \phi_{\epsilon,\sigma}(u_i) \Big] \chi_{D_\sigma}, \tag{2.29}$$ {eq:lapi}

where $\tilde{\mathcal{E}}(D_\sigma)$, d_ϵ is defined in Definition 2.1, and

$$\phi_{\epsilon,\sigma}(u_i) = \begin{cases} \dfrac{|\epsilon|}{d_\epsilon}(u_{i,\sigma} - u_{i,\sigma'}) & \text{if } \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}, \\ \\ \dfrac{|\epsilon|}{d_\epsilon} u_{i,\sigma} & \text{if } \epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma). \end{cases} \tag{2.30}$$

Note that we have the usual finite volume property of local conservativity of the flux through an interface $\epsilon = \sigma\ |\sigma'$:

$$\phi_{\epsilon,\sigma}(u_i) = -\phi_{\epsilon,\sigma'}(u_i), \quad \forall \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}. \tag{2.31}$$ {conservdiff}

Then the discrete Laplace operator of the full velocity vector is defined by

$$\begin{aligned} -\Delta_{\mathcal{E}} : \ & \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ & \boldsymbol{u} \mapsto -\Delta_{\mathcal{E}} \boldsymbol{u} = (-\Delta_{\mathcal{E}}^{(1)} u_1, \ldots, -\Delta_{\mathcal{E}}^{(d)} u_d)^t. \end{aligned} \tag{2.32}$$

Let us now recall the definition of the discrete H_0^1 inner product [5]; it is obtained by multiplying the discrete Laplace operator scalarly by a test function $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ and integrating over the computational domain. A simple reordering of the sums (which may be seen as a discrete integration by parts) yields, thanks to the conservativity of the diffusion flux (2.31):

$$\begin{aligned} &\forall (\boldsymbol{u}, \boldsymbol{v}) \in {\mathbf{H}_{\mathcal{E},0}}^2, \qquad \int_\Omega -\Delta_{\mathcal{E}} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} = \sum_{i=1}^d [u_i, v_i]_{1,\mathcal{E}^{(i)},0}, \\ &\text{with } [u_i, v_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon} \, (u_{i,\sigma} - u_{i,\sigma'}) \, (v_{i,\sigma} - v_{i,\sigma'}) + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} \, u_{i,\sigma} \, v_{i,\sigma} \end{aligned} \tag{2.33}$$ {ps}

The bilinear forms $\left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \times H_{\mathcal{E},0}^{(i)} \to \mathbb{R} \\ (u, v) \mapsto [u_i, v_i]_{1,\mathcal{E}^{(i)},0} \end{array} \right.$ and $\left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0} \to \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) \mapsto [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} \end{array} \right.$ are inner products on $H_{\mathcal{E},0}^{(i)}$ and $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following discrete H_0^1 norms:

$$\|u_i\|_{1,\mathcal{E}^{(i)},0}^2 = [u_i, u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)} \\ \epsilon = \sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon} \, (u_{i,\sigma} - u_{i,\sigma'})^2 + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \\ \epsilon \in \tilde{\mathcal{E}}(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} \, u_{i,\sigma}^2 \text{ for} \tag{2.34a}$$ {normi}

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0}^2 = [\boldsymbol{u}, \boldsymbol{u}]_{1,\mathcal{E},0} = \sum_{i=1}^d \|u_i\|_{1,\mathcal{E}^{(i)},0}^2. \tag{2.34b}$$ {normfull}

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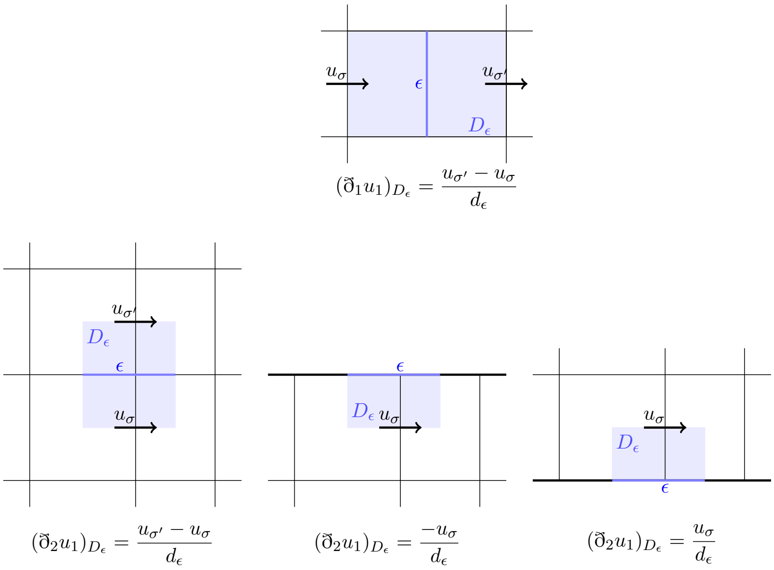

Figure 4: Notations for the definition of the partial space derivatives of the first component of the velocity, in two space dimensions.

We introduce the discrete gradient of velocity component u_i as follows:

$$\nabla_{\mathcal{E}^{(i)}} u_i = (\eth_1 u_i, \ldots, \eth_d u_i) \text{ with } \eth_j u_i = \sum_{\varepsilon \in \tilde{\mathcal{E}}^{(i,j)}} (\eth_j u_i)_{D_\epsilon}\, \mathcal{X}_{D_\epsilon} \tag{2.35}$$ {partialdiscrete}

where

$$(\eth_j u_i)_{D_\epsilon} = \begin{cases} \dfrac{u_{i,\sigma'} - u_{i,\sigma}}{d_\epsilon} \text{ if } \varepsilon = \overrightarrow{\sigma|\sigma'} \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\text{int}}, \\ -\frac{u_{i,\sigma}}{d_\varepsilon} \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\epsilon, D_\sigma} \text{ if } \varepsilon \in \tilde{\mathcal{E}}^{(i,j)} \cap \tilde{\mathcal{E}}^{(i)}_{\text{ext}} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases} \tag{2.36}$$ {Ddiscrete}

(see Figure 4). Recall that all notations used above are introduced in Definition 2.1. Note, that this definition is compatible with the definition of the discrete derivative $(\eth_i u_i)_K$ given by (2.25). Finally notice that the second line in (2.36) is equal to zero whenever $i = j$ (since $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$). With this definition, it is easily seen that

$$\int_\Omega \nabla_{\mathcal{E}^{(i)}} u \cdot \nabla_{\mathcal{E}^{(i)}} v \,\mathrm{d}x = [u, v]_{1,\mathcal{E}^{(i)},0}, \forall u, v \in H^{(i)}_{\mathcal{E},0}, \forall i = 1, \ldots, d. \tag{2.37}$$ {gradient-and-innerproduct}

where $[u, v]_{1,\mathcal{E}^{(i)},0}$ is the discrete H^1_0 inner product defined by (2.33). Now we define the discrete gradient of the velocity field $\boldsymbol{u}$,

$$\nabla_{\mathcal{E}} \boldsymbol{u} = (\nabla_{\mathcal{E}^{(1)}} u_1, \ldots, \nabla_{\mathcal{E}^{(d)}} u_d)$$

and verify easily that

$$\int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u} : \nabla_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}x = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0}.$$

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We will need discrete Sobolev inequalites for the discrete approximations. The following is proved in [5, Lemma 9.5].

Lemma 2.2. *[Discrete Sobolev inequalities]*
Let Ω be an open bounded subset of $\mathbb{R}^d$, $d=2$ or $d=3$, compatible with the MAC grid and let $\mathcal{D}=(\mathcal{T},\mathcal{E})$ be a MAC grid of Ω. Let $1 \le q < +\infty$ if $d=2$ and $q=6$ if $d=3$, $i=1,\dots,d$. Then there exists $c=c(q,|\Omega|)$ (independent of h) such that, for all $u \in H^{(i)}_{\mathcal{E},0}$,

$$\|u\|_{L^q(\Omega)} \le c\|u\|_{1,\mathcal{E}^{(i)},0}.$$

2.3 The numerical scheme

Given $\varrho^0 \in L_\mathcal{T}$, $\varrho^0 > 0$ and $\boldsymbol{u}^0 \in \mathbf{H}_{\mathcal{E},0}$, we consider an implicit-in-time scheme for unknown $\varrho^n \in L_\mathcal{T}$, $\boldsymbol{u}^n \in \mathbf{H}_{\mathcal{E},0}$, $1 \le n \le N$, which reads

$$\frac{1}{\delta t}(\varrho^n - \varrho^{n-1}) + \mathrm{div}^{\mathrm{up}}_\mathcal{T}(\varrho^n \boldsymbol{u}^n) = 0, \tag{2.38a}$$ {dcont}

$$\frac{1}{\delta t}(\widehat{\varrho^n}^{(i)} u_i^n - \widehat{\varrho^{n-1}}^{(i)} u_i^{n-1}) + \mathrm{div}^{(i)}_\mathcal{E}(\varrho^n \boldsymbol{u}^n u_i^n) - \mu \Delta^{(i)}_\mathcal{E} u_i^n$$
$$- (\mu+\lambda)\eth_i \,\mathrm{div}_\mathcal{T} \boldsymbol{u}^n + \frac{1}{\varepsilon^2}\eth_i p(\varrho^n) = 0, \qquad i=1,\dots,d. \tag{2.38b}$$ {dmom}

Equation (2.38a) is a finite volume discretization of the mass balance (1.1) over the primal mesh. Equation (2.38b) is the discretization of the momentum balance equation (1.2) on the dual cells associated to the faces of the mesh. The discrete spaces $\mathcal{L}_\mathcal{T}$, $\mathbf{H}_{\mathcal{E},0}$ are defined in Section 2.1.1. We recall that the quantities $\widehat{\varrho^n}^{(i)}$ are defined in (2.22), while the discrete differential operators appearing (2.38) are defined in Sections 2.2.1–2.2.4.

Of course, the quantities $(\varrho, \boldsymbol{u}) \in \mathbf{Y}_{\mathcal{M},\delta t} \times \mathbf{X}_{\mathcal{E},\delta t}$ (see Sections 2.1.1-2.1.2) depend tacitly on h, δt and ε, meaning that $(\varrho, \boldsymbol{u}) \equiv (\varrho_{h,\delta t,\varepsilon}, \boldsymbol{u}_{h,\delta t,\varepsilon})$. However, in order to avoid a cumbersome notation, we shall omit the subscripts $h, \delta t, \varepsilon$ in most formulas. We shall keep some of them only when a confusion could arise.

It is well known that the (2.38) admits at least one solution. Indeed, the following existence theorem is proved in [20, Appendix A].

Proposition 2.1. *Let $(\varrho^0, \boldsymbol{u}^0) \in L_\mathcal{T} \times \mathbf{H}_{\mathcal{E},0}$ such that $\varrho^0 > 0$ (meaning that $\varrho^0_K > 0$ for any $K \in \mathcal{T}$). There exists a solution $(\boldsymbol{u}, \varrho) \in \mathbf{H}_{\mathcal{E},0} \times L_\mathcal{T}$ of Problem (2.38). Moreover any solution is such that $\varrho > 0$ a.e in Ω (meaning that $\varrho^n_K > 0$ for any $n=1,...,N$ and for any $K \in \mathcal{T}$).*

Uniqueness remains an open problem.

2.4 Projection operators

In this section we introduce several projection operators. We first define the mean-value interpolator over $L_\mathcal{T}$:

$$\mathcal{P}_\mathcal{T} : \quad \left| \begin{array}{lcl} L^1_{\mathrm{loc}}(\Omega) & \longrightarrow & L_\mathcal{T} \\ \varphi & \mapsto & \mathcal{P}_\mathcal{T}\varphi = \sum_{K \in \mathcal{T}} \varphi_K \, \mathcal{X}_K, \end{array} \right. \tag{2.39}$$ {projprimalmesh}

with

$$\varphi_K = \frac{1}{|K|}\int_K \varphi(\boldsymbol{x})\mathbf{d}\boldsymbol{x}, \ \forall K \in \mathcal{T}. \tag{2.40}$$ {meanvalueprimal}

We also define over $H^{(i)}_{\mathcal{E},0}$ the following interpolation operator $\mathcal{P}^{(i)}_\mathcal{E}$:

$$\mathcal{P}_{\mathcal{E}}^{(i)}: \quad \left| \begin{array}{lll} H_0^1(\Omega) & \longrightarrow & H_{\mathcal{E},0}^{(i)} \\ \varphi & \mapsto & \mathcal{P}_{\mathcal{E}}^{(i)}\varphi = \sum_{\sigma\in\mathcal{E}_{\rm int}^{(i)}} \varphi_\sigma\, \mathcal{X}_{D_\sigma}, \end{array} \right. \tag{2.41}$$ {projdualmesh}

with

$$\varphi_\sigma = \frac{1}{|\sigma|}\int_\sigma \varphi(\boldsymbol{x})\mathrm{d}\gamma(\boldsymbol{x}),\ \forall\sigma\in\mathcal{E}_{\rm int}^{(i)}, \tag{2.42}$$ {meanvaluedual}

where $d\gamma$ is the $(d-1)-$Lebesgue measure on σ, and we denote

$$\mathcal{P}_{\mathcal{E}} = (\mathcal{P}_{\mathcal{E}}^{(1)},...,\mathcal{P}_{\mathcal{E}}^{(d)}) \in \mathcal{L}(H_0^1(\Omega)^d, \mathbf{H}_{\mathcal{E},0}) \tag{2.43}$$ {projdualmeshmac}

the vector valued extension. This operator preserves the divergence in the following sense, see [21].

Lemma 2.3.

$$\forall \boldsymbol{v}\in H_0^1(\Omega)^d, \forall q\in L_{\mathcal{T}}, \int_\Omega q\ \mathrm{div}_{\mathcal{T}}\,\mathcal{P}_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}x = \int_\Omega q\ \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x. \tag{2.44}$$ {L1-1}

In particular, if $\mathrm{div}\boldsymbol{v}=0$ *then* $\mathrm{div}_{\mathcal{T}}\mathcal{P}_{\mathcal{E}}(\boldsymbol{v}))=0$.

The next lemma deals with the properties of the projections defined by (2.39) and (2.41). It can be obtained by rescaling from the standard inequalities on the reference cell $[0,1]^d$, see e.g [5] or [20, Lemma 3].

Lemma 2.4. *[*Mean value inequalities*]*
Let $\mathcal{D}=(\mathcal{T},\mathcal{E})$ be a MAC grid of the computational domain Ω. Let $K\in\mathcal{T}$ and $\sigma\in\mathcal{E}(K)$. Let $1\le p\le\infty$. There exists c only depending on d and p such that $\forall v\in W^{1,p}(K)$.

$$||v-v_\sigma||_{L^p(K)} \le c\ \mathrm{diam}(K)\|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{2.45}$$ {L1-2}

$$||v-v_K||_{L^p(K)} \le c\ \mathrm{diam}(K)\|\nabla v\|_{L^p(K;\mathbb{R}^d)}, \tag{2.46}$$ {L1-3}

where v_K and v_σ are define in (2.40), (2.42).

From Lemma 2.4 on deduces in almost straightforward way the following "global" properties of projections $\mathcal{P}_{\mathcal{T}}$, $\mathcal{P}_{\mathcal{E}}$ (see [20, Lemma3]):

Lemma 2.5. *There exists $c>0$ such that for any $i=1,...d$, and for any $1\le p\le\infty$ one has*

$$\forall v\in L^p(\Omega),\quad \|\mathcal{P}_{\mathcal{T}}v\|_{L^p(\Omega)} \le c\|v\|_{L^p(\Omega)}, \tag{2.47}$$ {dod11}

$$\forall v\in W^{1,p}(\Omega),\quad \|\mathcal{P}_{\mathcal{T}}v-v\|_{L^p(\Omega)} \le ch\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)},$$

$$\forall v\in L^p(\Omega)\cap H_0^1(\Omega),\quad \|\mathcal{P}_{\mathcal{E}}^{(i)}v\|_{L^p(\Omega)} \le c\|v\|_{L^p(\Omega)}, \tag{2.48}$$ {dod12}

$$\forall v\in W^{1,p}\cap H_0^1(\Omega),\quad \|\mathcal{P}_{\mathcal{E}}^{(i)}v-v\|_{L^\infty(\Omega)} \le ch\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}.$$

Lemma 2.6. *There exists $c>0$ such that for any $i=1,...d$, and for any $1\le p\le\infty$ one has*

$$\forall v\in W^{1,p}(\Omega)\cap H_0^1(\Omega),\quad \|\nabla_{\mathcal{E}^{(i)}}\mathcal{P}_{\mathcal{E}}^{(i)}v\|_{L^p(\Omega;\mathbb{R}^d)} \le c\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}, \tag{2.49}$$ {dod13}

$$\forall v\in W^{2,\infty}(\Omega)\cap H_0^1(\Omega),\quad \|\eth_j\mathcal{P}_{\mathcal{E}}^{(i)}v-\partial_j v\|_{L^\infty(\Omega)} \le ch\|\nabla^2 v\|_{L^\infty(\Omega;\mathbb{R}^{d^2})}.$$

Next we introduce and recall some properties of different velocity interpolators.

Definition 2.3. *[*Velocity interpolators*]*

1. *Velocity reconstruction operator with respect to* (i,j)
   For a given MAC grid $\mathcal{D} = (\mathcal{T}, \mathcal{E})$, *we define, for* $i, j = 1, ..., d$, *the full grid velocity reconstruction operator with respect to* (i,j) *by*

$$\mathcal{R}_{\mathcal{E}}^{(i,j)} : H_{\mathcal{E},0}^{(i)} \to H_{\mathcal{E},0}^{(j)}, \quad v \mapsto \mathcal{R}_{\mathcal{E}}^{(i,j)} v = \sum_{\sigma \in \mathcal{E}_{\rm int}^{(j)}} \hat{v}_\sigma^{(i,j)} \mathcal{X}_{D_\sigma}, \tag{2.50}$$ {def:ufull}

where

$$\hat{v}_\sigma^{(i,j)} = v_\sigma \text{ if } \sigma \in \mathcal{E}_{\rm int}^{(i)}, \quad \hat{v}_\sigma^{(i,j)} = \frac{1}{\operatorname{card}(\mathcal{N}_\sigma)} \sum_{\sigma' \in \mathcal{N}_\sigma} v_{\sigma'} \text{ otherwise}, \quad \mathcal{N}_\sigma = \{\sigma' \in \mathcal{E}^{(i)}, D_\sigma \cap \sigma' \neq \emptyset\}. \tag{2.51}$$

2. *Velocity reconstruction to* $L_\mathcal{T}$
   For any $i = 1, ...d$, *we also define a projector from* $H_\mathcal{E}^{(i)}$ *into* $L_\mathcal{T}$ *by*

$$\mathcal{R}_\mathcal{T}^{(i)} : H_\mathcal{E}^{(i)} \to L_\mathcal{T}, \quad v \mapsto \mathcal{R}_\mathcal{T}^{(i)} v = \sum_{K \in \mathcal{T}} v_K \mathcal{X}_K, \tag{2.52}$$ {def:ufullprimal}

where

$$v_K = \frac{1}{|K|} \int_K v(x) \mathrm{d}x = \frac{1}{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} v_\sigma. \tag{2.53}$$ {eq:interpolate_primal}

We then set

$$\mathcal{R}_\mathcal{T} : \mathbf{H}_\mathcal{E} \to L_\mathcal{T}^d, \qquad \boldsymbol{v} = (v_1, ...v_d) \mapsto \mathcal{R}_\mathcal{T} \boldsymbol{v} = (\mathcal{R}_\mathcal{T}^{(1)} v_1, ..., \mathcal{R}_\mathcal{T}^{(d)} v_d). \tag{2.54}$$ {def:ufullprimalvect}

3. *Upwind velocity reconstruction operator with respect to* (i,j)
   Let $\sigma = K|L \in \mathcal{E}^{(j)}$ *and let* $u \in H_{\mathcal{E},0}^{(j)}$. *We define*

$$\sigma_u^{\rm up} = \left\{ \begin{array}{c} K \text{ if } u_{\sigma,K} \equiv u_{j,\sigma} \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} > 0, \\ \\ L \text{ if } u_{\sigma,K} \equiv u_{j,\sigma} \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K} \leq 0 \end{array} \right\} \in \mathcal{T}.$$

For any $v \in H_{\mathcal{E},0}^{(i)}$ *we define*

$$\mathcal{R}_\mathcal{E}^{(i,j,u)}(v) = \sum_{\sigma \in \mathcal{E}^{(j)}} v_{\sigma_u^{\rm up}} \chi(D_\sigma) \in H_{\mathcal{E},0}^{(j)}.$$

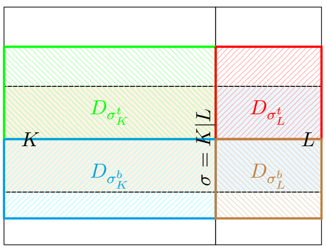

Figure 5: Full grid velocity interpolate.

The following Lemmas 2.7-2.9 are straightforward consequence of Definition 2.3, see [20, Lemma 4] for the proofs.

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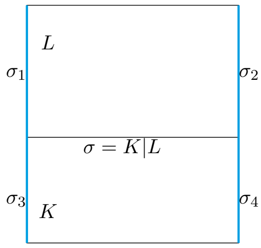

Figure 6: Set $\mathcal{N}_\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ with $\sigma \in \mathcal{E}_{\text{int}}^{(j)}(K)$, $j \neq i$ in two dimensions ($i = 1$, $j = 2$)

Lemma 2.7. *There exists $c > 0$ depnding on d, p such that for any $i = 1, ...d,$, for any $v \in H_{\mathcal{E},0}^{(i)}$ and for any $1 \leq p \leq \infty$ one has*

$$\| \mathcal{R}_{\mathcal{T}}^{(i)} v\|_{L^p(\Omega)} \leq c\|v\|_{L^p(\Omega)}, \quad \| \mathcal{R}_{\mathcal{T}}^{(i)} v - v\|_{L^p(\Omega)} \leq ch\|\eth_i v\|_{L^p(\Omega)}. \tag{2.55}$$ {ufullprimalest}

Lemma 2.8. *There exists $c > 0$, depending only on d, p and on the regularity of the mesh (defined by (2.8)) in a decreasing way, such that, for any $v \in L^p(\Omega)$, for any $1 \leq p \leq \infty$ and for any $i, j = 1, ..., d$,*

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j)} v\|_{L^p(\Omega)} \leq c\|v\|_{L^p(\Omega)}, \quad \|\mathcal{R}_{\mathcal{E}}^{(i,j)} v - v\|_{L^p(\Omega)} \leq ch\|\nabla_{\mathcal{E}}^{(i)} v\|_{L^p(\Omega);\mathbb{R}^d}. \tag{2.56}$$ {def:ufullest}

Lemma 2.9. *There exists $c > 0$, depending only on d, p and on the regularity of the mesh (defined by (2.8)) in a decreasing way, such that, for any $v \in L^p(\Omega)$, for any $1 \leq p \leq \infty$ and for any $i, j = 1, ..., d$,*

$$\|\mathcal{R}_{\mathcal{E}}^{(i,j,u)} v\|_{L^p(\Omega)} \leq c\|v\|_{L^p(\Omega)}, \quad \|\mathcal{R}_{\mathcal{E}}^{(i,j,u)} v - v\|_{L^p(\Omega)} \leq ch\|\nabla_{\mathcal{E}}^{(i)} v\|_{L^p(\Omega);\mathbb{R}^d}. \tag{2.57}$$ {R2up}

The following algebraic identity derived in [20, Lemma 5] in the spirit of [27] is useful to transform terms involving the dual fluxes into terms involving primal fluxes.

Lemma 2.1. *Let $\varrho \in L_{\mathcal{T}}$ and $\mathbf{u} \in \mathbf{H}_{\mathcal{E},0}$. Let $i \in \{1, \ldots, d\}$ Let $\varphi = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \varphi_\sigma \chi_{D_\sigma} \in H_{\mathcal{E},0}^{(i)}$ be a discrete scalar function. Let the primal fluxes be given by (2.12) and let the dual fluxes $F_{\epsilon,\sigma}$ be given by (2.19) or (2.20) (depending on the direction of $\mathbf{n}_\varepsilon$ with respect to $\mathbf{e}^{(i)}$). Then we have:*

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma} u_\epsilon \varphi_\sigma = \sum_{K \in \mathcal{T}} (\mathcal{R}_{\mathcal{T}}^{(i)} \varphi)\Big|_K \sum_{j=1}^{d} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(j)}(K)} F_{\sigma,K} (\mathcal{R}_{\mathcal{E}}^{(i,j)} u_i)\Big|_\sigma + R^i(u_i, \varphi)$$

where

$$R^i(u_i, \varphi) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}(K)} (\varphi_\sigma - (\mathcal{R}_{\mathcal{T}}^{(i)} \varphi)\Big|_K) F_{\sigma,K} (u_\sigma - (\mathcal{R}_{\mathcal{T}}^{(i)} u_i)_K)$$
$$+ \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}(K)} (\varphi_\sigma - (\mathcal{R}_{\mathcal{T}}^{(i)} \varphi)\Big|_K) \sum_{j=1, j \neq i}^{d} \sum_{\tau \in \mathcal{E}_{\text{int}}^{(j)}(K)} \sum_{\sigma' \in \mathcal{N}_{\tau,\sigma}} \frac{F_{\tau,K}}{2} \Big(\frac{u_{i,\sigma} + u_{i,\sigma'}}{2} - (\mathcal{R}_{\mathcal{T}}^{(i)} u_i)\Big|_K \Big).$$

In the last sum we have denoted

$$\mathcal{N}_{\tau,\sigma} = \{\sigma' \in \mathcal{E}^{(i)} \,|\, \text{int}_{d-1} \tau \cap \text{int}_{d-1}(D_\sigma | D_{\sigma'}) \neq \emptyset\},$$

where $\sigma \in \mathcal{E}_{\text{int}}^{(i)}(K)$, $\tau \in \mathcal{E}_{\text{int}}^{(j)}(K)$, $j \neq i$.

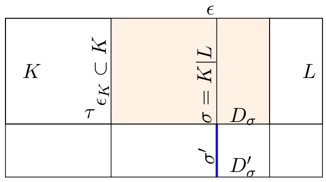

Figure 7: Set $\mathcal{N}_{\tau,\sigma} = \{\sigma'\}$ with $\tau \in \mathcal{E}^{(j)}_{\rm int}(K)$, $\sigma \in \mathcal{E}^{(i)}(K)$, $j \neq i$ in two dimensions ($i = 1$, $j = 2$)

2.5 Main result: asymptotic preserving error estimates

Now, we are ready to state the main results of this paper. For the sake of clarity, we shall state the theorem and perform the proofs only in the most interesting three dimensional case. The modifications to be done for the two dimensional case, which is in fact more simple, are mostly due to the different Sobolev embedings and are left to the interested reader.

2.5.1 Relative energy and relative energy functional

Before the announcement of the main theorems, we introduce relative energy function

$$E : [0,\infty) \times (0,\infty) \to [0,\infty), \quad E(\varrho|z) = \mathcal{H}(\varrho) - \mathcal{H}'(z)(\varrho - z) - \mathcal{H}(z), \text{ where } \mathcal{H}(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} {\rm d}s. \tag{2.58}$$ {E}

We notice that under assumption $p'(\varrho) > 0$, function $\varrho \mapsto H(\varrho)$ is strictly convex on $(0,\infty)$; whence

$$E(\varrho|z) \geq 0 \;\text{ and }\; E(\varrho|z) = 0 \;\Leftrightarrow\; \varrho = z.$$

In fact E obeys stronger coercivity property.

Lemma 2.10. *Let p satisfies assumptions (1.4). Let $\overline{\varrho} > 0$. Then there exists $c = c(\overline{\varrho}) > 0$ such that for all $\varrho \in [0,\infty)$ there holds*

$$E(\varrho|\overline{\varrho}) \geq c\Big(1_{R_+\setminus[\overline{\varrho}/2,2\overline{\varrho}]}(\varrho) + \varrho^\gamma 1_{R_+\setminus[\overline{\varrho}/2,2\overline{\varrho}]}(\varrho) + (\varrho - \overline{\varrho})^2 1_{[\overline{\varrho}/2,2\overline{\varrho}]}(\varrho)\Big). \tag{2.59}$$ {added}

Finally we introduce the corresponding relative energy functional,

$$\mathcal{E}_\varepsilon(\varrho, \boldsymbol{u}|z, \mathbf{v}) = \int_\Omega \Big(\varrho|\boldsymbol{u} - \mathbf{v}|^2 + \frac{1}{\varepsilon^2} E(\varrho|z)\Big){\rm d}x, \tag{2.60}$$ {calE}

where $\varrho \geq 0$, $z > 0$, $\boldsymbol{u}$, $\mathbf{v}$ are measurable functions on Ω.

2.5.2 Error estimates

We are at the point to announce the main theorem of the paper.

Theorem 2.1. [Error estimate in the low Mach number regime]
Let $\Omega \subset \mathbb{R}^3$ be a domain compatible with the MAC grid and let $\mathcal{D} = (\mathcal{T}, \mathcal{E})$ be a MAC grid of Ω (see Definition 2.1) with step size h (see (2.6)) and regularity $\theta_\mathcal{T}$ where $\theta_\mathcal{T}$ is defined in (2.8). Let us consider a partition $0 = t^0 < t^1 < ... < t^N = T$ of the time interval $[0,T]$, which, for the sake of simplicity, we suppose uniform where δt stands for the constant time step.

Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{T},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ (see Section 2.1.2) be a solution of the discrete problem (2.38) emanating from the initial data $(\varrho^0_\varepsilon, \boldsymbol{u}^0_\varepsilon) \in L_\mathcal{T} \times \mathbf{H}_{\mathcal{E},0}$ such that $\varrho^0_\varepsilon > 0$ and

$$M_{0,\mathcal{T},\varepsilon} = \int_\Omega \varrho^0_\varepsilon\,{\rm d}x, \qquad \mathcal{E}_{0,\mathcal{T},\varepsilon} = \int_\Omega \varrho^0_\varepsilon|\boldsymbol{u}^0_\varepsilon|^2\,{\rm d}x + \frac{1}{\varepsilon^2}\int_\Omega E(\varrho^0_\varepsilon|\overline{\varrho})\,{\rm d}x, \tag{2.61}$$ {energieinit}

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where

$$M_0/2 \le M_{0,\mathcal{T},\varepsilon} \le 2M_0,\ \overline{\varrho}|\Omega| = M_0, \quad \mathcal{E}_{0,\mathcal{T},\varepsilon} \le E_0,\ E_0 > 0 \tag{2.62}$$ {dod2}

(existence of which is guaranteed by Proposition 2.1).

Suppose that $[\Pi, \mathbf{V}]$ *is a classical solution to the initial-boundary value problem (1.7–1.9) in* $(0,T)\times\Omega$ *in the regularity class (1.10) with* $p = \max(2,\gamma')$*, emanating from the initial data* $\mathbf{V}(0) \equiv \boldsymbol{V}^0 \in L^2(\Omega)$.

Then there exists a positive number

$$C = C\Big(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}\|_{\mathcal{X}_T(\Omega)}, \|\Pi\|_{\mathcal{Y}^p_T(\Omega)}\Big)$$

depending on these parameters in a nondecreasing way, on $\theta_{\mathcal{T}}$ *in a nonincreasing way and dependent tacitly also on* Ω, T, γ, α, p_0, p_∞ *(and independent in particular on* h, δt, ε*) such that*

$$\sup_{1\le n\le N} \mathcal{E}_\varepsilon\Big(\varrho^n, \boldsymbol{u}^n\Big|\overline{\varrho}, \boldsymbol{V}^n_{\mathcal{E}}\Big) + \mu\delta t\sum_{n=1}^N ||\boldsymbol{u}^n - \boldsymbol{V}^n_{\mathcal{E}}||^2_{1,\mathcal{E},0} + (\mu+\lambda)\delta t\sum_{n=1}^N ||\operatorname{div}_{\mathcal{T}}(\boldsymbol{u}^n - \boldsymbol{V}^n_{\mathcal{E}})||^2_{L^2(\Omega)} \tag{2.63}$$ {M1}

$$\le C\left(\sqrt{\delta t} + h^A + \varepsilon + \mathcal{E}_\varepsilon\Big(\varrho^0_\varepsilon, \boldsymbol{u}^0_\varepsilon\Big|\overline{\varrho}, \boldsymbol{V}^0_{\mathcal{E}}\Big)\right),$$

where

$$A = \min(\frac{2\gamma-3}{\gamma}, 1). \tag{2.64}$$ {A1}

Here we have denoted $\boldsymbol{V}^n_{\mathcal{E}} = \mathcal{P}_{\mathcal{E}}(\boldsymbol{V}(t_n))$*, where* $\mathcal{P}_{\mathcal{E}}$ *is the projection to the discrete space* $\mathbf{H}_{\mathcal{E},0}$ *defined in Section 2.4. Operator* $\operatorname{div}_{\mathcal{T}}$ *is defined in (2.23) and the norm* $\|\cdot\|_{1,\mathcal{E},0}$ *is given in (2.34a–2.34b). Finally, in the above*

$$\|\boldsymbol{V}\|_{\mathcal{X}_T(\Omega)} = \|\boldsymbol{V}\|_{C^1([0,T]\times\overline{\Omega};\mathbb{R}^3)} + \|\nabla^2\boldsymbol{V}\|_{C([0,T]\times\overline{\Omega};\mathbb{R}^{27})} \tag{2.65}$$ {normspacestrong}

$$+\|\partial_t^2\boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^3))} + \|\partial_t\nabla\boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^9))},$$

$$\|\Pi\|_{\mathcal{Y}_T(\Omega)} = \|\Pi\|_{C^1([0,T]\times\overline{\Omega})} + \|\partial_t\Pi\|_{L^1(0,T;L^p(\Omega))}.$$

Remark 2.1.

1. *Due to (1.7) and (1.9–1.10),* $\mathbf{V}_0$ *belongs necessarily to* $C^1(\overline{\Omega};R^3)$ *and it is divergence free. If the initial data are ill prepared, meaning that*

$$\int_\Omega E(\varrho^0_\varepsilon|\overline{\varrho}) \lesssim \varepsilon^2, \quad \int_\Omega \varrho^0_\varepsilon|\mathbf{u}^0_\varepsilon - \mathbf{V}^0_{\mathcal{E}}|^2 \lesssim 1,$$

we obtain in Theorem 2.1 for the error solely a bound independent of ε*. On the other hand, if the initial data are well prepared, with a convergence rate,* ε^ξ, $\xi > 0$*, meaning*

$$\int_\Omega E(\varrho^0_\varepsilon|\overline{\varrho}) \lesssim \varepsilon^{2+\xi}, \quad \int_\Omega \varrho^0_\varepsilon|\boldsymbol{u}^0_\varepsilon - \mathbf{V}^0_{\mathcal{E}}|^2 \lesssim \varepsilon^\xi,$$

Theorem 2.1 gives uniform convergence as $(h,\delta t,\varepsilon) \to 0$ *of the numerical solution to the strong solution of the incompressible Navier-Stokes equations, provided the strong solution exists, including the rates of convergence. These results are in agreement with the theory of low Mach number limits in the continuous case.*

2. *In view of Lemma 2.10, formula (2.63) provides the bound for the "essential part" of the solution (where the numerical density remains bounded from above and from below outside zero):*

$$\|\varrho^m - \overline{\varrho}\|^2_{L^2(\Omega\cap\{\underline{\varrho}/2\le\varrho^m\le2\overline{\varrho}\})} + \|\boldsymbol{u}^m - \mathbf{V}^m_{\mathcal{E}}\|^2_{L^2(\Omega\cap\{\underline{\varrho}/2\le\varrho^m\le2\overline{\varrho}\})},$$

and for the "residual part" of the solution, where the numerical density can be "close" to zero or infinity:

$$|\{\varrho^m \le \underline{\varrho}/2\}| + |\{\varrho^m \ge 2\overline{\varrho}\}| + \|\varrho^m\|^\gamma_{L^\gamma(\Omega\cap\{\varrho^m\ge2\overline{r}\})} + \|\varrho^m|\boldsymbol{u}^m - \mathbf{V}^m_{\mathcal{E}}|^2\|_{L^1(\Omega\cap\{\varrho^m\ge2\overline{\varrho}\})}.$$

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In the above, for $B \subset \Omega$, $|B|$ denotes the Lebesgue measure of B.

Moreover, in the particular case of $p(\varrho) = \varrho^2$, we have $E(\varrho|r) = (\varrho - r)^2$ and the error estimate (2.63) provides a bound for the Lebesgue norms

$$\|\varrho^m - \overline{\varrho}\|^2_{L^2(\Omega)} + \|\varrho^m|\boldsymbol{u}^m - \mathbf{V}^m_{\mathcal{E}}|^2\|_{L^1(\Omega)}.$$

3. *Theorem 2.1 remains valid also for two dimensional bounded domains compatible with the MAC disretization described in Setion 2.2.1 with any $0 < A < \frac{2\gamma-2}{\gamma}$ if $\gamma \in (1,2]$, and $A = 1$ if $\gamma > 2$.*

3 Mesh independent estimates

3.1 Conservation of mass

Due to (2.15), summing (2.38a) over $K \in \mathcal{T}$, we obtain immediately the total conservation of mass,

$$\forall n = 1, ...N, \quad \int_\Omega \varrho^n \, \mathrm{d}x = \int_\Omega \varrho^0 \, \mathrm{d}x. \tag{3.1}$$ {masscons}

3.2 Energy Identity

In the next theorem we report the energy identity for any solution of the numerical scheme (2.38). This theorem shows that the scheme (2.38) is unconditionally stable meaning that the discrete energy inequality holds without any extra assumptions on the discrete solution. The theorem whose detailed proof can be find in [20, Theorem 4] reads

Lemma 3.1. *[Energy identity]*
Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{T},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution of with pressure obeying hypotheses $(1.4)_1$. Then there exists $\varrho^{n-1,n} \in L_{\mathcal{T}}$, $\min(\varrho^{n-1}_K, \varrho^n_K) \le \varrho^{n-1,n}_K \le \max(\varrho^{n-1}_K, \varrho_K r^n)$ and $\varrho^n_\sigma \in [\min(\varrho^n_K, \varrho^n_L), \max(\varrho^n_K, \varrho^n_L)]$, $\sigma = K|L \in \mathcal{E}_{\rm int}$, $n = 1, \dots, N$ such that for all $n = 1, \dots, n$ we have,

$$\begin{aligned}
&\frac{1}{\delta t}\frac{1}{\varepsilon^2}\int_\Omega \mathcal{H}(\varrho^n) - \mathcal{H}(\varrho^{n-1}) \, \mathrm{d}x + \frac{1}{2\delta t}\int_\Omega \varrho^n |\boldsymbol{u}^n|^2 - \varrho^{n-1}|\boldsymbol{u}^{n-1}|^2 \, \mathrm{d}x \\
&\qquad + \mu ||\boldsymbol{u}^n||^2_{1,\mathcal{E},0} + (\mu+\lambda)||\,\mathrm{div}_{\mathcal{T}}\, \boldsymbol{u}^n||^2_{L^2(\Omega)} + \frac{1}{2\delta t}\int_\Omega \varrho^{n-1}|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}|^2 \, \mathrm{d}x \\
&+ \frac{1}{2\delta t}\frac{1}{\varepsilon^2}\int_\Omega \mathcal{H}''(\varrho^{n-1,n})(\varrho^n - \varrho^{n-1})^2 \mathrm{d}x + \frac{1}{2}\frac{1}{\varepsilon^2}\sum_{\substack{\sigma \in \mathcal{E}_{\rm int} \\ \sigma = K|L}} |\sigma|\, \mathcal{H}''(\varrho^n_\sigma)(\varrho^n_K - \varrho^n_L)^2 |\boldsymbol{u}^n_{\sigma,K}| = 0.
\end{aligned} \tag{3.2}$$ {ied1g}

From now on, the letter c denotes a positive number that may tacitly depend on T, Ω, γ, α, p_0, p_∞, μ, λ, and on $\theta_{\mathcal{T}}$ in a nonincreasing way. This dependence on the above parameters will not be indicated in the argument of c. The number c may be dependent further in a nondecreasing way on M_0, E_0, $\overline{\varrho}$ (see (2.62)), and on given functions denoted $(\Pi, \boldsymbol{U}) \in \mathcal{X}_T(\Omega) \times \mathcal{Y}^p_T(\Omega)$, see (1.10). The dependence on these quantities (if any) is always explicitly indicated in the argument of c.

The numbers c can take different values even in the same formula. They are always independent of the size of the discretisation δt and h and on the Mach number ε.

Now, for fixed number $\overline{\varrho} > 0$ and fixed functions ϱ^n, $n = 0, \dots, N$, we introduce the residual and essential subsets of Ω (relative to ϱ^n) as follows:

$$\Omega^n_{\rm ess} = \{x \in \Omega \,\Big|\, \frac{1}{2}\overline{\varrho} \le \varrho^n(x) \le 2\overline{\varrho}\}, \ \Omega^n_{\rm res} = \Omega \setminus \Omega^n_{\rm ess}, \tag{3.3}$$ {essres}

and we set

$$[g]_{\rm ess}(x) = g(x)1_{\Omega^n_{\rm ess}}(x), \ [g]_{\rm res}(x) = g(x)1_{\Omega^n_{\rm res}}(x), \ \ x \in \Omega, \ \ g \in L^1(\Omega).$$

Corollary 3.1. *Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{T},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ be a solution of (2.38) with pressure p obeying (1.4) emanating from initial data (2.61-2.62). Then we have*

1. *Induced standard energy estimates:*

$$\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0}(\Omega))} \le c, \tag{3.4}$$ {est0}

$$\|\boldsymbol{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \le c, \tag{3.5}$$ {est1}

$$\|\varrho|\boldsymbol{u}|^2\|_{L^\infty(0,T;L^1(\Omega))} \le c, \tag{3.6}$$ {est2}

$$\max_{0\le n\le N} \int_\Omega E(\varrho^n|\overline{\varrho})\mathrm{d}x \le c\varepsilon^2 \tag{3.7}$$ {est3-}

$$\max_{0\le n\le N} \Big(\|\varrho^n\|^q_{L^q(\Omega^n_{\mathrm{res}})} \le c\varepsilon^2,\ 1\le q\le\gamma, \quad \max_{0\le n\le N}|\Omega^n_{\mathrm{res}}| \le c\varepsilon^2 \tag{3.8}$$ {est3}

$$\max_{0\le n\le N}\|\varrho^n-\overline{\varrho}\|_{L^q(\Omega^n_{\mathrm{ess}})} \le c(\overline{\varrho})\varepsilon^2,\ 2\le q<\infty.$$

2. *Estimates of numerical dissipation*

$$\delta t\sum_{n=1}^N \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} |\sigma| \frac{(\varrho^n_K-\varrho^n_L)^2}{[\max(\varrho^n_K,\varrho^n_L)]^{(2-\gamma)^+}} 1_{\{\varrho^n_\sigma\ge 1\}}\,|u^n_{\sigma,K}| + \delta t\sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} |\sigma|(\varrho^n_K-\varrho^n_L)^2 1_{\{\varrho^n_\sigma<1\}}\,|u^n_{\sigma,K}| \le c(M_0,E_0)\varepsilon^2, \tag{3.9}$$ {dissipative1}

$$\sum_{n=1}^N\sum_{K\in\mathcal{M}} |K| \frac{(\varrho^n_K-\varrho^{n-1}_K)^2}{[\max(\varrho^{n-1}_K,\varrho^n_K)]^{(2-\gamma)^+}} 1_{\{\varrho^{n-1,n}_K\ge 1\}} + \sum_{n=1}^N\sum_{K\in\mathcal{M}} |K|(\varrho^n_K-\varrho^{n-1}_K)^2 1_{\{\varrho^{n-1,n}<1\}} \le c(M_0,E_0)\varepsilon^2. \tag{3.10}$$ {dissipative2}

$$\sum_{n=1}^m\sum_{i=1}^3\sum_{\sigma\in\mathcal{E}^{(i)}_{\mathrm{int}}} |D_\sigma|\varrho^{n-1}_{D_\sigma}|u^n_\sigma-u^{n-1}_\sigma|^2 \le c. \tag{3.11}$$ {dis}

The quantities ϱ^n_σ and $\varrho^{n-1,n}$ are defined in Lemma 3.1.

Proof of 3.1
Lemma 3.1 in combination with the conservation of mass (3.1) and the definition (2.58) of $E(\cdot|\cdot)$, yields

$$\frac{1}{\delta t}\frac{1}{\varepsilon^2}\int_\Omega E(\varrho^n|\overline{\varrho}) - E(\varrho^{n-1}|\overline{\varrho})\,\mathrm{d}x + \frac{1}{2\delta t}\int_\Omega \varrho^n|\boldsymbol{u}^n|^2 - \varrho^{n-1}|\boldsymbol{u}^{n-1}|^2\,\mathrm{d}x + \mu||\boldsymbol{u}^n||^2_{1,\mathcal{E},0} + (\mu+\lambda)||\operatorname{div}_{\mathcal{T}}\boldsymbol{u}^n||^2_{L^2(\Omega)} + \frac{1}{2\delta t}\int_\Omega \varrho^{n-1}|\boldsymbol{u}^n-\boldsymbol{u}^{n-1}|^2\,\mathrm{d}x + \frac{1}{2\delta t}\frac{1}{\varepsilon^2}\int_\Omega \mathcal{H}''(\varrho^{n-1,n})(\varrho^n-\varrho^{n-1})^2\mathrm{d}x + \frac{1}{2}\frac{1}{\varepsilon^2}\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma|\,\mathcal{H}''(\varrho^n_\sigma)(\varrho^n_K-\varrho^n_L)^2|\boldsymbol{u}^n_{\sigma,K}| = 0. \tag{3.12}$$ {dodano4}

This yields immediately (after multiplication by δt and summation over $n=1,\dots,N$) estimates (3.4), (3.6), (3.7) and (3.11). We obtain (3.5) from (3.4) and the discrete Sobolev inequality reported in Lemma 2.2. We obtain (3.9) and (3.10) from the corresponding terms in Lemma 3.1 after employing (1.4)

Integrating inequality (2.59) we deduce

$$\int_\Omega \Big(\Big[1\Big]_{\mathrm{res}} + \Big[(\varrho^n)^\gamma\Big]_{\mathrm{res}} + \Big[\varrho^n-\overline{\varrho}\Big]^2_{\mathrm{ess}}\Big)\,\mathrm{d}x \ \le c(\overline{\varrho})\int_\Omega E(\varrho^n,\boldsymbol{u}^n\Big|\overline{\varrho},0)\mathrm{d}x; \tag{3.13}$$ {rentropy}

whence estimates (3.8) follow from (3.7). This completes the proof of 3.1.

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4 Relative energy inequality for the discrete problem

4.1 Exact relative energy inequality for the discrete problem

In this Section, we report the exact relative energy inequality for the numerical scheme (2.38). The proof of this inequality is available in [20, Proposition 2].

Theorem 4.1. *[Exact discrete relative energy inequality]*
Any solution $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{T},\delta t} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ of the discrete problem (2.38) with pressure p obeying hypotheses $(1.4)_1$ satisfies relative energy inequality that reads:

$$\frac{1}{\delta t}\Big(\mathcal{E}_\varepsilon(\varrho^n, \boldsymbol{u}^n\Big|r^n, \boldsymbol{U}^n) - \mathcal{E}_\varepsilon(\varrho^{n-1}, \boldsymbol{u}^{n-1}\Big|r^{n-1}, \boldsymbol{U}^{n-1})\Big) \tag{4.1}$$

{erelativeenergy}

$$+\mu||\boldsymbol{u}^n - \boldsymbol{U}^n||^2_{1,\mathcal{E},0} + (\mu+\lambda)||\operatorname{div}_{\mathcal{T}}(\boldsymbol{u}^n - \boldsymbol{U}^n)||^2_{L^2(\Omega)} \le \sum_{k=1}^{5} T_k^n$$

for any couple of discrete test functions $(r, \boldsymbol{U})$, $0 < r \in Y_{\mathcal{T},\delta t}$, $\boldsymbol{U} \in \boldsymbol{X}_{\mathcal{E},\delta t}$, where

$$T_1^n = \int_\Omega \varrho^{n-1}\frac{\boldsymbol{U}^{n-1}-\boldsymbol{U}^n}{\delta t}\cdot\Big(\boldsymbol{u}^{n-1} - \frac{1}{2}(\boldsymbol{U}^{n-1}+\boldsymbol{U}^n)\Big)\,\mathrm{d}x,$$

$$T_2^n = \sum_{i=1}^{3}\sum_{\sigma\in\mathcal{E}^{(i)}_{\rm int}}\sum_{\epsilon\in\widetilde{\mathcal{E}}(D_\sigma),\epsilon=D_\sigma|D_{\sigma'}} F_{\epsilon,\sigma}(\varrho^n,\boldsymbol{u}^n)\boldsymbol{U}^n_\sigma\cdot\Big(\boldsymbol{u}^n_\epsilon - \boldsymbol{U}^n_\varepsilon\Big),$$

$$T_3^n = \mu[\boldsymbol{U}^n - \boldsymbol{u}^n, \boldsymbol{U}^n]_{1,\mathcal{E},0} + (\mu+\lambda)\int_\Omega \operatorname{div}_{\mathcal{T}}(\boldsymbol{U}^n - \boldsymbol{u}^n)\operatorname{div}_{\mathcal{T}}\boldsymbol{U}^n\,\mathrm{d}x,$$

$$T_4^n = -\frac{1}{\varepsilon^2}\int_\Omega p(\varrho^n)\operatorname{div}_{\mathcal{T}}\boldsymbol{U}^n\,\mathrm{d}x,$$

$$T_5^n = \frac{1}{\varepsilon^2}\int_\Omega (r^n - \varrho^n)\frac{\mathcal{H}'(r^n) - \mathcal{H}'(r^{n-1})}{\delta t}\,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \operatorname{div}^{\rm up}_{\mathcal{T}}(\varrho^n\boldsymbol{u}^n)\,\mathcal{H}'(r^{n-1})\,\mathrm{d}x.$$

In the above formulas, flux $F_{\epsilon,\sigma}$ is defined in (2.19–2.20), $\boldsymbol{U}_\sigma = (U_{i,\sigma})_{i=1,2,3}$, see last alinea in Section 2.1.2, $\operatorname{div}_{\mathcal{T}}$ is defined in (2.23), the bilinear form $[\cdot,\cdot]_{1,\mathcal{E},0}$ and corresponding norm $\|\cdot\|_{1,\mathcal{E},0}$ are given in (2.33), (2.34a–2.34b). Finally, the operation denoted by $_\varepsilon$ is defined in (2.18), i.e. $u^n_{i,\varepsilon} = \frac{u_{i,\sigma}+u_{i,\sigma'}}{2}$ if $\sigma, \sigma' \in \mathcal{E}^{(i)}_{\rm int}$, $\varepsilon = D_\sigma|D_{\sigma'}$, $\boldsymbol{u}^n_\varepsilon = (u^n_{i,\varepsilon})_{i=1,2,3}$, and similarly for $\boldsymbol{U}^n_\varepsilon$.

4.2 Approximate relative energy inequality for the discrete problem

The exact relative energy inequality in Theorem 4.1 is an intrinsic inequality for the given MAC scheme. In what follows, we shall write this inequality with particular discrete test functions $r = \overline{\varrho} \in Y_{\mathcal{T},\delta t}$, $\boldsymbol{U} = \mathcal{P}_{\mathcal{E}}(\boldsymbol{V}^n)$, where $\boldsymbol{V}$ is divergence free function with zero traces in the regularity class $X_T(\Omega;\mathbb{R}^3)$. At the same time we shall transform some of the terms in the resulting inequality to the form convenient for comparison with an integral identity satisfied by any strong solution to problem (1.7–1.9). This identity will be derived later. The modified relative energy inequality and the latter mentioned inequality will give the wanted error estimate announced in Theorem 2.1. The lemma reads:

Lemma 4.1. *[Approximate discrete relative energy]*
Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{T},\delta t} \times \mathbf{X}_{\mathcal{E},\delta t}$ be a solution of the discrete problem (2.38) with pressure p satisfying relations $(1.4)_{\gamma\ge 3/2}$ emanating from initial data obeying (2.61–2.62) . Let $\boldsymbol{V} \in C^1([0,T]\times\overline{\Omega};\mathbb{R}^3)$ be such that

$$\boldsymbol{V}_{|\partial\Omega} = 0, \quad \operatorname{div}\boldsymbol{V} = 0.$$

Then there exists

$$c = c(M_0, E_0, \|\boldsymbol{V}, \nabla\boldsymbol{V}, \partial_t\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{15})})$$

such that for all $m = 1, \dots, N$ we have:

$$\mathcal{E}_\varepsilon(\varrho^m, \boldsymbol{u}^m \Big| \overline{\varrho}, \boldsymbol{V}_\mathcal{E}^m) - \mathcal{E}_\varepsilon(\varrho^0, \boldsymbol{u}^0 \Big| \overline{\varrho}, \boldsymbol{V}_\mathcal{E}^0)$$
$$+ \delta t \sum_{n=1}^m \Big(\mu||\boldsymbol{u}^n - \boldsymbol{V}_\mathcal{E}^n||_{1,\mathcal{E},0}^2 + (\mu+\lambda)||\operatorname{div}_\mathcal{T}(\boldsymbol{u}^n - \boldsymbol{V}_\mathcal{E}^n)||_{L^2(\Omega)}^2\Big) \le \sum_{k=1}^3 S_k + \mathcal{R}_{\mathcal{T},h,\delta t}^m + \mathcal{G}_{\mathcal{T},\delta t}^m, \quad (4.2)$$ {relativeenergy-}

where

$$S_1 = \delta t \sum_{n=1}^m \int_\Omega \varrho^{n-1}\Big(\frac{\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{V}_\mathcal{E}^{n-1}}{\delta t}\Big) \cdot \Big(\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{u}^n\Big),$$

$$S_2 = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}_{\rm int}^{(j)}} |\sigma|\Big(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n\Big)\, \varrho_\sigma^{n,\rm up} V_{j,\sigma}^n \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K}\, \Big(u_{i,K}^n - V_{i,K}^n\Big)$$

$$S_3 = \delta t \sum_{n=1}^m \mu[\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{u}^n, \boldsymbol{V}_\mathcal{E}^n]_{1,\mathcal{E},0},$$

for any divergence free vector field $\boldsymbol{V} \in \mathcal{X}_T(\Omega;\mathbb{R}^3)$ (see (1.10)) vanishing at the boundary of Ω. In the above inequality,

$$|\mathcal{G}_{\mathcal{T},\delta t}^m| \le c\delta t \sum_{n=1}^m \mathcal{E}_\varepsilon(\varrho^m, \boldsymbol{u}^m \Big| \overline{\varrho}, \boldsymbol{V}_\mathcal{E}^m) \quad (4.3)$$

$$|\,\mathcal{R}_{\mathcal{T},\delta t}^m\,| \le c(\sqrt{\delta t} + h^A), \quad (4.4)$$

and where A is by formula (2.64). Here, we have used the abbreviated notation $V_{i,\mathcal{E}} = \mathcal{P}_\mathcal{E}^{(i)}(V_i)$, $\boldsymbol{V}_\mathcal{E} = \mathcal{P}_\mathcal{E}\boldsymbol{V} = (V_{i,\mathcal{E}})_{i=1,2,3}$, where projections $\mathcal{P}_\mathcal{E}^{(i)}$, $\mathcal{P}_\mathcal{E}$ are defined in Section 2.4. Further, $V_{i,\mathcal{E},K} = [V_{i,\mathcal{E}}]_K = [\mathcal{R}_\mathcal{T}^{(i)} V_{i,\mathcal{E}}]|_K$, where the interpolator $\mathcal{R}_\mathcal{T} = (\mathcal{R}_\mathcal{T}^{(i)})_{i=1,2,3}$ is defined in Definition 2.3. Operator $\operatorname{div}_\mathcal{T}$ is defined in (2.23), the bilinear form $[\cdot,\cdot]_{1,\mathcal{E},0}$ and corresponding norm $\|\cdot\|_{1,\mathcal{E},0}$ are given in (2.33), (2.34a–2.34b).

Proof of Lemma 4.1
We shall use in the relative energy inequality (4.1) test functions $r = \overline{\varrho}$ and $\boldsymbol{U} = \boldsymbol{V}_\mathcal{E}$. Since $\overline{\varrho}$ is constant, term $T_5^n = 0$. According to Lemma 2.3, $\operatorname{div}_\mathcal{T}\boldsymbol{V}_\mathcal{E} = 0$; whence $T_4^n = 0$. Term T_3^n will be kept as it stays. It remains to transform terms T_1^n and T_2^n. This will be done in several steps.

Step 1: *Term T_1^n.*
We have

$$T_1^n = T_{1,1}^n + R_{1,1}^n + R_{1,2}^n, \quad (4.5)$$ {S1}

$$T_{1,1}^n = \int_\Omega \varrho^{n-1}\Big(\frac{\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{V}_\mathcal{E}^{n-1}}{\delta t}\Big) \cdot \Big(\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{u}^n\Big)\,{\rm d}x,$$

$$R_{1,1}^n = -\int_\Omega \frac{1}{2}\varrho^{n-1}\frac{\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{V}_\mathcal{E}^{n-1}}{\delta t} \cdot (\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{V}_\mathcal{E}^{n-1})\,{\rm d}x, \quad R_{1,2}^n = \int_\Omega \varrho^{n-1}\frac{\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{V}_\mathcal{E}^{n-1}}{\delta t} \cdot (\boldsymbol{u}^n - \boldsymbol{u}^{n-1})\,{\rm d}x.$$

By virtue of the first order integral Taylor formula applied to $\boldsymbol{V}$ in the interval (t_{n-1}, t_n), definition (2.41) of projection $\mathcal{P}_\mathcal{E}$, Cauchy-Schwartz inequality and numerical dissipation (3.11), we easily get

$$|\delta t \sum_{n=1}^m R_{1,1}^n| \le \delta t\, c(M_0, \|\partial_t\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}), \quad |\delta t \sum_{n=1}^m R_{1,2}^n| \le \sqrt{\delta t} c(M_0, E_0, \|\partial_t\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}). \quad (4.6)$$ {S1r}

Step 2: *Term* T_2^n
This step will consist of several successive transformations performed in four movements.

Step 2a:
Employing Lemma 2.1 we get

$$T_2^n = T_{2,1}^n + R_{2,1}^n \tag{4.7}$$ {S2}

$$T_{2,1}^n = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}_{\rm int}^{(j)}} \Big(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n\Big)\, F_{\sigma,K}(\varrho^n,\boldsymbol{u}^n)\,(\mathcal{R}_{\mathcal{E}}^{(i,j)}(u_i^n - V_{i,\mathcal{E}}^n))\Big|_{D_\sigma},$$

$$R_{2,1}^n = \sum_{i=1}^{3} R^i(u_i^n - V_{i,\mathcal{E}}^n, V_{i,\mathcal{E}}^n), \text{ where } R^i \text{ is defined in Lemma 2.1.}$$

We have used the local conservation of primal fluxes (2.15) in order to replace in $T_{2,1}^n$ expression $V_{i,\mathcal{E},K}^n$ by the difference $V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n$.

We easily see from definitions of the projections and interpolates in Section 2.4, and first order Taylor formula that

$$|V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n| \le ch\, \|\nabla V_i\|_{L^\infty(Q_T;\mathbb{R}^3)} \tag{4.8}$$ {F1}

Recalling definition of $F_{\sigma,K}(\varrho_n,\boldsymbol{u}_n)$ we obtain by Hölder's inequality

$$|\sum_{i=1}^{3} R^i(u_i^n - V_{i,\mathcal{E}}^n, V_{i,\mathcal{E}}^n)| \le c\|\varrho^n\|_{L^{\gamma_0}(\Omega)}\|\boldsymbol{u}^n\|_{L^6(\Omega;\mathbb{R}^3)}\|\mathcal{R}_{\mathcal{T}}(\boldsymbol{u}^n - \boldsymbol{V}_{\mathcal{E}}^n) - (\boldsymbol{u}^n - \boldsymbol{V}_{\mathcal{E}}^n)\|_{L^q(\Omega,\mathbb{R}^3)}, \tag{4.9}$$ {F1+}

where $\frac{1}{\gamma_0} + \frac{1}{q} = \frac{5}{6}, \gamma_0 = \min(\gamma, 3)$. Due to Lemmas 2.5 2.7, and Lemma 2.2

$$\|\mathcal{R}_{\mathcal{T}}\boldsymbol{V}^n - \boldsymbol{V}^n\|_{L^q(\Omega)} \le c\, h\|\nabla \boldsymbol{V}^n\|_{L^q(\Omega)},$$

$$\|\mathcal{R}_{\mathcal{T}}\boldsymbol{u}^n - \boldsymbol{u}^n\|_{L^2(\Omega)} \le c\, h\|\boldsymbol{u}^n\|_{1,\mathcal{E},0}, \quad \|\mathcal{R}_{\mathcal{T}}\boldsymbol{u}^n - \boldsymbol{u}^n\|_{L^6(\Omega)} \le c\,\|\boldsymbol{u}^n\|_{1,\mathcal{E},0};$$

whence interpolation of L^q between L^2 and L^6 yields

$$\|\mathcal{R}_{\mathcal{T}}\boldsymbol{u}^n - \boldsymbol{u}^n\|_{L^q(\Omega)} \le c\, h^{\frac{2\gamma_0-3}{\gamma_0}}\|\boldsymbol{u}^n\|_{1,\mathcal{E},0}. \tag{4.10}$$ {F2}

Consequently, coming back to formula (4.9), we arrive to the estimate

$$|\delta t \sum_{n=1}^{m} R_{2,1}^n| \le h^A\, c(M_0, E_0, \overline{\varrho}, \|\nabla_x \boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}) \tag{4.11}$$ {S2r}

after employing the known bounds (3.4–3.8) derived in Corollary 3.1. Here A is defined in (2.64).

Step 2b:
We rewrite $T_{2,1}^n$ by using the definition (2.12) of $F_{\sigma,K}$ as follows

$$T_{2,1}^n = T_{2,2}^n + R_{2,2}^n, \tag{4.12}$$ {S2+}

where

$$T_{2,2}^n = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}_{\rm int}^{(j)}} |\sigma|\Big(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n\Big)\, \varrho_\sigma^{n,\rm up} u_{j,\sigma}^n \boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K}\,\Big(u_{i,\sigma_{u_j^n}^{\rm up}} - V_{i,\sigma_{u_j^n}^{\rm up}}^n\Big)$$

$$R_{2,2}^n = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}_{\rm int}^{(j)}} |\sigma|\Big((\mathcal{R}_{\mathcal{T}}^{(i)}(V_{i,\mathcal{E}}^n))\underline{K} - V_{i,\sigma}^n\Big)\, \varrho_\sigma^{n,\rm up}\, u_{j,\sigma}^n \boldsymbol{e}^{(j)}\cdot \boldsymbol{n}_{\sigma,K}$$

$$\times\Big(\widehat{[u_i^n]}_\sigma^{(i,j)} - \widehat{[V_{i,\mathcal{E}}^n]}_\sigma^{(i,j)} - (u_{i,\sigma^{\rm up}_{u_j^n}} - V^n_{i,\mathcal{E},\sigma^{\rm up}_{u_j^n}})\Big).$$

Here, the number $\widehat{[\cdot]}_\sigma^{i,j}$, the primal cell $\sigma^{\rm up}_{u_j^n}$, and the related operators $\mathcal{R}_\mathcal{E}^{i,j}$, $\mathcal{R}_\mathcal{E}^{i,j,u_j^n}$ used in the next formulas are defined in items 1. and 3. of Definition 2.3. Using (4.8) and the Hölder's inequality, we get

$$|R_{2,2}^n| \le c\|\varrho^n\|_{L^{\gamma_0}(\Omega)}\|\boldsymbol{u}^n\|_{L^6(\Omega)}\sum_{i=1}^3\sum_{j=1}^3\|\mathcal{R}_\mathcal{E}^{i,j}(u_i^n - V_{i,\mathcal{E}}^n) - \mathcal{R}_\mathcal{E}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^q(\Omega)}, \tag{4.13}$$ {F3}

$\frac{1}{\gamma_0}+\frac{1}{q}=\frac{5}{6}$, $\gamma_0=\min(\gamma,3)$ as in (4.9). Due to Lemmas 2.8 and 2.9

$$\|\mathcal{R}_\mathcal{E}^{i,j}(u_i^n - V_{i,\mathcal{E}}^n) - \mathcal{R}_\mathcal{E}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^2(\Omega)} \le c\,h\,\|\nabla_{\mathcal{E}^{(i)}}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^2(\Omega;\mathbb{R}^3)},$$

$$\|\mathcal{R}_\mathcal{E}^{i,j}(u_i^n) - V_{i,\mathcal{E}}^n) - \mathcal{R}_\mathcal{E}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^6(\Omega)} \le c\|u_i^n - V_{i,\mathcal{E}}^n\|_{L^6(\Omega)},$$

where by the discrete Sobolev inequality evoked in Lemma 2.2

$$\|\boldsymbol{u}^n - \boldsymbol{V}_\mathcal{E}^n\|_{L^6(\Omega;\mathbb{R}^3)} \le \|\boldsymbol{u}^n - \boldsymbol{V}^n\|_{1,\mathcal{E},0}.$$

Now, by interpolation of L^q between L^2 and L^6,

$$\|\mathcal{R}_\mathcal{E}^{i,j}(u_i^n - V_{i,\mathcal{E}}^n) - \mathcal{R}_\mathcal{E}^{i,j,u_j^n}(u_i^n - V_{i,\mathcal{E}}^n)\|_{L^q(\Omega)} \le c\,h^{\frac{2\gamma_0-3}{\gamma_0}}(\|\boldsymbol{u}^n\|_{1,\mathcal{E},0} + \|\boldsymbol{V}_\mathcal{E}^n\|_{1,\mathcal{E},0}),$$

similarly as in (4.10). Consequently, employing formula (4.13), the above estimates and estimates (3.4), (3.8) from Corollary 3.1, we get

$$|\delta t\sum_{n=1^m} R_{2,2}^n| \le h^A\,c(M_0,E_0,\overline{\varrho},\|\nabla_x\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}). \tag{4.14}$$ {S2r+}

Step 2c:
In the next step, we write

$$T_{2,2}^n = T_{2,3}^n + R_{2,3}^n \tag{4.15}$$ {S2++}

with

$$T_{2,3}^n = \sum_{i=1}^3\sum_{j=1}^3\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}^{(j)}_{\rm int}}|\sigma|\Big(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n\Big)\,\varrho_\sigma^{n,\rm up}V_{j,\sigma}^n\boldsymbol{e}^{(j)}\cdot\boldsymbol{n}_{\sigma,K}\,\Big(u_{i,\sigma^{\rm up}_{u_j^n}} - V^n_{i,\sigma^{\rm up}_{u_j^n}}\Big)$$

and

$$R_{2,3}^n = \sum_{i=1}^3\sum_{j=1}^3\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}^{(j)}_{\rm int}}|\sigma|\Big(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n\Big)\,\varrho_\sigma^{n,\rm up}\Big(u_{j,\sigma}^n - V_{j,\sigma}^n\Big)\boldsymbol{e}^{(j)}\cdot\boldsymbol{n}_{\sigma,K}\,\Big(u_{i,\sigma^{\rm up}_{u_j^n}} - V^n_{i,\mathcal{E},\sigma^{\rm up}_{u_j^n}}\Big);$$

Noticing that

$$\int_\Omega\varrho^n|\mathbf{u}^n|^2{\rm d}x = \sum_{i=1}^3\sum_{K=\overrightarrow{[\sigma,\sigma']},\sigma\in\mathcal{E}^{(i)}}\Big(|D_{\sigma,K}|\varrho_K^n|u_{i,\sigma}^n|^2 + |D_{\sigma',K}|\varrho_K^n|u_{i,\sigma'}^n|^2\Big)$$

and recalling the definition of the primal cell $[\cdot]_{\sigma^{\rm up}_\cdot}$ in Definition 2.3, formula (4.8) and definition of relative energy $\mathcal{E}_\varepsilon(\cdot|\cdot)$ (see (2.60)), we conclude that

$$\delta t|\sum_{n=1}^m R_{2,3}^n| \le c(\|\nabla\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)})\ \ \delta t\sum_{n=1}^m\mathcal{E}_\varepsilon(\varrho^n,\boldsymbol{u}^n|\overline{\varrho},\boldsymbol{V}_\mathcal{E}^n). \tag{4.16}$$ {S2r++}

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Step 2d:
Finally,

$$T^n_{2,3} = T^n_{2,4} + R^n_{2,4}, \tag{4.17}$$ {S2+++}

where

$$T^n_{2,4} = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}^{(j)}_{\rm int}} |\sigma| \Big(V^n_{i,\mathcal{E},K} - V^n_{i,\sigma}\Big)\, \varrho^{n,{\rm up}}_\sigma V^n_{j,\sigma} \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K}\, \Big(u^n_{i,K} - V^n_{i,K}\Big)$$

and

$$R^n_{2,4} = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}^{(j)}(K)\cap\mathcal{E}^{(j)}_{\rm int}} |\sigma| \Big(V^n_{i,\mathcal{E},K} - V^n_{i,\sigma}\Big)\, \varrho^{n,{\rm up}}_\sigma V^n_{j,\sigma} \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K}$$

$$\times \Big(u^n_{i,\sigma^{\rm up}_{u^n_j}} - V^n_{i,\sigma^{\rm up}_{u^n_j}} - (u^n_{i,K} - V^n_{i,K})\Big).$$

Next, by the Hölder and Minkowski inequalities and (4.8),

$$|R^n_{2,4}| \le c(\|\mathbf{V}\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}) \|\varrho^n\|_{L^{\gamma_0}(\Omega)} \sum_{i=1}^3 \sum_{j=1}^3 \Big(\|\mathcal{R}^{(i,j,u^n_j)}_{\mathcal{E}} u^n_i - \mathcal{R}^{(i)}_{\mathcal{T}} u^n_i\|_{L^q(\Omega)} + \|\mathcal{R}^{(i,j,u^n_j)}_{\mathcal{E}} V^n_{i,\mathcal{E}} - \mathcal{R}^{(i)}_{\mathcal{T}} V^n_{i,\mathcal{E}}\|_{L^q(\Omega)}\Big)$$

where $\gamma_0 = \min(\gamma_0, 2)$, $\frac{1}{\gamma_0} + \frac{1}{q} = 1$. Now we estimate

$$\|\mathcal{R}^{(i,j,u^n_j)}_{\mathcal{E}} u^n_i - \mathcal{R}^{(i)}_{\mathcal{T}} u^n_i\|_{L^2(\Omega)} \le ch \|\nabla_{\mathcal{E}^{(i)}} u^n_i\|_{L^2(\Omega)}$$

and

$$\|\mathcal{R}^{(i,j,u^n_j)}_{\mathcal{E}} u^n_i - \mathcal{R}^{(i)}_{\mathcal{T}} u^n_i\|_{L^6(\Omega)} \le c\|u^n_i\|_{L^6(\Omega)} \le c\|\nabla_{\mathcal{E}^{(i)}} u^n_i\|_{L^2(\Omega)}$$

by virtue of Lemmas 2.7, 2.9 and 2.2. Similar estimates are true if we replace u^n_i by $V^n_{i,\mathcal{E}}$ in the argument of $\mathcal{R}^{(i,j,u^n_j)}_{\mathcal{E}}$ and of $\mathcal{R}^{(i)}_{\mathcal{T}}$. Consequently, by interpolation of L^q between L^2 and L^6,

$$\|\mathcal{R}^{(i,j,u^n_j)}_{\mathcal{E}} u^n_i - \mathcal{R}^{(i)}_{\mathcal{T}} u^n_i\|_{L^q(\Omega)} \le ch^{\frac{5\gamma_0-6}{2\gamma_0}} \|\nabla_{\mathcal{E}^{(i)}} u^n_i\|_{L^2(\Omega)}$$

Putting together these estimates, and employing in addition estimates for the numerical solution deduced in Corollary 3.1, we get

$$\delta t |\sum_{n=1}^m R^n_{2,4}| \le h^{\frac{5\gamma_0-6}{2\gamma_0}}\, c(M_0, E_0, \|\nabla\mathbf{V}\|_{L^\infty(Q_T;\mathbb{R}^9)}). \tag{4.18}$$ {S2r+++}

Now we put together formulas (4.5) and (4.17) together with estimates of remainders (4.6), (4.11), (4.14), (4.16), (4.18) in order to get the required result. Lemma 4.1 is proved.

5 An identity for the strong solution. Consistency error.

The goal of this section is to prove the following lemma.

Lemma 5.1. [Consistency error]
Let $(\varrho, \boldsymbol{u}) \in Y_{\mathcal{T},\delta t} \times \mathbf{X}_{\mathcal{E},\delta t}$ be a solution of the discrete problem (2.38) *with pressure p satisfying relations (1.4)$_{\gamma\ge 3/2}$ emanating from initial data obeying (2.61–2.62). Let the couple* $(\Pi, \boldsymbol{V})$ *belonging to the regularity class (1.10)$_{p=\max(2,\gamma')}$ be a strong solution to the incompressible Navier-Stokes equations (1.7–1.9).*

Then there exists

$$c = c(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \nabla\boldsymbol{V}, \nabla^2\boldsymbol{V}, \nabla\Pi\|_{L^\infty(Q_T;\mathbb{R}^{42})}, \|\partial_t\nabla\boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^9)}, \|\partial_t\Pi\|_{L^1(0,T;L^p(\Omega))}),$$

$p = \max(2, \gamma')$ such that for all $m = 1, \dots, N$ we have

$$\sum_{k=1}^{3} \mathcal{S}_k + \mathcal{R}^m_{h,\delta t} = 0, \tag{5.1}$$ {consistency}

where

$$\mathcal{S}_1 = \delta t \sum_{n=1}^{m} \int_\Omega \overline{\varrho}\Big(\frac{\boldsymbol{V}^n_{\mathcal{E}} - \boldsymbol{V}^{n-1}_{\mathcal{E}}}{\delta t}\Big) \cdot \Big(\boldsymbol{V}^n_{\mathcal{E}} - \boldsymbol{u}^n\Big),$$

$$\mathcal{S}_2 = \delta t \sum_{n=1}^{m} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}^{(j)}(K)} \Big(V^n_{i,\mathcal{E},K} - V^n_{i,\sigma}\Big)\, \overline{\varrho} V^n_{j,\sigma} \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K}(u^n_{i,K} - V^n_{i,\mathcal{E},K})$$

$$\mathcal{S}_3 = \delta t \sum_{n=1}^{m} \mu[\boldsymbol{V}^n_{\mathcal{E}} - \boldsymbol{u}^n, \boldsymbol{V}^n_{\mathcal{E}}]_{1,\mathcal{E},0},$$

and

$$|\mathcal{R}^m_{h,\delta t}| \le c(h + \delta t + \varepsilon).$$

Here we use the same notation as in Lemma 4.1.

The rest of this section is devoted to the proof of Lemma 4.1.

5.1 Getting started

Since $(\Pi, \boldsymbol{V})$ satisfies (1.7–1.9) on $(0,T)\times\Omega$ and belongs to the class (1.10), equation (1.7) can be rewritten in the form

$$\overline{\varrho}\partial_t \boldsymbol{V} + \overline{\varrho}\boldsymbol{V}\cdot\nabla\boldsymbol{V} + \nabla\Pi - \mu\Delta\boldsymbol{V} = 0 \ \text{ in } (0,T)\times\Omega.$$

From this fact, we deduce the identity

$$\sum_{s=1}^{4} \delta t \sum_{n=1}^{m} \mathcal{T}^n_s = 0, \quad m = 1, \dots, N, \tag{5.2}$$ {strong0}

where

$$\mathcal{T}^n_1 = \int_\Omega \overline{\varrho}[\partial_t \boldsymbol{V}]^n \cdot (\boldsymbol{V}^n - \boldsymbol{u}^n)\,\mathrm{d}x, \quad \mathcal{T}^n_2 = \int_\Omega \overline{\varrho}\boldsymbol{V}^n\cdot\nabla\boldsymbol{V}^n \cdot (\boldsymbol{V}^n - \boldsymbol{u}^n)\,\mathrm{d}x,$$

$$\mathcal{T}^n_3 = -\int_\Omega \Big(\mu\Delta\boldsymbol{V}^n\Big)\cdot(\boldsymbol{V}^n - \boldsymbol{u}^n)\,\mathrm{d}x, \quad \mathcal{T}^n_4 = -\int_\Omega \nabla\Pi^n\cdot\boldsymbol{u}^n\,\mathrm{d}x.$$

In the steps below, we deal with each of the terms $\mathcal{T}_s$.

5.2 Term with the time derivative

We proceed in two steps.
Step 1:

$$\mathcal{T}^n_1 = \mathcal{T}^n_{1,1} + \mathcal{R}^n_{1,1}, \tag{5.3}$$ {calS1}

where

$$\mathcal{T}^n_{1,1} = \int_\Omega \overline{\varrho}\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\delta t}\cdot(\boldsymbol{V}^n_{\mathcal{E}} - \boldsymbol{u}^n)\,\mathrm{d}x,$$

$$\mathcal{R}^n_{2,1} = \int_\Omega \overline{\varrho}[\partial_t \mathbf{V}]^n\cdot(\boldsymbol{V}^n - \boldsymbol{V}^n_{\mathcal{E}})\,\mathrm{d}x + \int_\Omega \overline{\varrho}\Big([\partial_t\boldsymbol{V}]^n - \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big)\cdot(\boldsymbol{V}^n_{\mathcal{E}} - \boldsymbol{u}^n)\,\mathrm{d}x.$$

Realizing that

$$[\partial_t\boldsymbol{V}]^n - \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} = \frac{1}{\delta t}\int_{t_{n-1}}^{t_n}\int_s^{t_n} \partial_t^2\boldsymbol{V}(z,\cdot)\mathrm{d}z\mathrm{d}s,$$

we get by using Hölder's inequality (in particular with $\boldsymbol{V}_{\mathcal{E}}^n - \boldsymbol{u}^n$ in $L^6(\Omega;\mathbb{R}^3)$), Lemma 2.5, Lemma 2.2

$$\delta t|\sum_{n=1}^{m}\mathcal{R}_{2,1}^n| \le (\delta t+h)\, c\Big(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \partial_t\boldsymbol{V}, \nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{15})}, \|\partial_t^2\boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^3))}\Big), \qquad (5.4) \quad \{calS1r\}$$

where we have used Lemma 2.2 and the energy bound (3.4) from Corollary 3.1 for $\boldsymbol{u}^n$.

Step 2:
Finally, we write

$$\mathcal{T}_{1,1}^n = \mathcal{T}_{1,2}^n + \mathcal{R}_{1,2}^n, \qquad (5.5) \quad \{calS1+\}$$

where

$$\mathcal{T}_{1,2}^n = \int_\Omega \overline{\varrho}\frac{\boldsymbol{V}_{\mathcal{E}}^n - \boldsymbol{V}_{\mathcal{E}}^{n-1}}{\delta t}\cdot(\boldsymbol{V}_{\mathcal{E}}^n - \boldsymbol{u}^n)\,\mathrm{d}x,$$

$$\mathcal{R}_{1,2}^n = \int_\Omega \overline{\varrho}\Big(\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\delta t} - \frac{\boldsymbol{V}_{\mathcal{E}}^n - \boldsymbol{V}_{\mathcal{E}}^{n-1}}{\delta t}\Big)\cdot(\boldsymbol{V}_{\mathcal{E}}^n - \boldsymbol{u}^n)\,\mathrm{d}x.$$

We have by Hölder's inequality and Lemma 2.5

$$|\mathcal{R}_{1,2}^n| \le h\,c\Big\|\nabla\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\delta t}\|_{L^{6/5}(\Omega;\mathbb{R}^9)}\|\boldsymbol{V}_{\mathcal{E}}^n - \boldsymbol{u}^n\|_{L^6(\Omega;\mathbb{R}^3)},$$

where $\nabla\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\delta t} = \frac{1}{\delta t}\int_{t_{n-1}}^{t_n}\partial_t\nabla\boldsymbol{V}(z,\cdot)\mathrm{d}z$; whence after taking into account Corollary 3.1, we deduce

$$|\mathcal{R}_{1,2}^n| \le h\,c(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^3)}, \|\partial_t\nabla\boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^9))}). \qquad (5.6) \quad \{calS1r+\}$$

5.3 Convective term

Step 1:
We decompose term T_2^n as follows:

$$\mathcal{T}_2^n = \mathcal{T}_{2,1}^n + \mathcal{R}_{2,1}^n, \qquad (5.7) \quad \{calS2\}$$

with

$$\mathcal{T}_{2,1}^n = \sum_{K\in\mathcal{T}}\int_K \overline{\varrho}\boldsymbol{V}_{\mathcal{E},K}^n\cdot\nabla\boldsymbol{V}^n\cdot(\boldsymbol{V}_{\mathcal{E},K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x,$$

$$\mathcal{R}_{2,1}^n = \sum_{K\in\mathcal{T}}\Big(\int_K \overline{\varrho}\boldsymbol{V}^n\cdot\nabla\boldsymbol{V}^n\cdot\Big(\boldsymbol{V}^n - \boldsymbol{V}_{\mathcal{E},K}^n - (\boldsymbol{u}^n - \boldsymbol{u}_K^n)\Big)\,\mathrm{d}x + \int_K \overline{\varrho}(\boldsymbol{V}^n - \boldsymbol{V}_{\mathcal{E},K}^n)\cdot\nabla\boldsymbol{V}^n\cdot(\boldsymbol{V}_{\mathcal{E},K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x.$$

Consequently, by Lemmas 2.7 2.5, 2.6 and estimate (3.4) in Corollary 3.1,

$$\delta t|\sum_{n=1}^{m}\mathcal{R}_{2,1}^n| \le h\; c(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \qquad (5.8) \quad \{calS2r\}$$

Step 2:
Integrating by parts in $\mathcal{T}_{2,1}^n$ while using the fact that $\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma \boldsymbol{V}_{\mathcal{E},K}^n\cdot\boldsymbol{n}_{\sigma,K} = 0$, we get

$$\mathcal{T}_{2,1}^n = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma |\sigma|\overline{\varrho}\boldsymbol{V}_{\mathcal{E},K}^n\cdot\boldsymbol{n}_{\sigma,K}(\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{\mathcal{E},K})\cdot(\boldsymbol{V}_{\mathcal{E},K}^n - \boldsymbol{u}_K^n).$$

Now, we rewrite the last expression as follows

$$\mathcal{T}_{2,1}^n = \mathcal{T}_{2,2}^n + \mathcal{R}_{2,2}^n, \qquad (5.9) \quad \{calS2+\}$$

where

$$\mathcal{T}_{2,2}^n = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma |\sigma|\overline{\varrho}\boldsymbol{V}_{\mathcal{E}}^n\cdot\boldsymbol{n}_{\sigma,K}(\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{\mathcal{E},K})\cdot(\boldsymbol{V}_{\mathcal{E},K}^n - \boldsymbol{u}_K^n)$$

and

$$\mathcal{R}_{2,2}^n = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\sigma|\overline{\varrho}\Big(\boldsymbol{V}_{\mathcal{E},K}^n - \boldsymbol{V}_{\mathcal{E}}\Big)\cdot \boldsymbol{n}_{\sigma,K}(\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{\mathcal{E},K})\cdot(\boldsymbol{V}_{\mathcal{E},K}^n - \boldsymbol{u}_K^n).$$

By Hölder's inequality, after application of Lemmas 2.5, 2.7 and 2.6, we get

$$\delta t|\sum_{n=1}^m \mathcal{R}_{2,2}^n| \le h\, c(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \tag{5.10}$$ {calS2r+}

Expression $\mathcal{T}_{2,2}^n$ written explicitly in coordinates is exactly term $\mathcal{S}_2$ in formula (5.1)

5.4 Viscous term

Step 1:

$$\mathcal{T}_3^n = \mathcal{T}_{3,1} + \mathcal{R}_{3,1}^n, \tag{5.11}$$ {calS3}

$$\mathcal{T}_{3,1}^n = \int_\Omega \mu\Delta\boldsymbol{V}^n\cdot(\boldsymbol{V}_{\mathcal{E}}^n - \boldsymbol{u}^n)\mathrm{d}x,$$

$$\mathcal{R}_{3,1}^n = \int_\Omega \mu\Delta\boldsymbol{V}^n\cdot(\boldsymbol{V} - \boldsymbol{V}_{\mathcal{E}}^n)\mathrm{d}x,$$

where by virtue of the Cauchy-Schwartz inequality and Lemma 2.5

$$\delta t|\sum_{n=1}^m \mathcal{R}_{3,1}^n| \le h\, c(\|\boldsymbol{V}, \nabla\boldsymbol{V})\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \tag{5.12}$$ {calS3r1}

Step 2:
In this step we decompose $\mathcal{T}_{3,1}^n$ as follows

$$\begin{aligned}\mathcal{T}_{3,1}^n &= \sum_{i=1}^3 \sum_{\sigma\in\mathcal{E}^{(i)}} \int_{D_\sigma} \mu\Delta V_i^n(V_{i,\sigma}^n - u_{i,\sigma}^n)\mathrm{d}x\\ &= \sum_{i=1}^3 \sum_{\sigma\in\mathcal{E}^{(i)}} \sum_{\varepsilon\in\tilde{\mathcal{E}}(D_\sigma)} \int_\varepsilon \mu\boldsymbol{n}_{\varepsilon,D_\sigma}\cdot\nabla V_i^n\cdot(V_{i,\sigma}^n - u_{i,\sigma}^n)\mathrm{d}\gamma\\ &= \sum_{i=1}^3\sum_{j=1}^3 \sum_{\varepsilon=\overrightarrow{\sigma|\sigma'}\in\tilde{\mathcal{E}}^{(i)},\varepsilon\perp\boldsymbol{e}^{(j)}} \int_\varepsilon \partial_j V_i\Big(V_{i,\sigma} - u_{i,\sigma} - (V_{i,\sigma'} - u_{i,\sigma'})\Big)\mathrm{d}\gamma.\\ &= \sum_{i=1}^3\sum_{j=1}^3 \sum_{\varepsilon=\overrightarrow{\sigma|\sigma'}\in\tilde{\mathcal{E}}^{(i)},\varepsilon\perp\boldsymbol{e}^{(j)}} |\varepsilon|d_\varepsilon\Big[\frac{1}{|\varepsilon|}\int_\varepsilon \partial_j V_i\mathrm{d}\gamma\Big]\eth_j(u_{i,\mathcal{E}} - V_{i,\mathcal{E}})\Big|_{D_\varepsilon},\end{aligned}$$

where we have used integration by parts and definition (2.35) of $\eth_j$. Here $\mathrm{d}\gamma$ is $d-1$ dimensional Hausdorff measure on σ. Consequently, we may write

$$\mathcal{T}_{3,1}^n = \mathcal{T}_{3,2} + \mathcal{R}_{3,2}^n, \tag{5.13}$$ {calS3+}

$$\mathcal{T}_{3,2} = \sum_{i=1}^3\sum_{j=1}^3 \int_\Omega \eth_j V_{i,\mathcal{E}}\eth_j(u_{i,\mathcal{E}} - V_{i,\mathcal{E}})\mathrm{d}x,$$

$$\mathcal{R}_{3,2}^n = \sum_{i=1}^3\sum_{j=1}^3 \sum_{\varepsilon=\overrightarrow{\sigma|\sigma'}\in\tilde{\mathcal{E}}^{(i)},\varepsilon\perp\boldsymbol{e}^{(j)}} |\varepsilon|d_\varepsilon\Big(\Big[\frac{1}{|\varepsilon|}\int_\varepsilon \partial_j V_i\mathrm{d}\gamma\Big] - \eth_j V_{i,\mathcal{E}}|_{D_\varepsilon}\Big)\eth_j(u_{i,\mathcal{E}} - V_{i,\mathcal{E}})\Big)\Big|_{D_\varepsilon},$$

where, due to the Cauchy-Schwartz inequality, Lemma 2.6 combined with the first order Taylor formula applied to $\Big[\frac{1}{|\varepsilon|}\int_\varepsilon \partial_j V_i\Big] - \eth_j V_{i,\mathcal{E}}|_{D_\varepsilon}$ and Corollary 3.1, we get

$$\delta t|\sum_{n=1}^m \mathcal{R}_{3,2}^n| \le h\, c(M_0, E_0\|\boldsymbol{V}, \nabla\boldsymbol{V}, \nabla^2\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{39})}). \tag{5.14}$$ {calS3r2}

5.5 Pressure term

Step 1: The following lemma about the consistency of the upwind discretization will be crucial.

Lemma 5.2. *For any $r, G \in \mathcal{L}_{\mathcal{T}}$, any $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$ and any $\phi \in C^1(\overline{\Omega})$ there holds*

$$\int_\Omega r\mathbf{u}\cdot\nabla\phi \mathrm{d}x + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(i)}} F_{\sigma,K}(r,\mathbf{u})G_K$$

$$= \sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma|(r_K - r_L)(\phi_\sigma - G_K)u_{i,\sigma} - \sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma|(r_L - r_\sigma^{\mathrm{up}})(G_K - G_L)u_{i,\sigma},$$

where the primal fluxes $F_{\sigma,K}$ are defined in (2.12).

Proof of Lemma 5.2
Using integration by parts,

$$\int_\Omega r\mathbf{u}\cdot\nabla\phi\mathrm{d}x = \sum_{K\in\mathcal{T}}\int_K r\mathbf{u}\cdot\nabla(\phi - G_K)\mathrm{d}x$$

$$= \sum_{i=1}^{3}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(i)}(K)} |\sigma| r_K\phi_\sigma u_{i,\sigma}\boldsymbol{n}^{(i)}\cdot\boldsymbol{n}_{\sigma,K} - \sum_{i=1}^{3}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(i)}(K)} |\sigma| r_K G_K u_{i,\sigma}\boldsymbol{n}^{(i)}\cdot\boldsymbol{n}_{\sigma,K}$$

$$= \sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}} |\sigma|(r_K - r_L)(\phi_\sigma - G_K)u_{i,\sigma}$$

$$- \sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}} |\sigma| r_L(G_K - G_L)u_{i,\sigma}$$

$$= \sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}} |\sigma|(r_K - r_L)(\phi_\sigma - G_K)u_{i,\sigma}$$

$$- \sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}} |\sigma|(r_L - r_\sigma^{\mathrm{up}})(G_K - G_L)u_{i,\sigma} - \sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}} |\sigma| r_\sigma^{\mathrm{up}}(G_K - G_L)u_{i,\sigma},$$

where for the latter term, we have

$$\sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}} |\sigma| r_\sigma^{\mathrm{up}}(G_K - G_L)u_{i,\sigma} = \sum_{i=1}^{3}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(i)}(K)} |\sigma| r_\sigma^{\mathrm{up}} G_K u_{i,\sigma}\boldsymbol{n}^{(i)}\cdot\boldsymbol{n}_{\sigma,K}$$

Lemma 5.2 is proved.

Step 2:
We shall now evaluate the error in the upwind disretization. We have

$$\mathcal{T}_4^n = -\frac{1}{\overline{\varrho}}\int_\Omega \varrho^n\mathbf{u}^n\cdot\nabla\Pi^n\mathrm{d}x + \frac{1}{\overline{\varrho}}\int_\Omega(\varrho^n - \overline{\varrho})\mathbf{u}^n\cdot\nabla\Pi^n\mathrm{d}x = \mathcal{T}_{4,1}^n + \mathcal{R}_{4,1}^n, \tag{5.15}$$ {calS41}

where

$$\delta t\Big|\sum_{n=1}^{N}\mathcal{R}_{4,1}^n\Big| \le \varepsilon\, c(M_0, E_0, \overline{\varrho}, \|\nabla\Pi\|_{L^\infty((0,T)\times\Omega)}), \tag{5.16}$$ {calS4r}

by virtue of Hölder's inequality and estimates (3.5), (3.8) from Corollary 3.1.

Next we deduce from the discrete continuity equation (2.38a) and Lemma 5.2

$$\mathcal{T}^n_{4,1} = \mathcal{J}^n_1 + \mathcal{J}^n_2 + \mathcal{J}^n_3, \tag{5.17}$$ {calS4+}

where

$$\mathcal{J}^n_1 = \frac{1}{\overline{\varrho}} \int_\Omega \frac{\varrho^n - \varrho^{n-1}}{\delta t} \Pi^n_{\mathcal{T}} \mathrm{dx},$$

$$\mathcal{J}^n_2 = \frac{1}{\overline{\varrho}} \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma|(\varrho^n_L - \varrho^n_K)(\Pi^n_\sigma - \Pi^n_K) u^n_{i,\sigma},$$

$$\mathcal{J}^n_3 = \frac{1}{\overline{\varrho}} \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma|(\varrho^n_L - \varrho^{n,\mathrm{up}}_\sigma)(\Pi^n_K - \Pi^n_L) u^n_{i,\sigma},$$

where $\Pi^n_{\mathcal{T}} = \mathcal{P}_{\mathcal{T}}\Pi^n$ is defined in (2.39).

Now we estimate each of terms $\mathcal{J}^n_1$, $\mathcal{J}^n_2$, $\mathcal{J}^n_3$ separately.

Step 2a:
We get by direct calculation

$$\delta t \sum_{n=1}^N \mathcal{J}^n_1 = \frac{\delta t}{\overline{\varrho}} \sum_{n=1}^N \int_\Omega \frac{\varrho^n - \varrho^{n-1}}{\delta t} \Pi^n_{\mathcal{T}} \mathrm{dx} = \frac{1}{\overline{\varrho}} \sum_{n=1}^N \int_\Omega \Big((\varrho^n - \overline{\varrho}) - (\varrho^{n-1} - \overline{\varrho}) \Big) \Pi^n_{\mathcal{T}} \mathrm{dx}$$

$$= \frac{1}{\overline{\varrho}} \sum_{n=1}^N \int_\Omega \Big((\varrho^n - \overline{\varrho})\Pi^n_{\mathcal{T}} - (\varrho^{n-1} - \overline{\varrho})\Pi^{n-1}_{\mathcal{T}} \Big) \mathrm{dx} + \frac{1}{\overline{\varrho}} \sum_{n=1}^N \int_\Omega (\varrho^{n-1} - \overline{\varrho})(\Pi^{n-1}_{\mathcal{T}} - \Pi^n_{\mathcal{T}}) \mathrm{dx}$$

$$= \frac{1}{\overline{\varrho}} \int_\Omega (\varrho^n - \overline{\varrho})\Pi^N_{\mathcal{T}} - \int_\Omega (\varrho^0 - \overline{\varrho})\Pi^0_{\mathcal{T}} + \frac{\delta t}{\overline{\varrho}} \sum_{n=1}^N \int_\Omega (\varrho^{n-1} - \overline{\varrho}) \frac{\Pi^{n-1}_{\mathcal{T}} - \Pi^n_{\mathcal{T}}}{\delta t} \mathrm{dx}.$$

Therefore, by virtue of Hölders inequality, Lemma 2.5, the first order Taylor formula

$$\delta t |\sum_{n=1}^N \mathcal{J}^n_1| = \leq \varepsilon(1+\delta t)\, c(M_0, E_0, \overline{\varrho}, \|\Pi\|_{L^\infty(Q_T)}, \|\partial_t \Pi\|_{L^1(0,T;L^p(\Omega))}), \quad p = \max(2, \gamma'),$$

where we have used estimates (3.5) and (3.8) in Corollary 3.1.

Step 2b:
First, we have by using Hölder's inequality,

$$\Big| \sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma|(\varrho^n_L - \varrho^n_K)(\Pi^n_\sigma - \Pi^n_K) u^n_{i,\sigma} \Big|$$

$$\leq \sqrt{h} \|\nabla_x \Pi\|_{L^\infty((0,T)\times\Omega)} \Big(\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma| 1_{E_i}(\sigma) \frac{\big[\varrho^n_K - \varrho^n_L\big]^2}{\max(\varrho^n_K, \varrho^n_L)^\delta} |u^n_i| \Big)^{1/2}$$

$$\times \Big(\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma| h \max(\varrho^n_K, \varrho^n_L)^\gamma \Big)^{\frac{\delta}{2\gamma}} \Big((\sum_{i=1}^3 \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)} |\sigma| h |u^n_i|^{\frac{\gamma}{\gamma-\delta}} \Big)^{\frac{\gamma-\delta}{2\gamma}}$$

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$$+\sqrt{h}\|\nabla_x \Pi\|_{L^\infty((0,T)\times\Omega)}\Big(\sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)}|\sigma|1_{\mathcal{E}^{(i)}\setminus E_i}(\sigma)\left[\varrho_K^n-\varrho_L^n\right]^2|u_i^n|\Big)^{1/2}$$

$$\times\Big(\sum_{i=1}^{3}\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}^{(i)}(K)}|\sigma|h|u_i^n|\Big)^{1/2}$$

with any $0\le\delta<\gamma$ and any $E_i\subset\mathcal{E}^{(i)}$, where we have used estimate Lemma 2.5 to evaluate the difference $\Pi_\sigma^n-\Pi_K^n$. Now employing estimates (3.5), (3.8), (3.9) in Corollary 3.1 we obtain

$$\delta t\sum_{n=1}^{N}|\mathcal{J}_2^n|\lesssim \varepsilon h^{1/2}c(M_0,E_0,\overline{\varrho},\|\nabla\Pi\|_{L^\infty((0,T)\times\Omega)}),$$

where $\varepsilon\sqrt{h}\le\frac{1}{2}(\varepsilon^2+h)$. The same estimate as above holds also for $\mathcal{J}_3^n$ by the same argument.

Resuming calculations in step 2, we get

$$\delta t\sum_{n=1}^{N}|\mathcal{T}_{4,1}^n|\le(\varepsilon+h+\delta t)\,c(M_0,E_0,\overline{\varrho},\|\nabla\Pi\|_{L^\infty((0,T)\times\Omega)},\|\partial_t\Pi\|_{L^1(0,T;L^p(\Omega))}),\quad p=\max(2,\gamma'). \tag{5.18}$$ {calS4++}

The statement of Lemma 4.1 follows when we put together principal terms (5.3),(5.5), (5.7),(5.9), (5.11), (5.13) and residual terms (5.4),(5.6), (5.8),(5.10), (5.12), (5.14) (5.16), (5.16),(5.17), (5.17).

6 A Gronwall inequality

In this Section we put together the relative energy inequality (4.2) and the identity (5.1) derived in the previous section. The final inequality resulting from this manipulation is formulated in the following lemma.

Lemma 6.1. *Let $(\varrho^n,\boldsymbol{u}^n)$ be a solution of the discrete problem (2.38a–2.38b) with the pressure satisfying (1.4), where $\gamma\ge 3/2$, emanating from initial data (2.61), (2.62). Then there exists a positive number*

$$c=c\Big(M_0,E_0,\overline{\varrho},\|\boldsymbol{V}\|_{X_T(\Omega)},\|\Pi\|_{Y_T^p(\Omega)}\Big),\quad p=\max(2,\gamma')$$

such that for all $m=1,\dots,N$, there holds:

$$\mathcal{E}_\varepsilon(\varrho^m,\boldsymbol{u}^m|\overline{\varrho},\boldsymbol{V}_{\mathcal{E}}^m)+\delta t\sum_{n=1}^{m}\Big(\mu||\boldsymbol{u}^n-\boldsymbol{V}_{\mathcal{E}}^n||_{1,\mathcal{E},0}^2+(\mu+\lambda)||\operatorname{div}_{\mathcal{T}}(\boldsymbol{u}^n-\boldsymbol{V}_{\mathcal{E}}^n)||_{L^2(\Omega)}^2\Big)$$

$$\le c\Big[h^A+\sqrt{\delta t}+\varepsilon+\mathcal{E}_\varepsilon(\varrho^0,\boldsymbol{u}^0|\overline{\varrho},\boldsymbol{V}_{\mathcal{E}}(0))\Big]+c\,\delta t\sum_{n=1}^{m}\mathcal{E}_\varepsilon(\varrho^n,\boldsymbol{u}^n|\overline{\varrho},\boldsymbol{V}_{\mathcal{E}}^n),$$

with any couple $(\Pi,\mathbf{V})$ belonging to (1.10) satisfying (1.7–1.9) on $[0,T)\times\Omega$, where A is defined in (2.64) and $\mathcal{E}_\varepsilon$ is given in (2.60).

Proof of Lemma 6.1
Gathering the formulae (4.2) and (5.1), one gets

$$\mathcal{E}_\varepsilon(\varrho^m,\boldsymbol{u}^m\Big|\overline{\varrho},\boldsymbol{V}_{\mathcal{E}}^m)-\mathcal{E}_\varepsilon(\varrho^0,\boldsymbol{u}^0\Big|\overline{\varrho},\boldsymbol{V}_{\mathcal{E}}(0)) \tag{6.1}$$ {relativeenergy-1}

$$+\delta t\sum_{n=1}^{m}\Big(\mu||\boldsymbol{u}^n-\boldsymbol{V}_{\mathcal{E}}^n||_{1,\mathcal{E},0}^2+(\mu+\lambda)||\operatorname{div}_{\mathcal{T}}(\boldsymbol{u}^n-\boldsymbol{V}_{\mathcal{E}}^n)||_{L^2(\Omega)}^2\Big)\le\mathcal{P}_1+\mathcal{P}_2+\mathcal{Q},$$

where

$$\mathcal{P}_1 = \delta t \sum_{n=1}^{m} \int_\Omega \Big(\varrho^{n-1} - \overline{\varrho}\Big)\Big(\frac{\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{V}_\mathcal{E}^{n-1}}{\delta t}\Big) \cdot \Big(\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{u}^n\Big),$$

$$\mathcal{P}_2 = \delta t \sum_{n=1}^{m}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}^{(j)}(K)} \Big(\varrho_\sigma^{n,\mathrm{up}} - \overline{\varrho}\Big)\Big(V_{i,\mathcal{E},K}^n - V_{i,\sigma}^n\Big) V_{j,\sigma}^n \boldsymbol{e}^{(j)} \cdot \boldsymbol{n}_{\sigma,K}(u_{i,K}^n - V_{i,\mathcal{E},K}^n),$$

$$\mathcal{Q} = \mathcal{R}_{\mathcal{T},h,\delta t}^m + \mathcal{G}_{\mathcal{T},\delta t}^m - \mathcal{R}_{h,\delta t}^m.$$

We use Hölder's inequality, together with the Taylor type formula (4.8) in order to get

$$|\mathcal{P}_1| \le \delta t \sum_{n=1}^{m} \Big(\|[\varrho^{n-1}]_{\mathrm{res}}\|_{L^q(\Omega)}|\Omega_{\mathrm{res}}|^{1/r} + \|[\varrho^{n-1} - \overline{\varrho}]_{\mathrm{ess}}\|_{L^2(\Omega)}\|\Big)\Big\|\frac{\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{V}_\mathcal{E}^{n-1}}{\delta t}\Big\|_{L^\infty(\Omega;\mathbb{R}^3)}\Big\|\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{u}^n\|_{L^6(\Omega;\mathbb{R}^3)},$$

$$|\mathcal{P}_2| \le c\|\nabla V\|_{L^\infty(Q_T;\mathbb{R}^3)}\delta t \sum_{n=1}^{m} \Big(\|[\varrho^n]_{\mathrm{res}}\|_{L^q(\Omega)}|\Omega_{\mathrm{res}}^n|^{1/r} + \|[\varrho^n - \overline{\varrho}]_{\mathrm{ess}}\|_{L^2(\Omega)}\|\Big)\Big\|\boldsymbol{V}_\mathcal{E}^n\Big\|_{L^\infty(\Omega;\mathbb{R}^3)}\Big\|\boldsymbol{V}_\mathcal{E}^n - \boldsymbol{u}^n\|_{L^6(\Omega;\mathbb{R}^3)},$$

where $q = \min(\gamma, 2)$, $\frac{1}{r} + \frac{1}{q} + \frac{1}{6} = 1$, and symbols $[\cdot]_{\mathrm{res}}$, $[\cdot]_{\mathrm{esss}}$ and the sets Ω_{res}^n are defined in (3.3). Evoking estimates (3.5) and (3.8) from Corollary 3.1, one gets

$$|\mathcal{P}_1| + |\mathcal{P}_2| \le \varepsilon\, c(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \nabla\boldsymbol{V}, \partial_t\boldsymbol{V}\|_{L^\infty(Q_T;\mathbb{R}^{15})}).$$

This formula, and the bounds of expressions $\mathcal{R}_{\mathcal{T},h,\delta t}^m$, $\mathcal{G}_{\mathcal{T},\delta t}^m$, $\mathcal{R}_{h,\delta t}^m$ evoked in (4.2), (5.1) yield the statement of Lemma 6.1.

Lemma 6.1 implies immediately error estimate (2.63) by the standard discrete Gronawll inequality. Theorem 2.1 is proved.

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Quelques résultats en analyse théorique et numérique des équations de Navier-Stokes compressibles

Dans cette thèse, nous nous intéressons à l'analyse mathématique théorique et numérique des équations de Navier-Stokes compressibles en régime barotrope. La plupart des travaux présentés ici combinent des méthodes d'analyse des équations aux dérivées partielles et des méthodes d'analyse numérique afin de clarifier la notion de solution faible ainsi que les mécanismes de convergence de méthodes numériques approximant ces solutions faibles. En effet les équations de Navier-Stokes compressibles sont fortement non linéaires et leur analyse mathématique repose nécessairement sur la structure de ces équations. Plus précisément, nous prouvons dans la partie théorique l'existence de solutions faibles pour un modèle d'écoulement compressible d'entropie variable où l'entropie du système est transportée. Nous utilisons les méthodes classiques permettant de prouver l'existence de solutions faibles aux équations de Navier-Stokes compressibles en regime barotrope. Nous étudions aussi dans cette partie la réduction de dimension 3D/2D dans les équations de Navier-Stokes compressibles en utilisant la méthode d'énergie relative. Dans la partie numérique nous nous intéressons aux estimations d'erreur inconditionnelles pour des schémas numériques approximant les solutions faibles des équations de Navier-Stokes compressibles. Ces estimations d'erreur sont obtenues à l'aide d'une version discrète de l'énergie relative satisfaite par les solutions discrètes de ces schémas. Ces estimations d'erreur sont obtenues pour un schéma numérique académique de type volumes finis/éléments finis ainsi que pour le schéma numérique Marker-and-Cell. Nous prouvons aussi que le schéma Marker-and-Cell est inconditionnellement et uniformément asymptotiquement stable en régime bas Mach. Ces résultats constituent les premiers résultats d'estimations d'erreur inconditionnelles pour des schémas numériques pour les équations de Navier-Stokes compressibles en régime barorope.

Mot clés : Navier-Stokes, Solutions faibles, Limites singulières, Energie relative, Schémas numériques, Estimations d'erreur.

Some theorical and numerical results for the compressible Navier-Stokes equations

In this thesis, we deal with mathematical and numerical analysis of compressible Navier-Stokes equations in barotropic regime. Most of these works presented here combine mathematical analysis of partial differential equations and numerical methods with aim to shred more light on the construction of weak solutions on one side and on the convergence mechanisms of numerical methods approximating these weak solutions on the other side. Indeed, the compressible Navier-Stokes equations are strongly nonlinear and their mathematical analysis necessarily relies on the structure of equations. More precisely, we prove in the theorical part the existence of weak solutions for a model a flow of compressible viscous fluid with variable entropy where the entropy is transported. We use the classical techniques to prove the existence of weak solutions for the compressible Navier-Stokes equations in barotropic regime. We also investigate the 3D/2D dimension reduction in the compressible Navier-Stokes equations using the relative energy method. In the numerical we deal with unconditionally error estimates for numerical schemes approximating weak solutions of the compressible Navier-Stokes equations. These error estimates are obtained by using the discrete version of the relative energy method. These error estimates are obtained for a academic finite volume/finite element scheme and for the Marker-and-Cell scheme. We also prove that the Marker-and-cell scheme is unconditionally and uniformly asymptotically stable at the Low Mach number regime. These are the first results on unconditionally error estimates for numerical schemes approximating the compressible Navier-Stokes equations in barotropic regime.

Keywords : Navier-Stokes, Weak solutions, Singular limits, Relative energy, Numerical schemes, Error estimates.