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THESIS

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A new discontinuous Galerkin formulation for time dependent Maxwell's equations: a priori and a posteriori error estimation

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Résumé

Dans la première partie de cette thèse, nous avons considéré les équations de Maxwell en temps et construit une formulation discontinue de Galerkin (DG). On a montré que cette formulation est bien posée et ensuite on a établi des estimateurs a priori pour cette formulation. On a obtenu des résultats numériques pour valider les estimateurs a priori obtenus théoriquement. Dans la deuxième partie de cette thèse, des estimateurs d'erreur a posteriori de cette formulation sont établis, pour le cas semi-discret et pour le système complètement discrétisé. Dans la troisième partie de cette thèse, on considére les équations de Maxwell en régime harmonique. On a développé une formulation discontinue de Galerkin mixte. On a établi des estimations d'erreur a posteriori pour cette formulation.

Mots-clés: Equations de Maxwell, Méthode de Galerkin discontinue (DG), *a posteriori*, *a priori*

Abstract

In the first part of this thesis, we have considered the time-dependent Maxwell's equations in second-order form and constructed discontinuous Galerkin (DG) formulation. We have established a priori error estimates for this formulation and carried out the numerical analysis to confirm our theoretical results. In the second part of this thesis, we have established a posteriori error estimates of this formulation for both semi discrete and fully discrete case. In the third part of the thesis we have considered the time-harmonic Maxwell's equations and we have developed mixed discontinuous Galerkin formulation. We showed the well posedness of this formulation and have established a posteriori error estimates.

Keywords: Maxwell's equations, Discontinuous Galerkin(DG), *a posteriori*, *a priori*

Contents

Chapter 0

Introduction

Motivation

The modeling of enviromental and physical phenomena arising in engineering and the sciences leads to partial differential equations in space and time, expressing the mathematical model of the problem to be solved. Unfortunately, in most cases the analytical solution of these problems does not exist, hence the numerical methods are developed and employed to find the approximate solutions. These methods provide approximate solutions belonging to a functional space of finite dimension.

The discretization of partial differential equations by the numerical methods generally leads to large systems. Solving large systems is costly in terms of computation time and computer resources. Here it comes the issue of determining the accuracy of the numerical method, the questions that arise are: how can the approximation error be mesured, controlled, and effectively minimized?

The theories and methods of a posteriori error estimation have been developed to answer these questions, where the approximate solution itself is used to assess the accuracy. Indeed, the goal of a posteriori error estimates is to give bounds on the error between the approximation and the exact solution in an appropriate norm, these bounds can be calculated in practice, once the approximate solution is known.

A posteriori error estimation formed the basis of adaptive mesh, refinement and coarsening techniques which are used to control and minimize the error. This procedure provide stopping criteria to ensure control overall error. The literature on a posteriori error estimation is vast. There exists different categories of a posteriori error estimates. We cite here the explicit residual estimates, cf. [BR78a], [VE96], and [CA97], the equilibrated residual method, cf. Ainsworth and Oden [OA00], equilibrated fluxes estimates, cf. [PS47], [BS08], [AI05], [VO13], functional a posteriori error estimates, cf. [RE08], hierarchical estimates, cf. [BS93].

In this thesis, we analyse the numerical approximation of Maxwell's equations by discontinuous Galerkin (DG) method, which is a very interesting method for adaptive mesh technique as it can easily handle complex geometries, irregular meshes with hanging nodes and approximations that have polynomials

of different degrees in different elements. In the first part of this thesis, we propose a new symmetric DG method for the spatial discretization of time-dependent Maxwell's equations in second order form. We derive a priori error estimates for this DG formulation and carried out some numerical experiments to confirm our theoretical results. In the second part of this thesis, we have derived a posteriori error estimates in the $L2$-norm for the DG formulation of this problem both for semi-discrete and fully discrete case. For Maxwell's equations, there exists the work on a posteriori error control, mostly for the time-harmonic case. For example, in [CD07], they have derived three types of a posteriori error estimates for finite element method. In [BHHW00], [SC08] residual type estimates have been developed for finite element method. In [HPD07], the a posteriori error estimates for DG method has been derived. To our knowledge, a posteriori error estimates have not been derived for the DG discretisation of time-dependent Maxwell's equations.
In the third part, we consider the time-harmonic Maxwell's equations in frequency domaine. We discretize this problem by the mixed DG method and we derive a posteriori error estimates for this DG formulation.
In this chapter, we first state the Maxwell's equations in time and frequency domain. Then we give a brief introduction of DG method. We end this general introduction by giving an outline of this thesis.

Maxwell's equations

In 1873 Maxwell introduced the equations that founded the modern theory of electromagnetism with the publication of his *Treatise on Electricity and Magnetics*, these equations now bear his name. A macroscopic electromagnetic field created by a distribution of static electric charges with charge density ϱ and the directed flow of electric charge with current density $\mathcal{J}$ is described by the four Maxwell's equations

$$\frac{\partial \mathcal{B}}{\partial t} + \nabla \times \mathcal{E} = \mathbf{0}, \tag{1a}$$

$$\nabla \cdot \mathcal{D} = \varrho, \tag{1b}$$

$$\frac{\partial \mathcal{D}}{\partial t} + \nabla \times \mathcal{H} = -\mathcal{J}, \tag{1c}$$

$$\nabla \cdot \mathcal{B} = 0, \tag{1d}$$

where the vector functions $\mathcal{E}, \mathcal{D}, \mathcal{H}, \mathcal{B}, \mathcal{J}$ and the scalar ϱ are functions of position $\boldsymbol{x} \in \mathbb{R}^3$ and time t.
Equation (1a) is Faraday's law and gives the effect of a changing magnetic

field $\mathcal{B}$ on the electric field $\mathcal{E}$. Equation (1b) is Gauss's law and describes the effect of the charge density ϱ on the electric displacement $\mathcal{D}$. Equation (1c) is Ampère's law and it gives the effect of a changing electric displacement $\mathcal{D}$ and a flow of electric charges $\mathcal{J}$ to the magnetic field $\mathcal{H}$. Finally, (1d) is Gauss's law and it shows that the magnetic induction $\mathcal{B}$ is solenoidal.
If charge is conserved, the divergence constraint (1b) and (1d) follows from (1a) and (1c) by taking the divergence of these two equations, we have

$$\nabla \cdot \frac{\partial \mathcal{B}}{\partial t} = 0 \text{ and } \nabla \cdot \frac{\partial \mathcal{D}}{\partial t} = -\nabla \cdot \mathcal{J}. \tag{2}$$

But it can be proved that from charge conservation ϱ and $\mathcal{J}$ are linked by the following relation

$$\nabla \cdot \mathcal{J} + \frac{\partial \varrho}{\partial t} = 0, \tag{3}$$

hence, we have

$$\frac{\partial}{\partial t} \nabla \cdot \mathcal{B} = \frac{\partial}{\partial t}(\nabla \cdot \mathcal{D} - \varrho) = 0. \tag{4}$$

Thus if the divergence constraints (1b) and (1d) hold at one time, they hold for all time. However, these divergence constraints can not be ignored while discretizing (1). A good numerical scheme must satify the discrete analogues of (1b)and (1d).
The equations (1)-(4) are completed by two constitutive laws that links $\mathcal{B}$ to $\mathcal{H}$ and $\mathcal{D}$ to $\mathcal{E}$ respectively. These laws depend on the characteristics of the matter in the domain of electromagnetic field. We assume that material occupying the domain is inhomogeneous, isotropic and linear, then the fields are related by

$$\mathcal{D} = \epsilon \mathcal{E} \text{ and } \mathcal{B} = \mu \mathcal{H}, \tag{5}$$

where μ, relative magnetic permittivity and ϵ, relative electric permeability are scalar, postive and bounded functions of $\boldsymbol{x} \in \mathbb{R}$. In conducting materials, we have another constitutive law, the electromagnetic field gives rise to currents. If we assume the field strength small, we can suppose that Ohms law holds

$$\mathcal{J} = \sigma \mathcal{E} + \mathcal{J}_s. \tag{6}$$

Here σ the conductivity of the medium is scalar function of position. It is positive in a conductor and vanishes in an insulator. $\mathcal{J}_s$ gives the applied current density.
By subsituting (5) and (6) in to (1a)-(1d), we obtain

$$\epsilon \frac{\partial \mathcal{E}}{\partial t} = \nabla \times \mathcal{H} - \sigma \mathcal{E} - \mathcal{J}_s, \tag{7}$$

$$\nabla \cdot (\epsilon \mathcal{E}) = \varrho, \tag{8}$$

$$\mu \frac{\partial \mathcal{H}}{\partial t} = \nabla \times \mathcal{E}, \tag{9}$$

$$\nabla \cdot (\mu \mathcal{H}) = 0 \tag{10}$$

which is the fundamental Maxwell time dependent system. By taking the time derivative of (7) and the rotation of (9), we eliminate the magnetic field $\mathcal{H}$, and we get the second order form of Maxwell system for the electric field $\mathcal{E}$

$$\begin{cases} \epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} + \sigma \frac{\partial \mathcal{E}}{\partial t} + \nabla \times (\mu^{-1} \nabla \times \mathcal{E}) = -\frac{\partial \mathcal{J}_s}{\partial t}, \\ \nabla \cdot (\epsilon \mathcal{E}) = \varrho. \end{cases} \tag{11}$$

A similar equation can be derived for $\mathcal{H}$, if $\sigma = 0$ or if $\epsilon, \sigma > 0$ are constants. Now we will derive the Maxwell's system for time harmonic fields. By taking the Fourier transform in time or considering the electromagnetic propagation at a single frequency $\omega > 0$, e.g. if the source currents $\mathcal{J}_s$ and charge density ϱ vary sinusoidally in time, the Maxwell's time dependent system (7)-(10) can be reduced to stationary equation in frequency domain. We subsitute in (7)-(9) the time-harmonic fields

$$\begin{aligned} \mathcal{E}(\boldsymbol{x}, t) = \mathrm{Re}(\exp(i\omega t)\boldsymbol{E}(\boldsymbol{x})),\ \mathcal{H}(\boldsymbol{x}, t) = \mathrm{Re}(\exp(i\omega t)\boldsymbol{H}(\boldsymbol{x})), \\ \mathcal{J}_s(\boldsymbol{x}, t) = \mathrm{Re}(\exp(i\omega t)\boldsymbol{J}_s(\boldsymbol{x})), \end{aligned} \tag{12}$$

to get

$$\epsilon i\omega \boldsymbol{E} = \nabla \times \boldsymbol{H} - \sigma \boldsymbol{E} - \boldsymbol{J}_s, \tag{13}$$

$$\mu i\omega \boldsymbol{H} = \nabla \times \boldsymbol{E}. \tag{14}$$

Combining the equations (13)-(14), we obtain the second order time-harmonic equation

$$\nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) - \omega^2 (\epsilon - i\sigma\omega^{-1}) \boldsymbol{E} = \boldsymbol{j}, \tag{15}$$

where $\boldsymbol{j} = -i\omega \mathcal{J}_s$. Now we derive the divergence constrain (8) in time-harmonic fields, Taking the derivative with respect to time of (8) and using the Ohm's law (6) and the relation (3), we get

$$\nabla \cdot (\epsilon \frac{\partial \mathcal{E}}{\partial t}) = -\nabla.(\sigma \mathcal{E} + \mathcal{J}_s).$$

Subsituting the time-harmonic fields (12) in the above equation, it reduces to,

$$\nabla \cdot (\omega^2 (\epsilon - i\sigma\omega^{-1}) \boldsymbol{E}) = -\frac{1}{\omega^2} \nabla . \boldsymbol{j}. \tag{16}$$

The system of equations (15)-(16) is Maxwell's time-harmonic system for electric field $\boldsymbol{E}$. The similar equations can be derived for magnetic field $\boldsymbol{H}$.
In the *low frequency case*, we have $\sigma >> \epsilon\omega$ and the term $\omega^2 \epsilon$ is neglected, the system (15)-(16) in this case is also referred to eddy current problem. Whereas in the *high frequency case*, we have $\sigma << \epsilon\omega$ and the term with $i\sigma\omega^{-1}$ is neglected.

Discontinuous Galerkin Method

The discontinuous Galerkin (DG) method is a numerical method which was first introduced by Reed and Hill [RH73] in 1973, as a tool to solve neutron transport problems. The advantage of DG methods are that they are locally conservative, stable, and high-order accurate. Recently this technique has become popular as a method for solving fluid dynamics or electromagnetic problems. An introduction to DG methods can be found in [CO99]. A history of their evolution can be found in [CKS00]. Finally, a fairly complete and updated review is given in [CS01].

Outline of thesis

The thesis is organised as follows: the chapter 1 consists of the preliminary results and notations, which we will use througout the thesis. In chapter 2, we first present the mathematical model problem and we derive the DG formulation for the semi and fully time discretization. We prove the continuity and coercivity of the bilinear form. In chapter 3, we derive a priori error estimates for the formulation proposed in previous chapter and we present some numerical results to confirm the convergence of our theoretical results. The chapter 4 is devoted to derive an abstract a posteriori error estimate for the seme-discrete case. To derive the a posteriori error estimates, we introduce the time-harmonic reconstruction of the approximate solution. In chapter 5, the a posteriori error estimate for fully discrete case is developed using an appropraite space-time reconstruction. In chapter 6, we generalize our time-harmonic reconstruction technique to obtain a posteriori error estimates for any time-dependent problem via the a posteriori error estimates of the corresponding stationary problem. In chapter 7, we consider the time harmonic Maxwell's equations and we derive a mixed DG formulation for this model probelm. We prove the well-posedness of our formulation and further we derive a posteriori error estimates for this DG formulation.
In chapter 8, we give the conclusion and perspectives of our work.
In the end, we have put the proof of some auxiliary results in appendix A.

CHAPTER 1

Preliminaries and notations

Contents

1.1 Elements of function spaces

Let Ω be a bounded Lipschitz polyhedron in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega$ and outward unit normal $\boldsymbol{n}$. We assume Ω to be simply connected and $\partial\Omega$ to be connected. We consider a class of spaces that consist of (Lebesgue-) integrable functions. Let p be a real number, $p \geq 1$, we denote by $L^p(\Omega)$ the set of all real-valued functions defined on Ω such that

$$\int_\Omega |u(x)|^p dx < \infty.$$

Any two functions which are equal almost everywhere (i.e. equal, except on a set of measure zero) on Ω are identified with each other. $L^p(\Omega)$ is equipped with the norm

$$\|u(x)\|_{L^p} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

A function u that is measurable on Ω is said to be essentially bounded on Ω if there is a constant M such that $u(x) < M$ a.e. on Ω. The greatest lower bound of such constants M is called the essential supremum of u on Ω, and is denoted by $\text{esssup}_{x\in\Omega} u(x)$. We denote by $L^\infty(\Omega)$ the vector space of all functions u that are essentially bounded on Ω, functions being once again identified if they are equal a.e. on Ω. $L^\infty(\Omega)$ is equipped with the norm

$$\|u(x)\|_{L^\infty} = \text{esssup}_{x\in\Omega} u(x).$$

A particularly important case corresponds to taking $p = 2$. The space $L^2(\Omega)$ is Hilbert space endowed with the inner product

$$(u, v)_\Omega = \int_\Omega u(x)v(x)dx.$$

The norm in $L^2(\Omega)$ is denoted by $\|.\|_{0,\Omega}$ or simply $\|.\|_0$ when no confusion about domain Ω is possible.

1.1.1 Sobolev spaces

Let m be a non-negative integer, and $1 \le p \le \infty$. We define with D^α denoting a weak or distributional derivative of order $|\alpha|$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

$$W_p^m(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \le m\},$$

$W_p^m(\Omega)$ is called Sobolev space of order m; it is a Banach space equipped with the norm

$$\|u\|_{W_p^m(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}} \text{ when } 1 \le p < \infty$$

and

$$\|u\|_{W_\infty^m(\Omega)} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty} \text{ when } p = \infty.$$

We define the semi norm on $W_p^m(\Omega)$, for $m \ge 1$

$$|u|_{W_p^m(\Omega)} = \left(\sum_{|\alpha| = m} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}} \text{ when } 1 \le p < \infty$$

and

$$|u|_{W_\infty^m(\Omega)} = \sum_{|\alpha| = m} \|D^\alpha u\|_{L^\infty} \text{ when } p = \infty.$$

A special case corresponds to taking $p = 2$; $W_2^m(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{W_2^m(\Omega)} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v).$$

We denote by $H^m(\Omega)$, the hilbertian Sobolev space of order $m \ge 0$ instead of $W_2^m(\Omega)$, we shall use the notation $\|.\|_{m,\Omega}$ and $|.|_{m,\Omega}$ for H^m norms and

semi norms respectively. In particular, for $m = 0$, we identify the space $H^0(\Omega) = L^2(\Omega)$, and so the norm $\|u\|_{0,\Omega} = \|u\|_{L^2(\Omega)}$.
Negative-order Sobolev spaces $H^{-m}(\Omega)$ for $m > 0$ are defined through duality. In the case $m = 1$, the definition of $\langle \cdot, \cdot \rangle_\Omega$ is extended to the standard duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
When considering vector-valued functions $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, we define the space

$$H(\text{curl}; \Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 : \nabla \times \boldsymbol{u} \in L^2(\Omega)^3\}.$$

It is endowed with the norm

$$\|\boldsymbol{u}\|_{H(\text{curl},\Omega)} = (\|\boldsymbol{u}\|_{0,\Omega}^2 + \|\nabla \times \boldsymbol{u}\|_{0,\Omega}^2)^{\frac{1}{2}}. \tag{1.1}$$

And the space

$$H(\text{div}; \Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 : \nabla \cdot \boldsymbol{u} \in L^2(\Omega)\}.$$

It is equipped with the graph norm, i.e.,

$$\|\boldsymbol{u}\|_{H(\text{div},\Omega)} = (\|\boldsymbol{u}\|_{0,\Omega}^2 + \|\nabla \cdot \boldsymbol{u}\|_{0,\Omega}^2)^{\frac{1}{2}}.$$

For space-time vector valued functions $\boldsymbol{v}(t, \boldsymbol{x}), (t, \boldsymbol{x}) \in (0, T) \times \Omega$, we introduce the space, for $1 \leq p < \infty$,

$$L^p(0, T; X) = \{\boldsymbol{v} : (0, T) \to X | \boldsymbol{v} \text{ is measurable and } \int_0^T \|v(t)\|_X^p dt < \infty\},$$

where X is a real Banach space with norm $\|.\|_X$. In similar way, we can define $L^\infty(0, T; X)$. $L^p(0, T, X)$ is endowed with the norm

$$\|\boldsymbol{v}\|_{L^p(I;X)} = \left(\int_0^T \|\boldsymbol{v}(t)\|_X^p dt \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty,$$

$$\|\boldsymbol{v}\|_{L^\infty(I;X)} = \text{esssup}_{0 \leq t \leq T} \|\boldsymbol{v}(t)\|_X \text{ for } p = \infty.$$

Theorem 1.1.1. *(Trace theorem) Let Ω be a bounded open set of $\mathbb{R}^d$ with Lipschitz continuous boundary and let $m > 1/2$.*

(a) There exists a unique linear continuous map $\gamma_0 : H^m(\Omega) \to H^{m-\frac{1}{2}}(\partial\Omega)$ such that $\gamma_0 v = v_{|\Omega}$ for each $v \in H^m(\Omega) \cap C^0(\bar{\Omega})$.

(b) There exists a linear continuous map $\mathcal{R}_0 : H^{m-\frac{1}{2}}(\partial\Omega) \to H^m(\Omega)$ such that $\gamma_0 \mathcal{R}_0 \phi = \phi$ for each $\phi \in H^{m-\frac{1}{2}}(\partial\Omega)$.

$H_0^1(\Omega)$ is space of functions in $H^1(\Omega)$ that vanish on the boundary $\partial\Omega$ (boundary values are taken in the sense of traces).

Remark 1.1.1. *Similar results as in theorem 1.1.1 can be obtained for vector functions belonging to $H(\text{div}, \Omega)$ and $H(\text{curl}, \Omega)$. See e.g. {[GR86], Chapter 1, Section 2}.*

$H_0(\text{curl}; \Omega)$ denotes the subspaces of functions in $H(\text{curl}; \Omega)$ which have zero tangential trace on $\partial\Omega$. $H_0(\text{div}; \Omega)$ denotes the subspaces of functions in $H(\text{div}; \Omega)$ which have zero normal trace on $\partial\Omega$.
We also denote

$$H(\text{curl}0; \Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 : \nabla \times \boldsymbol{u} \in L^2(\Omega)^3, \nabla \times \boldsymbol{u} = 0\},$$

and

$$H(\text{div}0; \Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 : \nabla \cdot \boldsymbol{u} \in L^2(\Omega), \nabla \cdot \boldsymbol{u} = 0\}.$$

We define the functional space, which we will use throughout this thesis,

$$\boldsymbol{X}(\Omega) = H_0(\text{curl}, \Omega) \cap \text{H}(\text{div}, \Omega).$$

With the canonical norm

$$\|\boldsymbol{u}\|_{\boldsymbol{X}(\Omega)} = \|\boldsymbol{u}\|_0 + \|\nabla \times \boldsymbol{u}\|_0 + \|\nabla \cdot \boldsymbol{u}\|_0. \tag{1.2}$$

The space $\boldsymbol{X}(\Omega)$ is compactly embedded in $L^2(\Omega)^3$ [WE80]. As a result when $\partial\Omega$ is connected, we have the following result for $\boldsymbol{X}(\Omega)$.

Proposition 1.1.1. *In $\boldsymbol{X}(\Omega)$, the semi-norm $|\boldsymbol{u}|_{\boldsymbol{X}} = (\|\nabla \times \boldsymbol{u}\|_0^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2)^{\frac{1}{2}}$ is a norm, which is equivalent to the canonical norm $\|.\|_{\boldsymbol{X}(\Omega)}$. In other words, there exist a constant $C > 0$, such that*

$$\|\boldsymbol{u}\|_0 \leq C(\|\nabla \times \boldsymbol{u}\|_0^2 + \|\nabla \cdot \boldsymbol{u}\|_0^2)^{\frac{1}{2}} \ \forall \boldsymbol{u} \in \boldsymbol{X}(\Omega). \tag{1.3}$$

1.1.2 Finite element spaces

Let $\mathscr{T}$ be a subdivision of Ω into disjoint open sets, which we call elements. We assume $\mathscr{T}$ to be parametrized by mappings F_κ , for each $\kappa \in \mathscr{T}$, where $F_\kappa : \hat{\kappa} \to \kappa$ is a diffeomorphism and $\hat{\kappa}$ is the reference element. The above mappings are such that $\overline{\Omega} = \cup_{\kappa \in \mathscr{T}} \overline{\kappa}$. We often use the word mesh for subdivision, and we say that a mesh is regular if it has no hanging nodes; otherwise the mesh is irregular. Unless otherwise stated, we allow the mesh to be $1-$irregular; i.e., for $\text{d} = 2$, there is at most one hanging node per edge, typically its center; for $\text{d} = 3$ a corresponding concept is available.

We define and characterise the sides of the subdivision $\mathscr{T}$, which we refer as "faces". An interior face of $\mathscr{T}$ is defined as the (non-empty) two-dimensional interior of $\partial\kappa_1 \cap \partial\kappa_2$, where κ_1 and κ_2 are two adjacents elements

of $\mathscr{T}$. A boundary face of $\mathscr{T}$ is defined as the (non-empty) two-dimensional interior of $\partial\kappa \cap \partial\Omega$, where κ is a boundary element of $\mathscr{T}$. We denote by $\mathscr{F}_I$ the union of all interior faces of $\mathscr{T}$, $\mathscr{F}_B$ the union of all boundary faces of $\mathscr{T}$ and
$\mathscr{F}_h = \mathscr{F}_I \cup \mathscr{F}_B$.

Let $H^s(\mathscr{T}) = \{v : v_{|\kappa} \in H^s(\kappa), \quad \forall\kappa \in \mathscr{T}\}$, for $s > \frac{1}{2}$ with the norm $\|v\|^2_{s,\mathscr{T}} = \sum_{\kappa\in\mathscr{T}} \|v\|^2_{s,\kappa}$. Then the elementwise traces of functions in $H^s(\mathscr{T})$ belong to $TR(\mathscr{F}_h) = \Pi_{\kappa\in\mathscr{T}} L^2(\partial\kappa)$; They are double-valued on $\mathscr{F}_I$ and single-valued on $\mathscr{F}_B$. The space $L^2(\mathscr{F}_h)$ can be identified with the functions in $TR(\mathscr{F}_h)$ for which two values coincide.
The diameter of element κ is denoted by h_κ, and the mesh size h is given by $h = \max_{\kappa\in\mathscr{T}_h} h_\kappa$.

We assume the mesh to be shape regular where the shape-regularity of the mesh $\mathscr{T}$ is defined as

$$\mu(\mathscr{T}) := \sup_{\kappa\in\mathscr{T}} \frac{h_\kappa}{r_\kappa} < \infty, \tag{1.4}$$

where r_κ is the radius of the largest ball that fits entirely in κ, see [CI78].
For approximation order $l \geq 1$, we introduce the following DG finite element space

$$V_h := \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}_{|\kappa} \in \mathcal{P}^l(\kappa)^d, \quad \forall\kappa \in \mathscr{T}_h\}, \tag{1.5}$$

where $\mathcal{P}^l(\kappa)$ denotes the space of real polynomials of total degree at most l on κ.
To this end, we fix $\boldsymbol{q} \in TR(\mathscr{F}_h)^3$ and $\phi \in TR(\mathscr{F}_h)$. Let $e \in \mathscr{F}_I$ be an interior side shared by κ_1 and κ_2. Let n_1 (resp. n_2) be the outer unit normal vector on e with respect to κ_1(resp. κ_2). Let $\boldsymbol{q}_i = \boldsymbol{q}_{|\kappa_i}$ and $\phi_i = \phi_{|\kappa_i}$, (i=1,2), we define for $x \in e$ the average, tangential and normal jumps of $\boldsymbol{q}$ as follows:

$$\{\!\{\boldsymbol{q}\}\!\} = \frac{\boldsymbol{q}_1 + \boldsymbol{q}_2}{2} \ , \quad [\![\boldsymbol{q}]\!]_T = \boldsymbol{n}_1 \times \boldsymbol{q}_1 + \boldsymbol{n}_2 \times \boldsymbol{q}_2 \ \text{ and } \ [\![\boldsymbol{q}]\!]_N = \boldsymbol{n}_1 \cdot \boldsymbol{q}_1 + \boldsymbol{n}_2 \cdot \boldsymbol{q}_2.$$

Similarly we define for $x \in e$ the average and normal jump of ϕ

$$\{\!\{\phi\}\!\} = \frac{\phi_1 + \phi_2}{2} \ , \quad \text{and} \ \ [\![\phi]\!]_N = \boldsymbol{n}_1 \cdot \phi_1 + \boldsymbol{n}_2 \cdot \phi_2.$$

Then, for any boundary face $e \subset \mathscr{F}_B$, we set for $x \in e$

$$\{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q}, \quad [\![\boldsymbol{q}]\!]_T = \boldsymbol{n}_1 \times \boldsymbol{q} \text{ and } \ [\![\phi]\!]_N = \boldsymbol{n}.\phi.$$

In order to define the average of $\nabla \times \boldsymbol{q}$, we set for $s > \frac{1}{2}$,

$$H^s(\text{curl}, \mathscr{T}) := \{\boldsymbol{v} : \boldsymbol{v}_{|\kappa} \in H^s(\kappa)^3 \text{ and } \nabla \times (\boldsymbol{v}_{|\kappa}) \in H^s(\kappa)^3, \forall\kappa \in \mathscr{T}\}.$$

1.1.3 Some functional inequalities

In this section, we present some relevent properties without proof of functions belonging to Sobolev spaces. The following inequalities are used at several places in this thesis. They can be found e.g. in [RI08] and the references therein.

Lemma 1.1.1. *(*Cauchy-Schwarz's inequality*). For all $f, g \in L^2(\Omega)$, the following inequality holds*

$$|(f, g)_\Omega| \leq \|f\|_{0,\Omega}\|g\|_{0,\Omega}. \tag{1.6}$$

Lemma 1.1.2. *(*Young's inequality*). For all $\theta > 0$, $\forall a, b \in \mathbb{R}$, the following inequality holds*

$$ab \leq \frac{\theta}{2}a^2 + \frac{1}{2\theta}b^2. \tag{1.7}$$

Lemma 1.1.3. *(*Poincaré inequality*). Assume that Ω is a bounded connected open set of $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. Then there exists a constant $C_\Omega > 0$, such that*

$$\|u\|_0 \leq C_\Omega\|\nabla u\|_0, \quad \forall u \in H_0^1(\Omega)$$

Corollary 1.1.1. *The mapping $u \to \|\nabla u\|_0$ is a norm on $H_0^1(\Omega)$ equivalent to the norm $\|u\|_1$.*

Lemma 1.1.4. *(*Friedrich's inequality*.) [KN84]*
Let $\Omega \subset \mathbb{R}^2$ is a bounded simply connected Lipschitz domain, then there exists a constant $C > 0$ such that

$$\|u\|_0 \leq C(\|\nabla.u\|_0 + \|\nabla \times u\|_0).$$

$\forall u \in \boldsymbol{X}(\Omega)$ *and* $\forall u \in \boldsymbol{Y}(\Omega) = H(\mathrm{curl}, \Omega) \cap \mathrm{H}_0(\mathrm{div}, \Omega)$.

We have the following discrete trace inequality (see e.g. in [PE12]).

Lemma 1.1.5. *(*Discrete trace inequality*). Let $\mathscr{T}$ be a shape and contact regular mesh with parameter r_κ defined in (1.4). Then, for all $\boldsymbol{v}_h \in P^k(\mathscr{T})$, $\forall \kappa \in \mathscr{T}$, $\forall e \in \mathscr{F}_h$, the following inequality holds*

$$h_\kappa^{\frac{1}{2}}\|v_h\|_{0,e} \leq C_{\mathrm{tr}}\|v_h\|_{0,\kappa}, \tag{1.8}$$

where C_{tr} only depends on r_κ and k. In particular, summing over $\in \mathscr{F}_h$, we infer from (1.8) and Cauchy-Schwarz inequality that

$$h_\kappa^{\frac{1}{2}}\|v_h\|_{0,\partial\kappa} \leq C_{\mathrm{tr}}N_\partial^{\frac{1}{2}}\|v_h\|_{0,\kappa}, \tag{1.9}$$

where N_∂ is the maximum number of mesh faces composing the boundary of mesh element, and they are bounded in h.

Remark 1.1.2. *Note that analogous bounds as in lemma 1.1.5 can be easily obtained for vector valued function.*

Lemma 1.1.6. (Inverse inequality.) *Let κ be an element of the mesh $\mathscr{T}$, then we have*

$$\|\boldsymbol{w}\|_{0,\partial\kappa} \leq C_{\mathrm{inv}} h_\kappa^{-1/2} \|\boldsymbol{w}\|_{0,\kappa}, \quad \forall \boldsymbol{w} \in (\mathcal{P}^l(\kappa))^3. \tag{1.10}$$

with a constant C_{inv} that depends only on the shape-regularity of the mesh, the approximation order l, and the dimension d.

The following result can be seen for example in [QV94].

Lemma 1.1.7. (Gronwall lemma.) *Let $f \in L^1(t_0, T)$ be a non-negative function, g and ϕ be continuous functions on $[t_0, T]$. If ϕ satisfies*

$$\phi(t) \leq g(t) + \int_{t_0}^t f(\tau)\phi(\tau) \quad \forall t \in [t_0, T],$$

then

$$\phi(t) \leq g(t) + \int_{t_0}^t f(s)g(s)\mathrm{exp}\big(\int_s^t f(\tau)\mathrm{d}\tau\big) \ \forall t \in [t_0, T].$$

If moreover g is no-decreasing, then

$$\phi(t) \leq g(t)\mathrm{exp}\big(\int_{t_0}^t f(\tau)\mathrm{d}\tau\big) \ \forall t \in [t_0, T].$$

Remark 1.1.3. *Often the Gronwall lemma will be used in the special case in which*

$$g(t) = \phi(0) + \int_0^t \psi(s), \ \ \psi(s) \ \geq 0.$$

1.1.4 Interpolation operators

Definition 1.1.1. (Clément interpolation.) *The Clément interpolation operator maps a function from $H_0^1(\Omega)$ to the usual space $S(\Omega, \mathscr{T})$, consisting of continuous piecewise linear functions on the triangulation which are zero on the boundary. The Clément interpolation operator of a function $v \in H_0^1(\Omega)$ is defined by $I_{cl} : H_0^1(\Omega) \longrightarrow S(\mathscr{T}, \Omega)$*

$$I_{cl}(v) := \sum_{x \in \mathcal{N}_\Omega} \frac{1}{|\omega_x|}\Big(\int_{\omega_x} v\Big)\phi_x, \tag{1.11}$$

where x denote a nodal point, and $\mathcal{N}_\Omega$ denote the set of internal nodes of the mesh. By ω_x we denote the union of all elements having x as node.

$\phi_x \in S(\Omega, \mathscr{T})$ is the nodal basis function associated with a node x, uniquely determined by the condition

$$\phi_x(y) = \delta_{x,y} \quad \forall y \in \mathcal{N}_\Omega.$$

In the next lemma, we state the interpolation estimates. For detail see in [CL75],[KU01],[NC03].

Lemma 1.1.8. *For any $v \in H_0^1(\Omega)$, we have*

$$\sum_{\kappa \in \mathscr{T}} h_\kappa^{-2} \|v - I_{cl}v\|_\kappa^2 \lesssim \|\nabla v\|^2, \tag{1.12}$$

$$\sum_{\kappa \in \mathscr{T}} \|\nabla(v - I_{cl}v)\|_\kappa^2 \lesssim \|\nabla v\|^2, \tag{1.13}$$

$$\sum_{e \in \mathscr{F}} h_e^{-1} \|v - I_{cl}v\|_e^2 \lesssim \|\nabla v\|^2. \tag{1.14}$$

In the following lemma we define the quasi interpolant operator for vector valued functions [BHHW00].

Lemma 1.1.9. *(Quasi-interpolant operator.) Let $\mathscr{N}\mathscr{D}_\kappa(\Omega, \mathscr{T})^3$ be the standard Nédélec's finite element space, setting $\mathscr{N}\mathscr{D}_{\kappa,0}(\Omega, \mathscr{T})^3 = \mathscr{N}\mathscr{D}_\kappa(\Omega, \mathscr{T})^3 \cap H_0(curl, \Omega)$. For $\kappa \in \mathscr{T}$, $e \in \mathscr{F}_h$, let D_κ, D_κ^1 and D_e^1 be given by*

$$D_\kappa = \cup\{\kappa \in \mathscr{T}, e \in \mathscr{F}_h(\kappa)\},$$

$$D_\kappa^1 = \cup\{D_\kappa, e \in \mathscr{F}_h(\kappa)\},$$

$$D_e^1 = \cup\{D_\kappa, e \in \mathscr{F}_B(\kappa)\}.$$

There exists a linear projection $\mathscr{B}^h : H^1(\Omega)^3 \cap H_0(curl, \Omega) \to \mathscr{N}\mathscr{D}_{\kappa,0}(\Omega, \mathscr{T})^3$, such that for all $\boldsymbol{\psi} \in H^1(\Omega)^3$

$$\|\mathscr{B}_h \boldsymbol{\psi}\|_{0,\kappa} \le c_1 \|\boldsymbol{\psi}\|_{H^1(D_\kappa^1)^3} \tag{1.15}$$

$$\|\mathscr{B}_h(\nabla \times \boldsymbol{\psi})\|_{0,\kappa} \le c_2 |\boldsymbol{\psi}|_{H^1(D_e^1)^3} \tag{1.16}$$

$$\|\boldsymbol{\psi} - \mathscr{B}_h \boldsymbol{\psi}\|_{0,\kappa} \le c_3 h_\kappa |\boldsymbol{\psi}|_{H^1(D_\kappa^1)^3} \tag{1.17}$$

$$\|\boldsymbol{\psi} - \mathscr{B}_h \boldsymbol{\psi}\|_{0,e} \le c_4 \sqrt{h_e} |\boldsymbol{\psi}|_{H^1(D_e^1)^3} \tag{1.18}$$

where the constants $c_1, c_2, c_3, c_4 > 0$ do neither depend on $\boldsymbol{\psi}$ nor on κ, but only on the shape regularity of the mesh $\mathscr{T}$.

In the following lemma we give some properties of L^2 Projection.

Lemma 1.1.10. *Let $\boldsymbol{\Pi_h}$ be the L^2 projection on to piecewise constant polynomial functions. Let $\kappa \in \mathscr{T}$ then*

- *For* $\boldsymbol{v} \in H^m(\kappa)^3, m \geq 0$, *we have*

$$\|\boldsymbol{v} - \boldsymbol{\Pi_h} \boldsymbol{v}\|_{0,\kappa} \leq C h_\kappa^{min\{m,l+1\}} \|\boldsymbol{v}\|_{s,\kappa}. \tag{1.19}$$

- *For* $\boldsymbol{v} \in H^m(\kappa)^3, m \geq \frac{1}{2}$, *we have*

$$\|\boldsymbol{v} - \boldsymbol{\Pi_h} \boldsymbol{v}\|_{0,\partial\kappa} \leq C h_\kappa^{min\{m-\frac{1}{2},l+\frac{1}{2}\}} \|\boldsymbol{v}\|_{s,\kappa}. \tag{1.20}$$

Chapter 2

Discontinuous Galerkin Formulation

Contents

2.1 Model problem

We consider an initial boundary value problem derived from the time dependent Maxwell's equations (11), with $\epsilon = 1$ *in a loseless medium i.e. with vanishing charge density* ϱ*, and conductivity* σ*. We augment the problem with a perfect electric conducting boundary condition*

$$\begin{cases} \partial^2_{tt}\boldsymbol{u} + \nabla \times (\mu^{-1}\nabla \times \boldsymbol{u}) = \boldsymbol{f}, \quad \text{in} \ \ \Omega \times I, \\ \nabla \cdot \boldsymbol{u} = 0, \quad \text{in} \ \ \Omega \times I, \\ \boldsymbol{n} \times \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}, \ \ on \ \partial\Omega \times I, \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \ \ \partial_t \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_1(\boldsymbol{x}), \ \ on \ \Omega. \end{cases} \tag{2.1}$$

Here $I = [0, T]$ *is a finite time interval,* $\boldsymbol{f}$ *is source function in* $L^2(0, T, L^2(\Omega)^3)$*.* $\boldsymbol{u}_0 \in \boldsymbol{X}(\Omega) = H_0(\text{curl}; \Omega) \cap \text{H}(\text{div}, \Omega)$ *and* $\boldsymbol{u}_1 \in L^2(\Omega)^3$*. We assume that* μ *is scalar positive function, which is bounded uniformly from below and above, i.e.*

$$0 < \mu_\star \leq \mu(\boldsymbol{x}) \leq \mu^\star < \infty, \tag{2.2}$$

for simplicity we assume that μ *is piecewise constant.*
Assume that the analytical solution $\boldsymbol{u}$ *of (2.1) satisfies* $\boldsymbol{u} \in H^1(0, T, L^2(\Omega)^3) \cap L^2(0, T, \boldsymbol{X}(\Omega)), \partial_t \boldsymbol{u} \in L^2(0, T, L^2(\Omega)^3), \partial^2_{tt}\boldsymbol{u} \in L^2(0, T, L^2(\Omega)^3)$*. The standard*

variational form of (2.1) *is given by:*

find $\boldsymbol{u} \in H^1(0,T,L^2(\Omega)^3) \cap L^2(0,T,\boldsymbol{X}(\Omega))$, *such that*

$$(\partial^2_{tt}\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{v}) = l(\boldsymbol{v}) \ \forall \boldsymbol{v} \in \boldsymbol{X}(\Omega), \ \text{a.e. in } I, \tag{2.3}$$

where the bilinear form $b(.,.)$ *is defined by*

$$b(\boldsymbol{u},\boldsymbol{v}) = (\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})_\Omega + (\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v})_\Omega;$$

and the linear form $l(.)$ *is given by*

$$l(\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_\Omega.$$

The problem (2.3) is well-posed and admits a unique solution which can be shown to be continuous in time,

$$\boldsymbol{u} \in \mathcal{C}^0(0,T,\boldsymbol{X}(\Omega)) \cap \mathcal{C}^1(0,T,L^2(\Omega)^3) \cap H^2(0,T,X^{'}(\Omega)),$$

where $X^{'}(\Omega)$ *is the dual space of* $\boldsymbol{X}(\Omega)$. *The existence and uniqueness is followed by the result of equivalence of norms in* $\boldsymbol{X}(\Omega)$ *stated in proposition 1.1.1 and the Lions varitional theory {[LM68], Chapter III, Th. 8.1 and Th. 8.2}. For more details see for example [RT92].*

Remark 2.1.1. *We identify a function* $\boldsymbol{v} \in \Omega \times [0,T] \to \mathbb{R}^3$ *with the function* $\boldsymbol{v} : [0,T] \to \boldsymbol{X}(\Omega)$ *and we use* $\boldsymbol{v}(t)$ *to indicate* $\boldsymbol{v}(\cdot,t)$.

2.2 Discretization in space:

In this section, first we will discretize the problem (2.1) *by interior penalty discontinuous Galerkin method in space. We propose a symmetric interior penalty discontinuous Galerkin (DG) formulation. Then we will show the well-posedness of the proposed formulation.*

2.2.1 Semi-discrete discontinuous Galerkin (DG) formulation

2.2.1.1 Derivation of DG formulation

Let $\mathscr{T}_h$, $h > 0$, *be a family of partitions of* Ω *into tetrahedra defined in section 1.1.2. To derive the DG formulation of (2.1), let* $\boldsymbol{v}_h$ *be a test function* $\boldsymbol{v}_h \in V_h$ *where the finite element space* V_h *is defined in (1.5). We multiply the first* *equation of (2.1) by* $\boldsymbol{v}_h|_\kappa$ *and integrate by parts on one element* $\kappa \in \mathscr{T}_h$, *we get the following identity*

$$\int_\kappa \partial^2_{tt}\boldsymbol{u}\cdot\boldsymbol{v}_h \mathrm{dx} + \int_\kappa (\mu^{-1}\nabla\times\boldsymbol{u}).(\nabla\times\boldsymbol{v}_h)\mathrm{dx} + \int_{\partial\kappa}(\mu^{-1}\nabla\times\boldsymbol{u})\times\boldsymbol{n}_\kappa\cdot\boldsymbol{v}_h \mathrm{ds} = \int_\kappa \boldsymbol{f}\cdot\boldsymbol{v}_h\mathrm{dx}, \quad (2.4)$$

where $\boldsymbol{n}_\kappa$ *is unit outward normal to* $\partial\kappa$. *Summing over all the elements* κ *in* $\mathscr{T}_h$, *we obtain*

$$\sum_{\kappa\in\mathscr{T}_h}\int_\kappa \partial^2_{tt}\boldsymbol{u}\cdot\boldsymbol{v}_h \mathrm{dx} + \sum_{\kappa\in\mathscr{T}_h}\int_\kappa (\mu^{-1}\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{v}_h)\mathrm{dx} + \sum_{\kappa\in\mathscr{T}_h}\int_{\partial\kappa}(\mu^{-1}\nabla\times\boldsymbol{u})\times\boldsymbol{n}_\kappa\cdot\boldsymbol{v}_h \mathrm{ds} = \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \boldsymbol{f}\cdot\boldsymbol{v}_h\mathrm{dx},$$

Using the following identity which holds true for $\boldsymbol{v}, \boldsymbol{w} \in \mathrm{TR}(\mathcal{F}_h)^3$,

$$\begin{aligned}\sum_{\kappa\in\mathscr{T}_h}\int_{\partial\kappa}\boldsymbol{v}\cdot\boldsymbol{n}_\kappa\times\boldsymbol{w}\mathrm{ds} &= -\sum_{\kappa\in\mathscr{T}_h}\int_{\partial\kappa}\boldsymbol{w}\cdot\boldsymbol{n}_\kappa\times\boldsymbol{v}\mathrm{ds} \\ &= -\int_{\mathscr{F}_h}[\![\boldsymbol{v}]\!]_T\{\!\{\boldsymbol{w}\}\!\}\mathrm{ds} + \int_{\mathscr{F}_h^I}[\![\boldsymbol{w}]\!]_T\{\!\{\boldsymbol{u}\}\!\}\mathrm{ds},\end{aligned} \quad (2.5)$$

we get

$$\sum_{\kappa\in\mathscr{T}_h}\int_\kappa \partial^2_{tt}\boldsymbol{u}\cdot\boldsymbol{v}_h \mathrm{dx} + \sum_{\kappa\in\mathscr{T}_h}\int_\kappa (\mu^{-1}\nabla\times\boldsymbol{u}).(\nabla\times\boldsymbol{v}_h)\mathrm{dx} - \sum_{e\in\mathscr{F}_h}\int_e [\![\boldsymbol{v}_h]\!]_T\{\!\{\mu^{-1}\nabla\times\boldsymbol{u}\}\!\}\mathrm{ds} + \sum_{e\in\mathscr{F}_h^I}\int_e [\![\mu^{-1}\nabla\times\boldsymbol{u}]\!]_T\{\!\{\boldsymbol{v}_h\}\!\}\mathrm{ds} = \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \boldsymbol{f}\cdot\boldsymbol{v}_h\mathrm{dx}.$$

Since $\boldsymbol{u}$ *is solution to equation (2.1), we have*

$$\sum_{\kappa\in\mathscr{T}_h}\int_\kappa \partial^2_{tt}\boldsymbol{u}\cdot\boldsymbol{v}_h \mathrm{dx} + \sum_{\kappa\in\mathscr{T}_h}\int_\kappa (\mu^{-1}\nabla\times\boldsymbol{u}).(\nabla\times\boldsymbol{v}_h)\mathrm{dx} - \sum_{e\in\mathscr{F}_h}\int_e [\![\boldsymbol{v}_h]\!]_T\{\!\{\mu^{-1}\nabla\times\boldsymbol{u}\}\!\}\mathrm{ds} = \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \boldsymbol{f}\cdot\boldsymbol{v}_h\mathrm{dx}. \quad (2.6)$$

Using the fact that $\boldsymbol{u} \in H(\mathrm{curl}, \Omega)$, *we have, the tangential jump* $[\![\boldsymbol{u}]\!]_T$ *vanishes on* $\mathcal{F}_h^I$. *Moreover,* $\boldsymbol{u}$ *satisfies the boundary condition in (2.1) thus,* $[\![\boldsymbol{u}]\!]_T = 0$

on $\mathcal{F}_h^B$. Hence, we have $[\![\boldsymbol{u}]\!]_T = 0$ on $\mathcal{F}_h$. We add the following continuity term to equation (2.6)

$$\sum_{e\in\mathscr{F}_h}\int_e [\![\boldsymbol{u}]\!]_T \{\!\{\mu^{-1}\nabla\times\boldsymbol{v}_h\}\!\}\mathrm{ds}.$$

We also add the following penality terms, which vanish for the exact solution $\boldsymbol{u}$ *to (2.6)*

$$J(\boldsymbol{u},\boldsymbol{v}_h) = \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \nabla\cdot\boldsymbol{u}\nabla\cdot\boldsymbol{v}_h\mathrm{dx} + \sum_{e\in\mathscr{F}_h}\int_e a[\![\boldsymbol{u}]\!]_T[\![\boldsymbol{v}_h]\!]_T\mathrm{ds} + \sum_{e\in\mathscr{F}_h^I}\int_e a[\![\boldsymbol{u}]\!]_N[\![\boldsymbol{v}_h]\!]_N\mathrm{ds}. \tag{2.7}$$

Hence we get the semi discrete discontinuous Galerkin formulation of (2.1), *which reads:*

find $\boldsymbol{u_h}\in[0,T]\to V_h$ such that

$$\begin{aligned}(\partial_{tt}^2\boldsymbol{u_h},\boldsymbol{v}_h)_\Omega + b_h(\boldsymbol{u_h},\boldsymbol{v}_h) = (\boldsymbol{f},\boldsymbol{v}_h)_\Omega,\ \forall\boldsymbol{v}_h\in V_h, t\in(0,T].\\ \boldsymbol{u_h}|_{t=0} = \Pi_h\boldsymbol{u}_0,\\ \partial_t\boldsymbol{u_h}|_{t=0} = \Pi_h\boldsymbol{u}_1.\end{aligned} \tag{2.8}$$

Here, Π_h denotes the L^2-projection onto V_h, and the discrete bilinear form b_h on $V_h\times V_h$ is given by

$$\begin{aligned}b_h(\boldsymbol{u},\boldsymbol{v}) = \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \mu^{-1}\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v}dx + \sum_{\kappa\in\mathscr{T}_h}\int_\kappa\nabla\cdot\boldsymbol{u}\nabla\cdot\boldsymbol{v}dx\\ -\sum_{e\in\mathscr{F}_h}\int_e[\![\boldsymbol{u}]\!]_T\cdot\{\!\{\mu^{-1}\nabla\times\boldsymbol{v}\}\!\}dA - \sum_{e\in\mathscr{F}_h}\int_e[\![\boldsymbol{v}]\!]_T\cdot\{\!\{\mu^{-1}\nabla\times\boldsymbol{u}\}\!\}dA\\ +\sum_{e\in\mathscr{F}_h}\int_e a[\![\boldsymbol{u}]\!]_T\cdot[\![\boldsymbol{v}]\!]_T dA + \sum_{e\in\mathscr{F}_h^I}\int_e a[\![\boldsymbol{u}]\!]_N\cdot[\![\boldsymbol{v}]\!]_N dA;\end{aligned} \tag{2.9}$$

The function a penalizes the tangential and normal jumps; it is referred to as the interior penalty stabilization function. To define it we first introduce the function h *by*

$$\mathrm{h}|_e = \begin{cases} min\{h_\kappa, h_{\kappa'}\}, & e\in\mathscr{F}_I, e=\partial\kappa\cap\partial\kappa',\\ h_\kappa, & e\in\mathscr{F}_B, e=\partial\kappa\cap\partial\Omega.\end{cases}$$

Now we define,

$$a = \alpha\mathrm{h}^{-1}, \tag{2.10}$$

where α is a positive parameter independent of the local mesh size.

2.2.2 Well-posedness of the Discrete problem

To show the well-posedness of DG formulation (2.8), we introduce the semi-norm

$$|\boldsymbol{v}|_h = \left(\sum_{\kappa \in \mathscr{T}_h} \|\nabla\times\boldsymbol{v}\|^2_{0,\kappa} + \sum_{\kappa \in \mathscr{T}_h} \|\nabla\cdot\boldsymbol{v}\|^2_{0,\kappa} + \sum_{e \in \mathscr{F}_h} \|a^{\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\|^2_{0,e} + \sum_{e \in \mathscr{F}_I} \|a^{\frac{1}{2}}[\![\boldsymbol{v}]\!]_N\|^2_{0,e} \right)^{\frac{1}{2}}, \quad \forall \boldsymbol{v} \in V_h. \tag{2.11}$$

where $\|.\|_{0,\kappa}$ and $\|.\|_{0,e}$ denote the L^2 norm over an element κ and a face e respectively.

Proposition 2.2.1. *The semi-norm $|.|_h$ given by (2.11) defines a norm on V_h.*

Proof. From [DZ09], we have

$$\|\boldsymbol{v}\|_0 \leq \left(\sum_{\kappa \in \mathscr{T}_h} \|\nabla \times \boldsymbol{v}\|^2_{0,\kappa} + \sum_{\kappa \in \mathscr{T}_h} \|\nabla \cdot \boldsymbol{v}\|^2_{0,\kappa} + \sum_{e \in \mathscr{F}_h} \|\frac{1}{\sqrt{h}}[\![\boldsymbol{v}]\!]_T\|^2_{0,e} + \sum_{e \in \mathscr{F}_I} \|\frac{1}{\sqrt{h}}[\![\boldsymbol{v}]\!]_N\|^2_{0,e} \right)^{\frac{1}{2}}, \quad \forall \boldsymbol{v} \in V_h. \tag{2.12}$$

Then the proof follows from the fact that $\sqrt{a} \geq \frac{1}{\sqrt{h}}$. □

We have the following lemma for the proof of the well-posedness of the DG semi-discrete problem.

Lemma 2.2.1. *1) There is a constant $C_{cont} > 0$, independant of mesh-size such that*

$$|b_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| \leq C_{\text{cont}} |\boldsymbol{u}_h|_h |\boldsymbol{v}_h|_h, \tag{2.13}$$

$\forall \boldsymbol{u}_h, \boldsymbol{v}_h \in V_h$.

2) There is a parameter $\alpha_0 > 0$ and a constant $C_{coer} > 0$, independent of mesh size such that for parameters α in (2.10) with $\alpha > \alpha_0$, we have that

$$b_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_{\text{coer}} |\boldsymbol{v}_h|^2_h, \quad \forall \boldsymbol{v}_h \in V_h. \tag{2.14}$$

Proof. 1) Applying the Cauchy-Schwarz's inequality, we have

$$
\begin{aligned}
|b_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| \leq{} & \mu_\star^{-1} \|\nabla \times \boldsymbol{u}_h\|_{0,\Omega} \|\nabla \times \boldsymbol{v}_h\|_{0,\Omega} + \|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega} \|\nabla \cdot \boldsymbol{v}_h\|_{0,\Omega} \\
& + \Big(\sum_{e \in \mathcal{F}_h} a \|\{\!\{\mu^{-1} \nabla \times \boldsymbol{u}_h\}\!\}\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{F}_h} a^{-1} \|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} \\
& + \Big(\sum_{e \in \mathcal{F}_h} a \|\{\!\{\mu^{-1} \nabla \times \boldsymbol{v}_h\}\!\}\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{F}_h} a^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} \\
& + \Big(\sum_{e \in \mathcal{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} \\
& + \Big(\sum_{e \in \mathcal{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2 \Big)^{\frac{1}{2}}.
\end{aligned}
\tag{2.15}
$$

Using the inverse inequality on third and fourth term of the above inequality and using Cauchy-Schwarz's inequality, we have the following bound

$$
|b_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| \leq \max\{2, \mu_\star^{-1} + \alpha^{-\frac{1}{2}} C_{\text{inv}} \mu_\star^{-1}\} |\boldsymbol{u}_h|_h |\boldsymbol{v}_h|_h, \tag{2.16}
$$

where C_{inv} is the constant from inverse inequality (1.10) and $\mu_\star$ is given by (2.2). Choosing $C_{\text{cont}} = max\{2, \mu_\star^{-1} + \alpha^{-\frac{1}{2}} C_{\text{inv}} \mu_\star^{-1}\}$, we obtain the continuity of bilinear form b_h.

2) Now, to show the coercivity of the bilinear form b_h, we have

$$
\begin{aligned}
b_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq{} & (\mu^\star)^{-1} \|\nabla \times \boldsymbol{u}_h\|_{0,\Omega}^2 + \|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega}^2 - 2 \Big| \sum_{e \in \mathcal{F}_h} \int_e \{\!\{\mu^{-1} \nabla \times \boldsymbol{u}_h\}\!\} [\![\boldsymbol{u}_h]\!]_T \Big| \\
& + \sum_{e \in \mathcal{F}_h} a \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 + \sum_{e \in \mathcal{F}_h} a \|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2.
\end{aligned}
\tag{2.17}
$$

Using Cauchy-Schwarz's and Young's inequality (1.7), we get the following bound

$$
\begin{aligned}
2 \Big| \sum_{e \in \mathcal{F}_h} \int_e \{\!\{\mu^{-1} \nabla \times \boldsymbol{u}_h\}\!\} [\![\boldsymbol{u}_h]\!]_T \Big| & \leq 2 \sum_{e \in \mathcal{F}_h} \Big(\|\{\!\{\mu^{-1} \nabla \times \boldsymbol{u}_h\}\!\}\|_{0,e} \Big) \Big(\|[\![\boldsymbol{u}_h]\!]_T\|_{0,e} \Big), \\
\leq \theta \mu_\star^2 C_{\text{inv}}^{-2} \sum_{e \in \mathcal{F}_h} h \|\{\!\{\mu^{-1} \nabla \times \boldsymbol{u}_h\}\!\}\|_{0,e}^2 & + \theta^{-1} \mu_\star^{-2} C_{\text{inv}}^2 \sum_{e \in \mathcal{F}_h} h^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2, \\
\leq \theta \sum_{\kappa \in \mathscr{T}_h} \|\nabla \times \boldsymbol{u}_h\|_{0,\kappa}^2 & + \theta^{-1} \mu_\star^{-2} C_{\text{inv}}^2 \sum_{e \in \mathcal{F}_h} h^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2,
\end{aligned}
\tag{2.18}
$$

where we have used the inverse inequality (1.10) to obtain the last bound. Now inserting the above bound in (2.17), we obtain

$$b_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \geq ((\mu^\star)^{-1}-\theta)\sum_{\kappa\in\mathscr{T}_h}\|\nabla\times\boldsymbol{u}_h\|^2_{0,\kappa}+\sum_{\kappa\in\mathscr{T}_h}\|\nabla\cdot\boldsymbol{u}_h\|^2_{0,\kappa}$$
$$+(1-\theta^{-1}\mu_\star^{-2}C^2_{\rm inv}\alpha^{-1})\sum_{e\in\mathcal{F}_h}\mathrm{a}\|[\![\boldsymbol{u}_h]\!]_T\|^2_{0,e}+\sum_{e\in\mathscr{F}_h}\mathrm{a}\|[\![\boldsymbol{u}_h]\!]_N\|^2_{0,e}. \quad (2.19)$$

Setting $\theta=\frac{(\mu^\star)^{-1}}{2}$ and $\alpha_0=4\mu_\star^{-2}C^2_{\rm inv}\mu^\star$, for $\alpha\geq\alpha_0$, we get the coercivity bound with $C_{\rm coer}=\min\{\frac{(\mu^\star)^{-1}}{2},\frac{1}{2}\}$.

□

Lemma 2.2.2. *(Consistency) The semi-discrete DG formulation (2.8) is consistent with the continuous problem. i.e. If $\boldsymbol{u}$ is the solution of problem (2.1), then $\boldsymbol{u}$ satisfies the DG problem (2.8).*

Proof. Let $\boldsymbol{u}\in H_0(curl,\Omega)\cap H(div,\Omega)$ be the solution of (2.1), this immediately follows that the terms in $b_h(,.,)$ involving $\nabla.\boldsymbol{u}$ and $[\![\boldsymbol{u}]\!]_N$ vanish. From the fact that $\boldsymbol{u},\mu^{-1}\nabla\times\boldsymbol{u}\in L^2(\Omega)^3$, we have

$$[\![\boldsymbol{u}]\!]_T=0 \text{ on } \mathcal{F}_h. \quad (2.20)$$

Thus we get,

$$b_h(\boldsymbol{u},\boldsymbol{v}_h)=(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}_h)_\Omega-\sum_{e\in\mathcal{F}_h}\int_e\{\!\{\mu^{-1}\nabla\times\boldsymbol{u}\}\!\}[\![\boldsymbol{v}_h]\!]_T.(2.21)$$

Using the identity (2.5), we can write the above equation in the form

$$b_h(\boldsymbol{u},\boldsymbol{v}_h)=(\mu^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}_h)_\Omega-\sum_{\kappa\in\mathscr{T}_h}\int_{\partial\kappa}\boldsymbol{v}_h.(\boldsymbol{n}_\kappa\times(\mu^{-1}\nabla\times\boldsymbol{u}))\mathrm{ds}$$
$$+\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\boldsymbol{v}_h\}\!\}[\![\mu^{-1}\nabla\times\boldsymbol{u}]\!]_T\mathrm{ds}. \quad (2.22)$$

Integration by parts yields

$$b_h(\boldsymbol{u},\boldsymbol{v}_h)=(\nabla\times(\mu^{-1}\nabla\times\boldsymbol{u}),\boldsymbol{v}_h)_\Omega$$

□

The stability result in lemma 2.2.1 implies that the discrete DG formulation (2.8) is well-posed and uniquely solvable provided that $\alpha>\alpha_0$.

2.3 Fully discrete scheme

To introduce the fully discrete implicit scheme approximating (2.8), we consider a subdivision of the time interval $[0,T]$ *into subintervals* $(t_{n-1},t_n], n=1,\ldots,N,$ *with* $t_0=0$ *and* $t_N=T$*, and we define* $k_n := t_n - t_{n-1}$*, the local time step. Associated with the time subdivision, let* $\mathscr{T}_h^n$*,* $n=0,\ldots,N$ *be a sequence of meshes. The meshes are assumed to be shape regular and compatible in the sense that for any two consective meshes* $\mathscr{T}^{n-1}$ *and* $\mathscr{T}^n$*,* $\mathscr{T}^n$ *is obtained by locally refining some elements and coersening some other ones. The finite element space corresponding to* $\mathscr{T}_h^n$ *will be denoted by* V_h^n*, defined by*

$$V_h^n := \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}_{|\kappa} \in \mathcal{P}^l(\kappa)^d, \quad \forall \kappa \in \mathscr{T}^n\}, \tag{2.23}$$

where $\mathcal{P}^l(\kappa)$ *denotes the space of real polynomials of total degree at most* l *on* κ*.*
Since the bilinear form b_h *depends on the mesh, we denote by* b_h^n *the bilinear form associated with the mesh* $\mathscr{T}_h^n$*. We consider the fully discrete scheme for the Maxwell's problem (2.1):*

for each $n=1,\ldots,N$*, find* $\boldsymbol{u}_h^n \in V_h^n$ *such that*

$$(\partial^2 \boldsymbol{u}_h^n, \boldsymbol{v})_\Omega + {b_h}^n(\boldsymbol{u}_h^n, \boldsymbol{v}) = (\boldsymbol{f}^n, \boldsymbol{v})_\Omega \text{ for all } \boldsymbol{v} \in V_h^n, \tag{2.24}$$

where $\boldsymbol{f}^n := \boldsymbol{f}(t^n, \cdot)$*.*
The backward second and first finite differences are given by

$$\partial^2 \boldsymbol{u}_h^n := \frac{\partial {\boldsymbol{u}_h}^n - \partial {\boldsymbol{u}_h}^{n-1}}{k_n}, \tag{2.25}$$

with

$$\partial \boldsymbol{u}_h^n := \begin{cases} \dfrac{\boldsymbol{u}_h^n - {\boldsymbol{u}_h}^{n-1}}{k_n} \text{ for } n = 1,2,\ldots N, \\ \boldsymbol{v}^0 = \boldsymbol{\pi}^0 \boldsymbol{u}_1 \text{ for } n=0. \end{cases} \tag{2.26}$$

Where ${\boldsymbol{u}_h}^0 := \boldsymbol{\pi}^0 \boldsymbol{u}^0$ *and* $\boldsymbol{\pi}^0 : L^2(\Omega)^d \to V_h^0$ *is a suitable projection onto the finite element space (for example the orthogonal* L^2-*projection operator). The continuity and coercivity of* $b_h^n(.,.)$ *implies that the problem (2.24) admits a unique solution* $(u_h^n)_{0\le n\le N}$*at each time step.*

2.4 Conclusion

In this chapter, a discontinuous Galerkin method for the discretization of the time-dependent Maxwell's equations in "stable medium" subject to the perfect *conducting boundary condition has been proposed. Both semi-discrete and fully discrete problems have been analysed.*

CHAPTER 3

A priori error estimates

Contents

3.1 Introduction

In this chapter, we derive the a priori error estimates in L^2-norm. These estimates are optimal in mesh size h, and slightly sub-optimal in the approximation degree p. To complete the convergence analysis some numerical experiments are given.
The outline of this chapter is as follows: in first section we give the definitions and some notations that we will need later in the chapter. In second section we derive a priori error estimates. In the third section, we show the numerical results and give some concluding remarks.

3.2 Preliminaries

We give an $hp-$approximation result to interpolate scalar function, see [BS87].

Proposition 3.2.1. *Let $\kappa \in \mathscr{T}_h$ and suppose that $u \in H^s(\kappa)$, $s \geq 1$. Then there exists a sequence of polynomials $\pi_p^{h_\kappa}(u) \in \mathcal{P}^p(\kappa)$, $p = 1, 2...$ satisfying, $\forall 0 \leq q \leq s$,*

$$\|u - \pi_p^{h_\kappa}(u)\|_{q,\kappa} \leq C \frac{h_\kappa^{min(p+1,s)-q}}{p^{s-q}} \|u\|_{s,\kappa},$$

and

$$\|u - \pi_p^{h_\kappa}(u)\|_{0,\partial\kappa} \leq C \frac{h_\kappa^{min(p+1,s)-\frac{1}{2}}}{p^{s-\frac{1}{2}}} \|u\|_{s,\kappa}.$$

The constant C is independent of $\boldsymbol{u}$, h_κ and p, but depends on the shape regularity of the mesh.

Definition 3.2.1. *In order to interpolate vector function, for $\boldsymbol{u} = (u_1, u_2, u_3)$, we define $\boldsymbol{\Pi}_p^h : H^s(\mathrm{curl}, \mathscr{T}_h) \to V_h$ by*

$$\boldsymbol{\Pi}_p^h(\boldsymbol{u}) = (\pi_p^h(u_1), \pi_p^h(u_2), \pi_p^h(u_3)),$$

with $\pi_p^{h_\kappa}(u_i)|_\kappa = \pi_p^{h_\kappa}(u_i|_\kappa), i = 1, ., 3$. Where $\pi_p^{h_\kappa}$ is given by proposition 3.2.1.

In this chapter $\boldsymbol{u}$ denotes the exact solution of (2.1) and $\boldsymbol{u}_h$ is its discrete solution by symmetric discontinuous Galerkin method. Thus $\boldsymbol{u_h} \in [0, T] \to V_h$ is the solution to the following problem:

$$\begin{array}{l} (\partial_{tt}^2 \boldsymbol{u_h}, \boldsymbol{v}_h)_\Omega + b_h(\boldsymbol{u_h}, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega, \ \forall \boldsymbol{v}_h \in V_h, t \in (0, T]. \\ \boldsymbol{u_h}|_{t=0} = \boldsymbol{\Pi}_p^h \boldsymbol{u}_0, \\ \partial_t \boldsymbol{u_h}|_{t=0} = \boldsymbol{\Pi}_p^h \boldsymbol{u}_1. \end{array} \tag{3.1}$$

Here, $\boldsymbol{\Pi}_p^h$ is given by definition 3.2.1, and the discrete bilinear form b_h on $V_h \times V_h$ is given by (2.9).

3.3 A priori error estimates

Let $\boldsymbol{e} = \boldsymbol{u}_h - \boldsymbol{u}$ denotes the approximation error. We decompose $\boldsymbol{e}$ as follows:

$$\boldsymbol{e} = \boldsymbol{\eta} - \boldsymbol{\vartheta}, \tag{3.2}$$

with $\boldsymbol{\vartheta} = \boldsymbol{\Pi}_p^h(\boldsymbol{u}) - \boldsymbol{u}$ and $\boldsymbol{\eta} = \boldsymbol{\Pi}_p^h(\boldsymbol{u}) - \boldsymbol{u}_h$, where $\boldsymbol{\Pi}_p^h$ is given by definition 3.2.1. We have the following error relation:

$$(\partial_{tt}^2 \boldsymbol{e}, \boldsymbol{v}_h)_\Omega + b_h(\boldsymbol{e}, \boldsymbol{v}_h) = 0, \ \forall \boldsymbol{v}_h \in V_h. \tag{3.3}$$

By using (3.2), we have

$$(\partial_{tt}^2 \boldsymbol{\eta}, \boldsymbol{v}_h)_\Omega + b_h(\boldsymbol{\eta}, \boldsymbol{v}_h) = (\partial_{tt}^2 \boldsymbol{\vartheta}, \boldsymbol{v}_h)_\Omega + b_h(\boldsymbol{\vartheta}, \boldsymbol{v}_h), \ \forall \boldsymbol{v}_h \in V_h, \tag{3.4}$$

Since $\boldsymbol{\eta}(t) \in V_h$, we have $\partial_t \boldsymbol{\eta} \in V_h$. Choosing $\boldsymbol{v}_h = \partial_t \boldsymbol{\eta}$ in (3.4), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t \boldsymbol{\eta}\|_0^2 + \frac{1}{2}\frac{d}{dt} b_h(\boldsymbol{\eta}, \boldsymbol{\eta}) = (\partial_{tt}^2 \boldsymbol{\vartheta}, \boldsymbol{\eta}_t)_\Omega + b_h(\boldsymbol{\vartheta}, \partial_t \boldsymbol{\eta}), \\
&\leq \frac{1}{2}\|\partial_{tt}^2 \boldsymbol{\vartheta}\|_0^2 + \frac{1}{2}\|\partial_t \boldsymbol{\eta}\|_0^2 + b_h(\boldsymbol{\vartheta}, \partial_t \boldsymbol{\eta}).
\end{aligned}$$

Thus, we have

$$\frac{d}{dt}\|\partial_t \boldsymbol{\eta}\|_0^2 + \frac{d}{dt} b_h(\boldsymbol{\eta}, \boldsymbol{\eta}) \leq \|\partial_{tt}^2 \boldsymbol{\vartheta}\|_0^2 + \|\partial_t \boldsymbol{\eta}\|_0^2 + 2 b_h(\boldsymbol{\vartheta}, \partial_t \boldsymbol{\eta}).$$

Integrating in time on $[0,t]$, and using the fact that $\boldsymbol{\eta}(0) = \boldsymbol{\eta}_t(0) = 0$, we get

$$\|\partial_t \boldsymbol{\eta}\|_0^2 + b_h(\boldsymbol{\eta}, \boldsymbol{\eta}) \leq \|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + \int_0^t \|\partial_t \boldsymbol{\eta}\|_0^2 dt + 2\int_0^t b_h(\boldsymbol{\vartheta}, \partial_t \boldsymbol{\eta}) dt.$$

Integrating the last term on right hand side, we get

$$\begin{aligned}
\|\partial_t \boldsymbol{\eta}\|_0^2 + b_h(\boldsymbol{\eta}, \boldsymbol{\eta}) \leq \|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + \int_0^t \|\partial_t \boldsymbol{\eta}\|_0^2 dt + 2|b_h(\boldsymbol{\vartheta}, \boldsymbol{\eta})| \\
+2\int_0^t |b_h(\partial_t \boldsymbol{\vartheta}, \boldsymbol{\eta})| dt,
\end{aligned} \tag{3.5}$$

using the continuity and coercivity of bilinear form b_h, we have

$$\begin{aligned}
&\|\partial_t \boldsymbol{\eta}\|_0^2 + C_{\mathrm{coer}}|\boldsymbol{\eta}|_h^2 \leq \|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + \int_0^t \|\partial_t \boldsymbol{\eta}\|_0^2 dt + C|\boldsymbol{\vartheta}|_h |\boldsymbol{\eta}|_h \\
&+ 2\int_0^t |b_h(\partial_t \boldsymbol{\vartheta}, \boldsymbol{\eta})| dt, \\
&\leq \|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + \int_0^t \|\partial_t \boldsymbol{\eta}\|_0^2 dt + C|\boldsymbol{\vartheta}|_h^2 + C|\boldsymbol{\eta}|_h^2 + C\int_0^t (|\partial_t \boldsymbol{\vartheta}|_h^2 + |\boldsymbol{\eta}|_h^2) dt, \\
&\leq C(\|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + sup_{t\in[0,t]}|\boldsymbol{\vartheta}|_h^2 + \int_0^t |\partial_t \boldsymbol{\vartheta}|_h^2 dt) + C|\boldsymbol{\eta}|_h^2 \\
&+ C(\int_0^t (\|\partial_t \boldsymbol{\eta}\|_0^2| + |\boldsymbol{\eta}|_h^2) dt.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\|\partial_t \boldsymbol{\eta}\|_0^2 + |\boldsymbol{\eta}|_h^2 \leq C(\|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + sup_{t\in[0,t]}|\boldsymbol{\vartheta}|_h^2 + \int_0^t |\partial_t \boldsymbol{\vartheta}|_h^2 dt) \\
+C(\int_0^t (\|\partial_t \boldsymbol{\eta}\|_0^2| + |\boldsymbol{\eta}|_h^2) dt.
\end{aligned} \tag{3.6}$$

As it holds for all $t \in [0,t]$, applying Gronwall's lemma 1.1.7, we get

$$\|\partial_t \boldsymbol{\eta}\|_0^2 + |\boldsymbol{\eta}|_h^2 \leq C(\|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + sup_{t\in[0,t]}|\boldsymbol{\vartheta}|_h^2 + \int_0^t |\boldsymbol{\vartheta}_t|_h^2 dt).$$

Since $\boldsymbol{e} = \boldsymbol{\vartheta} - \boldsymbol{\eta}$, we have

$$\begin{aligned}\|\partial_t \boldsymbol{e}\|_0^2 + |\boldsymbol{e}|_h^2 \leq C(\|\partial_{tt}^2 \boldsymbol{\vartheta}\|_{L^2(0,t,L^2(\Omega))} + sup_{t\in[0,t]}|\boldsymbol{\vartheta}|_h^2 + \int_0^t |\partial_t\boldsymbol{\vartheta}|_h^2 dt) \\ +C\|\partial_t\boldsymbol{\vartheta}\|_{L^\infty(0,t,L^2(\Omega))}.\end{aligned} \tag{3.7}$$

Hence, we conclude that error estimates for the DG approximation to the exact solution reduce to the error estimates for the piecewise polynomial interpolant. In the following proposition we give the estimates for $|\boldsymbol{\vartheta}|_h$.

Proposition 3.3.1. *Let $\boldsymbol{u}$ be the exact solution of (2.1) and suppose that $u(.,t)|_\kappa \in H^s(\kappa)^3$, with $s \geq 2$ and for any $t \in [0,t]$, then we have*

$$|\boldsymbol{u} - \boldsymbol{\Pi}_p^h \boldsymbol{u}|_h^2 \leq C \sum_{\kappa\in\mathscr{T}_h} \frac{h_\kappa^{2\mu_\kappa - 2}}{p^{2s-3}} \|\boldsymbol{u}\|_{s,\kappa}, \ \forall t \in [0,t] \tag{3.8}$$

and

$$|\boldsymbol{u} - \pi_p^h \boldsymbol{u}|_{q,\kappa}^2 \leq C \frac{h_\kappa^{\mu_\kappa - q}}{p^{s-q}} \|\boldsymbol{u}\|_{s,\kappa} \ \forall 0 \leq q \leq s \ , \ \forall t \in [0,t], \tag{3.9}$$

where $\mu_\kappa = \min\{\mathrm{p}+1, \mathrm{s}\}$ and C is independent of h and p.

By applying the previous proposition we obtain the following estimates for $\|\partial_t \boldsymbol{e}\|_0^2 + |\boldsymbol{e}|_h^2$.

Proposition 3.3.2.

$$\begin{aligned}\|\partial_t \boldsymbol{e}\|_0^2 + |\boldsymbol{e}|_h^2 \leq C \sum_{\kappa\in\mathscr{T}_h} \frac{h_\kappa^{2\mu_\kappa-2}}{p^{2s-3}} (\|\partial_{tt}^2 \boldsymbol{u}\|_{L^2(H^s(\kappa)^3)}^2 + \|\boldsymbol{u}\|_{L^\infty(H^s(\kappa)^3)}^2) \\ + C \sum_{\kappa\in\mathscr{T}_h} \frac{h_\kappa^{2\mu_\kappa-2}}{p^{2s-3}} (\|\partial_t \boldsymbol{u}\|_{L^2(H^s(\kappa)^3)}^2 + \|\partial_t \boldsymbol{u}\|_{L^\infty(H^s(\kappa)^3)}^2),\end{aligned} \tag{3.10}$$

where $\mu_\kappa = \min\{\mathrm{p}+1, \mathrm{s}\}$ and C is independent of h and p.

Remark 3.3.1. *In order to get the estimate for $\|\boldsymbol{e}(t)\|_0$, we could have used $\boldsymbol{v}_h = \boldsymbol{\eta}(t)$ in (3.3), but this can be deduced from propostion 3.3.2 since we have $\|\boldsymbol{e}(t)\|_0 \leq |\boldsymbol{e}(t)|_h$.*

3.4 Numerical experiments

In this section, we will present numerical results to confirm the theoretically established error bounds in proposition 3.3.2. To fully discretize our equation in time, we use the second order Newmark scheme in time, see for example [RT92]. In our example, we choose the DG stabilization parameter $\alpha = 10$.

3.4.1 Time discretization

We discretize (2.1) in space by the DG method (3.1) and we get linear second order system of ordinary differential equations as follows:

$$\mathbf{M}\partial_{tt}^2 \boldsymbol{u}_h(t) + \mathbf{A}\boldsymbol{u}_h(t) = \boldsymbol{f}_h(t), \quad t \in I, \tag{3.11}$$

with initial conditions

$$\mathbf{M}\boldsymbol{u}_h(0) = \boldsymbol{u}_h^0, \quad \mathbf{M}\partial_t \boldsymbol{u}_h(0) = \boldsymbol{u}_h^1. \tag{3.12}$$

Here, $\mathbf{M}$ *denote the mass matrix and* $\mathbf{A}$ *the stiffness matrix. To discretize (3.1) in time, we make use of the Newmark time stepping scheme; see, e.g. [RT92]. Let* k *denotes the time step and set* $t_n = n \cdot k$*. Then the Newmark method consists in finding approximation* $\{\boldsymbol{u}_h^n\}_n$ *to* $\boldsymbol{u}_h(t_n)$ *such that*

$$(\mathbf{M} + k^2\beta\mathbf{A})\boldsymbol{u}_h^1 = \left[\mathbf{M} - k^2(\frac{1}{2} - \beta)\mathbf{A}\right]\boldsymbol{u}_h^0 + k\mathbf{M}\boldsymbol{u}_h^1 + k^2\left[\beta\boldsymbol{f}_h^1 + (\frac{1}{2} - \beta)\boldsymbol{f}_h^0\right] \tag{3.13}$$

and

$$\begin{aligned}(\mathbf{M} + k^2\beta\mathbf{A})\boldsymbol{u}_h^{n+1} = \;& \left[2\mathbf{M} - k^2(\frac{1}{2} - 2\beta + \gamma)\mathbf{A}\right]\boldsymbol{u}_h^n \\ & - \left[\mathbf{M} + k^2(\frac{1}{2} + \beta - \gamma)\mathbf{A}\right]\boldsymbol{u}_h^{n-1} \\ & + k^2\left[\beta\boldsymbol{f}_h^{n+1} + (\tfrac{1}{2} - 2\beta + \gamma)\boldsymbol{f}_h^n + (\tfrac{1}{2} - \beta + \gamma)\boldsymbol{f}_h^{n-1}\right],\end{aligned} \tag{3.14}$$

for $n = 1,\ 2,\ \cdots\ N-1$*. Here* $\boldsymbol{f}_h^n := \boldsymbol{f}_h(t_n)$*, while* $\beta \geq 0$ *and* $\gamma \geq \frac{1}{2}$ *are free parameters that still can be chosen. We recall that for* $\gamma = \frac{1}{2}$ *the Newmark scheme is second order accurate in time, whereas it is only first order accurate for* $\gamma > \frac{1}{2}$*. For* $\beta = 0$*, the Newmark scheme (3.13)-(3.14) requires at each time step the solution of a linear system with the matrix* $\mathbf{M}$*. However, because individual elements decouples,* $\mathbf{M}$ *is a bloc diagonal with a bloc size equal to the number of degrees of freedom per element. It can be inverted at very low computational cost and the scheme is essentially explicit. In fact, if the basis functions are chosen mutually orthoghonal,* $\mathbf{M}$ *reduces to the identity; see [CKS00] and the references therein. Then, with* $\gamma = \frac{1}{2}$*, the explicit Newmark*

method corresponds to the standart leap-frog scheme.
For $\beta > 0$, the resulting scheme is implicit and involves the solution of a linear system with the symmetric positive definite stiffness matrix $\mathbf{A}$ at each time step. We finally note that the second order Newmark scheme with $\gamma = \frac{1}{2}$ is unconditionally stable for $\beta \geq \frac{1}{4}$, whereas for $\frac{1}{4} > \beta \geq 0$ the time step k has to be restricted by a CFL condition. In the case $\beta = 0$ the condition is $k^2 \lambda_{\max}(\mathbf{A}) \leq 4(1-\varepsilon)$, $\varepsilon \in (0,1)$, where $\lambda_{\max}(\mathbf{A})$ is the maximal eignenvalue of the DG stiffness matrix $\mathbf{A}$.
In our test, we will employ the implicit second order Newmark scheme, setting $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{2}$ in (3.13)-(3.14).

3.4.2 Example

We consider the three dimensional equation (2.1) in $\Omega \times I := (0,1)^3 \times (0,1)$ and data $\boldsymbol{f}$, $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ chosen such that the analytical solution is given by

$$\boldsymbol{u}(x,y,z,t) = \begin{pmatrix} \sin(t(y^2-y)(z^2-z)) \\ \sin(t(x^2-x)(z^2-z)) \\ \sin(t(x^2-x)(y^2-y)) \end{pmatrix}. \tag{3.15}$$

This solution is arbitrarily smooth so that our theoretical assumptions are satisfied. We discretize this problem using the polynomial spaces $\mathcal{P}^p(\kappa)^3$, $p = 1, 2$, on a sequence $\mathscr{T}_h$ of tetrahedral meshes. With decreasing meshsize h, smaller time step k is not necessary, because the scheme is unconditionally stable. In table Tab. 3.1, we show the relative errors at time $T = 1$ in the energy norm, as we decrease h. In (Fig 1) and (Fig 2) we see that the decrease of the energy norm as a function of the meshsize h is of order one for $p = 1$ and of order two for $p = 2$. Then the numerical results corroborate with the expected theoretical rates of $O(h^p)$ as we decrease the meshsize.

h	**Nbte**	**Nbtr**	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h,\ p=1$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h,\ p=2$
0.4367	12	30	0.2781E+00	0.1516E+00
0.2184	96	216	0.1867E+00	0.4129E-01
0.1733	192	432	0.1485E+00	0.2140E-01
0.1694	371	826	0.1122E+00	0.1935E-01
0.1379	660	1416	0.1002E+00	0.1176E-01
9.268E-02	2631	5502	0.6927E-01	0.4619E-02
7.703E-02	4682	9793	0.5584E-01	0.3152E-02

Table 3.1: Table of errors in the energy norm.

Here **Nbte** *is the number of tetrahedra on* Ω *and* **Nbtr** *is the number of triangles on the set of* $\mathscr{F}_h$*. We also give the errors of* $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$ *as a function of* h *in the following table.*

h	**Nbte**	**Nbtr**	$\|u-\boldsymbol{u}_h\|_{0,\Omega},\ p=1$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega},\ p=2$
0.4367	12	30	0.48550E-01	0.2109E-01
0.2184	96	216	0.2500E-01	0.2540E-02
0.1733	192	432	0.1513E-01	0.8845E-03
0.1694	371	826	0.1035E-01	0.8023E-03
0.1379	660	1416	0.8141E-02	0.3891E-03
9.26E-02	2631	5502	0.3868E-02	0.9734E-04
7.703E-02	4682	9793	0.2552E-02	0.5080E-04

Table 3.2: Table of errors in the $L^2(\Omega)$ norm.

Fig 1: Errors in the energy norm at time $T = 1$ for $p = 1$

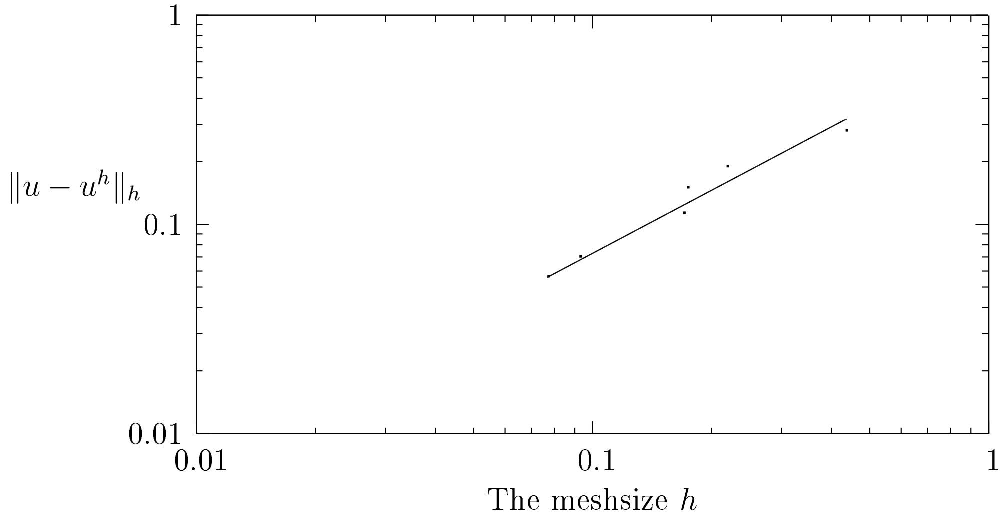
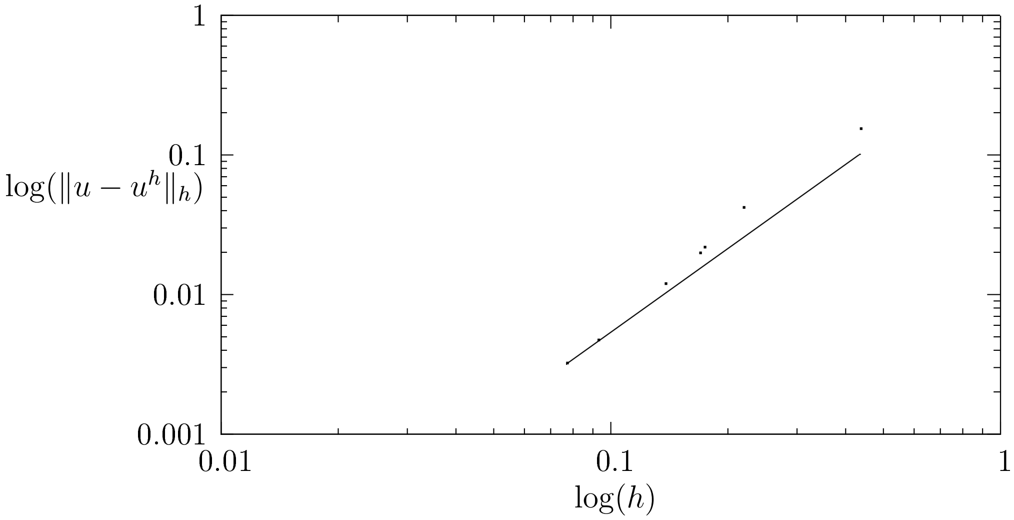


Fig 2: Error in the energy norm at $T = 1$ for $p = 2$



3.5 Conclusion

In this chapter, we have derived hp a priori error estimates for the DG formulation proposed in previous chapter. The hp−error estimates obtained are optimal in the mesh size and sub-optimal in the approximation degree. The numerical results are given to confirm the convergence rates as a function of the mesh size.

Chapter 4

A Posteriori Error estimation for Semi-discrete case

Contents

4.1 Introduction

In this chapter, we derive a posteriori error bound for the error $\|u - u_h\|_{L^\infty(0,T,L^2(\Omega)^3)}$ *between the exact solution of (2.1) and that of the semi-discrete scheme (2.8). The technique to derive these a posteriori error bounds is based on construction of an intermediate variable* $\boldsymbol{w}$*, which we refer as time-harmonic reconstruction of approximate solution* $\boldsymbol{u_h}$*, and then we approximate the error* $\boldsymbol{u} - \boldsymbol{u_h}$ *by means of error* $\boldsymbol{w} - \boldsymbol{u_h}$*. The time-harmonic reconstruction* $\boldsymbol{w}$ *is an auxiliary variable and is used as analysis tool, we do not need it in practical computation. The idea of this reconstruction follows from [GLV11] for parabolic problem, and [GLM13] for wave equation. This*

method allows us to estimate the a posteriori error of time dependent problem through the a posteriori error bounds of time-harmonic problem.
The outline of this chapter is as follows: in the first section we give the definition of time-harmonic reconstruction. In second section, we derive an error relation. In the third section, we obtain the abstract a posteriori error estimates using the error relation derived in previous section. In the forth section, we derive the a posteriori error estimates for the time-harmonic Maxwell's equations and we conclude the result using the estimates derived in the previous section.

4.2 Time-harmonic reconstruction

Definition 4.2.1. *Let $\boldsymbol{u_h}$ be the (semidiscrete) solution to the problem (2.8). Let also $\Pi_h : L^2(\Omega) \to V_h$ be the orthognal L^2-projection operator onto the finite element space V_h. We define the time-harmonic reconstruction, $\boldsymbol{w} = \boldsymbol{w}(t) \in \boldsymbol{X}(\Omega) = H_0(\mathrm{curl}, \Omega) \cap \mathrm{H}(\mathrm{div}, \Omega), \mathrm{t} \in \mathrm{I}$, of $\boldsymbol{u_h}$ to be the solution of the problem*

$$b(\boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v})_\Omega, \quad \forall \boldsymbol{v} \in \boldsymbol{X}(\Omega), \tag{4.1}$$

where

$$\boldsymbol{g} = A\boldsymbol{u_h} - \Pi_h \boldsymbol{f} + \boldsymbol{f}. \tag{4.2}$$

and $A : V_h \to V_h$ is the discrete operator defined by

$$(A\boldsymbol{q}, \boldsymbol{\chi})_\Omega = b_h(\boldsymbol{q}, \boldsymbol{\chi}), \text{ for all } \boldsymbol{q}, \boldsymbol{\chi} \in V_h.$$

The time-harmonic reconstruction is well defined. Indeed $A\boldsymbol{u_h} \in V_h$ is the unique L_2-Riesz representation of a linear functional on the finite-dimensional space V_h and the existence and uniqueness of (weak)solution of (4.1), with data $A\boldsymbol{u_h} - \Pi_h \boldsymbol{f} + \boldsymbol{f} \in L_2(\Omega)^d$ follows from the Lax-Milgram theorem.

4.2.1 Error splitting

We decompose the error as

$$\boldsymbol{e} := \boldsymbol{u}_h - \boldsymbol{u} = \boldsymbol{\rho} - \boldsymbol{\epsilon}, \tag{4.3}$$

where $\boldsymbol{\epsilon} := \boldsymbol{w} - \boldsymbol{u}_h$ and $\boldsymbol{\rho} := \boldsymbol{w} - \boldsymbol{u}$.

Remark 4.2.1. *(The role of $\boldsymbol{w}$) The DG solution $\boldsymbol{u_h}$ of the semi-discrete time dependent problem (2.8) is also the DG solution of the time-harmonic* *boundary-value problem (4.1). Indeed, let $\boldsymbol{w_h} \in V_h$ be the DG-approximation to $\boldsymbol{w}$, defined by the finite-dimensional linear system*

$$b_h(\boldsymbol{w_h}, \boldsymbol{v}) = (A\boldsymbol{u_h} - \Pi_h \boldsymbol{f} + \boldsymbol{f}, \boldsymbol{v})_\Omega, \tag{4.4}$$

for all $\boldsymbol{v} \in V_h$, which implies

$$b_h(\boldsymbol{w_h}, \boldsymbol{v}) = (A\boldsymbol{u_h}, \boldsymbol{v})_\Omega = b_h(\boldsymbol{u_h}, \boldsymbol{v}), \ \boldsymbol{v} \in V_h$$

i.e., $\boldsymbol{w_h} = \boldsymbol{u_h}$. Thus, by construction $\boldsymbol{w}$ is the exact solution of the time-harmonic Maxwell's equations with Dirichlet boundary

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \boldsymbol{w}) &= \boldsymbol{g}, \ \text{in } \Omega \\ \nabla \cdot \boldsymbol{w} &= 0, \ \text{in } \Omega \\ \boldsymbol{n} \times \boldsymbol{w} &= 0, \ \text{on } \partial\Omega \end{aligned} \tag{4.5}$$

with data $\boldsymbol{g}$ defined by (4.2) and the DG approximation of this problem is $\boldsymbol{u}_h$.

4.3 Error relation

Lemma 4.3.1. *With reference to the notation in (4.2.1), we have*

$$(\partial^2_{tt} \boldsymbol{e}, \boldsymbol{v})_\Omega + b(\boldsymbol{\rho}, \boldsymbol{v}) = \boldsymbol{0}, \ \text{for all } \boldsymbol{v} \in \boldsymbol{X}(\Omega). \tag{4.6}$$

Proof. We have

$$(\partial^2_{tt} \boldsymbol{e}, \boldsymbol{v})_\Omega + b(\boldsymbol{\rho}, \boldsymbol{v}) = (\partial^2_{tt} \boldsymbol{u_h}, \boldsymbol{v})_\Omega + b(\boldsymbol{w}, \boldsymbol{v}) - (\partial^2_{tt} \boldsymbol{u}, \boldsymbol{v})_\Omega - b(\boldsymbol{u}, \boldsymbol{v}).$$

Where we have used the error splitting defined in (4.3), now by using (2.3), we obtain

$$\begin{aligned} &= (\partial^2_{tt} \boldsymbol{u_h}, \boldsymbol{v})_\Omega + b(\boldsymbol{w}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_\Omega, \\ &= (\partial^2_{tt} \boldsymbol{u_h}, \Pi_h \boldsymbol{v})_\Omega + b(\boldsymbol{w}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_\Omega, \\ &= -b_h(\boldsymbol{u_h}, \Pi_h \boldsymbol{v}) + b(\boldsymbol{w}, \boldsymbol{v}) + (\Pi_h \boldsymbol{f} - \boldsymbol{f}, \boldsymbol{v})_\Omega, \end{aligned}$$

by the properties of L^2-orthognal projection and the formulation (2.8).
Now observing the identity $b_h(\boldsymbol{u_h}, \Pi_h \boldsymbol{v}) - (\Pi_h \boldsymbol{f} - \boldsymbol{f}, \boldsymbol{v})_\Omega = b(\boldsymbol{w}, \boldsymbol{v})$, due to the construction of $\boldsymbol{w}$, we get

$$(\partial^2_{tt} \boldsymbol{e}, \boldsymbol{v})_\Omega + b(\boldsymbol{\rho}, \boldsymbol{v}) = \boldsymbol{0}.$$

□

4.4 Abstract a posteriori error bounds for the semi-discrete problem

Theorem 4.4.1. *(Abstract semidiscrete error bound) Let $\boldsymbol{u}$ be the solution of equation (2.3) and $\boldsymbol{u}_h$ be the semi-discrete discontinuous Galerkin (DG) approximation of $\boldsymbol{u}$ obtained by (2.8), let $\boldsymbol{w}$ be the time-harmonic reconstruction of $\boldsymbol{u}_h$ as defined in definition (4.2.1), with the notations introduced in (4.3), the following error bound holds:*

$$\|\boldsymbol{e}\|_{L^\infty(0,T;L^2(\Omega)^d)} \leq \|\boldsymbol{\epsilon}\|_{L^\infty(0,T;L^2(\Omega)^d)} + \sqrt{2}(\|\boldsymbol{u}_0 - \boldsymbol{u_h}(0)\|_{0,\Omega} + \|\boldsymbol{\epsilon}(0)\|_{0,\Omega}) \\ +2\int_0^T \|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} + 2T\|\boldsymbol{u}_1 - \partial_t\boldsymbol{u_h}(0)\|_{0,\Omega}. \tag{4.7}$$

Proof. We use a testing procedure introduced by Baker [BA76]. Let $\hat{\boldsymbol{v}} : [0,T]\times\Omega \to \mathbb{R}^d$ with

$$\hat{\boldsymbol{v}}(t,\cdot) = \int_t^\tau \boldsymbol{\rho}(s,\cdot)ds,\ t \in [0,T], \tag{4.8}$$

for some fixed $\tau \in [0,T]$. Clearly $\hat{\boldsymbol{v}} \in \boldsymbol{X}(\Omega)$ as $\boldsymbol{\rho} \in \boldsymbol{X}(\Omega)$. It follows from definition of $\hat{\boldsymbol{v}}$ that

$$\hat{\boldsymbol{v}}(\tau,\cdot) = \boldsymbol{0}, \quad and \quad \partial_t\hat{\boldsymbol{v}}(t,\cdot) = -\boldsymbol{\rho}(t,\cdot) \quad a.e.\ \ in\ \ [0,T]. \tag{4.9}$$

Set $\boldsymbol{v} = \hat{\boldsymbol{v}}$ in (4.6), integrate between 0 and τ with respect to the variable t and integrate by parts the first term on the left-hand side to obtain

$$-\int_0^\tau (\partial_t\boldsymbol{e}, \partial_t\hat{\boldsymbol{v}})_\Omega + (\partial_t\boldsymbol{e}(\tau), \hat{\boldsymbol{v}}(\tau))_\Omega - (\partial_t\boldsymbol{e}(0), \hat{\boldsymbol{v}}(0))_\Omega + \int_0^\tau b(\boldsymbol{\rho}, \hat{\boldsymbol{v}}) = \boldsymbol{0}.$$

Using (4.9), we have

$$\int_0^\tau \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\rho}(t)\|^2_{0,\Omega}dt - \frac{1}{2}\int_0^\tau \frac{d}{dt}b(\hat{\boldsymbol{v}}(t), \hat{\boldsymbol{v}}(t))dt = \int_0^\tau (\partial_t\boldsymbol{\epsilon}, \boldsymbol{\rho})_\Omega + (\partial_t\boldsymbol{e}(0), \hat{\boldsymbol{v}}(0))_\Omega.$$

which implies

$$\frac{1}{2}\|\boldsymbol{\rho}(\tau)\|^2_{0,\Omega} - \frac{1}{2}\|\boldsymbol{\rho}(0)\|^2_{0,\Omega} + \frac{1}{2}b(\hat{\boldsymbol{v}}(0), \hat{\boldsymbol{v}}(0)) = \int_0^\tau (\partial_t\boldsymbol{\epsilon}, \boldsymbol{\rho})_\Omega + (\partial_t\boldsymbol{e}(0), \hat{\boldsymbol{v}}(0))_\Omega.$$

Hence, we deduce

$$\frac{1}{2}\|\boldsymbol{\rho}(\tau)\|^2_{0,\Omega} - \frac{1}{2}\|\boldsymbol{\rho}(0)\|^2_{0,\Omega} + \frac{1}{2}b(\hat{\boldsymbol{v}}(0), \hat{\boldsymbol{v}}(0)) \leq \max_{0\leq t\leq T}\|\boldsymbol{\rho}(t)\|_{0,\Omega}\int_0^\tau (\|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} \\ +\|\partial_t\boldsymbol{e}(0)\|_{0,\Omega}\|\hat{\boldsymbol{v}}(0)\|_{0,\Omega}). \tag{4.10}$$

Now, we select τ such that $\boldsymbol{\rho}(\tau) = \max_{0\le t\le T} \|\boldsymbol{\rho}(t)\|_{0,\Omega}$ (this is possible due to the continuity of $\boldsymbol{u}$ in the time variable under the data and domain regularity assumptions above see e.g. [RT92]. For this τ, we have $\|\hat{\boldsymbol{v}}(0)\|_{0,\Omega} \le \tau\|\boldsymbol{\rho}(\tau)\|_{0,\Omega}$, furthermore the positive semi-definiteness of the form b ensures that $b(\hat{\boldsymbol{v}}(0), \hat{\boldsymbol{v}}(0)) \ge 0$. This leads to the inequality,

$$\frac{1}{2}\|\boldsymbol{\rho}(\tau)\|^2_{0,\Omega} - \frac{1}{2}\|\boldsymbol{\rho}(0)\|^2_{0,\Omega} \le \|\boldsymbol{\rho}(\tau)\|_{0,\Omega}\left(\int_0^\tau \|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} + \tau\|\partial_t\boldsymbol{e}(0)\|_{0,\Omega}\right). \quad (4.11)$$

Now using the inequality $AB \le \frac{1}{4}A^2 + B^2$, with $A = \|\boldsymbol{\rho}(\tau)\|_{0,\Omega}$ and $B = \int_0^\tau \|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} + \tau\|\partial_t\boldsymbol{e}(0)\|_{0,\Omega}$, we get

$$\frac{1}{2}\|\boldsymbol{\rho}(\tau)\|^2_{0,\Omega} - \frac{1}{2}\|\boldsymbol{\rho}(0)\|^2_{0,\Omega} \le \frac{1}{4}\|\boldsymbol{\rho}(\tau)\|^2_{0,\Omega} + \left(\int_0^\tau \|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} + \tau\|\partial_t\boldsymbol{e}(0)\|_{0,\Omega}\right)^2,$$

which gives immediately

$$\frac{1}{4}\|\boldsymbol{\rho}(\tau)\|^2_{0,\Omega} - \frac{1}{2}\|\boldsymbol{\rho}(0)\|^2_{0,\Omega} \le \left(\int_0^\tau \|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} + \tau\|\partial_t\boldsymbol{e}(0)\|_{0,\Omega}\right)^2.$$

Using the bound $\|\boldsymbol{\rho}(0)\|_{0,\Omega} \le \|\boldsymbol{e}(0)\|_{0,\Omega} + \|\boldsymbol{\epsilon}(0)\|_{0,\Omega}$, $\boldsymbol{e}(0) = \boldsymbol{u_h}(0) - \boldsymbol{u}_0$, $\partial_t\boldsymbol{e}(0) = \partial_t\boldsymbol{u_h}(0) - \boldsymbol{u}_1$, and (4.11) for τ as above, we have

$$\|\boldsymbol{\rho}\|_{L^\infty(0,T;L^2(\Omega)^d)} \le \sqrt{2}(\|\boldsymbol{u}_0 - \boldsymbol{u_h}(0)\|_{0,\Omega} + \|\boldsymbol{\epsilon}(0)\|_{0,\Omega}) + 2\int_0^T \|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} + 2T\|\boldsymbol{u}_1 - \partial_t\boldsymbol{u_h}(0)\|_{0,\Omega}. \quad (4.12)$$

Now by using (4.3) and the above inequality (4.12), we get

$$\begin{aligned}\|\boldsymbol{e}\|_{L^\infty(0,T;L^2(\Omega)^d)} &\le \|\boldsymbol{\epsilon}\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|\boldsymbol{\rho}\|_{L^\infty(0,T;L^2(\Omega)^d)} \\ &\le \|\boldsymbol{\epsilon}\|_{L^\infty(0,T;L^2(\Omega)^d)} + \sqrt{2}(\|\boldsymbol{u}_0 - \boldsymbol{u_h}(0)\|_{0,\Omega} + \|\boldsymbol{\epsilon}(0)\|_{0,\Omega}) + 2\int_0^T \|\partial_t\boldsymbol{\epsilon}\|_{0,\Omega} \\ &\quad + 2T\|\boldsymbol{u}_1 - \partial_t\boldsymbol{u_h}(0)\|_{0,\Omega}.\end{aligned} \quad (4.13)$$

□

Remark 4.4.1. *The bound (4.7) is not yet explicitly a posteriori error bound: we still need to bound the norms involving the conforming error* $\boldsymbol{\epsilon} = \boldsymbol{w} - \boldsymbol{u_h}$ *by a computable quantity.*

Remark 4.4.2. *To this end, given* $\boldsymbol{q} \in L^2(\Omega)^3$, *consider the problem*
Find $\boldsymbol{z} \in \boldsymbol{X}(\Omega)$ *such that*

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \boldsymbol{z}) &= \boldsymbol{q}, \ \textit{in} \ \Omega, \\ \nabla \cdot \boldsymbol{z} &= 0 \ \textit{in} \ \Omega, \\ \boldsymbol{n} \times \boldsymbol{z} &= \boldsymbol{0} \ \textit{on} \ \partial\Omega, \end{aligned} \tag{4.14}$$

whose solution can be approximated by the following DG method:

find $\boldsymbol{z_h} \in V_h$

$$b_h(\boldsymbol{z_h}, \boldsymbol{v}) = (\boldsymbol{q}, \boldsymbol{v})_\Omega, \quad \forall \boldsymbol{v} \in V_h. \tag{4.15}$$

We assume that an a posteriori estimator functional $\mathscr{E}$ *exists, i.e.,*

$$\|\boldsymbol{z} - \boldsymbol{z_h}\|_{DG}^2 \leq C_0 \mathscr{E}(\boldsymbol{z_h}, \boldsymbol{q}, \mathscr{T}_h), \tag{4.16}$$

where $\|.\|_{DG}$ *is the corresponding energy norm in* V_h *and* C_0 *is a constant independent of* $\mathscr{T}_h$, $\boldsymbol{z}$, $\boldsymbol{z_h}$ *and mesh size.*

Now recalling the fact that $\boldsymbol{u_h}$*, which is the semi discrete DG approximation of (2.3) is also the DG approximation of the* $\boldsymbol{w}$ *satisfying (4.1) with data* $\boldsymbol{g} = A\boldsymbol{u_h} - \Pi_h \boldsymbol{f} + \boldsymbol{f}$ *as described in remark (4.2.1).*

Thus, if we have an a posteriori error estimator functional, which bounds the error between the exact solution $\boldsymbol{w}$ *of (4.1) and its DG approximation, as we see in above remark (4.4.2). Then the terms involving norms of* $\boldsymbol{\epsilon}$ *in (4.7) can be bounded as follows:*

Proposition 4.4.1. *Let* $\mathscr{E}$ *be an a posteriori error estimator functional which bounds the error between the exact and DG approximation of (4.1), with data* $\boldsymbol{g} = A\boldsymbol{u_h} - \Pi_h \boldsymbol{f} + \boldsymbol{f}$ *, and assume that* $\boldsymbol{f}$ *be differentiable in time then the terms in (4.7) can be computably bounded as:*

$$\boxed{\begin{aligned} \|\boldsymbol{\epsilon}\|_{L^\infty(0,T;L^2(\Omega)^d)} &\leq \|\mathscr{E}(\boldsymbol{u}_h, \boldsymbol{g}, \mathscr{T}_h)\|_{L^\infty(0,T)}, \\ \sqrt{2}\|\boldsymbol{\epsilon}(0)\|_{0,\Omega} &\leq \sqrt{2}\mathscr{E}(\boldsymbol{u}_h(0), \boldsymbol{g}(0), \mathscr{T}_h), \\ 2\int_0^T \|\partial_t \boldsymbol{\epsilon}\|_{0,\Omega} &\leq 2\int_0^T \mathscr{E}(\partial_t \boldsymbol{u}_h, \partial_t \boldsymbol{g}, \mathscr{T}_h). \end{aligned}} \tag{4.17}$$

In the following section we will derive the functional of type $\mathscr{E}$ *for time-harmonic Maxwell's problem.*

4.5 A posteriori residual bounds for the time-harmonic Maxwell problem

In this section, we present the method to derive a posteriori bounds in L^2*-norm of the error for the time-harmonic Maxwell's problem. We will extend* *the DG formulation with lifting operator to a larger finite element space to derive the error estimate in energy norm, which will give the L^2 norm of the error as well. This technique is widely used for error analysis in finite element method see e.g. [GSS06] for a priori error and [HPD07] for a posteriori error estimates.*

4.5.1 A posteriori error bounds in L^2-norm via energy norm

Now to establish our a posteriori error bounds, we define a larger finite element space $V(h) = \boldsymbol{X}(\Omega) + V_h$.
On $V(h)$ we define the DG energy norm by

$$|||\boldsymbol{v}|||^2 = \|\boldsymbol{v}\|_0^2 + |\boldsymbol{v}|_h^2. \tag{4.18}$$

with $|.|_h$ defined in (2.11). We will augment the bilinear form b_h in (2.9) to $V(h) \times V(h)$ in a non-consistent manner following [HPD07] by using the lifting operator. To this end, we give the definition and derive the bounds related to lifting operator in the following section.

4.5.2 Lifting operator

Definition 4.5.1. *$\forall \boldsymbol{v} \in V(h)$, the lifting operator $\mathscr{L}(\boldsymbol{v}) \in V_h$ is defined by*

$$\int_\Omega \mathscr{L}(\boldsymbol{v}).\boldsymbol{w} dx = \sum_{e \in \mathscr{F}_h} \int_e [\![\boldsymbol{v}]\!]_T.\{\!\{\mu^{-1}\boldsymbol{w}\}\!\} ds, \quad \forall \boldsymbol{w} \in V_h. \tag{4.19}$$

The operator $\mathscr{L}$ is well defined. We notice that the term on the right-hand-side of equation (4.19) is a linear operator over V_h, for each $\boldsymbol{v} \in V(h)$. Hence, $\mathscr{L}(\boldsymbol{v}) \in V_h$ is the representation of this linear operator obtained from Riesz representation theorem under L^2 scalar product in V_h, cf [TL00].

Lemma 4.5.1. *(Stability of lifting operator) Let $\mathscr{L}$ be the lifting operator defined by (4.19) and the stabilisation parameter a defined by (2.10), then under the assumptions on μ, we have for $\boldsymbol{v} \in V(h)$*

$$\|\mathscr{L}(\boldsymbol{v})\|_{0,\Omega}^2 \leq \alpha^{-1}(\mu_\star^{-1})^2 C_{\text{inv}}^2 \sum_{e \in \mathscr{F}} \|a^{\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\|_{0,e}^2. \tag{4.20}$$

Where the constant $C_{\text{inv}} > 0$ only depends on the shape regularity of the mesh and the approximation degree l.

Proof. Let $\boldsymbol{v} \in V(h)$, we have

$$\|\mu\mathscr{L}(\boldsymbol{v})\|_0 = \sup_{\boldsymbol{w}\in V_h} \frac{\int_\Omega \mathscr{L}(\boldsymbol{v}).\mu\boldsymbol{w}\mathrm{dx}}{\|\boldsymbol{w}\|_0}, \tag{4.21}$$

$$= \sup_{\boldsymbol{w}\in V_h} \frac{\sum_{e\in\mathscr{F}_h}\int_e [\![\boldsymbol{v}]\!]_T \{\!\{\mu^{-1}\boldsymbol{w}\}\!\}}{\|\boldsymbol{w}\|_0}, \tag{4.22}$$

$$\le \sup_{\boldsymbol{w}\in V_h} \frac{\left(\sum_{e\in\mathscr{F}_h}\int_e a[\![\boldsymbol{v}]\!]_T^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathscr{F}_h}\int_e a^{-1}|\{\!\{\mu^{-1}\boldsymbol{w}\}\!\}|^2\right)^{\frac{1}{2}}}{\|\boldsymbol{w}\|_0}, \tag{4.23}$$

$$\le \alpha^{-\frac{1}{2}}\mu_\star^{-1} \sup_{\boldsymbol{w}\in V_h} \frac{\left(\sum_{e\in\mathscr{F}_h}\int_e a[\![\boldsymbol{v}]\!]_T^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathscr{F}_h}\int_e h|\{\!\{\boldsymbol{w}\}\!\}|^2\right)^{\frac{1}{2}}}{\|\boldsymbol{w}\|_0}, \tag{4.24}$$

$$\le \alpha^{-\frac{1}{2}}\mu_\star^{-1} \sup_{\boldsymbol{w}\in V_h} \frac{\left(\sum_{e\in\mathscr{F}_h}\int_e a[\![\boldsymbol{v}]\!]_T^2\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathscr{T}_h}\int_{\partial\kappa} h_\kappa|\boldsymbol{w}|^2\right)^{\frac{1}{2}}}{\|\boldsymbol{w}\|_0}. \tag{4.25}$$

$$\tag{4.26}$$

Where we have used Cauchy-Schwarz inequality, and definition of a in (2.10). Now recalling the inverse inequality (1.10)

$$\|\boldsymbol{w}\|_{0,\partial\kappa} \le C_{\mathrm{inv}} h_\kappa^{-1/2}\|\boldsymbol{w}\|_{0,\kappa}, \text{ for all } \boldsymbol{w} \in (\mathcal{P}^l(\kappa))^3, \tag{4.27}$$

with a constant C_{inv} that depends only on the shape-regularity of the mesh, the approximation order l, and the dimension d.
Using this inverse inequality in above bound we get (4.20). □

We now introduce the non-consistent bilinear form

$$\begin{aligned}\tilde{b}_h(\boldsymbol{u},\boldsymbol{v}) = &\sum_{\kappa\in\mathscr{T}_h}\int_\kappa \mu^{-1}\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v}\mathrm{dx} + \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \nabla\cdot\boldsymbol{u}\nabla\cdot\boldsymbol{v}\mathrm{dx}\\ &- \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \mathscr{L}(\boldsymbol{u})\cdot(\nabla\times\boldsymbol{v})\mathrm{dx} - \sum_{\kappa\in\mathscr{T}_h}\int_\kappa \mathscr{L}(\boldsymbol{v})\cdot(\nabla\times\boldsymbol{u})\mathrm{dx}\\ &+ \sum_{e\in\mathscr{F}_h}\int_e a[\![\boldsymbol{u}]\!]_T\cdot[\![\boldsymbol{v}]\!]_T\mathrm{ds} + \sum_{e\in\mathscr{F}_h^I}\int_e a[\![\boldsymbol{u}]\!]_N\cdot[\![\boldsymbol{v}]\!]_N\mathrm{ds};\end{aligned} \tag{4.28}$$

$\tilde{b}_h = b_h$ on $V_h \times V_h$, and $\tilde{b}_h = b$ on $\boldsymbol{X}(\Omega)\times\boldsymbol{X}(\Omega)$, the bilinear form $\tilde{b}_h$ can be viewed as an extension of the two bilinear forms b_h and b to the space $V(h)\times V(h)$.

The semi discrete problem (2.8) in space can be written now as follows:

find $\boldsymbol{u}_h \in C^0(0,T;V_h)$ *such that*

$$(\partial^2_{tt}\boldsymbol{u}_h, \boldsymbol{v})_\Omega + \tilde{b}_h(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall \boldsymbol{v} \in V_h, t \in [0,T]. \tag{4.29}$$

Now we show the continuity and coercivity of $\tilde{b}_h$ *on* $V(h)$.

Lemma 4.5.2. *Let* α *be defined as in (2.10) and* $\alpha_0 > 0$ *be a constant, then there exists constants* $C_{\text{cont}} > 0$ *and* $C_{\text{coer}} > 0$, *independent of mesh size, such that* $\forall \alpha \geq \alpha_0$

$$|\tilde{b}_h(\boldsymbol{u}, \boldsymbol{v})| \leq C_{\text{cont}} |\boldsymbol{u}|_h |\boldsymbol{v}|_h, \tag{4.30}$$

$$|\tilde{b}_h(\boldsymbol{v}, \boldsymbol{v})| \geq C_{\text{coer}} |\boldsymbol{v}|_h^2, \tag{4.31}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V(h)$

Proof. The application of the Cauchy-Schwarz's inequality and the result in (4.20), readily gives in general case

$$|\tilde{b}_h(\boldsymbol{u}, \boldsymbol{v})| \leq \max\{(1 + \mu_\star^{-1}), ((\mu_\star^{-1})^2 \alpha^{-1} C_{\text{inv}}^2 + 1)\} |\boldsymbol{u}|_h |\boldsymbol{v}|_h.$$

Set $\alpha_0 = 4C_{\text{inv}}^2 (\mu_\star^{-1})^2$,
For $\alpha \geq \alpha_0$, the continuity of $\tilde{b}_h$ immediately follows. To show the coercivity of the form $\tilde{b}_h$, we note that

$$\begin{aligned} \tilde{b}_h(\boldsymbol{u}, \boldsymbol{u}) = &\sum_{\kappa \in \mathscr{T}_h} \|\mu^{-1} \nabla \times \boldsymbol{u}\|_{0,\kappa}^2 + \sum_{\kappa \in \mathscr{T}_h} \|\nabla \cdot \boldsymbol{u}\|_{0,\kappa}^2 \\ &- 2 \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \mathscr{L}(\boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u}) \mathrm{dx} + \sum_{e \in \mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{u}]\!]_T\|_{0,e}^2 + \sum_{e \in \mathscr{F}_h^I} \|a^{\frac{1}{2}} [\![\boldsymbol{u}]\!]_N\|_{0,e}^2. \end{aligned} \tag{4.32}$$

By using the weighted Cauchy-Schwarz's inequality, the geometric-arthmetic inequality $ab \leq \frac{\delta a^2}{2} + \frac{b^2}{2\delta}$, valid for any $\delta > 0$ and $\forall a, b \in \mathbb{R}$, on the third term of the above equation, we obtain

$$\begin{aligned} 2 \sum_{\kappa \in \mathscr{T}_h} \int_\kappa (\mathscr{L}(\boldsymbol{u})) \cdot (\nabla \times \boldsymbol{u}) \mathrm{dx} &\leq 2 \sum_{\kappa \in \mathscr{T}_h} \|\nabla \times \boldsymbol{u}\|_{0,\kappa} \|\mathscr{L}(\boldsymbol{u})\|_{0,\kappa}, \\ &\leq \delta \sum_{\kappa \in \mathscr{T}_h} \|\nabla \times \boldsymbol{u}\|_{0,\kappa}^2 + \delta^{-1} \sum_{\kappa \in \mathscr{T}_h} \|\mathscr{L}(\boldsymbol{u})\|_{0,\kappa}^2, \end{aligned}$$

Using the stability bound for the lifting operator in (4.20), we get

$$\leq \delta \sum_{\kappa \in \mathscr{T}_h} \|\nabla \times \boldsymbol{u}\|_{0,\kappa}^2 + \delta^{-1} \alpha^{-1} (\mu_\star^{-1})^2 C_{\text{inv}}^2 \sum_{e \in \mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{u}]\!]_T\|_{0,e}^2,$$

for a parameter $\delta > 0$, we conclude that

$$\begin{aligned}\tilde{b}_h(\boldsymbol{u},\boldsymbol{u}) &\geq ((\mu^\star)^{-1} - \delta) \sum_{\kappa \in \mathscr{T}_h} \|\nabla \times \boldsymbol{u}\|_{0,\kappa}^2 + \sum_{\kappa \in \mathscr{T}_h} \|\nabla \cdot \boldsymbol{u}\|_{0,\kappa}^2 \\ &+ (1 - \delta^{-1}(\mu_\star^{-1})^2 C_{\text{inv}}^2 \alpha^{-1}) \sum_{e \in \mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{u}]\!]_T\|_{0,e}^2 + \sum_{e \in \mathscr{F}_h^I} \|a^{\frac{1}{2}} [\![\boldsymbol{u}]\!]_N\|_{0,e}^2. \end{aligned} \tag{4.33}$$

Setting $\delta = (\mu^\star)^{-1}/2$, and $\alpha_0 = 4C_{\text{inv}}^2(\mu_\star^{-1})^2\mu^\star$, for $\alpha \geq \alpha_0$, we obtain the desired coercivity bound. □

4.5.3 A posteriori error estimate

In this section, For simplicity we assume that

$$\mu \equiv 1, \tag{4.34}$$

for the case of μ piecewise constant, the proof follows analogously.
The proof of a posteriori error is based on decomposing the error in two parts. One part is conforming and the second part is non-conforming. We first define the largest conforming space V_h^c underlying V_h that is,

$$V_h^c = V_h \cap \boldsymbol{X}(\Omega).$$

The space V_h^c is in fact Nédélec space of the second kind. The space V_h^c can be decomposed into

$$V_h^c = X_h \oplus \nabla S_h,$$

with the spaces X_h and S_h given by

$$S_h = \{q \in H_0^1(\Omega) : q|_\kappa \in \mathcal{P}^{l+1}(\kappa), \kappa \in \mathscr{T}_h\}, \tag{4.35}$$

$$X_h = \{\boldsymbol{v} \in V_h^c : (\boldsymbol{v}, \nabla q) = 0 \forall q \in S_h\}. \tag{4.36}$$

The space X_h is referred as the space of discrete divergence free functions. Let $\boldsymbol{z}$ be the exact solution to the time-harmonic Maxwell's equations (4.14) and $\boldsymbol{z_h}$ be the DG approxmation of $\boldsymbol{u}$ as in (4.15). We decompose the error $\boldsymbol{e}$ as

$$\boldsymbol{e} := \boldsymbol{z} - \boldsymbol{z_h} = \boldsymbol{z} - \boldsymbol{z}_{\boldsymbol{h}}^c + \boldsymbol{z}_{\boldsymbol{h}}^c - \boldsymbol{z}_h, \tag{4.37}$$

where $\boldsymbol{z}_{\boldsymbol{h}}^c \in V_h^c$, we denote

$$\boldsymbol{e}^c = \boldsymbol{z} - \boldsymbol{z}_{\boldsymbol{h}}^c, \ \boldsymbol{e}^d = \boldsymbol{z_h} - {\boldsymbol{z}_h}^c. \tag{4.38}$$

We will use the following lemma in the proof of our a posteriori error bound. From this lemma we can prove the existance of a conforming finite element function and can bound the non-conforming part of error $\boldsymbol{e}^d$.

Lemma 4.5.3. *Suppose $\mathscr{T}_h$ is a regular mesh. Then, for any function $\boldsymbol{z_h} \in V_h$ there exists a function $\boldsymbol{z_h^c} \in V_h^c$ such that*

$$\begin{aligned} \|\boldsymbol{z_h} - \boldsymbol{z_h^c}\|_{0,\Omega}^2 &\leq C_1 \sum_{e\in\mathscr{F}_h} h_e \|[\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2, \\ \sum_{\kappa\in\mathscr{T}_h} \|\nabla\times(\boldsymbol{z_h} - \boldsymbol{z_h^c})\|_{0,\kappa}^2 &\leq C_1 \sum_{e\in\mathscr{F}_h} h_e^{-1} \|[\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2. \end{aligned} \tag{4.39}$$

Where $C_1 > 0$ is a constant depending on shape regularity of the mesh.

Proof. The proof follows from [HPSD05]. □

Theorem 4.5.1. *Let $\mathscr{T}_h$ be a regular mesh. Let $\boldsymbol{z}$ be the exact solution to the time-harmonic Maxwell's equations (4.14) and $\boldsymbol{z_h}$ be the DG approxmation of $\boldsymbol{z}$ as in (4.15). Under the assumption (4.34) on μ the following bound holds*

$$\|\boldsymbol{z} - \boldsymbol{z_h}\|_{0,\Omega} \leq \mathscr{E}(\boldsymbol{z_h}, \boldsymbol{q}, \mathscr{T}_h),$$

where

$$\begin{aligned} \mathscr{E}(\boldsymbol{z_h}, \boldsymbol{q}, \mathscr{T}_h) = C_{up}\Bigg(&\sum_{\kappa\in\mathscr{T}_h} \Big(h_\kappa^2 \|\boldsymbol{q} + \nabla\times(\nabla\times\boldsymbol{z_h}) \\ &+ \nabla(\nabla\cdot\boldsymbol{z}_h)\|_{0,\kappa}^2 + \|\boldsymbol{q} + \nabla(\nabla\cdot\boldsymbol{z}_h)\|_{0,\kappa}^2 + \sum_{e\in\partial\kappa\setminus\Gamma} \frac{1}{2} h_\kappa \|[\![\nabla\times\boldsymbol{z_h}]\!]_T\|_{0,e}^2 \\ &+ \sum_{e\in\partial\kappa\setminus\Gamma} \frac{1}{2} h_\kappa \|[\![\nabla\cdot\boldsymbol{z_h}]\!]_N\|_{0,e}^2\Big) + \sum_{e\in\mathscr{F}_h} \|a^{\frac{1}{2}}[\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2 + \sum_{e\in\mathscr{F}_h} h_e^{-1}\|[\![\nabla\cdot\boldsymbol{z}_h]\!]\|_{0,e}^2\Bigg)^{\frac{1}{2}} \\ &+ max\{1, C_{\text{cont}}\}\Bigg(\sum_{e\in\mathscr{F}_h} (1 + \alpha^{-1}C_1(1 + h_e^2))\|a^{\frac{1}{2}}[\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2 + \sum_{e\in\mathscr{F}_I} \|a^{\frac{1}{2}}[\![\boldsymbol{z_h}]\!]_N\|_{0,e}^2 \\ &+ \sum_{\kappa\in\mathscr{T}_h} \|\nabla\cdot\boldsymbol{z}_h\|_{0,\kappa}^2\Bigg)^{\frac{1}{2}}, \end{aligned} \tag{4.40}$$

where $C_{up} > 0$ is a constant independent of $\boldsymbol{z}$, $\boldsymbol{z_h}$ and $\mathscr{T}_h$. C_1 is the constant from Lemma 4.5.3 and C_{cont} is the continuity constant of bilinear form $\tilde{b}_h$.

To prove this theorem we will use the following two lemmas.

Lemma 4.5.4. *Under the notations introduced in (4.37)-(4.38), the following a posteriori bound holds*

$$\|\boldsymbol{z} - \boldsymbol{z_h}\|_{0,\Omega} \leq \|\boldsymbol{e^c}\|_{0,\Omega} + \alpha^{-\frac{1}{2}} C_1^{\frac{1}{2}} \Big(\sum_{e\in\mathscr{F}_h} h_e^2 \|a^{\frac{1}{2}}[\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2\Big)^{\frac{1}{2}}, \tag{4.41}$$

where $C_1 > 0$ is the constant from (4.39).

Proof. We decompose the error $\boldsymbol{e} = \boldsymbol{z} - \boldsymbol{z_h}$ as in (4.37). By triangular inequality, we have

$$\|\boldsymbol{z} - \boldsymbol{z_h}\|_{0,\Omega} \leq \|\boldsymbol{z} - \boldsymbol{z_h^c}\|_{0,\Omega} + \|\boldsymbol{z_h^c} - \boldsymbol{z_h}\|_{0,\Omega} = \|\boldsymbol{e}^c\|_{0,\Omega} + \|\boldsymbol{e}^d\|_{0,\Omega}. \tag{4.42}$$

Now using the bound in lemma 4.5.3, we have

$$\|\boldsymbol{e}^d\|^2_{0,\Omega} \leq \alpha^{-1} C_1 \sum_{e\in\mathscr{F}_h} h_e^2 \|a^{\frac{1}{2}} [\![\boldsymbol{e}^d]\!]_T\|^2_{0,e}. \tag{4.43}$$

Using the fact that $[\![\boldsymbol{e}^d]\!]_T = [\![\boldsymbol{z_h}]\!]_T$, we obtain

$$\|\boldsymbol{e}^d\|_{0,\Omega} \leq \alpha^{-\frac{1}{2}} C_1^{\frac{1}{2}} \big(\sum_{e\in\mathscr{F}_h} h_e^2 \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_T\|^2_{0,e} \big)^{\frac{1}{2}}. \tag{4.44}$$

Combining (4.44) and (4.42) we get the result. □

Now to proof theorem 4.5.1 we derive the upper bound for $\|\boldsymbol{e}^c\|_{0,\Omega}$

Lemma 4.5.5. *Under the notations introduced in (4.37)-(4.38), the following bound holds*

$$\begin{aligned}
\|\boldsymbol{e}^c\|_{0,\Omega} \leq |\boldsymbol{e}^c|_h \leq \sqrt{2} C_{de} max\{1, C max\{1, \alpha^{-\frac{1}{2}} C_{\text{inv}}^{\frac{1}{2}}\}, C_{\text{tr}} N_\partial^{\frac{1}{2}}\} \times \\
\Bigg(\sum_{\kappa\in\mathscr{T}_h} \Big(h_\kappa^2 \|\boldsymbol{q} + \nabla \times (\nabla \times \boldsymbol{z_h}) \\
+ \nabla(\nabla \cdot \boldsymbol{z}_h)\|^2_{0,\kappa} + \|\boldsymbol{q} + \nabla(\nabla\cdot \boldsymbol{z}_h)\|^2_{0,\kappa} + \sum_{e\in\partial\kappa\setminus\Gamma} \frac{1}{2} h_\kappa \|[\![\nabla \times \boldsymbol{z_h}]\!]_T\|^2_{0,e} \\
+ \sum_{e\in\partial\kappa\setminus\Gamma} \frac{1}{2} h_\kappa \|[\![\nabla \cdot \boldsymbol{z_h}]\!]_N\|^2_{0,e} \Big) + \sum_{e\in\mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_T\|^2_{0,e} + \sum_{e\in\mathscr{F}_h} h_e^{-1} \|[\![\nabla\cdot\boldsymbol{z}_h]\!]\|^2_{0,e} \Bigg)^{\frac{1}{2}} \\
+ C_{\text{cont}} \Bigg((1+\alpha^{-1}C_1) \sum_{e\in\mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_T\|^2_{0,e} + \sum_{e\in\mathscr{F}_I} \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_N\|^2_{0,e} + \sum_{\kappa\in\mathscr{T}_h} \|\nabla\cdot\boldsymbol{z}_h\|^2_{0,\kappa} \Bigg)^{\frac{1}{2}}.
\end{aligned} \tag{4.45}$$

Proof. Clearly we have $\boldsymbol{e}^c \in \boldsymbol{X}(\Omega)$. Thus, we have

$$\tilde{b}_h(\boldsymbol{z}, \boldsymbol{e}^c) = b(\boldsymbol{z}, \boldsymbol{e}^c) = (\boldsymbol{q}, \boldsymbol{e}^c)_\Omega.$$

We decompose $\boldsymbol{e}^c$ by using regular decomposition from [CFL00], as follows

$$\boldsymbol{e}^c = \boldsymbol{\psi} + \nabla\gamma,$$

where $\boldsymbol{\psi} \in \boldsymbol{X}(\Omega) \cap H^1(\Omega)^3$ and $\gamma \in \{H_0^1(\Omega) : \Delta\gamma \in L^2(\Omega)\}$. Furthermore there exists a constant $C_{de} > 0$ such that

$$\begin{aligned} \|\boldsymbol{\psi}\|_{1,\Omega} &\leq C_{de}\|\boldsymbol{e}^c\|_{\boldsymbol{X}(\Omega)}, \\ \|\boldsymbol{\gamma}\|_{1,\Omega} &\leq C_{de}\|\boldsymbol{e}^c\|_{\boldsymbol{X}(\Omega)}, \end{aligned} \tag{4.46}$$

where the norm $\|.\|_{\boldsymbol{X}(\Omega)}$ is defined by (1.2). We choose $\boldsymbol{e}_h^c = \boldsymbol{\psi}_h + \nabla\gamma_h$, where $\boldsymbol{\psi}_h$ is the quasi interpolant of $\boldsymbol{\psi}$ defined in lemma 1.1.9 and γ_h is the Clément interpolation of γ in H_0^1 defined in lemma 1.1.8. In consequence of the properties in lemma 1.1.9 and lemma 1.1.8, we have the following bounds,

$$\sum_{\kappa\in\mathscr{T}_h}\left(\|\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\|_{0,\kappa}^2 + h_\kappa^{-2}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{0,\kappa}^2 + h_\kappa^{-1}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{0,\partial\kappa}^2\right) \leq C_1^2\|\boldsymbol{\psi}\|_{1,\Omega}^2, \tag{4.47}$$

$$\sum_{\kappa\in\mathscr{T}_h}\left(\|\nabla(\gamma-\gamma_h)\|_{0,\kappa}^2 + h_\kappa^{-2}\|\gamma-\gamma_h\|_{0,\kappa}^2 + h_\kappa^{-1}\|\gamma-\gamma_h\|_{0,\partial\kappa}^2\right) \leq C_2^2\|\gamma\|_{1,\Omega}^2, \tag{4.48}$$

where C_1 and C_2 are positive constants which only depends on shape regularity of the mesh. Now we have

$$\begin{aligned} \tilde{b}_h(\boldsymbol{e},\boldsymbol{e}^c) = \tilde{b}_h(\boldsymbol{z},\boldsymbol{e}^c) - \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{e}^c) &= (\boldsymbol{q},\boldsymbol{e}^c)_\Omega - \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{e}^c-\boldsymbol{e}_h^c) - \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{e}_h^c) \\ &= (\boldsymbol{q},\boldsymbol{e}^c-\boldsymbol{e}_h^c)_\Omega - \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{e}^c-\boldsymbol{e}_h^c), \end{aligned} \tag{4.49}$$

which implies

$$|\boldsymbol{e}^c|_h^2 = \tilde{b}_h(\boldsymbol{e}^c,\boldsymbol{e}^c) = (\boldsymbol{q},\boldsymbol{e}^c-\boldsymbol{e}_h^c)_\Omega - \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{e}^c-\boldsymbol{e}_h^c) - \tilde{b}_h(\boldsymbol{e}^d,\boldsymbol{e}^c).$$

Now we have

$$|\boldsymbol{e}^c|_h^2 \leq |(\boldsymbol{q},\boldsymbol{e}^c-\boldsymbol{e}_h^c)_\Omega - \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{e}^c-\boldsymbol{e}_h^c)| + |\tilde{b}_h(\boldsymbol{e}^d,\boldsymbol{e}^c)| = T_1 + T_2. \tag{4.50}$$

Now to bound T_1, we have

$$(\boldsymbol{q},\boldsymbol{e}^c-\boldsymbol{e}_h^c)_\Omega - \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{e}^c-\boldsymbol{e}_h^c) = T_{11} + T_{12},$$

where

$$T_{11} = \sum_{\kappa\in\mathscr{T}}\int_\kappa \boldsymbol{q}\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)dx + \tilde{b}_h(\boldsymbol{z_h},\boldsymbol{\psi}-\boldsymbol{\psi}_h),$$

$$T_{12} = \sum_{\kappa\in\mathscr{T}}\int_\kappa \boldsymbol{q}\cdot\nabla(\gamma-\gamma_h)\mathrm{dx} - \sum_{\kappa\in\mathscr{T}}\int_\kappa(\nabla\cdot\boldsymbol{z_h})\cdot\Delta(\gamma-\gamma_h)\mathrm{dx}.$$

Let us next bound T_{11} and T_{12},

$$T_{11} = \sum_{\kappa\in\mathscr{T}}\int_\kappa \boldsymbol{q}\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)dx - \sum_{\kappa\in\mathscr{T}}\int_\kappa(\nabla\times\boldsymbol{z_h})\cdot(\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h))\mathrm{dx}$$
$$-\sum_{\kappa\in\mathscr{T}}\int_\kappa(\nabla\cdot\boldsymbol{z_h})\cdot(\nabla\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h))\mathrm{dx} + \sum_{\kappa\in\mathscr{T}}\int_\kappa \mathscr{L}(\boldsymbol{z_h})\cdot(\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h)). \tag{4.51}$$

Integration by parts of second and third term on right hand side yields

$$(\boldsymbol{q},(\boldsymbol{\psi}-\boldsymbol{\psi}_h))_\Omega - \tilde{b}_h(\boldsymbol{z_h},(\boldsymbol{\psi}-\boldsymbol{\psi}_h)) = \sum_{\kappa\in\mathscr{T}}\int_\kappa\left(\boldsymbol{q}+\nabla\times(\nabla\times\boldsymbol{z_h})+\nabla(\nabla\cdot\boldsymbol{z}_h)\right)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\mathrm{dx}$$
$$-\sum_{\kappa\in\mathscr{T}_h}\int_{\partial\kappa}(\nabla\times\boldsymbol{z_h})\times\boldsymbol{n}_\kappa)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h) - \sum_{\kappa\in\mathscr{T}_h}\int_{\partial\kappa}(\nabla\cdot\boldsymbol{z_h})\cdot\boldsymbol{n}_\kappa)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)$$
$$+\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\mathscr{L}(\boldsymbol{z_h})\cdot(\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h))dx,$$
$$=\sum_{\kappa\in\mathscr{T}}\int_\kappa\left(\boldsymbol{q}+\nabla\times(\nabla\times\boldsymbol{z_h})+\nabla(\nabla\cdot\boldsymbol{z}_h)\right)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\mathrm{dx}$$
$$-\sum_{\kappa\in\mathscr{T}_h}\sum_{e\in\partial\kappa\setminus\Gamma}\frac{1}{2}\int_e[\![\nabla\times\boldsymbol{z_h}]\!]_T\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h) - \sum_{\kappa\in\mathscr{T}_h}\sum_{e\in\partial\kappa\setminus\Gamma}\frac{1}{2}\int_e[\![\nabla\cdot\boldsymbol{z_h}]\!]_N\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)$$
$$+\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\mathscr{L}(\boldsymbol{z_h})\cdot(\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h))dx, \tag{4.52}$$

here $\boldsymbol{n}_\kappa$ is the unit outward normal vector on $\partial\kappa$.

The terms on right hand side can be bounded as follows

$$\sum_{\kappa\in\mathscr{T}}\int_\kappa\left(\boldsymbol{q}+\nabla\times(\nabla\times\boldsymbol{z_h})+\nabla(\nabla\cdot\boldsymbol{z}_h)\right)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)dx \leq$$
$$\sum_{\kappa\in\mathscr{T}_h}h_\kappa\|\boldsymbol{q}+\nabla\times(\nabla\times\boldsymbol{z_h})+\nabla(\nabla\cdot\boldsymbol{z}_h)\|_{0,\kappa}h_\kappa^{-1}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{0,\kappa}. \tag{4.53}$$

$$\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\mathscr{L}(\boldsymbol{z_h})\cdot(\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h))dx \leq \left(\sum_{\kappa\in\mathscr{T}_h}\|\mathscr{L}(\boldsymbol{z_h})\|_{0,\kappa}^2\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathscr{T}_h}\|\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\|_{0,\kappa}^2\right)^{\frac{1}{2}}$$
$$\leq \alpha^{-\frac{1}{2}}C_{\mathrm{inv}}^{\frac{1}{2}}\left(\sum_{e\in\mathscr{F}_h}\|a^{\frac{1}{2}}[\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathscr{T}_h}\|\nabla\times(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\|_{0,\kappa}^2\right)^{\frac{1}{2}}. \tag{4.54}$$

Where we have used the stability of lifting operator (4.20).
Now to bound the second and third term on right hand side of (4.52), we apply the Cauchy-Schwarz inequality to get

$$\sum_{\kappa\in\mathscr{T}_h}\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}\int_e [\![\nabla\times\boldsymbol{z_h}]\!]_T\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\leq\sum_{\kappa\in\mathscr{T}_h}\left(\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}h_\kappa\|[\![\nabla\times\boldsymbol{z_h}]\!]_T\|_{0,e}^2\right)^{\frac{1}{2}}\times$$
$$\left(\sum_{e\in\partial\kappa}\frac{1}{2}h_\kappa^{-1}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{0,e}^2\right)^{\frac{1}{2}}$$
$$\leq\sum_{\kappa\in\mathscr{T}_h}\left(\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}h_\kappa\|[\![\nabla\times\boldsymbol{z_h}]\!]_T\|_{0,e}^2\right)^{\frac{1}{2}}h_\kappa^{-\frac{1}{2}}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{0,\partial\kappa}. \quad (4.55)$$

Similarly,

$$\sum_{\kappa\in\mathscr{T}_h}\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}\int_e [\![\nabla\cdot\boldsymbol{z_h}]\!]_N\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\leq\sum_{\kappa\in\mathscr{T}_h}\left(\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}h_\kappa\|[\![\nabla\cdot\boldsymbol{z_h}]\!]_N\|_{0,e}^2\right)^{\frac{1}{2}}\times$$
$$\left(\sum_{e\in\partial\kappa}\frac{1}{2}h_\kappa^{-1}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{0,e}^2\right)^{\frac{1}{2}}$$
$$\leq\sum_{\kappa\in\mathscr{T}_h}\left(\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}h_\kappa\|[\![\nabla\cdot\boldsymbol{z_h}]\!]_N\|_{0,e}^2\right)^{\frac{1}{2}}h_\kappa^{-\frac{1}{2}}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{0,\partial\kappa}. \quad (4.56)$$

Using Cauchy-Schwarz inequality, and the bounds in (4.53), (4.54), (4.55) and (4.56) we conclude that

$$T_{11}\leq Cmax\{1,\alpha^{-\frac{1}{2}}C_{\text{inv}}^{\frac{1}{2}}\}\left(\sum_{\kappa\in\mathscr{T}_h}\left(h_\kappa^2\|\boldsymbol{q}+\nabla\times(\nabla\times\boldsymbol{z_h})+\nabla(\nabla\cdot\boldsymbol{z}_h)\|_{0,\kappa}^2\right.\right.$$
$$\left.+\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}h_\kappa\|[\![\nabla\times\boldsymbol{z_h}]\!]_T\|_{0,e}^2+\sum_{e\in\partial\kappa\backslash\Gamma}\frac{1}{2}h_\kappa\|[\![\nabla\cdot\boldsymbol{z_h}]\!]_N\|_{0,e}^2\right)$$
$$\left.+\sum_{e\in\mathscr{F}_h}\|a^{\frac{1}{2}}[\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2\right)^{\frac{1}{2}}\|\boldsymbol{\psi}\|_{1,\Omega}. \quad (4.57)$$

Where we have used the bound (4.47).

Now to bound T_{12}, we integrated the second term to obtain

$$\begin{aligned}
&\sum_{\kappa\in\mathscr{T}}\int_\kappa \boldsymbol{q}\cdot\nabla(\gamma-\gamma_h)\mathrm{dx}-\sum_{\kappa\in\mathscr{T}}\int_\kappa(\nabla\cdot\boldsymbol{z}_h)\cdot\Delta(\gamma-\gamma_h))\mathrm{dx},\\
&=\sum_{\kappa\in\mathscr{T}}\Big(\int_\kappa(\boldsymbol{q}+\nabla(\nabla\cdot\boldsymbol{z}_h))\nabla(\gamma-\gamma_h))\mathrm{dx}-\int_{\partial\kappa}\nabla(\gamma-\gamma_h)\cdot\boldsymbol{n}_\kappa\nabla\cdot\boldsymbol{z}_h\mathrm{ds}\Big),\\
&=\sum_{\kappa\in\mathscr{T}}\int_\kappa(\boldsymbol{q}+\nabla(\nabla\cdot\boldsymbol{z}_h))\nabla(\gamma-\gamma_h)\mathrm{dx}-\sum_{e\in\mathscr{F}_h^I}\int_e[\![\nabla(\gamma-\gamma_h)\nabla\cdot\boldsymbol{z}_h]\!]\cdot\boldsymbol{n}_e\mathrm{ds}\\
&\qquad-\sum_{e\in\mathscr{F}_h^B}\int_e\nabla(\gamma-\gamma_h)\cdot\boldsymbol{n}_e\nabla\cdot\boldsymbol{z}_h\mathrm{ds},\\
&=\sum_{\kappa\in\mathscr{T}}\int_\kappa(\boldsymbol{q}+\nabla(\nabla\cdot\boldsymbol{z}_h))\nabla(\gamma-\gamma_h)\mathrm{dx}-\sum_{e\in\mathscr{F}_h}\int_e\{\!\{\nabla(\gamma-\gamma_h)\}\!\}\cdot\boldsymbol{n}_e[\![\nabla\cdot\boldsymbol{z}_h]\!]\mathrm{ds},\\
&=T_{21}+T_{22}.
\end{aligned}\tag{4.58}$$

T_{21} can be bounded by

$$\begin{aligned}
T_{21}&\leq\|\boldsymbol{q}+\nabla(\nabla\cdot\boldsymbol{z}_h)\|_{0,\Omega}\|\nabla(\gamma-\gamma_h)\|_{0,\Omega},\\
&\leq\|\boldsymbol{q}+\nabla(\nabla\cdot\boldsymbol{z}_h)\|_{0,\Omega}\|\gamma\|_{1,\Omega}.
\end{aligned}\tag{4.59}$$

For term T_{22}, using lemma 4.50 from [PE12], the trace inequality (1.9), and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
T_{22}&\leq\Big(\sum_{\kappa\in\mathscr{T}}\sum_{e\in\mathscr{F}_\kappa}h_e\|\nabla(\gamma-\gamma_h)|_\kappa.\boldsymbol{n}_e\|_{0,e}^2\Big)^{\frac12}\Big(\sum_{e\in\mathscr{F}_h}h_e^{-1}\|[\![\nabla\cdot\boldsymbol{z}_h]\!]\|_{0,e}^2\Big)^{\frac12},\\
&\leq\Big(\sum_{\kappa\in\mathscr{T}}h_\kappa\|\nabla(\gamma-\gamma_h)|_\kappa.\boldsymbol{n}_e\|_{0,\partial\kappa}^2\Big)^{\frac12}\Big(\sum_{e\in\mathscr{F}_h}h_e^{-1}\|[\![\nabla\cdot\boldsymbol{z}_h]\!]\|_{0,e}^2\Big)^{\frac12},\\
&\leq\Big(\sum_{\kappa\in\mathscr{T}}C_{\mathrm{tr}}^2\|\nabla(\gamma-\gamma_h)\|_{0,\kappa}^2\Big)^{\frac12}\Big(\sum_{e\in\mathscr{F}_h}h_e^{-1}\|[\![\nabla\cdot\boldsymbol{z}_h]\!]\|_{0,e}^2\Big)^{\frac12},\\
&\leq C_{\mathrm{tr}}\Big(\sum_{e\in\mathscr{F}_h}h_e^{-1}\|[\![\nabla\cdot\boldsymbol{z}_h]\!]\|_{0,e}^2\Big)^{\frac12}\|\gamma\|_{1,\Omega}.
\end{aligned}\tag{4.60}$$

Using Cauchy Schwarz inequality, the bounds in (4.59) and (4.60), we can bound T_{12} as follows

$$T_{12}\leq\Big(\sum_{\kappa\in\mathscr{T}}\|\boldsymbol{q}+\nabla(\nabla\cdot\boldsymbol{z}_h)\|_{0,\kappa}^2+C_{\mathrm{tr}}(\sum_{e\in\mathscr{F}_h}h_e^{-1}\|[\![\nabla\cdot\boldsymbol{z}_h]\!]\|_{0,e}^2\Big)^{\frac12}\|\gamma\|_{1,\Omega}.\tag{4.61}$$

We conclude from bounds (4.57) and (4.61) that

$$T_1 \leq max\{1, Cmax\{1, \alpha^{-\frac{1}{2}} C_{\text{inv}}^{\frac{1}{2}}\}, C_{\text{tr}}\} \times \Bigg(\sum_{\kappa \in \mathscr{T}_h} \Big(h_\kappa^2 \|\boldsymbol{q} + \nabla \times (\nabla \times \boldsymbol{z_h}) + \nabla(\nabla \cdot \boldsymbol{z}_h)\|_{0,\kappa}^2 + \|\boldsymbol{q} + \nabla(\nabla \cdot \boldsymbol{z}_h)\|_{0,\kappa}^2 + \sum_{e \in \partial\kappa \backslash \Gamma} \frac{1}{2} h_\kappa \|[\![\nabla \times \boldsymbol{z_h}]\!]_T\|_{0,e}^2 + \sum_{e \in \partial\kappa \backslash \Gamma} \frac{1}{2} h_\kappa \|[\![\nabla \cdot \boldsymbol{z_h}]\!]_N\|_{0,e}^2 \Big) + \sum_{e \in \mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2 + \sum_{e \in \mathscr{F}_h} h_e^{-1} \|[\![\nabla \cdot \boldsymbol{z}_h]\!]\|_{0,e}^2 \Bigg)^{\frac{1}{2}} \times (\|\boldsymbol{\psi}\|_{1,\Omega}^2 + \|\gamma\|_{1,\Omega}^2)^{\frac{1}{2}}. \tag{4.62}$$

Using the estimate in (4.46), we have

$$T_1 \leq \sqrt{2} C_{de} max\{1, Cmax\{1, \alpha^{-\frac{1}{2}} C_{\text{inv}}^{\frac{1}{2}}\}, C_{\text{tr}}\} \times \Bigg(\sum_{\kappa \in \mathscr{T}_h} \Big(h_\kappa^2 \|\boldsymbol{q} + \nabla \times (\nabla \times \boldsymbol{z_h}) + \nabla(\nabla \cdot \boldsymbol{z}_h)\|_{0,\kappa}^2 + \|\boldsymbol{q} + \nabla(\nabla \cdot \boldsymbol{z}_h)\|_{0,\kappa}^2 + \sum_{e \in \partial\kappa \backslash \Gamma} \frac{1}{2} h_\kappa \|[\![\nabla \times \boldsymbol{z_h}]\!]_T\|_{0,e}^2 + \sum_{e \in \partial\kappa \backslash \Gamma} \frac{1}{2} h_\kappa \|[\![\nabla \cdot \boldsymbol{z_h}]\!]_N\|_{0,e}^2 \Big) + \sum_{e \in \mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2 + \sum_{e \in \mathscr{F}_h} h_e^{-1} \|[\![\nabla \cdot \boldsymbol{z}_h]\!]\|_{0,e}^2 \Bigg)^{\frac{1}{2}} |\boldsymbol{e}^c|_h. \tag{4.63}$$

For bounding T_2, we use the continuity of $\tilde{b}_h$ and obtain

$$|\tilde{b}_h(\boldsymbol{e}^d, \boldsymbol{e}^c)| \leq C_{\text{cont}} |\boldsymbol{e}^d|_h |\boldsymbol{e}^c|_h.$$

Using the result in Lemma 4.5.3, we have

$$T_2 = |\tilde{b}_h(\boldsymbol{e}^d, \boldsymbol{e}^c)| \leq C_{\text{cont}} \Bigg((1 + \alpha^{-1} C_1) \sum_{e \in \mathscr{F}_h} \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_T\|_{0,e}^2 + \sum_{e \in \mathscr{F}_I} \|a^{\frac{1}{2}} [\![\boldsymbol{z_h}]\!]_N\|_{0,e}^2 + \sum_{\kappa \in \mathscr{T}_h} \|\nabla \cdot \boldsymbol{z}_h\|_{0,\kappa}^2 \Bigg)^{\frac{1}{2}} |\boldsymbol{e}^c|_h. \tag{4.64}$$

Combining the estimates in (4.63),(4.64) along with (4.50) , we get the bound for $|\boldsymbol{e}^c|_h$. Now using the proposition 2.2.1 we get the bound for $\|\boldsymbol{e}^c\|_{0,\Omega}$. $\square$

4.5.4 Proof of Theorem 4.5.1

Proof. Combining the result of lemma 4.5.4 and lemma 4.5.5, we get the desired result. □

Theorem 4.5.2. *(Semidiscrete residual-type a posteriori error bound). Assume that $\boldsymbol{f}$ is differentiable with respect to time and $\boldsymbol{g} = A\boldsymbol{u}_h - \Pi_h \boldsymbol{f} + \boldsymbol{f}$, then the following error bound holds:*

$$\|\boldsymbol{e}\|_{L^\infty(0,T;L^2(\Omega))} \leq C_0 \|\mathscr{E}(\boldsymbol{u}_h, \boldsymbol{g}, \mathscr{T}_h)\|_{L^\infty(0,T)} + 2C_0 \int_0^T \mathscr{E}(\partial_t \boldsymbol{u}_h, \partial_t \boldsymbol{g}, \mathscr{T}_h)$$

$$+\sqrt{2} C_0 \mathscr{E}(\boldsymbol{u}_h(0), \boldsymbol{g}(0), \mathscr{T}_h) + \sqrt{2}\|\boldsymbol{u}_0 - \boldsymbol{u}_h(0)\|_{0,\Omega} + 2T\|\boldsymbol{u}_1 - \partial_t \boldsymbol{u}_h(0)\|_{0,\Omega}.$$

where C_0 is a constant independent of $\boldsymbol{u}$, $\boldsymbol{u_h}$, and $\mathscr{T}_h$. And $\mathscr{E}$ is given by (4.40).

4.6 Conclusion

In this chapter, we derive a posteriori error estimates for semi-discrete DG formulation of Maxwell's problem by applying time-harmonic reconstruction technique, we control the error for the time-dependent Maxwell's problem via the error of the auxiliary time-harmonic Maxwell's problem.

Chapter 5

A posteriori error for fully discrete case

Contents

In this chapter, we derive a posteriori error estimates for fully discrete case. We bound the error $\|\boldsymbol{u}-\boldsymbol{u}_N\|_{L^\infty(0,T;L^2(\Omega)^d)}$ between the exact weak solution of (2.1) and its approximation $\boldsymbol{u}_N$, calculated from fully discrete solution $\boldsymbol{u}_h^n, n=0,\ldots,N$ in (2.24).
The outline of this chapter is as follows. We first define the space-time reconstruction for fully discrete scheme in section 5.1, which has a crucial zero-mean value *property in the time variable. This approach was studied by [GLM13] for the a posteriori error for conforming finite element approximation of the wave equation. We apply this technique for our DG formulation for Maxwell's time-dependent equations. In section 5.2, we derive the fully discrete error relation, and analogously to the semi-discrete case, we use the special testing procedure by [BA76] to derive the abstract fully a posteriori error bound in section 5.3.1. Section 5.4 is devoted to calculate explicitly the fully discrete a posteriori estimates, followed by the proof of the main theorem in section 5.5.*

5.1 The reconstructions in space and time

Analogues to time-harmonic reconstruction introduced in section 4.2, we will introduce here the time-harmonic reconstruction for fully discrete case. Note *that the time harmonic reconstruction here depends on n since finite element spaces change with time.*

Definition 5.1.1. *(The reconstruction in space) Let $\boldsymbol{u}_h^n, n = 0, \ldots, N$, be the fully discrete solution computed by the method (2.24), $\boldsymbol{\Pi}_h^n : L^2(\Omega)^d \to V_h^n$ be the orthogonal L^2-projection, and $A^n : V_h^n \to V_h^n$ to be the discrete operator defined by*

$$\text{for } \boldsymbol{\chi} \in V_h^n, (A^n \boldsymbol{\chi}, \boldsymbol{v})_\Omega = b_h^n(\boldsymbol{\chi}, \boldsymbol{v}), \ \forall \boldsymbol{v} \in V_h^n. \tag{5.1}$$

We define the time-harmonic reconstruction $\boldsymbol{w}^n \in \boldsymbol{X}(\Omega)$ of $\boldsymbol{u}_h^n$ to be the solution of the time-harmonic Maxwell problem

$$b_h^n(\boldsymbol{w}^n, \boldsymbol{v}) = (\boldsymbol{g}^n, \boldsymbol{v})_\Omega, \ \forall \boldsymbol{v} \in \boldsymbol{X}(\Omega),$$

with

$$\boldsymbol{g}^n = A^n \boldsymbol{u}_h^n - \boldsymbol{\Pi}_h^n \boldsymbol{f}^n + \overline{\boldsymbol{f}}^n. \tag{5.2}$$

where $\overline{\boldsymbol{f}}^0 = \boldsymbol{f}(0, \cdot)$, $\overline{\boldsymbol{f}}^n = k_n^{-1} \int_{t^{n-1}}^{t^n} \boldsymbol{f}(t, \cdot) dt$, for $n = 1, \ldots, N$. Finally, we define the time-harmonic reconstruction $\partial \boldsymbol{w}^0 \in \boldsymbol{X}(\Omega)$, of $\boldsymbol{u}_h^0$ to be the solution of the elliptic problem

$$b_h(\partial \boldsymbol{w}^0, \boldsymbol{v}) = (\partial \boldsymbol{g}^0, \boldsymbol{v})_\Omega, \ \forall \boldsymbol{v} \ \in \boldsymbol{X}(\Omega), \tag{5.3}$$

with

$$\partial \boldsymbol{g}^0 := A^0 \boldsymbol{u}_h^0 - \boldsymbol{\Pi}_h^0 \boldsymbol{f}^0 + \boldsymbol{f}^0.$$

where $\boldsymbol{\Pi}_h^0$ is L^2-orthognal projection on V_h^0.

Definition 5.1.2. *(The reconstruction in time) The time reconstruction $\boldsymbol{u}_N : [0, T] \times \Omega \to \mathbf{R}^d$ of $\{\boldsymbol{u}_h{}^n\}_{n=0}^N$ is defined by*

$$\boldsymbol{u}_N(t) := \frac{t - t^{n-1}}{k_n} \boldsymbol{u}_h^n + \frac{t^n - t}{k_n} \boldsymbol{u}_h^{n-1} - \frac{(t - t^{n-1})(t^n - t)^2}{k_n} \partial^2 \boldsymbol{u}_h^n, \tag{5.4}$$

for $t \in (t^{n-1}, t^n], n = 1, \ldots, N$, with $\partial^2 \boldsymbol{u}_h^n$ given in (2.25), noting that $\partial \boldsymbol{u}_h^0$ is well defined in (2.25). We note that $\boldsymbol{u}_N$ is a C^1 function in the time variable, with $\boldsymbol{u}_N(t^n) = \boldsymbol{u}_h^n$ and $\partial_t \boldsymbol{u}_N(t^n) = \partial \boldsymbol{u}_h^n$ for $n = 0, 1, \ldots, N$. We shall also use the time-continuous reconstruction $\boldsymbol{w}_N$, defined by

$$\boldsymbol{w}_N(t) := \frac{t - t^{n-1}}{k_n} \boldsymbol{w}^n + \frac{t^n - t}{k_n} \boldsymbol{w}^{n-1} - \frac{(t - t^{n-1})(t^n - t)^2}{k_n} \partial^2 \boldsymbol{w}^n, \tag{5.5}$$

Noting that $\partial \boldsymbol{w}^0$ is well defined. By construction, this is also a C^1 function in the time variable.

5.2 Fully discrete error relation

Definition 5.2.1. *We decompose the error as follows:*

$$\boldsymbol{e}_N := \boldsymbol{u}_N - \boldsymbol{u} = \boldsymbol{\rho}_N - \boldsymbol{\epsilon}_N,$$

where $\boldsymbol{\epsilon}_N := \boldsymbol{w}_N - \boldsymbol{u}_N$ *and* $\boldsymbol{\rho}_N := \boldsymbol{w}_N - \boldsymbol{u}$.

Theorem 5.2.1. *Under the notations introduced in definition 5.2.1, for* $t \in (t^{n-1}, t^n], n = 1, \ldots, N,$ *we have*

$$\begin{aligned}(\partial_{tt}^2 \boldsymbol{e}_N, \boldsymbol{v})_\Omega + b(\boldsymbol{\rho}_N, \boldsymbol{v}) = ((\boldsymbol{I} - \boldsymbol{\Pi}_h^n)\partial_{tt}^2 \boldsymbol{u}_N, \boldsymbol{v})_\Omega + \mu^n(t)(\partial^2 \boldsymbol{u}_h^n, \boldsymbol{\Pi}_h^n \boldsymbol{v})_\Omega \\ +b(\boldsymbol{w_N} - \boldsymbol{w}^n, \boldsymbol{v}) + (\overline{\boldsymbol{f}}^n - \boldsymbol{f}^n, \boldsymbol{v})_\Omega,\end{aligned} \tag{5.6}$$

for all $\boldsymbol{v} \in \boldsymbol{X}(\Omega)$, *with* $\boldsymbol{I}$ *being the identity mapping in* $L^2(\Omega)^d$, *and*

$$\mu^n(t) := -6k^{n-1}(t - \frac{t^n - t^{n-1}}{2}).$$

Proof. Noting that,

$$\partial_{tt}^2 \boldsymbol{u}_N(t) = (1 + \mu^n(t))\partial^2 \boldsymbol{u}_h^n, \tag{5.7}$$

for $t \in (t^{n-1}, t^n], n = 1, \ldots, N$. For all $\boldsymbol{v} \in \boldsymbol{X}(\Omega)$, we have

$$(\partial_{tt}^2 \boldsymbol{e}_N, \boldsymbol{v})_\Omega + b(\boldsymbol{\rho}_N, \boldsymbol{v}) = (\partial_{tt}^2 \boldsymbol{u}_N, \boldsymbol{v})_\Omega + b(\boldsymbol{w}_N, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_\Omega$$

where we have used the splitting in definition 5.2.1, and the fact that $\boldsymbol{u}$ is the solution of (2.3). Now using (5.7) and the properties of orthognal L^2 projection $\boldsymbol{\Pi}_h^n$ on the space V_h^n we get

$$\begin{aligned}(\partial_{tt}^2 \boldsymbol{e}_N, \boldsymbol{v})_\Omega + b(\boldsymbol{\rho}_N, \boldsymbol{v}) &= ((\boldsymbol{I} - \boldsymbol{\Pi}_h^n)\partial_{tt}^2 \boldsymbol{u}_N, \boldsymbol{v})_\Omega + (\partial_{tt}^2 \boldsymbol{u}_N, \boldsymbol{\Pi}_h^n \boldsymbol{v})_\Omega \\ &\quad +b(\boldsymbol{w}_N, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_\Omega, \\ &= ((\boldsymbol{I} - \boldsymbol{\Pi}_h^n)\partial_{tt}^2 \boldsymbol{u}_N, \boldsymbol{v})_\Omega + \mu^n(t)(\partial^2 \boldsymbol{u}_h^n, \boldsymbol{\Pi}_h^n \boldsymbol{v})_\Omega \\ &\quad -b_h^n(\boldsymbol{u}_h^n, \boldsymbol{\Pi}_h^n \boldsymbol{v}) + b(\boldsymbol{w}_N, \boldsymbol{v}) + (\boldsymbol{\Pi}_h^n \boldsymbol{f}^n - \overline{\boldsymbol{f}}^n, \boldsymbol{v})_\Omega.\end{aligned}$$

where we have used the fully discrete formulation given by (2.24). Now observing

$$\begin{aligned}b_h^n(\boldsymbol{u}_h^n, \boldsymbol{\Pi}_h^n \boldsymbol{v}) - (\boldsymbol{\Pi}_h^n \boldsymbol{f}^n - \overline{\boldsymbol{f}}^n, \boldsymbol{v})_\Omega &= (A^n \boldsymbol{u}_h^n, \boldsymbol{\Pi}_h^n \boldsymbol{v})_\Omega - (\boldsymbol{\Pi}_h^n \boldsymbol{f}^n - \overline{\boldsymbol{f}}^n, \boldsymbol{v})_\Omega \\ &= b(\boldsymbol{w}^n, \boldsymbol{v}).\end{aligned}$$

$\forall \boldsymbol{v} \in \boldsymbol{X}(\Omega)$, we then obtain

$$(\partial^2_{tt}\boldsymbol{e}_N, \boldsymbol{v})_\Omega + b(\boldsymbol{\rho}_N, \boldsymbol{v}) = ((\boldsymbol{I} - \boldsymbol{\Pi}^n_h)\partial^2_{tt}\boldsymbol{u}_N, \boldsymbol{v})_\Omega + \mu^n(t)(\partial^2\boldsymbol{u}^n_h, \boldsymbol{\Pi}^n_h\boldsymbol{v})_\Omega + b(\boldsymbol{w}_N - \boldsymbol{w}^n, \boldsymbol{v}) + (\overline{\boldsymbol{f}}^n - \boldsymbol{f}, \boldsymbol{v})_\Omega;$$

□

Remark 5.2.1. *(Zero mean-value property) The particular form of the remainder $\mu^n(t)$ satisfies the vanishing-moment property*

$$\int_{t^n-1}^{t^n} \mu^n(t)dt = 0, \tag{5.8}$$

This property is important for the derivation of the a posteriori error bound for the term $\mu^n(t)(\partial^2\boldsymbol{u}^n_h, \boldsymbol{\Pi}^n_h\boldsymbol{v})_\Omega$ in the above error relation. The proof of this property is given in the Appendix A.1

5.3 Abstract fully discrete a posteriori error bounds

To derive the a posteriori error bounds. We first define the error indicators that will form the error bounds in theorem 5.3.1.

Definition 5.3.1. *(A posteriori error indicators) For some fixed t^*, assume that $t^{m-1} \leq t^* \leq t^m$ for some integer m with $1 \leq m \leq N$, we define*

i) The mesh change indicator is given by $\xi_{\mathrm{MC}}(t^*) := \xi_{\mathrm{MC},1}(t^*) + \xi_{\mathrm{MC},2}(t^*)$, with

$$\xi_{\mathrm{MC},1}(t^*) := \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \|(\boldsymbol{I} - \boldsymbol{\Pi}^j_h)\partial_t\boldsymbol{u}_N\|_{0,\Omega}\mathrm{d}t + \int_{t^{m-1}}^{t^*} \|(\boldsymbol{I} - \boldsymbol{\Pi}^m_h)\partial_t\boldsymbol{u}_N\|_{0,\Omega}\mathrm{d}t, \tag{5.9}$$

and

$$\xi_{\mathrm{MC},2}(t^*) := \sum_{j=1}^{m-1}(t^* - t^j)\|(\boldsymbol{\Pi}^{j+1}_h - \boldsymbol{\Pi}^j_h)\partial\boldsymbol{u}^j_h\|_{0,\Omega}. \tag{5.10}$$

ii) The evolution error indicator

$$\xi_{\mathrm{evo}}(t^*) := \int_0^{t^*} \|\mathcal{G}\|_{0,\Omega}\mathrm{d}t, \tag{5.11}$$

where $\mathcal{G} : (0, t^N] \to \mathbb{R}$ *with* $\mathcal{G}|_{(t^{j-1},t^j]} := \mathcal{G}^j$, $j = 1, \dots, N$ *and*

$$\mathcal{G}^j(t) := \frac{(t^j - t)^2}{2}\partial \boldsymbol{g}^j - \left(\frac{(t^j - t)^4}{4k_j} - \frac{(t^j - t)^3}{3}\right)\partial^2 \boldsymbol{g}^j - \boldsymbol{\gamma}_j, \tag{5.12}$$

with $\boldsymbol{g}^j$ *as given by (5.2) and* $\boldsymbol{\gamma}_j := \boldsymbol{\gamma}_{j-1} + (k_j^2/2)\partial \boldsymbol{g}^j + (k_j^3/12)\partial^2 \boldsymbol{g}^j$, $j = 1, \dots, N$ *with* $\boldsymbol{\gamma}_0 = \boldsymbol{0}$*;*

iii) The data error indicators

$$\xi_{\mathrm{osc}}(t^*) := \frac{1}{2\pi}\sum_{j=1}^{m-1}\left(\int_{t^{j-1}}^{t^j} k_j^3 \|\overline{\boldsymbol{f}}^j - \boldsymbol{f}\|_{0,\Omega}^2 \mathrm{d}t\right)^{1/2} + \left(\int_{t^{m-1}}^{t^*} k_m^3 \|\overline{\boldsymbol{f}}^m - \boldsymbol{f}\|_{0,\Omega}^2 \mathrm{d}t\right)^{1/2}, \tag{5.13}$$

which can be viewed as an error estimator related to the time-oscillation of the source term.

iv) The time reconstruction error indicators

$$\xi_{\mathrm{tr}}(t^*) := \frac{1}{2\pi}\sum_{j=1}^{m-1}\left(\int_{t^{j-1}}^{t^j} k_j^3 \|\mu^j \partial^2 \boldsymbol{u}_h^j\|_{0,\Omega}^2 \mathrm{d}t\right)^{1/2} + \left(\int_{t^{m-1}}^{t^*} k_m^3 \|\mu^m \partial^2 \boldsymbol{u}_h^m\|_{0,\Omega}^2 \mathrm{d}t\right)^{1/2}. \tag{5.14}$$

Theorem 5.3.1. *(Abstract fully discrete error bound) Let* $\boldsymbol{u}$ *is the exact weak solution of (2.1),* $\boldsymbol{u}_N$ *and* $\boldsymbol{w}_N$ *are reconstructed from the fully discrete solution* $\{\boldsymbol{u}_h^n\}_{n=1}^N$ *and their time-harmonic reconstruction* $\{\boldsymbol{w}^n\}_{n=1}^N$*, respectively, as in (5.4) and (5.5). With the notation of indicators in definition 5.3.1, the following a posteriori error estimate holds*

$$\begin{aligned}\|\boldsymbol{e}_N\|_{L^\infty(0,t^N;L^2(\Omega)^d)} \leq{}& \|\boldsymbol{\epsilon}_N\|_{L^\infty(0,t^N;L^2(\Omega)^d)} + \sqrt{2}\|\boldsymbol{\epsilon}_N(0)\|_{0,\Omega} \\ &+ 2\int_0^{t^N} \|\partial_t \boldsymbol{\epsilon}_N\|_{0,\Omega}\mathrm{d}t + 2\big(\xi_{\mathrm{MC}}(t^N) + \xi_{\mathrm{evo}}(t^N) \\ &+ \xi_{\mathrm{osc}}(t^N) + \xi_{\mathrm{tr}}(t^N)\big) + \sqrt{2}\|\boldsymbol{u}_0 - \boldsymbol{u}_h^0\|_{0,\Omega} + 2T\|\boldsymbol{u}_1 - \partial \boldsymbol{u}_h^0\|_{0,\Omega},\end{aligned} \tag{5.15}$$

Proof. To derive this error bound, we proceed in the similar way as in semi-discrete case. We will use a test function $\boldsymbol{v} = \hat{\boldsymbol{v}}_N$ with $\hat{\boldsymbol{v}}_N$ is defined similarly as in (4.8)

$$\hat{\boldsymbol{v}}_N(t,\cdot) = \int_t^{t^*} \boldsymbol{\rho}_N(s,\cdot)ds, \; t \in [0, t^N], \tag{5.16}$$

assuming that $t^{m-1} \leq t^* \leq t^m$ for some integer m with $1 \leq m \leq N$, and $\boldsymbol{\rho}_N$ is defined as in (5.2.1), then $\hat{\boldsymbol{v}}_N \in \boldsymbol{X}(\Omega)$ as $\boldsymbol{\rho}_N \in \boldsymbol{X}(\Omega)$, we observe that

$$\hat{\boldsymbol{v}}_N(t^*,\cdot) = 0, \quad \text{and } \partial_t \hat{\boldsymbol{v}}_N(t,\cdot) = -\boldsymbol{\rho}_N(t,\cdot) \;\text{ a.e. in } [0, t^N].$$

Integration of the resulting equation with respect to t between 0 and t^*, yields

$$\int_0^{t^*} (\partial^2_{tt}\boldsymbol{e}_N, \hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t + \int_0^{t^*} b(\boldsymbol{\rho}_N, \hat{\boldsymbol{v}}_N)\mathrm{d}t = \sum_{i=1}^4 \mathcal{I}_i(t^*), \tag{5.17}$$

where

$$\begin{aligned}
\mathcal{I}_1(t^*) &:= \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} ((\boldsymbol{I} - \boldsymbol{\Pi}_h^j)\partial^2_{tt}\boldsymbol{u}_N, \hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t + \int_{t^{m-1}}^{t^*} ((\boldsymbol{I} - \boldsymbol{\Pi}_h^m)\partial^2_{tt}\boldsymbol{u}_N, \hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t,\\
\mathcal{I}_2(t^*) &:= \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} b(\boldsymbol{w}_N - \boldsymbol{w}^j, \hat{\boldsymbol{v}}_N)\mathrm{d}t + \int_{t^{m-1}}^{t^*} b(\boldsymbol{w}_N - \boldsymbol{w}^m, \hat{\boldsymbol{v}}_N)\mathrm{d}t,\\
\mathcal{I}_3(t^*) &:= \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} (\overline{\boldsymbol{f}}^j - \boldsymbol{f}, \hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t + \int_{t^{m-1}}^{t^*} (\overline{\boldsymbol{f}}^m - \boldsymbol{f}, \hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t,\\
\mathcal{I}_4(t^*) &:= \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \mu^j(\partial^2\boldsymbol{u}_h^j, \boldsymbol{\Pi}_h^j\hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t + \int_{t^{m-1}}^{t^*} \mu^m(\partial^2\boldsymbol{u}_h^m, \boldsymbol{\Pi}_h^m\hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t.
\end{aligned} \tag{5.18}$$

From Equation (5.17), using the splitting given in definition 5.2.1, we have

$$\int_0^{t^*} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\rho}_N\|^2_{0,\Omega}\mathrm{d}t - \int_0^{t^*} \frac{1}{2}\frac{d}{dt} b(\hat{\boldsymbol{v}}_N, \hat{\boldsymbol{v}}_N)\mathrm{d}t = \int_0^{t^*} (\partial^2_{tt}\boldsymbol{\epsilon}_N, \hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t + \sum_{i=1}^4 \mathcal{I}_i(t^*), \tag{5.19}$$

Integration in time and use of the properties of $\hat{\boldsymbol{v}}_N$ yields

$$\begin{aligned}
\frac{1}{2}\|\boldsymbol{\rho}_N(t^*)\|^2_{0,\Omega} - \frac{1}{2}\|\boldsymbol{\rho}_N(0)\|^2_{0,\Omega} + \frac{1}{2}b(\hat{\boldsymbol{v}}_N(0), \hat{\boldsymbol{v}}_N(0)) &= \int_0^{t^*} (\partial_t\boldsymbol{\epsilon}_N, \boldsymbol{\rho}_N)_\Omega \mathrm{d}t\\
&+ (\partial_t\boldsymbol{e}_N(0), \hat{\boldsymbol{v}}_N(0))_\Omega + \sum_{i=1}^4 \mathcal{I}_i(t^*).
\end{aligned} \tag{5.20}$$

Now to bound the last term on the right hand side, we have the following lemma.

Lemma 5.3.1. *Under the notations inroduced in definition 5.3.1, we have the following estimates:*

$$\begin{aligned}
\mathcal{I}_1(t^*) &\leq \xi_{\mathrm{MC}}(t^*) \max_{0\leq t\leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega};\\
\mathcal{I}_2(t^*) &\leq \xi_{\mathrm{evo}}(t^*) \max_{0\leq t\leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega};\\
\mathcal{I}_3(t^*) &\leq \xi_{\mathrm{osc}}(t^*) \max_{0\leq t\leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega};\\
\mathcal{I}_4(t^*) &\leq \xi_{\mathrm{tr}}(t^*) \max_{0\leq t\leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega};
\end{aligned} \tag{5.21}$$

The proof for the above lemma follows from the applications of integration by parts, the commutation of orthogonal L^2-projection with time differentiation and time integration, and the zero averages of μ^n and $\overline{\boldsymbol{f}}^n - \boldsymbol{f}^n$ on the interval $[t^{n-1}, t^n]$.
In [GLM13] the results were presented for the analysis of wave equation, for the sake of completeness, we present these proofs for Maxwell's problem in Appendix A.2.
Using the bounds stated in above lemma, we get

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\rho}_N(t^*)\|_{0,\Omega}^2 - \frac{1}{2}\|\boldsymbol{\rho}_N(0)\|_{0,\Omega}^2 + \frac{1}{2}b(\hat{\boldsymbol{v}}_N(0), \hat{\boldsymbol{v}}_N(0))\\
&\qquad\leq \max_{0\leq t\leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega}\bigg(\int_0^{t^*} \|\partial_t \boldsymbol{\epsilon}\|_{0,\Omega} \mathrm{d}t + \xi_{\mathrm{MC}}(t^*) + \xi_{\mathrm{evo}}(t^*)\\
&\qquad + \xi_{\mathrm{osc}}(t^*) + \xi_{\mathrm{tr}}(t^*)\bigg) + \|\partial \boldsymbol{e}_N(0)\|_{0,\Omega}\|\hat{\boldsymbol{v}}_N(0)\|_{0,\Omega}.
\end{aligned} \tag{5.22}$$

We select t^* such that $\|\boldsymbol{\rho}_N(t^*)\|_{0,\Omega} = \max_{0\leq t\leq t^N} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega}$. Then completely analogous to the proof in the semi-discrete case, we arrive at

$$\begin{aligned}
\|\boldsymbol{e}_N\|_{L^\infty(0,t^N;L^2(\Omega)^d)} &\leq \|\boldsymbol{\epsilon}_N\|_{L^\infty(0,T;L^2(\Omega)^d)} + \sqrt{2}(\|\boldsymbol{u}_0 - \boldsymbol{u}_h^0\|_{0,\Omega} + \|\boldsymbol{\epsilon}_N(0)\|_{0,\Omega})\\
&\quad + 2\Big(\int_0^{t^N} \|\partial_t \boldsymbol{\epsilon}_N\|_{0,\Omega} + \xi_{\mathrm{MC}}(t^N) + \xi_{\mathrm{evo}}(t^N)\\
&\quad + \xi_{\mathrm{osc}}(t^N) + \xi_{\mathrm{tr}}(t^N)\Big) + 2T\|\boldsymbol{u}_1 - \partial \boldsymbol{u}_h^0\|_{0,\Omega}.
\end{aligned} \tag{5.23}$$

This completes the proof. □

5.4 Explicit fully discrete a posteriori estimates

To get a practical a posteriori error bound, for the fully discrete scheme from theorem 5.3.1, we need to bound explicitely the terms $\|\boldsymbol{\epsilon}_N\|_{L^\infty(0,t^N;L^2(\Omega)^d)}$,

$\|\boldsymbol{\epsilon}_N(0)\|_{0,\Omega}$ *and* $\int_0^{t^N}\|\partial_t\boldsymbol{\epsilon}_N\|_{0,\Omega}$. *In this section, we will bound these quantities in the lemma 5.5.2, 5.5.1 and 5.5.4. The estimates derived in these lemmas enable us to prove the following error estimate, which is the main result of this chapter.*

Theorem 5.4.1. *(Explicit fully discrete a posteriori error bound). With the same hypotheses and notations as in theorem 4.4.1 and 5.3.1, we have the bound*

$$\|\boldsymbol{e}_N\|_{L^\infty(0,t^N;L^2(\Omega)^d)} \leq \big(\xi_{\text{sp}} + \xi_{\text{tm}}(t^N) + \xi_{\text{IC}}\big), \tag{5.24}$$

where ξ_{sp} *represents the spatial error,* ξ_{tm} *deals the terms corresponding to temporal error,* ξ_{IC} *represents the error related to the initial conditions of the problem, these quantities are defined as follows:*

$$\begin{aligned}\xi_{\text{sp}} &= \xi_{\text{sp},1} + \xi_{\text{sp},2} + \xi_{\text{sp},3},\\ \xi_{\text{tm}}(t^N) &= 2\Big(\xi_{\text{MC}}(t^N) + \xi_{\text{evo}}(t^N) + \xi_{\text{osc}}(t^N) + \xi_{\text{tr}}(t^N)\Big),\\ \xi_{\text{IC}} &= \sqrt{2}\|\boldsymbol{u}_0 - \boldsymbol{u}_h^0\|_{0,\Omega} + 2T\|\boldsymbol{u}_1 - \partial\boldsymbol{u}_h^0\|_{0,\Omega},\end{aligned} \tag{5.25}$$

with $\xi_{\text{MC}}, \xi_{\text{evo}}, \xi_{\text{osc}}, \xi_{\text{tr}}$ *are defined in definition 5.3.1, and*

$$\begin{aligned}\xi_{\text{sp},1} &= \sqrt{2}\mathscr{E}^0,\\ \xi_{\text{sp},2} &= \max\Bigg\{\frac{4k_1}{27}\mathscr{E}(\partial\boldsymbol{u}_h^0, \partial\boldsymbol{g}^0, \mathscr{T}^0),\\ &\qquad \left(\frac{31}{27} + \frac{31}{27}\max_{1\le j\le N}\frac{k_j}{k_{j-1}}\right)\max_{0\le j\le N}\left(\mathscr{E}^j + C_{\text{stab}}\|\overline{\boldsymbol{f}}^j - \boldsymbol{f}^j\|_{0,\Omega}\right)\Bigg\},\\ \xi_{\text{sp},3} &= 2\sum_{j=1}^N(\mathscr{E}^j + \mathscr{E}^{j-1}) + \sum_{j=1}^N 2k_jC_{\text{stab}}\|\partial\boldsymbol{f}^j - \partial\overline{\boldsymbol{f}}^j\|_{0,\Omega},\end{aligned} \tag{5.26}$$

with $\mathscr{E}^j := \mathscr{E}(\boldsymbol{u}_h^j, A^j\boldsymbol{u}_h^j - \boldsymbol{\Pi}_h^j\boldsymbol{f}^j + \boldsymbol{f}^j, \mathscr{T}^j)$*, for all* $0 \le j \le N$*, where* $\mathscr{E}$ *is defined from (4.40). Here* C_{stab} *is the stability constant of time-harmonic Maxwell's equations.*

5.5 Proof of theorem 5.4.1

This section is devoted to the proof of theorem 5.4.1, We will state and proof three lemmas to bound the terms $\|\boldsymbol{\epsilon}_N\|_{L^\infty(0,t^N;L^2(\Omega)^d)}$, $\|\boldsymbol{\epsilon}_N(0)\|_{0,\Omega}$ *and* $\int_0^{t^N}\|\partial_t\boldsymbol{\epsilon}_N\|_{0,\Omega}$, *and combining these results will lead to the proof of theorem 5.4.1.*

For the term $\|\boldsymbol{\epsilon}_N(0)\|_{0,\Omega}$ *, we clearly have the following bound.*

Lemma 5.5.1. *The following estimate holds*

$$\sqrt{2}\|\boldsymbol{\epsilon}_N(0)\|_{0,\Omega} \leq \xi_{\mathrm{sp},1}, \tag{5.27}$$

where $\xi_{\mathrm{sp},1}$ is introduced in theorem 5.4.1.

To bound the term $\|\boldsymbol{\epsilon}\|_{L^\infty(0,t^N;L^2(\Omega)^d)}$ we have:

Lemma 5.5.2. *The following estimate holds*

$$\|\boldsymbol{\epsilon}_N\|_{L^\infty(0,t^N;L^2(\Omega)^d)} \leq \xi_{\mathrm{sp},2},$$

where $\xi_{\mathrm{sp},2}$ is introduced in theorem 5.4.1.

Proof. From the construction of $\boldsymbol{u}_N$ and $\boldsymbol{w}_N$ in definition 5.1.1, for $t \in (t^{j-1}, t^j], j = 1, \ldots, N$, we have the following expression

$$\begin{aligned}\boldsymbol{\epsilon}_N(t) &= \boldsymbol{w}_N - \boldsymbol{u}_N \\ &= \frac{t-t^{j-1}}{k_j}(\boldsymbol{w}^j - \boldsymbol{u}_h^j) + \frac{t^j - t}{k_j}(\boldsymbol{w}^{j-1} - \boldsymbol{u}_h^{j-1}) \\ &\quad - \frac{(t-t^{j-1})(t^j-t)^2}{k_j}(\partial^2\boldsymbol{w}^j - \partial^2\boldsymbol{u}_h^j),\end{aligned} \tag{5.28}$$

which yields

$$\begin{aligned}&\|\boldsymbol{\epsilon}_N(t)\|_{0,\Omega} \\ &\leq \max\left\{\left(\frac{31}{27} + \frac{31}{27}\max_{1\leq j\leq N}\frac{k_j}{k_{j-1}}\right)\max_{0\leq j\leq N}\|\boldsymbol{w}^j - \boldsymbol{u}_h^j\|_{0,\Omega}; \frac{4k_1}{27}\|\partial\boldsymbol{w}^0 - \partial\boldsymbol{u}_h^0\|_{0,\Omega}\right\},\end{aligned} \tag{5.29}$$

noting that

$$\max_{t\in(t^{j-1},t^j]}\frac{(t-t^{j-1})(t^j-t)^2}{k_j} = \frac{4k_j^2}{27},$$

The explicit calculations are presented in Appendix A.1.

Now it remains to estimate the terms $\|\boldsymbol{w}^j - \boldsymbol{u}_h^j\|_{0,\Omega}$, $0 \leq j \leq N$ and $\|\partial\boldsymbol{w}^0 - \partial\boldsymbol{u}_h^0\|_{0,\Omega}$. To estimate the term $\|\boldsymbol{w}^j - \boldsymbol{u}_h^j\|_{0,\Omega}$: we will prove the following result

Lemma 5.5.3. *The following estimate holds for all $0 \leq j \leq N$*

$$\|\boldsymbol{w}^j - \boldsymbol{u}_h^j\|_{0,\Omega} \leq \mathscr{E}^j + C_{\mathrm{stab}}\|\overline{\boldsymbol{f}}^j - \boldsymbol{f}^j\|_{0,\Omega}, \tag{5.30}$$

with $\mathscr{E}^j := \mathscr{E}(\boldsymbol{u}_h^j, A^j\boldsymbol{u}_h^j - \boldsymbol{\Pi}_h^j\boldsymbol{f}^j + \boldsymbol{f}^j, \mathscr{T}^j)$, for all $0 \leq j \leq N$ where $\mathscr{E}$ is defined from (4.40).

Proof. To prove the result in lemma 5.5.3, we proceed as follows: First, we define $\underline{\boldsymbol{w}}^j \in \boldsymbol{X}(\Omega)$ to be the solution of the time-harmonic Maxwell's equation,

$$b(\underline{\boldsymbol{w}}^j, \boldsymbol{v}) = (A^j \boldsymbol{u}_h^j - \boldsymbol{\Pi}_h^j \boldsymbol{f}^j + \boldsymbol{f}^j, \boldsymbol{v})_\Omega \tag{5.31}$$

for $j = 0, 1, \ldots, N$.
Note that due to the fact that $\overline{\boldsymbol{f}}^0 = \boldsymbol{f}^0$, we have that $\underline{\boldsymbol{w}}^0 = \boldsymbol{w}^0$. On the other hand, we observe that $\boldsymbol{u}_h^j$ is the DG solution in $\mathbf{V}_h^j$ of time harmonic Maxwell's problem (5.31), noting that to prove this we follow the same proof as in remark 4.2.1. Applying theorem 4.5.1 implies that

$$\|\underline{\boldsymbol{w}}^j - \boldsymbol{u}_h^j\|_{0,\Omega} \leq C\mathscr{E}(\boldsymbol{u}_h^j, A^j \boldsymbol{u}_h^j - \boldsymbol{\Pi}_h^j \boldsymbol{f}^j + \boldsymbol{f}^j, \mathscr{T}^j), \tag{5.32}$$

for $j = 0, \ldots, N$.

Second, we need to estimate $\|\boldsymbol{w}^j - \underline{\boldsymbol{w}}^j\|_{0,\Omega}$. Observing that $\boldsymbol{w}^j - \underline{\boldsymbol{w}}^j$ is the solution of the time-harmonic Maxwell's problem with the load $\overline{\boldsymbol{f}}^j - \boldsymbol{f}^j$

$$b(\boldsymbol{w}^j - \underline{\boldsymbol{w}}^j, \boldsymbol{v}) = (\overline{\boldsymbol{f}}^j - \boldsymbol{f}^j, \boldsymbol{v})_\Omega,$$

we arrive at

$$\|\boldsymbol{w}^j - \underline{\boldsymbol{w}}^j\|_{0,\Omega} \leq C_{\text{stab}} \|\overline{\boldsymbol{f}}^j - \boldsymbol{f}^j\|_{0,\Omega}, \text{ for } j = 1, \ldots, N; \tag{5.33}$$

using the stability of time-harmonic Maxwell's problem. The constant C_{stab} is bounded by C_{coer}^{-1} where C_{coer} is coercivity constant of the bilinear form b. Now using the trianglular inequality, we have

$$\|\boldsymbol{w}^j - \boldsymbol{u}_h^j\|_{0,\Omega} \leq \|\boldsymbol{w}^j - \underline{\boldsymbol{w}}^j\|_{0,\Omega} + \|\underline{\boldsymbol{w}}^j - \boldsymbol{u}_h^j\|_{0,\Omega}, \tag{5.34}$$

along with the bounds (5.32), (5.33) imply lemma 5.5.3. □

Now to estimate $\|\partial \boldsymbol{w}^0 - \partial \boldsymbol{u}_h^0\|_{0,\Omega}$, We use the similar argument,

$$b_h^0(\partial \boldsymbol{u}_h^0, \boldsymbol{v}) = (A^0(\partial \boldsymbol{u}_h^0), \boldsymbol{v})_\Omega) = (A^0(\partial \boldsymbol{u}_h^0) - \boldsymbol{\Pi}_h^0 \boldsymbol{f}^0 + \boldsymbol{f}^0, \boldsymbol{v})_\Omega, \text{ for all } \boldsymbol{v} \in V_h^0,$$

hence, comparing with the construction (5.43), $\partial \boldsymbol{u}_h^0$ is the DG solution of $\partial \boldsymbol{w}^0$, which yields

$$\|\partial \boldsymbol{w}^0 - \partial \boldsymbol{u}_h^0\|_{0,\Omega} \leq \mathscr{E}(\partial \boldsymbol{u}_h^0, \partial \boldsymbol{g}^0, \mathscr{T}^0). \tag{5.35}$$

Thus the proof of lemma 5.5.2 is completed. □

Lemma 5.5.4.

$$2\int_0^{t^N} \|\partial_t \boldsymbol{\epsilon}_N\|_{0,\Omega} \mathrm{d}t \leq \xi_{\text{sp},3},$$

where $\xi_{\text{sp},3}$ *is defined as in 5.4.1.*

Proof. From the construction of $\boldsymbol{u}_N$ and $\boldsymbol{w}_N$ in definition 5.2.1 and from (5.28), for $t \in (t^{j-1}, t^j], j = 1, \ldots, N$, we have

$$\partial_t \boldsymbol{\epsilon}_N = \partial \boldsymbol{w}^j - \partial \boldsymbol{u}_h^j + k_j^{-1}(t^j - t)(3t - 2t^{j-1} - t^j)(\partial^2 \boldsymbol{w}^j - \partial^2 \boldsymbol{u}_h^j),$$

A simple calculation yields

$$\int_{t^{j-1}}^{t^j} k_j^{-2}(t^j - t)(3t - 2t^{j-1} - t^j)\mathrm{d}t = 0, \tag{5.36}$$

refer to appendix A.1.2 for explicit calculation. Hence, we get

$$\int_{t^{j-1}}^{t^j} \|\partial_t \boldsymbol{\epsilon}_N\|_{0,\Omega}\mathrm{d}t \leq k_j \|\partial \boldsymbol{w}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega}, \tag{5.37}$$

Summing up for $j = 1, \ldots, N$, we arrive at

$$\int_0^{t^N} \|\partial_t \boldsymbol{\epsilon}_N\|_{0,\Omega}\mathrm{d}t \leq \sum_{j=1}^{N} k_j \|\partial \boldsymbol{w}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega}. \tag{5.38}$$

Now we need to estimate the terms $\|\partial \boldsymbol{w}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega}$. We will use the triangular inequality by combining the bounds for $\|\partial \underline{\boldsymbol{w}}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega}$ and $\|\partial \boldsymbol{w}^j - \partial \underline{\boldsymbol{w}}^j\|_{0,\Omega}$, where $\partial \underline{\boldsymbol{w}}^j \in \boldsymbol{X}(\Omega)$, $j = 0, \ldots, N$ is the solution of the time-harmonic Maxwell's boundary value problem

$$b(\partial \underline{\boldsymbol{w}}^j, \boldsymbol{v}) = (A^j(\partial \boldsymbol{u}_h^j) - \partial \boldsymbol{\Pi}_h^j \boldsymbol{f}^j + \partial \boldsymbol{f}^j, \boldsymbol{v}) \tag{5.39}$$

$\forall \boldsymbol{v} \in \boldsymbol{X}(\Omega)$.
From the definition (5.1.1) of $\boldsymbol{w}^j, j = 1, , N$, we have

$$b(\partial \boldsymbol{w}^j, \boldsymbol{v}) = (A^j(\partial \boldsymbol{u}_h^j) - \partial \boldsymbol{\Pi}_h^j \boldsymbol{f}^j + \partial \boldsymbol{f}^j, \boldsymbol{v}), \tag{5.40}$$

$\forall \boldsymbol{v} \in \boldsymbol{X}(\Omega)$.
Thus using (5.39) and (5.40), For $j = 1, \ldots, N$, we have that $\partial \boldsymbol{w}^j - \partial \underline{\boldsymbol{w}}^j$ is the solution of the time-harmonic Maxwell's problem

$$b(\partial \boldsymbol{w}^j - \partial \underline{\boldsymbol{w}}^j, \boldsymbol{v}) = (\partial \overline{\boldsymbol{f}}^j - \partial \boldsymbol{f}^j, \boldsymbol{v}_\Omega),$$

$\forall \boldsymbol{v} \in \boldsymbol{X}(\Omega)$
then from the stability of time-harmonic Maxwell's problem, we have

$$\|\partial \boldsymbol{w}^j - \partial \underline{\boldsymbol{w}}^j\|_{0,\Omega} \leq C_{\mathrm{stab}} \|\partial \overline{\boldsymbol{f}}^j - \partial \boldsymbol{f}^j\|_{0,\Omega}, \text{ for } j = 1, \ldots, N. \tag{5.41}$$

Now we derive a bound for $\|\partial \underline{\boldsymbol{w}}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega}$ in the following lemma.

Lemma 5.5.5. *For $j = 1, \ldots, N$, let $\partial \underline{\boldsymbol{w}}^j \in \boldsymbol{X}(\Omega)$ be the solution of problem (5.39) and $\partial \boldsymbol{u}_h^j$ is defined by backward Euler scheme (2.26), with $\boldsymbol{u}_h^j$ be the fully discrete solution from (2.24). The following error bound holds*

$$\|\partial \underline{\boldsymbol{w}}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega} \leq k_j^{-1}(\mathscr{E}^j + \mathscr{E}^{j-1}), \tag{5.42}$$

with $\mathscr{E}^j := \mathscr{E}(\boldsymbol{u}_h^j, A^j \boldsymbol{u}_h^j - \boldsymbol{\Pi}_h^j \boldsymbol{f}^j + \boldsymbol{f}^j, \mathscr{T}^j)$, for all $0 \leq j \leq N$ and $\mathscr{E}$ is defined from (4.40).

Proof. We denote by $\underline{\boldsymbol{w}}^j$ the solution in (5.31), for $i = 0, 1, \ldots, N$. By employing (5.32) and from the backward finite differences we obtain the bounds as follows:

$$\|\partial \underline{\boldsymbol{w}}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega} \leq \frac{1}{k_j}(\|\underline{\boldsymbol{w}}^j - \boldsymbol{u}_h^j\|_{0,\Omega} + \|\underline{\boldsymbol{w}}^{j-1} - \boldsymbol{u}_h^{j-1}\|_{0,\Omega}), \tag{5.43}$$

for $j = 1, \ldots, N$. This completes the proof for lemma 5.5.5. □

The proof of proposition 5.5.4 is completed. □

Remark 5.5.1. *In case of stationary mesh (i.e. the same mesh is used between the initial time t_0 and the final t_N, or $\mathscr{T}^j = \mathscr{T}^{j-1}$, for all $j = 1, \ldots, N$), we have an alternative result for lemma 5.5.5 as follows:*

$$\|\partial \underline{\boldsymbol{w}}^j - \partial \boldsymbol{u}_h^j\|_{0,\Omega} \leq \mathscr{E}(\partial \boldsymbol{u}_h^j, \partial(\boldsymbol{A}^j \boldsymbol{u}_h^j) - \partial(\boldsymbol{\Pi}_h^j \boldsymbol{f}^j) + \partial \boldsymbol{f}^j, \mathscr{T}^j), \tag{5.44}$$

for all $1 \leq j \leq N$ and $\mathscr{E}$ is defined from (4.40).

Proof. Indeed, from the definitions of $\{\underline{\boldsymbol{w}}^j\}_{j=0}^N$ in (5.31), we deduce for all $j = 1, \ldots, N$ that:

$$b(\partial \underline{\boldsymbol{w}}^j, \boldsymbol{v}) = (\partial(\boldsymbol{A}^j \boldsymbol{u}_h^j) - \partial(\boldsymbol{\Pi}_h^j \boldsymbol{f}^j) + \partial \boldsymbol{f}^j, \boldsymbol{v})_\Omega, \tag{5.45}$$

$\forall \boldsymbol{v} \in \boldsymbol{X}(\Omega)$.
On the other hand, from the definitions of the discrete operators $\{\boldsymbol{A}^j\}_{j=0}^N$ in (5.1) and property of orthogonal L^2-projection, we have that for all $j = 1, \ldots, N$:

$$\begin{aligned} b_h^j(\partial \boldsymbol{u}_h^j, \boldsymbol{v}) &= (\partial(\boldsymbol{A}^j \boldsymbol{u}_h^j), \boldsymbol{v})_\Omega, \quad \forall \boldsymbol{v} \in \mathbf{V}_h^j \\ &= (\partial(\boldsymbol{A}^j \boldsymbol{u}_h^j) - \partial(\boldsymbol{\Pi}_h^j \boldsymbol{f}^j) + \partial \boldsymbol{f}^j, \boldsymbol{v})_\Omega, \quad \forall \boldsymbol{v} \in \mathbf{V}_h^j, \end{aligned} \tag{5.46}$$

which implies that $\partial \boldsymbol{u}_h^j$ is the DG approximation in $\mathbf{V}_h^j$ of the boundary value problem (5.45), so we obtain the error estimate (5.44). □

5.6 Conclusion

In this chapter, we have carried out an a posteriori error analysis for the DG method for the fully discretization of the time-dependent Maxwell's problem. The work is inspired from the method in [GLM13], which is used for wave equation for conforming finite element method and we have extented it to our problem for DG method. This method consists of using the special testing procedure introdcued by [BA76], and a suitable space-time reconstruction. These techniques allow us to derive the a posteriori error estimate for the time-dependent problem from the error estimates of the auxiliary time-harmonic problem. The numerical implementation of these computable a posteriori error bounds using adaptive algorithm strategy will be the subject of our future works.

Chapter 6

Generalization of a posteriori error results

Contents

6.1 Introduction

In this chapter, we generalize our a posteriori error analysis to any partial differential initial boundary value problem.

The main ingredient of our a posteriori error analysis is the time-harmonic reconstruction which allows us to separate the time discretization analysis from the spatial discretization analysis. If $\boldsymbol{u}$ denotes the exact solution to our partial differential initial boundary value problem, and $\boldsymbol{u}_h$ denotes its approximate solution obtained by a conforming or nonconforming numerical method. Then the main idea behind the time-harmonic reconstruction technique is to define an auxiliary function $\boldsymbol{w}$, which is called the time-harmonic reconstruction of $\boldsymbol{u}_h$. This function $\boldsymbol{w}$ satisfy two properties,

1) *an error relation which allows us to express the error $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h = (\boldsymbol{u} - \boldsymbol{w}) + (\boldsymbol{w} - \boldsymbol{u}_h)$, in terms of the quantities involving only $(\boldsymbol{w} - \boldsymbol{u}_h)$ and the given data of our problem.*

2) *$\boldsymbol{u}_h$ is Ritz-projection of $\boldsymbol{w}$ onto a given non-conforming finite element space V_h.*

6.2 Generalisation of a posteriori error estimates

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, with boundary $\partial\mathcal{O}$. We introduce the linear differential operators P and $Q_j, 1 \leq j \leq v$, in $\mathcal{O}$ and on $\partial\mathcal{O}$, respectively. Let

$\boldsymbol{f}$ and $\boldsymbol{g}_j$, $0 \leq j \leq v$, be given in functional spaces F and G_j , F being a space on $\mathcal{O}$ and the G_j are spaces on $\partial\mathcal{O}$. Then our model problem is as follows: find $\boldsymbol{u}$ in a function space $\mathcal{U}$ satisfying

$$\begin{aligned} P\boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{f} \text{ in } \mathcal{O}, \\ Q_j\boldsymbol{u} &= \boldsymbol{g}_j \text{ on } \partial\mathcal{O}, 0 \leq j \leq v. \end{aligned} \tag{6.1}$$

Q_j may be identically zero on part of $\partial\mathcal{O}$, so that the number of boundary conditions may depend on the parts of $\partial\mathcal{O}$ considered.
We consider $\mathcal{O} = \Omega\times]0,T[\subseteq \mathbb{R}^{n+1}$, where Ω is an open subset of $\mathbb{R}^n$, with boundary $\partial\Omega$; the part of the boundary $\partial\mathcal{O}$ on which boundary conditions are given splits up into $\overline{\Omega}$ and $\Sigma := \partial\Omega\times]0,T[$. We specify the system of operators $\{P, Q_j\}$ as follows:

a) *we consider the case where P is given by*

$$P = \frac{\partial}{\partial t} + \mathcal{X},$$

where $\mathcal{X}$ is an elliptic operator. Then the problem corresponding to (6.1) is

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{X}\boldsymbol{u}(\boldsymbol{x},t) &= f \text{ in } \mathcal{O}, \\ \boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0 \text{ in } \Omega, \\ B_j\boldsymbol{u} &= \boldsymbol{g}_j \text{ on } \Sigma, \end{aligned} \tag{6.2}$$

where B_j are suitable boundary operators.

b) *The case where P is an operator given by*

$$P = \frac{\partial^2}{\partial t^2} + \mathcal{X}.$$

In this case the problem corresponding to (6.1) is

$$\begin{aligned} \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \mathcal{X}\boldsymbol{u}(\boldsymbol{x},t) &= f \text{ in } \mathcal{O}, \\ \boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0 \text{ in } \Omega, \\ \frac{\partial \boldsymbol{u}}{\partial t}(\boldsymbol{x},0) &= \boldsymbol{u}_1 \text{ in } \Omega, \\ B_j u &= \boldsymbol{g}_j \text{ on } \Sigma, \end{aligned} \tag{6.3}$$

where B_j are suitable boundary operators, $\boldsymbol{u}_0$,$\boldsymbol{u}_1$ are given functions.
Let $\boldsymbol{u}$ be the exact solution of (6.1), and $\boldsymbol{u}_h$ be the solution obtained by discontinuous Galerkin(DG) approximation on a finite element space V_h. Let $\boldsymbol{\Pi}_h$ be

the L^2-orthognal projection onto V_h. We define $\boldsymbol{w}$ the solution of the following time harmonic problem

$$\begin{aligned} \mathcal{X}\boldsymbol{w} &= \boldsymbol{r} \text{ in } \mathcal{O}, \\ Q_j\boldsymbol{w} &= \boldsymbol{g}_j \text{ on } \partial\mathcal{O}, 0 \leq j \leq v, \end{aligned} \tag{6.4}$$

with $\boldsymbol{r} = A\boldsymbol{u}_h - \boldsymbol{\Pi}_h\boldsymbol{f} + \boldsymbol{f}$, where $A : V_h \longrightarrow V_h$ is the discrete operator defined by

$$(A\boldsymbol{z}_h, \boldsymbol{v})_\Omega = b_h(\boldsymbol{z}_h, \boldsymbol{v}), \ \forall \boldsymbol{v} \in V_h$$

for each $\boldsymbol{z}_h \in V_h$. Here b_h is bilinear form of corresponding DG formulation of (6.1).
Let $\boldsymbol{w_h}$ be the DG approximation of (6.4), then we have

$$b_h(\boldsymbol{w}_h, \boldsymbol{v}) = (A\boldsymbol{u}_h - \boldsymbol{\Pi}_h\boldsymbol{f} + \boldsymbol{f}, \boldsymbol{v})_\Omega = b_h(\boldsymbol{u}_h, \boldsymbol{v}).$$

Hence, $\boldsymbol{w}_h = \boldsymbol{u}_h$, which implies $\boldsymbol{w}$ is the exact solution of time-harmonic problem (6.4) whose DG approximation is $\boldsymbol{u}_h$, which is also DG approximation of (6.1).
We decompose the error $\boldsymbol{e}$ between the exact solution $\boldsymbol{u}$ and DG approximation $\boldsymbol{u}_h$ of (6.1) as $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{u} - \boldsymbol{w} + \boldsymbol{w} - \boldsymbol{u}_h = \boldsymbol{\epsilon} - \boldsymbol{\rho}$ where $\boldsymbol{\epsilon} := \boldsymbol{w} - \boldsymbol{u}_h$ and $\boldsymbol{\rho} := \boldsymbol{w} - \boldsymbol{u}$.
Using the above property of time-harmonic reconstruction $\boldsymbol{w}$, we conclude that the error $\boldsymbol{w} - \boldsymbol{u}_h$, is the error of the DG method in V_h for the time-harmonic problem (6.4).
By using some suitable techniques, we show that the $\boldsymbol{e}$ satisfies an error relation and using this error relation and the testing procedure introduced by [BA76], the error estimation can be reduced to the estimation of the quantities involving only the $\boldsymbol{\epsilon} = \boldsymbol{w} - \boldsymbol{u}_h$ and the given data of the problem.
Hence, if the a posteriori error estimates for DG approximation already exist for the corresponding time-harmonic or stationary problem (6.4) in L^2-norm, the time-harmonic reconstruction technique allows us to get the a posteriori error estimates of the DG approximation for the corresponding time-dependent problem (6.1). Furthermore a suitable space-time reconstruction allows us to get the a posteriori error estimates for fully discere case. This technique can be adopted to get the a posteriori error estimates of many problems for DG approximation. In our work we have used this technique for Maxwell's problem. We refer to the work of [LU14] for elasticity problem with DG approximation, [GLM13] for wave equation with finite element method, [GLV11] for the parabolic problems with DG approximation.

CHAPTER 7

Mixed DG formulation for time-harmonic Maxwell's equations and a posteriori error estimation.

Contents

7.1 Model Problem

Let Ω be a bounded polyhedron in $\mathbb{R}^3$, with a Lipschitz boundary $\partial\Omega$. We further assume that Ω is simply connected and that $\partial\Omega$ is connected. We consider the time-harmonic Maxwell equations in a heterogeneous insulating medium

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \boldsymbol{u}) - \nabla p = \boldsymbol{j} & in \quad \Omega, \\ \nabla.\boldsymbol{u} = 0 & in \quad \Omega, \\ \boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0} & on \quad \partial\Omega, \\ p = 0 & on \quad \partial\Omega. \end{cases} \tag{7.1}$$

Here, the unknowns are the electric field $\boldsymbol{u}$ which is a vector field, and the Lagrange multiplier p which is a scalar field and is related to the divergence free constraint. $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. $\mu = \mu(\boldsymbol{x})$ is the magnetic permeability of the medium, that we assume to be a real function in $L^\infty(\Omega)$ and it satisfies

$$0 < \mu_* \leq \mu(\boldsymbol{x}) \leq \mu^* < \infty,$$

For simplicity, we assume that μ is piecewise constant with respect to a partition of the domain Ω. The right-hand side $\boldsymbol{j} \in L^2(\Omega)^3$ is an external source field, which is divergence free that leads to $p \equiv 0$.
Let $V = H_0(\text{curl}, \Omega) \cap \text{H}(\text{div}, \Omega)$ and $Q = H_0^1(\Omega)$
The variational formulation of model problem (7.1) reads:

find $\boldsymbol{u} \in V$ and $p \in Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= (\boldsymbol{j}, \boldsymbol{v}), \\ b(\boldsymbol{v}, q) &= 0. \end{aligned} \tag{7.2}$$

$\forall(\boldsymbol{v}, q) \in V \times Q$

The bilinear forms a and b are given by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\mu^{-1}\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{v})\text{dx}; \;\; b(\boldsymbol{v}, q) = -\int_\Omega \nabla q \cdot \boldsymbol{v}\text{dx}; \tag{7.3}$$

The bilinear form a is continuous and coercive on kernal of b. The bilinear form b is continuous, coercive and satisfies an inf-sup condition. Hence the variational formulation (7.2) is well-posed. See for details [VD99],[BF91].

7.2 Discontinuous Galerkin(DG) Formulation

Let $\mathscr{T}_h$, $h > 0$, be a family of partitions of Ω into tetrahedra defined in section 1.1.2. For an element κ of $\mathscr{T}_h$, we denote by $\mathbb{P}^k(\kappa)$ the space of polynomials of total degree at most $k, k \geq 0$, on κ. The generic hp-finite element space of piecewise polynomials is given by

$$\mathbb{P}^k(\mathscr{T}_h) = \{u \in L^2(\Omega) : u_{|K} \in \mathbb{P}^k(\kappa), \quad \forall \kappa \in \mathscr{T}_h\}.$$

We use the finite element spaces, $V_h = \mathbb{P}^p(\mathscr{T}_h)^3$, $Q_h = \mathbb{P}^{l-1}(\mathscr{T}_h)$. Where we have considered the most general case in which different approximation orders can be used for DG spaces V_h and Q_h.
In this section, we derive the mixed discontinuous Galerkin formulation to (7.1), and we discuss the well-posedness of this formulation.
Let $\kappa \in \mathscr{T}_h$ fixed and $\boldsymbol{n}_\kappa$ the unit normal vector to $\partial\kappa$. Multiplying the first equation of (7.1) by $\boldsymbol{v}_h \in V_h$ and integrate by parts on κ using Green theorem,

$$\int_\kappa (\mu^{-1}\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{v}_h)\text{dx} + \int_\kappa \nabla p\cdot\boldsymbol{v}_h\text{dx} - \int_{\partial\kappa} \boldsymbol{v}_h\cdot(\mu^{-1}\widehat{\nabla \times \boldsymbol{u}}\times\boldsymbol{n}_\kappa)\text{ds} = \int_\kappa \boldsymbol{j}\cdot\boldsymbol{v}_h\text{dx}. \tag{7.4}$$

Multiplying the second equation of (7.1) by $q_h \in Q_h$ and integrate on κ,

$$-\int_{\kappa} \nabla q_h \cdot \boldsymbol{u} \mathrm{dx} + \int_{\partial\kappa} q_h(\widehat{\boldsymbol{u}} \cdot \boldsymbol{n}_\kappa)\mathrm{ds} = 0, \tag{7.5}$$

where $\widehat{\nabla \times \boldsymbol{u}}$ and $\widehat{\boldsymbol{u}}$ are the numerical fluxes of $\nabla \times \boldsymbol{u}$ and $\boldsymbol{u}$ respectively. We define these numerical fluxes in the following section.

7.2.1 The numerical fluxes

For any given vector function $\boldsymbol{\phi} \in H^s(\mathscr{T}_h)^3$, with $s > \frac{1}{2}$, the numerical fluxes $\widehat{\boldsymbol{\phi}}$ are the elements of $L^2(\mathscr{F}_h)^3$, in particular, for any $\kappa \in \mathscr{T}_h$, the numerical fluxes $\widehat{\boldsymbol{\phi}_\kappa}$ are the approximation of $\boldsymbol{\phi}$ on the boundary of κ. We say that the numerical fluxes are conservative if they are single valued on $\partial\Omega$. Similarly we can define the numerical fluxes for scalar functions. If $\varrho \in H^s(\mathscr{T}_h)$ is a scalar function, the numerical fluxes $\widehat{\varrho}$ are in $L^2(\mathscr{F}_h)$.
We define the numerical fluxes conservative and consistant in the sense of Arnold [ABCM02] in the following. For any $\kappa \in \mathscr{T}_h$

$$\begin{aligned}
\widehat{\nabla \times \boldsymbol{u}} &= \{\!\{\nabla \times \boldsymbol{u}\}\!\} - a[\![\boldsymbol{u}]\!]_T \ \textit{on}\ \partial\kappa\backslash\partial\Omega,\\
\widehat{\boldsymbol{u}} &= \{\!\{\boldsymbol{u}\}\!\} - a_p[\![p]\!]_N \ \textit{on}\ \partial\kappa\backslash\partial\Omega,\\
\widehat{p} &= \{\!\{p\}\!\} - a[\![\boldsymbol{u}]\!]_T \ \textit{on}\ \partial\kappa\backslash\partial\Omega.
\end{aligned}$$

$$\begin{aligned}
\widehat{\nabla \times \boldsymbol{u}} &= \nabla \times \boldsymbol{u} - a\boldsymbol{n}_\kappa \times \boldsymbol{u} \ \textit{on}\ \partial\kappa \cap \partial\Omega,\\
\widehat{\boldsymbol{u}} &= \boldsymbol{u} - a_p p \boldsymbol{n}_\kappa \ \textit{on}\ \partial\kappa \cap \partial\Omega,\\
\widehat{p} &= 0 \ \partial\kappa \cap \partial\Omega.
\end{aligned}$$

We have the following two identities which hold true $\forall \boldsymbol{v}, \boldsymbol{w} \in TR(F_h)^3$, $\forall q \in TR(F_h)$

$$\sum_{\kappa\in\mathscr{T}_h} \int_{\partial\kappa} \boldsymbol{v}\cdot\boldsymbol{n}_\kappa\times\boldsymbol{w}\mathrm{ds} = -\sum_{\kappa\in\mathscr{T}_h} \int_{\partial\kappa} \boldsymbol{w}\cdot\boldsymbol{n}_\kappa\times\boldsymbol{v}\mathrm{ds} = -\int_{\mathscr{F}_h} [\![\boldsymbol{v}]\!]_T\{\!\{\boldsymbol{w}\}\!\}\mathrm{ds} + \int_{\mathscr{F}_h^I} [\![\boldsymbol{w}]\!]_T\{\!\{\boldsymbol{u}\}\!\}\mathrm{ds}. \tag{7.6}$$

$$\sum_{\kappa\in\mathscr{T}_h} \int_{\partial\kappa} (\boldsymbol{w}\cdot\boldsymbol{n}_\kappa)q\mathrm{ds} = \int_{\mathscr{F}_h^I} ([\![\boldsymbol{w}]\!]_N\{\!\{q\}\!\} + [\![q]\!]_N\{\!\{\boldsymbol{w}\}\!\})\mathrm{ds} + \int_{\mathscr{F}_h^B} q(\boldsymbol{w}.\boldsymbol{n}_\kappa)\mathrm{ds}. \tag{7.7}$$

Summing the equations (7.4) and (7.5)over all the elements $\kappa \in \mathscr{T}_h$, using the definitions of the numercal fluxes and the identities (7.6),(7.7) and the *fact that $(\boldsymbol{u}, p)$ is the solution to (7.1), we get*

$$\sum_{\kappa\in\mathscr{T}_h}\int_\kappa(\mu^{-1}\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{v}_h)\mathrm{dx}+\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\nabla p\cdot\boldsymbol{v}_h\mathrm{dx}$$
$$-\int_{\mathscr{F}_h}[\![\boldsymbol{v}_h]\!]_T\{\!\{\mu^{-1}\nabla\times\boldsymbol{u}\}\!\}\mathrm{ds}-\int_{\mathscr{F}_h}[\![\boldsymbol{u}_h]\!]_T\{\!\{\mu^{-1}\nabla\times\boldsymbol{v}\}\!\}\mathrm{ds}=\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\boldsymbol{j}\cdot\boldsymbol{v}_h\mathrm{dx}, \tag{7.8}$$

$$-\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\nabla q_h\cdot\boldsymbol{u}\mathrm{dx}+\int_{\mathscr{F}_h^I}[\![q_h]\!]_N\{\!\{\boldsymbol{u}\}\!\}\mathrm{ds}=0, \tag{7.9}$$

where we have introduced the term $\int_{\mathscr{F}_h}[\![\boldsymbol{u}_h]\!]_T\{\!\{\mu^{-1}\nabla\times\boldsymbol{v}\}\!\}\mathrm{ds}$ to the equation (7.8) to get a symmetric formulation this term vanishes for exact solution. Now we intoduce the following interior penalty stabilizing terms to the equation (7.8), these terms vanish for the exact solution.

$$J_u(\boldsymbol{u}_h,\boldsymbol{v}_h)=c_u\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\nabla\cdot\boldsymbol{u}_h\nabla\cdot\boldsymbol{v}_h\mathrm{dx}+\sum_{e\in\mathscr{F}_h}\int_e a[\![\boldsymbol{u}_h]\!]_T[\![\boldsymbol{v}_h]\!]_T\mathrm{ds}$$
$$+\sum_{e\in\mathscr{F}_h^I}\int_e a[\![\boldsymbol{u}_h]\!]_N[\![\boldsymbol{v}_h]\!]_N\mathrm{ds}. \tag{7.10}$$

In the similar way, we introduce the following interior penalty stabilisation terms to the equation (7.9).

$$J_p(p_h,q_h)=c_p\sum_{\kappa\in\mathscr{T}_h}\int_\kappa\nabla p_h\cdot\nabla q_h\mathrm{dx}+\sum_{e\in\mathscr{F}_h^I}\int_e a_p[\![q_h]\!]_N[\![p_h]\!]_N\mathrm{ds}. \tag{7.11}$$

Here $c_u>0$ and $c_p>0$ are constants independent of h. And a, a_p are interior penalty parameters defined by

$$a=\alpha h^{-1}\in L^2(F_h)\ \textit{and}\ a_p=\gamma h^{-1}\in L^2(F_h), \tag{7.12}$$

where α and γ are positif real constants. The mixed discontinuous Galerkin finite element formulation to (7.1) is now defined as follows:

find $(\boldsymbol{u}_h,p_h)\in V_h\times Q_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,p_h)=L_h(\boldsymbol{v}_h), & \forall\boldsymbol{v}_h\in V_h. \\ b_h(\boldsymbol{u}_h,q_h)-c_h(p_h,q_h)=0, & \forall q_h\in Q_h. \end{cases} \tag{7.13}$$

Where a_h, b_h, c_h and L_h are defined by

$$
\begin{aligned}
a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \mu^{-1} \nabla_h \times \boldsymbol{u}_h . \nabla_h \times \boldsymbol{v}_h \mathrm{dx} + c_u \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h \cdot \boldsymbol{u}_h . \nabla_h \cdot \boldsymbol{v}_h \mathrm{dx} \\
- \int_{\mathscr{F}_h} [\![\boldsymbol{u}_h]\!]_T \{\!\{\mu^{-1} \nabla_h \times \boldsymbol{v}_h\}\!\} \mathrm{ds} - \int_{\mathscr{F}_h} [\![\boldsymbol{v}_h]\!]_T \{\!\{\mu^{-1} \nabla_h \times \boldsymbol{u}_h\}\!\} \mathrm{ds} \\
+ \int_{\mathscr{F}_h} \mathrm{a} [\![\boldsymbol{u}_h]\!]_T [\![\boldsymbol{v}_h]\!]_T \mathrm{ds} + \int_{\mathscr{F}_h^I} \mathrm{a} [\![\boldsymbol{u}_h]\!]_N [\![\boldsymbol{v}_h]\!]_N \mathrm{ds}; \quad (7.14)
\end{aligned}
$$

$$
b_h(\boldsymbol{v}_h, p_h) = \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h p_h \cdot \boldsymbol{v}_h \mathrm{dx} - \int_{\mathscr{F}_h} [\![p_h]\!]_N \{\!\{\boldsymbol{v}_h\}\!\} \mathrm{ds}; \quad (7.15)
$$

$$
c_h(p_h, q_h) = c_p \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h p_h \cdot \nabla_h q_h \mathrm{dx} + \int_{\mathscr{F}_h} \mathrm{a}_p [\![p_h]\!]_N [\![q_h]\!]_N \mathrm{ds}; \quad (7.16)
$$

$$
L_h(\boldsymbol{v}_h) = \int_\Omega \boldsymbol{j} . \boldsymbol{v}_h \mathrm{dx}. (7.17)
$$

where ∇_h denotes the elementwise ∇ operator.

Remark 7.2.1. *If we compare our DG formulation to the one obtained in [HPS05], in our DG formulation two additional terms appear which are*

$$
c_p \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h p_h \cdot \nabla_h q_h \mathrm{dx} + c_u \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h \cdot \boldsymbol{u}_h . \nabla_h \cdot \boldsymbol{v}_h \mathrm{dx}, \quad (7.18)
$$

these terms are required for stability of our DG formulation. Moreover these terms allows us to consider the DG formulation with different approximation order for V_h and Q_h. In [HPS05] only equal order approximation spaces were considered. But if we consider the general case of different order approximation spaces for V_h and Q_h, these terms can not be neglected.

Now we will discuss the stability and consistency of the DG formulation (7.13). To this end, we define the DG norm as

$$
\|(\boldsymbol{v}, q)\|_{DG}^2 = \|\boldsymbol{v}\|_{V_h}^2 + \|q\|_{Q_h}^2. \quad (7.19)
$$

where

$$
\|\boldsymbol{v}\|_{V_h}^2 = \|\mu^{-\frac{1}{2}} \nabla_h \times \boldsymbol{v}\|_{0,\Omega}^2 + c_u \|\nabla_h . \boldsymbol{v}\|_{0,\Omega}^2 + \|\sqrt{\mathrm{a}} [\![\boldsymbol{v}]\!]_T\|_{0,\mathscr{F}_h}^2 + \|\sqrt{\mathrm{a}} [\![\boldsymbol{v}]\!]_N\|_{0,\mathscr{F}_h^I}^2. \quad (7.20)
$$

$$
\|q\|_{Q_h}^2 = \|\nabla_h q\|_{0,\Omega}^2 + \|\sqrt{\mathrm{a}_p} [\![q]\!]_N\|_{0,\mathscr{F}_h}^2. \quad (7.21)
$$

We have the following continuity and coercivity result.

Proposition 7.2.1. *1)There exists a constant $C_{\text{cont}} > 0$ such that*

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \leq C_{\text{cont}} \|\boldsymbol{u}_h\|_{V_h} \|\boldsymbol{v}_h\|_{V_h}, \tag{7.22}$$

$\forall \boldsymbol{u}_h, \boldsymbol{v}_h \in V_h$

2) There exists a parameter $\alpha_0 > 0$ independent of mesh size and μ, such that for the stabilisation parameter a defined by (7.12) with $\alpha \geq \alpha_0$ we have that

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq C_{\text{coer}} \|\boldsymbol{u}_h\|^2_{V_h}, \ \forall \boldsymbol{u}_h \in V_h, \tag{7.23}$$

where $C_{\text{coer}} > 0$ is a constant independent of mesh size.

Proof. : To prove this propostion we follow the same steps as in Lemma 2.2.1.

7.2.2 Existence and uniqueness

Now we will show that the DG formulation (7.13) possesses a unique solution.

Proposition 7.2.2. *For the stabilisation parameters a and a_p defined by (7.12), there exists a parameter $\alpha > 0$, such that for $\alpha \geq \alpha_0$ the mixed DG formulation (7.13) is uniquely solvable.*

Proof. To proof the unicity of the solution, we will show that if $j = 0$ in (7.13) then we have $(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{0}, \boldsymbol{0})$.
Taking $\boldsymbol{v}|_{V_h} = \boldsymbol{u}_h$ and $q|_{Q_h} = p_h$ in (7.13) and subtracting the second equation from first equation of (7.13) yeilds

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(p_h, p_h) = 0, \tag{7.24}$$

by the definitions of a_h and c_h, we get

$$\begin{aligned} &\sum_{\kappa \in \mathscr{T}_h} \int_\kappa (\mu^{-1} \nabla_h \times \boldsymbol{u}_h)^2 \mathrm{dx} + c_u \sum_{\kappa \in \mathscr{T}_h} \int_\kappa (\nabla_h \cdot \boldsymbol{u}_h)^2 \mathrm{dx} \\ &-2 \int_{\mathscr{F}_h} [\![\boldsymbol{u}_h]\!]_T \{\!\{\mu^{-1} \nabla_h \times \boldsymbol{u}_h\}\!\} \mathrm{ds} + \int_{\mathscr{F}_h} a([\![\boldsymbol{u}_h]\!]_T)^2 \mathrm{ds} + \int_{\mathscr{F}_h^I} a([\![\boldsymbol{u}_h]\!]_N)^2 \mathrm{ds} + c_p \sum_{\kappa \in \mathscr{T}_h} \int_\kappa (\nabla_h p_h)^2 \mathrm{dx} \\ &+ \int_{\mathscr{F}_h} a_p([\![p_h]\!]_N)^2 \mathrm{ds} = 0. \end{aligned} \tag{7.25}$$

For the third term on left hand side, we have by using the arithematic inequality and Cauchy Schwarz inequality,

$$2 \int_{\mathscr{F}_h} [\![\boldsymbol{u}_h]\!]_T \{\!\{\nabla_h \times \boldsymbol{u}_h\}\!\} \mathrm{ds} \leq 2\delta \int_{\mathscr{F}_h} a [\![\boldsymbol{u}_h]\!]^2_T \mathrm{ds} + \frac{2}{\delta} \int_{\mathscr{F}_h} \frac{1}{a} |\{\!\{\mu^{-1} \nabla_h \times \boldsymbol{u}_h\}\!\}|^2 \mathrm{ds}.$$

Now using the inverse inequality (1.10) and the definition of stabilisation parameter a, we have

$$\int_{\mathscr{F}_h} |\frac{1}{\sqrt{a}}\{\!\{\mu^{-1}\nabla_h \times \boldsymbol{u}_h\}\!\}|^2 \mathrm{ds} \leq \frac{C}{\alpha}\int_{\Omega} |\{\!\{\mu^{-1}\nabla_h \times \boldsymbol{u}_h\}\!\}|^2 \mathrm{dx},$$

hence, we have

$$\begin{aligned} a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)+c_h(p_h,p_h) \geq (1-\frac{2C}{\alpha\delta})\int_{\Omega}(\mu^{-1}\nabla_h \times \boldsymbol{u}_h)^2\mathrm{dx} + c_u\int_{\Omega}(\nabla_h\cdot\boldsymbol{u}_h)^2\mathrm{dx} \\ +(1-2\delta C)\int_{\mathscr{F}_h} a([\![\boldsymbol{u}_h]\!]_T)^2\mathrm{ds} + \int_{\mathscr{F}_h^I} a([\![\boldsymbol{u}_h]\!]_N)^2\mathrm{ds} \\ +c_p\sum_{\kappa\in\mathscr{T}_h}\int_{\kappa}(\nabla_h p_h)^2\mathrm{dx} + \int_{\mathscr{F}_h} a_p([\![p_h]\!]_N)^2\mathrm{ds} = 0. \end{aligned} \tag{7.26}$$

Thus, we have

$$\nabla_h \times \boldsymbol{u}_h = \mathbf{0},\ \nabla_h\cdot\boldsymbol{u}_h = 0 \text{ in } \Omega,\ [\![\boldsymbol{u}_h]\!]_T = 0 \text{ on } \mathscr{F}_h,\ [\![\boldsymbol{u}_h]\!]_N = 0 \text{ on } \mathscr{F}_h. \tag{7.27}$$

$$\nabla_h p_h = \mathbf{0} \text{ in } \Omega,\ [\![p_h]\!]_N \text{ on } \mathscr{F}_h \tag{7.28}$$

From (7.27), we have

$$\boldsymbol{u}_h \in H_0(\mathrm{curl}0,\Omega)\cap H_0(\mathrm{div}0,\Omega)$$

This implies that $\boldsymbol{u}_h = 0$ in Ω.
From (7.28), we conclude that $p_h \in H_0^1(\Omega)$ and

$$\nabla p_h = 0,$$

and since $p_h = 0$ on Γ, we get $p_h = 0$ on Ω.

□

7.3 A posteriori error estimation

We establish an a posteriori estimator for the error measured in terms of the energy norm $\|.\|_{DG}$.
First we state a result which we will use in the derivation of the a posteriori error.

Definition 7.3.1. *Let us denote the tangential component of the numerical flux* $\widehat{\mu^{-1}\nabla\times\boldsymbol{u}}$

$$\sigma_t(\boldsymbol{u}) = \boldsymbol{n}_\kappa \times (\mu^{-1}\nabla\times\boldsymbol{u}) \text{ on } \partial\kappa, \tag{7.29}$$

Now introducing the discrete numerical flux $\Sigma_t(\boldsymbol{u})$ of $\sigma_t(\boldsymbol{u})$, as follows

$$\Sigma_t(\boldsymbol{u}) = \begin{cases} \boldsymbol{n}_\kappa \times (\{\!\{\mu^{-1}\nabla \times \boldsymbol{u}\}\!\} - a[\![\boldsymbol{u}]\!]_T) & \text{on } \partial\kappa\backslash\partial\Omega, \\ \boldsymbol{n}_\kappa \times (\mu^{-1}\nabla \times \boldsymbol{u} - a\boldsymbol{n}_\kappa \times \boldsymbol{u}) & \text{on } \partial\kappa \cap \partial\Omega. \end{cases} \tag{7.30}$$

we get the elementwise conservation property

$$\int_{\partial\kappa} \Sigma_t(\boldsymbol{u}_h) + \int_{\partial\kappa\backslash\partial\Omega} a[\![\boldsymbol{u}_h]\!]_N + \int_\kappa \nabla_h p_h - \int_{\partial\kappa} [\![p_h]\!]_N - \int_\kappa \boldsymbol{j} = 0. \tag{7.31}$$

for all $\kappa \in \mathscr{T}_h$ by setting $\boldsymbol{v}_h$ equal to characteristic function ($\boldsymbol{v}_h = 1$ on κ and $\boldsymbol{v}_h = 0$ on $\Omega\backslash\kappa$) of κ in the first equation of (7.13). We will use this property in derivation of the a posteriori error.

For simplicity we assume that $\mu \equiv 1$ in the following.

Theorem 7.3.1. *Let $(\boldsymbol{u}, p)$ be the solution of (7.1), we assume that $\boldsymbol{j}$ is divergence free source field, so that $p \equiv 0$. Let $(\boldsymbol{u}_h, p_h)$ be the discontinuous Galerkin approximation obtained by (7.13). Then, there is a constant $C > 0$ independent of mesh size h, such that*

$$\| (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|_{DG} \leq C(\sum_{\kappa\in\mathscr{T}_h} \eta_\kappa^2)^{\frac{1}{2}},$$

where the element error indicator η_κ is given by

$$\begin{aligned} \eta_\kappa^2 = h_\kappa^{2s} \| \boldsymbol{j} - \nabla_h \times (\nabla_h \times \boldsymbol{u}_h) + \nabla_h p_h\|_{0,\kappa}^2 + h_\kappa^{2s-1} \| \Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h)\|_{0,\partial\kappa}^2 \\ + h_\kappa^{-1} \| [\![\boldsymbol{u}_h]\!]_T\|_{0,\partial\kappa}^2 + h_\kappa^{-1} \| [\![\boldsymbol{u}_h]\!]_N\|_{0,\partial\kappa\backslash\partial\Omega}^2 + c_u \| \nabla_h.\boldsymbol{u}_h\|_{0,\kappa}^2 \\ + \|\nabla_h p_h\|_{0,\kappa}^2 + h_\kappa^{-1} \| [\![p_h]\!]_N\|_{0,\partial\kappa}^2 . \end{aligned} \tag{7.32}$$

Proof. From the definition of the norm $\| .\|_{DG}$, the continuity of the tangential trace of u at the inter-element boundaries, and the fact that $p \equiv 0$, we have,

$$\begin{aligned} \| (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|_{DG} = \| \nabla_h \times (\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}^2 + c_u \| \nabla_h.\boldsymbol{u}_h\|_{0,\Omega} + \Big\| \sqrt{a}[\![\boldsymbol{u}_h]\!]_T\Big\|_{0,\mathscr{F}_h}^2 \\ + \Big\| \sqrt{a}[\![\boldsymbol{u}_h]\!]_N\Big\|_{0,\mathscr{F}_h^I}^2 + \| \nabla_h p_h\|_{0,\Omega}^2 + \Big\| \sqrt{a_p}[\![p_h]\!]_N\Big\|_{0,\mathscr{F}_h}^2 . \end{aligned} \tag{7.33}$$

We observe that in the above equation all the terms depend on the approximate solution of equation (7.13) except the first term on the right hand side. Hence, we need to estimate the term $\sum_{\kappa\in\mathscr{T}_h} \| \nabla_h \times (\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\kappa}^2$ in order to get an explicit a posteriori error bound. Let us denote $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$.
We follow the approach of [BHL03] and we write L^2-orthogonal decomposition

of elementwise flux $\nabla_h \times \boldsymbol{e} \in L^2(\kappa)^3, \forall \kappa \in \mathscr{T}_h$ as

$$\nabla_h \times \boldsymbol{e} = \nabla \phi + \nabla \times \boldsymbol{\chi}, \tag{7.34}$$

with $\phi \in H^1(\kappa) \setminus \mathbb{R}$ and $\boldsymbol{\chi} \in H_0(\text{curl};\kappa) \cap \text{H}(\text{div}0;\kappa), \forall \kappa \in \mathscr{T}_\text{h}$ with $\sum_{\kappa \in \mathscr{T}_h} \| \boldsymbol{\chi}\|_{\text{curl},\kappa} \leq C \sum_{\kappa \in \mathscr{T}_h} \| \nabla \times \boldsymbol{\chi}\|_{0,\kappa}$. From the embedding property $H_0(\text{curl};\kappa) \cap H(\text{div}0;\kappa) \hookrightarrow H^s(\kappa)^3, \forall \kappa \in \mathscr{T}_h$, we have that $\boldsymbol{\chi} \in H^s(\kappa)^3; \forall \kappa \in \mathscr{T}_h$ and satisfies $\sum_{\kappa \in \mathscr{T}_h} \| \boldsymbol{\chi}\|_{s,\kappa} \leq C \sum_{\kappa \in \mathscr{T}_h} \| \nabla \times \boldsymbol{\chi}\|_{0,\kappa}$. Therefore, the following stability estimate of the decomposition (7.34) holds

$$\sum_{\kappa \in \mathscr{T}_h} \| \nabla \phi\|_{0,\kappa}^2 + \sum_{\kappa \in \mathscr{T}_h} \| \boldsymbol{\chi}\|_{s,\kappa}^2 \leq C \sum_{\kappa \in \mathscr{T}_h} \| \nabla_h \times \boldsymbol{e}\|_{0,\kappa}^2 . \tag{7.35}$$

Using the above decomposition of $\boldsymbol{e}$, we have that

$$\sum_{\kappa \in \mathscr{T}_h} \| \nabla_h \times \boldsymbol{e}\|_{0,\kappa}^2 = \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h \times \boldsymbol{e}.\nabla \phi \text{dx} + \sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h \times \boldsymbol{e}.\nabla \times \boldsymbol{\chi} \text{dx} = I + II. \tag{7.36}$$

We first deal with term I: using the definition of $\boldsymbol{e}$, the smoothness of $\boldsymbol{u}$ and ϕ, integration by parts and the fact that $\boldsymbol{n} \times \boldsymbol{u} = 0$ on $\partial\kappa$, where $\boldsymbol{n}$ is unit outward normal vector on $\partial\kappa$, we have

$$\sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h \times \boldsymbol{e}.\nabla \phi \text{dx} = - \sum_{\kappa \in \mathscr{T}_h} \int_{\partial\kappa} \boldsymbol{n}_\kappa \times \boldsymbol{u}_h.\nabla \phi \text{ds}, \tag{7.37}$$

now if we fix $\boldsymbol{v} \in V_h \cap H_0(\text{curl};\Omega)$ we observe that

$$\sum_{\kappa \in \mathscr{T}_h} \int_{\partial\kappa} \boldsymbol{n}_\kappa \times \boldsymbol{v}.\nabla \phi \text{ds} = 0.$$

Hence, we have

$$\sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h \times \boldsymbol{e}.\nabla \phi \text{dx} = \sum_{\kappa \in \mathscr{T}_h} \int_{\partial\kappa} \boldsymbol{n}_\kappa \times (\boldsymbol{v} - \boldsymbol{u}_h).\nabla \phi \text{ds}, \tag{7.38}$$

$$\leq (\sum_{\kappa \in \mathscr{T}_h} h_\kappa^2 \| \nabla \phi\|_{0,\partial\kappa}^2)^{\frac{1}{2}} (\sum_{\kappa \in \mathscr{T}_h} h_\kappa^{-2} \| \boldsymbol{n}_\kappa \times (\boldsymbol{v} - \boldsymbol{u}_h)\|_{0,\partial\kappa}^2)^{\frac{1}{2}}. \tag{7.39}$$

Using the inverse estimate (1.10), we have

$$\sum_{\kappa \in \mathscr{T}_h} \int_\kappa \nabla_h \times \boldsymbol{e}.\nabla \phi \text{dx} \leq \sum_{\kappa \in \mathscr{T}_h} \| \nabla \phi\|_{0,\kappa} (\sum_{\kappa \in \mathscr{T}_h} h_\kappa^{-2} \| \boldsymbol{n}_\kappa \times (\boldsymbol{v} - \boldsymbol{u}_h)\|_{0,\partial\kappa}^2)^{\frac{1}{2}}, \tag{7.40}$$

together with (7.35), we get

$$\sum_{\kappa\in\mathscr{T}_h}\int_\kappa \nabla_h\times \boldsymbol{e}.\nabla\phi \mathrm{dx} \le \sum_{\kappa\in\mathscr{T}_h} \| \nabla_h\times\boldsymbol{e}\|_{0,\kappa} (\sum_{\kappa\in\mathscr{T}_h} h_\kappa^{-2} \| \boldsymbol{n}_\kappa\times(\boldsymbol{v}-\boldsymbol{u}_h)\|^2_{0,\partial\kappa})^{\frac{1}{2}}. \tag{7.41}$$

Inverse estimate and the discrete trace inequality, yields

$$\sum_{\kappa\in\mathscr{T}_h} h_\kappa^{-1}\| \boldsymbol{n}_\kappa\times(\boldsymbol{v}-\boldsymbol{u}_h)\|^2_{0,\partial\kappa} \le C\sum_{\kappa\in\mathscr{T}_h} h_\kappa^{-2}\| (\boldsymbol{v}-\boldsymbol{u}_h)\|^2_{0,\partial\kappa}, \tag{7.42}$$

$$\le C\sum_{\kappa\in\mathscr{T}_h}(h_\kappa^{-2}\| (\boldsymbol{v}-\boldsymbol{u}_h)\|^2_{0,} + \| \nabla_h\times(\boldsymbol{v}-\boldsymbol{u}_h)\|^2_{0,\kappa}). \tag{7.43}$$

Using the result stated in lemma 4.5.3, we have

$$\inf_{v\in V_h\cap H_0(curl;\Omega)}\sum_{\kappa\in\mathscr{T}_h}(h_\kappa^{-2}\| (\boldsymbol{v}-\boldsymbol{u}_h)\|^2_{0,\kappa}+\| \nabla_h\times(\boldsymbol{v}-\boldsymbol{u}_h)\|^2_{0,\kappa}) \le C\left\| h^{-1/2}[\![\boldsymbol{u}_h]\!]_T\right\|^2_{0,\mathscr{F}_h}. \tag{7.44}$$

As a consequence, we get

$$\sum_{\kappa\in\mathscr{T}_h}\int_\kappa \nabla_h\times\boldsymbol{e}.\nabla\phi\mathrm{dx} \le C\sum_{\kappa\in\mathscr{T}_h}\| \nabla_h\times\boldsymbol{e}\|_{0,\kappa}\Big(\sum_{\kappa\in\mathscr{T}_h} h_\kappa^{-1}\| [\![\boldsymbol{u}_h]\!]_T\|^2_{0,\mathscr{F}_h}\Big)^{\frac{1}{2}}. \tag{7.45}$$

In order to deal with term II, we first write $\boldsymbol{\chi}_h$ to denote the L_2-projection of $\boldsymbol{\chi}$ onto $P^0(\mathscr{T}_h)$, so that

$$\| \boldsymbol{\chi}-\boldsymbol{\chi}_h\|_{0,\kappa} + h_\kappa^{1/2}\| \boldsymbol{\chi}-\boldsymbol{\chi}_h\|_{0,\partial\kappa} \le Ch_\kappa^s\| \boldsymbol{\chi}\|_{s,\kappa}, \ \forall\kappa\in\mathscr{T}_h. \tag{7.46}$$

Therefore, replacing $\boldsymbol{\chi}$ by $\boldsymbol{\chi}-\boldsymbol{\chi}_h$ in term II, integrating by parts and using the fact that $\boldsymbol{u}$ satisfies (7.1) with $p\equiv 0$, we obtain

$$\begin{aligned}\sum_{\kappa\in\mathscr{T}_h}\int_\kappa \nabla_h\times\boldsymbol{e}.\nabla\times\boldsymbol{\chi}\mathrm{dx} &= \sum_{\kappa\in\mathscr{T}_h}\int_\kappa(\boldsymbol{j}-\nabla_h\times(\nabla_h\times\boldsymbol{u}_h)).(\boldsymbol{\chi}-\boldsymbol{\chi}_h)\mathrm{dx}\\ &-\sum_{\kappa\in\mathscr{T}_h} < \sigma_t(\boldsymbol{e}),\boldsymbol{\chi}>_{\partial\kappa} + \sum_{\kappa\in\mathscr{T}_h} < \sigma_t(\boldsymbol{e}),\boldsymbol{\chi}_h>_{\partial\kappa}.\end{aligned} \tag{7.47}$$

For the second term on right hand side of (7.47), using the continuity properties of $\boldsymbol{u}$ and $\boldsymbol{\chi}$, and the fact that the numerical fluxes $\Sigma_t(\boldsymbol{u}_h)$ are continuous across the edges, we have

$$\sum_{\kappa\in\mathscr{T}_h} < \sigma_t(\boldsymbol{u}),\boldsymbol{\chi}>_{\partial\kappa} = \sum_{\kappa\in\mathscr{T}_h} < \Sigma_t(\boldsymbol{u}_h),\boldsymbol{\chi}>_{\partial\kappa}.$$

Hence, we have

$$\sum_{\kappa\in\mathscr{T}_h} < \sigma_t(\boldsymbol{e}), \boldsymbol{\chi} >_{\partial\kappa} = \sum_{\kappa\in\mathscr{T}_h} < \Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h), \boldsymbol{\chi} >_{\partial\kappa} .$$

For the third term on the right hand side of (7.47), we have

$$\begin{aligned} \sum_{\kappa\in\mathscr{T}_h} < \sigma_t(\boldsymbol{e}), \boldsymbol{\chi}_h >_{\partial\kappa} &= \sum_{\kappa\in\mathscr{T}_h} < \sigma_t(\boldsymbol{u}) - \sigma_t(\boldsymbol{u}_h), \boldsymbol{\chi}_h >_{\partial\kappa} \\ = \Big(\sum_{\kappa\in\mathscr{T}_h} < \sigma_t(\boldsymbol{u}) - \Sigma_t(\boldsymbol{u}_h), \boldsymbol{\chi}_h >_{\partial\kappa} \Big) &+ \Big(\sum_{\kappa\in\mathscr{T}_h} < \Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h), \boldsymbol{\chi}_h >_{\partial\kappa} \Big) \\ = \sum_{\kappa\in\mathscr{T}_h} \Big(\int_{\partial\kappa\setminus\partial\Omega} a[\![\boldsymbol{u}_h]\!]_N \{\!\{\boldsymbol{\chi}_h\}\!\} \mathrm{ds} &+ \int_\kappa \nabla_h p_h \boldsymbol{\chi}_h \mathrm{dx} - \int_{\partial\kappa} [\![p_h]\!]_N \{\!\{\boldsymbol{\chi}_h\}\!\} \mathrm{ds} \Big) \\ &+ \Big(\sum_{\kappa\in\mathscr{T}_h} < \Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h), \boldsymbol{\chi}_h >_{\partial\kappa} \Big) \end{aligned} \tag{7.48}$$

where in the last equality we have used the elementwise conservation property (7.31) and the fact that from continuous problem, we have

$$\int_{\partial\kappa} \sigma_t(\boldsymbol{u}) \boldsymbol{\chi}_h \mathrm{ds} = \int_\kappa \boldsymbol{j} \boldsymbol{\chi}_h \mathrm{dx}.$$

Hence, we get

$$\begin{aligned} \sum_{\kappa\in\mathscr{T}_h} \int_\kappa \nabla_h \times \boldsymbol{e} . \nabla \times \boldsymbol{\chi} \mathrm{dx} = \sum_{\kappa\in\mathscr{T}_h} \int_\kappa (\boldsymbol{j} - \nabla_h \times (\nabla_h \times \boldsymbol{u}_h) + \nabla_h p_h).(\boldsymbol{\chi} - \boldsymbol{\chi}_h) \mathrm{dx} \\ - \sum_{\kappa\in\mathscr{T}_h} < \Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h), \boldsymbol{\chi} - \boldsymbol{\chi}_h >_{\partial\kappa} \\ + \sum_{\kappa\in\mathscr{T}_h} \Big(\int_{\partial\kappa\setminus\partial\Omega} a[\![\boldsymbol{u}_h]\!]_N \{\!\{\boldsymbol{\chi} - \boldsymbol{\chi}_h\}\!\} \mathrm{ds} - \int_{\partial\kappa} [\![p_h]\!]_N \{\!\{\boldsymbol{\chi} - \boldsymbol{\chi}_h\}\!\} \mathrm{ds} \Big), \end{aligned} \tag{7.49}$$

where we also have added and subtracted $\sum_{\kappa\in\mathscr{T}_h} \int_\kappa \nabla_h p_h . \boldsymbol{\chi} \mathrm{dx}$, integrated by parts and used the fact that $\nabla_h . \boldsymbol{\chi} = 0$. Finally using estimates (7.46) and (7.34), we obtain the following bound:

$$\begin{aligned} \sum_{\kappa\in\mathscr{T}_h} \int_\kappa \nabla_h \times \boldsymbol{e} . \nabla \times \boldsymbol{\chi} \mathrm{dx} \le C \Big(\sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s} \parallel \boldsymbol{j} - \nabla_h \times (\nabla_h \times \boldsymbol{u}_h) + \nabla_h p_h \|_{0,\kappa}^2 \\ + \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s-1} \|\Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h)\|_{0,\partial\kappa}^2 \\ + \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s-1} (\parallel [\![p_h]\!]_N \|_{0,\partial\kappa}^2 + h_\kappa^{-1} \parallel [\![\boldsymbol{u}_h]\!]_N \|_{0,\partial\kappa\setminus\partial\Omega}^2) \Big)^{\frac{1}{2}} \times \sum_{\kappa\in\mathscr{T}_h} \parallel \nabla_h \times \boldsymbol{e} \|_{0,\kappa} . \end{aligned} \tag{7.50}$$

Subsituting (7.45) and (7.50) into (7.36), we get

$$\sum_{\kappa\in\mathscr{T}_h} \|\nabla_h \times \boldsymbol{e}\|_{0,\kappa}^2 \leq C\big(\sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s} \|\boldsymbol{j} - \nabla_h \times (\nabla_h \times \boldsymbol{u}_h) + \nabla_h p_h\|_{0,\kappa}^2$$
$$+ \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s-1} \|\Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h)\|_{0,\partial\kappa}^2 + \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s-1} (\| [\![p_h]\!]_N\|_{0,\partial\kappa}^2 + h_\kappa^{-1} \| [\![\boldsymbol{u}_h]\!]_N\|_{0,\partial\kappa\setminus\partial\Omega}^2)$$
$$+ \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{-1} \| [\![\boldsymbol{u}_h]\!]_T\|_{0,\partial\kappa}^2 \big)^{1/2} \times \sum_{\kappa\in\mathscr{T}_h} \|\nabla_h \times \boldsymbol{e}\|_{0,\kappa}. \quad (7.51)$$

Now dividing throughout by $\sum_{\kappa\in\mathscr{T}_h} \| \nabla_h \times \boldsymbol{e}\|_{0,\kappa}$, we conclude that

$$\sum_{\kappa\in\mathscr{T}_h} \| \nabla_h \times \boldsymbol{e}\|_{0,\kappa} \leq C\big(\sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s} \| \boldsymbol{j} - \nabla_h \times (\nabla_h \times \boldsymbol{u}_h) + \nabla_h p_h\|_{0,\kappa}^2$$
$$+ \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s-1} \| \Sigma_t(\boldsymbol{u}_h) - \sigma_t(\boldsymbol{u}_h)\|_{0,\partial\kappa}^2 + \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{2s-1} (\| [\![p_h]\!]_N\|_{0,\partial\kappa}^2 + h_\kappa^{-1} \| [\![\boldsymbol{u}_h]\!]_N\|_{0,\partial\kappa\setminus\partial\Omega}^2)$$
$$+ \sum_{\kappa\in\mathscr{T}_h} h_\kappa^{-1} \| [\![\boldsymbol{u}_h]\!]_T\|_{0,\partial\kappa}^2 \big)^{1/2}. \quad (7.52)$$

□

CHAPTER 8

Conclusion and perspectives

8.1 Conclusion

In this thesis, we have studied discontinuous Galerkin method for the Maxwell's equations both in time harmonic and time dependent case. We have developped the *a priori* and *a posteriori* error estimates.

In first part, we have considered the time dependent Maxwell's equations and we have established an interior penalty DG formulation and we have derived a priori error estimates for this formulation. We carried out numerical analysis to validate these estimates obtained theoratically. Then, we have derived the *a posteriori* error bounds for semi-discrete and fully discrete formulation using *time-harmonic reconstruction* technique. For fully discrete scheme, we make use of the backward-Euler scheme and an appropriate space-time reconstruction.

In the end, we consider the time-harmonic Maxwell operator with lagrange multiplier and we developped mixed DG formulation for this problem. We proved the well-posedness and existence of unique solution. We derive a posteriori error estimates for our formulation using the technique of Helmoltz decomposition.

8.2 Perspectives

For the a posteriori error bounds the efficiency bounds are not presented. We will consider this in future work. Furthermore the a posteriori error bounds should allow us to predict the error at each simulation in time and in each part of the simulation domain (error localization), in order to adjust the calculation parameters during the simulation to improve the accuracy (adaptivity). So in the future, we will work out for an adaptive algorithm with numerical implementation of the proposed bounds.

Appendix A

Appendix A

A.1 Calculations

Firstly, we give the proof of property 5.8:

Proof.

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \mu^n(t)\mathrm{d}t &= k_n^{-1}(-3t^2 + (3t^{n-1} + 3t^n)t)\Big|_{t^{n-1}}^{t^n} \\ &= k_n^{-1}(-3(t^n)^2 - (t^{n-1})^2) + (3t^{n-1} + 3t^n)k_n) \\ &= -3(t^n + t^{n-1}) + 3t^{n-1} + 3t^n = 0.\end{aligned} \tag{A.1}$$

□

Lemma A.1.1. *(*Concerning the term $\boldsymbol{\epsilon}$*). The following formula holds*

$$A := \int_{t^{n-1}}^{t^n} (t^n - t)(3t - 2t^{n-1} - t^n)\mathrm{d}t = 0.$$

Proof. Indeed,

$$\begin{aligned}A &= \int_{t^{n-1}}^{t^n} (t^n - t)(3t - 3t^n + 2(t^n - t^{n-1}))\mathrm{d}t \\ &= \int_{t^{n-1}}^{t^n} (-3(t - t^n)^2 + 2(t^n - t)k_n)\mathrm{d}t \\ &= -(t - t^n)^3\Big|_{t^{n-1}}^{t^n} - (t - t^n)^2 k_n\Big|_{t^{n-1}}^{t^n} \\ &= (t^{n-1} - t^n)^3 + (t^{n-1} - t^n)^2 k_n = -k_n^3 + k_n^3 = 0.\end{aligned} \tag{A.2}$$

□

Lemma A.1.2. *(*Concerning the term $\partial_t \boldsymbol{\epsilon}$*). The following formula holds*

$$\max_{t \in [t^{n-1}, t^n]} (t - t^{n-1})(t^n - t)^2 = \frac{4k_n^3}{27}.$$

Proof. Set $M = (t - t^{n-1})(t^n - t)^2$, defined on $[t^{n-1}, t^n]$. We have that

$$M'(t) = (t^n - t)^2 + 2(t - t^n)(t - t^{n-1}) = (t - t^n)(3t - t^n - 2t^{n-1}),$$

this yields that $M(t)$ obtains its maximum value at $\bar{t} = \frac{t^n + 2t^{n-1}}{3}$

$$\begin{aligned} M(\bar{t}) &= \left(\frac{t^n + 2t^{n-1}}{3} - t^{n-1}\right)\left(t^n - \frac{t^n + 2t^{n-1}}{3}\right)^2 \\ &= \left(\frac{t^n - t^{n-1}}{3}\right)\left(\frac{2t^n - 2t^{n-1}}{3}\right)^2 = \frac{4k_n^3}{27}. \end{aligned} \tag{A.3}$$

□

A.2 Proof of lemma 5.3.1

In this section we carry out the proofs of proposition 5.3.1 in order to prove theorem 5.3.1. The proof is carried out for the wave equation in [GLM13], and here we have slightly modified for Maxwell's problem (vector valued function).

Proposition A.2.1. *(*Mesh change error estimate*). Under the assumptions of theorem 5.3.1 and with the notation in (5.18), we have*

$$\mathcal{I}_1(t^*) \leq \xi_{\mathrm{MC}}(t^*) \max_{0 \leq t \leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega}.$$

Proof. Observing that the projections $\boldsymbol{\Pi}_h^j$, $j = 1, \ldots, N$ commute with the time differentiation, we integrate by parts with respect to t, arriving at

$$\begin{aligned} \mathcal{I}_1(t^*) := &\sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} ((\boldsymbol{I} - \boldsymbol{\Pi}_h^j)\partial_t \boldsymbol{u}_N, \boldsymbol{\rho}_N)_\Omega \mathrm{d}t + \int_{t^{m-1}}^{t^*} \langle(\boldsymbol{I} - \boldsymbol{\Pi}_h^m)\partial_t \boldsymbol{u}_N, \boldsymbol{\rho}_N)_\Omega \mathrm{d}t \\ &+ \sum_{j=1}^{m-1} ((\boldsymbol{\Pi}_h^{j+1} - \boldsymbol{\Pi}_h^j)\partial \boldsymbol{u}_h^j, \hat{\boldsymbol{v}}(t^j))_\Omega - ((\boldsymbol{I} - \boldsymbol{\Pi}_h^0)\partial \boldsymbol{u}_h^0, \hat{\boldsymbol{v}}(0))_\Omega. \end{aligned} \tag{A.4}$$

The first two term on the right hand side of (A.4) are bounded by

$$\max_{0 \leq t \leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega} \left(\sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \|(\boldsymbol{I} - \boldsymbol{\Pi}_h^j)\partial_t \boldsymbol{u}_N\|_{0,\Omega} \mathrm{d}t + \int_{t^{m-1}}^{t^*} \|(\boldsymbol{I} - \boldsymbol{\Pi}_h^m)\partial_t \boldsymbol{u}_N\|_{0,\Omega} \mathrm{d}t\right). \tag{A.5}$$

Recalling the definition of $\hat{\boldsymbol{v}}$ and that of $\partial_t \boldsymbol{u}_N(t^j) = \partial \boldsymbol{u}_h^j$, $j = 0, 1, \ldots, N$, we can bound the last two terms on the right hand side of (A.4) by

$$\max_{0 \leq t \leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega} \left(\sum_{j=1}^{m-1} (t^* - t^j)\|(\boldsymbol{\Pi}_h^{j+1} - \boldsymbol{\Pi}_h^j)\partial \boldsymbol{u}_h^j\|_{0,\Omega} + t^*\|(\boldsymbol{I} - \boldsymbol{\Pi}_h^0)\partial \boldsymbol{u}_h^0\|_{0,\Omega}\right). \tag{A.6}$$

noting that $(\boldsymbol{I} - \boldsymbol{\Pi}_h^0)\partial \boldsymbol{u}_h^0 = 0$ because $\partial \boldsymbol{u}_h^0 \in \mathbf{V}_h^0$, we get the result. □

Proposition A.2.2. *(*Evolution error bound*). Under the assumptions of theorem 5.3.1 and with the notation in (5.18), we have*

$$\mathcal{I}_2(t^*) \leq \xi_{\text{evo}}(t^*) \max_{0\leq t\leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega}.$$

Proof. First we observe the identity

$$\boldsymbol{w}_N - \boldsymbol{w}^j = -(t^j - t)\partial \boldsymbol{w}^j + \Big(k_j^{-1}(t^j - t)^3 - (t^j - t^2)\Big)\partial^2 \boldsymbol{w}^j \tag{A.7}$$

on each $(t^{j-1}, t^j]$, $j = 2, \ldots, m$. Hence from definition 5.1.1, we deduce

$$b(\boldsymbol{w}_N - \boldsymbol{w}^j, \hat{\boldsymbol{v}}) = (-(t^j - t)\partial \boldsymbol{g}^j + \Big(k_j^{-1}(t^j - t)^3 - (t^j - t^2)\Big)\partial^2 \boldsymbol{g}^j, \hat{\boldsymbol{v}})_\Omega. \tag{A.8}$$

The integral of the first component in the inner product on the right-hand side of (A.8) with respect to t between t^{j-1} and t^j is then given by $\mathcal{G}^j$ in (5.12).

Hence integrate by parts on each interval $(t^{j-1}, t^j]$, $j = 1, \ldots, m$, we obtain

$$\mathcal{I}_2(t^*) = \sum_{j=1}^{m} \int_{t^{j-1}}^{t^j} (\mathcal{G}^j, \boldsymbol{\rho}_N)_\Omega \mathrm{d}t + \int_{t^{m-1}}^{t^*} (\mathcal{G}^m, \boldsymbol{\rho}_N)_\Omega \mathrm{d}t, \tag{A.9}$$

which now implies the result. Note that the choice of the constants $\boldsymbol{\gamma}_j$ as in (5.12) makes the family of $\mathcal{G}^j$ continuous on $t^j, j = 1, , N$, and we also have $\mathcal{G}(0) = \mathbf{0}$. □

Proposition A.2.3. *(*Data approximation error bound*). Under the assumptions of theorem 5.3.1 and with the notation 5.18, we have*

$$\mathcal{I}_3(t^*) \leq \xi_{\text{osc}}(t^*) \max_{0\leq t\leq T} \|\boldsymbol{\rho}_N(t)\|_{0,\Omega}. \tag{A.10}$$

Proof. We begin by observing the zero mean value of $\overline{\boldsymbol{f}}^j - \boldsymbol{f}$ on $[t^{j-1}, t^j]$ as follows

$$\int_{t^{j-1}}^{t^j} (\overline{\boldsymbol{f}}^j - \boldsymbol{f})\mathrm{d}t = 0 \tag{A.11}$$

for all $j = 1, \ldots, m-1$. Hence we have

$$\sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} (\overline{\boldsymbol{f}}^j - \boldsymbol{f}, \hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t = \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} (\overline{\boldsymbol{f}}^j - \boldsymbol{f}, \hat{\boldsymbol{v}}_N - \overline{\hat{\boldsymbol{v}}_N}^j)_\Omega \mathrm{d}t, \tag{A.12}$$

where $\overline{\hat{\boldsymbol{v}}_N}^j(\cdot) := k_j^{-1} \int_{t^{j-1}}^{t^j} \hat{\boldsymbol{v}}_N(t, \cdot)\mathrm{d}t$. Using the Friedrich-Poincaré inequality with respect to the variable t we have

$$\int_{t^{j-1}}^{t^j} \|\hat{\boldsymbol{v}}_N - \overline{\hat{\boldsymbol{v}}_N}^j\|_{0,\Omega}^2 \mathrm{d}t \leq \frac{k_j^2}{4\pi^2} \int_{t^{j-1}}^{t^j} \|\partial_t \hat{\boldsymbol{v}}_N\|_{0,\Omega}^2 \mathrm{d}t \tag{A.13}$$

and recalling that $\partial_t \hat{\boldsymbol{v}}_N = -\boldsymbol{\rho}_N$, we have

$$\begin{aligned}\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}(\overline{\boldsymbol{f}}^j-\boldsymbol{f},\hat{\boldsymbol{v}}_N)_\Omega \mathrm{d}t &\le \sum_{j=1}^{m-1}\left(\int_{t^{j-1}}^{t^j}\|\overline{\boldsymbol{f}}^j-\boldsymbol{f}\|^2_{0,\Omega \mathrm{d}t}\right)^{1/2}\left(\int_{t^{j-1}}^{t^j}\|\hat{\boldsymbol{v}}_N-\overline{\hat{\boldsymbol{v}}_N}\|^2_{0,\Omega}\mathrm{d}t\right)^{1/2}\\ &\le \frac{1}{2\pi}\sum_{j=1}^{m-1}\left(\int_{t^{j-1}}^{t^j}\|\overline{\boldsymbol{f}}^j-\boldsymbol{f}\|^2_{0,\Omega}\mathrm{d}t\right)^{1/2}\left(\int_{t^{j-1}}^{t^j}k_j^2\|\boldsymbol{\rho}_N\|^2_{0,\Omega}\mathrm{d}t\right)^{1/2}\\ &\le \frac{1}{2\pi}\sum_{j=1}^{m-1}\left(k_j^3\int_{t^{j-1}}^{t^j}\|\overline{\boldsymbol{f}}^j-\boldsymbol{f}\|^2_{0,\Omega}\mathrm{d}t\right)^{1/2}\max_{0\le t\le T}\|\boldsymbol{\rho}_N(t)\|_{0,\Omega}.\end{aligned} \tag{A.14}$$

For the remaining term in $\mathcal{I}_{\text{osc}}$, we first observe that

$$\int_{t^{m-1}}^{t^*}\|\hat{\boldsymbol{v}}_N\|^2_{0,\Omega}\mathrm{d}t \le \int_{t^{m-1}}^{t^*}k_m\int_t^{t^*}\|\boldsymbol{\rho}_N\|^2_{0,\Omega}\mathrm{d}s\mathrm{d}t \le k_m^3\max_{0\le s\le T}\|\boldsymbol{\rho}_N(s)\|^2_{0,\Omega}, \tag{A.15}$$

which implies

$$\int_{t^{m-1}}^{t^*}(\overline{\boldsymbol{f}}^m-\boldsymbol{f},\hat{\boldsymbol{v}}_N)_\Omega\mathrm{d}t \le \left(\int_{t^{m-1}}^{t^*}k_m^3\|\overline{\boldsymbol{f}}^m-\boldsymbol{f}\|^2_{0,\Omega}\mathrm{d}t\right)^{1/2}\max_{0\le t\le T}\|\boldsymbol{\rho}_N(t)\|_{0,\Omega}, \tag{A.16}$$

this gives the bound A.2.3.

□

Proposition A.2.4. *(*Time-reconstruction error bound*). Under the assumptions of theorem 5.3.1 and with the notation (5.18), we have*

$$\mathcal{I}_4(t^*) \le \xi_{\text{tr}}(t^*)\max_{0\le t\le T}\|\boldsymbol{\rho}_N(t)\|_{0,\Omega}. \tag{A.17}$$

Proof. Recalling the zero-mean value (5.8) and noting that $\partial^2\boldsymbol{u}_h^j$ is piecewise constant, and μ^j has zero mean value on $[t^{j-1},t^j]$, $j=1,\dots,n$, we have

$$\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\mu^j(\partial^2\boldsymbol{u}_h^j,\boldsymbol{\Pi}_h^j\hat{\boldsymbol{v}}_N)_\Omega\mathrm{d}t = \sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\mu^j(\partial^2\boldsymbol{u}_h^j,\boldsymbol{\Pi}_h^j(\hat{\boldsymbol{v}}_N-\overline{\hat{\boldsymbol{v}}_N}^j))_\Omega\mathrm{d}t, \tag{A.18}$$

where $\overline{\hat{\boldsymbol{v}}_N}^j(\cdot) := k_j^{-1}\int_{t^{j-1}}^{t^j}\hat{\boldsymbol{v}}_N(t,\cdot)\mathrm{d}t$. The projections $\boldsymbol{\Pi}_h^j$, $j=1,\dots,N$ commute with the time integration, we obtain

$$\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\mu^j(\partial^2\boldsymbol{u}_h^j,\boldsymbol{\Pi}_h^j(\hat{\boldsymbol{v}}_N-\overline{\hat{\boldsymbol{v}}_N}^j))_\Omega \mathrm{d}t$$
$$\leq \frac{1}{2\pi}\sum_{j=1}^{m-1}\left(\int_{t^{j-1}}^{t^j}\|\mu^j\partial^2\boldsymbol{u}_h^j\|_{0,\Omega}^2\mathrm{d}t\right)^{1/2}\left(\int_{t^{j-1}}^{t^j}k_j^2\ \|\boldsymbol{\Pi}_h^j\boldsymbol{\rho}_N\|_{0,\Omega}^2\mathrm{d}t\right)^{1/2}$$
$$\leq \frac{1}{2\pi}\sum_{j=1}^{m-1}\left(\int_{t^{j-1}}^{t^j}k_j^3\|\mu^j\partial^2\boldsymbol{u}_h^j\|_{0,\Omega}^2\mathrm{d}t\right)^{1/2}\max_{0\leq t\leq T}\|\boldsymbol{\rho}_N(t)\|_{0,\Omega}. \tag{A.19}$$

For the remaining term in $\mathcal{I}_4$, upon using an argument similar to (A.15), we have

$$\int_{t^{m-1}}^{t^*}(\mu^m\partial^2\boldsymbol{u}_h^m,\boldsymbol{\Pi}_h^m\hat{\boldsymbol{v}}_N)_\Omega\mathrm{d}t \leq \left(\int_{t^{m-1}}^{t^*}k_j^3\|\mu^m\partial^2\boldsymbol{u}_h^m\|_{0,\Omega}^2\mathrm{d}t\right)^{1/2}\max_{0\leq t\leq T}\|\boldsymbol{\rho}_N(t)\|_{0,\Omega}, \tag{A.20}$$

this completes the proof. □

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